Fixed point theorems for mappings in fuzzy type metric spaces

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Preface

According to Stefan Banach(1922), on a complete metric space, every contraction has a unique fixed point. In order to broaden the Banach fixed point theorem, several writers developed a number of contractive type constraints. By extending the concept of contraction from single-valued to multi-valued mappings, Nadler applied the Banach contraction principle. Several authors have now broadened the range of contractive type constraints to include multi-valued mappings.
In order to be the first work to present fixed point theory in fuzzy metric spaces [in the sense of Kramosil and Michalek], Grabiec (1988) produced a fuzzy metric version of the Banach and Edelstein fixed point theorems(1975).
One of the more appealing generalisations of the Banach contraction theorem, which establishes metric completeness, is Caristi's(1976) fixed point theorem, which is well recognised. With regard to fuzzy metric spaces, an intriguing generalisation of the fixed point theorem by Caristi (1976) and the variational principle by Ekeland (1972) was recently reported by Abbasi and Golshan (2016). However, their findings do not address the Kirk's dilemma or the accompanying fuzzy metric's completeness characterisation (1976). J. Martinez-Moreno *et. al* develops a class of Caristi type mappings with a fixed point and describes the completeness of the appropriate fuzzy metric to solve these problems.

The results of fuzzy metric space on fuzzy b-metric space are expanded in this dissertation. Basic results and definitions that are required for later chapters are provided in chapter 1. In the second chapter, we define a class of Caristi type mappings with fixed points and describe the completeness of the related fuzzy metric. Some results from chapter 2 are extended in chapter 3 on fuzzy b-metric space.

Contents

Chapter 1

Preliminaries

This chapter provides a review of few concepts and findings that are necessary for the study of sequal chapters. $\mathfrak{R}$ and $\mathfrak{Z}$ are the sets of real numbers and integers respectively, used in this chapter.

1.1 Metric spaces

The notion of metric, which Frechet proposed in 1906, is a generalisation of the calculus concept of how far two points are from one another. The study of metric is helpful in the development of the concept of convergence and continuity in abstract spaces. Metric is important in geometry and analysis. In this section we will define some basic definitions from [1]

Definition 1.1.1. Suppose $\mathfrak{U} = \mathfrak{R} - \{0\}$, and $\zeta : \mathfrak{U} \times \mathfrak{U} \longrightarrow \mathfrak{R}$ is satisfying the properties given below

1. $\zeta(u, v) \geq 0$
2. $\zeta(u, v) = 0 \Longleftrightarrow u = v$
3. $\zeta(u, v) = \zeta(v, u)$
4. $\zeta(u, w) \leq \zeta(u, v) + \zeta(v, w) \;\; \forall \;\; u, v, w \in \mathfrak{U}$.

If so, ζ is referred to as metric on $\mathfrak{U}$, and $(\mathfrak{U}, \zeta)$ is known as metric space.

Example 1.1.2. Define $\zeta : \mathfrak{U} \times \mathfrak{U} \longrightarrow \mathfrak{R}$ by

$$\zeta(u, v) = |\frac{1}{u} - \frac{1}{v}|.$$

Then ζ is metric on $\mathfrak{U}$.

Example 1.1.3. For $\mathfrak{U} = \mathfrak{R}^s$, Define $\zeta : \mathfrak{U} \times \mathfrak{U} \longrightarrow \mathfrak{R}$ by

$$\zeta(u, v) = \sqrt{\sum_{p=1}^{s} |u_p - v_p|^2},$$

where $u = (u_1, u_2, u_3, ...u_s)$, and $v = (v_1, v_2, v_3, ..., v_s)$. Then ζ is a metric on $\mathfrak{U}$.

Example 1.1.4. Suppose $\mathfrak{U} = B[u, v]$ comprises all real-valued, bounded functions defined on [u,v]. Define $\zeta : \mathfrak{U} \times \mathfrak{U} \longrightarrow \mathfrak{R}$ by

$$\zeta(h, k) = \int_u^v |h(u) - k(u)| \mathrm{d}\mathfrak{U}.$$

Then ζ is metric on $\mathfrak{U}$.

Definition 1.1.5. Assume that $(\mathfrak{U}, \zeta)$ is a metric space, let $u_0 \in \mathfrak{U}$ and $t \in \mathfrak{R}^+$, then open and closed ball having centre at u_0 and radius t in $\mathfrak{U}$ are denoted and defined as

1. $S(u_0, t) = \{u \in \mathfrak{U} : \zeta(u, u_0) < t\}$

   (Open Ball)

2. $S[u_0, t] = \{u \in \mathfrak{U} : \zeta(u,\ u_0) \leq t\}$

   (Closed Ball)

Definition 1.1.6. Suppose $(\mathfrak{U}, \zeta)$ is a metric space, and $\mathfrak{V} \subseteq \mathfrak{U}$.
i- The open set in $\mathfrak{U}$ is $\mathfrak{V} \Longleftrightarrow$ for any u $\in \mathfrak{U}$ $\exists\ t > 0$ in a way that $u \in S(u, t) \subseteq \mathfrak{V}$
ii- The Closed set in $\mathfrak{U}$ is $\mathfrak{V} \Longleftrightarrow \mathfrak{V}^c$ is open in $(\mathfrak{U}, \zeta)$

Definition 1.1.7. In a metric space $(\mathfrak{U}, \zeta)$
i- u_n is stated to be converge to a point $u \in \mathfrak{U}$. $\Longleftrightarrow$ for some positive integer n_0 and for every $\varepsilon > 0$ regardless of how little such that $\zeta(u_r, u_s) < \varepsilon$ whenever $r, s \geq n_0$.
ii- u_r is stated to be a cauchy sequence in $(\mathfrak{U}, \zeta) \Longleftrightarrow$ for some positive integer r_o and for every $\varepsilon > 0$, $\zeta(u_s, u_r) < \varepsilon$ whenever $s, r \geq r_o$.
iii- $\{u_r\}$ is stated to be a bounded sequence $\Longleftrightarrow$ $\exists$ some positive real no μ, however large to the effect that for each positive integer r

$$\zeta(u_r, u) \leq \mu.$$

Here u is some fixed element of $\mathfrak{U}$.

Remark 1.1.8. i- When $u_r \longrightarrow u$ then $\underset{r \longrightarrow \infty}{lim} d(u_r, u) = 0$. We also write it as

$$\underset{r \longrightarrow \infty}{lim} u_r = u.$$

ii- Every convergent sequence is Cauchy in the metric space $(\mathfrak{U}, \zeta)$. Nevertheless, not all Cauchy sequence converges.

Definition 1.1.9. $(\mathfrak{U}, \zeta)$ is complete metric space provided in $\mathfrak{U}$ each and every Cauchy sequence converges.

Definition 1.1.10. Suppose $(\mathfrak{U}, d_u)$ and $(\mathfrak{V}, d_v)$ are metric spaces and h is function from $\mathfrak{U}$ to $\mathfrak{V}$.
i- A point t $\in \mathfrak{V}$ is said to be limit of function h at point $u_0 \in \mathfrak{U}$ provided whenever $\varepsilon > 0$, a ponit exists for $\delta > 0$ such that $\zeta_u(u, u_0) < \delta \implies \zeta_v(h(u), l) < \varepsilon$ or

$$\lim_{u \longrightarrow u_0} h(u) = l.$$

ii- h is called continuous at $u_0 \in \mathfrak{U}$ provided for each open ball $S(v, r)$ in $\mathfrak{V}$ containing $h(u_0)$ $\exists$ some open ball $S(u, t)$ in $\mathfrak{U}$ containing u_0 such that $h(S(u, t)) \subseteq S(v, r)$.

Remark 1.1.11. If a function h is continuous $\forall u \in \mathfrak{U}$, then it is said to be continuous at $\mathfrak{U}$.

1.2 Fixed point

A fixed point is a value that does not change as a result of a specific transformation (sometimes referred to as an invariant point or fixed point). A fixed point of a function in mathematics is a particular element that the function maps to itself.

Definition 1.2.1. [1] Assume that $h : \mathfrak{U} \longrightarrow \mathfrak{U}$ is a mapping and $(\mathfrak{U}, \zeta)$ is a metric space. If a point u is mapped into itself, i.e. $h(u) = u$, it is then referred to as a fixed point of h.

Remark 1.2.2. There are four different types of fixed points that can exist in a mapping: none, one, several, and infinite.

Example 1.2.3. i- Define a mapping $h : \mathfrak{R} \longrightarrow \mathfrak{R}$ as

$$h(u) = \frac{u}{2}$$

has unique fixed point 0.
ii- There are two fixed points 0 and 1 in the mapping $h : \mathfrak{R} \longrightarrow \mathfrak{R}$ defined as

$$h(u) = u^2$$

.
iii- The number of fixed points in the mapping $h : \mathfrak{R}^2 \longrightarrow \mathfrak{R}^2$ described by

$$h(u, v) = u$$

are infinite.
iv- Suppose $\mathfrak{U} \neq \phi$ and defne $h : \mathfrak{U} \longrightarrow \mathfrak{U}$ as

$$h(u) = u + a$$

where a is arbitrary constant. No fixed point exists for h.

Remark 1.2.4. Points where the graph of h, whose equation is $v = h(u)$, crosses the diagonal, whose equation is $u = v$, are graphically known as fixed points.

1.3 Fixed point theorems

Theorems describing fixed point's existence and properties are known as fixed point theorems. Informally, the field of mathematics known as fixed point theory seeks to locate all self-maps (also known as self correspondence) where at least one element is left invariant.

i- Metric fixed point theory
Although other people were aware of the metric fixed point theory's core concepts earlier, it is ascribed to the polish mathematician **Stefan Banach** with popularising and making use of the idea.
ii- Banach fixed point theorem
The contraction mapping theorem, also called contraction mapping principle, is another name for the Bannach fixed point theorem and is a crucial tool when researching metric spaces. In particular self-mapping metric spaces, it guarantees the existence of fixed points and are unique and offers a useful technique for locating those fixed points. It states that
"Assume that $(\mathfrak{U}, \zeta)$ is a complete metric space . If γ is a mapping from $\mathfrak{U}$ into itself satisafying $\zeta(\gamma u, \gamma v) < \alpha\zeta(u, v)$ (called Banach Contraction) for each u and v belongs to $\mathfrak{U}$, where α is any real no such that $0 \leq \alpha < 1$ then fixed point of γ in $\mathfrak{U}$ is distinct".

1.4 b-metric spaces

There are numerous metric and metric space extensions in addition to fuzzy metric spaces. In order to broaden the application of the Banach contraction principle, Bakhtin(1989) and Czerwik(1993) developed a space where a weaker condition was found than the triangle inequality. b-metric spaces was the name given to these spaces.

Definition 1.4.1. [6] Let $\mathfrak{U} \neq \emptyset$, a mapping $\zeta_b : \mathfrak{U} \times \mathfrak{U} \longrightarrow \mathfrak{R}^+$ is called b-metric if it meets the requirements listed below;

1. $\zeta_b(u, v) \geq 0$
2. $\zeta_b(u, v) = 0 \Longleftrightarrow u = v$
3. $\zeta_b(u, v) = \zeta_b(v, u)$
4. $\zeta_b(u, w) \leq \zeta_b(\frac{u}{b}, \frac{v}{b}) + \zeta_b(\frac{v}{b}, \frac{w}{b})\ \forall\ u, v, w \in \mathfrak{U}$ and $b \geq 1$

$(\mathfrak{U}, \zeta_b)$ here stands for b-metric space.

Remark 1.4.2. For b=1, any b-metric becomes a metric, although the opposite is generally incorrect.

Example 1.4.3. If $\zeta_b(u, v) = |v - u|^2$, $u, v \in \mathfrak{R}$, then ζ_b is a b-metric for $b = 2$. For $u = 5, v = 3$ and $w = 4$ it is not a metric because the condition $\zeta_b(u, w) \leq \zeta_b(u, v) + \zeta_b(v, w)$ fails.

1.5 Fuzzy set

A group of items known as a fuzzy set has a range of membership grades. A membership (characteristic) function, which assigns each object a membership grade between zero and one, is one technique to recognise such a set. In this section we will define some basic definitions from [2].

Definition 1.5.1. If $\mathfrak{U}$ is a set that is not empty, one way to describe a fuzzy set is $\mathfrak{A} = \{(u, \sigma_A(u))|u \in \mathfrak{U}\}$ in which u is a particular element of $\mathfrak{U}$ and $\sigma_A : \mathfrak{U} \longrightarrow [0, 1]$ referred to as the membership function and $\sigma_A(u)$ is said to be degree of membership of u.

Example 1.5.2. Let $\mathfrak{U} = \{1, 2, 3, 4, 5, 6\}$ then fuzzy set of $\mathfrak{U}$ is

$$\mathfrak{A} = \{(1, 0.9), (2, 0.5), (3, 0.4), (4, 0.6), (5, 0.2), (6, 0.7)\}$$

or also we write it as

$$\sigma_A(1) = 0.9$$
$$\sigma_A(2) = 0.5$$
$$\sigma_A(3) = 0.4$$
$$\sigma_A(4) = 0.6$$
$$\sigma_A(5) = 0.2$$
$$\sigma_A(6) = 0.7$$

Definition 1.5.3. Suppose σ_A and σ_B be two fuzzy sets of $\mathfrak{U}$.
i- Intersection of σ_A and σ_B is denoted by $\sigma_A \cap \sigma_B$ and defined as

$$(\sigma_A \cap \sigma_B)(u) = min\{\sigma_A(u), \sigma_B(u)\}$$

ii- Union of σ_A and σ_B is asserted as $\sigma_A \cup \sigma_B$ and defined by

$$(\sigma_A \cup \sigma_B)(x) = max\{\sigma_A(x), \sigma_B(x)\}$$

iii- A fuzzy set's complement is asserted by $\sigma_A^c(x)$ and defined as

$$\sigma_A^c(x) = 1 - \sigma_A(x)$$

Example 1.5.4. Let $\mathfrak{U} = \{1, 2, 3\}$

$$\sigma_A = \{(1, 0.8), (2, 0.3), (3, 0.7)\},$$

$$\sigma_B = \{(1, 0.2), (2, 0.7), (3, 0.3)\}.$$

$$
\begin{aligned}
i - \sigma_A \cap \sigma_B &= \{(1, min(0.8, 0.2)), (2, min(0.3, 0.7)), (3, min(0.7, 0.3))\} \\
&= \{(1, 0.2), (2, 0.3), (3, 0.3)\}
\end{aligned}
$$

$$
\begin{aligned}
ii - \sigma_A \cup \sigma_B &= \{(1, max(0.8, 0.2)), (2, max(0.3, 0.7)), (3, max(0.7, 0.3))\} \\
&= \{(1, 0.8), (2, 0.7), (3, 0.7)\}
\end{aligned}
$$

Definition 1.5.5. i- A set whose degree of membership is 1 called universal set.
ii- A set whose degree of membership is 0 called empty set.

