A study on Three-way decision methods based on uncertain and irresolute information

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A Thesis submitted to the Department of Mathematics, Quaid-i-Azam University, Islamabad, in the partial fulfillment of the requirement for the degree of

Doctor of Philosophy in Mathematics

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A THESIS SUBMITTED IN THE PARTIAL FULFILLMENT OF THE REQUIREMENTS FOR THE DEGREE OF THE

DOCTOR OF PHILOSOPHY IN MATHEMATICS

We accept this thesis as conforming to the required standard

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Certificate of Approval

This is to certify that the research work presented in this thesis entitled **A study on Three-way decision methods based on uncertain and irresolute information** was conducted by Ms. Saima Mahnaz under the kind supervision of **Prof. Dr. Asif Ali**. No part of this thesis has been submitted anywhere else for any other degree. This thesis is submitted to the Department of Mathematics, Quaid-i-Azam University, Islamabad in partial fulfillment of the requirements for the degree of Doctor of Philosophy in field of Mathematics from Department of Mathematics, Quaid-i-Azam University Islamabad, Pakistan.

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Dedication

This thesis is dedicated to my beloved father (late), who always supported and encouraged me throughout my studies in his life, my husband Waqas Mumtaz who continually provided his moral, emotional, and financial support. And lastly, I dedicate this thesis to Almighty Allah. Thank you for the guidance, strength, power of mind and skills and giving us a healthy life.

Abstract

Three-way decision (TWD) theory is based on human cognition and plays a vital role in everyday multi-criteria decision-making (MCDM) problems based on uncertain and irresolute information. Conflict, being an instinctive feature of human civilizations, exists in diverse real-life problems. Three-way decisions are more advantageous to resolve conflict problems, and its structure correlates with conflict analysis naturally. The current study mainly aims to improve TWD analysis to address the complex issues more significantly. For this, we propose a three-way conflict study analysis from a trade-off view point by incorporating game theory with rough set data analysis approach. Further, we present a three-way conflict analysis model to deal with hesitant fuzzy information system (HFIS). To exemplify the viability of proposed techniques, we resolve some real-life problems including the Syrian conflict, Middle East conflict, and the development problem of Gansu province China. Besides, detailed result analysis and comparative study are also provided. Another novel contribution of this thesis is to define some generalized operation rules for T-spherical fuzzy numbers (T-SFNs) by using Frank t-norm and Frank t-conorm, which can make more provision of options for the decision-makers. Additionally, we originate some T-spherical fuzzy frank aggregation operators and investigate some basic characteristics of these operators. By proposing the entropy measure for T-spherical fuzzy information, we generate a possibility to obtain the unknown weights information of the criteria. Then it is further used for criteria weight determination in the proposed aggregation based MCDM and TWD models. To validate the potentiality of the suggested approaches, we consider two practical cases concerning the investment problem. Moreover, in-depth comparative study and sensitivity analysis are also delineated.

List of Abbreviations

MCDM	Multi-criteria decision making
TWD	Three-way decision
GTRS	Game-theoretic rough set
CS	Conflict space
IS	Information system
DTRS	Decision-theoretic rough sets
FS	Fuzzy set
HFE	Hesitant fuzzy element
HFN	Hesitant fuzzy number
HFS	Hesitant fuzzy set
HFIS	Hesitant fuzzy information system
PyFN	Pythagorean fuzzy number
PyFS	Pythagorean fuzzy set
PyFIS	Pythagorean fuzzy information system
HFLTS	Hesitant fuzzy linguistic term set

IFS	Intuitionistic fuzzy set
SFN	Spherical fuzzy number
T-SFNs	T-spherical fuzzy numbers
qROFS	q-rung orthopair fuzzy set
PFS	Picture fuzzy set
SFS	Spherical fuzzy set
T-SFS	T-spherical fuzzy set
T-SFFWA	T-spherical fuzzy Frank weighted average
T-SFFHA	T-spherical fuzzy Frank hybrid averaging
T-SFFOWA	T-spherical fuzzy Frank ordered weighted averaging
T-SFFWG	T-spherical fuzzy Frank weighted geometric
T-SFFOWG	T-spherical fuzzy Frank ordered weighted geometric
T-SFFHG	T-spherical fuzzy Frank hybrid geometric
T-SFEHIA	T-spherical fuzzy Einstein hybrid interactive averaging
T-SFEHIG	T-spherical fuzzy Einstein hybrid interactive geometric
TSFHHA	T-spherical fuzzy Hamacher averaging
TSFHHG	T-spherical fuzzy Hamacher geometric
SFNWA	Spherical fuzzy number weighted averaging
SFNWG	Spherical fuzzy number weighted geometric
PULWG	Probabilistic uncertain linguistic weighted geometric

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Chapter 1

Introduction

This chapter aims to present an overview of the background knowledge and motivations of current study. The novel contributions are also outlined.

1.1 Background knowledge and Motivations

1.1.1 Three-way conflict study based on game-theoretic rough set model

TWD theory is considered as one of the significant ways to analyse and solve the decision making problems under uncertainty. The essential idea of TWD was originally introduced in the framework of rough sets [1, 2]. Later, Yao developed TWD based on the partition of a whole into three pair-wise disjoint parts labeled as the positive, boundary and negative regions [3]. Further, these three regions enable us to produce different decision rules for acceptance, indecision, or delayed decision and rejection. In past decades, numerous authors have successfully applied it to several domains, which can be formulated as three-way recommendation system [4, 5], three-way concept learning [6, 7], three-way decision space [8, 9], three-way data classification [10, 11], three-way email spam filtering [12, 13], three-way fuzzy clustering [14], three-way multi-attribute decision-making [15], three-way government decisions [16], medical diagnosis [17] and three-way conflict analysis [18, 19].

Game-theoretic rough set (GTRS) model was initially established by Yao and Herbert [20–22] as a new extension of traditional rough sets. GTRS has proved to be more advantageous in solving three-way classification decision problems, specifically for MCDM problems where multiple evaluation functions or criteria are involved in the procedure of decision-making. GTRS have been successfully applied to numerous real-life problems like MCDM [23, 24], feature selection [25], the architecture of a web-based medical decision support system [26], analyzing the uncertainties of rough set regions for finding the suitable thresholds [27], recommended system to analyze the capacity of an intelligent component [28], determining three-way decision conditions for evaluation functions (measures) [29], three-way email spam filtering [31] and studying shadowed sets by utilizing a three-way trade-off viewpoint [30].

Conflict analysis [32, 33] is of undeniable importance in the practical as well as theoretical areas. Preliminary work about conflicts was done by Pawlak [32–34]. In literature, Pawlak's model is widely studied and several developments have been made to enable a far better understanding of conflicts. Stoyo et al. [35] described an alternative technique to manage conflicts by using the idea of the co-occurrence of parameters in soft set theory. Afterward, by emerging a matrix technique for examining conflict structure, Sun et al. [36] developed a conflict resolution study model founded on rough set theory over two universes. Lately, Lang et al. [37] presented a decision-theoretic rough set (DTRS) model of conflict resolution with incremental approaches to obtain the maximal coalition in dynamic ISs. Also, Lang et al. [38] studied conflict resolution with both rough set theory and formal concept analysis and designed a unified model that resolves conflict both quantitatively and qualitatively. Then, Sun et al. [39] modified the Pawlak model by employing the principle of TWD based on a probabilistic rough set over two universes. Further, Zhi et al. [40] developed a conflict study model based on one vote and three-way formal concept analysis by utilizing an inconsistency measure for inducing a partition of attributes set and formulated a three-way concept lattice to determine differences among several participants.

Despite of aforementioned notable studies, the research on three-way conflict resolution based on GTRS is still a blank. Taking into account this flaw of existing conflicts study, we designed a novel conflict study model based on GTRS. By constructing a game mechanism among all participants (players), computing payoffs of all strategies, and following equilibrium rules, the more realistic and accurate results are obtained. To further validate the proposed technique, some real-life conflict problems are resolved and a detailed comparative study analysis with existing models is provided.

1.1.2 Motivations for using GTRS for three-way conflict study

- Pawlak's model [33, 34], is rigid and possesses some certain limitations as with the strict condition of the threshold value 1/2, model is unable to deal the values in the distance matrix which are very close to 0,1 and 0.5. The complexities and imprecise data in different real-life scenarios would require adjustment in the bench-mark value 1/2. Thus, we should be flexible in our approach to decision making and carefully explore all possibilities.

- Most of existing models [36–38] are unilateral and use a single measure (aggregated difference of opinion) to evaluate the opinion of an object independently, without taking into account the opinion of other objects, which may lead to a wrong conclusion. Hence, there is a severe need to inspect the given informtion system (IS) more closely.

- In a general sense, some recently designed models [38–40] are advantageous in finding feasible sets of issues/attributes for a given conflict situation with minimum conflict or maximum collations for all the objects. Nevertheless, considering the opinion of an object in isolation to opinions of other objects could be troublesome in real world conflict situations.

- To determine the relative positions of two objects, in TWD models based on Bayesian risk decision-making model [41, 42], the simultaneous actions (conflict,alliance), (alliance,conflict), (neutral,conflict) and others are not taken into account for objects.

Therefore, there is a need for more careful study on the gains or losses of each object subject to their possible actions.

1.1.3 Contributions

Given the motivations mentioned above, the main novelties of the current study are delineated as follows:

- The proposed GTRS based model allows decision makers to separately set parameters for alliance, neutrality, and conflict based on their priorities in different circumstances. By constructing a game mechanism among all participants (players), computing payoffs of all strategies, and following equilibrium rules, the resulting outcomes are more realistic and accurate in classifying the objects in the conflict, neutral and allied sets.
- To address the deficiencies of the existing models, the proposed model explores all possibilities and is flexible in determining different threshold values relative to the complexities of real-life problems.
- In comparison to present techniques, the GTRS based proposed model adopts a bilateral approach. By considering the actions of all objects simultaneously, computing the respective gains (payoffs) accordingly for all possible actions, and submitting the difference of opinions on all issues, more accurate results are obtained.
- GTRS based game settings provide a compromise and trade-off mechanism for combining and balancing the differences between the opinions of different objects. In particular, we demonstrate that equilibrium analysis can be used to construct conflict, allied, and neutral sets.

  To enhance the supremacy of the provided approach, three real-life conflict problems are solved with the proposed model, and a comprehensive analysis with existing models is provided.

1.1.4 Three-way conflict study in hesitant fuzzy setting

To bargain with the fuzziness of assessment data, Zadeh [43] coined the idea of fuzzy set (FS), till date, numerous extensions of FSs have been provided for better depiction of the real world's imprecision. As a recent enhancement of FS, hesitant FS (HFSs) proposed by Torra and Narukawa [44, 45], have great capability to reduce the complexity of initiating the membership values for decision makers. HFSs are more resourceful to express the hesitant or irresolute attitudes of decision makers. Many researchers have applied HFS theory in many fields. For instance, Xia and Xu [46] and Xia et al. [47] derived several aggregation operators in the hesitant fuzzy framework and provided an approach to solve issues related to decision analysis. Xu and Xia [48] profoundly investigated the similarity measures of HFS established on the distance, correlation, and entropy, respectively. Rodrıguez et al. [49] provided the idea of the hesitant fuzzy linguistic term set (HFLTS), this new approach is used in several multi-criterion linguistic decision making problems. Afterward, for the sake of adjusting the foremost questionable and complicated environment, Zhu et al. [50] presented the idea of dual HFSs (DHFSs) composed of membership degree and non-membership degree. Additionally, Qian et al. [51] investigated generalized HFSs and their utilization in the problems associated with decision study support systems. Further, Zhu et al. [52] derived a ranking procedure for group decision analysis by using hesitant fuzzy preference relations. Then, Liang and Liu [53] examined a risk decision procedure founded on DTRS in the hesitant fuzzy environment.

HFS have been widely used in the recent past, to understand and resolve multi criteria TWD problems. By allowing decision makers to depict their opinions with multiple values instead of a single one, HFS theory provides an effective approach to reduce the complexity of decision analysis. Liu et al. [54] adopted dual hesitant fuzzy sets to develop a new TWD method founded on DTRS with the dual hesitant fuzzy setting. Qiao et al. [55] described the notion of hesitant relations and discussed its novel properties and applications in TWD. Moreover, Zhang et al. [56] examined multi-granularity TWD with adjustable hesitant fuzzy linguistic multi-granulation DTRS over two universes. The concept of tri-partition of set of agents or issues in three pair wise disjoint sets in conflict analysis is similar to the idea of TWD. Yao [57] initiated a three-way conflict resolution model based on reformulation of Pawlak model. This work bridges the gap between TWD theory and conflict analysis, and some noteworthy researches are published afterwards. Such as, G. Lang [58] explored the three-way group conflict study for the Pythagorean fuzzy IS (PyFIS). Soon after, Li et al. [59] put forward a three-way conflict analysis model founded on a triangular fuzzy IS (TFIS) and obtained a tri-partition of agents by determining the total attitude of agents to all issues. Bashir et al. [60] established a three-way conflict resolution model by using game theory and rough sets. Recently, Yi et al. [61] presented a three-way conflict analysis by formulating the alliance and conflict sets on a single issue and then on compound issues respectively, under hesitant fuzzy information setting. Most of existing techniques have some drawbacks as they failed to cope complex conflicts with multi-valued data sets. Till now, little research work has been done on conflict resolution based on hesitant fuzzy IS(HFIS), with agents' opinion as HFNs.

Our study aims to devise a novel three-way conflict study model under a hesitant fuzzy setting. Our research work contributes in three aspects. Firstly, we reset our initial IS to a HFIS. Secondly, we construct a three-way conflict analysis model based on HFIS that depicts agents' opinion as HFEs and loss functions as real numbers. In last, we drive three-way decision by utilising two different techniques, the first one is general method based on average of score functions and the second one is ranking method of possibility degree founded on a stochastic way. To enhance the validity and advantages of the proposed model, we use it to solve the Middle East conflict problem and a detailed comparison with existing literature is also outlined.

1.1.5 Motivations for studying conflicts under hesitant fuzzy setting

- Most of the existing conflict study models [34, 36, 37, 39] are not flexible in their approach and do not allow decision makers to have a certain viewpoint. Therefore, a more adjustable model is required to evaluate the complex hesitant attitudes of decision makers.

- Some recent studies, [58,59] used PyFSs and TFSs respectively, to describe the opinion of agents towards given issues. Both models have their advantages, but the scales employed to indicate the agents' opinions concerning agreement and disagreement are not self-sufficient. To analyze ambiguous and irresolute opinions of decision makers, a more flexible approach is required.
- Present conflict studies are not capable of modeling complex data sets based on both positive and negative values. To circumvent this concern, a more reasonable approach is required to investigate the complex hesitant setup of conflicts.

1.1.6 Contributions

To negate the challenges specified above, the main contributions of this work are listed in a nutshell below:

- To adress the flaws of existing models, our proposed model provides leverage to decision makers for submitting their opinions using values in a range from -1 to 1, instead of considering strict values and thus proves to be more effective to comprehend complex scenarios.
- To drive a three-way classification of objects, our proposed technique utilized the general idea based on the average of score functions and the second one, the ranking method of possibility degree founded on a stochastic way that examines all the possibilities more accurately.
- We construct conflict, neutral and allied sets of objects more precisely by using aggregated opinion functions based on HFEs and associated loss functions.

1.1.7 Application of T-spherical fuzzy Frank aggregation operators in multicriteria TWD problems

Owing to the uncertainty of decision data, many theoretical developments on FS including intuitionistic FS (IFS), Pythagorean FS (PyFS), q-rung orthopair FS (q-ROFS), Picture FS (PFS) have been put forward, till now [62–65]. But, there were still some difficulties; when a person provides such numbers in $[0, 1]$, the total of which exceeds the unit interval, the PFS is unable to handle it. Mahmood et al. [66] initiated spherical FS (SFS) for dealing with such difficulties by changing the PFS rule such that the sum of the squares of the truth, abstinence, and falsehood degrees is confined to $[0, 1]$. Compared to existing PFS, SFS is a more robust approach for dealing with complex and untrustworthy information in decision analysis. Furthermore, Mahmood et al. [66] changed the SFS condition to examine the theory of T-spherical FS (T-SFS) owing a requirement that the total of the t-powers of the truth, abstinence, and falsehood degrees is not greater than the unit interval. Due to its generalized structure, the T-SFS has been widely used and drawn more interest from many researchers [67–69].

The geometrical analysis of PFS, SFS, and T-SFS spaces is presented in Fig. 1.1.

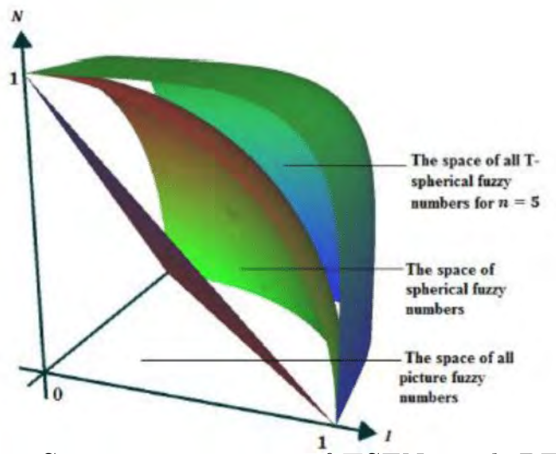

Figure 1.1: Spaces comparison of TSFNs with PFNs and SFNs

Aggregation operators play a key role in decision-making issues hence numerous scholars have made significant contributions to introducing aggregation operators for spherical fuzzy environments. Ashraf and Abdullah [70] presented some families of aggregation operators founded on Archimedean t-norm and t-conorm with spherical fuzzy information. Chinram et al. [71] worked on the uncertainty to probe the best power plant location in Pakistan using SFS and Yager aggregation operators. But, over time, it is noticed that these operators [70, 71] cannot model decision makers opinions when we obtain information in the form of a triplet like $(0.6, 0.7, 0.6\rangle$ where the sum of squares of truth, abstinence, and falsehood degrees is greater than 1, i.e., $0.6^2 + 0.7^2 + 0.6^2 \nleq 1$. To handle such issues, Garg et al. [72] studied some geometric aggregation operators founded on their developed T-spherical fuzzy operational rules. Quek and his coworkers [73] investigated some generalized T-spherical fuzzy weighted aggregation operators and then used them effectively to a problem related to the degree of pollution. Liu et al. [74] initiated the concept of normal T-SFNs and their relevant theory. They further put forward a normal T-spherical fuzzy Maclaurin symmetric mean operator and explored a novel MCDM approach. Guleria and Bajaj [75] presented several averaging and geometric aggregation operators for T-spherical fuzzy soft numbers. Based on their proposed operators, they provided an MCDM technique for handling complex problems regarding decision analysis.

The MCDM approach allows decision makers to rate alternatives according to a specific set based on criteria and then opt for the best one. While, TWD methods generate the possibility to classify the alternatives into three domains, namely, positive domain, boundary domain, and negative domain. Zeng et al. [76] highlighted the drawbacks of the existing T-spherical fuzzy Einstein aggregation operators and intuitionistic fuzzy operators and studied some novel T-spherical fuzzy Einstein aggregation operators along with their desired properties. Recently, Ullah et al. [77] combined the concept of T-SFNs and Hamacher aggregation operators. However, the aforementioned proposed operators still have several weak points and are not capable of solving complex decision problems. (Detailed discussion in Chapter 5, Section 5.5).

This study aims to explore some generalized operational rules of T-SFNs to develop T-spherical fuzzy aggregation operators that comply with the principles of Frank t-norm and t-conorm. Most of existing decision making models only rank the alternatives and classification of alternatives is not provided. We make contributions by establishing novel MCDM and TWD model based on the proposed Frank operators. MCDM technique rank the alternatives while TWD model also provides an effective approach to classify all alternatives in the complex decision making problems.

1.1.8 Motivations

- Existing aggregation operators [70, 71] still have several weak points and cannot model decision makers opinions for obtained information in the form of T-SFNs.
- Though several information fusion techniques [73–77] have been explored to aggregate spherical fuzzy data. Nevertheless, all these techniques are limited to algebraic, Einstein or Hamacher t-norm, and t-conorm, and some of them have certain limitations.
- Generally, these operators can only handle the MCDM problems expressed by PFNs or SFNs. These operators are not capable of dealing with complex MCDM and TWD problems with T-spherical fuzzy data sets.
- Frank sum and product are suitable substitutes of the algebraic, Einstein and Hamacher product for a union and intersection and can deliver an even assessment of the algebraic sum and product. In the present literature, there is no research on aggregation operators utilizing these operations on T-SFSs.
- The existing decision methods based on aggregation operators basically rank the alternatives and opt the best alternative, three-way classification of alternatives is not provided.

1.1.9 Contributions

The core contributions of the present study are summarized as follows:

- To explore novel generalized operational rules of T-SFNs by using beneficial characteristics of Frank t-norm and Frank t-conorm, which can provide more choices for the decision makers.
- To originate T-spherical fuzzy Frank arithmetic and geometric aggregation operators based on the proposed frank operations. Further, some basic properties like idempotency, monotonicity, boundedness, homogeneity and some limiting cases of these operators are also investigated.
- To propose the entropy measure for T-spherical fuzzy data sets, which can help to obtain the unknown weights information of the criteria.
- To develop MCDM model and TWD model based on the proposed T-spherical fuzzy operators to handle the T-spherical fuzzy decision problems with unknown weight information.

  To manifest the implementation of the suggested approaches, two practical investment problem are solved and detailed comparison with existing literature is provided.

1.2 Arrangement of the thesis

This thesis contains six chapters. Excluding the current chapter, other parts of the thesis are arranged as follows:

- Chapter 2 describes the general frame work and some basic concepts related of this dissertation, including TWD, Pawlak's conflict study model, GTRS model, HFSs, and T-SFSs and T-spherical fuzzy aggregation operators. These concepts will help to understand the core purpose of the research work presented in the current thesis.

- Chapter 3 introduces a conflict resolution model based on GTRS. The game settings and mechanism are explained in detail. GTRS model is employed to some real-life conflict problems to validate the proposed work and detailed comparative analysis with existing models is provided.

- Chapter 4 proposes a three-way conflict study model to manage the complexity of the hesitant fuzzy environment. We present the conflict, neutral, and alliance sets in hesitant fuzzy settings. A detailed comparison is done with the existing conflict study models under hesitant fuzzy framework to enhance the virtually of the presented method.

- Chapter 5 explores the novel generalized operational rules of T-SFNs to build T-spherical fuzzy aggregation operators that comply with the principles of Frank t-norm and t-conorm. Besides, a T-spherical fuzzy entropy measure is proposed along with detailed proof of its characteristics. Then, to handle complex decision-making problems, MCDM and TWD methods based on the proposed operators are established. Two real-life project investment cases are provided to elaborate on the implication of the suggested MCDM and TWD approaches. Finally, a detailed analysis of comparison is conducted with some existing approaches to highlight the feasibility and supremacy of the presented study.

- Chapter 6 concisely reviews some novel contributions and concludes the current study. Some possible future research directions are also discussed and summarised.

Chapter 2

Some related foundations

This chapter overviews some core fundamental concepts related to this thesis, namely, TWD theory, Pawlak model for conflict analysis, GTRS model, HFSs, T-SFSs, and T-spherical fuzzy aggregation operators, that are required to understand the work proposed in succeeding chapters.

2.1 Three-way decision theory

The idea of three-way decisions is initially outlined by Yao [1, 2]. The core concept of TWD theory is introduced as a trinal classification based on evaluation of a set of criteria. Consider U as a finite nonempty set of alternatives or objects and C is a nonempty finite set of concepts. Every concept belonging to C may be an idea, constraint, or criterion. The universal set U is divided into three parts that are pair-wise disjoint labelled as, acceptance, non-commitment and rejection regions denoted as $POS(C), BON(C)$ and $NEG(C)$ regions, respectively. TWD rules are induced by using these three regions. The decision for the inclusion of any object in a specific region is according to the degrees or levels to which objects meet the concept C.

- An object is considered in $POS(C)$ i.e, acceptance region of the criteria if the degree to which it meets the concept C is above to a certain level of acceptance.

- An object is considered in $NEG(C)$ i.e, rejection region of the criteria if the degree to which it meets the concept C is above a certain level of acceptance.
- An object is considered in $BON(C)$ i.e, boundary region of the criteria if the degree to which it meets the concept C is between the rejection level and acceptance level.

2.2 Pawlak's conflict analysis model

Pawlak [34] used an IS to get the initial opinions of decision makers.

Definition 2.2.1 *A set-based IS is defined as a quadruple, $S = (U, At, V_a, \eta_a)$, where finite non-empty set U consists of elements $k_1, k_2, ..., k_n$, At is a finite non-empty set of attributes, V_a is the nonempty set of values of attribute $a \in At$, and cardinality of V_a is always greater than one, η_a represents an information function for each attribute, from U to 2^{V_a}.*

For Pawlak's study of Middle East Conflict Model $V_a = \{-1, 0, 1\}$, thus $\eta_a(k)$ can take three values -1, 0 and 1 giving the opinion of object k as disagreement, neutrality and agreement, respectively, as shown in Table 2.1. The IS will be written as $IS = (U, A)$ in short.

Table 2.1: Information system for the Middle East conflict.

U	a_1	a_2	a_3	a_4	a_5
k_1	-1	$+1$	$+1$	$+1$	$+1$
k_2	$+1$	0	-1	-1	-1
k_3	$+1$	-1	-1	-1	0
k_4	0	-1	-1	0	-1
k_5	$+1$	-1	-1	-1	-1
k_6	0	$+1$	-1	0	$+1$

Example 2.2.1 *. [34]. The Middle East Conflict situation is represented in Table 2.1, where rows of the table depict the agents and columns depict the issues involved in the conflict.*
We denote $U = \{k_1, k_2, k_3, k_4, k_5, k_6\}$ as the universe of six agents, where
k_1: Israel k_2: Egypt k_3: Palestine k_4: Jordan k_5: Syria k_6: Saudi Arabia.

Let $A = \{a_1, a_2, a_3, a_4, a_5\}$ denotes the universe of five issues of the conflict problem, where:

a_1: refers to an Autonomous State In The West Bank And Gaza;

a_2: represents an Israeli military outpost along the Jordan River;

a_3: denotes Israel retaining East Jerusalem;

a_4: stands for Israeli military outposts on the Golan Heights;

a_5: represents Arab countries granting citizenship to Palestinians who choose to remain within their borders.

Definition 2.2.2 . *[34]. Consider $IS = (U, A)$, an auxiliary function denoted by $\varphi_a(k, l)$ for each attribute $a \in A$ is defined as follows:*

$$\varphi_a(k,l) = \begin{cases} 1, & \text{if } \eta_a(k).\eta_a(l) = 1 \vee k = l \ ; \\ 0, & \text{if } \eta_a(k).\eta_a(l) = 0 \wedge k \neq l; \\ -1, & \text{if } \eta_a(k).\eta_a(l) = -1. \end{cases} \tag{2.1}$$

When the auxiliary function $\varphi_a(k, l) = 1$, it means that the objects k and l have the same judgement about issue a; when the auxiliary function $\varphi_a(k, l) = 0$, it shows that at least one of the objects has a neutral judgement about issue a; and if the auxiliary function $\varphi_a(k, l) = -1$, then it means that objects k and l have different judgements on issue a.

Afterward, Pawlak suggests the idea of distance function for any two objects (k, l) for conflict situation as follows:

Definition 2.2.3 . *[34]. Consider $IS = (U, A)$, the distance function $\sigma_A(k, l)$ for any two objects $k, l \in U$, is defined as follows*

$$\sigma_A(k,l) = \frac{\sum_{a \in A \varphi_a^*(k,l)}}{|A|},$$

where

$$\varphi_a^*(k,l) = \frac{1-\varphi_a(k,l)}{2} = \begin{cases} 0, & \textit{if } \eta_a(k).\eta_a(l) = 1 \vee k = l \ ; \\ 0.5, & \textit{if } \eta_a(k).\eta_a(l) = 0 \wedge k \neq l \ ; \\ 1, & \textit{if } \eta_a(k).\eta_a(l) = -1. \end{cases}$$

Table 2.2: Distance matrix for the Middle East conflict.

U	k_1	k_2	k_3	k_4	k_5	k_6
k_1						
k_2	0.9					
k_3	0.9	0.2				
k_3	0.8	0.3	0.3			
k_5	1.0	0.1	0.1	0.2		
k_6	0.4	0.5	0.5	0.4	0.6	

Thus, by using Definition 2.2.3, we obtain the conflict space (CS), $CS = (U, \sigma_A)$, where σ_A represents the distance function. By using the distance function σ_A, Pawlak introduced the conflict, neutral and allied relations to study conflicts as follows:

Definition 2.2.4 . *[34]. Consider $CS = (U, \sigma_A)$, where $\sigma_A(k,l)$ represents the distance function, then:*

i. $(k,l) \in U$ is said to be conflict if $\sigma_A(k,l) > 0.5$;

ii. $(k,l) \in U$ is said to be neutral if $\sigma_A(k,l) = 0.5$;

iii. $(k,l) \in U$ is said to be allied if $\sigma_A(k,l) < 0.5$.

By using Definition 2.2.4, Pawalk introduced the conflict, neutral and allied sets as follows:

Definition 2.2.5 . *[34]. Consider $CS = (U, \sigma_A)$, the conflict, neutral and allied sets of $k \in U$ is defined as follows;*

i. $CON(k) = \{l \in U : \sigma_A(k,l) > 0.5\}$;

ii. $NEU(k) = \{l \in U : \sigma_A(k,l) = 0.5\}$;

iii. $ALL(k) = \{l \in U : \sigma_A(k,l) < 0.5\}$.

By using the Definition 2.2.5, pawlak formulated the conflict, neutral and allied sets of each object, in this way a relationship is defined between two objects.

2.3 Game-theoretic rough sets model

As a novel extension of traditional rough sets, the GTRS model combines the game theory with rough sets [20, 21]. These sets are widely used in decision analysis in the situation of conflict or cooperation. GTRS model proves to be more resourceful in solving different types of decision-making problems. GTRS provides a deeper and a wider perspective to assist in problem analysis by facilitating the set of possible outcomes. It lists both non-cooperative and cooperative outcomes by considering all possible actions or strategies and computing respective payoffs for each player. Whereas, equilibrium analysis can be used by scrutinizing payoffs. The model is used in many problems to establish a game among multiple agents for analyzing the interactive situation of conflict. A game mechanism is established among the agents to obtain an optimal solution by seeking dominance over the rest of the agents or trying to team up with others. GTRS model comprises a player's set, a set of strategies for each player, and a set of payoff functions for respective strategies.

The normal form also named as the strategic form is usually considered as the most formal and fundamental formulation of game theory. The mathematical expression of normal form of game theory as a tuple (P, S, F) [78], where:

- $P = \{p_1, p_2, ..., p_n\}$ is a finite set of n players;
- $S = \{s_1, s_2, s_3, ..., s_n\}$ is a strategy profile for player;
- $F = \{u_1, u_2, ..., u_n\}$, where $u_i : S \to \Re$ is a real-valued pay off function for n players.

A formal two-player GTRS based game is presented in Table 2.3. The two players are denoted as p_1 and p_2, respectively. In the game setting, each player has assigned a strategy profile $\{s_1, s_2, s_3\}$. In table, each block is congruous with a strategy profile comprising a pair of payoff values based on that strategy profile. Each block of the table gives the payoffs to both players for each combination of actions or strategies. To give an instance, for player p_1 strategy profile (s_1, s_1) containing payoffs functions $u_1(s_1, s_1)$ and $u_2(s_1, s_1)$ appears on the top left block, for player p_2 on the right. The game solution is a formal rule for predicting the best strategy profile in which the participants adopt their revered actions.

Table 2.3: Table of Payoffs for two-player GTRS based game

		player p_2		
		s_1	s_2	s_3
	s_1	$u_1(s_1, s_1), u_2(s_1, s_1)$	$u_1(s_1, s_2), u_2(s_1, s_2)$	$u_1(s_1, s_3), u_2(s_1, s_3)$
player p_1	s_2	$u_1(s_2, s_1), u_2(s_2, s_1)$	$u_1(s_2, s_2), u_2(s_2, s_2)$	$u_1(s_2, s_3), u_2(s_2, s_3)$
	s_3	$u_1(s_3, s_1), u_2(s_3, s_1)$	$u_1(s_3, s_2), u_2(s_3, s_2)$	$u_1(s_3, s_3), u_2(s_3, s_3)$

The Nash equilibrium is utilized by GTRS for analyzing the payoff tables to determine the possible outcomes of the games. In the game mechanism, the equilibrium point suggests that no player may obtain a better gain or payoff by switching to some other strategy while knowledge has been provided to him about the strategies followed by other players. Nash equilibrium is specifically employed for game solutions to opt for desirable game outcomes in GTRS. If for two players, s_i and s_j are the best responses to each other then the strategy profile (s_i, s_j) is a pure strategy or Nash equilibrium. Mathematically this equilibrium point can be expressed as [79]

$$\forall s'_i \in S_1, u_1(s_i, s_j) \geq u_1(s'_i, s_j) \text{ where } s_i \in S_1 \text{ and } s'_i \neq s_i \tag{2.2}$$

$$\forall s'_j \in S_2, u_1(s_i, s_j) \geq u_1(s_i, s'_j) \text{ where } s_j \in S_2 \text{ and } s'_j \neq s_j \tag{2.3}$$

Equations 2.2-2.3 may be presented as a strategy profile as no benefit can be secured by any player if he changes his respective strategy. Intuitively, a strategy profile is a stable state as no player can obtain a higher benefit from a unilateral deviation to some other strategy. Deviating from the strategies to increase the payoff of one player results in a decrease in the other player's payoff. To get the possible solution, equilibrium analysis may be used by examining payoff tables carefully. Subsequently, we may struggle to find a steady solution such that each player can achieve the highest payoff knowing about their opponent's chosen strategies. Equilibrium analysis is used to acquire the optimal solution for the conflict problem.

2.4 Hesitant fuzzy sets

The well-known concept of HFS is given by Torra [44, 45] and has been applied effectively to deal vagueness of real life. HFEs allow the membership degrees of elements to a set to be provided by various possible values between 0 and 1. The concepts of HFEs and a brief review of associated operations are provided in this section.

Definition 2.4.1 *Let Y be a fixed set then a HFS on Y is in terms of a function that when applied to Y returns a subset of $[0,1]$, which is mathematically denoted as follows:*

$$E = \{\langle s, \tau_E(s)\rangle | s \in S\}$$

where τ_E is a set of some values in $[0,1]$, which denotes the possible membership degrees of the element $s \in S$ to the set E [45, 46]. For comfort, we say τ_E as an HFE.

Definition 2.4.2 *Let three HFEs be denoted as τ, τ_1, and τ_2, and then some associated operations are expressed as follows:*

- $\tau^c = \bigcup_{\mu\in\tau}\{1-\mu\}$
- $\zeta\tau = \bigcup_{\mu\in\tau}\{1-(1-\mu)^\zeta\}$
- $\tau_1 \oplus \tau_2 = \bigcup_{\mu_1\in\tau_1,\mu_2\in\tau_2}\{\mu_1+\mu_2-\mu_1\mu_2\}$

- $\tau_1 \otimes \tau_2 = \bigcup_{\mu_1 \in \tau_1, \mu_2 \in \tau_2} \{\mu_1 \mu_2\}$

where ζ *is constant, For instance, if* $\tau = \mu, \tau_1 = \{\mu_1\}$ *and* $\tau_2 = \{\mu_2\}$ *then* $\tau^c = \{1 - \mu\}$; $\zeta\tau = \{1 - (1 - \mu)^\zeta\}; \tau_1 \oplus \tau_2 = \{\mu_1 + \mu_2 - \mu_1\mu_2\}; \tau_1 \otimes \tau_2 = \{\mu_1\mu_2\}$.

Xia and Xu [46] introduced a criterion to compare HFEs as follows:

Definition 2.4.3 *Let* τ *be a HFE, the score function of* τ *is denoted as* $s(\tau) = \frac{1}{l_\tau} \sum_{\mu \in \tau} \mu$, *here* $l(\tau)$ *indicates the number of elements in* τ*. For* τ_1 *and* τ_2*:*

if, $sc(\tau_1) < sc(\tau_2) \Rightarrow \tau_1 \prec \tau_2$*;*
if, $sc(\tau_1) = sc(\tau_2) \Rightarrow \tau_1 \approx \tau_2$*;*
if, $sc(\tau_1) \preceq sc(\tau_2) \Rightarrow \tau_1 \preceq h_2$*.*
Conversely,
if, $sc(\tau_1) > sc(\tau_2) \Rightarrow \tau_1 \succ \tau_2$*;*
if, $sc(\tau_1) \geq sc(\tau_2) \Rightarrow \tau_1 \succeq \tau_2$*.*

2.5 T-spherical fuzzy sets

The notion of T-SFS is propounded by Mahmood et al. [66] as a synthesis of SFS to offer a broader range of preferences for decision makers and enable them to express their hesitation about an alternative. Some basic definitions of T-SFS and terms relevant to planned work are delineated as follows:

Definition 2.5.1 *[66] Let* Y *be a given nonempty set. A T-spherical fuzzy set (SFS)* $\mathcal{S}$ *on* Y *is given by*

$$\mathcal{S} = \{(y, \sigma(y), \vartheta(y), \varrho(y)) \,|y \in Y\}, \tag{2.4}$$

where $\sigma(y)$*,* $\vartheta(y)$*,* $\varrho(y) \in [0, 1]$ *indicate the membership, neutral and non-membership grades of* $y \in Y$ *to the set* $\mathcal{S}$*, respectively, with the restriction that* $0 \leq \sigma^t(y) + \vartheta^t(y) + \varrho^t(y) \leq 1$*. The degree of refusal is* $\pi(y) = \sqrt[t]{1 - \sigma^t(y) - \vartheta^t(y) - \varrho^t(y)}$*. For convince,* $(\sigma(y), \vartheta(y), \varrho(y))$

is called a T-spherical fuzzy number (T-SFN), labelled by $\mathcal{S} = (\sigma, \vartheta, \varrho)$.

Definition 2.5.2 *[80] Let* $\mathcal{S}_1 = (\sigma_1, \vartheta_1, \varrho_1)$ *and* $\mathcal{S}_2 = (\sigma_2, \vartheta_2, \varrho_2)$ *be two T-SFNs and* $\eta > 0$, *then*

i. $\mathcal{S}_1 \oplus \mathcal{S}_2 = \left(\sqrt[t]{\sigma_1^t + \sigma_2^t - \sigma_1^t \sigma_2^t}, \vartheta_1 \vartheta_2, \varrho_1 \varrho_2 \right);$

ii. $\mathcal{S}_1 \otimes \mathcal{S}_2 = \left(\sigma_1 \sigma_2, \sqrt[t]{\vartheta_1^t + \vartheta_2^t - \vartheta_1^t \vartheta_2^t}, \sqrt[t]{\varrho_1^t + \varrho_2^t - \varrho_1^t \varrho_2^t} \right);$

iii. $\mathcal{S}_1^\eta = \left(\sigma_1^\eta, \sqrt[t]{1 - (1 - \vartheta_1^t)^\eta}, \sqrt[t]{1 - (1 - \varrho_1^t)^\eta} \right);$

iv. $\eta \mathcal{S}_1 = \left(\sqrt[t]{1 - (1 - \sigma_1^t)^\eta}, \vartheta_1^\eta, \varrho_1^\eta \right);$

v. $\mathcal{S}_1^c = (\varrho_1, \vartheta_1, \sigma_1)$.

Definition 2.5.3 *[66, 81]* $\mathcal{S}_1 = (\sigma_1, \vartheta_1, \varrho_1)$ *and* $\mathcal{S}_2 = (\sigma_2, \vartheta_2, \varrho_2)$ *be any two T-SFNs, let* $S(\mathcal{S}_1) = \sigma_1^t - \vartheta_1^t - \varrho_1^t + \left(\frac{\exp^{\sigma_1^t - \vartheta_1^t - \varrho_1^t}}{\exp^{\sigma_1^t - \vartheta_1^t - \varrho_1^t} + 1} - \frac{1}{2} \right) \pi^t$ *and* $S(\mathcal{S}_2) = \sigma_2^t - \vartheta_2^t - \varrho_2^t + \left(\frac{\exp^{\sigma_2^t - \vartheta_2^t - \varrho_2^t}}{\exp^{\sigma_2^t - \vartheta_2^t - \varrho_2^t} + 1} - \frac{1}{2} \right) \pi^t$ *be the score values of* $\mathcal{S}_1$ *and* $\mathcal{S}_2$, *respectively, and let* $A(\mathcal{S}_1) = \sigma_1^t + \vartheta_1^t + \varrho_1^t$ *and* $A(\mathcal{S}_2) = \sigma_2^t + \vartheta_2^t + \varrho_2^t$ *be the accuracy values of* $\mathcal{S}_1$ *and* $\mathcal{S}_2$, *respectively. Then,*

i. If $S(\mathcal{S}_1) < S(\mathcal{S}_1)$, *then* $\mathcal{S}_1 < \mathcal{S}_2$;

ii. If $S(\mathcal{S}_1) = S(\mathcal{S}_1)$, *then*

a. *If* $A(\mathcal{S}_1) < A(\mathcal{S}_1)$, *then* $\mathcal{S}_1 < \mathcal{S}_2$;

b. *If* $A(\mathcal{S}_1) = A(\mathcal{S}_2)$, *then* $\mathcal{S}_1 = \mathcal{S}_2$.

2.6 T-spherical fuzzy aggregation operators

As an important tool in information fusion, the T-spherical fuzzy aggregation operator has received much attention, Mahmood et al. [82] propounded the T-SFWA operator and the T-SFWG operator as follows:

Definition 2.6.1 *[82] Let* $\mathcal{S}_i\,(i = 1, 2, ..., n)$ *be a collection of SFNs, then the T-SFWA* *operator is a mapping* $\mathcal{S}^n \longrightarrow \mathcal{S}$ *such that*

$$\begin{aligned} &T-SFWA\left(\mathcal{S}_1,\mathcal{S}_2,...,\mathcal{S}_n\right)=w_1\mathcal{S}_1\oplus w_2\mathcal{S}_2\oplus...\oplus w_n\mathcal{S}_n \\ &=\left(\left(1-\prod_{k=1}^{n}\left(1-\sigma_k^t\right)^{w_k}\right)^{1/t},\prod_{k=1}^{n}\left(\vartheta_k\right)^{w_k},\prod_{k=1}^{n}\left(\varrho_k\right)^{w_k}\right), \end{aligned} \tag{2.5}$$

where $w=\{w_1,w_2,...,w_n\}^T$ *is the weight vector of* $(\mathcal{S}_1,\mathcal{S}_2,...,\mathcal{S}_n)$ *such that* $0\leq w_i\leq 1$ *and* $\sum_{i=1}^{n} w_i = 1$.

Definition 2.6.2 *[82] Let* $\mathcal{S}_i\,(i=1,2,...,n)$ *be a collection of SFNs, then the T-spherical fuzzy weighted geometric(T-SFWG) operator is a mapping* $\mathcal{S}^n \longrightarrow \mathcal{S}$ *such that*

$$\begin{aligned} &T-SFWG\left(\mathcal{S}_1,\mathcal{S}_2,...,\mathcal{S}_n\right)=w_1\mathcal{S}_1\otimes w_2\mathcal{S}_2\otimes...\otimes w_n\mathcal{S}_n \\ &=\left(\prod_{k=1}^{n}\left(\sigma_k\right)^{w_k},\left(1-\prod_{k=1}^{n}\left(1-\vartheta_k^t\right)^{w_k}\right)^{1/t},\left(1-\prod_{k=1}^{n}\left(1-\varrho_k^t\right)^{w_k}\right)^{1/t}\right), \end{aligned} \tag{2.6}$$

where $w=(w_1,w_2,...,w_n)^T$ *is the weight vector of* $(\mathcal{S}_1,\mathcal{S}_2,...,\mathcal{S}_n)$ *such that* $0\leq w_i\leq 1$ *and* $\sum_{i=1}^{n} w_i = 1$.

The study of triangular norms is extensive, initiating from Zadeh presented max and min operation as a pair of the triangular norm and triangular conorm. We can make reference to several triangular norms and corresponding triangular conorms, like product t-norm and probabilistic sum t-conorm [83], Einstein t-norm and t-conorm [84], Lukasiewicz t-norm and t-conorm [85], Hamacher t-norm and t-conorm [86] etc., are vehicles for operations on FSs. Frank operations include Frank's product and Frank's sum, which are examples of triangular norms and triangular conorms, respectively.

Frank t-norm T_F and Frank t-conorm S_F are defined as follows.

$$T_F\left(y_1,y_2\right)=\log_\tau\left(1+\frac{(\tau^{y_1}-1)(\tau^{y_2}-1)}{\tau-1}\right)\ \ \forall\ \ (y_1,y_2)\in[0,1]^2,$$

$$S_F\left(y_1,y_2\right)=1-\log_\tau\left(1+\frac{\left(\tau^{1-y_1}-1\right)\left(\tau^{1-y_2}-1\right)}{\tau-1}\right)\ \ \forall\ \ (y_1,y_2)\in[0,1]^2.$$

Frank t-norm and Frank t-conorm have the properties described as follows [87].

$T_F(y_1, y_2) + S_F(y_1, y_2) = y_1 + y_2$, $\frac{\partial T_F(y_1,y_2)}{\partial y_1} + \frac{\partial S_F(y_1,y_2)}{\partial y_1} = 1$.

Using the limit theory, one can easily verify the following desirable results [87].

1). If $\tau \longrightarrow 1$, then $T_F(y_1, y_2) \longrightarrow y_1 + y_2 - y_1 y_2$, $S_F(y_1, y_2) \longrightarrow y_1 y_2$, the Frank t-norm and Frank t-conorm are reduced to probabilistic product and probabilistic sum.

2). If $\tau \longrightarrow \infty$, then $T_F(y_1, y_2) \longrightarrow \min(y_1 + y_2, 1)$, $S_F(y_1, y_2) \longrightarrow \max(0, y_1 + y_2 - 1)$, the Frank t-norm and Frank t-conorm are reduced to the Lukasiewicz product and Lukasiewicz sum, respectively.

Chapter 3

Conflict resolution using game theory and rough sets

The core aim of the current chapter is to address the real-life conflict problems with the GTRS approach. Conflicts occur naturally in the real world at all levels of society, individually, in groups, or society as a whole. Almost all the existing conflict resolution models are unilateral in their decision-making process. They do not consider the actions of the involved parties simultaneously. Therefore, in this chapter, a novel three-way conflict resolution model is presented which is founded on GTRS, by formulating a game mechanism among all the concerned parties(players), computing the payoff of different strategies, and classifying them following equilibrium rules. The proposed model yields more realistic and accurate results as it explores all possibilities and is flexible in determining different threshold values relative to the complexities of real-life problems. Three real-life conflict situations are solved with the proposed model, and a comprehensive analysis is presented to validate the practicality of the proposed method. The research work presented in this chapter is published in [60].

3.1 A novel conflict study model based on GTRS

This section presents the novel GTRS based conflict study model. GTRS is more efficient in determining the conflict, alliance, or neutrality between two objects by playing a game between them because all the possibilities are explored by the calculating payoff of all strategies of conflict, alliance, and neutrality. The payoff functions are designed based on initial opinions of objects/agents on given issues/attributes.

