ON QUOTIENTS OF
HECKE GROUPS H (λ_q), q = 3, 6

By

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Islamabad, Pakistan
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A thesis Submitted in Partial Fulfillment of the
Requirements for the Degree of
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In

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CERTIFICATE

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We accept this thesis as conforming to the required standard

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ABSTRACT

The Hecke group $H(\lambda_q)$ is a finitely generated discrete subgroup of $PSL(2,\mathbb{R})$ generated by the transformations: $z \longmapsto -1/z$ and $z \longmapsto -1/(z+\lambda_q)$ of order 2 and q respectively, where $\lambda_q = 2cos(\pi/q)$, q is an integer > 2. When $q = 3$, the group $H(\lambda_3)$ is known as the modular group. It is not difficult to see that the triangle group $\Delta(2,3,n) = \langle x, y : x^2 = y^3 = (xy)^n = 1\rangle$ is a quotient of the modular group $H(\lambda_3)$. The triangle group $\Delta(2,3,13)$ is an infinite group and a subgroup of the group $\Delta^*(2,3,13) = \langle x, y, t : x^2 = y^3 = t^2 = (xt)^2 = (yt)^2 = (xy)^{13} = 1\rangle$ of index 2. In this thesis, we investigate finite quotients of the triangle group $\Delta(2,3,13)$ by employing coset diagrams.

The group $G^{k,l,m} = \langle x, y, t : x^2 = y^k = t^2 = (xt)^2 = (yt)^2 = (xy)^l = (xyt)^m = 1\rangle$ is known as the three generator Coxeter group ([17]). G. Higman raised a question: how small can the integers k, l and m be made while maintaining the property that all but finitely many alternating groups A_n and symmetric groups S_n are factor groups of $G^{k,l,m}$? We answer the question by proving that for all but finitely many positive integers n, both A_n and S_n occur as quotients of the group $G^{3,13,252}$.

The other group we deal with in this thesis is the group $H(\lambda_6) = H(\sqrt{3})$. We investigate action of the group $H(\sqrt{3})$ on real quadratic fields. We make use of coset diagrams drawn for orbits of the group $H(\sqrt{3})$ acting on projective line over $\mathbb{Q}^*(\sqrt{n}) = \{(a+\sqrt{n})/3c : n$ is a non-square positive integer, and $(a^2-n)/3c$, a, $3c$ are relatively prime integers $\}$ to answer the question: when does an orbit of the group $H(\sqrt{3})$ comprising a circuit (closed path) of a given type exist? In case the orbit exists, we find a condition for existence of a real quadratic irrational number $(a+\sqrt{n})/3c$ and its algebraic conjugate $(a-\sqrt{n})/3c$ in the same orbit.

One relator quotients of the Hecke group $H(\lambda_q)$ are the groups obtained by adding one more relation to the existing ones. The quotients of the Hecke group $H(\lambda_q)$ when $q = 3$,

4 and 5, are already investigated (see for instance [12], [24], [40], and [46]). Lastly, we investigate one relator quotients of the group $H(\sqrt{3})$. We find structure of the quotients through GAP.

Dedicated to

My sweet mother and my great father(late)

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CONTENTS

INTRODUCTION

Erich Hecke in 1936 introduced the Hecke group $H(\lambda_q)$ in his paper [25]. The Hecke group $H(\lambda_q)$ is a finitely generated discrete subgroup of $PSL(2,\mathbb{R})$, which is generated by two linear fractional transformations:

$$z \longmapsto \frac{-1}{z} \text{ and } z \longmapsto \frac{-1}{z+\lambda_q}, \text{where } \lambda_q = 2cos(\pi/q) \text{ and } q \text{ is an integer} > 2.$$

The modular group $H(\lambda_3) = PSL(2,\mathbb{Z})$ seems to be the most discussed Hecke group so far. An extensive literature (see for instance [12], [24], [29], [30]) based on various techniques of study is available on the modular group. One of the techniques is based on the graphs known as 'coset diagrams'. In this thesis, we mostly use this technique. Q. Mushtaq ([29]) introduced coset diagrams for the modular group $H(\lambda_3)$. He used the diagrams to investigate actions of the modular group $H(\lambda_3)$ on real quadratic fields, and structure of words in the group over finite and real quadratic fields (see [30], [36]).

The triangle group $\Delta\,(2,3,n)$ is a well known quotient of the modular group $H(\lambda_3)$. The group $\Delta\,(2,3,n)$ has a finite presentation $\langle x, y : x^2 = y^3 = (xy)^n = 1\rangle$. Finite groups in this class of triangle groups are A_5, S_4, A_4, S_3 and a trivial group for $n = 5$, 4, 3, 2 and 1, respectively. We study coset diagrams drawn from linear fractional transformations for the triangle group $\Delta\,(2,3,13)$. We investigate finite quotients of infinite triangle group $\Delta\,(2,3,13)$ and show how more quotients of large size can be obtained by connecting two or more coset diagrams representing the quotients. Using the technique introduced by M. D. E. Conder (see [9], [13]), we draw coset diagrams of another type for the same group directly by connecting vertices through edges without involving linear fractional transformations, and answer the question of G. Higman.

The Hecke group $H(\lambda_q)$ when $q = 5$ is the group $H(\lambda_5) = H(\frac{1+\sqrt{5}}{2})$. The group $H(\lambda_5)$ is investigated for its congruence subgroups and one relator quotients by İ. N. Cangül et. al. (see [19] and [46]). Many similarities to the modular group are observed. After the modular group, next two important groups of this class of Hecke groups are obtained when $q = 4$ and $q = 6$. These are the groups $H(\lambda_4) = H(\sqrt{2})$ and $H(\lambda_6) = H(\sqrt{3})$. We use the notation $H(\sqrt{2})$ and $H(\sqrt{3})$ for these two groups in this thesis. One of the reasons for considering the groups $H(\sqrt{2})$ and $H(\sqrt{3})$ to be the next important Hecke groups comes from the fact that these are the only ones whose elements can be described completely. These groups remained an object of study from many aspects (see for instance [2], [7], [40], [41]). One direction for study of the Hecke groups is to study quotients of the groups (see for instance [12], [40], [46], [24]). Addition of a new relation to presentation of any group leads to formation of a new group which is called a one relator quotient of the group. One relator quotients of the group $H(\sqrt{2})$ are a part of discussion in [40]. The group $H(\sqrt{3})$ has a finite presentation $\langle a, b : a^2 = b^6 = 1\rangle$. Addition of a new relation to the presentation results in presentation of a new group which is a one relator quotient of the group $H(\sqrt{3})$. We investigate one relator quotients of the group $H(\sqrt{3})$ in this thesis.

Hecke groups are also studied by investigating properties of closed paths and structure of words through action of the groups on finite and infinite fields. In this thesis, we are interested in use of coset diagrams for the study. Let the real quadratic irrational number $\frac{(a+\sqrt{n})}{3c}$, where n is a non-square positive integer and $\frac{(a^2-n)}{3c}$, a, $3c$ are relatively prime integers, be denoted by γ and algebraic conjugate $\frac{(a-\sqrt{n})}{3c}$ of the number be denoted by $\bar{\gamma}$. In [37], it was proved that a closed path can be found in an orbit of the group $H(\sqrt{3})$ such that the closed path contains all the numbers γ. In this thesis, we extend this study of coset diagrams for the group $H(\sqrt{3})$ to investigation of word structure of those elements of the group $H(\sqrt{3})$, which generate the numbers γ. In fact, we investigate certain characteristics of one class of circuits containing the numbers γ.

In chapter 1, we give relevant definitions and results concerning the Hecke group $H(\lambda_q)$ and its quotients. We define the triangle group $\Delta(l, m, n)$; discuss when the group is finite or infinite and significance of the group. We discuss the triangle group $\Delta(2, 3, n)$ as a quotient of the Hecke group $H(\lambda_3)$, quotients of Fuchsian groups, projective lines over the finite field, and coset diagrams. In fact, some connected results are shared on alternating groups as quotients of Fuchsian groups. A detailed introduction of coset diagrams including a brief history along with an example of a coset diagram is also a part of this chapter.

We investigate quotients of the group $\Delta(2, 3, 13)$ using coset diagrams in chapter 2. Suppose q is a power of a prime p and $\theta \in F_q$. We know from [34] that coset diagrams can be used to represent conjugacy classes of actions of $PGL(2, \mathbb{Z})$ on $F_q \cup \{\infty\}$. We parametrize conjugacy classes of actions of $\Delta(2, 3, 13)$ on $F_q \cup \{\infty\}$, using the technique devised in [34]. In fact, we relate a coset diagram $D(\theta, q)$ to each $\theta \in F_q$. One diagram depicts one of the conjugacy classes. We find conditions in terms of q and θ, which guarantee only the coset diagrams depicting finite quotients of $(2, 3, 13)$. We also investigate conditions for existence of some special types of fragments in the coset diagrams representing the quotients. Finally, we give a list of linear fractional transformations, which provides complete information for drawing the coset diagrams when $q = p < 1300$.

In chapter 3, we draw more coset diagrams for the triangle group $\Delta(2, 3, 13)$ and investigate the Coxeter group $G^{3,13,252}$ as its quotients. These diagrams are drawn independent of linear fractional transformations that are used in chapter 2 (see [9], [13]). Some suitable diagrams are then chosen and connected to prove our result. In fact, we answer the question of G. Higman by proving that for all but finitely many positive integers n, S_n and A_n are quotients of $G^{k,l,m}$, where $(k, l, m) = (3, 13, 252)$. We apply Jordan's Theorem in the proof.

In chapter 4, we investigate action of the group $H(\sqrt{3})$ on projective line over

$\mathbb{Q}^*(\sqrt{n}) = \{\frac{(a+\sqrt{n})}{3c} : n$ is a non-square positive integer and $\frac{(a^2-n)}{3c}$, a, $3c$ are relatively prime integers $\}$, using coset diagrams. Results in papers [2] and [37] provide a basis for this study. The diagrams for orbits of the group $H(\sqrt{3})$ acting on projective line over $\mathbb{Q}^*(\sqrt{n})$ give useful information for study of word structure of the group. We answer the question: 'when does an orbit of the group $H(\sqrt{3})$, containing a circuit (closed path) of a given type exist?'. Moreover, in case of existence of the circuit we find a condition for existence of γ and algebraic conjugate $\bar{\gamma}$ of the number in the same orbit. We use the same method to investigate the circuits existing in the orbits of the group $H(\sqrt{3})$ as used in [36] for the circuits existing in the orbits of the modular group.

In chapter 5, we are concerned with finding the structure of one relator quotients of the group $H(\sqrt{3})$. We use GAP in this chapter. By including a new relation $w = R(a,b) = 1$ to the existing ones, in terms of a and b for a cyclically reduced word $w = ab^{\epsilon_1}ab^{\epsilon_2}ab^{\epsilon_3}...ab^{\epsilon_n}$, where $1 \leq \epsilon_i \leq 5$, we get a one relator quotient of the group $H(\sqrt{3})$. Let k be the sum of exponents of a in w, l be the sum of exponents of b in w, and $N_{k,l}$ be the total number of non-equivalent cyclically reduced words w. We observe that

(a) if $k = 0$ then $1 \leq l \leq 5$, and if $k = n$ then $n \leq l \leq 5n$,

(b) $N_{n,n} = N_{n,n+1} = N_{n,5n} = N_{n,5n-1} = 1$.

Moreover, we investigate all one relator quotients of the group $H(\sqrt{3})$ that can be obtained up to $k = 4$ and $l = 20$.

Chapter 1

PRELIMINARIES

1.1 Introduction

This chapter includes some basic definitions, relevant notions and important connected results to give an idea of what is already achieved and what is still to be investigated in this area of research. We discuss the group $H(\lambda)$, the triangle group $\Delta(l, m, n)$, quotients of Fuchsian groups, finite fields and projective line over the fields. We also give an intuitive description of graph of a group and certain types of graphs. In particular, we describe coset diagrams. We give a brief history of the coset diagrams and discuss some examples of use of coset diagrams to show how they are employed up to the level of a useful technique to solve problems in group theory.

1.2 Hecke Groups

Erich Hecke ([25]) introduced the group $H(\lambda)$ generated by two linear-fractional transformations defined by

$$U(z) = z + \lambda \text{ and } T(z) = \frac{-1}{z}$$

where λ is a fixed positive real number. He showed that the set

$$F_\lambda = \{z\colon \ \mathrm{Im}\, z > 0, \mid \mathrm{Re}\, z \mid < \lambda/2, \mid z \mid > 1\}$$

is a fundamental region for the group $H(\lambda)$ when $\lambda \geq 2$ and real, or when

$$\lambda = \lambda_q = 2cos(\frac{\pi}{q}),$$

where q is an integer > 2, when $\lambda < 2$, and also that F_λ fails to be fundamental region for all other $\lambda > 0$. It implies that $H(\lambda)$ is Fuchsian if and only if $\lambda = \lambda_q$ or λ is a real number > 2. In both the cases, the group $H(\lambda)$ is called a Hecke group. We are interested in the case when $\lambda = \lambda_q$. Let $S = TU$, i.e., $S(z) = \dfrac{-1}{z+\lambda}$. In this case, the Hecke group $H(\lambda_q)$ is a finitely generated discrete subgroup of $PSL(2, \mathbb{R})$ generated by the transformations T and S of order 2 and q respectively. For small values of q, we get $\lambda_3 = 1$, $\lambda_4 = \sqrt{2}$, $\lambda_5 = \dfrac{1+\sqrt{5}}{2}$, $\lambda_6 = \sqrt{3}$. When $q = 3$, we get the group $H(\lambda_3) = H(1) = PSL(2, \mathbb{Z})$; the most popular discrete group named modular group. When $q = 6$ and $q = 4$, we obtain the Hecke groups $H(\sqrt{3})$ and $H(\sqrt{2})$ respectively. The groups are useful in study of Modular forms, Kloosterman sums, Dirichlet series, etc.

Throughout this thesis we are concerned with the groups $PSL(2, \mathbb{Z})$ and $H(\sqrt{3})$. It is known that the group $H(\sqrt{3})$ is generated by two linear-fractional transformations defined by

$$y(z) = \frac{-1}{3(z+1)} \text{ and } x(z) = \frac{-1}{3z}$$

which satisfy the relations: $x^2 = y^6 = 1$ ([37]). The modular group $PSL(2, \mathbb{Z})$ also can be expressed as a group generated by the transformations defined by

$$y(z) = \frac{z-1}{z} \text{ and } x(z) = \frac{-1}{z}$$

which satisfy the relations: $x^2 = y^3 = 1$ ([20]). The linear-fractional transformation t defined by $z \longmapsto \frac{1}{z}$ is an involution and inverts both x and y, that is, $t^2 = (xt)^2 = (yt)^2 = 1$. Therefore, inclusion of t to $PSL(2, \mathbb{Z})$ results in extension to the group $PGL(2, \mathbb{Z})$. Thus, x, t and y are generators of the extended modular group $PGL(2, \mathbb{Z})$ and the relations: $x^2 = t^2 = y^3 = (xt)^2 = (yt)^2 = 1$ are its defining relations. Every countable group is a subgroup of some quotient of the modular group. The symmetric group of degree k $(k = 5, 6, 8)$ is itself a quotient of the group. The triangle group $\Delta(2, 3, n) = \langle x, y : x^2 = y^3 = (xy)^n = 1 \rangle$, where n is an integer ≥ 1, is another quotient of the group. Now we define the triangle group $\Delta(l, m, n)$ and give some known facts about the group since a large part of the thesis involves the triangle group $\Delta(2, 3, 13)$.

1.3 Triangle Groups

The triangle group is represented by $\Delta(l, m, n) = \langle x, y : x^l = y^m = (xy)^n = 1 \rangle$, where l, m, n are integers ≥ 1. The group is independent of the order in which l, n, m are listed. It is known that the triangle group $\Delta(l, m, n)$ is finite when

$$\delta = \frac{1}{l} + \frac{1}{m} + \frac{1}{n} - 1 > 0.$$

The group in this case is called a spherical triangle group. The groups satisfying the inequality are:

$\Delta(1, n, n) \cong C_n$	Cyclic group of order n,
$\Delta(2, 2, n) \cong D_{2n}$	Dihedral group of order $2n$,

$\Delta(2,3,3) \cong A_4$	Tetrahedral group,
$\Delta(2,3,4) \cong S_4$	Octahedral group, and
$\Delta(2,3,5) \cong A_5$	Icosahedral group.

The triangle group $\Delta(l,m,n)$ comprises a fundamental group of an orientable surface of positive genus as a subgroup of finite index whenever $\delta \leq 0$; in particular the group $\Delta(l,m,n)$ is infinite.

If $\delta = \frac{1}{l} + \frac{1}{m} + \frac{1}{n} - 1 = 0$, that is, $(l,m,n) = (3,3,3)$, $(2,4,4)$ or $(2,3,6)$, then the triangle group $\Delta(l,m,n)$ is infinite but soluble; the commutator subgroup is a free Abelian group of rank two, and the factor commutator group is cyclic of order n. The triangle group $\Delta(l,m,n)$ in this case is called a Euclidean triangle group.

If $\delta = \frac{1}{l} + \frac{1}{m} + \frac{1}{n} - 1 < 0$, then the triangle group $\Delta(l,m,n)$ is called a hyperbolic triangle group. Any hyperbolic triangle group is a Fuchsian group.

Significance of the triangle groups is evident from the existing literature on different dimensions of these groups (See for instance [5], [13], [14], [15]). The literature regarding construction of coset diagrams of the triangle group $\Delta(l,m,n)$ shows that it is convenient to use finite presentation of the group $\Delta^*(l,m,n)$ instead of finite presentation of $\Delta(l,m,n)$ to maintain vertical symmetry in the coset diagrams. Adjoining of an involution t to the finite presentation of the triangle group $\Delta(l,m,n)$ results in an extension of the group to the group $\Delta^*(l,m,n) = \langle x, y, t : x^l = y^m = t^2 = (xt)^2 = (yt)^2 = (xy)^n = 1 \rangle$. Since in the coset diagrams t maintains only vertical symmetry so there is no mathematical harm in making use of $\Delta^*(l,m,n)$ while drawing the diagrams for $\Delta(l,m,n)$.

The triangle group $\Delta(2,3,n)$ is specially important as a quotient of the group $PSL(2,\mathbb{Z})$. The group $\Delta(2,3,n)$ is infinite if and only if $n \geq 6$. When $n = 6$, the group $\Delta(2,3,6)$ is an infinite group but soluble. The most investigated infinite group in this class of triangle groups is probably the group $\Delta(2,3,7)$, when $n = 7$. A non-trivial finite quotient of $\Delta(2,3,7)$ is known as the Hurwitz group. There exists a bulk of literature on finite quotients of $\Delta\,(2,3,7)$ (see for instance [14], [29], and [45]). A. M. Macbeath ([27]) proved that the group $PSL(2,q)$ is a Hurwitz group when $q=7$, and when $q=p$ for any prime $p\equiv\pm1$ $(mod\ 7)$, and when $q=p^3$ for any prime $p\equiv\pm2$ or $\pm\ 3$ $(mod\ 7)$, and for no other values of q. M. D. E. Conder discussed quotients of $\Delta\,(2,3,7)$ in his paper ([15]). Including M. D. E. Conder, some authors used the notation '$(2,3,7)$ triangle group' instead of $\Delta\,(2,3,7)$. The smallest known Hurwitz group is the simple group $PSL(2,7)$ of order 168. Conder ([14]) gave a brief survey of the Hurwitz group including properties, importance and description with some examples. W. W. Stothers ([45]) associated with a subgroup of finite index u in the triangle group $\Delta\,(2,3,7)$, a quintuple of non-negative integers (u,p,e,f,g), $u\geq1$ and $u=84(p-1)+21e+28f+36g$. He proved that any such specification (u,p,e,f,g) is the specification of some subgroup of the triangle group $\Delta\,(2,3,7)$ with the exception of $(16,0,0,1,2)$, $(21,1,1,0,0)$ and $(36,1,0,0,1)$. Q. Mushtaq ([32]) used coset diagrams for study of the Hurwitz group. He parametrized actions of the triangle group $\Delta\,(2,3,7)$ on projective line over the finite field F_q.

P. C. R. Stephenson ([44]) in his Ph.D. thesis investigated subgroups of finite index in the triangle group $\Delta\,(2,3,n)$ for $n=9,\ 11,\ 13$. He also used the notation '$(2,3,n)$ triangle group' for the triangle group $\Delta\,(2,3,n)$. He employed coset diagrams in his work. He associated with a subgroup of finite index u, in $\Delta\,(2,3,9)$, a sextuple of non-negative integers (u,p,e,f,g_1,g_3), with $u\geq1$, $u\equiv f$ $(mod\ 3)$ and $u=36(p-1)+9e+12f+16g_1+12g_3$, and proved that each sextuple satisfying the conditions corresponds to a subgroup of $\Delta\,(2,3,9)$ with the exception of $(12n+9,0,1,0,0,n+3)$, $\forall\ n\geq0$, $(24,0,0,0,0,5)$, $(24,0,0,0,3,1)$, and $(24,0,0,3,0,2)$. He associated with a subgroup of finite index u, in $\Delta\,(2,3,11)$, a quintuple of non-negative integers (u,p,e,f,g), with $u\geq1$ and $5u=132(p-1)+33e+44f+60g$, and proved that each quintuple satisfying the conditions corresponds to a subgroup of the group $\Delta\,(2,3,11)$. He associated with a subgroup of finite index u, in $\Delta\,(2,3,13)$, a quintuple of non-negative integers (u,p,e,f,g), with $u\geq1$ and $7u=156(p-1)+39e+52f+72g$, and proved that each quintuple satisfying the conditions corresponds to a subgroup of $\Delta(2,3,13)$. A detailed study of coset diagrams for $\Delta(2,3,11)$ is available in [39].

1.4 Quotients of Fuchsian Groups

The Fuchsian group is a discrete subgroup of $PSL(2,\mathbb{R})$. Almost all alternating groups are quotients of the Fuchsian group. The existence of alternating groups as quotients of the Fuchsian group is evident from the following important results.

Miller ([28]) proved that $PSL(2,\mathbb{Z})$ has every alternating group among its homomorphic images, except A_8, A_7, A_6 and A_3.

In 1980, Conder ([10]) proved that the alternating group A_n is a Hurwitz group for $n \geq 168$.

In 1981, Conder ([11]) also proved that

(a) All but finitely many alternating groups can be presented as quotient groups of the triangle group $\Delta(2,3,k)$ for every $k \geq 7$.

(b) All but finitely many alternating groups can be generated by two elements v, u with $u^2 = v^k = 1$.

In 1992, Q. Mushtaq and G. C. Rota ([33]) proved: For nearly all natural numbers n, A_n is a homomorphic image of $\Delta(2,k,l)$ with even $k \geq 6$ and $l \geq 5k-3$.

B. Everitt ([21]) proved: For all $r \geq 6$, nearly all alternating groups A_n are factor groups of the triangle group $\Delta(2,4,r)$.

In 1997, B. Everitt ([22]) showed:

(a) For every $r \geq 40$ there is a number N so that $G = \Delta(3,5,r)$ has among its quotients the group A_n or S_n for all $n > N$.

(b) For every prime $q \geq 7$ and every $r \geq 4q$, $\Delta(3, q, r)$ has the same property.

In 2000, B. Everitt ([23]) proved G. Higman's thirty years old conjecture:

Any Fuchsian group has among its homomorphic images all but finitely many alternating groups.

1.5 Finite Fields and Projective Line over the Finite Field

Finite Fields are very important from group theoretic aspect. The most familiar examples of such fields are the field $\mathbb{Z}_p$ for prime p, but these are not all. A finite field is uniquely determined up to isomorphism by the number of elements it contains; that the number must be a power of a prime; and that for every prime p and integer $r > 0$ there exists a field with p^r elements. The ring of integers $\mathbb{Z}$ induces a natural ring structure on $\mathbb{Z}_n = \mathbb{Z}/n\mathbb{Z}$, the set of integers modulo n. If n is a prime p, then $\mathbb{Z}_p$ is in fact a field. $(\mathbb{Z}_p)^r = \{(a_o, a_1, ..., a_{r-1}) : a_i \in \mathbb{Z}_p\}$, is also a field. We identify the sequence $(a_0, a_1, ..., a_{r-1})$ with the polynomial $a_0 + a_1 t + ... + a_{r-1}t^{r-1}$ in the ring of polynomials $\mathbb{Z}_p[t]$, and then choose a polynomial $f(t)$ of degree r which is irreducible in $\mathbb{Z}_p[t]$ (that is, $f(t)$ has no zeros in $\mathbb{Z}_p$). The multiplication of two sequences is defined by multiplying the corresponding polynomials in $\mathbb{Z}_p[t]$, and then reducing modulo $f(t)$. It is always possible to choose $f(t)$ such that the non-zero elements of the field are just the powers $t, t^2, ..., t^{p^r-1}$, and the last being the multiplicative identity 1. The field constructed in this way is called the Galois field with p^r elements. The field with $q = p^r$ elements is written by $GF(q)$ or F_q.

Suppose V is a vector space over a field F. Put $V^* = V \setminus \{0\}$, if x, $y \in V^*$ then the statement 'for some $\lambda \in F^* = F \setminus \{0\}$, $x = \lambda y$' defines an equivalence relation on V^*.

The set of equivalence classes obtained this way is called the projective space $PG(V)$. We denote the class of $x \in V^*$ by $[x] \in PG(V)$, and define a subspace $[U]$ of $PG(V)$ to be the image of a subspace U of V under the map $x \longrightarrow [x]$. For geometric reasons it is convenient to say; if U has dimension k then $[U]$ has (projective) dimension $k-1$; in particular if $V = V(n, q)$, we write $PG(V) = PG(n-1, q)$.

If V is a vector space of dimension 2 over a finite field F_q, we write $V = V(2, q)$, and V has q^2 elements. The projective space over $V = V(2, q)$ is written as $PG(1, q)$ (called the projective line $PL(F_q)$) and it has $q+1$ points. It may be represented by q symbols $[1, z]$ (where z runs through F_q), and the additional symbol $[0, 1]$. We often think of $PG(1, q) = PL(F_q)$ as the set $F_q \cup \{\infty\}$, where ∞ is image of $[0, 1]$ under the bijection $[x_0, x_1] \longleftrightarrow \frac{x_1}{x_0}$. Thus $PL(F_q) = PG(1, q) = F_q \cup \{\infty\} = \{0, 1, 2, 3, ..., q-1\} \cup \{\infty\}$.

1.6 Coset Diagrams

Action of a group can be represented by a graph, and this way of representation has a long history. We can find applications of graphs in many branches of mathematics. Actually, graphs provide methods by which several algebraic and topological structures can be visualized. Graphical methods played an explicit role in study of finitely generated groups. The graphs proved to be an economical mathematical technique to prove certain important results (see for instance, [8], [13], and [15]). For finite groups of small order, the graphs can be used instead of multiplication tables. The graphs give the same information but in a much more efficient way (see for instance, [42], [43]). The first paper with explicit use of graphs is authored by A. Cayley ([8]) in 1878. After A. Cayley, in 1893, Hurwitz used the graphs to represent groups. Then in 1896, H. Maschke made use of the graphs given by A. Cayley to prove some significant results about the representation of finite groups; particularly on the rotation groups of the regular bodies in three and four-dimensional spaces. In 1910, Dehn extensively used Cayley's graphs. Later, mathematicians like O. Schreier, J. H. C. Whitehead ([47]), H. S. M. Coxeter and W. O. J. Moser ([18]), W. Burnside, et al. ([6]), contributed seminal papers containing graphical representations of groups.

The Cayley diagram for a given group is a graph containing the vertices representing elements of the group, which are cosets of the trivial subgroup. O. Schreier generalized the notion of Cayley diagrams by introducing a graph with vertices representing cosets of any subgroup. Coxeter and Moser ([18]) used both Schreier and Cayley diagrams to give proofs of some results on finitely generated groups.

In 1978, G. Higman propounded the idea of coset diagrams for the modular group. In 1983, under the supervision of G. Higman; Q. Mushtaq initiated the theory of coset diagrams for the group. M. D. E. Conder ([10] and [11]) used the diagrams to solve certain identification problems.

A coset diagram is also a graph having vertices which are (right) cosets of a subgroup of finite index in a finitely generated group. The vertices representing cosets u and v (say), are joined by an S_i-edge, of 'colour i' directed from vertex u to vertex v, whenever $uS_i = v$.

$$u \rightarrow u\,\mathrm{S}_i = v$$

It may well happen that $uS_i = u$, in which case the $u-$vertex is joined to itself by a S_i-loop or a fixed point. Formally, a coset diagram corresponding to a subgroup H of finite index in a finitely generated group G is a directed edge coloured graph containing the vertices which are (right) cosets of H in G, and the edges which are defined as follows: we take a specific set of generators for G, and for each generator x and each vertex Hg for some g in G draw an edge of colour E^x from Hg to Hgx. The notion of a coset diagram is similar to that of a Schreier coset graph whose vertices represent the cosets of any given subgroup in a finitely generated group, and also to that of a Cayley diagram containing the vertices which are the group elements themselves, with trivial stabilizer. It is possible to draw these diagrams for any finitely generated group acting on any arbitrary set or space.

We now define some relevant terms that are frequently used in this thesis without any more comments. These definitions can be found in [18].

An edge whose both vertices namely initial and final coincide with each other is called a loop.

If $\pi = \{v_0,\ e_1,\ v_1,\ e_2,\ v_2, ..., e_k,\ v_k\}$ is a sequence of vertices v_i and edges e_i of a graph, then π is called a path. The path connects v_0 to v_k, where e_i connects v_{i-1} to v_i for each i and $e_i \neq e_j$ $(i \neq j)$. The path P described backwards is called the inverse path. P is called a closed path if its initial vertex coincides with its terminal vertex.

If a word C satisfies the relation $C = 1$, where 1 is the identity element, then any path corresponding to C is called a circuit. In other words, a circuit is a closed path. So, loop is an example of a circuit. A circuit in which elements are fixed just by one word, and its inverse is called a simple circuit. Otherwise, it is called a non-simple or a connected circuit.

If any two vertices in a coset diagram are joined by a path, then the diagram is called a connected coset diagram.

Every connected coset diagram for a finitely generated group G action on a set of n points represents a transitive permutation representation of G on that set, which is in fact equivalent to the natural action of G on the cosets of some subgroup H of index n. Coxeter and Moser ([18]) attribute these diagrams to Schreier. Steinberg ([43]) proved that all finite simple groups of Lie type are two generator groups. It is also generally known that many, if not all, known finite simple groups are of Lie type. This means that all but a finite number of finite simple groups are two generator groups.

Coset diagrams defined by G. Higman for the actions of $PSL(2,\mathbb{Z})$ are special in a number of ways. First, they are defined for a particular group, namely $PSL(2,\mathbb{Z})$, which has a representation in two generators x and y. Since there are only two generators, there is a possibility of avoiding use of colours as well as the orientation of edges connected with x. Since y has order 3, so it is required to distinguish y^2 from y. Small triangles are therefore used to represent $3-$cycles of y such that y permutes their vertices anticlockwise, while heavy dots are used to denote the fixed points (if any) of y and x. Thus, the geometry of the figure makes it obvious to distinguish between $y-$ edges and $x-$ edges.

Q. Mushtaq ([29]) employed coset diagrams for the modular group extensively to prove that for each element θ of a finite field F_q, where q is a prime power, there exists a coset diagram for the natural action of $PGL(2,\mathbb{Z})$ on $PL(F_q)$. His thesis also contains some partial answers concerning the 'Reconstruction Conjecture'. That is, the way a diagram is reproducible from certain types of fragments. If we have certain $fragments$ of a coset diagram, we can find the conditions for the existence of those fragments in the respective coset diagram. The condition in fact is a polynomial in $\mathbb{Z}[z]$. For many reasons connected with $PGL(2,q)$ actions on surfaces, it is important to know when $PGL(2,q)$ is an image of $PGL(2,\mathbb{Z})$. The solution to that is given in [31].

Coset diagrams may be employed to explain several aspects of combinatorial group theory, such as the Reidemeister-Schreier procedure, proof of the Ree-Singerman theorem (on the cycle structures of generating-permutations for a transitive group), diagrammatically. The diagrams can also be used as an equivalent to the Abelianized form of the Reidemeister-Schreier process. The same sort of method is also useful in constructing infinite families of finite quotients of any finitely-presented group. Use of the diagrams in finding torsion-free subgroups of finitely-presented groups has been influential in building of small volume hyperbolic $3-$manifolds. The diagrams are also employed in construction of maximal automorphism groups of Riemann surfaces and arc-transitive graphs ([16]). Coset diagrams can be applied to find a presentation for a subgroup H of finite index n in a finitely presented group $G = \langle X \mid R \rangle$. Moreover, these diagrams can also be used to prove certain groups to be infinite, by joining diagrams together to get an arbitrarily large degree permutation representation of a given group.