Definition 1.5.6. Let $\sigma_1, \sigma_2, \sigma_3, ...\sigma_n$ be n fuzzy sets of $\mathfrak{U}_1, \mathfrak{U}_2, \mathfrak{U}_3, ..., \mathfrak{U}_n$ respectively then their cross product or cartesian product is denoted and defined as

$$\sigma_1 \times \sigma_2 \times \sigma_3 \times .... \times \sigma_n(u_1, u_2, u_3, ..., u_n) = min\{\sigma_1(u_1), \sigma_2(u_2), \sigma_3(u_3), ..., \sigma_n(u_n)\}$$

1.6 t-norm

A t-norm is a type of binary operation used in multi-valued logic, particularly in fuzzy logic, as well as regarding probabilistic metric spaces. The term "triangular norm" refers to the way that t-norms are used to generalise the triangle inequality of fuzzy metric spaces.

Definition 1.6.1. [3] A mapping of the form $* : [0, 1] \times [0, 1] \longrightarrow [0, 1]$ is referred to as a continuous and triangular norm (t-norm) if it meets the criteria below.
$*$ is

1. Continuous
2. Commutative and Associative
3. $1 * p = p \, \forall$ p $\in [0, 1]$.
4. For $p, q, r, s \in [0, 1]$ $p * q \leq r * s$ when $p \leq r$ and $q \leq s$

Example 1.6.2. i- Minimum t-norm For $p, q \in [0, 1], p * q = \min\{p, q\}$.
ii- Product t-norm Suppose $p, q \in [0, 1], p * q = pq$.
iii- Lukasiewicz t-norm Suppose $p, q \in [0, 1], p * q = max\{p + q - 1, 0\}$ $\forall$ p , $q \in [0, 1]$.
iv. Archimedean t-norm A t-norm $*$ is known as Archimedean, if $\forall$ p , $q \in [0, 1]$,

$$p * q \geq p \Longrightarrow q = 1.$$

1.7 Fuzzy metric spaces

As stated in the definition of fuzzy metric spaces provided by Kaleva and Seikkala(1984), if the distance between the elements is not a precise number, the imprecision is included in the metric. The concept of a fuzzy metric space was then introduced, initially by Kramosil and Michalek(1975) and later by George and Veeramani(1994). In this chapter, we examine the fuzzy metric space theory proposed by George and Veeramani(1994).

Definition 1.7.1. [4]
Suppose $\mathfrak{Y} \neq \phi$ and F_m is a fuzzy set on $\mathfrak{Y} \times \mathfrak{Y} \times (0, \infty)$ and $*$ is a t-norm. Then an ordered triplet $(\mathfrak{Y}, F_m, *)$ is a fuzzy metric space satisfying the conditions given below $\forall$ p , $q, r \in \mathfrak{Y}$ and $u, w > 0$.

1. $F_m(p, q, w) > 0$
2. $F_m(p, q, w) = 1 \Longleftrightarrow p = q$
3. $F_m(p, q, w) = F_m(q, p, w)$
4. $F_m(p, r, u + w) \geq F_m(p, q, u) * F_m(q, r, w)$
5. $F_m(p, q, .) : (0, \infty) \longrightarrow (0, 1]$ is continuous.

Remark 1.7.2. This is what the second condition of fuzzy metric space is equivalent to

$$F_m(p, p, w) = 1$$

$\forall$ p $\in \mathfrak{Y}$ and $w > 0$ and

$$F_m(p, q, w) < 1$$

$\forall$ p $\neq q$ and $w > 0$.

Definition 1.7.3. Think of $(\mathfrak{Y}, F_m, *)$ as a fuzzy metric space. The definition of open ball $\mathfrak{S}(p, q, w)$ with centre at $p \in \mathfrak{Y}$ and radius $0 < q < 1$ is defined as

$$\mathfrak{S}(p, q, w) = \{r \in \mathfrak{Y} : F_m(p, r, w) > 1 - q\}.$$

Definition 1.7.4. Suppose $(\mathfrak{Y}, F_m, *)$ is a fuzzy metric space. Suppose τ is a collection of all $\mathfrak{X} \subset \mathfrak{Y}$ with $p \in \mathfrak{X} \Longleftrightarrow \exists\ w > 0$ and $0 < q < 1$ in a way that $\mathfrak{S}(p, q, w) \subset \mathfrak{A}$. Consequentally τ is a topology on $\mathfrak{Y}$ (resulting from fuzzy metric F_m). This topology is 1st Countable and Hausdorff.

Definition 1.7.5. Consider a fuzzy metric space $(\mathfrak{Y}, F_m, *)$.
i- $\{p_n\} \subseteq \mathfrak{Y}$ is referred to as convergent to a point $p \in \mathfrak{Y}$ when a sequence. If

$$\lim_{n \longrightarrow \infty} F_m(p_n, p, w) = 1 \qquad \forall \text{ w } > 0$$

ii- $\{p_n\} \subseteq \mathfrak{Y}$ is referred to as a Cauchy sequence when every $w > 0$ and $\varepsilon \in (0,1)$, a positive integer $u \in N$ in a way that

$$F_m(p_n, p_m, w) > 1 - \varepsilon \quad \forall \text{ m }, n \geq u.$$

iii- A fuzzy metric space $(\mathfrak{Y}, F_m, *)$ is regarded as complete if all Cauchy sequences are convergent.

Remark 1.7.6. If the metric space $(\mathfrak{Y}, F_m, *)$ is fuzzy, then F_m is a continuous function on $\mathfrak{Y} \times \mathfrak{Y} \times (0, \infty)$.

Definition 1.7.7. Consider a fuzzy metric space $(\mathfrak{Y}, F_m, *)$, a mapping F_m on $\mathfrak{Y} \times \mathfrak{Y} \times (0, \infty)$, is regarded as continuous if

$$\lim_{n \longrightarrow \infty} F_m(p_n, w_n, q_n) = F_m(p, w, q)$$

whenever $\{(p_n, q_n, w_n)\}$ is a sequence in $\mathfrak{Y} \times \mathfrak{Y} \times (0, \infty)$ this converges about a point $(p, q, w) \in \mathfrak{Y} \times \mathfrak{Y} \times (0, \infty)$
i.e.

$$\lim_{n \longrightarrow \infty} F_m(p_n, p, w) = \lim_{n \longrightarrow \infty} F_m(q_n, q, w) = 1$$

and

$$\lim_{n \longrightarrow \infty} F_m(p, q, w_n) = F_m(p, q, w).$$

Definition 1.7.8 (5)**.** When a sequence $\{u_n\}_{n \in \mathfrak{N}}$ in metric space F_m is such that,

$$\gamma^k u_n \longrightarrow \gamma v \implies \gamma^{k-1} u_n \longrightarrow v.$$

The term "k-continuous" refers to a self mapping of that space where k = 1, 2, 3,.......... etc.

Remark 1.7.9. Although the opposite isn't always true, any continuous map is k-continuous

1.8 Fuzzy b-metric spaces

Fuzzy metric spaces are just one of several metric and metric space expansions. Fuzzy b-metric spaces, in which the triangle inequality is substituted with a weaker one, were first described by Sedghi S., Shobe N. (2012)

Definition 1.8.1. [7]
Suppose $\mathfrak{Y} \neq \phi$ and F_{bm} be a fuzzy set on $\mathfrak{Y} \times \mathfrak{Y} \times (0, \infty)$ and $*$ is a t-norm. Then fuzzy b-metric space is an ordered triplet $(\mathfrak{Y}, F_{bm}, *)$ satisafying the following conditions $\forall$ p $, q, r \in \mathfrak{Y}$, $u, w > 0$ and $b \geq 1$

1. $F_{bm}(p,q,w) > 0$
2. $F_{bm}(p,q,w) = 1 \Longleftrightarrow p = q$
3. $F_{bm}(p,q,w) = F_{bm}(q,p,w)$
4. $F_{bm}(p,r,u+w) \geq F_{bm}(p,q,\frac{u}{b}) * F_{bm}(q,r,\frac{w}{b})$
5. $F_{bm}(p,q,.) : (0,\infty) \longrightarrow (0,1]$ is continuous.

Remark 1.8.2. In general, the reverse is not true. When $b = 1$, any fuzzy b-metric space is a fuzzy metric space.

Example 1.8.3. Consider the case where $F_{bm}(p,q,u) = e^{\frac{-|p-q|^p}{u}}$ and $b > 1$ be a real number. F_{bm} is therefore a fuzzy b-metric with $p = 2^{b-1}$. However, when $b = 2$ it is not a fuzzy metric.

Chapter 2

Completeness of Archimedian type fuzzy metric spaces using Caristi type mappings

In this chapter, we review some basic results of [8]. We first construct a fixed point theorem that generalises Abbasi and Golshan's central theorem, and then we establish a theory that defines the completeness of an Archimedian type fuzzy metric space.

Definition 2.0.1. Assume that the mapping $\xi : [0,1] \longrightarrow [0,1]$ is a self mapping. Then

1. ξ is called amenable if
$$\xi^{-1}(1) = 1$$
2. ξ is called $*$- superadditive if
$$\xi(u * v) \geq \xi(u) * \xi(v) \forall \text{ u }, v \in [0,1]$$

Lemma 2.0.2. *Assume that the mapping $\xi : [0,1] \longrightarrow [0,1]$ is continuous and non-decreasing. If $*$ is Archimedean and $\xi(u) = 1$ for some $u \in (0,1)$ then $\xi(v) = 1 \;\forall$ v $\in [0,1]$*

Proof. Due to fact that $u \in (0,1)$ and $*$ is Archimedian so $\exists$ some w such that $*^n(u) < v$. Then

$$\xi(v) \geq *^w(u) = 1$$

because ξ is continuous and non-decreasing, this implies

$$\xi(v) \geq 1$$

$$\Longrightarrow \xi(v) = 1 \qquad \forall \text{ v } \in [0,1].$$

In particular

$$\xi(0) = \xi(1) = 1$$

□

2.1 Common point of self mappings in fuzzy metric spaces

In this section, first we define a Caristi-Kirk ball in fuzzy metric space $(\mathfrak{Y}, F_m, *)$ and then by using the amenability of mapping $\xi : [0,1] \longrightarrow [0,1]$ and the contractive condition

$$\xi(F_m(\beta y, \alpha y, u)) * \gamma(\alpha y) \geq \gamma(\beta y)$$

we prove that the self mappings α and β have a common point in $\mathfrak{Y}$ provided $\beta(\mathfrak{Y})$ is complete.

Definition 2.1.1. Assume that $(\mathfrak{Y}, F_m, *)$ is a fuzzy metric space, $\gamma : \mathfrak{Y} \longrightarrow [0,1]$ and $\xi : [0,1] \longrightarrow [0,1]$. The Caristi-Kirk ball is defined as follows for each $y_1 \in \mathfrak{Y}$ such that .

$$\mathfrak{C}(y_1) = \{y_2 \in \mathfrak{Y} : \xi(F_m(y_1, y_2, w) * \gamma(y_2) \geq \gamma(y_1), \quad \forall \text{ w } > 0\}$$

Theorem 2.1.2. *Consider a fuzzy metric space* $(\mathfrak{Y}, F_m, *)$ *in which* $*$ *is*

1. *Continuous*
2. *Archimedean*

and where $\alpha, \beta : \mathfrak{Y} \longrightarrow \mathfrak{Y}$ *are self mappings,* $\gamma : \mathfrak{Y} \longrightarrow [0,1]$ *is*

1. *Non-trivial on* β $(i.e. y \in \mathfrak{Y}$ *such that* $\gamma(\beta y) \neq 0)$
2. *Upper semi-continuous functions.*

Consider the following $\xi : [0,1] \longrightarrow [0,1]$ *is continuous, non-decreasing mapping that meets the conditions*

$$\xi(F_m(\beta y, \alpha y, u)) * \gamma(\alpha y) \geq \gamma(\beta y) \tag{2.1.1}$$

$\forall\, y \in \mathfrak{Y}$ *and* $u > 0$ *and that* $\xi(u * v) \geq \xi(u) * \xi(v)$ *and* $\xi^{-1}(1) = \{1\}$. *Consequentally* α *and* β *share a common point in* $\mathfrak{Y}$ *provided* $\beta(\mathfrak{Y})$ *is complete.*

Proof. Set the Caristi-Kirk balls for each $y \in \mathfrak{Y}$ so that $\gamma(y) \neq 0$,

$$\mathfrak{C}(y) = \{y' \in \mathfrak{Y} : \xi(F_m(y, y', u)) * \gamma(y') \geq \gamma(y) \quad \forall \text{ u } > 0\}$$

and

$$\delta(y) = \sup_{y' \in C(y)} \gamma(y')$$

Then $\forall$ y' $\in \mathfrak{C}(y)$

$$1 \geq \delta(y) \geq \gamma(y')$$

Clearly $\mathfrak{C}(\beta y) \neq \emptyset$ $\forall$ y , because by (2.1.1)

$$\alpha y \in \mathfrak{C}(\beta y)$$

Suppose $y_1 = y$, then

$$\beta y_1 \in \mathfrak{C}(\beta y)$$

Similarly

$$\beta y_2 \in \mathfrak{C}(\beta y_1)$$

$$\beta y_3 \in \mathfrak{C}(\beta y_2)$$

$$\vdots$$

$$\beta y_{n+1} \in \mathfrak{C}(\beta y_n)$$

and

$$\gamma(\beta y_{n+1}) \geq \delta(\beta y_n) - \frac{1}{n}, \ \ \forall \text{ u } \geq 0$$

Now as

$$\beta y_{n+1} \in \mathfrak{C}(\beta y_n)$$

$$\Longrightarrow \gamma(\beta y_{n+1}) \geq \xi(F_m(\beta y_n, \beta y_{n+1}, u)) * \gamma(\beta y_{n+1}) \geq \gamma(\beta y_n)$$

$\forall$ u > 0. So $\{\gamma(\beta y_n)\}$ is an increasing sequence and hence it converges.
Now as

$$\delta(\beta y_n) \geq \gamma(\beta y_{n+1}) \geq \delta(\beta y_n) - \frac{1}{n}$$

So

$$\lim_{n \longrightarrow \infty} \delta(\beta y_n) = \lim_{n \longrightarrow \infty} \gamma(\beta y_n)$$

exists.
Suppose

$$l = \lim_{n \longrightarrow \infty} \delta(\beta y_n) = \operatorname{Lim}_{n \longrightarrow \infty} \gamma(\beta y_n). \quad (2.1.2)$$

Next, we demonstrate the following inequality by induction.

$$\xi(F_m(\beta y_n, \beta y_m, u)) * \gamma(\beta y_m) \geq \gamma(\beta y_n) \qquad \forall \text{ u } > 0, \forall \text{ m } > n. \quad (2.1.3)$$

