Construction of formal Lagangian for Dynamical systems

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In The Name Of

Allah

The Most Beneficent and

The Most Merciful

DEDICATED TO MY
BELOVED MOTHER
SURAYYA BAIGUM

Abstract

An integrating factor is a function that can be multiplied by a given differential equation to make it exact. In mathematical analysis and optimization problems, adjoint equations are often used to find solutions or optimize certain quantities related to a given system. It's important to note that both integrating factors and adjoint equations are powerful techniques used in specific contexts to simplify or analyze differential equations. The concept we are using involves adjoint equations to construct a Lagrangian for systems described by arbitrary differential equations, where the number of equations is equal to the number of dependent variables. This method uses adjoint equations and the concept of Lagrangians to analyze and solve equations that might not traditionally be associated with Lagrangian formulations. This approach can provide insights into the underlying symmetries and conservation laws of these systems. Let's break down the steps involved in this process and how Noether's theorem can be applied to the Maxwell equations as an example.

Declaration

I claim that **"Construction of formal Langangian for Dynamical systems"** is my work and all the material that I have handed-down have been presented in the form of references.

Asad Ali

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May **Allah Almighty** all of above cited personalities with success and honor in their life.

Asad Ali

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Chapter 1

Introduction

1.1 Historical background

Adjoint equations are typically associated with linear equations and integrating factors are commonly discussed for first-order nonlinear ordinary differential equations (ODEs). Traditionally, adjoint equations have been associated with linear differential equations. These equations capture information about the sensitivity and gradients of solutions. Integrating factors are often used to solve first-order linear differential equations by transforming them into exact equations. The introduction suggests that integrating factors can also be discussed in the context of non-linear ordinary differential equations. This implies that the concept of integrating factors can have relevance beyond just linear equations and first-order equations. Noether's theorem establishes a deep connection between the symmetries of a physical system and the conservation laws that arise from those symmetries. The theorem is classically applied to variational problems and Lagrangian mechanics. The introduction implies that Noether's theorem can be extended to broader settings, allowing for the exploration of conservation laws associated with symmetries in various types of equations and systems. The introduction outlines that the subsequent discussion will delve into the definitions and results related to this extended approach. This likely includes defining adjoint equations in a broader context, discussing how integrating factors can apply to non-linear equations, and explaining how Noether's theorem can be adapted to systems beyond traditional variational problems.

1.2 Integrating factor

Indeed, the integrating factor method is a powerful technique for solving certain types of first-order ordinary differential equations (ODEs) by transforming them into equations that are more amenable to direct integration. A first order ordinary differential equation is,

$$a(z_1, z_2)z_2' + b(z_1, z_2) = 0, \tag{1.1}$$

The derivative of a function z_2 with respect to z_1, denoted as $z_2' = \frac{dz_2}{dz_1}$ is represented in terms of differentials:

$$a(z_1, z_2)dz_2 + b(z_1, z_2)dz_1 = 0. \tag{1.2}$$

(1.2) is considered to be exact if the left-hand side of that equation can be expressed as the differential of a certain function. In mathematical terms, this is known as an exact differential equation.

$$a(z_1, z_2)dz_2 + b(z_1, z_2)dz_1 = d\phi(z_1, z_2), \tag{1.3}$$

When an equation of the form (1.2) is exact, meaning its left-hand side can be expressed as the total differential of a function $\phi(z_1, z_2)$. The solutions to the equation are curves or surfaces in the z_1z_2-plane that satisfy the condition $\phi(z_1, z_2) = C = cons$, where c is the constant.

In general, the equation (1.2) is not initially exact, an integrating factor to transform a non-exact equation (1.2) into an exact equation.

$$\nu(adz_2 + bdz_1) = d\phi \equiv \phi_{z_2}dz_2 + \phi_{z_1}dz_1, \tag{1.4}$$

where,

$$\phi_{z_2} = \frac{\partial \phi}{\partial z_2}, \qquad \phi_{z_1} = \frac{\partial \phi}{\partial z_1}.$$

The function $\nu(z_1, z_2)$ is referred to as an integrating factor for equation (1.2). It follows from equation (1.4) that,

$$\phi_{z_2} = \nu a, \qquad \phi_{z_1} = \nu b. \tag{1.5}$$

The given system of equations (1.5), the mixed partial derivatives of a function $\phi_{z_1 z_2} = \phi_{z_2 z_1}$ with respect to x and z_2 are equated and this equation is used to find the integrating factors associated with a given system of differential equations,

$$\frac{\partial(\nu a)}{\partial z_1} = \frac{\partial(\nu b)}{\partial z_2}. \tag{1.6}$$

Equation (1.6) theoratically provides an infinite number of integrating factors for equation (1.2). However, in reality integrating the original differential equation (1.2) is frequently not any easier than integrating (1.6). However, the idea of an integrating factor is still because individual integrating factors can be discovered by means of certain methods. If we have two linearly independent integrating factors, $\nu_1(z_1, z_2)$ and $\nu_2(z_1, z_2)$, for the differential equation (1.2), it means that a powerful tool to directly obtain the general solution of (1.2) without needing to perform additional complex integration steps.

$$\frac{\nu_1(z_1, z_2)}{\nu_2(z_1, z_2)} = C. \tag{1.7}$$

1.3 Adjoint linear differential operators

Let the notation $z_1 = ({z_1}^1, ..., {z_1}^n)$ represents a vector of n independent variable and $\mu^\alpha_\epsilon = (\mu^1, ..., \mu^m)$ represents a vector of m dependent variables with the partial derivatives of the dependent variables $\mu_{(1)} = \mu^\alpha_\epsilon$,$\mu_{(2)} = \mu^\alpha_{\epsilon\kappa}, ...$ of the first, second, etc. indicating how the rates of change themselves change, where, $\mu^\alpha_\epsilon = \frac{\partial \mu^\alpha}{\partial z_1^\epsilon}$, $\mu^\alpha_{\epsilon\kappa} = \frac{\partial^2 \mu^\alpha}{\partial z_1^i \partial z_1^\kappa}$. Denoting,

$$\boldsymbol{D}_\epsilon = \frac{\partial}{\partial z_{1i}} + \mu^\alpha_\epsilon \frac{\partial}{\partial \mu^\alpha} + \mu^\alpha_{\epsilon\kappa} \frac{\partial}{\partial \mu^\alpha_\kappa} + ... \tag{1.8}$$

Taking the total differentiation with respect to ${z_1}^\epsilon$ provides insights into how the function changes as a result of altering the $i-th$ independent variable,

$$\mu^\alpha_\epsilon = \boldsymbol{D}_\epsilon(\mu^\alpha), \qquad \mu^\alpha_{\epsilon\kappa} = \boldsymbol{D}_\epsilon(\mu^\alpha_\kappa) = \boldsymbol{D}_\epsilon \boldsymbol{D}_\kappa(\mu^\alpha), ...$$

Previously discussed in the adjoint linear operator's definition. Take the scalar field (m=1) as an example, an equation involving a function of multiple variables and its partial derivatives up to second order,

$$\boldsymbol{L}[\mu] \equiv a^\epsilon \kappa(z_1)\mu_{\epsilon\kappa} + b^\epsilon(z_1)\mu_\epsilon + c(z_1)u = \boldsymbol{f(z_1)}, \tag{1.9}$$

where L is associated with a following linear differential equation:

$$\boldsymbol{L} = a^{\epsilon\kappa}(z_1)\boldsymbol{D}_\epsilon \boldsymbol{D}_\kappa + b^\epsilon(z_1)\boldsymbol{D}_\epsilon + c(z_1). \tag{1.10}$$

It is assumed that summation is taking place over that index, the indices i and j range from 1 to n. For any i and j, the coefficient $a^{\epsilon\kappa}(z_1)$ is equal to the coefficient $a^{\kappa\epsilon}$. $\boldsymbol{L}^*$ is a second-order linear differential operator called the adjoint operator, and it possesses certain properties related to L,

$$\nu\boldsymbol{L}[\mu] - uL * [\nu] = \boldsymbol{D}_\epsilon(p^\epsilon) \equiv divP(z_1), \tag{1.11}$$

The adjoint operator $\boldsymbol{L}^*$ possesses specific properties and a unique form when applied to all functions u and v, where $P(z_1) = (p^1(z_1), ..., p^n(z_1)$ is any vector. Its characteristics and behavior are completely defined based on the given conditions,

$$\boldsymbol{L}^*[v] = \boldsymbol{D}_\epsilon \boldsymbol{D}_\kappa(a^{\epsilon\kappa}v) - \boldsymbol{D}_\epsilon(b^\epsilon v) + cv. \tag{1.12}$$

An operator L is labeled as self-adjoint if the action of the operator L on a function u is same as the action of its adjoint operator $\boldsymbol{L}^*$ on the same function u, in mathematical notation , this is represented as $\boldsymbol{L}[\mu] = \boldsymbol{L}^*[\mu]$ for any function $u(z_1)$. A previously mentioned operator (1.10) and mention that this operator is self-adjoint if and only if a certain condition is met,

$$b^\epsilon(z_1) = \boldsymbol{D}_\kappa(a^{\epsilon\kappa}), \qquad i = 1, ..., n. \tag{1.13}$$

A type of mathematical equation that is linear in nature and involves only homogeneous terms,

$$\boldsymbol{L}^*[v] \equiv \boldsymbol{D}_\epsilon \boldsymbol{D}_\kappa (a^{\epsilon\kappa} v) - \boldsymbol{D}_\epsilon (b^\epsilon v) + cv = 0, \tag{1.14}$$

The equation referred to as the "adjoint equation" is associated with a linear differential equation (1.9, where, $\boldsymbol{L}[\mu] = \boldsymbol{f(z_1)}$.