3.1.1 Components of GTRS model

Let $IS = (U, A)$ be an IS describing the opinions of objects/agents in $U = \{k_1, k_2, k_3, ..., k_n\}$ regarding given issues/attributes $A = \{a_1, a_2, ..., a_m\}$, as shown in Table 2.1.

The Players Set: U is the set of players and game is played among all the players ($k_1, k_2, k_3, ...,$
k_n) taking two at a time. Each player will participate in a game in an effective way to get a final decision for the formulation of conflict, neutral and allied sets objects.

The Strategy Set: $S = \{s_{all}, s_{neu}, s_{con}\}$ be the set of strategies/actions for each player in U, where:

s_{all} is a strategy of making alliance with the other player;

s_{neu} is a strategy of abstaining in making any decision of making alliance or conflict with the other player;

s_{con} is a strategy of making conflict with the other player.

The Payoff Functions: Consider two players $k_r, k_t \in U$, the utility functions denoted u_{k_r} and u_{k_t} for k_r and k_t, respectively, measure the outcomes of opting a certain action/strategy in S by players. The two payoff sets are formulated as follows:

$$u_{k_r} = \{u_{k_r}(s_{all}, s_{all}), u_{k_r}(s_{all}, s_{neu}), u_{k_r}(s_{all}, s_{con}), u_{k_r}(s_{neu}, s_{all}), u_{k_r}(s_{neu}, s_{neu}),$$

$u_{k_r}(s_{neu}, s_{con}), u_{k_r}(s_{con}, s_{all}), u_{k_r}(s_{con}, s_{neu}), u_{k_r}(s_{con}, s_{con})\}.$

$u_{k_t} = \{u_{k_t}(s_{all}, s_{all}), u_{k_t}(s_{all}, s_{neu}), u_{k_t}(s_{all}, s_{con}), u_{k_t}(s_{neu}, s_{all}), u_{k_t}(s_{neu}, s_{neu}),$

$u_{k_t}(s_{neu}, s_{con}), u_{k_t}(s_{con}, s_{all}), u_{k_t}(s_{con}, s_{neu}), u_{k_t}(s_{con}, s_{con})\}.$

3.1.2 Game formulation

Now, the payoff/utility functions are designed that lead to a final decision for each player. Let k_r and k_t be two players in U. First, the values of auxiliary functions $\varphi_a(k_r, k_t)$ are calculated for all $a \in A$ according to Equation 2.1. Using these functions, cumulative degree of agreement denoted as $CO_A(k_r, k_t)$ is obtained as the sum of auxiliary functions with value +1 *i.e.* $\varphi_a^+(k_r, k_t)$ and cumulative degree of neutrality denoted as $CO_N(k_r, k_t)$ is computed as the sum of auxiliary functions with value 0 *i.e.* $\varphi_a^0(k_r, k_t)$. Similarly, cumulative degree of disagreement between k_r and k_t denoted as $CO_D(k_r, k_t)$ is obtained as the sum of auxiliary functions with value −1 *i.e.* $\varphi_a^-(k_r, k_t)$. Based on these cumulative degrees, normalized/mean cumulative degrees of agreement (alliance), neutrality and disagreement (conflict) are defined, respectively, as follows:

$$\overline{CO}_A(k_r, k_t) = \frac{CO_A(k_r, k_t)}{m} = \frac{1}{m}\sum_{a\in K}\varphi_a^+(k_r, k_t); \tag{3.1}$$

$$\overline{CO}_N(k_r, k_t) = \frac{CO_A(k_r, k_t)}{m} = \frac{1}{m}\sum_{a\in K}\varphi_a^0(k_r, k_t); \tag{3.2}$$

$$\overline{CO}_D(k_r, k_t) = \frac{CO_A(k_r, k_t)}{m} = \frac{1}{m}\sum_{a\in K}\varphi_a^-(k_r, k_t). \tag{3.3}$$

For decision making, decision-theoretic rough set model (DTRS) used probabilities [41, 42]. A pair of threshold values is used to represent the required level of precision for the classification of objects [1]. In contrast to DTRS, we used normalised/mean cumulative degrees of three involved measures *i.e.* agreement, neutrality and disagreement instead of probabilities. The pair of threshold values can be determined by Bayesian decision-theoretic analysis using notions of risks and losses [88]. This threshold pair helps to obtain a three-way classification of objects effectively. Intuitively, an object is classified into a particular concept or set, if its conditional probability within the bounds of the set is equal to or greater than threshold α. The object will be rejected for the concept or set if the conditional probability is equal to or less than the threshold β. When the conditional probability remains between the two thresholds α and β, then a final decision for the object can not be made and hence abstained or delayed. Thus, three pairs of threshold values (α_A, β_A), (α_N, β_N) and (α_D, β_D) are decided by decision makers based on how much leverage they want to give to make alliance, neutrality and conflict, respectively. For players k_r and k_t in U, the preliminary weighted payoff denoted by ω_{k_r,k_t} for all the strategies is defined as follows:

$$\omega_{k_r,k_t}(s_{all}) = \begin{cases} 1, & \overline{CO}_A(k_r, k_t) \geq \alpha_A \\ 0.5, & \beta_A < \overline{CO}_A(k_r, k_t) < \alpha_A \\ 0, & \overline{CO}_A(k_r, k_t) \leq \beta_A \end{cases} \tag{3.4}$$

$$\omega_{k_r,k_t}(s_{neu}) = \begin{cases} 1, & \overline{CO}_N(k_r, k_t) \geq \alpha_N \\ 0.5, & \beta_N < \overline{CO}_N(k_r, k_t) < \alpha_N \\ 0, & \overline{CO}_N(k_r, k_t) \leq \beta_N \end{cases} \tag{3.5}$$

$$\omega_{k_r,k_t}(s_{con}) = \begin{cases} 1, & \overline{CO}_D(k_r, k_t) \geq \alpha_D \\ 0.5, & \beta_D < \overline{CO}_D(k_r, k_t) < \alpha_D \\ 0, & \overline{CO}_D(k_r, k_t) \leq \beta_D \end{cases} \tag{3.6}$$

The function $\omega_{k_r,k_t}(s_{all})$ signifies the utility/function of alliance of a player k_r with the other player k_t. If the value of $\overline{CO}_A(k_r, k_t)$ for a player k_r with the other player k_t is equal to or greater than threshold α_A, then the function attains its maximum value 1. This shows the highest interest of a player to go for an alliance with the other player. If the value of $\overline{CO}_A(k_r, k_t)$ for a player k_r with the other player k_t is lesser than or equal to threshold β_A, then the function attains its minimum value 0. This reflects a player's lowest interest in accepting the other player for alliance (conflict). When the value of $\overline{CO}_A(k_r, k_t)$ for a player with the other player is between thresholds α_A and β_A, the function attains the value of 0.5, suggesting an uncertainty for acceptance or rejection of the other player for the alliance or conflict. Similarly, other two functions $\omega_{k_r,k_t}(s_{neu})$ and $\omega_{k_r,k_t}(s_{con})$ can be described.

Here, we need interaction within the game among different players. Players' personal beliefs and utilities would be affected by preferred strategies of the other players. By considering it, we employ a mean value or average of utilities of involved players in securing their payoff functions. It is a rational choice as we try to obtain coordination between players in deciding an object's inclusion in allied, neutral or conflict sets. Additionally, the judgements of both players are considered as beneficial without any distinction. The preliminary payoff serves as a basis to define payoffs of both the players k_r and k_t. Let s_1, s_2 be strategies in $S = \{s_{all}, s_{neu}, s_{con}\}$, payoffs of both the players are defined as follows:

$$u_{k_r}(s_1, s_2) = \frac{\omega_{k_r,k_t}(s_1) + \omega_{k_t,k_r}(s_2)}{2}; \tag{3.7}$$

$$u_{k_t}(s_1, s_2) = \frac{\omega_{k_t,k_r}(s_1) + \omega_{k_r,k_t}(s_2)}{2}. \tag{3.8}$$

Finally, we construct a game mechanism between two players k_r and k_t. The respective payoffs are presented in Table 3.1, where each block corresponding to a strategy outline comprises a pair of payoff values based on that strategy outline. Each block of the table gives the payoffs to both players for each combination of actions. For example, top left block of the table contains two payoffs $u_{k_r}(s_{all}, s_{all})$ and $u_{k_t}(s_{all}, s_{all})$ for players k_r and k_t with strategy outline (s_{all}, s_{all}), respectively. This implies that each cell of the table represents a payoff pair $(u_{k_r}(s_j, s_h), u_{k_t}(s_j, s_h))$ corresponding to strategy s_j of player k_r and s_h of the other player k_t. To construct this game, we have total nine pairs of payoffs, listed in Table 3.1.

Table 3.1: Table of payoffs for two-player GTRS based game

		k_t		
		s_{all}	s_{neu}	s_{con}
	s_{all}	$u_{k_r}(s_{all}, s_{all}), u_{k_t}(s_{all}, s_{all})$	$u_{k_r}(s_{all}, s_{neu}), u_{k_t}(s_{all}, s_{neu})$	$u_{k_r}(s_{all}, s_{con}), u_{k_t}(s_{all}, s_{con})$
k_r	s_{neu}	$u_{k_r}(s_{neu}, s_{all}), u_{k_t}(s_{neu}, s_{all})$	$u_{k_r}(s_{neu}, s_{neu}), u_{k_t}(s_{neu}, s_{neu}),$	$u_{k_r}(s_{neu}, s_{con}), u_{k_t}(s_{neu}, s_{con})$
	s_{con}	$u_{k_r}(s_{con}, s_{all}), u_{k_t}(s_{con}, s_{all})$	$u_{k_r}(s_{con}, s_{neu}), u_{k_t}(s_{con}, s_{neu}),$	$u_{k_r}(s_{con}, s_{con}), u_{k_t}(s_{con}, s_{con})$

The game solution is a formal rule for predicting the best strategy outline in which the participants adopt their preferred actions. The game solutions are then used in determining three-way decisions. A strategic portfolio and a steady condition of a game is Nash equilibrium, when every player understands all possible predictions about the other player's move. Moreover, it is the best response to the other player's possible choice. Nash equilibrium is typically employed for game solution to conclude possible game outcomes in GTRS. In the proposed two-player game between the players k_r and k_t, a payoff pair $(u'_{k_r}(s_{all}, s_{all}), u'_{k_t}(s_{all}, s_{all}))$ is an equilibrium point if for any strategy s_e,

$$u'_{k_r}(s_{all}, s_{all}) \geq u_{k_r}(s_e, s_v) \forall s_e, s_v \in S \tag{3.9}$$

$$u'_{k_t}(s_{all}, s_{all}) \geq u_{k_t}(s_e, s_o) \forall s_e, s_o \in S \tag{3.10}$$

This payoff pair $(u'_{k_r}(s_{all}, s_{all}), u'_{k_t}(s_{all}, s_{all}))$ is said to be the optimal solution or a balanced point in determining actions for the players and any variation in it would not be beneficial for any player. Similar descriptions are valid for the other two payoff pairs $(u'_{k_r}(s_{neu}, s_{neu}), u'_{k_t}(s_{neu}, s_{neu}))$ and $(u'_{k_r}(s_{con}, s_{con}), u'_{k_t}(s_{con}, s_{con}))$.

The proposed GTRS model utilizes Nash equilibrium [89] for analyzing payoff tables in order to find possible outcomes for games. Further, this Equilibrium analysis suggests the inclusion of an object/agent into one of the sets *i.e.*, *CON(k)*, *NEU(k)*, *ALL(k)*.

Algorithm: To formulate *CON(k)*, *NEU(k)*, *ALL(k)*

Input: IS for conflict situation $IS = (U, A)$

Step 1: Choose two players k_r, k_t in U.

Step 2: Compute $\varphi_a(k_r, k_t)$ for all $a \in A$ based on Equation 2.2.

Step 3: Compute $\overline{CO}_A(k_r, k_t), \overline{CO}_N(k_r, k_t)$ and $\overline{CO}_D(k_r, k_t)$ using Equations 3.1-3.3.

Step 4: Set the threshold values for involved measures, $i.e.$,$(\alpha_A, \beta_A), (\alpha_N, \beta_N)$ and (α_D, β_D).

Step 5: Compute values of weighted payoff, $i.e.$, $\omega_{k_r,k_t}(s_{all}), \omega_{k_r,k_t}(s_{neu}), \omega_{k_r,k_t}(s_{con})$ by using Equations 3.4-3.6.

Step 6: Compute payoffs of both players (k_r, k_t) for all strategies by using Equations 3.7-3.8.

Step 7: Calculate the equilibrium point in payoffs tables by using Equations 3.9–3.10.

Step 8: Classify k_r, k_t as ally, neutral or enemy based on Nash equilibrium.

Step 9: Repeat the steps $1 - 8$ unless the game is played between all the objects of U.

Output:*CON(k), NEU(k), ALL(k).*

We present the flow chart of the proposed algorithm in Figure 3.1.

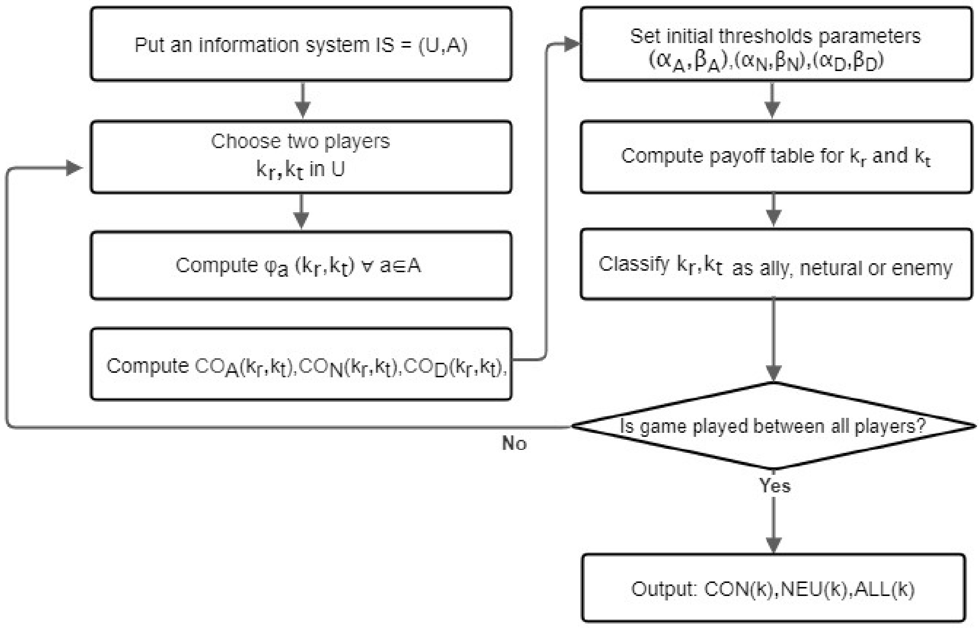

Figure 3.1: Flow chart of proposed conflict resolution model

3.1.3 A case study on Syrian conflict

In this section to practically demonstrate the proposed model, we solve the Syrian Conflict problem [90] with IS given in Table 3.2.

Table 3.2: Information system for the Syrian conflict

U	a_1	a_2	a_3	a_4	a_5	a_6	a_7
k_1	-1	$+1$	$+1$	-1	$+1$	0	-1
k_2	0	$+1$	-1	-1	-1	$+1$	-1
k_3	$+1$	-1	-1	-1	$+1$	$+1$	-1
k_4	$+1$	-1	0	$+1$	0	-1	$+1$
k_5	-1	$+1$	$+1$	-1	0	-1	$+1$
k_6	-1	$+1$	0	-1	0	-1	-1
k_7	$+1$	-1	-1	$+1$	0	0	$+1$
k_8	-1	$+1$	$+1$	-1	$+1$	$+1$	-1

Here, we denote $U = \{k_1, k_2, k_3, k_4, k_5, k_6, k_7, k_8\}$ as the universe of eight agents, where k_1 : USA k_2 : Turkey k_3 : Russia k_4 : Iran k_5 : Saudi Arabia k_6 : Qatar k_7 : Iraq k_8 : Israel.

Let $A = \{a_1, a_2, a_3, a_4, a_5, a_6, a_7\}$ denotes the universe of seven issues of the conflict problem, where:

a_1: refers to support to retain the government of Bashar Al-Assad;

a_2: represents safe zone;

a_3: denotes Kurdish federalism (Rojava);

a_4: stands for naming the moderate opposition factions ;

a_5: represents Syrian Democratic Forces (SDF) ;

a_6: denotes useful Syria project;

a_7: Hezbollah involvement in the Syrian war.

To illustrate our proposed GTRS model Algorithm, we explain and compute all steps for player k_1:

Step 1: Choose two players k_r, k_t in U.

We choose two player k_1 and k_i for $i = 2, 3, ..., 8$, alternatively.

Step 2: Compute $\varphi_a(k_r, k_t)$ for all $a \in A$ based on Equation 2.1.

Table 3.3: Auxiliary functions of objects

$\varphi_a(k_r, k_t)$	a_1	a_2	a_3	a_4	a_5	a_6	a_7
$\varphi_a(k_1, k_2)$	0	+1	−1	+1	−1	0	+1
$\varphi_a(k_1, k_3)$	−1	−1	0	−1	0	0	−1
$\varphi_a(k_1, k_4)$	+1	+1	0	+1	0	0	+1
$\varphi_a(k_1, k_5)$	+1	+1	0	+1	0	0	+1
$\varphi_a(k_1, k_6)$	−1	−1	−1	−1	0	0	−1
$\varphi_a(k_1, k_7)$	+1	+1	+1	+1	+1	0	+1
$\varphi_a(k_1, k_8)$	0	+1	−1	+1	−1	0	+1

Auxiliary functions for player k_1 and other players are calculated by using Equation 2.1, presented in Table 3.3.

Step 3: Compute $\overline{CO}_A(k_r, k_t), \overline{CO}_N(k_r, k_t)$ and $\overline{CO}_D(k_r, k_t)$ by using Equations 3.1-3.3.

In this step, we need to calculate mean cumulative degree of opinion for three measures, *i.e.*, $\overline{CO}_A$, $\overline{CO}_N$ and $\overline{CO}_D$. They can be calculated with the help of Table 3.3 using their respective definitions in Equations 3.1–3.3. For example, $\overline{CO}_A(k_1, k_2)$ is computed as a mean cumulative degree of agreement for objects k_1 and k_2 as their total number of agreement (+1 on various issues divided by the number of total issues within the conflict, *i.e.*, $\frac{\sum \varphi_a^+(k_1,k_2)}{m} = \frac{3}{7} = 0.4287$. Similarly, $\overline{CO}_N(k_1, k_2)$ and $\overline{CO}_D(k_1, k_2)$ are calculated as $\frac{\sum \varphi_a^0(k_1,k_2)}{m} = \frac{2}{7} = 0.2857$ and $\frac{\sum \varphi_a^-(k_1,k_2)}{m} = \frac{2}{7} = 0.2857$, respectively. We present $\overline{CO}_A(k_1, k_t)$, $\overline{CO}_N(k_1, k_t)$ and $\overline{CO}_D(k_1, k_t)$ for $t = 2, ...8$ in Table 3.4.

Step 4: Set the threshold values for involved measures, *i.e.*, $(\alpha_A, \beta_A), (\alpha_N, \beta_N)$ and (α_D, β_D).

By analysing the mean cumulative degree values, we can define a pair of threshold for each player in order to obtain the payoff functions. A pair of threshold can be calculated in several ways, *e.g.*, by inspecting all the objects' mean cumulative degree values for the involved measures, *i.e.*, agreement, neutrality and disagreement, selector's perception of tolerance levels for choosing an object for conflict, neutral or alliance. In this example, we obtain the threshold pair as $(\alpha_A, \beta_A) = (0.54, 0.29)$ by using DTRS model [41] for agreement between two objects. Same pair of threshold values can also be used for the other two measures, *i.e.*, neutrality (α_N, β_N) and disagreement (α_D, β_D).

Step 5: Compute weighted payoff, *i.e.*, $\omega_{k_r,k_t}(s_{all}), \omega_{k_r,k_t}(s_{neu}), \omega_{k_r,k_t}(s_{con})$ based on Equations 3.4-3.6.

Table 3.4: Mean cumulative degrees of objects

(k_r, k_t)	CO_A	CO_N	CO_D	$\overline{CO}_A$	$\overline{CO}_N$	$\overline{CO}_D$
(k_1, k_2)	3	2	2	0.4287	0.2857	0.2857
(k_1, k_3)	0	3	4	0.0000	0.4287	0.5714
(k_1, k_4)	0	3	4	0.0000	0.4287	0.5714
(k_1, k_5)	4	2	1	0.5714	0.2857	0.4128
(k_1, k_6)	4	3	0	0.5714	0.4287	0.000
(k_1, k_7)	0	2	5	0.0000	0.2857	0.7142
(k_1, k_8)	6	1	0	0.8571	0.1428	0.0000

For any particular player k_i, we determine its payoff table by calculating all respective payoff pairs of individual cells of the table. Recalling the representation of Table 3.1, the payoff pair $(u_{k_1}(s_{all}, s_{all}), u_{k_2}(s_{all}, s_{all}))$ is corresponding pair of the first cell of the table. From Table 3.4, We can see that $\overline{CO}_A(k_1, k_2) = 0.4287$, $\overline{CO}_N(k_1, k_2) = 0.2857$ and $\overline{CO}_D(k_1, k_2) = 0.2857$. By utilising the function $\omega_{k_r,k_t}(s_{all})$, presented in Equations 3.4-3.6, we can calculate the utility or gain of player k_r if he makes an alliance with the other player k_t. The utility function for player k_1, in making an alliance with player k_2, is denoted as $\omega_{k_1,k_2}(s_{all})$, is 0.5 as $\beta_A < \overline{CO}_A(k_1, k_2) = 0.42875 < \alpha_A$. This means that player k_1 has a moderate desire to make an alliance with the player k_2. Similarly, the utility function for player k_2 in making alliance with the player k_1 within the conflict will also be 0.5 as $\beta < \overline{CO}_A(k_2, k_1) = 0.42875 < \alpha_A$. This suggests that k_2 has also a moderate desire to accept the player k_1 for an alliance.

Step 6: Compute payoffs of both players for all strategies by using Equations 3.7-3.8.

Now, the payoff function corresponding to pair $(u_{k_1}(s_{all}, s_{all}), u_{k_2}(s_{all}, s_{all}))$ can be calculated as follows:

$$u_{k_1}(s_{all}, s_{all}) = \frac{\omega_{k_1,k_2}(s_{all}) + \omega_{k_2,k_1}(s_{all})}{2} = \frac{0.5 + 0.5}{2} = 0.5$$

$$u_{k_2}(s_{all}, s_{all}) = \frac{\omega_{k_2,k_1}(s_{all}) + \omega_{k_1,k_2}(s_{all})}{2} = \frac{0.5 + 0.5}{2} = 0.5$$

Table 3.5: Table of payoffs for game played between players k_1 and k_2

		k_2		
		s_{all}	s_{neu}	s_{con}
	s_{all}	**(0.50,0.50)**	$(0.25, 0.25)$	$(0.25, 0.25)$
k_1	s_{neu}	$(0.25, 0.25)$	$(0.0, 0.0)$	$(0.0, 0.0)$
	s_{con}	$(0.25, 0.25)$	$(0.0, 0.0)$	$(0.0, 0.0)$

Similarly, by utilising the functions presented in Equations 3.7- 3.8, $u_{k_r}(s_{all}, s_{neu})$ and $u_{k_r}(s_{all}, s_{con})$ are calculated, respectively. Now, the utility for player k_1 in getting a neutral position with the player k_2 is 0 as $\overline{CO}_N(k_1, k_2) = 0.2857 < \beta_N$. This suggests that k_1 has the least desire to get a neutral position with k_2 within the conflict. Similar description is valid for k_2. Now, the payoff function corresponding to the pair $(u_{k_1}(s_{all}, s_{neu}), u_{k_2}(s_{all}, s_{neu}))$ are calculated as follows:

$$u_{k_1}(s_{all}, s_{neu}) = \frac{\omega_{k_1,k_2}(s_{all}) + \omega_{k_2,k_1}(s_{neu})}{2} = \frac{0.5 + 0}{2} = 0.25.$$
$$u_{k_2}(s_{all}, s_{neu}) = \frac{\omega_{k_2,k_1}(s_{all}) + \omega_{k_1,k_2}(s_{neu})}{2} = \frac{0 + 0.5}{2} = 0.25.$$

Following the same pattern, the other payoff pairs are calculated to get a payoff table for a game between two players k_1 and k_2, shown in Table 3.5.

Step 7: Calculate equilibrium point in payoffs tables using Equations 3.9–3.10.

If we analyze the payoff values in Table 3.5, then equilibrium analysis suggests the strategy s_{all} for player k_1 and s_{all} for player k_2 is a Nash equilibrium, represented in bold. It is also observed that no player can obtain a better payoff, provided with the knowledge of the other player's chosen strategy. For instance, if player k_1 changes its strategy from s_{all} to s_{neu} or s_{con}, its payoff will decrease from 0.5 to a lower value. Same description is also true for k_2.

Step 8: Thus, the object k_2 is included in the set $ALL(k_1)$ using equilibrium analysis.

Step 9: We repeat steps(1-8) until we obtain classification for all objects.

3.1.4 Further analysis

Similar to the game between players k_1 and k_2, we can examine the competitions/games of player k_1 with the other objects k_3, k_4, k_5, k_6, k_7, k_8. In this section, we will examine them one-by-one.

Table 3.6: Table of payoffs for game played between players k_1 and k_3

		k_3		
		s_{all}	s_{neu}	s_{con}
	s_{all}	$(0.00, 0.00)$	$(0.25, 0.25)$	$(0.50, 0.50)$
k_1	s_{neu}	$(0.25, 0.25)$	$(0.50, 0.50)$	$(0.75, 0.75)$
	s_{con}	$(0.50, 0.50)$	$(0.75, 0.75)$	$(\mathbf{1.00}, \mathbf{1.00})$

For the player k_3: it has the lowest mean cumulative degree of agreement, *i.e.*, $\overline{CO}_A(k_1, k_3) = 0$, a moderate degree of neutrality, *i.e.*, $\overline{CO}_N(k_1, k_3) = 0.4287$, and the mean cumulative degree of disagreement is higher, *i.e.*, $\overline{CO}_D(k_1, k_3) = 0.5714$. The payoff table for game between players k_1 and k_3 is shown in Table 3.6. Equilibrium analysis suggests strategy s_{con} for player k_1 and s_{con} for player k_3 is Nash equilibrium. Thus, player k_3 is included in the set $CON(k_1)$.

Table 3.7: Table of payoffs for game played between players k_1 and k_4

		k_4		
		s_{all}	s_{neu}	s_{con}
	s_{all}	$(0.00, 0.00)$	$(0.25, 0.25)$	$(0.50, 0.50)$
k_1	s_{neu}	$(0.25, 0.25)$	$(0.50, 0.50)$	$(0.75, 0.75)$
	s_{con}	$(0.5, 0.5)$	$(0.75, 0.75)$	$(\mathbf{1.00}, \mathbf{1.00})$

Next, we consider the game between players k_1 and k_4. The object k_4 has the lowest mean cumulative degree of agreement, a moderate mean cumulative degree of neutrality and a higher mean cumulative degree of disagreement, shown in Table 3.4. Payoff table for this game is shown in Table 3.7. The equilibrium states are s_{con} for player k_1 and s_{con} for player k_4. Hence, the object k_4 is also considered to be in the set $CON(k_1)$.

Considering the object k_5, it has a higher mean cumulative degree of agreement, lower mean cumulative degrees of neutrality and disagreement. In this case, player k_1 has the highest desire to ally

Table 3.8: Table of payoffs for game played between players k_1 and k_5

		k_5		
		s_{all}	s_{neu}	s_{con}
	s_{all}	(**1.00,1.00**)	$(0.50, 0.50)$	$(0.75, 0.75)$
k_1	s_{neu}	$(0.50, 0.50)$	$(0.0, 0.0)$	$(0.25, 0.25)$
	s_{con}	$(0.75, 0.75)$	$(0.25, 0.25)$	$(0.50, 0.50)$

with the player k_5. Similarly, k_5 has also the highest desire to accept the player k_1 in an alliance. By using utility functions, we can construct the payoff table of the game between players k_1 and k_5 as presented in Table 3.8. Equilibrium analysis suggests the inclusion of object k_5 in the set $ALL(k_1)$.

Table 3.9: Table of payoffs for game played between players k_1 and k_6

		k_6		
		s_{all}	s_{neu}	s_{con}
	s_{all}	(**1.00,1.00**)	$(0.75, 0.75)$	$(0.50, 0.50)$
k_1	s_{neu}	$(0.75, 0.75)$	$(0.50, 0.50)$	$(0.25, 0.25)$
	s_{con}	$(0.50, 0.50)$	$(0.25, 0.25)$	$(0.0, 0.0)$

Now, we consider the object k_6. It has a higher mean cumulative degree of agreement, a moderate mean cumulative degree of neutrality and the lowest mean cumulative degree of disagreement. Table 3.9 shows its payoff table. Equilibrium analysis suggests strategy s_{all} for player k_1 and s_{all} for player k_6 as a Nash equilibrium and its inclusion in the set $ALL(k_1)$.

Table 3.10: Table of payoffs for game played between players k_1 and k_7

		k_7		
		s_{all}	s_{neu}	s_{con}
	s_{all}	$(0.00, 0.00)$	$(0.00, 0.00)$	$(0.50, 0.50)$
k_1	s_{neu}	$(0.00, 0.00)$	$(0.00, 0.00)$	$(0.50, 0.50)$
	s_{con}	$(0.50, 0.50)$	$(0.50, 0.50)$	(**1.00,1.00**)

Now, considering the object k_7, it has lowest the mean cumulative degree of agreement, a low cumulative degree of neutrality and the highest mean cumulative degree of disagreement. Table 3.10 shows its payoff table. Equilibrium analysis suggests that strategy s_{con} for player k_1 and s_{con} for player k_7 is a Nash equilibrium. Hence, object k_7 is included in the set $CON(k_1)$. Finally, we

Table 3.11: Table of payoffs for game played between players k_1 and k_8

		k_8		
		s_{all}	s_{neu}	s_{con}
	s_{all}	(**1.00,1.00**)	$(0.50, 0.50)$	$(0.50, 0.50)$
k_1	s_{neu}	$(0.50, 0.50)$	$(0.00, 0.00)$	$(0.00, 0.00)$
	s_{con}	$(0.50, 0.50)$	$(0.00, 0.00)$	$(0.00, 0.00)$

observe the last object k_8. It has the highest mean cumulative degree of agreement and lower mean cumulative degrees of neutrality and disagreement. All payoffs are shown in Table 3.11. Equilibrium analysis suggests strategy s_{all} for player k_1 and s_{all} for player k_8 as a Nash equilibrium. Hence, object k_8 is included in the set $CON(k_1)$.

From the payoff tables of games between players k_1 and k_i for $i = 2, 3, ..., 8$, we can construct the sets $ALL(k_1)$, $CON(k_1)$ and $NEU(k_1)$. We need to note, if equilibrium analysis suggests for both players strategies (s_{all}, s_{all}) or (s_{con}, s_{con}) in a game, this reflects that they both are agreed for an alliance or a conflict respectively. Thus, they are included in the allied set $ALL(k)$ or the conflict set $CON(k)$. An object will be in the neutral set if one player agrees for an alliance, and the other has a disagreement or both want to be neutral within the conflict. On the basis of Nash equilibrium analysis, we can formulate the Conflict, Neutral and Allied sets of the first object k_1 as follows:

$CON(k_1) = \{k_7, k_3, k_4\}$; $NEU(k_1) = \{\} = \emptyset$; $ALL(k_1) = \{k_1, k_2, k_5, k_6, k_8\}$.

Now, we repeat the steps $1-9$ unless the game is played between the all possible pairs of objects of U. Following the steps $1-8$ for objects k_i for $i = 2, 3, ..., 8$, we can establish games between all possible pairs of objects in U and consequently, can construct payoff tables for each game. Finally using Nash equilibrium, objects are classified in conflict, neutral and allied sets of k_i for $i = 1, 2, 3....., 8$.

Output: *CON(k), NEU(k), ALL(k)*, presented in Table 3.12.

Table 3.12: Allied, Neutral and Conflict sets for Syrian conflict Problem

U	CON(k_i)	NEU(k_i)	ALL(k_i)
k_1	$\{k_3, k_4, k_7\}$	$\emptyset$	$\{k_2, k_5, k_6, k_8\}$
k_2	$\{k_3, k_7\}$	$\{k_4\}$	$\{k_1, k_5, k_6, k_8\}$
k_3	$\{k_1, k_2, k_5, k_6, k_8\}$	$\emptyset$	$\{k_4, k_7\}$
k_4	$\{k_1, k_5, k_6, k_8\}$	$\{k_2\}$	$\{k_3, k_7\}$
k_5	$\{k_3, k_4, k_7\}$	$\emptyset$	$\{k_1, k_2, k_6, k_8\}$
k_6	$\{k_3, k_4, k_7\}$	$\emptyset$	$\{k_1, k_2, k_5, k_8\}$
k_7	$\{k_1, k_2, k_5, k_6, k_8\}$	$\emptyset$	$\{k_3, k_4\}$
k_8	$\{k_3, k_4, k_7\}$	$\emptyset$	$\{k_1, k_2, k_5, k_6\}$

3.1.5 Algorithmic analysis

In the proposed model, $IS = (U, A)$ describes the opinion of n agents $U = \{k_1, k_2, k_3, ..., k_n\}$ regarding the given m issues/attributes $A = \{a_1, a_2, a_3, ..., a_m\}$. Each player/agent in U will play a game against every other player $k_i \in U$ to determine their conflict, alliance or neutrality. For each game between two players, we need to determine their conflict, alliance or neutrality for each $a_j \in A$ for $j = 1, 2, 3, ..., m$. Thus, the complexity of playing one game is $O(m)$ to determine the conflict, alliance or neutrality between two players for m attributes/issues. The other factor that contributes towards the complexity of the proposed algorithm is the total number of games played. To calculate the total number of games, we need to determine how many games will be played by each player. For example, k_1 will play $(n-1)$ games with the remaining player in U. Player k_2 will play $(n-2)$ games with players $\{k_3, k_4, ..., k_n\}$ because he has already played a game with k_1. Similarly, Player k_3 will play $(n-3)$ games with players $\{k_4, k_5, ..., k_n\}$ because he has already played games with k_1 and k_2. Thus, players $k_1, k_2, k_3, ..., k_{(n-3)}, k_{(n-2)}, k_{(n-1)}$, and k_n will play $(n-1), (n-2), (n-3), ..., 3, 2, 1, 0$ games, respectively. We can calculate the total games by adding these numbers as follows:

$$\text{Total games} = \sum_{i-0}^{n-1} i = \frac{n(n-1)}{2}. \tag{3.11}$$

Based on the total number of games, the proposed algorithm will have time complexity as follows:

$$Algorithm_{(\text{time complexity})} = \frac{n(n-1)}{2} \times (\text{time for one game}). \quad (3.12)$$

Consequently, the proposed algorithm has the complexity $O(n^2m)$.

3.1.6 A case study on development plans for Gansu province in China

To further validate the proposed algorithm and make a comparison with recent studies on conflict analysis, we present and solve another case study on development plans for Gansu province in China [38, 39]. In China, for local government, economic development and social stabilisation are great

Table 3.13: Information system of development plans for Gansu province China

U	a_1	a_2	a_3	a_4	a_5	a_6	a_7	a_8	a_9	a_{10}	a_{11}
k_1	+1	−1	0	−1	+1	−1	0	−1	+1	−1	+1
k_2	0	+1	−1	0	0	+1	−1	0	0	+1	−1
k_3	−1	0	−1	−1	−1	+1	+1	−1	−1	0	0
k_4	0	0	−1	+1	+1	−1	−1	+1	0	−1	−1
k_5	−1	+1	−1	0	−1	+1	0	0	−1	+1	+1
k_6	0	+1	0	−1	−1	−1	−1	−1	0	+1	−1
k_7	+1	+1	0	+1	0	+1	0	+1	+1	+1	0
k_8	−1	0	−1	+1	−1	0	+1	+1	−1	0	+1
k_9	+1	+1	0	−1	+1	+1	−1	−1	+1	+1	−1
k_{10}	−1	−1	−1	0	+1	−1	+1	0	−1	−1	+1
k_{11}	−1	0	−1	−1	−1	−1	−1	−1	−1	0	−1
k_{12}	0	+1	0	−1	+1	+1	+1	−1	0	+1	0
k_{13}	−1	0	−1	+1	0	0	0	+1	−1	0	+1
k_{14}	−1	−1	−1	0	−1	−1	−1	0	−1	−1	−1

challenges. For the implementation of a new policy, local government faces various conflicts among involved cities. The local government of Gansu province needs to implement a new development plan for the new year. Gansu Province consists of fourteen cities, namely Lanzhou, Jinchang, Baiyin, Tianshui, Jiayuguan, Wuwei, Zhangye, Pingliang, Jiuquan, Qingyang, Dingxi, Longnan, Linxia, and Gannan, which are denoted as $k_1, k_2, k_3, k_4, k_5, k_6, k_7, k_8, k_9, k_{10}, k_{11}, k_{12}, k_{13}, k_{14}$, respectively. The development plan mainly involves eleven issues, namely, the construction of roads, factories, entertainment, educational institutions, total population of residence, ecology environment, the number of senior intellectuals, the traffic capacity, mineral resources, sustainable development capacity and water resources carrying capacity, which are denoted as $a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8, a_9, a_{10}, a_{11}$, respectively. The opinions of each agent on all issues are shown in Table 3.13.

All cities have different positions on related issues, which is natural as every city has its own geographical, political and economic needs, so their priorities are different. We solve this problem with our proposed model based on GTRS and present the results in Table 3.14.

Table 3.14: Solution of development problem of Gansu province China with proposed model

U	ALL(k_i)	NEU(k_i)	CON(k_i)
k_1	$\{k_9, k_{10}\}$	$\{k_2, k_4, k_{12}, k_{13}\}$	$\{k_3, k_5, k_6, k_7, k_8, k_{11}, k_{14}\}$
k_2	$\{\}$	$\{k_1, k_3, k_4, k_5, k_6, k_7, k_8, k_9, k_{11}, k_{12}, k_{13}, k_{14}\}$	$\{k_{10}\}$
k_3	$\{k_8, k_{10}, k_{11}, k_{12}\}$	$\{k_2, k_5, k_6, k_7, k_{13}, k_{14}\}$	$\{k_1, k_4, k_9\}$
k_4	$\{k_{11}, k_{14}\}$	$\{k_1, k_2, k_5, k_7, k_8, k_{10}, k_{13}\}$	$\{k_3, k_6, k_9, k_{12}\}$
k_5	$\{k_8, k_{10}, k_{11}\}$	$\{k_2, k_3, k_4, k_6, k_7, k_9, k_{12}, k_{13}, k_{14}\}$	$\{k_1\}$
k_6	$\{k_9, k_{11}, k_{12}\}$	$\{k_2, k_3, k_5, k_7, k_8, k_{13}, k_{14}\}$	$\{k_1, k_4, k_{10}\}$
k_7	$\{k_9\}$	$\{k_2, k_3, k_4, k_5, k_6, k_7, k_8, k_{10}, k_{11}, k_{12}, k_{13}, k_{14}\}$	$\{k_1\}$
k_8	$\{k_3, k_5, k_{10}, k_{11}, k_{13}\}$	$\{k_2, k_4, k_6, k_7, k_{12}, k_{14}\}$	$\{k_1, k_9\}$
k_9	$\{k_1, k_6, k_7, k_{12}\}$	$\{k_2, k_5, k_{11}, k_{13}\}$	$\{k_3, k_4, k_8, k_{10}, k_{14}\}$
k_{10}	$\{k_1, k_3, k_5, k_8, k_{14}\}$	$\{k_4, k_7, k_{11}, k_{12}, k_{13}\}$	$\{k_2, k_6, k_9\}$
k_{11}	$\{k_3, k_4, k_5, k_6, k_8, k_{14}\}$	$\{k_2, k_7, k_9, k_{10}, k_{12}, k_{13}\}$	$\{k_1\}$
k_{12}	$\{k_3, k_6, k_9\}$	$\{k_1, k_2, k_5, k_7, k_8, k_{10}, k_{11}, k_{13}\}$	$\{k_4, k_{14}\}$
k_{13}	$\{k_8\}$	$\{k_1, k_2, k_3, k_4, k_5, k_6, k_7, k_9, k_{10}, k_{11}, k_{12}, k_{14}\}$	$\{\}$
k_{14}	$\{k_4, k_{10}, k_{11}\}$	$\{k_2, k_3, k_5, k_6, k_7, k_8, k_{13}\}$	$\{k_1, k_9, k_{12}\}$

3.2 Comparative analysis with the existing conflict study models

For comparison and to validate the proposed model, we solve the Middle East conflict presented in Table 2.1 using the proposed model. Results are shown in Table 3.15. To compare our solution with Pawlak's solution for the Middle East conflict, Table 3.16 is computed based on Pawlak's model, described in Chapter 2.

Table 3.15: Solution of Middle East conflict with the proposed model

U	CON(k_i)	NEU(k_i)	ALL(k_i)
k_1	$\{k_2, k_3, k_4, k_5\}$	$\{k_6\}$	$\emptyset$
k_2	$\{k_1\}$	$\{k_4, k_6\}$	$\{k_3, k_5\}$
k_3	$\{k_1\}$	$\{k_4, k_6\}$	$\{k_2, k_5\}$
k_4	$\{k_1\}$	$\{k_2, k_3, k_6\}$	$\{k_5\}$
k_5	$\{k_1\}$	$\{k_6\}$	$\{k_2, k_3, k_4\}$
k_6	$\emptyset$	$\{k_1, k_2, k_3, k_4, k_5\}$	$\emptyset$

Table 3.16: Pawlak's solution of Middle East conflict

U	CON(k_i)	NEU(k_i)	ALL(k_i)
k_1	$\{k_2, k_3, k_4, k_5\}$	$\emptyset$	$\{k_6\}$
k_2	$\{k_1\}$	$\{k_6\}$	$\{k_3, k_4, k_5\}$
k_3	$\{k_1\}$	$\{k_6\}$	$\{k_2, k_4, k_5\}$
k_4	$\{k_1\}$	$\emptyset$	$\{k_2, k_3, k_5, k_6\}$
k_5	$\{k_1, k_6\}$	$\emptyset$	$\{k_2, k_3, k_4\}$
k_6	$\{k_5\}$	$\{k_2, k_3\}$	$\{k_1, k_4\}$

Despite of evident contributions of Pawlak model concerning to conflict resolution, there are some certain deficiencies in it. In Pawlak model, quadruple $IS = (U, A, V_A, f)$ be an IS that describes the opinions of objects in U regarding the issues in A utilizing a function $f : U \times A \to V_A$. Here, $V_A = \{-1, 0, 1\}$, typically $f(k_i, a_j) = -1, 0$ or 1 means object k_i is against, neutral or in agreement on issue a_j, respectively. However, there are issues in the classification of objects. Subsequently, objects k_i and k_j are classified as against (in the conflict set), neutral (in the neutral set) or allied (in the allied set) if $\sigma_A(k_i, k_j) > 1/2$, $\sigma_A(k_i, k_j) = 1/2$ or $\sigma_A(k_i, k_j) < 1/2$, respectively, where $\sigma_A(k_i, k_j)$ is the distance among k_i and k_j (defined in the Chapter refch1). If distance of two objects is near 1/2 in either side *let say 4.99 or 5.05*. No doubt, It seems unrealistic to classify objects in conflict or allied sets strictly, by giving it a second thought, we should inspect the given IS more closely. In real-world problems, if an object k that is actually in conflict with the other object l, is classified into the allied set of the object l with a threshold 0.5 then object l bears an exceptional loss, for instance, Pawlak' model put objects k_5 and k_6 in conflict as $\sigma_A(k_5, k_6) = 0.6 > 0.5$, whereas, from Table 2.2 we may observe that objects k_5 and k_6 are in conflict on attributes a_2 and a_5 and neutral on attributes a_1 and a_4 but on attribute a_3 they are allied, therefore, there are some losses if we put k_5 and k_6 in conflict. The complex and imprecise information in various real-world scenarios would need some adjustment in the bench-mark value 1/2. Hence, it is vital to find the optimal threshold values for classification of objects. The proposed model, with optimal threshold values and balanced solution, addressed this issue effectively and classified all objects more precisely.

GTRS based model, setting up games between all involved parties (players), computing the payoff of different strategies and classifying them following equilibrium rules, yields more realistic and accurate results as it explores all possibilities and is flexible in determining different threshold values relative to the complexities of real-life problems.

Table 3.17: Informtion system

U	a_1	a_2	a_3	a_4	a_5	a_6	a_7	a_8
k_1	$+1$	$+1$	$+1$	$+1$	-1	-1	-1	-1
k_2	-1	-1	-1	-1	$+1$	$+1$	$+1$	$+1$

Lang's model [37] uses a single measure σ_A (aggregated difference of opinion) which may lead to loss of important information in many scenarios. Sun et al. [39] used two measures $|F^+(k_i)|$ and $|F^-(k_i)|$ (count of $+1$ and -1 for object k_i in IS, respectively), but for an object k_i counting $+1$ or -1 individually, without taking into consideration the opinion of rest of objects may provide wrong conclusion. The inspection of IS(Table 3.17) reveals that k_1 and k_2 must be in conflict set by all accounts. But for $B = \{a_3, a_4, a_5, a_6\}$, we have $P(B|F^+(k1)) = \frac{|B \cap F^+(k1)|}{|F^+(k1)|} = 1/2$ and $P(B|F^+(k2)) = \frac{|B \cap F^+(k2)|}{|F^+(k2)|} = 1/2$. Similarly, $P(B|F^-(k1)) = 1/2$ and $P(B|F^-(k2)) = 1/2$, so their model will most likely classify them as neutral or with respect to B both k_1 and k_2 have neutral attitude. Also, Lang et al. [38] designed a unified model with both rough set theory and formal concept analysis that resolve conflict both quantitatively and qualitatively. Their work generalised the work examined by Sun et al. [39] with the use of two evaluation functions (positive and negative).

Generally, both models are accurate in pointing out the feasible sets of issues(attributes) with minimum conflict or maximum collations for all the objects. However, measuring the opinion of an object in isolation to opinions of other objects could be problematic in some cases, usually when some issues are more important than others or both positive and negative measures are proportional, as discussed earlier. More recently, Zhi et al [40] proposed an inconsistency measure for any $X \in U$ that leads to the partition of issues/attributes set A and build a three-way concept lattice to access differences among members of X. In our suggested model, instead of using a single measure, it uses the mean cumulative degree of three measures: agreement, neutrality and disagreement, with a pair of threshold values for each measure to signify the utilities of the players. Each object uses its respective payoff functions in analysing another object for its inclusion in conflict, neutral and allied sets of the object. The strategies of players are used to drive a balanced solution by using Nash equilibrium for classifying the conflict, neutral and allied objects.

We summarise our analysis as follows.

