On the Regularity of Solutions of PDEs via Harmonic Analysis Methods

By

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Department of Mathematics

Quaid-I-Azam University Islamabad, Pakistan

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A THESIS SUBMITTED IN CONFORMITY WITH THE

REQUIREMENTS FOR THE DEGREE OF

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IN

MATHEMATICS

Supervised By

Dr. Amjad Hussain

Quaid-I-Azam University Islamabad, Pakistan

2023

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On the Regularity of Solutions of PDEs via Harmonic Analysis Methods

By

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Abstract

The gist of this thesis is that, by employing the pure mathematical machinery of functional analytical methods, harmonic analysis, measure theory, distribution theory, and energy methods, we analyze the regularity of generalized solutions of weakly formulated PDEs and present improved regularity criteria that will ensure the smoothness of their weak solutions. Different kinds of regularity criteria involving pressure, vorticity, velocity, logarithmic, component reduction, one-directional derivatives, etc. are proved in Besov, critical Besov, and anisotropic Lorentz spaces for various fluid dynamical systems. One of Clay's millennium problems, the smoothness of the Navier-Stokes equation, is intimately related to all the systems that we are going to analyze in this thesis and is a fundamental open problem of well-posedness and regularity that arises from the turbulent behavior of flows over a period of time. Our goal is to obtain the regularity in more general critical function spaces that will ensure the regularity of the systems in that particular time interval. In this research work, we deal with the unsteady fluid problems on the entire three-dimensional spatial domain and in the finite-time interval.

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Even though it is inspiring, pursuing unified diversity in mathematics is a mentally taxing endeavor. An exhausting task that causes you to sense the crucial "vibes" of an excellent investigation is undoubtedly priceless. One requires solitude and a clear understanding of the ultimate aims in order to write down thoughts (original or existing) about this three-year-long adventure. If one is allowed the freedom to organize whatever scattered thoughts they come upon, it will always lead to maximum productivity. My supervisor, Dr. Amjad Hussain, has actually given me flexibility and independence to pursue whatever problems I deem necessary. I heartily owe him for that matter.

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DEDICATED

TO

MY INDEFATIGABLE

FATHER

Contents

Chapter 1

Introduction to the regularity of PDEs and harmonic analysis methods

1.1 Introduction

Nearly all the physical phenomena in the universe have been modeled by well-posed partial differential equations (PDEs), which have three key characteristics. First and foremost, the PDE problem must have a solution, i.e., a solution exists; secondly, it must be unique; and thirdly, it must be regular or continuously dependent on the data given in the problem. Previously, in the nineteenth century, PDE's solution problem was investigated by finding it in the form of explicit formulas or exact solutions. This method generated substantial challenges, considering that there is a fairly small group of PDEs with exact solutions. In the earlier nineteenth century, with the introduction of Sobolev spaces, weak solutions, and other harmonic analysis tools, the mathematical theory of PDEs advanced effectively. Although the understanding of the differentiability properties of solutions has improved, the problem of the regularity of weak solutions, whose existence is only known, remains open and is considered one of the millennium problems [1]. Serrin [3] pioneered to show the regularity of such Leray-Hopf [2, 4] weak solutions in terms of velocity and proved the result for the Navier-Stokes equations (NSE). For the NSE the problem of finding unique solution for each smooth initial condition for arbitrarily large times is still open in higher dimensions. The same problem appears to be true for more general dynamical systems.

The rigorous harmonic analysis tools of decompositions, Fourier analysis methods, Besov spaces, Lorentz spaces, functional inequalities, and energy methods together with *a priori* estimates, will help us proving our mains results also the embedding properties of function spaces will be useful in establishing inequalities. This approach encouraged new problems in the mathematical theory of turbulence, the global-in-time regularity problem, the boundary value problem for steady state, the Liouville problem of steady state with finite energy, and the boundary layer description at the zero viscosity limit.

1.2 Function spaces

We recall definitions of several function spaces that helped us support our findings.

Definition 1.2.1 For a measurable subset $\hat{\mu}$ of $\mathbb{R}^n$ and $1 \leq p < \infty$, the norm of $L^p(X, \hat{\mu})$ space of a p-integrable functions is

$$\|f\|_{L^p} := \Big(\int_{\hat{\mu}} |f(s)|^p ds\Big)^{\frac{1}{p}} < \infty,$$

whereas, for $p = 2$, L^2 is called Hilbert space. The norm of its weak form L^∞ is given as

$$\|f\|_{L^\infty} := \text{ess sup } |f|.$$

Definition 1.2.2 Let $f \in L^p(\mathbb{R}^3)$ with $1 \leq p < \infty$. The space $W^{1,p}$ consists of all those functions whose first weak derivatives exists and are in $L^p(\mathbb{R}^3)$. Whereas, the norm is given as

$$\|f\|_{W^{1,p}} := \Big(\int_{\mathbb{R}^3} |f(s)|^p + \sum_{j=1}^{n} \int_{\mathbb{R}^3} |\partial_j f(s)|^p ds\Big)^{\frac{1}{p}}.$$

For $\hat{\eta} \geq 2$, the Sobolev norm is defined as

$$\|f\|^p_{W^{\hat{\eta},p}} := \sum_{|\hat{\alpha}| \leq \hat{\eta}} \|\partial^{\hat{\alpha}} f\|^p_{L^p}.$$

Throughout the thesis we often use L^2-based Sobolev spaces with norm defined as

$$\|f\|^2_{H^{\hat{\eta}}} := \sum_{0 \leq |\hat{\alpha}| \leq \hat{\eta}} \|\partial^{\hat{\alpha}} f\|^2,$$

whereas, $H^{\hat{\eta}} = W^{\hat{\eta},2}$.

Definition 1.2.3 A function $f \in BMO$, if $\|f\|_{BMO} < \infty$, where

$$\|f\|_{BMO} = \sup_B \frac{1}{m(B)} \int_B |f(x) - f_B| dx,$$

and $f_B = \dfrac{1}{m(B)} \displaystyle\int_B f(x) dx$ with B a ball over $\mathbb{R}^n$.

Now, we define the homogeneous Besov space.

Definition 1.2.4 Let $\hat{\sigma} \in \mathbb{R}$, $1 \leq \hat{l}, \hat{m} \leq \infty$, the homogeneous Besov space $\dot{B}^{\hat{\sigma}a}_{\hat{l},\hat{m}}(\mathbb{R}^3)$ is defined by the full dyadic decomposition such as

$$\dot{B}^{\hat{\sigma}}_{\hat{l},\hat{m}} = \{f \in \mathbb{Z}'(\mathbb{R}^3); \|f\|_{\dot{B}^{\hat{\sigma}}_{\hat{l},\hat{m}}} < \infty\},$$

where

$$\|f\|_{\dot{B}^{\hat{\sigma}}_{\hat{l},\hat{m}}} = \|\{2^{\hat{j}\hat{\sigma}} \|\Delta_j f\|_{L^{\hat{l}}}\}^{\infty}_{\hat{j}=-\infty}\|_{l^{\hat{m}}}.$$

The details on dyadic decomposition can be found in [33].

Definition 1.2.5 Let $\hat{l} = (\hat{l}_1, \hat{l}_2, \hat{l}_3)$ and $\hat{m} = (m_1, m_2, m_3)$ with $0 < \hat{l}_i \leq \infty$, $0 < m_i \leq \infty$. If $\hat{l}_i = \infty$ then $m_i = \infty$ for every $i = 1, 2, 3$. An anisotropic Lorentz space $L^{\hat{l}_1, m_1}(\mathbb{R}_{x_1}; L^{\hat{l}_2, m_2}(\mathbb{R}_{x_2}; L^{\hat{l}_3, m_3}((\mathbb{R}_{x_3})))$ is the set of functions defined as

$$\left\| \left\| \left\| f \right\|_{L^{\hat{l}_1, m_1}_{x_1}} \right\|_{L^{\hat{l}_2, m_2}_{x_2}} \right\|_{L^{\hat{l}_3, m_3}_{x_3}} := \Big(\int_0^\infty \Big(\int_0^\infty \Big(\int_0^\infty [\hat{t}_1^{\frac{1}{\hat{l}_1}} \hat{t}_2^{\frac{1}{\hat{l}_2}} \hat{t}_3^{\frac{1}{\hat{l}_3}} f^{*_1, *_2, *_3}(\hat{t}_1, \hat{t}_2, \hat{t}_3)]^{m_1} \frac{d\hat{t}_1}{\hat{t}_1} \Big)^{\frac{m_2}{m_1}} \frac{d\hat{t}_2}{\hat{t}_2} \Big)^{\frac{m_3}{m_2}} \frac{d\hat{t}_3}{\hat{t}_3} \Big)^{\frac{1}{m_3}} < \infty.$$

Definition 1.2.6 For complete Banach space X, and $1 \leq p < \infty$. The space of all strongly measurable functions denoted by $L^p(0, T; X)$ and has norm

$$\|f\|_{L^p(0,T;X)} := \Big(\int_0^T \|f(s)\|_X^p ds \Big)^{\frac{1}{p}} < \infty$$

is called Bochner space. For the weak form $L^\infty(0, T; X)$ the norm

$$\|f\|_{L^\infty(0,T;X)} := \text{ess sup}_{t \in [0,T]} \ \|f(s)\|_X$$

is finite.

Definition 1.2.7 For the Banach space X, let $f \in L^1(0, T; X)$ has a weak time derivative $\partial_t f \in L^1(0, T; X)$ if

$$\int_0^T f(s) \partial_t \Phi(s) ds = - \int_0^T \partial_t f(s) \Phi(s)$$

$\forall\ \phi \in C_c^\infty(0, T)$.

To prove our results we use the following inequalities given in [90–93], for $1 \leq r < \infty$ and $\hat{s} > 0$ we have that

$$\|f\|_{L^{2r}}^2 \leq C \|f\|_{L^r} \|f\|_{BMO} \tag{1.1}$$

$$\|f\|_{BMO} \leq C \big(1 + \|f\|_{\dot{B}^0_{\infty,\infty}} \log^{\frac{1}{2}} (1 + \|f\|_{H^{s-1}}) \big) \tag{1.2}$$

$$\|(\mathcal{V} \cdot \nabla) \mathcal{V}\|_{L^r} \leq C \|\mathcal{V}\|_{L^r} \|\nabla \mathcal{V}\|_{BMO} \tag{1.3}$$

$$\|\nabla \mathcal{U}\|_{L^4}^2 \leq C \|\Delta \mathcal{U}\|_{L^2} \|\nabla \mathcal{U}\|_{\dot{B}^{-1}_{\infty,\infty}}. \tag{1.4}$$

The following proposition will be effective in proving results for the last chapter:

Proposition 1.2.8 [88] Let $(\mathfrak{h}, \mathfrak{D}, \mathfrak{B})$ is weak solution of the system (7.1). Then $\forall\ t \in [0,T]$

$$\|\Pi^{\hat{s}}\mathcal{W}\|_{L^2}^2+\|\Pi^{\hat{s}}\mathcal{W}\|_{L^2}^2+\|\Pi^{\hat{s}}\mathcal{W}\|_{L^2}^2+\int_0^T \left(\|\Pi^{\hat{s}+\hat{\alpha_1}}\mathcal{U}\|_{L^2}^2+\|\Pi^{\hat{s}+\hat{\alpha_2}}\mathcal{W}\|_{L^2}^2+\|\Pi^{\hat{s}+\hat{\alpha_3}}\mathcal{V}\|_{L^2}^2\right) dt \leq C. \tag{1.5}$$

1.3 Main results

We now present a brief synopsis of the few key results reported in this thesis. All the results are proved for the 3D incompressible fluid models on the whole space $\mathbb{R}^3$ and in the smooth finite time interval subject to the various conditions.

The first regularity criteria was proved for the NSE via velocity. But the system of three equations that is the NSE is governed by the time evolution of three components of velocity $(\mathcal{U}_1, \mathcal{U}_2, \mathcal{U}_3)$ and by the scalar pressure. Therefore, the question of finding regularity conditions for these components is natural. This brought about the work on the regularity via reduced components, one component, and pressure. Similarly, the work on the more geometrically complex systems, in which the NSE is just a subpart, initiated. Our first conditional regularity result is proved for the 3D magnetic Bénard system. The result given in terms of pressure "Ψ" that guarantee the conditional smoothness in the interval $[0,T]$ is given as

$$\int_0^T \frac{\|\Psi\|^2_{\dot{B}^{-1}_{\infty,\infty}}}{\left(1+\ln\left(e+\|\Psi\|_{\dot{B}^{-1}_{\infty,\infty}}\right)\right.} dt < \infty,$$

and is followed by the gradient pressure result

$$\int_0^T \frac{\|\nabla\Psi\|^{\frac{2}{3}}_{\dot{B}^{0}_{\infty,\infty}}}{\left(1+\ln(e+\|\nabla\Psi\|_{\dot{B}^{0}_{\infty,\infty}})\right)^{\frac{3}{2}}} dt < \infty.$$

Previously, these results were also given for the NSE. The detailed analysis of these conditions and background of the problem has been done in the upcoming chapters.

As the mathematical theory of PDEs advanced, more improved and new criteria were presented. In this regard, we proved a new component reduction one directional derivative result in terms of velocity and magnetic field given as

$$\int_0^T \left\| \left\| \left\| (\partial_3 \mathcal{U}, \partial_3 \mathcal{V}) \right\|_{L^{l,\infty}_{x_1}} \right\|_{L^{m,\infty}_{x_2}} \right\|_{L^{n,\infty}_{x_3}}^{\frac{2}{1-(\frac{1}{l}+\frac{1}{m}+\frac{1}{n})}} < \infty. \tag{1.6}$$

Result (1.6) is proved for the 3D magneto-micropolar system.

Employing the techniques of the component reduction regularity. It has been further improved and demonstrated for the Navier-Stokes-Nernst-Planck system via velocity i.e.,

$$\int_0^T \left\| \left\| \left\| \partial_3 \mathcal{U} \right\|_{L^{l,\infty}_{x_1}} \right\|_{L^{m,\infty}_{x_2}} \right\|_{L^{n,\infty}_{x_3}}^{\frac{2}{1-(\frac{1}{l}+\frac{1}{m}+\frac{1}{n})}} dt < \infty.$$

The harmonic analysis tools including the dyadic decomposition and Besov spaces enabled us to discuss different kinds of regularity results for each and every component of the system. Following this line, one of the last results we presented is for the fractional magneto-micropolar system. Due to its generality, we explored it for the Beale-Kato-Majda type result provided via vorticity i.e., $\Omega = \nabla \times \mathcal{U}$. The proved conditions that controls to blow-up of weak solution until T is given as

$$\int_0^T \frac{\|\Omega\|_{\dot{B}^0_{\infty,\infty}}}{\sqrt{(1+\log(e+\|\Omega\|_{\dot{B}^0_{\infty,\infty}})}} dt < \infty.$$

1.4 Overview of the thesis

This thesis aims to presents the results on the geometric constraints of various fluid systems, including the Cauchy problems of the 3D magnetic Bénard system, the 3D Boussinesq equations, the magneto-micropolar system, the Navier-Stokes-Poisson-Nernst-Planck system, and the fractional or generalized magneto-micropolar system in the more regular function spaces.

In Chapter 2, we discuss the regularity of the weak solutions to the Cauchy problem of the 3D magnetic Bénard system. These results are given for the pressure term, i.e., for "Ψ" in the critical Besov space $\dot{B}^{-1}_{\infty,\infty}$ and for "$\nabla\Psi$" in the homogenous Besov space $\dot{B}^0_{\infty,\infty}$. Thus improving numerous previously established regularity results for this system. The contents of this Chapter have been published in [35].

In Chapter 3, we explore a new kind of result presented on the one directional derivative of velocity for the Cauchy problem of 3D Boussinesq equations in the negative index Besov spaces. The contents of this Chapter have been submitted for publication [5].

In Chapter 4, we present the blow-up criteria that deals with the component reduction improvement of the regularity for the 3D incompressible magneto-micropolar and Navier–Stokes-Poisson–Nernst–Planck systems. These results are evaluated employing energy methods in the anisotropic Lorentz spaces. The contents of this Chapter are published in [6].

In Chapter 5, we again use energy inequality to prove the finite-time regularity via of pressure. The pressure and gradient pressure results are proved in the anisotropic Lorentz spaces. The contents of this Chapter have been submitted for publication [8].

In Chapter 6, the focus is on the scale-invariant critical Besov spaces. The double-logarithmic result for various fluid systems is proved in such function spaces. The contents of this Chapter have been submitted for publication [7] and available online on researchgate with DOI: 10.13140/RG.2.2.18887.37284.

In Chapter 7, for the fractional magneto-micropolar system, the first result is demonstrated for the system's voricity $\|\Omega\|_{\dot{B}^0_{\infty,\infty}}$, and the second result is demonstrated for the system's velocity $\|\nabla\mathfrak{h}\|^2_{\dot{B}^{-1}_{\infty,\infty}}$. The contents of this Chapter have been submitted for publication [9].

1.4.1 References of Contribution

The contribution is cited in the reference list which include [5–9, 35].

1.5 General notation

For notational convenience, throughout the thesis, $\mathcal{U}$ and $\mathcal{V}$ are taken as a velocity and magnetic fields, Ψ as pressure, and $\mathcal{W}$ as a micro-rotational velocity. $\int_{\mathbb{R}^3}$ and $\int_0^T$ are spatial and time integrals. All through the thesis, "C" is a generic constant that could vary from line to line.

The $\hat{\theta}$ in chapter 2 is different from θ used in chapter 4. The $\hat{\vartheta}$ in chapter 2 is not same as ϑ in chapter 4. For non-verbose notations, systems (4.1) and (4.2) in chapter 6 are rewritten with new symbols. For fractional system in chapter 7, we use new notations to analyze system (7.1).

The terms smoothness and regularity are interchangeable. Similarly, the terms weak solutions, generalized solutions, and Leray-Hopf weak solutions mean the same.