The idea of adjoint equation and adjoint operator, initially introduced for linear differential equations of the form (1.9) are generalized to systems of second-order equations. In this generalization, the function u becomes an m-dimensional vector function, along with the operator's coefficients (1.10) becomes $m \times m - matrices$. When both the number of independent variables (n) and the number of dependent variables (m) are equal to 1 (i.e.,$n = m = 1$), then the definition of the adjoint operator aligns with the familiar concept of the adjoint operator. Consider the function u as $u = y$, and focus on a first order equation,

$$\boldsymbol{L}[z_2] \equiv a_0(z_1) z_2' + a_1(z_1) y = \boldsymbol{f(z_1)}. \tag{1.15}$$

The adjoint operator $\boldsymbol{L}^*[z]$ has a specific form that relates to the original operator to $\boldsymbol{L}[z_2]$ is,

$$\boldsymbol{L}^*[z] = -(a_0 z)' + a_1 z. \tag{1.16}$$

The idea of Higher-order equations can be used with the adjoint operator, and they introduce the example of a second-order equation to illustrate how the concept applies in practice. The details of the example equation will likely be provided in the following context.

$$\boldsymbol{L}[z_2] \equiv a_0 {z_2}'' + a_1 z_2' + a_2 z_2 = \boldsymbol{f(z_1)}, \tag{1.17}$$

a second order equation with variable coefficients $a_0(z_1), a_1(z_1), a_2(z_1)$, the adjoint operator $\boldsymbol{L}^*[z]$ corresponds to the operator $\boldsymbol{L}[z_2]$ takes on a specific form,

$$\boldsymbol{L}^*[z] = (a_0 z)^{''} - (a_1 z)^{'} + a_2 z. \tag{1.18}$$

Similarly, just as discussed for second-order equations, the concept apply to third-order equations with variable coefficients, the adjoint operator,

$$\boldsymbol{L}[z_2] \equiv a_0 z_2^{'''} + a_1 z_2{''} + a_2 z_2' = a_3 z_2 = \boldsymbol{f(z_1)}. \tag{1.19}$$

Using the adjoint operator $\boldsymbol{L}^*[z]$ is equivalent to $\boldsymbol{L}[z_2]$ is going to be presented,

$$\boldsymbol{L}^*[z] = -(a_0 z)^{'''} + (a_1 z)^{''} - (a_2 z)^{'} + a_3 z. \tag{1.20}$$

The homogeneous equation $\boldsymbol{L}^*[z] = 0$ is referred to as the adjoint equation to the primary linear differential equation. $\boldsymbol{L}[z_2] = \boldsymbol{f(z_1)}$.

1.4 Noether's theorem

Symmetries and conservation laws are fundamentally connected by Noether's theorem in the setting of variational issues, notably for systems modeled by Euler-Lagrange equations. When the equations of motion display specific symmetries, this theorem offers a systematic approach for determining conservation laws. Following is an outline of the process. Consider a Lagrangian, denoted as $L(z_1, u, \mu_{(1)})$ which depends on independent variables involving, $z_1 = ({z_1}^1, {z_1}^2, ..., {z_1}^n)$ dependent variables a$u = (\mu^1, \mu^2, ..., \mu^m)$, the first-order derivatives $\mu_1 = (\mu_1^1, \mu_2^2, ..., \mu_1^m)$. The system's equations of motion, known as Euler-Lagrange equations, are derived from the principle of stationary action and have the general form:

$$\frac{\delta \mathfrak{L}}{\delta \mu^\alpha} \equiv \frac{\partial \mathfrak{L}}{\partial \mu^\alpha} - \boldsymbol{D}_\epsilon\left(\frac{\partial \mathfrak{L}}{\partial \mu_\epsilon^\alpha}\right) = 0, \qquad \alpha = 1, ..., m. \tag{1.21}$$

The Euler-Lagrangian $\int \mathfrak{L}(z_1, u, \mu_{(1)}) dz_1$ with regard to the independent variables z_1 is varied over an arbitrary n-dimensional domain in the space of these variables to produce the Euler-Lagrange equations. This variation process leads to the derivation of the equations that govern the system's behavior. Noether's theorem establishes a profound connection between symmetries of a system and the corresponding conservation laws. It specifically focuses on the symmetries arising from continuous transformation groups, denoted as G, which play a fundamental role in the behavior of physical systems.

$$z_1 = \xi^{\epsilon}(z_1, u)\frac{\partial}{\partial z_1^{\epsilon}} + \eta^{\alpha}(z_1, u)\frac{\partial}{\partial \mu^{\alpha}}, \tag{1.22}$$

the vector field $C = (C^1, ..., C^n)$ is a mathematical construct that represents the infinitesimal generator of the continuous transformation group G associated with the symmetry of the Lagrangian $\mathfrak{L}(z_1, u, \mu_{(1)})$ is defined by,

$$C^{\epsilon} = \xi^{\epsilon}\mathfrak{L} + (\eta^{\alpha} - z_1^{\epsilon} z_2^{\kappa} \mu_{\kappa}^{\alpha})\frac{\partial \mathfrak{L}}{\partial \mu_{\kappa}^{\alpha}}, \qquad i = 1, ..., n, \tag{1.23}$$

Noether's theorem establishes that if a Lagrangian $\mathfrak{L}(z_1, u, \mu_{(1)})$ possesses a continuous symmetry G with an associated generator vector field $C = (C^1, ..., C^n)$ then there exists a corresponding conservation law for the Euler-Lagrange equations. This conservation law is expressed mathematically as $\equiv D^{\epsilon}(C^{\epsilon}) = divC = 0$ which means that the divergence of the vector field C is zero for all solutions of the Euler-Lagrange equation. The Euler-Lagrange equations arise in the calculus of variations as the equations that describe the stationary points of the action functional. If the variational integral is invariant under the action of a symmetry group G, it implies that the underlying Euler-Lagrange equations also possess the same symmetry. This means that if you apply the symmetry transformation to the solution of the Euler-Lagrange equations, the transformed solution still satisfies the equations. Noether's theorem establishes a connection between symmetries and conserved quantities in the context of Lagrangian and Hamiltonian mechanics. In the context you're describing, to apply Noether's theorem, you need to identify the symmetries of the system, particularly the Euler-Lagrange equations (1.22). These symmetries are transformations that leave the form of the equations invariant. This can be done by the infinitesimal test serves as a criterion to determine whether a symmetry transformation maintains the form of the integral.

$$X(\mathfrak{L}) + \mathfrak{L}\boldsymbol{D}_{\epsilon}(\xi^{\epsilon}) = 0, \tag{1.24}$$

The formula for prolonging the generator$z_1$to the first derivatives $\mu_{(1)}$ involves applying the Lie derivative operation to the generator with respect to the dependent variables u and their first-order derivatives $\mu_{(1)}$.

$$z_1 = \xi^{\epsilon}\frac{\partial}{\partial z_1^{\epsilon}} + \eta^{\alpha}\frac{\partial}{\partial \mu^{\alpha}} + [\boldsymbol{D}_{\epsilon}(\eta^{\alpha}) - \mu_{\kappa}^{\alpha}\boldsymbol{D}_{\epsilon}(\xi^{\kappa})]\frac{\partial}{\partial \mu_{\epsilon}^{\alpha}}. \tag{1.25}$$

In the context of Noether's theorem and the derivation of conservation laws, the invariance condition is a requirement that the Lagrangian remains unchanged under the action of the symmetry transformation (1.25) defined by the generator z_1,

$$X(\mathfrak{L}) + \mathfrak{L}\boldsymbol{D}_{\epsilon}(\xi^{\epsilon}) = \boldsymbol{D}_{\epsilon}(B^{\epsilon}). \tag{1.26}$$

Then (1.22) has a conservation law $\boldsymbol{D}_{\epsilon}(C^{\epsilon}) = 0$, where (1.24) is replaced by,

$$C^{\epsilon} = \xi^{\epsilon}\mathfrak{L} + (\eta^{\alpha} - \xi^{\kappa}\mu_{\kappa}^{\alpha})\frac{\partial \mathfrak{L}}{\partial \mu_{\epsilon}^{\alpha}} - B^{\epsilon}. \tag{1.27}$$

Noether's theorem is more general than previously believed, and linkages between symmetries and conservation laws can be formed in a wider range of situations than previously anticipated.

Chapter 2

Main constructions

By making the problem more receptive to conventional solution methods, an integrating factor is a function that can be utilized to make the process of solving differential equations more straightforward. In specifically, in the context of optimization, sensitivity analysis, or discovering conservation laws, an adjoint equation is a mathematical construct that is frequently employed to examine features of differential equation solutions.