- In comparison to existing models, the GTRS based proposed model adopts a bilateral approach and considers the actions of all objects simultaneously, and calculates the respective gains (payoffs) accordingly. We use three measures alliance ($\overline{CO}_A$), neutrality ($\overline{CO}_N$), and conflict ($\overline{CO}_D$). Whereas, using an aggregated difference of opinions or two measures (alliance and conflict) only, cannot support the computation of payoffs for all possible actions. Also, we register the difference of opinions on all issues with care to obtain correct results.
- The proposed model is more flexible as it allows decision-makers to separately set parameters (α_A, β_A), (α_N, β_N) and (α_D, β_D) for alliance, neutrality and conflict, respectively. Depending upon the priorities in different circumstances, we can reduce and increase the margins of all three measures alliance, conflict and neutrality. We determine the relative position of an object by comparing its payoffs with the principle of the Nash equilibrium, which is quite

accurate in pointing out the right choice with maximum benefits for any object.

- The inspection of Table 3.14 reveals that most of the cities have a neutral attitude towards others because of given issues. In order to remove conflict among cities and come to a partial agreement, we can persuade the cities in the neutral zone to reconsider their opinions. If the issues are not critical, they can agree to development plans. Also, in our model government can set lower values of alliance parameters (α_A, β_A) to increase collation among cities. Based on these observations, in the set $\{k_2, k_3, k_5, k_6, k_7, k_8, k_{11}, k_{12}, k_{13}\}$ no two cities are in conflict with each other, therefore, government can implement a plan for these cites.

- Generally, if the objects with a neutral attitude can tolerate, we can find the maximum possible set as a conflict resolution solution, in which no two objects conflict with each other. Our solution has a different prospect than the models [38,39]. Their work is related to finding a sub-collection of issues according to which a certain number of objects are in agreement. Their study has its own merits.

In light of the above discussion, the proposed model has many advantages and merits compared to the existing models. We analyzed the conflict scenarios more critically and tried to be flexible in our decision-making process to come up with accurate results.

Chapter 4

Three-way conflict analysis based on hesitant fuzzy environment

HFSs, as a generalization of FSs, have been extensively practiced in many areas of risk decision analysis. Whenever we deal with an uncertain and complex conflict situation, we need to reduce the complexity and fuzziness of the conflict problems. HFSs can manage complex situations more effectively. This study presents a three-way conflict analysis model founded on HFIS where evaluation functions are HFEs while associated loss functions are real numbers. Further, we drive three-way decisions and costs associated with each object. Moreover, we use two methods to derive three-way conflict analysis. The first method is general and based on the scores of loss functions while the other method is founded on the ranking method of possibility degrees with a stochastic strategy and encapsulates all comparisons among expected losses. To validate the rationality of the presented model, the example of the Middle East conflict problem is solved and a result analysis is presented. A comparison with existing conflict analysis models is also provided to enhance the viability of the suggested approach.

4.1 A new conflict study model based on hesitant fuzzy frame work

In the present section, we introduced a novel method of conflict study founded on the TWD in a hesitant fuzzy environment. The study of conflict analysis is mainly relies on an IS. In Pawlak's model, an IS is represented in a matrix form where rows represent agents while columns represent issues of conflict problem. Pawlak's model restricted the opinions of agents in $\{-1, 0, +1\}$ as opposite, neutral, and favorable towards the issues, respectively. In real-life situations, we often have to deal with complex conflict problems, where a participant is hesitant to be in favor or against an issue. To deal with such situations, we need a more complex IS, founded on subset values of $(-1, 0, +1)$. For instance, an object may be hesitant to be in favor or against an issue then the values $+1$ and -1 are not enough. We need more than one value to depict the possible degree to which a participant is against or in support of an issue. With this aim, we need to introduce a new kind of IS by extending the domain of attributes of V_b to $[-1, +1]$ for all b, where values between $(-1, 0)$ show the disagreement and values between $(0, 1)$ show agreement while 0 shows the neutrality.

Table 4.1: Information system

X	b_1	b_2	.	.	.	b_m
x_1	ξ_{11}	ξ_{12}	.	.	.	ξ_{1m}
x_2	ξ_{21}	ξ_{22}	.	.	.	ξ_{2m}
x_3	ξ_{31}	ξ_{32}	.	.	.	ξ_{3m}
.	.	.	.	.	.	.
.	.	.	.	.	.	.
.	.	.	.	.	.	.
x_n	ξ_{n1}	ξ_{n2}	.	.	.	ξ_{nm}

Definition 4.1.1 *Let* $X = \{x_1, x_2, x_3, ..., x_n\}$ *and* $I = \{b_1, b_2, ..., b_m\}$ *be the set of agents and issues (attributes) respectively then* $S = (X, I) = [\xi_{kl}]_{n\times m}$ *for* $k = 1, 2, ..., n,\ l = 1, 2, ..., m,$ *be an IS where* $\xi_{kl} = \{\xi^s_{kl} \in [-1, 1] : s = 1, 2, ..., \#\xi_{kl}\}$ *describing the opinion of kth object regarding the lth issue.*

Here the object (x_k) provides all the possible evaluated values based on three measures i.e; agreement, neutrality and disagreement, under the attribute(issue) (b_l) denoted by ξ_{kl}, as shown in Table 4.1. Here all the values of ξ_{kl} are arranged in increasing order. It is noteworthy that different values of $[\xi_{kl}]$ may not be necessarily in order e.g., $\#\xi_{12} \neq \#\xi_{13}$. We need to ensure that they are of the same length [48] so we may arrange them in any order for convenience. For the said purpose, Zhu et al. [52] defined a technique to normalize the HFE. Keeping in mind that the values of ξ_{kl} are like HFNs but the domain is $[-1, 1]$ instead of $[0, 1]$, we may use the same technique. This defined technique may be used for adding elements in HFEs.

Let $\xi_{kl} = \{\xi_{kl}^s : s = 1, 2, ..., \#\xi_{kl}\}$, then we may normalize it by adding the values. For optimized parameter $0 \leq \kappa \leq 1$, the added membership degree of ξ_{kl} is $\kappa\delta^+ + (1-\kappa)\delta^-$, where δ^+ is the largest while δ^- is the smallest value of ξ_{kl}. The decision maker can opt the argument $\kappa(0 \leq \kappa \leq 1)$ according to his risk preferences. Then by Zhu et al. [52] added value would be δ^+ and δ^- for $\kappa = 1$ and $\kappa = 0$ respectively and $\bar{\xi_{kl}} = \{\overline{\xi_{kl}}_{n\times n}\}$ be a normalized set of values with κ as optimised parameter and $\overline{\xi_{kl}}$ is a normalized value. Here we may use $\kappa_z(0 < \kappa_z < 1, z = 1, 2, ..., g)$ to convert initial IS $S = [\xi_{kl}]_{n\times m} = \{\xi_{kl}^s : s = 1, 2, ..., \#\xi_{kl}\}$ into the normalized IS $\overline{S} = [\overline{\xi_{kl}}]_{n\times n} = \{\overline{\xi_{kl}^s} : s = 1, 2, ..., d\}$ by using the following equations:

$$\bar{\xi}_{kl} = \begin{cases} \xi_{kl}, & \#\xi_{kl} = d; \\ \left\{ \underbrace{\xi_{kl}^{(1)},....,\xi_{kl}^{(t_{kl})}}_{t_{kl}}, \underbrace{\kappa_z\xi_{kl}^{(\#\xi_{kl})} + (1-\kappa_z)\xi_{kl}^{(1)}, ..., \kappa_z\xi_{kl}^{(\#\xi_{kl})} + (1-\kappa_z)\xi_{kl}^{(1)}}_{d-(\#\xi_{kl})}, \underbrace{\xi_{kl}^{(t_{kl}+1)}, ..., \xi^{(\#\xi_{kl})}}_{(\#\xi kl)-t_{kl}} \right\}, & \#\xi_{kl} < d. \end{cases} \tag{4.1}$$

where

$d = max\{\#\xi_{kl} | k = 1, 2, ..., n, l = 1, 2, ..., m\}$ and

$t_{kl} = max\{s \in \{1, 2, ..., \#\xi_{kl}\} | \xi_{kl}^s \leq \kappa_z(\xi_{kl}^{(\#\xi_{kl})} + (1-\kappa_z)\xi_{kl}^{(1)})\}$.

According to new kind of IS, $\overline{S} = [\overline{\xi_{kl}}]_{n\times m}$, we define an auxiliary function as follows:

Definition 4.1.2 *Let $\overline{S} = [\overline{\xi_{kl}}]_{n\times m}$ be the normalized IS then auxiliary function denoted by $\varphi_{b_l}(x_k, x_q) = \{\varphi_{b_l}^s(x_k, x_q) : s = 1, 2, 3, ...\}$ is defined as follows:*

$$\varphi_{b_l}(x_k, x_q) = \begin{cases} \frac{|\overline{\xi^s_{kl}} - \overline{\xi^s_{ql}})|}{2} & if\ \overline{\xi^s_{kl}}.\overline{\xi^s_{ql}} > 0 \vee x_k = x_q; \\ \frac{1}{2} & if \overline{\xi^s_{kl}}.\overline{\xi^s_{ql}} = 0 \wedge x_k \neq x_q; \\ \frac{1}{2} + \frac{|\overline{\xi^s_{kl}} - \overline{\xi^s_{ql}}|}{4} & if\ \overline{\xi^s_{kl}}.\overline{\xi^s_{ql}} < 0; \end{cases} \tag{4.2}$$

for all $s = 1, 2, 3....\#\overline{\xi_{kl}}$

The defined auxiliary function $\varphi_{b_l}(x_k, x_q)$ works in two ways, firstly it measures the similarity or difference in judgments or opinions of two objects (x_k,x_q) about issue b_l, secondly the scale of opinion difference is shifted from $[-1, 1]$ to $[0, 1]$. The value near 1 shows the strong agreement of two objects on an issue while the value near -1 depicts the strong disagreement and values near 0 represent the neutrality of opinion about an issue. Furthermore, the values of $\varphi_{b_l}(x_k, x_q)$ are HFEs, hence we may accumulate the opinion of all objects on different issues by using any feasible operator.

4.1.1 Hesitant fuzzy information system

The conflict problem is investigated through a new normalized HFIS.

Table 4.2: Normalized Information system

X	b_1	b_2	.	.	b_m	
x_1	(χ_{11})	(χ_{12})	.	.	.	(χ_{1m})
x_2	(χ_{21})	(χ_{22})	.	.	.	(χ_{2m})
x_3	(χ_{31})	(χ_{32})	.	.	.	(χ_{3m})
.	.	.	.	.	.	.
.	.	.	.	.	.	.
.	.	.	.	.	.	.
x_n	(χ_{n1})	(χ_{n2})	.	.	.	(χ_{nm})

Definition 4.1.3 *A HFIS is a quadruple $S = (X, I, V_b, \chi)$ consisting a non empty finite set of agents $X = \{x_1, x_2, ..., x_n\}$, a non empty finite set of issues, $I = \{b_1, b_2, ..., b_m\}$ and $V_b = \{V_b | b \in I\}$,*

where V_b is the set of issue values on b based on HFNs and χ is function from $X \times I$ into V.

It is obvious that the HFIS is a generalization of Pawlak's IS which depicts all the knowledge where the agents yield their judgments about alternatives(agreement,disagreement,neutrality) under the attributes(issues) (b_l) denoted as $\{\xi_{kl}\}_{n\times m}$ for $k = 1, 2, ..., n$, $l = 1, 2, ..., m$.

Example 4.1.1 *A HFIS represented by Table 4.2 is employed to depict the Middle East Conflict, where six agents and five issues are denoted by $x_1, x_2, ..., x_6$ and $b_1, b_2, ..., b_5$ respectively. For instance, $(\chi_{11}) = (u_{11}, v_{11}, w_{11})$, where this triplet evaluates the degree of similarity of agents x_1 and x_2 regarding issue b_1.*

As far as a concern to conflict analysis, all involved attributes(issues) are not of equal importance as some issues could have more weight than others. Therefore we assign the importance degree $w_l(l = 1, 2, ..., m)$ for attributes according to different focuses, advantages, and preferences of analysts. Here for convenience, we have assigned equal weights to all issues.

Definition 4.1.4 *Let x_k and x_q be two objects and $\varphi_{b_l}(x_k, x_q)$ is the auxiliary function depicting the difference of opinion between objects x_k and x_q then weighted average operator denoted as $\varrho(x_k, x_q)$ is defined as follows:*

$$\varrho(x_k, x_q) = \sum_{l=1}^{m} \overline{w_l}\varphi^s_{b_l}(x_k, x_q) : s = 1, 2, 3, ..., d \text{ for } k < q$$

where $\overline{w} = (w_1, w_2, ..., w_m)^T$ is the normalized weight vector of $\varphi_{b_l}(x_k, x_q)$ with $\overline{w_l} \in [0, 1]$ and $\Sigma_{l=1}^{m}\overline{w_l} = 1$, if $\overline{w} = (\frac{1}{m}, \frac{1}{m}, ..., \frac{1}{m})^T$, then

$$\varrho(x_k, x_q) = \sum_{l=1}^{m} \frac{1}{m}\varphi_{b_l}(x_k, x_q)] \text{ for } k < q$$

For better analysis, we may aggregate the opinion of any two individuals, holding the same or opposite attitude towards described issues.

Based on the above definition, we may aggregate the opinion of any two individuals x_k and x_q. To aggregate our normalized HFIS, we define some new operations on HFEs. Let three HFEs be τ, τ_1, and τ_2, and then, $\zeta \geq 0$ and τ^c be the complementary set of τ, then new operations for

HFEs are defined as follows:

- $\tau^c = \bigcup_{\mu \in \tau} \{1 - \mu\}$
- $\zeta\tau = \bigcup_{\mu \in \tau} \{\zeta\mu\}$
- $\tau_1 \oplus \tau_2 = \bigcup_{\mu_1 \in \tau_1, \mu_2 \in \tau_2} \{\mu_1 + \mu_2\}$
- $\tau_1 \otimes \tau_2 = \bigcup_{\mu_1 \in \tau_1, \mu_2 \in \tau_2} \{\mu_1 \mu_2\}$
- $\tau^\zeta = \bigcup_{\mu \in \tau} \{\tau^\zeta\}$

4.1.2 Deriving three-way decisions in hesitant fuzzy environment

Taking into consideration the new normalized HFIS of a conflict problem provided in Table 4.1, in the present section, we develop a novel conflict study model under a hesitant fuzzy environment. Practically, we employ the new measurement of HFEs. We consider the losses as real numbers while the evaluation functions are in form of HFEs. This new conflict study is based on DTRS and Bayesian decision procedure [53]. In [53], the decision-making method is concretely based on losses of DTRS of HFEs. In our proposed model, the Bayesian decision procedure is applied to choose the minimum risk after calculating and comparing the conditional risks of all states by using the given information. Moreover, we drive a three-way decisions [2] and consequently, a mechanism is found for rules of classification of all objects. The conditional probability is also considered as one of the main components, while we employ the Bayesian decision procedure. Here, we replace conditional probability with aggregated evaluation function $\varrho(x_k, x_q)$ which is a HFE. Let $\varrho(x_k, x_q)$ and $1 - (\varrho(x_k, x_q))$ be the probability of conflict and allied space respectively, therefore we have a relationship $\varrho(x_k, x_q) + (\varrho(x_k, x_q))^c = 1$.

The model is based on two states and three actions. Let the set of states be $\Pi = \{C, A\}$ indicating the inclusion of an object in the conflict or allied state respectively. Let a set of three actions $\{a_C, a_N, a_A\}$ where a_C refers action of accepting an object into conflict set i.e $CO(x_k)$, a_N refers action of accepting an object into neutral set i.e; $NE(x_k)$ and a_A refers action of accepting an object into allied set i.e; $AL(x_k)$. We express the losses of accepting these actions regarding two states C and A in Table 4.3, where $(\zeta_{a_C \backslash C}), (\zeta_{a_N \backslash C}), (\zeta_{a_A \backslash C})$ designate the associated losses to

Table 4.3: Loss functions related to different actions

Action	Loss	State
a_C	$(\zeta_{a_C\backslash C})$	C
a_N	$(\zeta_{a_N\backslash C})$	C
a_A	$(\zeta_{a_A\backslash C})$	C
a_C	$(\zeta_{a_C\backslash A})$	A
a_N	$(\zeta_{a_N\backslash A})$	A
a_A	$(\zeta_{a_A\backslash A})$	A

take actions of a_C, a_N, a_A respectively. Similarly $(\zeta_{a_C\backslash A}), (\zeta_{a_N\backslash A}), (\zeta_{a_A\backslash A})$ designate the associated losses to take the same actions when an object be in A. We have losses a_C, a_N, a_A, for conflict and allied states, these losses satisfy the following conditions:

$$\zeta(a_C|C) \leq \zeta(a_N|C) < \zeta(a_A|C) \text{ and } \zeta(a_C|A) \leq \zeta(a_N|A) < \zeta(a_A|A)$$

That is, the loss of including an object x_k that belongs to C into the $CO(x_k)$ is equal to or less than the loss of including x_k into the neutral set $NE(x_k)$, and these two losses are strictly less than the loss of including x_k into the allied set $AL(x_k)$. For these losses, the reverse orders are also applicable for the inclusion of an object not in C. For an object x_k, the expected losses $R(a_C|x_k), R(a_N|x_k), R(a_A|x_k)$ related to the corresponding actions can be calculated as:

$$R(a_C|x_k) = \zeta(a_C|C)((x_k, x_q)) + \zeta(a_C|A)(1 - (x_k, x_q)) \tag{4.3}$$

$$R(a_N|x_k) = \zeta(a_N|C)((x_k, x_q)) + \zeta(a_N|A)(1 - (x_k, x_q)) \tag{4.4}$$

$$R(a_A|x_k) = \zeta(a_A|C)((x_k, x_q)) + \zeta(a_A|A)(1 - (x_k, x_q)) \tag{4.5}$$

where $\zeta(a_p|C) \geq 0, \zeta(a_p|A) \geq 0, p \in (C, N, A)$ and $0 \leq ((x_k, x_q)) \leq 1, 0 \leq ((x_k, x_q))^c \leq 1$

In this situation losses are real numbers but $\varrho(x_k, x_q)$ for $k \neq q$ are in the form of HFNs so we need to consider the situation under the new defined hesitant fuzzy environment. With the accordance of new defined operations of HFEs, Equations 4.3-4.5 can be calculated as:

$$R(a_C|x_k) = \bigcup_{\mu_1 \in (\varrho(x_k,x_q))} \{\zeta(a_C|C)\mu_1\} \oplus \bigcup_{\mu_2 \in (\varrho(x_k,x_q))^c} \{\zeta(a_C|A)\mu_2\} = w_1 + w_2 \tag{4.6}$$

$$R(a_N|x_k) = \bigcup_{\mu_3 \in (\varrho(x_k,x_q))} \{\zeta(a_N|C)\mu_3\} \oplus \bigcup_{\mu_4 \in (\varrho(x_k,x_q))^c} \{\zeta(a_N|A)\mu_4\} = w_3 + w_4 \tag{4.7}$$

$$R(a_A|x_k) = \bigcup_{\mu_5 \in (\varrho(x_k,x_q))} \{\zeta(a_A|C)\mu_5\} \oplus \bigcup_{\mu_6 \in (\varrho(x_k,x_q))^c} \{\zeta(a_A|A)\mu_6\} = w_5 + w_6 \tag{4.8}$$

where

$$w_1 = \bigcup_{\mu_1 \in (\varrho(x_k,x_q))} \{\zeta(a_C|C)\mu_1\}, w_2 = \bigcup_{\mu_2 \in (\varrho(x_k,x_q))^c} \{\zeta(a_C|A)\mu_2\},$$

$$w_3 = \bigcup_{\mu_3 \in (\varrho(x_k,x_q))} \{\zeta(a_N|C)\mu_3\}, w_4 = \bigcup_{\mu_4 \in (\varrho(x_k,x_q))^c} \{\zeta(a_N|A)\mu_4\},$$

$$w_5 = \bigcup_{\mu_5 \in (\varrho(x_k,x_q))} \{\zeta(a_A|C)\mu_5\}, w_6 = \bigcup_{\mu_6 \in ((x_k,x_q))^c} \{\zeta(a_A|A)\mu_6\}.$$

By using the new operations of HFEs, these expected losses can be re-expressed as follows:

$$R(a_C|x_k) = w_1 + w_2 = \bigcup_{\mu_1' \in w_1, \mu_2' \in w_2} \{\mu_1' + \mu_2'\}$$

$$R(a_N|x_k) = w_3 + w_4 = \bigcup_{\mu_3' \in w_3, \mu_4' \in w_4} \{\mu_3' + \mu_4'\}$$

$$R(a_A|x_k) = w_5 + w_6 = \bigcup_{\mu_5' \in w_5, \mu_6' \in w_6} \{\mu_5' + \mu_6'\}$$

Hence it may be concluded that $R(a_p|x_k)(p = C, N, A)$ are HFEs with the same cardinality i.e; $\#R(a_C|x_k) = \#R(a_N|x_k) = \#R(a_A|x_k)$. Now, if the value of $Pr(C|x_k)$ is constant, the variations of the expected losses with the losses were discussed by Lang and Liu in [53]. Inspired by that techniques, we may deduce the following result.

Preposition 1: For $R(a_C|x_k)$, let $\tau_1 = \zeta(a_C|C)\mu_1 + \zeta(a_C|A)\mu_2$ where $\mu_1 \in (\varrho(x_k, x_q))$ and $\mu_2 \in (\varrho(x_k, x_q))^c$, then τ_1 is non-monotone decreasing with any increase in μ_1 and μ_2, when the loss functions $\zeta(a_C|C), \zeta(a_C|A)$ are constant.

Proof : Here in this preposition, we consider μ_1, μ_2 are two independent variables of τ_1. To examine the variation of τ_1, we evaluate the partial derivative of τ_1 with respect to μ_1 and μ_2, respectively. When μ_2 is constant, we evaluate the partial derivative of τ_1 with respect to μ_1 as $\frac{\partial \tau_1}{\partial \mu_1} = \zeta(a_C|C)$. As $\zeta(a_C|A) \geq 0, \zeta(a_C|C) \geq 0$.

Hence, it may proved that $\frac{\partial \tau_1}{\partial \mu_1} \geq 0$.

Now consider μ_1 as constant, by taking partial derivative of τ_1 with respect to μ_2 can be calculated as $\frac{\partial \tau_1}{\partial \mu_2} = \zeta(a_C|A)$. As $\zeta(a_C|A) \geq 0, \zeta(a_C|C) \geq 0$. Hence, it may proved that $\frac{\partial \tau_1}{\partial \mu_2} \geq 0$. By analysing partial derivative of τ_1 w.r.t μ_1 and μ_2, it may deduce that $\frac{\partial \tau_1}{\partial \mu_1} \geq 0$ and $\frac{\partial \tau_1}{\partial \mu_2} \geq 0$.

Therefore, the statement of Preposition 1 is satisfied.

Proposition 1 demonstrates that if the values of μ_1 and μ_2 increase, then we have variation in value of τ_1. According to results of Proposition 1, we can deduce two more corollaries based on Equations 4.7 and Equation 4.8.

Corollary 1: For $R(a_N|x_k)$, let $\tau_1 = \zeta(a_N|C)\mu_3 + \zeta(a_N|A)\mu_4$ where $\mu_3 \in (\varrho(x_k, x_q))$ and $\mu_4 \in (\varrho(x_k, x_q))^c$, then τ_1 is non-monotone decreasing with any increase in μ_3 and μ_4, when the loss functions $\zeta(a_N|C), \zeta(a_N|A)$ are constant.

Corollary 2: For $R(a_A|x_k)$, $\tau_1 = \zeta(a_A|C)\mu_5 + \zeta(a_A|A)\mu_6$ where $\mu_5 \in (\varrho(x_k, x_q))$ and $\mu_6 \in (\varrho(x_k, x_q))^c$, then τ_1 is non-monotone decreasing with any increase in μ_5 and μ_6, when the loss functions $\zeta(a_A|C), \zeta(a_A|A)$ are constant.

By using the results reported in [41,88] the Bayesian decision procedure provides the minimum-cost decision laws are described as follows:

If $R(a_C|x_k) \preceq R(a_N|x_k)$ and $R(a_C|x_k) \preceq R(a_A|x_k)$ then $x_k \in CO\{x_q\}$

If $R(a_N|x_k) \preceq R(a_C|x_k)$ and $R(a_N|x_k) \preceq R(a_A|x_k)$ then $x_k \in NE\{x_q\}$

If $R(a_A|x_k) \preceq R(a_C|x_k)$ and $R(a_A|x_k) \preceq R(a_N|x_k)$ then $x_k \in AL\{x_q\}$

Where $\preceq$ is the equivalent or inferior.

These are decision rules for conflict analysis. To compare the expected losses we may re-express these relations as follows:

$(C1)$ If $sc(R(a_C|x_k)) \leq sc(R(a_N|x_k))$ and $sc(R(a_C|x_k)) \leq sc(R(a_A|x_k))$ then $x_k \in CO\{x_q\}$

$(N1)$ If $sc(R(a_N|x_k)) \leq sc(R(a_C|x_k))$ and $sc(R(a_N|x_k)) \leq sc(R(a_A|x_k))$ then $x_k \in NE\{x_q\}$

$(A1)$ If $sc(R(a_A|x_k)) \leq sc(R(a_C|x_k))$ and $sc(R(a_A|x_k)) \leq sc(R(a_N|x_k))$ then $x_k \in AL\{x_q\}$

where

$$sc(R(a_C|x_k)) = \frac{1}{l(R(a_C|x_k)} \sum_{u_1 \in R(a_C|x_k)} u_1 \tag{4.9}$$

$$sc(R(a_N|x_k)) = \frac{1}{l(R(a_N|x_k)} \sum_{u_2 \in R(a_N|x_k)} u_2 \tag{4.10}$$

$$sc(R(a_A|x_k)) = \frac{1}{l(R(a_A|x_k)} \sum_{u_3 \in R(a_A|x_i)} u_3 \tag{4.11}$$

where $l(R(a_C|x_k)), l(R(a_N|x_k)), l(R(a_A|x_k))$ indicate the number of elements found in the $(R(a_C|x_k))$, $(R(a_N|x_k))$ and $(R(a_A|x_k))$. Using score functions, $(R(a_C|x_k))$, $(R(a_N|x_k))$ and $(R(a_A|x_k))$ are transformed into precise real values and the hesitancy of the expected losses is eliminated. Now, the decision rule for an object x_k is derived by evaluating score i.e; $(R(a_C|x_k)), (R(a_N|x_k))$ and $(R(a_A|x_k)$.

Algorithmic description to formulate $CO(x_k), NE(x_k)$ and $AL(x_k)$.

Input: IS for conflict situation $S = [(\xi_{kl})]_{n\times m}$

Step 1. Calculate normalized HFIS $\bar{S} = [\bar{\xi_{kl}}]_{n\times m}$.

Step 2. Compute the values of auxiliary function $\varphi_l(x_k, x_q)$ for $k \neq q, k = 1, 2, ..., n, l = 1, 2, ..., m$ for all objects of X based on Equation 4.2.

Step 3. Compute the aggregated opinion $\varrho_I(x_k, x_q)$ for $k \neq q, k = 1, 2, ..., n, l = 1, 2, ..., m$ by using Definition 4.1.4.

Step 4. Compute the aggregated opinion matrix $A_g = |\varrho_I(x_k, x_q)|_{m\times n}$ for $k \neq q$.

Step 5. Calculate the losses $\zeta(a_C|C), \zeta(a_N|C), \zeta(a_A|C)$ and $\zeta(a_C|A), \zeta(a_N|A), \zeta(a_A|A)$ by using Bayesian decision procedure.

Step 6. Calculate successively expected loss functions $R(a_C|x_k), R(a_N|x_k), R(a_A|x_k)$ by using Equations 4.6-4.8 for the object x_k under a given value of $(\varrho(x_k, x_q))$.

Step 7. Calculate the scores of expected losses by using Equations 4.9 -4.11.

Step 8. Classify all objects into allied, neutral and conflict sets based on the three-way classification rules given by $(C1 - A1)$.

4.2 Application of proposed three-way hesitant fuzzy conflict analysis model

To validate the proposed algorithm, we consider the case of the Middle East conflict problem [36,38] with HFIS given in Table 4.4.

4.2.1 Application of general method based on the scores functions

Step 1. We normalize the initial IS $[\xi_{kl}]_{n\times m}$. The normalized IS is shown in Table 4.5.

Table 4.4: Information system for conflict problem

U	b_1	b_2	b_3	b_4	b_5
x_1	$(0.09, 0.26, 0.93)$	$(0.31, 0.42, 1.00)$	$(-0.23, 0.57, 0.95$	$(-0.78, -0.70)$	$(0.41, 0.93)$
x_2	$(-0.10, -0.10)$	$(0.05, 0.57, 1.00)$	$(-0.80, 0.30)$	$(-0.08, -0.17, 0.32)$	$(-0.80, 0.89)$
x_3	$(-1.00, -0.43)$	$(-0.99, -0.48, 0.07)$	$(-0.97, -0.46, 0.25)$	$(-0.75, -0.74, -0.41)$	$(0.22, 0.22)$
x_4	$(-0.60, -0.36, 0.20)$	$(0.68, 0.98, 1.00)$	$(-1.00, -0.05, 0.10)$	$(0.59, 0.75, 0.99)$	$(0.98, 1.00)$
x_5	$(-0.40, 0.75)$	$(-0.39, 0.38, 0.59)$	$(-0.87, -0.35)$	$(0.48, 0.87, 0.96)$	$(0.18, 0.77, 0.86)$
x_6	$(-0.30, -0.26)$	$(0.10, 0.80)$	$(-0.20, 0.00, 0.10)$	$(-0.90, 0.43, 0.55)$	$(-1.00, -0.10, 0.30)$

Table 4.5: Normalized Information system for conflict problem.

U	b_1	b_2	b_3	b_4	b_5
x_1	$(0.09, 0.26, 0.93)$	$(0.31, 0.42, 1.00)$	$(-0.23, 0.57, 0.95)$	$(-0.78, -0.70, -0.70)$	$(0.41, 0.93, 0.93)$
x_2	$(-0.10, -0.10, -0.10)$	$(0.05, 0.57, 1.00)$	$(-0.80, 0.30, 0.30)$	$(-0.08, -0.17, 0.32)$	$(-0.80, 0.89, 0.89)$
x_3	$(-1.00, -0.43, -0.43)$	$(-0.99, -0.48, 0.07)$	$(-0.97, -0.46, 0.25)$	$(-0.75, -0.74, -0.41)$	$(0.22, 0.22, 0.22)$
x_4	$(-0.60, -0.36, 0.20)$	$(0.68, 0.98, 1.00)$	$(-1.00, -0.05, 0.10)$	$(0.59, 0.75, 0.99)$	$(0.98, 1.00, 1.00)$
x_5	$(-0.40, 0.75, 0.75)$	$(-0.39, 0.38, 0.59)$	$(-0.87, -0.35, -0.35)$	$(0.48, 0.87, 0.96)$	$(0.18, 0.77, 0.86)$
x_6	$(-0.30, -0.26, -0.26)$	$(0.10, 0.80, 0.80)$	$(-0.20, 0.00, 0.10)$	$(-0.90, 0.43, 0.55)$	$(-1.00, -0.10, 0.30)$

Step 2. We compute the values of auxiliary function $\varphi_I(x_k, x_q)$ for all $x_k \neq x_q$ of X based on Equations 4.2.

Let us consider two objects x_1 and x_2 from set X of IS $S = (X, I)$ shown in Table 4.4. The auxiliary function $\varphi_I(x_1, x_2)$ can be calculated by using Definition 4.2 as follows:

$\varphi_{b_1}(x_1, x_2) = (0.55, 0.59, 0.76)$; $\varphi_{b_2}(x_1, x_2) = (0.13, 0.08, 0.00)$; $\varphi_{b_3}(x_1, x_2) = (0.29, 0.14, 0.33)$;
$\varphi_{b_4}(x_1, x_2) = (0.35, 0.27, 0.76)$; $\varphi_{b_5}(x_1, x_2) = (0.80, 0.02, 0.02)$.
Following same manners, all other auxiliary functions can be evaluated as depicted in Table 4.6.

Step 3. We calculate the aggregated opinion function $\varrho_I(x_k, x_q)$ by using Definition 4.1.4.

For the IS depicted in Table 4.4, by using Definition 4.1.4, we may calculate the aggregated opinion of objects x_1 and x_2, for $n = 6$ and $I = \{b_1, b_2, b_3, b_4, b_5\}$, as follows:

$$\varrho_I(x_1, x_2) = \{\tfrac{1}{5}(0.55 + 0.13 + 0.29 + 0.35 + 0.80), \tfrac{1}{5}(0.59 + 0.08 + 0.14 + 0.27 + 0.02),$$
$$\tfrac{1}{5}(0.76 + 0 + 0.33 + 0.76 + 0.02)\}$$
$$= (0.424, 0.220, 0.374)$$

Table 4.6: Auxiliary functions for object x_k

U	b_1	b_2	b_3	b_4	b_5
$\varphi_b(x_1, x_2)$	$(0.55, 0.59, 0.76)$	$(0.13, 0.08, 0.00)$	$(0.29, 0.14, 0.33)$	$(0.35, 0.27, 0.76)$	$(0.80, 0.02, 0.02)$
$\varphi_b(x_1, x_3)$	$(0.77, 0.67, 0.84)$	$(0.83, 0.73, 0.47)$	$(0.37, 0.85, 0.35)$	$(0.02, 0.02, 0.15)$	$(0.10, 0.36, 0.36)$
$\varphi_b(x_1, x_4)$	$(0.67, 0.66, 0.37)$	$(0.19, 0.28, 0.00)$	$(0.39, 0.75, 0.43)$	$(0.84, 0.86, 0.92)$	$(0.29, 0.04, 0.04)$
$\varphi_b(x_1, x_5)$	$(0.62, 0.25, 0.09)$	$(0.68, 0.02, 0.21)$	$(0.32, 0.83, 0.83)$	$(0.82, 0.89, 0.92)$	$(0.12, 0.08, 0.04)$
$\varphi_b(x_1, x_6)$	$(0.60, 0.63, 0.80)$	$(0.11, 0.19, 0.10)$	$(0.02, 0.50, 0.43)$	$(0.06, 0.78, 0.81)$	$(0.85, 0.76, 0.32)$
$\varphi_b(x_2, x_3)$	$(0.50, 0.17, 0.17)$	$(0.76, 0.76, 0.47)$	$(0.09, 0.69, 0.03)$	$(0.34, 0.29, 0.68)$	$(0.76, 0.34, 0.34)$
$\varphi_b(x_2, x_4)$	$(0.50, 0.13, 0.58)$	$(0.32, 0.21, 0.00)$	$(0.10, 0.59, 0.10)$	$(0.67, 0.73, 0.34)$	$(0.95, 0.06, 0.06)$
$\varphi_b(x_2, x_5)$	$(0.50, 0.71, 0.71)$	$(0.61, 0.10, 0.21)$	$(0.04, 0.66, 0.66)$	$(0.64, 0.76, 0.32)$	$(0.75, 0.06, 0.02)$
$\varphi_b(x_2, x_6)$	$(0.50, 0.08, 0.08)$	$(0.03, 0.12, 0.10)$	$(0.30, 0.50, 0.10)$	$(0.41, 0.65, 0.12)$	$(0.10, 0.75, 0.30)$
$\varphi_b(x_3, x_4)$	$(0.20, 0.04, 0.66)$	$(0.92, 0.87, 0.47)$	$(0.02, 0.21, 0.08)$	$(0.84, 0.87, 0.85)$	$(0.38, 0.39, 0.39)$
$\varphi_b(x_3, x_5)$	$(0.30, 0.80, 0.80)$	$(0.30, 0.72, 0.26)$	$(0.05, 0.06, 0.65)$	$(0.81, 0.90, 0.84)$	$(0.02, 0.28, 0.32)$
$\varphi_b(x_3, x_6)$	$(0.35, 0.09, 0.09)$	$(0.77, 0.82, 0.37)$	$(0.39, 0.50, 0.08)$	$(0.08, 0.79, 0.74)$	$(0.81, 0.58, 0.04)$
$\varphi_b(x_4, x_5)$	$(0.10, 0.78, 0.28)$	$(0.77, 0.30, 0.21)$	$(0.07, 0.15, 0.61)$	$(0.06, 0.06, 0.02)$	$(0.40, 0.12, 0.07)$
$\varphi_b(x_4, x_6)$	$(0.15, 0.05, 0.62)$	$(0.29, 0.09, 0.10)$	$(0.40, 0.50, 0.00)$	$(0.87, 0.16, 0.22)$	$(1.00, 0.78, 0.35)$
$\varphi_b(x_5, x_6)$	$(0.05, 0.75, 0.75)$	$(0.62, 0.21, 0.11)$	$(0.34, 0.50, 0.61)$	$(0.85, 0.22, 0.21)$	$(0.80, 0.74, 0.28)$

We can calculate aggregated opinion of all other objects by using the same manners.

Step 4. The aggregated opinion matrix of the IS shown in Table 4.5 is provided in Table 4.7.

Table 4.7: Matrix of aggregated opinions of objects for conflict situation

X	x_1	x_2	x_3	x_4	x_5	x_6
x_1						
x_2	$(0.424, 0.22, 0.374)$					
x_3	$(0.418, 0.526, 0.434)$	$(0.490, 0.450, 0.338)$				
x_4	$(0.476, 0.702, 0.352)$	$(0.508, 0.344, 0.220)$	$(0.472, 0.476, 0.490)$			
x_5	$(0.512, 0.414, 0.418)$	$(0.508, 0.458, 0.384)$	$(0.296, 0.552, 0.574)$	$(0.280, 0.282, 0.238)$		
x_6	$(0.328, 0.572, 0.492)$	$(0.268, 0.420, 0.140)$	$(0.480, 0.556, 0.264)$	$(0.542, 0.316, 0.258)$	$(0.532, 0.484, 0.392)$	

Step 5. We calculate the losses $\zeta(a_C|C), \zeta(a_N|C), \zeta(a_A|C)$ and $\zeta(a_C|A), \zeta(a_N|A), \zeta(a_A|A)$ by using DTRSs.

Step 6. We calculate expected loss functions $R(a_C|x_k), R(a_N|x_k), R(a_A|x_k)$ by using Equations 4.6-4.8.

Table 4.8: Loss functions for IS given in Table 4.5

Action\states	C	A
a_C	$(\zeta_{a_C \backslash C}) = 0.1$	$(\zeta_{a_C \backslash A}) = 0.7$
a_N	$(\zeta_{a_N \backslash C}) = 0.3$	$(\zeta_{a_N \backslash A}) = 0.5$
a_A	$(\zeta_{a_A \backslash C}) = 0.9$	$(\zeta_{a_A \backslash A}) = 0.0$

For instance, consider object x_1 then incurred loss functions for taking object x_2 as conflict, neutral and ally, can be represented as follows:

$$R(a_C|x_2) = \zeta(a_C|C)\varrho_I(x_1, x_2) + \zeta(a_C|A)\varrho_I^c(x_1, x_2)$$

$$R(a_N|x_2) = \zeta(a_N|C)\varrho_I(x_1, x_2) + \zeta(a_N|A)\varrho_I^c(x_1, x_2)$$

$$R(a_A|x_2) = \zeta(a_A|C)\varrho_I(x_1, x_2) + \zeta(a_A|A)\varrho_I^c(x_1, x_2)$$

Hence

$$R(a_C|x_2) = 0.1(0.424, 0.22, 0.374) + 0.7(0.576, 0.780, 0.626)$$

$$R(a_N|x_2) = 0.3(0.424, 0.22, 0.374) + 0.5(0.576, 0.780, 0.626)$$

$$R(a_A|x_2) = 0.9(0.424, 0.22, 0.374) + 0.0(0.576, 0.780, 0.626)$$

We have

$$R(a_C|x_2) = \{0.4456, 0.5680, 0.4756\}$$

$$R(a_N|x_2) = \{0.4152, 0.3540, 0.4255\}$$

$$R(a_A|x_2) = \{0.3816, 0.1980, 0.3366\}$$

Similarly, all other losses can be obtained as depicted in Table 4.9

Step 7. We compute the score of expected losses by using Equations 4.9-4.11.

$$sc(R(a_C|x_2)) = \frac{0.4456, 0.5680, 0.4756}{3} = 0.4964$$

Table 4.9: Expected loss functions for all objects

objects	$R(a_C\|x_k)$	$R(a_N\|x_k)$	$R(a_A\|x_k)$
(x_1, x_2)	$(0.4456, 0.5680, 0.4756)$	$(0.4152, 0.3540, 0.4255)$	$(0.3816, 0.1980, 0.3366)$
(x_1, x_3)	$(0.4492, 0.3844, 0.4396)$	$(0.4164, 0.3948, 0.4132)$	$(0.3762, 0.4734, 0.3906)$
(x_1, x_4)	$(0.4144, 0.3892, 0.4888)$	$(0.4048, 0.3964, 0.4296)$	$(0.4284, 0.4662, 0.3168)$
(x_1, x_5)	$(0.3928, 0.4516, 0.2764)$	$(0.3976, 0.4172, 0.4164)$	$(0.4608, 0.3726, 0.3726)$
(x_1, x_6)	$(0.5032, 0.3568, 0.4048)$	$(0.4344, 0.3856, 0.4016)$	$(0.2952, 0.5148, 0.4428)$
(x_2, x_3)	$(0.4060, 0.4300, 0.4972)$	$(0.4020, 0.4100, 0.4323)$	$(0.4410, 0.4050, 0.3042)$
(x_2, x_4)	$(0.3952, 0.4936, 0.5680)$	$(0.3984, 0.43120.4560)$	$(0.4572, 0.3096, 0.1980)$
(x_2, x_5)	$(0.3952, 0.4249, 0.4696)$	$(0.3984, 0.4084, 0.5464)$	$(0.4572, 0.4122, 0.3456)$
(x_2, x_6)	$(0.5392, 0.4480, 0.6160)$	$(0.4464, 0.4160, 0.4720)$	$(0.2412, 0.3780.0.1260)$
(x_3, x_4)	$(0.4168, 0.4144, 0.4046)$	$(0.4168, 0.4018, 0.4020)$	$(0.4248, 0.4284, 0.4410)$
(x_3, x_5)	$(0.5224, 0.3688, 0.3556)$	$(0.4408, 0.3896, 0.3852)$	$(0.2664, 0.1490, 0.1540)$
(x_3, x_6)	$(0.4120, 0.3664, 0.5416)$	$(0.4040, 0.3888, 0.4472)$	$(0.432, 0.5004, 0.2376)$
(x_4, x_5)	$(0.5320, 0.5308, 0.5572)$	$(0.4440, 0.4436, 0.4524)$	$(0.2520, 0.2538, 0.2142)$
(x_4, x_6)	$(0.3748, 0.5104, 0.5452)$	$(0.4832, 0.5736, 0.4484)$	$(0.4878, 0.2844, 0.2322)$
(x_5, x_6)	$(0.3808, 0.4096, 0.4648)$	$(0.3936, 0.4032, 0.4216)$	$(0.4788, 0.4356, 0.3528)$

$$sc(R(a_N|x_2)) = \frac{0.4152, 0.3540, 0.4255}{3} = 0.3982$$

$$sc(R(a_A|x_2)) = \frac{0.3816, 0.1980, 0.3366}{3} = 0.3054$$

By following same manners, we compute all scores of expected loss functions as shown in Table 4.10.

Step 8. We classify all objects by using TWD rules $(C_1)-(A_1)$. Hence for deciding about the inclusion of object x_2, we get $sc(R(a_A|x_2)) \preceq sc(R(a_C|x_2))$ and $sc(R(a_A|x_2)) \preceq sc(R(a_N|x_2))$ then $x_2 \in AL\{x_1\}$

Hence $x_2 \in AL\{x_1\}$

Output: Following same manners we obtain $CON(x_k)$, $NEU(x_k)$, $ALL(x_k)$, for $k = 1, 2, ..., 6$, as shown in Table 4.11.

Table 4.10: Score functions for all objects

Qbjects	$R(a_C\|x_k)$	$R(a_N\|x_k)$	$R(a_A\|x_k)$
(x_1, x_2)	0.4964	0.3982	0.3054
(x_1, x_3)	0.4244	0.4081	0.4134
(x_1, x_4)	0.4308	0.4102	0.4037
(x_1, x_5)	0.3736	0.4104	0.4020
(x_1, x_6)	0.4216	0.4072	0.4176
(x_2, x_3)	0.4444	0.4147	0.3834
(x_2, x_4)	0.4856	0.4285	0.3216
(x_2, x_5)	0.4299	0.4510	0.4050
(x_2, x_6)	0.5344	0.4448	0.2484
(x_3, x_4)	0.4119	0.4068	0.4314
(x_3, x_5)	0.4156	0.4052	0.1898
(x_3, x_6)	0.4400	0.4133	0.3900
(x_4, x_5)	0.5400	0.4475	0.2400
(x_4, x_6)	0.4768	0.5017	0.3348
(x_5, x_6)	0.4184)	0.4061	0.4224

Table 4.11: Allied, Neutral and Conflict sets for the Middle East conflict with proposed model

U	CO(x_k)	NE(x_k)	AL(x_k)
x_1	$\{x_5\}$	$\{x_3, x_6\}$	$\{x_2, x_4\}$
x_2	$\{\}$	$\{\}$	$\{x_1, x_3, x_4, x_5, x_6\}$
x_3	$\{\}$	$\{x_1, x_4\}$	$\{x_2, x_5, x_6\}$
x_4	$\{\}$	$\{x_3\}$	$\{x_1, x_2, x_5, x_6\}$
x_5	$\{x_1\}$	$\{x_6\}$	$\{x_2, x_3, x_4\}$
x_6	$\{\}$	$\{x_1, x_5\}$	$\{x_2, x_3, x_4\}$

4.2.2 A ranking method of possibility degrees

We have computed the score functions of expected losses by using the idea of average. The expected losses $R(a_C), R(a_N), R(a_A)$ are the HFEs as represented by Equations 4.6-4.8. Hence, when we use the method of score functions then we may lose some information about expected losses so we need a more effective method to compare the expected losses of decision rules $(C_1 - N_1)$ expressed in Section 4.1. We employ the technique of degree of possibility ranking by assuming that the membership degree of the element x_k is stochastic. This method was initially introduced by Lahdelma and Salminen [91], motivated by a stochastic multi-criteria acceptability analysis (SMAA) method. Furthermore, the method was effectively applied by Zhu and Xu [92] into analytic hierarchy process-hesitant group decision analysis and numerical preference relations. With the fast development in the field of information technology, we may attain relevant technical support. To deal with HFEs, we may utilize the stochastic technique to obtain a sequence of exact values of the expected losses. By using Equations 4.6-4.8 and outcomes derived in the previous section, ultimately, we can generate a set of expected losses from $R(a_C/x_k) = \tau_C, R(a_N/x_k) = \tau_N$ and $R(a_A/x_k) = \tau_A$ i.e; $R(a_C/x_k) = \tau_1, R(a_N/x_k) = \tau_2, R(a_A/x_k) = \tau_3$, where $\tau_1 \in \tau_C, \tau_2 \in \tau_N$ and $\tau_3 \in \tau_A$ and

$$\tau_1 = \zeta(a_C/C)t_1 + \zeta(a_C/C)t_2, \tag{4.12}$$

$$\tau_2 = \zeta(a_N/C)t_3 + \zeta(a_N/C)t_4, \tag{4.13}$$

$$\tau_3 = \zeta(a_A/C)t_5 + \zeta(a_A/C)t_6. \tag{4.14}$$

where $t_1 \in \varrho_I(x_k, x_q), t_2 \in \varrho_I^c(x_k, x_q), t_3 \in \varrho_I(x_k, x_q), t_4 \in \varrho_I^c(x_k, x_q), t_5 \in \varrho_I(x_k, x_q)$, and $t_6 \in \varrho_I^c(x_k, x_q)$. Here, τ_1, τ_2 and τ_3 are treated as numeric not as sets. When we consider the different combination of the values τ_1, τ_2 and τ_3, we can derive different decision rules.