Chapter 2

An improved regularity criterion for the 3D magnetic Bénard system in Besov spaces

2.1 Introduction

This chapter notably targets the more general (extended) function spaces by investigating the regularity of the weak solutions or turbulent solutions to the Cauchy problem of the 3D magnetic Bénard system by converting it into mathematical symmetric form, in the absence of thermal diffusion, in terms of pressure. In that regard, we successfully improved the results by obtaining sufficient integrable regularity conditions for the pressure and gradient pressure in the homogeneous Besov spaces. We analyse the following 3D magnetic Bénard system in $\mathbb{R}^3 \times \mathbb{R}_+$:

$$\begin{cases} \frac{\partial \mathcal{U}}{\partial t} + \mathcal{U} \cdot \nabla \mathcal{U} - \beta_1 \triangle \mathcal{U} + \nabla \Psi - \mathcal{V} \cdot \nabla \mathcal{V} - \hat{\theta} e_3 = 0, \\ \frac{\partial \mathcal{V}}{\partial t} + \mathcal{U} \cdot \nabla \mathcal{V} - \beta_2 \triangle \mathcal{V} - \mathcal{V} \cdot \nabla \mathcal{U} = 0, \\ \frac{\partial \hat{\theta}}{\partial t} + \mathcal{U} \cdot \nabla \hat{\theta} - \beta_3 \triangle \hat{\theta} - \mathcal{U} \cdot e_3 = 0, \\ \mathrm{div} \mathcal{U} = 0, \quad \mathrm{div} \mathcal{V} = 0, \\ (\mathcal{U}, \mathcal{V}, \hat{\theta})|_{t=0} = (\mathcal{U}_0, \mathcal{V}_0, \hat{\theta}_0), \end{cases} \tag{2.1}$$

where $\mathcal{U}(x,t)$, $\mathcal{V}(x,t)$, $\hat{\theta}(x,t)$ are the velocity field vector, magnetic field vector and scalar temperature field, respectively, while $\Psi(x,t)$ is the scalar pressure. β_1 and β_2 are the viscosity and diffusivity with β_3 as the thermal diffusion, $e_3 = (0,0,1)$ and $\hat{\theta} e_3$ reports the acting buoyancy force on the fluid motion, $\mathcal{U} \cdot e_3$ imitates the Rayleigh–Bénard convection in a heated inviscid fluid. Equation $(2.1)_4$ describes the divergence free velocity and magnetic fields with $(2.1)_5$ tells about the prescribed initial conditions $\mathcal{U}_0, \mathcal{V}_0$ and $\hat{\theta}_0$.

As described by Mulone and Rionero [10] and Nakamura [11], the 3D magnetic Bénard system models the heat convection phenomenon influenced by velocity, magnetic field and temperature. The magnetic Bénard problem has sparked interest due to the thermal instability caused by the magnetic field. Although in 2D, the well-posedness problem has been resolved but the 3D case is still an unresolved issue in the whole space $\mathbb{R}^3$. When we ignore $\hat{\theta}$ system (2.1) is simplified to MHD system. System (2.1) is reduced to Boussinesq equations if $\mathcal{V}$ is neglected and to Navier-Stokes equations (NSE) by taking $\mathcal{V} = 0$ and $\hat{\theta} = 0$. System (2.1) also studies chemotaxis model, an important biological model, which has been extensively studied by [12–14] in the bounded domains.

In 1934, Leray [2] founded the concept of weak solutions (turbulent solutions), i.e., the solutions with finite kinetic energy belongs to a class $L^\infty(0,T;L^2)\cap L^2(0,T;H^1)$, for the proper definition of weak solution and its properties see [15, 16], and the first finite time regularity criteria were given by Serrin [3] for the incompressible NSE, i.e., $\mathcal{U}$ becomes Leray-Hopf weak solution, if

$$\mathcal{U}\in L^m(0,T;L^l(\mathbb{R}^3)),\qquad \frac{2}{m}+\frac{3}{l}=1,\quad 3<l\leq\infty,\quad 1<m\leq\infty,$$

then smoothness of solution remains in the interval $(0,T]$. Later on, the regularity problem has been extensively explored by establishing various geometrically important constraints on the velocity, vorticity, pressure, strain tensor, etc.

In this chapter, our interest is to explore the regularity in pressure terms for the system (2.1) because pressure controls the solutions of the whole system (2.1) by taking the divergence by test function, we can decouple velocity, magnetic field, and temperature from pressure. Therefore, it plays a significant role in understanding fluid flows. The NSE's regularity criteria for pressure and its gradient were demonstrated by Chae and Lee [17], Berselli and Galdi [18], and Zhou [19–21], given as

$$\Psi\in L^{\frac{2}{2-l}}(0,T,L^{\frac{3}{l}})\ \text{ with }\ 0<l\leq 1,$$

and

$$\nabla\Psi\in L^{\frac{2}{3-l}}(0,T,L^{\frac{3}{l}})\ \text{ with }\ 0<l\leq 1.$$

Duan [22] has obtained similar conditions for the MHD system.

For system (2.1), the global existence problem was addressed by Ma in [23], and the blow-up and regularity problem in terms of $\mathcal{U}$ and $\nabla\mathcal{U}$ in [24] for the multiplier space. The Serrin-type criteria $\Psi^{\frac{2}{2-l}}(0.T;L^{\frac{3}{l}})$ with $0<l\leq 1$, for the pressure, was given by Liu [25] in Lebesgue space. Recently, Chen et al. [26] established numerous important regularity results for the system (2.1), without thermal diffusion, based on pressure and its gradient in various function spaces, i.e., in Lebesgue spaces

$$\Psi\in L^2(0,T;L^{\frac{3}{l}})\ \text{ with }\ 0<l\leq 1,$$

$$\nabla\Psi \in L^{\frac{9-2l}{2l}}(0,T;L^{\frac{3}{l}}) \;\; \text{with} \;\; 0 < l \leq 1.$$

In Morrey-Companato and Multiplier spaces

$$\Psi \in L^{\frac{4l}{4l-6}}(0,T;\dot{M}_{l,m}) \;\; \text{with} \;\; \frac{3}{2} < l \leq \infty,$$

$$\Psi \in L^{2}(0,T;\dot{X}^{-l}) \;\; \text{with} \;\; 0 < l \leq 1.$$

In BMO and Besov spaces

$$\nabla\Psi \in L^{2}(0,T;BMO), \tag{2.2}$$

$$\Psi \in L^{2}(0,T;\dot{B}^{-1}_{\infty,\infty}). \tag{2.3}$$

Motivated by the above discussions and results, we will present improved integrable regularity conditions for the following 3D magnetic Bénard system with zero thermal diffusion:

$$\begin{cases} \frac{\partial \mathcal{U}}{\partial t} + \mathcal{U}\cdot\nabla\mathcal{U} - \beta_1\triangle\mathcal{U} + \nabla\Psi - \mathcal{V}\cdot\nabla\mathcal{V} - \hat{\theta}e_3 = 0, \quad \text{in} \;\; \mathbb{R}^3\times\mathbb{R}_+, \\ \frac{\partial \mathcal{V}}{\partial t} + \mathcal{U}\cdot\nabla\mathcal{V} - \beta_2\triangle\mathcal{V} - \mathcal{V}\cdot\nabla\mathcal{U} = 0, \quad \text{in} \;\; \mathbb{R}^3\times\mathbb{R}_+, \\ \frac{\partial \hat{\theta}}{\partial t} + \mathcal{U}\cdot\nabla\hat{\theta} = 0, \quad \text{in} \;\; \mathbb{R}^3\times\mathbb{R}_+, \\ \text{div}\mathcal{U} = 0, \;\; \text{div}\mathcal{V} = 0, \qquad \text{in} \;\; \mathbb{R}^3\times\mathbb{R}_+, \\ (\mathcal{U},\mathcal{V},\hat{\theta})|_{t=0} = (\mathcal{U}_0,\mathcal{V}_0,\hat{\theta}_0) \quad \text{in} \;\; \mathbb{R}^3. \end{cases} \tag{2.4}$$

We will convert system (2.4) into mathematical symmetric form by putting $\mathcal{Q}^+ = \mathcal{U}+\mathcal{V}$ and $\mathcal{Q}^- = \mathcal{U}-\mathcal{V}$, as it will be useful in calculations and to apply certain inequalities such as (2.6) for the prove of our desired regularity conditions.

The very first log improvement in $\mathcal{U}$ for the 3D NSE system was given by Montgomery-Smith [27]

$$\int_0^T \frac{\|\mathcal{U}\|^l_{L^m}}{1+\ln(e+\|\mathcal{U}\|_{L^m})}dt < \infty, \quad \frac{2}{l}+\frac{3}{m} = 1, \quad 2 < l \leq \infty, \;\; \text{and} \;\; 3 < m \leq \infty. \tag{2.5}$$

Later on, such types of criteria were enhanced by (see, [28–30]) and also established for other fluid models (see [31, 32] and references therein).

Similar to the log-criterion for weak solutions, we established improved logarithmic regularity condition for the system (2.4) based on pressure and its gradient. Our results naturally generalise the result (2.5). Throughout the calculations, the non-negative parameters β_1, β_2, and β_3 are taken 1.

The well-known pressure-velocity relations by the Calderon-Zygmund are given as:

$$\begin{cases} \|\Psi\|_{L^\alpha} \leq \|\mathcal{U}\|_{L^{2\alpha}}, \\ \|\nabla\Psi\|_{L^\alpha} \leq \|\mathcal{U}\cdot\nabla\mathcal{U}\|_{L^\alpha}, \\ \|\Psi\|_{L^\alpha} \leq C\|\mathcal{Q}^+\|_{L^{2\alpha}}\|\mathcal{Q}^-\|_{L^{2\alpha}}, \\ \|\nabla\Psi\|_{L^\alpha} \leq C\|\mathcal{Q}^+\cdot\nabla\mathcal{Q}^-\|_{L^\alpha}, \\ \|\nabla\Psi\|_{L^\alpha} \leq C\|\mathcal{Q}^-\cdot\nabla\mathcal{Q}^+\|_{L^\alpha}. \end{cases} \tag{2.6}$$

2.2 Main result and proof I

This section presents result (2.7) and its proof by using well-known energy methods.

Theorem 2.2.1 Assume that $(\mathcal{U}_0, \mathcal{V}_0, \hat{\theta}_0) \in H^1(\mathbb{R}^3)$ with $\nabla\cdot\mathcal{U}_0 = 0$, $\nabla\cdot\mathcal{V}_0 = 0$ in the sense of distributions. Let $T > 0$ and $(\mathcal{U}, \mathcal{V}, \hat{\theta})$ is a weak solution of system (2.1) in the interval $(0, T]$. If pressure Ψ satisfies

$$\int_0^T \frac{\|\Psi\|^2_{\dot{B}^{-1}_{\infty,\infty}}}{\left(1+\ln\left(e+\|\Psi\|_{\dot{B}^{-1}_{\infty,\infty}}\right)\right.} dt < \infty, \tag{2.7}$$

then $(\mathcal{U}, \mathcal{V}, \hat{\theta})$ remains its smoothness on $\mathbb{R}^3 \times (0, T]$, and there are no moving singular points or blow-ups in the area under consideration, i.e, the interval (0,T].

Proof. Firstly, we will convert the system (2.4) into a symmetric form:

$$\begin{cases} \frac{\partial\mathcal{Q}^+}{\partial t} + \mathcal{Q}^-\cdot\nabla\mathcal{Q}^+ - \triangle\mathcal{Q}^+ + \nabla\Psi - \hat{\theta}e_3 = 0, \\ \frac{\partial\mathcal{Q}^-}{\partial t} + \mathcal{Q}^+\cdot\nabla\mathcal{Q}^- - \triangle\mathcal{Q}^- + \nabla\Psi - \hat{\theta}e_3 = 0, \\ \frac{\partial\hat{\theta}}{\partial t} + \frac{1}{2}(\mathcal{Q}^+ + \mathcal{Q}^-)\cdot\nabla\hat{\theta} = 0, \\ \mathrm{div}\mathcal{Q}^+ = 0, \quad \mathrm{div}\mathcal{Q}^- = 0, \\ (\mathcal{Q}^+, \mathcal{Q}^-, \hat{\theta})|_{t=0} = (\mathcal{Q}_0^+, \mathcal{Q}_0^-, \hat{\theta}_0). \end{cases} \tag{2.8}$$

Now, testing $(2.8)_1$ with $\mathcal{Q}^+|\mathcal{Q}^+|^2$, $(2.8)_2$ with $\mathcal{Q}^-|\mathcal{Q}^-|^2$ and $(2.8)_3$ with $\hat{\theta}|\hat{\theta}|^2$, integrating over $\mathbb{R}^3$, adding all the equations, we finally get an L^4-estimates for $\mathcal{Q}^+$,

$\mathcal{Q}^-$ and for $\hat{\theta}$, given as

$$\frac{1}{4}\frac{d}{dt}\Big(\|\mathcal{Q}^+\|_{L^4}^4+\|\mathcal{Q}^-\|_{L^4}^4+\|\hat{\theta}\|_{L^4}^4\Big)+\frac{1}{2}\Big(\|\nabla|\mathcal{Q}^+|^2\|_{L^2}^2+\|\nabla|\mathcal{Q}^-|^2\|_{L^2}^2\Big)+$$

$$\Big(\||\mathcal{Q}^+||\nabla\mathcal{Q}^+|\|_{L^2}^2+\||\mathcal{Q}^-||\nabla\mathcal{Q}^-|\|_{L^2}^2\Big)$$

$$=-\int_{\mathbb{R}^3}\nabla\Psi\big(\mathcal{Q}^+|\mathcal{Q}^+|^2+\mathcal{Q}^-|\mathcal{Q}^-|^2\big)dx+\int_{\mathbb{R}^3}\hat{\theta}e_3\mathcal{Q}^+|\mathcal{Q}^+|^2dx+\int_{\mathbb{R}^3}\hat{\theta}e_3\mathcal{Q}^-|\mathcal{Q}^-|^2dx$$

$$=I_1+I_2+I_3. \tag{2.9}$$

For I_2 and I_3, we derive that

$$I_2\leq C\|\hat{\theta}\|_{L^4}^4+\|\mathcal{Q}^+\|_{L^4}^4.$$

$$I_3\leq C\|\hat{\theta}\|_{L^4}^4+\|\mathcal{Q}^-\|_{L^4}^4.$$

I_1 is estimated as in (5.2) by Chen et al. [26].

Putting all the estimates in (2.9), using $\|\mathcal{Q}^+\|_{L^4}^4+\|\mathcal{Q}^-\|_{L^4}^4=\|\mathcal{U}\|_{L^4}^4+\|\mathcal{V}\|_{L^4}^4$, we get

$$\frac{1}{4}\frac{d}{dt}\Big(\|\mathcal{U}\|_{L^4}^4+\|\mathcal{V}\|_{L^4}^4+1\Big)+\frac{1}{4}\Big(\|\nabla|\mathcal{U}|^2\|_{L^2}^2+\|\nabla|\mathcal{V}|^2\|_{L^2}^2\Big)+\frac{1}{2}(\|\mathcal{U}\cdot\nabla\mathcal{U}\|_{L^2}^2$$

$$+\|\mathcal{V}\cdot\nabla\mathcal{U}\|_{L^2}^2+\|\mathcal{U}\cdot\nabla\mathcal{V}\|_{L^2}^2+\|\mathcal{V}\cdot\nabla\mathcal{V}\|_{L^2}^2)$$

$$\leq C(\|\Psi\|_{\dot{B}^{-1}_{\infty,\infty}}^2+1)\Big(\|\mathcal{U}\|_{L^4}^4+\|\mathcal{V}\|_{L^4}^4+1\Big) \tag{2.10}$$

$$\leq C\Big(1+\frac{\|\Psi\|_{\dot{B}^{-1}_{\infty,\infty}}^2}{1+\ln(e+\|\Psi\|_{\dot{B}^{-1}_{\infty,\infty}})}\Big)(1+\ln(e+\|\Psi\|_{\dot{B}^{-1}_{\infty,\infty}})\Big(\|\mathcal{U}\|_{L^4}^4+\|\mathcal{V}\|_{L^4}^4+1\Big).$$

Using inequality $(2.6)_1$, we deduce

$$\leq C\Big(1+\frac{\|\Psi\|_{\dot{B}^{-1}_{\infty,\infty}}^2}{1+\ln(e+\|\Psi\|_{\dot{B}^{-1}_{\infty,\infty}})}\Big)(1+\ln(e+\|\mathcal{U}\|_{L^6}^2)\Big(\|\mathcal{U}\|_{L^4}^4+\|\mathcal{V}\|_{L^4}^4+1\Big).$$

$$\leq C\Big(1+\frac{\|\Psi\|_{\dot{B}^{-1}_{\infty,\infty}}^2}{1+\ln(e+\|\Psi\|_{\dot{B}^{-1}_{\infty,\infty}})}\Big)(1+\ln(e+Z(t))\Big(\|\mathcal{U}\|_{L^4}^4+\|\mathcal{V}\|_{L^4}^4+1\Big).$$

$\forall\ t\in[T_*,T]$, define $Z(t):=\sup_{T_*\leq s\leq t}\|\Lambda^3\mathcal{U}\|_{L^2}^2+\|\Lambda^3\mathcal{V}\|_{L^2}^2+\|\Lambda^3\hat{\theta}\|_{L^2}^2$.

Applying Gronwall's lemma on the interval $[T_*,t]$, we have

$$\Big(\|\mathcal{U}\|_{L^4}^4+\|\mathcal{V}\|_{L^4}^4+1\Big)\leq C_0\exp\Big(C\int_{T_*}^t\Big(1+\frac{\|\Psi\|_{\dot{B}^{-1}_{\infty,\infty}}^2}{1+\ln(e+\|\Psi\|_{\dot{B}^{-1}_{\infty,\infty}})}\Big)ds(1+\ln(e+Z(t))\Big),$$

$$\text{where } C_0=\Big(\|\mathcal{U}(\cdot,T_*)\|_{L^4}^4+\|\mathcal{V}(\cdot,T_*)\|_{L^4}^4+1\Big).$$

$$\Big(\|\mathcal{U}\|_{L^4}^4+\|\mathcal{V}\|_{L^4}^4+1\Big)\le C_0\exp(2C\epsilon\ln(e+Z(t)))\le C_0(e+Z(t))^{2C\epsilon}. \tag{2.11}$$

If there were a sufficiently small constant $\epsilon>0$, $\exists\, T_*<T$, such that

$$\int_{T_*}^{T}\Big(1+\frac{\|\Psi\|_{\dot{B}^{-1}_{\infty,\infty}}^2}{1+\ln(e+\|\Psi\|_{\dot{B}^{-1}_{\infty,\infty}})}\Big)dt<\epsilon.$$

Now, we get bounds for $Z(t)$.