2.5 Preliminaries

the sequence z and the concept of differential functions,

$$z = (z_1, u, \mu_{(1)}, \mu_{(2)}, ...), \tag{2.28}$$

The sequence z is a collection of elements representing various quantities. Each element is denoted by z^v with $v \geq 1$, where $z^\epsilon = {z_1}^\epsilon$ for $1 \leq i \leq n$ where ${z_1}^\epsilon$ are the independent variables, and $z^n + \alpha = \mu^\alpha$ for $1 \leq \alpha \leq m$, where μ^α are dependent variables with an added constant term α. The remaining elements represent the derivatives of u with respect to the independent variable x. A differential function ($\boldsymbol{f}$), which can be locally expanded into a Taylor series with regard to all of its arguments (variables), is introduced as a differential function. The claims of $\boldsymbol{f}$ are selected from the sequence's finite number of variables. The derivatives in the highest order that appear in Its order is decided by f, represented as $ord(\boldsymbol{f})$. For example, $Ord(\boldsymbol{f}) = s$ suggests that depending on its derivatives u and other varieties is a locally analytic function. The set of all differential functions of finite order is denoted as A. This set A is treated as a vector space, and differential functions within it can be multiplied and manipulated using standard algebraic operations.

$$a\boldsymbol{f} + b\boldsymbol{g} \in A, \qquad ord(a\boldsymbol{f} + b\boldsymbol{g}) \leq max\{ord(\boldsymbol{f}), ord(\boldsymbol{g})\},$$

$$fg \in A, \qquad ord(fg) = max\{ord(\boldsymbol{f}), ord(\boldsymbol{g})\}.$$

If a differential function $\boldsymbol{f}$ is a member of the set A, which contains all locally analytic differential functions of finite order, then the result of applying a total derivative operation to $\boldsymbol{f}$ will also be in the same set A, i.e.,

$$\boldsymbol{D}_\epsilon(\boldsymbol{f}) \in A, \qquad ord(\boldsymbol{D}_\epsilon(\boldsymbol{f})) = ord(\boldsymbol{f}) + 1.$$

In order to create the operator, terms must be combined in a precise way and according to a predetermined mathematical procedure.

$$\frac{\partial}{\partial \mu^\alpha} = \frac{\partial}{\partial \mu^\alpha} - \boldsymbol{D}_\epsilon \frac{\partial}{\partial \mu^\alpha_\epsilon} + \boldsymbol{D}_\epsilon \boldsymbol{D}_\kappa \frac{\partial}{\partial \mu^\alpha_{\epsilon\kappa}} + ..., \qquad \alpha = 1, ..., m, \tag{2.29}$$

For every value of s, the expression involves summation over repeated indices $i, j, ...$ which run from 1 to n, where n represents the number of independent variables. The operator $\frac{\partial}{\partial \mu^\alpha}$ is introduced and referred to as the variational derivative. The expression (2.29) which defines the Euler-Lagrange operator, can be written in a more explicit form when there's only one independent variable z_1,

$$\frac{\partial}{\partial \mu^\alpha} = \frac{\partial}{\partial \mu^\alpha} - \boldsymbol{D}_{z_1} \frac{\partial}{\partial \mu^\alpha_{z_1}} + \boldsymbol{D}^2_{z_1} \frac{\partial}{\partial \mu^\alpha_{z_1 z_1}} - \boldsymbol{D}^3_{z_1} \frac{\partial}{\partial \mu^\alpha_{z_1 z_1 z_1}} + ... \tag{2.30}$$

When there's only one independent variable$z_1$and one dependent variable z_2, the sequence z is defined with elements $z(z_1, z_2, z_2', ..., z_2'', ..., z_2(s), ...)$. Then the total differentiation formula (1.8) which involves taking the total derivative of a function, can be expressed in terms of the sequence z and its elements is written as follows:

$$\boldsymbol{D}_{z_1} = \frac{\partial}{\partial z_1} + z_2' \frac{\partial}{\partial z_2} + z_2'' \frac{\partial}{\partial z_2'} + ..., \tag{2.31}$$

and the Euler-Lagrange operator, represented by the expression (2.30) is written in a specific way,

$$\frac{\partial}{\partial z_2} = \frac{\partial}{\partial z_2} - \boldsymbol{D}_{z_1} \frac{\partial}{\partial z_2'} + \boldsymbol{D}_{z_1}^2 \frac{\partial}{\partial z_2''} - \boldsymbol{D}_{z_1}^3 \frac{\partial}{\partial z_2'''} + ... \tag{2.32}$$

The primary content presented in this section relies on the utilization of multipliers, and it is supported by a series of lemmas. The purpose of these lemmas is to establish specific mathematical results that are integral to the concepts being discussed. The source for the proofs of these lemmas is referenced as [6,Section 8.4]).

Lemma 2.1. For a differentiable function $f(z_1, z_2, z_2', ..., z_2(s))$ belong to the set A, If the total derivatives $\boldsymbol{D}_{z_1}(\boldsymbol{f})$ with respect to all variables $z_1, z_2, z_2', ..., z_2(s), and z_2(s+1)$, is identically zero, then the function $\boldsymbol{f}$ must be constant c. Similarly if $f(z_1, u, \mu_{(1)}, ..., \mu_{(s)}$ is a differential function with one independent variables z_1 and multiple dependent variables $u = (\mu^1, ..., \mu^m)$, and the total derivative $\boldsymbol{D}_{z_1}(\boldsymbol{f})$ is zero, then $\boldsymbol{f}$ must be constant c.

Lemma 2.2. Total derivative is a term used to describe a differential function $f(z_1, u, ..., \mu_{(s)})$ that belongs to the set A and has one independent variable z_1.

$$f = \boldsymbol{D}_{z_1}(\boldsymbol{g}), \qquad g(z_1, u, ..., \mu_{(s-1)} \in A, \tag{2.33}$$

a differential function $f(z_1, u, \mu_{(1)}, ...)$ is a total derivative if and only if, a set of equations are satisfied without exception for all possible values of the variables $z_1, u, \mu_{(1)}, ...$,

$$\frac{\partial f}{\partial \mu^\alpha} = 0, \qquad \alpha = 1, ..., m. \tag{2.34}$$

Lemma 2.3. If a function $f(z_1, u, ..., \mu_{(s)}$ belongs to set A and involves several independent variables $z_1 = ({z_1}^1, ..., {z_1}^n)$ and several dependent variables $u = (\mu^1, ..., \mu^m)$ then $\boldsymbol{f}$ can be expressed as the divergence of a vector field $H = (h^1, ..., h^n)$, where each component h^ϵ also belongs to the set A.

$$f = divH \equiv \boldsymbol{D}_\epsilon(h^\epsilon), \tag{2.35}$$

the specified condition applies to the function if the mentioned equations are consistently and universally true for $z_1, u, \mu_{(1)}, ...$:

$$\frac{\delta f}{\delta \mu^\alpha} = 0, \qquad \alpha = 1, ..., m. \tag{2.36}$$

2.6 Integrating factor for higher-order equations

Definition 2.1. Consider ordinary differential equations involves the sth-derivative of the dependent variable,

$$a(z_1, z_2, z_2', ..., z_2(s-1))z_2(s) + b(z_1, z_2, z_2', ..., z_2(s-1)) = 0. \tag{2.37}$$

A differentiable function ν depends on the independent variable z_1 and the dependent variable z_2, as well as its derivatives up to $(s-1)th$ order. The differentiable function ν is termed an integrating factor for the differential equation represented by (2.37). the left-hand side of the equation becomes equal to the total derivative of a function $\phi(z_1, z_2, z_2', ..., z_2(s-1))$ belongs to the set A:

$$\nu a z_2(s) + \nu b = \boldsymbol{D}_{z_1}(\phi). \tag{2.38}$$

The integrating factor of the given differential equation represented by (2.37) enable to lower the order of the equation. The equation (2.37) and (2.38) are rewritten as $\boldsymbol{D}_{z_1}(\phi) = 0$. The application of Lemma 2.1 yields an equation of lower order, specifically an $(s-1)$-order equation,

$$\phi(z_1, z_2, z_2', ..., z_2(s-1)) = C. \tag{2.39}$$

Definition 2.1 refers to a concept that can be applied to systems of ordinary differential equations (ODEs) regardless of their order.

Theorem 2.1. A specific equation that can be used to find or calculate the integrating factors for the given differential equation (2.37),

$$\frac{\delta}{\delta z_2}(\nu a z_2(s) + \nu b) = 0, \tag{2.40}$$

The symbol $\frac{\delta}{\delta z_2}$ represents the variational derivative in equation (2.31). Equation (2.40) is introduced as a specific equation involving the variables $z_1, z_2, z_2', ..., z_2(2s-2)$ and this equation holds true for all possible values of these variables, without exception.

Proof. Equation (2.40) is a result obtained using Lemma 2.2. After applying the variational derivative process, $2s-1$ is the highest order of differentiation that takes place.

$$(-1)^s \boldsymbol{D}_{z_1}^s(\nu a) \qquad \textit{and} \qquad (-1)^{s-1}\boldsymbol{D}_{z_1}^{s-1}[z_2(s)\frac{\partial(\nu a)}{\partial z_2^{(s-1)}}].$$

certain terms are being disregarded from consideration $z_2(2s-1)$:

$$(-1)^s \boldsymbol{D}_{z_1}^s(\nu a) = -(-1)^{s-1}\boldsymbol{D}_{z_1}^{s-1}[z_2(s)\frac{\partial(\nu a)}{\partial z_2^{(s-1)}}] + ...$$

The terms in equation (2.40) that includes $z_2(2s-1)$ are found to mutually cancel each other out. As a consequence of the cancellation, equation (2.40) is simplified. It now involves only the variables $z_1, z_2, z_2', ..., z_2(2s-2)$,

For the first-order ordinary differential equation represented by (1.1) which is $a(z_1, z_2)z_2' + b(z_1, z_2) = 0$, equation (2.40) can be expresses in particular form:

$$\frac{\delta}{\delta z_2}(\nu a z_2' + \nu b) = \frac{\partial}{\partial z_2}(\nu a z_2' + \nu b) - \boldsymbol{D}_{z_1}[\frac{\partial}{\partial z_2'}(\nu a z_2' + \nu b)].$$

Where the integrating factor ν is a function that relies on both the independent variable z_1 and the dependent variable z_2,

$$\frac{\delta}{\delta z_2}(\nu a z_2' + \nu b) = z_2'(\nu a)_{z_2} + (\nu b)_{z_2} - \boldsymbol{D}_{z_1}(\nu a + 0),$$

$$\frac{\delta}{\delta z_2}(\nu a z_2' + \nu b) = z_2'(\nu a)_{z_2} + (\nu b)_{z_2} - (\nu a)_{z_1} - z_2'(\nu a)_{z_2}.$$

We arrive at (1.6), $(\nu b)_{z_2} - (\nu a)_{z_1} = 0$.

