Unsteady MHD Axisymmetric Flow of a Power-law Fluid over Radially Stretching Sheet

By

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MASTER OF PHILOSOPHY

IN

<u>MATHEMATICS</u>

Supervised By

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CERTIFICATE

A DISSERTATION SUBMITTED IN THE PARTIAL FULFILLMENT OF THE REQUIREMENTS FOR THE DEGREE OF THE MASTER OF PHILOSOPHY

We accept this dissertation as conforming to the required standard

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Preface

In recent years, the analysis of non-Newtonian fluids has acquired great importance. In industrial and technological applications, non-Newtonian fluids are now acknowledged as more appropriate than Newtonian fluids. The governing equations which arise in case of non-Newtonian fluids are more complicated, of higher order and more non-linear compared to Navier-Stokes equations. In such case, one needs additional boundary conditions for obtaining a unique solution. Rajagopal [1,2] discussed the issue of additional boundary conditions regarding the existence and uniqueness of the solution. Although the flow characteristics of non-Newtonian fluids are more complex, still such fluids are on the leading edge of research in fluid mechanics.

The boundary layer flows over a stretching sheet have attracted the attention of many investigators because of their wide applications. Such flows encounter in metal extrusion, metal spinning, continuous stretching of plastic films and artificial fibers, in the manufacture of crystalline materials, polymeric sheets and sheet glass. Sakiadis [3] was probably the first to study the boundary layer flow over a stretched surface moving with a constant velocity. Later on, Crane [4] reported analytical solution of boundary layer flow of an incompressible viscous fluid over a stretching sheet. Schowalter [5] considered the case of the boundary layer flow of non-Newtonian power-law fluid and gave the equations governing the self similar flow of pseudoplastic fluid. Followed by this, many investigators such as Chen and Char [6], Rajagopal [7] and Banks [8] considered the various aspects of the related problem of a stretched sheet with a linear velocity and different thermal boundary conditions. The work on the unsteady boundary layer flows is very scarce in the literature. Much importance has been given to steady flow problems. However, unsteady boundary-layer flows due to an impulsively stretching surface were investigated by some researchers [9-12].

Magnetohydrodynamics (MHD) flow is continuing to be an interesting area of research due to its practical applications in chemical engineering, electrochemistry and polymer processing. Initially, the MHD flows of non-Newtonian fluids were studied by Sarpkaya [13]. The MHD flow over a stretching surface was studied by a number of researchers [14-17].

The work on axisymmetric flow over a radial stretching sheet [18-20] is very scarce in the literature. Motivated by the aforementioned facts, the objective of the present dissertation is therefore to investigate the unsteady MHD axisymmetric flows of power-law fluid over a radially stretching sheet.

The entire work in the dissertation has been divided into two chapters.

Chapter 1 contains the review of the work by Xu and Liao [21]. In this chapter the analytical solutions for unsteady magnetohydrodynamic flows of non-Newtonian fluids over a stretching plate are constructed. The analytical solutions are obtained by using the homotopy analysis method (HAM) [22-24], which are valid for all values of the dimensionless time. The impact of the emerging flow parameters on the velocity is highlighted and examined graphically.

Chapter 2 discusses the unsteady MHD axisymmetric flow of power-law fluid over a radially stretching sheet. The axisymmetric flow equations are reduced by the help of similarity transformations and then solved analytically via homotopy analysis method. Finally, the influence of the emerging flow parameters are plotted and discussed.

Contents

Chapter 1

Series solutions of unsteady magnetohydrodynamic flow of non-Newtonian fluids caused by an impulsively stretching plate

1.1 Introduction

This chapter describes the unsteady, incompressible, magnetohydrodynamic viscous flow of non-Newtonain fluids caused by an impulsively stretching plate under the influence of a transverse magnetic field. By using similarity transformations, the modeled non-linear partial differential equations in three independent variables are reduced to a single partial differential equation in two independent variables. An analytical technique, namely the homotopy analysis method (HAM) is used to give analytic solution. The analytic series solutions are uniformly valid for all non-dimensional time $0 \leq \tau < +\infty$, in the whole spatial region $0 \leq \eta < \infty$. Finally, the influence of various emerging flow parameters are shown and discussed. This chapter is a detailed review of a paper by Xu and Liao [21] .

1.2 Formulation of the problem

Consider the unsteady, two-dimensional magnetohydrodynamic flow of non-Newtonian fluids caused by an impulsively stretching plate. The fluid is assumed to be electrically conducting, obeying the power-law model in the presence of a transverse magnetic field. The flat rigid plate is in $xz-$plane and the flow being confined to $y > 0$.

The governing equations for unsteady MHD flows are given as follows

$$\operatorname{div} \mathbf{V} = 0, \tag{1.1}$$

$$\rho \frac{d\mathbf{V}}{dt} = \boldsymbol{\nabla} \cdot \mathbf{T} + \mathbf{J} \times \mathbf{B}, \tag{1.2}$$

where $\mathbf{V}$ is the velocity vector, $\mathbf{T}$ the Cauchy stress tensor, $\mathbf{J}$ the current density and $\mathbf{B} = \mathbf{B_0}+\mathbf{b}$, the total magnetic field with $\mathbf{b}$ the induced magnetic field and $\mathbf{B_0}$ the applied magnetic field. It is assumed that the magnetic Reynolds number is small so that the induced magnetic field is neglected in comparison to the applied magnetic field, so that $\mathbf{B} = \mathbf{B_0}$.

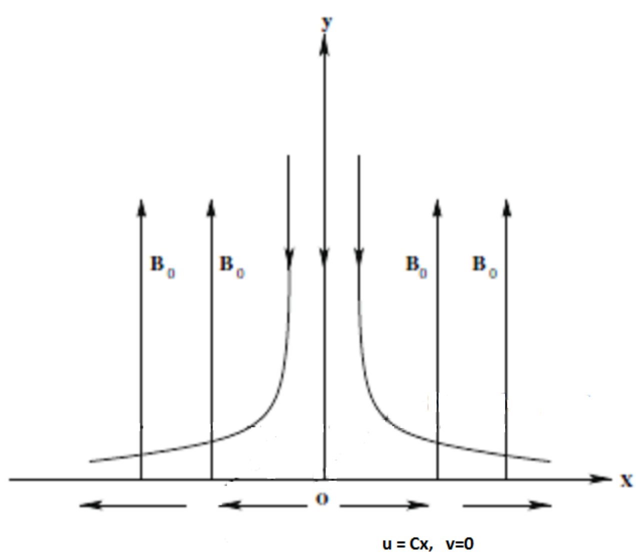

Figure 1.1: Physical model for the planar stretching sheet.

By generalized Ohm's law

$$\mathbf{J} = \sigma\left(\mathbf{E} + \mathbf{V} \times \mathbf{B}\right), \tag{1.3}$$

where $\boldsymbol{\sigma}$ and $\mathbf{E}$ denote the electrical conductivity and the total electrical field, respectively.

In the present case, we have assumed that the electric field is absent, therefore, Eq. (1.3) becomes

$$\mathbf{J} = \sigma\left(\mathbf{V} \times \mathbf{B}\right). \tag{1.4}$$

For an electrically conducting fluid, the Maxwell equations are

$$\boldsymbol{\nabla} \cdot \mathbf{B} = 0, \quad \boldsymbol{\nabla} \times \mathbf{B} = \mu_0 \mathbf{J}, \quad \boldsymbol{\nabla} \times \mathbf{E} = 0, \tag{1.5}$$

where μ_0 is the magnetic permeability.

The velocity profile in the present case will be

$$\mathbf{V} = \left[u\left(x, y, t\right), v\left(x, y, t\right), 0\right], \tag{1.6}$$

where t denotes the time while u and v are the velocity components in the $x-$ and $y-$directions, respectively.

The transverse magnetic field is applied in the positive $y-$direction, normal to stretching surface throughout the fluid flow. The magnetic field is of the form

$$\mathbf{B} = \left[0, B_0, 0\right], \tag{1.7}$$

where B_0 is the magnitude of $\mathbf{B_0}$.

Now

$$\mathbf{V} \times \mathbf{B} = \begin{bmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ u & v & 0 \\ 0 & B_0 & 0 \end{bmatrix}, \tag{1.8}$$

where $\hat{\imath}$, $\hat{\jmath}$ and $\hat{k}$ are the unit vectors along the $x-$, $y-$ and $z-$directions, respectively, and therefore

$$\mathbf{V} \times \mathbf{B} = uB_0\hat{k}, \tag{1.9}$$

and

$$
(\mathbf{V}\times\mathbf{B})\times\mathbf{B}=\begin{bmatrix}\hat{\imath} & \hat{\jmath} & \hat{k}\\ 0 & 0 & uB_0\\ 0 & B_0 & 0\end{bmatrix}, \tag{1.10}
$$

or

$$
(\mathbf{V}\times\mathbf{B})\times\mathbf{B}=-B_0^2u\hat{\imath}. \tag{1.11}
$$

The Lorentz force in Eq. (1.2) becomes

$$
\mathbf{J}\times\mathbf{B}=\sigma\left(\mathbf{V}\times\mathbf{B}\right)\times\mathbf{B}. \tag{1.12}
$$

Using Eq. (1.11) in Eq. (1.12), it follows that

$$
\mathbf{J}\times\mathbf{B}=-\sigma B_0^2u\hat{\imath}. \tag{1.13}
$$

The Cauchy stress tensor for non-Newtonian power-law fluids is defined as

$$
\mathbf{T}=-p\mathbf{I}+\mathbf{S}, \tag{1.14}
$$

where p is the pressure, $\mathbf{I}$ the identity tensor and $\mathbf{S}$ the extra stress tensor given by

$$
\mathbf{S}=K\left[\left|\sqrt{\frac{1}{2}tr\left(\mathbf{A}_1^2\right)}\right|^{n-1}\right]\mathbf{A}_1. \tag{1.15}
$$

In Eq. (1.15), K is the consistency coefficient, n the power-law index and $\mathbf{A}_1$ the first Rivlin-Erickson tensor which is defined as

$$
\mathbf{A}_1=\mathrm{grad}\,\mathbf{V}+\left(\mathrm{grad}\,\mathbf{V}\right)^T, \tag{1.16}
$$

with

$$
\mathrm{grad}\,\mathbf{V}=\begin{bmatrix}\frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & 0\\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & 0\\ 0 & 0 & 0\end{bmatrix}, \tag{1.17}
$$

and

$$(\operatorname{grad}\mathbf{V})^T = \begin{bmatrix} \frac{\partial u}{\partial x} & \frac{\partial v}{\partial x} & 0 \\ \frac{\partial u}{\partial y} & \frac{\partial v}{\partial y} & 0 \\ 0 & 0 & 0 \end{bmatrix}. \tag{1.18}$$

Using Eqs. (1.17) and (1.18) in Eq. (1.16), we get

$$\mathbf{A}_1 = \begin{bmatrix} 2\frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} & 0 \\ \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} & 2\frac{\partial v}{\partial y} & 0 \\ 0 & 0 & 0 \end{bmatrix}, \tag{1.19}$$

$$\mathbf{A}_1^2 = \begin{bmatrix} 4\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)^2 & 2\frac{\partial u}{\partial x}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) + 2\frac{\partial v}{\partial y}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) & 0 \\ 2\frac{\partial u}{\partial x}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) + 2\frac{\partial v}{\partial y}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) & 4\left(\frac{\partial v}{\partial y}\right)^2 + \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)^2 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \tag{1.20}$$

$$tr\left(\mathbf{A}_1^2\right) = 4\left(\frac{\partial u}{\partial x}\right)^2 + 2\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)^2 + 4\left(\frac{\partial v}{\partial y}\right)^2, \tag{1.21}$$

$$\left|\frac{1}{2}tr\left(\mathbf{A}_1^2\right)\right|^{\frac{n-1}{2}} = \left|2\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)^2 + 2\left(\frac{\partial v}{\partial y}\right)^2\right|^{\frac{n-1}{2}}. \tag{1.22}$$

Now putting Eqs. (1.19) and (1.22) in Eq. (1.15)

$$\mathbf{S} = K\left|2\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)^2 + 2\left(\frac{\partial v}{\partial y}\right)^2\right|^{\frac{n-1}{2}} \begin{bmatrix} 2\frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} & 0 \\ \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} & 2\frac{\partial v}{\partial y} & 0 \\ 0 & 0 & 0 \end{bmatrix}. \tag{1.23}$$

With the help of the above relation, Eq. (1.14) becomes

$$\mathbf{T} = -p\mathbf{I} + K\left|2\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)^2 + 2\left(\frac{\partial v}{\partial y}\right)^2\right|^{\frac{n-1}{2}} \begin{bmatrix} 2\frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} & 0 \\ \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} & 2\frac{\partial v}{\partial y} & 0 \\ 0 & 0 & 0 \end{bmatrix}. \tag{1.24}$$

The Cauchy stress tensor can also be written as

$$\mathbf{T} = \begin{bmatrix} \sigma_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_{zz} \end{bmatrix}, \tag{1.25}$$

where σ_{xx}, σ_{yy}, σ_{zz} are the normal stresses and τ_{xy}, τ_{yz}, τ_{xz} are the shear stresses.

The comparison of Eqs. (1.24) and (1.25) gives

$$\tau_{xy} = \tau_{yx} = K \left| 2\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)^2 + 2\left(\frac{\partial v}{\partial y}\right)^2 \right|^{\frac{n-1}{2}} \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right), \tag{1.26}$$

$$\tau_{xz} = \tau_{yz} = \tau_{zx} = \tau_{zy} = \sigma_{zz} = 0, \tag{1.27}$$

$$\sigma_{xx} = -p + 2K \left| 2\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)^2 + 2\left(\frac{\partial v}{\partial y}\right)^2 \right|^{\frac{n-1}{2}} \frac{\partial u}{\partial x}, \tag{1.28}$$

$$\sigma_{yy} = -p + 2K \left| 2\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)^2 + 2\left(\frac{\partial v}{\partial y}\right)^2 \right|^{\frac{n-1}{2}} \frac{\partial v}{\partial y}. \tag{1.29}$$

For two-dimensional flow, Eq. (1.1) becomes

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0. \tag{1.30}$$

The $x-$ and $y-$components of Eq. (1.2) are

$$\rho\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\right) = \frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} - \sigma B_0^2 u, \tag{1.31}$$

$$\rho\left(\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y}\right) = \frac{\partial \tau_{yx}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y}. \tag{1.32}$$

Using the relations mentioned in Eqs. (1.26) to (1.29), the $x-$ and $y-$components of the momentum equation becomes

$$\rho\left(\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}\right) = -\frac{\partial p}{\partial x}+2K\frac{\partial}{\partial x}\left[\frac{\partial u}{\partial x}\left|4\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)^2\right|^{\frac{n-1}{2}}\right]$$
$$+K\frac{\partial}{\partial y}\left[\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)\left|4\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)^2\right|^{\frac{n-1}{2}}\right]$$
$$-\sigma B_0^2 u, \tag{1.33}$$

$$\rho\left(\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}\right) = -\frac{\partial p}{\partial y}+K\frac{\partial}{\partial x}\left[\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)\left|4\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)^2\right|^{\frac{n-1}{2}}\right]$$
$$+2K\frac{\partial}{\partial y}\left[\frac{\partial v}{\partial y}\left|4\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)^2\right|^{\frac{n-1}{2}}\right]. \tag{1.34}$$

Equations (1.30), (1.33) and (1.34) can be written in dimensionless form by using the following relations

$$u^*=\frac{u}{U},\ v^*=\frac{v}{U},\ x^*=\frac{x}{L},\ y^*=\frac{y}{L},\ p^*=\frac{p}{\rho U^2}\text{ and }t^*=\frac{tU}{L}, \tag{1.35}$$

where L is the characteristic length and U the stretching velocity.