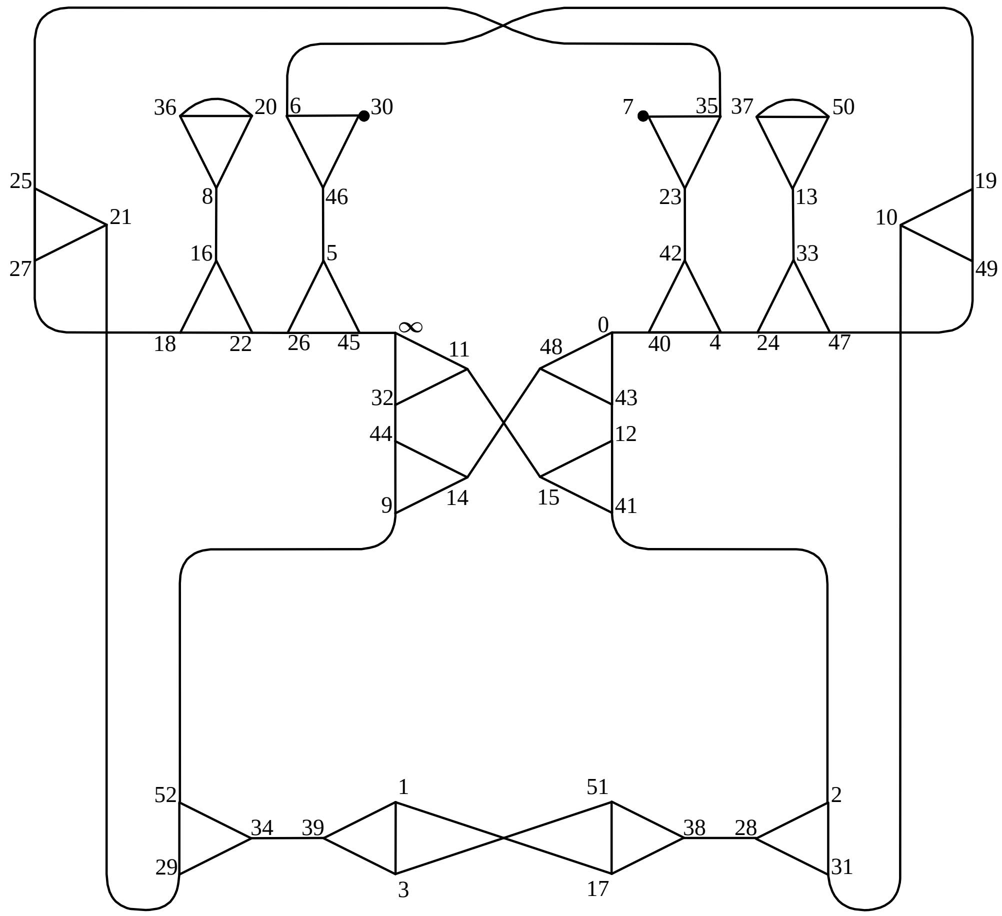
Now we give an example of a coset diagram representing permutation representation of an action of an infinite group on a finite set.

Example 1 *Consider the following coset diagram (Fig.1.1). It is a diagram of permutation representation of a quotient or homomorphic image of the group* $\Delta(2, 3, 13)$ *acting on* $PL(F_{53})$*. Each vertex of the diagram satisfies the relations:* $\bar{x}^2 = \bar{t}^2 = \bar{y}^3 = (\bar{x}\bar{t})^2 = (\bar{y}\bar{t})^2 = (\bar{x}\bar{y})^{13} = 1$*. The linear fractional transformations* $\bar{x}$*,* $\bar{t}$*, and* $\bar{y}$ *, which are employed to get the permutations to draw the required diagram are obtained through parameterization process that is discussed in detail in the next chapter. The transformation* $\bar{t}$ *maintains symmetry about vertical axis in the diagram.*

$\bar{x}$: (0 40)(1 17)(2 41)(3 51)(4 24)(5 46)(6 27)(7)(8 16)(9 52)(10 31)(11 15)(12 43)(13 33)(14 48)(18 19)(20 36)(21 29)(22 26)(23 42)(25 47)(28 38)(30)(32 44)(34 39)(35 49)(37 50)(45 ∞)

$\bar{y}$: (0 48 43)(1 39 3)(2 28 31)(4 42 40)(5 26 45)(6 46 30)(7 23 35)(8 20 36)(9 14 44)(10 49 19)(11 ∞ 32)(12 15 41)(13 50 37)(16 18 22)(17 38 51)(21 25 27)(24 47 33)(29 34 52)

$\bar{t}$: (0 ∞)(1 51)(2 52)(3 17)(4 26)(5 42)(6 35)(7 30)(8 13)(9 41)(10 21)(11 48)(12 44)(14 15)(16 33)(18 47)(19 25)(20 37)(22 24)(23 46)(27 49)(28 34)(29 31)(32 43)(36 50)(38 39)(40 45).



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$Fig.1.1$

Chapter 2

QUOTIENTS OF THE TRIANGLE GROUP $\Delta(2,3,13)$

2.1 Introduction

The triangle group $\Delta(2,3,13)$ has a fundamental domain containing two copies of a hyperbolic triangle having angles $\frac{\pi}{2}, \frac{\pi}{3}, \frac{\pi}{13}$. Suppose vertices of the hyperbolic triangle are denoted by a, b and c, and R_i represent hyperbolic reflections in the hyperbolic sides M_i (i = 1, 2 and 13). Put $x = R_{13}R_3$ and $y = R_2R_{13}$, so that $yx = R_2R_{13}R_{13}R_3 = R_2R_3$. Then $R_{13}R_3$ is an anticlockwise hyperbolic rotation of π about a, R_2R_{13} is an anticlockwise hyperbolic rotation of $\frac{2\pi}{3}$ about b, and R_2R_3 is an anticlockwise hyperbolic rotation of $\frac{2\pi}{13}$ about c. Hence $x^2 = y^3 = (xy)^{13} = 1$, which are defining relations of the group $\Delta(2,3,13)$. We see that $\Delta(2,3,13)$ is a quotient group of $PSL(2,\mathbb{Z})$.

As mentioned in Chapter 1, $PSL(2,\mathbb{Z})$ is generated by the transformations x and y satisfying the relations:

$$x^2 = y^3 = 1.$$

Addition of the linear-fractional transformation $t : z \longrightarrow \frac{1}{z}$ to the group $PSL(2,\mathbb{Z})$

results in extension to the group $PGL(2,\mathbb{Z})$. The group $PGL(2,\mathbb{Z})$ is known as extended modular group and it has a finite presentation:

$$\langle x,t,y : x^2 = t^2 = y^3 = (xt)^2 = (yt)^2 = 1\rangle.$$

If q denotes a prime p power, then $PGL(2,q)$ is defined to be the group of all the linear-fractional transformations: $z \longmapsto \dfrac{az+b}{cz+d}$, where $ad-bc \neq 0$ and a, d, b, c are in F_q. The group $PSL(2,q)$ is a subgroup of $PGL(2,q)$, containing all the linear-fractional transformations: $z \longmapsto \dfrac{az+b}{cz+d}$, where $ad-bc$ is a non-zero square in F_q.

Suppose projective line over F_q is denoted by $PL(F_q)$. If $PGL(2,\mathbb{Z})$ acts on $PL(F_q)$, then every element of $PGL(2,q)$ is a permutation on $PL(F_q)$. The group $PGL(2,q)$ is then obviously a subgroup of S_{q+1}. Since elements of $PSL(2,q)$ provide even permutations, therefore, $PSL(2,q)$ is a subgroup of $A_{q+1.}$

It is a known fact that each of the conjugacy classes of the group $PGL(2,\mathbb{Z})$ actions on $PL(F_q)$ can be depicted by a coset diagram $D(\theta,q)$, where $\theta \in F_q$ and q is a prime p power. In this chapter, we investigate conditions in terms of θ and q, which ensure emergence of coset diagrams which represent quotients of the triangle group $\Delta(2,3,13)$ on $PL(F_q)$. We also find conditions for existence of some particular fragments in the coset diagrams as found in [39] for coset diagrams in case of the group $\Delta(2,3,11)$ actions on $PL(F_q)$. We employ the technique used in [39] to stitch together small coset diagrams representing the quotients of $\Delta(2,3,13)$ on $PL(F_q)$ to obtain more quotients of the same triangle group but of larger degree through the fragments.

There are three more sections in this chapter. In first section we discuss parametrization process of the group $PGL(2,\mathbb{Z})$ actions on $PL(F_q)$ as given in [34]. We explicitly describe the coset diagrams and construction of the diagrams through the parametrization process. Since we are interested in the diagrams which are permutation representations of quotients of $\Delta(2,3,13)$, therefore, first we obtain conditions in terms of θ and q which ensure emergence of only the required coset diagrams. In second section we discuss some fragments that frequently exist in the diagrams. Then we explain how new diagrams representing the homomorphic images can be obtained just by stitching together two or more diagrams with the help of available fragments in the diagrams. In the last section we find conditions for existence of some particular fragments in the diagrams. Finally, we give a list of linear-fractional transformations which can be employed to draw the diagrams which are quotients or homomorphic image representations of the group on prime fields F_p, where $p < 1300$.

2.2 Parametrization and Coset Diagrams

We use the parametrization process introduced in [34]. We recall the process here. A homomorphism $\alpha : PGL(2,\mathbb{Z}) \longrightarrow PGL(2,q)$ is called a non-degenerate homomorphism if none of the generators x, y and t of $PGL(2,\mathbb{Z})$ lies in the kernel of α, so that $\bar{x} = x\alpha$, $\bar{y} = y\alpha$ and $\bar{t} = t\alpha$ are of orders 2, 3 and 2 respectively ([34]). Any two non-degenerate homomorphisms α and β are called conjugate if there exists an inner automorphism ρ of $PGL(2,q)$ such that $\beta = \alpha\rho$ ([34]).

In [34] it was proved that conjugacy classes of non-degenerate homomorphisms from $PGL(2,\mathbb{Z})$ into $PGL(2,q)$ are in one to one correspondence with the elements $\theta \neq 0$, 3 of F_q, under the correspondence which maps each class to its parameter. This means it is possible to parametrize the conjugacy classes of non-degenerate homomorphisms $\alpha : PGL(2,\mathbb{Z}) \longrightarrow PGL(2,q)$, with the non-trivial elements of F_q. In other words, actions of $PGL(2,\mathbb{Z})$ on $PL(F_q)$ can be parametrized.

Let α be a non-degenerate homomorphism from the group $PGL(2,\mathbb{Z})$ into $PGL(2,q)$, and X, T and Y be elements of $GL(2,q)$ corresponding to the linear-fractional transformations $\bar{x}$, $\bar{t}$ and $\bar{y}$ in the group $PGL(2,q)$, where $\bar{x} = x\alpha$, $\bar{t} = t\alpha$, $\bar{y} = y\alpha$, and F_q is not of characteristic 2 or 3. Since orders of $\bar{x}$, $\bar{t}$ and $\bar{y}$ are 2, 2, and 3 respectively, therefore, the matrices X, T and Y are taken to be

$$X = \begin{bmatrix} a & kc \\ c & -a \end{bmatrix}, T = \begin{bmatrix} 0 & -k \\ 1 & 0 \end{bmatrix} \text{ and } Y = \begin{bmatrix} d & kf \\ f & -d-1 \end{bmatrix}$$

, where k, f, d, c, $a \in F_q$ with $k \neq 0$.

We put

$$a^2 + kc^2 = -\triangle \tag{2.1}$$

and require that

$$d^2 + d + kf^2 + 1 = 0. \tag{2.2}$$

This produces elements which satisfy the relations: $X^2 = \lambda_1 I$, $Y^3 = \lambda_2 I$ and $T^2 = \lambda_3 I$, where λ_1, λ_2 and λ_3 are non-zero scalars and I is the identity matrix. The class containing $\bar{x}\bar{y}$ is assigned to α by one-to-one correspondence. Therefore, α is determined by $\bar{x}\bar{y}$ and it is enough to investigate only the conjugacy class of $\bar{x}\bar{y}$. The matrix XY has the trace

$$r = a\,(2d+1) + 2kcf \tag{2.3}$$

If $tr\,(XYT) = ks$, then

$$s = 2af - c\,(2d+1) \tag{2.4}$$

so that

$$3\triangle = r^2 + ks^2 \tag{2.5}$$

and set

$$\theta = \frac{r^2}{\triangle}. \tag{2.6}$$

For a given pair of θ and q we can always find the matrices X, Y and T by the equations (2.1) to (2.6). The group $PGL(2,\mathbb{Z})$ action on $PL(F_q)$ involves $PGL(2,q)$, and the corresponding coset diagram yields a permutation representation of $PGL(2,q)$. In

[34], a mechanism was developed to find a unique coset diagram $D(\theta, q)$ corresponding to each $\theta \in F_q$. It is unique in the sense that actions corresponding to one conjugacy class produce exactly one coset diagram except labelling of the vertices. Such a diagram is defined as follows: small triangles are used to represent 3-cycles of y; vertices of the triangles are permuted anticlockwise by y; an edge is used to connect an two vertices which are interchanged by x; reflection about the vertical line of symmetry represents action of t; heavy dots denote the fixed points of x and y (if exist). Notice that $(yt)^2 = 1$ is equivalent to $tyt = y^{-1}$, which means that t reverses orientation of the triangles representing 3-cycles of y (as reflection does); because of this, there is no need to make the diagram more complicated by introducing t-edges. These diagrams are called coset diagrams because vertices in the diagrams are identifiable with the right cosets in $PGL(2,\mathbb{Z})$ of the stabilizer N of any given point of $PL(F_q)$, so that y or x connects the coset Ngy or Ngx $(g \in PGL(2,\mathbb{Z}))$.

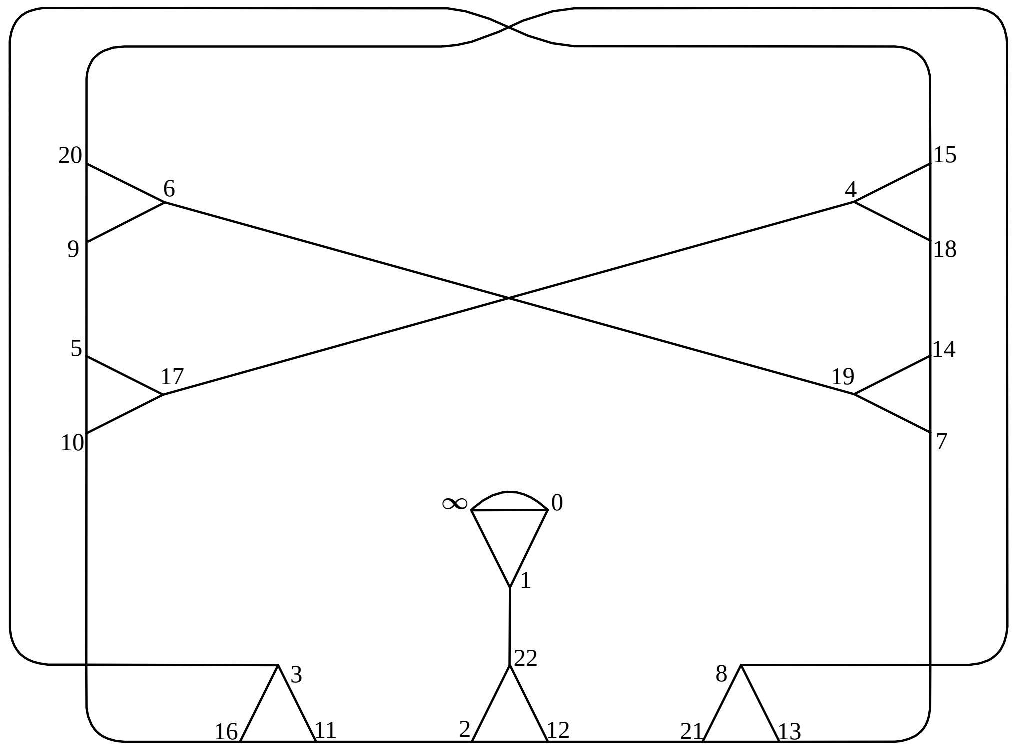
For instance, consider an action of the group $PGL(2,\mathbb{Z})$ on $PL(F_{23})$. The linear-fractional transformations:

$$\bar{x} : z \longrightarrow \frac{-1}{z},\ \bar{t} : z \longrightarrow \frac{1}{z} \text{ and } \bar{y} : z \longrightarrow \frac{z-1}{z}$$

give the permutation representation:

$$\begin{aligned}
\bar{x} &: (0\ \infty)(1\ 22)(2\ 11)(3\ 15)(4\ 17)(5\ 9)(6\ 19)(7\ 13)(8\ 20)(10\ 16)(12\ 21)(14\ 18) \\
\bar{t} &: (0\ \infty)(1)(2\ 12)(3\ 8)(4\ 6)(5\ 14)(7\ 10)(9\ 18)(11\ 21)(13\ 16)(15\ 20)(17\ 19)(22) \\
\bar{y} &: (0\ \infty\ 1)(2\ 12\ 22)(3\ 16\ 11)(4\ 18\ 15)(5\ 10\ 17)(6\ 20\ 9)(7\ 14\ 19)(8\ 21\ 13).
\end{aligned}$$

Coset diagram of the action is as follows:



$Fig.2.1$

We look for those conjugacy classes of non-degenerate homomorphisms α, which evolve pairs of the linear-fractional transformations $\bar{y}$ and $\bar{x}$ satisfying the relations: $\bar{y}^3 = \bar{x}^2 = (\bar{x}\bar{y})^{13} = 1$. So we establish a condition in the next theorem that ensures emergence of only our required pairs of linear-fractional transformations as established in [39] for the group $\Delta(2, 3, 11)$.

Theorem 2 *For each zero of* $f(z) = z^6 - 11z^5 + 45z^4 - 84z^3 + 70z^2 - 21z + 1$ *in* F_p, *where* $f(z) \in \mathbb{Z}[z]$ *and* p *is a prime number such that* $p = 13$ *or* $p \equiv \pm 1 \pmod{13}$, *there exists a conjugacy class of non-degenerate homomorphisms* α *such that* $\alpha(PGL(2, \mathbb{Z})) = \langle \bar{x}, \bar{y} : \bar{x}^2 = \bar{y}^3 = (\bar{x}\bar{y})^{13} = 1 \rangle$.

Proof. *Since* $p = 13$ *or* $p \equiv \pm 1 \,(\mathrm{mod}\, 13)$*, therefore, there exist six distinct traces* r_6*,* r_5*,* r_4*,* r_3*,* r_2*,* r_1 *of elements of the group* $SL(2,p)$*, which produce* 13 *order elements in* $PGL\,(2,\ p)$ *([27]). Thus, there are six conjugacy classes of non-degenerate homomorphisms from* $\Delta\,(2,3,13)$ *into the group* $PGL\,(2,\ p)$*. Every element of* $PSL\,(2,p)$*, which comes from an element of* $SL(2,p)$ *having trace* r_6*,* r_5*,* r_4*,* r_3*,* r_2*, or* r_1 *has order* 13*. Now suppose* A *is any element of* $SL(2,p)$ *having trace* $r = r_6, r_5, r_4, r_3, r_2,$ *or* r_1*. Since* A *is a conjugate in* $GL\,(2,\ p^2)$ *to a matrix* B *of the form*

$$\begin{bmatrix} \rho & 0 \\ 0 & \rho^{-1} \end{bmatrix}, \text{ where } \rho \text{ is a primitive } 13th \text{ root of unity in } F_{p^2}$$

, therefore, $tr(B) = tr(A) = r = \rho + \rho^{-1}$*. Next,* $r^2 = \left(\rho + \rho^{-1}\right)^2 = \rho^2 + \rho^{-2} + 2$*, so* $r^2 - 2 = \rho^2 + \rho^{-2}$*, which is the trace of* B^2*, and* $r^3 = \left(\rho + \rho^{-1}\right)^3 = \rho^3 + 3\rho + 3\rho^{-1} + \rho^{-3}$*, so* $r^3 - 3r = \rho^3 + \rho^{-3}$*, which is the trace of* B^3*, and* $\left(r^2 - 2\right)^2 = \left(\rho^2 + \rho^{-2}\right)^2$ *implies that* $r^4 - 4r^2 + 4 = \rho^4 + \rho^{-4} + 2$*, so* $r^4 - 4r^2 + 2 = \rho^4 + \rho^{-4}$*, which is the trace of* B^4*. Similarly we get* $r^5 - 5r^3 + 5r = \rho^5 + \rho^{-5}$*, which is the trace of* B^5*, and* $r^6 - 6r^4 + 9r^2 - 2 = \rho^6 + \rho^{-6}$*, which is the trace of* B^6*.*

Since $\rho^{13} = 1$*, therefore,* $(\rho - 1)\left(\rho^{12} + \rho^{11} + \rho^{10} + \rho^9 + \rho^8 + \rho^7 + \rho^6 + \rho^5 + \rho^4 + \rho^3 + \rho^2 + \rho + 1\right) = 0$*. But* $\rho \neq 1$*, which implies that* $\rho^{12} + \rho^{11} + \rho^{10} + \rho^9 + \rho^8 + \rho^7 + \rho^6 + \rho^5 + \rho^4 + \rho^3 + \rho^2 + \rho + 1 = 0$*. We have* $\rho^{12} = \rho^{-1}$*,* $\rho^{11} = \rho^{-2}$*,* $\rho^{10} = \rho^{-3}$*,* $\rho^9 = \rho^{-4}$*,* $\rho^8 = \rho^{-5}$*,* $\rho^7 = \rho^{-6}$*. So by substituting in the equation* $\rho^{12} + \rho^{11} + \rho^{10} + \rho^9 + \rho^8 + \rho^7 + \rho^6 + \rho^5 + \rho^4 + \rho^3 + \rho^2 + \rho + 1 = 0$*, we get* $\left(\rho^6 + \rho^{-6}\right) + \left(\rho^5 + \rho^{-5}\right) + \left(\rho^4 + \rho^{-4}\right) + \left(\rho^3 + \rho^{-3}\right) + \left(\rho^2 + \rho^{-2}\right) + \left(\rho + \rho^{-1}\right) + 1 = 0$*, which implies that* $r^6 + r^5 - 5r^4 - 4r^3 + 6r^2 + 3r - 1 = 0$*. Since* $det(B) = \triangle = 1$ *and* $tr(B) = r$*, we put* $\theta = r^2$ *in the equation to obtain* $\theta^6 - 11\theta^5 + 45\theta^4 - 84\theta^3 + 70\theta^2 - 21\theta + 1 = 0$*. Thus* $\bar{x}\bar{y}$ *has order* 13 *if and only if* $f\,(\theta) = \theta^6 - 11\theta^5 + 45\theta^4 - 84\theta^3 + 70\theta^2 - 21\theta + 1 = 0$*.*

Above theorem can be proved alternatively as: ■

Proof. Let X, Y, and XY be the matrices in $GL\,(2,\ p^2)$ corresponding to the elements $\bar{x}$, $\bar{y}$ and $\bar{x}\bar{y}$ respectively. Now characteristic equation of XY can be written as:

$$(XY)^2 - rXY + \triangle I = 0$$

This implies that:

$$\begin{aligned}
(XY)^2 &= rXY - \triangle I \\
(XY)^4 &= \left(r^2 - \triangle\right)(XY)^2 - r \triangle XY \\
&= \left(r^2 - \triangle\right)(rXY - \triangle I) - r \triangle XY \\
&= \left(r^3 - 2r\triangle\right) XY + \left(- \triangle r^2 + \triangle^2\right) I \\
(XY)^5 &= \left(r^3 - 2r\triangle\right)(XY)^2 + \left(- \triangle r^2 + \triangle^2\right) XY \\
&= \left(r^3 - 2r\triangle\right)(rXY - \triangle I) + \left(- \triangle r^2 + \triangle^2\right) XY \\
&= \left(r^4 - 3 \triangle r^2 + \triangle^2\right) XY + \left(-r^3 \triangle + 2r\triangle^2\right) I \\
(XY)^6 &= \left(r^5 - 4r^3 \triangle + 3r\triangle^2\right) XY + \left(-r^4 \triangle + 3r^2 \triangle^2 - \triangle^3\right) I \\
(XY)^7 &= \left(r^6 - 5r^4 \triangle + 6r^2 \triangle^2 - \triangle^3\right) XY + \left(-r^5 \triangle + 4r^3 \triangle^2 - 3r\triangle^3\right) I \\
(XY)^8 &= \left(r^7 - 6r^5 \triangle + 10r^3 \triangle^2 - 4r\triangle^3\right) XY + (- \triangle r^6 + 5r^4 \triangle^2 - 6r^2 \triangle^3 + \triangle^4) I \\
(XY)^9 &= \left(r^8 - 7 \triangle r^6 + 15r^4 \triangle^2 - 10r^2 \triangle^3 + \triangle^4\right) XY + (-r^7 \triangle + 6r^5 \triangle^2 - \\
&\quad 10r^3 \triangle^3 + 4r\triangle^4) I \\
(XY)^{10} &= \left(r^9 - 8r^7 \triangle + 21r^5 \triangle^2 - 20r^3 \triangle^3 + 5r\triangle^4\right) XY + (-r^8 \triangle + 7 \triangle^2 r^6 - \\
&\quad 15r^4 \triangle^3 + 10r^2 \triangle^4 - \triangle^5) I \\
(XY)^{11} &= \left(r^{10} - 9r^8 \triangle + 28 \triangle^2 r^6 - 35r^4 \triangle^3 + 15r^2 \triangle^4 - \triangle^5\right) XY + (-r^9 \triangle + \\
&\quad 8r^7 \triangle^2 - 21r^5 \triangle^3 + 20r^3 \triangle^4 - 5r\triangle^5) I
\end{aligned}$$

Continuing in the similar way we get

$$
\begin{aligned}
(XY)^{12} &= \left(r^{11} - 10r^{9}\triangle + 36r^{7}\triangle^{2} - 56r^{5}\triangle^{3} + 35r^{3}\triangle^{4} - 6r\triangle^{5}\right) XY + (-r^{10}\triangle + \\
&\quad 9r^{8}\triangle^{2} - 28\triangle^{3} r^{6} + 35r^{4}\triangle^{4} - 15r^{2}\triangle^{5} + \triangle^{6})I \\
(XY)^{13} &= \left(r^{12} - 11r^{10}\triangle + 45r^{8}\triangle^{2} - 84r^{6}\triangle^{3} + 70r^{4}\triangle^{4} - 21r^{2}\triangle^{5} + \triangle^{6}\right) XY - \\
&\quad \left(r^{11}\triangle - 10r^{9}\triangle^{2} + 36r^{7}\triangle^{3} - 56r^{5}\triangle^{4} + 35r^{3}\triangle^{5} - 6r\triangle^{6}\right) I
\end{aligned}
$$

But $(XY)^{13} = \lambda I$, so we must have:

$$r^{12} - 11r^{10}\triangle + 45r^{8}\triangle^{2} - 84r^{6}\triangle^{3} + 70r^{4}\triangle^{4} - 21r^{2}\triangle^{5} + \triangle^{6} = 0.$$

But $r^2 = \triangle\theta$, so

$$\theta^{6}\triangle^{6} - 11\theta^{5}\triangle^{6} + 45\theta^{4}\triangle^{6} - 84\theta^{3}\triangle^{6} + 70\theta^{2}\triangle^{6} - 21\theta\triangle^{6} + \triangle^{6} = 0$$

or

$$\theta^{6} - 11\theta^{5} + 45\theta^{4} - 84\theta^{3} + 70\theta^{2} - 21\theta + 1 = 0,$$

which is the required condition for $(\bar{x}\bar{y})^{13} = 1$. The zeros of the polynomial $f(\theta) = \theta^{6} - 11\theta^{5} + 45\theta^{4} - 84\theta^{3} + 70\theta^{2} - 21\theta + 1$ lie in F_p if $p = 13$ or $p = 13m \pm 1$, where $m \in \mathbb{Z}$. ■

We employ together Theorem 2 and the technique used in [34] to obtain the coset diagrams depicting only those actions of the group $PGL(2,\mathbb{Z})$ on $PL(F_q)$, which for a suitable q, evolve finite quotients or homomorphic images of the group $\Delta(2,3,13)$ as subgroups of $PGL(2,q)$.

For each zero of $f(\theta)$ in F_p, we can draw a coset diagram. For example, $f(\theta) = 0$ has one repeated root 4 in F_{13}, six roots 10, 11, 13, 15, 28 and 40 in F_{53}, and six roots 8, 13, 20, 36, 42 and 50 in F_{79}. We draw a coset diagram for each of these zeros of $f(\theta)$. If we take $\theta = 4$ in F_{13}, then the linear fractional transformations $\bar{x}$, $\bar{t}$ and $\bar{y}$ are $z \longmapsto \dfrac{1}{z}$,

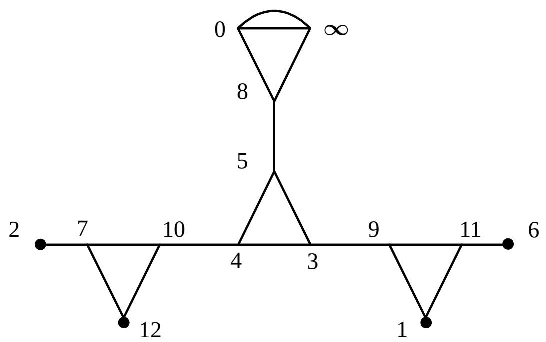
$z \longmapsto \dfrac{-1}{z}$ and $z \longmapsto \dfrac{5}{5z-1}$ respectively. The linear fractional transformations $\bar{x}$, $\bar{t}$ and $\bar{y}$ give the permutation representation:

$\bar{x} : (0\ \infty)\,(1)\,(7\ 2)\,(8\ 5)\,(9\ 3)\,(10\ 4)\,(11\ 6)\,(12)$

$\bar{y} : (0\ 8\ \infty)(1\ 11\ 9)(2)(3\ 5\ 4)(5\ 4\ 3)(6)(7\ 12\ 10)(11\ 9\ 1)(12\ 10\ 7)$

$\bar{t} : (0\ \infty)\,(4\ 3)\,(5)\,(6\ 2)\,(8)\,(10\ 9)\,(6\ 44)\,(11\ 7)\,(12\ 1)$

We therefore draw the coset diagram $D(4, 13)$ (shown as $(Fig.2.2)$), which has each vertex fixed by $(\bar{x}\bar{y})^{13}$. Thus, it is a diagram depicting $\alpha\,(\Delta\,(2, 3, 13))$.