Assume (2.1.3) is accurate $\forall \ m > n$.
We provide proof for m+1:

$$\begin{aligned} \xi(F_m(\beta y_n, \beta y_{m+1}, u)) * \gamma(\beta y_{m+1}) &\geq \xi(F_m(\beta y_n, \beta y_m, \frac{u}{2})) * \xi(F_m(\beta y_m, \beta y_{m+1}, \frac{u}{2})) * \gamma(\beta y_{m+1}) \\ &\geq \xi(F_m(\beta y_n, \beta y_m, \frac{u}{2}) * \xi(F_m(\beta y_m, \beta y_{m+1}, \frac{u}{2})) * \gamma(\beta y_{m+1}) \\ &\geq \xi(F_m(\beta y_n, \beta y_m, \frac{u}{2})) * \gamma(\beta y_m) \\ &\geq \gamma(\beta y_n) \end{aligned}$$

$\Longrightarrow$ (2.1.3) is correct for m+1. Consequently, it holds accurate for every $m \in \mathfrak{N}$.
We shall now demonstrate that $\{\beta y_n\}$ is a Cauchy sequence. On the other hand, if $\{\beta y_n\}$ is not a Cauchy sequence, then $\exists, 0 < \epsilon < 1$ and $u > 0$ in a way that $\forall \; n \in \mathfrak{N}$, $\exists \; m \in \mathfrak{N}$

$$F_m(\beta y_n, \beta y_m, u) < 1 - \epsilon.$$

By (2.1.2) for every $0 < \epsilon' < 1 \; \exists$ N $\in \mathfrak{N}$ with

$$l \geq \gamma(\beta y_n) \geq l(1 - \epsilon') \qquad \forall \text{ n } > N.$$

From (2.1.3) and properties of ξ, we can conclude

$$\begin{aligned} l * \xi((1-\epsilon)) &\geq \xi(F_m(\beta y_n, \beta y_m, u)) * l \\ &\geq \xi(F_m(\beta y_n, \beta y_m, u)) * \gamma(\beta y_m) \\ &\geq \gamma(\beta y_n) \\ &\geq l(1-\epsilon') \end{aligned}$$

valid $\forall$ m $> n > N$
i.e.

$$l * \xi((1-\epsilon)) \geq l(1-\epsilon').$$

Hence, because to the amenability of ξ, contradicts the Archimedean condition. Hence $\{\beta y_n\}$ converges to $p = \beta(w) \in \beta(\mathfrak{Y})$.
Since γ is upper semi-continuous and by (2.1.2) we have $l = \underset{n \longrightarrow \infty}{lim} Sup(\gamma(\beta y_n)) < \gamma(\beta w)$. Taking the limit from both sides of (2.1.3) now, we get

$$\begin{aligned} \gamma(\beta y_n) &\leq \underset{m \longrightarrow \infty}{lim} Sup(\xi(F_m(\beta y_n, \beta y_m, u)) * \gamma(\beta y_m)) \\ &\leq \xi(F_m(\beta y_n, w, u)) * \gamma(\beta w) \end{aligned}$$

$\forall$ u > 0. Thus

$$\beta w \in \mathfrak{C}(\beta y_n).$$

Therefore

$$\delta(\beta y_n) > \gamma(\beta w).$$

So by (2.1.2), $k \geq \gamma(\beta w)$ and hence

$$l = \gamma(\beta w) = \gamma(p)$$

Since $\beta w \in \mathfrak{C}(\beta y_n)$ and (2.1.1) holds so $\alpha w \in \mathfrak{C}(\beta w)$. Note that

$$\begin{aligned} \xi(F_m(\beta y_n, \alpha w, u)) * \gamma(\alpha w) &\geq \xi(F_m(\beta y_n, \beta w, \frac{u}{2})) * \xi(F_m(\beta w, \alpha w, \frac{u}{2})) * \gamma(\beta w) \\ &\geq \xi(F_m(\beta y_n, \beta w, \frac{u}{2})) * \gamma(\beta y) \\ &> \gamma(\beta y_n), \qquad \forall \text{ u } > 0. \end{aligned}$$

Hence

$$\alpha w \in \mathfrak{C}(\beta y_n) \quad \forall \text{ n } \in \mathfrak{N}$$

$$\implies \gamma(\alpha w) \leq \delta_n(y_n) \quad \forall \text{ n } \in \mathfrak{N}.$$

Hence by (2.1.2) we get

$$\gamma(\alpha w) \leq l.$$

.
Since (2.1.1) holds and $\gamma(\beta w) = l$, we have that

$$\gamma(\beta w) = l \geq \gamma(\alpha w) \geq \gamma(\beta w).$$

Thus

$$\gamma(\beta w) = \gamma(\alpha w) = l.$$

Also (2.1.2) shows that

$$l * \xi(F_m(\beta w, \alpha w, u)) \geq l \quad \forall \text{ u } > 0. \tag{2.1.4}$$

It means that

$$\xi(F_m(\beta w, \alpha w, u)) = 1$$

and from previous lemma

$$F_m(\beta w, \alpha w, u) = 1 \quad \forall \text{ u } > 0$$

and hence

$$\beta w = \alpha w$$

□

Remark 2.1.3. We get a fixed point result if we take β as identity.

2.2 Fixed point of self mappings in complete fuzzy metric spaces

In this section, we prove that if fuzzy metric space $(\mathfrak{Y}, F_m, *)$ is complete then the self mapping α has a fixed point in $\mathfrak{Y}$ by using the similar contractive condition as in the above section.

Corollary 2.2.1. *Consider a fuzzy metric space $(\mathfrak{Y}, F_m, *)$ which is complete with $*$ is*

1. *Continuous*

2. *Archimedean*

$\alpha : \mathfrak{Y} \longrightarrow \mathfrak{Y}$ *be a self mapping,* $\gamma : \mathfrak{Y} \longrightarrow [0,1]$ *is*

1. *Non-trivial* $(i.e. y \in \mathfrak{Y} \implies \gamma(y) \neq 0)$.

2. *Upper semi-continuous function*

Consider $\xi : [0,1] \longrightarrow [0,1]$ *which is*

1. *Continuous*

2. *Non-decreasing mapping that meets the conditions*

$$\xi(F_m(y, \alpha y, u)) * \gamma(\alpha y) \geq \gamma(y) \tag{2.2.1}$$

$\forall$ *y* $\in \mathfrak{Y}$ *and* $u > 0$ *with* $\xi(u * v) \geq \xi(u) * \xi(v)$ *and* $\xi^{-1}(1) = \{1\}$*. As a result* $\mathfrak{Y}$ *contains the fixed point of* α*.*

Proof. Set the Caristi-Kirk ball for each $y \in \mathfrak{Y}$ so that $\gamma(y) \neq 0$,

$$\mathfrak{C}(y) = \{y' \in \mathfrak{Y} : \xi(F_m(y, y', u) * \gamma(y') \geq \gamma(y) \quad \forall \text{ u } > 0\}$$

and

$$\delta(y) = \sup_{y' \in \mathfrak{C}(y)} \gamma(y').$$

Then $\forall$ y' $\in \mathfrak{C}(y)$

$$1 \geq \delta(y) \geq \gamma(y').$$

Clearly $\mathfrak{C}(y) \neq \emptyset$ $\forall$ y, because by (2.2.1)

$$\alpha y \in \mathfrak{C}(y).$$

Suppose $y_1 = y$, then

$$y_1 \in \mathfrak{C}(y).$$

Similarly

$$y_2 \in \mathfrak{C}(y_1)$$

$$y_3 \in \mathfrak{C}(y_2)$$

$$\vdots$$

$$y_{n+1} \in \mathfrak{C}(y_n)$$

and

$$\gamma(y_{n+1}) \geq \delta(y_n) - \frac{1}{n}, \forall \text{ u } \geq 0.$$

Now as $y_{n+1} \in \mathfrak{C}(y_n)$

$$\implies \gamma(y_{n+1}) \geq \xi(F_m(y_n, y_{n+1}, u)) * \gamma(y_{n+1}) \geq \gamma(y_n), \qquad \forall \text{ u } > 0.$$

So $\{\gamma(y_n)\}$ is an increasing sequence and hence it converges.
Now as

$$\delta(y_n) \geq \gamma(y_{n+1}) \geq \delta(y_n) - \frac{1}{n}.$$

So

$$\lim_{n \longrightarrow \infty} \delta(y_n) = \lim_{n \longrightarrow \infty} \gamma(y_n)$$

exists.
Suppose

$$l = \lim_{n \longrightarrow \infty} \delta(y_n) = \underset{n \longrightarrow \infty}{Lim} \gamma(y_n). \tag{2.2.2}$$

Now the following inequality is demonstrated through induction

$$\xi(F_m(y_n, y_m, u)) * \gamma(y_m) \geq \gamma(y_n) \forall u > 0, \forall m > n. \tag{2.2.3}$$

Assume that (2.2.3) holds true if $m > n$.
We establish it for m+1:

$$\begin{aligned}
\xi(F_m(y_n, y_{m+1}, u)) * \gamma(y_{m+1}) &\geq \xi(F_m(y_n, y_m, \frac{u}{2})) * \xi(F_m(y_m, y_{m+1}, \frac{u}{2})) * \gamma(y_{m+1}) \\
&\geq \xi(F_m(y_n, y_m, \frac{u}{2})) * \xi(F_m(y_m, y_{m+1}, \frac{u}{2})) * \gamma(y_{m+1}) \\
&\geq \xi(F_m(y_n, y_m, \frac{u}{2})) * \gamma(y_m) \\
&\geq \gamma(y_n)
\end{aligned}$$

$\implies$ $(2.2.3)$ is accurate for m+1 and hence it holds for any $m \in \mathfrak{N}$.
We shall now demonstrate that $\{y_n\}$ is a Cauchy sequence. $\{y_n\}$ must be a Cauchy sequence otherwise, $\exists\, 0 < \epsilon < 1$ and $u > 0$ in a way that $\forall\ n \in \mathfrak{N}, \exists\, m \in \mathfrak{N}$

$$F_m(y_n, y_m, u) < 1 - \epsilon.$$

By (2.2.2) for every $0 < \epsilon' < 1 \ \exists\, N \in \mathfrak{N}$ with

$$l \geq \gamma(y_n) \geq l(1 - \epsilon') \qquad \forall \text{ n } > N.$$

From (2.2.3) and properties of ξ, we can conclude

$$\begin{aligned}
l * \xi((1 - \epsilon)) &\geq \xi(F_m(y_n, y_m, u)) * l \\
&\geq \xi(F_m(y_n, y_m, u)) * \gamma(y_m) \\
&\geq \gamma(y_n) \\
&\geq l(1 - \epsilon')
\end{aligned}$$

valid $\forall\, m > n > N$
i.e.

$$l * \xi((1-\epsilon)) \geq l(1-\epsilon').$$

Hence, because of the amenability of ξ, contradicts the Archimedean condition. Hence $\{y_n\}$ converges to $p = w \in \mathfrak{Y}$.
Since γ is upper semi-continuous and by (2.2.2) we have

$$l = \lim_{n \longrightarrow \infty} Sup \gamma(y_n) < \gamma(w).$$

Taking a limit now from both sides of (2.2.3) , we get

$$\begin{aligned} \gamma(y_n) &\leq \lim_{m \longrightarrow \infty} Sup(\xi(F_m(y_n, y_m, u)) * \gamma(y_m)) \\ &\leq \xi(F_m(y_n, w, u)) * \gamma(u) \end{aligned}$$

$\forall$ u > 0. Thus $w \in C(y_n)$.
Therefore

$$\delta(y_n) > \gamma(w).$$

So by (2.2.2), $k \geq \gamma(w)$ and so

$$l = \gamma(w) = \gamma(p).$$

Since $w \in C(y_n)$ and (2.2.1) holds $\alpha w \in C(w)$. Note that

$$\begin{aligned} \xi(F_m(y_n, \alpha w, u)) * \gamma(\alpha w) &\geq \xi(F_m(y_n, w, \frac{u}{2})) * \xi(F_m(w, \alpha w, \frac{u}{2})) * \gamma(w) \\ &\geq \xi(F_m(y_n, w, \frac{u}{2})) * \gamma(y) > \gamma(y_n) \qquad \forall \text{ u } > 0. \end{aligned}$$

Hence $\alpha w \in C(y_n)$ $\forall$ n $\in \mathfrak{N}$

$$\Longrightarrow \gamma(\alpha w) \leq \delta_n(y_n), \qquad \forall \text{ n } \in \mathfrak{N}.$$

Hence by (2.2.2) we get

$$\gamma(\alpha w) \leq l.$$

Since (2.2.1) holds and $\gamma(w) = l$, we possess that

$$\gamma(w) = l \geq \gamma(\alpha w) \geq \gamma(w).$$

Thus

$$\gamma(w) = \gamma(\alpha w) = l.$$

Also (2.2.2) shows that

$$l * \xi(F_m(w, \alpha w, u)) \geq l \qquad \forall \text{ u } > 0. \tag{2.2.4}$$

It means that

$$\xi(F_m(w, \alpha w, u)) = 1$$

and from lemma(2.0.1)

$$F_m(w, \alpha w, u) = 1 \qquad \forall \text{ u } > 0$$

and hence

$$w = \alpha w.$$

$\Longrightarrow$ The required fixed point of α is w. □

Example 2.2.2. Suppose $\{y_n\}$ represents strictly increasing sequence of real numbers, where $0 < y \leq 1 \forall n \in \mathfrak{N}$ with $\underset{n \longrightarrow \infty}{Lim} y_n = 1$. Suppose

$$\mathfrak{Y} = \{y_n : n \in \mathfrak{N}\} \cup \{1\}$$

and

$$p *_{\frac{1}{2}} q = \frac{2pq}{1 + p + q - pq} \qquad \forall \ p, q \in [0, 1].$$

On $\mathfrak{Y} \times \mathfrak{Y} \times (0, \infty)$ define a fuzzy set F_m by

$$F_m(1, 1, u) = 1 = F_m(y_n, y_n, u) \qquad \forall \text{ n } \in \mathfrak{N}$$

$$F_m(y_1, y_2, u) = F_m(y_2, y_1, u) = \frac{1}{7}$$

$$\begin{aligned} F_m(y_1, y_3, u) &= F_m(y_3, y_1, u) \\ &= F_m(y_2, y_3, u) \\ &= F_m(y_3, y_2, u) \\ &= F_m(y_2, y_4, u) \\ &= F_m(y_4, y_2, u) \\ &\vdots \\ &= \frac{1}{49} \end{aligned}$$