Using Eq. (1.35) in Eqs. (1.30), (1.33) and (1.34) we get

$$\frac{\partial u^*}{\partial x^*}+\frac{\partial v^*}{\partial y^*}=0, \tag{1.36}$$

$$\frac{\partial u^*}{\partial t^*}+u^*\frac{\partial u^*}{\partial x^*}+v^*\frac{\partial u^*}{\partial y^*} = -\frac{\partial p^*}{\partial x^*}+2\epsilon\frac{\partial}{\partial x^*}\left[\frac{\partial u^*}{\partial x^*}\left|\begin{array}{c}4\left(\frac{\partial u^*}{\partial x^*}\right)^2+\left(\frac{\partial u^*}{\partial y^*}\right)^2\\ +\left(\frac{\partial v^*}{\partial x^*}\right)^2+2\frac{\partial u^*}{\partial y^*}\frac{\partial v^*}{\partial x^*}\end{array}\right|^{\frac{n-1}{2}}\right]$$
$$+\epsilon\frac{\partial}{\partial y^*}\left[\left(\frac{\partial u^*}{\partial y^*}+\frac{\partial v^*}{\partial x^*}\right)\left|\begin{array}{c}4\left(\frac{\partial u^*}{\partial x^*}\right)^2+\left(\frac{\partial u^*}{\partial y^*}\right)^2\\ +\left(\frac{\partial v^*}{\partial x^*}\right)^2+2\frac{\partial u^*}{\partial y^*}\frac{\partial v^*}{\partial x^*}\end{array}\right|^{\frac{n-1}{2}}\right]$$
$$-Hau^*, \tag{1.37}$$

$$\frac{\partial v^*}{\partial t^*} + u^*\frac{\partial v^*}{\partial x^*} + v^*\frac{\partial v^*}{\partial y^*} = -\frac{\partial p^*}{\partial y^*} + \epsilon\frac{\partial}{\partial x^*}\left[\left(\frac{\partial u^*}{\partial y^*} + \frac{\partial v^*}{\partial x^*}\right)\left|\begin{array}{c} 4\left(\frac{\partial u^*}{\partial x^*}\right)^2 + \left(\frac{\partial u^*}{\partial y^*}\right)^2 \\ +\left(\frac{\partial v^*}{\partial x^*}\right)^2 + 2\frac{\partial u^*}{\partial y^*}\frac{\partial v^*}{\partial x^*}\end{array}\right|^{\frac{n-1}{2}}\right]$$
$$+2\epsilon\frac{\partial}{\partial y^*}\left[\frac{\partial v^*}{\partial y^*}\left|\begin{array}{c} 4\left(\frac{\partial u^*}{\partial x^*}\right)^2 + \left(\frac{\partial u^*}{\partial y^*}\right)^2 \\ +\left(\frac{\partial v^*}{\partial x^*}\right)^2 + 2\frac{\partial u^*}{\partial y^*}\frac{\partial v^*}{\partial x^*}\end{array}\right|^{\frac{n-1}{2}}\right], \tag{1.38}$$

where

$$\epsilon = \frac{\frac{K}{\rho}}{LU}\left(\frac{U}{L}\right)^{n-1}, \tag{1.39}$$

and

$$Ha = \frac{\sigma}{\rho}\frac{L}{U}B_0^2, \tag{1.40}$$

are the dimensionless parameters.

In standard boundary layer assumptions, t, x, p and u are of order 1 while v and y are of order δ, where δ is the boundary layer thickness. The dimensionless parameter ϵ is of order δ^{n+1}. Thus the boundary layer approximation of Eqs. (1.36), (1.37) and (1.38) in the dimensional form yields

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \tag{1.41}$$

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \frac{K}{\rho}\frac{\partial}{\partial y}\left(\left|\frac{\partial u}{\partial y}\right|^{n-1}\frac{\partial u}{\partial y}\right) - \left(\frac{\sigma B_0^2}{\rho}\right)u, \tag{1.42}$$

$$0 = -\frac{1}{\rho}\frac{\partial p}{\partial y}. \tag{1.43}$$

As $y \to \infty$, no disturbance is found above the boundary layer so we take velocity equal to zero; that is,

$$t > 0: \qquad u \to 0 \text{ as } y \to +\infty. \tag{1.44}$$

Using Eq. (1.44) in Eq. (1.42) , we finally have

$$0 = -\frac{1}{\rho}\frac{\partial p}{\partial x}. \tag{1.45}$$

Now the pressure gradient is set equal to zero in Eq. (1.42), by using Eq. (1.45) , and we have

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=\frac{K}{\rho}\frac{\partial}{\partial y}\left(\left|\frac{\partial u}{\partial y}\right|^{n-1}\frac{\partial u}{\partial y}\right)-\left(\frac{\sigma B_0^2}{\rho}\right)u. \tag{1.46}$$

In the present case we take $\frac{\partial u}{\partial y}<0$ (a detailed discussion regarding the sign of $\left|\frac{\partial u}{\partial y}\right|$ is made by Mahapatra et al. [25]).

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=-\frac{K}{\rho}\frac{\partial}{\partial y}\left(-\frac{\partial u}{\partial y}\right)^{n}-\left(\frac{\sigma B_0^2}{\rho}\right)u. \tag{1.47}$$

The resulting form of mass and momentum conservation equations are as follows

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0, \tag{1.48}$$

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=-\frac{K}{\rho}\frac{\partial}{\partial y}\left(-\frac{\partial u}{\partial y}\right)^{n}-\left(\frac{\sigma B_0^2}{\rho}\right)u. \tag{1.49}$$

The corresponding initial and boundary conditions are

$$t\leq 0:\qquad u=v=0\qquad \text{at } y\geq 0,\ -\infty<x<+\infty, \tag{1.50}$$

$$t>0:\qquad u=Cx,\ v=0\qquad \text{at } y=0, \tag{1.51}$$

$$t>0:\qquad u\to 0\qquad \text{as } y\to+\infty, \tag{1.52}$$

where C is a positive constant.

We use the following similarity transformations

$$\Psi=\left(\frac{K\xi}{\rho C^{1-2n}}\right)^{\frac{1}{n+1}}x^{\frac{2n}{(n+1)}}F(\eta,\xi), \tag{1.53}$$

and

$$\eta=y\left(\frac{\rho C^{2-n}}{K\xi}\right)^{\frac{1}{n+1}}x^{\frac{(1-n)}{(1+n)}}, \tag{1.54}$$

$$\xi=1-\exp(-\tau),\quad \tau=Ct, \tag{1.55}$$

where Ψ denotes the stream function defined as

$$u = \frac{\partial \Psi}{\partial y} \text{ and } \; v = -\frac{\partial \Psi}{\partial x}, \tag{1.56}$$

and η and ξ are the transformed dimensionless independent variables and τ the dimensionless time.

Using the above transformations, Eq. (1.56) yields

$$u = Cx\frac{\partial F(\eta,\xi)}{\partial \eta}, \tag{1.57}$$

$$v = -\left(\frac{2n}{n+1}\right)\left(\frac{K\xi}{\rho C^{1-2n}}\right)^{\frac{1}{n+1}} x^{\frac{(n-1)}{(n+1)}} F(\eta,\xi) - \left(\frac{1-n}{1+n}\right) Cy\frac{\partial F(\eta,\xi)}{\partial \eta}. \tag{1.58}$$

From Eqs. (1.57) and (1.58), we can calculate the following quantities

$$\frac{\partial u}{\partial t} = C^2 x (1-\xi) \frac{\partial^2 F(\eta,\xi)}{\partial \eta \partial \xi} - \left(\frac{1}{1+n}\right) C^2 x \eta \xi^{-1} (1-\xi) \frac{\partial^2 F(\eta,\xi)}{\partial \eta^2}, \tag{1.59}$$

$$\frac{\partial u}{\partial x} = C\frac{\partial F(\eta,\xi)}{\partial \eta} + C\eta\left(\frac{1-n}{1+n}\right)\frac{\partial^2 F(\eta,\xi)}{\partial \eta^2}, \tag{1.60}$$

$$\frac{\partial u}{\partial y} = Cx\left(\frac{\rho C^{2-n}}{K\xi}\right)^{\frac{1}{n+1}} x^{\frac{(1-n)}{(1+n)}} \frac{\partial^2 F(\eta,\xi)}{\partial \eta^2}, \tag{1.61}$$

$$\frac{\partial \tau_{xy}}{\partial y} = nC^2 x \rho \xi^{-1}\left(-\frac{\partial^2 F(\eta,\xi)}{\partial \eta^2}\right)^{n-1} \frac{\partial^3 F(\eta,\xi)}{\partial \eta^3}. \tag{1.62}$$

Using Eqs. (1.57) to (1.62), the continuity equation (1.48) is identically satisfied while Eq. (1.49) takes the following form

$$(1-\xi)\left(\frac{\eta}{n+1}\frac{\partial^2 F}{\partial \eta^2} - \xi\frac{\partial^2 F}{\partial \eta \partial \xi}\right) + \xi\left[\frac{2n}{n+1}F\frac{\partial^2 F}{\partial \eta^2} - \left(\frac{\partial F}{\partial \eta}\right)^2 - M\frac{\partial F}{\partial \eta}\right]$$

$$+n\left(-\frac{\partial^2 F}{\partial \eta^2}\right)^{n-1}\frac{\partial^3 F}{\partial \eta^3} = 0, \tag{1.63}$$

where F is the dimensionless stream function, $\frac{\partial F}{\partial \eta}$ the dimensionless velocity and M the magnetic parameter defined as

$$M = \frac{\sigma B_0^2}{\rho C}. \tag{1.64}$$

The boundary conditions are transformed as follows

$$F(0, \xi) = 0, \tag{1.65}$$

$$\left.\frac{\partial F(\eta, \xi)}{\partial \eta}\right|_{\eta=0} = 1, \tag{1.66}$$

$$\left.\frac{\partial F(\eta, \xi)}{\partial \eta}\right|_{\eta \to +\infty} = 0. \tag{1.67}$$

The skin friction coefficient $C_f(\xi)$ at the wall is

$$C_f(\xi) = \frac{\tau_w}{\rho (Cx)^2} = \xi^{-\frac{n}{(n+1)}} [-F_{\eta\eta}(0, \xi)]^n \, \mathrm{Re}^{-\frac{1}{(n+1)}}, \tag{1.68}$$

where τ_w is the local wall shear stress and Re the local Reynolds number defined as

$$\mathrm{Re} = (Cx)^{2-n} \frac{x^n}{\left(\frac{K}{\rho}\right)}. \tag{1.69}$$

1.2.1 Exact solution of Rayleigh type equation

When $\xi = 0$ and $n = 1$, Eq. (1.63) becomes Rayleigh type of equation

$$\frac{\partial^3 F}{\partial \eta^3} + \frac{\eta}{2} \frac{\partial^2 F}{\partial \eta^2} = 0, \tag{1.70}$$

subject to boundary conditions

$$F(0, 0) = 0, \tag{1.71}$$

$$\frac{\partial F(\eta, \xi)}{\partial \eta} = 1 \quad \text{at } \eta = 0,\ \xi = 0, \tag{1.72}$$

$$\frac{\partial F(\eta, \xi)}{\partial \eta} = 0 \quad \text{as } \eta \to +\infty,\ \xi = 0. \tag{1.73}$$

Thus, Eq.(1.70) has the exact solution given by

$$F(\eta,0)=\eta\ \operatorname{erf}c\left(\frac{\eta}{2}\right)+\frac{2}{\sqrt{\pi}}\left[1-\exp\left(-\frac{\eta^2}{4}\right)\right]. \tag{1.74}$$

1.2.2 Exact solution of Crane type differential equation

When $\xi=1$, corresponding to $\tau\to+\infty$ and $n=1$, Eq. (1.63) becomes Crane type differential equation

$$\frac{\partial^3 F}{\partial\eta^3}+F\frac{\partial^2 F}{\partial\eta^2}-\left(\frac{\partial F}{\partial\eta}\right)^2-M\frac{\partial F}{\partial\eta}=0, \tag{1.75}$$

with the corresponding boundary conditions

$$F(0,1)=0, \tag{1.76}$$

$$\frac{\partial F(\eta,\xi)}{\partial\eta}=0 \quad \text{at } \eta=0,\ \xi=1, \tag{1.77}$$

$$\frac{\partial F(\eta,\xi)}{\partial\eta}=0 \quad \text{as } \eta\to+\infty,\ \xi=1. \tag{1.78}$$

The exact solution of Eq.(1.75) is

$$F(\eta,1)=\frac{1-\exp\left(-\sqrt{1+M}\eta\right)}{\sqrt{1+M}}. \tag{1.79}$$

1.3 Analytic solution by homotopy analysis method

We can express $F(\eta,\xi)$ by a set of base functions

$$\left\{\xi^k\eta^m\exp(-n\eta)\mid k\geq 0,\ n\geq 0,\ m\geq 0\right\}, \tag{1.80}$$

in the form of following series

$$F(\eta,\xi)=\sum_{k=0}^{+\infty}\sum_{m=0}^{+\infty}\sum_{n=1}^{+\infty}a_{m,n}^k\xi^k\eta^m\exp(-n\eta), \tag{1.81}$$

in which $a_{m,n}^k$ is a coefficient.