For $R(a_C/x) = \tau_1, R(a_N/x) = \tau_2, R(a_A/x) = \tau_3$, the decision laws can be re-stated as follows:

(C2) if $(\tau_1 < \tau_2) \vee (\tau_1 < \tau_3) \Longrightarrow x_k \in CO\{x_q\}$

(N2) if $(\tau_2 < \tau_1) \vee (\tau_2 < \tau_3) \Longrightarrow x_k \in NE\{x_q\}$

(A2) if $(\tau_3 < \tau_1) \vee (\tau_3 < \tau_2) \Longrightarrow x_k \in AL\{x_q\}$

By taking into account the new decision rules, for inclusion of an object x_k in the allied, neutral or conflict set of object x_q, the extract results can be found based on these rules when the values of expected losses are determined. We consider loss functions as real numbers and expected losses as HFEs. By using Equations 4.12- 4.14. We can obtain different values of the expected losses $R(a_C/x) = \tau_1, R(a_N/x) = \tau_2, R(a_A/x) = \tau_3$. For any element x_k, the number of maximum simulation is

$$M(x_k) = \#(\tau_C) \times \#(\tau_N) \times \#(\tau_A)$$

For each element x_k, each simulation provides an extract result based on the decision laws $(C2 - N2)$. Through this process, we obtain different decisions rules based on the inclusion of elements in conflict, neutral and allied sets. We consider that number of rules for conflict be $M_C(x_k)$, number of rules for neutral be $M_N(x_k)$ and rules for alliance be $M_A(x_k)$. Here we remove the redundancy, we obtain with the values of expected losses having the same elements. After calculating the values of $M(x_k)$,$M_C(x_k)$,$M_N(x_k)$ and $M_A(x_k)$ with the use of stochastic simulation, by exerting the enumerated results, we may acquire the proportions of the object x_k classifying into each decision rule.

Let $P_C(x_k)$, $P_N(x_k)$ and $P_A(x_k)$ refer the proportion of x_k to the conflict rule $CO(x_k)$, neutral rule $NE(x_k)$ and alliance $AL(x_k)$ rule respectively then

$$P_C(x_k) = \frac{M_C(x_k)}{M(x_k)} \times 100\% \tag{4.15}$$

$$P_N(x_k) = \frac{M_N(x_k)}{M(x_k)} \times 100\% \tag{4.16}$$

$$P_A(x_k) = \frac{M_A(x_k)}{M(x_k)} \times 100\% \tag{4.17}$$

Hence, the final decision rules for inclusion of an object in $CO(x_k)$, $NE(x_k)$ and $AL(x_k)$ are calculated as follows:

(C3) if $(P_C(x_k)) \geq (P_N(x_k))$ and $(P_C(x_k)) \geq (P_A(x_k)) \Longrightarrow x_k \in CO\{x_q\}$

(N3) if $(P_N(x_k)) \geq (P_C(x_k))$ and $(P_N(x_k)) \geq (P_A(x_k)) \Longrightarrow x_k \in NE\{x_q\}$

(A3) if $(P_A(x_k)) \geq (P_C(x_k))$ and $(P_A(x_k)) \geq (P_N(x_k)) \Longrightarrow x_k \in AL\{x_q\}$

The decision rules ($C3-A3$) are utilized the enumerated results of the stochastic process to formulate the final criteria for the inclusion of objects in three regions. Consequently, the supreme proportion of an object decides the inclusion of that object in a specific set.

Algorithm2: Algorithmic description to formulate $CO(x_k)$, $NE(x_k)$ and $AL(x_k)$.

Input: $R(a_C/x) = \tau_C, R(a_N/x) = \tau_N, R(a_A/x) = \tau_A$

Step 1. Compute the total number of simulation $M(x_k)$ based on the expected losses.

Step 2. Calculate $M_C(x_k), M_N(x_k), M_A(x_k)$ by using the corresponding extract values for expected losses.

Step 3. Formulate the extract results for the object x_k by using the decision rules ($C2 - A2$).

Step 4. By striving the enumerated results, acquire the proportions $P_C(x_k)$, $P_N(x_k)$ and $P_A(x_k)$ by using Equations 4.15-4.17.

Step 5. Classify x_k into $CO(x_k)$, $NE(x_k)$ and $AL(x_k)$ by using the decision rules ($C3 - A3$).

Step 6. Repeat the steps 1 to 5 for all the objects.

Output: $CO(x_k)$, $NE(x_k)$ and $AL(x_k)$.

4.2.3 Application of ranking method of possibility degree

Now we utilize the ranking method of possibility degree for conflict situation provided by Table 4.5, which is mainly based on the algorithmic description of the ranking method of possibility degree. Following the steps of Algorithm 2, if we first consider the object x_2 for deciding its inclusion in ally, neutral or conflict sets of object x_1 then we obtain the results as follows:

$M(x_2) = 27, M_{AL}(x_2) = 24, M_{NE}(x_2) = 03$ and $M_{CO}(x_2) = 0$

Hence, $P_{AL}(x_2) = 88.88\%, P_{NE}(x_2) = 11.11\%, P_{CO}(x_2) = 0\%$

Based on the above results, the proportion of object x_2 classifying in the conflict rule is greater than the neutral and alliance rule. By employing the conditions provided by Equations 4.15-4.17, we may include object x_2 in $AL(x_1)$. By repeating the same steps for the rest of the objects, we obtained the same results as shown in Table 4.11.

4.3 A comparison with existing conflict study models

Pawlak's conflict study model [34] uses three values $+1, 0, -1$ to rate the attitudes(positive, negative, neutral) of agents towards an issue. The model failed to demonstrate the degree of three attitudes of agents. In many practical situations, agents prefer to depict their attitudes with a combination of positive and negative values. Most of the recent models are failed to manage such scenarios as the involved agents have no freedom to partially agreed or disagreed and are restricted to take some definite viewpoint. Therefore, to deal the hesitancy of partial agreement or disagreement, a more flexible model is needed to resolve such conflict problems. Our proposed model proves to be more realistic as the IS with uncertain attitudes of agents can be restructured by using an auxiliary function that limits the agents' opinion in $[0, 1]$. Then, the hesitancy of agents' attitudes can be dealt more sensibly by using the aggregated opinion function and associated loss functions. In the conflict analysis model investigated by Lang et al. [58] based on the pythagorean IS, the positive(negative) attitude of an agent towards an issue is presented by a pair of real numbers that is certainly not sufficient to describe the vantage point of an agent towards some particular issue. Later on, Li et al. [59] presented a conflict study model by employing a range of fuzzy numbers denoted by $\tilde{A} = (l, m, u)$ to depict the attitude of agents and described the total attitude $\triangle s$ of agents about issues. Though the model claims that it uses the three most indicative fuzzy numbers to illustrate the agent's attitude but in fact, it is not enough to deal with an IS in which agents' opinions are based on both positive and negative values. By analysing the aforementioned aspects, this study established a novel three-way conflict study model that allows decision makers to provide their opinions independently using the multiple values in range from -1 to $+1$ instead of providing their opinion with strict values. This setting nicely models the hesitancy of agents' attitude. In real world conflict problems, all issues have not same worth. Consequently, we redefine the difference measure of opinion associated to any pair of objects by employing aggregated opinion operator. Most recently, Yi et al. [61] proposed a three-way conflict study model founded on HFIS. The model used the preliminary idea of auxiliary function and a threshold pair to analyze the conflict and showed the coalitions by using graph theory. Our proposed model works differently as it mainly relies on aggregated opinion functions and the associated loss functions, calculation of threshold pair is not a task here. Secondly, while we deal with huge data sets it is a not convenient to investigate the conflict, model the coalition based on binary relation, and convert these coalitions to maximal sub-graphs. On the other hand, our proposed model yields more realistic and less time-consuming. Particularly, when ISs are based on more complex agents' attitude, our proposed model derives more accurate results and validate the efficiency of the proposed approach in dealing with complex real-life conflict problems.

Chapter 5

T-spherical fuzzy Frank aggregation operators and their application to decision making with unknown weight information

This chapter presents virtual MCDM and TWD techniques to deal with decision analysis with provided information based on T-spherical fuzzy data with all fully unknown weights of criteria. For this, we introduced some generalized operational laws, namely Frank operational laws for T-SFNs using Frank t-norm and t-conorm. Moreover, by employing these developed operations, a range of T-spherical fuzzy aggregation operators is provided to aggregate T-spherical fuzzy information effectively. Considering the significance of ordered position and argument itself, the notions of T-SFFHA and T-SFFHG are provided. Some advisable properties and particular cases related to these operators are also investigated comprehensibly. Afterward, we examined the entropy measure and its potential worth to fulfill the desirable properties. We used it to determine criteria weight in the proposed aggregation-based MCDM and TWD methods. Later, we examined the impact of the parameters τ and t in the decision procedure and reported the stability stage of sorting results. To enhance the superiority and viability of the suggested approach two descriptive examples are provided. The research work presented in this chapter is published in [93]

5.1 Frank operations of T-spherical fuzzy sets

This section is dedicated to presenting Frank operations of T-SFNs and studying some interesting properties of these operations.

Definition 5.1.1 *Let $\mathcal{S}_1 = (\sigma_1, \vartheta_1, \varrho_1)$ and $\mathcal{S}_2 = (\sigma_2, \vartheta_2, \varrho_2)$ be two T-SFNs and $\eta > 0$, then*

$$i.\ \mathcal{S}_1 \oplus \mathcal{S}_2 = \left(\begin{array}{c} \sqrt[t]{1 - \log_\tau \left(1 + \frac{\left(\tau^{1-\sigma_1^t} - 1\right)\left(\tau^{1-\sigma_2^t} - 1\right)}{\tau - 1} \right)}, \\ \sqrt[t]{\log_\tau \left(1 + \frac{\left(\tau^{\vartheta_1^t} - 1\right)\left(\tau^{\vartheta_2^t} - 1\right)}{\tau - 1} \right)}, \\ \sqrt[t]{\log_\tau \left(1 + \frac{\left(\tau^{\varrho_1^t} - 1\right)\left(\tau^{\varrho_2^t} - 1\right)}{\tau - 1} \right)} \end{array} \right);$$

$$ii.\ \mathcal{S}_1 \otimes \mathcal{S}_2 = \left(\begin{array}{c} \sqrt[t]{\log_\tau \left(1 + \frac{\left(\tau^{\sigma_1^t} - 1\right)\left(\tau^{\sigma_2^t} - 1\right)}{\tau - 1} \right)}, \\ \sqrt[t]{1 - \log_\tau \left(1 + \frac{\left(\tau^{1-\vartheta_1^t} - 1\right)\left(\tau^{1-\vartheta_2^t} - 1\right)}{\tau - 1} \right)}, \\ \sqrt[t]{1 - \log_\tau \left(1 + \frac{\left(\tau^{1-\varrho_1^t} - 1\right)\left(\tau^{1-\varrho_2^t} - 1\right)}{\tau - 1} \right)} \end{array} \right);$$

$$iii.\ \mathcal{S}_1^\eta = \left(\begin{array}{c} \sqrt[t]{\log_\tau \left(1 + \frac{\left(\tau^{\sigma_1^t} - 1\right)^\eta}{(\tau - 1)^{\eta - 1}} \right)}, \\ \sqrt[t]{1 - \log_\tau \left(1 + \frac{\left(\tau^{1-\vartheta_1^t} - 1\right)^\eta}{(\tau - 1)^{\eta - 1}} \right)}, \\ \sqrt[t]{1 - \log_\tau \left(1 + \frac{\left(\tau^{1-\vartheta_1^t} - 1\right)^\eta}{(\tau - 1)^{\eta - 1}} \right)} \end{array} \right);$$

$$iv.\ \eta\mathcal{S}_1 = \left(\begin{array}{c} \sqrt[t]{1-\log_\tau\left(1+\frac{\left(\tau^{1-\sigma_1^t}-1\right)^\eta}{(\tau-1)^{\eta-1}}\right)}, \\ \sqrt[t]{\log_\tau\left(1+\frac{\left(\tau^{\vartheta_1^t}-1\right)^\eta}{(\tau-1)^{\eta-1}}\right)}, \\ \sqrt[t]{\log_\tau\left(1+\frac{\left(\tau^{\varrho_1^t}-1\right)^\eta}{(\tau-1)^{\eta-1}}\right)} \end{array} \right);$$

v. $\mathcal{S}_1^c = (\varrho_1, \vartheta_1, \sigma_1)$.

By using the operational laws given in Definition 5.1.1, we examine the subsequent results.

Theorem 5.1.1 *Let* $\mathcal{S}_\ell = (\sigma_\ell, \vartheta_\ell, \varrho_\ell)$ $(\ell = 1, 2)$ *and* $\mathcal{S} = (\sigma, \vartheta, \varrho)$ *be three T-SFNs, and* $\eta, \eta_1, \eta_2 > 0$, *then*

i. $\mathcal{S}_1 \oplus \mathcal{S}_2 = \mathcal{S}_2 \oplus \mathcal{S}_1$;

ii. $\mathcal{S}_1 \otimes \mathcal{S}_2 = \mathcal{S}_2 \otimes \mathcal{S}_1$;

iii. $\eta(\mathcal{S}_1 \oplus \mathcal{S}_2) = \eta\mathcal{S}_1 \oplus \eta\mathcal{S}_2$;

iv. $(\mathcal{S}_1 \otimes \mathcal{S}_2)^\eta = \mathcal{S}_1^\eta \otimes \mathcal{S}_2^\eta$;

v. $\eta_1\mathcal{S} \oplus \eta_2\mathcal{S} = (\eta_1 + \eta_2)\mathcal{S}$;

vi. $\mathcal{S}^{\eta_1} \otimes \mathcal{S}^{\eta_2} = \mathcal{S}^{\eta_1+\eta_2}$;

vii. $(\eta_1\eta_2)\mathcal{S} = \eta_1(\eta_2\mathcal{S})$.

PROOF: We prove only parts 1, 3, 5 and 7 and similarly for others.

1. It is obvious.

3.

$$\mathcal{S}_1 \oplus \mathcal{S}_2 = \left(\begin{array}{c} \sqrt[t]{1-\log_\tau\left(1+\frac{\left(\tau^{1-\sigma_1^t}-1\right)\left(\tau^{1-\sigma_2^t}-1\right)}{\tau-1}\right)}, \\ \sqrt[t]{\log_\tau\left(1+\frac{\left(\tau^{\vartheta_1^t}-1\right)\left(\tau^{\vartheta_2^t}-1\right)}{\tau-1}\right)}, \\ \sqrt[t]{\log_\tau\left(1+\frac{\left(\tau^{\varrho_1^t}-1\right)\left(\tau^{\varrho_2^t}-1\right)}{\tau-1}\right)} \end{array} \right),$$

by the Frank operational law (4) in Definition 5.1.1, it follows that

$$\eta\left(\mathcal{S}_1 \oplus \mathcal{S}_2\right) = \left(\begin{array}{c} \sqrt[t]{1-\log_\tau\left(1+\frac{\left(\tau^{\log_\tau\left(1+\frac{\left(\tau^{1-\sigma_1^t}-1\right)\left(\tau^{1-\sigma_2^t}-1\right)}{\tau-1}\right)}-1\right)^\eta}{(\tau-1)^{\eta-1}}\right)}, \\ \sqrt[t]{\log_\tau\left(1+\frac{\left(\tau^{\log_\tau\left(1+\frac{\left(\tau^{\vartheta_1^t}-1\right)\left(\tau^{\vartheta_2^t}-1\right)}{\tau-1}\right)}-1\right)^\eta}{(\tau-1)^{\eta-1}}\right)}, \\ \sqrt[t]{\log_\tau\left(1+\frac{\left(\tau^{\log_\tau\left(1+\frac{\left(\tau^{\varrho_1^t}-1\right)\left(\tau^{\varrho_2^t}-1\right)}{\tau-1}\right)}-1\right)^\eta}{(\tau-1)^{\eta-1}}\right)} \end{array} \right),$$

$$= \begin{pmatrix} \sqrt[t]{1-\log_\tau\left(1+\frac{\left(\tau^{1-\sigma_1^t}-1\right)^\eta\left(\tau^{1-\sigma_2^t}-1\right)^\eta}{(\tau-1)^{2\eta-1}}\right)}, \\ \sqrt[t]{\log_\tau\left(1+\frac{\left(\tau^{\vartheta_1^t}-1\right)^\eta\left(\tau^{\vartheta_2^t}-1\right)^\eta}{(\tau-1)^{2\eta-1}}\right)}, \\ \sqrt[t]{\log_\tau\left(1+\frac{\left(\tau^{\varrho_1^t}-1\right)^\eta\left(\tau^{\varrho_2^t}-1\right)^\eta}{(\tau-1)^{2\eta-1}}\right)} \end{pmatrix}. \tag{5.1}$$

Now

$$\eta\mathcal{S}_1 \oplus \eta\mathcal{S}_2 = \begin{pmatrix} \sqrt[t]{1-\log_\tau\left(1+\frac{\left(\tau^{1-\sigma_1^t}-1\right)^\eta}{(\tau-1)^{\eta-1}}\right)}, \\ \sqrt[t]{\log_\tau\left(1+\frac{\left(\tau^{\vartheta_1^t}-1\right)^\eta}{(\tau-1)^{\eta-1}}\right)}, \\ \sqrt[t]{\log_\tau\left(1+\frac{\left(\tau^{\varrho_1^t}-1\right)^\eta}{(\tau-1)^{\eta-1}}\right)} \end{pmatrix} \oplus \begin{pmatrix} \sqrt[t]{1-\log_\tau\left(1+\frac{\left(\tau^{1-\sigma_2^t}-1\right)^\eta}{(\tau-1)^{\eta-1}}\right)}, \\ \sqrt[t]{\log_\tau\left(1+\frac{\left(\tau^{\vartheta_2^t}-1\right)^\eta}{(\tau-1)^{\eta-1}}\right)}, \\ \sqrt[t]{\log_\tau\left(1+\frac{\left(\tau^{\varrho_2^t}-1\right)^\eta}{(\tau-1)^{\eta-1}}\right)} \end{pmatrix},$$

$$= \left(\begin{array}{c} \sqrt[t]{1-\log_{\tau}\left(1+\frac{\left(\tau^{\log_{\tau}\left(1+\frac{\left(\tau^{1-\sigma_1^t}-1\right)^{\eta}}{(\tau-1)^{\eta-1}}\right)}-1\right)\left(\tau^{\log_{\tau}\left(1+\frac{\left(\tau^{1-\sigma_2^t}-1\right)^{\eta}}{(\tau-1)^{\eta-1}}\right)}-1\right)}{\tau-1}\right)}, \\ \sqrt[t]{\log_{\tau}\left(1+\frac{\left(\tau^{\log_{\tau}\left(1+\frac{\left(\tau^{\vartheta_1^t}-1\right)^{\eta}}{(\tau-1)^{\eta-1}}\right)}-1\right)\left(\tau^{\log_{\tau}\left(1+\frac{\left(\tau^{\vartheta_2^t}-1\right)^{\eta}}{(\tau-1)^{\eta-1}}\right)}-1\right)}{\tau-1}\right)}, \\ \sqrt[t]{\log_{\tau}\left(1+\frac{\left(\tau^{\log_{\tau}\left(1+\frac{\left(\tau^{\varrho_1^t}-1\right)^{\eta}}{(\tau-1)^{\eta-1}}\right)}-1\right)\left(\tau^{\log_{\tau}\left(1+\frac{\left(\tau^{\varrho_2^t}-1\right)^{\eta}}{(\tau-1)^{\eta-1}}\right)}-1\right)}{\tau-1}\right)} \end{array} \right)$$

$$= \left(\begin{array}{c} \sqrt[t]{1-\log_{\tau}\left(1+\frac{\left(\tau^{1-\sigma_1^t}-1\right)^{\eta}\left(\tau^{1-\sigma_2^t}-1\right)^{\eta}}{(\tau-1)^{2\eta-1}}\right)}, \\ \sqrt[t]{\log_{\tau}\left(1+\frac{\left(\tau^{\vartheta_1^t}-1\right)^{\eta}\left(\tau^{\vartheta_2^t}-1\right)^{\eta}}{(\tau-1)^{2\eta-1}}\right)}, \\ \sqrt[t]{\log_{\tau}\left(1+\frac{\left(\tau^{\varrho_1^t}-1\right)^{\eta}\left(\tau^{\varrho_2^t}-1\right)^{\eta}}{(\tau-1)^{2\eta-1}}\right)} \end{array} \right). \tag{5.2}$$

From Equation (5.1) and (5.2), we get

$$\eta\left(\mathcal{S}_1 \oplus \mathcal{S}_2\right) = \eta\mathcal{S}_1 \oplus \eta\mathcal{S}_2.$$

5.

$$\eta_1\mathcal{S}\oplus\eta_2\mathcal{S}=\begin{pmatrix}\sqrt[t]{1-\log_\tau\left(1+\frac{\left(\tau^{1-\sigma^t}-1\right)^{\eta_1}}{(\tau-1)^{\eta_1-1}}\right)},\\ \sqrt[t]{\log_\tau\left(1+\frac{\left(\tau^{\vartheta^t}-1\right)^{\eta_1}}{(\tau-1)^{\eta_1-1}}\right)},\\ \sqrt[t]{\log_\tau\left(1+\frac{\left(\tau^{\varrho^t}-1\right)^{\eta_1}}{(\tau-1)^{\eta_1-1}}\right)}\end{pmatrix}\oplus\begin{pmatrix}\sqrt[t]{1-\log_\tau\left(1+\frac{\left(\tau^{1-\sigma^t}-1\right)^{\eta_2}}{(\tau-1)^{\eta_2-1}}\right)},\\ \sqrt[t]{\log_\tau\left(1+\frac{\left(\tau^{\vartheta^t}-1\right)^{\eta_2}}{(\tau-1)^{\eta_2-1}}\right)},\\ \sqrt[t]{\log_\tau\left(1+\frac{\left(\tau^{\varrho^t}-1\right)^{\eta_2}}{(\tau-1)^{\eta_2-1}}\right)}\end{pmatrix}$$

$$=\begin{pmatrix}\sqrt[t]{1-\log_\tau\left(1+\frac{\left(\tau^{\log_\tau\left(1+\frac{\left(\tau^{1-\sigma^t}-1\right)^{\eta_1}}{(\tau-1)^{\eta_1-1}}\right)}-1\right)\left(\tau^{\log_\tau\left(1+\frac{\left(\tau^{1-\sigma^t}-1\right)^{\eta_2}}{(\tau-1)^{\eta_2-1}}\right)}-1\right)}{\tau-1}\right)},\\ \sqrt[t]{\log_\tau\left(1+\frac{\left(\tau^{\log_\tau\left(1+\frac{\left(\tau^{\vartheta^t}-1\right)^{\eta_1}}{(\tau-1)^{\eta_1-1}}\right)}-1\right)\left(\tau^{\log_\tau\left(1+\frac{\left(\tau^{\vartheta^t}-1\right)^{\eta_2}}{(\tau-1)^{\eta_2-1}}\right)}-1\right)}{\tau-1}\right)},\\ \sqrt[t]{\log_\tau\left(1+\frac{\left(\tau^{\log_\tau\left(1+\frac{\left(\tau^{\varrho^t}-1\right)^{\eta_1}}{(\tau-1)^{\eta_1-1}}\right)}-1\right)\left(\tau^{\log_\tau\left(1+\frac{\left(\tau^{\varrho^t}-1\right)^{\eta_2}}{(\tau-1)^{\eta_2-1}}\right)}-1\right)}{\tau-1}\right)}\end{pmatrix}$$

$$=\begin{pmatrix}\sqrt[t]{1-\log_\tau\left(1+\frac{\left(\tau^{1-\sigma^t}-1\right)^{\eta_1+\eta_2}}{(\tau-1)^{\eta_1+\eta_2-1}}\right)},\sqrt[t]{\log_\tau\left(1+\frac{\left(\tau^{\vartheta^t}-1\right)^{\eta_1+\eta_2}}{(\tau-1)^{\eta_1+\eta_2-1}}\right)},\\ \sqrt[t]{\log_\tau\left(1+\frac{\left(\tau^{\varrho^t}-1\right)^{\eta_1+\eta_2}}{(\tau-1)^{\eta_1+\eta_2-1}}\right)}\end{pmatrix}. \tag{5.3}$$

And

$$(\eta_1+\eta_2)\,\mathcal{S}=\left(\begin{array}{c}\sqrt[t]{1-\log_\tau\left(1+\frac{\left(\tau^{1-\sigma^t}-1\right)^{\eta_1+\eta_2}}{(\tau-1)^{\eta_1+\eta_2-1}}\right)},\sqrt[t]{\log_\tau\left(1+\frac{\left(\tau^{\vartheta^t}-1\right)^{\eta_1+\eta_2}}{(\tau-1)^{\eta_1+\eta_2-1}}\right)},\\ \sqrt[t]{\log_\tau\left(1+\frac{\left(\tau^{\varrho^t}-1\right)^{\eta_1+\eta_2}}{(\tau-1)^{\eta_1+\eta_2-1}}\right)}\end{array}\right). \tag{5.4}$$

Thus, from Equations (5.3) and (5.4), we get the required result.

7. Since

$$\eta_2\mathcal{S}=\left(\begin{array}{c}\sqrt[t]{1-\log_\tau\left(1+\frac{\left(\tau^{1-\sigma^t}-1\right)^{\eta_2}}{(\tau-1)^{\eta_2-1}}\right)},\sqrt[t]{\log_\tau\left(1+\frac{\left(\tau^{\vartheta^t}-1\right)^{\eta_2}}{(\tau-1)^{\eta_2-1}}\right)},\\ \sqrt[t]{\log_\tau\left(1+\frac{\left(\tau^{\varrho^t}-1\right)^{\eta_2}}{(\tau-1)^{\eta_2-1}}\right)}\end{array}\right).$$

From this, we can further write $\eta_1\left(\eta_2\mathcal{S}\right)$

$$=\left(\begin{array}{c}\sqrt[t]{1-\log_\tau\left(1+\frac{\left(\tau^{\log_\tau\left(1+\frac{\left(\tau^{1-\sigma^t}-1\right)^{\eta_2}}{(\tau-1)^{\eta_2-1}}\right)}-1\right)^{\eta_1}}{(\tau-1)^{\eta_1-1}}\right)},\sqrt[t]{\log_\tau\left(1+\frac{\left(\tau^{\log_\tau\left(1+\frac{\left(\tau^{\vartheta^t}-1\right)^{\eta_2}}{(\tau-1)^{\eta_2-1}}\right)}-1\right)^{\eta_1}}{(\tau-1)^{\eta_1-1}}\right)},\\ \sqrt[t]{\log_\tau\left(1+\frac{\left(\tau^{\log_\tau\left(1+\frac{\left(\tau^{\varrho^t}-1\right)^{\eta_2}}{(\tau-1)^{\eta_2-1}}\right)}-1\right)^{\eta_1}}{(\tau-1)^{\eta_1-1}}\right)}\end{array}\right)$$

$$=\left(\begin{array}{c}\sqrt[t]{1-\log_\tau\left(1+\frac{\left(\tau^{1-\sigma^t}-1\right)^{\eta_1\eta_2}}{(\tau-1)^{\eta_1\eta_2-1}}\right)},\sqrt[t]{\log_\tau\left(1+\frac{\left(\tau^{\vartheta^t}-1\right)^{\eta_1\eta_2}}{(\tau-1)^{\eta_1\eta_2-1}}\right)},\\ \sqrt[t]{\log_\tau\left(1+\frac{\left(\tau^{\varrho^t}-1\right)^{\eta_1\eta_2}}{(\tau-1)^{\eta_1\eta_2-1}}\right)}\end{array}\right)$$

$$=(\eta_1\eta_2)\,\mathcal{S}$$

□

5.2 Aggregation operators about T-SFNs based Frank operations

By employing the proposed Frank operation rules, in what follows, we propose a series of weighted aggregation operators for T-SFNs.

5.2.1 T-spherical fuzzy Frank averaging operators

Definition 5.2.1 *Let $\mathcal{S}_j = (\sigma_j, \vartheta_j, \varrho_j)\,(j = 1, 2, ..., n)$ be a family of T-SFNs, then the T-spherical fuzzy Frank weighted averaging operator (T-SFFWA) is:*

$$T - SFFWA\left(\mathcal{S}_1, \mathcal{S}_2, ..., \mathcal{S}_n\right) = \oplus_{j=1}^{n}\left(\varpi_j \mathcal{S}_j\right), \tag{5.5}$$

where $\varpi = (\varpi_1, \varpi_2, ..., \varpi_n)^T$ is the weight vector of $\mathcal{S}_j\,(j = 1, 2, .., n)$ such that $\varpi_j > 0$ and $\sum_{j=1}^{n} \varpi_j = 1$. Especially, if $\varpi = \left(\frac{1}{n}, \frac{1}{n}, ..., \frac{1}{n}\right)^T$, then the T-SFFWA operator reduces to the T-spherical fuzzy Frank averaging (T-SFFA) operator of dimension n, which is given as follows:

$$T - SFFA\left(\mathcal{S}_1, \mathcal{S}_2, ..., \mathcal{S}_n\right) = \frac{1}{n} \oplus_{j=1}^{n}\left(\mathcal{S}_j\right). \tag{5.6}$$

Theorem 5.2.1 *Let $\mathcal{S}_j = (\sigma_j, \vartheta_j, \varrho_j)\,(j = 1, 2, ..., n)$ be a family of T-SFNs, then the result obtained by using the T-SFFWA operator is still a T-SFN, and*

$$T - SFFWA\left(\mathcal{S}_1, \mathcal{S}_2, ..., \mathcal{S}_n\right)$$

$$= \left(\begin{array}{c} \sqrt[t]{1 - \log_\tau\left(1 + \Pi_{j=1}^{n}\left(\tau^{1-\sigma_j^t} - 1\right)^{\varpi_j}\right)}, \sqrt[t]{\log_\tau\left(1 + \Pi_{j=1}^{n}\left(\tau^{\vartheta_j^t} - 1\right)^{\varpi_j}\right)}, \\ \sqrt[t]{\log_\tau\left(1 + \Pi_{j=1}^{n}\left(\tau^{\varrho_j^t} - 1\right)^{\varpi_j}\right)} \end{array}\right). \tag{5.7}$$

Proof: We use mathematical induction on n to verify it.

For $n = 2$, $T - SFFWA\left(\mathcal{S}_1, \mathcal{S}_2\right) = \varpi_1 \mathcal{S}_1 \oplus \varpi_2 \mathcal{S}_2$

$$
= \left(\begin{array}{c} \sqrt[t]{1-\log_{\tau}\left(1+\frac{\left(\tau^{1-\left(1-\log_{\tau}\left(1+\frac{\left(\tau^{1-\sigma_1^t}-1\right)^{\varpi_1}}{(\tau-1)^{\varpi_1-1}}\right)\right)}-1\right)\left(\tau^{1-\left(1-\log_{\tau}\left(1+\frac{\left(\tau^{1-\sigma_2^t}-1\right)^{\varpi_2}}{(\tau-1)^{\varpi_2-1}}\right)\right)}-1\right)}{\tau-1}\right)}, \\ \sqrt[t]{\log_{\tau}\left(1+\frac{\left(\tau^{1-\left(1-\log_{\tau}\left(1+\frac{\left(\tau^{\vartheta_1^t}-1\right)^{\varpi_1}}{(\tau-1)^{\varpi_1-1}}\right)\right)}-1\right)\left(\tau^{1-\left(1-\log_{\tau}\left(1+\frac{\left(\tau^{\vartheta_2^t}-1\right)^{\varpi_2}}{(\tau-1)^{\varpi_2-1}}\right)\right)}-1\right)}{\tau-1}\right)}, \\ \sqrt[t]{\log_{\tau}\left(1+\frac{\left(\tau^{1-\left(1-\log_{\tau}\left(1+\frac{\left(\tau^{\varrho_1^t}-1\right)^{\varpi_1}}{(\tau-1)^{\varpi_1-1}}\right)\right)}-1\right)\left(\tau^{1-\left(1-\log_{\tau}\left(1+\frac{\left(\tau^{\varrho_2^t}-1\right)^{\varpi_2}}{(\tau-1)^{\varpi_2-1}}\right)\right)}-1\right)}{\tau-1}\right)} \end{array} \right)
$$

$$
= \left(\begin{array}{c} \sqrt[t]{1-\log_{\tau}\left(1+\frac{\left(1+\frac{\left(\tau^{1-\sigma_1^t}-1\right)^{\varpi_1}}{(\tau-1)^{\varpi_1-1}}-1\right)\left(1+\frac{\left(\tau^{1-\sigma_2^t}-1\right)^{\varpi_2}}{(\tau-1)^{\varpi_2-1}}-1\right)}{\tau-1}\right)}, \\ \sqrt[t]{\log_{\tau}\left(1+\frac{\left(1+\frac{\left(\tau^{\vartheta_1^t}-1\right)^{\varpi_1}}{(\tau-1)^{\varpi_1-1}}-1\right)\left(1+\frac{\left(\tau^{\vartheta_2^t}-1\right)^{\varpi_2}}{(\tau-1)^{\varpi_2-1}}-1\right)}{\tau-1}\right)}, \\ \sqrt[t]{\log_{\tau}\left(1+\frac{\left(1+\frac{\left(\tau^{\varrho_1^t}-1\right)^{\varpi_1}}{(\tau-1)^{\varpi_1-1}}-1\right)\left(1+\frac{\left(\tau^{\varrho_2^t}-1\right)^{\varpi_2}}{(\tau-1)^{\varpi_2-1}}-1\right)}{\tau-1}\right)} \end{array} \right)
$$

$$= \left(\begin{array}{c} \sqrt[t]{1 - \log_\tau \left(1 + \frac{\left(1+\frac{\left(\tau^{1-\sigma_1^t}-1\right)^{\varpi_1}}{(\tau-1)^{\varpi_1-1}}-1\right)\left(1+\frac{\left(\tau^{1-\sigma_2^t}-1\right)^{\varpi_2}}{(\tau-1)^{\varpi_2-1}}-1\right)}{\tau-1} \right)}, \\ \sqrt[t]{\log_\tau \left(1 + \frac{\left(1+\frac{\left(\tau^{\vartheta_1^t}-1\right)^{\varpi_1}}{(\tau-1)^{\varpi_1-1}}-1\right)\left(1+\frac{\left(\tau^{\vartheta_2^t}-1\right)^{\varpi_2}}{(\tau-1)^{\varpi_2-1}}-1\right)}{\tau-1} \right)}, \\ \sqrt[t]{\log_\tau \left(1 + \frac{\left(1+\frac{\left(\tau^{\varrho_1^t}-1\right)^{\varpi_1}}{(\tau-1)^{\varpi_1-1}}-1\right)\left(1+\frac{\left(\tau^{\varrho_2^t}-1\right)^{\varpi_2}}{(\tau-1)^{\varpi_2-1}}-1\right)}{\tau-1} \right)} \end{array} \right)$$

$$= \left(\begin{array}{c} \sqrt[t]{1 - \log_\tau \left(1 + \left(\left(\tau^{1-\sigma_1^t} - 1\right)^{\varpi_1}\right)\left(\left(\tau^{1-\sigma_2^t} - 1\right)^{\varpi_2}\right)\right)}, \\ \sqrt[t]{\log_\tau \left(1 + \left(\left(\tau^{\vartheta_1^t} - 1\right)^{\varpi_1}\right)\left(\left(\tau^{\vartheta_2^t} - 1\right)^{\varpi_2}\right)\right)}, \\ \sqrt[t]{\log_\tau \left(1 + \left(\left(\tau^{\varrho_1^t} - 1\right)^{\varpi_1}\right)\left(\left(\tau^{\varrho_2^t} - 1\right)^{\varpi_2}\right)\right)} \end{array} \right).$$

Thus, result holds for $n = 2$.

If Equation (5.7) satisfies for $n = k$, then for $n = k + 1$, we get

$$T - SFFWA\left(\mathcal{S}_1, \mathcal{S}_2, ..., \mathcal{S}_{k+1}\right) = T - SFFWA\left(\mathcal{S}_1, \mathcal{S}_2, ..., \mathcal{S}_k\right) \oplus \varpi_{k+1}\mathcal{S}_{k+1}$$

$$= \left(\begin{array}{c} \sqrt[t]{1 - \log_\tau \left(1 + \Pi_{j=1}^k \left(\tau^{1-\sigma_j^t} - 1\right)^{\varpi_j}\right)}, \\ \sqrt[t]{\log_\tau \left(1 + \Pi_{j=1}^k \left(\tau^{\vartheta_j^t} - 1\right)^{\varpi_j}\right)}, \\ \sqrt[t]{\log_\tau \left(1 + \Pi_{j=1}^k \left(\tau^{\varrho_j^t} - 1\right)^{\varpi_j}\right)} \end{array} \right) \oplus \left(\begin{array}{c} \sqrt[t]{1 - \log_\tau \left(1 + \frac{\left(\tau^{1-\sigma_{(k+1)}^t} - 1\right)^{\varpi(k+1)}}{(\tau-1)^{\varpi(k+1)^{-1}}}\right)}, \\ \sqrt[t]{\log_\tau \left(1 + \frac{\left(\tau^{\vartheta_{(k+1)}^t} - 1\right)^{\varpi(k+1)}}{(\tau-1)^{\varpi(k+1)^{-1}}}\right)}, \\ \sqrt[t]{\log_\tau \left(1 + \frac{\left(\tau^{\varrho_{(k+1)}^t} - 1\right)^{\varpi(k+1)}}{(\tau-1)^{\varpi(k+1)^{-1}}}\right)} \end{array} \right)$$

$$= \left(\begin{array}{c} \sqrt[t]{1-\log_{\tau}\left(1+\frac{\Pi_{j=1}^{k}\left(\tau^{1-\sigma_{j}^{t}}-1\right)^{\varpi_j}\frac{\left(\tau^{1-\sigma_{(k+1)}^{t}}-1\right)^{\varpi(k+1)}}{(\tau-1)^{\varpi(k+1)^{-1}}}}{\tau-1}\right)}, \\ \sqrt[t]{\log_{\tau}\left(1+\frac{\Pi_{j=1}^{k}\left(\tau^{\vartheta_{j}^{t}}-1\right)^{\varpi_j}\frac{\left(\tau^{\vartheta_{(k+1)}^{t}}-1\right)^{\varpi(k+1)}}{(\tau-1)^{\varpi(k+1)^{-1}}}}{\tau-1}\right)}, \\ \sqrt[t]{\log_{\tau}\left(1+\frac{\Pi_{j=1}^{k}\left(\tau^{\varrho_{j}^{t}}-1\right)^{\varpi_j}\frac{\left(\tau^{\varrho_{(k+1)}^{t}}-1\right)^{\varpi(k+1)}}{(\tau-1)^{\varpi(k+1)^{-1}}}}{\tau-1}\right)} \end{array} \right)$$

$$= \left(\begin{array}{c} \sqrt[t]{1-\log_{\tau}\left(1+\frac{\Pi_{j=1}^{k}\left(\tau^{1-\sigma_{j}^{t}}-1\right)^{\varpi_j}\left(\tau^{1-\sigma_{(k+1)}^{t}}-1\right)^{\varpi(k+1)}}{(\tau-1)^{\sum_{j=1}^{k}\varpi(j)^{-1}}(\tau-1)(\tau-1)^{\varpi(k+1)^{-1}}}\right)}, \\ \sqrt[t]{\log_{\tau}\left(1+\frac{\Pi_{j=1}^{k}\left(\tau^{\vartheta_{j}^{t}}-1\right)^{\varpi_j}\left(\tau^{\vartheta_{(k+1)}^{t}}-1\right)^{\varpi(k+1)}}{(\tau-1)^{\sum_{j=1}^{k}\varpi(j)^{-1}}(\tau-1)(\tau-1)^{\varpi(k+1)^{-1}}}\right)}, \\ \sqrt[t]{\log_{\tau}\left(1+\frac{\Pi_{j=1}^{k}\left(\tau^{\varrho_{j}^{t}}-1\right)^{\varpi_j}\left(\tau^{\varrho_{(k+1)}^{t}}-1\right)^{\varpi(k+1)}}{\sum_{j=1}^{k}(\tau-1)^{\varpi(j)^{-1}}(\tau-1)(\tau-1)^{\varpi(k+1)^{-1}}}\right)} \end{array} \right)$$

$$= \left(\begin{array}{c} \sqrt[t]{1-\log_{\tau}\left(1+\frac{\Pi_{j=1}^{k}\left(\tau^{1-\sigma_{j}^{t}}-1\right)^{\varpi_j}\left(\tau^{1-\sigma_{(k+1)}^{t}}-1\right)^{\varpi(k+1)}}{(\tau-1)^{\sum_{j=1}^{k+1}\varpi(j)^{-1}}}\right)}, \\ \sqrt[t]{\log_{\tau}\left(1+\frac{\Pi_{j=1}^{k}\left(\tau^{\vartheta_{j}^{t}}-1\right)^{\varpi_j}\left(\tau^{\vartheta_{(k+1)}^{t}}-1\right)^{\varpi(k+1)}}{(\tau-1)^{\sum_{j=1}^{k+1}\varpi(j)^{-1}}}\right)}, \\ \sqrt[t]{\log_{\tau}\left(1+\frac{\Pi_{j=1}^{k}\left(\tau^{\varrho_{j}^{t}}-1\right)^{\varpi_j}\left(\tau^{\varrho_{(k+1)}^{t}}-1\right)^{\varpi(k+1)}}{\sum_{j=1}^{k+1}(\tau-1)^{\varpi(j)^{-1}}}\right)} \end{array} \right)$$

$$= \begin{pmatrix} \sqrt[t]{1 - \log_\tau \left(1 + \Pi_{j=1}^{k} \left(\tau^{1-\sigma_j^t} - 1\right)^{\varpi_j} \left(\tau^{1-\sigma_{(k+1)}^t} - 1\right)^{\varpi_{(k+1)}}\right)} \\ \sqrt[t]{\log_\tau \left(1 + \Pi_{j=1}^{k} \left(\tau^{\vartheta_j^t} - 1\right)^{\varpi_j} \left(\tau^{\vartheta_{(k+1)}^t} - 1\right)^{\varpi_{(k+1)}}\right)} \\ \sqrt[t]{\log_\tau \left(1 + \Pi_{j=1}^{k} \left(\tau^{\varrho_j^t} - 1\right)^{\varpi_j} \left(\tau^{\varrho_{(k+1)}^t} - 1\right)^{\varpi_{(k+1)}}\right)} \end{pmatrix}.$$

Thus, results are valid for $n = k+1$ and therefore, by the principle of mathematical induction, result obtained in Equations (5.7) is valid for all positive integer n. □

Example 5.2.2 *Let $\mathcal{S}_1 = (0.4, 0.3, 0.5)$, $\mathcal{S}_2 = (0.7, 0.3, 0.4)$, $\mathcal{S}_3 = (0.6, 0.7, 0.8)$ be three T-SFNs, and $\varpi = (0.4, 0.3, 0.3)^T$ be the weight vector of $\mathcal{S}_j$ $(j = 1, 2, 3)$. Suppose $\tau = 2$, then by Definition 5.2.1 and Theorem 5.2.1, we can get $(t = 4)$:*

$$T - SFFWA(\mathcal{S}_1, \mathcal{S}_2, \mathcal{S}_3) = \begin{pmatrix} \sqrt[4]{1 - \log_2 \left(1 + \prod_{j=1}^{3} \left(2^{1-\sigma_j^4} - 1\right)^{\varpi_j}\right)}, \\ \sqrt[4]{\log_2 \left(1 + \prod_{j=1}^{3} \left(2^{\vartheta_j^4} - 1\right)^{\varpi_j}\right)}, \\ \sqrt[4]{\log_2 \left(1 + \prod_{j=1}^{3} \left(2^{\varrho_j^4} - 1\right)^{\varpi_j}\right)} \end{pmatrix}$$

$$= \begin{pmatrix} \sqrt[4]{1 - \log_2 \left(1 + \left(2^{1-.4^4} - 1\right)^{.4} \left(2^{1-.7^4} - 1\right)^{.3} \left(2^{1-.6^4} - 1\right)^{.3}\right)}, \\ \sqrt[4]{\log_2 \left(1 + \left(2^{.3^4} - 1\right)^{.4} \left(2^{.3^4} - 1\right)^{.3} \left(2^{.7^4} - 1\right)^{.3}\right)}, \\ \sqrt[4]{\log_2 \left(1 + \left(2^{.5^4} - 1\right)^{.4} \left(2^{.4^4} - 1\right)^{.3} \left(2^{.8^4} - 1\right)^{.3}\right)} \end{pmatrix}$$

$$= (0.5940, 0.3887, 0.5418)$$

Theorem 5.2.3 *Let $\mathcal{S}_j = (\sigma_j, \vartheta_j, \varrho_j)$ $(j = 1, 2, ..., n)$ be a family of T-SFNs, and $\tau > 1$. As $\tau \longrightarrow 1$, the T-SFFWA operator proceeds towards the following limit*

$$\lim_{\tau \longrightarrow 1} T - SFFWA(\mathcal{S}_1, \mathcal{S}_2, ..., \mathcal{S}_n) = \left(\sqrt[t]{1 - \Pi_{j=1}^{n} \left(1 - \sigma_j^t\right)^{\varpi_j}}, \sqrt[t]{\Pi_{j=1}^{n} \left(\vartheta_j^t\right)^{\varpi_j}}, \sqrt[t]{\Pi_{j=1}^{n} \left(\varrho_j^t\right)^{\varpi_j}} \right). \quad (5.8)$$

Proof: As $\tau \longrightarrow 1$, then

$$\left(\Pi_{j=1}^{n} \left(\tau^{1-\sigma_j^t} - 1\right)^{\varpi_j}, \Pi_{j=1}^{n} \left(\tau^{\vartheta_j^t} - 1\right)^{\varpi_j}, \Pi_{j=1}^{n} \left(\tau^{\varrho_j^t} - 1\right)^{\varpi_j} \right) \longrightarrow (0,0,0)$$

by log property and the rule of infinitesimal changes, we have

$$\log_\tau \left(1 + \Pi_{j=1}^{n} \left(\tau^{1-\sigma_j^t} - 1\right)^{\varpi_j}\right) = \frac{\ln\left(1+\Pi_{j=1}^{n}\left(\tau^{1-\sigma_j^t}-1\right)^{\varpi_j}\right)}{\ln\tau} \longrightarrow \frac{\Pi_{j=1}^{n}\left(\tau^{1-\sigma_j^t}-1\right)^{\varpi_j}}{\ln\tau}$$

$$\log_\tau \left(1 + \Pi_{j=1}^{n} \left(\tau^{\vartheta_j^t} - 1\right)^{\varpi_j}\right) = \frac{\ln\left(1+\Pi_{j=1}^{n}\left(\tau^{\vartheta_j^t}-1\right)^{\varpi_j}\right)}{\ln\tau} \longrightarrow \frac{\Pi_{j=1}^{n}\left(\tau^{\vartheta_j^t}-1\right)^{\varpi_j}}{\ln\tau}$$

$$\log_\tau \left(1 + \Pi_{j=1}^{n} \left(\tau^{\varrho_j^t} - 1\right)^{\varpi_j}\right) = \frac{\ln\left(1+\Pi_{j=1}^{n}\left(\tau^{\varrho_j^t}-1\right)^{\varpi_j}\right)}{\ln\tau} \longrightarrow \frac{\Pi_{j=1}^{n}\left(\tau^{\varrho_j^t}-1\right)^{\varpi_j}}{\ln\tau}$$