Multiply $\Lambda^3=(-\Delta)^{\frac{3}{2}}$ with $(2.8)_1$ and taking the inner product with $\Lambda^3\mathcal{Q}^+$, Multiply Λ^3 with $(2.8)_2$ and taking the inner product with $\Lambda^3\mathcal{Q}^-$, Multiply Λ^3 with $(2.8)_3$ and taking the inner product with $\Lambda^3\hat{\theta}$, and using $(2.4)_4$, adding all the equations. We finally obtain

$$\frac{1}{2}\frac{d}{dt}\big(\|\Lambda^3\mathcal{Q}^+\|_{L^2}^2+\|\Lambda^3\mathcal{Q}^-\|_{L^2}^2+\|\Lambda^3\hat{\theta}\|_{L^2}^2\big)+\|\Lambda^4\mathcal{Q}^+\|_{L^2}^2+\|\Lambda^4\mathcal{Q}^-\|_{L^2}^2$$

$$\begin{aligned}
&=-\int_{\mathbb{R}^3}(\Lambda^3(\mathcal{Q}^-\cdot\nabla\mathcal{Q}^+)\Lambda^3\mathcal{Q}^+))dx-\int_{\mathbb{R}^3}(\Lambda^3(\mathcal{Q}^+\cdot\nabla\mathcal{Q}^-)\Lambda^3\mathcal{Q}^-))dx+\int_{\mathbb{R}^3}\Lambda^3(\hat{\theta}e_3)\Lambda^3\mathcal{Q}^+dx\\
&\quad+\int_{\mathbb{R}^3}\Lambda^3(\hat{\theta}e_3)\Lambda^3\mathcal{Q}^-dx-\int_{\mathbb{R}^3}\Lambda^3\big((\mathcal{Q}^++\mathcal{Q}^-)\cdot\nabla\hat{\theta}\big)\Lambda^3\hat{\theta}dx.\\
&=P_1+P_2+P_3+P_4+P_5,
\end{aligned} \tag{2.12}$$

where we used integration by parts, $\Lambda^s=(-\Delta)^{\frac{s}{2}}$ for $s\in\mathbb{R}$, and property of differentiating distributions. Now, we get estimate for P_3+P_4

$$P_3+P_4=\int_{\mathbb{R}^3}\Lambda^3(\hat{\theta}e_3)\Lambda^3\mathcal{U}dx$$

$$\le C\big(\|\Lambda^3\hat{\theta}\|_{L^2}^2+\|\Lambda^3\mathcal{U}\|_{L^2}^2\big)\le C\big(e+\|\Lambda^3\hat{\theta}\|_{L^2}^2+\|\Lambda^3\mathcal{U}\|_{L^2}^2++\|\Lambda^3\mathcal{V}\|_{L^2}^2\big)\le C_1(e+Z(t))^2,$$

where C_1 is a positive constant.

Similarly,

$$\begin{aligned}
|P_5|&=\int_{\mathbb{R}^3}\Lambda^3(\mathcal{U}\cdot\nabla\hat{\theta})\Lambda^3\hat{\theta}dx\\
&\le C\big(\|\Lambda^4\mathcal{U}\|_{L^2}^2+\|\Lambda^4\hat{\theta}\|_{L^2}^2\big)+C_1(e+Z(t))^{\frac{3}{2}+\frac{13}{2}C\epsilon},
\end{aligned}$$

here we use $\mathcal{Q}^++\mathcal{Q}^-=\mathcal{U}$.

For P_1 and P_2, Due to Kato and Ponce [34], we shall utilize the commutator estimate that follows:

$$\|\nabla^\alpha(fg)-f\nabla^\alpha g\|_{L^l}\le C\big(\|\Lambda^{\alpha-1}g\|_{L^{m_1}}\|\nabla f\|_{L^{l_1}}+\|\Lambda^\alpha f\|_{L^{l_2}}\|g\|_{L^{m_2}}\big), \tag{2.13}$$

for $\alpha > 1$ and $\frac{1}{l} = \frac{1}{l_1} + \frac{1}{m_1} = \frac{1}{l_2} + \frac{1}{m_2}$.

$$|P_1 + P_2| \leq \Big| \int_{\mathbb{R}^3} (\Lambda^3(\mathcal{Q}^- \cdot \nabla \mathcal{Q}^+) - \mathcal{Q}^- \cdot \nabla \Lambda^3 \mathcal{Q}^+)) \Lambda^3 \mathcal{Q}^+ dx$$

$$+ \int_{\mathbb{R}^3} (\Lambda^3(\mathcal{Q}^+ \cdot \nabla \mathcal{Q}^-) - \mathcal{Q}^+ \cdot \nabla \Lambda^3 \mathcal{Q}^-)) \Lambda^3 \mathcal{Q}^- dx \Big|.$$

Using (2.13) with these inequalities

$$\|\nabla \mathcal{U}\|_{L^3} \leq C \|\nabla \mathcal{U}\|_{L^2}^{\frac{3}{4}} \|\nabla \Delta \mathcal{U}\|_{L^2}^{\frac{1}{4}}, \quad \|\nabla \Delta \mathcal{U}\|_{L^3} \leq C \|\nabla \mathcal{U}\|_{L^2}^{\frac{1}{6}} \|\Delta^2 \mathcal{U}\|_{L^2}^{\frac{5}{6}},$$

we deduce the final estimate that is given as

$$\begin{aligned}
|P_1 + P_2| &\leq C\big(\|\nabla \mathcal{Q}^-\|_{L^3} \|\Lambda^3 \mathcal{Q}^+\|_{L^3}^2 + \|\nabla \mathcal{Q}^+\|_{L^3} \|\Lambda^3 \mathcal{Q}^+\|_{L^3} \|\Lambda^3 \mathcal{Q}^-\|_{L^3}\big) \\
&\quad + C\big(\|\nabla \mathcal{Q}^+\|_{L^3} \|\Lambda^3 \mathcal{Q}^-\|_{L^3}^2 + \|\nabla \mathcal{Q}^-\|_{L^3} \|\Lambda^3 \mathcal{Q}^+\|_{L^3} \|\Lambda^3 \mathcal{Q}^-\|_{L^3}\big) \\
&\leq C\big(\|\nabla \mathcal{Q}^+\|_{L^2}^{\frac{13}{2}} + \|\nabla \mathcal{Q}^+\|_{L^2}^2 \|\nabla \mathcal{Q}^-\|_{L^2}^{\frac{9}{2}} + \|\nabla \mathcal{Q}^+\|_{L^2}^{\frac{9}{2}} \|\nabla \mathcal{Q}^-\|_{L^2}^2 + \|\nabla \mathcal{Q}^-\|_{L^2}^{\frac{13}{2}}\big) \\
&\quad \cdot \big(\|\Lambda^3 \mathcal{Q}^-\|_{L^2}^{\frac{3}{2}} + \|\Lambda^3 \mathcal{Q}^+\|_{L^2}^{\frac{3}{2}}\big) + \frac{1}{2}\big(\|\Lambda^3 \nabla \mathcal{Q}^-\|_{L^2}^2 + \|\Lambda^3 \nabla \mathcal{Q}^+\|_{L^2}^2\big) \\
&\leq \frac{1}{2}\big(\|\Lambda^4 \mathcal{Q}^+\|_{L^2}^2 + \|\Lambda^4 \mathcal{Q}^-\|_{L^2}^2\big) + C\big(\|\nabla \mathcal{Q}^+\|_{L^2}^2 + \|\nabla \mathcal{Q}^-\|_{L^2}^2\big)^{\frac{13}{4}} Z^{\frac{3}{2}}(t).
\end{aligned}$$

Now, testing $(2.8)_1$ with $-\Delta \mathcal{Q}^+$ and $(2.8)_2$ with $-\Delta \mathcal{Q}^-$, the weak form is derived as

$$\frac{1}{2} \frac{d}{dt}\big(\|\nabla \mathcal{Q}^+\|_{L^2}^2 + \|\nabla \mathcal{Q}^-\|_{L^2}^2\big) + \|\Delta \mathcal{Q}^+\|_{L^2}^2 + \|\Delta \mathcal{Q}^-\|_{L^2}^2$$

$$= -\int_{\mathbb{R}^3} (\mathcal{Q}^- \cdot \nabla \mathcal{Q}^+) \cdot \Delta \mathcal{Q}^+ dx + \int_{\mathbb{R}^3} \hat{\theta} e_3 \cdot \Delta \mathcal{Q}^+ dx - \int_{\mathbb{R}^3} (\mathcal{Q}^+ \cdot \nabla \mathcal{Q}^-) \cdot \Delta \mathcal{Q}^- dx + \int_{\mathbb{R}^3} \hat{\theta} e_3 \cdot \Delta \mathcal{Q}^- dx.$$

$$\leq \|\Delta \mathcal{Q}^+\|_{L^2}^2 + \|\Delta \mathcal{Q}^-\|_{L^2}^2 + \frac{1}{2}\big(\|\Delta \mathcal{Q}^+\|_{L^2}^2 + \|\Delta \mathcal{Q}^-\|_{L^2}^2\big) + C\big(\|\mathcal{Q}^+\|_{L^6}^8 + \|\mathcal{Q}^-\|_{L^6}^8\big), \quad (2.14)$$

where we employed the following maximum principle frequently used and presented in [26] for system (2.8)

$$\|\hat{\theta}\|_{L^l} \leq \|\hat{\theta}_0\|_{L^l} \leq 1, \quad \text{where} \quad 1 < l \leq \infty. \quad (2.15)$$

Integrating (2.14) in $[T_*, t]$, we deduce that

$$\big(\|\nabla \mathcal{Q}^+\|_{L^2}^2 + \|\nabla \mathcal{Q}^-\|_{L^2}^2\big) \leq C(1 + Z(t))^{\frac{4C\epsilon}{3}} (t - T_*) + \|\nabla \mathcal{Q}^+(T_*)\|_{L^2}^2 + \|\nabla \mathcal{Q}^-(T_*)\|_{L^2}^2. \quad (2.16)$$

Putting all the estimates into (2.12), absorbing dissipative terms together with (2.16) we have final H^3-bounds by applying Gronwall's inequality providing that ϵ must be sufficiently small. We get

$$\|\Lambda^3\mathcal{Q}^+\|_{L^2}^2 + \|\Lambda^3\mathcal{Q}^-\|_{L^2}^2 + \|\Lambda^3\hat{\theta}\|_{L^2}^2 \leq C. \tag{2.17}$$

Bounds (2.17) and (2.16) together with (2.11) implies that

$$\Big(\|\mathcal{U}\|_{L^4}^4 + \|\mathcal{V}\|_{L^4}^4 + 1\Big) \leq C.$$

Thus, by providing sufficient estimates that ensure smoothness up to time T, Theorem 2.2.1 has been proved.

Corollary 2.2.2 One of the foremost outcomes of above theorem is the result (2.3).

2.3 Main result and proof II

Theorem 2.3.1 Suppose that $(\mathcal{U}_0, \mathcal{V}_0, \hat{\theta}_0) \in H^3(\mathbb{R}^3)$ with $\nabla \cdot \mathcal{U}_0 = 0$, $\nabla \cdot \mathcal{V}_0 = 0$ in distributional sense. For $T > 0$, $(\mathcal{U}, \mathcal{V}, \hat{\theta})$ is a weak solution of system (2.1). If pressure Ψ satisfies an integrable regularity condition

$$\int_0^T \frac{\|\nabla\Psi\|_{\dot{B}^0_{\infty,\infty}}^{\frac{2}{3}}}{\left(1 + \ln(e + \|\nabla\Psi\|_{\dot{B}^0_{\infty,\infty}})\right)^{\frac{3}{2}}} dt < \infty, \tag{2.18}$$

then $(\mathcal{U}, \mathcal{V}, \hat{\theta})$ shows its smoothness in the interval $\mathbb{R}^3 \times (0, T]$, and there are no moving singular points or blow-ups in the area under consideration, i.e, the interval (0,T].

Proof. To prove this theorem we established *a priori* estimate for the weakly formulated equation (2.9).

For I_3

$$\begin{aligned} I_3 &\leq \|\hat{\theta}\|_{L^4}\|\mathcal{Q}^+\|_{L^4}\|\mathcal{Q}^+\|_{L^4}^2 \leq \frac{1}{2}\big(\|\hat{\theta}\|_{L^4}^2\|\mathcal{Q}^+\|_{L^4}^2\big) + C\|\mathcal{Q}^+\|_{L^4}^4 \\ &\frac{1}{4}\|\hat{\theta}\|_{L^4}^4 + C\|\mathcal{Q}^+\|_{L^4}^4 + C\|\mathcal{Q}^+\|_{L^4}^4 \leq \frac{1}{4}\|\hat{\theta}\|_{L^4}^4 + C\|\mathcal{Q}^+\|_{L^4}^4. \end{aligned} \tag{2.19}$$

Similarly,

$$I_2 \leq \frac{1}{4}\|\hat{\theta}\|_{L^4}^4 + C\|\mathcal{Q}^-\|_{L^4}^4. \tag{2.20}$$

$$I_1 = -\int_{\mathbb{R}^3} \nabla\Psi(\mathcal{Q}^+|\mathcal{Q}^+|^2 + \mathcal{Q}^-|\mathcal{Q}^-|^2)dx = -\int_{\mathbb{R}^3} \nabla\Psi(\mathcal{Q}^+|\mathcal{Q}^+|^2)dx - \int_{\mathbb{R}^3} \nabla\Psi(\mathcal{Q}^-|\mathcal{Q}^-|^2)dx$$
$$= P_1 + P_2. \tag{2.21}$$
$$|P_1| \leq \Big| -\int_{\mathbb{R}^3} \nabla\Psi \cdot \mathcal{Q}^+|\mathcal{Q}^+|^2 dx\Big| \leq \|\nabla\Psi\|_{L^4}\|\mathcal{Q}^+\|_{L^4}^3 \leq C\|\nabla\Psi\|_{L^2}^{\frac{1}{2}}\|\nabla\Psi\|_{BMO}^{\frac{1}{2}}\|\mathcal{Q}^+\|_{L^4}^3.$$

Similarly,

$$|P_2| \leq \Big| -\int_{\mathbb{R}^3} \nabla\Psi \cdot \mathcal{Q}^-|\mathcal{Q}^-|^2 dx\Big| \leq \|\nabla\Psi\|_{L^4}\|\mathcal{Q}^-\|_{L^4}^3 \leq C\|\nabla\Psi\|_{L^2}^{\frac{1}{2}}\|\nabla\Psi\|_{BMO}^{\frac{1}{2}}\|\mathcal{Q}^-\|_{L^4}^3.$$

Putting estimates (2.19), (2.20) and (2.21) into (2.9), and using $\|\mathcal{Q}^+\|_{L^4}^3 + \|\mathcal{Q}^-\|_{L^4}^3 = \|\mathcal{U}\|_{L^4}^3 + \|\mathcal{V}\|_{L^4}^3$, we are down to

$$\frac{1}{4}\frac{d}{dt}\Big(\|\mathcal{U}\|_{L^4}^4 + \|\mathcal{V}\|_{L^4}^4 + 1\Big) + \frac{1}{4}\Big(\|\nabla|\mathcal{U}|^2\|_{L^2}^2 + \|\nabla|\mathcal{V}|^2\|_{L^2}^2\Big) + \frac{1}{2}(\|\mathcal{U}\cdot\nabla\mathcal{U}\|_{L^2}^2 + \|\mathcal{V}\cdot\nabla\mathcal{U}\|_{L^2}^2$$
$$+\|\mathcal{U}\cdot\nabla\mathcal{V}\|_{L^2}^2 + \|\mathcal{V}\cdot\nabla\mathcal{V}\|_{L^2}^2)$$
$$\leq \|\mathcal{U}\|_{L^4}^3(\|\mathcal{U}\cdot\nabla\mathcal{U}\|_{L^2}^{\frac{1}{2}}\|\nabla\Psi\|_{BMO}^{\frac{1}{2}}) + \|\mathcal{V}\|_{L^4}^3(\|\mathcal{U}\cdot\nabla\mathcal{U}\|_{L^2}^{\frac{1}{2}}\|\nabla\Psi\|_{BMO}^{\frac{1}{2}}) + (\|\mathcal{U}\|_{L^4}^4 + \|\mathcal{V}\|_{L^4}^4 + \|\hat{\theta}\|_{L^4}^4)$$
$$\leq \frac{1}{2}\||\mathcal{U}||\nabla\mathcal{U}|\|_{L^2}^2 + C\|\nabla\Psi\|_{BMO}^{\frac{2}{3}}\|\mathcal{U}\|_{L^4}^4 + \frac{1}{2}\||\mathcal{U}||\nabla\mathcal{U}|\|_{L^2}^2 + C\|\nabla\Psi\|_{BMO}^{\frac{2}{3}}\|\mathcal{V}\|_{L^4}^4$$
$$+\big(\|\mathcal{U}\|_{L^4}^4 + \|\mathcal{V}\|_{L^4}^4 + \|\hat{\theta}\|_{L^4}^4\big)$$
$$\leq C\big(\|\mathcal{U}\|_{L^4}^4 + \|\mathcal{V}\|_{L^4}^4 + \|\hat{\theta}\|_{L^4}^4\big)\big(1 + \|\nabla\Psi\|_{BMO}^{\frac{2}{3}}\big).$$

Using (2.6) for $\nabla\Psi$, we get that

$$\leq C\big(\|\mathcal{U}\|_{L^4}^4 + \|\mathcal{V}\|_{L^4}^4 + \|\hat{\theta}\|_{L^4}^4\big)\big(1 + \|\nabla\Psi\|_{\dot{B}^0_{\infty,\infty}}^{\frac{2}{3}} \ln^{\frac{1}{3}}(1 + \|\nabla\Psi\|_{H^2})\big)$$

$$\leq \big(\|\mathcal{U}\|_{L^4}^4 + \|\mathcal{V}\|_{L^4}^4 + \|\hat{\theta}\|_{L^4}^4\big)\Big(1 + \frac{\|\nabla\Psi\|_{\dot{B}^0_{\infty,\infty}}^{\frac{2}{3}}}{(1 + \ln(1 + \|\nabla\Psi\|_{\dot{B}^0_{\infty,\infty}})^{\frac{2}{3}}}\Big)\ln(1 + \|\Lambda^3\mathcal{U}\|_{L^2})\big). \tag{2.22}$$

For $\hat{\theta}$ we use (2.15), which implies that

$$\leq \big(\|\mathcal{U}\|_{L^4}^4 + \|\mathcal{V}\|_{L^4}^4 + 1\big)\Big(1 + \frac{\|\nabla\Psi\|_{\dot{B}^0_{\infty,\infty}}^{\frac{2}{3}}}{(1 + \ln(1 + \|\nabla\Psi\|_{\dot{B}^0_{\infty,\infty}})^{\frac{2}{3}}}\Big)\ln(1 + \kappa(t)\big).$$

Because of (2.18), $\exists\, T_* < T$, such that

$$\int_{T_*}^{T} \frac{\|\nabla\Psi\|_{\dot{B}^0_{\infty,\infty}}^{\frac{2}{3}}}{1 + \ln(1 + \|\nabla\Psi\|_{\dot{B}^0_{\infty,\infty}})^{\frac{2}{3}}} < \epsilon.$$

We set

$$\kappa(t) := (\|\Lambda^3\mathcal{U}\|_{L^2} + \|\Lambda^3\mathcal{V}\|_{L^2} + \|\Lambda^3\hat{\theta}\|_{L^2}).$$

$\kappa(t)$ is bounded by the same process as $Z(t)$.

Due to the application of Gronwall's Lemma to (2.22), we obtain

$$\sup_{T_*<t\leq T}\left(\|\mathcal{U}\|_{L^4}^4+\|\mathcal{V}\|_{L^4}^4+1\right)\leq C_*(e+\kappa(t))^{C\epsilon}$$

This proves Theorem 2.3.1.