$Fig.2.2$

If we take $\theta = 10$ in F_{53}, then the linear fractional transformations $\bar{x}$, $\bar{t}$ and $\bar{y}$ are $\dfrac{34z+3}{3z-34}$, $\dfrac{-1}{z}$ and $\dfrac{23}{23z-1}$ respectively. The linear fractional transformations $\bar{x}$, $\bar{t}$ and $\bar{y}$ give the permutation representation:

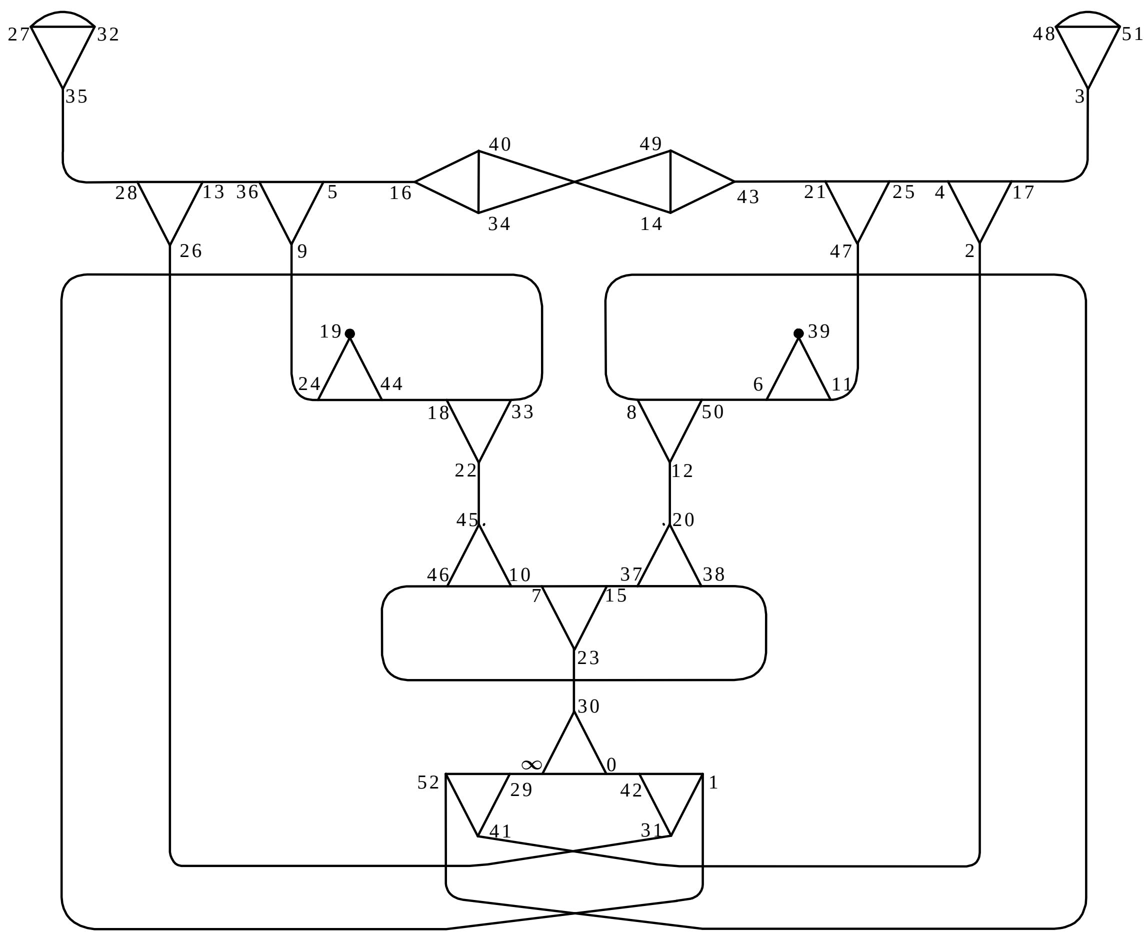
$\bar{x} : (10\ 7)(16\ 5)(17\ 3)(19)(20\ 12)(24\ 9)(25\ 4)(29\ \infty)(30\ 23)(31\ 26)(32\ 27)(33\ 1)$
$(35\ 28)(36\ 13)(37\ 15)(39)(40\ 14)(41\ 2)(42\ 0)(43\ 21)(44\ 18)(45\ 22)(46\ 38)(47\ 11)$
$(49\ 34)(50\ 6)(51\ 48)(52\ 8)$

$\bar{y} : (0\ 30\ \infty)\,(1\ 42\ 31)(2\ 17\ 4)(3\ 51\ 48)(5\ 36\ 9)(7\ 23\ 15)(11\ 39\ 6)(12\ 50\ 8)(13\ 28\ 26)$

(21 47 25)(22 33 18)(24 44 19)(29 52 41)(34 40 16)(35 32 27)(37 38 20)(43 49 14)
(45 46 10)

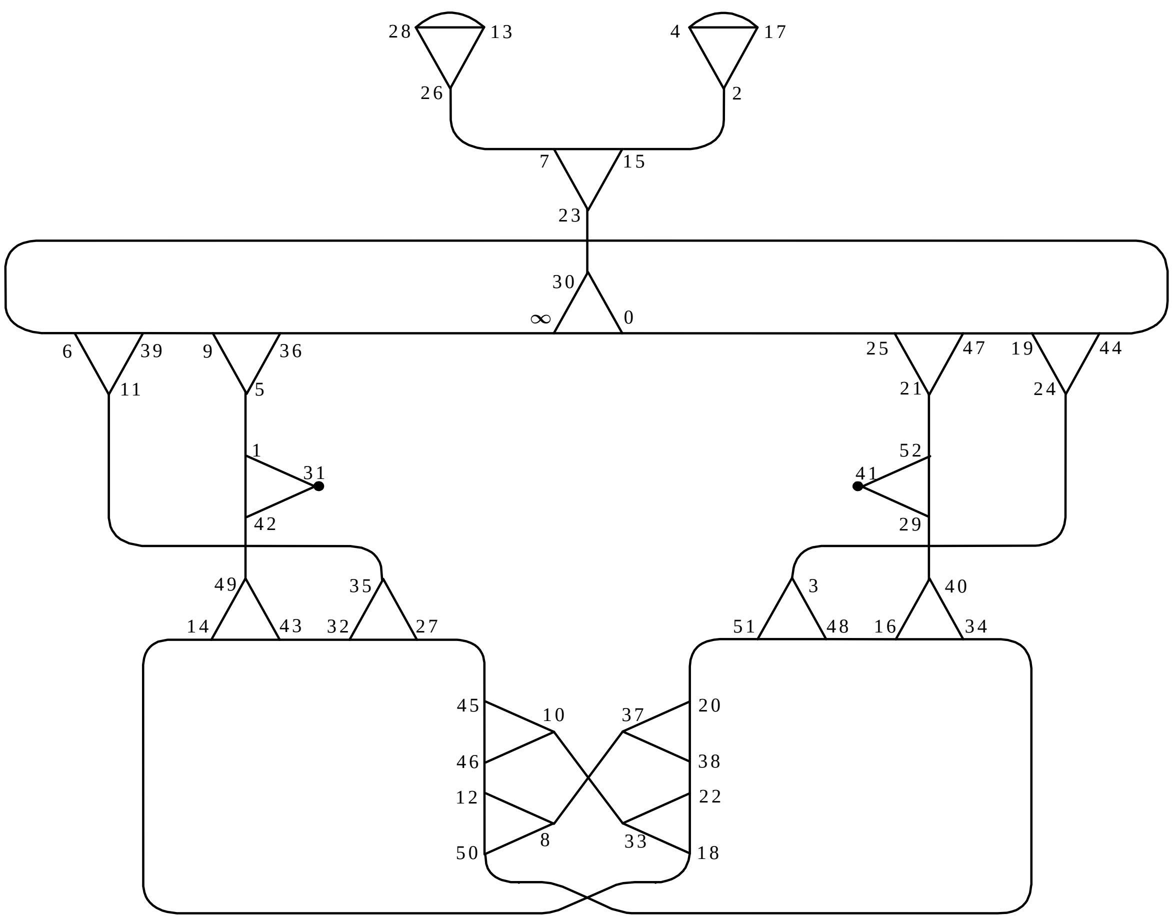
$\bar{t}$: $(0\ \infty)$ (13 4)(15 7)(21 5)(22 12)(23)(24 11)(26 2)(28 17)(30)(33 8)(34 14)(35 3)
(36 25)(37 10)(39 19)(41 31)(42 29)(43 16)(44 6)(45 20)(46 38)(47 9)(48 32)(49 40)
(50 18)(51 27)(52 1)

We therefore draw the coset diagram $D(10, 53)$ (shown as $(Fig.2.3)$) which has each vertex fixed by $(\bar{x}\bar{y})^{13}$. Thus, it is a diagram depicting $\alpha\,(\Delta\,(2, 3, 13))$.

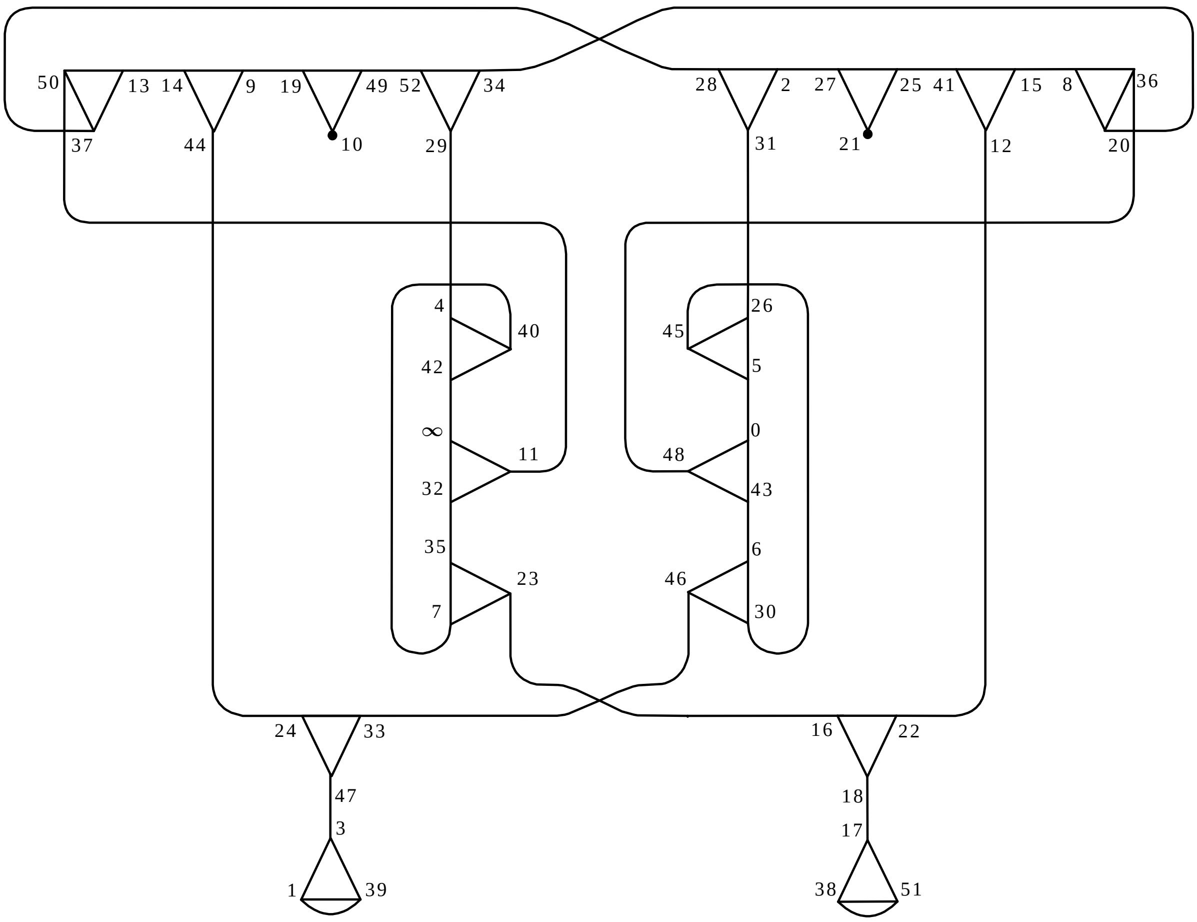


$Fig.2.3$

We repeat the process for each zero of $f(\theta)$ in F_{53} and F_{79} to get the transformations $\bar{x}$, $\bar{t}$ and $\bar{y}$, and permutation representation for each of the zeros. Then we draw a coset diagram for each permutation representation. We get the diagrams $D(11, 53)$ (shown as $Fig.1.1$ in example 1 in chapter 1), $D(13, 53)$ (shown as $Fig.2.4$), $D(15, 53)$ (shown as $Fig.2.5$), $D(28, 53)$ (shown as $Fig.2.6$), $D(40, 53)$ (shown as $Fig.2.7$), $D(8, 79)$ (shown as $Fig.2.8$), $D(13, 79)$ (shown as $Fig.2.9$), $D(20, 79)$ (shown as $Fig.2.10$), $D(36, 79)$ (shown as $Fig.2.11$), $D(40, 79)$ (shown as $Fig.2.12$), and $D(50, 79)$ (shown as $Fig.2.13$).



$Fig.2.4$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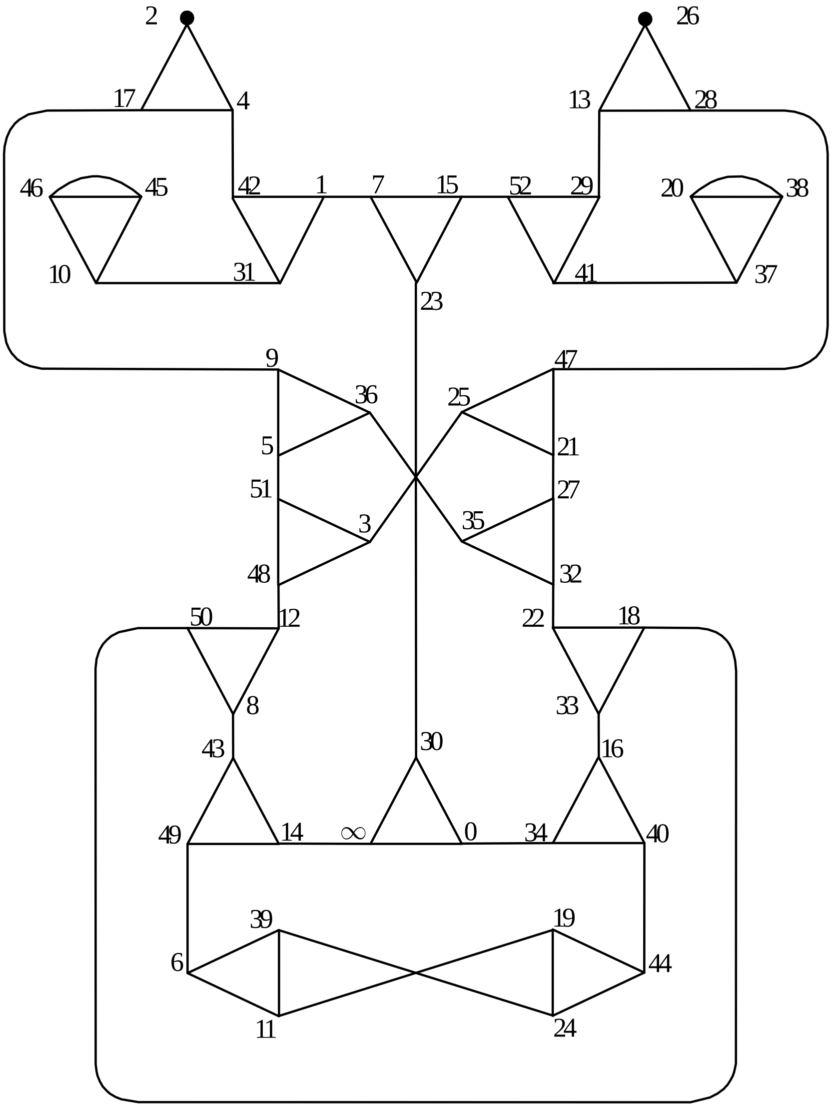
$Fig.2.5$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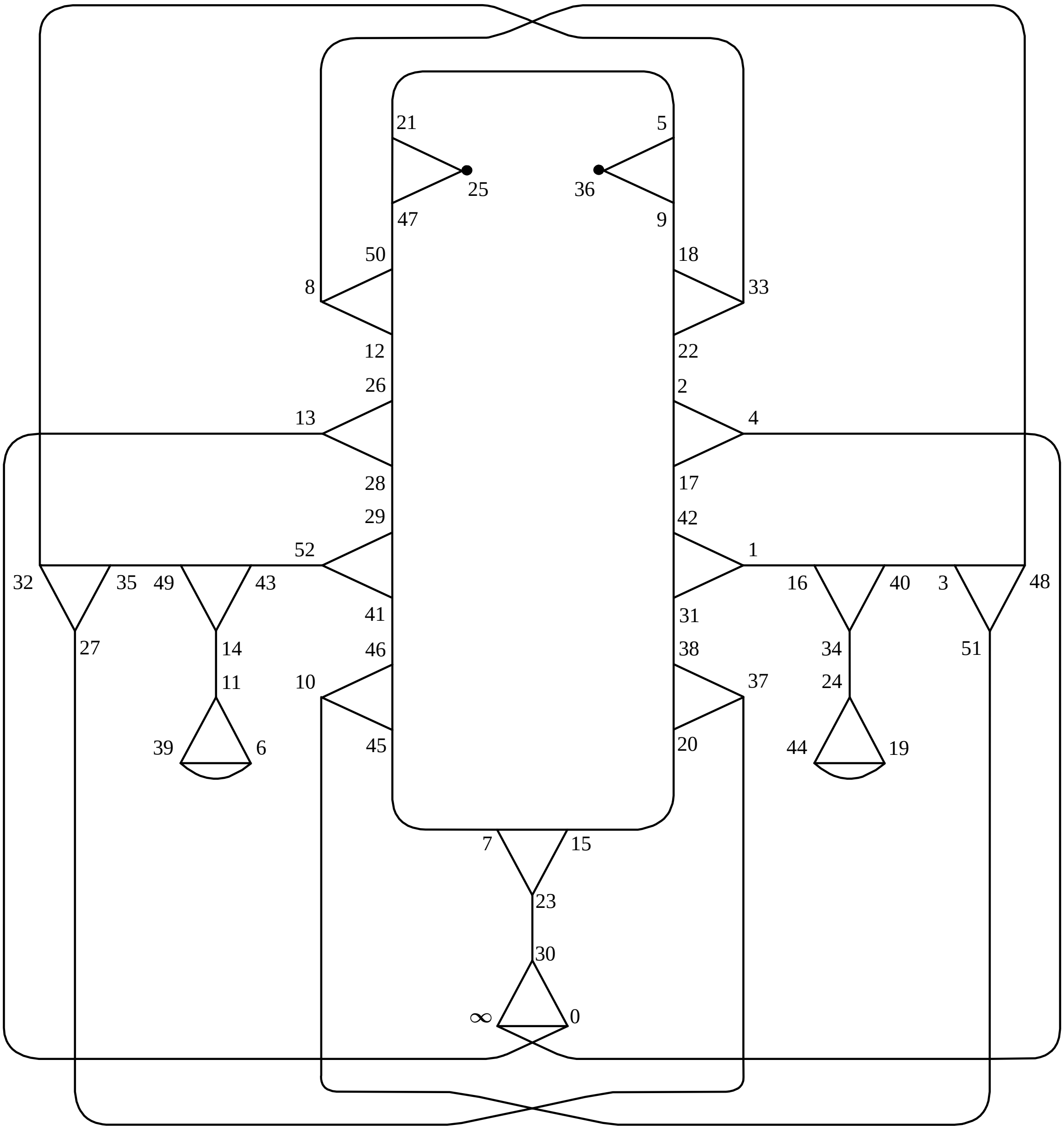
$Fig. 2.6$



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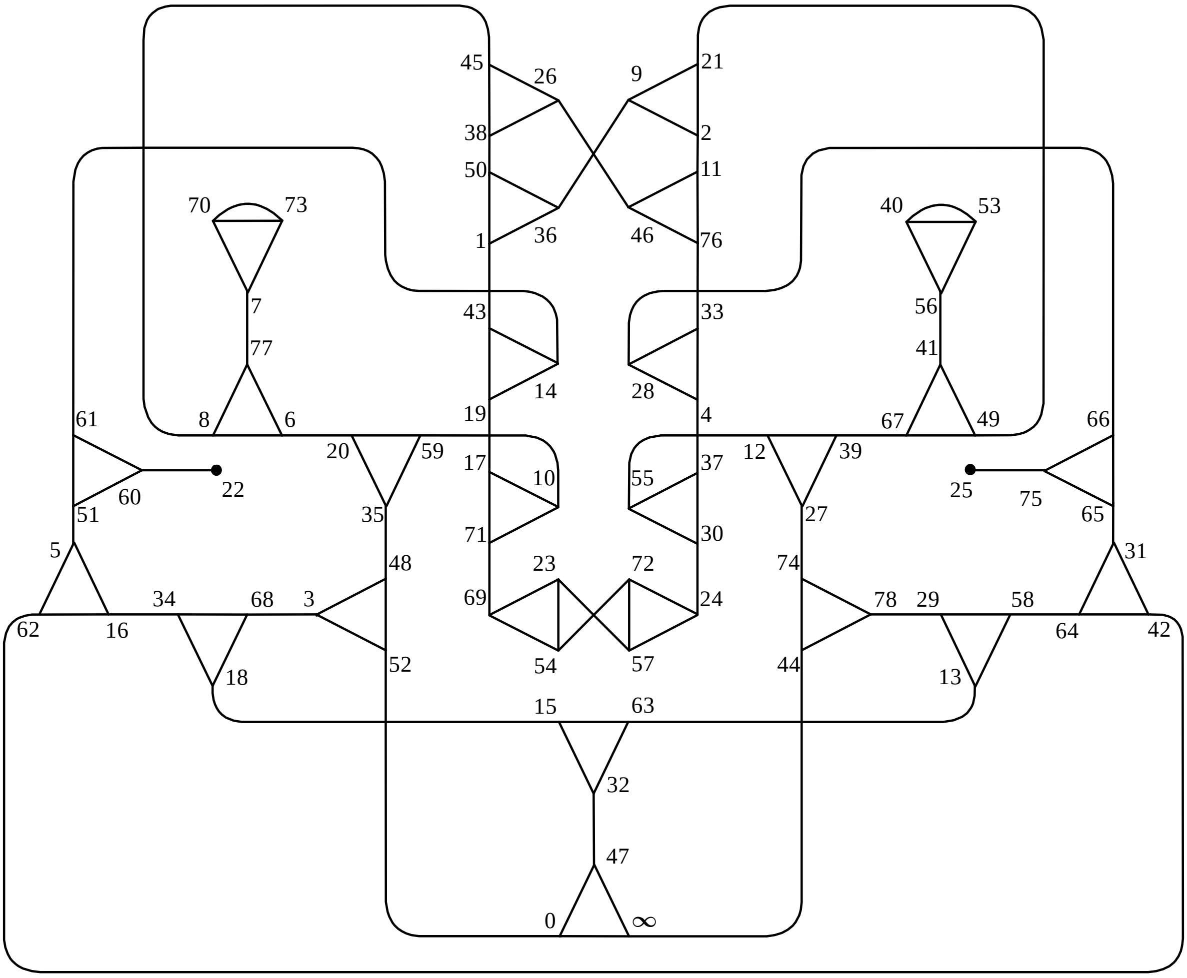
$Fig.2.7$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$Fig.2.8$

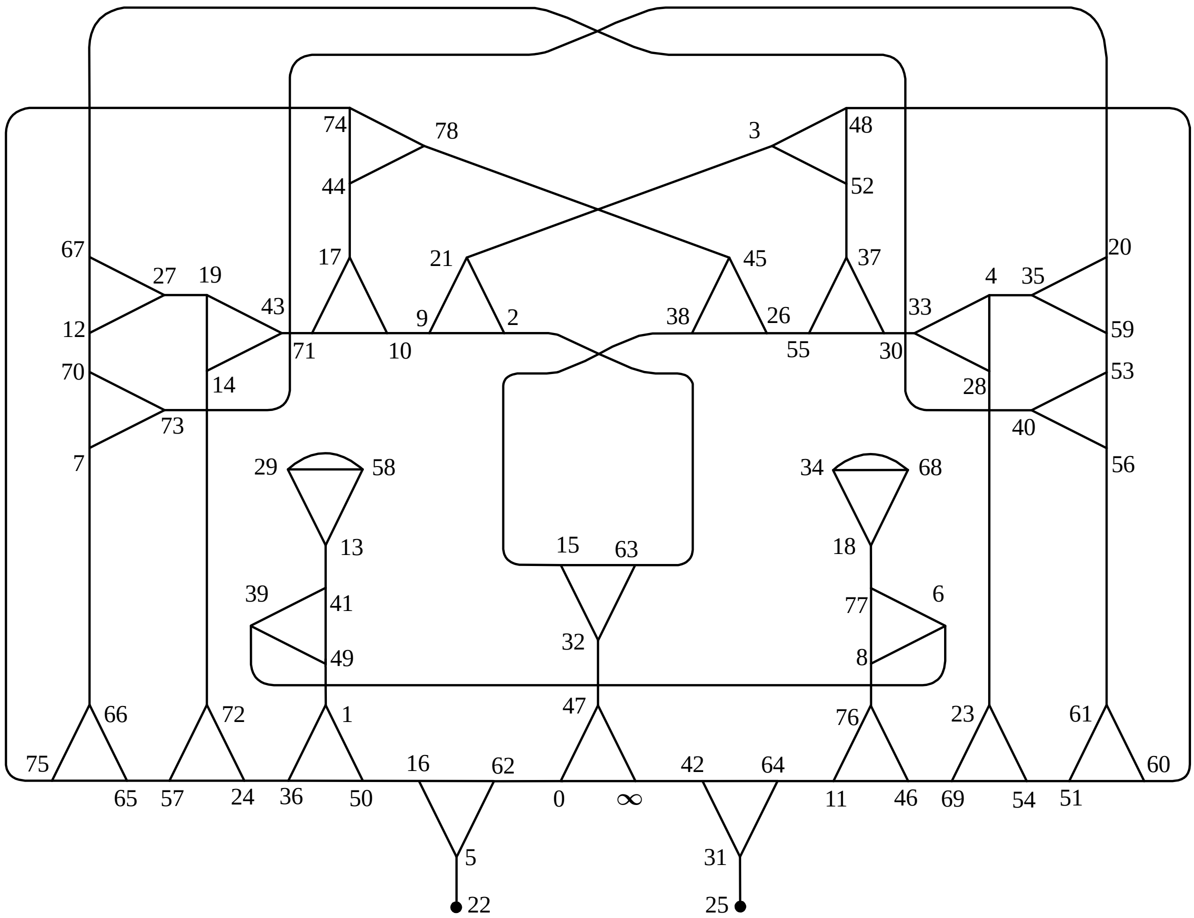
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*Fig.*2.9