By using Taylor's expansion formula, we have

$$\tau^{1-\sigma_j^t} = 1 + \left(1-\sigma_j^t\right)\ln\tau + \frac{\left((1-\sigma_j^t)\ln\tau\right)^2}{2!} + ...$$

$$\tau^{\vartheta_j^t} = 1 + \left(\vartheta_j^t\right)\ln\tau + \frac{\left((\vartheta_j^t)\ln\tau\right)^2}{2!} + ...$$

$$\tau^{\varrho_j^t} = 1 + \left(\varrho_j^t\right)\ln\tau + \frac{\left((\varrho_j^t)\ln\tau\right)^2}{2!} + ...$$

Also, since $\tau > 1$, then

$$\ln\tau > 0,\ \tau^{1-\sigma_j^t} = 1 + \left(1-\sigma_j^t\right)\ln\tau + O\left(\ln\tau\right),\ \tau^{\vartheta_j^t} = 1 + \left(\vartheta_j^t\right)\ln\tau + O\left(\ln\tau\right),$$

$$\tau^{\varrho_j^t} = 1 + \left(\varrho_j^t\right)\ln\tau + O\left(\ln\tau\right).$$

It follows that

$$\left(\tau^{1-\sigma_j^t} - 1\right)^{\varpi_j} \longrightarrow \left(\left(1-\sigma_j^t\right)\ln\tau\right)^{\varpi_j}$$

$$\Pi_{j=1}^{n}\left(\tau^{1-\sigma_j^t} - 1\right)^{\varpi_j} \longrightarrow \Pi_{j=1}^{n}\left(1-\sigma_j^t\right)\Pi_{j=1}^{n}\left(\ln\tau\right)^{\varpi_j}$$

$$\Pi_{j=1}^{n}\left(\tau^{1-\sigma_j^t} - 1\right)^{\varpi_j} \longrightarrow \Pi_{j=1}^{n}\left(1-\sigma_j^t\right)\ln\left(\tau\right)^{\sum_{j=1}^{n}\varpi_j}$$

$$\frac{\Pi_{j=1}^{n}\left(\tau^{1-\sigma_j^t}-1\right)^{\varpi_j}}{\ln\tau} \longrightarrow \Pi_{j=1}^{n}\left(1-\sigma_j^t\right).$$

Analogously, we can get

$$\frac{\Pi_{j=1}^{n}\left(\tau^{\vartheta_j^t}-1\right)^{\varpi_j}}{\ln\tau} \longrightarrow \Pi_{j=1}^{n}\left(\vartheta_j^t\right) \text{ and } \frac{\Pi_{j=1}^{n}\left(\tau^{\varrho_j^t}-1\right)^{\varpi_j}}{\ln\tau} \longrightarrow \Pi_{j=1}^{n}\left(\varrho_j^t\right)$$

Then, we have

$\lim_{\tau\longrightarrow 1} T-SFFWA\left(\mathcal{S}_1,\mathcal{S}_2,...,\mathcal{S}_n\right)$

$$= \lim_{\tau\longrightarrow 1}\left(\begin{array}{c}\sqrt[t]{1-\log_\tau\left(1+\Pi_{j=1}^n\left(\tau^{1-\sigma_j^t}-1\right)^{\varpi_j}\right)}, \sqrt[t]{\log_\tau\left(1+\Pi_{j=1}^n\left(\tau^{\vartheta_j^t}-1\right)^{\varpi_j}\right)}, \\ \sqrt[t]{\log_\tau\left(1+\Pi_{j=1}^n\left(\tau^{\varrho_j^t}-1\right)^{\varpi_j}\right)}\end{array}\right)$$

$$= \lim_{\tau\longrightarrow 1}\left(\begin{array}{c}\sqrt[t]{1-\frac{\ln\left(1+\Pi_{j=1}^n\left(\tau^{1-\sigma_j^t}-1\right)^{\varpi_j}\right)}{\ln\tau}}, \sqrt[t]{\frac{\ln\left(1+\Pi_{j=1}^n\left(\tau^{\vartheta_j^t}-1\right)^{\varpi_j}\right)}{\ln\tau}}, \\ \sqrt[t]{\frac{\ln\left(1+\Pi_{j=1}^n\left(\tau^{\varrho_j^t}-1\right)^{\varpi_j}\right)}{\ln\tau}}\end{array}\right)$$

$$= \lim_{\tau\longrightarrow 1}\left(\begin{array}{c}\sqrt[t]{1-\frac{\Pi_{j=1}^n\left(\tau^{1-\sigma_j^t}-1\right)^{\varpi_j}}{\ln\tau}}, \sqrt[t]{\frac{\Pi_{j=1}^n\left(\tau^{\vartheta_j^t}-1\right)^{\varpi_j}}{\ln\tau}}, \\ \sqrt[t]{\frac{\Pi_{j=1}^n\left(\tau^{\varrho_j^t}-1\right)^{\varpi_j}}{\ln\tau}}\end{array}\right)$$

$$= \left(\begin{array}{c}\sqrt[t]{1-\Pi_{j=1}^n\left(1-\sigma_j^t\right)^{\varpi_j}}, \sqrt[t]{\Pi_{j=1}^n\left(\vartheta_j^t\right)^{\varpi_j}}, \\ \sqrt[t]{\Pi_{j=1}^n\left(\varrho_j^t\right)^{\varpi_j}}\end{array}\right).$$

which completes the proof. □

Theorem 5.2.4 *Let $\mathcal{S}_j = (\sigma_j, \vartheta_j, \varrho_j)\,(j = 1, 2, ..., n)$ be a family of T-SFNs, and $\tau > 1$. As $\tau \longrightarrow \infty$, the T-SFFWA operator proceeds towards the following limit*

$$\lim_{\tau\longrightarrow\infty} T-SFFWA\left(\mathcal{S}_1,\mathcal{S}_2,...,\mathcal{S}_n\right) = \left(\begin{array}{c}\sqrt[t]{\left(\sum_{j=1}^n\varpi_j\left(\sigma_j^t\right)\right)}, \sqrt[t]{1-\left(\sum_{j=1}^n\varpi_j\left(\vartheta_j^t\right)\right)}, \\ \sqrt[t]{1-\left(\sum_{j=1}^n\varpi_j\left(\varrho_j^t\right)\right)}\end{array}\right). \quad (5.9)$$

Proof: According to Theorem 5.2.1, we have

$\lim_{\tau\longrightarrow\infty} T-SFFWA\left(\mathcal{S}_1,\mathcal{S}_2,...,\mathcal{S}_n\right)$

$$= \left(\begin{array}{c}\lim_{\tau\longrightarrow\infty}\sqrt[t]{1-\log_\tau\left(1+\Pi_{j=1}^n\left(\tau^{1-\sigma_j^t}-1\right)^{\varpi_j}\right)}, \\ \lim_{\tau\longrightarrow\infty}\sqrt[t]{\log_\tau\left(1+\Pi_{j=1}^n\left(\tau^{\vartheta_j^t}-1\right)^{\varpi_j}\right)}, \\ \lim_{\tau\longrightarrow\infty}\sqrt[t]{\log_\tau\left(1+\Pi_{j=1}^n\left(\tau^{\varrho_j^t}-1\right)^{\varpi_j}\right)}\end{array}\right)$$

Using limit rules, logarithmic transform and L'Hospital's rule, it follows that,

$$= \left(\begin{array}{c} \sqrt[t]{1 - \lim_{\tau \longrightarrow \infty} \frac{\ln\left(1+\Pi_{j=1}^{n}\left(\tau^{1-\sigma_j^t}-1\right)^{\varpi_j}\right)}{\ln \tau}}, \\ \sqrt[t]{\lim_{\tau \longrightarrow \infty} \frac{\ln\left(1+\Pi_{j=1}^{n}\left(\tau^{\vartheta_j^t}-1\right)^{\varpi_j}\right)}{\ln \tau}}, \\ \sqrt[t]{\lim_{\tau \longrightarrow \infty} \frac{\ln\left(1+\Pi_{j=1}^{n}\left(\tau^{\varrho_j^t}-1\right)^{\varpi_j}\right)}{\ln \tau}} \end{array} \right)$$

$$= \left(\begin{array}{c} \sqrt[t]{1 - \lim_{\tau \longrightarrow \infty} \frac{\frac{\Pi_{j=1}^{n}\left(\tau^{1-\sigma_j^t}-1\right)^{\varpi_j}}{1+\Pi_{j=1}^{n}\left(\tau^{1-\sigma_j^t}-1\right)^{\varpi_j}}\left(\sum_{j=1}^{n}\varpi_j\left(1-\sigma_j^t\right)\frac{\tau^{-\sigma_j^t}}{\tau^{1-\sigma_j^t}-1}\right)}{\frac{1}{\tau}}}, \\ \sqrt[t]{1 - \lim_{\tau \longrightarrow \infty} \frac{\frac{\Pi_{j=1}^{n}\left(\tau^{\vartheta_j^t}-1\right)^{\varpi_j}}{1+\Pi_{j=1}^{n}\left(\tau^{\vartheta_j^t}-1\right)^{\varpi_j}}\left(\sum_{j=1}^{n}\varpi_j\left(\vartheta_j^t\right)\frac{\tau^{\vartheta_j^t-1}}{\tau^{\vartheta_j^t}-1}\right)}{\frac{1}{\tau}}}, \\ \sqrt[t]{1 - \lim_{\tau \longrightarrow \infty} \frac{\frac{\Pi_{j=1}^{n}\left(\tau^{\varrho_j^t}-1\right)^{\varpi_j}}{1+\Pi_{j=1}^{n}\left(\tau^{\varrho_j^t}-1\right)^{\varpi_j}}\left(\sum_{j=1}^{n}\varpi_j\left(\varrho_j^t\right)\frac{\tau^{\varrho_j^t-1}}{\tau^{\varrho_j^t}-1}\right)}{\frac{1}{\tau}}}, \end{array} \right)$$

$$= \left(\begin{array}{c} \sqrt[t]{1 - \lim_{\tau \longrightarrow \infty} \frac{\Pi_{j=1}^{n}\left(\tau^{1-\sigma_j^t}-1\right)^{\varpi_j}}{1+\Pi_{j=1}^{n}\left(\tau^{1-\sigma_j^t}-1\right)^{\varpi_j}}\left(\sum_{j=1}^{n}\varpi_j\left(1-\sigma_j^t\right)\frac{\tau^{1-\sigma_j^t}}{\tau^{1-\sigma_j^t}-1}\right)}, \\ \sqrt[t]{1 - \lim_{\tau \longrightarrow \infty} \frac{\Pi_{j=1}^{n}\left(\tau^{\vartheta_j^t}-1\right)^{\varpi_j}}{1+\Pi_{j=1}^{n}\left(\tau^{\vartheta_j^t}-1\right)^{\varpi_j}}\left(\sum_{j=1}^{n}\varpi_j\left(\vartheta_j^t\right)\frac{\tau^{\vartheta_j^t}}{\tau^{\vartheta_j^t}-1}\right)}, \\ \sqrt[t]{1 - \lim_{\tau \longrightarrow \infty} \frac{\Pi_{j=1}^{n}\left(\tau^{\varrho_j^t}-1\right)^{\varpi_j}}{1+\Pi_{j=1}^{n}\left(\tau^{\varrho_j^t}-1\right)^{\varpi_j}}\left(\sum_{j=1}^{n}\varpi_j\left(\varrho_j^t\right)\frac{\tau^{\varrho_j^t}}{\tau^{\varrho_j^t}-1}\right)} \end{array} \right)$$

$$= \left(\sqrt[t]{1 - \left(\sum_{j=1}^{n}\varpi_j\left(1-\sigma_j^t\right)\right)}, \sqrt[t]{1 - \left(\sum_{j=1}^{n}\varpi_j\left(\vartheta_j^t\right)\right)}, \sqrt[t]{1 - \left(\sum_{j=1}^{n}\varpi_j\left(\varrho_j^t\right)\right)} \right)$$

$$= \left(\sqrt[t]{\left(\sum_{j=1}^{n}\varpi_j\left(\sigma_j^t\right)\right)}, \sqrt[t]{1 - \left(\sum_{j=1}^{n}\varpi_j\left(\vartheta_j^t\right)\right)}, \sqrt[t]{1 - \left(\sum_{j=1}^{n}\varpi_j\left(\varrho_j^t\right)\right)} \right)$$

which completes the proof of Theorem 5.2.4. □

Theorem 5.2.5 *(Idempotency) Let $\mathcal{T}_j = (\sigma_j, \vartheta_j, \varrho_j)\,(j = 1, 2, ..., n)$ be a family of T-SFNs, if $\mathcal{S}_j = \mathcal{S}_0\ \forall\ j$, then*

$$T - SFFWA\,(\mathcal{S}_1, \mathcal{S}_2, ..., \mathcal{S}_n) = \mathcal{S}_0. \tag{5.10}$$

PROOF: Since for all j, $\mathcal{S}_j = \mathcal{S}_0 = (\sigma_0, \vartheta_0, \varrho_0)$, and $\sum_{j=1}^n \varpi_j = 1$ so by Theorem 5.2.1, we have

$$T - SFFWA\,(\mathcal{S}_1, \mathcal{S}_2, ..., \mathcal{S}_n) = \begin{pmatrix} \sqrt[t]{1 - \log_\tau\left(1 + \Pi_{j=1}^n \left(\tau^{1-\sigma_0^t} - 1\right)^{\varpi_j}\right)}, \\ \sqrt[t]{\log_\tau\left(1 + \Pi_{j=1}^n \left(\tau^{\vartheta_0^t} - 1\right)^{\varpi_j}\right)}, \\ \sqrt[t]{\log_\tau\left(1 + \Pi_{j=1}^n \left(\tau^{\varrho_0^t} - 1\right)^{\varpi_j}\right)} \end{pmatrix}$$
$$= \left(\sqrt[t]{1 - \log_\tau \tau^{1-\sigma_0^t}}, \sqrt[t]{\log_\tau \tau^{\vartheta_0^t}}, \sqrt[t]{\log_\tau \tau^{\varrho_0^t}}\right)$$
$$= (\sigma_0, \vartheta_0, \varrho_0) = \mathcal{S}_0.$$

Thus, proof is completed. □

Theorem 5.2.6 *(Monotonicity) Let $\mathcal{S}_j = (\sigma_j, \vartheta_j, \varrho_j)\,(j = 1, 2, ..., n)$ and $\dot{\mathcal{S}}_j = \left(\dot{\sigma}_j, \dot{\vartheta}_j, \dot{\varrho}_j\right)$ $(j = 1, 2, ..., n)$ be two families of T-SFNs such that $\sigma_j \geq \dot{\sigma}_j, \vartheta_j \leq \dot{\vartheta}_j$ and $\varrho_j \leq \dot{\varrho}_j\ \forall\ j$, then*

$$T - SFFWA\,(\mathcal{S}_1, \mathcal{S}_2, ..., \mathcal{S}_n) \geq T - SFFWA\left(\dot{\mathcal{S}}_1, \dot{\mathcal{S}}_2, ..., \dot{\mathcal{S}}_n\right). \tag{5.11}$$

PROOF: According to Definition 2.5.3, when $\sigma_j \geq \dot{\sigma}_j, \vartheta_j \leq \dot{\vartheta}_j$ and $\varrho_j \leq \dot{\varrho}_j\ \forall\ j$, then

$$\sqrt[t]{1 - \log_\tau\left(1 + \Pi_{j=1}^n \left(\tau^{1-\sigma_j^t} - 1\right)^{\varpi_j}\right)} \geq \sqrt[t]{1 - \log_\tau\left(1 + \Pi_{j=1}^n \left(\tau^{1-\dot{\sigma}_j^t} - 1\right)^{\varpi_j}\right)},$$

$$\sqrt[t]{\log_\tau\left(1 + \Pi_{j=1}^n \left(\tau^{\vartheta_j^t} - 1\right)^{\varpi_j}\right)} \leq \sqrt[t]{\log_\tau\left(1 + \Pi_{j=1}^n \left(\tau^{\dot{\vartheta}_j^t} - 1\right)^{\varpi_j}\right)}$$

and

$$\sqrt[t]{\log_\tau\left(1 + \Pi_{j=1}^n \left(\tau^{\varrho_j^t} - 1\right)^{\varpi_j}\right)} \leq \sqrt[t]{\log_\tau\left(1 + \Pi_{j=1}^n \left(\tau^{\dot{\varrho}_j^t} - 1\right)^{\varpi_j}\right)}$$

Thus, $S\,(T - SFFWA\,(\mathcal{S}_1, \mathcal{S}_2, ..., \mathcal{S}_n)) \geq S\left(T - SFFWA\left(\dot{\mathcal{S}}_1, \dot{\mathcal{S}}_2, ..., \dot{\mathcal{S}}_n\right)\right)$

Hence, $T-SFFWA\left(\mathcal{S}_1,\mathcal{S}_2,...,\mathcal{S}_n\right)\geq T-SFFWA\left(\dot{\mathcal{S}}_1,\dot{\mathcal{S}}_2,...,\dot{\mathcal{S}}_n\right).$ □

Theorem 5.2.7 *(Boundedness) Let* $\mathcal{S}_j=(\sigma_j,\vartheta_j,\varrho_j)\,(j=1,2,...,n)$ *be a family of T-SFNs, and let* $\mathcal{S}^-=(\min_{1\leq j\leq n}\sigma_j,\ \max_{1\leq j\leq n}\vartheta_j,\max_{1\leq j\leq n}\varrho_j)$, $\mathcal{S}^+=(\max_{1\leq j\leq n}\sigma_j,\min_{1\leq j\leq n}\vartheta_j,\min_{1\leq j\leq n}\varrho_j)$, *then*

$$\mathcal{S}^-\leq T-SFFWA\left(\mathcal{S}_1,\mathcal{S}_2,...,\mathcal{S}_n\right)\leq\mathcal{S}^+. \tag{5.12}$$

PROOF: Since for all j, $\min_{1\leq j\leq n}\sigma_j\leq\sigma_j\leq\max_{1\leq j\leq n}\sigma_j$, $\min_{1\leq j\leq n}\vartheta_j\leq\vartheta_j\leq\max_{1\leq j\leq n}\vartheta_j$ and $\min_{1\leq j\leq n}\varrho_j\leq\varrho_j\leq\max_{1\leq j\leq n}\varrho_j$, thereby on the basis of idempotency and monotonicity, we get

$$\mathcal{S}^-\leq T-SFFWA\left(\mathcal{S}_1,\mathcal{S}_2,...,\mathcal{S}_n\right)\leq\mathcal{S}^+.$$ □

Theorem 5.2.8 *(Shift-invariance) Let* $\mathcal{S}_j=(\sigma_j,\vartheta_j,\varrho_j)\,(j=1,2,...,n)$ *be a family of T-SFNs and* $\dot{\mathcal{S}}=\left(\dot{\sigma},\dot{\vartheta},\dot{\varrho}\right)$ *be any other T-SFNs, then*

$$T-SFFWA\left(\mathcal{S}_1\oplus\dot{\mathcal{S}},\mathcal{S}_2\oplus\dot{\mathcal{S}},...,\mathcal{S}_n\oplus\dot{\mathcal{S}}\right)=T-SFFWA\left(\mathcal{S}_1,\mathcal{S}_2,...,\mathcal{S}_n\right)\oplus\dot{\mathcal{S}}. \tag{5.13}$$

Theorem 5.2.9 *(Homogeneity) Let* $\mathcal{S}_j=(\sigma_j,\vartheta_j,\varrho_j)\,(j=1,2,...,n)$ *be a family of T-SFNs and* $\eta>0$ *be any real number, then*

$$T-SFFWA\left(\eta\mathcal{S}_1,\eta\mathcal{S}_2,...,\eta\mathcal{S}_n\right)=\eta T-SFFWA\left(\mathcal{S}_1,\mathcal{S}_2,...,\mathcal{S}_n\right). \tag{5.14}$$

By using the proposed Frank operational laws of T-SFNs, the above two theorems can be easily verified. Due to the space limitations, it is omitted here.

Definition 5.2.2 *Let* $\mathcal{S}_j=(\sigma_j,\vartheta_j,\varrho_j)\,(j=1,2,...,n)$ *be a family of T-SFNs, then the T-spherical fuzzy Frank ordered weighted averaging (T-SFFOWA) operator is:*

$$T-SFFOWA\left(\mathcal{S}_1,\mathcal{S}_2,...,\mathcal{S}_n\right)=\oplus_{j=1}^n\left(w_j\mathcal{S}_{\delta(j)}\right), \tag{5.15}$$

where $w=(w_1,w_2,...,w_n)^T$ *is the position weights of* $\mathcal{S}_j\,(j=1,2,..,n)$ *such that* $w_j>0$ *and* $\sum_{j=1}^n w_j=1$. $(\delta(1),\delta(2),...,\delta(n))$ *is a permutation of* $(1,2,3,...,n)$ *such that* $\mathcal{S}_{\delta(j-1)}\geq\mathcal{S}_{\delta(j)}$ *for*

$j = 2, 3.., n.$

Theorem 5.2.10 *Let $\mathcal{S}_j = (\sigma_j, \vartheta_j, \varrho_j)\,(j = 1, 2, ..., n)$ be a family of T-SFNs, then the result obtained by using the T-SFFOWA operator is still a T-SFN, and*

$$T - SFFOWA\,(\mathcal{S}_1, \mathcal{S}_2, ..., \mathcal{S}_n) = \begin{pmatrix} \sqrt[t]{1 - \log_\tau \left(1 + \Pi_{j=1}^{n} \left(\tau^{1-\sigma^t_{\delta(j)}} - 1\right)^{w_j}\right)}, \\ \sqrt[t]{\log_\tau \left(1 + \Pi_{j=1}^{n} \left(\tau^{\vartheta^t_{\delta(j)}} - 1\right)^{w_j}\right)}, \\ \sqrt[t]{\log_\tau \left(1 + \Pi_{j=1}^{n} \left(\tau^{\varrho^t_{\delta(j)}} - 1\right)^{w_j}\right)} \end{pmatrix}. \tag{5.16}$$

PROOF: This result has similar proof to that of Theorem 5.2.1, and so is omitted here. □

Example 5.2.11 *Let $\mathcal{S}_1 = (0.3, 0.3, 0.5)$, $\mathcal{S}_2 = (0.7, 0.4, 0.5)$, $\mathcal{S}_3 = (0.6, 0.7, 0.8)$ be three T-SFNs, then according to Definition 2.5.3, we can get (t=4):*

$S(\mathcal{S}_1) = -0.0769,\ S(\mathcal{S}_2) = 0.1775,\ S(\mathcal{S}_3) = -0.5482$

Since $S(\mathcal{S}_2) > S(\mathcal{S}_1) > S(\mathcal{S}_3)$, we have

$\mathcal{S}_{\delta(1)} = (0.7, 0.4, 0.5), \mathcal{S}_{\delta(2)} = (0.3, 0.3, 0.5), \mathcal{S}_{\delta(3)} =, (0.6, 0.7, 0.8)$ *and $w = (0.3, 0.4, 0.3)^T$ is the weight vector associated with the T-SFFOWA operator. Suppose $\tau = 2$, then by Definition 5.2.2 and Theorem 5.2.10, we can get:*

$$T - SFFOWA\left(\mathcal{S}_{\delta(1)}, \mathcal{S}_{\delta(2)}, \mathcal{S}_{\delta(3)}\right) = \begin{pmatrix} \sqrt[4]{1 - \log_2 \left(1 + \prod_{j=1}^{3} \left(2^{1-\sigma^4_{\delta(j)}} - 1\right)^{w_j}\right)}, \\ \sqrt[4]{\log_2 \left(1 + \prod_{j=1}^{3} \left(2^{\vartheta^4_{\delta(j)}} - 1\right)^{w_j}\right)}, \\ \sqrt[4]{\log_2 \left(1 + \prod_{j=1}^{3} \left(2^{\varrho^4_{\delta(j)}} - 1\right)^{w_j}\right)} \end{pmatrix}$$

$$= \begin{pmatrix} \sqrt[4]{1 - \log_2 \left(1 + \left(2^{1-.7^4} - 1\right)^{.3} \left(2^{1-.3^4} - 1\right)^{.4} \left(2^{1-.6^4} - 1\right)^{.3}\right)}, \\ \sqrt[4]{\log_2 \left(1 + \left(2^{.4^4} - 1\right)^{.3} \left(2^{.3^4} - 1\right)^{.4} \left(2^{.7^4} - 1\right)^{.3}\right)}, \\ \sqrt[4]{\log_2 \left(1 + \left(2^{.5^4} - 1\right)^{.3} \left(2^{.5^4} - 1\right)^{.4} \left(2^{.8^4} - 1\right)^{.3}\right)} \end{pmatrix}$$

$= (0.5861, 0.4236, 0.5786)$.

Theorem 5.2.12 *Let $\mathcal{S}_j = (\sigma_j, \vartheta_j, \varrho_j)\,(j = 1, 2, ..., n)$ be a family of T-SFNs, and $\tau > 1$. As $\tau \longrightarrow 1$, the T-SFFOWA operator proceeds towards the following limit*

$$\lim_{\tau \longrightarrow 1} T - SFFOWA\,(\mathcal{S}_1, \mathcal{S}_2, ..., \mathcal{S}_n) = \begin{pmatrix} \sqrt[t]{1 - \Pi_{j=1}^{n}\left(1 - \sigma_{\delta(j)}^{t}\right)^{w_j}}, \sqrt[t]{\Pi_{j=1}^{n}\left(\vartheta_{\delta(j)}^{t}\right)^{w_j}}, \\ \sqrt[t]{\Pi_{j=1}^{n}\left(\varrho_{\delta(j)}^{t}\right)^{w_j}} \end{pmatrix}. \quad (5.17)$$

Theorem 5.2.13 *Let $\mathcal{S}_j = (\sigma_j, \vartheta_j, \varrho_j)\,(j = 1, 2, ..., n)$ be a family of T-SFNs, and $\tau > 1$. As $\tau \longrightarrow \infty$, the T-SFFOWA proceeds towards the following limit*

$$\lim_{\tau \longrightarrow \infty} T - SFFOWA\,(\mathcal{S}_1, \mathcal{S}_2, ..., \mathcal{S}_n) = \begin{pmatrix} \sqrt[t]{\left(\sum_{j=1}^{n} w_j\left(\sigma_{\delta(j)}^{t}\right)\right)}, \sqrt[t]{1 - \left(\sum_{j=1}^{n} w_j\left(\vartheta_{\delta(j)}^{t}\right)\right)}, \\ \sqrt[t]{1 - \left(\sum_{j=1}^{n} w_j\left(\varrho_{\delta(j)}^{t}\right)\right)} \end{pmatrix}. \quad (5.18)$$

To be same as T-SFFWA operator, the T-SFFOWA operator also follows the boundedness, idempotency and monotonicity, shift-invariance, homogeneity properties. Besides the aforementioned properties, the T-SFFOWA operator has some other useful results, as follows:

Theorem 5.2.14 *Let $\mathcal{S}_j = (\sigma_j, \vartheta_j, \varrho_j)\,(j = 1, 2, ..., n)$ be a family of T-SFNs, then we have the following:*

i). If $w = (1, 0, ..., 0)^T$ then $T - SFFOWA\,(\mathcal{S}_1, \mathcal{S}_2, ..., \mathcal{S}_n) = \max\{\mathcal{S}_1, \mathcal{S}_2, ..., \mathcal{S}_n\}$.

ii). If $w = (0, 0, ..., 1)^T$ then $T - SFFOWA\,(\mathcal{S}_1, \mathcal{S}_2, ..., \mathcal{S}_n) = \min\{\mathcal{S}_1, \mathcal{S}_2, ..., \mathcal{S}_n\}$.

iii). If $w_j = 1$ and $\varpi_i = 0\,(i \neq j)$ then $T - SFFOWA\,(\mathcal{S}_1, \mathcal{S}_2, ..., \mathcal{S}_n) = \mathcal{S}_{\delta(j)}$ where $\mathcal{S}_{\delta(j)}$ is the jth largest of $\mathcal{S}_j$, $(j = 1, 2, ..., n)$.

Based on the definition of T-SFFWA and T-SFFOWA operators, we can see that the T-SFFWA operator can weights only the SFNs while T-SFFOWA operator weights only the ordered position of SFNs. In real-life practical scenarios, we should take under consideration the both aspects at the same time. Consequently, to remove this flaw, we define the hybrid averaging operator [95] based on Frank t-norm and t-conorm, which weight both the given T-SFNs and their ordered positions.

Definition 5.2.3 *Let $\mathcal{S}_j = (\sigma_j, \vartheta_j, \varrho_j)\,(j = 1, 2, ..., n)$ be a family of T-SFNs, then the T-spherical fuzzy Frank hybrid averaging (T-SFFHA) operator is:*

$$T - SFFHA\,(\mathcal{S}_1, \mathcal{S}_2, ..., \mathcal{S}_n) = \oplus_{j=1}^{n}\left(w_j \hat{\mathcal{S}}_{\delta(j)}\right), \tag{5.19}$$

where $w = (w_1, w_2, ..., w_n)^T$ is the weight vector associated with T-SFFHA such that $w_j > 0$ and $\sum_{j=1}^{n} w_j = 1$, $\varpi = (\varpi_1, \varpi_2, ..., \varpi_n)^T$ is the weight vector of $\mathcal{S}_j\,(j = 1, 2, .., n)$ such that $\varpi_j > 0$ and $\sum_{j=1}^{n} \varpi_j = 1$. $\hat{\mathcal{S}}_{\delta(j)}$ is the jth largest of the weighted T-SFNs $\hat{\mathcal{S}}_j$ $\left(\hat{\mathcal{S}}_j = (n\varpi_j)\,\mathcal{S}_j, j = 1, 2, .., n\right)$ and n is the balancing coefficient.

Theorem 5.2.15 *Let $\mathcal{S}_j = (\sigma_j, \vartheta_j, \varrho_j)\,(j = 1, 2, ..., n)$ be a family of T-SFNs, then the result obtained by using the T-SFFHA operator is still a T-SFN, and*

$$T - SFFHA\,(\mathcal{S}_1, \mathcal{S}_2, ..., \mathcal{S}_n) = \left(\begin{array}{c} \sqrt[t]{1 - \log_\tau\left(1 + \Pi_{j=1}^{n}\left(\tau^{1-\hat{\sigma}^t_{\delta(j)}} - 1\right)^{w_j}\right)}, \sqrt[t]{\log_\tau\left(1 + \Pi_{j=1}^{n}\left(\tau^{\hat{\vartheta}^t_{\delta(j)}} - 1\right)^{w_j}\right)}, \\ \sqrt[t]{\log_\tau\left(1 + \Pi_{j=1}^{n}\left(\tau^{\hat{\varrho}^t_{\delta(j)}} - 1\right)^{w_j}\right)} \end{array}\right), \tag{5.20}$$

Proof: The proof of this result is same as of Theorem 5.2.1, and so is omitted here. □

Example 5.2.16 *Let $\mathcal{S}_1 = (0.4, 0.5, 0.2)$, $\mathcal{S}_2 = (0.6, 0.7, 0.8)$ $\mathcal{S}_3 = (0.3, 0.6, 0.5)$, be three T-SFNs (t=4), and $\varpi = (0.4, 0.4, 0.2)^T$ is the weight vector of $\mathcal{S}_j\,(j = 1, 2, 3)$. Suppose $\tau = 2$, then according to Definition 5.1.1, we can get the weighted T-SFNs:*

$$\hat{\mathcal{S}}_1 = 3 \times 0.4 \times \mathcal{S}_1 = \left(\begin{array}{c} \sqrt[4]{1 - \log_2\left(1 + \frac{\left(2^{1-0.4^4} - 1\right)^{3\times 0.4}}{(2-1)^{3\times 0.4-1}}\right)}, \sqrt[4]{\log_2\left(1 + \frac{\left(2^{0.5^4} - 1\right)^{3\times 0.4}}{(2-1)^{3\times 0.4-1}}\right)}, \\ \sqrt[4]{\log_2\left(1 + \frac{\left(2^{0.2^4} - 1\right)^{3\times 0.4}}{(2-1)^{3\times 0.4-1}}\right)} \end{array}\right)$$

$$= (0.4185, 0.4289, 0.1423);$$

$$\hat{\mathcal{S}}_2 = 3\times 0.4 \times \mathcal{S}_2 = \left(\begin{array}{c} \sqrt[4]{1-\log_2\left(1+\frac{\left(2^{1-0.6^4}-1\right)^{3\times 0.4}}{(2-1)^{3\times 0.4-1}}\right)}, \sqrt[4]{\log_2\left(1+\frac{\left(2^{0.7^4}-1\right)^{3\times 0.4}}{(2-1)^{3\times 0.4-1}}\right)}, \\ \sqrt[4]{\log_2\left(1+\frac{\left(2^{0.8^4}-1\right)^{3\times 0.4}}{(2-1)^{3\times 0.4-1}}\right)} \end{array} \right)$$

$$= (0.6264, 0.6464, 0.7617)$$

$$\hat{\mathcal{S}}_3 = 3\times 0.2 \times \mathcal{S}_3 = \left(\begin{array}{c} \sqrt[4]{1-\log_2\left(1+\frac{\left(2^{1-0.3^4}-1\right)^{3\times 0.2}}{(2-1)^{3\times 0.4-1}}\right)}, \sqrt[4]{\log_2\left(1+\frac{\left(2^{0.6^4}-1\right)^{3\times 0.2}}{(2-1)^{3\times 0.4-1}}\right)}, \\ \sqrt[4]{\log_2\left(1+\frac{\left(2^{0.5^4}-1\right)^{3\times 0.2}}{(2-1)^{3\times 0.4-1}}\right)} \end{array} \right)$$

$$= (0.2641, 0.7478, 0.6743).$$

According to Definitions 2.5.3, we can get the score of $\hat{\mathcal{S}}_j\ (j=1,2,3)$:

$$S\left(\hat{\mathcal{S}}_1\right) = -0.0044,\ S\left(\hat{\mathcal{S}}_2\right) = -0.3868,\ S\left(\hat{\mathcal{S}}_3\right) = -0.5745.$$

Since $S\left(\hat{\mathcal{S}}_1\right) > S\left(\hat{\mathcal{S}}_2\right) > S\left(\hat{\mathcal{S}}_3\right)$, *we have*

$$\hat{\mathcal{S}}_{\delta(1)} = (0.4185, 0.4289, 0.1423),\ \hat{\mathcal{S}}_{\delta(2)} = (0.6264, 0.6464, 0.7617),\ \hat{\mathcal{S}}_{\delta(3)} = (0.2641, 0.7478, 0.6743).$$

Suppose $w = (0.3, 0.4, 0.3)^T$ *is the weight vector associated with the T-SFFHA operator. Then by Definition 5.2.3 and Theorem 5.2.15, we can get:*

$$T-SFFHA\left(\hat{\mathcal{S}}_{\delta(1)}, \hat{\mathcal{S}}_{\delta(2)}, \hat{\mathcal{S}}_{\delta(3)}\right)$$

$$= \left(\begin{array}{c} \sqrt[4]{1-\log_2\left(1+\prod_{j=1}^{3}\left(2^{1-\hat{\sigma}^4_{\delta(j)}}-1\right)^{w_j}\right)}, \sqrt[4]{\log_2\left(1+\prod_{j=1}^{3}\left(2^{\hat{\vartheta}^4_{\delta(j)}}-1\right)^{w_j}\right)}, \\ \sqrt[4]{\log_2\left(1+\prod_{j=1}^{3}\left(2^{\hat{\varrho}^4_{\delta(j)}}-1\right)^{w_j}\right)} \end{array} \right)$$

$$= \left(\begin{array}{c} \sqrt[4]{1-\log_2\left(1+\left(2^{1-0.4185^4}-1\right)^{.3}\left(2^{1-0.6264^4}-1\right)^{.4}\left(2^{1-0.2641^4}-1\right)^{.3}\right)}, \\ \sqrt[4]{\log_2\left(1+\left(2^{0.4289^4}-1\right)^{.3}\left(2^{0.6464^4}-1\right)^{.4}\left(2^{0.7478^4}-1\right)^{.3}\right)}, \\ \sqrt[4]{\log_2\left(1+\left(2^{0.1423^4}-1\right)^{.3}\left(2^{0.7617^4}-1\right)^{.4}\left(2^{0.6743^4}-1\right)^{.3}\right)} \end{array} \right)$$

$$= (0.5216, 0.5995, 0.4501).$$

Theorem 5.2.17 *Let $\mathcal{S}_j = (\sigma_j, \vartheta_j, \varrho_j)\,(j = 1, 2, ..., n)$ be a family of T-SFNs, and $\tau > 1$. As $\tau \longrightarrow 1$, the T-SFFHA operator proceeds towards the following limit*

$$\lim_{\tau \longrightarrow 1} T - SFFHA\,(\mathcal{S}_1, \mathcal{S}_2, ..., \mathcal{S}_n) = \left(\begin{array}{c} \sqrt[t]{1 - \Pi_{j=1}^{n} \left(1 - \hat{\sigma}_{\delta(j)}^{t}\right)^{w_j}}, \sqrt[t]{\Pi_{j=1}^{n} \left(\hat{\vartheta}_{\delta(j)}^{t}\right)^{w_j}}, \\ \sqrt[t]{\Pi_{j=1}^{n} \left(\hat{\varrho}_{\delta(j)}^{t}\right)^{w_j}} \end{array} \right). \quad (5.21)$$

Theorem 5.2.18 *Let $\mathcal{S}_j = (\sigma_j, \vartheta_j, \varrho_j)\,(j = 1, 2, ..., n)$ be a family of T-SFNs, and $\tau > 1$. As $\tau \longrightarrow \infty$, the T-SFFHA operator approaches the following limit*

$$\lim_{\tau \longrightarrow \infty} T - SFFHA\,(\mathcal{S}_1, \mathcal{S}_2, ..., \mathcal{S}_n) = \left(\begin{array}{c} \sqrt[t]{\left(\sum_{j=1}^{n} w_j \left(\hat{\sigma}_{\delta(j)}^{t}\right)\right)}, \sqrt[t]{1 - \left(\sum_{j=1}^{n} w_j \left(\hat{\vartheta}_{\delta(j)}^{t}\right)\right)}, \\ \sqrt[t]{1 - \left(\sum_{j=1}^{n} w_j \left(\hat{\varrho}_{\delta(j)}^{t}\right)\right)} \end{array} \right). \quad (5.22)$$

To be similar as T-SFFWA operator, the T-SFFHA operator also follows the boundedness, idempotency and monotonicity, shift-invariance, homogeneity properties. Besides the aforementioned properties, the T-SFFHA operator has the following special cases.

Corollary 5.2.19 *T-SFFWA operator is a special case of the T-SFFHA operator.*

PROOF: Let $w = \left(\frac{1}{n}, \frac{1}{n}, ..., \frac{1}{n}\right)^T$, then

$$T - SFFHA\,(\mathcal{S}_1, \mathcal{S}_2, ..., \mathcal{S}_n) = w_1 \hat{\mathcal{S}}_{\delta(1)} \oplus w_2 \hat{\mathcal{S}}_{\delta(2)} \oplus ... \oplus w_n \hat{\mathcal{S}}_{\delta(n)}$$

$$= \frac{1}{n} \left(\hat{\mathcal{S}}_{\delta(1)} \oplus \hat{\mathcal{S}}_{\delta(2)} \oplus ... \oplus \hat{\mathcal{S}}_{\delta(n)}\right) = \varpi_1 \mathcal{S}_1 \oplus \varpi_2 \mathcal{S}_2 \oplus ... \oplus \varpi_n \mathcal{S}_n = T - SFFWA\,(\mathcal{S}_1, \mathcal{S}_2, ..., \mathcal{S}_n).$$

□

Corollary 5.2.20 *T-SFFOWA operator is a special case of the T-SFFHA operator.*

PROOF: Let $\varpi = \left(\frac{1}{n}, \frac{1}{n}, ..., \frac{1}{n}\right)^T$, then

$$T - SFFHA\left(\mathcal{S}_1, \mathcal{S}_2, ..., \mathcal{S}_n\right) = w_1\hat{\mathcal{S}}_{\delta(1)} \oplus w_2\hat{\mathcal{S}}_{\delta(2)} \oplus ... \oplus w_n\hat{\mathcal{S}}_{\delta(n)}$$

$$= w_1\mathcal{S}_{\delta(1)} \oplus w_2\mathcal{S}_{\delta(2)} \oplus ... \oplus w_n\mathcal{S}_{\delta(n)} = T - SFFOWA\left(\mathcal{S}_1, \mathcal{S}_2, ..., \mathcal{S}_n\right).$$

□

5.2.2 T-spherical fuzzy Frank geometric operators

This section is devoted to providing a series of T-spherical fuzzy Frank geometric aggregation operators founded on proposed Frank operations. In geometric aggregation operators, we will further discuss the T-SFFWG, T-SFFOWG and T-SFFHWG and also the basic definitions, remarks, and results, corollary for these operators which are based on the Frank t-norm and t-conorm.

Definition 5.2.4 *Let $\mathcal{S}_j = (\sigma_j, \vartheta_j, \varrho_j)\,(j = 1, 2, ..., n)$ be a family of T-SFNs, then the T-spherical fuzzy Frank weighted geometric operator (T-SFFWG) is:*

$$T - SFFWG\left(\mathcal{S}_1, \mathcal{S}_2, ..., \mathcal{S}_n\right) = \otimes_{j=1}^{n}\left(\mathcal{S}_j\right)^{\varpi_j}, \tag{5.23}$$

where $\varpi = (\varpi_1, \varpi_2, ..., \varpi_n)^T$ is the weight vector of $\mathcal{S}_j\,(j = 1, 2, .., n)$ such that $\varpi_j > 0$ and $\sum_{j=1}^{n}\varpi_j = 1$. Especially, if $\varpi = \left(\frac{1}{n}, \frac{1}{n}, ..., \frac{1}{n}\right)^T$, then the T-SFFWG operator reduces to the T-spherical fuzzy Frank geometric (T-SFFG) operator of dimension n, which is given as follows:

$$T - SFFA\left(\mathcal{S}_1, \mathcal{S}_2, ..., \mathcal{S}_n\right) = \otimes_{j=1}^{n}\left(\mathcal{S}_j\right)^{\frac{1}{n}}. \tag{5.24}$$

Theorem 5.2.21 *Let $\mathcal{S}_j = (\sigma_j, \vartheta_j, \varrho_j)\,(j = 1, 2, ..., n)$ be a family of T-SFNs, then the result obtained by usingT-SFFWG operator is still a T-SFN, and*

$$T-SFFWG\left(\mathcal{S}_1,\mathcal{S}_2,...,\mathcal{S}_n\right)=\left(\begin{array}{c}\sqrt[t]{\log_\tau\left(1+\Pi_{j=1}^n\left(\tau^{\sigma_j^t}-1\right)^{\varpi_j}\right)},\\ \sqrt[t]{1-\log_\tau\left(1+\Pi_{j=1}^n\left(\tau^{1-\vartheta_j^t}-1\right)^{\varpi_j}\right)},\\ \sqrt[t]{1-\log_\tau\left(1+\Pi_{j=1}^n\left(\tau^{1-\varrho_j^t}-1\right)^{\varpi_j}\right)}\end{array}\right). \tag{5.25}$$

PROOF: We prove it by mathematical induction on n.

For $n=2$, we have

$T-SFFWG\left(\mathcal{S}_1,\mathcal{S}_2\right)=\mathcal{S}_1^{\varpi_1}\otimes\mathcal{S}_2^{\varpi_2}$

$$=\left(\begin{array}{c}\sqrt[t]{\log_\tau\left(1+\frac{\left(\tau^{1-\left(1-\log_\tau\left(1+\frac{\left(\tau^{\sigma_1^t}-1\right)^{\varpi_1}}{(\tau-1)^{\varpi_1-1}}\right)\right)}-1\right)\left(\tau^{1-\left(1-\log_\tau\left(1+\frac{\left(\tau^{\sigma_2^t}-1\right)^{\varpi_2}}{(\tau-1)^{\varpi_2-1}}\right)\right)}-1\right)}{\tau-1}\right)},\\ \sqrt[t]{1-\log_\tau\left(1+\frac{\left(\tau^{1-\left(1-\log_\tau\left(1+\frac{\left(\tau^{1-\vartheta_1^t}-1\right)^{\varpi_1}}{(\tau-1)^{\varpi_1-1}}\right)\right)}-1\right)\left(\tau^{1-\left(1-\log_\tau\left(1+\frac{\left(\tau^{1-\vartheta_2^t}-1\right)^{\varpi_2}}{(\tau-1)^{\varpi_2-1}}\right)\right)}-1\right)}{\tau-1}\right)},\\ \sqrt[t]{1-\log_\tau\left(1+\frac{\left(\tau^{1-\left(1-\log_\tau\left(1+\frac{\left(\tau^{1-\varrho_1^t}-1\right)^{\varpi_1}}{(\tau-1)^{\varpi_1-1}}\right)\right)}-1\right)\left(\tau^{1-\left(1-\log_\tau\left(1+\frac{\left(\tau^{1-\varrho_2^t}-1\right)^{\varpi_2}}{(\tau-1)^{\varpi_2-1}}\right)\right)}-1\right)}{\tau-1}\right)}\end{array}\right)$$

$$= \begin{pmatrix} \sqrt[t]{\log_\tau \left(1 + \frac{\left(1+\frac{\left(\tau^{\sigma_1^t}-1\right)^{\varpi_1}}{(\tau-1)^{\varpi_1-1}}-1\right)\left(1+\frac{\left(\tau^{\sigma_2^t}-1\right)^{\varpi_2}}{(\tau-1)^{\varpi_2-1}}-1\right)}{\tau-1}\right)}, \\ \sqrt[t]{1-\log_\tau \left(1 + \frac{\left(1+\frac{\left(\tau^{1-\vartheta_1^t}-1\right)^{\varpi_1}}{(\tau-1)^{\varpi_1-1}}-1\right)\left(1+\frac{\left(\tau^{1-\vartheta_2^t}-1\right)^{\varpi_2}}{(\tau-1)^{\varpi_2-1}}-1\right)}{\tau-1}\right)}, \\ \sqrt[t]{1-\log_\tau \left(1 + \frac{\left(1+\frac{\left(\tau^{1-\varrho_1^t}-1\right)^{\varpi_1}}{(\tau-1)^{\varpi_1-1}}-1\right)\left(1+\frac{\left(\tau^{1-\varrho_2^t}-1\right)^{\varpi_2}}{(\tau-1)^{\varpi_2-1}}-1\right)}{\tau-1}\right)} \end{pmatrix}$$

$$= \begin{pmatrix} \sqrt[t]{\log_\tau \left(1 + \left(\left(\tau^{\sigma_1^t}-1\right)^{\varpi_1}\right)\left(\left(\tau^{\sigma_2^t}-1\right)^{\varpi_2}\right)\right)}, \\ \sqrt[t]{1-\log_\tau \left(1 + \left(\left(\tau^{1-\vartheta_1^t}-1\right)^{\varpi_1}\right)\left(\left(\tau^{1-\vartheta_2^t}-1\right)^{\varpi_2}\right)\right)}, \\ \sqrt[t]{1-\log_\tau \left(1 + \left(\left(\tau^{1-\varrho_1^t}-1\right)^{\varpi_1}\right)\left(\left(\tau^{1-\varrho_2^t}-1\right)^{\varpi_2}\right)\right)} \end{pmatrix}.$$

Thus, result holds for $n = 2$.