Now we prove that $(\mathfrak{Y}, F_m, *)$ be a fuzzy metric space.
i- Clearly for $y_1, y_2, \in \mathfrak{Y}$ and $u \in [0, 1]$

$$F_m(y_1, y_2, u) > 0$$

ii-

$$F_m(y, y, u) = 1$$

and

$$F_m(y_1, y_2, u) < 1$$

$\forall\ y_1, y_2 \in \mathfrak{Y}$ and $u \in [0, 1]$.
iii- Also for any $y_1, y_2 \in \mathfrak{Y}$ and $u \in [0, 1]$

$$F_m(y_1, y_2, u) = F_m(y_2, y_1, u).$$

iv- Let $y_1, y_2, y_3 \in \mathfrak{Y}$, $u, v \in [0, 1]$.
Now

$$\begin{aligned} F_m(y_1, y_2, u) * F_m(y_2, y_3, v) &= \frac{1}{7} * \frac{1}{49} \\ &= \frac{2(\frac{1}{7})(\frac{1}{49})}{1 + \frac{1}{7} + \frac{1}{49} - (\frac{1}{7})(\frac{1}{49})} \\ &= \frac{1}{199} \end{aligned}$$

and

$$F_m(y_1, y_3, u) = \frac{1}{49}.$$

Clearly

$$F_m(y_1, y_3, u) > F_m(y_1, y_2, u) * F_m(y_2, y_3, v).$$

Hence for any $y_1, y_2, y_3 \in \mathfrak{Y}$ with $u, v \in [0, 1]$

$$F_m(y_1, y_3, u) \geq F_m(y_1, y_2, u) * F_m(y_2, y_3, v).$$

v- Also $F_m(y_1, y_2, .) : (0, \infty) \longrightarrow (0, 1]$ is continuous.
Consequently, the metric space $(\mathfrak{Y}, F_m, *)$ is fuzzy . Define a mapping $\alpha : \mathfrak{Y} \longrightarrow \mathfrak{Y}$ by

$$\alpha y_n = y_{n+1} \qquad \forall \text{ n } \in \mathfrak{N},$$

$$\alpha(1) = 1.$$

Define $\gamma(r) = r$ and $\xi(w) = 1$ next.
Then, α has a single fixed point with the value $j = 1$ because it meets all the requirements of corrollary (2.2.1). α, however, does not fulfil the Abbasi criterion. Contrarily assume that α satisafy the Abbasi condition, then

$$\underset{n \longrightarrow \infty}{Lim} F_m(y_n, \alpha y_n, u) * \underset{n \longrightarrow \infty}{Lim} \gamma(\alpha y_n) \geq \underset{n \longrightarrow \infty}{Lim} Sup \gamma(y_n).$$

i.e.

$$\frac{1}{49} * \underset{n \longrightarrow \infty}{Lim} Sup \gamma(y_{n+1}) \geq \gamma(y_n)$$

or

$$\frac{1}{7} * \underset{n \longrightarrow \infty}{Lim} Sup \gamma(y_{n+1}) \geq \gamma(y_n).$$

Since γ is upper semi-continuous and hence

$$k = \underset{n \longrightarrow \infty}{Lim} \gamma(y_{n+1}) \leq \gamma(j)$$

where

$$\underset{n \longrightarrow \infty}{Lim} y_n = j.$$

The aforementioned inequality thus becomes

$$\frac{1}{49} * k \geq k$$

or

$$\frac{1}{7} * k \geq k$$

that contradicts the Archimedean requirement of $*$.

Remark 2.2.3. The following Abbasi and Golshan conclusion follows if we assume that ξ and β are identity maps in theorem (2.1.2).

Corollary 2.2.4. *Let us assume that $(\mathfrak{Y}, F_m, *)$ is complete fuzzy metric space, in which $*$ is*

1. *Archimedean*
2. *Continuous.*

A self mapping is $\alpha : \mathfrak{Y} \longrightarrow \mathfrak{Y}$. $\gamma : \mathfrak{Y} \longrightarrow [0, 1]$ is

1. *Non-trivial (i.e. $y \in \mathfrak{Y} \implies \gamma(y) \neq 0$)*
2. *Upper semi-continuous functions.*

Suppose

$$F_m(y, \alpha y, u) * \gamma(\alpha y) \geq \gamma(y) \tag{2.2.5}$$

$\forall\, y \in \mathfrak{Y}$ and $u > 0$. A fixed point for α is then found in $\mathfrak{Y}$.

Proof. Assume that $\xi : [0,1] \longrightarrow [0,1]$ is an identity map. Also ξ is amenable and $*$-supperadditive. For each $y \in \mathfrak{Y}$ such that $\gamma(y) \neq 0$, set the Caristi-Kirk ball

$$\mathfrak{C}(y) = \{y' \in \mathfrak{Y} : \xi(F_m(y, y', u)) * \gamma(y') \geq \gamma(y) \quad \forall \text{ u } > 0\}$$
$$\mathfrak{C}(y) = \{y' \in \mathfrak{Y} : F_m(y, y', u) * \gamma(y') \geq \gamma(y) \quad \forall \text{ u } > 0\}$$

and

$$\delta(y) = \sup_{y' \in \mathfrak{C}(y)} \gamma(y')$$

Then $\forall$ y' $\in \mathfrak{C}(y)$

$$1 \geq \delta(y) \geq \gamma(y')$$

Clearly $\mathfrak{C}(y) \neq \emptyset$ $\forall$ y , because by (2.2.5)

$$\alpha y \in \mathfrak{C}(y)$$

Suppose $y_1 = y$ then

$$y_1 \in \mathfrak{C}(y)$$

Similarly

$$y_2 \in \mathfrak{C}(y_1)$$

$$y_3 \in \mathfrak{C}(y_2)$$

$$\vdots$$

$$y_{n+1} \in \mathfrak{C}(y_n)$$

and

$$\gamma(y_{n+1}) \geq \delta(y_n) - \frac{1}{n}, \quad \forall \text{ u } \geq 0.$$

Now as

$$y_{n+1} \in \mathfrak{C}(y_n)$$

$$\Longrightarrow \gamma(y_{n+1}) \geq F_m(y_n, y_{n+1}, u) * \gamma(y_{n+1}) \geq \gamma(y_n) \quad \forall \text{ u } > 0.$$

So $\{\gamma(y_n)\}$ is an increasing sequence and hence it converges.
Now as

$$\delta(y_n) \geq \gamma(y_{n+1}) \geq \delta(y_n) - \frac{1}{n}.$$

So

$$\lim_{n\longrightarrow\infty}\delta(y_n) = \lim_{n\longrightarrow\infty}\gamma(y_n)$$

exists.
Suppose

$$l = \lim_{n\longrightarrow\infty}\delta(y_n) = \underset{n\longrightarrow\infty}{Lim}\gamma(y_n). \tag{2.2.6}$$

Now we establish the following inequality using induction.

$$F_m(y_n, y_m, u) * \gamma(y_m) \geq \gamma(y_n) \quad \forall \text{ u } > 0, \ \forall \text{ m } > n. \tag{2.2.7}$$

Assume (2.2.7) is accurate when $m > n$.
We demonstrate it for m+1:

$$\begin{aligned} F_m(y_n, y_{m+1}, u) * \gamma(y_{m+1}) &= F_m(y_n, y_{m+1}, \frac{u}{2} + \frac{u}{2}) * \gamma(y_{m+1}) \\ &\geq F_m(y_n, y_m, \frac{u}{2}) * F_m(y_m, y_{m+1}, \frac{u}{2}) * \gamma(y_{m+1}) \\ &\geq F_m(y_n, y_m, \frac{u}{2}) * F_m(y_m, y_{m+1}, \frac{u}{2}) * \gamma(y_{m+1}) \\ &\geq F_m(y_n, y_m, \frac{u}{2}) * \gamma(y_m) \\ &\geq \gamma(y_n) \end{aligned}$$

$\Longrightarrow$ (2.2.7) is correct for m+1 and hence it is accurate for any $m \in \mathfrak{N}$.
$\{y_n\}$ will now be demonstrated to be a Cauchy sequence. On the other hand, let's say that $\{y_n\}$ is not a Cauchy sequence. So $\forall \ n \in \mathfrak{N}, \exists$ m $\in \mathfrak{N}$, it follows that for $0 < \epsilon < 1$ and $u > 0$

$$F_m(y_n, y_m, u) < 1 - \epsilon.$$

By (2.2.6) for every $0 < \epsilon' < 1$ $\exists$ N $\in \mathfrak{N}$ in a way that

$$l \geq \gamma(y_n) \geq l(1 - \epsilon') \quad \forall \text{ n } > N.$$

From (2.2.7) and properties of ξ, we can conclude

$$\begin{aligned} l * \xi((1 - \epsilon)) &\geq \xi(F_m(y_n, y_m, u)) * l \\ &\geq F_m(y_n, y_m, u) * \gamma(y_m) \\ &\geq \gamma(y_n) \\ &\geq l(1 - \epsilon') \end{aligned}$$

valid $\forall$ m $> n > N$ i.e.

$$l * \xi((1 - \epsilon)) \geq l(1 - \epsilon').$$

It is in conflict with the Archimedean condition because it is amenable to ξ. Hence $\{y_n\}$ converges to $p = w \in \mathfrak{Y}$.
Since γ is upper semi-continuous and by (2.2.6) we have $l = \lim_{n\longrightarrow\infty} Sup\gamma(y_n) < \gamma(w)$.

Currently, we obtain simply taking the limit from both sides of (2.2.7)

$$\begin{aligned}\gamma(y_n) &\leq \lim_{m\longrightarrow\infty} Sup F_m(y_n, y_m, u) * \gamma(y_m)\\ &\leq F_m(y_n, w, u) * \gamma(u) \qquad \forall \text{ u } > 0.\end{aligned}$$

Thus $w \in C(y_n)$.
Therefore $\delta(y_n) > \gamma(w)$. So by (2.2.6), $k \geq \gamma(w)$ and so $l = \gamma(w) = \gamma(p)$.
Since $w \in C(y_n)$ and (2.2.5) holds $\alpha w \in C(w)$. Note that

$$\begin{aligned}F_m(y_n, \alpha w, u) * \gamma(\alpha w) &\geq F_m(y_n, w, \frac{u}{2}) * F_m(w, \alpha w, \frac{u}{2}) * \gamma(w)\\ &\geq F_m(y_n, w, \frac{u}{2}) * \gamma(y)\\ &> \gamma(y_n) \qquad \forall \text{ u } > 0.\end{aligned}$$

Hence $\alpha w \in C(y_n)\ \forall \text{ n } \in \mathfrak{N}$

$$\Longrightarrow \gamma(\alpha w) \leq \delta_n(y_n) \qquad \forall \text{ n } \in \mathfrak{N}.$$

Hence by (2.2.6) we get

$$\gamma(\alpha w) \leq l.$$

Since (2.2.5) holds and $\gamma(w) = l$, we possess that

$$\gamma(w) = l \geq \gamma(\alpha w) \geq \gamma(w).$$

Thus

$$\gamma(w) = \gamma(\alpha w) = l.$$

Also (2.2.6) shows that

$$l * F_m(w, \alpha w, u) \geq l \qquad \forall \text{ u } > 0. \tag{2.2.8}$$

It means that

$$F_m(w, \alpha w, u) = 1.$$

As a result $w = \alpha w$.
$\Longrightarrow$ The required fixed point of α is w. □

Remark 2.2.5. The Abbasi theorem will be generalised in the theorem that follows.

Theorem 2.2.6. *Consider a fuzzy metric space $(\mathfrak{Y}, F_m, *)$ under the operation $*$ that fullfils the criterion*
$*$ is

1. *Continuous*

2. *Archimedean*

and $\alpha : \mathfrak{Y} \longrightarrow \mathfrak{Y}$ should be a k-continuous self mapping that meets the requirement

$$F_m(y, \alpha y, u) * \gamma(\alpha y) \geq \gamma(y) \tag{2.2.9}$$

$\forall\, y \in \mathfrak{Y}$ and $u > u_0$ for some $u_0 > 0$, where $\gamma : \mathfrak{Y} \longrightarrow [0, 1]$ to the extent it is not trivial. (i.e $y \in \mathfrak{Y} \implies \gamma(y) \neq 0$). Subsequentally $\mathfrak{Y}$ contains a fixed point for α.

Proof. Suppose $\xi : [0, 1] \longrightarrow [0, 1]$ is an identity map. Also ξ is amenable and $*$- supper-additive. If $y \in \mathfrak{Y} \implies \gamma(y) \neq 0$, then set a Caristi-Kirk ball

$$\mathfrak{C}(y) = \{y' \in \mathfrak{Y} : \xi(F_m(y, y', u)) * \gamma(y') \geq \gamma(y) \quad \forall\, \mathbf{u} > 0\}$$

$$\mathfrak{C}(y) = \{y' \in \mathfrak{Y} : F_m(y, y', u) * \gamma(y') \geq \gamma(y) \quad \forall\, \mathbf{u} > 0\}$$

and

$$\delta(y) = \sup_{y' \in \mathfrak{C}(y)} \gamma(y')$$

.
Then $\forall\; y' \in \mathfrak{C}(y)$

$$1 \geq \delta(y) \geq \gamma(y').$$

Clearly $\mathfrak{C}(y) \neq \emptyset \;\; \forall$ y , because by (2.2.9) $\alpha y \in \mathfrak{C}(y)$.
Suppose $y_1 = y$ then

$$y_1 \in \mathfrak{C}(y).$$

Similarly

$$y_2 \in \mathfrak{C}(y_1)$$

$$y_3 \in \mathfrak{C}(y_2)$$

$$\vdots$$

$$y_{n+1} \in \mathfrak{C}(y_n)$$

and

$$\gamma(y_{n+1}) \geq \delta(y_n) - \frac{1}{n}, \quad \forall\, \mathbf{u} \geq 0.$$

Now as $y_{n+1} \in \mathfrak{C}(y_n)$

$$\implies \gamma(y_{n+1}) \geq F_m(y_n, y_{n+1}, u) * \gamma(y_{n+1}) \geq \gamma(y_n) \quad \forall\, \mathbf{u} > 0.$$

So $\{\gamma(y_n)\}$ is an increasing sequence and hence it converges.
Now as

$$\delta(y_n) \geq \gamma(y_{n+1}) \geq \delta(y_n) - \frac{1}{n}.$$

So

$$\lim_{n\longrightarrow\infty}\delta(y_n)=\lim_{n\longrightarrow\infty}\gamma(y_n)$$

exists.
Suppose

$$l=\lim_{n\longrightarrow\infty}\delta(y_n)=\operatorname{Lim}_{n\longrightarrow\infty}\gamma(y_n). \tag{2.2.10}$$

Now we establish the following inequality using induction

$$F_m(y_n,y_m,u)*\gamma(y_m)\geq\gamma(y_n)\qquad\forall\text{ u }>0,\forall\text{ m }>n. \tag{2.2.11}$$

Suppose (2.2.11) is accurate $\forall$ m $>n$.
We provide proof for m+1:

$$\begin{aligned}F_m(y_n,y_{m+1},u)*\gamma(y_{m+1}) &= F_m(y_n,y_{m+1},\frac{u}{2}+\frac{u}{2})*\gamma(y_{m+1})\\ &\geq F_m(y_n,y_m,\frac{u}{2})*F_m(y_m,y_{m+1},\frac{u}{2})*\gamma(y_{m+1})\\ &\geq F_m(y_n,y_m,\frac{u}{2})*\gamma(y_m)\\ &\geq \gamma(y_n).\end{aligned}$$

Since (2.2.11) holds true for m+1 it is accurate $\forall$ m $\in\mathfrak{N}$.
We shall now demonstrate that $\{y_n\}$ be a Cauchy sequence. If however, $\{y_n\}$ be not a Cauchy sequence, then $\exists\ 0<\epsilon<1$ and $u>0$, there is a $m\in\mathfrak{N}\ \forall\ n\in\mathfrak{N}$ in a way that

$$F_m(y_n,y_m,u)<1-\epsilon.$$

By (2.2.10) for every $0<\epsilon'<1\quad\exists$ N $\in\mathfrak{N}$ with

$$l\geq\gamma(y_n)\geq l(1-\epsilon')\qquad\forall\text{ n }>N.$$

From (2.2.11) and properties of ξ, we can conclude

$$\begin{aligned}l*\xi((1-\epsilon)) &\geq \xi(F_m(y_n,y_m,u))*l\\ &\geq F_m(y_n,y_m,u)*\gamma(y_m)\\ &\geq \gamma(y_n)\\ &\geq l(1-\epsilon')\end{aligned}$$

valid $\forall$ m $>n>N$
i.e.

$$l*\xi((1-\epsilon))\geq l(1-\epsilon').$$

Hence, because to the amenability of ξ contradicts the Archimedean condition. Consequently, $\{y_n\}$ must be Cauchy.