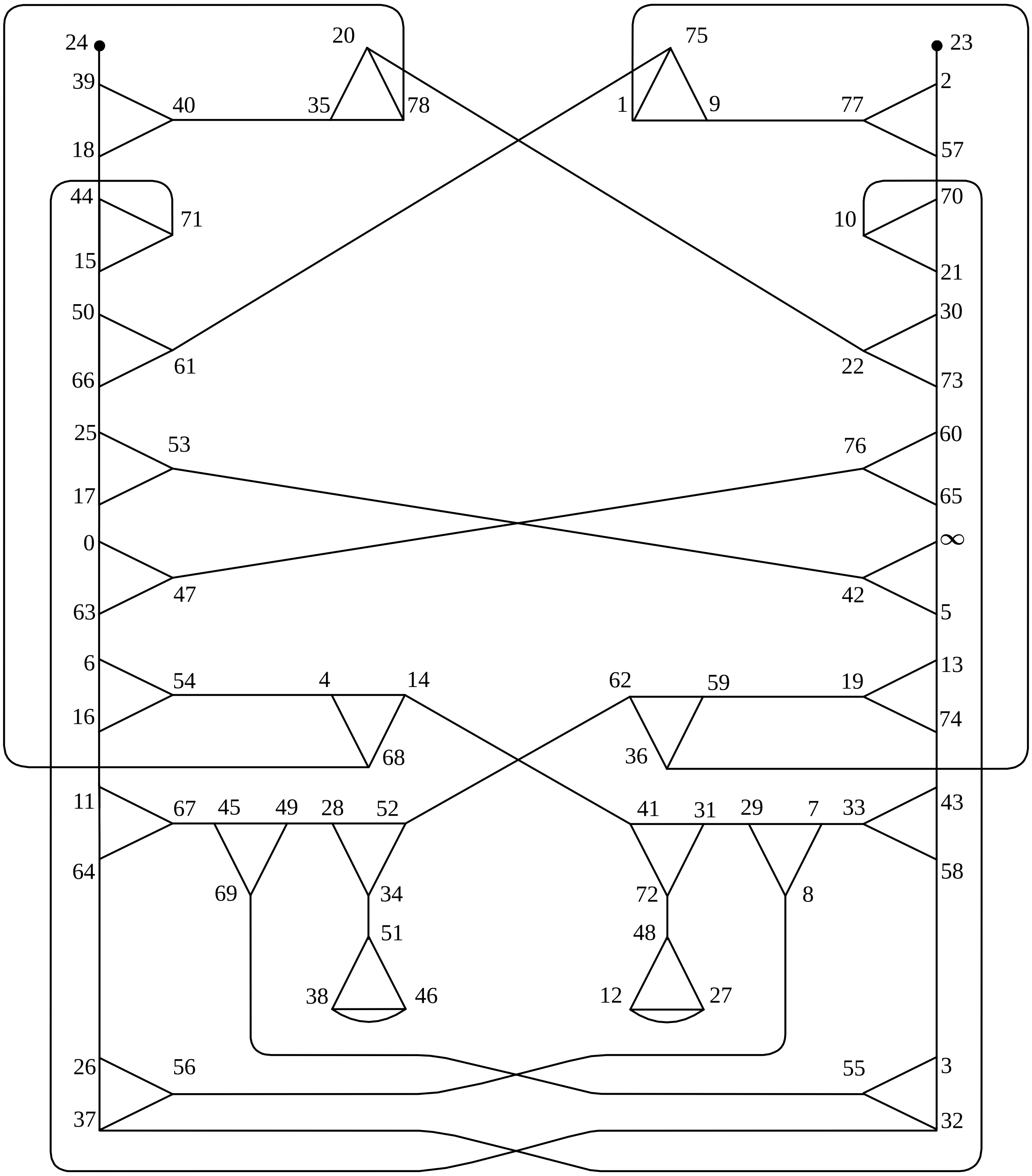

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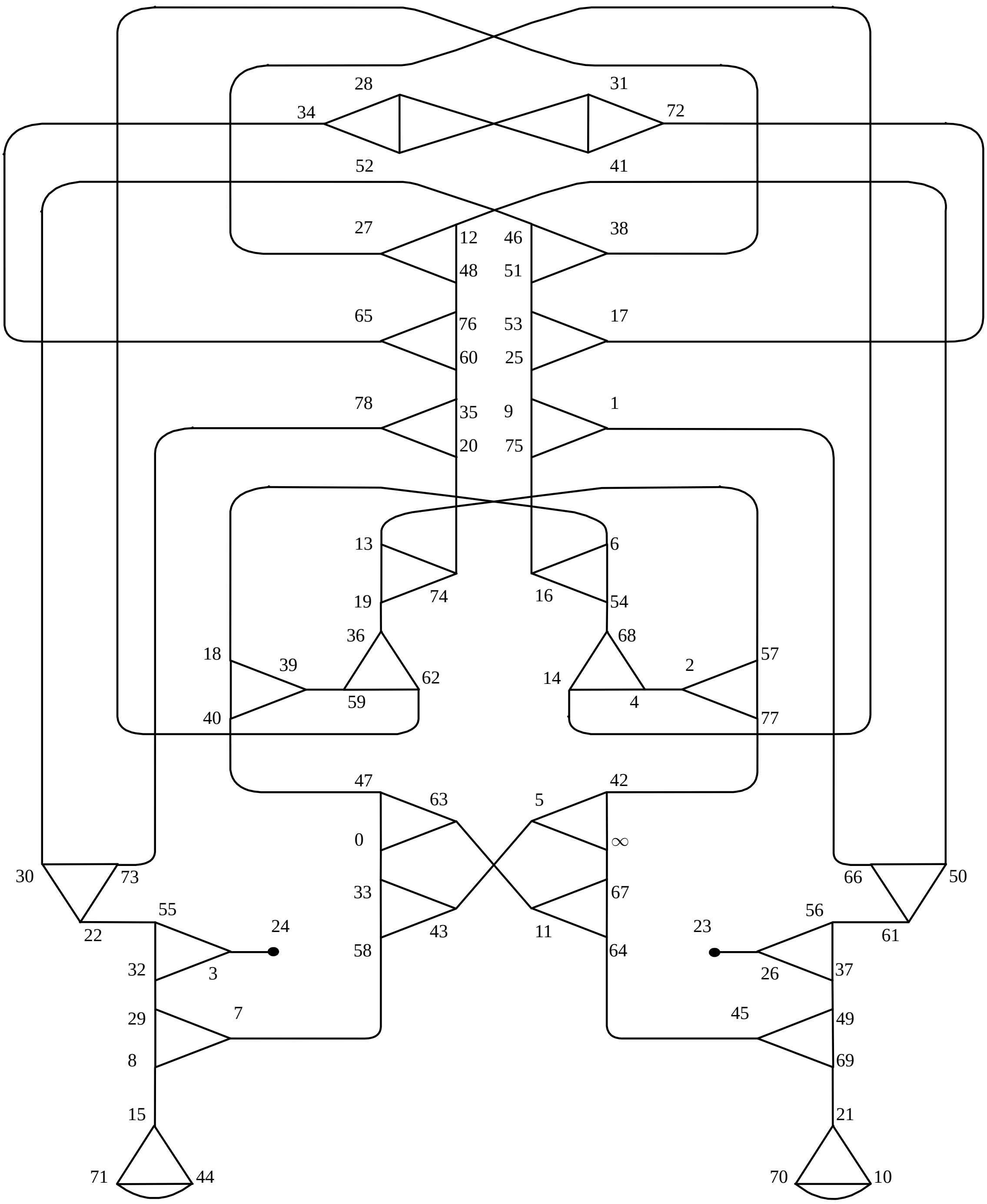
*Fig.*2.10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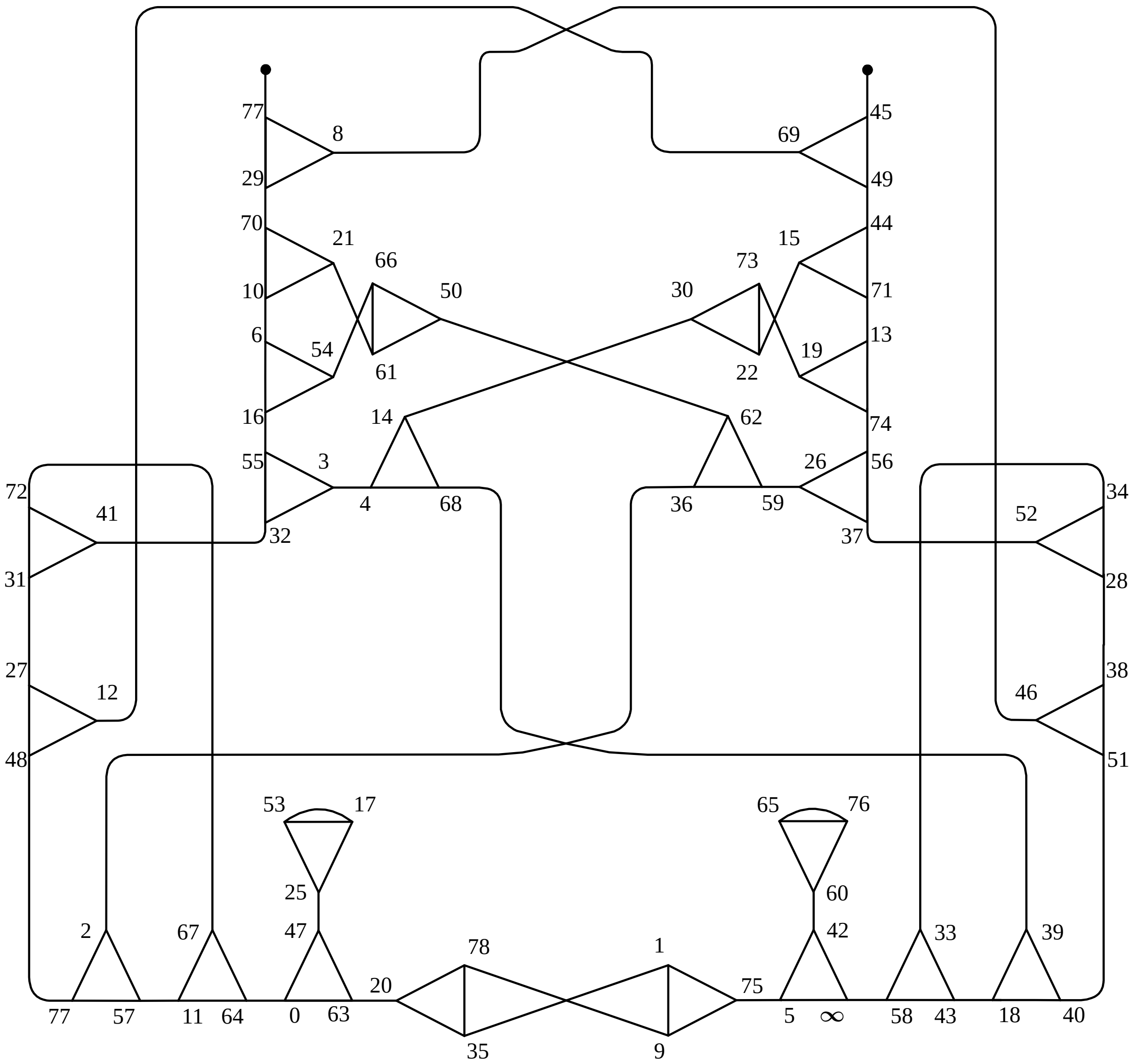
$Fig.2.11$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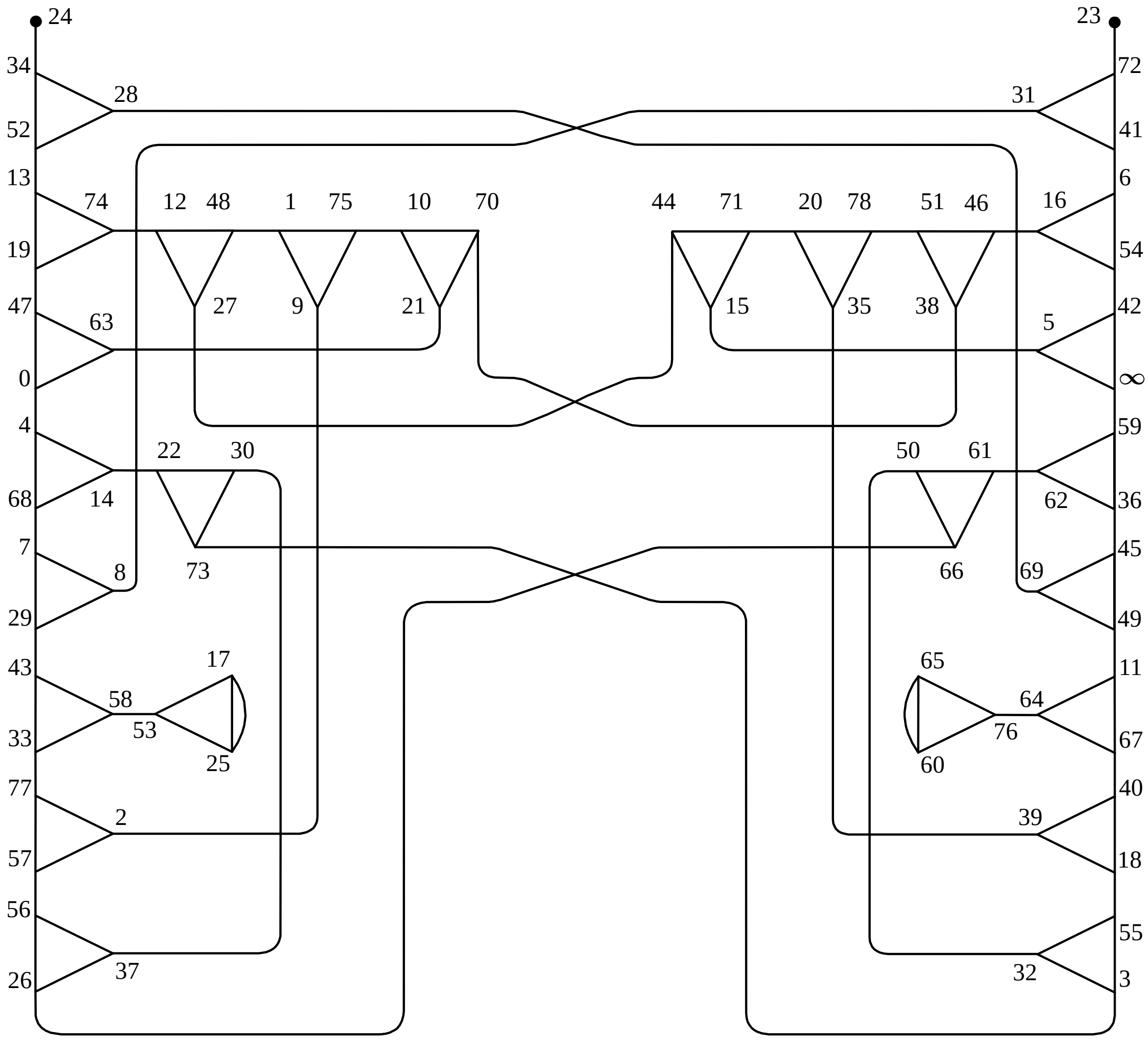
$Fig.2.12$

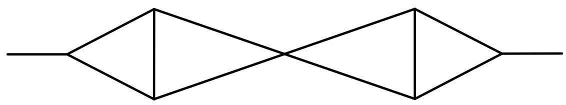


$Fig.2.13$

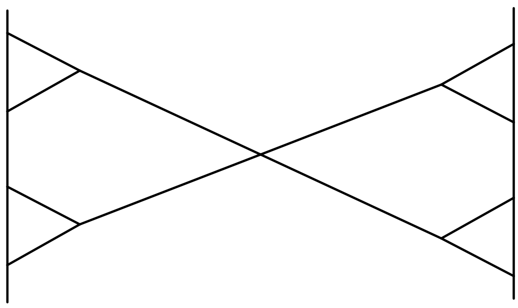
We can see some specific fragments exist in the above diagrams. If we become able to connect two or more diagrams somehow in such a way that the resulting diagram also satisfies the existing relations: $x^2 = y^3 = (xy)^{13} = 1$, then more homomorphic images of $\Delta(2, 3, 13)$ can be got directly. In other words, by connecting smaller graphs representing groups of smaller degree we can obtain a bigger graph representing a group of larger degree. This is possible with the help of the fragments. In the next section we explain in detail how to connect the diagrams using the fragments.

2.3 Connecting Coset Diagrams

Any two coset diagrams can be joined together to obtain a coset diagram of an arbitrary size provided that they are connected in a special way. The new coset diagram thus produced still preserves all the inherited properties of the component coset diagrams. To connect two or more coset diagrams together we need the diagrams containing fragments $\gamma_1(Fig.2.14)$ and $\gamma_2(Fig.2.15)$. These fragments can be observed in some of the coset diagrams drawn for quotients of the group $\Delta(2,3,13)$ in section 2.2. They do not exist in all the coset diagrams. So it is important to know when do they exist in a coset diagram. The conditions for existence of the fragments γ_1 and γ_2 in a coset diagram are given in [31].



$Fig.2.14$



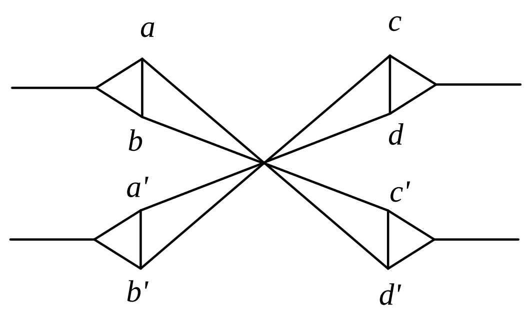
$Fig.2.15$

We note that $D(11,53)\,(Fig.1.1)$ and $D(36,79)\,(Fig.2.11)$ are the coset diagrams each containing both the fragments γ_1 and γ_2. These two coset diagrams can be connected using the fragments γ_1 and γ_2 to get more more quotients of $\Delta(2,3,13)$. We connect them in the next theorem.

By $|D(\theta,q)|$ we mean the number of vertices in $D(\theta,q)$, and by $PL(F_{q_1}) \cup PL(F_{q_2})$ we mean a set having q_1+q_2+2 elements.

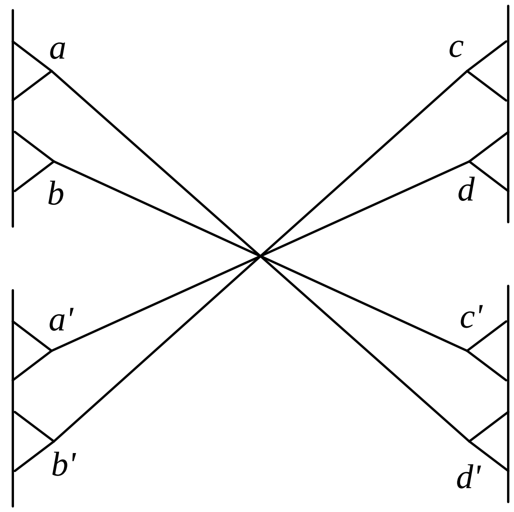
Theorem 3 *Let $\acute{l}$ and $\acute{m}$ be the number of copies of the coset diagrams $D(11,53)$ and $D(36,79)$ respectively, containing both γ_1 and γ_2. Then for all n which are expressible as $n = \acute{l}\,|D(11,53)| + \acute{m}\,|D(36,79)|$, the coset diagram $D(n)$ of n vertices depicts $\alpha(\Delta(2,3,13))$.*

Proof. We pick one of the coset diagrams $D(11,53)$ and $D(36,79)$, and label vertices of its fragment γ_1 with a, b, c and d. We pick another copy of $D(11,53)$ or $D(36,79)$, and label vertices of its fragment γ_1 with a', b', c' and d'. Then we put one diagram below the other on a common vertical axis of symmetry, and connect a to d', b to c', c to b', and d to a' by $x-$edges (as shown below in $Fig.2.16$).



$Fig.2.16$

Similarly, we put one copy of $D(11,53)$ or $D(36,79)$ containing the fragment γ_2 labelled a, b, c and d, below another copy of $D(11,53)$ or $D(36,79)$ containing the fragment γ_2 labelled a', b', c' and d', on a common vertical axis of symmetry, and connect a to d', b to c', c to b' and d to a' by $x-$ edges (as shown below in $Fig.2.17$).



$Fig.2.17$

The resulting coset diagram is one of the following:

1. $D(11,53)+D(11,53)$
2. $D(11,53)+D(36,79)$
3. $D(36,79)+D(36,79)$

and it depicts an action of $PGL(2,\mathbb{Z})$ on $PL(F_{53})\cup PL(F_{53})$ having 108 elements or on $PL(F_{53})\cup PL(F_{79})$ having 134 elements or on $PL(F_{79})\cup PL(F_{79})$ having 160 elements respectively. Let $(\tau,c_1,c_2,...,c_{j-1},\sigma,c_j,...,c_{q-2})$ and $(\mu,d_1,d_2,...,d_{j-1},\lambda,d_j,...,d_{q-2})$ be appropriate cycles of $\bar{x}\bar{y}$ in $\alpha(\Delta(2,3,13))$ depicted by $D(11,53)$ and $D(11,53)$ or $D(11,53)$ and $D(36,79)$ or $D(36,79)$ and $D(36,79)$. Then $(\mu,c_1,c_2,...,c_{j-1},\sigma,d_j,...,d_{q-2})$ and $(\tau,d_1,d_2,...,d_{j-1},\lambda,c_j,...,c_{q-2})$ become the cycles of $\bar{x}\bar{y}$ after connection of two diagrams. Other cycles of $\bar{x}\bar{y}$ are unchanged, that is to say each of length 13. Thus, the resulting coset diagram $D(11,53)+D(11,53)$ or $D(11,53)+D(36,79)$ or $D(36,79)+D(36,79)$ is again a coset diagram for $\alpha(\Delta(2,3,13))$.

We can join $\acute{l}$ copies of $D(11,53)$ and $\acute{m}$ copies of $D(36,79)$ in the similar way. After joining these coset diagrams together, we obtain the coset diagram $D(n)$ with $\acute{l}\,|D(11,53)| + \acute{m}\,|D(36,79)|$ vertices representing $\alpha(\Delta(2,3,13))$. ■

Remark 4 *The vertices of a coset diagram $D(\theta, q)$ can be relabelled by different symbols, because any two finite fields of the same size are isomorphic. Therefore, the vertices of the coset diagram $D(n)$ can be relabelled avoiding repetition of labels.*

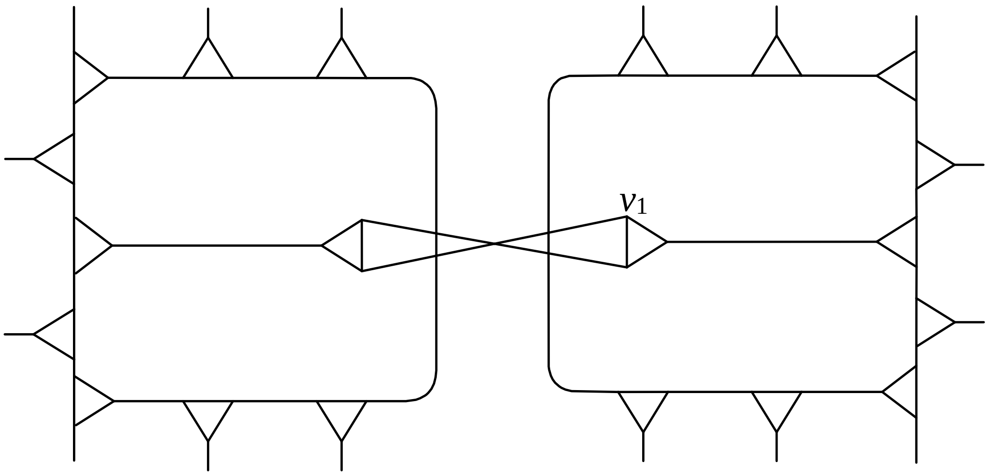
Remark 5 *Note that $D(n)$ still contains γ_1 and γ_2. Therefore, stitching of more diagrams containing γ_1 or γ_2 or both, can be continued but possibly different from $D(11,53)$ and $D(36,79)$.*

We see that same fragments do not exist in all coset diagrams. Moreover, it is not necessary that a coset diagram has always some fragment as a part of it. However, since the fragments are key to connect two or more coset diagrams, therefore, we look for existence of more fragments in the coset diagrams for quotients of $\Delta(2,3,13)$. The next section is devoted to finding conditions for existence of different fragments in these coset diagrams.

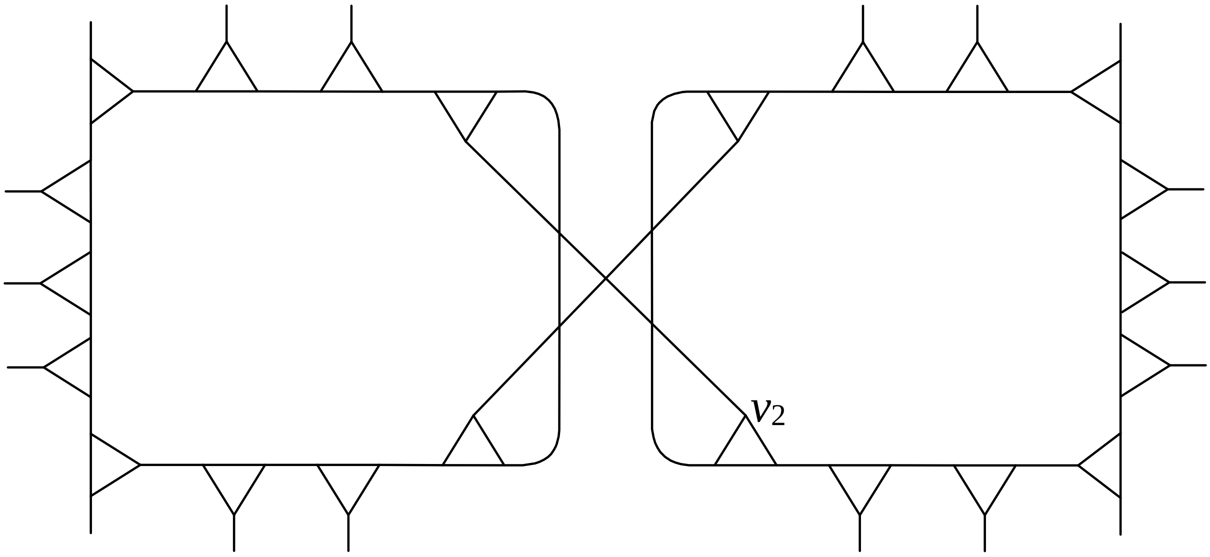
2.4 Conditions for Existence of Fragments

In coset diagrams for the group $PGL(2,\mathbb{Z})$ action on $PL(F_q)$, where q is a prime power, every non-trivial linear-fractional transformation does not fix more than two vertices ([31]). Thus, we look for the fragments of coset diagrams in which a linear fractional transformation fixes more than two vertices. Existence of these fragments in a coset diagram for the group $PGL(2,\mathbb{Z})$ action on $PL(F_q)$ ensures that linear fractional transformation $(\bar{x}\bar{y})^{13}$ is trivial. Their existence therefore ensures coset diagrams for quotients of $\Delta(2,3,13)$.

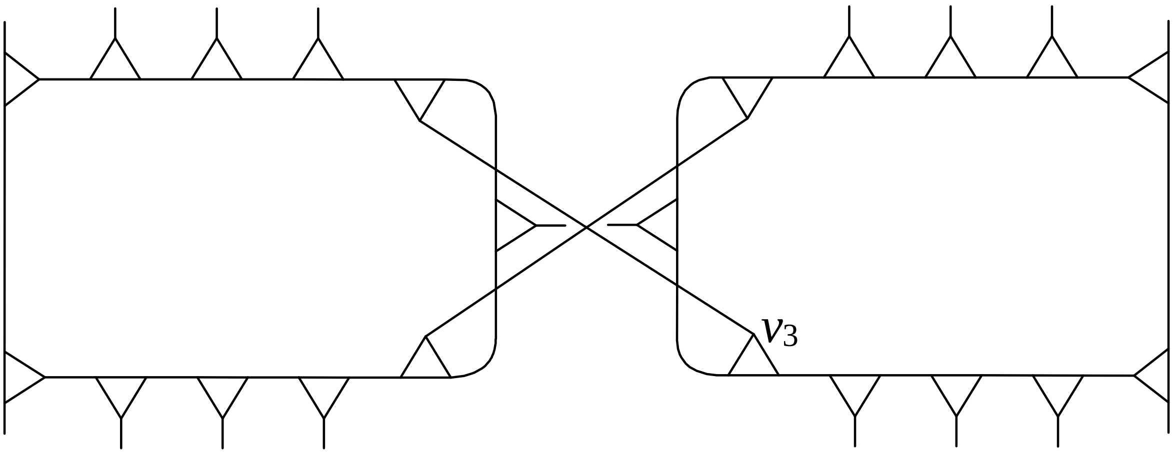
Existence of fragments in coset diagrams for the extended modular group action on projective line over the finite field F_q was discussed in [31]. The coset diagrams for $\alpha(\Delta(2,3,13))$ also contain some special fragments. In these fragments more than two vertices are fixed by $(\bar{x}\bar{y})^{13}$. In the next theorem we determine conditions in terms of θ and q, for existence of the fragments in $D(\theta,q)$ depicting homomorphic images of $\Delta(2,3,13)$.



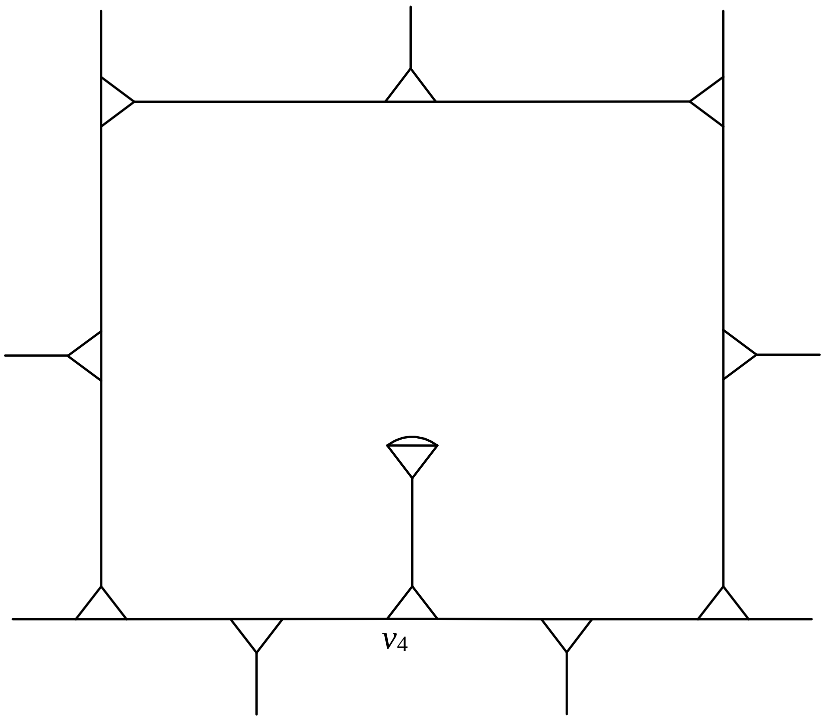
$Fig.2.18$



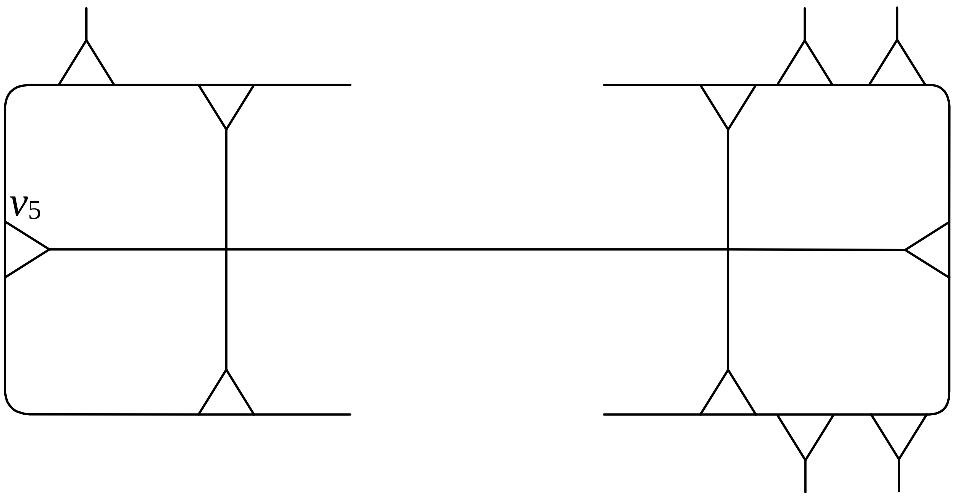
$Fig.2.19$



$Fig.2.20$



$Fig.2.21$



$Fig.2.22$

Theorem 6

(i) $\theta^2 - 2\theta - 3$ *is a square in* F_q *if the fragment* γ_3 $(Fig.\ 2.18)$ *occurs in* $D(\theta, q)$.

(ii) $\theta(\theta - 3)\left(\theta^2 - 3\theta + 4\right)$ *is a square in* F_q *if the fragment* γ_4 $(Fig.\ 2.19)$ *occurs in* $D(\theta, q)$.

(iii) $(\theta - 1)^2(\theta - 3)(\theta^3 - 5\theta^2 + 7\theta + 1)$ *is a square in* F_q *if the fragment* γ_5 $(Fig.2.20)$ *occurs in* $D(\theta, q)$.

(iv) $\theta(\theta - 4)$ *is a square in* F_q *if the fragment* γ_6 $(Fig.2.21)$ *occurs in* $D(\theta, q)$.

(v) $\left(\theta^4 - 6\theta^3 + 11\theta^2 - 6\theta - 3\right)$ *is a square in* F_q *if the fragment* γ_7 $(Fig.2.22)$ *occurs in* $D(\theta, q)$.

Proof. The vertices v_1, v_2, v_3, v_4 and v_5 are fixed by the elements $\bar{x}\bar{y}\bar{x}\bar{y}^{-1}$, $\bar{x}\bar{y}\bar{x}\bar{y}\bar{x}\bar{y}^{-1}\bar{x}\bar{y}^{-1}$, $\bar{x}\bar{y}\bar{x}\bar{y}\bar{x}\bar{y}\bar{x}\bar{y}^{-1}\bar{x}\bar{y}^{-1}\bar{x}\bar{y}^{-1}$, $\bar{y}^{-1}\bar{x}\bar{y}\bar{x}\bar{y}\bar{x}\bar{y}$ and, $\bar{x}\bar{y}\bar{x}\bar{y}\bar{x}\bar{y}\bar{x}\bar{y}^{-1}$ of $\alpha(PGL(2, \mathbb{Z}))$.

Notice that $\det(X) = \triangle$, $tr(X) = 0$, $\det(Y) = 1$, $tr(Y) = -1$, $\det(XY) = \triangle$, and $tr(XY) = r$. Following equations can be derived easily.

$$XYX = rX + \triangle I + \triangle Y \tag{2.7}$$

$$YXY = rY + X \tag{2.8}$$

$$YX = rI - X - XY \tag{2.9}$$

We also have the equations:

$$X^2 + \triangle I = 0$$

$$(XY)^2 - r(XY) + \triangle I = 0$$

$$Y^2 + Y + I = 0$$

, where the matrices Y and X correspond to the linear fractional transformations $\bar{y}$ and $\bar{x}$ respectively.

In fragment γ_3, vertex v_1 is fixed by $\bar{x}\bar{y}\bar{x}\bar{y}^{-1}$ and its corresponding matrix is $M_1 = XYXY^{-1}$, and $\det(M_1) = \det(X)\det(Y)\det(X)\det(Y^2) = \det(X)\det(Y)\det(X)(\det(Y))^2 = \triangle^2$. As $Y^{-1} = Y^2$ so M_1 is written as $M_1 = XYXY^2$. On substituting the value of Y^2, we get
$M_1 = XYX(-Y - I) = -(XY)^2 - XYX$. Now by substituting values of $(XY)^2$ and XYX obtained from the above equations in the equation $M_1 = -(XY)^2 - XYX$, we get $M_1 = -rXY + \triangle I - rX - \triangle I - \triangle Y = -rXY - rX - \triangle Y$.
So $tr(M_1) = tr(-rXY) - tr(rX) - tr(\triangle Y) = -r^2 + \triangle$.

The characteristic equation of M_1 has the discriminant
$(-r^2 + \triangle)^2 - 4\triangle^2 = r^4 - 3\triangle^2 - 2r^2\triangle$. But $r^2 = \triangle\theta$. That is, the discriminant is $\theta^2\triangle^2 - 3\triangle^2 - 2\theta\triangle^2$. Since θ is a square iff $\triangle$ is a square, we eliminate $\triangle^2$, as we are dealing with F_q. The discriminant simplifies to $\theta^2 - 2\theta - 3$.

Similarly, we can find the discriminants of the characteristic equations for matrices corresponding to the transformations $\bar{x}\bar{y}\bar{x}\bar{y}\bar{x}\bar{y}^{-1}\bar{x}\bar{y}^{-1}$, $\bar{x}\bar{y}\bar{x}\bar{y}\bar{x}\bar{y}\bar{x}\bar{y}^{-1}\bar{x}\bar{y}^{-1}\bar{x}\bar{y}^{-1}$, $\bar{y}^{-1}\bar{x}\bar{y}\bar{x}\bar{y}\bar{x}\bar{y}$ and $\bar{x}\bar{y}\bar{x}\bar{y}\bar{x}\bar{y}\bar{x}\bar{y}^{-1}$ as $\theta(\theta-3)(\theta^2-3\theta+4)$, $(\theta-1)^2(\theta-3)(\theta^3-5\theta^2+7\theta+1)$, $\theta(\theta-4)$ and $\left(\theta^4-6\theta^3+11\theta^2-6\theta-3\right)$ respectively. Thus,

(i) $\theta^2 - 2\theta - 3$ is a square in F_q if the fragment γ_3 ($Fig.$ 2.18) occurs in $D(\theta, q)$.

(ii) $\theta(\theta-3)\left(\theta^2-3\theta+4\right)$ is a square in F_q if the fragment γ_4 ($Fig.$ 2.19) occurs in $D(\theta, q)$.

(iii) $(\theta-1)^2(\theta-3)(\theta^3-5\theta^2+7\theta+1)$ is a square in F_q if the fragment γ_5 ($Fig.$2.20) occurs in $D(\theta, q)$.

(iv) $\theta(\theta-4)$ is a square in F_q if the fragment γ_6 ($Fig.$2.21) occurs in $D(\theta, q)$.