Invoking the Rule of solution expressions for $F(\eta,\xi)$ and with the help of boundary conditions we choose

$$F_0(\eta,\xi) = 1 - \exp(-\eta), \tag{1.82}$$

as the initial approximation to $F(\eta,\xi)$. We define the auxiliary linear operator as

$$\mathcal{L}[\phi(\eta,\xi;q)] = \frac{\partial^3\phi}{\partial\eta^3} - \frac{\partial\phi}{\partial\eta}, \tag{1.83}$$

which satisfies the following property

$$\mathcal{L}[C_1\exp(-\eta) + C_2\exp(\eta) + C_3] = 0, \tag{1.84}$$

where C_i $(i = 1-3)$ are the arbitrary constants.

1.3.1 The zeroth-order deformation equation

Let $\hbar$ be a non-zero auxiliary parameter and $q \in [0,1]$ is an embedding parameter then the zeroth order deformation problem is

$$(1-q)\,\mathcal{L}[\phi(\eta,\xi;q) - F_0(\eta,\xi)] = q\hbar N_1[\phi(\eta,\xi;q)], \tag{1.85}$$

$$\phi(0,\xi;q) = 0, \ \frac{\partial\phi(\eta,\xi;q)}{\partial\eta} = 1 \quad \text{at } \eta = 0, \tag{1.86}$$

$$\frac{\partial\phi(\eta,\xi;q)}{\partial\eta} = 0 \quad \text{as } \eta \to +\infty, \tag{1.87}$$

where N_1 is the non-linear operator defined as

$$N_1[\phi(\eta,\xi;q)] = (1-\xi)\left(\frac{\eta}{n+1}\frac{\partial^2\phi}{\partial\eta^2} - \xi\frac{\partial^2\phi}{\partial\eta\partial\xi}\right) + n\left(-\frac{\partial^2\phi}{\partial\eta^2}\right)^{n-1}\frac{\partial^3\phi}{\partial\eta^3}$$

$$+\xi\left[\frac{2n}{n+1}\phi\frac{\partial^2\phi}{\partial\eta^2} - \left(\frac{\partial\phi}{\partial\eta}\right)^2 - M\frac{\partial\phi}{\partial\eta}\right]. \tag{1.88}$$

Obviously for $q = 0$ and $q = 1$, we, respectively, have

$$\phi(\eta,\xi;0) = F_0(\eta,\xi) \text{ and } \phi(\eta,\xi;1) = F(\eta,\xi). \tag{1.89}$$

As q increases from zero to unity, $\phi(\eta, \xi; q)$ varies from initial guess $F_0(\eta, \xi)$ to final solution $F(\eta, \xi)$. By using the Taylor's theorem and Eq.(1.89) we can expand $\phi(\eta, \xi; q)$ in the Taylor's series of an embedding parameter q as follows

$$\phi(\eta, \xi; q) = \phi(\eta, \xi; 0) + \sum_{m=1}^{+\infty} F_m(\eta, \xi) q^m, \tag{1.90}$$

where

$$F_m(\eta, \xi) = \frac{1}{m!} \left. \frac{\partial^m \phi(\eta, \xi; q)}{\partial q^m} \right|_{q=0}. \tag{1.91}$$

The convergence of the series in Eq. (1.90) depends upon the value of $\hbar$. We properly choose the value of $\hbar$ such that the series mentioned in Eq. (1.90) is convergent at $q = 1$ then due to Eq. (1.89), the solution series is

$$F(\eta, \xi) = F_0(\eta, \xi) + \sum_{m=1}^{+\infty} F_m(\eta, \xi). \tag{1.92}$$

1.3.2 The mth-order deformation equation

We can obtain the mth-order deformation equation by differentiating the zeroth-order deformation Eq. (1.85) m times with respect to q and then dividing by $m!$ and finally setting $q = 0$, as follows

$$\mathcal{L}[F_m(\eta, \xi) - \chi_m F_{m-1}(\eta, \xi)] = \hbar R_{m1}(\vec{F}_{m-1}). \tag{1.93}$$

The boundary conditions for mth-order deformation equation are

$$F_m(0, \xi) = 0, \ \frac{\partial F_m(\eta, \xi)}{\partial \eta} = 0 \quad \text{at } \eta = 0, \tag{1.94}$$

$$\frac{\partial F_m(\eta, \xi)}{\partial \eta} = 0 \quad \text{as } \eta \to +\infty, \tag{1.95}$$

where in Eq. (1.93), $R_{m1}(\vec{F}_{m-1})$ and χ_m are given as

$$R_{m1}(\vec{F}_{m-1}) = \frac{1}{(m-1)!} \left. \frac{\partial^{m-1} N_1[\phi(\eta, \xi; q)]}{\partial q^{m-1}} \right|_{q=0}, \tag{1.96}$$

and

$$\chi_m = \begin{cases} 0, \ m \leq 1, \\ 1, \ m > 1. \end{cases} \tag{1.97}$$

As $R_{m1}(\vec{F}_{m-1})$ depends on the integer power-law index n. When $n = 1$, we have

$$\begin{aligned} R_{m1}(\vec{F}_{m-1}) \quad &= \quad (1-\xi)\left(\frac{\eta}{2}\frac{\partial^2 F_{m-1}}{\partial \eta^2} - \xi\frac{\partial^2 F_{m-1}}{\partial \eta \partial \xi}\right) + \frac{\partial^3 F_{m-1}}{\partial \eta^3} + \xi\left[\sum_{i=0}^{m-1} F_i \frac{\partial^2 F_{m-1-i}}{\partial \eta^2}\right. \\ &\left. - \sum_{i=0}^{m-1} \frac{\partial F_i}{\partial \eta}\frac{\partial F_{m-1-i}}{\partial \eta} - M\frac{\partial F_{m-1}}{\partial \eta}\right], \end{aligned} \tag{1.98}$$

when $n = 2$, it reads

$$\begin{aligned} R_{m1}(\vec{F}_{m-1}) \quad &= \quad (1-\xi)\left(\frac{\eta}{3}\frac{\partial^2 F_{m-1}}{\partial \eta^2} - \xi\frac{\partial^2 F_{m-1}}{\partial \eta \partial \xi}\right) - 2B^1_{m-1} + \xi\left[\sum_{i=0}^{m-1} \frac{4}{3} F_i \frac{\partial^2 F_{m-1-i}}{\partial \eta^2}\right. \\ &\left. - \sum_{i=0}^{m-1} \frac{\partial F_i}{\partial \eta}\frac{\partial F_{m-1-i}}{\partial \eta} - M\frac{\partial F_{m-1}}{\partial \eta}\right], \end{aligned} \tag{1.99}$$

when $n = 3$, it holds

$$\begin{aligned} R_{m1}(\vec{F}_{m-1}) \quad &= \quad (1-\xi)\left(\frac{\eta}{4}\frac{\partial^2 F_{m-1}}{\partial \eta^2} - \xi\frac{\partial^2 F_{m-1}}{\partial \eta \partial \xi}\right) + 3\sum_{i=0}^{m-1} A^1_i \frac{\partial^3 F_{m-1-i}}{\partial \eta^3} + \xi\left[\sum_{i=0}^{m-1} \frac{3}{2} F_i \frac{\partial^2 F_{m-1-i}}{\partial \eta^2}\right. \\ &\left. - \sum_{i=0}^{m-1} \frac{\partial F_i}{\partial \eta}\frac{\partial F_{m-1-i}}{\partial \eta} - M\frac{\partial F_{m-1}}{\partial \eta}\right], \end{aligned} \tag{1.100}$$

when $n = 4$, we get

$$\begin{aligned} R_{m1}(\vec{F}_{m-1}) \quad &= \quad (1-\xi)\left(\frac{\eta}{5}\frac{\partial^2 F_{m-1}}{\partial \eta^2} - \xi\frac{\partial^2 F_{m-1}}{\partial \eta \partial \xi}\right) - 4\sum_{i=0}^{m-1} A^1_i B^1_{m-1-i} + \xi\left[\sum_{i=0}^{m-1} \frac{8}{5} F_i \frac{\partial^2 F_{m-1-i}}{\partial \eta^2}\right. \\ &\left. - \sum_{i=0}^{m-1} \frac{\partial F_i}{\partial \eta}\frac{\partial F_{m-1-i}}{\partial \eta} - M\frac{\partial F_{m-1}}{\partial \eta}\right], \end{aligned} \tag{1.101}$$

where

$$A^1_j = \sum_{i=0}^{j} \frac{\partial^2 F_i}{\partial \eta^2}\frac{\partial^2 F_{j-1}}{\partial \eta^2}, \tag{1.102}$$

$$B_j^1 = \sum_{i=0}^{j} \frac{\partial^2 F_i}{\partial \eta^2} \frac{\partial^3 F_{j-1}}{\partial \eta^3}. \tag{1.103}$$

Let $F_m^* (\eta, \xi)$ be a special solution of Eq. (1.93) corresponding to the boundary conditions in Eqs. (1.94) and (1.95). We have its general solution given as

$$F_m (\eta, \xi) = F_m^* (\eta, \xi) + C_1 \exp(-\eta) + C_2 \exp(\eta) + C_3, \tag{1.104}$$

where the coefficients C_i $(i = 1 - 3)$ are determined by

$$C_2 = 0,\ C_1 = \left. \frac{\partial F_m^* (\eta, \xi)}{\partial \eta} \right|_{\eta=0},\ C_3 = -C_1 - F_m^* (0, \xi). \tag{1.105}$$

The linear Eq. (1.93) with the boundary conditions in Eq. (1.94) and (1.95), can be solved in the order $m = 1,\ 2,\ 3,\ ...$ by the help of the symbolic computation software like Mathematica.

1.4 Results and discussion

In figures 1.2 to 1.4, the effects of different values of the magnetic parameter M on the velocity profile are observed for different values of the power-law index. It is observed that the fluid velocity decreases near the plate by increasing the value of M. Thus, the boundary layer decreases for increasing value of the magnetic parameter.

The physical layout of the boundary layer structure which developes near the sheet can be observe by having a glimpse at the velocity profile. The behavior of velocity profile as a function of τ for some different values of M and n is shown in figures 1.5 to 1.8. These figures reveal that with the increase in the dimensionless time τ, the velocity profile and boundary layer thickness increases. Thus it is seen that as τ increases from zero to ∞, the velocity profile develop rapidly.

The variations of velocity profiles for different values of M at the same dimensionless time $\tau = 0.1$ and $\tau = 0.5$ when $n = 3$ are shown in figures 1.9 and 1.10, respectively. These figures elucidate that the velocity profile decreases for increasing values of the magnetic parameter M. Thus, it reveals that the flow for large value of the magnetic parameter develops more slowly.

The influence of the power-law index n on the velocity distribution for different values of the magnetic parameter M are shown in figures 1.11 and 1.12. These figures show the velocity profiles at $\tau = 0.1$ for some fixed value of M with increasing power-law index n. These figures depict that the velocity profile tends to steady state more quickly as n enlarges.

Figures 1.13 and 1.14 are plotted to see the variation of the skin friction coefficient for fixed values of either the power-law index n or the magnetic parameter M, respectively. Figure 1.13 indicates that as the values of the power-law index n increases, the skin friction coefficient increases at the same dimensionless time $\tau \in (0, +\infty)$. Further, it is observed through figure 1.14 that for fixed value of n and at the same dimensionless time τ, the skin friction coefficient increases as the value of magnetic parameter M enlarges. All the above analytic results are plotted at $\hbar = -1/4$.

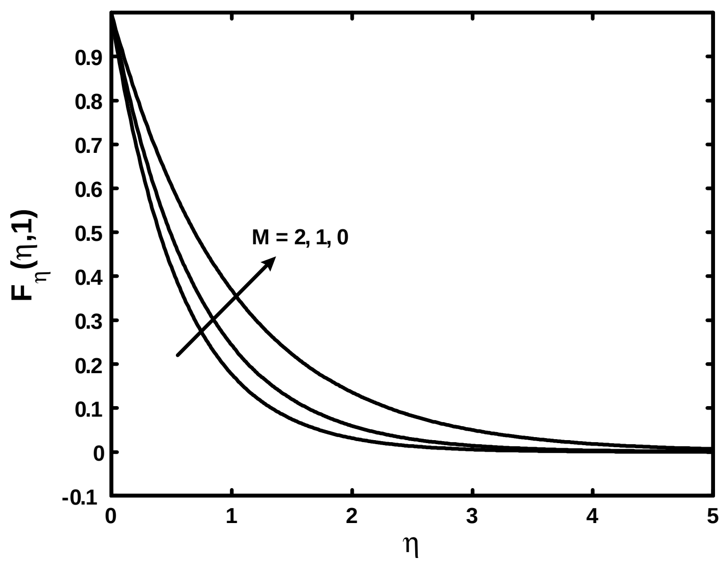

Figure 1.2: The variation of $\mathrm{F}_{\eta}(\eta, 1)$ when $M = 0,\ 1,\ 2$ and $n = 1$.

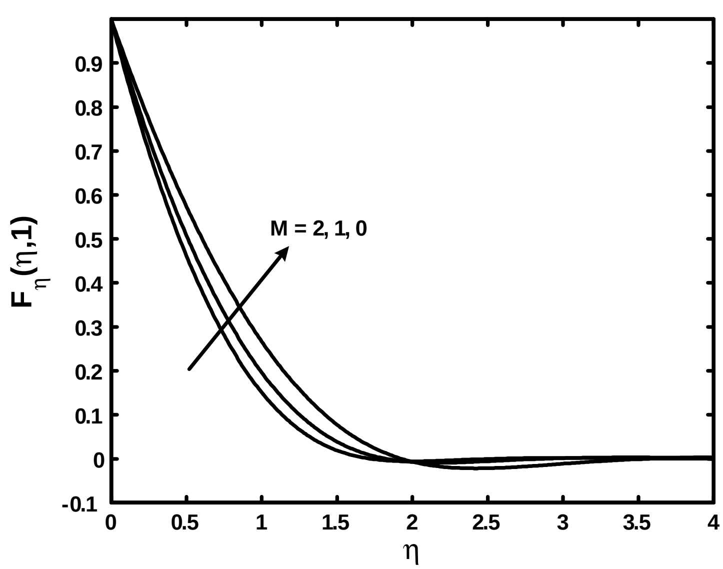

Figure 1.3: The variation of $\mathrm{F}_{\eta}(\eta, 1)$ when $M = 0,\ 1,\ 2$ and $n = 2$.