Corollary 2.3.2 The continuous embedding $BMO \hookrightarrow \dot{B}^0_{\infty,\infty}$ results in very important consequence of Theorem 2.3.1 that is the condition

$$\nabla\Psi \in L^{\frac{2}{3}}\left(0,T;\dot{B}^0_{\infty,\infty}\right),$$

which improves the criteria (2.2) by taking it from BMO (Bounded mean oscillations) space to larger Besov space $\dot{B}^0_{\infty,\infty}$.

The other very important aspect of the non-linear differential system (2.1), i.e., the 3D magnetic Bénard system, is the occurrence of movable singularities, i.e., starting from smooth initial data, the solution becomes infinite in finite time due to the cumulative effect of the nonlinearities. Such types of singularity formations in non-linear differential systems are also known as blow-ups. In the framework of the regularity theory of weak solutions, the blow-up or singularity occurs if the solution becomes infinite at some (or many) points as t approaches a certain finite time T. The singularity or blow-up problem states that the solution with some smooth initial data is well-defined in some function space for some time $0<t<T$. Such type of singularities explicitly depend upon the type of function space and time. The alternative interpretation of conditions (2.7) and (2.18) is let $T=T^{\dagger}<\infty$ is the maximal time for the existence of a smooth solutions, then the solution blows up (also called the first time blow up) to create finite time singularity, and condition (2.7) takes the form shown as

$$\int_0^{T^{\dagger}}\frac{\|\Psi\|^2_{\dot{B}^{-1}_{\infty,\infty}}}{\left(1+\ln\left(e+\|\Psi\|_{\dot{B}^{-1}_{\infty,\infty}}\right)\right.}dt=\infty,$$

similarly, the condition (2.18) becomes

$$\int_0^{T^{\dagger}}\frac{\|\nabla\Psi\|^{\frac{2}{3}}_{\dot{B}^0_{\infty,\infty}}}{\left(1+\ln(e+\|\nabla\Psi\|_{\dot{B}^0_{\infty,\infty}})\right)^{\frac{3}{2}}}dt=\infty.$$

Therefore, the blow-up is exactly the inability to continue the weak solution up to or past a given time.

2.4 Conclusions

The mathematical significance of our results lies in wider spaces, i.e., Besov spaces of a negative index. Such spaces are important due to their criticality defined by their scale invariance because the local regularity results by using scale invariance pro- perty could be taken to global regularity results. The criteria (2.18) replace BMO space with larger space, i.e., $\dot{B}^0_{\infty,\infty}$, consequently, improving the regularity of solutions. Our results that are proved in the finite-time interval $C^\infty(\mathbb{R}^3 \times (0,T])$ constitute vital work on the millennium clay mathematical problem [1] which requires the solutions to be regular in $C^\infty(\mathbb{R}^3 \times (0,\infty))$ i.e., for all time. We use pressure, which has remarkable properties, to control the solutions of the system (2.4) by imposing sufficient integrable regularity conditions that improve numerous previously established results.

Chapter 3

One directional derivative regularity

3.1 Introduction

In this chapter, we present two new regularity criterion for the 3D Boussinesq equations in terms of one directional derivative of velocity. These criterion are established within the framework of negative index Besov spaces that is if weak solution satisfies the conditions

$$\int_0^T \|\partial_3 \mathcal{U}\|^2_{\dot{B}^{-1}_{\infty,\infty}} dt < \infty,$$

and

$$\int_0^T \|\partial_3 \mathcal{U}\|^{\frac{2}{1-r}}_{\dot{B}^{-r}_{\infty,\infty}} dt < \infty,$$

where $-1 < r < 1$, then it remains regular in the interval (0,T].

We investigate the following Cauchy problem of 3D Boussinesq equations:

$$\begin{cases} \frac{\partial \mathcal{U}}{\partial t} + (\mathcal{U} \cdot \nabla)\mathcal{U} - \chi_1 \triangle \mathcal{U} + \nabla \Psi = \hat{\vartheta} e_3, \\ \frac{\partial \hat{\vartheta}}{\partial t} + (\mathcal{U} \cdot \nabla)\hat{\vartheta} - \chi_2 \triangle \hat{\vartheta} = 0, \\ \nabla \cdot \mathcal{U} = 0, \nabla \cdot \hat{\vartheta} = 0, \\ \mathcal{U}(x,0) = \mathcal{U}_0(x), \quad \hat{\vartheta}(x,0) = \hat{\vartheta}_0(x), \end{cases} \tag{3.1}$$

where $\mathcal{U}(x,t)$, $\hat{\vartheta}(x,t)$ and $\Psi(x,t)$ are unknown velocity vector, temperature function and pressure having domain in $\mathbb{R}^3 \times (0,T)$ and $e_3 = (0,0,1)^T$. The initial data $\mathcal{U}_0$, $\hat{\vartheta}_0$ satisfies $\nabla \cdot \mathcal{U}_0 = 0$ and $\nabla \cdot \hat{\vartheta}_0 = 0$ in the distributional sense. With no loss of generality, we fix parameters $\chi_1 = \chi_2 = 1$, see [35].

Before stating the main theorems, we now give a brief background on the problem. The Boussinesq equations arise as a result of Boussinesq approximations from the incompressible Navier-Stokes equations (NSE) an models oceanic and atmospheric motions [36]. To analyse the weak solutions of a system (3.1), we have in hand two strategies: analysing the partial regularity of appropriate weak solutions is the first and second is to propose different improved criteria; thus, the weak solutions' regularity will be ensured. This research pursuit relies on improving regularity criteria concerned with the second approach. Our results improve numerous previously established results on this problem in the one-directional derivative of velocity. The contextual analysis is given as follow:

Cao and wu [37] presented the following result for the 3D MHD equations

$$\int_0^T \|\partial_3\mathcal{U}\|_{L^l}^m dt < \infty, \tag{3.2}$$

where $l \geq 3$, and $\frac{3}{l} + \frac{2}{m} \leq 1$, in the time domain $(0, T]$.

Later on, the refined form of (3.2) for NSE in Morrey–Campanato space was given by Liu [38], that is

$$\int_0^T \|\partial_3\mathcal{U}\|_{\dot{M}_{l,\frac{3}{r}}}^{\frac{2}{1-r}} dt < \infty, \tag{3.3}$$

where $0 < r < 1, \quad 2 \leq l \leq \frac{3}{r}$.

The above result (3.3) was improved for NSE by Gala [39] in bounded mean oscillation space to obtain the condition

$$\int_0^T \|\partial_3\mathcal{U}\|_{BMO}^2 dt < \infty. \tag{3.4}$$

Although, not much has been done to obtain results in terms of one-directional derivatives. Recently, for the initial smooth data in $H^1(\mathbb{R}^3)$ for the system (3.1), Wu [40] improved the result (3.4) in Besov spaces of index zero, namely, $\dot{B}^0_{\infty,\infty}$. It satisfies the condition

$$\int_0^T \|\partial_3\mathcal{U}\|_{\dot{B}^0_{\infty,\infty}}^2 dt < \infty. \tag{3.5}$$

For the system (3.1), Zhang [41] proved the result

$$\int_0^T \left(\|\nabla\mathcal{U}\|_{\dot{B}^{-1}_{\infty,\infty}}^2 + \|\nabla\hat{\vartheta}\|_{\dot{B}^{-1}_{\infty,\infty}}^2\right) dt < \infty,$$

and its improved form was presented by Barbagallo et al. in [42]

$$\int_0^T \|\nabla\mathcal{U}\|_{\dot{B}^{-1}_{\infty,\infty}}^2 dt < \infty. \tag{3.6}$$

Inspired by the above results, we improve conditions (3.5) and (3.6) via one directional derivative in the function spaces $\dot{B}^{-1}_{\infty,\infty}$ and $\dot{B}^{-r}_{\infty,\infty}$.

3.2 Energy estimates for $\|\partial_3\mathcal{U}\|_{\dot{B}^{-1}_{\infty,\infty}}^2$ criteria

Theorem 3.2.1. For $(\mathcal{U}_0, \hat{\vartheta}_0) \in H^1(\mathbb{R}^3)$ with $\nabla \cdot \mathcal{U}_0 = 0$ in distributional form. Let $(\mathcal{U}, \hat{\vartheta})$ be the weak solution to (3.1) in $(\mathbb{R}^3 \times (0, T])$ and satisfy strong energy inequality together with

$$\int_0^T \|\partial_3\mathcal{U}\|_{\dot{B}^{-1}_{\infty,\infty}}^2 dt < \infty, \tag{3.7}$$

then the weak solution is regular on (0,T].

Proof. Multiply $\mathcal{U}$ and $\hat{\vartheta}$ to the equations $(3.1)_1$ and $(3.1)_2$ respectively and integrating them by parts in $\mathbb{R}^3$. The Holder's and Young's inequalities helps us to write:

$$\frac{1}{2}\frac{d}{dt}(\|\mathcal{U}\|_{L^2}^2+\|\hat{\vartheta}\|_{L^2}^2)+(\|\nabla\mathcal{U}\|_{L^2}^2+\|\nabla\hat{\vartheta}\|_{L^2}^2)=\int_{\mathbb{R}^3}\hat{\vartheta}e_3\mathcal{U}dx \ \leq \ \|\hat{\vartheta}\|^2\|\mathcal{U}\|^2\leq \ \|\hat{\vartheta}\|^2+\|\mathcal{U}\|^2.$$

Gronwall's Lemma results in

$$\sup_{0\leq t\leq T}(\ \|\mathcal{U}\|_{L^2}^2+\|\hat{\vartheta}\|_{L^2}^2)+2\int_0^T \ \|\nabla\hat{\vartheta}\|_{L^2}^2+\|\nabla\mathcal{U}\|_{L^2}^2dt\leq C.$$

Now, differentiating $(3.1)_1$ with respect to space variable x_3, then multiplying with $\partial_3\mathcal{U}$, we have that

$$\frac{1}{2}\frac{d}{dt}\|\partial_3\mathcal{U}\|_{L^2}^2+\|\nabla\partial_3\mathcal{U}\|_{L^2}^2=-\int_{\mathbb{R}^3}\partial_3(\mathcal{U}\cdot\nabla)\mathcal{U}\cdot\partial_3\mathcal{U}\ dx+\int_{\mathbb{R}^3}\partial_3\hat{\vartheta}\cdot\partial_3\mathcal{U}\ dx=I_1+I_2. \quad (3.8)$$

Again differentiating $(3.1)_2$ with respect to x_3 and multiplying with $\partial_3 u$, we derive that

$$\frac{1}{2}\frac{d}{dt}(\|\partial_3\hat{\vartheta}\|_{L^2}^2+\|\nabla\partial_3\hat{\vartheta}\|_{L^2}^2)=-\int_{\mathbb{R}^3}\partial_3(\mathcal{U}\cdot\nabla)\hat{\vartheta}\cdot\partial_3\hat{\vartheta}\ dx=I_3. \quad (3.9)$$

Now, we focus on estimating I_1 and I_2 by using techniques from harmonic analysis specifically energy methods, functional analysis, and inequalities from the theory of function spaces, the corresponding estimates are given by employing Holder's, Young's, Geometric mean inequalities and by definition of Besov spaces.

In case of I_1

$$I_1=-\int_{\mathbb{R}^3}\partial_3(\mathcal{U}\cdot\nabla)\mathcal{U}\cdot\partial_3\mathcal{U}\ dx$$

$$|I_1|\leq C\|\partial_3\mathcal{U}\|_{L^4}\|\nabla\mathcal{U}\|_{L^2}\|\partial_3\mathcal{U}\|_{L^4}$$

$$\leq C\|\partial_3\mathcal{U}\|_{L^4}^2\|\nabla\mathcal{U}\|_{L^2}$$

$$\text{By} \ \ \|f\|_{L^4}\leq C\|f\|_{\dot{H}^1}^{\frac{1}{2}}\|f\|_{\dot{B}^{-1}_{\infty,\infty}}^{\frac{1}{2}}$$

$$\leq C\|\partial_3\mathcal{U}\|_{\dot{B}^{-1}_{\infty,\infty}}\|\nabla\mathcal{U}\|_{L^2}\|\nabla\partial_3\mathcal{U}\|_{L^2}.$$

By Young's inequality

$$|I_1|\leq C\|\partial_3\mathcal{U}\|_{\dot{B}^{-1}_{\infty,\infty}}^2\|\nabla\mathcal{U}\|_{L^2}^2+\|\nabla\partial_3\mathcal{U}\|_{L^2}^2. \quad (3.10)$$

In case of I_2,

$$|I_2| \le C(\|\partial_3\hat{\vartheta}\|_{L^2}^2 + \|\partial_3\mathcal{U}\|_{L^2}^2). \tag{3.11}$$

By (3.10) and (3.11), we obtain

$$|I_1| + |I_2| \le C\|\partial_3\mathcal{U}\|_{\dot{B}^{-1}_{\infty,\infty}}^2 \|\nabla\mathcal{U}\|_{L^2}^2 + \|\nabla\partial_3\mathcal{U}\|_{L^2}^2 + C(\|\partial_3\hat{\vartheta}\|_{L^2}^2 + \|\partial_3\mathcal{U}\|_{L^2}^2). \tag{3.12}$$

Similarly, for I_3

$$\begin{aligned}|I_3| &\le C\|\partial_3\mathcal{U}\|_{L^4}\|\nabla\hat{\vartheta}\|_{L^2}\|\partial_3\hat{\vartheta}\|_{L^4}\\ &\le C\|\nabla\partial_3\mathcal{U}\|_{L^2}^{\frac{1}{2}}\|\partial_3\mathcal{U}\|_{\dot{B}^{-1}_{\infty,\infty}}^{\frac{1}{2}}\|\nabla\hat{\vartheta}\|_{L^2}\|\nabla\partial_3\hat{\vartheta}\|_{L^2}^{\frac{1}{2}}\|\partial_3\hat{\vartheta}\|_{\dot{B}^{-1}_{\infty,\infty}}^{\frac{1}{2}}\\ &\le \|\nabla\partial_3\mathcal{U}\|_{L^2}^2 + \|\nabla\partial_3\hat{\vartheta}\|_{L^2}^2 + C\|\partial_3\mathcal{U}\|_{\dot{B}^{-1}_{\infty,\infty}}^2\|\nabla\hat{\vartheta}\|_{L^2}^2 + \|\nabla\hat{\vartheta}\|_{L^2}^2\|\partial_3\hat{\vartheta}\|_{\dot{B}^{-1}_{\infty,\infty}}^2.\end{aligned} \tag{3.13}$$

Now, combining (3.12) and (3.13) into the addition of (3.8) and (3.9), we have that

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|\partial_3\mathcal{U}\|_{L^2}^2 + \|\nabla\partial_3\mathcal{U}\|_{L^2}^2 + \frac{1}{2}\frac{d}{dt}\|\partial_3\hat{\vartheta}\|_{L^2}^2 + \|\nabla\partial_3\hat{\vartheta}\|_{L^2}^2\\ &\le C\|\partial_3\mathcal{U}\|_{\dot{B}^{-1}_{\infty,\infty}}^2\|\nabla\mathcal{U}\|_{L^2}^2 + \|\nabla\partial_3\mathcal{U}\|_{L^2}^2 + C(\|\partial_3\hat{\vartheta}\|_{L^2}^2 + \|\partial_3\mathcal{U}\|_{L^2}^2)\\ &\|\nabla\partial_3\mathcal{U}\|_{L^2}^2 + \|\nabla\partial_3\hat{\vartheta}\|_{L^2}^2 + C\|\partial_3\mathcal{U}\|_{\dot{B}^{-1}_{\infty,\infty}}^2\|\nabla\vartheta\|_{L^2}^2 + \|\nabla\vartheta\|_{L^2}^2\|\partial_3\vartheta\|_{\dot{B}^{-1}_{\infty,\infty}}^2\\ &\le C(\|\partial_3\mathcal{U}\|_{\dot{B}^{-1}_{\infty,\infty}}^2 + \|\partial_3\hat{\vartheta}\|_{\dot{B}^{-1}_{\infty,\infty}}^2)(\|\nabla(\mathcal{U},\hat{\vartheta})\|_{L^2}^2) + C(\|\partial_3\hat{\vartheta}\|_{L^2}^2 + \|\partial_3\mathcal{U}\|_{L^2}^2).\end{aligned}$$

Employing the fact that $\forall x > 0$, we have $x < x + 1$,

$$\le C(1 + \|\partial_3\mathcal{U}\|_{\dot{B}^{-1}_{\infty,\infty}}^2 + \|\partial_3\hat{\vartheta}\|_{\dot{B}^{-1}_{\infty,\infty}}^2)(1 + \|\nabla(\mathcal{U},\hat{\vartheta})\|_{L^2}^2)(\|\partial_3\hat{\vartheta}\|_{L^2}^2 + \|\partial_3\mathcal{U}\|_{L^2}^2). \tag{3.14}$$

Integrating (3.14) and by energy inequality, we derive that

$$\sup_{0\le t\le T}\|\partial_3(\mathcal{U},\hat{\vartheta})\|_{L^2}^2 + \int_0^T \|\nabla\partial_3(\mathcal{U},\hat{\vartheta})\|_{L^2}^2 \le C.$$

By the definition of Lebesgue space and weak Lebesuge space the Gronwalls's inequality results in Bochner space. Therefore, by energy argument for the weak solutions and by Sobolev embedding theorem our proof has been completed.

3.3 Energy estimates for $\|\partial_3\mathcal{U}\|^{\frac{2}{1-r}}_{\dot{B}^{-r}_{\infty,\infty}}$ criteria

Theorem 3.3.1. Suppose $(\mathcal{U}_0, \hat{\vartheta}_0) \in H^1(\mathbb{R}^3)$ with $\nabla \cdot \mathcal{U}_0 = 0$ in distributional form. Let $(\mathcal{U}, \hat{\vartheta})$ be the weak solution to equation (3.1) in $(\mathbb{R}^3 \times (0,T])$ and satisfy strong energy inequality together with

$$\int_0^T \|\partial_3\mathcal{U}\|^{\frac{2}{1-r}}_{\dot{B}^{-r}_{\infty,\infty}} dt < \infty, \tag{3.15}$$

where $-1 < r < 1$,then the weak solution is regular on (0,T].