(v) $\left(\theta^4-6\theta^3+11\theta^2-6\theta-3\right)$ is a square in F_q if the fragment γ_7 ($Fig.$2.22) occurs in $D(\theta, q)$. ■

For $p < 1300$, we give a table of triplets $\bar{x}$, $\bar{t}$, $\bar{y}$ such that $\bar{x}^2 = \bar{t}^2 = \bar{y}^3 = (\bar{x}\bar{t})^2 = (\bar{y}\bar{t})^2 = (\bar{x}\bar{y})^{13} = 1$. Here p denotes a prime number and $p = 13$ or $p \equiv \pm 1 \pmod{13}$, and $f(\theta)$ denotes the polynomial $\theta^6 - 11\theta^5 + 45\theta^4 - 84\theta^3 + 70\theta^2 - 21\theta + 1$. Information provided in the list is sufficient to draw the coset diagrams representing finite quotients of the triangle group $\Delta(2, 3, 13)$ on $PL(F_p)$. The coset diagrams can be employed to investigate further characteristics of the quotients and reach some generalized facts about the triangle group $\Delta(2, 3, n)$.

p	Roots of $f(\theta) = 0$	$\bar{x}(z)$	$\bar{y}(z)$	$\bar{t}(z)$
13	4	$\frac{1}{z}$	$\frac{5}{5z-1}$	$\frac{-1}{z}$
53	10	$\frac{34z+3}{3z-34}$	$\frac{23}{23z-1}$	$\frac{-1}{z}$
	11	$\frac{31z-4}{2z-31}$	$\frac{4z-28}{14z-5}$	$\frac{2}{z}$
	13	$\frac{4z+6}{6z-4}$	$\frac{23}{23z-1}$	$\frac{-1}{z}$
	15	$\frac{46z+35}{44z-46}$	$\frac{z+10}{5z-2}$	$\frac{-2}{z}$
	28	$\frac{35z+29}{29z-35}$	$\frac{23}{23z-1}$	$\frac{-1}{z}$
	40	$\frac{36z+9}{9z-36}$	$\frac{23}{23z-1}$	$\frac{-1}{z}$
79	8	$\frac{60z+40}{66z-60}$	$\frac{32}{37z-1}$	$\frac{-3}{z}$
	13	$\frac{57z+21}{7z-57}$	$\frac{32}{37z-1}$	$\frac{-3}{z}$
	20	$\frac{47z+70}{70z-47}$	$\frac{z+32}{32z-2}$	$\frac{-1}{z}$
	36	$\frac{20z+51}{51z-20}$	$\frac{z+32}{32z-2}$	$\frac{-1}{z}$
	42	$\frac{6z+11}{11z-6}$	$\frac{z+32}{32z-2}$	$\frac{-1}{z}$
	50	$\frac{67z+48}{48z-67}$	$\frac{z+32}{32z-2}$	$\frac{-1}{z}$

103	14	$\frac{87z+19}{19z-87}$	$\frac{z+10}{10z-2}$	$\frac{-1}{z}$
	41	$\frac{28z+23}{42z-28}$	$\frac{93}{31z-1}$	$\frac{-3}{z}$
	46	$\frac{22z+66}{66z-22}$	$\frac{z+10}{10z-2}$	$\frac{-1}{z}$
	58	$\frac{18z+40}{82z-18}$	$\frac{93}{31z-1}$	$\frac{-3}{z}$
	79	$\frac{7z+57}{19z-7}$	$\frac{93}{31z-1}$	$\frac{-3}{z}$
	82	$\frac{28z+80}{61z-28}$	$\frac{93}{31z-1}$	$\frac{-3}{z}$
131	15	$\frac{80z+125}{128z-80}$	$\frac{28}{14z-1}$	$\frac{-2}{z}$
	38	$\frac{84z+77}{104z-84}$	$\frac{28}{14z-1}$	$\frac{-2}{z}$
	45	$\frac{41z+105}{105z-41}$	$\frac{5z+10}{10z-6}$	$\frac{-1}{z}$
	64	$\frac{69z+68}{34z-69}$	$\frac{28}{14z-1}$	$\frac{-2}{z}$
	117	$\frac{8z+1}{66z-8}$	$\frac{28}{14z-1}$	$\frac{-2}{z}$
	125	$\frac{96z+79}{79z-96}$	$\frac{5z+10}{10z-6}$	$\frac{-1}{z}$
157	3	$\frac{81z+8}{4z-81}$	$\frac{2z+92}{46z-3}$	$\frac{-2}{z}$
	68	$\frac{5z+133}{145z-5}$	$\frac{2z+92}{46z-3}$	$\frac{-2}{z}$
	117	$\frac{114z+15}{86z-114}$	$\frac{2z+92}{46z-3}$	$\frac{-2}{z}$
	126	$\frac{140z+131}{144z-140}$	$\frac{2z+92}{46z-3}$	$\frac{-2}{z}$
	144	$\frac{26z+39}{39z-26}$	$\frac{28}{28z-1}$	$\frac{-1}{z}$
	147	$\frac{154z+124}{124z-154}$	$\frac{28}{28z-1}$	$\frac{-1}{z}$
181	34	$\frac{110z+140}{70z-110}$	$\frac{2z+158}{79z-3}$	$\frac{-2}{z}$
	55	$\frac{125z+170}{170z-125}$	$\frac{19}{19z-1}$	$\frac{-1}{z}$
	94	$\frac{128z+57}{119z-128}$	$\frac{2z+158}{79z-3}$	$\frac{-2}{z}$
	114	$\frac{91z+166}{166z-91}$	$\frac{19}{19z-1}$	$\frac{-1}{z}$
	119	$\frac{128z+57}{119z-128}$	$\frac{2z+158}{79z-3}$	$\frac{-2}{z}$
	138	$\frac{119z+104}{104z-119}$	$\frac{19}{19z-1}$	$\frac{-1}{z}$

233	9	$\frac{167z+26}{164z-167}$	$\frac{z+34}{89z-2}$	$\frac{-3}{z}$
	49	$\frac{50z+166}{166z-50}$	$\frac{89}{89z-1}$	$\frac{-1}{z}$
	91	$\frac{94z+98}{188z-94}$	$\frac{z+34}{89z-2}$	$\frac{-3}{z}$
	112	$\frac{164z+197}{197z-164}$	$\frac{89}{89z-1}$	$\frac{-1}{z}$
	217	$\frac{30z+168}{168z-30}$	$\frac{89}{89z-1}$	$\frac{-1}{z}$
	232	$\frac{7z+79}{79z-7}$	$\frac{89}{89z-1}$	$\frac{-1}{z}$
311	42	$\frac{194z+197}{131z-194}$	$\frac{250}{51z-1}$	$\frac{-11}{z}$
	45	$\frac{249z+1}{198z-249}$	$\frac{250}{51z-1}$	$\frac{-11}{z}$
	50	$\frac{42z+76}{120z-42}$	$\frac{250}{51z-1}$	$\frac{-11}{z}$
	75	$\frac{109z+177}{214z-109}$	$\frac{250}{51z-1}$	$\frac{-11}{z}$
	127	$\frac{301z+167}{167z-301}$	$\frac{5z+88}{88z-6}$	$\frac{-1}{z}$
	294	$\frac{301z+144}{144z-301}$	$\frac{5z+88}{88z-6}$	$\frac{-1}{z}$
313	26	$\frac{187z+276}{243z-187}$	$\frac{2z+252}{113z-3}$	$\frac{-5}{z}$
	79	$\frac{91z+13}{13z-91}$	$\frac{25}{25z-1}$	$\frac{-1}{z}$
	87	$\frac{131z+162}{95z-131}$	$\frac{2z+252}{113z-3}$	$\frac{-5}{z}$
	200	$\frac{134z+86}{86z-134}$	$\frac{25}{25z-1}$	$\frac{-1}{z}$
	263	$\frac{276z+307}{124z-276}$	$\frac{2z+252}{113z-3}$	$\frac{-5}{z}$
	295	$\frac{165z+5}{z-165}$	$\frac{2z+252}{113z-3}$	$\frac{-5}{z}$
337	26	$\frac{258z+145}{29z-258}$	$\frac{5z+70}{14z-6}$	$\frac{-5}{z}$
	75	$\frac{308z+13}{13z-308}$	$\frac{148}{148z-1}$	$\frac{-1}{z}$
	181	$\frac{333z+40}{8z-333}$	$\frac{5z+70}{14z-6}$	$\frac{-5}{z}$
	227	$\frac{17z+261}{261z-17}$	$\frac{148}{148z-1}$	$\frac{-1}{z}$
	239	$\frac{233z+138}{95z-233}$	$\frac{5z+70}{14z-6}$	$\frac{-5}{z}$
	274	$\frac{259z+286}{192z-259}$	$\frac{5z+70}{14z-6}$	$\frac{-5}{z}$

389	178	$\frac{371z+381}{381z-371}$	$\frac{115}{115z-1}$	$\frac{-1}{z}$
	193	$\frac{54z+388}{194z-54}$	$\frac{z+108}{54z-2}$	$\frac{-2}{z}$
	245	$\frac{159z+361}{375z-159}$	$\frac{z+108}{54z-2}$	$\frac{-2}{z}$
	304	$\frac{41z+332}{166z-41}$	$\frac{z+108}{54z-2}$	$\frac{-2}{z}$
	310	$\frac{103z+211}{300z-103}$	$\frac{z+108}{54z-2}$	$\frac{-2}{z}$
	337	$\frac{107z+328}{328z-107}$	$\frac{115}{115z-1}$	$\frac{-1}{z}$
443	13	$\frac{431z+228}{114z-431}$	$\frac{422}{211z-1}$	$\frac{-2}{z}$
	75	$\frac{419z+75}{75z-419}$	$\frac{2z+128}{128z-3}$	$\frac{-1}{z}$
	121	$\frac{266z+225}{225z-266}$	$\frac{2z+128}{128z-3}$	$\frac{-1}{z}$
	289	$\frac{124z+427}{435z-124}$	$\frac{422}{211z-1}$	$\frac{-2}{z}$
	414	$\frac{4z+267}{355z-4}$	$\frac{422}{211z-1}$	$\frac{-2}{z}$
	428	$\frac{150z+782}{391z-150}$	$\frac{422}{211z-1}$	$\frac{-2}{z}$
467	23	$\frac{214z+419}{419z-214}$	$\frac{5z+48}{48z-6}$	$\frac{-1}{z}$
	83	$\frac{190z+77}{77z-190}$	$\frac{5z+48}{48z-6}$	$\frac{-1}{z}$
	221	$\frac{122z+153}{153z-122}$	$\frac{5z+48}{48z-6}$	$\frac{-1}{z}$
	317	$\frac{276z+280}{280z-276}$	$\frac{5z+48}{48z-6}$	$\frac{-1}{z}$
	327	$\frac{406z+106}{53z-406}$	$\frac{126}{63z-1}$	$\frac{-2}{z}$
	441	$\frac{332z+446}{223z-332}$	$\frac{126}{63z-1}$	$\frac{-2}{z}$
521	5	$\frac{447z+485}{485z-447}$	$\frac{235}{235z-1}$	$\frac{-1}{z}$
	9	$\frac{495z+121}{214z-495}$	$\frac{z+184}{235z-2}$	$\frac{-3}{z}$
	20	$\frac{33z+306}{102z-33}$	$\frac{z+184}{235z-2}$	$\frac{-3}{z}$
	49	$\frac{199z+32}{358z-199}$	$\frac{z+184}{235z-2}$	$\frac{-3}{z}$
	125	$\frac{32z+287}{443z-32}$	$\frac{z+184}{235z-2}$	$\frac{-3}{z}$
	324	$\frac{386z+518}{518z-386}$	$\frac{235}{235z-1}$	$\frac{-1}{z}$

547	66	$\frac{343z+99}{99z-343}$	$\frac{z+81}{81z-2}$	$\frac{-1}{z}$
	183	$\frac{41z+507}{507z-41}$	$\frac{z+81}{81z-2}$	$\frac{-1}{z}$
	209	$\frac{348z+227}{387z-348}$	$\frac{190}{95z-1}$	$\frac{-2}{z}$
	267	$\frac{490z+539}{543z-490}$	$\frac{190}{95z-1}$	$\frac{-2}{z}$
	439	$\frac{262z+215}{215z-262}$	$\frac{z+81}{81z-2}$	$\frac{-1}{z}$
	488	$\frac{171z+106}{106z-171}$	$\frac{z+81}{81z-2}$	$\frac{-1}{z}$
571	66	$\frac{17z+515}{515z-17}$	$\frac{z+219}{219z-2}$	$\frac{-1}{z}$
	99	$\frac{464z+234}{117z-464}$	$\frac{418}{209z-1}$	$\frac{-2}{z}$
	273	$\frac{329z+524}{262z-329}$	$\frac{418}{209z-1}$	$\frac{-2}{z}$
	353	$\frac{23z+129}{350z-23}$	$\frac{418}{209z-1}$	$\frac{-2}{z}$
	436	$\frac{457z+43}{43z-457}$	$\frac{z+219}{219z-2}$	$\frac{-1}{z}$
	497	$\frac{458z+282}{141z-458}$	$\frac{418}{209z-1}$	$\frac{-2}{z}$
599	8	$\frac{343z+258}{208z-343}$	$\frac{259}{37z-1}$	$\frac{-7}{z}$
	36	$\frac{525z+232}{232z-525}$	$\frac{2z+259}{259z-3}$	$\frac{-1}{z}$
	139	$\frac{246z+402}{143z-246}$	$\frac{259}{37z-1}$	$\frac{-7}{z}$
	200	$\frac{53z+571}{571z-53}$	$\frac{2z+259}{259z-3}$	$\frac{-1}{z}$
	269	$\frac{565z+380}{380z-565}$	$\frac{2z+259}{259z-3}$	$\frac{-1}{z}$
	557	$\frac{518z+564}{564z-518}$	$\frac{2z+259}{259z-3}$	$\frac{-1}{z}$
677	126	$\frac{550z+546}{273z-550}$	$\frac{z+632}{316z-2}$	$\frac{-2}{z}$
	134	$\frac{545z+543}{610z-545}$	$\frac{z+632}{316z-2}$	$\frac{-2}{z}$
	220	$\frac{25z+472}{472z-25}$	$\frac{26}{26z-1}$	$\frac{-1}{z}$
	482	$\frac{207z+402}{402z-207}$	$\frac{26}{26z-1}$	$\frac{-1}{z}$
	499	$\frac{174z+460}{230z-174}$	$\frac{z+632}{316z-2}$	$\frac{-2}{z}$
	581	$\frac{100z+454}{227z-100}$	$\frac{z+632}{316z-2}$	$\frac{-2}{z}$

701	76	$\frac{198z+412}{206z-198}$	$\frac{z+380}{190z-2}$	$\frac{-2}{z}$
	132	$\frac{224z+102}{51z-224}$	$\frac{z+380}{190z-2}$	$\frac{-2}{z}$
	379	$\frac{473z+599}{599z-473}$	$\frac{135}{135z-1}$	$\frac{-1}{z}$
	431	$\frac{436z+281}{491z-436}$	$\frac{z+380}{190z-2}$	$\frac{-2}{z}$
	527	$\frac{336z+109}{109z-336}$	$\frac{135}{135z-1}$	$\frac{-1}{z}$
	569	$\frac{458z+550}{275z-458}$	$\frac{z+380}{190z-2}$	$\frac{-2}{z}$
727	15	$\frac{531z+29}{29z-531}$	$\frac{z+164}{164z-2}$	$\frac{-1}{z}$
	169	$\frac{495z+492}{492z-495}$	$\frac{z+164}{164z-2}$	$\frac{-1}{z}$
	263	$\frac{437z+493}{493z-437}$	$\frac{z+164}{164z-2}$	$\frac{-1}{z}$
	510	$\frac{440z+593}{593z-440}$	$\frac{z+164}{164z-2}$	$\frac{-1}{z}$
	529	$\frac{575z+190}{548z-575}$	$\frac{164}{297z-1}$	$\frac{-3}{z}$
	706	$\frac{566z+101}{276z-566}$	$\frac{164}{297z-1}$	$\frac{-3}{z}$
857	92	$\frac{678z+437}{717z-678}$	$\frac{z+621}{207z-2}$	$\frac{-3}{z}$
	282	$\frac{19z+810}{810z-19}$	$\frac{207}{207z-1}$	$\frac{-1}{z}$
	387	$\frac{730z+529}{462z-730}$	$\frac{z+621}{207z-2}$	$\frac{-3}{z}$
	413	$\frac{112z+654}{218z-112}$	$\frac{z+621}{207z-2}$	$\frac{-3}{z}$
	587	$\frac{147z+569}{569z-147}$	$\frac{207}{207z-1}$	$\frac{-1}{z}$
	821	$\frac{685z+588}{196z-685}$	$\frac{z+621}{207z-2}$	$\frac{-3}{z}$
859	20	$\frac{832z+680}{340z-832}$	$\frac{296}{148z-1}$	$\frac{-2}{z}$
	249	$\frac{557z+411}{635z-557}$	$\frac{296}{148z-1}$	$\frac{-2}{z}$
	324	$\frac{142z+313}{586z-142}$	$\frac{296}{148z-1}$	$\frac{-2}{z}$
	604	$\frac{289z+64}{64z-289}$	$\frac{z+338}{338z-2}$	$\frac{-2}{z}$
	626	$\frac{83z+29}{29z-83}$	$\frac{z+338}{338z-2}$	$\frac{-1}{z}$
	765	$\frac{512z+803}{831z-512}$	$\frac{296}{148z-1}$	$\frac{-2}{z}$

883	38	$\frac{240z+380}{190z-240}$	$\frac{42}{21z-1}$	$\frac{-2}{z}$
	116	$\frac{789z+723}{723z-789}$	$\frac{z+208}{208z-2}$	$\frac{-1}{z}$
	268	$\frac{24z+703}{793z-24}$	$\frac{42}{21z-1}$	$\frac{-2}{z}$
	308	$\frac{29z+636}{318z-29}$	$\frac{42}{21z-1}$	$\frac{-2}{z}$
	413	$\frac{876z+673}{673z-876}$	$\frac{z+208}{208z-2}$	$\frac{-1}{z}$
	634	$\frac{205z+145}{514z-205}$	$\frac{42}{21z-1}$	$\frac{-2}{z}$
911	53	$\frac{200z+846}{251z-200}$	$\frac{579}{343z-1}$	$\frac{-7}{z}$
	251	$\frac{476z+636}{221z-476}$	$\frac{579}{343z-1}$	$\frac{-7}{z}$
	405	$\frac{702z+415}{415z-702}$	$\frac{2z+332}{332z-3}$	$\frac{-1}{z}$
	609	$\frac{6z+390}{390z-6}$	$\frac{2z+332}{332z-3}$	$\frac{-1}{z}$
	647	$\frac{827z+267}{267z-827}$	$\frac{2z+332}{332z-3}$	$\frac{-1}{z}$
	779	$\frac{496z+209}{209z-496}$	$\frac{2z+332}{332z-3}$	$\frac{-1}{z}$
937	12	$\frac{885z+818}{818z-885}$	$\frac{196}{196z-1}$	$\frac{-1}{z}$
	35	$\frac{271z+139}{139z-271}$	$\frac{196}{196z-1}$	$\frac{-1}{z}$
	100	$\frac{377z+360}{360z-377}$	$\frac{196}{196z-1}$	$\frac{-1}{z}$
	152	$\frac{649z+444}{651z-649}$	$\frac{2z+366}{448z-3}$	$\frac{-5}{z}$
	234	$\frac{360z+377}{377z-360}$	$\frac{196}{196z-1}$	$\frac{-1}{z}$
	415	$\frac{180z+62}{762z-180}$	$\frac{2z+366}{448z-3}$	$\frac{-5}{z}$
1013	240	$\frac{360z+679}{846z-360}$	$\frac{z+542}{271z-2}$	$\frac{-2}{z}$
	305	$\frac{55z+822}{822z-55}$	$\frac{45}{45z-1}$	$\frac{-1}{z}$
	368	$\frac{141z+286}{143z-141}$	$\frac{z+542}{271z-2}$	$\frac{-2}{z}$
	569	$\frac{483z+355}{684z-483}$	$\frac{z+542}{271z-2}$	$\frac{-2}{z}$
	639	$\frac{780z+656}{328z-780}$	$\frac{z+542}{271z-2}$	$\frac{-2}{z}$
	929	$\frac{113z+733}{733z-113}$	$\frac{45}{45z-1}$	$\frac{-1}{z}$

1039	395	$\frac{622z+55}{711z-622}$	$\frac{281}{440z-1}$	$\frac{-3}{z}$
	543	$\frac{33z+923}{923z-33}$	$\frac{z+281}{281z-2}$	$\frac{-1}{z}$
	677	$\frac{291z+215}{418z-291}$	$\frac{281}{440z-1}$	$\frac{-3}{z}$
	722	$\frac{19z+646}{646z-19}$	$\frac{z+281}{281z-2}$	$\frac{-1}{z}$
	852	$\frac{572z+10}{10z-572}$	$\frac{z+281}{281z-2}$	$\frac{-1}{z}$
	978	$\frac{953z+857}{857z-953}$	$\frac{z+281}{281z-2}$	$\frac{-1}{z}$
1091	21	$\frac{863z+535}{535z-863}$	$\frac{6z+219}{219z-7}$	$\frac{-1}{z}$
	76	$\frac{431z+931}{1011z-431}$	$\frac{1058}{529z-1}$	$\frac{-2}{z}$
	143	$\frac{853z+577}{834z-853}$	$\frac{1058}{529z-1}$	$\frac{-2}{z}$
	243	$\frac{169z+629}{860z-169}$	$\frac{1058}{529z-1}$	$\frac{-2}{z}$
	258	$\frac{253z+277}{277z-253}$	$\frac{6z+219}{219z-7}$	$\frac{-1}{z}$
	361	$\frac{630z+283}{687z-630}$	$\frac{1058}{529z-1}$	$\frac{-2}{z}$
1093	103	$\frac{936z+644}{644z-936}$	$\frac{530}{530z-1}$	$\frac{-1}{z}$
	364	$\frac{1072z+460}{460z-1072}$	$\frac{530}{530z-1}$	$\frac{-1}{z}$
	394	$\frac{827z+1069}{1081z-827}$	$\frac{6z+436}{1081z-827}$	$\frac{-2}{z}$
	644	$\frac{1030z+266}{133z-1030}$	$\frac{6z+436}{218z-7}$	$\frac{-2}{z}$
	808	$\frac{737z+417}{755z-737}$	$\frac{6z+436}{218z-7}$	$\frac{-2}{z}$
	977	$\frac{360z+1048}{524z-360}$	$\frac{6z+436}{218z-7}$	$\frac{-2}{z}$
1117	12	$\frac{987z+852}{852z-987}$	$\frac{214}{214z-1}$	$\frac{-1}{z}$
	100	$\frac{61z+989}{989z-61}$	$\frac{214}{214z-1}$	$\frac{-1}{z}$
	107	$\frac{1023z+246}{123z-1023}$	$\frac{6z+420}{210z-7}$	$\frac{-2}{z}$
	386	$\frac{721z+611}{864z-721}$	$\frac{6z+420}{210z-7}$	$\frac{-2}{z}$
	668	$\frac{87z+1057}{1057z-87}$	$\frac{214}{214z-1}$	$\frac{-1}{z}$
	972	$\frac{449z+1064}{1064z-449}$	$\frac{214}{214z-1}$	$\frac{-1}{z}$

1171	84	$\frac{366z+753}{962z-366}$	$\frac{278}{139z-1}$	$\frac{-2}{z}$
	754	$\frac{715z+636}{636z-715}$	$\frac{z+330}{330z-2}$	$\frac{-1}{z}$
	828	$\frac{425z+201}{686z-425}$	$\frac{278}{139z-1}$	$\frac{-2}{z}$
	869	$\frac{688z+107}{107z-688}$	$\frac{z+330}{330z-2}$	$\frac{-1}{z}$
	1078	$\frac{749z+304}{304z-749}$	$\frac{z+330}{330z-2}$	$\frac{-1}{z}$
	1082	$\frac{421z+751}{751z-421}$	$\frac{z+330}{330z-2}$	$\frac{-1}{z}$
1223	157	$\frac{432z+782}{401z-432}$	$\frac{424}{574z-1}$	$\frac{-5}{z}$
	181	$\frac{57z+673}{673z-57}$	$\frac{5z+78}{78z-6}$	$\frac{-1}{z}$
	243	$\frac{36z+273}{273z-36}$	$\frac{5z+78}{78z-6}$	$\frac{-1}{z}$
	488	$\frac{1109z+708}{1120z-1109}$	$\frac{424}{574z-1}$	$\frac{-5}{z}$
	600	$\frac{957z+671}{671z-957}$	$\frac{5z+78}{78z-6}$	$\frac{-1}{z}$
	788	$\frac{845z+255}{255z-845}$	$\frac{5z+78}{78z-6}$	$\frac{-1}{z}$
1249	225	$\frac{1202z+600}{600z-1202}$	$\frac{585}{585z-1}$	$\frac{-1}{z}$
	278	$\frac{155z+1048}{685z-155}$	$\frac{2z+348}{585z-3}$	$\frac{-7}{z}$
	419	$\frac{1236z+783}{783z-1236}$	$\frac{585}{585z-1}$	$\frac{-1}{z}$
	582	$\frac{302z+696}{1170z-302}$	$\frac{2z+348}{585z-3}$	$\frac{-7}{z}$
	1018	$\frac{220z+996}{996z-220}$	$\frac{585}{585z-1}$	$\frac{-1}{z}$
	1236	$\frac{846z+608}{608z-846}$	$\frac{585}{585z-1}$	$\frac{-1}{z}$

It is natural to ask whether it is the only way we can draw our required coset diagrams or there is some other way as well. The answer is yes, we can do so. The method to draw another type of diagrams was introduced in 1980 by M. D. E. Conder ([9]). We use the same method for the group we are interested in. A coset diagram holding the relations: $x^2 = t^2 = y^3 = (xt)^2 = (yt)^2 = (xy)^{13} = 1$, can also be drawn directly, independent of the procedure followed in this chapter. We just join vertices through edges keeping in mind that each vertex of the diagram needs to be satisfying the required relations. Although we can not control the number of vertices of the diagram completely but still we can find some diagrams which are quite useful. To make them useful we need to connect them and to do so we need partial fragments (a pair of special type of vertices) existing in the diagrams. So it becomes a rather difficult task to draw a diagram satisfying the relations and having partial fragments which can be utilized to connect two or more diagrams. To obtain such type of diagrams in a suitable number is also necessary to make a useful connection of the diagrams. The partial fragments which are used to connect these diagrams have a specific name 'handle' in literature. In the next chapter, we draw the diagrams and connect them to prove that both the groups S_n and A_n are quotients of $G^{3,13,252}$ for all but finitely many positive integers n.

Some part of research work presented in this chapter has been published in [4].

Chapter 3

THE COXETER GROUP $G^{3,13,252}$

3.1 Introduction

The group $G^{k,l,m} = \langle x, t, y : x^2 = t^2 = y^k == (xt)^2 = (yt)^2 = (xy)^l = (xyt)^m = 1\rangle$ was studied by H. S. M. Coxeter in his paper [17]. As he explained, the group $G^{k,l,m}$ is isomorphic to the group $G^{r,q,p}$ for any rearrangement (r, q, p) of (k, l, m), and there is no loss of generality in assuming $m \geq l \geq k$. Geometrically the Coxeter group is a symmetric group of the maps $\{\Im, \hbar\}_\lambda$, which is constructed from the tessellations $\{\Im, \hbar\}$ of the hyperbolic plane by identifying two points at a distance λ apart along a Petrie path. A $j-$th order hole is a cyclic sequence of edges; each two consecutive sharing a vertex so that at each vertex the adjacent edges subtend j faces on one side; either the right or left but consistently throughout. A $j-$th order Petrie is a cyclic sequence of edges, but at each vertex j faces are enclosed on the right and on the left alternately. A first order Petrie path is called simply a Petrie path and a first order hole is just a face. Let $\Im$ and $\hbar$ be two points along a Petrie path at a distance λ apart. Then the map $\{\Im, \hbar\}_\lambda$ is formed from the tessellations $\{\Im, \hbar\}$ of the hyperbolic plane by identifying the points. The group acts on the maps with parameters $\{\Im, \hbar\}_\lambda$ by automorphisms. The actual symmetry groups of these maps are finite images rather than the group itself.

Coxeter showed that the group $G^{k,l,m}$ is infinite and insoluble if $\frac{1}{k}+\frac{1}{l}+\frac{1}{m}\leq 1$ ([17]). Suppose q denotes a prime p power. He also proved that $G^{k,l,m}$ is isomorphic to either $PGL(2,q)$ or $PSL(2,q)$ for small values of k, l and m ([17]). S.E. Wilson ([49]) proved that $PSL(2,q)$ is a quotient of $G^{k,l,m}$ if -1 is a quadratic residue in F_q, and $PGL(2,q)$ is a quotient of $G^{k,l,m}$ otherwise. As mentioned in chapter 1, the triangle group $\Delta(l,m,n)=\langle x,y : x^l=y^m=(xy)^n=1\rangle$ can be extended to the group $\Delta^*(l,m,n)=\langle x,y,t : x^l=y^m=t^2=(xt)^2=(yt)^2=(xy)^n=1\rangle$ by adding an involution t. It is apparent from the presentation of $G^{k,l,m}$ that these groups are quotients groups of $\Delta^*(l,m,n)$. In case m is even, the subgroup $\langle x,y\rangle$ of $G^{k,l,m}$ is isomorphic to the group $\Delta(2,k,l;m/2)=\langle x,y : x^2=y^k=(xy)^l=(xyx^{-1}y^{-1})^{m/2}=1\rangle$ and has index 2.

G. Higman raised a question: how small can the integers k, l and m be made while maintaining the property that all but finitely many alternating groups A_n and symmetric groups S_n are factor groups of $G^{k,l,m}$?

3.2 G. Higman's Problem

G. Higman proved: all but finitely many alternating groups A_n of finite degree are homomorphic images of the triangle group $\Delta(2,3,7)$. He mentioned in [26] that A_n is a homomorphic image of $G^{k,l,m}$ when $k=3$, $l=7$, and $m=19$. Coxeter investigated the group $G^{k,l,m}$ structure when $k=3$, $l=7$ and $m\leq 15$. The group $G^{k,l,m}$ has order 1 or 2 when $m\leq 7$ and $m=10$ or 11; and it is isomorphic to $PSL(2,29)$ for $m=15$. In [11], M. D. E. Conder answered G. Higman's question for $k=3$, and $l=7$, where m turned out to be 720720. He refined this result in [13] for $k=6$ and $l=6$, where m turned out to be 6. Later, he proved that all but finitely many A_n and S_n are quotients of $G^{3,7,168}$([14]). In [1], it was shown that A_n and S_n are quotients of $G^{4,5,120}$. In [38], it was proved: all but finitely many S_n and A_n, where n is congruent to 2 or 11 (mod 20), are quotients of the group $G^{5,5,24}$.

Since in this thesis we are studying coset diagrams for quotients of the triangle group $\Delta(2,3,13)$, which satisfy the relations: $x^2 = y^3 = (xy)^{13} = 1$ and maintain symmetry about the vertical line of axis; therefore, we investigate both the symmetric group S_n and the alternating group A_n as quotients of $G^{3,13,m}$, where $m = 252$. The number m depends on the suitable diagrams found and connected to prove the result. In the last section of this chapter (Theorem 8) we give an answer to the question of G. Higman.

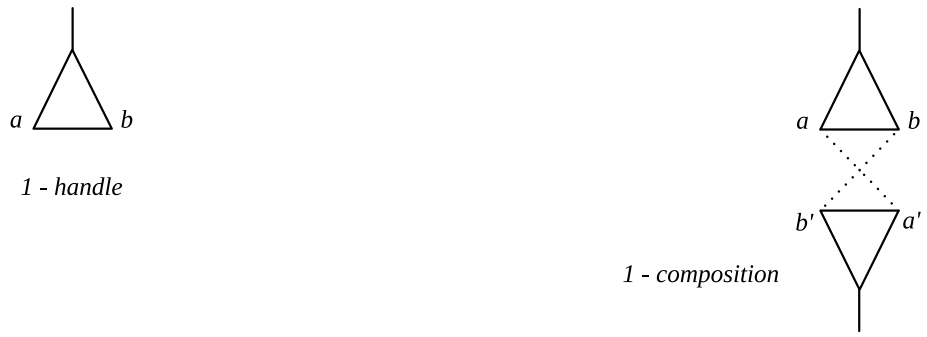
3.3 Diagrams and terminology

We use coset diagrams attributed to G. Higman. Detailed information about this type of diagrams is available in [9]. These diagrams are drawn directly by joining vertices through edges satisfying the relations: $x^2 = t^2 = y^3 = (xt)^2 = (yt)^2 = (xy)^{13} = 1$. Each of the diagrams depicts the group $\Delta^*(2,3,13) = \langle x, y, t : x^2 = t^2 = y^3 = (xt)^2 = (yt)^2 = (xy)^{13} = 1\rangle$ acting a finite set. The diagrams are defined as follows:

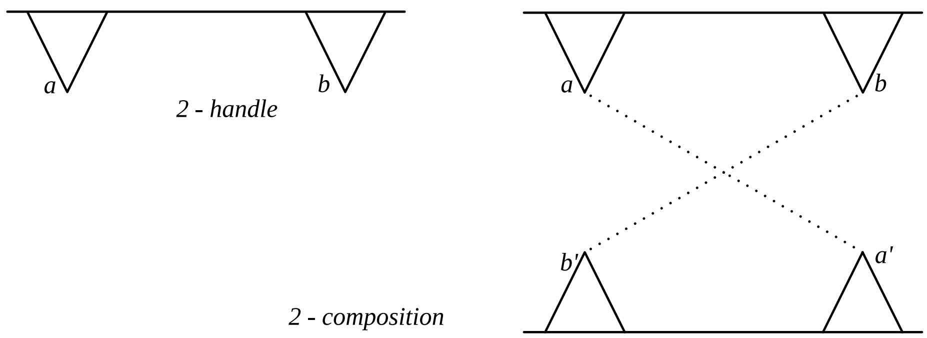
Triangles are used to represent $3-cycles$ of y. Vertices of the triangles are permuted anticlockwise by y. Any two vertices interchanged by x are connected by an edge. Every vertex of these diagrams is fixed by $(xy)^{13}$. Reflection about the vertical line of axis represents the action of t . So t is nowhere seen in the diagrams. Examples of such diagrams can be seen in section 3.4.

We require connected coset diagrams to prove our result. Technique of connecting the coset diagrams together is available in [13]. Two diagrams can be connected provided each of them has a special pair of vertices; called a handle. A pair of points a and b in a diagram is a $k-handle$ ($k = 1, 2$ *or* 3) if $(xy)^k$ takes a to b and both the points are fixed by x. Such a pair is usually represented by h_a^b or $[a,b]_k$. We use $[a,b]_k$ throughout this thesis.