Since $\mathfrak{Y}$ is complete, so $y \in \mathfrak{Y}$ must be in a way that

$$\underset{n \longrightarrow \infty}{Lim}(y_n) = y$$

and

$$\underset{n \longrightarrow \infty}{lim}(\alpha^w y_n) = y \qquad \forall \text{ w } \geq 1$$

k-continuity of α then suggests that

$$\underset{n \longrightarrow \infty}{Lim}(\alpha^k y_n) \longrightarrow y.$$

As a result, y is the required fixed point of α. □

2.3 Invariance of fuzzy metric under certain mappings

In this section, we exhibit that if the mapping $\xi : [0,1] \longrightarrow [0,1]$ is satisfying some properties then it is fuzzy metric preserving.

Lemma 2.3.1. *Assume $\xi : [0,1] \longrightarrow [0,1]$ fullfills the criteria given below*

1. $\xi(u) \geq \xi(v)$ *whenever* $u \geq v$
2. $\xi(u * v) \geq \xi(u) * \xi(v)$
3. $\xi^{-1}(1) = 1$
4. $\xi(u) > 0 \; \forall \; u > 0.$

Subsequentally ξ is a fuzzy metric-preserving function.

Proof. Consider the fuzzy metric space $(\mathfrak{Y}, F_m, *)$. Currently we establish that $F'_m = \xi \circ F_m$ is a fuzzy metric space.
i- Suppose $y_1, y_2 \in \mathfrak{Y}$ and $u > 0$ then

$$\begin{aligned} F'_m(y_1, y_2, u) &= \xi \circ F_m(y_1, y_2, u) \\ &= \xi(F_m(y_1, y_2, u)) \quad \forall y_1, y_2 \in \mathfrak{Y} \quad and \quad u \in F_m \\ &= \xi(u') \quad where \quad u' > 0. \\ &> 0 \quad \because \xi(u) > 0 \;\; when \;\; u > 0. \\ \Longrightarrow F'_m(y_1, y_2, u) &> 0. \end{aligned}$$

ii- For $y_1, y_2 \in \mathfrak{Y}$ and $u > 0$

$$\begin{aligned} F'_m(y_1, y_2, u) &= \xi \circ F_m(y_1, y_2, u) \\ &= \xi(F_m(y_1, y_2, u)). \end{aligned}$$

Now as F_m is a fuzzy metric, so

$$\begin{aligned} F_m(y_1, y_2, u) &= 1 \iff y_1 = y_2 \\ \implies F'_m(y_1, y_2, u) &= \xi(1) \iff y_1 = y_2 \\ &= 1 \quad \forall\ y_1, y_2 \in \mathfrak{Y} \quad and \quad u > 0. \end{aligned}$$

iii- For any $y_1, y_2 \in \mathfrak{Y}$ and $u > 0$

$$\begin{aligned} F'_m(y_1, y_2, u) &= \xi \circ F_m(y_1, y_2, u) \\ &= \xi(F_m(y_1, y_2, u)) \\ &= \xi(F_m(y_2, y_1, u)) \\ &= \xi \circ F_m(y_2, y_1, u) \\ &= F'_m(y_2, y_1, u). \end{aligned}$$

iv- Let $y_1, y_2, y_3 \in \mathfrak{Y}$ with $u, v > 0$ then

$$\begin{aligned} F'_m(y_1, y_3, u+v) &= \xi \circ F_m(y_1, y_3, u+v) \\ &\geq \xi(F_m(y_1, y_2, u)) * F_m(y_2, y_3, v)) \\ &\geq \xi(F_m(y_1, y_2, u)) * \xi(F_m(y_2, y_3, v)) \\ &= F'_m(y_1, y_2, u) * F'_m(y_2, y_3, v) \\ \implies F'_m(y_1, y_3, u+v) &\geq F'_m(y_1, y_2, u) * F'_m(y_2, y_3, v). \end{aligned}$$

v- Obviously $F'_m(y_1, y_2, .) : (0, \infty) \longrightarrow (0, 1]$ is continuous because
$F_m(y_1, y_2, .) : (0, \infty) :\longrightarrow (0, 1]$ as well as ξ is continuous.
ξ' is thus a fuzzy metric space. □

Remark 2.3.2. We show a kind of Archimedean t-norm $*$ in the case below, such that $u *_m v > u * v > uv$.

Example 2.3.3. Suppose that $\mathfrak{Y} = \{1, \frac{1}{2}, \frac{2}{7}\}$,

$$u *_{\frac{1}{2}} v = \frac{\{2uv\}}{1 + u + v - uv} \qquad \forall\ \mathbf{u}\ , v \in [0, 1]$$

and
$F_m : \mathfrak{Y} \times \mathfrak{Y} \times (0, \infty) \longrightarrow [0, 1]$ defined for each $u > 0$ as

$$\begin{aligned} F_m(1, 1, u) = F_m(\frac{1}{2}, \frac{2}{7}, u) &= 1 \\ F_m(1, \frac{2}{7}, u) = F_m(\frac{2}{7}, 1, u) = F_m(\frac{1}{2}, \frac{2}{7}, u) = F_m(\frac{2}{7}, \frac{1}{2}, u) &= \frac{1}{2} \end{aligned}$$

and

$$F_m(1, \frac{1}{2}, u) = F_m(\frac{1}{2}, 1, u) = \frac{2}{7}.$$

i- Clearly

$$F_m(y_1, y_2, u) > 0 \qquad \forall\ y_1, y_2 \in \mathfrak{Y}$$

ii-

$$F_m(y_1, y_2, u) = 1 \iff y_1 = y_2.$$

iii- Clearly

$$F_m(y_1, y_2, u) = F_m(y_2, y_1, u) \qquad \forall\ y_1, y_2 \in \mathfrak{Y}\ \ and\ \ u > 0.$$

iv- Also in particular for $1, \frac{1}{2}, \frac{2}{7} \in \mathfrak{Y}$

$$\begin{aligned}
F_m(1, \frac{2}{7}, u) &= \frac{1}{2} \\
F_m(1, \frac{1}{2}, u) * F_m(\frac{1}{2}, \frac{2}{7}, u) &= \frac{2}{7} * \frac{1}{2} \\
&= \frac{2(\frac{2}{7})(\frac{1}{2})}{1 + \frac{2}{7} + \frac{1}{2} - (\frac{2}{7})(\frac{1}{2})} \\
&= \frac{\frac{2}{7}}{\frac{14+4+7-2}{14}} \\
&= \frac{\frac{2}{7}}{\frac{23}{14}} \\
&= \frac{2}{7} \times \frac{14}{23} \\
&= \frac{4}{23} \\
&< \frac{1}{2} \\
\implies F_m(1, \frac{1}{2}, u) * F_m(\frac{1}{2}, \frac{2}{7}, u) &< F_m(1, \frac{2}{7}, u).
\end{aligned}$$

Hence in general $\forall\ \ y_1, y_2, y_3 \in \mathfrak{Y}$ with $u, v \in [0, 1]$

$$F_m(y_1, y_3, u + v) \geq F_m(y_1, y_2, u) * F_m(y_2, y_3, v)$$

v- Obviously $F_m(y_1, y_2, .) : (0, \infty) \longrightarrow (0, 1]$ is continuous.
As a result $(\mathfrak{Y}, F_m, *)$ be a fuzzy metric space.
Now consider a mapping $\alpha : \mathfrak{Y} \longrightarrow \mathfrak{Y}$ by

$$\alpha(1) = 1$$

$$\alpha(\frac{1}{2}) = \frac{2}{7}$$

$$\alpha(\frac{2}{7}) = \frac{2}{7}$$

and

$$\gamma(1) = 1$$

$$\gamma(\frac{1}{2}) = \frac{4}{19}$$

$$\gamma(\frac{2}{7}) = \frac{2}{7}.$$

Then clearly α satisafies all the conditions of corrollary (2.2.4) having fixed point as a result $y_1 = 1, y_2 = \frac{2}{7}$.

Remark 2.3.4. Clearly $u *_{\frac{1}{2}} v > uv$. t-norm $*$ is an Archimedean.

2.4 Completeness of Archimedean type fuzzy metric spaces

In this section, we prove the completeness of fuzzy metric space when it is satisfying the Archimedean condition which is the main objective of this chapter.

Theorem 2.4.1. *Consider a fuzzy metric space* $(\mathfrak{Y}, F_m, *)$*, and* $*$ *satisafies the conditions.* $*$ *is*

1. *Continuous*
2. *Archimedean.*

If the condition

$$F_m(\alpha y, \alpha^2 y, u) > F_m(y, \alpha y, u) \implies F_m(\alpha y, \alpha^2 y, u)^2 \geq F_m(y, \alpha y, u) \tag{2.4.1}$$

$\forall\, y \neq \alpha y$ *and* $u > 0$
and each k-continuous self-mappings of $\mathfrak{Y}$ *that meets the requirement of theorem (2.2.6) contains a fixed point. Consequentally* $\mathfrak{Y}$ *is complete.*

Proof. Assume that each k-continuous self-mappings of $\mathfrak{Y}$ that meets the requirements of the theorem (2.2.6) contains a fixed point. We demonstrate the completeness of $\mathfrak{Y}$. On contrary let $\mathfrak{Y}$ is not complete. Following that, $\mathfrak{Y}$ has a Cauchy equence, say $G = \{v_1, v_2, v_3, \dots\}$, is up of different points that don't converge. Assume that $w \in \mathfrak{Y}$ is provided. As w is not the Cauchy sequence G's limit point, $\exists\ N(w)$ a least positive integer, like that $w \neq v_{N(w)}$. Thus we have for any case where $t > N(w)$ and $u > 0$

$$F_m(w, v_{N(w)}, u) < F_m(v_{N(w)}, w_t, u). \tag{2.4.2}$$

Let us define a mapping $\alpha : \mathfrak{Y} \longrightarrow \mathfrak{Y}$ by $\alpha(w) = v_{N(w)}$. Then $\alpha w \neq w$ for each w and using equation (2.4.1), for any $w \in \mathfrak{Y}$ and $u > 0$ we get

$$F_m(\alpha w, \alpha^2 w, u) = F_m(v_{N(w)}, v_{N(\alpha w)}, u) > F_m(v_N(w), w_t, u) = F_m(w, \alpha w, u) \quad (2.4.3)$$

then by equation (2.4.1), we have

$$F_m(\alpha w, \alpha^2 w, u)^2 \geq \mathfrak{U}(w, \alpha w, u).$$

Setting

$$\gamma(w) = F_m(w, \alpha w, u_0)^2,$$

we have

$$\begin{aligned} F_m(w, \alpha w, u_0) * \gamma(\alpha w) &= F_m(w, \alpha w, u_0) * F_m(\alpha w, \alpha^2 w, u_0)^2 \\ &\geq F_m(w, \alpha w, u_0) * F_m(w, \alpha w, u_0) \\ &= \gamma(w). \end{aligned}$$

Moreover

$$\begin{aligned} F_m(w, \alpha w, u) * \gamma(\alpha w) &\geq F_m(w, \alpha w, u_0) * \gamma(\alpha w) \\ &\geq \gamma(w) \quad \forall \text{ u } \geq u_0. \end{aligned}$$

Therefore, the mapping α satisfies the theorem's (2.2.6) contractive criterion. Moreover, the non-convergent cauchy sequence $G = \{v_n\}_{n\in\mathfrak{N}}$ contains the range of the fixed point free mapping α. Thus, the sequence $\{y_n\}_{n\in\mathfrak{N}}$ in $\mathfrak{Y}$ is not present for which the condition $\{\alpha y_n\}_{n\in\mathfrak{Y}}$ converges, i.e the sequence $\{y_n\}_{n\in\mathfrak{N}}$ in $\mathfrak{Y}$ does not exist in which the condition $\alpha y_n \longrightarrow z \quad \Longleftrightarrow \quad \alpha^2 y_n \longrightarrow \alpha z$ is not met. Algorithm α is 2-continuous as a result. We therefore have a self-mapping α of $\mathfrak{Y}$. It meets every requirement of the theorem (2.2.6), but lacks a fixed point.This is counter to the theorem's premise. As a result, $\mathfrak{Y}$ is complete. □

Remark 2.4.2. The following example serves as a proof of the theorem.

Example 2.4.3. Suppose $\mathfrak{Y} = (0, 1]$,

$$u * v = uv \quad \forall \text{ u }, v \in [0, 1]$$

and

$$F_m(y_1, y_2, u) = \frac{min\{y_1, y_2\}}{max\{y_1, y_2\}} \quad \forall\ y_1, y_2 \in \mathfrak{Y},\ \forall \text{ u } > 0.$$

Clearly the fuzzy metric space $(\mathfrak{Y}, F_m, *)$ is $\mathfrak{Y}$-complete.. Establish the mapping $\alpha : \mathfrak{Y} \longrightarrow \mathfrak{Y}$ by

$$\alpha(y) = y^{\frac{1}{2}} \quad \forall \text{ y } \in \mathfrak{Y}.$$

If γ is defined by

$$\gamma(y) = \begin{cases} y \text{ if } y \in (0, \frac{1}{4}], \\ 1 \text{ if } y \in (\frac{1}{4}, 1]. \end{cases}$$

then the fixed point of α is 1 since it fullfills all of the conditions of theorem (2.2.6). Be aware that at $\frac{1}{4}$, γ is not an upper semi-continuous mapping.