Now, consider a new equation that is of second-order,

$$a(z_1, z_2, z_2')z_2'' + b(z_1, z_2, z_2') = 0. \tag{2.41}$$

The integrating factors ν is a function is a function that depends on the variables z_1, z_2, z_2', The equation (2.40) is then used to determine the expression for $\nu(z_1, z_2, z_2')$:

$$\frac{\delta}{\delta z_2}(\nu a z_2'' + \nu b) = \frac{\partial}{\partial z_2}(\nu a z_2'' + \nu b) - \boldsymbol{D}_{z_1}[\frac{\partial}{\partial z_2'}(\nu a z_2'' + \nu b)] + \boldsymbol{D}_{z_1}^2[\frac{\partial}{\partial z_2''}(\nu a z_2'' + \nu b)],$$

$$\frac{\delta}{\delta z_2}(\nu a z_2'' + \nu b) = z_2''(\nu a)_{z_2} + (\nu b)_{z_2} - \boldsymbol{D}_{z_1}[z_2''(\nu a)_{z_2'} + (\nu b)_{z_2'}] + \boldsymbol{D}_{z_1}^2(\nu a + 0),$$

$$\begin{aligned}\frac{\delta}{\delta z_2}(\nu a z_2'' + \nu b) &= z_2''(\nu a)_{z_2} + (\nu b)_{z_2} - z_2''[(\nu a)_{z_1 z_2'} + z_2'(\nu a)_{z_2 z_2'} + z_2''(\nu a)_{z_2' z_2'}] \\ &\quad - [(\nu b)_{z_1 z_2'} + z_2'(\nu b)_{z_2 z_2'} + z_2''(\nu b)_{z_2' z_2'}] + \boldsymbol{D}_{z_1}[(\nu a)_{z_1} + z_2'(\nu a)_{z_2} + z_2''(\nu a)_{z_2'}],\end{aligned}$$

$$\begin{aligned}\frac{\delta}{\delta z_2}(\nu a z_2'' + \nu b) &= z_2''(\nu a)_{z_2} + (\nu b)_{z_2} - z_2''(\nu a)_{z_1 z_2'} - z_2' z_2''(\nu a)_{z_2 z_2'} - z_2''2(\nu a)_{z_2' z_2'} \\ &\quad - (\nu b)_{z_1 z_2'} - z_2'(\nu b)_{z_2 z_2'} - z_2''(\nu b)_{z_2' z_2'} + (\nu a)_{z_1 z_1} + z_2'(\nu a)_{z_1 z_2} + z_2''(\nu a) z_1 z_2' \\ &\quad + z_2'(\nu a)_{z_1 z_2} + z_2\prime 2(\nu a)_{z_2 z_2} + z_2' z_2''(\nu a)_{z_2 z_2'} + z_2''(\nu a)_{z_1 z_2'} + z_2'' z_2'(\nu a)_{z_2 z_2'} \\ &\quad + z_2''2(\nu a)_{z_2' z_2'},\end{aligned}$$

$$\begin{aligned}\frac{\delta}{\delta z_2}(\nu a z_2'' + \nu b) &= z_2''[(\nu a)_{z_2} - (\nu a)_{z_1 z_2'} - z_2'(\nu a)_{z_2 z_2'} - (\nu b)_{z_2' z_2'} + 2(\nu a) z_1 z_2' \\ &\quad + z_2'(\nu a)_{z_2 z_2'} + z_2'(\nu a)_{z_2 z_2}] + (\nu b)_{z_2} - (\nu b) z_1 z_2' + (\nu a)_{z_1 z_1},\end{aligned}$$

and hence,

$$\frac{\delta}{\delta z_2}(\nu a z_2'' + \nu b) = z_2''[z_2'(\nu a)_{z_2 z_2'} + (\nu a)_{z_1 z_2'} + 2(\nu a)_{z_2} - (\nu b)_{z_2' z_2'}] + z_2\prime 2(\nu a)_{z_2 z_2} + 2z_2'(\nu a)_{z_1 z_2}$$
$$+ (\nu a)_{z_1 z_1} - z_2'(\nu b)_{z_2 z_2'} - (\nu b)_{z_1 z_2'} + (\nu b)_{z_2}.$$

Since The expression being discussed is expected to become identically equal to zero when evaluated for all possible values of $z_1, z_2, z_2' and z_2''$, The analysis of the expression's behavior leads to a specific conclusion.

Theorem 2.2. The funtion ν that act as integrating factors for the given second-order equation (2.41), are found by solving a system of two equations.

$$z_2'(\nu a)_{z_2 z_2'} + (\nu a)_{z_1 z_2'} + 2(\nu a)_{z_2} - (\nu b) z_2' z_2' = 0, \tag{2.42}$$

$$z_2'^2(\nu a)_{z_2 z_2} + 2z_2'(\nu a)_{z_1 z_2} + (\nu a)_{z_1 z_1} - z_2'(\nu b)_{z_2 z_2'} - (\nu b)_{z_1 z_2'} + (\nu b)_{z_2} = 0. \tag{2.43}$$

Theorem 2.2 demonstrates that second-order equations may not always have integrating factors available to simplify them, unlike first-order equations. $\nu(z_1, z_2)$ For first-order equations, the integrating factor $\nu(z_1, z_2)$ can be found by solving a single linear partial differential equation, as represented by (1.6). This equation has infinite number of solutions. When dealing with second-order equations (2.41), to find $\nu(z_1, z_2, z_2')$ as an integrating factor, one must solve a system of two second-order linear partial differential equations (2.42)-(2.43) is compatible. The compatibility of these equations is crucial for the presence of an integrating factor.

Remark 2.1. For a differential equation of second order represented by (2.41) if it possesses two integrating factors, then it is possible to find the general solution of the equation without needing any extra step for integration.

Example 2.1. Using integrating factors, let's determine the particular differential equation:

$$z_2'' + \frac{z_2\prime 2}{z_2} + 3\frac{z_2'}{z_1} = 0. \tag{2.44}$$

consider second-order equation,

$$a(z_1, z_2, z_2')z_2'' + b(z_1, z_2, z_2') = 0,$$

from above equation we can write,

$$a = 1, \qquad b = \frac{z_2\prime 2}{z_2} + 3\frac{z_2'}{z_1}.$$

To simplify the process by searching for integrating factors that have a specific form: $\nu = \nu(z_1, z_2)$. The equation (2.42) is transformed to $2\nu_{z_2} - (\nu b)_{z_2' z_2'} = 0$. The term $(\nu b)_{z_2' z_2'}$ represents a partial derivative of the product νb with respect to z_2' twice. The value of this derivative is $\frac{2\nu}{z_2}$,

$$\frac{\partial \nu}{\partial z_2} - \frac{\nu}{z_2} = 0,$$

$$when \nu = \phi(z_1)y, \qquad \mu_y = \phi(z_1) \qquad \nu_{z_2 z_2} = 0, \qquad \nu_{z_1 z_2} = \phi'(z_1)y \qquad \nu_{z_1 z_2} = \phi',$$
$$\nu_{z_1 z_1} = \phi'' z_2, \qquad \nu b = \phi z_2\prime 2 + 3\frac{\phi}{z\,1} z_2 z_2',$$

$$(\nu b)_{z_2} = 3\frac{\phi}{z_1} z_2', \qquad (\nu b)_{z_2 z_2'} = 3\frac{\phi}{z_1}, \qquad (\nu b)_{z_1 z_2'} = 2\phi' z_2' + 3(\frac{\phi'}{z_1} - \frac{\phi}{{z_1}^2})z_2.$$