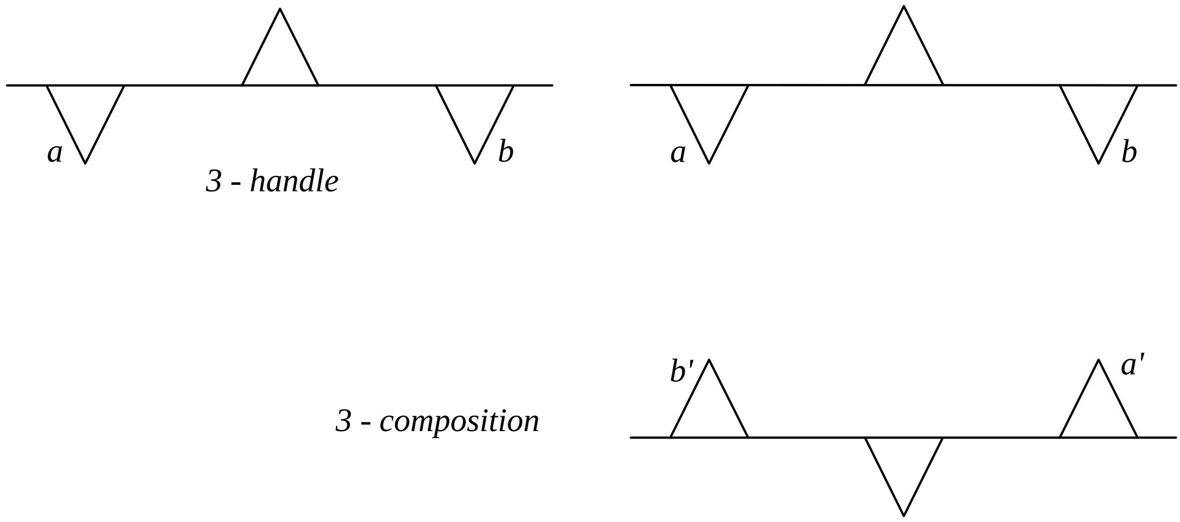
Given any two coset diagrams; one with $[a,b]_k$ and the other with $[a',b']_k$, a new coset diagram can be drawn. This is done by putting one diagram above the other on a common vertical axis of symmetry, and joining a' to a and b' to b by $x-edges$ as shown below.



$Fig.3.1$



$Fig.3.2$



$Fig.3.3$

This is the way how we can connect as many coset diagrams as we want. This method of connecting the diagrams is called $k-composition$ ($k = 1$, 2 or 3). If two or more coset diagrams for the group $\Delta(2, 3, 13)$ acting on some finite sets are connected this way; the resulting diagram represents the group $\Delta(2, 3, 13)$ acting on a larger set. The relations: $x^2 = y^3 = (xy)^{13} = t^2 = (xt)^2 = (yt)^2 = 1$ still hold.

Moreover, if (a, $\acute{e}_1$, $\acute{e}_2$, $\acute{e}_3$, $\acute{e}_4$, $\acute{e}_5$, $\acute{e}_6$, $\acute{e}_7$, $\acute{e}_8$, $\acute{e}_9$, $\acute{e}_{10}$, $\acute{e}_{11}$, $\acute{e}_{12}$) and ($\acute{a}$, $\acute{f}_1$, $\acute{f}_2$, $\acute{f}_3$, $\acute{f}_4$, $\acute{f}_5$, $\acute{f}_6$, $\acute{f}_7$, $\acute{f}_8$, $\acute{f}_9$, $\acute{f}_{10}$, $\acute{f}_{11}$, $\acute{f}_{12}$), where $\acute{e}_k = b$, $\acute{f}_k = \acute{b}$ ($k = 1, 2$ or 3), represent the appropriate cycles of xy in the group $\Delta(2, 3, 13)$ representation depicted by two coset diagrams, then these cycles are replaced by (a, σ_1, σ_2, σ_3, $\acute{e}_4$, $\acute{e}_5$, $\acute{e}_6$, $\acute{e}_7$, $\acute{e}_8$, $\acute{e}_9$, $\acute{e}_{10}$, $\acute{e}_{11}$, $\acute{e}_{12}$) and ($\acute{a}$, η_1, η_2, η_3, $\acute{f}_4$, $\acute{f}_5$, $\acute{f}_6$, $\acute{f}_7$, $\acute{f}_8$, $\acute{f}_9$, $\acute{f}_{10}$, $\acute{f}_{11}$, $\acute{f}_{12}$), where $\sigma_i = \acute{f}_i$, $\eta_i = \acute{e}_i$, for $i \leq k$, and $\sigma_i = \acute{e}_i$, $\eta_i = \acute{f}_i$ for $i > k$, in the diagram obtained after $k - composition$ ($k = 1$, 2 or 3). There is no change in other cycles of xy, so order of xy is still 13. This composition has the same type of effect on cycles of xyt and $xyxyt$.

Information gathered from a coset diagram is composed in a specific way. In other words, each coset diagram to be used is given a specification. This specification includes degree of permutation representation of the group, structure of cycles of both $xyxyt$ and xyt, parity of the action of x, t and y, and the number of handles.

We describe this as follows:
(i) $D(n)$ means a diagram with n vertices satisfying the
given relations: $x^2 = t^2 = y^3 = (xt)^2 = (yt)^2 = (xy)^{13} = 1$.
(ii) $[a, b]_k$ means a $k - handle$ with vertices a and b.
(iii) $xyt/xyxyt$: $(a, \acute{n})(b, \acute{m})$ means vertices a and b lie in the
cycles of $xyt/xyxyt$ of lengths $\acute{n} + 1$ and $\acute{m} + 1$ respectively.

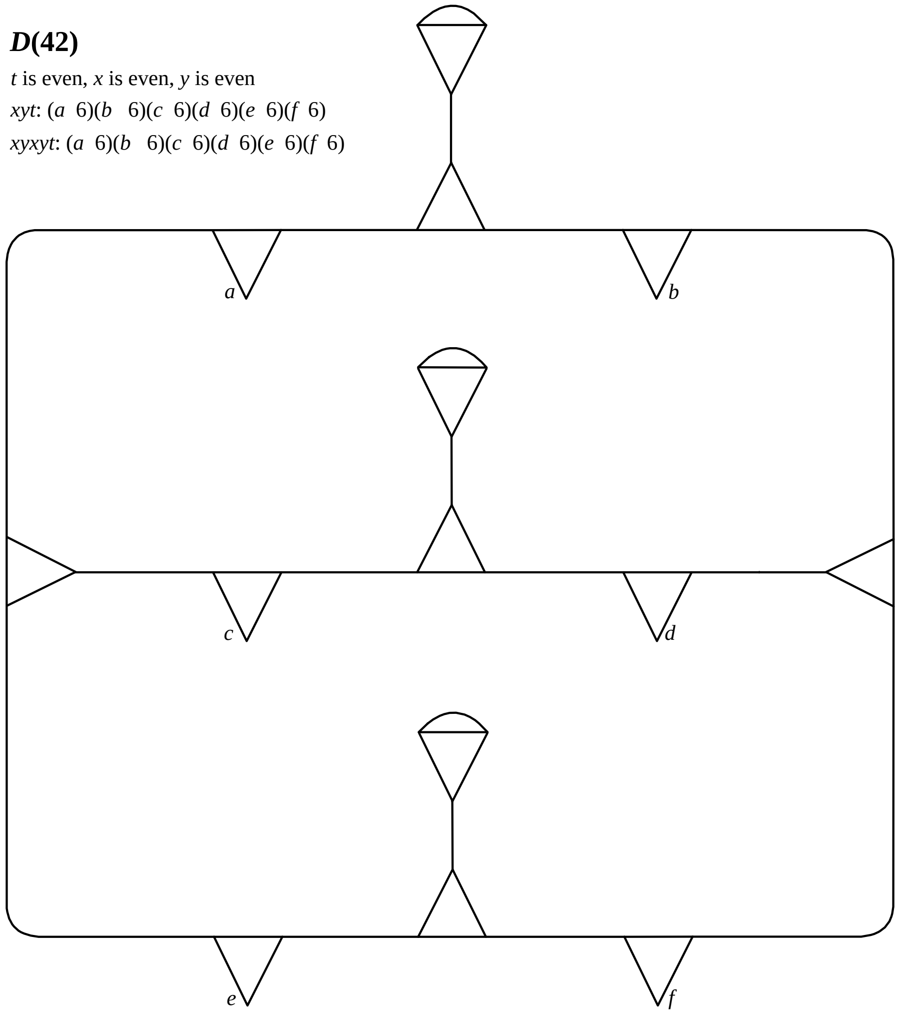
Now we recall some definitions which are needed in the last section of this chapter. Suppose $G \neq 1$ denotes a permutation group on a set X. Degree of G means the number of points actually moved by G. Degree of a permutation $g \neq 1$ is the degree of the cyclic group $\langle g \rangle$.

A block B of the permutation group G is a subset B of X such that for all $g \in G$, B and $(B)g$ are either disjoint or identical. It is not difficult to see that empty set ϕ, $\{x\}$ and X, where $x \in X$, are blocks for every permutation group G. These blocks are called trivial blocks.

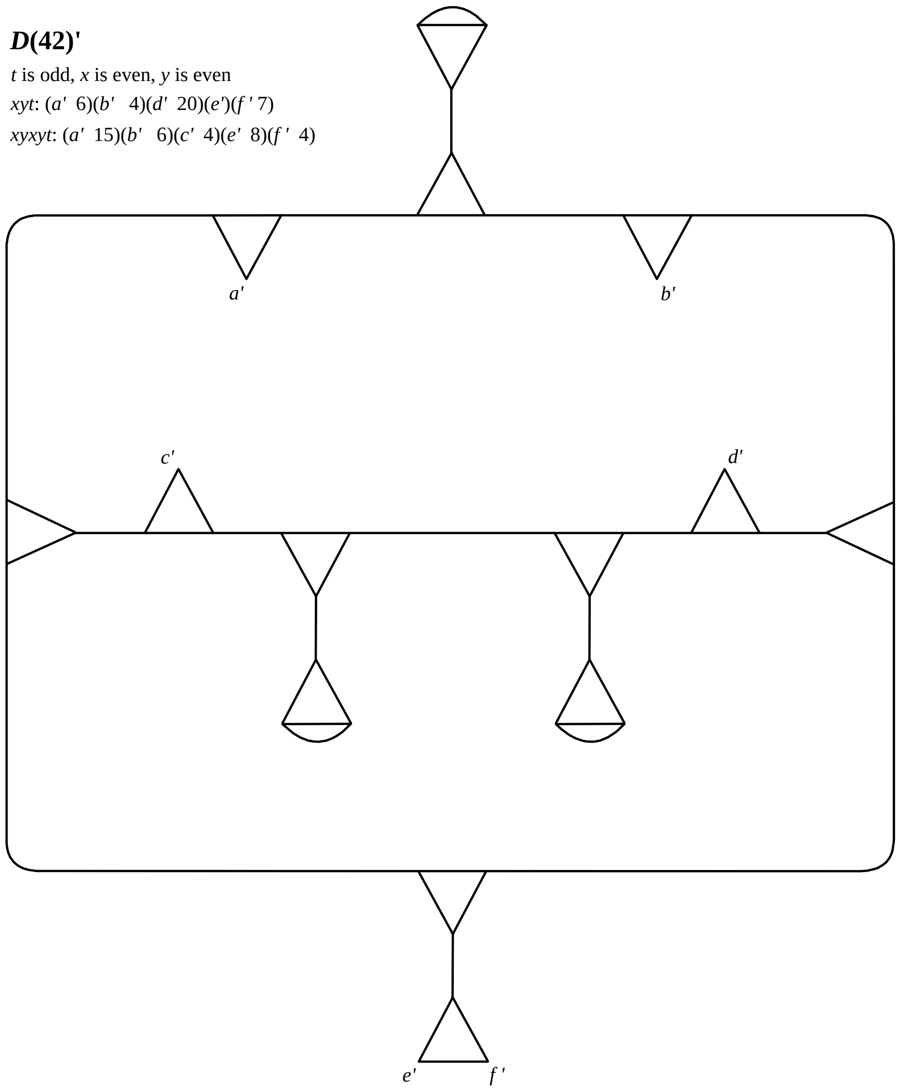
A transitive group G is called imprimitive if there is at least one non-trivial block B. If G has only trivial blocks, then G is called primitive.

3.4 Basic diagrams

We use $D(42)(Fig.3.4)$, $D(42)'(Fig.3.5)$, $D(52)(Fig.3.6)$ and $D(53)(Fig.3.7)$, as basic diagrams to prove our result.


D(42)
t is even, x is even, y is even
xyt: (a 6)(b   6)(c 6)(d 6)(e 6)(f 6)
xyxyt: (a 6)(b   6)(c 6)(d 6)(e 6)(f 6)
a
b
c
d
e
f

$Fig.3.4$


D(42)'
t is odd, x is even, y is even
xyt: (a' 6)(b' 4)(d' 20)(e')(f' 7)
xyxyt: (a' 15)(b' 6)(c' 4)(e' 8)(f' 4)
a'
b'
c'
d'
e'
f'

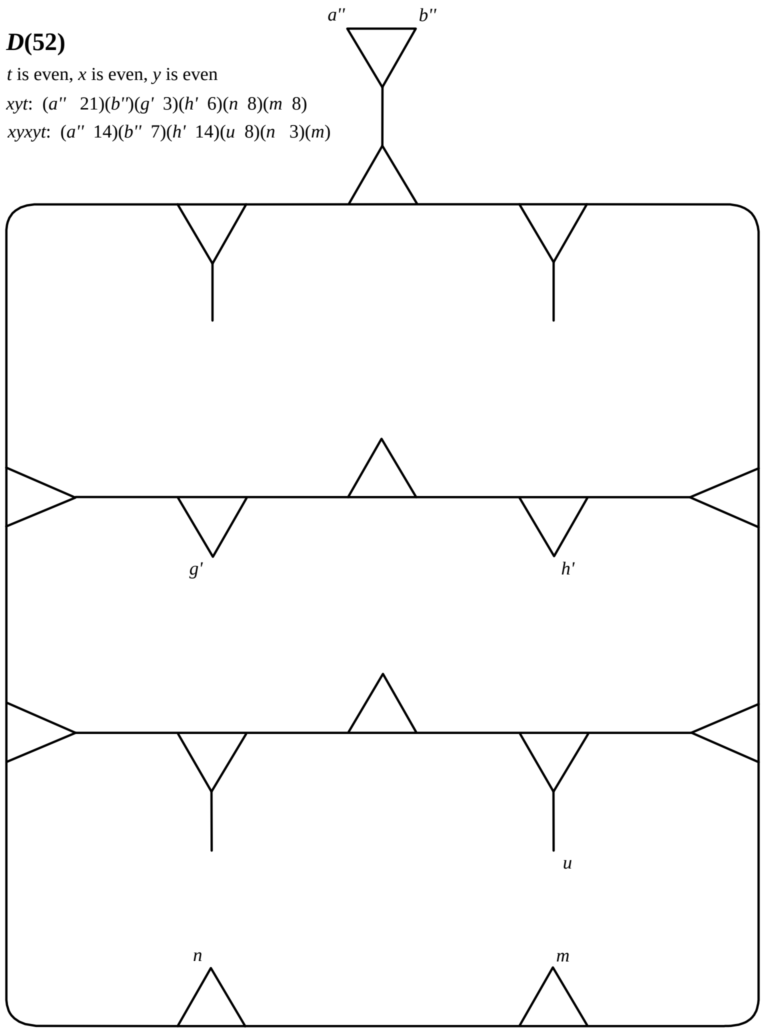
$Fig. 3.5$

D(52)

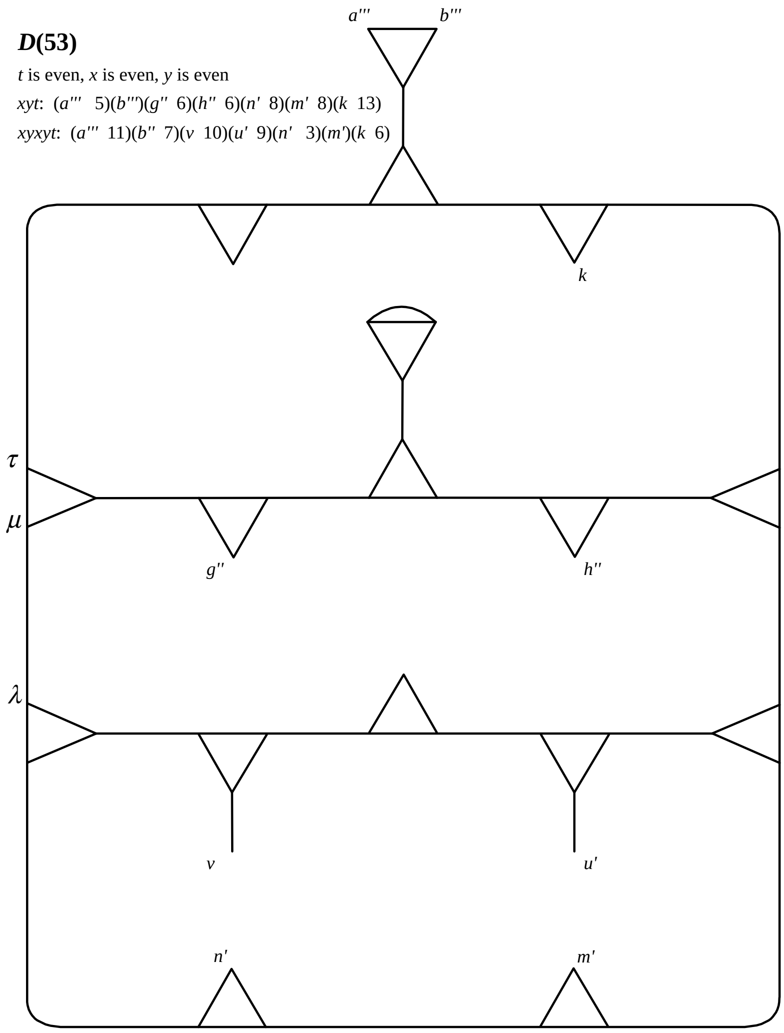
t is even, x is even, y is even

xyt: $(a''\ \ 21)(b'')(g'\ \ 3)(h'\ \ 6)(n\ \ 8)(m\ \ 8)$

$xyxyt$: $(a''\ \ 14)(b''\ \ 7)(h'\ \ 14)(u\ \ 8)(n\ \ 3)(m)$



$Fig.3.6$



$Fig.3.7$

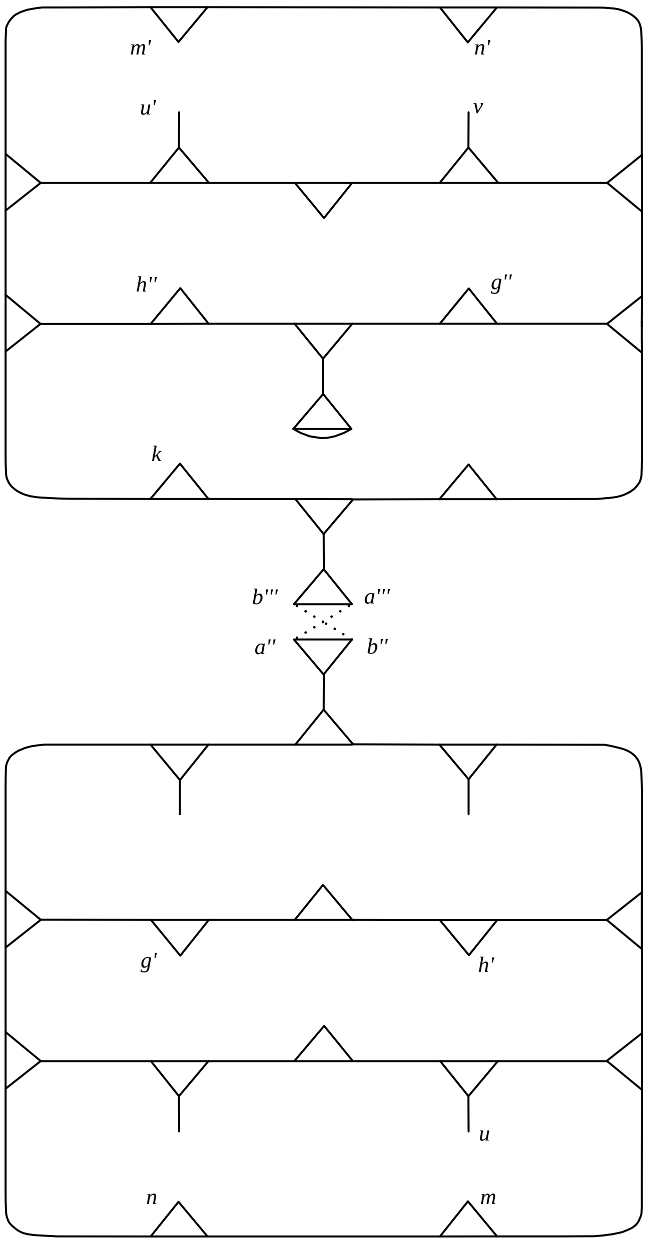
The vertices labelled λ, μ and τ in the diagram $D(53)$ are pointed out for a reason which becomes clear in section 3.5. These vertices are in the same cycle of $xyxyt$ of length 12. Now we join one copy of $D(52)$ to one copy of $D(53)$ by connecting $[a'', b'']_1$ to $[a''', b''']_1$, as shown in $Fig.3.8$.

D(105)

t is even, x is even, y is even

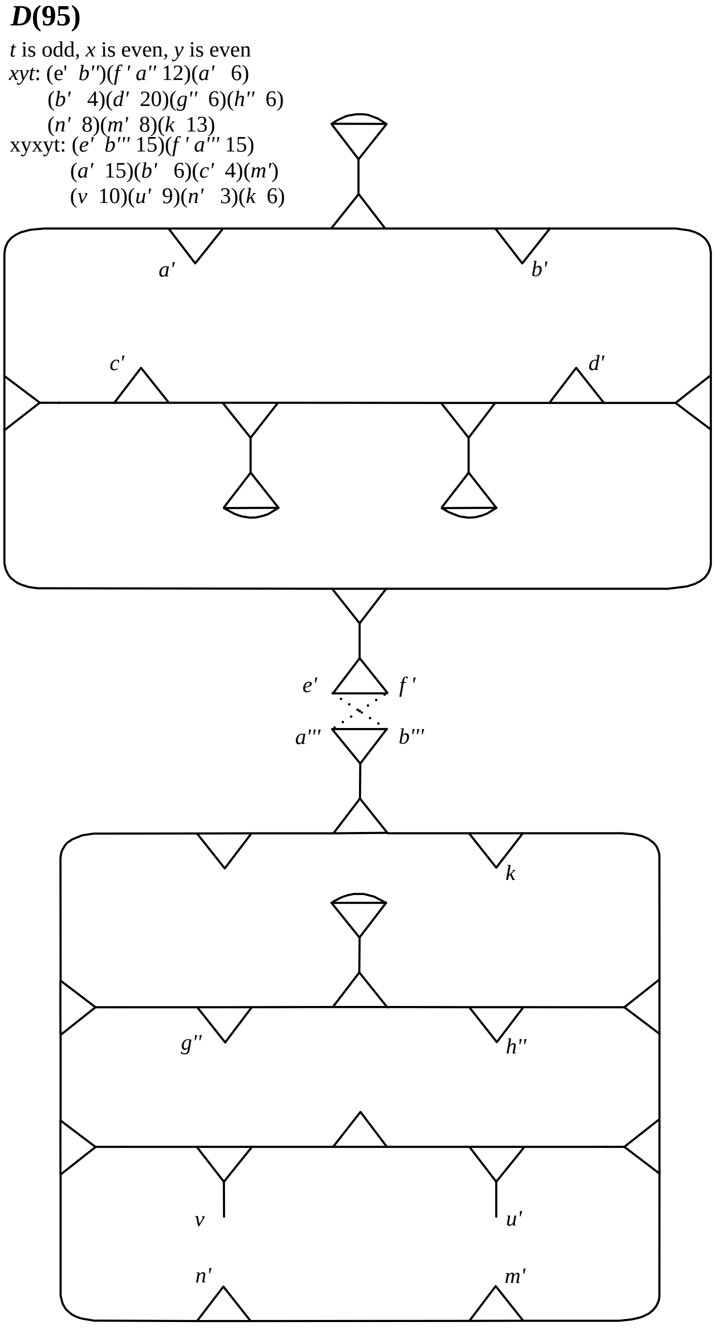
xyt: $(a''' a''\ 26)(b''' b'')(g''\ 6)(h''\ 6)(n'\ 8)$
$(m'\ 8)(k\ 13)(g'\ 3)(h'\ 6)(n\ 8)(m\ 8)$

$xyxyt$: $(a''' a''\ 25)(b''' b''\ 14)(v\ 10)(u'\ 9)(m')$
$(n'\ 3)(k\ 6)(h'\ 14)(u\ 8)(n\ 3)(m)$



$Fig.3.8$

Next we join one copy of $D(42)'$ to one copy of $D(53)$, by connecting $[e', f']_1$ to $[a''', b''']_1$, as shown in $Fig.3.9$.



$Fig.3.9$

3.5 Quotients of $G^{3,13,252}$

We employ Jordan's Theorem in our proof. Statement of the theorem is given here.

Theorem 7 *(Theorem 13.9, page 39, [48]). Let p be a prime number and G a primitive group of degree* $n = p + k$ *with* $k \geq 3$*. If G contains an element of degree and order p then G is either alternating or symmetric.*

Now we state and prove our result.

Theorem 8 *For all but finitely many positive integers n,* A_n *and* S_n *occur as quotients of the group* $G^{3,13,252}$*.*

Proof. We use four diagrams $D(42)(Fig.3.4)$, $D(53)(Fig.3.5)$, $D(105)(Fig.3.6)$, and $D(95)(Fig.3.7)$. These diagrams are connected by using $1-composition$, $2-composition$ or $3-composition$ to get the required diagram. Each of these diagrams carries a specification. $D(42)$ has three $3-handles$ ($[e,f]_3$, $[c,d]_3$ and $[a,b]_3$). $D(53)$ has one $1-handle$ ($[a''',b''']_1$), one $2-handle$ ($[n',m']_2$), and one $3-handle$ ($[g'',h'']_3$). $D(105)$ has two $2-handles$ ($[n,m]_2$ and $[n',m']_2$), and two $3-handles$ ($[g',h']_3$ and $[g'',h'']_3$). $D(95)$ has one $2-handle$ ($[n',m']_2$), and two $3-handles$ ($[a',b']_3$ and $[g'',h'']_3$).

Let us take u copies of $D(42)$, v copies of $D(53)$ and w copies of $D(105)$, and connect them together by the order $uD(42) + vD(53) + wD(105)$, where u, v, w are positive integers. First, u copies of $D(42)$ are joined by connecting any two diagrams by one $3-handle$ of each diagram. After u copies of $D(42)$ are linked, we have $u + 2$ number of free $3-handles$ in this diagram. This way of joining u copies of $D(42)$ is always possible for any positive integer u. In this chain of u copies of $D(42)$, every cycle of $xyxyt$ is of length 7 or 14. Every $3-handle$ has two cycles of $xyxyt$, each of length 7 or each of length 14. Now we join v copies of $D(53)$. To join any two copies of $D(53)$ we use $1-handle$ of each diagram or $2-handle$ of each diagram. This way of joining v copies of

$D(53)$ is always possible for any positive integer v. After v copies of $D(53)$ are linked in this way, we have v number of free $3-handles$ in this diagram. In this chain of v copies of $D(53)$, every cycle of $xyxyt$ is of length $1, 2, 4, 7, 8, 11, 12, 16$ or 24. Every $3-handle$ has two cycles of $xyxyt$: one is of length 8 and the other is of 12, or one is of length 16 and the other is of 24. Now by using $3-composition$ we connect this chain of v copies of $D(53)$ to the diagram consisting of chain of u copies of $D(42)$. While doing so we make sure that $3-handle$ involved in this $3-composition$ chosen from chain of u copies of $D(42)$ is one which has two cycles of $xyxyt$ of length 7. This diagram has $u+1$ number of free $3-handles$ in chain of u copies of $D(42)$ and $v-1$ number of free $3-handles$ in chain of v copies of $D(53)$. This composition introduces two new cycles of $xyxyt$: one of length 15 and the other of length 19, or one of length 23 and the other of length 31. We may fix that new introduced cycles of $xyxyt$ have lengths 15 and 19. It is worth noting that in this diagram there is exactly one place where $D(42)$ and $D(53)$ are linked. Now we draw one more diagram by joining w copies of $D(105)$ together. Each of these w diagrams has two $2-handles$. We use only $2-composition$ to connect these w copies of $D(105)$ such that any two diagrams are linked by one $2-handle$ of each diagram. After w copies of $D(105)$ are linked, we have w number of free $3-handles$ $[g', h']_3$, and also w number of free $3-handles$ $[g'', h'']_3$ in this diagram. Now we connect this diagram to the diagram we already have with $u+v$ number of free $3-handles$ by using one $3-handle$ $[g'', h'']_3$.

The resulting diagram $D(n)$ has n $(= 42u + 53v + 105w)$ vertices. It is important to note that the numbers 42, 53 and 105 have no common prime factor. First, take $v = 2$ and $w = 1$, we can get any integer of the form $42u+106+105 = 42(u+5)+1$ with $u > 0$. In other words we can get any $n \geq 253$ that is congruent to 1 mod 42. Next, take $v = 4$ and $w = 2$, we can get any integer of the form $42u+212+210 = 42(u+10)+2$ with $u > 0$. In other words we can get any $n \geq 464$ that is congruent to 2 mod 42. We can continue this way till we get such an expression for every positive integer n greater than a certain limit. The largest positive integer n which can not be expressed in this way is 1323, which is congruent to 21 mod 42. Any positive integer $n \geq 1324$ can be expressed in this way. Moreover, permutations associated with x, t and y are even. Length of each cycle of xyt in $D(n)$ is a divisor of 252, so $D(n)$ is a permutation representation of $G^{3,13,252}$. Every cycle of $xyxyt$ in $D(n)$ is of length $2, 4, 7, 8, 9, 10, 11, 12, 14, 15, 16, 24, 27$ or 34, except for one, which has length 19. If we define $g = (xyxyt)^{14137200}$, then obviously $g \in \langle x, y \rangle$ and g induces a single cycle of length 19 containing λ, μ and τ.

Now we prove that the permutation representation of the group $G^{3,13,252}$ is primitive. We assume that the representation is imprimitive, so that all vertices of the cycle must belong to the same block of imprimitivity; say $\acute{B}$. Now τ, λ, $\mu \in \acute{B}$ and $\tau y = \mu$, $\lambda x = \mu$. This shows that $\acute{B}$ is preserved by both x and y. But $\langle x, y \rangle$ is transitive on vertices of the diagram $D(n)$; so $\acute{B}$ has n vertices. It is a contradiction to the assumption of imprimitivity. Thus the permutation representation is primitive.

There exists an element of order and degree 19, and the group $G^{3,13,252}$ is primitive on n vertices; therefore, by Theorem 7 permutations x, t and y must be representing either S_n or A_n. Since x, t and y induce only even permutations, $D(n)$ is a representation of A_n. Thus A_n is a quotient of $G^{3,13,252}$.

Now we connect one copy of $D(95)$ to the existing diagram $D(n)$ using the $3-handle$ $[a', b']_3$ by $3-composition$ to prove that S_n is also a quotient of $G^{3,13,252}$. Every cycle of $xyxyt$ in $D(95)$ is of length $1, 4, 5, 7, 10, 11, 16$ or 17. We link the handle $[a', b']_3$ to any of $u + 1$ number of free $3 - handles$ available in the diagram in chain of u copies of $D(42)$. The composition of only one copy of $D(95)$ results in two cycles of $xyxyt$ of lengths 14 and 23 (or 30 and 21, depending on which handle of $D(42)$ is used). Now the resulting diagram $D(n)$ is a permutation representation of $G^{3,13,252}$ with n $(= 42u+53v+105w+95)$ vertices. This diagram is a transitive representation of $\langle x, y \rangle$ on n points again. But now t is an odd permutation while both x and y are even . By using the same single cycle of length 19 and the above given argument $D(n)$ is a representation of S_n now. ■

Corollary 9 *For all but finitely many positive integers n, A_n is a quotient of $\Delta(2,3,13;126)$.*

Theorem 8 holds for all $n > 1323$. One can further investigate $G^{3,13,m}$ for any $m < 252$ by finding more suitable coset diagrams for the group $\Delta^*(2,3,13)$ action on finite sets. Then possibility of a general statement about quotients of $G^{3,13,m}$ can also be investigated.

Chapter 4

ACTION OF THE HECKE GROUP $H(\sqrt{3})$ ON REAL QUADRATIC FIELDS

4.1 Introduction

The triangle group $\Delta(2, 3, 13)$ is an infinite quotient of the modular group. One of the techniques applied to investigate the modular group is 'coset diagrams'. The technique is useful particularly in study of action of the group on finite and infinite fields (for instance see [30], [31], [36]). Detailed information on construction of the diagrams for the group is available in [29]. The group action on real quadratic fields is discussed in [30]. It has been proved that a finite number of ambiguous numbers of the form $(a + \sqrt{n})/c$ exist for a fixed value of a non-square positive integer n; the part of a coset diagram comprising these numbers makes up a circuit and it is the only circuit in the orbit. In [36], these circuits have been further classified by finding a condition for existence of an orbit of the modular group comprising a circuit of a given type. Moreover, necessary and sufficient conditions were also found for existence of two orbits of the modular group: one containing both $(a + \sqrt{n})/c$ and its conjugate $(a - \sqrt{n})/c$; the other containing both $(a + \sqrt{n})/c$ and $1/\dfrac{(a - \sqrt{n})}{c}$.