Remark 2.4.4. A further generalisation of the Abbasi theorem is the following theorem.

Theorem 2.4.5. *Assume $(\mathfrak{Y}, F_m, *)$ as a fuzzy metric space with $\alpha, \beta : \mathfrak{Y} \longrightarrow \mathfrak{Y}$ are continuous and Archimedean. Suppose $\exists$ a mapping $\gamma : \mathfrak{Y} \longrightarrow [0,1]$ such that*

1. $F_m(\alpha y, \beta y, u) * \gamma(\beta y) \geq \gamma(\alpha y) \ \forall u \ \geq 0$ *and* $y \in \mathfrak{Y}$*,*
2. $F_m(\beta y_1, \beta y_2, u)^2 > min\{F_m(\alpha y_1, \alpha y_2, u)^2, F_m(\alpha y_1, \beta y_2, u) * F_m(\alpha y_1, \beta y_2, u)\}$
   $\forall \ y_1 \neq y_2$ *and* $u \geq 0$
3. $\beta(\mathfrak{Y}) \subset \alpha(\mathfrak{Y})$
4. $\beta(\mathfrak{Y})$ *or* $\alpha(\mathfrak{Y})$ *are complete.*

As a result in $\mathfrak{Y}$, α and β shares a common point.

Proof. Assume that $y_1 = y$ and select y_n in such a way that

$$\alpha y_n = \beta y_{n-1}.$$

Take that into consideration without losing any generality.

$$\alpha y_n \neq \alpha y_n \qquad \forall \text{ n} .$$

In other case, $\alpha y_n = \beta y_n$
By (i)

$$\gamma(\alpha y_{n+1}) \geq F_m(\alpha y_n, \alpha y_{n+1}, u) * \gamma(\alpha y_{n+1}) \geq \gamma(\alpha y_n).$$

So $\{\gamma(\alpha y_n)\}$ is an increasing sequence which causes it to converge $\forall$ u ≥ 0.
We may determine that $\{\alpha y_n\}$ be a Cauchy sequence with similar justification using theorem (2.1.2).
Algorithm $\alpha(\mathfrak{Y})$ is complete, therefore $\{\alpha y_n\}$ converges to $\alpha w = p \in \mathfrak{Y}$.
Additionally, $\{\beta y_n\}$ converges to $\alpha w = p$.
We will now demonstrate that $\alpha w = \beta w$.
Imagine instead that $\alpha w \neq \beta w$, then by using (ii)

$$F_m(\beta y_n, \beta w, u)^2 > min\{F_m(\alpha y_n, \alpha w, u)^2, F_m(\alpha y_n, \beta w, u) * F_m(\beta y_n, \alpha w, u)\}$$

Letting $n \longrightarrow \infty$,

$$F_m(\alpha w, \beta w, u)^2 \geq F_m(\alpha w, \beta w, u).$$

This gives a contradiction. Hence

$$\alpha w = \beta w.$$

□

2.5 Compatible mappings and their common points

In this section, first we define compatible mappings then we will prove that the compatible mappings have a unique common fixed point if is satisfying some properties.

Definition 2.5.1. Two maps $\alpha, \beta : \mathfrak{Y} \longrightarrow \mathfrak{Y}$ if commutes at coincidence points, they are said to be weakly compatible.
i.e.

$$\beta\alpha w = \alpha\beta w,$$

$\forall$ w such that

$$\beta w = \alpha w.$$

Theorem 2.5.2. *Similar to the theorem's requirements (2.4.5).They are weakly compatible mappings if β and α share only one fixed point between them.*

Proof. Suppose $p = \beta w = \alpha w$. Since β and α are weakly compatible,

$$\alpha p = \alpha\beta w = \beta\alpha w = \beta p.$$

i.e. Another point that both β and α share is p.
Imagine that $\beta p \neq p$, then

$$\begin{aligned} F_m(\beta p, p, u)^2 &= F_m(\beta^2 w, \beta w, u)^2 \\ &> min\{F_m(\alpha\beta w, \alpha w, u)^2, F_m(\alpha\beta w, \beta w, u) * F_m(\beta^2 w, \alpha w, u)\} \\ &= min\{F_m(\alpha\beta w, \beta w, u)^2, F_m(\beta p, p, u)^2\} \\ &= F_m(\beta p, p, u)^2 \end{aligned}$$

a contradiction. Hence

$$\alpha p = \beta p = p.$$

□

Chapter 3

Fixed point results in fuzzy b-metric spaces

With regard to fuzzy metric spaces, Abbasi and Golshan [15] provided an intriguing generalisation of Caristi's [16] fixed point theorem. However, their findings do not address the characterization of the relevant fuzzy metric's completeness. J.Martinez-Mereno *et. al* [8] develops a class of Caristi type mappings with fixed points to address these problems and characterises the completeness of the appropriate fuzzy metric. For interested readers we have a little important literature related to Martinez-Mereno's study ([9], [10], [11], [12], [13], [14]). However, these results are not yet examined for extended fuzzy metric spaces. In order to validate these results for extended fuzzy metric spaces, here we use fuzzy b-metric space. Therefore, in this chapter we prove that same class of Caristi type mapping also have fixed point in fuzzy b-metric space.

3.1 Common point of self mappings in fuzzy b-metric spaces

The Caristi-Kirk ball is defined in this section first using the fuzzy b-metric space $(\mathfrak{Y}, F_{bm}, *)$, and then by utilising the amenability of mapping $\xi : [0,1] \longrightarrow [0,1]$ with the help of the contractive condition

$$\xi(F_{bm}(\beta y, \alpha y, u)) * \gamma(\alpha y) \geq \gamma(\beta y).$$

We are able to demonstrate that the self mappings α and β share a common point in $\mathfrak{Y}$, provided that $\beta(\mathfrak{Y})$ is complete.

Definition 3.1.1. Assume that $(\mathfrak{Y}, F_{bm}, *)$ is a fuzzy b-metric space, $\gamma : \mathfrak{Y} \longrightarrow [0,1]$ and $\xi : [0,1] \longrightarrow [0,1]$. The Caristi-Kirk ball is defined as follows for each $y_1 \in \mathfrak{Y}$ such that .

$$\mathfrak{C}(y_1) = \{y_2 \in \mathfrak{Y} : \xi(F_m(y_1, y_2, w) * \gamma(y_2) \geq \gamma(y_1), \quad \forall \text{ w } > 0\}$$

Theorem 3.1.2. *Consider a fuzzy b-metric space $(\mathfrak{Y}, F_{bm}, *)$ satisafying the conditions. $*$ is*

1. *Continuous*
2. *Archimedean*

$\alpha, \beta : \mathfrak{Y} \longrightarrow \mathfrak{Y}$ be a self mapping, $\gamma : \mathfrak{Y} \longrightarrow [0,1]$ in order for γ to be non-trivial on β be such that $(i.e. y \in \mathfrak{Y}$ such that $\gamma(\beta y) \neq 0)$ and upper semi-continuous functions. Suppose $\xi : [0,1] \longrightarrow [0,1]$ is

1. *Continuous*
2. *Non-decreasing*
3. *$*$-supper-additive*
4. *Amenable*

and satisfying

$$\xi(F_{bm}(\beta y, \alpha y, u)) * \gamma(\alpha y) \geq \gamma(\beta y) \tag{3.1.1}$$

$\forall\, y \in \mathfrak{Y}$ and $u > 0$. α and β share a common point in $\mathfrak{Y}$ provided $\beta(y)$ is complete.

Proof. Set the Caristi-Kirk ball for any $y \in \mathfrak{Y}$ such that $\gamma(y) \neq 0$,

$$\mathfrak{C}(y) = \{y' \in \mathfrak{Y} : \xi(F_{bm}(y, y', u)) * \gamma(y') \geq \gamma(y) \quad \forall \text{ u } > 0\}$$

and

$$\delta(y) = \sup_{y' \in \mathfrak{C}(y)} \gamma(y').$$

Then ∀ y' $\in \mathfrak{C}(y)$

$$1 \geq \delta(y) \geq \gamma(y').$$

Clearly $\mathfrak{C}(\beta y) \neq \emptyset$ ∀ y because by (3.1.1)

$$\alpha y \in \mathfrak{C}(\beta y).$$

Suppose $y_1 = y$ then

$$\beta y_1 \in \mathfrak{C}(\beta y).$$

Similarly

$$\beta y_2 \in \mathfrak{C}(\beta y_1)$$

$$\beta y_3 \in \mathfrak{C}(\beta y_2)$$

$$\vdots$$

$$\beta y_{n+1} \in \mathfrak{C}(\beta y_n)$$

and

$$\gamma(\beta y_{n+1}) \geq \delta(\beta y_n) - \frac{1}{n}, \qquad \forall \text{ u } \geq 0.$$

Now as

$$\beta y_{n+1} \in \mathfrak{C}(\beta y_n)$$

$$\implies \gamma(\beta y_{n+1}) \geq \xi(F_{bm}(\beta y_n, \beta y_{n+1}, u)) * \gamma(\beta y_{n+1}) \geq \gamma(\beta y_n)$$

$\forall$ u > 0 and $b \geq 1$.
So $\{\gamma(\beta y_n)\}$ is an increasing sequence and hence it converges.
Now as

$$\delta(\beta y_n) \geq \gamma(\beta y_{n+1}) \geq \delta(\beta y_n) - \frac{1}{n}.$$

So

$$\lim_{n \longrightarrow \infty} \delta(\beta y_n) = \lim_{n \longrightarrow \infty} \gamma(\beta y_n)$$

exists.
Suppose

$$l = \lim_{n \longrightarrow \infty} \delta(\beta y_n) = \operatorname*{Lim}_{n \longrightarrow \infty} \gamma(\beta y_n). \tag{3.1.2}$$

We now demonstrate the following inequality via induction

$$\xi(F_{bm}(\beta y_n, \beta y_m, u)) * \gamma(\beta y_m) \geq \gamma(\beta y_n) \qquad \forall \text{ u } > 0, \forall \text{ m } > n. \tag{3.1.3}$$

Assume that (3.1.3) holds true for $m > n$.
We establish it for m+1:

$$\begin{aligned}
\xi(F_{bm}(\beta y_n, \beta y_{m+1}, u)) * \gamma(\beta y_{m+1}) &= \xi(F_{bm}(\beta y_n, \beta y_{m+1}, \frac{u}{2} + \frac{u}{2})) * \gamma(\beta y_{m+1}) \\
&\geq \xi(F_{bm}(\beta y_n, \beta y_{m+1}, \frac{u}{2b}) * F_{bm}(\beta y_m, \beta y_{m+1}, \frac{u}{2b})) * \gamma(\beta y_{m+1}) \\
&\geq \xi(F_{bm}(\beta y_n, \beta y_m, \frac{u}{2b})) * \xi(F_{bm}(\beta y_m, \beta y_{m+1}, \frac{u}{2b})) * \gamma(\beta y_{m+1}) \\
&\geq \xi(F_{bm}(\beta y_n, \beta y_m, \frac{u}{2b})) * \gamma(\beta y_m) \\
&\geq \gamma(\beta y_n)
\end{aligned}$$

$\implies$ $(3.1.3)$ is correct for m+1 and hence it holds for any $m \in \mathfrak{N}$.

We shall now demonstrate that βy_n is a Cauchy sequence. However let's assume that $\{\beta y_n\}$ be not a Cauchy sequence, in which case, $\exists\ 0 < \epsilon < 1$ with $u > 0$ $\forall$ n $\in \mathfrak{N}$, $\exists$ m $\in \mathfrak{N}$ in a way that

$$F_{bm}(\beta y_n, \beta y_m, u) < 1 - \epsilon.$$

By (3.1.2) for each $0 < \epsilon' < 1$, $\exists$ N $\in \mathfrak{N}$ with

$$l \geq \gamma(\beta y_n) \geq l(1 - \epsilon') \qquad \forall \text{ n } > N.$$

From (3.1.3) and the characteristics of ξ, we can say

$$\begin{aligned} l * \xi((1-\epsilon)) &\geq \xi(F_{bm}(\beta y_n, \beta y_m, u)) * l \\ &\geq \xi(F_{bm}(\beta y_n, \beta y_m, u)) * \gamma(\beta y_m) \\ &\geq \gamma(\beta y_n) \\ &\geq l(1-\epsilon') \end{aligned}$$

valid $\forall$ m $> n > N$
i.e.

$$l * \xi((1-\epsilon)) \geq l(1-\epsilon').$$

Due to the amenability of ξ , it conflicts with the Archimedean condition. Hence $\{\beta y_n\}$ converges to $p = \beta(w) \in \beta(\mathfrak{Y})$.
Since γ is upper semi-continuous and by (3.1.2) we have

$$l = \lim_{n \longrightarrow \infty} Sup(\gamma(\beta y_n)) < \gamma(\beta w).$$

Taking the limit on each side of (3.1.3), we get

$$\begin{aligned} \gamma(\beta y_n) &\leq \lim_{m \longrightarrow \infty} Sup(\xi(F_{bm}(\beta y_n, \beta y_m, u)) * \gamma(\beta y_m)) \\ &\leq \xi(F_{bm}(\beta y_n, w, u)) * \gamma(\beta w) \qquad \forall \text{ u } > 0. \end{aligned}$$

Thus

$$\beta w \in \mathfrak{C}(\beta y_n).$$

Therefore

$$\delta(\beta y_n) > \gamma(\beta w).$$

So by (3.1.2),

$$k \geq \gamma(\beta w)$$

and so

$$l = \gamma(\beta w) = \gamma(p).$$

Since $\beta w \in \mathfrak{C}(\beta y_n)$ and (3.1.1) holds

$$\alpha w \in \mathfrak{C}(\beta w).$$

Note that

$$\begin{aligned}
\xi(F_{bm}(\beta y_n, \alpha w, u)) * \gamma(\alpha w) &= \xi(F_{bm}(\beta y_n, \alpha w, \frac{u}{2} + \frac{u}{2})) * \gamma(\alpha w) \\
&\geq \xi(F_{bm}(\beta y_n, \beta w, \frac{u}{2b}) * F_{bm}(\beta w, \alpha w, \frac{u}{2b})) * \gamma(\alpha w) \\
&\geq \xi(F_{bm}(\beta y_n, \beta w, \frac{u}{2b})) * \xi(F_{bm}(\beta w, \alpha w, \frac{u}{2b})) * \gamma(\beta w) \\
&\geq \xi(F_{bm}(\beta y_n, \beta w, \frac{u}{2b})) * \gamma(\beta y) > \gamma(\beta y_n) \quad \forall \text{ u } > 0, \text{ b } \geq 1
\end{aligned}$$

Hence

$$\alpha w \in \mathfrak{C}(\beta y_n) \quad \forall \text{ n } \in \mathfrak{N}$$

$$\implies \gamma(\alpha w) \leq \delta_n(y_n) \quad \forall \text{ n } \in \mathfrak{N}.$$

Hence by (3.1.2) we get

$$\gamma(\alpha w) \leq l.$$

Since (3.1.1) holds and $\gamma(\beta w) = l$, we have that

$$\gamma(\beta w) = l \geq \gamma(\alpha w) \geq \gamma(\beta w).$$

Thus

$$\gamma(\beta w) = \gamma(\alpha w) = l.$$

Also (3.1.2) shows that

$$l * \xi(F_{bm}(\beta w, \alpha w, u)) \geq l \quad \forall \text{ u } > 0. \tag{3.1.4}$$

It means that

$$\xi(F_{bm}(\beta w, \alpha w, u)) = 1$$

as ξ is amenable, so

$$F_{bm}(\beta w, \alpha w, u) = 1 \quad \forall \text{ u } > 0$$

and hence

$$\beta w = \alpha w.$$

□

Remark 3.1.3. The **Theorem 3.3 of Martinez-Moreno** *et. al* **2021** follows directly from the previous theorem for $b = 1$.