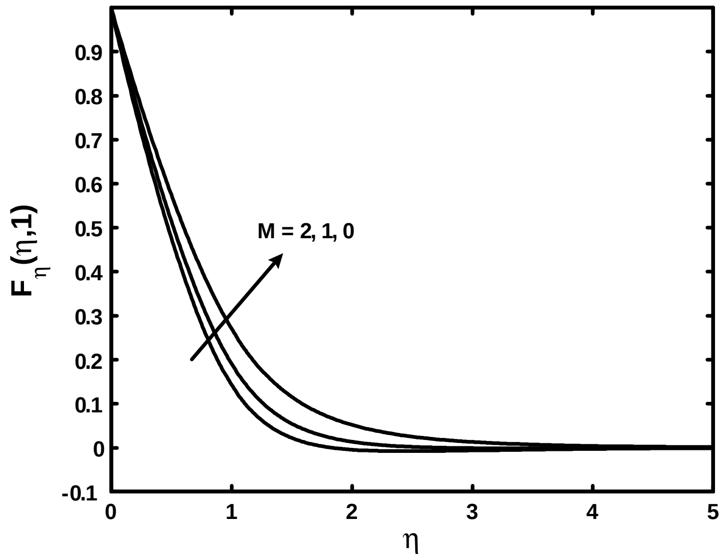

Figure 1.4: The variation of $F_\eta(\eta, 1)$ when $M = 0$, 1, 2 and $n = 3$.

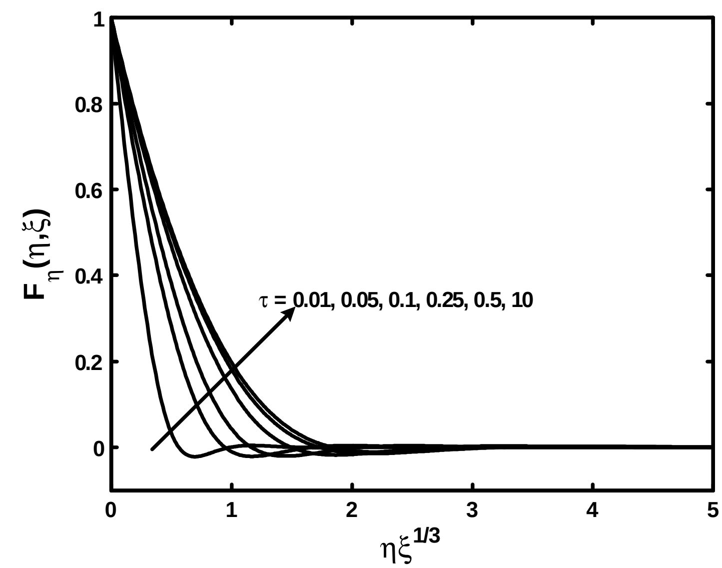

Figure 1.5: The variation of the velocity profile $F_\eta(\eta, \xi)$ when $n = 2$ and $M = 1$.

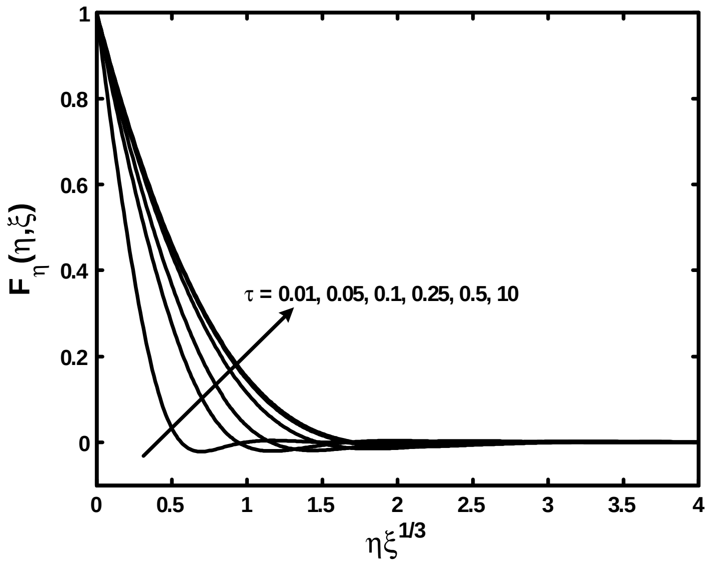

Figure 1.6: The variation of the velocity profile $F_\eta(\eta, \xi)$ when $n = 2$ and $M = 2$.

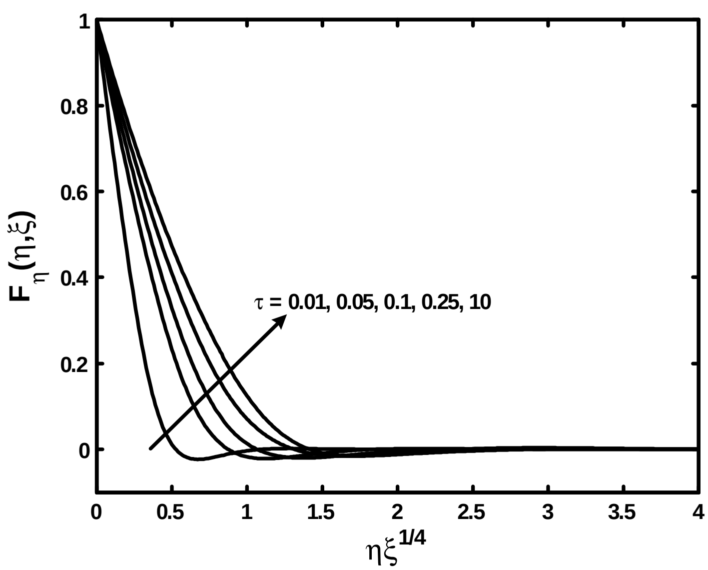

Figure 1.7: The variation of the velocity profile $F_\eta(\eta, \xi)$ when $n = 3$ and $M = 1$.

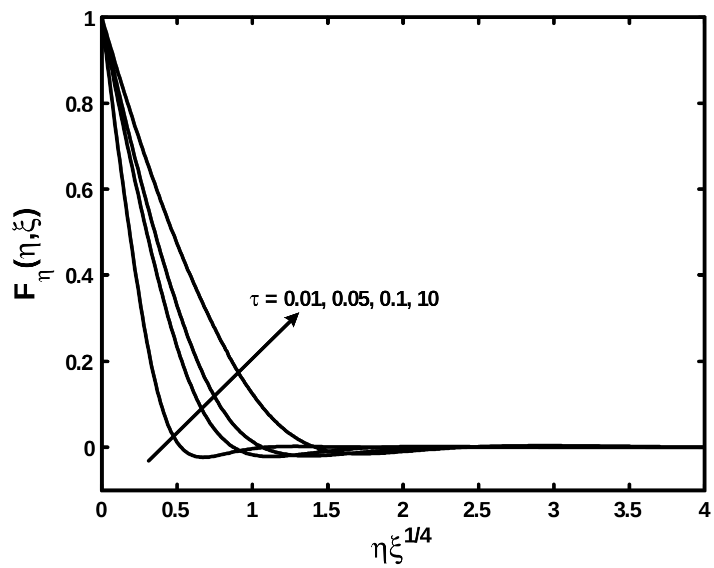

Figure 1.8: The variation of the velocity profile $\mathrm{F}_\eta(\eta, \xi)$ when $n = 3$ and $M = 2$.

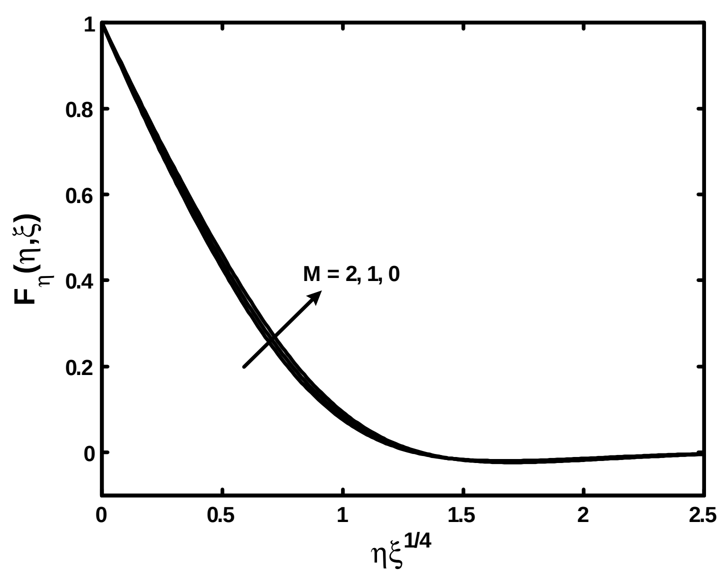

Figure 1.9: The variation of the velocity profile $\mathrm{F}_\eta(\eta, \xi)$ when $n = 3$ and $M = 0$, 1, 2 at $\tau = 0.1$.

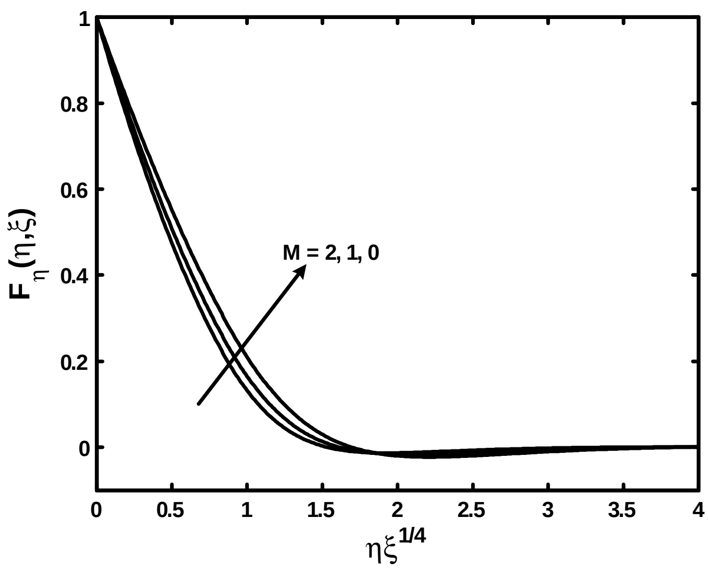

Figure 1.10: The velocity profile $\mathrm{F}_\eta(\eta, \xi)$ when $n = 3$ and $M = 0,\ 1,\ 2$ at $\tau = 0.5$.

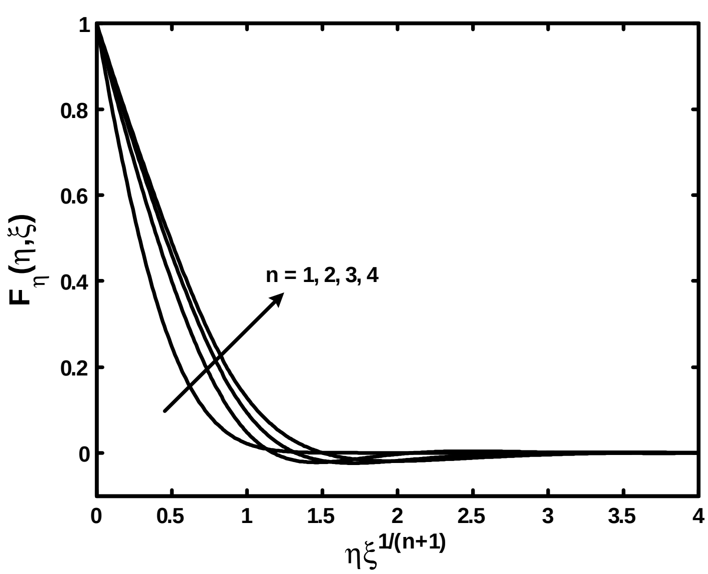

Figure 1.11: The velocity profile $\mathrm{F}_\eta(\eta, \xi)$ when $M = 0$ and $n = 1,\ 2,\ 3,\ 4$ at $\tau = 0.1$.

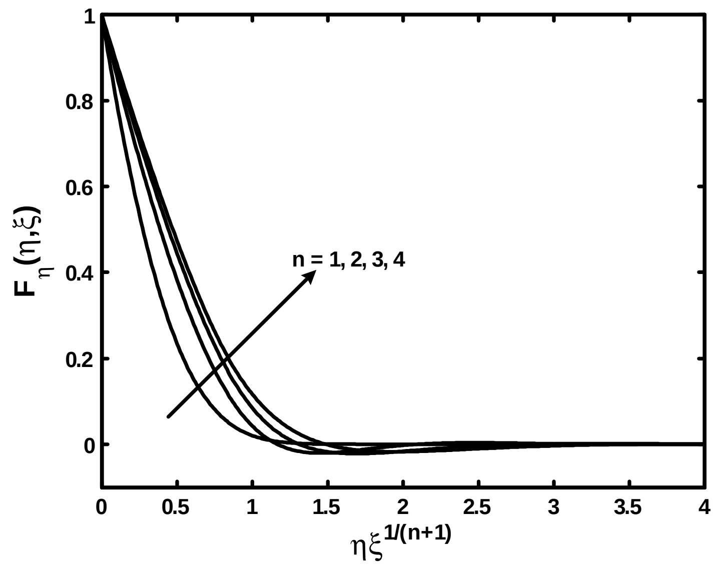

Figure 1.12: The velocity profile $\mathrm{F}_\eta(\eta, \xi)$ when $M = 1$ and $n = 1,\ 2,\ 3,\ 4$ at $\tau = 0.1$.

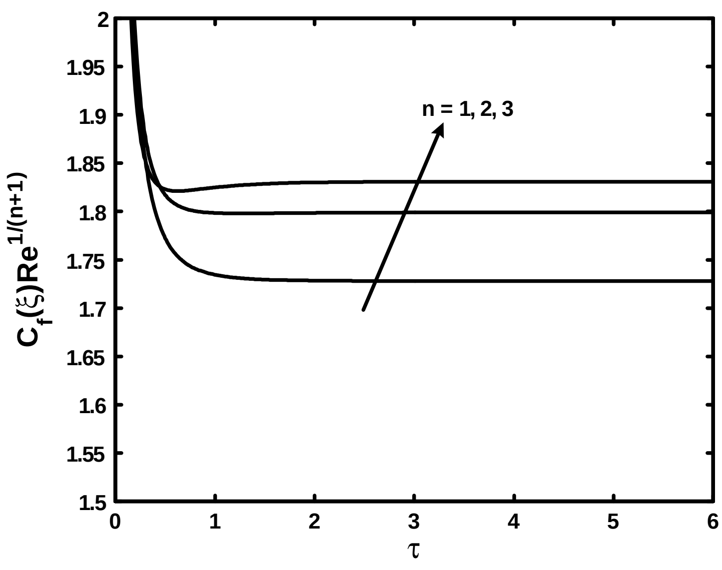

Figure 1.13: The variation of the skin friction coefficient as a function of τ for different power-law index n when $M = 2$.