Suppose m denotes a positive integer. The modular group belongs to the class of a group G generated by two linear-fractional transformations x and y with a finite presentation: $\langle x, y : x^2 = y^m = 1 \rangle$. If G is to act on all real quadratic fields then possible values of m are $6, 4, 3, 2$ and 1. G is a cyclic group of order 2, when $m = 1$. When $m = 2$, G is an infinite order dihedral group. Action of the group on set of real quadratic irrational numbers does not provide any inspiring information. When $m = 4$, action of G on real quadratic fields was discussed in [35]. If some results hold for one group belonging to a specific class of groups then it is an immediate question to ask whether the results hold for some other groups of the same class or for the whole class. It is a common way of generalizing the existing explored facts in any field and in the field of group theory as well. When $m = 6$, G is the group $H(\sqrt{3})$. In this chapter, we investigate the group $H(\sqrt{3})$ for some facts which already exist for the modular group. In fact, we discuss structure of words in the group $H(\sqrt{3})$ through action of the group on real quadratic fields.

Let the group $H(\sqrt{3})$ be denoted by H for this chapter. As mentioned earlier, the linear-fractional transformations x and y defined by $z \longmapsto -1/3z$ and $z \longmapsto -1/3(z+1)$ respectively, generate the group H and satisfy the relations: $x^2 = y^6 = 1$ [37]. Suppose γ denotes the number $(a + \sqrt{n})/c$, where a, $(a^2 - n)/c$, c are relatively prime integers and n is a non-square positive integer. Let the algebraic conjugate of γ be denoted by $\bar{\gamma} = (a - \sqrt{n})/c$. In case both γ and $\bar{\gamma}$ have different signs, the number γ is called ambiguous. It is proved in [37] that n does not change its value in the orbit γ^H of the group H acting on rational projective line and real quadratic fields. The ambiguous numbers obtained are finite in number and form a single closed path in a coset diagram containing such numbers. Moreover, the closed path is the only in the orbit. In [2], it is shown: if γ is of the form $(a + \sqrt{n})/3c$ (a, $(a^2 - n)/3c$, $3c$, are relatively prime integers and n is a non-square positive integer), then a closed path can be found in a coset diagram for γ^H. In the path all numbers are of the form $(a + \sqrt{n})/3c$, and they all belong to projective line over $\mathbb{Q}^*(\sqrt{n})$.

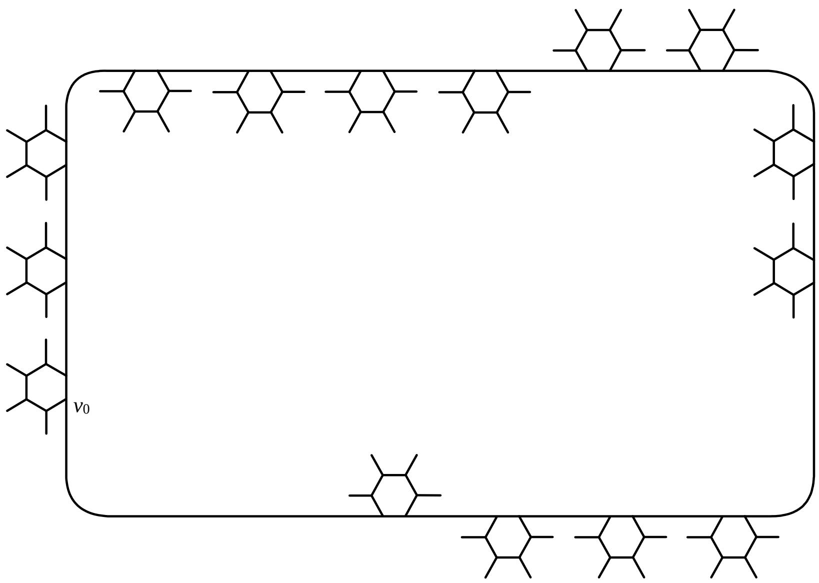
As a step towards generalization we investigate existence of some results regarding word structure of the group H, which already exist for the modular group ([36]). Motivation behind the work is the above mentioned results which were explored while studying the group H action on rational projective line and real quadratic fields. In particular, we are concerned with structure of words of the elements of H generating the number γ. We also discuss characteristics of one class of the circuits comprising these numbers. We associate a condition with existence of an orbit of H comprising a circuit of a given type. In case of existence of such a circuit, we find a condition for combined existence of both the real quadratic irrational number γ and its algebraic conjugate $\bar{\gamma}$ in the orbit.

This chapter contains two more sections. In the first section, we define coset diagrams for the group H, explain the relevant terms and give an example of the diagrams. In the second section, we recall linked existing results, discuss the group H action on projective line over $\mathbb{Q}^*(\sqrt{n})$, elaborate the class of circuits we are interested in and prove the results.

4.2 Coset Diagrams for $H(\sqrt{3})$

In a coset diagram for H six vertices of a hexagon permuted anticlockwise by y denote $6-$cycles of y as in [37]. Two vertices interchanged by x are joined by an edge in the diagram. A sequence $v_0, e_1, v_1, e_2, . . ., e_t, v_t$, containing vertices and edges of a coset diagram is called a path if e_i joins v_{i-1} and v_i for $i = 1, 2, 3, . . ., t$, where $e_i \neq e_j$ (for $i \neq j$), as in [36]. A circuit is meant to be a closed path comprising edges and hexagons. For example, one can refer to $Fig.5.5$ in [37]. There are different types of circuits depending upon different number of vertices of a hexagon outside/inside a circuit. Here we discuss a class of circuits such that each circuit in the class is composed of only the hexagons having four vertices outside/inside the circuit.

Let $n_1, n_2, \ldots, n_{2t}$ be a sequence containing positive integers. A $(n_1, n_2, \ldots, n_{2t})-type$ circuit is meant to be the circuit in which; n_1 hexagons are such that four vertices of each lie outside the circuit; n_2 hexagons are such that four vertices of each lie inside the circuit; and this pattern is continued till n_{2t}. An element $g = (yx)^{n_1}(y^{-1}x)^{n_2} \ldots (y^{-1}x)^{n_{2t}}$ of H is induced from the circuit, which fixes a specific vertex of a hexagon. As an example, a circuit is shown in $Fig.4.1$. An element $g = (yx)^3(y^{-1}x)^4(yx)^2(y^{-1}x)^2(yx)^3(y^{-1}x)$ of H is induced from the circuit, which fixes the vertex v_0 as shown in the diagram. In the class of circuits under discussion, this circuit is $(3, 4, 2, 2, 3, 1) - type$. If a circuit belonging to the class exists in an orbit of H for the sequence $n_1, n_2, ..., n_{2t}$, we find a condition which ensures existence of both γ and its conjugate $\bar{\gamma}$ in the same orbit as found in [36] for an orbit of the modular group.



$Fig.4.1$

4.3 Action of $H(\sqrt{3})$ on Projective Line over $\mathbb{Q}^*(\sqrt{n})$

The ambiguous numbers of the form $(a+\sqrt{n})/c$ (a, $(a^2-n)/c$, c are relatively prime integers and n is a non-square positive integer) are finite for a fixed value of n ([37]). Moreover, the ambiguous numbers form a single circuit in a coset diagram for orbit of $(a+\sqrt{n})/c$ and it is the only circuit contained in the orbit. As defined before in introduction of the thesis $\mathbb{Q}^*(\sqrt{n})$ is the set $\{(a+\sqrt{n})/3c : a, (a^2-n)/3c, 3c$ are relatively prime integers and n is a non-square positive integer$\}$ and projective line over the set is $\mathbb{Q}^*(\sqrt{n}) \cup \{\infty\}$. It was proved further in [2] that in an orbit a circuit can be found in which all the ambiguous numbers are of the form $(a+\sqrt{n})/3c$ (a, $(a^2-n)/3c$, $3c$ are relatively prime integers and n is a non-square positive integer). We are considering here those circuits, each of which consists of the hexagons having four vertices outside/inside the circuit. If $\acute{t}$ is used to denote the number of sets of hexagons in the circuit with four vertices inside the circuit and t is used to denote the number of sets of hexagons in the circuit with four vertices outside the circuit, then $t=\acute{t}$. The total number of sets of hexagons in the circuit then becomes $2t$. These sets of hexagons with four vertices outside/inside occur alternately in these circuits.

Theorem 10 *Every finite order element of H, except the (group theoretic) conjugates of x, y^2, $y^{\pm1}$, and $(yx)^n$, $n>0$, has real quadratic irrational numbers as fixed points.*

Proof. Let $\acute{g} : z \longrightarrow (az+b)/(cz+d)$ be an element of H and v_0 be a fixed point of $\acute{g}$. Then, $cv_0^2 + (d-a)v_0 - b = 0$ (4.1)

Roots of equation 4.1 are real iff $(d-a)^2+4bc \geq 0$. Since $\acute{g}$ belong to H, therefore $ad-bc = 1$ *or* 3.

Let us discuss these two cases separately. If determinant $ad-bc=3$, then $d^2+a^2-2ad+4(ad-3) \geq 0$. It implies that $(d+a)^2-12 \geq 0$, where $d+a$ is trace of the matrix associated with $\acute{g}$. Thus, for complex roots of (4.1), we have $(d+a)^2 < 12$ and the possible values of $d+a$ are 0, ±1, ±2, ±3.

If $d+a=0$, then $\acute{g}: z \longrightarrow (az+b)/(cz-a)$ and $\acute{g}^2 = 1$. Since each of the order 2 elements in discussion is a conjugate of x; therefore, so is $\acute{g}$. Hence, conjugates of x have complex fixed points. Put $d+a = \pm 1$; since $-a, -b, -c, -d$ can replace a, b, c, d in $\acute{g}$; therefore, $d+a$ is considered equal to -1 only. Now possible values of order of $\acute{g}$ are 1, 2, 3, 6; therefore, $A^n = \lambda I$ only when $n = 1, 2, 3, 6$, where A is a matrix corresponding to $\acute{g}$. For a 2×2 matrix A we know that

$$A^2 = \lambda I \text{ iff } tr(A) = 0 \tag{4.2}$$

$$A^3 = \lambda I \text{ iff } tr^2(A) = det(A) \tag{4.3}$$

$$A^6 = \lambda I \text{ iff } tr^2(A) = 3det(A). \tag{4.4}$$

If $d+a=-1$, this implies that $det(A) = 1$. Thus no matrix, with trace -1 and determinant 3, having finite order exists. The matrix corresponding to $\acute{g}$ has infinite order.

If $d+a=\pm 2$, then again none of the equations (4.2), (4.3), (4.4) holds for A with determinant 3. It implies that such matrices correspond to an element of H having infinite order. Thus $\acute{g}$, having infinite order, has complex numbers as fixed points.

If $d+a=\pm 3$, then (4.4) holds for A with determinant 3. Thus $\acute{g}$ is of order 6 and it is a conjugate of y because each of the elements of order 6 in H is a conjugate of y.

It implies that the fixed points of the conjugates of $y^{\pm 1}$are complex numbers. If $d+a = \acute{m} \geq \sqrt{12}$, then $(d+a)^2 - 12 \geq 0$; the roots are real.

If $(d+a)^2 - 12$ is a perfect square, then we shall be handling rational numbers in a coset diagram. In that situation, we already know that only fixed point is ∞ ([37]). Hence, it is not possible to consider $(d+a)^2 - 12$ to be a perfect square. Moreover, the fixed points are irrational numbers.

If determinant $ad - bc = 1$, then (4.1) has real roots for $d^2 + a^2 - 2ad + 4(ad - 1) \geq 0 \implies (d + a)^2 - 4 \geq 0$. It suggests that $(d + a)^2 < 4$ for complex roots. In this case, the possibilities are $d + a = \pm 1$ and $d + a = 0$. If $d + a = 0$, then $\acute{g}$ is defined by $z \longmapsto (az + b)/(cz - a)$ and $\acute{g}^2 = 1$. By the argument given earlier for the other case, again $\acute{g}$ is proved to be a conjugate of x. Hence, conjugates of x have complex fixed points.

If $d + a = \pm 1$, we can consider only $d + a = -1$ as discussed in other case. It implies that only possibility is (4.2); the order of $\acute{g}$ is 3 and $\acute{g}$ is a conjugate of y^2. This shows that conjugates of y^2 have complex fixed points.

If $d + a = \pm 2$, then A has characteristic equation: $A^2 - 2A + I = 0$. By repeated multiplication of this equation by A and substitution of $2A - I$ for A^2, we obtain the equation $A^n - nA + (n-1)I = 0$ for a positive integer n. Thus, $\acute{g}$ proves to be a conjugate of $(yx)^n : z \longrightarrow z + n$ in this case and ∞ is the only fixed point of it.

If $d + a = \pm 3$, then none of (4.2), (4.3), (4.4) holds for A with determinant 1. This completes the proof. ■

Let $g = (yx)^{n_1}(y^{-1}x)^{n_2} \ . \ . \ . \ (y^{-1}x)^{n_{2t}}$, $n_i > 0$ for all $i = 1, 2, . . ., 2t$, be an element of H, which fixes γ (a real quadratic irrational number). If the matrix corresponding to g is denoted by $B(g)$, then trace of the matrix $B(g)$ fixes size of the circuit (n_1, n_2, . . ., n_{2t}) comprising γ. Infact, the trace and the sequence n_1, n_2, . . ., n_{2t} are strongly related. We establish this relationship in the next Theorem as established in [36] for the modular group.

Theorem 11 *Let $E = \{1, 2, . . ., 2t\}$ be the cyclically ordered set of positive integers and the orbit of γ contains a $(n_1, n_2, . . ., n_{2t}) -$ type circuit, where $n_i > 0$. Let S be the collection of non-empty subsets of E obtained by striking out any number of adjacent pairs of elements of E. Let $n_J = \prod_{i \in J} n_i$ for $J \in S$. Then, the trace of $B(g)$ is $2 + \sum_{J \in S} \lambda_{i_J} n_J, \lambda_{i_J} = 3^{k_J}$, k_J is some positive integer.*

Proof. Consider an element $g = (yx)^{n_1}(y^{-1}x)^{n_2} \ . \ . \ . \ (y^{-1}x)^{n_{2t}}$, $n_i > 0$, of H corresponding to a $(n_1, n_2, \ldots, n_{2t})-type$ circuit such that g fixes γ. Since the transformations $yx : z \longrightarrow z+1$ and $y^{-1}x : z \longrightarrow z/(3z+1)$ represent the matrices

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \text{ and } \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix}$$

respectively, therefore, the matrix $B(g)$ corresponding to g is of the form:

$$\begin{bmatrix} 1 & n_1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 3n_2 & 1 \end{bmatrix} \cdots \begin{bmatrix} 1 & n_i \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 3n_{i+1} & 1 \end{bmatrix} \cdots \begin{bmatrix} 1 & n_{2t-1} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 3n_{2t} & 1 \end{bmatrix} \tag{4.5}$$

Let A be a matrix written in the form:

$$\begin{bmatrix} a_{11}^{(1)} & a_{12}^{(1)} \\ a_{21}^{(1)} & a_{22}^{(1)} \end{bmatrix} \begin{bmatrix} a_{11}^{(2)} & a_{12}^{(2)} \\ a_{21}^{(2)} & a_{22}^{(2)} \end{bmatrix} \cdots \begin{bmatrix} a_{11}^{(m)} & a_{12}^{(m)} \\ a_{21}^{(m)} & a_{22}^{(m)} \end{bmatrix}.$$

Then trace of A, of course, is of the type:

$$\sum a_{\lambda m\lambda 1}^{(1)} a_{\lambda 1\lambda 2}^{(2)} \cdots \ a_{\lambda m-1\lambda m}^{(m)}. \tag{4.6}$$

The factor matrices in case of the matrix $B(g)$ are alternately

$$\begin{bmatrix} 1 & n_i \\ 0 & 1 \end{bmatrix} \text{ and } \begin{bmatrix} 1 & 0 \\ 3n_j & 1 \end{bmatrix}$$

; therefore, in any term of (4.6) if some $a_{\lambda i\lambda j}^{(l)} = 0$, the whole term is zero. We can ignore any $a_{\lambda i\lambda j}^{(l)}$ which is 1. We consider only those terms which are neither 1 nor 0.

Now we pick a part of (4.5) containing three matrices as shown below:

$$\begin{bmatrix} 1 & 0 \\ 3n_p & 1 \end{bmatrix} \begin{bmatrix} 1 & n_q \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 3n_r & 1 \end{bmatrix}.$$

Trace of $B(g)$ is sum of the certain products got by picking one number from each matrix in (4.5); therefore, suppose we pick n_q from the middle matrix. In order to get a non-zero product we pick 1 from first column of the first matrix and $3n_r$ from the third matrix. If $3n_p$ is picked instead of 1 from the first matrix, then we require to pick 1 from second column of the third matrix.

In the same way, we consider another portion of three matrices from (4.5) as shown below:

$$\begin{bmatrix} 1 & n_p \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 3n_q & 1 \end{bmatrix} \begin{bmatrix} 1 & n_r \\ 0 & 1 \end{bmatrix}.$$

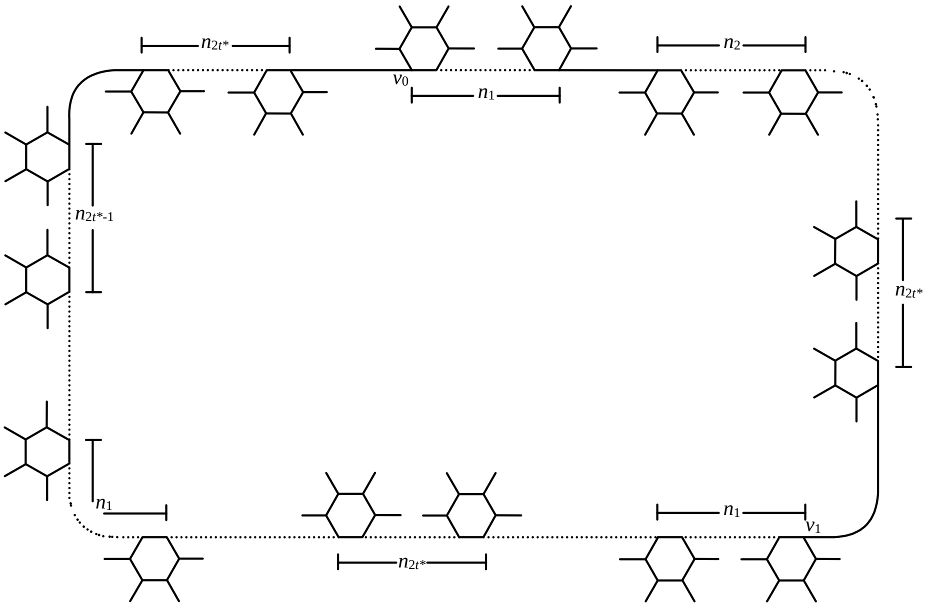
Suppose we pick $3n_q$ from the middle matrix of the three matrices. Then we require to pick n_p from the first matrix and 1 from first row in the last matrix, or if 1 from second row in the first matrix is picked, then we require to pick n_r from the last matrix. Actually, we are picking out adjacent pairs of elements in E. Hence, for $J \in S$, if we put $n_J = \prod_{i \in J} n_i$, then $B(g)$ has trace $2 + \sum_{J \in S} \lambda_{i_J} n_J, \lambda_{i_J} = 3^{k_J}$, k_J is some positive integer. ■

Let $n_1, n_2, \ldots, n_{2t}$ be a sequence of positive integers. A $(n_1, n_2, \ldots, n_{2t^*}, n_1, n_2, \ldots, n_{2t^*}, \ldots, n_1, n_2, \ldots, n_{2t^*}) - type$ circuit, where t^* divides t, is stated to have a period of length $2t^*$. Next we prove that existence of this type of circuit in an orbit of H acting on projective line over $\mathbb{Q}^*(\sqrt{n})$ is not possible as in case of the modular group ([36]).

Theorem 12 *For a given sequence of positive integers $n_1, n_2, \ldots, n_{2t}$, there does not exist a circuit which has a period of length $2t^*$, where t^* divides t, in an orbit of H.*

Proof. Suppose a circuit having a period of length $2t^*$ for the given sequence exists. Then, the circuit will be a $(n_1, n_2, \ldots, n_{2t^*}, n_1, n_2, \ldots, n_{2t^*}, \ldots, n_1, n_2, \ldots, n_{2t^*}) - type$ circuit as shown in $Fig.$ 4.2. The orientation of the hexagons in the circuit may be reversed. It does not make any difference.

If v_0, v_1, . . ., v_{t/t^*} are the vertices of the hexagons in the circuit as shown in $Fig.4.2$ and $g = (yx)^{n_1}(y^{-1}x)^{n_2}. . .(y^{-1}x)^{n_{2t^*}} \neq 1$, then $v_{i+1} = v_i g$, where $i \in \{0, 1, 2, . . ., (t/t^* - 1)\}$. It implies that $v_0 \neq v_0 g$. Since for all i, $v_i = v_i(g)^{t/t^*}$ and $(g)^{t/t^*} \neq 1$; therefore, it contradicts the fact: if $g \in H$ and $g \neq 1$, then g has exactly one or two fixed points, unless $g^n = 1$ for an appropriate n. Hence, no γ^H comprises a $(n_1, n_2, . . ., n_{2t^*}, n_1, n_2, . . ., n_{2t^*}, . . ., n_1, n_2, . . ., n_{2t^*}) - type$ circuit. ■



$Fig.4.2$

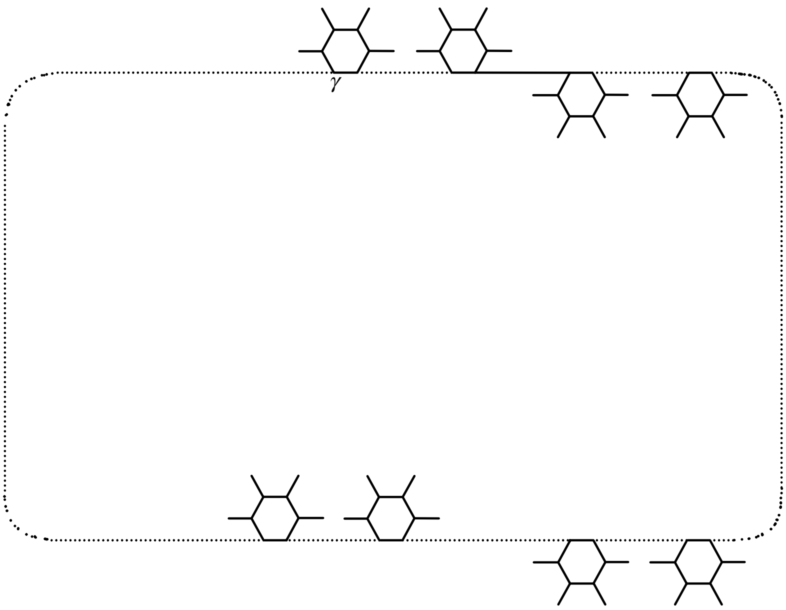
Theorem 13 *For a given sequence* $n_1, n_2, . . ., n_{2t}$ *of positive integers there exists a real quadratic irrational number* γ *such that a circuit in the orbit of* γ *under* H *is* $(n_1, n_2, . . ., n_{2t}) - type$ *if the circuit does not have a period of even length.*

Proof. In Theorem 12, it is established that a sequence containing positive integers cannot be associated to a circuit with repetitions. In order to prove this condition to be sufficient, we are to just show that a fixed point k of $g = (yx)^{n_1}(y^{-1}x)^{n_2} . . . (y^{-1}x)^{n_{2t}}$ is a real quadratic irrational number. Since g fixes k; therefore, by Theorem 10 g cannot be a conjugate of x, y^2, $y^{\pm1}$, and $(yx)^n$, where $n > 0$. By using Theorem 11, trace of the matrix

$B(g)$ is $r = 2 + \sum_{J \in S} \lambda_{i_J} n_J, \lambda_{i_J} = 3^{k_J}$ (k_J is some positive integer, where $n_J = \prod_{i \in J} n_i$). It implies that $\sqrt{r^2 - 4}$ is not a complex number. Moreover, $r^2 - 4$ is not a perfect square. If it would be a perfect square, then k must be ∞ because of being fixed point of g [37]. We know that k is not ∞ and $det(B(g)) = 1$; therefore, $k \in \gamma^H$ is a real quadratic irrational number. We also know that ambiguous numbers in case of a coset diagram for γ^H form a set of circuits [37]; so Theorem 12 suggests that the orbit comprises a $(n_1, n_2, \ldots, n_{2t}) - type$ circuit. ■

In Theorem 14 we find a necessary and sufficient condition for exixtense of both γ and its conjugate $\bar{\gamma}$ in a circuit.

Theorem 14 *A circuit contains γ with its conjugate $\bar{\gamma}$ if and only if the circuit is $(n_1, n_2, \ldots, n_{t-1}, n_t, n_t, \ldots, n_2, n_1) - type$.*



$Fig.4.3$

Proof. We notice that $\bar{\gamma}g$ and γg are conjugates for every $g \in H$ if γ and $\bar{\gamma}$ are conjugates. It implies that it is enough to prove the result for any one element instead of every element in H.

Suppose both γ and $\bar{\gamma}$ exist in a circuit and γ is fixed by $g = (yx)^{n_1}(y^{-1}x)^{n_2} \ldots (y^{-1}x)^{n_{2t}}$, where $n_i > 0$, $i = 1, 2, \ldots, 2t$. The vertices which belong to the circuit shown in $Fig.4.3$ are indexed by the finite set $\{1, 2, \ldots, n\}$.

If γ occupy odd labelled vertices, then $\bar{\gamma}$ cannot occupy any odd labelled vertex. If any $\bar{\gamma}$ occupies such a vertex, then $\bar{\gamma} = \gamma(yx)^{n_1}(y^{-1}x)^{n_2} \ldots (y^{-1}x)^{n_r}$ for some $r < t$. This suggests that both γ and $\bar{\gamma}$ are fixed points of $g = (yx)^{n_1}(y^{-1}x)^{n_2} \ldots (y^{-1}x)^{n_{2t}}$ and $h = (yx)^{n_{r+1}}(y^{-1}x)^{n_{r+2}} \ldots (y^{-1}x)^{n_{2t}}(yx)^{n_1}(y^{-1}x)^{n_2} \ldots (y^{-1}x)^{n_r}$, respectively. But γ and $\bar{\gamma}$ being conjugates are fixed by the same element of H. So g must be equal to h. If it is the case, then $g = f^m$ for some $m > 1$; γ is a fixed point of f. By Theorem 12 this can not happen except for g^t, $t \geq 1$. It is a contradiction. Thus, no $\bar{\gamma}$ occupies an odd labelled vertex. All $\bar{\gamma}$ occupy only the vertices labelled with even numbers; so $\bar{\gamma}$ is fixed by the element $h = (yx)^{n_t}(y^{-1}x)^{n_{t-1}} \ldots (yx)^{n_1}(y^{-1}x)^{n_{2t}} \ldots (y^{-1}x)^{n_{t+1}}$of H. This shows that $\bar{\gamma}$ corresponds to $(n_t, n_{t-1}, \ldots, n_1, n_{2t}, n_{2t-1}, \ldots, n_{t+1})$ but with reversed order orientation of the hexagons and γ corresponds to $(n_1, n_2, \ldots, n_{2t})$. In other words, type of the circuit which corresponds to $\bar{\gamma}$ is same as that of the circuit which corresponds to γ but starting at a different point with reversed signs. Since both the types are same, therefore, any circuit which comprises both $\bar{\gamma}$ and γ must be $(n_1, n_2, \ldots, n_{k-1}, n_k, n_k, \ldots, n_2, n_1) - type$. ■

It is worth noting that the conditions determined here for the circuits existing in the orbit γ^H are similar to those determined in [36] for the circuits existing in the orbit γ^M (M stands for the modular group).

Chapter 5

ONE RELATOR QUOTIENTS OF THE HECKE GROUP $H(\sqrt{3})$

5.1 Introduction

One relator quotients of Hecke groups have been a matter of concern for many group theorists. For instance one can refer to [24] and [46]. One relator quotients of the group $H(\sqrt{2})$, the modular group and the group $H(\frac{1+\sqrt{5}}{2})$ were discussed in [12], [24], [40] and [46]. In this chapter, we investigate one relator quotients of the Hecke group $H(\sqrt{3})$. We obtain the one relator quotients by adding a new relation to the existing ones. As mentioned earlier in the thesis, the group $H(\sqrt{3})$ has a finite presentation $\langle a, b : a^2 = b^6 = 1\rangle$. Addition of a new relation to the presentation gives rise to the formation of a new group which is a quotient of the group $H(\sqrt{3})$. We add a new relation $w = R(a, b) = 1$ in terms of a and b for a cyclically reduced word $w = ab^{\epsilon_1}ab^{\epsilon_2}ab^{\epsilon_3}...ab^{\epsilon_n}$, where $5 \geq \epsilon_i \geq 1$, and get a finite representation of the one relator quotient of the group $H(\sqrt{3})$. Then we find order and structure of the quotient group when finite or we prove infiniteness otherwise. We use some GAP built in functions about properties of a group to find the quotients.

5.2 Notations

Throughout this chapter, we denote by k the sum of exponents of a in w and by l the sum of exponents of b in w. All the other notations used are standard group theoretical or GAP notations.

Theorem 15 *If $k = 0$ then $1 \leq l \leq 5$ and if $k = n$ then $n \leq l \leq 5n$.*

Proof. Suppose that $k = 0$, then w is equal to a power of b, and $l = \epsilon_i$ ($\because b^6 = 1$), so $1 \leq l \leq 5$.

If $k = n$, we suppose that $n > 0$. Since $1 \leq \epsilon_i \leq 5$, therefore, the minimum value of l (the sum of exponents of b in w) is n and it happens when $\epsilon_i = 1$ for each i. Similarly, when $\epsilon_i = 5$ for each i, we get the maximum value of l and it is $5n$. Hence in this case $n \leq l \leq 5n$. ■

A word w' is equivalent to w if w' can be obtained by cutting some part of w from the beginning and pasting it to the end in same order and vice versa. Let $N_{k,l}$ be the total number of non equivalent cyclically reduced words w with k and l as defined above. Then we have the following theorem.

Theorem 16 $N_{n,n} = N_{n,n+1} = N_{n,5n} = N_{n,5n-1} = 1$.

Proof. We prove that each expressions $N_{n,n}, N_{n,n+1}, N_{n,5n}, N_{n,5n-1}$ is equal to 1. Consider $N_{n,n}$, now $k = n$ and $l = n$, therefore, it follows from the definition of w that no expression except of the type $abababab...ab$, is possible for w in this case.

Consider $N_{n,n+1}$, now $k = n$ and $l = n + 1$, therefore, it again follows from the definition of w that no expression except of the type $abababab...ab^2$, is possible for w in this case. Some more expressions e.g. $ababab^2ab...ab$ are also possible but they all are equivalent to $abababab...ab^2$.

Consider $N_{n,5n}$, now $k = n$ and $l = 5n$, therefore, it follows from the definition of w that no expression except of the type $ab^5ab^5ab^5ab^5...ab^5$, is possible for w in this case.

Consider $N_{n,5n-1}$, now $k = n$ and $l = 5n - 1$, therefore, it follows from the definition of w that no expression except of the type $ab^5ab^5ab^5ab^5...ab^4$, is possible for w in this case. Some more expressions, for instance $ab^5ab^5ab^4ab^5...ab^5$, are also possible but they all are equivalent to $ab^5ab^5ab^5ab^5...ab^4$. ■

5.3 Methodology and Table Construction:

We consider all possible non-equivalent cyclically reduced words for a given pair of integers k and l. We pick one of the words and put it equal to identity element to establish a new relation. This relation is then added to the finite presentation of the group as an extra relation to get a one relator quotient of the group. Then we use GAP to investigate structure of the quotient. We construct a finitely presented group in GAP using finite presentation of the one relator quotient starting from a free group generated by two elements. Then we try to find order of the quotient group by using built in function 'Size'. GAP uses coset enumeration to determine order of a finite group. Once we have found order of a one relator quotient group then it is not difficult to determine structure of the group. If coset enumeration fails and we are unable to find order of a one relator quotient then we investigate whether the group is infinite or coset enumeration fails due to limitations of GAP. To prove a group to be infinite we use the function 'AbelianInvariants' or 'IsInfiniteAbelianizationGroup'.