If Equation (5.25) holds for $n = k$, then for $n = k + 1$, we have

$$T - SFFWG\left(\mathcal{S}_1, \mathcal{S}_2, ..., \mathcal{S}_{k+1}\right) = T - SFFWG\left(\mathcal{S}_1, \mathcal{S}_2, ..., \mathcal{S}_k\right) \otimes \mathcal{S}_{k+1}^{\varpi_{k+1}}$$

$$= \begin{pmatrix} \sqrt[t]{\log_\tau \left(1 + \Pi_{j=1}^k \left(\tau^{\sigma_j^t}-1\right)^{\varpi_j}\right)}, \\ \sqrt[t]{1-\log_\tau \left(1 + \Pi_{j=1}^k \left(\tau^{1-\vartheta_j^t}-1\right)^{\varpi_j}\right)}, \\ \sqrt[t]{1-\log_\tau \left(1 + \Pi_{j=1}^k \left(\tau^{1-\varrho_j^t}-1\right)^{\varpi_j}\right)} \end{pmatrix} \otimes \begin{pmatrix} \sqrt[t]{\log_\tau \left(1 + \frac{\left(\tau^{\sigma_{(k+1)}^t}-1\right)^{\varpi_{(k+1)}}}{(\tau-1)^{\varpi_{(k+1)}-1}}\right)}, \\ \sqrt[t]{1-\log_\tau \left(1 + \frac{\left(\tau^{1-\vartheta_{(k+1)}^t}-1\right)^{\varpi_{(k+1)}}}{(\tau-1)^{\varpi_{(k+1)}-1}}\right)}, \\ \sqrt[t]{1-\log_\tau \left(1 + \frac{\left(\tau^{1-\varrho_{(k+1)}^t}-1\right)^{\varpi_{(k+1)}}}{(\tau-1)^{\varpi_{(k+1)}-1}}\right)} \end{pmatrix}$$

$$= \left(\begin{array}{c} \sqrt[t]{\log_\tau\left(1+\frac{\Pi_{j=1}^{k}\left(\tau^{\sigma_j^t}-1\right)^{\varpi_j}\frac{\left(\tau^{\sigma_{(k+1)}^t}-1\right)^{\varpi(k+1)}}{(\tau-1)^{\varpi(k+1)^{-1}}}}{\tau-1}\right)}, \\ \sqrt[t]{1-\log_\tau\left(1+\frac{\Pi_{j=1}^{k}\left(\tau^{1-\vartheta_j^t}-1\right)^{\varpi_j}\frac{\left(\tau^{1-\vartheta_{(k+1)}^t}-1\right)^{\varpi(k+1)}}{(\tau-1)^{\varpi(k+1)^{-1}}}}{\tau-1}\right)}, \\ \sqrt[t]{1-\log_\tau\left(1+\frac{\Pi_{j=1}^{k}\left(\tau^{1-\varrho_j^t}-1\right)^{\varpi_j}\frac{\left(\tau^{1-\varrho_{(k+1)}^t}-1\right)^{\varpi(k+1)}}{(\tau-1)^{\varpi(k+1)^{-1}}}}{\tau-1}\right)} \end{array} \right)$$

$$= \left(\begin{array}{c} \sqrt[t]{\log_\tau\left(1+\frac{\Pi_{j=1}^{k}\left(\tau^{\sigma_j^t}-1\right)^{\varpi_j}\left(\tau^{\sigma_{(k+1)}^t}-1\right)^{\varpi(k+1)}}{(\tau-1)^{\sum_{j=1}^{k}\varpi(j)^{-1}}(\tau-1)(\tau-1)^{\varpi(k+1)^{-1}}}\right)}, \\ \sqrt[t]{1-\log_\tau\left(1+\frac{\Pi_{j=1}^{k}\left(\tau^{1-\vartheta_j^t}-1\right)^{\varpi_j}\left(\tau^{1-\vartheta_{(k+1)}^t}-1\right)^{\varpi(k+1)}}{(\tau-1)^{\sum_{j=1}^{k}\varpi(j)^{-1}}(\tau-1)(\tau-1)^{\varpi(k+1)^{-1}}}\right)}, \\ \sqrt[t]{1-\log_\tau\left(1+\frac{\Pi_{j=1}^{k}\left(\tau^{1-\varrho_j^t}-1\right)^{\varpi_j}\left(\tau^{1-\varrho_{(k+1)}^t}-1\right)^{\varpi(k+1)}}{(\tau-1)^{\sum_{j=1}^{k}\varpi(j)^{-1}}(\tau-1)(\tau-1)^{\varpi(k+1)^{-1}}}\right)} \end{array} \right)$$

$$= \left(\begin{array}{c} \sqrt[t]{\log_\tau\left(1+\frac{\Pi_{j=1}^{k}\left(\tau^{\sigma_j^t}-1\right)^{\varpi_j}\left(\tau^{\sigma_{(k+1)}^t}-1\right)^{\varpi(k+1)}}{(\tau-1)^{\sum_{j=1}^{k+1}\varpi(j)^{-1}}}\right)}, \\ \sqrt[t]{1-\log_\tau\left(1+\frac{\Pi_{j=1}^{k}\left(\tau^{1-\vartheta_j^t}-1\right)^{\varpi_j}\left(\tau^{1-\vartheta_{(k+1)}^t}-1\right)^{\varpi(k+1)}}{(\tau-1)^{\sum_{j=1}^{k+1}\varpi(j)^{-1}}}\right)}, \\ \sqrt[t]{1-\log_\tau\left(1+\frac{\Pi_{j=1}^{k}\left(\tau^{1-\varrho_j^t}-1\right)^{\varpi_j}\left(\tau^{1-\varrho_{(k+1)}^t}-1\right)^{\varpi(k+1)}}{(\tau-1)^{\sum_{j=1}^{k+1}\varpi(j)^{-1}}}\right)} \end{array} \right)$$

$$= \left(\begin{array}{c} \sqrt[t]{\log_\tau \left(1 + \Pi_{j=1}^{k} \left(\tau^{\sigma_j^t} - 1\right)^{\varpi_j} \left(\tau^{\sigma_{(k+1)}^t} - 1\right)^{\varpi_{(k+1)}}\right)}, \\ \sqrt[t]{1 - \log_\tau \left(1 + \Pi_{j=1}^{k} \left(\tau^{1-\vartheta_j^t} - 1\right)^{\varpi_j} \left(\tau^{1-\vartheta_{(k+1)}^t} - 1\right)^{\varpi_{(k+1)}}\right)}, \\ \sqrt[t]{1 - \log_\tau \left(1 + \Pi_{j=1}^{k} \left(\tau^{1-\varrho_j^t} - 1\right)^{\varpi_j} \left(\tau^{1-\varrho_{(k+1)}^t} - 1\right)^{\varpi_{(k+1)}}\right)} \end{array} \right).$$

Thus, results holds for $n = k+1$ and hence, by the principle of mathematical induction, result given in Equation (5.25) holds for all positive integer n. □

Example 5.2.22 *(Continued from Example 5.2.2)*

According to Definition 5.2.4 and Theorem 5.2.21, we have

$$T - SFFWG(\mathcal{S}_1, \mathcal{S}_2, \mathcal{S}_3) = \left(\begin{array}{c} \sqrt[4]{\log_2 \left(1 + \prod_{j=1}^{3} \left(2^{\sigma_j^4} - 1\right)^{\varpi_j}\right)}, \\ \sqrt[4]{1 - \log_2 \left(1 + \prod_{j=1}^{3} \left(2^{1-\vartheta_j^4} - 1\right)^{\varpi_j}\right)}, \\ \sqrt[4]{1 - \log_2 \left(1 + \prod_{j=1}^{3} \left(2^{1-\varrho_j^4} - 1\right)^{\varpi_j}\right)} \end{array} \right)$$

$$= \left(\begin{array}{c} \sqrt[4]{\log_2 \left(1 + \left(2^{.4^4} - 1\right)^{.4} \left(2^{.7^4} - 1\right)^{.3} \left(2^{.6^4} - 1\right)^{.3}\right)}, \\ \sqrt[4]{1 - \log_2 \left(1 + \left(2^{1-.3^4} - 1\right)^{.4} \left(2^{1-.3^4} - 1\right)^{.3} \left(2^{1-.7^4} - 1\right)^{.3}\right)}, \\ \sqrt[4]{1 - \log_2 \left(1 + \left(2^{1-.5^4} - 1\right)^{.4} \left(2^{1-.4^4} - 1\right)^{.3} \left(2^{1-.8^4} - 1\right)^{.3}\right)} \end{array} \right)$$

$$= (0.5361, 0.5358, 0.6417).$$

Theorem 5.2.23 *Let $\mathcal{S}_j = (\sigma_j, \vartheta_j, \varrho_j)\,(j = 1, 2, ..., n)$ be a family of T-SFNs, and $\tau > 1$. As $\tau \longrightarrow 1$, the T-SFFWG operator proceeds towards the following limit*

$$\lim_{\tau \longrightarrow 1} T - SFFWG(\mathcal{S}_1, \mathcal{S}_2, ..., \mathcal{S}_n) = \left(\begin{array}{c} \sqrt[t]{\Pi_{j=1}^{n} \left(\sigma_j^t\right)^{\varpi_j}}, \sqrt[t]{1 - \Pi_{j=1}^{n} \left(1 - \vartheta_j^t\right)^{\varpi_j}}, \\ \sqrt[t]{1 - \Pi_{j=1}^{n} \left(1 - \varrho_j^t\right)^{\varpi_j}} \end{array} \right). \qquad (5.26)$$

PROOF: As $\tau \longrightarrow 1$, then

$$\left(\Pi_{j=1}^{n} \left(\tau^{\sigma_j^t} - 1\right)^{\varpi_j},\ \Pi_{j=1}^{n} \left(\tau^{1-\vartheta_j^t} - 1\right)^{\varpi_j}, \Pi_{j=1}^{n} \left(\tau^{1-\varrho_j^t} - 1\right)^{\varpi_j}\right) \longrightarrow (0, 0, 0)$$

by log property and the rule of infinitesimal changes, we have

$$\log_\tau\left(1+\Pi_{j=1}^n\left(\tau^{\sigma_j^t}-1\right)^{\varpi_j}\right)=\frac{\ln\left(1+\Pi_{j=1}^n\left(\tau^{\sigma_j^t}-1\right)^{\varpi_j}\right)}{\ln\tau}\longrightarrow\frac{\Pi_{j=1}^n\left(\tau^{\sigma_j^t}-1\right)^{\varpi_j}}{\ln\tau}$$

$$\log_\tau\left(1+\Pi_{j=1}^n\left(\tau^{1-\vartheta_j^t}-1\right)^{\varpi_j}\right)=\frac{\ln\left(1+\Pi_{j=1}^n\left(\tau^{1-\vartheta_j^t}-1\right)^{\varpi_j}\right)}{\ln\tau}\longrightarrow\frac{\Pi_{j=1}^n\left(\tau^{1-\vartheta_j^t}-1\right)^{\varpi_j}}{\ln\tau}$$

$$\log_\tau\left(1+\Pi_{j=1}^n\left(\tau^{1-\varrho_j^t}-1\right)^{\varpi_j}\right)=\frac{\ln\left(1+\Pi_{j=1}^n\left(\tau^{1-\varrho_j^t}-1\right)^{\varpi_j}\right)}{\ln\tau}\longrightarrow\frac{\Pi_{j=1}^n\left(\tau^{1-\varrho_j^t}-1\right)^{\varpi_j}}{\ln\tau}$$

Based on Taylor's expansion formula, we have

$$\tau^{\vartheta_j^t}=1+\left(\sigma_j^t\right)\ln\tau+\frac{\left((\sigma_j^t)\ln\tau\right)^2}{2!}+...$$

$$\tau^{1-\sigma_j^t}=1+\left(1-\vartheta_j^t\right)\ln\tau+\frac{\left((1-\vartheta_j^t)\ln\tau\right)^2}{2!}+...$$

$$\tau^{1-\sigma_j^t}=1+\left(1-\varrho_j^t\right)\ln\tau+\frac{\left((1-\varrho_j^t)\ln\tau\right)^2}{2!}+...$$

Also, since $\tau>1$, then $\ln\tau>0$,

$$\tau^{\sigma_j^t}=1+\left(\sigma_j^t\right)\ln\tau+O\left(\ln\tau\right),$$

$$\tau^{1-\vartheta_j^t}=1+\left(1-\vartheta_j^t\right)\ln\tau+O\left(\ln\tau\right),$$

$$\tau^{1-\varrho_j^t}=1+\left(1-\varrho_j^t\right)\ln\tau+O\left(\ln\tau\right).$$

It follows that

$$\left(\tau^{\sigma_j^t}-1\right)^{\varpi_j}\longrightarrow\left(\left(\sigma_j^t\right)\ln\tau\right)^{\varpi_j}$$

$$\Pi_{j=1}^n\left(\tau^{\sigma_j^t}-1\right)^{\varpi_j}\longrightarrow\Pi_{j=1}^n\left(\sigma_j^t\right)\Pi_{j=1}^n\left(\ln\tau\right)^{\varpi_j}$$

$$\Pi_{j=1}^n\left(\tau^{\sigma_j^t}-1\right)^{\varpi_j}\longrightarrow\Pi_{j=1}^n\left(\sigma_j^t\right)\ln\left(\tau\right)^{\sum_{j=1}^n\varpi_j}$$

$$\frac{\Pi_{j=1}^n\left(\tau^{\sigma_j^t}-1\right)^{\varpi_j}}{\ln\tau}\longrightarrow\Pi_{j=1}^n\left(\sigma_j^t\right).$$

Analogously, we can get

$$\frac{\Pi_{j=1}^n\left(\tau^{1-\vartheta_j^t}-1\right)^{\varpi_j}}{\ln\tau}\longrightarrow\Pi_{j=1}^n\left(1-\vartheta_j^t\right)\text{ and }\frac{\Pi_{j=1}^n\left(\tau^{1-\varrho_j^t}-1\right)^{\varpi_j}}{\ln\tau}\longrightarrow\Pi_{j=1}^n\left(1-\varrho_j^t\right).$$

Then, we have $\lim_{\tau\longrightarrow1}T-SFFWG\left(\mathcal{S}_1,\mathcal{S}_2,...,\mathcal{S}_n\right)$

$$= \lim_{\tau \longrightarrow 1} \begin{pmatrix} \sqrt[t]{\log_\tau \left(1 + \Pi_{j=1}^{n} \left(\tau^{\sigma_j^t} - 1\right)^{\varpi_j}\right)}, \sqrt[t]{1 - \log_\tau \left(1 + \Pi_{j=1}^{n} \left(\tau^{1-\vartheta_j^t} - 1\right)^{\varpi_j}\right)}, \\ \sqrt[t]{1 - \log_\tau \left(1 + \Pi_{j=1}^{n} \left(\tau^{1-\varrho_j^t} - 1\right)^{\varpi_j}\right)} \end{pmatrix}$$

$$= \lim_{\tau \longrightarrow 1} \begin{pmatrix} \sqrt[t]{\frac{\ln\left(1+\Pi_{j=1}^{n}\left(\tau^{\sigma_j^t}-1\right)^{\varpi_j}\right)}{\ln \tau}}, \sqrt[t]{1 - \frac{\ln\left(1+\Pi_{j=1}^{n}\left(\tau^{1-\vartheta_j^t}-1\right)^{\varpi_j}\right)}{\ln \tau}}, \\ \sqrt[t]{1 - \frac{\ln\left(1+\Pi_{j=1}^{n}\left(\tau^{1-\varrho_j^t}-1\right)^{\varpi_j}\right)}{\ln \tau}} \end{pmatrix}$$

$$= \lim_{\tau \longrightarrow 1} \begin{pmatrix} \sqrt[t]{\frac{\Pi_{j=1}^{n}\left(\tau^{\sigma_j^t}-1\right)^{\varpi_j}}{\ln \tau}}, \sqrt[t]{1 - \frac{\Pi_{j=1}^{n}\left(\tau^{1-\vartheta_j^t}-1\right)^{\varpi_j}}{\ln \tau}}, \\ \sqrt[t]{1 - \frac{\Pi_{j=1}^{n}\left(\tau^{1-\vartheta_j^t}-1\right)^{\varpi_j}}{\ln \tau}} \end{pmatrix}$$

$$= \begin{pmatrix} \sqrt[t]{\Pi_{j=1}^{n}\left(\sigma_j^t\right)^{\varpi_j}}, \sqrt[t]{1 - \Pi_{j=1}^{n}\left(1-\vartheta_j^t\right)^{\varpi_j}}, \\ \sqrt[t]{1 - \Pi_{j=1}^{n}\left(1-\varrho_j^t\right)^{\varpi_j}} \end{pmatrix}$$

which completes the proof. □

Theorem 5.2.24 *Let $\mathcal{S}_j = (\sigma_j, \vartheta_j, \varrho_j)\,(j = 1, 2, ..., n)$ be a family of T-SFNs, and $\tau > 1$. As $\tau \longrightarrow \infty$, the T-SFFWG operator proceeds towards the following limit*

$$\lim_{\tau \longrightarrow \infty} T - SFFWG\left(\mathcal{S}_1, \mathcal{S}_2, ..., \mathcal{S}_n\right) = \begin{pmatrix} \sqrt[t]{1 - \left(\sum_{j=1}^{n} \varpi_j \left(\sigma_j^t\right)\right)}, \sqrt[t]{\left(\sum_{j=1}^{n} \varpi_j \left(\vartheta_j^t\right)\right)}, \\ \sqrt[t]{\left(\sum_{j=1}^{n} \varpi_j \left(\varrho_j^t\right)\right)} \end{pmatrix}. \tag{5.27}$$

Proof: According to Theorem 5.2.21, we have

$$\lim_{\tau \longrightarrow \infty} T - SFFWG\left(\mathcal{S}_1, \mathcal{S}_2, ..., \mathcal{S}_n\right)$$

$$= \begin{pmatrix} \lim_{\tau \longrightarrow \infty} \sqrt[t]{\log_\tau \left(1 + \Pi_{j=1}^{n} \left(\tau^{\sigma_j^t} - 1\right)^{\varpi_j}\right)}, \\ \lim_{\tau \longrightarrow \infty} \sqrt[t]{1 - \log_\tau \left(1 + \Pi_{j=1}^{n} \left(\tau^{1-\vartheta_j^t} - 1\right)^{\varpi_j}\right)}, \\ \lim_{\tau \longrightarrow \infty} \sqrt[t]{1 - \log_\tau \left(1 + \Pi_{j=1}^{n} \left(\tau^{1-\varrho_j^t} - 1\right)^{\varpi_j}\right)} \end{pmatrix}.$$

Using limit rules, logarithmic transform and L'Hospital's rule, it follows that,

$$\left(\begin{array}{c} \sqrt[t]{\lim_{\tau \longrightarrow \infty} \frac{\ln\left(1+\Pi_{j=1}^{n}\left(\tau^{\sigma_j^t}-1\right)^{\varpi_j}\right)}{\ln \tau}}, \\ \sqrt[t]{1-\lim_{\tau \longrightarrow \infty} \frac{\ln\left(1+\Pi_{j=1}^{n}\left(\tau^{1-\vartheta_j^t}-1\right)^{\varpi_j}\right)}{\ln \tau}}, \\ \sqrt[t]{1-\lim_{\tau \longrightarrow \infty} \frac{\ln\left(1+\Pi_{j=1}^{n}\left(\tau^{1-\varrho_j^t}-1\right)^{\varpi_j}\right)}{\ln \tau}} \end{array} \right)$$

$$= \left(\begin{array}{c} \sqrt[t]{1-\lim_{\tau \longrightarrow \infty} \frac{\frac{\Pi_{j=1}^{n}\left(\tau^{\sigma_j^t}-1\right)^{\varpi_j}}{1+\Pi_{j=1}^{n}\left(\tau^{\sigma_j^t}-1\right)^{\varpi_j}}\left(\sum_{j=1}^{n}\varpi_j\left(\vartheta_j^t\right)\frac{\tau^{\sigma_j^t-1}}{\tau^{\sigma_j^t}-1}\right)}{\frac{1}{\tau}}}, \\ \sqrt[t]{1-\lim_{\tau \longrightarrow \infty} \frac{\frac{\Pi_{j=1}^{n}\left(\tau^{1-\vartheta_j^t}-1\right)^{\varpi_j}}{1+\Pi_{j=1}^{n}\left(\tau^{1-\vartheta_j^t}-1\right)^{\varpi_j}}\left(\sum_{j=1}^{n}\varpi_j\left(1-\vartheta_j^t\right)\frac{\tau^{-\vartheta_j^t}}{\tau^{1-\vartheta_j^t}-1}\right)}{\frac{1}{\tau}}}, \\ \sqrt[t]{1-\lim_{\tau \longrightarrow \infty} \frac{\frac{\Pi_{j=1}^{n}\left(\tau^{1-\varrho_j^t}-1\right)^{\varpi_j}}{1+\Pi_{j=1}^{n}\left(\tau^{1-\varrho_j^t}-1\right)^{\varpi_j}}\left(\sum_{j=1}^{n}\varpi_j\left(1-\varrho_j^t\right)\frac{\tau^{-\varrho_j^t}}{\tau^{1-\varrho_j^t}-1}\right)}{\frac{1}{\tau}}}, \end{array} \right)$$

$$= \left(\begin{array}{c} \sqrt[t]{1-\lim_{\tau \longrightarrow \infty} \frac{\Pi_{j=1}^{n}\left(\tau^{\sigma_j^t}-1\right)^{\varpi_j}}{1+\Pi_{j=1}^{n}\left(\tau^{\sigma_j^t}-1\right)^{\varpi_j}}\left(\sum_{j=1}^{n}\varpi_j\left(\sigma_j^t\right)\frac{\tau^{\sigma_j^t}}{\tau^{\sigma_j^t}-1}\right)}, \\ \sqrt[t]{1-\lim_{\tau \longrightarrow \infty} \frac{\Pi_{j=1}^{n}\left(\tau^{1-\vartheta_j^t}-1\right)^{\varpi_j}}{1+\Pi_{j=1}^{n}\left(\tau^{1-\vartheta_j^t}-1\right)^{\varpi_j}}\left(\sum_{j=1}^{n}\varpi_j\left(1-\vartheta_j^t\right)\frac{\tau^{1-\vartheta_j^t}}{\tau^{1-\vartheta_j^t}-1}\right)}, \\ \sqrt[t]{1-\lim_{\tau \longrightarrow \infty} \frac{\Pi_{j=1}^{n}\left(\tau^{1-\varrho_j^t}-1\right)^{\varpi_j}}{1+\Pi_{j=1}^{n}\left(\tau^{1-\varrho_j^t}-1\right)^{\varpi_j}}\left(\sum_{j=1}^{n}\varpi_j\left(1-\varrho_j^t\right)\frac{\tau^{1-\varrho_j^t}}{\tau^{1-\varrho_j^t}-1}\right)} \end{array} \right)$$

$$= \left(\begin{array}{c} \sqrt[t]{1-\left(\sum_{j=1}^{n}\varpi_j\left(\sigma_j^t\right)\right)}, \\ \sqrt[t]{1-\left(\sum_{j=1}^{n}\varpi_j\left(1-\vartheta_j^t\right)\right)}, \\ \sqrt[t]{1-\left(\sum_{j=1}^{n}\varpi_j\left(1-\varrho_j^t\right)\right)} \end{array} \right)$$

$$= \begin{pmatrix} \sqrt[t]{1-\left(\sum_{j=1}^{n}\varpi_j\left(\sigma_j^t\right)\right)}, \\ \sqrt[t]{\left(\sum_{j=1}^{n}\varpi_j\left(\vartheta_j^t\right)\right)}, \\ \sqrt[t]{\left(\sum_{j=1}^{n}\varpi_j\left(\varrho_j^t\right)\right)} \end{pmatrix}$$

which completes the proof of Theorem 5.2.24. □

Theorem 5.2.25 *(Idempotency)*

Let $\mathcal{T}_j = (\sigma_j, \vartheta_j, \varrho_j)\,(j = 1, 2, ..., n)$ be a family of T-SFNs, if $\mathcal{S}_j = \mathcal{S}_0$ for all j, then

$$T - SFFWG\,(\mathcal{S}_1, \mathcal{S}_2, ..., \mathcal{S}_n) = \mathcal{S}_0. \tag{5.28}$$

PROOF: Since for all j $\mathcal{S}_j = \mathcal{S}_0 = (\sigma_0, \vartheta_0, \varrho_0)$, and $\sum_{j=1}^{n}\varpi_j = 1$ so by Theorem 5.2.21, we have

$$T - SFFWG\,(\mathcal{S}_1, \mathcal{S}_2, ..., \mathcal{S}_n)$$

$$= \begin{pmatrix} \sqrt[t]{\log_\tau\left(1+\Pi_{j=1}^{n}\left(\tau^{\sigma_0^t}-1\right)^{\varpi_j}\right)}, \sqrt[t]{1-\log_\tau\left(1+\Pi_{j=1}^{n}\left(\tau^{1-\vartheta_0^t}-1\right)^{\varpi_j}\right)}, \\ \sqrt[t]{1-\log_\tau\left(1+\Pi_{j=1}^{n}\left(\tau^{1-\varrho_0^t}-1\right)^{\varpi_j}\right)} \end{pmatrix}$$

$$= \left(\sqrt[t]{\log_\tau \tau^{\sigma_0^t}}, \sqrt[t]{1-\log_\tau \tau^{1-\vartheta_0^t}}, \sqrt[t]{1-\log_\tau \tau^{1-\varrho_0^t}} \right)$$

$$= (\sigma_0, \vartheta_0, \varrho_0) = \mathcal{S}_0.$$

Thus, proof is completed. □

Theorem 5.2.26 *(Monotonicity) Let $\mathcal{S}_j = (\sigma_j, \vartheta_j, \varrho_j)\,(j = 1, 2, ..., n)$ and $\dot{\mathcal{S}}_j = \left(\dot{\sigma}_j, \dot{\vartheta}_j, \dot{\varrho}_j\right)$ $(j = 1, 2, ..., n)$ be two families of T-SFNs such that $\sigma_j \geq \dot{\sigma}_j, \vartheta_j \leq \dot{\vartheta}_j$ and $\varrho_j \leq \dot{\varrho}_j$ $\forall$ j, then*

$$T - SFFWG\,(\mathcal{S}_1, \mathcal{S}_2, ..., \mathcal{S}_n) \geq T - SFFWG\left(\dot{\mathcal{S}}_1, \dot{\mathcal{S}}_2, ..., \dot{\mathcal{S}}_n\right). \tag{5.29}$$

PROOF: According to Definition 2.5.3, when $\sigma_j \geq \dot{\sigma}_j, \vartheta_j \leq \dot{\vartheta}_j$ and $\varrho_j \leq \dot{\varrho}_j$ $\forall$ j, then

$$\sqrt[t]{\log_\tau\left(1+\Pi_{j=1}^{n}\left(\tau^{\sigma_j^t}-1\right)^{\varpi_j}\right)} \leq \sqrt[t]{\log_\tau\left(1+\Pi_{j=1}^{n}\left(\tau^{\dot{\sigma}_j^t}-1\right)^{\varpi_j}\right)},$$

$$\sqrt[t]{1-\log_{\tau}\left(1+\Pi_{j=1}^{n}\left(\tau^{1-\vartheta_j^t}-1\right)^{\varpi_j}\right)} \geq \sqrt[t]{1-\log_{\tau}\left(1+\Pi_{j=1}^{n}\left(\tau^{1-\dot{\vartheta}_j^t}-1\right)^{\varpi_j}\right)}$$

and

$$\sqrt[t]{1-\log_{\tau}\left(1+\Pi_{j=1}^{n}\left(\tau^{1-\varrho_j^t}-1\right)^{\varpi_j}\right)} \geq \sqrt[t]{1-\log_{\tau}\left(1+\Pi_{j=1}^{n}\left(\tau^{1-\dot{\varrho}_j^t}-1\right)^{\varpi_j}\right)}.$$

Thus, $S\left(T-SFFWG\left(\mathcal{S}_1,\mathcal{S}_2,...,\mathcal{S}_n\right)\right) \geq S\left(T-SFFWG\left(\dot{\mathcal{S}}_1,\dot{\mathcal{S}}_2,...,\dot{\mathcal{S}}_n\right)\right)$

Hence, $T-SFFWG\left(\mathcal{S}_1,\mathcal{S}_2,...,\mathcal{S}_n\right) \geq T-SFFWG\left(\dot{\mathcal{S}}_1,\dot{\mathcal{S}}_2,...,\dot{\mathcal{S}}_n\right)$. □

Theorem 5.2.27 *(Boundedness) Let $\mathcal{S}_j=(\sigma_j,\vartheta_j,\varrho_j)\,(j=1,2,...,n)$ be a family of T-SFNs, and let $\mathcal{S}^-=(\min_{1\leq j\leq n}\sigma_j,\max_{1\leq j\leq n}\vartheta_j,\ \max_{1\leq j\leq n}\varrho_j)$,*

$\mathcal{S}^+=(\max_{1\leq j\leq n}\sigma_j,\min_{1\leq j\leq n}\vartheta_j,\ \min_{1\leq j\leq n}\varrho_j)$, then

$$\mathcal{S}^- \leq T-SFFWG\left(\mathcal{S}_1,\mathcal{S}_2,...,\mathcal{S}_n\right) \leq \mathcal{S}^+. \tag{5.30}$$

Proof: Since for all j, $\min_{1\leq j\leq n}\sigma_j \leq \sigma_j \leq \max_{1\leq j\leq n}\sigma_j$, $\min_{1\leq j\leq n}\vartheta_j \leq \vartheta_j \leq \max_{1\leq j\leq n}\vartheta_j$ and $\min_{1\leq j\leq n}\varrho_j \leq \varrho_j \leq \max_{1\leq j\leq n}\varrho_j$, thereby on the basis of idempotency and monotonicity, we get $\mathcal{S}^- \leq T-SFFWG\left(\mathcal{S}_1,\mathcal{S}_2,...,\mathcal{S}_n\right) \leq \mathcal{S}^+$. □

Theorem 5.2.28 *(Shift-invariance) Let $\mathcal{S}_j=(\sigma_j,\vartheta_j,\varrho_j)\,(j=1,2,...,n)$ be a family of T-SFNs and $\dot{\mathcal{S}}=\left(\dot{\sigma},\dot{\vartheta},\dot{\varrho}\right)$ be any other T-SFNs, then*

$$T-SFFWG\left(\mathcal{S}_1\otimes\dot{\mathcal{S}},\mathcal{S}_2\otimes\dot{\mathcal{S}},...,\mathcal{S}_n\otimes\dot{\mathcal{S}}\right)=T-SFFWG\left(\mathcal{S}_1,\mathcal{S}_2,...,\mathcal{S}_n\right)\otimes\dot{\mathcal{S}}. \tag{5.31}$$

Theorem 5.2.29 *(Homogeneity) Let $\mathcal{S}_j=(\sigma_j,\vartheta_j,\varrho_j)$*
$(j=1,2,...,n)$ be a family of T-SFNs and $\eta>0$ be any real number, then

$$T-SFFWG\left(\eta\mathcal{S}_1,\eta\mathcal{S}_2,...,\eta\mathcal{S}_n\right)=\eta T-SFFWG\left(\mathcal{S}_1,\mathcal{S}_2,...,\mathcal{S}_n\right). \tag{5.32}$$

The above two theorems can be easily verified from the proposed Frank operational laws of T-SFNs; thus, we omit here due to the space limitations.

Definition 5.2.5 *Let $\mathcal{S}_j = (\sigma_j, \vartheta_j, \varrho_j)\,(j = 1, 2, ..., n)$ be a family of T-SFNs, then the T-spherical fuzzy Frank ordered weighted geometric (T-SFFOWG) operator is:*

$$T - SFFOWG\,(\mathcal{S}_1, \mathcal{S}_2, ..., \mathcal{S}_n) = \otimes_{j=1}^{n} \left(\mathcal{S}_{\delta(j)}^{w_j}\right), \tag{5.33}$$

where $w = (w_1, w_2, ..., w_n)^T$ is the position weights of $\mathcal{S}_j\,(j = 1, 2, .., n)$ such that $w_j > 0$ and $\sum_{j=1}^{n} w_j = 1$. $(\delta(1), \delta(2), ..., \delta(n))$ is a permutation of $(1, 2, 3, ..., n)$ such that $\mathcal{S}_{\delta(j-1)} \geq \mathcal{S}_{\delta(j)}$ for $j = 2, 3.., n$.

Theorem 5.2.30 *Let $\mathcal{S}_j = (\sigma_j, \vartheta_j, \varrho_j)\,(j = 1, 2, ..., n)$ be a family of T-SFNs, then the result obtained by using T-SFFOWG operator is still a T-SFN, and*

$$T - SFFOWG\,(\mathcal{S}_1, \mathcal{S}_2, ..., \mathcal{S}_n) = \begin{pmatrix} \sqrt[t]{\log_\tau \left(1 + \Pi_{j=1}^{n} \left(\tau^{\sigma_{\delta(j)}^t} - 1\right)^{w_j}\right)}, \\ \sqrt[t]{1 - \log_\tau \left(1 + \Pi_{j=1}^{n} \left(\tau^{1-\vartheta_{\delta(j)}^t} - 1\right)^{w_j}\right)}, \\ \sqrt[t]{1 - \log_\tau \left(1 + \Pi_{j=1}^{n} \left(\tau^{1-\varrho_{\delta(j)}^t} - 1\right)^{w_j}\right)} \end{pmatrix}. \tag{5.34}$$

PROOF: The proof of this result is similar to that of Theorem 5.2.21, and so we omit here. □

Example 5.2.31 *(Continued from Example 5.2.11) According to Definition 5.2.5 and Theorem 5.2.30, we can get:*

$$T - SFFOWG\left(\mathcal{S}_{\delta(1)}, \mathcal{S}_{\delta(2)}, \mathcal{S}_{\delta(3)}\right)$$
$$= \begin{pmatrix} \sqrt[4]{\log_2 \left(1 + \prod_{j=1}^{3} \left(2^{\sigma_{\delta(j)}^4} - 1\right)^{w_j}\right)}, \\ \sqrt[4]{1 - \log_2 \left(1 + \prod_{j=1}^{3} \left(2^{1-\vartheta_{\delta(j)}^4} - 1\right)^{w_j}\right)}, \\ \sqrt[4]{1 - \log_2 \left(1 + \prod_{j=1}^{3} \left(2^{1-\varrho_{\delta(j)}^4} - 1\right)^{w_j}\right)} \end{pmatrix}$$

$$= \begin{pmatrix} \sqrt[4]{\log_2\left(1+\left(2^{.7^4}-1\right)^{.3}\left(2^{.3^4}-1\right)^{.4}\left(2^{.6^4}-1\right)^{.3}\right)}, \\ \sqrt[4]{1-\log_2\left(1+\left(2^{1-.4^4}-1\right)^{.3}\left(2^{1-.3^4}-1\right)^{.4}\left(2^{1-.7^4}-1\right)^{.3}\right)}, \\ \sqrt[4]{1-\log_2\left(1+\left(2^{1-.5^4}-1\right)^{.3}\left(2^{1-.5^4}-1\right)^{.4}\left(2^{1-.8^4}-1\right)^{.3}\right)} \end{pmatrix}$$

$$= (0.4788, 0.5438, 0.6509).$$

Theorem 5.2.32 *Let $\mathcal{S}_j = (\sigma_j, \vartheta_j, \varrho_j)\,(j = 1, 2, ..., n)$ be a family of T-SFNs, and $\tau > 1$. As $\tau \longrightarrow 1$, the T-SFFOWG operator approaches the following limit*

$$\lim_{\tau \longrightarrow 1} T - SFFOWG(\mathcal{S}_1, \mathcal{S}_2, ..., \mathcal{S}_n)$$

$$= \begin{pmatrix} \sqrt[t]{\Pi_{j=1}^{n}\left(\sigma_{\delta(j)}^{t}\right)^{w_j}}, \sqrt[t]{1-\Pi_{j=1}^{n}\left(1-\vartheta_{\delta(j)}^{t}\right)^{w_j}}, \\ \sqrt[t]{1-\Pi_{j=1}^{n}\left(1-\varrho_{\delta(j)}^{t}\right)^{w_j}} \end{pmatrix}. \quad (5.35)$$

Theorem 5.2.33 *Let $\mathcal{S}_j = (\sigma_j, \vartheta_j, \varrho_j)\,(j = 1, 2, ..., n)$ be a family of T-SFNs, and $\tau > 1$. As $\tau \longrightarrow \infty$, the T-SFFOWG operator approaches the following limit*

$$\lim_{\tau \longrightarrow \infty} T - SFFOWG(\mathcal{S}_1, \mathcal{S}_2, ..., \mathcal{S}_n)$$

$$= \begin{pmatrix} \sqrt[t]{1-\left(\sum_{j=1}^{n} w_j\left(\sigma_{\delta(j)}^{t}\right)\right)}, \sqrt[t]{\left(\sum_{j=1}^{n} w_j\left(\vartheta_{\delta(j)}^{t}\right)\right)}, \\ \sqrt[t]{\left(\sum_{j=1}^{n} w_j\left(\varrho_{\delta(j)}^{t}\right)\right)} \end{pmatrix}. \quad (5.36)$$

As similar to those of the T-SFFWG operator, the T-SFFOWG operator also follows the boundedness, idempotency and monotonicity, shift-invariance, homogeneity properties. Besides the aforementioned properties, the T-SFFOWG operator has the following desirable results.

Theorem 5.2.34 *Let $\mathcal{S}_j = (\sigma_j, \vartheta_j, \varrho_j)\,(j = 1, 2, ..., n)$ be a family of T-SFNs, then we have the following:*

i). If $w = (1, 0, ..., 0)^T$ then $T - SFFOWG(\mathcal{S}_1, \mathcal{S}_2, ..., \mathcal{S}_n) = \max\{\mathcal{S}_1, \mathcal{S}_2, ..., \mathcal{S}_n\}$.

ii). If $w = (0, 0, ..., 1)^T$ then $T - SFFOWG(\mathcal{S}_1, \mathcal{S}_2, ..., \mathcal{S}_n) = \min\{\mathcal{S}_1, \mathcal{S}_2, ..., \mathcal{S}_n\}$.

iii). If $w_j = 1$ and $\varpi_i = 0\,(i \neq j)$ then $T - SFFOWG\,(\mathcal{S}_1, \mathcal{S}_2, ..., \mathcal{S}_n) = \mathcal{S}_{\delta(j)}$ where $\mathcal{S}_{\delta(j)}$ is the jth largest of $\mathcal{S}_j$, $(j = 1, 2, ..., n)$.

Based on the definition of T-SFFWG and T-SFFOWG operators, we can see that the T-SFFWG operator can weigh only the SFNs while the T-SFFOWG operator weights only the ordered position of SFNs. In practical real-life scenarios, both aspects should be considered at the same time. Hence, to overcome this shortcoming, we define the hybrid geometric operator [95] founded on Frank t-norm and t-conorm, which weights both the given T-SFNs and also their ordered positions.

Definition 5.2.6 *Let $\mathcal{S}_j = (\sigma_j, \vartheta_j, \varrho_j)\,(j = 1, 2, ..., n)$ be a family of T-SFNs, then the T-spherical fuzzy Frank hybrid geometric (T-SFFHG) operator is:*

$$T - SFFHG\,(\mathcal{S}_1, \mathcal{S}_2, ..., \mathcal{S}_n) = \otimes_{j=1}^{n} \left(\hat{\mathcal{S}}_{\delta(j)}\right)^{w_j}, \tag{5.37}$$

where $w = (w_1, w_2, ..., w_n)^T$ is the weight vector associated with T-SFFHG such that $w_j > 0$ and $\sum_{j=1}^{n} w_j = 1$, $\varpi = (\varpi_1, \varpi_2, ..., \varpi_n)^T$ is the weight vector of $\mathcal{S}_j\,(j = 1, 2, .., n)$ such that $\varpi_j > 0$ and $\sum_{j=1}^{n} \varpi_j = 1$. $\hat{\mathcal{S}}_{\delta(j)}$ is the jth largest of the weighted T-SFNs $\hat{\mathcal{S}}_j$, where $\hat{\mathcal{S}}_j = (\mathcal{S}_j)^{n\varpi_j}, (j = 1, 2, .., n)$ and n is the balancing coefficient.

Theorem 5.2.35 *Let $\mathcal{S}_j = (\sigma_j, \vartheta_j, \varrho_j)\,(j = 1, 2, ..., n)$ be a family of T-SFNs, then the result derived by using the T-SFFHG operator is still a T-SFN, and*

$$T - SFFHG\,(\mathcal{S}_1, \mathcal{S}_2, ..., \mathcal{S}_n) = \begin{pmatrix} \sqrt[t]{\log_\tau \left(1 + \Pi_{j=1}^{n} \left(\tau^{\hat{\sigma}^t_{\delta(j)}} - 1\right)^{w_j}\right)}, \sqrt[t]{1 - \log_\tau \left(1 + \Pi_{j=1}^{n} \left(\tau^{1-\hat{\vartheta}^t_{\delta(j)}} - 1\right)^{w_j}\right)}, \\ \sqrt[t]{1 - \log_\tau \left(1 + \Pi_{j=1}^{n} \left(\tau^{1-\hat{\varrho}^t_{\delta(j)}} - 1\right)^{w_j}\right)} \end{pmatrix}, \tag{5.38}$$

PROOF: This proof is similar to that of Theorem 5.2.21, hence omitted here. □

Example 5.2.36 *(Continued from Example 5.2.16) According to Definition 5.1.1, we can get the weighted T-SFNs:*

$$\hat{\mathcal{S}}_1 = \mathcal{S}_1^{3\times 0.4} = \left(\begin{array}{c} \sqrt[4]{\log_2\left(1+\frac{\left(2^{0.4^4}-1\right)^{3\times 0.4}}{(2-1)^{3\times 0.4-1}}\right)}, \sqrt[4]{1-\log_2\left(1+\frac{\left(2^{1-0.5^4}-1\right)^{3\times 0.4}}{(2-1)^{3\times 0.4-1}}\right)}, \\ \sqrt[4]{1-\log_2\left(1+\frac{\left(2^{1-0.2^4}-1\right)^{3\times 0.4}}{(2-1)^{3\times 0.4-1}}\right)} \end{array} \right)$$

$$= (0.3275, 0.5227, 0.2093)\,;$$

$$\hat{\mathcal{S}}_2 = \mathcal{S}_2^{3\times 0.4} = \left(\begin{array}{c} \sqrt[4]{\log_2\left(1+\frac{\left(2^{0.6^4}-1\right)^{3\times 0.4}}{(2-1)^{3\times 0.4-1}}\right)}, \sqrt[4]{1-\log_2\left(1+\frac{\left(2^{1-0.7^4}-1\right)^{3\times 0.4}}{(2-1)^{3\times 0.4-1}}\right)}, \\ \sqrt[4]{1-\log_2\left(1+\frac{\left(2^{1-0.8^4}-1\right)^{3\times 0.4}}{(2-1)^{3\times 0.4-1}}\right)} \end{array} \right)$$

$$= (0.5353, 0.7290, 0.8292)\,;$$

$$\hat{\mathcal{S}}_3 = \mathcal{S}_3^{3\times 0.2}$$

$$= \left(\begin{array}{c} \sqrt[4]{\log_2\left(1+\frac{\left(2^{0.3^4}-1\right)^{3\times 0.2}}{(2-1)^{3\times 0.4-1}}\right)}, \sqrt[4]{1-\log_2\left(1+\frac{\left(2^{1-0.6^4}-1\right)^{3\times 0.2}}{(2-1)^{3\times 0.4-1}}\right)}, \\ \sqrt[4]{1-\log_2\left(1+\frac{\left(2^{1-0.5^4}-1\right)^{3\times 0.2}}{(2-1)^{3\times 0.4-1}}\right)} \end{array} \right)$$

$$= (0.5012, 0.5307, 0.4410)\,.$$

According to Definition 2.5.3, we can get the score of $\hat{\mathcal{S}}_j\ (j=1,2,3)$:

$S\left(\hat{\mathcal{S}}_1\right) = -0.0799,\ S\left(\hat{\mathcal{S}}_2\right) = -0.6995,\ S\left(\hat{\mathcal{S}}_3\right) = -0.0651.$

Since $S\left(\hat{\mathcal{S}}_3\right) > S\left(\hat{\mathcal{S}}_1\right) > S\left(\hat{\mathcal{S}}_2\right)$, *we have*

$$\hat{\mathcal{S}}_{\delta(1)} = (0.5012, 0.5307, 0.4410)\,,$$

$$\hat{\mathcal{S}}_{\delta(2)} = (0.3275, 0.5227, 0.2093)\,,$$

$$\hat{\mathcal{S}}_{\delta(3)} = (0.5353, 0.7290, 0.8292)\,.$$

Suppose $w = (0.3, 0.4, 0.3)^T$ *is the weight vector associated with the T-SFFHG operator. Then by Definition 5.2.6 and Theorem 5.2.35, we can get:*

$$T-SFFHG\left(\hat{\mathcal{S}}_{\delta(1)},\hat{\mathcal{S}}_{\delta(2)},\hat{\mathcal{S}}_{\delta(3)}\right)=\left(\begin{array}{c}\sqrt[4]{\log_2\left(1+\prod_{j=1}^{3}\left(2^{\hat{\sigma}^4_{\delta(j)}}-1\right)^{w_j}\right)},\\ \sqrt[4]{1-\log_2\left(1+\prod_{j=1}^{3}\left(2^{1-\hat{\vartheta}^4_{\delta(j)}}-1\right)^{w_j}\right)},\\ \sqrt[4]{1-\log_2\left(1+\prod_{j=1}^{3}\left(2^{1-\hat{\varrho}^4_{\delta(j)}}-1\right)^{w_j}\right)}\end{array}\right)$$

$$=\left(\begin{array}{c}\sqrt[4]{\log_2\left(1+\left(2^{0.5012^4}-1\right)^{.3}\left(2^{0.3275^4}-1\right)^{.4}\left(2^{0.5353^4}-1\right)^{.3}\right)},\\ \sqrt[4]{1-\log_2\left(1+\left(2^{1-0.5307^4}-1\right)^{.3}\left(2^{1-0.5227^4}-1\right)^{.4}\left(2^{1-0.7290^4}-1\right)^{.3}\right)},\\ \sqrt[4]{1-\log_2\left(1+\left(2^{1-0.4410^4}-1\right)^{.3}\left(2^{1-0.2093^4}-1\right)^{.4}\left(2^{1-0.8292^4}-1\right)^{.3}\right)}\end{array}\right)$$

$$=(0.4317, 0.6143, 0.6486).$$

Theorem 5.2.37 *Let $\mathcal{S}_j=(\sigma_j,\vartheta_j,\varrho_j)\,(j=1,2,...,n)$ be a family of T-SFNs, and $\tau>1$. As $\tau\longrightarrow 1$, the T-SFFHG operator proceeds towards the following limit*

$$\lim_{\tau\longrightarrow 1} T-SFFHG(\mathcal{S}_1,\mathcal{S}_2,...,\mathcal{S}_n)$$
$$=\left(\begin{array}{c}\sqrt[t]{\Pi_{j=1}^{n}\left(\hat{\sigma}^t_{\delta(j)}\right)^{w_j}},\sqrt[t]{1-\Pi_{j=1}^{n}\left(1-\hat{\vartheta}^t_{\delta(j)}\right)^{w_j}},\\ \sqrt[t]{1-\Pi_{j=1}^{n}\left(1-\hat{\varrho}^t_{\delta(j)}\right)^{w_j}}\end{array}\right). \quad (5.39)$$

Theorem 5.2.38 *Let $\mathcal{S}_j=(\sigma_j,\vartheta_j,\varrho_j)\,(j=1,2,...,n)$ be a family of T-SFNs, and $\tau>1$. As $\tau\longrightarrow\infty$, the T-SFFHG operator approaches the following limit*

$$\lim_{\tau\longrightarrow\infty} T-SFFHG(\mathcal{S}_1,\mathcal{S}_2,...,\mathcal{S}_n)$$
$$=\left(\begin{array}{c}\sqrt[t]{1-\left(\sum_{j=1}^{n}w_j\left(\hat{\sigma}^t_{\delta(j)}\right)\right)},\sqrt[t]{\left(\sum_{j=1}^{n}w_j\left(\hat{\vartheta}^t_{\delta(j)}\right)\right)},\\ \sqrt[t]{\left(\sum_{j=1}^{n}w_j\left(\hat{\varrho}^t_{\delta(j)}\right)\right)}\end{array}\right). \quad (5.40)$$

To be similar as T-SFFWG operator, the T-SFFHG operator also follows the boundedness, idempotency and monotonicity, shift-invariance, homogeneity properties. Besides the aforementioned properties, the T-SFFHG operator has the following special cases.