Proof. To prove this theorem, multiply $(3.1)_1$ with $\mathcal{U}$, then integrating by parts, using Holder's and Young's inequality, we derive that

$$\int_{\mathbb{R}^3} \partial_t\mathcal{U} \cdot \mathcal{U} + \int_{\mathbb{R}^3} (\mathcal{U} \cdot \nabla)\mathcal{U} \cdot \mathcal{U} - \int_{\mathbb{R}^3} \Delta\mathcal{U} \cdot \mathcal{U} + \int_{\mathbb{R}^3} \nabla\Psi \cdot \mathcal{U} = \int_{\mathbb{R}^3} \hat{\vartheta} e_3 \mathcal{U}$$

$$\frac{1}{2}\frac{d}{dt}(\|\mathcal{U}\|^2_{L^2}) + (\|\nabla\mathcal{U}\|^2_{L^2}) = \int_{\mathbb{R}^3} \hat{\vartheta} e_3 \mathcal{U} dx \;\leq\; \|\hat{\vartheta}\|^2\|\mathcal{U}\|^2 \leq\; \|\hat{\vartheta}\|^2 + \|\mathcal{U}\|^2. \tag{3.16}$$

Similarly, multiply $(3.1)_2$ with $\hat{\vartheta}$, we get

$$\int_{\mathbb{R}^3} \partial_t\hat{\vartheta} \cdot \hat{\vartheta} + \int_{\mathbb{R}^3} (\mathcal{U} \cdot \nabla)\hat{\vartheta} \cdot \hat{\vartheta} - \int_{\mathbb{R}^3} \Delta\hat{\vartheta} \cdot \hat{\vartheta} = 0$$

$$\frac{1}{2}\frac{d}{dt}(\|\hat{\vartheta}\|^2_{L^2}) + (\|\nabla\hat{\vartheta}\|^2_{L^2}) = 0. \tag{3.17}$$

combining (3.16) and (3.17), we have

$$\frac{1}{2}\frac{d}{dt}(\|\mathcal{U}\|^2_{L^2}) + (\|\hat{\vartheta}\|^2_{L^2}) + (\|\nabla\mathcal{U}\|^2_{L^2}) + (\|\nabla\hat{\vartheta}\|^2_{L^2}) \;\leq\; \|\hat{\vartheta}\|^2\|\mathcal{U}\|^2 \leq\; \|\hat{\vartheta}\|^2 + \|\mathcal{U}\|^2.$$

By Gronwalls's lemma

$$\Longrightarrow \quad \sup_{0\leq t\leq T}(\; \|\hat{\vartheta}\|^2_{L^2} + \|\mathcal{U}\|^2_{L^2}) + 2\int_0^T \|\nabla\hat{\vartheta}\|^2_{L^2} + \|\nabla\mathcal{U}\|^2_{L^2} dt \leq C. \tag{3.18}$$

Differentiating $(3.1)_1$ in the direction x_3, multiplying with $\partial_3 u$ and integrating over $\mathbb{R}^3$, we obtain

$$\frac{1}{2}\frac{d}{dt}(\|\partial_3\mathcal{U}\|^2_{L^2}) + (\|\nabla\partial_3\mathcal{U}\|^2_{L^2}) = -\int_{\mathbb{R}^3} \partial_3(\mathcal{U}\cdot\nabla\mathcal{U})\cdot\partial_3\mathcal{U}\, dx + \int_{\mathbb{R}^3} \partial_3\hat{\vartheta}\cdot\partial_3\mathcal{U}\, dx = I_1 + I_2. \tag{3.19}$$

Now, we get the following estimates

$$|I_1| \leq \int_{\mathbb{R}^3} \partial_3(\mathcal{U} \cdot \nabla\mathcal{U}) \cdot \partial_3\mathcal{U}\; dx \leq \|\partial_3\mathcal{U}\|_{L^4}\|\partial_3\mathcal{U}\|_{L^4}\|\nabla\mathcal{U}\|_{L^2}.$$

$$\leq \|\partial_3\mathcal{U}\|^2_{L^4}\|\nabla\mathcal{U}\|_{L^2}$$

$$\leq \|\partial_3\mathcal{U}\|_{\dot{B}^{-r}_{\infty,\infty}}\|\partial_3\mathcal{U}\|_{\dot{H}^r}\|\nabla\mathcal{U}\|_{L^2}$$

$$\leq \|\partial_3\mathcal{U}\|_{\dot{B}^{-r}_{\infty,\infty}}\|\partial_3\mathcal{U}\|^{1-r}_{L^2}\|\nabla\partial_3\mathcal{U}\|^r_{L^2}\|\nabla\mathcal{U}\|_{L^2}$$

$$\leq \frac{1}{4}\|\nabla\partial_3\mathcal{U}\|^2_{L^2} + C[(\|\partial_3\mathcal{U}\|^{2/1-r}_{\dot{B}^{-r}_{\infty,\infty}})^{1-r/2-r}\|\nabla\mathcal{U}\|^2_{L^2})^{1/2-r}\|\partial_3\mathcal{U}\|^{2(1-r)/2-r}_{L^2}]$$

$$|I_1| \leq \frac{1}{4}\|\nabla\partial_3\mathcal{U}\|^2_{L^2} + C(1+\|\partial_3\mathcal{U}\|^2_{L^2})(\|\partial_3\mathcal{U}\|^{2/1-r}_{\dot{B}^{-r}_{\infty,\infty}} + \|\nabla\mathcal{U}\|^2_{L^2}). \tag{3.20}$$

Similarly, for I_2, we use Poincare inequality and achieve

$$|I_2| \leq \|\partial_3\mathcal{U}\|^2_{L^2}\|\partial_3\hat{\vartheta}\|^2_{L^2} \leq \|\nabla\partial_3\mathcal{U}\|^2_{L^2} + \|\partial_3\hat{\vartheta}\|^2_{L^2}. \tag{3.21}$$

Now, combining (3.20) and (3.21) into (3.19) together with (3.18), which by employing Gronwall's Lemma results in final inequality that ensure the required bounds, i.e.,

$$\Longrightarrow \quad \sup_{0\leq t\leq T}(\ \|\partial_3\mathcal{U}\|^2_{L^2}) + 2\int_0^T \ \|\nabla\partial_3\mathcal{U}\|^2_{L^2}dt \leq C.$$

Which completes the proof of Theorem 3.3.1.

3.4 Conclusions

The well-posedness problems in higher dimensions i.e., for $n \geq 3$ remained partially solved. Among them the regularity in one-directional derivative is rarely tackled by the researchers. Recently, Wu presented the one-directional derivative result (3.5) in Besov spaces. We have improved it by presenting the new result (3.7), result (3.15) improves (3.6). This work, for sure, will open new dimensions for researchers as the one-directional results are not yet tackled in Besov space with a negative index.

Chapter 4

Blow-up criteria for different fluid models in anisotropic Lorentz spaces

4.1 Introduction

This chapter establishes new blow-up criteria, in anisotropic Lorentz spaces, via one-directional derivatives of the velocity and magnetic fields for the Cauchy problem to the 3D magneto-micropolar model and via one-directional derivative of velocity for the Cauchy problem to the 3D nonlinear dissipative system. The first model consists of five equations governing the unsteady, viscous, incompressible magneto-micropolar flow:

$$\begin{cases} \frac{\partial \mathcal{U}}{\partial t} + \mathcal{U} \cdot \nabla \mathcal{U} - \Delta \mathcal{U} + \nabla(\Psi + \mathcal{V}^2) - \nabla \times \mathcal{W} - \mathcal{V} \cdot \nabla \mathcal{V} = 0, \\ \frac{\partial \mathcal{W}}{\partial t} - \Delta \mathcal{W} + \mathcal{U} \cdot \nabla \mathcal{W} - \nabla \times \mathcal{U} + 2\mathcal{W} - \nabla div\ \mathcal{W} = 0, \\ \frac{\partial \mathcal{V}}{\partial t} - \Delta \mathcal{V} + \mathcal{U} \cdot \nabla \mathcal{V} - \mathcal{V} \cdot \nabla \mathcal{U} = 0, \\ \nabla \cdot \mathcal{U} = 0, \quad \nabla \cdot \mathcal{V} = 0, \\ (\mathcal{U}, \mathcal{W}, \mathcal{V})|_{t=0} = (\mathcal{U}_0, \mathcal{W}_0, \mathcal{V}_0). \end{cases} \tag{4.1}$$

In the system (4.1), $\mathcal{U}(x,t)$ and $\mathcal{V}(x,t)$ are the velocity and magnetic fields. The micro-rotational velocity and hydrostatic pressure are given the notations $\mathcal{W}(x,t)$, $\Psi(x,t)$, while $\mathcal{U}_0$, $\mathcal{V}_0$ and $\mathcal{W}_0$ are the given initial velocity, magnetic field and micro-rotation velocity with $\nabla \cdot \mathcal{U}_0 = 0$ and $\nabla \cdot \mathcal{V}_0 = 0$ in the distributional sense. Galdi and Rionero [43] were the first who suggested the model (4.1). Rojas-Medar and Boldrini [44] established the existence of global weak solutions to the system (4.1). Later on, the authors in [45] and [46], respectively, considered the problem of the existence of local and global strong solutions to the same system for small initial data. However, concerning the weak solutions to the system (4.1), there arises a question of the regularity of these solutions. In this regard, several publications discussing the regularity of weak solutions of system (4.1) have appeared in the literature, see for instance [47–53] and references therein. In this article, we also choose to discuss the blow-up criteria for the system (4.1) that guarantees the regularity of local smooth solutions for all time $(0, \infty]$. In view of the physical importance of system (4.1), it models the flow of microelements under the influence of a magnetic field. These micropolar fluids have a diluted suspension of tiny, stiff, cylindrical macromolecules that move independently and are affected by spin inertia. Such types of flows are significant in analysing animal and human blood, polymer fluids, liquid crystals, etc. Recently, enormous studies have been conducted on studying such fluids on different surfaces, including bounded and unbounded domains.

The second system we consider here for analysis is the
Navier–Stokes-Poisson–Nernst–Planck system:

$$\begin{cases} \frac{\partial \mathcal{U}}{\partial t} + \mathcal{U} \cdot \nabla \mathcal{U} - \Delta \mathcal{U} + \nabla \Psi - \Delta \psi \nabla \psi = 0, \\ \nabla \cdot \mathcal{U} = 0, \\ \frac{\partial \theta}{\partial t} + \mathcal{U} \cdot \nabla \theta - \nabla \cdot (\nabla \theta + \theta \nabla \psi) = 0, , \\ \frac{\partial \vartheta}{\partial t} + \mathcal{U} \cdot \nabla \vartheta - \nabla \cdot (\nabla \vartheta - \vartheta \nabla \psi) = 0, \\ \Delta \psi = \theta - \vartheta, \quad \text{in } \mathbb{R}^3 \times \mathbb{R}_+, \\ (\mathcal{U}, \theta, \vartheta)|_{t=0} = (\mathcal{U}_0, \theta_0, \vartheta_0). \end{cases} \tag{4.2}$$

In the system (4.2), $\mathcal{U}(x,t)$ and $\Psi(x,t)$ are the velocity and pressure, $\theta(x,t)$ and $\vartheta(x,t)$ are the densities of binary diffusive negative and positive charges, ψ is the electric potential, respectively. Rubinstein [54] proposed system (4.2), which can describe the drift, diffusion, and convection process for the charged ions in incompressible viscous fluids (see [55–58], and the references cited therein). The well-posedness problem of the system (4.2) has been tackled by Jerome [59] based on Kato's semigroup framework. The global existence of strong solutions for small initial data and the local existence of strong solutions for arbitary initial data has been established by Zhao et al. [60–62] in various function spaces. However, for arbitary initial data, the all time existence of local smooth solutions is one of the key open problem that we will investigate and present new blow-up conditions in anisotropic Lorentz space. Similar to system (4.1) the electro diffusion model covers various fluid models and could be considered as general formulation to Navier-Stokes, Micropolar, MHD, and Boussinesq systems. The momentum and mass conservation equations for the flow are $(4.2)_1$ and $(4.2)_2$, respectively, while the balance between diffusion and convective transport of charges by the flow and electric fields is modelled by $(4.2)_3$ and $(4.2)_4$, respectively, and the Poisson equation for the electrostatic potential is $(4.2)_5$. Keep in mind that the Lorentz force produced by the charges is represented in $(4.2)_1$. To learn more about the physical backdrop of this issue, we direct the reader to [63–66] and the references therein.

In that regard, for the system (4.1), Yuan [67] presented the regularity critreia (4.3), (4.4), Lorenz et al.[68] presented conditions (4.5), (4.6) and Wang [69] established the regularity criteria (4.7)

$$\mathcal{U} \in L^m(0,T;L^l(\mathbb{R}^3)), \quad \text{where } \frac{3}{l}+\frac{2}{m}=1, \quad 3<l\leq\infty, \tag{4.3}$$

$$\nabla\mathcal{U} \in L^m(0,T;L^l(\mathbb{R}^3)), \quad \text{where } \frac{3}{l}+\frac{2}{m}=2, \quad \frac{3}{2}<l\leq\infty, \tag{4.4}$$

$$\nabla\mathcal{U}_3, \nabla_h\mathcal{V}, \nabla_h\mathcal{W} \in L^{\frac{32}{7}}(0,T;L^2(\mathbb{R}^3)), \tag{4.5}$$

$$\partial_3\mathcal{U}_3, \partial_3\mathcal{V}, \partial_3\mathcal{W} \in L^{\infty}(0,T;L^2(\mathbb{R}^3)), \tag{4.6}$$

$$\partial_3\mathcal{U} \in L^m(0,T;L^l(\mathbb{R}^3)) \quad \text{where } \frac{3}{l}+\frac{2}{m}\leq 1, \quad 3<l\leq\infty, \tag{4.7}$$

where $\nabla=(\partial_1,\partial_2,\partial_3)$ and $\nabla_h=(\partial_1,\partial_2)$.

For the system (4.2), Zhao and Bai [70] proved the regularity criteria (4.8), (4.9)

$$\mathcal{U} \in L^m(0,T;L^l(\mathbb{R}^3)), \quad \text{where } \frac{3}{l}+\frac{2}{m}\leq 2, \quad \frac{3}{2}<l\leq\infty, \tag{4.8}$$

$$\nabla\mathcal{U} \in L^m(0,T;L^l(\mathbb{R}^3)), \quad \text{where } \frac{3}{l}+\frac{2}{m}\leq 3, \quad 1<l\leq\infty. \tag{4.9}$$

Remark 4.1.1 The embedding relation $L^p \hookrightarrow L^{p,\infty}$ ensures that the anisotropic Lorentz space is larger than the anisotropic Lebesgue space and classical(simple) Lebesgue space. Furthermore, dropping ∞ and setting $l=m=n$ in the anisotropic Lorentz space we get anisotorpic Lebesgue space and simple Lebesgue space. This important observation is very useful because the results in anisotropic Lorentz sapces hold and improve numerous previous results in smaller spaces.

Remark 4.1.2 Throughout the paper the notation $\left\| \left\| \left\| (f,g) \right\|_{L^{l,\infty}_{x_1}} \right\|_{L^{m,\infty}_{x_2}} \right\|_{L^{n,\infty}_{x_3}}^{\frac{2}{1-(\frac{1}{l}+\frac{1}{m}+\frac{1}{n})}}$ is expanded as $\left\| \left\| \left\| f \right\|_{L^{l,\infty}_{x_1}} \right\|_{L^{m,\infty}_{x_2}} \right\|_{L^{n,\infty}_{x_3}}^{\frac{2}{1-(\frac{1}{l}+\frac{1}{m}+\frac{1}{n})}} + \left\| \left\| \left\| g \right\|_{L^{l,\infty}_{x_1}} \right\|_{L^{m,\infty}_{x_2}} \right\|_{L^{n,\infty}_{x_3}}^{\frac{2}{1-(\frac{1}{l}+\frac{1}{m}+\frac{1}{n})}}$.

As the blow-up of solution of the system (4.1) is controlled by four unknowns that is $\mathcal{U}$, $\mathcal{V}$, $\mathcal{W}$, Ψ. The important question regarding the regularity of weak solutions arises here. Can we propose a blow-up criteria for the system (4.1) only by controlling velocity and magnetic fields. In this paper, we give positive answer. Motivated by the above discussion, Remark 4.1.1 and conditions (4.5), (4.6) and (4.7), we present the following blow-up criteria in anisotropic Lorentz space for the system (4.1).

4.2 Controlling regularity via velocity and magnetic field

In this section, we state the condition that controls the blow-up of the system (4.1) via velocity and magnetic field.

Theorem 4.2.1. Assume that $(\mathcal{U}_0, \mathcal{V}_0, \mathcal{W}_0) \in H^1(\mathbb{R}^3)$ with $\nabla \cdot \mathcal{U}_0 = \nabla \cdot \mathcal{V}_0 = 0$ in the sense of distributions. The Leray-Hopf weak solution $(\mathcal{U}, \mathcal{V}, \mathcal{W})$ of the system (4.1) is smooth on the interval (0,T], if

$$\int_0^T \left\| \left\| \left\| (\partial_3 \mathcal{U}, \partial_3 \mathcal{V}) \right\|_{L^{l,\infty}_{x_1}} \right\|_{L^{m,\infty}_{x_2}} \right\|_{L^{n,\infty}_{x_3}}^{\frac{2}{1-(\frac{1}{l}+\frac{1}{m}+\frac{1}{n})}} < \infty, \tag{4.10}$$

where $2 < l, m, n \leq \infty$ and $1 - (\frac{1}{l} + \frac{1}{m} + \frac{1}{n}) \geq 0$. Otherwise, if $T = T^\star < \infty$ is the maximal time for the existence of smooth solution then the solution blows up in finite time i.e.

$$\int_0^{T^\star} \left\| \left\| \left\| (\partial_3 \mathcal{U}, \partial_3 \mathcal{V}) \right\|_{L^{l,\infty}_{x_1}} \right\|_{L^{m,\infty}_{x_2}} \right\|_{L^{n,\infty}_{x_3}}^{\frac{2}{1-(\frac{1}{l}+\frac{1}{m}+\frac{1}{n})}} = \infty.$$

4.2.1 *A priori* estimates for velocity and magnetic field

Proof of Theorem 4.2.1 In order to get the fundamental energy estimates of the system (4.1), taking inner product of $(4.1)_1$, $(4.1)_2$, $(4.1)_3$ over $\mathbb{R}^3$ with $\mathcal{U}$, $\mathcal{W}$, $\mathcal{V}$, respectively, then adding the resulting equations and integrating in time, we get

$$\begin{aligned}\|(\mathcal{U}, \mathcal{W}, \mathcal{V})\|_{L^2}^2 + 2\int_0^t (\|\nabla \mathcal{U}\|_{L^2}^2 + \|\nabla \mathcal{W}\|_{L^2}^2 + \|\nabla \mathcal{V}\|_{L^2}^2) d\tau + 2\int_0^t (\|\nabla \cdot \mathcal{W}\|_{L^2}^2 + \|\mathcal{W}\|_{L^2}^2) d\tau \\ \leq \|(\mathcal{U}_0, \mathcal{W}_0, \mathcal{V}_0)\|_{L^2}^2. \end{aligned} \tag{4.11}$$

In order to find L^2-estimates for one-directional derivative of the velocity, take derivative of $(4.1)_1$ with respect to x_3, then multiply resulting equation with $\partial_3 \sqcap$ in

$L^2(\mathbb{R}^3)$ inner product and integrating, we get the resulting equation as