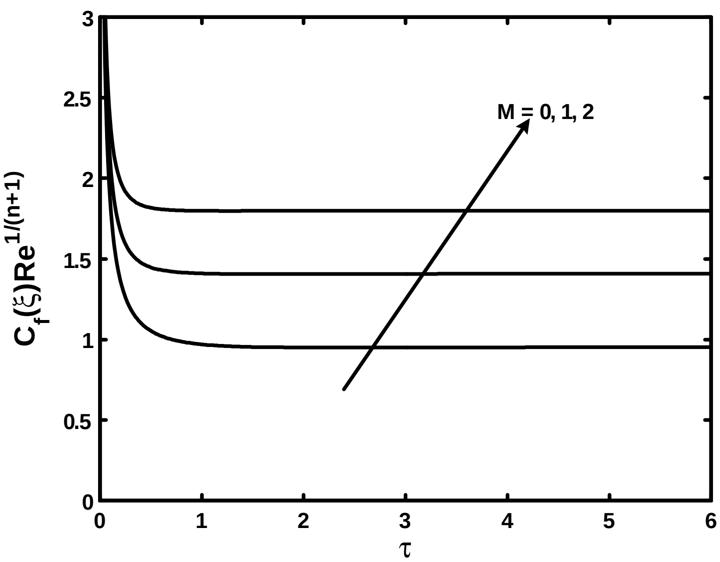

Figure 1.14: The variation of the skin friction coefficient as a function of τ for different parameter M when $n = 2$.

Chapter 2

Unsteady MHD axisymmetric flow of power-law fluids over a radially stretching sheet

2.1 Introduction

The aim of this chapter is to analyze the unsteady, axisymmetric MHD flow of power-law fluids over a radially stretching sheet. The governing partial differential equations in three independent variables are converted into a single highly non-linear partial differential equation in two independent variables. Analytic solution valid for all time, in the whole spatial region $0 \leq \eta < \infty$, has been derived by employing the homotopy analysis method (HAM). Finally, the influence of various emerging parameters are plotted and discussed in detail.

2.2 Mathematical formulation

Let we consider the unsteady, two-dimensional magnetohydrodynamic flow of an incompressible non-Newtonian fluid over a stretching sheet. The fluid obeys the power-law model in the presence of a transverse magnetic field. We take the cylindrical polar coordinates $(r,\ \theta,\ z)$ for mathematical modelling. The sheet is placed in the plane $z = 0$, i.e., in $r\theta-$plane and is stretched in the radial direction and the flow is confined to $z > 0$. The flow will have rotational symmetry due to which all the physical quantities will be independent of θ.

The governing equations of the problem are given by Eqs. (1.1) and (1.2) whereas the Cauchy stress tensor for power-law fluids is defined through Eqs. (1.4) and (1.5).

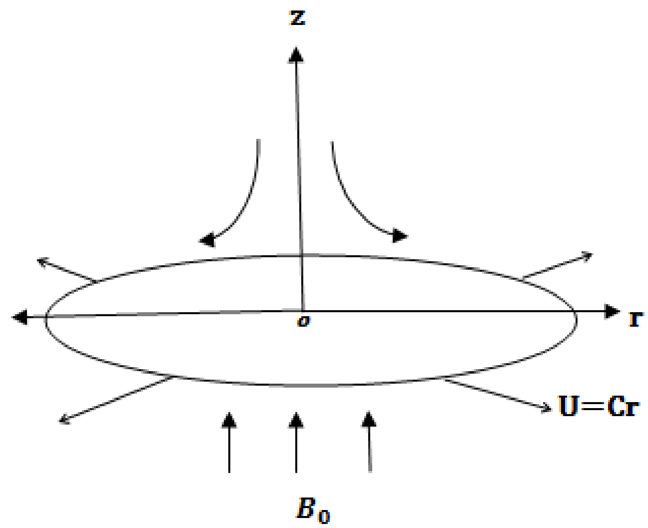

Figure 2.1: Physical model for the radial stretching sheet.

According to the assumption of axisymmetric flow, $v = 0$ so velocity field is defined as

$$\mathbf{V} = [u(r, z, t), 0, w(r, z, t)], \tag{2.1}$$

where u and w are the velocity components in the $r-$ and $z-$ directions, respectively, while t denotes the time.

As

$$\operatorname{grad}\mathbf{V} = \begin{bmatrix} \frac{\partial u}{\partial r} & \frac{1}{r}\frac{\partial u}{\partial \theta} - \frac{v}{r} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial r} & \frac{1}{r}\frac{\partial v}{\partial \theta} + \frac{u}{r} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial r} & \frac{1}{r}\frac{\partial w}{\partial \hat{\theta}} & \frac{\partial w}{\partial z} \end{bmatrix}. \tag{2.2}$$

For the present case, we have

$$\operatorname{grad}\mathbf{V}=\begin{bmatrix}\frac{\partial u}{\partial r} & 0 & \frac{\partial u}{\partial z}\\ 0 & \frac{u}{r} & 0\\ \frac{\partial w}{\partial r} & 0 & \frac{\partial w}{\partial z}\end{bmatrix}, \tag{2.3}$$

$$(\operatorname{grad}\mathbf{V})^{T}=\begin{bmatrix}\frac{\partial u}{\partial r} & 0 & \frac{\partial w}{\partial r}\\ 0 & \frac{u}{r} & 0\\ \frac{\partial u}{\partial z} & 0 & \frac{\partial w}{\partial z}\end{bmatrix}. \tag{2.4}$$

Using Eqs. (2.3) and (2.4) in Eq. (1.16) , we obtain

$$\mathbf{A}_1=\begin{bmatrix}2\frac{\partial u}{\partial r} & 0 & \frac{\partial u}{\partial z}+\frac{\partial w}{\partial r}\\ 0 & 2\frac{u}{r} & 0\\ \frac{\partial u}{\partial z}+\frac{\partial w}{\partial r} & 0 & 2\frac{\partial w}{\partial z}\end{bmatrix}, \tag{2.5}$$

$$\mathbf{A}_1^2=\begin{bmatrix}4\left(\frac{\partial u}{\partial r}\right)^2+\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial r}\right)^2 & 0 & 2\frac{\partial u}{\partial r}\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial r}\right)+2\frac{\partial w}{\partial z}\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial r}\right)\\ 0 & 4\left(\frac{u}{r}\right)^2 & 0\\ 2\frac{\partial u}{\partial r}\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial r}\right)+2\frac{\partial w}{\partial z}\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial r}\right) & 0 & 4\left(\frac{\partial w}{\partial z}\right)^2+\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial r}\right)^2\end{bmatrix}, \tag{2.6}$$

$$tr\left(\mathbf{A}_1^2\right)=4\left(\frac{\partial u}{\partial r}\right)^2+4\left(\frac{u}{r}\right)^2+4\left(\frac{\partial w}{\partial z}\right)^2+2\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial r}\right)^2, \tag{2.7}$$

$$\left|\frac{1}{2}tr\left(\mathbf{A}_1^2\right)\right|^{\frac{n-1}{2}}=\left|2\left(\frac{\partial u}{\partial r}\right)^2+2\left(\frac{u}{r}\right)^2+2\left(\frac{\partial w}{\partial z}\right)^2+\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial r}\right)^2\right|^{\frac{n-1}{2}}. \tag{2.8}$$

Putting Eqs. (2.5) and (2.8) in Eq. (1.15), we get

$$\mathbf{S}=K\left|2\left(\frac{\partial u}{\partial r}\right)^2+2\left(\frac{u}{r}\right)^2+2\left(\frac{\partial w}{\partial z}\right)^2+\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial r}\right)^2\right|^{\frac{n-1}{2}}\begin{bmatrix}2\frac{\partial u}{\partial r} & 0 & \frac{\partial u}{\partial z}+\frac{\partial w}{\partial r}\\ 0 & 2\frac{u}{r} & 0\\ \frac{\partial u}{\partial z}+\frac{\partial w}{\partial r} & 0 & 2\frac{\partial w}{\partial z}\end{bmatrix}. \tag{2.9}$$

By using the above result, Eq. (1.14) becomes

$$\mathbf{T}=-p\mathbf{I}+K\left|2\left(\frac{\partial u}{\partial r}\right)^2+2\left(\frac{u}{r}\right)^2+2\left(\frac{\partial w}{\partial z}\right)^2+\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial r}\right)^2\right|^{\frac{n-1}{2}}\begin{bmatrix} 2\frac{\partial u}{\partial r} & 0 & \frac{\partial u}{\partial z}+\frac{\partial w}{\partial r} \\ 0 & 2\frac{u}{r} & 0 \\ \frac{\partial u}{\partial z}+\frac{\partial w}{\partial r} & 0 & 2\frac{\partial w}{\partial z} \end{bmatrix}. \tag{2.10}$$

The Cauchy stress tensor can also be written as

$$\mathbf{T}=\begin{bmatrix} \sigma_{rr} & \tau_{r\theta} & \tau_{rz} \\ \tau_{\theta r} & \sigma_{\theta\theta} & \tau_{\theta z} \\ \tau_{zr} & \tau_{z\theta} & \sigma_{zz} \end{bmatrix}, \tag{2.11}$$

where, σ_{rr}, $\sigma_{\theta\theta}$, σ_{zz} denote the normal stresses and $\tau_{r\theta}$, $\tau_{\theta r}$, $\tau_{z\theta}$ are the shear stresses.

On comparing Eqs. (2.10) and (2.11), we get the following results

$$\tau_{rz}=\tau_{zr}=K\left|2\left(\frac{\partial u}{\partial r}\right)^2+2\left(\frac{u}{r}\right)^2+2\left(\frac{\partial w}{\partial z}\right)^2+\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial r}\right)^2\right|^{\frac{n-1}{2}}\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial r}\right), \tag{2.12}$$

$$\tau_{r\theta}=\tau_{\theta r}=\tau_{\theta z}=\tau_{z\theta}=0, \tag{2.13}$$

$$\sigma_{rr}=-p+2K\left|2\left(\frac{\partial u}{\partial r}\right)^2+2\left(\frac{u}{r}\right)^2+2\left(\frac{\partial w}{\partial z}\right)^2+\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial r}\right)^2\right|^{\frac{n-1}{2}}\frac{\partial u}{\partial r}, \tag{2.14}$$

$$\sigma_{zz}=-p+2K\left|2\left(\frac{\partial u}{\partial r}\right)^2+2\left(\frac{u}{r}\right)^2+2\left(\frac{\partial w}{\partial z}\right)^2+\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial r}\right)^2\right|^{\frac{n-1}{2}}\frac{\partial w}{\partial z}, \tag{2.15}$$

$$\sigma_{\theta\theta}=p+2K\left|2\left(\frac{\partial u}{\partial r}\right)^2+2\left(\frac{u}{r}\right)^2+2\left(\frac{\partial w}{\partial z}\right)^2+\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial r}\right)^2\right|^{\frac{n-1}{2}}\frac{u}{r}. \tag{2.16}$$

For the case of two-dimensional flow, the equation of continuity (1.1) reduces to

$$\frac{\partial u}{\partial r}+\frac{u}{r}+\frac{\partial w}{\partial z}=0. \tag{2.17}$$

From Eq. (1.2), we find the equation of radial momentum as

$$\rho\left(\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial r}+w\frac{\partial u}{\partial z}\right)=\frac{1}{r}\frac{\partial\left(r\sigma_{rr}\right)}{\partial r}+\frac{\partial\tau_{rz}}{\partial z}-\sigma B_0^2u, \tag{2.18}$$

and the equation of axial momentum as

$$\rho\left(\frac{\partial w}{\partial t}+u\frac{\partial w}{\partial r}+v\frac{\partial w}{\partial z}\right)=\frac{1}{r}\frac{\partial\left(r\tau_{zr}\right)}{\partial r}+\frac{\partial\sigma_{zz}}{\partial z}. \tag{2.19}$$

Using the relations mentioned in Eqs. (2.12) to (2.16), the $r-$ and $z-$components of the momentum equation become

$$\begin{aligned}\rho\left(\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial r}+w\frac{\partial u}{\partial z}\right) &= -\frac{\partial p}{\partial r}+2\frac{K}{r}\frac{\partial}{\partial r}\left[r\frac{\partial u}{\partial r}\left|\begin{array}{c}4\left(\frac{\partial u}{\partial r}\right)^2+4\left(\frac{u}{r}\right)^2\\ +4\frac{u}{r}\frac{\partial u}{\partial r}+\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial r}\right)^2\end{array}\right|^{\frac{n-1}{2}}\right]\\ &+K\frac{\partial}{\partial z}\left[\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial r}\right)\left|\begin{array}{c}4\left(\frac{\partial u}{\partial r}\right)^2+4\left(\frac{u}{r}\right)^2\\ +4\frac{u}{r}\frac{\partial u}{\partial r}+\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial r}\right)^2\end{array}\right|^{\frac{n-1}{2}}\right]\\ &-\sigma B_0^2 u,\end{aligned} \tag{2.20}$$

$$\begin{aligned}\rho\left(\frac{\partial w}{\partial t}+u\frac{\partial w}{\partial r}+w\frac{\partial w}{\partial z}\right) &= -\frac{\partial p}{\partial z}+\frac{K}{r}\frac{\partial}{\partial r}\left[r\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial r}\right)\left|\begin{array}{c}4\left(\frac{\partial u}{\partial r}\right)^2+4\left(\frac{u}{r}\right)^2\\ +4\frac{u}{r}\frac{\partial u}{\partial r}+\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial r}\right)^2\end{array}\right|^{\frac{n-1}{2}}\right]\\ &+2K\frac{\partial}{\partial z}\left[\frac{\partial w}{\partial z}\left|\begin{array}{c}4\left(\frac{\partial u}{\partial r}\right)^2+4\left(\frac{u}{r}\right)^2+4\frac{u}{r}\frac{\partial u}{\partial r}\\ +\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial r}\right)^2\end{array}\right|^{\frac{n-1}{2}}\right].\end{aligned} \tag{2.21}$$

The above equations can be written in dimensionless form by using the following relations

$$u^*=\frac{u}{U},\ w^*=\frac{w}{U},\ r^*=\frac{r}{L},\ z^*=\frac{z}{L},\ p^*=\frac{p}{\rho U^2}\text{ and }t^*=\frac{tU}{L}, \tag{2.22}$$

where L is the characteristic length and U the stretching velocity.