Corollary 5.2.39 *T-SFFWG operator is a special case of the T-SFFHG operator.*

PROOF: Let $w = \left(\frac{1}{n}, \frac{1}{n}, ..., \frac{1}{n}\right)^T$, then

$$T - SFFHG\left(\mathcal{S}_1, \mathcal{S}_2, ..., \mathcal{S}_n\right) = \hat{\mathcal{S}}_{\delta(1)}^{w_1} \otimes \hat{\mathcal{S}}_{\delta(2)}^{w_2} \otimes ... \otimes \hat{\mathcal{S}}_{\delta(n)}^{w_n}$$

$$= \left(\hat{\mathcal{S}}_{\delta(1)} \otimes \hat{\mathcal{S}}_{\delta(2)} \otimes ... \otimes \hat{\mathcal{S}}_{\delta(n)}\right)^{\frac{1}{n}} = \mathcal{S}_1^{\varpi_1} \otimes \mathcal{S}_2^{\varpi_2} \otimes ... \otimes \mathcal{S}_n^{\varpi_n}$$

$$= T - SFFWG\left(\mathcal{S}_1, \mathcal{S}_2, ..., \mathcal{S}_n\right).$$

□

Corollary 5.2.40 *T-SFFOWG operator is a special case of the T-SFFHG operator.*

PROOF: Let $\varpi = \left(\frac{1}{n}, \frac{1}{n}, ..., \frac{1}{n}\right)^T$, then

$$T - SFFHG\left(\mathcal{S}_1, \mathcal{S}_2, ..., \mathcal{S}_n\right) = \hat{\mathcal{S}}_{\delta(1)}^{w_1} \otimes \hat{\mathcal{S}}_{\delta(2)}^{w_2} \otimes ... \otimes \hat{\mathcal{S}}_{\delta(n)}^{w_n}$$

$$= \mathcal{S}_{\delta(1)}^{w_1} \otimes \mathcal{S}_{\delta(2)}^{w_2} \otimes ... \otimes \mathcal{S}_{\delta(n)}^{w_n} = T - SFFOWG\left(\mathcal{S}_1, \mathcal{S}_2, ..., \mathcal{S}_n\right).$$

□

5.3 Entropy measure for T-spherical fuzzy set

In this part, the entropy measure for T-spherical fuzzy set is given in detail and the required proof in terms of satisfying given properties is shared.

Definition 5.3.1 *Let $\mathcal{S}_1$ and $\mathcal{S}_2$ be two T-SFSs on Y. A real-valued function E :T-SFS $\longrightarrow [0,1]$ is an entropy measure for SFSs if it is provided with the following properties:*

p_1. *$E\left(\mathcal{S}_1\right) = 0$ iff $\mathcal{S}_1$ is a crisp set;*

p_2. *$E\left(\mathcal{S}_1\right) = 1$ iff $\sigma_1(y) = \varrho_1(y)$ and $\vartheta_1(y) = \sqrt[t]{0.25}$ for all $y \in Y$;*

p_3. *$E\left(\mathcal{S}_1\right) = E\left(\mathcal{S}_1^c\right)$;*

p_4. *$E\left(\mathcal{S}_1\right) \leq E\left(\mathcal{S}_2\right)$ if $\vartheta_2^t(y) \leq \vartheta_1^t(y)$ and $\sigma_1^t(y) \leq \sigma_2^t(y) \leq \varrho_2^t(y) \leq \varrho_1^t(y)$ or $\varrho_1^t(y) \leq \varrho_2^t(y) \leq \sigma_2^t(y) \leq \sigma_1^t(y)$ for all $y \in Y$.*

Theorem 5.3.1 *Let $\mathcal{S}$ be a T-SFS on Y. The mapping*

$$E\left(\mathcal{S}\right) = \frac{1}{n}\sum_{i=1}^{n}\left(1 - \frac{4}{5}\left[\left|\sigma^t(y_i) - \varrho^t(y_i)\right| + \left|\vartheta^t(y_i) - 0.25\right|\right]\right), \tag{5.41}$$

is an entropy measure for T-SFS.

PROOF:

p_1. Let $\mathcal{S}$ be a crisp set. Then, we have $\sigma^t(y_i) = 1$, $\varrho^t(y_i) = 0$, $\vartheta^t(y_i) = 0$ or $\sigma^t(y_i) = 0$, $\varrho^t(y_i) = 1$, $\vartheta^t(y_i) = 0$ $\forall y_i \in Y$. If $\sigma^t(y_i) = 1$, $\varrho^t(y_i) = 0$, $\vartheta^t(y_i) = 0$, then

$E(\mathcal{S}) = \frac{1}{n}\sum_{i=1}^{n}\left(1 - \frac{4}{5}\left[\left|\sigma^t(y_i) - \varrho^t(y_i)\right| + \ \left|\vartheta^t(y_i) - 0.25\right|\right]\right)$
$= \frac{1}{n}\sum_{i=1}^{n}\left(1 - \frac{4}{5}\left[|1-0| + \ |0 - 0.25|\right]\right)$
$= \frac{1}{n}\sum_{i=1}^{n}\left(1 - \frac{4}{5}\left[1.25\right]\right) = 0.$

Analogously, when $\sigma^t(y_i) = 0$, $\varrho^t(y_i) = 1$, $\vartheta^t(y_i) = 0$ $\forall y_i \in Y$, we can easily show that $E(\mathcal{S}) = 0$. Conversely, suppose that $E(\mathcal{S}) = 0$. Thus,

$$\frac{4}{5}\left[\left|\sigma^t(y_i) - \varrho^t(y_i)\right| + \left|\vartheta^t(y_i) - 0.25\right|\right] = 1, \tag{5.42}$$

Eq. (5.42) $\Rightarrow$ $\left|\sigma^t(y_i) - \varrho^t(y_i)\right| + \left|\vartheta^t(y_i) - 0.25\right| = 1.25.$

There are four possibilities:

The first one is $\left(\sigma^t(y_i) - \varrho^t(y_i)\right) + \left(\vartheta^t(y_i) - 0.25\right) = 1.25$
$\Rightarrow \sigma^t(y_i) - \varrho^t(y_i) + \vartheta^t(y_i) = 1.5 \Rightarrow \sigma^t(y_i) + \vartheta^t(y_i) = \varrho^t(y_i) + 1.5.$

The second is $\left(\sigma^t(y_i) - \varrho^t(y_i)\right) - \left(\vartheta^t(y_i) - 0.25\right) = 1.25 \Rightarrow \sigma^t(y_i) - \varrho^t(y_i) - \vartheta^t(y_i) = 1$
$\Rightarrow \sigma^t(y_i) = \vartheta^t(y_i) + \varrho^t(y_i) + 1.$

The third is $-\left(\sigma^t(y_i) - \varrho^t(y_i)\right) + \left(\vartheta^t(y_i) - 0.25\right) = 1.25 \Rightarrow -\sigma^t(y_i) + \varrho^t(y_i) + \vartheta^t(y_i) = 1.5$
$\Rightarrow \varrho^t(y_i) + \vartheta^t(y_i) = \sigma^t(y_i) + 1.5.$

The last one is $-\left(\sigma^t(y_i) - \varrho^t(y_i)\right) - \left(\vartheta^t(y_i) - 0.25\right) = 1.25$
$\Rightarrow -\sigma^t(y_i) + \varrho^t(y_i) - \vartheta^t(y_i) = 1 \Rightarrow \varrho^t(y_i) = \sigma^t(y_i) + \vartheta^t(y_i) + 1.$

In all these possibilities, the inequality $0 \leq \sigma^t(y_i) + \vartheta^t(y_i) + \varrho^t(y_i) \leq 1$ is not satisfied for all $y_i \in Y$. So Eq. (5.42) can hold when $\sigma^t(y_i) = 1$, $\varrho^t(y_i) = 0$, $\vartheta^t(y_i) = 0$ or $\sigma^t(y_i) = 0$, $\varrho^t(y_i) = 1$, $\vartheta^t(y_i) = 0$ $\forall y_i \in Y$. Consequently, it is proved that $\mathcal{S}$ is a crisp set.

p_2. Let $\sigma^t(y_i) = \varrho^t(y_i)$ $\vartheta^t(y_i) = 0.25$ $\forall y_i \in Y$. Then

$E(\mathcal{S}) = \frac{1}{n}\sum_{i=1}^{n}\left(1 - \frac{4}{5}\left[\left|\sigma^t(y_i) - \varrho^t(y_i)\right| \ + \left|\vartheta^t(y_i) - 0.25\right|\right]\right) = \frac{1}{n}\sum_{i=1}^{n}\left(1 - \frac{4}{5}\left[0 + \ 0\right]\right) = 1.$

Conversely, suppose that $E(\mathcal{S}) = 1$. Then $\frac{4}{5}\left[\left|\sigma^t(y_i) - \varrho^t(y_i)\right| + \left|\vartheta^t(y_i) - 0.25\right|\right] = 0$

$\Rightarrow \left|\sigma^t(y_i) - \varrho^t(y_i)\right| + \left|\vartheta^t(y_i) - 0.25\right| = 0.$

Hence, $\sigma(y_i) = \varrho(y_i)$ and $\vartheta(y_i) = \sqrt[t]{0.25}\ \forall\ y_i \in Y$.

p_3. $E(\mathcal{S}) = \frac{1}{n}\sum_{i=1}^{n}\left(1 - \frac{4}{5}\left[\left|\sigma^t(y_i) - \varrho^t(y_i)\right| + \left|\vartheta^t(y_i) - 0.25\right|\right]\right)$

$= \frac{1}{n}\sum_{i=1}^{n}\left(1 - \frac{4}{5}\left[\left|\varrho^t(y_i) - \sigma^t(y_i)\right| + \left|\vartheta^t(y_i) - 0.25\right|\right]\right) = E(\mathcal{S}^c)$.

p_4. Since $\vartheta_2^t(y) \leq \vartheta_1^t(y)$ and $\sigma_1^t(y) \leq \sigma_2^t(y) \leq \varrho_2^t(y) \leq \varrho_1^t(y)$, we have

$\left|\sigma_2^t(y_i) - \varrho_2^t(y_i)\right| \leq \left|\sigma_1^t(y_i) - \varrho_1^t(y_i)\right|$ and $\left|\vartheta_2^t(y_i) - 0.25\right| \leq \left|\vartheta_1^t(y_i) - 0.25\right|$.

Therefore

$E(\mathcal{S}_1) = \frac{1}{n}\sum_{i=1}^{n}\left(1 - \frac{4}{5}\left[\left|\sigma_1^t(y_i) - \varrho_1^t(y_i)\right| + \left|\vartheta_1^t(y_i) - 0.25\right|\right]\right)$
$\leq \frac{1}{n}\sum_{i=1}^{n}\left(1 - \frac{4}{5}\left[\left|\sigma_2^t(y_i) - \varrho_2^t(y_i)\right| + \left|\vartheta_2^t(y_i) - 0.25\right|\right]\right)$
$= E(\mathcal{S}_2)$.

Analogously, if $\vartheta_2^t(y) \leq \vartheta_1^t(y)$ and $\varrho_1^t(y) \leq \varrho_2^t(y) \leq \sigma_2^t(y) \leq \sigma_1^t(y)$ for all $y \in Y$, then $E(\mathcal{S}_1) \leq E(\mathcal{S}_2)$.

□

5.4 Proposed method of SFS for MCDM problems

This section focuses on presenting an MCDM method based on the proposed Frank aggregation operators to handle decision-making problems with T-spherical fuzzy information.

5.4.1 Decision-making method

Let $o = \{o_1, o_2, ..., o_m\}$ be a set of alternatives, and $\kappa = \{\kappa_1, \kappa_2, ..., \kappa_n\}$ be the set of criteria, whose weight vector is unknown. The characteristics of each alternative $o_i\ (i = 1, 2, .., m)$ with respect to each criteria is characterized in terms of T-SFNs $\mathcal{S}_{ij} = (\sigma_{ij} + \vartheta_{ij} + \varrho_{ij})$; $0 \leq \sigma_{ij}^t + \vartheta_{ij}^t + \varrho_{ij}^t \leq 1$ Then, in the following, we construct an MCDM method based on the proposed operator to cope with the decision-making problems with T-spherical fuzzy information, which mainly relies on following steps.

Step 1: Formation of decision matrix:

Collect the T-spherical fuzzy information from experts about the finite set of alternatives observing the criteria in the form of matrix $M = [\mathcal{S}_{ij}]$. Also, decide the least value of t for which every triplet of the provided information lies in the frame of T-SFNs.

Step 2: Normalization:

Transform the decision matrix $M = [\mathcal{S}_{ij}]$ into the normalized form $\widetilde{M} = \left[\widetilde{\mathcal{S}}_{ij}\right]$ by the below formula:

$$\widetilde{\mathcal{S}}_{ij} = \begin{cases} \mathcal{S}_{ij}, & \text{if for benefit criteria} \\ \left(\mathcal{S}_{ij}\right)^c, & \text{for cost criteria.} \end{cases}$$

where $\left(\mathcal{S}_{ij}\right)^c$ is the complement of $\mathcal{S}_{ij}$.

Step 3: Criteria weight determination:

The stated entropy measure for T-SFSs is used to derive the weights of criteria. First, Eq. (5.41) is used for each jth criteria to get the entropy measure that represents the information dispersion in handled criteria, and then all the criteria's entropies are used to derive their weights (ϖ_j).

$$E_j = \frac{1}{n}\sum_{i=1}^{m}\left(1 - \frac{4}{5}\left[\left|\sigma_{ij}^t - \varrho_{ij}^t\right| + \left|\vartheta_{ij}^t - 0.25\right|\right]\right) \tag{5.43}$$

Divergence representing the intrinsic information of jth criteria is computed through $div_j = 1 - E_j$. The objective criteria weights can be computed as follows [96]:

$$\varpi_j = \frac{div_j}{\sum_{j=1}^{n} div_j}. \tag{5.44}$$

Step 4: Aggregation:

Aggregate the T-SFNs $\mathcal{S}_{ij}\,(j = 1, 2, ..., n)$ for each alternative $o_i\,(i = 1, 2, ..., n)$ into the overall preference value $\mathcal{S}_i$ either by using the proposed T-SFFHWA or T-SFFHWG operators. Mathematically,

$$\mathcal{S}_i = T - SFFHWA_{\varpi,w}\left(\mathcal{S}_{i1}, \mathcal{S}_{i2}, ..., \mathcal{S}_{im}\right), \tag{5.45}$$

$$\mathcal{S}_i = T - SFFHWG_{\varpi,w}\left(\mathcal{S}_{i1}, \mathcal{S}_{i2}, ..., \mathcal{S}_{im}\right), \tag{5.46}$$

where $\varpi = (\varpi_1, \varpi_2, ..., \varpi_n)$ is the weight vector of criteria derived in Step 3. And $w = (w_1, w_2, ..., w_n)$ is the weight vector associated the aggregation operator such that $w_j > 0$ and $\sum_{j=1}^{n} w_j = 1$.

Step 5: Score values:

Compute the score values of $S(\mathcal{S}_i)$
$(i = 1, 2, ..., m)$ of the overall values $\mathcal{S}_i\ (i = 1, 2, ..., m)$.

Step 6: Ranking:

Rank the alternatives $o_i\ (i = 1, 2, ..., m)$ according to the score values $S(\mathcal{S}_i)$ and select the best one.

5.5 Illustrative example

This section first uses a numerical example to demonstrate the working procedure of the suggested MCDM method, then performs a series of experiments to examine the effects of different specific operators and parameter values on the obtained aggregation results.

5.5.1 Example

The presented MCDM method is illustrated by using a numerical example to determine the best industry for investment from five possible industries (adopted from Ref. [74]).

To attain maximum use of idle capital, the board of directors of the company decided to invest in a new industry. Following the preliminary investigation, four industries were selected as potential investment targets. The four alternative industries are manufacturing industry (o_1), real estate development industry (o_2), education and training industry (o_3) and medical industry (o_4). To determine the best choice for investment, the directors have set up a panel of experts. These experts were called to rate the four alternatives industries based on four criteria namely, level of capital gain (κ_1), market potential (κ_2), growth potential (κ_3) and political stability (κ_4), whose

Table 5.1: T-spherical fuzzy decision matrix

	κ_1	κ_2	κ_3	κ_4
o_1	$(0.7, 0.3, 0.4)$	$(0.3, 0.3, 0.5)$	$(0.6, 0.7, 0.6)$	$(0.3, 0.4, 0.2)$
o_2	$(0.5, 0.5, 0.4)$	$(0.6, 0.3, 0.4)$	$(0.3, 0.2, 0.8)$	$(0.7, 0.2, 0.4)$
o_3	$(0.6, 0.9, 0.2)$	$(0.7, 0.3, 0.3)$	$(0.5, 0.4, 0.3)$	$(0.4, 0.7, 0.5)$
o_4	$(0.8, 0.2, 0.2)$	$(0.5, 0.6, 0.2)$	$(0.4, 0.1, 0.3)$	$(0.5, 0.7, 0.4)$

weights are unknown. To offer sufficient freedom in the evaluation of five criteria values for each alternative industry, the experts were allowed to use T-SFNs. The evaluation information of experts is detailed in the matrix as follows.

5.5.2 The process of solving

In the following, we utilize T-SFFHA operator and T-SFFHG operator in the provided MCDM approach with T-spherical fuzzy information.

Using T-SFFHA operator

Step 1. The provided decision matrix is listed in Table 5.1. On account of this, we determine the value of 't' for which the given data lie in T-spherical fuzzy information.

As $0.6+0.7+0.6 = 1.9 > 1$, for $t = 2$, $0.6^2+0.7^2+0.6^2 = 1.21 > 1$, for $t = 3$, $0.6^3+0.7^3+0.6^3 = 0.775 < 1$. Analogously, for $t = 3$, all the given data lie in the T-spherical fuzzy information.

Step 2. Consider all the criteria $\kappa_j\,(j = 1, 2, 3, 4)$ be the benefit criteria, hence, the criteria values of alternatives $o_i\,(i = 1, 2, 3, 4)$ do not require normalization.

Step 3. In the light of Equation (5.41), the criteria weight is calculated as given below.

$$\varpi = (0.3576, 0.2337, 0.2344, 0.1743),$$

$$n\varpi = (1.4304, 0.9348, 0.9376, 0.6972)\,.$$

Step 4. Using the T-spherical fuzzy information listed in Table 5.1, the values of $\hat{\mathcal{S}}_{ij} = (n\varpi_j)\,\mathcal{S}_{ij}$, (suppose n=4, $\tau = 2$), are worked out as given below.

$\hat{\mathcal{S}}_{11} = (0.7713, 0.1702, 0.2580)\,,\ \hat{\mathcal{S}}_{12} = (0.2934, 0.3267, 0.5256)\,,\ \hat{\mathcal{S}}_{13} = (0.5883, 0.7172, 0.6214)\,,$

$\hat{\mathcal{S}}_{14} = (0.2663, 0.5406, 0.3366)\,,\ \hat{\mathcal{S}}_{21} = (0.5595, 0.3575, 0.2580)\,,\ \hat{\mathcal{S}}_{22} = (0.5878, 0.3267, 0.4271)\,,$

$\hat{\mathcal{S}}_{23} = (0.2937, 0.2227, 0.8120)\,,\ \hat{\mathcal{S}}_{24} = (0.6303, 0.3366, 0.5406)\,,\ \hat{\mathcal{S}}_{31} = (0.6675, 0.8578, 0.0950)\,,$

$\hat{\mathcal{S}}_{32} = (0.6867, 0.3267, 0.3267)\,,\ \hat{\mathcal{S}}_{33} = (0.4899, 0.4259, 0.3255)\,,\ \hat{\mathcal{S}}_{34} = (0.3555, 0.7857, 0.6280)\,,$

$\hat{\mathcal{S}}_{41} = (0.8675, 0.0950, 0.0950)\,,\ \hat{\mathcal{S}}_{42} = (0.4894, 0.6224, 0.2238)\,,\ \hat{\mathcal{S}}_{43} = (0.3917, 0.1163, 0.3255)\,,$

$\hat{\mathcal{S}}_{44} = (0.4455, 0.7857, 0.5406)\,.$

Based on the score function, we have

$\hat{\mathcal{S}}_{\delta(11)} = \hat{\mathcal{S}}_{11} = (0.7713, 0.1702, 0.2580)\,,\ \hat{\mathcal{S}}_{\delta(12)} = \hat{\mathcal{S}}_{12} = (0.2934, 0.3267, 0.5256)\,,$

$\hat{\mathcal{S}}_{\delta(13)} = \hat{\mathcal{S}}_{14} = (0.2663, 0.5406, 0.3366)\,,\ \hat{\mathcal{S}}_{\delta(14)} = \hat{\mathcal{S}}_{13} = (0.5883, 0.7172, 0.6214)\,,$

$\hat{\mathcal{S}}_{\delta(21)} = \hat{\mathcal{S}}_{21} = (0.5595, 0.3575, 0.2580)\,,\ \hat{\mathcal{S}}_{\delta(22)} = \hat{\mathcal{S}}_{22} = (0.5878, 0.3267, 0.4271)\,,$

$\hat{\mathcal{S}}_{\delta(23)} = \hat{\mathcal{S}}_{24} = (0.6303, 0.3366, 0.5406)\,,\ \hat{\mathcal{S}}_{\delta(24)} = \hat{\mathcal{S}}_{23} = (0.2937, 0.2227, 0.8120)\,,$

$\hat{\mathcal{S}}_{\delta(31)} = \hat{\mathcal{S}}_{32} = (0.6867, 0.3267, 0.3267)\,,\ \hat{\mathcal{S}}_{\delta(32)} = \hat{\mathcal{S}}_{33} = (0.4899, 0.4259, 0.3255)\,,$

$\hat{\mathcal{S}}_{\delta(33)} = \hat{\mathcal{S}}_{31} = (0.6675, 0.8578, 0.0950)\,,\ \hat{\mathcal{S}}_{\delta(34)} = \hat{\mathcal{S}}_{34} = (0.3555, 0.7857, 0.6280)\,,$

$\hat{\mathcal{S}}_{\delta(41)} = \hat{\mathcal{S}}_{41} = (0.8675, 0.0950, 0.0950)\,,\ \hat{\mathcal{S}}_{\delta(42)} = \hat{\mathcal{S}}_{43} = (0.3917, 0.1163, 0.3255)\,,$

$\hat{\mathcal{S}}_{\delta(43)} = \hat{\mathcal{S}}_{42} = (0.4894, 0.6224, 0.2238)\,,\ \hat{\mathcal{S}}_{\delta(44)} = \hat{\mathcal{S}}_{44} = (0.4455, 0.7857, 0.5406)\,.$

Now applying the T-SFFHA operator Equation (5.19), having associated weight vector $w = (0.3, 0.2, 0.3, 0.2)^T$ to get the overall preference values $\mathcal{S}_i$ of the alternative $o_i\,(i = 1, 2, ..., n)$:

$\mathcal{S}_1 = (0.5922, 0.3691, 0.3857)\,,\ \mathcal{S}_2 = (0.5604, 0.3138, 0.4526)\,,$

$\mathcal{S}_3 = (0.6082, 0.5579, 0.2583)\,,\ \mathcal{S}_4 = (0.6747, 0.2700, 0.2233)\,.$

Step 5: By Definition 2.5.3, we calculate the score values $S(\mathcal{S}_i)\ (i = 1, 2, 3, 4)$ of the overall preference values $\mathcal{S}_i\ (i = 1, 2, 3, 4)$ as follows:

$S(\mathcal{S}_1) = 0.1171,\ S(\mathcal{S}_2) = 0.0615,\ S(\mathcal{S}_3) = 0.0391,\ S(\mathcal{S}_4) = 0.3218.$

Step 6: Based on the above score values the ranking order of alternatives is: $o_4 \succ o_1 \succ o_2 \succ o_3$, where the symbol "$\succ$" implies "superior to". Hence, the most preferable company is o_4.

Using T-SFFHG operator

Step 4. Using the T-spherical fuzzy information listed in Table 5.1, and $\hat{\mathcal{S}}_{ij} = (\mathcal{S}_{ij})^{n\varpi_j}$, (suppose n=4, $\tau = 2$), the results are worked out as given below.

$$\hat{\mathcal{S}}_{11} = (0.5893, 0.3376, 0.4492), \hat{\mathcal{S}}_{12} = (0.3267, 0.2934, 0.4894), \hat{\mathcal{S}}_{13} = (0.6214, 0.6873, 0.5883),$$

$$\hat{\mathcal{S}}_{14} = (0.4447, 0.3555, 0.1774), \hat{\mathcal{S}}_{21} = (0.3575, 0.5595, 0.4492), \hat{\mathcal{S}}_{22} = (0.6224, 0.2934, 0.3913),$$

$$\hat{\mathcal{S}}_{23} = (0.3255, 0.1958, 0.7873), \hat{\mathcal{S}}_{24} = (0.7857, 0.1774, 0.3555), \hat{\mathcal{S}}_{31} = (0.4681, 0.9496, 0.2253),$$

$$\hat{\mathcal{S}}_{32} = (0.7179, 0.2934, 0.2934), \hat{\mathcal{S}}_{33} = (0.5244, 0.3917, 0.2937), \hat{\mathcal{S}}_{34} = (0.5406, 0.6303, 0.4455),$$

$$\hat{\mathcal{S}}_{41} = (0.7199, 0.2253, 0.2253), \hat{\mathcal{S}}_{42} = (0.5256, 0.5878, 0.1956), \hat{\mathcal{S}}_{43} = (0.4259, 0.0979, 0.2937),$$

$$\hat{\mathcal{S}}_{44} = (0.6280, 0.6303, 0.3555).$$

Based on the score function, we have

$$\hat{\mathcal{S}}_{\delta(11)} = \hat{\mathcal{S}}_{11} = (0.5893, 0.3376, 0.4492), \hat{\mathcal{S}}_{\delta(12)} = \hat{\mathcal{S}}_{14} = (0.4447, 0.3555, 0.1774),$$

$$\hat{\mathcal{S}}_{\delta(13)} = \hat{\mathcal{S}}_{12} = (0.3267, 0.2934, 0.4894), \hat{\mathcal{S}}_{\delta(14)} = \hat{\mathcal{S}}_{13} = (0.6214, 0.6873, 0.5883),$$

$$\hat{\mathcal{S}}_{\delta(21)} = \hat{\mathcal{S}}_{24} = (0.7857, 0.1774, 0.3555), \hat{\mathcal{S}}_{\delta(22)} = \hat{\mathcal{S}}_{22} = (0.6224, 0.2934, 0.3913),$$

$$\hat{\mathcal{S}}_{\delta(23)} = \hat{\mathcal{S}}_{21} = (0.3575, 0.5595, 0.4492), \hat{\mathcal{S}}_{\delta(24)} = \hat{\mathcal{S}}_{23} = (0.3255, 0.1958, 0.7873),$$

$$\hat{\mathcal{S}}_{\delta(31)} = \hat{\mathcal{S}}_{32} = (0.7179, 0.2934, 0.2934), \hat{\mathcal{S}}_{\delta(32)} = \hat{\mathcal{S}}_{33} = (0.5244, 0.3917, 0.2937),$$

$$\hat{\mathcal{S}}_{\delta(33)} = \hat{\mathcal{S}}_{34} = (0.5406, 0.6303, 0.4455), \hat{\mathcal{S}}_{\delta(34)} = \hat{\mathcal{S}}_{31} = (0.4681, 0.9496, 0.2253),$$

$$\hat{\mathcal{S}}_{\delta(41)} = \hat{\mathcal{S}}_{41} = (0.7199, 0.2253, 0.2253), \hat{\mathcal{S}}_{\delta(42)} = \hat{\mathcal{S}}_{43} = (0.4259, 0.0979, 0.2937),$$

$$\hat{\mathcal{S}}_{\delta(43)} = \hat{\mathcal{S}}_{44} = (0.6280, 0.6303, 0.3555), \hat{\mathcal{S}}_{\delta(44)} = \hat{\mathcal{S}}_{42} = (0.5256, 0.5878, 0.1956).$$

Applying the T-SFFHG operator Equation (5.37), having the associated weight vector $w = (0.3, 0.2, 0.3, 0.2)^T$ to get the overall preference values $\mathcal{S}_i$ of the alternative o_i $(i = 1, 2, ..., n)$:

$$\mathcal{S}_1 = (0.4734, 0.46298, 0.4730), \mathcal{S}_2 = (0.5018, 0.3979, 0.5499),$$

$\mathcal{S}_3 = (0.5699, 0.7152, 0.3471)\,,\ \mathcal{S}_4 = (0.5863, 0.4996, 0.2867)\,.$

Step 5: By definition 2.5.3, we calculate the score values $S(\mathcal{S}_i)$ $(i = 1, 2, 3, 4)$ of the overall preference values $\mathcal{S}_i$ $(i = 1, 2, 3, 4)$ as follows:

$S(\mathcal{S}_1) = -0.1160,\ S(\mathcal{S}_2) = -0.1195,$

$S(\mathcal{S}_3) = -0.2451,\ S(\mathcal{S}_4) = 0.0619.$

Step 6: Based on the above score values the ranking order of alternatives is: $o_4 \succ o_1 \succ o_2 \succ o_3$, where the symbol "$\succ$" means "superior to". Thus, the most desirable company is o_4.

5.5.3 Parameter analysis

The two parameters (τ, t) associated with the established model certainly have an effect on the final outcomes; therefore, we attempt to examine their effects.

The influence of the parameter τ

To represent the impact of different values of parameter τ, we employ several values of τ in our suggested methodology to rank the alternatives. The ranking results are depicted in Table 5.2 and Figures 5.1 and 5.2.

Table 5.2: Ranking of alternatives with different values of τ

τ	T-SFFHA		T-SFFHG	
	Score values	Ranking	Score values	Ranking
$\tau \longrightarrow 1$	0.1219,0.0656,0.0525,0.3291	$o_4 \succ o_1 \succ o_2 \succ o_3$	-0.1222,-0.1286,-0.2659,0.0523	$o_4 \succ o_1 \succ o_2 \succ o_3$
$\tau = 2$	0.1171,0.0615,0.0391,0.3218	$o_4 \succ o_1 \succ o_2 \succ o_3$	-0.1160,-0.1195,-0.2451,0.0619	$o_4 \succ o_1 \succ o_2 \succ o_3$
$\tau = 5$	0.1119,0.0568,0.0210,0.3138	$o_4 \succ o_1 \succ o_2 \succ o_3$	-0.0797,-0.1101,-0.2208,0.0732	$o_4 \succ o_1 \succ o_2 \succ o_3$
$\tau = 10$	0.1160,0.0582,0.0195,0.3205	$o_4 \succ o_1 \succ o_2 \succ o_3$	-0.1046,-0.1049,-0.2056,0.0806	$o_4 \succ o_1 \succ o_2 \succ o_3$
$\tau = 50$	0.1255,0.0634,0.0156,0.3136	$o_4 \succ o_1 \succ o_2 \succ o_3$	-0.0956,-0.0971,-0.1793,0.0951	$o_4 \succ o_1 \succ o_2 \succ o_3$

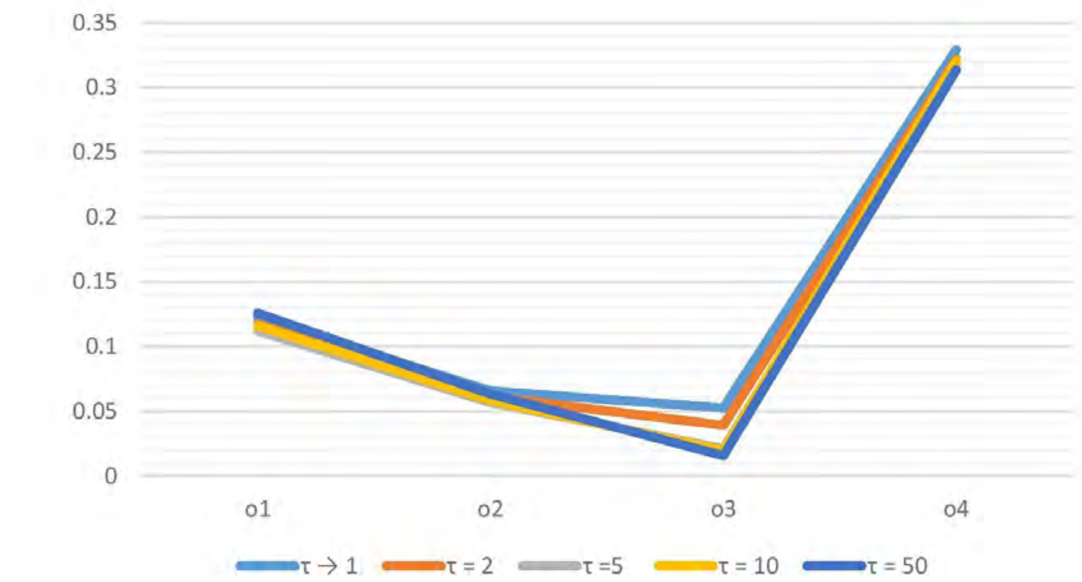

Figure 5.1: Ranking of alternatives by T-SFFHA operator for different values of τ

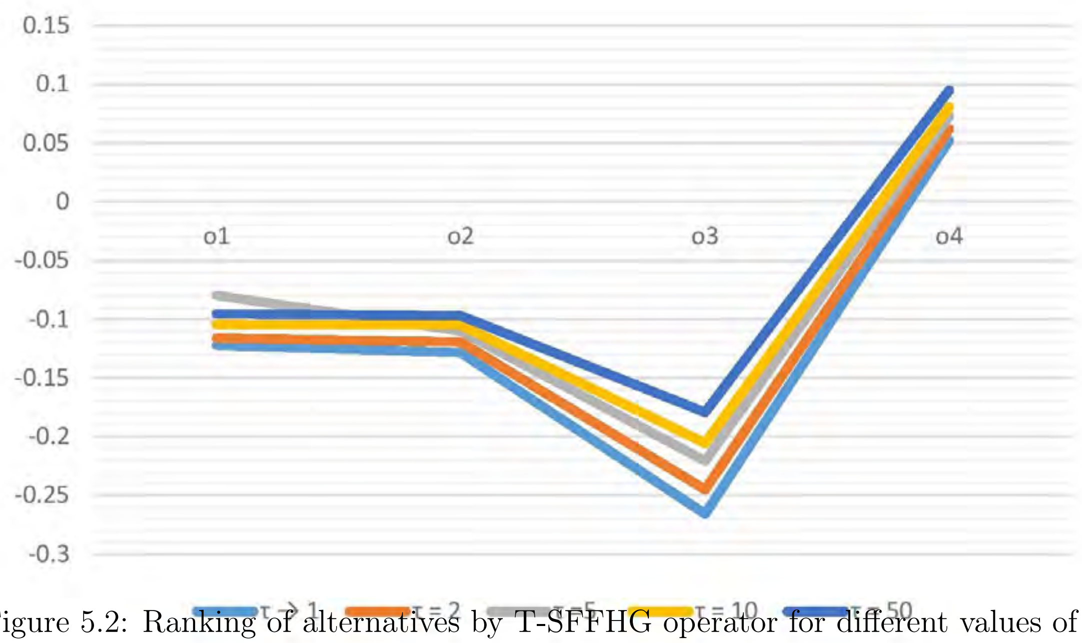

Figure 5.2: Ranking of alternatives by T-SFFHG operator for different values of τ

As shown in Table 5.2, the score values with various parameters τ alter, but the order of ranking for the alternatives are changeless, indicating that the suggested approach has the feature of isotonicity, allowing decision makers to select the adequate value based on their preferences. By examining Figures 5.1 and 5.2, we can see that the score values generated by the T-SFFHWG operator rise as the parameter grow for the same option, but the score values produced by T-SFFHWA operator get lower as the parameter τ increases within the interval $(1, 5]$. Further from Table 5.2, one can notice that the aggregated score values of alternatives o_2 and o_3 obtained by T-SFFHWA operator are quite close for smaller values of τ, but as the value of τ increases, their gap of difference also becomes enlarge. This implies that the T-SFFHWA operator with the larger value of τ has a strong sense of differentiation.

The influence of the parameter t

To examine the influence of various values of the parameter t on the ranking order of alternatives, we adapt different values of t in Step 4 of the suggested MCDM approach. The derived results are shown in Tables 5.3 and Figures 5.3 and 5.4 (setting $\tau = 2$).

Table 5.3: Ranking of alternatives with different values of t

t	T-SFFHA		T-SFFHG	
	Score values	Ranking	Score values	Ranking
$t = 3$	0.1171,0.0615,0.0391,0.3218	$o_4 \succ o_1 \succ o_2 \succ o_3$	-0.1160,-0.1195,-0.2451,0.0619	$o_4 \succ o_1 \succ o_2 \succ o_3$
$t = 5$	0.0914,0.0530,0.0433,0.1979	$o_4 \succ o_1 \succ o_2 \succ o_3$	-0.0505,-0.06351,-0.2261,0.0169	$o_4 \succ o_1 \succ o_2 \succ o_3$
$t = 7$	0.0484,0.0275,0.0268,0.1196	$o_4 \succ o_1 \succ o_2 \succ o_3$	-0.0179,-0.0400,-0.1810,0.0089	$o_4 \succ o_1 \succ o_2 \succ o_3$
$t = 9$	0.0237,0.0151,0.0144,0.0742	$o_4 \succ o_1 \succ o_2 \succ o_3$	-0.0114,-0.0315,-0.1549,-0.0035	$o_4 \succ o_1 \succ o_2 \succ o_3$
$t = 12$	0.0079,0.0042,0.0052,0.0373	$o_4 \succ o_1 \succ o_2 \succ o_3$	-0.0037,-0.0162,-0.1098,-0.0023	$o_4 \succ o_1 \succ o_2 \succ o_3$

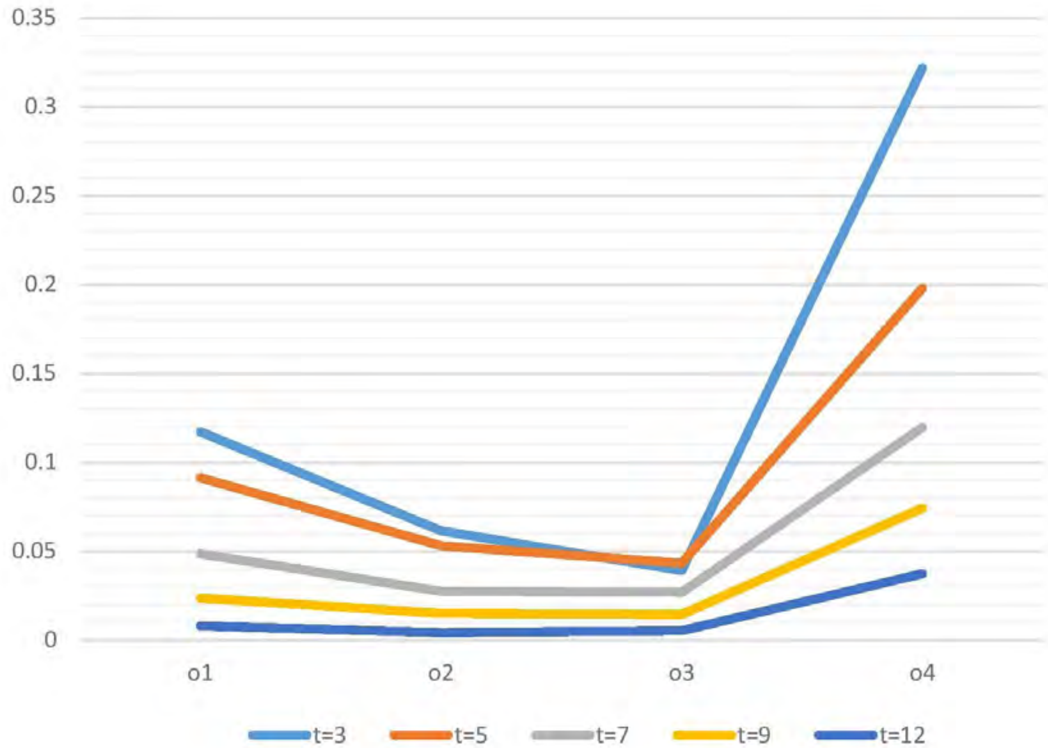

Figure 5.3: Ranking of alternatives by T-SFFHA operator for different values of t

Figure 5.4: Ranking of alternatives by T-SFFHG operator for different values of t

From Table 5.3, we can observe that the score values secured by T-SFFHA operator become smaller and smaller as the value of t increases while those obtained by T-SFFHG operator rise when the value of t increases. However, the rating results are same; the best alternative is all o_4. The parameter t may be considered as the "DM's attitude." The operator is T-SFFHA suitable for modelling pessimistic decision makers, whereas the T-SFFHA operator is considered useful in reflecting optimistic decision makers. Suppose we use T-SFFHA operator as an aggregation tool for the decision process. In that case, the higher value of t indicates that decision makers have a more pessimistic attitude and vice versa in the case of the T-SFHG operator. So, different decision makers can choose the value of t based on their attitude.

5.5.4 The analysis of comparison

To validate the effectiveness of the developed approach, in what follows, we compare our established approach with three previous MCDM approach based on T-spherical fuzzy Einstein hybrid interactive averaging (T-SFEHIA) [76], T-spherical fuzzy Einstein hybrid interactive geometric (T-SFEHIG) [76], T-spherical fuzzy Hamacher averaging (TSFHHA) [77], T-spherical fuzzy Hamacher geometric (TSFHHG) [77], spherical fuzzy number weighted averaging operator (SFNWA) [70], spherical fuzzy number weighted geometric operator (SFNWG) [70] operators. The comparison results obtained by these existing methods and our proposed methods are depicted in Table 5.4.

Table 5.4: Ranking results for different existing aggregation operators

Parameter	score values	Ranking
T-SFEHIA operator [76]	-0.0207,-0.2146,-0.9255,0.0230	$o_4 \succ o_1 \succ o_2 \succ o_3$
T-SFEHIG operator [76]	0.0296,-0.0676,-0.1309,0.1639	$o_4 \succ o_1 \succ o_2 \succ o_3$
TSFHHA operator [77]	0.1525,0.1048,0.1091,0.3257	$o_4 \succ o_1 \succ o_3 \succ o_2$
TSFHHG operator [77]	-0.1520,-0.1529,-0.2851,-0.0041	$o_4 \succ o_1 \succ o_2 \succ o_3$
SFNWA operator [70]	Not applicable	
SFNWG operator [70]	Not applicable	

Based on Table 5.4, we can see that the ranking results secured by applying T-SFEHIA and T-SFEHIG operators [76] are consistent with the result obtained by the proposed operators. But these operators are based on Einstein operational rules [76] which are not authentic. For instance, if we take $\mathcal{S} = (0.6, 0.9, 0.2)$, then clearly $0.6^3 + 0.9^3 + 0.2^3 = 0.953 < 1$. Now accordingly to the operational rules of [76], if we take scalar multiple of $\mathcal{S}$ with scalar $\eta = 1.4304$, we get $\eta\mathcal{S} = (0.6724, 0.9540, 0.1463)$. From this, we have $0.6724^3 + 0.9540^3 + 0.1463^3 = 1.1754 \nless 1$. Analogously, if we take scalar power with η, we get $\mathcal{S}^\eta = (0.5135, 0.9540, 0.2253)$. Again we have $0.5135^3 + 0.9540^3 + 0.2253^3 = 1.015 \nless 1$. This seems unsuitable because the operating results are not valid for $t = 3$, and may cause unfruitful results in the decision process. In addition, the provided operators include a parameter, which effectively adjust the aggregate value according to the genuine decision needs and encapsulates many existing hesitant fuzzy aggregation operators. Hence, the usefulness of the proposed operators evident with their higher universality and feasibility.

In the second comparison TSFHHA and TSFHHG operators [77] are adopted to address the problem presented in subsection 5.5.1. From Table 5.4, it can be observed that likewise T-SFEHIA and T-SFEHIG operators, the ranking result obtained by utilizing TSFHHG operator is consistent with our proposed operators' results. However, in the case that TSFHHA operator is applied we have o_4, o_1, o_3, o_2. The alternatives o_2 and o_3 have interchanged their positions. Thus the ranking of alternatives is slightly different from that derived by the proposed and other existing operators. However, the best alternative remains the same in all cases. This verifies the validity of the developed operators.

Finally, we compare our developed approach with the method from Ashraf and Abdullah [70] for the considered problem, where the method from [70] is structured on the Archimedean t-norm and t-conorm based spherical fuzzy weighted averaging and geometric operators. Obviously, these operators have a general representation by Archimedean t-norm and t-conorm. But, Ashraf and Abdullah's method also has some failure cases because it has some certain limitations. It can only handle the MCDM problems demonstrated by PFNs or SFNs. Therefore it cannot be employed to solve the aforementioned MCDM problem. On the contrary, the proposed method has the parameter t, which can express the spherical fuzzy information more flexible. Hence, in the real world practical problems, the suggested approach is more appropriate.

5.6 TWD model for T-spherical fuzzy decision system

This section aims to construct a three-way decision model in a T-spherical fuzzy decision system. Let (X, BUD, S) be a T-spherical fuzzy decision systems with $B \cap D = \emptyset$, where $X = \{x_1, x_2, ..., x_k\}$ is a set of objects, $B = \{b_1, b_2, b_3, ..., b_l\}$ is a set of attributes with unknown weight vectors, S is a fuzzy measure of X, D is a decision set and $\Omega_D = \{D_1, D_2, ..., D_e\}$ is a partition of decision set. The T-spherical fuzzy decision system is described in Table 5.5 in which all attribute values are T-SFNs. The element $x_q(b_r) = S_{qr}$ implies the evaluation of object x_q with respect to attribute b_r.