So, (2.43) becomes,

$$z_2\prime 2\nu_{z_2 z_2} + 2z_2'\nu_{z_1 z_2} + \nu_{z_1 z_1} - z_2'\frac{3\phi}{z_1} - 2\phi' z_2' - 3y(\frac{\phi'}{z_1} - \frac{\phi}{{z_1}^2}) + 3z_2'\frac{\phi}{z_1} = 0,$$

$$y({z_1}^2\phi'' - 3x\phi' + 3\phi) = 0,$$

$${z_1}^2\phi'' - 3x\phi' + 3\phi = 0.$$

Let $\phi = {z_1}^r$ $\phi' = r{z_1}^{r-1}$ $\phi'' = r(r-1){z_1}^{r-2}$ so,

$$
\begin{aligned}
{z_1}^2.r(r-1){z_1}^{r-2} - 3x.r{z_1}^{r-1} + 3{z_1}^r = 0, \\
r^2 - r - 3r + 3 = 0, \\
r - 3 = 0, \qquad r - 1 = 0, \\
\phi(z_1) = c_1 z_1 + c_2 {z_1}^3,
\end{aligned}
$$

we find two separate solutions for a certain equation. In this case, the solutions are denoted as, $\phi = z_1$, and $\phi = {z_1}^3$. These two integrating factors has associated with these solutions:

$$
\nu_1 = z_1 z_2, \qquad \nu_2 = {z_1}^3 z_2, \tag{2.45}
$$

and the equation (2.44) can be solved without the need of extra integration step (see Remark 2.1). first integrating factor is applied by multiplying the equation (2.44), we have

$$
z_1 z_2({z_2}'' + \frac{z_2\prime 2}{z_2} + 3\frac{y\prime}{z_1}) = 0,
$$

$$
z_1 z_2 {z_2}'' + z_1 z_2'^2 + 3 z_2 z_2' = 0.
$$

Substituting $z_1 z_2 {z_2}'' = \boldsymbol{D}_{z_1}(z_1 z_2 z_2') - z_2 z_2' - z_1 z_2'^2$, we reduce it to as,

$$
\boldsymbol{D}_{z_1} = \frac{\partial}{\partial z_1} + z_2' \frac{\partial}{\partial z_2} + {z_2}'' \frac{\partial}{\partial z_2'},
$$

$$
\boldsymbol{D}_{z_1}(z_1 z_2 z_2') = z_2 z_2' + z_2' z_1 z_2' + {z_2}'' z_1 z_2,
$$

$$
\boldsymbol{D}_{z_1}(z_1 z_2 z_2') = z_2 z_2' + z_1 z_2'^2 + z_1 z_2 {z_2}'',
$$

$$
\boldsymbol{D}_{z_1}(z_1 z_2 z_2') + 2 z_2 z_2' = 0,
$$

$$
\boldsymbol{D}_{z_1}(z_1 z_2 z_2') + \boldsymbol{D}_{z_1}(z_2 2) = 0,
$$

$$\boldsymbol{D}_{z_1}(z_1 z_2 z_2' + z_2 2)+ = 0,$$

where,

$$z_1 z_2 z_2' + z_2 2 = C_1. \tag{2.46}$$

Now, $\nu_2 = {z_1}^3 y,$

$$\begin{aligned}
{z_1}^3 y {z_2}'' + {z_1}^3 z_2\prime 2 + 3{z_1}^2 z_2 z_2' = 0,\\
\boldsymbol{D}_{z_1} = \frac{\partial}{\partial z_1} + z_2' \frac{\partial}{\partial z_2} + {z_2}'' \frac{\partial}{\partial z_2'},\\
3{z_1}^2 z_2 z_2' + z_2'({z_1}^3 z_2') + {z_2}''({z_1}^3 y) = 0,\\
3{z_1}^2 z_2 z_2' + {z_1}^3 z_2\prime 2 + {z_1}^3 z_2 {z_2}'' = 0,\\
\boldsymbol{D}_{z_1}({z_1}^3 z_2 z_2') - {z_1}^3 z_2\prime 2 - 3{z_1}^2 z_2 z_2',\\
\boldsymbol{D}_{z_1}({z_1}^3 z_2 z_2') = 3{z_1}^2 z_2 y\prime + {z_1}^3 y' 2 + {z_1}^3 z_2 {z_2}'',\\
\boldsymbol{D}_{z_1}({z_1}^3 z_2 z_2') - 3{z_1}^2 z_2 y\prime - z_2\prime 2 {z_1}^3 + {z_1}^3 z_2\prime 2 + 3{z_1}^2 z_2 y\prime,
\end{aligned}$$

$${z_1}^3 z_2 z_2' = C_2. \tag{2.47}$$

From the two equations (2.46)-(2.47), the variable z_2' can be eliminated. then general solution to the original equation (2.44) can be obtained:

$$z_2 = \pm\sqrt{C_1} - \frac{C_2}{{z_1}^2}. \tag{2.48}$$

2.7 Adjoint equations

Definition 2.2. Consider a collection partial differential equations of sth-order,

$$F_\alpha(z_1, u, ..., \mu_{(s)}) = 0, \qquad \alpha = 1, ..., m, \tag{2.49}$$

This function F_α depends on various variables, including n independent variables denoted as $x = ({z_1}^1, ..., {z_1}^n)$ and m dependent variables denoted as $u = (\mu^1, ..., \mu^m)$. The notation

$u = \mu_{(x)}$ suggests that the variable u depends on the derivatives of x. This implies that the dependent variables u are functions of x and its derivatives.

$$F^*_\alpha(z_1, u, v, ..., \mu_{(s)}, v_{(s)}) \equiv \frac{\delta(v^\beta F_\beta)}{\delta\mu^\alpha} = 0, \qquad \alpha = 1, ..., m, \tag{2.50}$$

where $v = (v^1, ..., v^m)$ consist of new dependent variables ,$v = v(z_1)$.

Remark 2.2. In the context of linear differential equations, the adjoint equations introduced in Definition 2.2 correspond precisely to the classical adjoint equations discussed earlier. Therefore, a system of linear equations $F(z_1, u, ..., \mu_{(s)}) = 0$ involving $u(z_1)$ then it's noted that the adjoint equation $F^*(z_1, v, ..., v_{(s)}) = 0$ involving $v(z_1)$ is also linear. i.e., $F^{**} = F$. Specifically, if the adjoint equation to $F^*(z_1, v, ..., v_{(s)}) = 0$ is $F^{**}(z_1, w, ..., w_{(s)}) = 0$, then setting $w = u$ in the latter equation result in the original equation.

Definition 2.3. A system of equations (2.49) is said to be self-adjoint if a specific condition involving its adjoint equations (2.50) is met. perform a substitution replacing v with u:

$$F^*_\alpha(z_1, u, u, ..., \mu_{(s)}) = 0, \qquad \alpha = 1, ..., m,$$

the system of equations derived by substituting $v = u$, the adjoint equations is exactly the same as the system of equations (2.49).

Example 2.2. Considering the first-order linear ordinary differential equation of $n = 1, m = 1$, set $u = y, v = z$, and the first-order linear ODE might resemble this:(1.15):

$$F(z_1, z_2, z_2') \equiv a_0 z_2' + a_1 z_2 - \boldsymbol{f(z_1)} = 0.$$

As Euler Lagrange operator is,

$$F^*_\alpha(z_1, u, v, \mu_{(s)}, v_{(s)}) = \frac{\delta}{(v^\beta F_\beta)} = 0,$$

$$\frac{\delta(vF)}{\delta z_2} = \frac{\partial(vF)}{\partial z_2} - \boldsymbol{D}_{z_1}\frac{\partial(vF)}{\partial z_2'} + \boldsymbol{D}^2_{z_1}\frac{\partial(vF)}{\partial z_2'} - ....$$

$$\boldsymbol{D}_{z_1} = \frac{\partial}{\partial z_2} + z_2'\frac{\partial}{\partial z_2} + z_2''\frac{\partial}{\partial z_2'} + ...$$

Equation (2.50) that definines the adjoint equation represented as:

$$\frac{\delta(zF)}{\delta z_2} = (\frac{\partial}{\partial z_2} - \boldsymbol{D}_{z_1}\frac{\partial}{\partial z_2'})(z[a_0 z_2' + a_1 z_2 - \boldsymbol{f(z_1)}]) = 0.$$

since,

$$\frac{\partial}{\partial z_2}(z[a_0 z_2' + a_1 z_2 - \boldsymbol{f(z_1)}]) = a_1 z, \qquad \frac{\partial}{\partial z_2'}(z[a_0 z_2' + a_1 z_2 - \boldsymbol{f(z_1)}]) = a_0 z,$$

(2.50) yields the adjoint equation $a_1 z - \boldsymbol{D}_{z_1}(a_0 z) = 0$, or

$$a_1 z - (a_0 z)' = 0$$

the left-hand side of the equation being discussed is exactly the same as the previously defined adjoint operator in equation (1.16).

Example 2.3. In the equation of second order (1.17),

$$a_0 z_2'' + a_1 z_2' + a_2 z_2 = \boldsymbol{f(z_1)},$$

By using definition 2.2 we can obtain the adjoint equation,

$$(\frac{\partial}{\partial z_2} - \boldsymbol{D}_{z_1}\frac{\partial}{\partial z_2'} + \boldsymbol{D}^2_{z_1}\frac{\partial}{\partial z_2''})(z[a_0 z_2'' + a_1 z_2' + a_2 z_2 - \boldsymbol{f(z_1)}]) = 0.$$

similarly by using the same idea of the last example, one can get the adjoint equation (1.18):

$$(a_0 z)'' - (a_1 z)' + a_2 z = 0.$$

Example 2.4. Consider a certain kind of linear partial differential equation. (1.8);

$$\boldsymbol{L}[\mu] \equiv a^{\epsilon\kappa}(z_1)\mu_{\epsilon\kappa} + b^{\epsilon}(z_1)\mu_{\epsilon} + cu = \boldsymbol{f(z_1)}.$$