Upon making use of Eq. (2.22) in Eqs. (2.17), (2.20) and (2.21) we obtain

$$\frac{\partial u^*}{\partial r^*}+\frac{u^*}{r^*}+\frac{\partial w^*}{\partial z^*}=0, \tag{2.23}$$

$$\frac{\partial u^*}{\partial t^*}+u^*\frac{\partial u^*}{\partial r^*}+w^*\frac{\partial u^*}{\partial z^*} = -\frac{\partial p^*}{\partial r^*}+2\epsilon\frac{1}{r^*}\frac{\partial}{\partial r^*}\left[r^*\frac{\partial u^*}{\partial r^*}\left|\begin{array}{c}4\left(\frac{\partial u^*}{\partial r^*}\right)^2+4\left(\frac{u^*}{r^*}\right)^2\\+4\frac{u^*}{r^*}\frac{\partial u^*}{\partial r^*}+\left(\frac{\partial u^*}{\partial z^*}+\frac{\partial w^*}{\partial r^*}\right)^2\end{array}\right|^{\frac{n-1}{2}}\right]$$

$$+\epsilon\frac{\partial}{\partial z^*}\left[\left(\frac{\partial u^*}{\partial z^*}+\frac{\partial w^*}{\partial r^*}\right)\left|\begin{array}{c}4\left(\frac{\partial u^*}{\partial r^*}\right)^2+4\left(\frac{u^*}{r^*}\right)^2\\+4\frac{u^*}{r^*}\frac{\partial u^*}{\partial r^*}+\left(\frac{\partial u^*}{\partial z^*}+\frac{\partial w^*}{\partial r^*}\right)^2\end{array}\right|^{\frac{n-1}{2}}\right]$$

$$-Hau^*, \tag{2.24}$$

$$\frac{\partial w^*}{\partial t^*}+u^*\frac{\partial w^*}{\partial r^*}+w^*\frac{\partial w^*}{\partial z^*} = -\frac{\partial p^*}{\partial z^*}+\epsilon\frac{1}{r^*}\frac{\partial}{\partial r^*}\left[r^*\left(\frac{\partial u^*}{\partial z^*}+\frac{\partial w^*}{\partial r^*}\right)\left|\begin{array}{c}4\left(\frac{\partial u^*}{\partial r^*}\right)^2+4\left(\frac{u^*}{r^*}\right)^2\\+4\frac{u^*}{r^*}\frac{\partial u^*}{\partial r^*}+\left(\frac{\partial u^*}{\partial z^*}+\frac{\partial w^*}{\partial r^*}\right)^2\end{array}\right|^{\frac{n-1}{2}}\right]$$

$$+2\epsilon\frac{\partial}{\partial z^*}\left[\frac{\partial w^*}{\partial z^*}\left|\begin{array}{c}4\left(\frac{\partial u^*}{\partial r^*}\right)^2+4\left(\frac{u^*}{r^*}\right)^2\\+4\frac{u^*}{r^*}\frac{\partial u^*}{\partial r^*}+\left(\frac{\partial u^*}{\partial z^*}+\frac{\partial w^*}{\partial r^*}\right)^2\end{array}\right|^{\frac{n-1}{2}}\right], \tag{2.25}$$

where the dimensionless parameters ϵ and Ha are defined as

$$\epsilon=\frac{K}{\rho LU}\left(\frac{U}{L}\right)^{n-1}, \tag{2.26}$$

$$Ha=\frac{\sigma}{\rho}\frac{L}{U}B_0^2. \tag{2.27}$$

Under the standard boundary layer assumptions, t, r, p and u are of order 1 while w and z are of order δ, where δ is the boundary layer thickness. The non-dimensional parameter ϵ is of order δ^{n+1}. Therefore the boundary layer approximation of Eqs. (2.23), (2.24) and (2.25) in the dimensional form yields

$$\frac{\partial u}{\partial r}+\frac{u}{r}+\frac{\partial w}{\partial z}=0, \tag{2.28}$$

,

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial r}+w\frac{\partial u}{\partial z}=-\frac{1}{\rho}\frac{\partial p}{\partial r}+\frac{K}{\rho}\frac{\partial}{\partial z}\left(\left|\frac{\partial u}{\partial z}\right|^{n-1}\frac{\partial u}{\partial z}\right)-\left(\frac{\sigma B_0^2}{\rho}\right)u, \tag{2.29}$$

$$0=-\frac{1}{\rho}\frac{\partial p}{\partial z}. \tag{2.30}$$

As $z\to\infty$, no disturbance is found above the boundary layer and the velocity is zero there and hence

$$t > 0: \qquad u \to 0 \text{ as } z \to +\infty, \tag{2.31}$$

and

$$0 = -\frac{1}{\rho}\frac{\partial p}{\partial r}. \tag{2.32}$$

By using Eq. (2.32), the pressure gradient term in Eq. (2.29) is set equal to zero. Thus we obtain the following governing equation

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial r} + w\frac{\partial u}{\partial z} = \frac{K}{\rho}\frac{\partial}{\partial z}\left(\left|\frac{\partial u}{\partial z}\right|^{n-1}\frac{\partial u}{\partial z}\right) - \left(\frac{\sigma B_0^2}{\rho}\right)u. \tag{2.33}$$

Here we take $\frac{\partial u}{\partial z} < 0$ (c.f. Chapter 1), so Eq. (2.33) result in

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial r} + w\frac{\partial u}{\partial z} = -\frac{K}{\rho}\frac{\partial}{\partial z}\left(-\frac{\partial u}{\partial z}\right)^n - \left(\frac{\sigma B_0^2}{\rho}\right)u. \tag{2.34}$$

Thus the governing equations for the present problem are as follows

$$\frac{\partial u}{\partial r} + \frac{u}{r} + \frac{\partial w}{\partial z} = 0, \tag{2.35}$$

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial r} + w\frac{\partial u}{\partial z} = -\frac{K}{\rho}\frac{\partial}{\partial z}\left(-\frac{\partial u}{\partial z}\right)^n - \left(\frac{\sigma B_0^2}{\rho}\right)u. \tag{2.36}$$

The relevant initial and boundary conditions are:

$$t \le 0: \qquad u = w = 0 \quad \text{ at } z \ge 0,\ -\infty < r < +\infty, \tag{2.37}$$

$$t > 0: \qquad u = Cr,\ w = 0 \quad \text{ at } z = 0, \tag{2.38}$$

$$t > 0: \qquad u \to 0 \quad \text{ as } z \to +\infty, \tag{2.39}$$

where C is a positive constant.

Let Ψ denotes the stream function defined as

$$\Psi = \left(\frac{K\xi}{\rho C^{1-2n}}\right)^{\frac{1}{n+1}} r^{\frac{(3n+1)}{(n+1)}} F(\eta, \xi), \tag{2.40}$$

with

$$\eta = z\left(\frac{\rho C^{2-n}}{K\xi}\right)^{\frac{1}{n+1}} r^{\frac{(1-n)}{(1+n)}}, \tag{2.41}$$

$$\xi = 1-\exp\left(-\tau\right), \quad \tau = Ct, \tag{2.42}$$

where η and ξ are the transformed non-dimensional independent variables and τ the non-dimensional time and

$$u = -\frac{1}{r}\frac{\partial \Psi}{\partial z} \text{ and } w = \frac{1}{r}\frac{\partial \Psi}{\partial r}. \tag{2.43}$$

Using the transformations defined in Eqs. (2.40) – (2.42), Eq. (2.43) results in

$$u = Cr\frac{\partial F\left(\eta,\xi\right)}{\partial \eta}, \tag{2.44}$$

$$w = -\left(\frac{3n+1}{n+1}\right)\left(\frac{K\xi}{\rho C^{1-2n}}\right)^{\frac{1}{n+1}} r^{\frac{(n-1)}{(n+1)}} F\left(\eta,\xi\right) - \left(\frac{1-n}{1+n}\right) Cz\frac{\partial F\left(\eta,\xi\right)}{\partial \eta}. \tag{2.45}$$

By the help of Eqs. (2.44) and (2.45), we can calculate the following quantities

$$\frac{\partial u}{\partial t} = C^2 r\left(1-\xi\right)\frac{\partial^2 F\left(\eta,\xi\right)}{\partial \eta \partial \xi} - \left(\frac{1}{1+n}\right) C^2 r\eta\xi^{-1}\left(1-\xi\right)\frac{\partial^2 F\left(\eta,\xi\right)}{\partial \eta^2}, \tag{2.46}$$

$$\frac{\partial u}{\partial r} = C\frac{\partial F\left(\eta,\xi\right)}{\partial \eta} + C\eta\left(\frac{1-n}{1+n}\right)\frac{\partial^2 F\left(\eta,\xi\right)}{\partial \eta^2}, \tag{2.47}$$

$$\frac{\partial u}{\partial z} = Cr\left(\frac{\rho C^{2-n}}{K\xi}\right)^{\frac{1}{n+1}} r^{\frac{(1-n)}{(1+n)}}\frac{\partial^2 F\left(\eta,\xi\right)}{\partial \eta^2}, \tag{2.48}$$

$$\frac{\partial \tau_{rz}}{\partial z} = -nC^2\rho r\xi^{-1}\left(-\frac{\partial^2 F\left(\eta,\xi\right)}{\partial \eta^2}\right)^{n-1}\frac{\partial^3 F\left(\eta,\xi\right)}{\partial \eta^3}. \tag{2.49}$$

In view of Eqs. (2.44) to (2.49), the continuity equation (2.35) is identically satisfied and Eq. (2.36) takes the form

$$\left(1-\xi\right)\left(\frac{\eta}{n+1}\frac{\partial^2 F}{\partial \eta^2} - \xi\frac{\partial^2 F}{\partial \eta \partial \xi}\right) + \xi\left[\frac{3n+1}{n+1}F\frac{\partial^2 F}{\partial \eta^2} - \left(\frac{\partial F}{\partial \eta}\right)^2 - M\frac{\partial F}{\partial \eta}\right]$$

$$+n\left(-\frac{\partial^2 F}{\partial \eta^2}\right)^{n-1}\frac{\partial^3 F}{\partial \eta^3} = 0, \tag{2.50}$$

where F is the dimensionless stream function, $\frac{\partial F}{\partial \eta}$ the dimensionless velocity and M the magnetic parameter defined by

$$M = \frac{\sigma B_0^2}{\rho C}. \tag{2.51}$$

The transformed boundary conditions are as follows

$$F(0, \xi) = 0, \tag{2.52}$$

$$\frac{\partial F(\eta, \xi)}{\partial \eta} = 1 \quad \text{at } \eta = 0, \tag{2.53}$$

$$\frac{\partial F(\eta, \xi)}{\partial \eta} = 0 \quad \text{as } \eta \to \infty. \tag{2.54}$$

We know that the skin friction coefficient $C_f(\xi)$ at the wall is given by

$$C_f(\xi) = \frac{\tau_w}{\rho (Cr)^2} = \xi^{-\frac{n}{(n+1)}} \left[-F_{\eta\eta}(0, \xi)\right]^n \mathrm{Re}^{-\frac{1}{(n+1)}}. \tag{2.55}$$

where τ_w is the local wall shear stress and Re the local Reynolds number defined by

$$\mathrm{Re} = (Cr)^{2-n} \frac{r^n}{\left(\frac{K}{\rho}\right)}. \tag{2.56}$$

2.3 Analytic solution by homotopy analysis method

We can express the velocity distribution $F(\eta, \xi)$ by a set of base functions of the form

$$\left\{\xi^k \eta^m \exp(-n\eta) \mid k \geq 0, \, n \geq 0, \, m \geq 0\right\}, \tag{2.57}$$

in the form of following series

$$F(\eta, \xi) = \sum_{k=0}^{+\infty} \sum_{m=0}^{+\infty} \sum_{n=1}^{+\infty} a_{m,n}^k \xi^k \eta^m \exp(-n\eta), \tag{2.58}$$

in which $a_{m,n}^k$ is a coefficient. By the Rule of solution expressions for $F(\eta, \xi)$ and with the help of boundary conditions, we choose $F_0(\eta, \xi)$ as the initial approximation to $F(\eta, \xi)$

$$F_0(\eta, \xi) = 1 - \exp(-\eta). \tag{2.59}$$

We further define the following auxiliary linear operator

$$\mathcal{L}\left[\phi\left(\eta,\xi;q\right)\right]=\frac{\partial^3\phi}{\partial\eta^3}-\frac{\partial\phi}{\partial\eta}, \tag{2.60}$$

which satisfies the property given as

$$\mathcal{L}\left[C_4\exp\left(-\eta\right)+C_5\exp\left(\eta\right)+C_6\right]=0, \tag{2.61}$$

where C_i $(i=4-6)$ are the arbitrary constants.

2.3.1 The zeroth-order deformation equation

Let $\hbar$ be a non-zero auxiliary parameter and $q\in[0,1]$ is an embedding parameter then the zeroth order deformation problem satisfies

$$\left(1-q\right)\mathcal{L}\left[\phi\left(\eta,\xi;q\right)-F_0\left(\eta,\xi\right)\right]=q\hbar N_2\left[\phi\left(\eta,\xi;q\right)\right], \tag{2.62}$$

$$\phi\left(0,\xi;q\right)=0,\ \frac{\partial\phi\left(\eta,\xi;q\right)}{\partial\eta}=1\quad\text{at } \eta=0, \tag{2.63}$$

$$\frac{\partial\phi\left(\eta,\xi;q\right)}{\partial\eta}=0\quad\text{as } \eta\rightarrow+\infty. \tag{2.64}$$

where N_2 is the non-linear operator given by

$$\begin{aligned}N_2\left[\phi\left(\eta,\xi;q\right)\right] &= \left(1-\xi\right)\left(\frac{\eta}{n+1}\frac{\partial^2\phi}{\partial\eta^2}-\xi\frac{\partial^2\phi}{\partial\eta\partial\xi}\right)+n\left(-\frac{\partial^2\phi}{\partial\eta^2}\right)^{n-1}\frac{\partial^3\phi}{\partial\eta^3}\\ &\quad+\xi\left[\frac{3n+1}{n+1}\phi\frac{\partial^2\phi}{\partial\eta^2}-\left(\frac{\partial\phi}{\partial\eta}\right)^2-M\frac{\partial\phi}{\partial\eta}\right].\end{aligned} \tag{2.65}$$

For $q=0$ and $q=1$, we, respectively, have

$$\phi\left(\eta,\xi;0\right)=F_0\left(\eta,\xi\right),\ \ \phi\left(\eta,\xi;1\right)=F\left(\eta,\xi\right). \tag{2.66}$$

As q increases from zero to unity, $\phi\left(\eta,\xi;q\right)$ deforms from initial guess $F_0\left(\eta,\xi\right)$ to final solution $F\left(\eta,\xi\right)$. Then by Taylor's theorem and Eq.(2.66) we can expand $\phi\left(\eta,\xi;q\right)$ in the Taylor's series of an embedding parameter q as

$$\phi(\eta,\xi;q)=\phi(\eta,\xi;0)+\sum_{m=1}^{+\infty}F_m(\eta,\xi)\,q^m, \tag{2.67}$$

where

$$F_m(\eta,\xi)=\frac{1}{m!}\left.\frac{\partial^m\phi(\eta,\xi;q)}{\partial q^m}\right|_{q=0}. \tag{2.68}$$