3.2 Fixed point of self mappings in complete fuzzy b-metric spaces

This section uses the similar contractive condition as in the part before to demonstrate that if the fuzzy b-metric space $(\mathfrak{Y}, F_{bm}, *)$ is complete, then the self mapping α has a fixed point in this space.

Corollary 3.2.1. *Assume that $(\mathfrak{Y}, F_{bm}, *)$ be a complete fuzzy b-metric space in which $*$ is*

1. *Continuous*
2. *Archimedean*

$\alpha : \mathfrak{Y} \longrightarrow \mathfrak{Y}$ be a self mapping, $\gamma : \mathfrak{Y} \longrightarrow [0,1]$ to the extent that γ is non-trivial (i.e. $y \in \mathfrak{Y}$ that way $\gamma(y) \neq 0$) together with the upper semi-continuous functions. Suppose $\xi : [0,1] \longrightarrow [0,1]$ is

1. *Continuous*
2. *Non-decreasing*

mapping satisafying

$$\xi(u * v) \geq \xi(u) * \xi(v),$$
$$\xi^{-1}(\{1\}) = \{1\}$$

and

$$\xi(F_{bm}(y, \alpha y, u)) * \gamma(\alpha y) \geq \gamma(y) \tag{3.2.1}$$

$\forall\, y \in \mathfrak{Y}$ and $u > 0$. Hence $\mathfrak{Y}$ contains a fixed point for α.

Proof. Define a Caristi-Kirk ball for $y \in \mathfrak{Y}$ in a way that $\gamma(y) \neq 0$,

$$\mathfrak{C}(y) = \{y' \in \mathfrak{Y} : \xi(F_{bm}(y, y', u)) * \gamma(y') \geq \gamma(y) \quad \forall \text{ u } > 0\}$$

and

$$\delta(y) = \sup_{y' \in \mathfrak{C}(y)} \gamma(y').$$

Then $\forall$ y' $\in \mathfrak{C}(y)$

$$1 \geq \delta(y) \geq \gamma(y').$$

Clearly $\mathfrak{C}(y) \neq \emptyset$ $\forall$ y because by (3.2.1)

$$\alpha y \in \mathfrak{C}(y).$$

Suppose $y_1 = y$ then

$$y_1 \in \mathfrak{C}(y).$$

Similarly

$$y_2 \in \mathfrak{C}(y_1)$$

$$y_3 \in \mathfrak{C}(y_2)$$

$$\vdots$$

$$y_{n+1} \in \mathfrak{C}(y_n)$$

and

$$\gamma(y_{n+1}) \geq \delta(y_n) - \frac{1}{n}, \;\; \forall \text{ u } \geq 0.$$

Now as

$$y_{n+1} \in \mathfrak{C}(y_n)$$

$$\implies \gamma(y_{n+1}) \geq \xi(F_{bm}(y_n, y_{n+1}, u)) * \gamma(y_{n+1}) \geq \gamma(y_n) \qquad \forall \text{ u } > 0.$$

So $\{\gamma(y_n)\}$ is an increasing sequence and hence it converges.
Now as

$$\delta(y_n) \geq \gamma(y_{n+1}) \geq \delta(y_n) - \frac{1}{n}.$$

So

$$\underset{n\longrightarrow\infty}{lim} \delta(y_n) = \underset{n\longrightarrow\infty}{lim} \gamma(y_n)$$

exists.
Suppose

$$l = \underset{n\longrightarrow\infty}{lim} \delta(y_n) = \underset{n\longrightarrow\infty}{Lim} \gamma(y_n). \tag{3.2.2}$$

The following inequality is now demonstrated via induction.

$$\xi(F_{bm}(y_n, y_m, u)) * \gamma(y_m) \geq \gamma(y_n) \qquad \forall \text{ u } > 0, \forall \text{ m } > n. \tag{3.2.3}$$

Assume that (3.2.3) is accurate when $m > n$.

We demonstrate it for m+1:

$$\begin{aligned}
\xi(F_{bm}(y_n, y_{m+1}, u)) * \gamma(y_{m+1}) &= \xi(F_{bm}(y_n, y_{m+1}, \frac{u}{2} + \frac{u}{2})) * \gamma(y_{m+1}) \\
&\geq \xi(F_{bm}(y_n, y_{m+1}, \frac{u}{2b})) * F_{bm}(y_m, y_{m+1}, \frac{u}{2b})) * \gamma(y_{m+1}) \\
&= \xi(F_{bm}(y_n, y_m, \frac{u}{2b})) * \xi(F_{bm}(y_m, y_{m+1}, \frac{u}{2b})) * \gamma(y_{m+1}) \\
&\geq \xi(F_{bm}(y_n, y_m, \frac{u}{2b}) * \xi(F_{bm}(y_m, y_{m+1}, \frac{u}{2b})) * \gamma(y_{m+1}) \\
&\geq \xi(F_{bm}(y_n, y_m, \frac{u}{2b})) * \gamma(y_m) \\
&\geq \gamma(y_n).
\end{aligned}$$

Since (3.2.3) holds for m+1, it holds for any $m \in \mathfrak{N}$.
To demonstrate that $\{y_n\}$ is a Cauchy sequence, read on. On the other hand, imagine that $\{y_n\}$ be not a Cauchy sequence, in which case $\exists\ 0 < \epsilon < 1$ and $u > 0$ in a way that $\forall$ n $\in \mathfrak{N}, \exists$ m $\in \mathfrak{N}$ this way

$$F_{bm}(y_n, y_m, u) < 1 - \epsilon.$$

From (3.2.2) $\forall\ 0 < \epsilon' < 1\ \exists$ N $\in \mathfrak{N}$ with

$$l \geq \gamma(y_n) \geq l(1 - \epsilon') \quad \forall \text{ n } > N.$$

Inferring from (3.2.3) and the characteristics of ξ

$$\begin{aligned}
l * \xi((1 - \epsilon)) &\geq \xi(F_{bm}(y_n, y_m, u)) * l \\
&\geq \xi(F_{bm}(y_n, y_m, u)) * \gamma(y_m) \\
&\geq \gamma(y_n) \\
&\geq l(1 - \epsilon')
\end{aligned}$$

valid $\forall$ m $> n > N$
i.e.

$$l * \xi((1 - \epsilon)) \geq l(1 - \epsilon')$$

which, because ξ is amenable, contradicts the Archimedean condition. Hence $\{y_n\}$ converges to $p = w \in \mathfrak{Y}$.
Since γ is upper semi-continuous and by (3.2.2) we have

$$l = \lim_{n \longrightarrow \infty} Sup \gamma(y_n) < \gamma(w).$$

Taking the maximum from both sides of (3.2.3). Now by taking limit from both sides of (3.2.3), in our case

$$\begin{aligned}
\gamma(y_n) &\leq \lim_{m \longrightarrow \infty} Sup(\xi(F_{bm}(y_n, y_m, u)) * \gamma(y_m)) \\
&\leq \xi(F_{bm}(y_n, w, u)) * \gamma(u) \quad \forall \text{ u } > 0.
\end{aligned}$$

Thus

$$w \in \mathfrak{C}(y_n).$$

Therefore

$$\delta(y_n) > \gamma(w).$$

So by (3.2.2),

$$k \geq \gamma(w)$$

and so

$$l = \gamma(w) = \gamma(p).$$

Since $w \in \mathfrak{C}(y_n)$ and (3.2.1) holds

$$\alpha w \in \mathfrak{C}(w).$$

Note that

$$\begin{aligned}
\xi(F_{bm}(y_n, \alpha w, u)) * \gamma(\alpha w) &= \xi(F_{bm}(y_n, \alpha w, \frac{u}{2} + \frac{u}{2})) * \gamma(\alpha w) \\
&\geq \xi(F_{bm}(y_n, w, \frac{u}{2b}) * F_{bm}(w, \alpha w, \frac{u}{2b})) * \gamma(\alpha w) \\
&\geq \xi(F_{bm}(y_n, w, \frac{u}{2b})) * \xi(F_{bm}(w, \alpha w, \frac{u}{2b})) * \gamma(w) \\
&\geq \xi(F_{bm}(y_n, w, \frac{u}{2b})) * \gamma(y) > \gamma(y_n) \quad \forall \text{ u } > 0, \text{ b } \geq 1.
\end{aligned}$$

Hence

$$\alpha w \in \mathfrak{C}(y_n) \quad \forall \text{ n } \in \mathfrak{N}$$

$$\implies \gamma(\alpha w) \leq \delta_n(y_n) \quad \forall \text{ n } \in \mathfrak{N}.$$

Hence by (3.2.2) we get

$$\gamma(\alpha w) \leq l.$$

Since (3.2.1) holds and $\gamma(w) = l$, we possess that

$$\gamma(w) = l \geq \gamma(\alpha w) \geq \gamma(w).$$

Thus

$$\gamma(w) = \gamma(\alpha w) = l.$$

Also (3.2.2) shows that

$$l * \xi(F_{bm}(w, \alpha w, u)) \geq l \quad \forall \text{ u } > 0. \tag{3.2.4}$$

It means that

$$\xi(F_{bm}(w, \alpha w, u)) = 1$$

as ξ is amenable

$$F_{bm}(w, \alpha w, u) = 1 \quad \forall \text{ u } > 0$$

and hence

$$w = \alpha w$$

$\implies$ The required fixed point of α is w. $\square$

Remark 3.2.2. The **Corollary 3.4 of Martinez-Moreno** *et. al* **2021** is a direct consequence of the above corollary for $b = 1$.

Corollary 3.2.3. *Assume that* $(\mathfrak{Y}, F_{bm}, *)$ *is fuzzy* b*-metric space which is complete under the operation* $*$*, which is*

1. *Continuous*
2. *Archimedean*

$\gamma : \mathfrak{Y} \longrightarrow [0, 1]$ *be a mapping satisafying the following*
γ *is*

1. *Non-trivial*
2. *Upper semi-continuous function*

$\alpha : \mathfrak{Y} \longrightarrow \mathfrak{Y}$ *is a self mapping.*
Suppose that

$$F_{bm}(y, \alpha y, u) * \gamma(\alpha y) \geq \gamma(y) \tag{3.2.5}$$

$\forall$ *y* $\in \mathfrak{Y}$ *and* $u > 0$*. As a result* $\mathfrak{Y}$ *carries a fixed point for* α*.*

Proof. Let $\xi : [0, 1] \longrightarrow [0, 1]$ is an identity map. Also ξ is amenable and $*$- supperadditive. If $\gamma(y) \neq 0$ for each $y \in \mathfrak{Y}$, set the Caristi-Kirk balls

$$\mathfrak{C}(y) = \{y' \in \mathfrak{Y} : \xi(F_{bm}(y, y', u)) * \gamma(y') \geq \gamma(y) \quad \forall \text{ u } > 0\}$$

$$\mathfrak{C}(y) = \{y' \in \mathfrak{Y} : F_{bm}(y, y', u) * \gamma(y') \geq \gamma(y) \quad \forall \text{ u } > 0\}$$

and

$$\delta(y) = \sup_{y' \in \mathfrak{C}(y)} \gamma(y').$$

Then $\forall$ y ' $\in \mathfrak{C}(y)$

$$1 \geq \delta(y) \geq \gamma(y').$$

Clearly $\mathfrak{C}(y) \neq \emptyset$ $\forall$ y' , because by (3.2.5)

$$\alpha y \in \mathfrak{C}(y).$$

Suppose $y_1 = y$ then

$$y_1 \in \mathfrak{C}(y).$$

Similarly

$$y_2 \in \mathfrak{C}(y_1)$$

$$y_3 \in \mathfrak{C}(y_2)$$

$$\vdots$$

$$y_{n+1} \in \mathfrak{C}(y_n)$$

and

$$\gamma(y_{n+1}) \geq \delta(y_n) - \frac{1}{n}, \forall \text{ u } \geq 0.$$

Now as $y_{n+1} \in \mathfrak{C}(y_n)$

$$\implies \gamma(y_{n+1}) \geq F_{bm}(y_n, y_{n+1}, u) * \gamma(y_{n+1}) \geq \gamma(y_n) \qquad \forall \text{ u } > 0.$$

So $\{\gamma(y_n)\}$ is an increasing sequence and hence it converges.
Now as

$$\delta(y_n) \geq \gamma(y_{n+1}) \geq \delta(y_n) - \frac{1}{n}.$$

So

$$\lim_{n \longrightarrow \infty} \delta(y_n) = \lim_{n \longrightarrow \infty} \gamma(y_n)$$

exists.
Suppose

$$l = \lim_{n \longrightarrow \infty} \delta(y_n) = \operatorname{Lim}_{n \longrightarrow \infty} \gamma(y_n). \tag{3.2.6}$$

We demonstrate the following inequality via induction

$$F_{bm}(y_n, y_m, u) * \gamma(y_m) \geq \gamma(y_n) \qquad \forall \text{ u } > 0, \text{ b } \geq 1, \text{ m } > n. \tag{3.2.7}$$

Assume (3.2.7) is accurate for $m > n$.
We demonstrate it for m+1:

$$\begin{aligned} F_{bm}(y_n, y_{m+1}, u) * \gamma(y_{m+1}) &= F_{bm}(y_n, y_{m+1}, \frac{u}{2} + \frac{u}{2}) * \gamma(y_{m+1}) \\ &\geq F_{bm}(y_n, y_m, \frac{u}{2b}) * F_{bm}(y_m, y_{m+1}, \frac{u}{2b}) * \gamma(y_{m+1}) \\ &\geq F_{bm}(y_n, y_m, \frac{u}{2b}) * F_{bm}(y_m, y_{m+1}, \frac{u}{2b}) * \gamma(y_{m+1}) \\ &\geq F_{bm}(y_n, y_m, \frac{u}{2b}) * \gamma(y_m) \\ &\geq \gamma(y_n). \end{aligned}$$