T-spherical fuzzy decision systems depict all available information and knowledge, where

Table 5.5: A T-spherical fuzzy decision system

x_1	$(\sigma_{11}, \vartheta_{11}, \varrho_{11})$	$(\sigma_{12}, \vartheta_{12}, \varrho_{12})$	.	.	.	.	.	$(\sigma_{1k}, \vartheta_{1k}, \varrho_{1k})$
x_2	$(\sigma_{21}, \vartheta_{21}, \varrho_{21})$	$(\sigma_{22}, \vartheta_{22}, \varrho_{22})$	.	.	.	.	.	$(\sigma_{2k}, \vartheta_{2k}, \varrho_{2k})$
.	.	.	.	.	.	.	.	.
.	.	.	.	.	.	.	.	.
.	.	.	.	.	.	.	.	.
x_k	$(\sigma_{k1}, \vartheta_{k1}, \varrho_{k1})$	$(\sigma_{k2}, \vartheta_{k2}, \varrho_{k2})$	.	.	.	.	.	$(\sigma_{kl}, \vartheta_{kl}, \varrho_{kl})$

objects are considered by using a finite number of attributes, and attribute values are T-SFNs based on three evaluation measures i.e; degree of support, abstain, and objection.T-SF decision systems depict all available information and knowledge, where objects are considered by using a finite number of attributes and attributes values are T-SFNs based on three evaluation measures i.e; degree of support, abstain, and objection. In Table 5.5 $S_{qr} = x_q(b_r) = (\sigma_{qr}, \vartheta_{qr}, \varrho_{qr}), (q = \{1, 2, \ldots, k\}, r = \{1, 2, \ldots, l\})$, where $\sigma_{qr}, \vartheta_{qr}, \varrho_{qr} \in [0, 1]$, satisfy the condition $0 \leq \sigma_{qr}^t + \vartheta_{qr}^t + \varrho_{qr}^t \leq 1$. σ_{qr} is a degree of support, which is used to describe the degree that object x_q satisfies the attribute a_r, ϑ_{qr} is abstain degree, which is used to describe the degree that object x_q dose not satisfies the attribute a_r, ϱ_{qr} indicates the degree that object x_q dose not fulfill the attribute a_r. To aggregate the evaluation measures of an object x_q on issues (b_r) for $(r = 1, 2, ..., l)$, we use T-SFFHA operator or T-SFFHG operator based on frank fuzzy operations, defined by Equations 5.19 and 5.37 in Section 5.2.

Let $\mathcal{S}_{qr} = (\sigma_{qr}, \vartheta_{qr}, \varrho_{qr})$ be evaluation measures of object x_q on issues b_r, $(q = 1, 2, ..., k)$, $(r = 1, 2, ..., l)$, then the T-SFFHA and T-SFFHG operators aggregate the evaluation measures as follows:

$$T - SFFHA\left(\mathcal{S}_{q1}, \mathcal{S}_{q2}, ..., \mathcal{S}_{ql}\right) = \begin{pmatrix} \sqrt[t]{1 - \log_\tau\left(1 + \Pi_{r=1}^{n}\left(\tau^{1-\hat{\sigma}_{\delta(r)}^t} - 1\right)^{w_r}\right)}, \\ \sqrt[t]{\log_\tau\left(1 + \Pi_{r=1}^{n}\left(\tau^{\hat{\vartheta}_{\delta(r)}^t} - 1\right)^{w_r}\right)}, \\ \sqrt[t]{\log_\tau\left(1 + \Pi_{r=1}^{n}\left(\tau^{\hat{\varrho}_{\delta(r)}^t} - 1\right)^{w_r}\right)} \end{pmatrix}.$$

$$T - SFFHG\left(\mathcal{S}_{q1}, \mathcal{S}_{q2}, ..., \mathcal{S}_{ql}\right) = \begin{pmatrix} \sqrt[t]{\log_\tau\left(1 + \Pi_{r=1}^{l}\left(\tau^{\hat{\sigma}_{\delta(r)}^t} - 1\right)^{w_r}\right)}, \\ \sqrt[t]{1 - \log_\tau\left(1 + \Pi_{r=1}^{l}\left(\tau^{1-\hat{\vartheta}_{\delta(r)}^t} - 1\right)^{w_r}\right)}, \\ \sqrt[t]{1 - \log_\tau\left(1 + \Pi_{r=1}^{l}\left(\tau^{1-\hat{\varrho}_{\delta(r)}^t} - 1\right)^{w_r}\right)} \end{pmatrix}.$$

where $w=(w_1,w_2,...,w_l)^T$ is the weight vector associated with T-SFFHA and T-SFFHG such that $w_r>0$ and $\sum_{r=1}^{l} w_r=1$, $\varpi=(\varpi_1,\varpi_2,...,\varpi_l)^T$ is the weight vector of $\mathcal{S}_{qr}$ $(r=1,2,..,l)$ such that $\varpi_r>0$ and $\sum_{r=1}^{n}\varpi_r=1$, $\hat{\mathcal{S}}_{\delta(qr)}$ is the rth largest of the weighted $\hat{\mathcal{S}}_{qr}=l\varpi_r\mathcal{S}_{qr})$, $(r=1,2,...,l)$ where l is the balancing coefficient.

An aggregated evaluation measure for an object x_q on attribute set B is obtained by using the following formula:

$$E(x_q,B)=\oplus_{r=1}^{l}\varpi_r\mathcal{S}_{qr}$$

where, $\varpi\mathcal{S}_{qr}=\varpi x_q(b_r)$, ϖ is the attribute weights.

By using T-spherical fuzzy frank operators, we obtain aggregated evaluation of an object x_q on B in the form of T-SFN. Further the aggregated evaluation matrix may be defined as follows:

$$M(E)=|E(x_q,B)|_{k\times k}$$

Moreover, on the basis of $E(x_q,B)$, we have the decision rules as follows:

(P1) If $(E(x_q,B))\geq\alpha$ then decide $x_q\in POS(X)$;
(B1) If $\beta<(E(x_q,B))<\alpha$ then decide $x_q\in BND(X)$;
(N1) If $(E(x_q,B))\leq\beta$ then decide $x_q\in NEG(X)$.

where, (α,β) is a pair of threshold based on T-SFNs. Then the positive, neutral and negative regions are given by rules $(P_1),(B_1)$ and (N_1) respectively, as follows:

$POS_{(\alpha,\beta)}(X)=\{x_q\in X\,|\,(E(x_q,B))\geq\alpha\}$;
$BND_{(\alpha,\beta)}(X)=\{x_q\in X\,|\,\alpha<(E(x_q,B)<\beta\}$;
$NEG_{(\alpha,\beta)}(X)=\{x_q\in X\,|\,(E(x_q,B)\leq\beta\}$.

We can get a tri-partition based on positive, neutral, and negative regions by using the threshold pair based on T-SFNs. For these sets, in particular, $POS(\alpha,\beta)(X)\cup BND(\alpha,\beta)(X)\cup NEG(\alpha,\beta)(X)=X$ i.e., these are pairwise disjoint sets. Moreover, we may compare T-SFNs by using the score function defined by Definition 2.5.3 as (x_q,B) and (α,β) are T-SFNs.

5.6.1 Decision making process

We construct a TWD method based on the proposed T-spherical fuzzy frank operators to deal with multi-criteria decision problems with T-spherical fuzzy information, the main steps of the procedure are as follows:

Step 1: Consider a T-spherical fuzzy decision system:

Consider a T-spherical fuzzy decision system founded on the finite set of objects observing the attributes in the form of matrix $M(S) = [\mathcal{S}_{qr}]$. Also, choose the least value of t for which every triplet of the considered decision system lies in the frame of T-SFNs.

Step 2: Normalization:

Transform the decision matrix $M(S) = [\mathcal{S}_{qr}]$ into the normalized form $\widetilde{M(S)} = \left[\widetilde{\mathcal{S}}_{qr}\right]$ by the below formula:

$$\widetilde{\mathcal{S}}_{qr} = \begin{cases} \mathcal{S}_{qr}, & \text{if for benefit attribute} \\ (\mathcal{S}_{qr})^c, & \text{for cost attribute.} \end{cases}$$

where $(\mathcal{S}_{qr})^c$ is the complement of $\mathcal{S}_{qr}$.

Step 3: Attributes' weight determination:

To derive the weights of attributes, the stated entropy measure for T-SFSs is used. First, Equation (5.41) is applied for each rth attribute to compute the entropy measure which depicts the dispersion in provided information of attributes, further these attributes' entropies are utilized to assign their weights (ϖ_r).

$$E_r = \frac{1}{l} \sum_{q=1}^{k} \left(1 - \frac{4}{5}\left[\left|\sigma_{qr}^t - \varrho_{qr}^t\right| + \left|\vartheta_{qr}^t - 0.25\right|\right]\right) \tag{5.47}$$

Divergence is expressing the actual information of rth attribute obtained by using relation

$div_r = 1 - E_r$. The objective attribute weights can be determined as follows:

$$\varpi_r = \frac{div_r}{\sum_{r=1}^{l} div_r}. \tag{5.48}$$

Step 4: Aggregation:

Aggregate the T-SFNs based evaluation measures $x_q(b_r)= \mathcal{S}_{qr}$ $(r = 1, 2, ..., l)$ for each object x_q $(q = 1, 2, ..., k)$ into the overall aggregated value $\mathcal{S}_q = E(x_q, B)$ either by using the proposed T-SFFHA or T-SFFHG operators. Mathematically,

$$\mathcal{S}_q = T - SFFHWA_{\varpi,w}\left(\mathcal{S}_{q1}, \mathcal{S}_{q2}, ..., \mathcal{S}_{ql}\right),$$

$$\mathcal{S}_q = T - SFFHWG_{\varpi,w}\left(\mathcal{S}_{q1}, \mathcal{S}_{q2}, ..., \mathcal{S}_{qk}\right),$$

where $\varpi = (\varpi_1, \varpi_2, ..., \varpi_l)$ is the weight vector of attributes obtained in Step 3, and $w = (w_1, w_2, ..., w_l)$ is the weight vector associated the aggregation operator such that $w_r > 0$ and $\sum_{r=1}^{l} w_r = 1$.

Step 5: Score values:

Compute the score values of the aggregated evaluation measures $s\left(\mathcal{S}_q\right)$ $(q = 1, 2, ..., k)$.

Step 6: Set threshold values:

A pair of threshold based on T-SFNs is chosen by experts. These two numbers represent desired levels of precision for deciding about inclusion of objects in $POS(X), BND(X)$ and $NEG(X)$.

Step 7: Construction of $POS(X), BND(X)$ and $NEG(X)$.

By using decision rules, $(P1), (B1)$ and $(N1)$, we formulate the $POS(X)$, $BND(X)$ and $NEG(X)$.

5.7 Illustrative example

This section aims to resolve a practical example of project investment to validate the procedure of the proposed multi-criteria TWD method. Moreover, we perform a series of experiments to examine the influence of various specific operators and parameter values on the aggregation results.

5.7.1 Example

An investment company needs to opt some investment projects for reasonable use of idle funds. There are five alternatives (where $X = (x_1, x_2, x_3, x_4, x_5)$ can be selected: Two projects are associated with internet education (denoted as x_1 and x_2) and three projects are associated with film studio investments (represent as x_3, x_4 and x_5). The company's board of directors acquired the services of some experts(decision makers). According to the project investment books, the decision makers evaluate the alternatives with respect to four attributes including, human resources, social benefits, marketing management and expected benefits (denoted as b_1, b_2, b_3 and b_4, respectively), with unknown weights. Also $\Omega_D = \{accepted, abstain, rejected\}$ is the partition of decision set. To offer sufficient freedom in the evaluation of the values of the five attributes related to each alternative project, the decision makers were allowed to use T-SFNs. The evaluation information of decision makers is detailed in the following matrix. [97]

Table 5.6: T-spherical fuzzy decision matrix

	b_1	b_2	b_3	b_4
x_1	$(0.47, 0.66, 0.75)$	$(0.81, 0.30, 0.37)$	$(0.57, 0.51, 0.39)$	$(0.34, 0.56, 0.78)$
x_2	$(0.11, 0.11, 0.11)$	$(0.59, 0.66, 0.66)$	$(0.91, 0.34, 0.68)$	$(0.68, 0.46, 0.88)$
x_3	$(0.35, 0.45, 0.61)$	$(0.42, 0.56, 0.71)$	$(0.27, 0.59, 0.72)$	$(0.41, 0.73, 0.41)$
x_4	$(0.59, 0.45, 0.90)$	$(0.44, 0.55, 0.77)$	$(0.46, 0.46, 0.45)$	$(0.76, 0.46, 0.85)$
x_5	$(0.82, 0.46, 0.69)$	$(0.16, 0.33, 0.42)$	$(0.55, 0.44, 0.29)$	$(0.21, 0.43, 0.13)$

Table 5.7: Normalized T-spherical fuzzy decision matrix

	b_1	b_2	b_3	b_4
x_1	$(0.75, 0.66, 0.47)$	$(0.81, 0.30, 0.37)$	$(0.39, 0.51, 0.57)$	$(0.34, 0.56, 0.78)$
x_2	$(0.11, 0.11, 0.11)$	$(0.59, 0.66, 0.66)$	$(0.68, 0.34, 0.91)$	$(0.68, 0.46, 0.88)$
x_3	$(0.61, 0.45, 0.35)$	$(0.42, 0.56, 0.71)$	$(0.72, 0.59, 0.27)$	$(0.41, 0.73, 0.41)$
x_4	$(0.90, 0.45, 0.59)$	$(0.44, 0.55, 0.77)$	$(0.45, 0.46, 0.46)$	$(0.76, 0.46, 0.85)$
x_5	$(0.69, 0.46, 0.82)$	$(0.16, 0.33, 0.42)$	$(0.29, 0.44, 0.55)$	$(0.21, 0.43, 0.13)$

5.7.2 The procedure of solution

In the following procedure, we use T-SFFHA operator and T-SFFHG operator to solve TWD problem with T-spherical fuzzy information.

Using T-SFFHA operator

Step 1. The decision matrix of described problem is provided in Table 5.6. We need to determine the value of 't' for which the given data lies in T-spherical fuzzy information.

As for $t = 3$, $0.90^3 + 0.45^3 + 0.59^3 = 1.025 > 1$, for $t = 4$, $0.90^4 + 0.45^4 + 0.59^4 = 0.8182 < 1$,. Hence, all the given values in Table 5.6 lie in T-spherical fuzzy information for $n = 4$.

Step 2. We may observe that the social benefits and expected benefits are benefit attributes while human resources, and marketing management are associated with cost attributes. The normalized T-spherical fuzzy decision system is provided in Table 5.7.

Step 3. By using Equation 5.41, the attribute weight is calculated as shown below.

$$\varpi = (0.3509, 0.2070, 0.2627, 0.1794) .$$

$$l\varpi = (1.4036, 0.8280, 1.0508, 0.7176) .$$

Step 4. Using the normalized T-spherical fuzzy information listed in Table 5.7, the values of $\hat{\mathcal{S}}_{qr} = (l\varpi_r)\,\mathcal{S}_{qr}$, (for $l = 4$, $\tau = 2$), are listed as shown below.

Table 5.8: The values of $\hat{\mathcal{S}}_{qr}$ ($l = 4$, $\tau = 2$)

$\hat{\mathcal{S}}_{11} = (0.7486, 0.9693, 0.9957)$	$\hat{\mathcal{S}}_{12} = (0.7795, 0.9950, 0.9900)$	$\hat{\mathcal{S}}_{13} = (0.3948, 0.9851, 0.9758)$	$\hat{\mathcal{S}}_{14} = (0.3129, 0.9455, 0.8409)$
$\hat{\mathcal{S}}_{21} = (0.1197, 0.9999, 0.9999)$	$\hat{\mathcal{S}}_{22} = (0.5639, 0.9276, 0.9276)$	$\hat{\mathcal{S}}_{23} = (0.6877, 0.9973, 0.7567)$	$\hat{\mathcal{S}}_{24} = (0.6295, 0.9696, 0.7416)$
$\hat{\mathcal{S}}_{31} = (0.6603, 0.9975, 0.9994)$	$\hat{\mathcal{S}}_{32} = (0.4008, 0.9595, 0.9052)$	$\hat{\mathcal{S}}_{33} = (0.7279, 0.9719, 0.9989)$	$\hat{\mathcal{S}}_{34} = (0.3775, 0.8739, 0.9782)$
$\hat{\mathcal{S}}_{41} = (0.9420, 0.9975, 0.9881)$	$\hat{\mathcal{S}}_{42} = (0.4199, 0.9619, 0.8700)$	$\hat{\mathcal{S}}_{43} = (0.4555, 0.9904, 0.9904)$	$\hat{\mathcal{S}}_{44} = (0.7066, 0.9696, 0.7780)$
$\hat{\mathcal{S}}_{51} = (0.7439, 0.9971, 0.9090)$	$\hat{\mathcal{S}}_{52} = (0.1526, 0.9932, 0.9847)$	$\hat{\mathcal{S}}_{53} = (0.2936, 0.9921, 0.9793)$	$\hat{\mathcal{S}}_{54} = (0.1932, 0.9750, 0.9990)$

By using the score function, we have obtained ordered values of $\hat{\mathcal{S}}_{qr}$ as follows:

Table 5.9: The values of $\hat{\mathcal{S}}_{\delta(qr)}$ ($l = 4$, $\tau = 2$)

$\hat{\mathcal{S}}_{\delta(11)} = (0.3129, 0.9455, 0.8409)$	$\hat{\mathcal{S}}_{\delta(12)} = (0.7486, 0.9693, 0.9957)$	$\hat{\mathcal{S}}_{\delta(13)} = (0.7795, 0.9950, 0.9900)$	$\hat{\mathcal{S}}_{\delta(14)} = (0.3948, 0.9851, 0.9758)$
$\hat{\mathcal{S}}_{\delta(21)} = (0.6295, 0.9696, 0.7416)$	$\hat{\mathcal{S}}_{\delta(22)} = (0.6877, 0.9973, 0.7567)$	$\hat{\mathcal{S}}_{\delta(23)} = (0.5639, 0.9276, 0.9276)$	$\hat{\mathcal{S}}_{\delta(24)} = (0.1197, 0.9999, 0.9999)$
$\hat{\mathcal{S}}_{\delta(31)} = (0.3775, 0.8739, 0.9782)$	$\hat{\mathcal{S}}_{\delta(32)} = (0.4008, 0.9595, 0.9052)$	$\hat{\mathcal{S}}_{\delta(33)} = (0.7279, 0.9719, 0.9989)$	$\hat{\mathcal{S}}_{\delta(34)} = (0.6603, 0.9975, 0.09994)$
$\hat{\mathcal{S}}_{\delta(41)} = (0.7066, 0.9696, 0.7780)$	$\hat{\mathcal{S}}_{\delta(42)} = (0.9420, 0.9975, 0.9881)$	$\hat{\mathcal{S}}_{\delta(43)} = (0.4199, 0.9619, 0.8700)$	$\hat{\mathcal{S}}_{\delta(44)} = (0.4555, 0.9904, 0.9904)$
$\hat{\mathcal{S}}_{\delta}51 = (0.7439, 0.9971, 0.9090)$	$\hat{\mathcal{S}}_{\delta}52 = (0.2936, 0.9921, 0.9793)$	$\hat{\mathcal{S}}_{\delta}53 = (0.1932, 0.9750, 0.9990)$	$\hat{\mathcal{S}}_{\delta}54 = (0.1526, 0.9932, 0.9847)$

Now we apply the T-SFFHA operator by using Equation (5.19), taking associated weight vector $w = (0.3, 0.2, 0.3, 0.2)^T$ to get the aggregated evaluation measures $\mathcal{S}_q$ of the objects $x_q\,(q = 1, 2, ..., k)$:

$E(x_1, B) = \mathcal{S}_1 = (0.8813, 0.4057, 0.4912)$, $E(x_2, B) = \mathcal{S}_2 = (0.8603, 0.4225, 0.6254)$,
$E(x_3, B) = \mathcal{S}_3 = (0.8684, 0.4859, 0.4031)$, $E(x_4, B) = \mathcal{S}_4 = (0.9136, 0.3896, 0.5799)$,
$E(x_5, B) = \mathcal{S}_5 = (0.80760, 0.3263, 0.4335)$.

Step 5: By Definition 2.5.3, we calculate the score values $s\,(E(x_q, B)) = s\,(\mathcal{S}_q)\,(q = 1, 2, 3, 4, 5)$ of the aggregated evaluation measures $\mathcal{S}_q\,(q = 1, 2, 3, 4, 5)$ as follows:

$s(E(x_1, B)) = s\,(\mathcal{S}_1) = 0.0916$, $s(E(x_2, B)) = s\,(\mathcal{S}_2) = 0.3868$, $s(E(x_3, B)) = s\,(\mathcal{S}_3) = 0.5699$,
$s(E(x_4, B)) = s\,(\mathcal{S}_4) = 0.5833$, $s(E(x_5, B)) = s\,(\mathcal{S}_4) = 0.4280$.

Step 6: Two T-SFNs (α, β) based on T-SFNs are chosen by experts as threshold pair. Here, $\alpha = (0.87, 0.65, 0.33)$ and $\beta = (0.73, 0.65, 0.43)$. On the basis of these threshold values, decisions are made about inclusion of objects in $POS(X), BND(X)$ and $NEG(X)$.

Step 7: By using decision rule $(P1), (B1)$ and $(N1)$, we formulate the $POS(X)$, $BND(X)$ and $NEG(X)$.

As $(E(x_q, B))$ and (α, β) are T-SFNs so we compare these by using score functions provided by Definition 2.5.3. $s(\alpha) = 0.4408, s(\beta) = 0.1565$. On the basis of the score values of $(E(x_q, B))$ and (α, β), we formulate the $POS(X), BND(X)$ and $NEG(X)$ as follows:

$$POS(X) = \{x_3, x_4\}; BND(X) = \{x_2, x_5\}; NEG(X) = \{x_1\}.$$

Using T-SFFHG operator

Step 4. Using the normalized T-spherical fuzzy information listed in Table 5.7, and $\hat{\mathcal{S}}_{qr} = (\mathcal{S}_{qr})^{l\varpi_r}$, (for $l = 4$, $\tau = 2$), the derived results are listed as shown below:

Table 5.10: The values of $\hat{\mathcal{S}}_{qr}$ ($l = 4$, $\tau = 2$)

$\hat{\mathcal{S}}_{11} = (0.6583, 0.7128, 0.5106)$	$\hat{\mathcal{S}}_{12} = (0.8420, 0.2862, 0.3530)$	$\hat{\mathcal{S}}_{13} = (0.3702, 0.5161, 0.5768)$	$\hat{\mathcal{S}}_{14} = (0.4715, 0.5167, 0.7264)$
$\hat{\mathcal{S}}_{21} = (0.0434, 0.1197, 0.1197)$	$\hat{\mathcal{S}}_{22} = (0.6517, 0.6316, 0.6316)$	$\hat{\mathcal{S}}_{23} = (0.6653, 0.3442, 0.9164)$	$\hat{\mathcal{S}}_{24} = (0.7651, 0.4238, 0.8291)$
$\hat{\mathcal{S}}_{31} = (0.4873, 0.4890, 0.6392)$	$\hat{\mathcal{S}}_{32} = (0.4938, 0.5350, 0.6803)$	$\hat{\mathcal{S}}_{33} = (0.7068, 0.5970, 0.2733)$	$\hat{\mathcal{S}}_{34} = (0.5382, 0.6775, 0.3776)$
$\hat{\mathcal{S}}_{41} = (0.8599, 0.4890, 0.6392)$	$\hat{\mathcal{S}}_{42} = (0.5130, 0.5254, 0.7394)$	$\hat{\mathcal{S}}_{43} = (0.4304, 0.4656, 0.4656)$	$\hat{\mathcal{S}}_{44} = (0.8258, 0.4238, 0.7972)$
$\hat{\mathcal{S}}_{51} = (0.5827, 0.4998, 0.8725)$	$\hat{\mathcal{S}}_{52} = (0.2227, 0.3148, 0.4008)$	$\hat{\mathcal{S}}_{53} = (0.2711, 0.4454, 0.5566)$	$\hat{\mathcal{S}}_{54} = (0.3345, 0.3961, 0.1196)$

By using the score function, we have obtained ordered values of $\hat{\mathcal{S}}_{qr}$ as follows:

Table 5.11: The values of $\hat{\mathcal{S}}_{\delta(qr)}$ ($l=4,\ \tau=2$)

$\hat{\mathcal{S}}_{\delta(11)}=(0.8420,0.2862,0.3530)$	$\hat{\mathcal{S}}_{\delta(12)}=(0.3702,0.5161,0.5768)$	$\hat{\mathcal{S}}_{\delta(13)}=(0.6583,0.7128,0.5106)$	$\hat{\mathcal{S}}_{\delta(14)}=(0.4715,0.5167,0.7264)$
$\hat{\mathcal{S}}_{\delta(21)}=(0.0434,0.1197,0.1197)$	$\hat{\mathcal{S}}_{\delta(22)}=(0.6517,0.6316,0.6316)$	$\hat{\mathcal{S}}_{\delta(23)}=(0.5639,0.9276,0.9276)$	$\hat{\mathcal{S}}_{\delta(24)}=(0.6653,0.3442,0.9164)$
$\hat{\mathcal{S}}_{\delta(31)}=(0.5382,0.6775,0.3776)$	$\hat{\mathcal{S}}_{\delta(32)}=(0.7068,0.5970,0.2733)$	$\hat{\mathcal{S}}_{\delta(33)}=(0.4873,0.4890,0.6392)$	$\hat{\mathcal{S}}_{\delta(34)}=(0.4938,0.5350,0.6803)$
$\hat{\mathcal{S}}_{\delta(41)}=(0.8599,0.4890,0.6392)$	$\hat{\mathcal{S}}_{\delta(42)}=(0.4304,0.4656,0.4656)$	$\hat{\mathcal{S}}_{\delta(43)}=(0.8258,0.4238,0.7972)$	$\hat{\mathcal{S}}_{\delta(44)}=(0.5130,0.5254,0.7394)$
$\hat{\mathcal{S}}_{\delta(51)}=(0.3345,0.3961,0.1196)$	$\hat{\mathcal{S}}_{\delta(52)}=(0.2227,0.3148,0.4008)$	$\hat{\mathcal{S}}_{\delta(53)}=(0.2711,0.4454,0.5566)$	$\hat{\mathcal{S}}_{\delta(54)}=(0.5827,0.4998,0.8725)$

Now we apply the T-SFFHG operator Eq. (5.19), taking associated weight vector $w=(0.3,0.2,0.3,0.2)^T$ to get the aggregated evaluation measures $\mathcal{S}_q$ of the objects $x_q\,(q=1,2,...,k)$:

$E(x_1,B)=\mathcal{S}_1=(0.5959,0.5789,0.5715)$; $E(x_2,B)=\mathcal{S}_2=(0.2804,0.7616,0.8316)$;
$E(x_3,B)=\mathcal{S}_3=(0.5430,0.5932,0.5668)$; $E(x_4,B)=\mathcal{S}_4=(0.6743,0.4765,0.7066)$;
$E(x_5,B)=\mathcal{S}_5=(0.3240,0.4286,0.6491)$.

Step 5: By Definition 2.5.3, we calculate the score values $s\left((x_q,B)\right)=s\left(\mathcal{S}_q\right)(q=1,2,3,4,5)$ of the aggregated evaluation measures $\mathcal{S}_q(q=1,2,3,4,5)$ are listed as follows:

$s(E(x_1,B))=s\left(\mathcal{S}_1\right)=-0.8479$; $s(E(x_2,B))=s\left(\mathcal{S}_2\right)=-0.5328$; $s(E(x_3,B))=s\left(\mathcal{S}_3\right)=-0.1639$;
$s(E(x_4,B))=s\left(\mathcal{S}_4\right)=-0.1056$; $s(E(x_5,B))=s\left(\mathcal{S}_4\right)=-0.2389$.

Step 6: A threshold pair (α,β) based on T-SFNs is chosen by experts. Here, $\alpha=(0.33,0.45,0.63)$ and $\beta=(0.47,0.58,0.87)$. Based on these threshold values, decisions are made about the inclusion of objects in $POS(X),BND(X)$, and $NEG(X)$.

Step 7: By using decision rule $(P1),(B1)$ and $(N1)$, we formulate the $POS(X),BND(X)$ and $NEG(X)$.

As $(E(x_q,B))$ and (α,β) are T-SFNs so we compare these by using score functions provided by Definition 2.5.3. $s(\alpha)=0.-0.2233, s(\beta)=-0.6780$. On the basis of score values of $(E(x_q,B))$ and (α,β). we formulate the $POS(X),BND(X)$ and $NEG(X)$ as follows:

$$POS(X)=\{x_3,x_4\};BND(X)=\{x_2,x_5\};NEG(X)=\{x_1\}.$$

5.7.3 Parameter analysis

The influence of two parameters (τ, t) on final decisions of three-way classification of objects, is obvious so we may investigate their effects.

The influence of the parameter τ

We employ several values of τ to find the effect of different values of parameter τ, on our established three-way decisions approach. The obtained outcomes are depicted in Table 5.12.

Table 5.12: TWD outcomes with different values of τ by using T-SFFHA operator

τ	Score values	outcomes
$\tau \longrightarrow 1$	0.0988,0.3916,0.4711,0.5935,0.4299	$POS(X) = \{x_3, x_4\}, BND(X) = \{x_2, x_5\}, NEG(X) = \{x_1\}$
$\tau = 2$	0.0916,0.3868,0.5699,0.5833,0.4280	$POS(X) = \{x_3, x_4\}, BND(X) = \{x_2, x_5\}, NEG(X) = \{x_1\}$
$\tau = 5$	0.0891,0.3714,0.5539,0.5725,0.4167	$POS(X) = \{x_3, x_4\}, BND(X) = \{x_2, x_5\}, NEG(X) = \{x_1\}$
$\tau = 10$	0.1028,0.3853,0.5436,0.5661,0.4058	$POS(X) = \{x_3, x_4\}, BND(X) = \{x_2, x_5\}, NEG(X) = \{x_1\}$
$\tau = 50$	0.1103,0.3987,0.5521.0.5535,0.4169	$POS(X) = \{x_3, x_4\}, BND(X) = \{x_2, x_5\}, NEG(X) = \{x_1\}$

Table 5.13: TWD outcomes with different values of τ by using T-SFFHG operator

τ	Score values	outcomes
$\tau \longrightarrow 1$	-0.8587,-0.5465,-0.1739,-0.1172,-0.2729	$POS(X) = \{x_3, x_4\}, BND(X) = \{x_2, x_5\}, NEG(X) = \{x_1\}$
$\tau = 2$	-0.8479,-0.5328,-0.1639,-0.1056,-0.2689	$POS(X) = \{x_3, x_4\}, BND(X) = \{x_2, x_5\}, NEG(X) = \{x_1\}$
$\tau = 5$	-0.7901,-0.5224,-0.1597,-0.1167,-0.2758	$POS(X) = \{x_3, x_4\}, BND(X) = \{x_2, x_5\}, NEG(X) = \{x_1\}$
$\tau = 10$	-0.8299,-0.5328,-0.1558,-0.0985,-0.2574	$POS(X) = \{x_3, x_4\}, BND(X) = \{x_2, x_5\}, NEG(X) = \{x_1\}$
$\tau = 50$	-0.8153,-0.5232,-0.1429,-0.0873,-0.2364	$POS(X) = \{x_3, x_4\}, BND(X) = \{x_2, x_5\}, NEG(X) = \{x_1\}$

As shown in Tables 5.12, 5.13, the score values with different parameter τ vary, but the decision results are same, indicating that the suggested method has an isotonic approach, and decision makers are allowed to opt the adequate value based on their preferences. We may observe that the score values derived by the T-SFFHWG operator enlarge as the parameter enlarges for the same option, but the score values generated by T-SFFHWA operator decrease as the parameter τ increases within the interval $(1, 5]$.

The influence of the parameter t

To explore the impact of different values of the parameter t on three-way decisions outcomes, we use various values of t in Step 4 of the proposed TWD approach. The obtained outcomes are depicted in Tables 5.14 and Table 5.15. (setting $\tau = 2$).

Table 5.14: TWD outcomes with different values of t by using T-SFFHA operator

t	Score values	outcomes
$t = 4$	0.0916,0.3868,0.5699,0.5833,0.4280	$POS(X) = \{x_3, x_4\}, BND(X) = \{x_2, x_5\}, NEG(X) = \{x_1\}$
$t = 6$	0.0683,0.3628,0.5418,0.5765,0.4159	$POS(X) = \{x_3, x_4\}, BND(X) = \{x_2, x_5\}, NEG(X) = \{x_1\}$
$t = 8$	0.0529,0.3463,0.5233,0.5671,0.4050	$POS(X) = \{x_3, x_4\}, BND(X) = \{x_2, x_5\}, NEG(X) = \{x_1\}$
$t = 10$	0.0465,0.3253,0.5038,0.5412,0.3967	$POS(X) = \{x_3, x_4\}, BND(X) = \{x_2, x_5\}, NEG(X) = \{x_1\}$
$t = 12$	0.0316,0.3081,0.4966,0.5239,0.3724	$POS(X) = \{x_3, x_4\}, BND(X) = \{x_2, x_5\}, NEG(X) = \{x_1\}$

Table 5.15: TWD outcomes with different values of t by using T-SFFHG operator

t	Score values	outcomes
$t = 4$	-0.8479,-0.5328,-0.1639,-0.1056,-0.2689	$POS(X) = \{x_3, x_4\}, BND(X) = \{x_2, x_5\}, NEG(X) = \{x_1\}$
$t = 6$	-0.8376,-0.5293,-0.1548,-0.1021,-0.2501	$POS(X) = \{x_3, x_4\}, BND(X) = \{x_2, x_5\}, NEG(X) = \{x_1\}$
$t = 8$	-0.8265,-0.5187,-0.1499,-0.1008,-0.2443	$POS(X) = \{x_3, x_4\}, BND(X) = \{x_2, x_5\}, NEG(X) = \{x_1\}$
$t = 10$	-0.8173,-0.5062,-0.1345,-0.1001,-0.2399	$POS(X) = \{x_3, x_4\}, BND(X) = \{x_2, x_5\}, NEG(X) = \{x_1\}$
$t = 12$	-0.8064,-0.4986,-0.1299,-0.0972,-0.2258	$POS(X) = \{x_3, x_4\}, BND(X) = \{x_2, x_5\}, NEG(X) = \{x_1\}$

Table 5.14 shows that the score values derived by using T-SFFHA operator become smaller and smaller as the value of t increases while Table 5.15 depicts that the score values derived by T-SFFHG operator rise as the value of t increases. However, the decision results are same. The

parameter t can be viewed as the "DM's attitude". Same as for MCDM approach, different decision makers can opt the value of t based on their optimistic or pessimistic attitude.

5.7.4 Comparative analysis

In fact, the existing decision methods based on aggregation operators [70, 76, 77] mainly focus on ranking of the alternatives and then selection of the best one, classification of alternatives is not provided. Nevertheless, our suggested approach classifies these alternatives into three domains i.e; positive domain, central domain and negative domain. Nevertheless, it is a strenuous task here to directly compare our suggested approach with some existing methodologies. Although, we may attain the ranking of alternatives by utilizing the proposed T-SFFA operators. To validate the viability and supremacy of the suggested approach, this study provides a collective comparative analysis with some previous decision approaches that includes the MCDM approach founded on T-SFEHIA and T-SFEHIG [76], TSFHHA and TSFHHG [77], SFNWA and SFNWG [70] operators. The comparison results obtained by these existing methods and our designed method are depicted in Table 5.16.

Table 5.16: Ranking results for different existing aggregation operators

Parameter	score values	Ranking
Proposed T-SFFHA operator	0.0916,0.3868,0.5699,0.5833,0.4280	$x_4 \succ x_3 \succ x_5 \succ x_2 \succ x_1$
Proposed T-SFFHG operator	-0.8479,-0.5328,-0.1639,-0.1056,-0.2689	$x_4 \succ x_3 \succ x_5 \succ x_2 \succ x_1$
T-SFEHIA operator [76]	-0.5637,-0.1059,0.3521,0.5023,0.2779	$x_4 \succ x_3 \succ x_5 \succ x_2 \succ x_1$
T-SFEHIG operator [76]	-0.5967,-0.1344,0.2360,0.4019,-0.2064	$x_4 \succ x_3 \succ x_5 \succ x_2 \succ x_1$
T-SFHHA operator [77]	-0.6352,-0.1643,0.2971,0.3623,-0.1481	$x_4 \succ x_3 \succ x_2 \succ x_5 \succ x_1$
T-SFHHG operator [77]	-0.4579,-0.3288,-0.2361,-0.1757,-0.3048	$x_4 \succ x_3 \succ x_5 \succ x_2 \succ x_1$
SFNWA operator [70]	Not applicable	
SFNWG operator [70]	Not applicable	

It is observed that ranking results of alternatives attained by using the previous methods and our designed method are same. However, the ranking results for the alternatives x_2 and

x_5 are slightly changed by using TSFHHA operator [77]. But the interesting thing is, that the best alternative x_1 remains same in all cases. This demonstrates the validity and feasibility of the developed operators. The proposed T-spherical frank fuzzy operators with the spherical fuzzy set environment provide an effective generalized methodology to deal with uncertainty in decision analysis. These operators with the T-spherical fuzzy environment are more flexible and virtual to manage MCDM and TWD problems of real world.

In comparison to the aforementioned aggregation operators, the proposed work has the following key advantages during implementation.

1). Unlike Ullah et al. [77], the proposed approach remains unchanged on the same output ranking for different operators.

2). The developed method includes two parameters, which seems more suitable to adjust the aggregate value according to the real-world decision needs and encapsulate the various spherical fuzzy aggregation operators. Consequently, the proposed method proved to be more general and applicable.

3). Unlike the previous methods [70,76,77], the stated method employ the proposed entropy measure for the criteria weight determination, and the derived weights are then used in the decision process. Thus the proposed method is more useful in the situation where the criteria weights are completely unknown.

4). T-spherical fuzzy Frank operators can tackle the problems considered in the present day literature [70, 98, 99], but the existing operators of IFSs, PyFSs, PFSs and q-ROFSs cannot address the problems described in T-spherical fuzzy environment.

Chapter 6

Concluding Remarks

In this chapter, the proposed doctoral work is concluded, the contributions of this thesis are summarized. Further, some possible future research directions are presented.

6.1 Conclusions

In the real world, decision-making activity occurs frequently and most of the decision problems have imprecise and ambiguous information. Due to the presence of various types of uncertainties, classical methods are not as resourceful to address these decision problems. In last few decades, several decision making techniques have been established to reduce the complexity of MCDM problems. But still, there are some imperfections in these provided techniques, therefore, there is a need to develop some advanced approaches. TWD theory has been proved more inventive in diverse decision making activities as it provides sufficient flexibility and minimizes the decision risks. Conflict analysis study aims to lessen the complexity of conflicts, that occur naturally in many aspects of real-life. Numerous studies have been established to seek effective procedures to upgrade conflict management. TWD concept is closely related to the idea of conflict analysis, hence recent studies have combined both ideas significantly to manage more complex real-life scenarios. In the current thesis, novel three-way conflict resolution models are established to facilitate the intricate conflict situations more precisely. In addition, MCDM and TWD models are proposed to solve decision problems with T-spherical fuzzy data, whose weights of criteria are fully unknown.

At the beginning of this thesis, we initiated a novel three-way conflict study model based on

GTRS. In comparison to current conflict studies, the proposed model adopts a bilateral approach because almost all existing conflict analysis models are unilateral and did not take into account the concerns of all the objects. In reality, the actions of objects are interlinked with severe consequences. Therefore, the actions of all objects should be considered simultaneously so that the respective gains (payoffs) can be calculated accordingly. The proposed model overcome the imperfections of existing models effectively and improved the understandability of conflict resolution.

Our proposed GTRS approach is more fruitful as it explores all possible actions for each player concerning the others and is not as conservative as Pawlak's model. It uses the mean cumulative degree of three measures: agreement, neutrality, and disagreement, with a pair of threshold values for each measure to signify the utilities of the players. Each object uses its respective payoff functions in analyzing another object for its inclusion in conflict, neutral and allied sets of the object. The strategies of players are used to drive a balanced solution by using Nash equilibrium for classifying the conflict, neutral and allied objects. GTRS is more efficient in solving three-way conflict resolution and investigating the varying probabilistic thresholds to improve the rough set decision-making. We demonstrate that the proposed game mechanism is more useful for reducing the complexity experienced in the context of multi-agent-based conflict situations. We elaborated the game representation for conflict situations, including a detailed description of all game components. The game settings provide a compromise and trade-off mechanism for combining and balancing the differences between the opinions of different objects. In particular, we demonstrate that equilibrium analysis can be used to construct conflict, allied, and neutral sets. We have also utilized the proposed model to solve the Syrian conflict problem with a detailed analysis and illustrate its effectiveness. The proposed model is further used for Middle East conflict resolution. It is also applied to a case study on development plans for Gansu province in China. A detailed comparison with existing models is given to validate the strength of the proposed model. As a result, we expect twofold advantages: at first, it will enhance the current understandability of the conflict study in the GTRS framework, and secondly, it will provide a better perception of the GTRS theory, its wide scope, and appropriateness to deal with various real-life conflict problems.

The upcoming chapter focused on another approach to address the conflict resolution problems when the given IS is based on agents' opinions with multiple positive and negative values rather than a single value. When facing a conflict situation it seems more reasonable to depict the agents' attitude by multiple positive and negative real numbers. Most of the current conflict study models are rigid in their approach and decision makers have no freedom to take a definite stance. Though these models claim that the most indicative scales are employed to depict the agents' attitude of partial agreement or disagreement but in fact, they are failed to deal with an IS based on agents' opinions with multiple values (positive and negative). Hence, there is a need for an adjustable model to evaluate the hesitancy of agents' opinions. Compared with existing studies, we put efforts to promote the conflict analysis study under a hesitant fuzzy environment. Our research work contributes in three aspects. Firstly, we have reset the initial IS by limiting the domain of agents' opinions from $[-1, +1]$ to $[0, 1]$. Secondly, we construct a three-way conflict analysis model based on a hesitant fuzzy IS by utilizing aggregated opinion functions as HFEs and associated loss functions as real numbers. Instead of using traditional hesitant fuzzy operations we have defined and utilized some new hesitant fuzzy operations to calculate the risk loss functions based on the Bayesian decision procedure. The proposed model does not rely on threshold values to formulate allied, neutral, and conflict sets of agents. In last, we drive three-way decisions by utilizing two different techniques, the first one is the general method based on the average of score functions and the second one is the ranking method of possibility degree founded on a stochastic way. To demonstrate the vitality of the proposed model the Middle East conflict problem is considered, and the results are summarized. A comparison with existing models is also provided.

The fifth chapter enhanced the contribution of this thesis regarding the provision of the MCDM method and TWD method based on the proposed Frank aggregation operators to handle decision-making problems with T-spherical fuzzy information. In present-day studies, the existing aggregation operators concentrated on the algebraic, Einstein, and Hamacher norms under T-SFSs to formulate the combination process. Algebraic, Einstein, and Hamacher product and sum are not only fundamental TSFS operations that characterize the union and intersection of two T-SFSs. A generalized norm may be utilized to build a general union and intersection under T-spherical fuzzy information; that is, instances of deferent-norms families can be used to execute the corresponding intersections and unions under a T-spherical fuzzy environment. Frank product and sum are suitable substitutes of the algebraic, Einstein, and Hamacher product for an intersection and union and can deliver smooth estimates of the algebraic product and sum. But it seems that in the literature, there is no research on aggregation operators utilizing these operations on T-SFSs. Motivated by these defects and beneficial characteristics of Frank t-norm and t-conorm, we explored novel generalized operational rules of T-SFNs to build T-spherical fuzzy aggregation operators that comply with the principles of Frank t-norm and t-conorm. Keeping in mind the significance of ordered position and argument itself, the notions of T-SFFHA and T-SFFHG are provided. Some desirable properties and special cases of these operators are also studied comprehensibly. Besides, the T-spherical fuzzy entropy measure is proposed along with detailed proof of its characteristics. Then, based on the proposed operators and entropy measure, the MCDM method and TWD method are established to handle the decision-making problems with T-SF information. We compare our developed approach with the method structured on the Archimedean t-norm and t-conorm based spherical fuzzy weighted averaging and geometric operators. These operators have a general representation by Archimedean t-norm and t-conorm. But, due to its limited scope of practical application, it can only handle the MCDM problems demonstrated by PFNs or SFNs. On the contrary, the proposed study has the parameter t, which can express the spherical fuzzy information more flexibly. The presented study has a good improvement in terms of criteria weights, that is, it utilizes the proposed entropy measure to find the criteria weights and can address the completely unknown weight information problems accurately. Two practical cases are provided to elaborate on the implication of the suggested MCDM and TWD methods for selecting the best investment company. Further, we examined the impact of the parameters τ and t in the decision procedure and reported the stability stage of sorting results. Finally, a comparative analysis is conducted with some existing approaches to highlight the feasibility and supremacy of the presented work.

6.2 Future research work suggestions

This section presents some significant future research guidelines for researchers in this area by analyzing the subject matter of this study. For possible future research lines, see our suggestions as follows:

i. In the proposed approach for conflict resolution, we examined Nash equilibrium game analysis for the GTRS model. Some other types of game analysis, for instance, the trembling-hand perfect equilibrium [100], correlated equilibrium [101], and ϵ-Nash equilibrium [102], may also be applied in the GTRS model. In this context, it would be a useful and interesting task to fix the relationship between the decision thresholds and a specific kind of game analysis under observation. Another significant issue would be to analyze the implications of various game solutions for decision-making.

ii. Other kinds of games like matrix games and in some cases zero-sum games can be used to study conflict resolution. To devise more practical ways to remove conflict, we suggest using GTRS in combination with formal three-way concept analysis.

iii. GTRS generally considers a direct modification of thresholds as a possible game strategy. Very often, this proves to be troublesome in exactly finding out and illustrating the strategies of players as a decision framework from players' viewpoint. More significant approaches may be utilized to develop strategies that may incite far better and more purposeful interpretations.

iv. One can work on incomplete ISs as it is possible for some issues an object cannot take a concrete decision. Therefore, it is a challenging problem to deal with such scenarios. We suggest considering an IS with fuzzy values that allow objects more freedom in expressing their opinions as an object does not need to fully agree or disagree on a certain issue.

v. Apart from working on conflict resolution under a hesitant fuzzy environment, the proposed approach can be utilized for other multi-criteria decision problems, especially in three-way decision problems.

vi. In the future, the proposed methods can be employed in diverse fields like spam email filtering, document classification, and web mining under uncertain environments.

vii. T-Spherical fuzzy frank aggregation operators can also be examined with fuzzy, hesitant fuzzy, and dual fuzzy weight information.

viii. As future research study, the established MCDM and TWD methods for the T-spherical fuzzy system can be applied in other domains, such as three-way conflict analysis, project selection, risk analysis, and in other fields with uncertain and incomplete information.

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