Since the convergence of the series (2.67) depends upon $\hbar$ so we properly choose the value of $\hbar$ in such a way that the series (2.67) is convergent at $q=1$ and due to Eq. (2.66), the solution is

$$F(\eta,\xi)=F_0(\eta,\xi)+\sum_{m=1}^{+\infty}F_m(\eta,\xi). \tag{2.69}$$

2.3.2 The mth-order deformation equation

Differentiating the zeroth-order deformation Eq. (2.62) m times with respect to q and then dividing by $m!$ and finally setting $q=0$, we obtain the mth-order deformation equation given as

$$\mathcal{L}[F_m(\eta,\xi)-\chi_m F_{m-1}(\eta,\xi)]=\hbar R_{m2}(\vec{F}_{m-1}). \tag{2.70}$$

The boundary conditions for mth order deformation equation are

$$F_m(0,\xi)=0,\quad \left.\frac{\partial F_m(\eta,\xi)}{\partial\eta}\right|_{\eta=0}=0, \tag{2.71}$$

$$\left.\frac{\partial F_m(\eta,\xi)}{\partial\eta}\right|_{\eta\to+\infty}=0, \tag{2.72}$$

where in the above equation, $R_{m2}(\vec{F}_{m-1})$ is given by

$$R_{m2}(\vec{F}_{m-1})=\frac{1}{(m-1)!}\left.\frac{\partial^{m-1}N_2[\phi(\eta,\xi;q)]}{\partial q^{m-1}}\right|_{q=0}, \tag{2.73}$$

Since $R_{m2}(\vec{F}_{m-1})$ depends on the power-law index n. For $n = 1$, we have

$$\begin{aligned} R_{m2}(\vec{F}_{m-1}) &= (1-\xi)\left(\frac{\eta}{2}\frac{\partial^2 F_{m-1}}{\partial\eta^2} - \xi\frac{\partial^2 F_{m-1}}{\partial\eta\partial\xi}\right) + \frac{\partial^3 F_{m-1}}{\partial\eta^3} + \xi\left[\sum_{i=0}^{m-1} 2F_i\frac{\partial^2 F_{m-1-i}}{\partial\eta^2}\right. \\ &\left. - \sum_{i=0}^{m-1}\frac{\partial F_i}{\partial\eta}\frac{\partial F_{m-1-i}}{\partial\eta} - M\frac{\partial F_{m-1}}{\partial\eta}\right], \end{aligned} \tag{2.74}$$

when $n = 2$, it gives

$$\begin{aligned} R_{m2}(\vec{F}_{m-1}) &= (1-\xi)\left(\frac{\eta}{3}\frac{\partial^2 F_{m-1}}{\partial\eta^2} - \xi\frac{\partial^2 F_{m-1}}{\partial\eta\partial\xi}\right) - 2B_{m-1}^2 + \xi\left[\sum_{i=0}^{m-1}\frac{7}{3}F_i\frac{\partial^2 F_{m-1-i}}{\partial\eta^2}\right. \\ &\left. - \sum_{i=0}^{m-1}\frac{\partial F_i}{\partial\eta}\frac{\partial F_{m-1-i}}{\partial\eta} - M\frac{\partial F_{m-1}}{\partial\eta}\right], \end{aligned} \tag{2.75}$$

when $n = 3$, it reads

$$\begin{aligned} R_{m2}(\vec{F}_{m-1}) &= (1-\xi)\left(\frac{\eta}{4}\frac{\partial^2 F_{m-1}}{\partial\eta^2} - \xi\frac{\partial^2 F_{m-1}}{\partial\eta\partial\xi}\right) + 3\sum_{i=0}^{m-1} A_i^2\frac{\partial^3 F_{m-1-i}}{\partial\eta^3} + \xi\left[\sum_{i=0}^{m-1}\frac{5}{2}F_i\frac{\partial^2 F_{m-1-i}}{\partial\eta^2}\right. \\ &\left. - \sum_{i=0}^{m-1}\frac{\partial F_i}{\partial\eta}\frac{\partial F_{m-1-i}}{\partial\eta} - M\frac{\partial F_{m-1}}{\partial\eta}\right], \end{aligned} \tag{2.76}$$

when $n = 4$, we obtain

$$\begin{aligned} R_{m2}(\vec{F}_{m-1}) &= (1-\xi)\left(\frac{\eta}{5}\frac{\partial^2 F_{m-1}}{\partial\eta^2} - \xi\frac{\partial^2 F_{m-1}}{\partial\eta\partial\xi}\right) - 4\sum_{i=0}^{m-1} A_i^2 B_{m-1-i}^2 + \xi\left[\sum_{i=0}^{m-1}\frac{13}{5}F_i\frac{\partial^2 F_{m-1-i}}{\partial\eta^2}\right. \\ &\left. - \sum_{i=0}^{m-1}\frac{\partial F_i}{\partial\eta}\frac{\partial F_{m-1-i}}{\partial\eta} - M\frac{\partial F_{m-1}}{\partial\eta}\right], \end{aligned} \tag{2.77}$$

where

$$A_j^2 = \sum_{i=0}^{j}\frac{\partial^2 F_i}{\partial\eta^2}\frac{\partial^2 F_{j-1}}{\partial\eta^2}, \tag{2.78}$$

$$B_j^2 = \sum_{i=0}^{j}\frac{\partial^2 F_i}{\partial\eta^2}\frac{\partial^3 F_{j-1}}{\partial\eta^3}. \tag{2.79}$$

The general solution of Eq. (2.70) corresponding to the boundary conditions in Eqs. (2.71) and (2.72) is given by

$$F_m(\eta,\xi) = F_m^*(\eta,\xi) + C_4 \exp(-\eta) + C_5 \exp(\eta) + C_6, \tag{2.80}$$

where $F_m^*(\eta,\xi)$ be a special solution and the constants C_i $(i = 4-6)$ are determined by

$$C_5 = 0,\;\; C_4 = \left.\frac{\partial F_m^*(\eta,\xi)}{\partial \eta}\right|_{\eta=0},\; C_6 = -C_4 - F_m^*(0,\xi). \tag{2.81}$$

The linear differential Eq. (2.70) with the boundary conditions (2.71) and (2.72) can be solved in the order $m = 1, 2, 3, \ldots$ with the help of any symbolic computation software like Mathematica.

In this work the optimal values of the convergence control parameter is choosen by minimizing the discrete squared residual given by

$$E_{f,m} = \frac{1}{N+1}\sum_{j=0}^{N}\left[N_f\left(\sum_{i=0}^{m} F_j(i \triangle \eta)\right)\right]^2. \tag{2.82}$$

2.4 Results and discussion

In this chapter, we have considered the axisymmetric flow of an MHD power-law fluids over a radially stretching sheet. The analytical solution of the time-dependent non-linear problem is constructed using the homotopy analysis method. The effects of various physical parameters are studied and shown graphically.

The effects of different values of the magnetic parameter M on the velocity profile are observed for the different value of the power-law index through the figures 2.2 to 2.4. An inspection of these figures reveals that the magnetic parameter M plays an important role. From the figures, it is noted that the velocity profile decreases near the plate by increasing value of M for fixed value of n. Consequently, the boundary layer thickness decreases substantially.

Figures 2.5 to 2.8 depict the behavior of velocity profile as a function of τ for different values of M and n. By analyzing the graphs, it reveals that as τ increases from zero to ∞, the velocity profiles develop rapidly. These figures show that the velocity profile increases with the increase in the dimensionless time τ. It is further noticed that the effect of increasing value of τ also increases the boundary layer thickness.

In figures 2.9 and 2.10 the variations of velocity profile for different values of M, when $n = 3$, at the dimensionless time $\tau = 0.1$ and $\tau = 0.5$ are shown, respectively. These figures also portrait the boundary layer structure for different values of the magnetic parameter M. It is evident from these figures that the velocity profile decreases with increasing values of M for both times. These figures reveal that the flow develops more slowly for higher value of the magnetic parameter for small time.

Figures 2.11 and 2.12 display the effects of the power-law index n on the velocity profile $F_\eta(\eta, \xi)$ for two different values of the magnetic parameter M. Further, the velocity profiles at $\tau = 0.1$ for some fixed value of M with increasing power-law index n are shown in these figures. From these figures, it is clear that as n increases the velocity profile tends to steady state more quickly for both hydrodynamic and hydromagnetic cases.

The variation of the skin friction coefficient with several sets of physical parameters is shown in figures 2.13 and 2.14. In order to see the variation of the skin friction coefficient for a fixed value of either the power-law index n or the magnetic parameter M figures 2.13 and 2.14 are respectively, plotted. These figures, elucidate that as the values of the power-law index n

increases, the skin friction coefficient increases at the same dimensionless time $\tau \in (0, +\infty)$. Moreover, the skin friction coefficient increases as the value of the magnetic parameter M increases for fixed value of n and at the same dimensionless time τ.

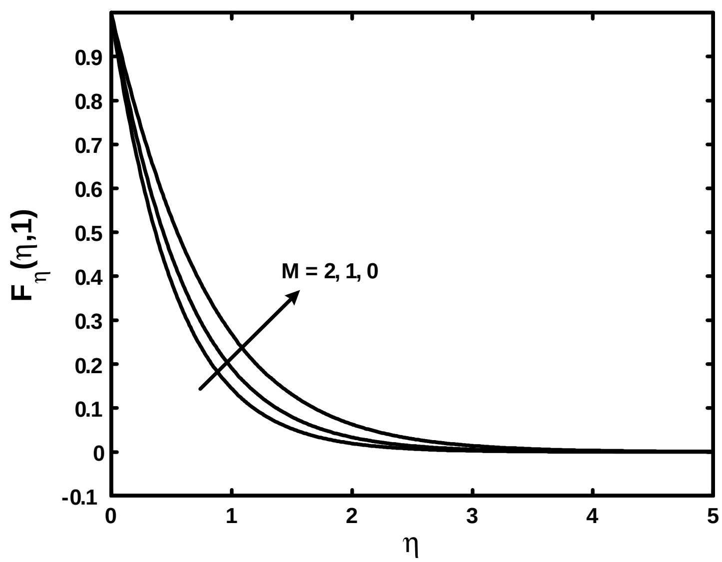

Figure 2.2: The variation of $\mathrm{F}_\eta(\eta, 1)$ when $M = 0$, 1, 2 and $n = 1$.

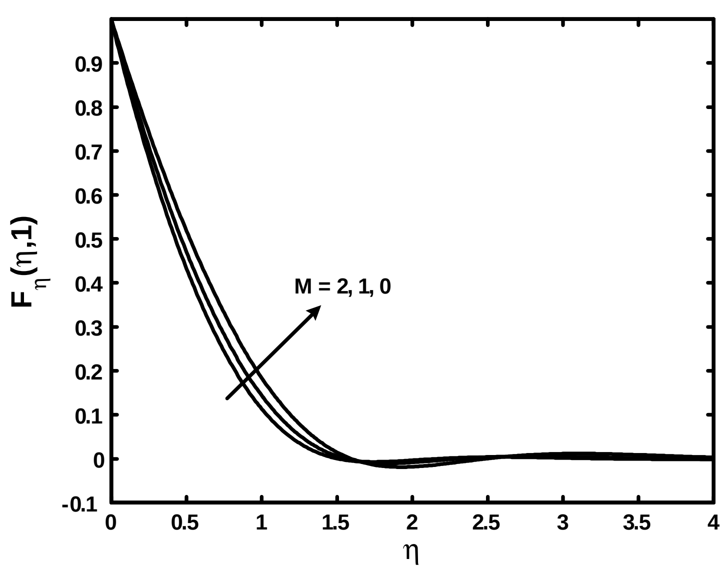

Figure 2.3: The variation of $\mathrm{F}_\eta(\eta, 1)$ when $M = 0$, 1, 2 and $n = 2$.

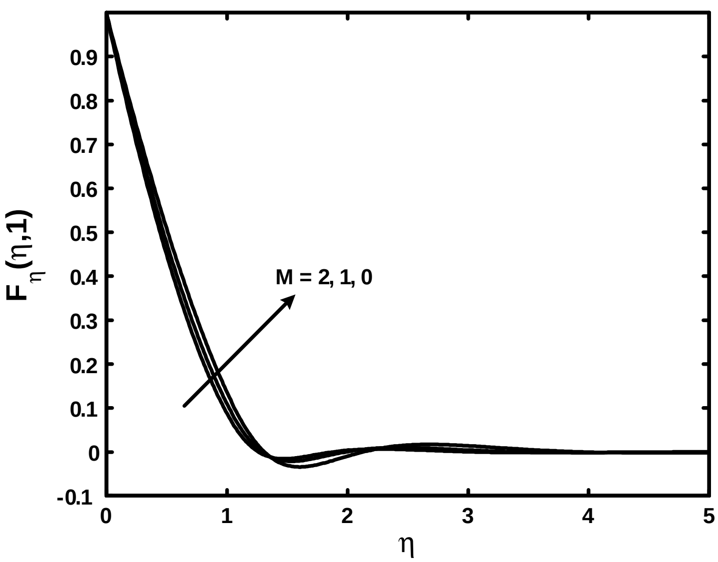

Figure 2.4: The variation of $\mathrm{F}_{\eta}(\eta, 1)$ when $M = 0,\ 1,\ 2$ and $n = 3$.

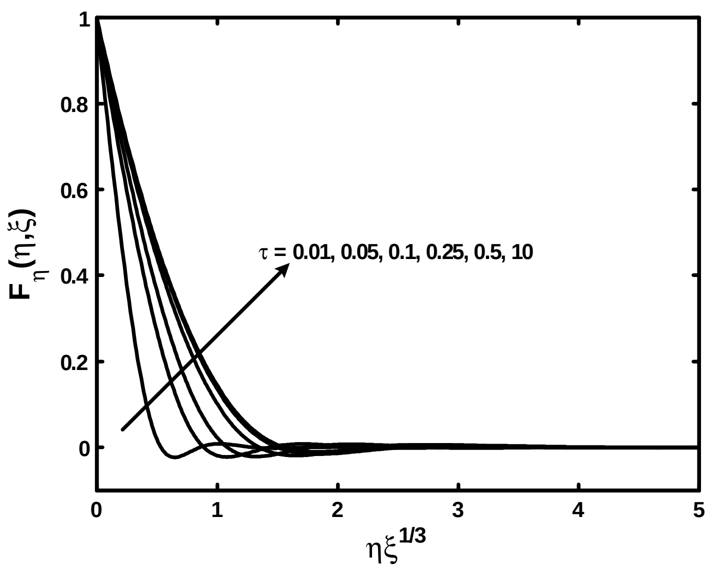

Figure 2.5: The variation of the velocity profile $\mathrm{F}_{\eta}(\eta, \xi)$ when $n = 2$ and $M = 1$.