The adjoint equation is defined as follows in (2.50) ,

$$(\frac{\partial}{\partial u}-\boldsymbol{D}_\epsilon\frac{\partial}{\partial \mu_\epsilon}+\boldsymbol{D}_\epsilon\boldsymbol{D}_\kappa\frac{\partial}{\partial \mu_{\epsilon\kappa}})(v[a^{\epsilon\kappa}(z_1)\mu_{\epsilon\kappa}+b^\epsilon(z_1)\mu_\epsilon+cu-\boldsymbol{f(z_1)}])=0,$$

$$\frac{\partial}{\partial u}(v[a^{\epsilon\kappa}(z_1)\mu_{\epsilon\kappa}+b^\epsilon(z_1)\mu_\epsilon+cu-\boldsymbol{f(z_1)}])=cv,$$

$$\frac{\partial}{\partial \mu_{\epsilon\kappa}}(v[a^{\epsilon\kappa}(z_1)\mu_{\epsilon\kappa}+b^\epsilon(z_1)\mu_\epsilon+cu-\boldsymbol{f(z_1)}])=b^\epsilon v,$$

$$\frac{\partial}{\partial \mu_{\epsilon\kappa}}(v[a^{\epsilon\kappa}(z_1)\mu_{\epsilon\kappa}+b^\epsilon(z_1)\mu_\epsilon+cu-\boldsymbol{f(z_1)}])=a^{\epsilon\kappa}v,$$

so, calculation is leading to the derivation of the adjoint equation (1.14),

$$\boldsymbol{L}^*[\mu]\equiv\boldsymbol{D}_\epsilon\boldsymbol{D}_\kappa(a^{\epsilon\kappa}v)-\boldsymbol{D}_\epsilon(b^\epsilon v)+cv=0.$$

Example 2.5. Consider the heat equation

$$\mu_t-c(z_1)\mu_{z_1z_1}=0,$$

where a constant coefficient $c(z_1)$ is used. Equation (2.50) can be written as, (see (2.29)):

$$\frac{\delta}{\delta u}(v[c(z_1)\mu_{z_1z_1}-\mu_t])=(-\boldsymbol{D}_t\frac{\partial}{\partial \mu_t}+\boldsymbol{D}^2_{z_1}\frac{\partial}{\partial \mu_{z_1z_1}})(v[c(z_1)\mu_{z_1z_1}-\mu_t])=0,$$

$$\frac{\partial}{\partial \mu_t}(v[c(z_1)\mu_{z_1z_1}-\mu_t])=-v,$$

$$\frac{\partial}{\partial \mu_{z_1z_1}}(v[c(z_1)\mu_{z_1z_1}-\mu_t])=vc(z_1),$$

which produces the adjoint equation $\boldsymbol{D}^2_{z_1}(c(z_1)v)+\boldsymbol{D}_t(v)=0$,

$$v_t+(cv)_{z_1z_1}=0.$$

Definition 2.2 the adjoint equation likely refer to a specific concept in a particular mathematics or physics context.

Example 2.6. Consider the Korteweg-de Vries equation

$$\mu_t = \mu_{z_1 z_1 z_1} + u\mu_x. \tag{2.51}$$

We take partial differential equation involving several variables and partial derivatives, i.e., $F(t, z_1, u, ..., \mu_{(3)}) = \mu_t - \mu_{z_1 z_1 z_1} - u\mu_x$ and write the left-hand side of (2.50) in the form,

$$\begin{aligned}\frac{\delta}{\delta u}(v[\mu_t - \mu_{z_1 z_1 z_1} - u\mu_x]) &= \frac{\partial}{\partial u}(v[\mu_t - \mu_{z_1 z_1 z_1} - u\mu_x]) - \boldsymbol{D}_{z_1}\frac{\partial}{\partial \mu_x}(v[\mu_t - \mu_{z_1 z_1 z_1} - u\mu_x]) \\ &- \boldsymbol{D}_t \frac{\partial}{\partial \mu_t}(v[\mu_t - \mu_{z_1 z_1 z_1} - u\mu_x]) \\ &- \boldsymbol{D}_{z_1}^3 \frac{\partial}{\partial \mu_{z_1 z_1 z_1}}(v[\mu_t - \mu_{z_1 z_1 z_1} - u\mu_x]),\end{aligned}$$

$$= -v\mu_{z_1} - \boldsymbol{D}_{z_1}(-vu) - \boldsymbol{D}_t(v) - \boldsymbol{D}_{z_1}^3(-v),$$

$$= -v\mu_{z_1} - [-v\mu_{z_1} - v_{z_1}u] - v_t + v_{z_1 z_1 z_1},$$

$$= -v_t + v_{z_1 z_1 z_1} + uv_{z_1}.$$

Hence, $F^*(t, z_1, u, v, ..., \mu_{(3)}, v_{(3)}) = -(v_t - v_{z_1 z_1 z_1} - uv_{z_1})$, and the adjoint equation to the Korteweg-de Vries equation(2.51) is,

$$v_t = v_{z_1 z_1 z_1} + uv_{z_1}. \tag{2.52}$$

we have,

$$F^*(t, z_1, u, u, ..., \mu_{(3)}, \mu_{(3)}) = -(\mu_t - \mu_{z_1 z_1 z_1} - u\mu_x) \equiv -F(t, z_1, u, ..., \mu_{(3)}).$$

Thus, The equation (2.51) is self-adjoint implies that the Korteweg-de Vries (KdV) equation possesses a certain symmetry in relation to its adjoint equation (see Definition 2.3). Let us finding the adjoint equation associated with it (ref50). We have,

$$\begin{aligned}
\frac{\delta}{\delta v}(w[v_t - v_{z_1z_1z_1} - uv_{z_1}]) = \frac{\partial}{\partial v}(w[v_t - v_{z_1z_1z_1} - uv_{z_1}]) - \boldsymbol{D}_{z_1}\frac{\partial}{\partial v_{z_1}}(w[v_t - v_{z_1z_1z_1} - uv_{z_1}]) \\
- \boldsymbol{D}_t \frac{\partial}{\partial v_t}(w[v_t - v_{z_1z_1z_1} - uv_{z_1}]) \\
+ \boldsymbol{D}_{z_1}^3 \frac{\partial}{\partial v_{z_1z_1z_1}}(w[v_t - v_{z_1z_1z_1} - uv_{z_1}]), \\
= \boldsymbol{D}_{z_1}(uw) - \boldsymbol{D}_t(w) + \boldsymbol{D}_{z_1}^3(w), \\
= -w_t + w_{z_1z_1z_1} + uw_{z_1} + w\mu_x.
\end{aligned}$$

Hence, the adjoint equation to equation (2.52), is $w_t = w_{z_1z_1z_1} + uw_{z_1} + w\mu_x$. Substituting $w = u$, we arrive at a new equation that has to do with how a function u is derived,

$$\mu_t = \mu_{z_1z_1z_1} + 2u\mu_{z_1},$$

is distinct from the original KdV equation (2.51) ,The specific differences and implications could be further explained in the context of (cf.Remark 2.2).

Example 2.7. Take the Burgers equation, for example,

$$\mu_t = u\mu_{z_1} + u_{z_1z_1}. \tag{2.53}$$

Equation (2.50) is written as follows on it's left hand side:

$$\begin{aligned}
\frac{\delta}{\delta u}(v[\mu_t - u\mu_{z_1} - \mu_{z_1z_1}]) = \frac{\partial}{\partial u}(v[\mu_t - u\mu_{z_1} - \mu_{z_1z_1}]) = -\boldsymbol{D}_{z_1}\frac{\delta}{\delta \mu_x}(v[\mu_t - u\mu_{z_1} - \mu_{z_1z_1}]) \\
- \boldsymbol{D}_t \frac{\delta}{\delta \mu_t}(v[\mu_t - u\mu_{z_1} - \mu_{z_1z_1}]) + \boldsymbol{D}_{z_1}^2 \frac{\delta}{\delta \mu_{z_1z_1}}(v[\mu_t - u\mu_{z_1} - \mu_{z_1z_1}]) \\
- \boldsymbol{D}_t^2 \frac{\delta}{\delta \mu_{tt}}(v[\mu_t - u\mu_{z_1} - \mu_{z_1z_1}]), \\
= -v\mu_{z_1} - \boldsymbol{D}_{z_1}(-vu) - \boldsymbol{D}_t(v) + \boldsymbol{D}_{z_1}^2(-v) - \boldsymbol{D}_t^2(0),
\end{aligned}$$

Hence, by taking the Lagrangian's variation with respect to the adjoint variable, the adjoint equation for the Burgers equation can be produced. (2.53) (is see also [7])

$$v_t = uv_{z_1} - v_{z_1 z_1}. \tag{2.54}$$

Example 2.8. Consider the non-linear heat equation:

$$\mu_t = [k(u))\mu_x]_{z_1}. \tag{2.55}$$

The left-hand side of Eq.(2.50) is written:

$$\begin{aligned} &\frac{\partial}{\partial u}(v[\mu_t - k(u)\mu_{z_1 z_1} - k'(u)\mu_{z_1}^2]) \\ &\qquad = -v_t - k'(u)v\mu_{z_1 z_1} - k''(u)v\mu_{z_1}^2 - \boldsymbol{D}_{z_1}^2(k(u)v) \\ &\qquad + 2\boldsymbol{D}_{z_1}(k'(u)v\mu_x). \end{aligned} \tag{2.56}$$

We have $\boldsymbol{D}_{z_1}(k(u)v) = kv_{z_1} + k'v\mu_x$ and therefore,

$$-\boldsymbol{D}_{z_1}^2(k(u)v) + 2\boldsymbol{D}_{z_1}(k'(u)v\mu_x) = -\boldsymbol{D}_{z_1}(kv_{z_1}) + \boldsymbol{D}_{z_1}(k'v\mu_x).$$

Using the Lagrangian and its variations, we insert this into the adjoint equation for the nonlinear heat equation, referred to another equation (2.56) and a source for further information (see also [7]):

$$v_t + k(u)v_{z_1 z_1} = 0. \tag{2.57}$$

Let's locate the adjoint equation to (2.57). We have,

$$\frac{\partial}{\partial v}(w[v_t + k(u)v_{z_1 z_1}]) = -w_t + \boldsymbol{D}_{z_1}^2[k(u)w].$$

Hence, the adjoint equation for the non-linear heat equation (2.57) and discovered that it is differ from original equation $w_t = [k(u)w]_{z_1 z_1}$. The fact that the adjoint equation doesn't coincide with the original equation when setting $w = u$.