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^3}|\partial_3\mathcal{U}|^2dx+\int_{\mathbb{R}^3}|\nabla\partial_3\mathcal{U}|^2dx-\int_{\mathbb{R}^3}\partial_3\mathcal{U}\cdot\mathcal{V}\cdot\nabla\partial_3\mathcal{V}dx=-\int_{\mathbb{R}^3}\partial_3\mathcal{U}\cdot\partial_3\mathcal{U}\cdot\nabla\mathcal{U}dx$$
$$+\int_{\mathbb{R}^3}\partial_3\mathcal{U}\cdot\partial_3\mathcal{V}\cdot\nabla\mathcal{V}dx+\int_{\mathbb{R}^3}\partial_3\mathcal{U}\cdot\nabla\times\partial_3\mathcal{W}dx. \tag{4.12}$$

Similarly, multiplying $(4.1)_2$ with $\partial_3\mathcal{W}$ and $(4.1)_3$ with $\partial_3\mathcal{V}$, integrating by parts, we get

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^3}|\partial_3\mathcal{W}|^2dx+\int_{\mathbb{R}^3}|\nabla\partial_3\mathcal{W}|^2dx+\int_{\mathbb{R}^3}|\nabla\cdot\partial_3\mathcal{W}|^2dx+2\int_{\mathbb{R}^3}|\partial_3\mathcal{W}|^2dx$$
$$=-\int_{\mathbb{R}^3}\partial_3\mathcal{W}\cdot\partial_3\mathcal{U}\cdot\nabla\mathcal{W}dx+\int_{\mathbb{R}^3}\partial_3\mathcal{W}\cdot\nabla\times\partial_3\mathcal{U}dx. \tag{4.13}$$

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^3}|\partial_3\mathcal{V}|^2dx+\int_{\mathbb{R}^3}|\nabla\partial_3\mathcal{V}|^2dx-\int_{\mathbb{R}^3}\partial_3\mathcal{V}\cdot\mathcal{V}\cdot\nabla\partial_3\mathcal{U}dx=-\int_{\mathbb{R}^3}\partial_3\mathcal{V}\cdot\partial_3\mathcal{U}\cdot\nabla\mathcal{V}dx$$
$$+\int_{\mathbb{R}^3}\partial_3\mathcal{V}\cdot\partial_3\mathcal{V}\cdot\nabla\mathcal{U}dx. \tag{4.14}$$

Adding (4.12), (4.13) and (4.14), we obtain

$$\frac{1}{2}\frac{d}{dt}\big(\|\partial_3\mathcal{U}\|_{L^2}^2+\|\partial_3\mathcal{W}\|_{L^2}^2+\|\partial_3\mathcal{V}\|_{L^2}^2\big)+\big(\|\nabla\partial_3\mathcal{U}\|_{L^2}^2+\|\nabla\partial_3\mathcal{W}\|_{L^2}^2+\|\nabla\partial_3\mathcal{V}\|_{L^2}^2\big)$$
$$+\|div\ \partial_3\mathcal{W}\|_{L^2}^2+2\|\partial_3\mathcal{W}\|_{L^2}^2$$
$$=-\int_{\mathbb{R}^3}\partial_3\mathcal{U}\cdot\partial_3\mathcal{U}\cdot\nabla\mathcal{U}dx+\int_{\mathbb{R}^3}\partial_3\mathcal{U}\cdot\partial_3\mathcal{V}\cdot\nabla\mathcal{V}dx-\int_{\mathbb{R}^3}\partial_3\mathcal{W}\cdot\partial_3\mathcal{U}\cdot\nabla\mathcal{W}dx$$
$$+\int_{\mathbb{R}^3}\partial_3\mathcal{W}\cdot\nabla\times\partial_3\mathcal{U}dx-\int_{\mathbb{R}^3}\partial_3\mathcal{V}\cdot\partial_3\mathcal{U}\cdot\nabla\mathcal{V}dx+\int_{\mathbb{R}^3}\partial_3\mathcal{V}\cdot\partial_3\mathcal{V}\cdot\nabla\mathcal{U}dx$$
$$=P_1+P_2+P_3+P_4+P_5+P_6. \tag{4.15}$$

Now, we will find estimates for every term of (4.15), one by one, taking C as a generic constant.

$$|P_1|=\Big|\int_{\mathbb{R}^3}\partial_3\mathcal{U}\cdot\partial_3\mathcal{U}\cdot\nabla\mathcal{U}dx\Big|.$$

Using Holder's inequality, we obtain

$$|P_1|\le C\Big\|\Big\|\Big\|\partial_3\mathcal{U}\Big\|_{L^{l,\infty}_{x_1}}\Big\|_{L^{m,\infty}_{x_2}}\Big\|_{L^{n,\infty}_{x_3}}\Big\|\Big\|\Big\|\partial_3\mathcal{U}\Big\|_{L^{\frac{2l}{l-2},2}_{x_1}}\Big\|_{L^{\frac{2m}{m-2},2}_{x_2}}\Big\|_{L^{\frac{2n}{n-2},2}_{x_3}}\|\nabla\mathcal{U}\|_{L^2}$$

$$\le C\Big\|\Big\|\Big\|\partial_3\mathcal{U}\Big\|_{L^{l,\infty}_{x_1}}\Big\|_{L^{m,\infty}_{x_2}}\Big\|_{L^{n,\infty}_{x_3}}\|\partial_3\mathcal{U}\|_{L^2}^{1-(\frac{1}{l}+\frac{1}{m}+\frac{1}{n})}\|\partial_1\partial_3\mathcal{U}\|_{L^2}^{\frac{1}{l}}\|\partial_2\partial_3\mathcal{U}\|_{L^2}^{\frac{1}{m}}\|\partial_3\partial_3\mathcal{U}\|_{L^2}^{\frac{1}{n}}\|\nabla\mathcal{U}\|_{L^2}$$

$$\leq C\left\|\left\|\left\|\partial_3\mathcal{U}\right\|_{L^{l,\infty}_{x_1}}\right\|_{L^{m,\infty}_{x_2}}\right\|_{L^{n,\infty}_{x_3}}\|\partial_3\mathcal{U}\|_{L^2}^{1-(\frac{1}{l}+\frac{1}{m}+\frac{1}{n})}\|\nabla\partial_3\mathcal{U}\|_{L^2}^{\frac{1}{l}+\frac{1}{m}+\frac{1}{n}}\|\nabla\mathcal{U}\|_{L^2}.$$

Applying Young's inequality

$$\leq C\left(\left\|\left\|\left\|\partial_3\mathcal{U}\right\|_{L^{l,\infty}_{x_1}}\right\|_{L^{m,\infty}_{x_2}}\right\|_{L^{n,\infty}_{x_3}}^{\frac{2}{2-(\frac{1}{l}+\frac{1}{m}+\frac{1}{n})}}\|\partial_3\mathcal{U}\|_{L^2}^{2.\frac{1-(\frac{1}{l}+\frac{1}{m}+\frac{1}{n})}{2-(\frac{1}{l}+\frac{1}{m}+\frac{1}{n})}}\|\nabla\mathcal{U}\|_{L^2}^{\frac{2}{2-(\frac{1}{l}+\frac{1}{m}+\frac{1}{n})}}\right)+\|\nabla\partial_3\mathcal{U}\|_{L^2}^2.$$

Adjusting above inequality's exponents to apply again Young's inequality

$$\leq C\left(\left\|\left\|\left\|\partial_3\mathcal{U}\right\|_{L^{l,\infty}_{x_1}}\right\|_{L^{m,\infty}_{x_2}}\right\|_{L^{n,\infty}_{x_3}}^{\frac{2}{1-(\frac{1}{l}+\frac{1}{m}+\frac{1}{n})}}\|\partial_3\mathcal{U}\|_{L^2}^2\right)^{\frac{1-(\frac{1}{l}+\frac{1}{m}+\frac{1}{n})}{2-(\frac{1}{l}+\frac{1}{m}+\frac{1}{n})}}\left(\|\nabla\mathcal{U}\|_{L^2}^2\right)^{\frac{1}{2-(\frac{1}{l}+\frac{1}{m}+\frac{1}{n})}}+\|\nabla\partial_3\mathcal{U}\|_{L^2}^2$$

$$\leq C\left(\left\|\left\|\left\|\partial_3\mathcal{U}\right\|_{L^{l,\infty}_{x_1}}\right\|_{L^{m,\infty}_{x_2}}\right\|_{L^{n,\infty}_{x_3}}^{\frac{2}{1-(\frac{1}{l}+\frac{1}{m}+\frac{1}{n})}}\|\partial_3\mathcal{U}\|_{L^2}^2+\|\nabla\mathcal{U}\|_{L^2}^2\right)+\|\nabla\partial_3\mathcal{U}\|_{L^2}^2.$$

Finally, we get an estimate for P_1 as

$$|P_1|\leq C(1+\|\partial_3\mathcal{U}\|_{L^2}^2)\left(\left\|\left\|\left\|\partial_3\mathcal{U}\right\|_{L^{l,\infty}_{x_1}}\right\|_{L^{m,\infty}_{x_2}}\right\|_{L^{n,\infty}_{x_3}}^{\frac{2}{1-(\frac{1}{l}+\frac{1}{m}+\frac{1}{n})}}+\|\nabla\mathcal{U}\|_{L^2}^2\right)+\|\nabla\partial_3\mathcal{U}\|_{L^2}^2. \quad (4.16)$$

Similarly, we get bound for P_6 as

$$|P_6|\leq C(1+\|\partial_3\mathcal{V}\|_{L^2}^2)\left(\left\|\left\|\left\|\partial_3\mathcal{V}\right\|_{L^{l,\infty}_{x_1}}\right\|_{L^{m,\infty}_{x_2}}\right\|_{L^{n,\infty}_{x_3}}^{\frac{2}{1-(\frac{1}{l}+\frac{1}{m}+\frac{1}{n})}}+\|\nabla\mathcal{U}\|_{L^2}^2\right)+\|\nabla\partial_3\mathcal{V}\|_{L^2}^2. \quad (4.17)$$

In case of P_4, using Holder's and Young's inequalities

$$|P_4|\leq\frac{1}{4}\|\nabla\partial_3\mathcal{U}\|_{L^2}^2+C\|\partial_3\mathcal{W}\|_{L^2}^2. \quad (4.18)$$

Estimating P_2, P_3 and P_5

$$|P_2|\leq C\left\|\left\|\left\|\partial_3\mathcal{V}\right\|_{L^{l,\infty}_{x_1}}\right\|_{L^{m,\infty}_{x_2}}\right\|_{L^{n,\infty}_{x_3}}\left\|\left\|\left\|\partial_3\mathcal{U}\right\|_{L^{\frac{2l}{l-2},2}_{x_1}}\right\|_{L^{\frac{2m}{m-2},2}_{x_2}}\right\|_{L^{\frac{2n}{n-2},2}_{x_3}}\|\nabla\mathcal{V}\|_{L^2}$$

$$\leq C\left\|\left\|\left\|\partial_3\mathcal{V}\right\|_{L^{l,\infty}_{x_1}}\right\|_{L^{m,\infty}_{x_2}}\right\|_{L^{n,\infty}_{x_3}}\|\partial_3\mathcal{U}\|_{L^2}^{1-(\frac{1}{l}+\frac{1}{m}+\frac{1}{n})}\|\partial_1\partial_3\mathcal{U}\|_{L^2}^{\frac{1}{l}}\|\partial_2\partial_3\mathcal{U}\|_{L^2}^{\frac{1}{m}}\|\partial_3\partial_3\mathcal{U}\|_{L^2}^{\frac{1}{n}}\|\nabla\mathcal{V}\|_{L^2}$$

$$\leq C\left\|\left\|\left\|\partial_3\mathcal{V}\right\|_{L^{l,\infty}_{x_1}}\right\|_{L^{m,\infty}_{x_2}}\right\|_{L^{n,\infty}_{x_3}}\|\partial_3\mathcal{U}\|_{L^2}^{1-(\frac{1}{l}+\frac{1}{m}+\frac{1}{n})}\|\nabla\partial_3\mathcal{U}\|_{L^2}^{\frac{1}{l}+\frac{1}{m}+\frac{1}{n}}\|\nabla\mathcal{V}\|_{L^2}.$$

Following on the same steps as for (4.16)

$$\leq C\left(\left\|\left\|\left\|\partial_3\mathcal{V}\right\|_{L^{l,\infty}_{x_1}}\right\|_{L^{m,\infty}_{x_2}}\right\|_{L^{n,\infty}_{x_3}}^{\frac{2}{1-(\frac{1}{l}+\frac{1}{m}+\frac{1}{n})}}\|\partial_3\mathcal{U}\|_{L^2}^2+\|\nabla\mathcal{V}\|_{L^2}^2\right)+\|\nabla\partial_3\mathcal{U}\|_{L^2}^2.$$

$$|P_2|\leq C(1+\|\partial_3\mathcal{U}\|_{L^2}^2)\left(\left\|\left\|\left\|\partial_3\mathcal{V}\right\|_{L^{l,\infty}_{x_1}}\right\|_{L^{m,\infty}_{x_2}}\right\|_{L^{n,\infty}_{x_3}}^{\frac{2}{1-(\frac{1}{l}+\frac{1}{m}+\frac{1}{n})}}+\|\nabla\mathcal{V}\|_{L^2}^2\right)+\|\nabla\partial_3\mathcal{U}\|_{L^2}^2. \quad (4.19)$$

$$|P_3|\leq C(1+\|\partial_3\mathcal{W}\|_{L^2}^2)\left(\left\|\left\|\left\|\partial_3\mathcal{U}\right\|_{L^{l,\infty}_{x_1}}\right\|_{L^{m,\infty}_{x_2}}\right\|_{L^{n,\infty}_{x_3}}^{\frac{2}{1-(\frac{1}{l}+\frac{1}{m}+\frac{1}{n})}}+\|\nabla\mathcal{W}\|_{L^2}^2\right)+\|\nabla\partial_3\mathcal{W}\|_{L^2}^2. \quad (4.20)$$

$$|P_5|\leq C(1+\|\partial_3\mathcal{V}\|_{L^2}^2)\left(\left\|\left\|\left\|\partial_3\mathcal{U}\right\|_{L^{l,\infty}_{x_1}}\right\|_{L^{m,\infty}_{x_2}}\right\|_{L^{n,\infty}_{x_3}}^{\frac{2}{1-(\frac{1}{l}+\frac{1}{m}+\frac{1}{n})}}+\|\nabla\mathcal{U}\|_{L^2}^2\right)+\|\nabla\partial_3\mathcal{V}\|_{L^2}^2. \quad (4.21)$$

Now we will find L^2-estimates for the gradient of velocity, magnetic field and micro-rotational velocity. In order to get required estimates, multiply $(4.1)_1$, $(4.1)_2$, $(4.1)_3$ with $-\Delta\mathcal{U}$, $-\Delta\mathcal{W}$, $-\Delta\mathcal{V}$, respectively, then integrating over $\mathbb{R}^3$, adding the resulting three equations, we obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}(\|\nabla\mathcal{U}\|_{L^2}^2+\|\nabla\mathcal{W}\|_{L^2}^2+\|\nabla\mathcal{V}\|_{L^2}^2)+(\|\Delta\mathcal{U}\|_{L^2}^2+\|\Delta\mathcal{W}\|_{L^2}^2+\|\Delta\mathcal{V}\|_{L^2}^2)\\ +\|\nabla div\mathcal{W}\|_{L^2}^2+2\|\nabla\mathcal{W}\|_{L^2}^2\\ \leq(\Delta\mathcal{U},\mathcal{U}\cdot\nabla\mathcal{U})-(\Delta\mathcal{U},\mathcal{V}\cdot\nabla\mathcal{V})+(\Delta\mathcal{V},\mathcal{V}\cdot\nabla\mathcal{V})-(\Delta\mathcal{V},\mathcal{V}\cdot\nabla\mathcal{U})\\ +(\Delta\mathcal{W},\mathcal{W}\cdot\nabla\mathcal{W})-2(\Delta\mathcal{W},\nabla\times\mathcal{U})\\ =\hat{\beta}_1+\hat{\beta}_2+\hat{\beta}_3+\hat{\beta}_4+\hat{\beta}_5+\hat{\beta}_6.\end{aligned} \quad (4.22)$$

The terms in (4.22) are bounded by Tang et al. [73] in inequality (33). For detailed prove see [73].

$$\Longrightarrow \|\nabla\mathcal{U}\|_{L^2}^2+\|\nabla\mathcal{W}\|_{L^2}^2+\|\nabla\mathcal{V}\|_{L^2}^2<\infty. \quad (4.23)$$

This implies the fact

$$(\mathcal{U},\mathcal{V},\mathcal{W})\in L^\infty(0,T,H^1(\mathbb{R}^3))\cap L^2(0,T,H^2(\mathbb{R}^3)).$$

Putting all estimates in (4.15), after simplifications, it yields

$$\frac{d}{dt}\big(\|\partial_3\mathcal{U}\|_{L^2}^2+\|\partial_3\mathcal{W}\|_{L^2}^2+\|\partial_3\mathcal{V}\|_{L^2}^2\big)+2\big(\|\nabla\partial_3\mathcal{U}\|_{L^2}^2+\|\nabla\partial_3\mathcal{W}\|_{L^2}^2+\|\nabla\partial_3\mathcal{V}\|_{L^2}^2\big)$$

$$+2\|div\ \partial_3\mathcal{W}\|_{L^2}^2+2\|\partial_3\mathcal{W}\|_{L^2}^2$$

$$\leq C\Big(1+\|\partial_3\mathcal{U}\|_{L^2}^2+\|\partial_3\mathcal{V}\|_{L^2}^2+\|\partial_3\mathcal{W}\|_{L^2}^2\Big)\Bigg(\Bigg\|\Big\|\Big\|(\partial_3\mathcal{U},\partial_3\mathcal{V})\Big\|_{L^{l,\infty}_{x_1}}\Big\|_{L^{m,\infty}_{x_2}}\Bigg\|_{L^{n,\infty}_{x_3}}^{\frac{2}{1-(\frac{1}{l}+\frac{1}{m}+\frac{1}{n})}}$$

$$+\|\nabla\mathcal{U}\|_{L^2}^2+\|\nabla\mathcal{V}\|_{L^2}^2+\|\nabla\mathcal{W}\|_{L^2}^2\Bigg).$$

Invoking Gronwall's inequality with (4.23), we get

$$\sup_{0\leq t\leq T}\big(\|\partial_3\mathcal{U}\|_{L^2}^2+\|\partial_3\mathcal{W}\|_{L^2}^2+\|\partial_3\mathcal{V}\|_{L^2}^2\big)+2\int_0^t\big(\|\nabla\partial_3\mathcal{U}\|_{L^2}^2+\|\nabla\partial_3\mathcal{W}\|_{L^2}^2+\|\nabla\partial_3\mathcal{V}\|_{L^2}^2\big)d\tau$$

$$+2\int_0^t\|div\ \partial_3\mathcal{W}\|_{L^2}^2d\tau+2\int_0^t\|\partial_3\mathcal{W}\|_{L^2}^2d\tau$$

$$\leq C\Big(1+\|\partial_3\mathcal{U}_0\|_{L^2}^2+\|\partial_3\mathcal{V}_0\|_{L^2}^2+\|\partial_3\mathcal{W}_0\|_{L^2}^2\Big)\Bigg(\Bigg\|\Big\|\Big\|(\partial_3\mathcal{U},\partial_3\mathcal{V})\Big\|_{L^{l,\infty}_{x_1}}\Big\|_{L^{m,\infty}_{x_2}}\Bigg\|_{L^{n,\infty}_{x_3}}^{\frac{2}{1-(\frac{1}{l}+\frac{1}{m}+\frac{1}{n})}}$$

$$+\|\nabla\mathcal{U}\|_{L^2}^2+\|\nabla\mathcal{V}\|_{L^2}^2+\|\nabla\mathcal{W}\|_{L^2}^2\Bigg)$$

$$\sup_{0\leq t\leq T}\big(\|\partial_3\mathcal{U}\|_{L^2}^2+\|\partial_3\mathcal{W}\|_{L^2}^2+\|\partial_3\mathcal{V}\|_{L^2}^2\big)+2\int_0^t\big(\|\nabla\partial_3\mathcal{U}\|_{L^2}^2+\|\nabla\partial_3\mathcal{W}\|_{L^2}^2+\|\nabla\partial_3\mathcal{V}\|_{L^2}^2\big)d\tau$$

$$+2\int_0^t\|div\ \partial_3\mathcal{W}\|_{L^2}^2d\tau+2\int_0^t\|\partial_3\mathcal{W}\|_{L^2}^2d\tau\leq C.$$

Which completes the desired proof.