Example 17 *For $k = 4$ and $l = 11$, we obtain the following non-equivalent cyclically reduced words.*

$$ababab^4ab^5$$
$$ababab^5ab^4$$

$$abab^4abab^5$$

$$abab^2ab^3ab^5$$

$$abab^2ab^5ab^3$$

$$abab^3ab^2ab^5$$

$$abab^3ab^5ab^2$$

$$abab^5ab^2ab^3$$

$$abab^5ab^3ab^2$$

$$abab^2ab^4ab^4$$

$$abab^4ab^4ab^2$$

$$abab^4ab^2ab^4$$

$$abab^3ab^3ab^4$$

$$abab^3ab^4ab^3$$

$$abab^4ab^3ab^3$$

$$ab^2ab^2ab^2ab^5$$

$$ab^2ab^2ab^3ab^4$$

$$ab^2ab^2ab^4ab^3$$

$$ab^2ab^3ab^2ab^4$$

$$ab^2ab^3ab^3ab^3$$

Let us consider the first word $ababab^4ab^5$, other words $ab^5ababab^4$, ab^4ab^5abab and $abab^4ab^5ab$ are omitted since these are equivalent to it. We add a relation $ababab^4ab^5 = 1$ to the group $H(\sqrt{3}) = \langle a, b : a^2 = b^6 = 1\rangle$ and get a one relator quotient $\langle a, b : a^2 = b^6 = ababab^4ab^5 = 1\rangle$. Using all these three relations we simplify as:

$$ababab^4ab^5 = 1$$

$$ababab^4ab^5a = a$$

$$(ababab^4ab^5a)(ababab^4ab^5a) = 1$$

$$ababab^5ab^4ab^5a = 1$$

$$bab^5ab^4ab^5a = ab^5a$$

$$bab^5ab^4 = 1$$
$$ab^5ab^4 = b^5$$
$$ab^5a = b$$
$$abab = 1$$

Thus we get $\langle a, b : a^2 = b^6 = ababab^4ab^5 = (ab)^2 = 1\rangle$ which is a finite presentation of a quotient of the triangle group $\Delta(2, 6, 2)$. We investigate that the quotient is isomorphic to C_2. We do this by using GAP built in functions available for finitely presented groups.

We gather our investigated one relator quotients of the group in form of a table. First two columns of the table indicate values of k and l, third column shows all non-equivalent words for each pair of values of k and l. Fourth column shows the finite presentation of the one relator quotient obtained by adding an extra relation got through placing the word in second column equal to identity element. Last column shows structure of the quotient investigated through use of GAP. The table is given below:

k	l	Word	Quotient Group	Abstract Structure
0	1	b	$\langle a, b : a^2 = b^6 = b = 1\rangle$	C_2
0	2	b^2	$\langle a, b : a^2 = b^6 = b^2 = 1\rangle$	$infinite\ group$
0	3	b^3	$\langle a, b : a^2 = b^6 = b^3 = 1\rangle$	$infinite\ group$
0	4	b^4	$\langle a, b : a^2 = b^6 = b^4 = 1\rangle$	$infinite\ group$
0	5	b^5	$\langle a, b : a^2 = b^6 = b^5 = 1\rangle$	C_2

k	l	Word	Quotient Group	Abstract Structure
1	0	a	$\langle a, b : a^2 = b^6 = a = 1\rangle$	C_6
1	1	ab	$\langle a, b : a^2 = b^6 = ab = 1\rangle$	C_2
1	2	ab^2	$\langle a, b : a^2 = b^6 = ab^2 = 1\rangle$	C_2
1	3	ab^3	$\langle a, b : a^2 = b^6 = ab^3 = 1\rangle$	C_6
1	4	ab^4	$\langle a, b : a^2 = b^6 = ab^4 = 1\rangle$	C_2
1	5	ab^5	$\langle a, b : a^2 = b^6 = ab^5 = 1\rangle$	C_2
2	2	$abab$	$\langle a, b : a^2 = b^6 = (ab)^2 = 1\rangle$	D_{12}
2	3	$abab^2$	$\langle a, b : a^2 = b^6 = abab^2 = 1\rangle$	C_6
2	4	$abab^3$	$\langle a, b : a^2 = b^6 = abab^3 = 1\rangle$	$C_2 \times C_2$
		ab^2ab^2	$\langle a, b : a^2 = b^6 = ab^2ab^2 = 1\rangle$	*infinite group*
2	5	$abab^4$	$\langle a, b : a^2 = b^6 = abab^4 = 1\rangle$	S_3
		ab^2ab^3	$\langle a, b : a^2 = b^6 = ab^2ab^3 = 1\rangle$	C_2
2	6	$abab^5$	$\langle a, b : a^2 = b^6 = abab^5 = 1\rangle$	$C_6 \times C_2$
		ab^2ab^4	$\langle a, b : a^2 = b^6 = ab^2ab^4 = 1\rangle$	*infinite group*
		ab^3ab^3	$\langle a, b : a^2 = b^6 = ab^3ab^3 = 1\rangle$	*infinite group*
2	7	ab^2ab^5	$\langle a, b : a^2 = b^6 = ab^2ab^5 = 1\rangle$	S_3
		ab^3ab^4	$\langle a, b : a^2 = b^6 = ab^3ab^4 = 1\rangle$	C_2
2	8	ab^3ab^5	$\langle a, b : a^2 = b^6 = ab^3ab^5 = 1\rangle$	$C_2 \times C_2$
		ab^4ab^4	$\langle a, b : a^2 = b^6 = ab^4ab^4 = 1\rangle$	*infinite group*
2	9	ab^4ab^5	$\langle a, b : a^2 = b^6 = ab^4ab^5 = 1\rangle$	C_6
2	10	ab^5ab^5	$\langle a, b : a^2 = b^6 = ab^5ab^5 = 1\rangle$	D_{12}
3	3	$ababab$	$\langle a, b : a^2 = b^6 = ababab = 1\rangle$	$\Delta(2, 6, 3)$
3	4	$ababab^2$	$\langle a, b : a^2 = b^6 = ababab^2 = 1\rangle$	C_2
3	5	$ababab^3$	$\langle a, b : a^2 = b^6 = ababab^3 = 1\rangle$	S_3
		$abab^2ab^2$	$\langle a, b : a^2 = b^6 = abab^2ab^2 = 1\rangle$	C_2

k	l	Word	Quotient Group	Abstract Structure
3	6	$ababab^4$	$\langle a, b : a^2 = b^6 = ababab^4 = 1 \rangle$	$C_2 \times A_4$
		$abab^2ab^3$	$\langle a, b : a^2 = b^6 = abab^2ab^3 = 1 \rangle$	C_6
		$abab^3ab^2$	$\langle a, b : a^2 = b^6 = abab^3ab^2 = 1 \rangle$	C_6
		$ab^2ab^2ab^2$	$\langle a, b : a^2 = b^6 = ab^2ab^2ab^2 = 1 \rangle$	$infinite\ group$
3	7	$ababab^5$	$\langle a, b : a^2 = b^6 = ababab^5 = 1 \rangle$	S_3
		$abab^2ab^4$	$\langle a, b : a^2 = b^6 = abab^2ab^4 = 1 \rangle$	C_2
		$abab^4ab^2$	$\langle a, b : a^2 = b^6 = abab^4ab^2 = 1 \rangle$	C_2
		$abab^3ab^3$	$\langle a, b : a^2 = b^6 = abab^3ab^3 = 1 \rangle$	S_3
		$ab^2ab^2ab^3$	$\langle a, b : a^2 = b^6 = ab^2ab^2ab^3 = 1 \rangle$	C_2
3	8	$abab^2ab^5$	$\langle a, b : a^2 = b^6 = abab^2ab^5 = 1 \rangle$	C_2
		$abab^5ab^2$	$\langle a, b : a^2 = b^6 = abab^5ab^2 = 1 \rangle$	C_2
		$abab^3ab^4$	$\langle a, b : a^2 = b^6 = abab^3ab^4 = 1 \rangle$	C_2
		$abab^4ab^3$	$\langle a, b : a^2 = b^6 = abab^4ab^3 = 1 \rangle$	C_2
		$ab^2ab^2ab^4$	$\langle a, b : a^2 = b^6 = ab^2ab^2ab^4 = 1 \rangle$	C_2
		$ab^2ab^3ab^3$	$\langle a, b : a^2 = b^6 = ab^2ab^3ab^3 = 1 \rangle$	C_2
3	9	$abab^3ab^5$	$\langle a, b : a^2 = b^6 = abab^3ab^5 = 1 \rangle$	$(C_9 : C_3) : C_2$
		$abab^5ab^3$	$\langle a, b : a^2 = b^6 = abab^5ab^3 = 1 \rangle$	$(C_9 : C_3) : C_2$
		$ab^2ab^2ab^5$	$\langle a, b : a^2 = b^6 = ab^2ab^2ab^5 = 1 \rangle$	$C_2 \times A_4$
		$abab^4ab^4$	$\langle a, b : a^2 = b^6 = abab^4ab^4 = 1 \rangle$	$C_2 \times A_4$
		$ab^2ab^3ab^4$	$\langle a, b : a^2 = b^6 = ab^2ab^3ab^4 = 1 \rangle$	$(C_7 : C_3) : C_2$
		$ab^2ab^4ab^3$	$\langle a, b : a^2 = b^6 = ab^2ab^4ab^3 = 1 \rangle$	$(C_7 : C_3) : C_2$
		$ab^3ab^3ab^3$	$\langle a, b : a^2 = b^6 = ab^3ab^3ab^3 = 1 \rangle$	$infinite\ group$

k	l	Word	Quotient Group	Abstract Structure
3	10	$abab^4ab^5$	$\langle a, b : a^2 = b^6 = abab^4ab^5 = 1\rangle$	C_2
		$abab^5ab^4$	$\langle a, b : a^2 = b^6 = abab^5ab^4 = 1\rangle$	C_2
		$ab^2ab^3ab^5$	$\langle a, b : a^2 = b^6 = ab^2ab^3ab^5 = 1\rangle$	C_2
		$ab^2ab^5ab^3$	$\langle a, b : a^2 = b^6 = ab^2ab^5ab^3 = 1\rangle$	C_2
		$ab^2ab^4ab^4$	$\langle a, b : a^2 = b^6 = ab^2ab^4ab^4 = 1\rangle$	C_2
		$ab^3ab^3ab^4$	$\langle a, b : a^2 = b^6 = ab^3ab^3ab^4 = 1\rangle$	C_2
3	11	$abab^5ab^5$	$\langle a, b : a^2 = b^6 = abab^5ab^5 = 1\rangle$	S_3
		$ab^2ab^4ab^5$	$\langle a, b : a^2 = b^6 = ab^2ab^4ab^5 = 1\rangle$	C_2
		$ab^3ab^3ab^5$	$\langle a, b : a^2 = b^6 = ab^3ab^3ab^5 = 1\rangle$	S_3
		$ab^2ab^5ab^4$	$\langle a, b : a^2 = b^6 = ab^2ab^5ab^4 = 1\rangle$	C_2
		$ab^3ab^4ab^4$	$\langle a, b : a^2 = b^6 = ab^3ab^4ab^4 = 1\rangle$	C_2
3	12	$ab^2ab^5ab^5$	$\langle a, b : a^2 = b^6 = ab^2ab^5ab^5 = 1\rangle$	$C_2 \times A_4$
		$ab^3ab^4ab^5$	$\langle a, b : a^2 = b^6 = ab^3ab^4ab^5 = 1\rangle$	C_6
		$ab^3ab^5ab^4$	$\langle a, b : a^2 = b^6 = ab^3ab^5ab^4 = 1\rangle$	C_6
		$ab^4ab^4ab^4$	$\langle a, b : a^2 = b^6 = ab^4ab^4ab^4 = 1\rangle$	$infinite\ group$
3	13	$ab^3ab^5ab^5$	$\langle a, b : a^2 = b^6 = ab^3ab^5ab^5 = 1\rangle$	S_3
		$ab^4ab^4ab^5$	$\langle a, b : a^2 = b^6 = ab^4ab^4ab^5 = 1\rangle$	C_2
3	14	$ab^4ab^5ab^5$	$\langle a, b : a^2 = b^6 = ab^4ab^5ab^5 = 1\rangle$	C_2
3	15	$ab^5ab^5ab^5$	$\langle a, b : a^2 = b^6 = ab^5ab^5ab^5 = 1\rangle$	$infinite\ group$
4	4	$abababab$	$\langle a, b : a^2 = b^6 = abababab = 1\rangle$	$\Delta(2, 6, 4)$
4	5	$abababab^2$	$\langle a, b : a^2 = b^6 = abababab^2 = 1\rangle$	C_2
4	6	$abababab^3$	$\langle a, b : a^2 = b^6 = abababab^3 = 1\rangle$	$C_3 \times D_8$
		$ababab^2ab^2$	$\langle a, b : a^2 = b^6 = ababab^2ab^2 = 1\rangle$	$C_6 \times S_3$
		$abab^2abab^2$	$\langle a, b : a^2 = b^6 = abab^2abab^2 = 1\rangle$	$infinite\ group$

k	l	Word	Quotient Group	Abstract Structure
4	7	$abababab^4$	$\langle a, b : a^2 = b^6 = abababab^4 = 1 \rangle$	$GL(2, 3)$
		$ababab^2ab^3$	$\langle a, b : a^2 = b^6 = ababab^2ab^3 = 1 \rangle$	C_2
		$ababab^3ab^2$	$\langle a, b : a^2 = b^6 = ababab^3ab^2 = 1 \rangle$	C_2
		$abab^2abab^3$	$\langle a, b : a^2 = b^6 = abab^2abab^3 = 1 \rangle$	S_3
		$abab^2ab^2ab^2$	$\langle a, b : a^2 = b^6 = abab^2ab^2ab^2 = 1 \rangle$	C_2
4	8	$abababab^5$	$\langle a, b : a^2 = b^6 = abababab^5 = 1 \rangle$	D_8
		$ababab^2ab^4$	$\langle a, b : a^2 = b^6 = ababab^2ab^4 = 1 \rangle$	$C_2 \times C_2$
		$ababab^4ab^2$	$\langle a, b : a^2 = b^6 = ababab^4ab^2 = 1 \rangle$	$infinite\ group$
		$abab^2abab^4$	$\langle a, b : a^2 = b^6 = abab^2abab^4 = 1 \rangle$	$infinite\ group$
		$ababab^3ab^3$	$\langle a, b : a^2 = b^6 = ababab^3ab^3 = 1 \rangle$	$(C_6 \times C_2) : C_2$
		$abab^3abab^3$	$\langle a, b : a^2 = b^6 = abab^3abab^3 = 1 \rangle$	$infinite\ group$
		$abab^2ab^2ab^3$	$\langle a, b : a^2 = b^6 = abab^2ab^2ab^3 = 1 \rangle$	$C_2 \times C_2$
		$abab^2ab^3ab^2$	$\langle a, b : a^2 = b^6 = abab^2ab^3ab^2 = 1 \rangle$	$infinite\ group$
		$abab^3ab^2ab^2$	$\langle a, b : a^2 = b^6 = abab^3ab^2ab^2 = 1 \rangle$	$C_2 \times C_2$
		$ab^2ab^2ab^2ab^2$	$\langle a, b : a^2 = b^6 = ab^2ab^2ab^2ab^2 = 1 \rangle$	$infinite\ group$

k	l	Word	Quotient Group	Abstract Structure
4	9	$ababab^2ab^5$	$\langle a,b : a^2 = b^6 = ababab^2ab^5 = 1\rangle$	$C_3 \times S_3$
		$ababab^5ab^2$	$\langle a,b : a^2 = b^6 = ababab^5ab^2 = 1\rangle$	$C_3 \times S_3$
		$abab^2abab^5$	$\langle a,b : a^2 = b^6 = abab^2abab^5 = 1\rangle$	$SL(2,3) : C_2$
		$ababab^3ab^4$	$\langle a,b : a^2 = b^6 = ababab^3ab^4 = 1\rangle$	C_6
		$ababab^4ab^3$	$\langle a,b : a^2 = b^6 = ababab^4ab^3 = 1\rangle$	C_6
		$abab^3abab^4$	$\langle a,b : a^2 = b^6 = abab^3abab^4 = 1\rangle$	C_6
		$abab^2ab^2ab^4$	$\langle a,b : a^2 = b^6 = abab^2ab^2ab^4 = 1\rangle$	$C_3 \times S_3$
		$abab^2ab^4ab^2$	$\langle a,b : a^2 = b^6 = abab^2ab^4ab^2 = 1\rangle$	$SL(2,3) : C_2$
		$abab^4ab^2ab^2$	$\langle a,b : a^2 = b^6 = abab^4ab^2ab^2 = 1\rangle$	$C_3 \times S_3$
		$abab^2ab^3ab^3$	$\langle a,b : a^2 = b^6 = abab^2ab^3ab^3 = 1\rangle$	C_6
		$abab^3ab^2ab^3$	$\langle a,b : a^2 = b^6 = abab^3ab^2ab^3 = 1\rangle$	$infinite\ group$
		$abab^3ab^3ab^2$	$\langle a,b : a^2 = b^6 = abab^3ab^3ab^2 = 1\rangle$	C_6
		$ab^2ab^2ab^2ab^3$	$\langle a,b : a^2 = b^6 = ab^2ab^2ab^2ab^3 = 1\rangle$	C_6

k	l	Word	Quotient Group	Abstract Structure
4	10	$ababab^3ab^5$	$\langle a,b : a^2 = b^6 = ababab^3ab^5 = 1\rangle$	D_8
		$ababab^5ab^3$	$\langle a,b : a^2 = b^6 = ababab^5ab^3 = 1\rangle$	D_8
		$abab^3abab^5$	$\langle a,b : a^2 = b^6 = abab^3abab^5 = 1\rangle$	$(C_6 \times C_2) : C_2$
		$ababab^4ab^4$	$\langle a,b : a^2 = b^6 = ababab^4ab^4 = 1\rangle$	$(C_2 \times SL(2,3)) : C_2$
		$abab^4abab^4$	$\langle a,b : a^2 = b^6 = abab^4abab^4 = 1\rangle$	$infinite\ group$
		$abab^2ab^3ab^4$	$\langle a,b : a^2 = b^6 = abab^2ab^3ab^4 = 1\rangle$	$infinite\ group$
		$abab^2ab^4ab^3$	$\langle a,b : a^2 = b^6 = abab^2ab^4ab^3 = 1\rangle$	D_{12}
		$abab^3ab^2ab^4$	$\langle a,b : a^2 = b^6 = abab^3ab^2ab^4 = 1\rangle$	$C_2 \times C_2$
		$abab^3ab^4ab^2$	$\langle a,b : a^2 = b^6 = abab^3ab^4ab^2 = 1\rangle$	D_{12}
		$abab^4ab^2ab^3$	$\langle a,b : a^2 = b^6 = abab^4ab^2ab^3 = 1\rangle$	$C_2 \times C_2$
		$abab^4ab^3ab^2$	$\langle a,b : a^2 = b^6 = abab^4ab^3ab^2 = 1\rangle$	$infinite\ group$
		$ab^2ab^2ab^2ab^4$	$\langle a,b : a^2 = b^6 = ab^2ab^2ab^2ab^4 = 1\rangle$	$infinite\ group$
		$abab^3ab^3ab^3$	$\langle a,b : a^2 = b^6 = abab^3ab^3ab^3 = 1\rangle$	D_8
		$ab^2ab^2ab^3ab^3$	$\langle a,b : a^2 = b^6 = ab^2ab^2ab^3ab^3 = 1\rangle$	D_{12}
		$ab^2ab^3ab^2ab^3$	$\langle a,b : a^2 = b^6 = ab^2ab^3ab^2ab^3 = 1\rangle$	$infinite\ group$

k	l	Word	Quotient Group	Abstract Structure
4	11	$ababab^4ab^5$	$\langle a, b : a^2 = b^6 = ababab^4ab^5 = 1\rangle$	C_2
		$ababab^5ab^4$	$\langle a, b : a^2 = b^6 = ababab^5ab^4 = 1\rangle$	C_2
		$abab^4abab^5$	$\langle a, b : a^2 = b^6 = abab^4abab^5 = 1\rangle$	C_2
		$abab^2ab^3ab^5$	$\langle a, b : a^2 = b^6 = abab^2ab^3ab^5 = 1\rangle$	S_3
		$abab^2ab^5ab^3$	$\langle a, b : a^2 = b^6 = abab^2ab^5ab^3 = 1\rangle$	C_2
		$abab^3ab^2ab^5$	$\langle a, b : a^2 = b^6 = abab^3ab^2ab^5 = 1\rangle$	C_2
		$abab^3ab^5ab^2$	$\langle a, b : a^2 = b^6 = abab^3ab^5ab^2 = 1\rangle$	C_2
		$abab^5ab^2ab^3$	$\langle a, b : a^2 = b^6 = abab^5ab^2ab^3 = 1\rangle$	C_2
		$abab^5ab^3ab^2$	$\langle a, b : a^2 = b^6 = abab^5ab^3ab^2 = 1\rangle$	S_3
		$ab^2ab^2ab^2ab^5$	$\langle a, b : a^2 = b^6 = ab^2ab^2ab^2ab^5 = 1\rangle$	$GL(2,3)$
		$abab^2ab^4ab^4$	$\langle a, b : a^2 = b^6 = abab^2ab^4ab^4 = 1\rangle$	C_2
		$abab^4ab^2ab^4$	$\langle a, b : a^2 = b^6 = abab^4ab^2ab^4 = 1\rangle$	C_2
		$abab^4ab^4ab^2$	$\langle a, b : a^2 = b^6 = abab^4ab^4ab^2 = 1\rangle$	C_2
		$abab^3ab^3ab^4$	$\langle a, b : a^2 = b^6 = abab^3ab^3ab^4 = 1\rangle$	S_3
		$abab^3ab^4ab^3$	$\langle a, b : a^2 = b^6 = abab^3ab^4ab^3 = 1\rangle$	C_2
		$abab^4ab^3ab^3$	$\langle a, b : a^2 = b^6 = abab^4ab^3ab^3 = 1\rangle$	S_3
		$ab^2ab^2ab^3ab^4$	$\langle a, b : a^2 = b^6 = ab^2ab^2ab^3ab^4 = 1\rangle$	C_2
		$ab^2ab^2ab^4ab^3$	$\langle a, b : a^2 = b^6 = ab^2ab^2ab^4ab^3 = 1\rangle$	C_2
		$ab^2ab^3ab^2ab^4$	$\langle a, b : a^2 = b^6 = ab^2ab^3ab^2ab^4 = 1\rangle$	S_3
		$ab^2ab^3ab^3ab^3$	$\langle a, b : a^2 = b^6 = ab^2ab^3ab^3ab^3 = 1\rangle$	C_2

k	l	Word	Quotient Group	Abstract Structure
4	12	$ababab^5ab^5$	$\langle a,b : a^2 = b^6 = ababab^5ab^5 = 1\rangle$	$C_3 \times ((C_6 \times C_2) : C_2)$
		$abab^5abab^5$	$\langle a,b : a^2 = b^6 = abab^5abab^5 = 1\rangle$	*infinite group*
		$abab^2ab^4ab^5$	$\langle a,b : a^2 = b^6 = abab^2ab^4ab^5 = 1\rangle$	*infinite group*
		$abab^2ab^5ab^4$	$\langle a,b : a^2 = b^6 = abab^2ab^5ab^4 = 1\rangle$	*infinite group*
		$abab^4ab^2ab^5$	$\langle a,b : a^2 = b^6 = abab^4ab^2ab^5 = 1\rangle$	$((C_7 : C_3) : C_2) \times S_3$
		$abab^4ab^5ab^2$	$\langle a,b : a^2 = b^6 = abab^4ab^5ab^2 = 1\rangle$	*infinite group*
		$abab^5ab^2ab^4$	$\langle a,b : a^2 = b^6 = abab^5ab^2ab^4 = 1\rangle$	$((C_7 : C_3) : C_2) \times S_3$
		$abab^5ab^4ab^2$	$\langle a,b : a^2 = b^6 = abab^5ab^4ab^2 = 1\rangle$	*infinite group*
		$ab^2ab^2ab^3ab^5$	$\langle a,b : a^2 = b^6 = ab^2ab^2ab^3ab^5 = 1\rangle$	$C_2 \times ((C_7 : C_3) : C_2)$
		$ab^2ab^2ab^5ab^3$	$\langle a,b : a^2 = b^6 = ab^2ab^2ab^5ab^3 = 1\rangle$	$C_2 \times ((C_7 : C_3) : C_2)$
		$ab^2ab^3ab^2ab^5$	$\langle a,b : a^2 = b^6 = ab^2ab^3ab^2ab^5 = 1\rangle$	*infinite group*
4	13	$abab^2ab^5ab^5$	$\langle a,b : a^2 = b^6 = abab^2ab^5ab^5 = 1\rangle$	C_2
		$ab^2abab^5ab^5$	$\langle a,b : a^2 = b^6 = ab^2abab^5ab^5 = 1\rangle$	C_2
		$abab^5ab^2ab^5$	$\langle a,b : a^2 = b^6 = abab^5ab^2ab^5 = 1\rangle$	C_2
		$ab^2ab^2ab^4ab^5$	$\langle a,b : a^2 = b^6 = ab^2ab^2ab^4ab^5 = 1\rangle$	C_2
		$ab^2ab^2ab^5ab^4$	$\langle a,b : a^2 = b^6 = ab^2ab^2ab^5ab^4 = 1\rangle$	C_2
		$ab^2ab^4ab^2ab^5$	$\langle a,b : a^2 = b^6 = ab^2ab^4ab^2ab^5 = 1\rangle$	C_2
		$ab^3ab^3ab^2ab^5$	$\langle a,b : a^2 = b^6 = ab^3ab^3ab^2ab^5 = 1\rangle$	S_3
		$ab^3ab^3ab^5ab^3$	$\langle a,b : a^2 = b^6 = ab^3ab^3ab^5ab^2 = 1\rangle$	S_3
		$ab^3ab^2ab^3ab^5$	$\langle a,b : a^2 = b^6 = ab^3ab^2ab^3ab^5 = 1\rangle$	C_2

k	l	Word	Quotient Group	Abstract Structure
4	14	$abab^3ab^5ab^5$	$\langle a, b : a^2 = b^6 = abab^3ab^5ab^5 = 1\rangle$	D_8
		$ab^3abab^5ab^5$	$\langle a, b : a^2 = b^6 = ab^3abab^5ab^5 = 1\rangle$	D_8
		$abab^5ab^3ab^5$	$\langle a, b : a^2 = b^6 = abab^5ab^3ab^5 = 1\rangle$	$(C_6 \times C_2) : C_2$
		$ab^2ab^2ab^5ab^5$	$\langle a, b : a^2 = b^6 = ab^2ab^2ab^5ab^5 = 1\rangle$	$(C_2 \times SL(2, 3)) : C_2$
		$ab^2ab^5ab^2ab^5$	$\langle a, b : a^2 = b^6 = ab^2ab^5ab^2ab^5 = 1\rangle$	*infinite group*
		$ab^4ab^4abab^5$	$\langle a, b : a^2 = b^6 = ab^4ab^4abab^5 = 1\rangle$	$C_2 \times C_2$
		$ab^4ab^4ab^5ab$	$\langle a, b : a^2 = b^6 = ab^4ab^4ab^5ab = 1\rangle$	$C_2 \times C_2$
		$ab^4abab^4ab^5$	$\langle a, b : a^2 = b^6 = ab^4abab^4ab^5 = 1\rangle$	*infinite group*
		$ab^2ab^3ab^4ab^5$	$\langle a, b : a^2 = b^6 = ab^2ab^3ab^4ab^5 = 1\rangle$	*infinite group*
		$ab^2ab^3ab^5ab^4$	$\langle a, b : a^2 = b^6 = ab^2ab^3ab^5ab^4 = 1\rangle$	D_{12}
		$ab^2ab^4ab^3ab^5$	$\langle a, b : a^2 = b^6 = ab^2ab^4ab^3ab^5 = 1\rangle$	$C_2 \times C_2$
		$ab^2ab^4ab^5ab^3$	$\langle a, b : a^2 = b^6 = ab^2ab^4ab^5ab^3 = 1\rangle$	D_{12}
		$ab^2ab^5ab^4ab^3$	$\langle a, b : a^2 = b^6 = ab^2ab^5ab^4ab^3 = 1\rangle$	*infinite group*
		$ab^2ab^5ab^3ab^4$	$\langle a, b : a^2 = b^6 = ab^2ab^5ab^3ab^4 = 1\rangle$	$C_2 \times C_2$
		$ab^3ab^3ab^4ab^4$	$\langle a, b : a^2 = b^6 = ab^3ab^3ab^4ab^4 = 1\rangle$	D_{12}
		$ab^3ab^4ab^3ab^4$	$\langle a, b : a^2 = b^6 = ab^3ab^4ab^3ab^4 = 1\rangle$	*infinite group*
4	15	$ab^3ab^2ab^5ab^5$	$\langle a, b : a^2 = b^6 = ab^3ab^2ab^5ab^5 = 1\rangle$	C_6
		$ab^2ab^3ab^5ab^5$	$\langle a, b : a^2 = b^6 = ab^2ab^3ab^5ab^5 = 1\rangle$	C_6
		$ab^2ab^5ab^3ab^5$	$\langle a, b : a^2 = b^6 = ab^2ab^5ab^3ab^5 = 1\rangle$	C_6
		$ab^3ab^3ab^4ab^5$	$\langle a, b : a^2 = b^6 = ab^3ab^3ab^4ab^5 = 1\rangle$	C_6
		$ab^3ab^3ab^5ab^4$	$\langle a, b : a^2 = b^6 = ab^3ab^3ab^5ab^4 = 1\rangle$	C_6
		$ab^3ab^4ab^3ab^5$	$\langle a, b : a^2 = b^6 = ab^3ab^4ab^3ab^5 = 1\rangle$	*infinite group*

k	l	Word	Quotient Group	Abstract Structure
4	16	$abab^5ab^5ab^5$	$\langle a,b : a^2 = b^6 = abab^5ab^5ab^5 = 1\rangle$	D_8
		$ab^2ab^4ab^5ab^5$	$\langle a,b : a^2 = b^6 = ab^2ab^4ab^5ab^5 = 1\rangle$	$C_2 \times C_2$
		$ab^4ab^2ab^5ab^5$	$\langle a,b : a^2 = b^6 = ab^4ab^2ab^5ab^5 = 1\rangle$	$C_2 \times C_2$
		$ab^2ab^5ab^4ab^5$	$\langle a,b : a^2 = b^6 = ab^2ab^5ab^4ab^5 = 1\rangle$	$infinite\ group$
		$ab^3ab^3ab^5ab^5$	$\langle a,b : a^2 = b^6 = ab^3ab^3ab^5ab^5 = 1\rangle$	$(C_6 \times C_2) : C_2$
		$ab^3ab^5ab^3ab^5$	$\langle a,b : a^2 = b^6 = ab^3ab^5ab^3ab^5 = 1\rangle$	$infinite\ group$
		$ab^4ab^4ab^3ab^5$	$\langle a,b : a^2 = b^6 = ab^4ab^4ab^3ab^5 = 1\rangle$	$C_2 \times C_2$
		$ab^4ab^4ab^5ab^3$	$\langle a,b : a^2 = b^6 = ab^4ab^4ab^5ab^3 = 1\rangle$	$C_2 \times C_2$
		$ab^4ab^3ab^4ab^5$	$\langle a,b : a^2 = b^6 = ab^4ab^3ab^4ab^5 = 1\rangle$	$infinite\ group$
		$ab^4ab^4ab^4ab^4$	$\langle a,b : a^2 = b^6 = ab^4ab^4ab^4ab^4 = 1\rangle$	$infinite\ group$
4	17	$ab^2ab^5ab^5ab^5$	$\langle a,b : a^2 = b^6 = ab^2ab^5ab^5ab^5 = 1\rangle$	$GL(2,3)$
		$ab^3ab^4ab^5ab^5$	$\langle a,b : a^2 = b^6 = ab^3ab^4ab^5ab^5 = 1\rangle$	C_2
		$ab^4ab^3ab^5ab^5$	$\langle a,b : a^2 = b^6 = ab^4ab^3ab^5ab^5 = 1\rangle$	C_2
		$ab^4ab^5ab^3ab^5$	$\langle a,b : a^2 = b^6 = ab^4ab^5ab^3ab^5 = 1\rangle$	S_3
		$ab^4ab^4ab^4ab^5$	$\langle a,b : a^2 = b^6 = ab^4ab^4ab^4ab^5 = 1\rangle$	C_2
4	18	$ab^3ab^5ab^5ab^5$	$\langle a,b : a^2 = b^6 = ab^3ab^5ab^5ab^5 = 1\rangle$	$C_3 \times D_8$
		$ab^4ab^4ab^5ab^5$	$\langle a,b : a^2 = b^6 = ab^4ab^4ab^5ab^5 = 1\rangle$	$C_6 \times S_3$
		$ab^4ab^5ab^4ab^5$	$\langle a,b : a^2 = b^6 = ab^4ab^5ab^4ab^5 = 1\rangle$	$infinite\ group$
4	19	$ab^4ab^5ab^5ab^5$	$\langle a,b : a^2 = b^6 = ab^4ab^5ab^5ab^5 = 1\rangle$	C_2
4	20	$ab^5ab^5ab^5ab^5$	$\langle a,b : a^2 = b^6 = ab^5ab^5ab^5ab^5 = 1\rangle$	$infinite\ group$

The information we get about the quotient groups investigated in the list is a step leading towards general results about quotients of the Hecke group in question. We see that most of the finite quotients are cyclic groups of order 2 or 6. Infinite groups are also there in a significant number. All finite groups in the list are soluble. One might extend the list to reach some generalized statements regarding structure of the quotients, frequency of occurrence of the cyclic groups or solubility of the quotients. To investigate common structural properties of infinite groups in the list may also be an interesting question.

Research work presented in this chapter has been published in [3].

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