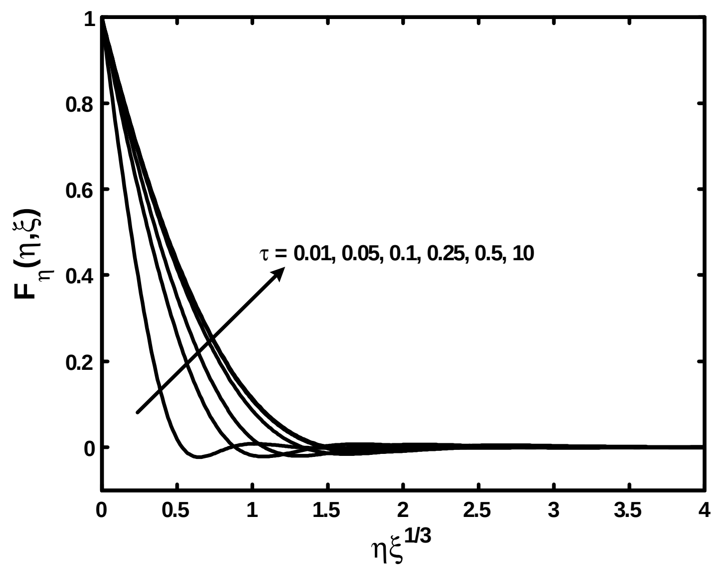

Figure 2.6: The variation of the velocity profile $\mathrm{F}_\eta(\eta, \xi)$ when $n = 2$ and $M = 2$.

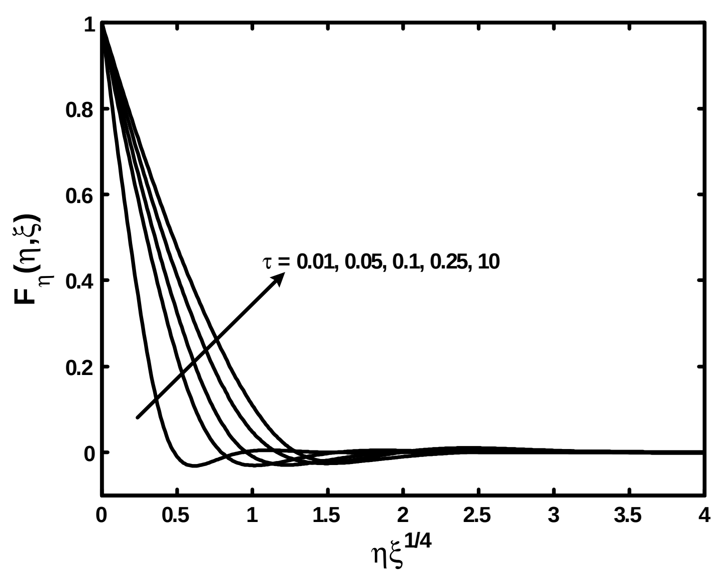

Figure 2.7: The variation of the velocity profile $\mathrm{F}_\eta(\eta, \xi)$ when $n = 3$ and $M = 1$.

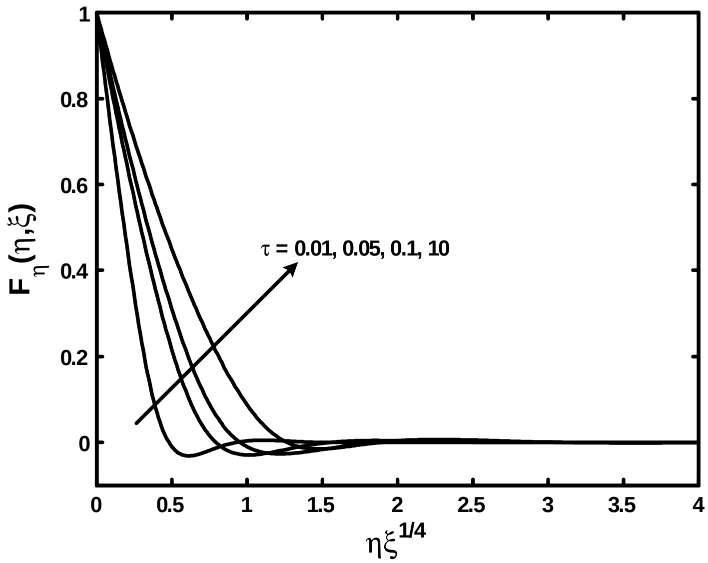

Figure 2.8: The variation of the velocity profile $\mathrm{F}_\eta(\eta,\xi)$ when $n = 3$ and $M = 2$.

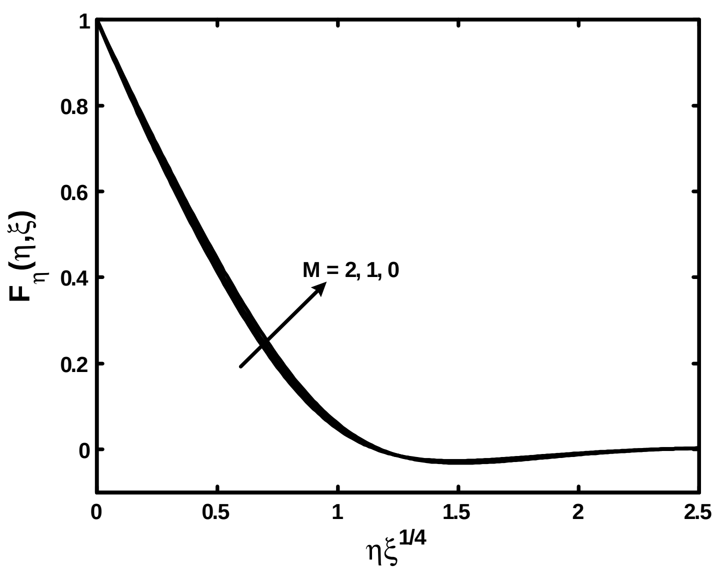

Figure 2.9: The variation of the velocity profile $\mathrm{F}_\eta(\eta,\xi)$ when $n = 3$ and $M = 0,\ 1,\ 2$ at $\tau = 0.1$.

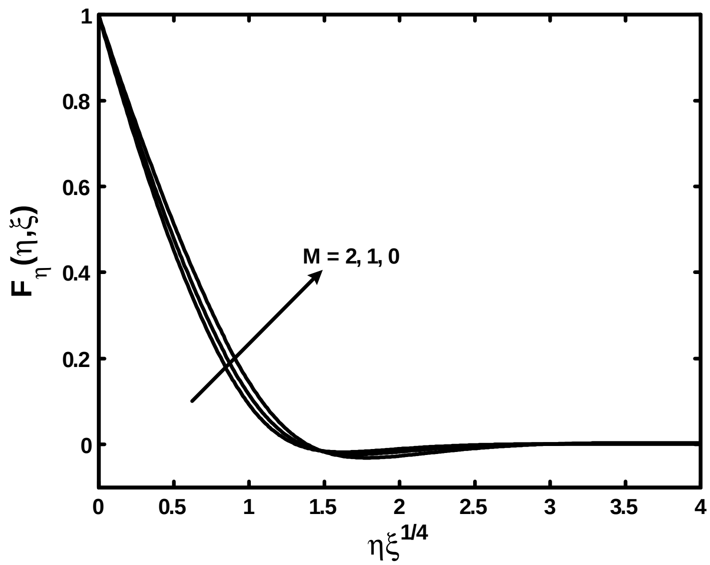

Figure 2.10: The velocity profile $F_\eta(\eta, \xi)$ when $n = 3$ and $M = 0,\ 1,\ 2$ at $\tau = 0.5$.

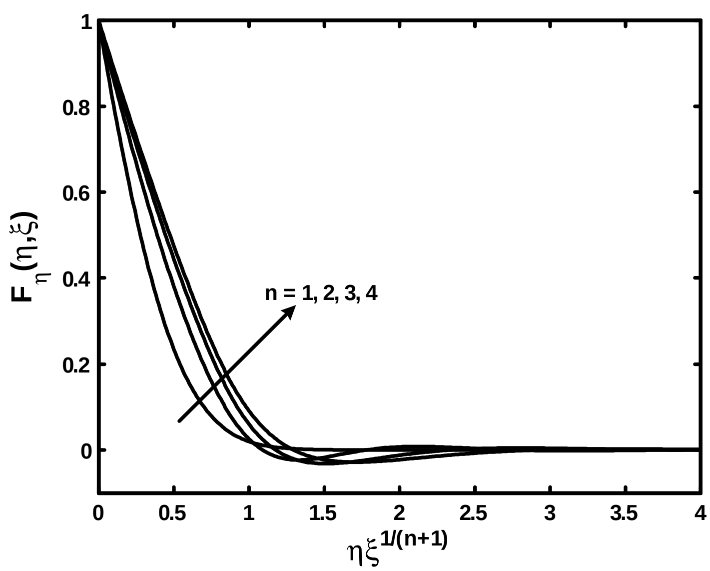

Figure 2.11: The velocity profile $F_\eta(\eta, \xi)$ when $M = 0$ and $n = 1,\ 2,\ 3,\ 4$ at $\tau = 0.1$.

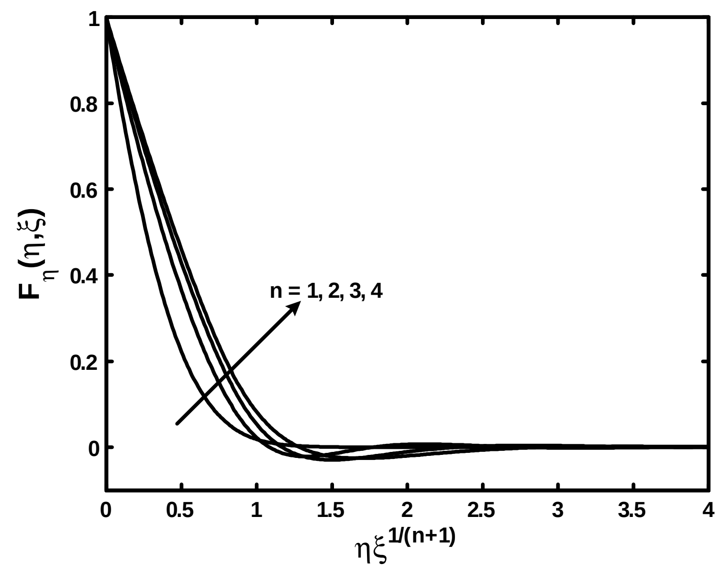

Figure 2.12: The velocity profile $F_\eta(\eta, \xi)$ when $M = 1$ and $n = 1,\ 2,\ 3,\ 4$ at $\tau = 0.1$.

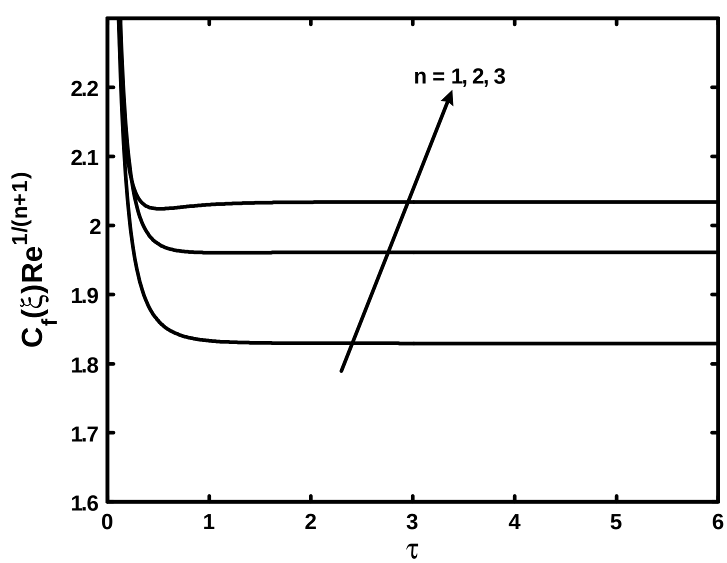

Figure 2.13: The variation of the skin friction coefficient as a function of τ for different power-law index n when $M = 2$.

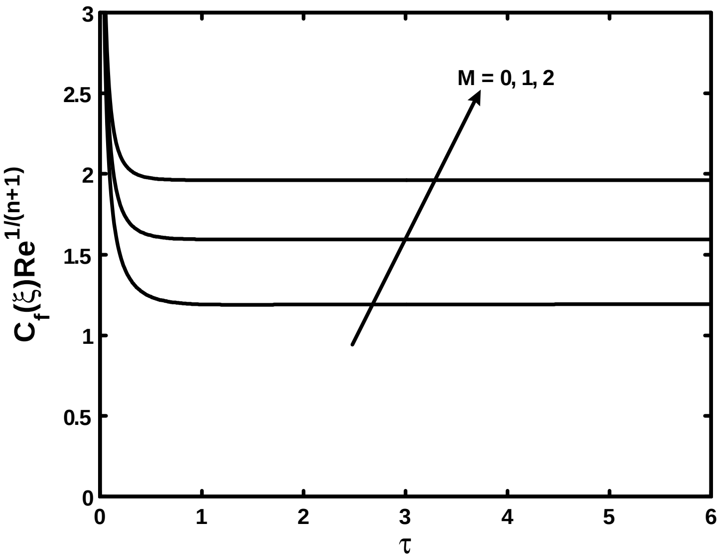

Figure 2.14: The variation of the skin friction coefficient as a function of τ for different parameter M when $n = 2$.

2.5 Conclusions

The work in this chapter dealt with non-linear problem for unsteady MHD flow of non-Newtonian fluids over a stretching sheet. By using suitable similarity transformations, the modelled non-linear partial differential equations in three-independent variables are converted into a single partial differential equation in two-independent variables. The analytical solution for problem of unsteady, axisymmetric MHD flow of power-law fluids over a radially stretching sheet was found by using the HAM. The effects of various emerging parameters were observed from several graphs.

The main findings could be listed as:

- As τ varies from zero to ∞, the velocity profiles developed more rapidly.
- The increasing values of the magnetic parameter showed that the flow developed more slowly.
- The velocity profiles tended to steady state more quickly as the value of power-law index n enlarged.
- The skin friction coefficient has increased when the value of the power-law index n increased for fixed values of the magnetic parameter M and the dimensionless time τ.
- The skin friction coefficient has increased as the values of the magnetic parameter M enlarged at the same dimensionless time and for the same power-law index n.

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