Since (3.2.7) is accurate for m+1, it is also correct for any $m \in \mathfrak{N}$.
To demonstrate that $\{y_n\}$ be a Cauchy sequence. Contrarily consider the scenario where $\{y_n\}$ is not a Cauchy sequence , so $\exists\ 0 < \epsilon < 1$ and $u > 0$ in a way that $\forall$ n $\in \mathfrak{N}, \exists$ m$\in \mathfrak{N}$ as such

$$F_{bm}(y_n, y_m, u) < 1 - \epsilon.$$

Using (3.2.6) $\forall\ 0 < \epsilon' < 1\ \exists$ N $\in \mathfrak{N}$ in a way that

$$l \geq \gamma(y_n) \geq l(1 - \epsilon') \quad \forall \text{ n } > N.$$

A conclusion can be drawn from (3.2.7) and the characteristics of ξ

$$\begin{aligned} l * \xi((1-\epsilon)) &\geq \xi(F_{bm}(y_n, y_m, u)) * l \\ &\geq F_{bm}(y_n, y_m, u) * \gamma(y_m) \\ &\geq \gamma(y_n) \\ &\geq l(1-\epsilon') \end{aligned}$$

valid $\forall$ m $> n > N$
i.e.

$$l * \xi((1-\epsilon)) \geq l(1-\epsilon').$$

This, because ξ is amenable, is in conflict with the archimedean condition. Hence $\{y_n\}$ converges to $p = w \in \mathfrak{Y}$.
Since γ is upper semi-continuous and by (3.2.6) we have $l = \lim_{n \longrightarrow \infty} Sup \gamma(y_n) < \gamma(w)$.
As a result of applying the limit on each side side of (3.2.7), we get

$$\begin{aligned} \gamma(y_n) &\leq \lim_{m \longrightarrow \infty} Sup F_{bm}(y_n, y_m, u) * \gamma(y_m) \\ &\leq F_{bm}(y_n, w, u) * \gamma(u) \quad \forall \text{ u } > 0. \end{aligned}$$

Thus

$$w \in \mathfrak{C}(y_n)$$

.
Therefore

$$\delta(y_n) > \gamma(w).$$

So by (3.2.6),

$$k \geq \gamma(w)$$

and so

$$l = \gamma(w) = \gamma(p).$$

Since $w \in \mathfrak{C}(y_n)$ and (3.2.5) holds

$$\alpha w \in \mathfrak{C}(w).$$

Note that

$$\begin{aligned}
F_{bm}(y_n, \alpha w, u) * \gamma(\alpha w) &= F_{bm}(y_n, \alpha w, \frac{u}{2} + \frac{u}{2}) * \gamma(\alpha w) \\
&\geq F_{bm}(y_n, w, \frac{u}{2b}) * F_{bm}(w, \alpha w, \frac{u}{2b}) * \gamma(w) \\
&\geq F_{bm}(y_n, w, \frac{u}{2b}) * \gamma(y) \\
&\geq \gamma(y_n) \quad \forall \text{ u } > 0.
\end{aligned}$$

Hence

$$\alpha w \in \mathfrak{C}(y_n) \quad \forall \text{ n } \in \mathfrak{N}$$

$$\implies \gamma(\alpha w) \leq \delta_n(y_n) \quad \forall \text{ n } \in \mathfrak{N}.$$

Hence by (3.2.6) we get

$$\gamma(\alpha w) \leq l.$$

Since (3.2.5) holds and $\gamma(w) = l$, we possess that

$$\gamma(w) = l \geq \gamma(\alpha w) \geq \gamma(w).$$

Thus

$$\gamma(w) = \gamma(\alpha w) = l.$$

Also (3.2.6) shows that

$$l * F_{bm}(w, \alpha w, u) \geq l \quad \forall \text{ u } > 0. \tag{3.2.8}$$

It means that

$$F_{bm}(w, \alpha w, u) = 1.$$

As a result

$$w = \alpha w$$

$\implies$ w is required fixed point of α. □

Remark 3.2.4. The **Corollary 3.6 of Martinez-Moreno** *et. al* **2021** is a direct consequence of the above corollary for $b = 1$.

Theorem 3.2.5. *Suppose $(\mathfrak{Y}, F_{bm}, *)$ be a fuzzy b-metric space where $*$ is*

1. *Continuous*
2. *Archimedean*

$\alpha : \mathfrak{Y} \longrightarrow \mathfrak{Y}$ be a k-continuous self mapping. Consider $\gamma : \mathfrak{Y} \longrightarrow [0,1]$ satisafies $(\gamma(y) \neq 0)$ and the condition

$$F_{bm}(y, \alpha y, u) * \gamma(\alpha y) \geq \gamma(y) \tag{3.2.9}$$

$\forall\, y \in \mathfrak{Y}$ and $u > u_0$ for some $u_0 > 0$. Hence $\mathfrak{Y}$ contains a fixed point for α.

Proof. Consider $\xi : [0,1] \longrightarrow [0,1]$ is an identity map. Also ξ is amenable and $*$- supper-additive. Any $y \in \mathfrak{Y} \implies \gamma(y) \neq 0$ is considered, set a Caristi-Kirk balls

$$\mathfrak{C}(y) = \{y' \in \mathfrak{Y} : \xi(F_{bm}(y, y', u)) * \gamma(y') \geq \gamma(y) \quad \forall \text{ u } > 0\}$$

$$\mathfrak{C}(y) = \{y' \in \mathfrak{Y} : F_{bm}(y, y', u) * \gamma(y') \geq \gamma(y) \quad \forall \text{ u } > 0\}$$

and

$$\delta(y) = \sup_{y' \in \mathfrak{C}(y)} \gamma(y').$$

Then $\forall$ y' $\in \mathfrak{C}(y)$

$$1 \geq \delta(y) \geq \gamma(y').$$

Clearly $\mathfrak{C}(y) \neq \emptyset$ $\forall$ y , because by (3.2.9)

$$\alpha y \in \mathfrak{C}(y).$$

Suppose $y_1 = y$ then

$$y_1 \in \mathfrak{C}(y).$$

Similarly

$$y_2 \in \mathfrak{C}(y_1)$$

$$y_3 \in \mathfrak{C}(y_2)$$

$$\vdots$$

$$y_{n+1} \in \mathfrak{C}(y_n)$$

and

$$\gamma(y_{n+1}) \geq \delta(y_n) - \frac{1}{n}, \quad \forall \text{ u } \geq 0.$$

Now as

$$y_{n+1} \in \mathfrak{C}(y_n).$$

Hence

$$\gamma(y_{n+1}) \geq F_{bm}(y_n, y_{n+1}, u) * \gamma(y_{n+1}) \geq \gamma(y_n) \quad \forall \text{ u } > 0.$$

So $\{\gamma(y_n)\}$ is an increasing sequence and hence it converges.
Now as

$$\delta(y_n) \geq \gamma(y_{n+1}) \geq \delta(y_n) - \frac{1}{n}.$$

So

$$\underset{n\longrightarrow\infty}{lim}\delta(y_n) = \underset{n\longrightarrow\infty}{lim}\gamma(y_n)$$

exists.
Suppose

$$l = \underset{n\longrightarrow\infty}{lim}\delta(y_n) = \underset{n\longrightarrow\infty}{Lim}\gamma(y_n). \tag{3.2.10}$$

Now we demonstrate the next inequality using induction

$$F_{bm}(y_n, y_m, u) * \gamma(y_m) \geq \gamma(y_n) \quad \forall \text{ u } > 0, \forall \text{ m } > n. \tag{3.2.11}$$

Assume (3.2.11) is accurate $\forall\ m > n$.
We provide proof for m+1:

$$\begin{aligned}
F_{bm}(y_n, y_{m+1}, u) * \gamma(y_{m+1}) &= F_{bm}(y_n, y_{m+1}, \frac{u}{2} + \frac{u}{2}) * \gamma(y_{m+1}) \\
&\geq F_{bm}(y_n, y_m, \frac{u}{2b}) * F_{bm}(y_m, y_{m+1}, \frac{u}{2b}) * \gamma(y_{m+1}) \\
&\geq F_{bm}(y_n, y_m, \frac{u}{2b}) * F_{bm}(y_m, y_{m+1}, \frac{u}{2b}) * \gamma(y_{m+1}) \\
&\geq F_{bm}(y_n, y_m, \frac{u}{2b}) * \gamma(y_m) \\
&\geq \gamma(y_n)
\end{aligned}$$

$\implies$ $(3.2.11)$ is correct for m+1 as a result it also correct for any $m \in \mathfrak{N}$.
In this section, $\{y_n\}$ will be demonstrated to be a Cauchy sequence. Suppose, however, that $\{y_n\}$ is not a Cauchy sequence, so $\exists$, $0 < \epsilon < 1$ and $u > 0$ in a way that $\forall$ n $\in \mathfrak{N}$, $\exists$ $m \in \mathfrak{N}$ in a way that

$$F_{bm}(y_n, y_m, u) < 1 - \epsilon.$$

From (3.2.10) $\forall\ 0 < \epsilon' < 1 \quad \exists$ N $\in \mathfrak{N}$ in a way that

$$l \geq \gamma(y_n) \geq l(1 - \epsilon') \quad \forall \text{ n } > N.$$

From (3.2.11) and the characteristics of ξ, we may say

$$\begin{aligned}
l * \xi((1 - \epsilon)) &\geq \xi(F_{bm}(y_n, y_m, u)) * l \\
&\geq F_{bm}(y_n, y_m, u) * \gamma(y_m) \\
&\geq \gamma(y_n) \\
&\geq l(1 - \epsilon')
\end{aligned}$$

valid $\forall$ m $> n > N$
i.e.

$$l * \xi((1-\epsilon)) \geq l(1-\epsilon').$$

Due to the amenability of ξ, which is in conflict with the Archimedean condition. Therefore, $\{y_n\}$ must be Cauchy.
Given that $\mathfrak{Y}$ is complete, so there should be a point $y \in \mathfrak{Y}$ in a way that

$$\underset{n \longrightarrow \infty}{Lim}(y_n) = y$$

and

$$\underset{n \longrightarrow \infty}{lim}(\alpha^w y_n) = y \quad \forall \text{ w } \geq 1$$

k-continuity of α thus suggests that

$$\underset{n \longrightarrow \infty}{Lim}(\alpha^k y_n) \longrightarrow y.$$

As a result y is the required fixed point of α. □

Remark 3.2.6. The **Theorem 3.6 of Martinez-Moreno** *et. al* **2021** is a direct consequence of the above theorem for $b = 1$.

3.3 Invariance of fuzzy b-metric under certain mappings

In this section, we demonstrate that the mapping $\xi : [0,1] \longrightarrow [0,1]$ is fuzzy b-metric preserving if certain conditions are satisfied.

Lemma 3.3.1. *Consider* $\xi : [0,1] \longrightarrow [0,1]$ *is a mapping satisafying the following* ξ *is*

1. *Continuous*
2. *Non-decreasing*
3. $*$*-superadditive*
4. *Amenable*
5. $\xi(u) > 0 \;\; \forall \;\; u > 0.$

Afterwards ξ *is fuzzy b-metric preserving.*

Proof. Suppose a fuzzy b-metric space is $(\mathfrak{Y}, F_{bm}, *)$. Now we demonstrate that F'_{bm} is a fuzzy b-metric space, defined as $\xi \circ F_{bm}$.
i- Suppose $y_1, y_2 \in \mathfrak{Y}$ and $u > 0$ then

$$\begin{aligned} F'_{bm}(y_1, y_2, u) &= \xi \circ F_{bm}(y_1, y_2, u) \\ &= \xi(F_{bm}(y_1, y_2, u)) \quad \forall y_1, y_2 \in \mathfrak{Y},\ \text{u} \in F_{bm} \\ &= \xi(u') \quad where\ \text{u'} > 0 \end{aligned}$$

because $(\mathfrak{Y}, F_{bm}, *)$ is a fuzzy b-metric space.
$\because \xi(u) > 0$ when $u > 0$.

$$\implies F'_{bm}(y_1, y_2, u) > 0.$$

ii- For $y_1, y_2 \in \mathfrak{Y}$ and $u > 0$

$$\begin{aligned} F'_{bm}(y_1, y_2, u) &= \xi \circ F_{bm}(y_1, y_2, u) \\ &= \xi(F_{bm}(y_1, y_2, u)). \end{aligned}$$

Now as F_{bm} is a fuzzy b-metric, so

$$\begin{aligned} F_{bm}(y_1, y_2, u) = 1 \iff y_1 &= y_2 \\ \implies F'_{bm}(y_1, y_2, u) &= \xi(1) \iff y_1 = y_2 \quad \forall\ y_1, y_2 \in \mathfrak{Y}, u > 0 \\ &= 1 \end{aligned}$$

$\because \xi$ is amenable.
iii- For any $y_1, y_2 \in \mathfrak{Y}$ and $u > 0$

$$\begin{aligned} F'_{bm}(y_1, y_2, u) &= \xi \circ F_{bm}(y_1, y_2, u) \\ &= \xi(F_{bm}(y_1, y_2, u)) \\ &= \xi(F_{bm}(y_2, y_1, u)) \\ &= \xi \circ F_{bm}(y_2, y_1, u) \\ &= F'_{bm}(y_2, y_1, u) \end{aligned}$$

iv- Let $y_1, y_2, y_3 \in \mathfrak{Y}$ and $u, v > 0$ then

$$\begin{aligned} F'_{bm}(y_1, y_3, u+v) &= \xi \circ F_{bm}(y_1, y_3, u+v) \\ &\geq \xi(F_{bm}(y_1, y_2, \frac{u}{b}) * F_{bm}(y_2, y_3, \frac{v}{b})) \\ &\geq \xi(F_{bm}(y_1, y_2, \frac{u}{b})) * \xi F_{bm}(y_2, y_3, \frac{v}{b})) \\ &= \xi \circ F_{bm}(y_1, y_2, \frac{u}{b}) * \xi \circ F_{bm}(y_2, y_3, \frac{v}{b}) \\ &= F'_{bm}(y_1, y_2, \frac{u}{b}) * F'_{bm}(y_2, y_3, \frac{v}{b}). \end{aligned}$$

v- Obviously
$F'_{bm}(y_1, y_2, .) : (0, \infty) \longrightarrow (0, 1]$ is continuous because $F_{bm}(y_1, y_2, .) : (0, \infty) :\longrightarrow (0, 1]$ as well as ξ is continuous.
Consequently fuzzy b-metric space is ξ' . □

Remark 3.3.2. The **Lemma 3.7 of Martinez-Moreno***et. al***2021** is a direct consequence of the above lemma for $b = 1$.