As the structure of the systems (4.1) and (4.2) suggests that the velocity plays more dominant role in the regularity of weak solutions than other unknowns. In view of these observations, we pose another problem. Can we prove a blow-up criterion that is only controlled by the one-directional derivative of velocity "$\partial_3\mathcal{U}$"?. Thanks to the distributional methods, we give positive answer to this question and prove this criteria for the system (4.2). Because system (4.2) is also important for the theoretical and mathematical purposes having wide range of applications in electro-chemical and fluid-mechanical transport.

4.3 An improved regularity result via velocity

Now, we present an improvement of our previous result.

Theorem 4.3.1 Assume that $(\mathcal{U}_0, \theta_0, \vartheta_0) \in H^1(\mathbb{R}^3)$ with $\nabla \cdot \mathcal{U}_0 = 0$ in the sense of distributions. The Leray-Hopf weak solution $(\mathcal{U}, \theta, \vartheta)$ to system (4.2) is regular on the interval (0,T], if

$$\int_0^T \left\| \left\| \left\| \partial_3 \mathcal{U} \right\|_{L^{l,\infty}_{x_1}} \right\|_{L^{m,\infty}_{x_2}} \right\|_{L^{n,\infty}_{x_3}}^{\frac{2}{1-(\frac{1}{l}+\frac{1}{m}+\frac{1}{n})}} dt < \infty, \tag{4.24}$$

where $2 < l, m, n \leq \infty$ and $1 - (\frac{1}{l} + \frac{1}{m} + \frac{1}{n}) \geq 0$. Otherwise, if $T = T^\star < \infty$ is the maximal time for the existence of smooth solution then the solution blows up to create finite time singularity that is

$$\int_0^{T^\star} \left\| \left\| \left\| \partial_3 \mathcal{U} \right\|_{L^{l,\infty}_{x_1}} \right\|_{L^{m,\infty}_{x_2}} \right\|_{L^{n,\infty}_{x_3}}^{\frac{2}{1-(\frac{1}{l}+\frac{1}{m}+\frac{1}{n})}} = \infty.$$

Result (4.24) is refinement of the result (4.10).

Result (4.24) is also true for the system (4.1) and refines the result (4.10).

4.3.1 Establishing estimates in anisotropic Lorentz space

The proof of Theorems 4.3.1 is based on distributional methods and setting up of a priori estimates under the blow-up condition (4.24).

Proof of Theorem 4.3.1 As a first step we will find L^2-estimates for $\mathcal{U}$, θ, ϑ and $\nabla\psi$. Multiplying $(4.2)_3$ with θ and $(4.2)_4$ with ϑ, integrating over $\mathbb{R}^3$, using divergence free condition $(4.2)_2$ and $(4.2)_5$, we obtain

$$\frac{1}{2}\frac{d}{dt}(\|\theta\|_{L^2}^2 + \|\vartheta\|_{L^2}^2) + (\|\nabla\theta\|_{L^2}^2 + \|\nabla\vartheta\|_{L^2}^2) + \int_{\mathbb{R}^3} (\theta + \vartheta)(\theta - \vartheta)^2 dx = 0. \tag{4.25}$$

As masses of θ and ϑ are conserved, θ and ϑ are non-negative, we infer from (4.25) that for all $0 \leq t \leq T$

$$(\|\theta\|_{L^2}^2 + \|\vartheta\|_{L^2}^2) + 2\int_0^t (\|\nabla\theta\|_{L^2}^2 + \|\nabla\vartheta\|_{L^2}^2) d\tau \leq \|\theta_0\|_{L^2}^2 + \|\vartheta_0\|_{L^2}^2. \tag{4.26}$$

Now, multiplying $(4.2)_1$ with $\mathcal{U}$, $(4.2)_3$, $(4.2)_4$ with ψ, integrating over $\mathbb{R}^3$, and using $(4.2)_5$, it gives

$$\frac{1}{2}\frac{d}{dt}\|\mathcal{U}\|_{L^2}^2 + \|\nabla\mathcal{U}\|_{L^2}^2 - \int_{\mathbb{R}^3}(\theta-\vartheta)\mathcal{U}\cdot\nabla\psi dx = 0, \tag{4.27}$$

$$\int_{\mathbb{R}^3}\Big[\frac{\partial\theta}{\partial t}\psi + \nabla\cdot(\theta\nabla\psi)\psi - \Delta\theta\psi + (\mathcal{U}\cdot\nabla)\theta\psi\Big]dx = 0, \tag{4.28}$$

$$\int_{\mathbb{R}^3}\Big[\frac{\partial\vartheta}{\partial t}\psi + \nabla\cdot(\vartheta\nabla\psi)\psi - \Delta\vartheta\psi + (\mathcal{U}\cdot\nabla)\vartheta\psi\Big]dx = 0. \tag{4.29}$$

Subtracting (4.29) from (4.28), using integration by parts and $\Delta\psi = \theta - \vartheta$, we get

$$\frac{1}{2}\frac{d}{dt}\|\nabla\psi\|_{L^2}^2 + \int_{\mathbb{R}^3}(\theta+\vartheta)|\nabla\psi|^2dx + \int_{\mathbb{R}^3}|\Delta\psi|^2dx + \int_{\mathbb{R}^3}(\theta-\vartheta)\mathcal{U}\cdot\nabla\psi dx = 0. \tag{4.30}$$

Adding (4.27) and (4.30), it follows that

$$\frac{1}{2}\frac{d}{dt}(\|\mathcal{U}\|_{L^2}^2 + \|\nabla\psi\|_{L^2}^2) + \|\nabla\mathcal{U}\|_{L^2}^2 + \|\Delta\psi\|_{L^2}^2 + \int_{\mathbb{R}^3}(\theta+\vartheta)|\nabla\psi|^2dx = 0. \tag{4.31}$$

Because of the non-negativity of θ and ϑ, we obtained the final bound

$$\|\mathcal{U}\|_{L^2}^2 + \|\nabla\psi\|_{L^2}^2 + 2\int_0^t\|\nabla\mathcal{U}\|_{L^2}^2 + \|\Delta\psi\|_{L^2}^2 d\tau \le C. \tag{4.32}$$

Now, we will find H^1-estimates for $\mathcal{U}$, θ and ϑ. For required bounds multiply $-\Delta\mathcal{U}$ with $(4.2)_1$, integrating over $\mathbb{R}^3$, we get

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\nabla\mathcal{U}\|_{L^2}^2 + \|\Delta\mathcal{U}\|_{L^2}^2 &= \int_{\mathbb{R}^3}(\mathcal{U}\cdot\nabla)\mathcal{U}\cdot\Delta\mathcal{U}dx - \int_{\mathbb{R}^3}\Delta\psi\nabla\psi\cdot\Delta\mathcal{U}dx \\ &= Q_1 + Q_2.\end{aligned} \tag{4.33}$$

For Q_2, using Holder's and Young's inequalities, using $\Delta\psi = \theta - \vartheta$, interpolation inequality $\|\nabla f\|_{L^4} \le \|f\|_{L^4}^{\frac{1}{8}}\|\Delta f\|_{L^4}^{\frac{7}{8}}$, and combining (4.26), (4.32), we obtain

$$\begin{aligned}|Q_2| &\le \|\Delta\psi\|_{L^4}\|\nabla\psi\|_{L^4}\|\Delta\mathcal{U}\|_{L^2} \\ &\le \frac{1}{4}\|\Delta\mathcal{U}\|_{L^2}^2 + C\|\nabla\psi\|_{L^4}^2\|(\theta,\vartheta)\|_{L^4}^2 \\ &\le \frac{1}{4}\|\Delta\mathcal{U}\|_{L^2}^2 + C\|\nabla\psi\|_{L^2}^2\|(\theta,\vartheta)\|_{L^2}^2 + C\|(\nabla\theta,\nabla\vartheta)\|_{L^2}^2\|(\theta,\vartheta)\|_{L^2}^2 \\ &\le \frac{1}{4}\|\Delta\mathcal{U}\|_{L^2}^2 + C(\|\nabla\theta\|_{L^2}^2 + \|\nabla\vartheta\|_{L^2}^2 + 1).\end{aligned} \tag{4.34}$$

For Q_1

$$|Q_1| \le \int_{\mathbb{R}^3}\nabla\mathcal{U}\nabla\mathcal{U}\nabla\mathcal{U}dx$$

$$\leq C\|\nabla\mathcal{U}\|_{L^3}^3 \leq C\|\nabla\mathcal{U}\|_{L^3}^{\frac{3}{2}}\|\nabla\mathcal{U}\|_{L^6}^{\frac{3}{2}} \qquad \text{(Interpolation inequality)}$$

$$\leq C\|\nabla\mathcal{U}\|_{L^2}^{\frac{3}{2}}\|\nabla\partial_1\mathcal{U}\|_{L^2}^{\frac{1}{2}}\|\nabla\partial_2\mathcal{U}\|_{L^2}^{\frac{1}{2}}\|\nabla\partial_3\mathcal{U}\|_{L^2}^{\frac{1}{2}} \qquad \text{(Lemma 2.5., for } \alpha_4 = 2)$$

$$\leq \|\nabla\mathcal{U}\|_{L^2}^{\frac{3}{2}}\|\nabla^2\mathcal{U}\|_{L^2}\|\nabla\partial_3\mathcal{U}\|_{L^2}^{\frac{1}{2}}$$

$$\leq \frac{1}{4}\|\Delta\mathcal{U}\|_{L^2}^2 + C\|\nabla\mathcal{U}\|_{L^2}^3\|\nabla\partial_3\mathcal{U}\|_{L^2}. \qquad \text{(Young's inequality)} \tag{4.35}$$

Putting (4.34) and (4.35) into (4.33), and employing Gronwall's inequality, it yields

$$\sup_{0\leq t\leq T}\|\nabla\mathcal{U}\|_{L^2}^2 + 2\int_0^t\|\Delta\mathcal{U}\|_{L^2}^2 d\tau \leq (\|\nabla\mathcal{U}_0\|_{L^2}^2 + e)\exp(C\int_0^t(\|\nabla\partial_3\mathcal{U}\|_{L^2}^2 + \|\nabla\mathcal{U}\|_{L^2}^2 + \|\nabla\theta\|_{L^2}^2$$

$$+\|\nabla\vartheta\|_{L^2}^2 + 1)d\tau). \tag{4.36}$$

$$\Longrightarrow \mathcal{U} \in L^\infty(0,T,H^1(\mathbb{R}^3)) \cap L^2(0,T,H^2(\mathbb{R}^3)).$$

To get similar results for θ and ϑ. Multiply $-\Delta\theta$ with $(4.2)_3$ and $-\Delta\vartheta$ with $(4.2)_4$, we achieve

$$\sup_{0\leq t\leq T}(\|\nabla\theta\|_{L^2}^2 + \|\nabla\vartheta\|_{L^2}^2) + 2\int_0^t(\|\Delta\theta\|_{L^2}^2 + \|\Delta\vartheta\|_{L^2}^2)d\tau \leq C. \tag{4.37}$$

For our desired results, differentiate $(4.2)_1$ with respect to x_3, then multiply by $\partial_3\mathcal{U}$ and integrating by parts to get

$$\frac{1}{2}\frac{d}{dt}\|\partial_3\mathcal{U}\|_{L^2}^2 + \|\nabla\partial_3\mathcal{U}\|_{L^2}^2 = -\int_{\mathbb{R}^3}\partial_3(\mathcal{U}\cdot\nabla\mathcal{U})\cdot\partial_3\mathcal{U}dx + \int_{\mathbb{R}^3}\partial_3(\Delta\psi\nabla\psi)\partial_3\mathcal{U}dx$$

$$= D_1 + D_2. \tag{4.38}$$

Estimating D_2 as (4.34), we obtain

$$|D_2| \leq \int_{\mathbb{R}^3}\partial_3(\Delta\psi\nabla\psi)\partial_3\mathcal{U}dx$$

$$\leq \frac{1}{4}\|\nabla\partial_3\mathcal{U}\|_{L^2}^2 + C\|(\theta,\vartheta)\|_{L^2}^2\|(\nabla\theta,\nabla\vartheta)\|_{L^2}^2 + C\|(\theta,\vartheta)\|_{L^2}^2\|\nabla\psi\|_{L^2}^2$$

$$\leq \frac{1}{4}\|\nabla\partial_3\mathcal{U}\|_{L^2}^2 + C(\|\nabla\theta\|_{L^2}^2 + \|\nabla\vartheta\|_{L^2}^2 + 1). \tag{4.39}$$

Similar to (4.16), D_1 is estimated as

$$|D_1| \leq C(1 + \|\partial_3\mathcal{U}\|_{L^2}^2)\left(\left\|\left\|\left\|\partial_3\mathcal{U}\right\|_{L^{l,\infty}_{x_1}}\right\|_{L^{m,\infty}_{x_2}}\right\|_{L^{n,\infty}_{x_3}}^{\frac{2}{1-(\frac{1}{l}+\frac{1}{m}+\frac{1}{n})}} + \|\nabla\mathcal{U}\|_{L^2}^2\right) + \|\nabla\partial_3\mathcal{U}\|_{L^2}^2. \tag{4.40}$$

putting (4.39), (4.40) into (4.38)

$$\frac{1}{2}\frac{d}{dt}(1 + \|\partial_3\mathcal{U}\|_{L^2}^2) + \|\nabla\partial_3\mathcal{U}\|_{L^2}^2$$

$$\leq C(1+\|\partial_3\mathcal{U}\|_{L^2}^2)\left(\left\|\left\|\left\|\partial_3\mathcal{U}\right\|_{L^{l,\infty}_{x_1}}\right\|_{L^{m,\infty}_{x_2}}\right\|_{L^{n,\infty}_{x_3}}^{\frac{2}{1-(\frac{1}{l}+\frac{1}{m}+\frac{1}{n})}}+\|(\nabla\mathcal{U},\nabla\theta,\nabla\vartheta)\|_{L^2}^2+1\right).$$

Applying Gronwall's inequality together with (4.36) and (4.37) yields

$$(1+\|\partial_3\mathcal{U}\|_{L^2}^2)+2\int_0^T\|\nabla\partial_3\mathcal{U}\|_{L^2}^2d\tau\leq(1+\|\partial_3\mathcal{U}_0\|_{L^2}^2)\exp\left(C\int_0^T\left(\left\|\left\|\left\|\partial_3\mathcal{U}\right\|_{L^{l,\infty}_{x_1}}\right\|_{L^{m,\infty}_{x_2}}\right\|_{L^{n,\infty}_{x_3}}^{\frac{2}{1-(\frac{1}{l}+\frac{1}{m}+\frac{1}{n})}}\right.\right.$$

$$\left.\left.+\|(\nabla\mathcal{U},\nabla\theta,\nabla\vartheta)\|_{L^2}^2+1\right)d\tau.\right.$$

$$\sup_{0\leq t\leq T}(1+\|\partial_3\mathcal{U}\|_{L^2}^2)+2\int_0^T\|\nabla\partial_3\mathcal{U}\|_{L^2}^2d\tau\leq C. \tag{4.41}$$

The bound (4.41) ensures the smoothness of weak solutions of system (4.2) on the interval $(0,T]$. Hence proved.

Chapter 5

Regularity in anisotropic Lorentz spaces

5.1 Introduction

This chapter focuses on two new regularity criterion based on pressure and its gradient to the Cauchy problem of the 3D magneto-micropolar system i.e., for the system (4.1) in anisotropic Lorentz spaces.

Before presenting our main findings, first, we will go over the problem's history. Ignoring the micro-rotational effects in (4.1) models the magnetohydrodynamics flows. The finite-time singularity problem for MHD flows has been extensively tackled by different authors (see [74–80]) but it is still an important open problem. For the magneto-hydrodynamics system Zhou [81] obtained the conditions

$$\Psi \in L^{l,m}, \quad \mathcal{V} \in L^{2l,2m}, \text{ or } \|\Psi\|_{L^{\infty,\frac{3}{l}}}, \quad \|\mathcal{V}\|_{L^{\infty,3}}, \tag{5.1}$$

$$\text{where } \frac{2}{l} + \frac{3}{m} \leq 2, \quad \frac{3}{2} < m \leq \infty,$$

and

$$\nabla\Psi \in L^{l,m}, \quad \mathcal{V} \in L^{3l,3m}, \text{ or } \|\nabla\Psi\|_{L^{\infty,3}}, \quad \|\nabla\mathcal{V}\|_{L^{\infty,3}}, \tag{5.2}$$

$$\text{where } \frac{2}{l} + \frac{3}{m} \leq 3, \quad 1 < m \leq \infty.$$

This important result in Lorentz space for micropolar fluid system was presented by Yuan [82] as

$$\nabla\Psi \in L^m(0,T,L^{l,\infty}) \text{ with } \frac{2}{m} + \frac{3}{l} \leq 3, \quad 1 < l \leq \infty. \tag{5.3}$$

Feng-Ping and Guang-Xia [83] presented the following criteria

$$\nabla\Psi \in L^m(0,T,L^{l,\infty}) \text{ with } \frac{2}{m} + \frac{3}{l} \leq 3, \quad m \geq 2, \quad l > 1, \tag{5.4}$$

$$\nabla\Psi \in L^{\frac{2}{3}}(0,T,BMO). \tag{5.5}$$

Recently, Li and Niu [84] presented the regularity criteria in Lorentz spaces

$$\Psi \in L^{m,\infty}(0,T,L^{l,\infty}) \text{ with } \frac{2}{m} + \frac{3}{l} = 2, \quad m \geq 2, \quad \frac{3}{2} < l \leq \infty. \tag{5.6}$$

Motivated by the above results specifically in Lebesgue and Lorentz spaces we establish new conditions in generalize Lorentz spaces that is in anisotropic Lorentz spaces.