2.8 Lagrangians

Theorem 2.3. A set of second-order differential equations with general form is denoted by (2.22),

$$F_\alpha(z_1, u, ..., \mu_{(s)}) = 0, \qquad \alpha = 1, ..., m, \tag{2.22}$$

considering a differential equation alongside its adjoint equation (2.50),

$$F^*_\alpha(z_1, u, v, ..., \mu_{(s)}, v_{(s)}) \equiv \frac{\delta(v^\beta F_\beta)}{\delta\mu^\alpha} = 0, \qquad \alpha = 1, ..., m, \tag{2.23}$$

has a Lagrangian. A system of second-order differential equations, specifically a simultaneous system involving equations (2.49)-(2.50) with $2m$ dependent variables represented by $u = (\mu^1, ..., \mu^m)$ and $v = (v^1, ..., v^m)$ is the system of Euler-Lagrange equations and Lagrangian is defined based on a certain equation (1.21),

$$\mathfrak{L} = v^\beta F_\beta. \tag{2.58}$$

Proof. we have,

$$\frac{\delta\mathfrak{L}}{\delta v^\alpha} = F_\alpha(z_1, u, ..., \mu_{(s)}), \tag{2.59}$$

and

$$\frac{\delta\mathfrak{L}}{\delta\mu^\alpha} = F^*_\alpha(z_1, u, v, ..., \mu_{(s)}, v_{(s)}). \tag{2.60}$$

The homogeneous linear second-order partial differential equation will be discussed (1.8):

$$\boldsymbol{L}[\mu] \equiv a^{\epsilon\kappa}(z_1)\mu_{\epsilon\kappa} + b^\epsilon(z_1)\mu_\epsilon + c(z_1)u = 0. \tag{2.61}$$

The Lagrangian (2.58) is written:

$$\mathfrak{L} = v\boldsymbol{L}[\mu] = v(a^{\epsilon\kappa}(z_1)\mu_{\epsilon\kappa} + b^\epsilon(z_1)\mu_\epsilon + c(z_1)u). \tag{2.62}$$

We have,

$$\frac{\delta \mathfrak{L}}{\delta v} = \frac{\delta \mathfrak{L}}{\delta v} = \boldsymbol{L}[\mu], \tag{2.63}$$

and

$$\frac{\delta \mathfrak{L}}{\delta \mu^{\alpha}} = \boldsymbol{D}_{\epsilon}\boldsymbol{D}_{\kappa}(\frac{\partial \mathfrak{L}}{\partial \mu_{\epsilon\kappa}}) - \boldsymbol{D}_{\epsilon}(\frac{\partial \mathfrak{L}}{\partial \mu_{\epsilon}}) + \frac{\partial \mathfrak{L}}{\partial u},$$

$$= \boldsymbol{D}_{\epsilon}\boldsymbol{D}_{\kappa}(a^{\epsilon\kappa} v) - \boldsymbol{D}_{\epsilon}(b^{\epsilon} v) + cv = \boldsymbol{L}^{*}[v]. \tag{2.64}$$

Theorem 2.4. A linear operator $\boldsymbol{L}[\mu]$ is self-adjoint, if its adjoint (or conjugate transpose) is the same as the operator itself, i.e., $\boldsymbol{L}^{*}[\mu] = \boldsymbol{L}[\mu]$. Then equation (2.61) is derived from the Lagrangian,

$$\mathfrak{L} = \frac{1}{2}[c(z_1)\mu^2 - a^{\epsilon\kappa}(z_1)\mu_{\epsilon}\mu_{\kappa}]. \tag{2.65}$$

Proof. To write a given Lagrangian (2.62) in the specific form,

$$\mathfrak{L} = v(a^{\epsilon\kappa}\mu_{\epsilon\kappa} + b^{\epsilon}\mu_{\epsilon} + cu) = \boldsymbol{D}_{\kappa}(va^{\epsilon\kappa}\mu_{\epsilon}) - v\mu_{\epsilon}\boldsymbol{D}_{\kappa}(a^{\epsilon\kappa}) + vb^{\epsilon}\mu_{\epsilon} - a^{\epsilon\kappa}\mu_{\epsilon}v_{\kappa} + cuv.$$

The first step involves dropping the first term on the right-hand side of an equation, likely due to a result stated in Lemma 2.3, due to a condition stated in equation (1.13), the second and third terms cancel each other. Finally, set a variable v equal to u, and then divide the equation by two, after following the above steps, reached the desired Lagrangian, denoted as (2.65).

Example 2.9. The Helmholtz equation is given by, $\Delta u + k^2 u = 0$, where δ is the Laplace operator, k is a constant, and u is the independent variable representing the wave-like function. The Lagrangian is given by (2.65)$\mathfrak{L} = (k^2\mu^2 - \mid \backslash\nabla u \mid^2)/2$.

Making a Lagrangian formulation for non-linear or linear non-self-adjoint equations requires taking into account both the original equation and its adjoint equation. This approach provides a versatile framework for understanding, analyzing, and solving complex mathematical and physical systems.

Example 2.10. One method to deal with the linear heat equation's non-self-adjointness is to take into account both it and its adjoint equation. A system of two equations is produced as a result, and it offers a more comprehensive knowledge of the behavior, symmetries, and sensitivities related to the diffusion process:

$$\mu_t - c(z_1)\mu_{z_1 z_1} = 0, \qquad v_t + (cv)_{z_1 z_1}, = 0 \tag{2.66}$$

which is the linear heat equation and its adjoint equation, is derived from a Lagrangian,

$$\mathfrak{L} = v\mu_t - c(z_1)v\mu_{z_1 z_1}. \tag{2.67}$$

Example 2.11. According to Example 2.6, the Lagrangian,

$$\mathfrak{L} = v[\mu_t - \mu_{z_1 z_1 z_1} - u\mu_x], \tag{2.68}$$

by taking into count the conjugate of the Korteweg-de Vries equation (2.51), a combined system of equations is obtained. This combined system likely describes the dynamics and behavior of both equations as a unified whole.

$$\mu_t = \mu_{z_1 z_1 z_1} + u\mu_{z_1}, \qquad v_t = v_{z_1 z_1 z_1} + uv_{z_1}. \tag{2.69}$$

Example 2.12. From the context of Example 2.8 it's indicated that a Lagrangian is derived,

$$\mathfrak{L} = v[\mu_t - k(u)\mu_{z_1 z_1} - k'(u)\mu_{z_1}^2], \tag{2.70}$$

that leads to the non-linear heat equation (2.55) and its conjugate (2.57) a combined system of equation is obtained:

$$\mu_t = [k(u)\mu_x]_{z_1}, \qquad v_t + k(u)v_{z_1 z_1} = 0. \tag{2.71}$$

Example 2.13. The Dirac equation is a fundamental equation in quantum mechanics,

$$\gamma^k \frac{\partial \psi}{\partial z_1^k} + m\psi = 0, \qquad m = const. \tag{2.72}$$

A four-dimensional column vector (psi) represents the variable, and its components are complex valued quantities denoted as $\psi^1, \psi^2, \psi^3, \psi^4$. The first three components ${z_1}^1$, ${z_1}^2$, ${z_1}^3$ correspond to real-valued spatial variables, while the fourth component ${z_1}^4$ is a complex variable defined as ${z_1}^4 = ict$, where t is time and c is the speed of light. Furthermore, the Dirac matrices γ^k are a set of 4×4 complex matrices, where k denotes the matrix index:

$$\gamma^1 = \begin{pmatrix} 0 & 0 & 0 & -\iota \\ 0 & 0 & -\iota & 0 \\ 0 & \iota & 0 & 0 \\ \iota & 0 & 0 & 0 \end{pmatrix},$$

$$\gamma^2 = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix},$$

$$\gamma^2 = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix},$$

$$\gamma^3 = \begin{pmatrix} 0 & 0 & -\iota & 0 \\ 0 & 0 & 0 & \iota \\ \iota & 0 & 0 & 0 \\ 0 & -\iota & 0 & 0 \end{pmatrix},$$

$$\gamma^4 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

Equation (2.72) lacks a Lagrangian. Therefore, a common approach is to consider it together with its conjugate equation,

$$\frac{\partial \widetilde{\psi}}{\partial z_1^k}\gamma^k - m\widetilde{\psi} = 0. \tag{2.73}$$

Here $\widetilde{\psi} = \overline{\psi}^T \gamma^4$ is the complex-conjugate used to create the row vector, where $\overline{\psi}$ represents the complex conjugate of original coulmn vector ψ and T stands for transposition operation. The system of equation (2.72)-(2.73) has a corresponding Lagrangian,

$$\mathfrak{L} = \frac{1}{2}[\widetilde{\psi}(\gamma^k \frac{\partial \psi}{\partial z_1^k} + m\psi) - (\frac{\partial \widetilde{\psi}}{\partial z_1^k}\gamma^k - m\widetilde{\psi})\psi].$$

Indeed, we have,

$$\frac{\delta \mathfrak{L}}{\delta \psi} = -(\frac{\partial \widetilde{\psi}}{\partial z_1^k}\gamma^k - m\widetilde{\psi}), \qquad \frac{\delta \mathfrak{L}}{\delta \widetilde{\psi}} = \gamma^k \frac{\partial \psi}{\partial z_1^k} + m\psi.$$

Chapter 3

Application to the Maxwell equations

This section focuses on illustrating the use of Noether's theorem by applying it to the Maxwell equations in vacuum. By doing so, it aims to uncover conserved quantities associated with symmetries present in the equations, leading to a more comprehensive understanding of electromagnetic field behavior.

$$\frac{1}{c}\frac{\partial \mathbf{R}}{\partial t} = curl\mathbf{S}, \qquad div\mathbf{R} = 0,$$

$$\frac{1}{c}\frac{\partial \mathbf{S}}{\partial t} = -curl\mathbf{R}, \qquad div\mathbf{S} = 0. \tag{3.74}$$

The system (3.74) involves six dependent variables,which correspond to the components of the electric field $\mathbf{R}$ and the magnetic field $\mathbf{S}$ in three dimensional space. The system contains eight equations. However, the system is too heavily determined since there are more equations than dependent variables. This situation is where you have more equations than variables to solve for. The Euler-Lagrange equations represents by (1.21) are used to derive equations of motion from a Lagrangian. The number of equations in this context matches the number of dependent variables. Despite the absence of Lagrangian for system (3.74) the literature offers a way to establish a variational problem in electrodynamics, This Lagrangian allows for the application of variational principles to understand the behavior of wave phenomena,

$$\Delta\mathbf{A} - \frac{1}{c^2}\frac{\partial^2 \mathbf{A}}{\partial t^2} = 0,$$

Theorem 2.3 allows for the derivation of the Lagrangian for the electromagnetic field, which is likely related to the principles of electromagnetic theory and Lagrangian mechanics. The equations $div\mathbf{R} = 0$, and $div\mathbf{S} = 0$ represents the divergence free condition for the electric field and magnetic field. It is stated that these requirements must be met initially in order for them to hold at any time $t = 0$. Due to the divergence-free conditions acting as initial conditions, the system of Maxwell's equations (3.74) can be reduced to a determined system of differential equations. The time variable is also transformed to $t' = ct$. The use of the new time variable t' (in units of distance divided by the speed of light) simplifies the equations by incorporating the speed of light directly into the equations, which is a common practice in relativistic physics.

$$curl\mathbf{R} + \frac{\partial \mathbf{S}}{\partial t} = 0, \qquad curl\mathbf{S} - \frac{\partial \mathbf{R}}{\partial t} = 0. \tag{3.75}$$

The process involves introducing six new dependent variables, specifically the components of two vectors $\mathbf{U} = (V^1, V^2, V^3)$ and $\mathbf{T} = (W^1, W^2, W^3)$, and the subsequent introduction of a Lagrangian,

$$\mathfrak{L} = \mathbf{U}.(curl\mathbf{R} + \frac{\partial \mathbf{S}}{\partial t}) + \mathbf{T}.(curl\mathbf{S} - \frac{\partial \mathbf{R}}{\partial t}). \tag{3.76}$$

the following explanation or action is consistent with the definition in equation (2.58).

the provided Lagrangian (3.76) that can be used to drive the system (3.75) with its adjoint:

$$\frac{\partial \mathfrak{L}}{\partial \mathbf{U}} \equiv curl\mathbf{R} + \frac{\partial \mathbf{S}}{\partial t} = 0, \qquad \frac{\partial \mathfrak{L}}{\partial \mathbf{T}} \equiv curl\mathbf{S} - \frac{\partial \mathbf{R}}{\partial t} = 0. \tag{3.77}$$

$$\frac{\partial \mathfrak{L}}{\partial \mathbf{R}} \equiv curl\mathbf{U} + \frac{\partial \mathbf{T}}{\partial t} = 0, \qquad \frac{\partial \mathfrak{L}}{\partial \mathbf{S}} \equiv curl\mathbf{T} - \frac{\partial \mathbf{U}}{\partial t} = 0. \tag{3.78}$$

when $\mathbf{U} = \mathbf{R}$, and $\mathbf{T} = \mathbf{S}$, the expression (3.77) coincides with the previous equation (3.74) that represent the Maxwell equation and this equation is self-adjoint. Therefore set $\mathbf{U} = \mathbf{R}$, and $\mathbf{T} = \mathbf{S}$ in the derived equation, the equation is then divided by two. The result is the Lagrangian for the Maxwell equations (3.74) (cf. Theorem 2.4):

$$\mathfrak{L} = \frac{1}{2}[\mathbf{R}.(curl\mathbf{R} + \frac{\partial \mathbf{S}}{\partial t}) + \mathbf{S}.(curl\mathbf{S} - \frac{\partial \mathbf{R}}{\partial t})], \tag{3.79}$$

the Lagrangian (3.79) will be presented in a mathematical form:

$$\begin{aligned} \mathfrak{L} = &E^1(E_{z_2}3 - E_z^2 + H_t^1) + E^2(E_z^1 - E_{z_1}^3 + H_t^2) + E^3(E_{z_1}^2 - E_{z_2}1 + H_t^3) \\ &+ H^1(H_{z_2}3 - H_z^2 - E_t^1) + H^2(H_z^1 - H_{z_1}^3 - E_t^2) + H^3(H_{z_1}^2 - H_{z_2}1 - E_t^3). \end{aligned} \tag{3.80}$$

The Maxwell equations, which describe electromagnetic phenomena, have well-known symmetries, the Lagrangian (3.79) is employed as the mathematical framework to apply Noether's theorem. The invariance of the system (3.75) with respect to a particular group of transformation.

$$\mathbf{S}' = \mathbf{S}cos\theta + \mathbf{R}sin\theta, \qquad \mathbf{R}' = \mathbf{R}cos\theta - \mathbf{S}sin\theta, \tag{3.81}$$

with the generator,

$$z_1 = \mathbf{R}\frac{\partial}{\partial \mathbf{S}} - \mathbf{S}\frac{\partial}{\partial \mathbf{R}} \equiv \sum_{i=1}^{3}(E^\epsilon \frac{\partial}{\partial H^\epsilon} - H^\epsilon \frac{\partial}{\partial E^\epsilon}). \tag{3.82}$$

There is a specific expression associated with (1.25),

$$z_1 = \mathbf{R}\frac{\partial}{\partial \mathbf{S}} - \mathbf{S}\frac{\partial}{\partial \mathbf{R}} + \mathbf{R}_t\frac{\partial}{\partial \mathbf{S}_t} - \mathbf{S}_t\frac{\partial}{\partial \mathbf{R}_t} + \mathbf{R}_{z_1}\frac{\partial}{\partial \mathbf{S}_{z_1}} - \mathbf{S}_{z_1}\frac{\partial}{\partial \mathbf{R}_{z_1}} + ... \tag{3.83}$$

Applying a specific operator (3.83) to the Lagrangian (3.79), here's a concise explanation:

$$\begin{aligned} X(\mathfrak{L}) = &\frac{1}{2}[-\mathbf{S}.(curl\mathbf{R} + \mathbf{S}_t) + \mathbf{R}.(curl\mathbf{S} - \mathbf{R}_t) \\ &+ \mathbf{R}.(-curl\mathbf{S} + \mathbf{R}_t) + \mathbf{S}.(curl\mathbf{R} + \mathbf{S}_t)] = 0. \end{aligned}$$

Hence, the derivation of conservation law using the formula (1.23) and then presents the conservation law in a particular format,

$$\boldsymbol{D}_t(\tau) + div\chi = 0, \tag{3.84}$$

where $\chi = (\chi^1, \chi^2, \chi^3), div\chi = \boldsymbol{D}_{z_1}(\chi^1) + \boldsymbol{D}_{z_2}(\chi^2) + \boldsymbol{D}_z(\chi^3)$. Equation (1.23) yields

$$\tau = \mathbf{R}.\frac{\partial \mathfrak{L}}{\partial \mathbf{S}_t} - \mathbf{S}.\frac{\partial \mathfrak{L}}{\partial \mathbf{R}_t} = \frac{1}{2}[\mathbf{R}.\mathbf{R} - \mathbf{S}. - (\mathbf{S})] = \frac{1}{2}[E^2 + H^2].$$

Consequently, τ stands for the energy density.. The poynting vector χ is calculated as a result of determining the spatial coordinates of the conserved vector (1.23). The Poynting vector is a fundamental concept in electromagnetics and represents the directional energy flux (power per unit area) of electromagnetic waves. Thus, by establishing energy density and is the poynting vector the passage concludes that the conservation of energy has been obtained.

$$\boldsymbol{D}_t(\frac{E^2 + H^2}{2}) + div(\mathbf{R} \times \mathbf{S}) = 0. \tag{3.85}$$

3.9 Conclusion:

Maxwell's Equation had a significant influence on modern science and technology. Their uses are extensive, ranging from energy production and communication to materials science and medical imaging. These equations continue to influence how we think about the fundamentals of electromagnetism and propel technological advancement in a variety of fields.

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