5.2 Pressure regularity of weak solutions

Theorem 5.2.1. Suppose $(\mathcal{U}_0, \mathcal{V}_0, \mathcal{W}_0) \in H^1(\mathbb{R}^3)$ with $\nabla \cdot \mathcal{U}_0 = \nabla \cdot \mathcal{V}_0 = 0$. Suppose that $(\mathcal{U}, \mathcal{V}, \mathcal{W})$ is the weak solution of the system (4.1). If

$$\int_0^T \left\| \left\| \left\| \Psi \right\|_{L^{l,\infty}_{x_1}} \right\|_{L^{m,\infty}_{x_2}} \right\|_{L^{n,\infty}_{x_3}}^{\frac{2}{2-(\frac{1}{l}+\frac{1}{m}+\frac{1}{n})}} dt < \infty, \tag{5.7}$$

then the solution remains its smoothness upto T. Where $2 \leq l, m, n \leq \infty$ and 1-$(\frac{1}{l} + \frac{1}{m} + \frac{1}{n}) \geq 0$.

Proof. For finding L^4-estimates, take inner product of $(4.1)_1$ with $|\mathcal{U}|^2\mathcal{U}$, $(4.1)_2$ with $|\mathcal{W}|^2\mathcal{W}$, and $(4.1)_3$ with $|\mathcal{V}|^2\mathcal{V}$. After adding all the equations, we finally get that

$$\frac{1}{4}\frac{d}{dt}(\|\mathcal{U}\|_{L^4}^4 + \|\mathcal{W}\|_{L^4}^4 + \|\mathcal{V}\|_{L^4}^4) + \||\nabla\mathcal{U}||\mathcal{U}|\|_{L^2}^2 + \frac{1}{2}\|\nabla|\mathcal{U}|^2\|_{L^2}^2$$

$$+\||\nabla\mathcal{W}||\mathcal{W}|\|_{L^2}^2 + \frac{1}{2}\|\nabla|\mathcal{W}|^2\|_{L^2}^2 + \|div\ \mathcal{W}\|_{L^2}^2 + 2\|\mathcal{W}\|_{L^4}^4 + \||\nabla\mathcal{V}||\mathcal{V}|\|_{L^2}^2$$

$$+2\|\nabla|\mathcal{V}||\mathcal{V}|\|_{L^2}^2$$

$$\leq 2\int_{\mathbb{R}^3} |\Psi||\mathcal{U}|^2|\nabla\mathcal{U}|dx + 3\int_{\mathbb{R}^3} |\mathcal{W}||\mathcal{U}|^2|\nabla\mathcal{U}|dx + 3\int_{\mathbb{R}^3} |\mathcal{U}||\mathcal{W}|^2|\nabla\mathcal{W}|dx$$

$$- \int_{\mathbb{R}^3} |\mathcal{V}||\nabla(|\mathcal{U}|^2\mathcal{U})||\mathcal{V}|dx + \int_{\mathbb{R}^3} |\mathcal{V}||\nabla(|\mathcal{V}|^2\mathcal{V})|\mathcal{U}||dx$$

$$= L_1 + L_2 + L_3 + L_4 + L_5. \tag{5.8}$$

Now we estimate L_1

$$2\int_{\mathbb{R}^3} |\Psi||\mathcal{U}|^2|\nabla\mathcal{U}|dx \leq \frac{1}{4}\|\nabla|\mathcal{U}|^2\|_{L^2}^2 + C\int_{\mathbb{R}^3} |\Psi||\Psi||\mathcal{U}|^2dx = P_1 + P_2.$$

For P_2, we obtain

$$P_2 = C\int_{\mathbb{R}^3} |\Psi||\Psi||\mathcal{U}|^2dx \leq C\left\| \left\| \left\| \Psi \right\|_{L^{l,\infty}_{x_1}} \right\|_{L^{m,\infty}_{x_2}} \right\|_{L^{n,\infty}_{x_3}} \left\| \left\| \left\| \Psi \right\|_{L^{\frac{2l}{l-2},2}_{x_1}} \right\|_{L^{\frac{2m}{m-2},2}_{x_2}} \right\|_{L^{\frac{2n}{n-2},2}_{x_3}} \||\mathcal{U}|^2\|_{L^2}$$

$$\leq C\left\| \left\| \left\| \Psi \right\|_{L^{l,\infty}_{x_1}} \right\|_{L^{m,}_{x_2}} \right\|_{L^{n,\infty}_{x_3}} \|\Psi\|_{L^2}^{1-(\frac{1}{l}+\frac{1}{m}+\frac{1}{n})} \|\partial_1\Psi\|_{L^2}^{\frac{1}{l}} \|\partial_2\Psi\|_{L^2 L^2}^{\frac{1}{m}} \|\partial_3\Psi\|_{L^2}^{\frac{1}{n}} \|\mathcal{U}\|_{L^4}^2$$

$$\leq C\left\| \left\| \left\| \Psi \right\|_{L^{l,\infty}_{x_1}} \right\|_{L^{m,\infty}_{x_2}} \right\|_{L^{n,\infty}_{x_3}} \|\Psi\|_{L^2}^{1-(\frac{1}{l}+\frac{1}{m}+\frac{1}{n})} \|\nabla\Psi\|_{L^2}^{\frac{1}{l}+\frac{1}{m}+\frac{1}{n}} \|\mathcal{U}\|_{L^4}^2$$

$$\leq C\left\| \left\| \left\| \Psi \right\|_{L^{l,\infty}_{x_1}} \right\|_{L^{m,\infty}_{x_2}} \right\|_{L^{n,\infty}_{x_3}} \||\mathcal{U}|\nabla\mathcal{U}\|_{L^2}^{\frac{1}{l}+\frac{1}{m}+\frac{1}{n}} \|\mathcal{U}\|_{L^4}^{4-(\frac{2}{l}+\frac{2}{m}+\frac{2}{n})}$$

$$P_2 \leq \frac{1}{4}\||\mathcal{U}|\nabla\mathcal{U}\|_{L^2}^2 + C\left\|\left\|\left\|\Psi\right\|_{L^{l,\infty}_{x_1}}\right\|_{L^{m,\infty}_{x_2}}\right\|_{L^{n,\infty}_{x_3}}^{\frac{2}{2-(\frac{1}{l}+\frac{1}{m}+\frac{1}{n})}}\|\mathcal{U}\|_{L^4}^4.$$

The final estimates for L_1 are

$$L_1 \leq \frac{1}{4}\|\nabla|\mathcal{U}|^2\|_{L^2}^2 + \frac{1}{4}\||\mathcal{U}|\nabla\mathcal{U}\|_{L^2}^2 + C\left\|\left\|\left\|\Psi\right\|_{L^{l,\infty}_{x_1}}\right\|_{L^{m,\infty}_{x_2}}\right\|_{L^{n,\infty}_{x_3}}^{\frac{2}{2-(\frac{1}{l}+\frac{1}{m}+\frac{1}{n})}}\|\mathcal{U}\|_{L^4}^4. \tag{5.9}$$

Assessing L^4-estimates for L_2 and L_3

$$L_2 \leq \frac{1}{2}\||\mathcal{U}||\nabla\mathcal{U}|\|_{L^2}^2 + C\Big(\|\mathcal{U}\|_{L^4}^4 + \|\mathcal{W}\|_{L^4}^4\Big). \tag{5.10}$$

$$L_3 \leq \frac{1}{2}\||\mathcal{W}||\nabla\mathcal{W}|\|_{L^2}^2 + C\Big(\|\mathcal{U}\|_{L^4}^4 + \|\mathcal{W}\|_{L^4}^4\Big). \tag{5.11}$$

Now, assessing L^2-estimates for L_4 and L_5

$$\begin{aligned}
L_4 &\leq C\||\mathcal{V}|^2|\mathcal{U}|\|_{L^2}\|\nabla|\mathcal{U}|^2\|_{L^2} \leq C\||\mathcal{V}|^2|\mathcal{U}|\|_{L^2}^2 + \frac{1}{4}\|\nabla|\mathcal{U}|^2\|_{L^2}^2 \\
&\leq C\||\mathcal{V}|^2\|_{L^6}^2\||\mathcal{U}|\|_{L^3}^2 + \frac{1}{4}\|\nabla|\mathcal{U}|^2\|_{L^2}^2 \leq C\|\nabla|\mathcal{V}|^2\|_{L^2}^2\|\nabla\mathcal{U}|\|_{L^2}\||\mathcal{U}|\|_{L^2} + \frac{1}{4}\|\nabla|\mathcal{U}|^2\|_{L^2}^2 \\
&\leq C\||\mathcal{V}|\nabla|\mathcal{V}|\|_{L^2}^2 + \frac{1}{4}\|\nabla|\mathcal{U}|^2\|_{L^2}^2
\end{aligned} \tag{5.12}$$

$$\begin{aligned}
L_5 &\leq C\||\mathcal{V}^2|\mathcal{U}|\|_{L^2}^2 + \frac{1}{8}\|\nabla|\mathcal{V}|^2\|_{L^2} \\
&\leq C\||\mathcal{V}|\nabla|\mathcal{V}|\|_{L^2}^2.
\end{aligned} \tag{5.13}$$

Putting all the estimates in (5.8) results as

$$\begin{aligned}
&\frac{1}{4}\frac{d}{dt}(\|\mathcal{U}\|_{L^4}^4 + \|\mathcal{W}\|_{L^4}^4 + \|\mathcal{V}\|_{L^4}^4) + \||\nabla\mathcal{U}||\mathcal{U}|\|_{L^2}^2 + \frac{1}{2}\|\nabla|\mathcal{U}|^2\|_{L^2}^2 \\
&+\||\nabla\mathcal{W}||\mathcal{W}|\|_{L^2}^2 + \frac{1}{2}\|\nabla|\mathcal{W}|^2\|_{L^2}^2 + \|div\ \mathcal{W}\|_{L^2}^2 + 2\|\mathcal{W}\|_{L^4}^4 + \||\nabla\mathcal{V}||\mathcal{V}|\|_{L^2}^2 \\
&+2\|\nabla|\mathcal{V}||\mathcal{V}|\|_{L^2}^2 \\
&\leq \frac{1}{4}\|\nabla|\mathcal{U}|^2\|_{L^2}^2 + \frac{1}{4}\||\mathcal{U}|\nabla\mathcal{U}\|_{L^2}^2 + C\left\|\left\|\left\|\Psi\right\|_{L^{l,\infty}_{x_1}}\right\|_{L^{m,\infty}_{x_2}}\right\|_{L^{n,\infty}_{x_3}}^{\frac{2}{2-(\frac{1}{l}+\frac{1}{m}+\frac{1}{n})}}\|\mathcal{U}\|_{L^4}^4 + \frac{1}{2}\||\mathcal{U}||\nabla\mathcal{U}|\|_{L^2}^2 \\
&+\frac{1}{2}\||\mathcal{W}||\nabla\mathcal{W}|\|_{L^2}^2 + C\Big(\|\mathcal{U}\|_{L^4}^4 + \|\mathcal{W}\|_{L^4}^4\Big) + C\||\mathcal{V}|\nabla|\mathcal{V}|\|_{L^2}^2 + \frac{1}{4}\|\nabla|\mathcal{U}|^2\|_{L^2}^2 \\
&+C\||\mathcal{V}|\nabla|\mathcal{V}|\|_{L^2}^2.
\end{aligned} \tag{5.14}$$

Simplification yields

$$\frac{1}{4}\frac{d}{dt}(\|\mathcal{U}\|_{L^4}^4 + \|\mathcal{W}\|_{L^4}^4 + \|\mathcal{V}\|_{L^4}^4) + \||\nabla\mathcal{U}||\mathcal{U}|\|_{L^2}^2 + \frac{1}{2}\|\nabla|\mathcal{U}|^2\|_{L^2}^2$$

$$+\||\nabla\mathcal{W}||\mathcal{W}|\|_{L^2}^2+\frac{1}{2}\|\nabla|\mathcal{W}|^2\|_{L^2}^2+\|div\ \mathcal{W}\|_{L^2}^2+2\|\mathcal{W}\|_{L^4}^4+\||\nabla\mathcal{V}||\mathcal{V}|\|_{L^2}^2$$

$$+2\|\nabla|\mathcal{V}||\mathcal{V}|\|_{L^2}^2$$

$$\leq C\Big(1+\Big\|\Big\|\Big\|\Psi\Big\|_{L^{l,\infty}_{x_1}}\Big\|_{L^{m,\infty}_{x_2}}\Big\|_{L^{n,\infty}_{x_3}}^{\frac{2}{2-(\frac{1}{l}+\frac{1}{m}+\frac{1}{n})}}\Big)\Big(\|\mathcal{U}\|_{L^4}^4+\|\mathcal{W}\|_{L^4}^4+\|\mathcal{V}\|_{L^4}^4\Big). \tag{5.15}$$

Gronwall's Lemma results in

$$\sup_{0\leq t\leq T}(\|\mathcal{U}\|_{L^4}^4+\|\mathcal{W}\|_{L^4}^4+\|\mathcal{V}\|_{L^4}^4)$$

$$\leq C\exp\int_0^t\{\Big(1+\Big\|\Big\|\Big\|\Psi\Big\|_{L^{l,\infty}_{x_1}}\Big\|_{L^{m,\infty}_{x_2}}\Big\|_{L^{n,\infty}_{x_3}}^{\frac{2}{2-(\frac{1}{l}+\frac{1}{m}+\frac{1}{n})}}\Big)\}\Big(\|\mathcal{U}_0\|_{L^4}^4+\|\mathcal{W}_0\|_{L^4}^4+\|\mathcal{V}_0\|_{L^4}^4\Big)$$

$$\Longrightarrow \sup_{0\leq t\leq T}\Big(\|\mathcal{U}\|_{L^4}^4+\|\mathcal{W}\|_{L^4}^4+\|\mathcal{V}\|_{L^4}^4\Big)<\infty. \tag{5.16}$$

Which proves Theorem 5.2.1. as desired.

5.3 Gradient pressure regularity of weak solutions

Theorem 5.3.1 Let $(\mathcal{U}_0,\mathcal{V}_0,\mathcal{W}_0)\in H^1(\mathbb{R}^3)$ with $\nabla\cdot\mathcal{U}_0=\nabla\cdot\mathcal{V}_0=0$ in the distributional sense. Suppose $(\mathcal{U},\mathcal{V},\mathcal{W})$ be the weak solution to system (4.1). If

$$\int_0^T\Big\|\Big\|\Big\|\nabla\Psi\Big\|_{L^{l,\infty}_{x_1}}\Big\|_{L^{m,\infty}_{x_2}}\Big\|_{L^{n,\infty}_{x_3}}^{\frac{2}{3-(\frac{1}{l}+\frac{1}{m}+\frac{1}{n})}}dt<\infty, \tag{5.17}$$

then the solution remains its smoothness upto T. Where $1\leq l,m,n\leq\infty$ and
1-$(\frac{1}{2l}+\frac{1}{2m}+\frac{1}{2n})\geq 0$.

Proof. We will prove Theorem 5.3.1 by finding a priori estimates, in that regards, we continue our calculations from equation (5.8), and obtain new bounds for gradient pressure.

Estimating L_1 by employing Holder's and Young's inequality.

$$L_1=2\int_{\mathbb{R}^3}|\Psi||\mathcal{U}|^2|\nabla\mathcal{U}|dx\leq C\int_{\mathbb{R}^3}|\nabla\Psi||\mathcal{U}|^3dx\leq C\int_{\mathbb{R}^3}|\nabla\Psi|^{\frac{1}{2}}|\nabla\Psi|^{\frac{1}{2}}|\mathcal{U}|^2|\mathcal{U}|dx$$

$$\leq C\Big\|\Big\|\Big\||\nabla\Psi|^{\frac{1}{2}}\Big\|_{L^{2l,\infty}_{x_1}}\Big\|_{L^{2m,\infty}_{x_2}}\Big\|_{L^{2n,\infty}_{x_3}}\Big\||\nabla\Psi|^{\frac{1}{2}}\Big\|_{L^4}\Big\|\Big\|\Big\||\mathcal{U}|^2\Big\|_{L^{\frac{2l}{l-1},2}_{x_1}}\Big\|_{L^{\frac{2m}{m-1},2}_{x_2}}\Big\|_{L^{\frac{2n}{n-1},2}_{x_3}}\|\mathcal{U}\|_{L^4}$$

$$\leq C\Bigg\|\Big\|\big\|\nabla\Psi\big\|_{L^{l,\infty}_{x_1}}\Big\|_{L^{m,\infty}_{x_2}}\Bigg\|^{\frac{1}{2}}_{L^{n,\infty}_{x_3}}\|\nabla|\mathcal{U}|^2\|^{\frac{1}{2l}+\frac{1}{2m}+\frac{1}{2n}}_{L^2}\||\mathcal{U}||\nabla\mathcal{U}|\|^{\frac{1}{2}}_{L^2}\|\mathcal{U}\|^{3-(\frac{1}{l}+\frac{1}{m}+\frac{1}{n})}_{L^4}.$$

$$\leq C\Bigg\|\Big\|\big\|\nabla\Psi\big\|_{L^{l,\infty}_{x_1}}\Big\|_{L^{m,\infty}_{x_2}}\Bigg\|^{\frac{2}{3-(\frac{1}{l}+\frac{1}{m}+\frac{1}{n})}}_{L^{n,\infty}_{x_3}}\|\mathcal{U}\|^4_{L^4}+\frac{1}{4}(\|\nabla|\mathcal{U}|^2\|^2_{L^2}+\||\mathcal{U}||\nabla\mathcal{U}|\|^2_{L^2}).\tag{5.18}$$

For L_2

$$\Gamma_2\leq\frac{1}{2}\||\mathcal{U}||\nabla\mathcal{U}|\|^2_{L^2}+C\Big(\|\mathcal{U}\|^4_{L^4}+\|\mathcal{W}\|^4_{L^4}\Big).\tag{5.19}$$

For L_3

$$\Gamma_3\leq\frac{1}{2}\||\mathcal{W}||\nabla\mathcal{W}|^2\|^2_{L^2}+C\Big(\|\mathcal{U}\|^4_{L^4}+\|\mathcal{W}\|^4_{L^4}\Big).\tag{5.20}$$

Γ_4 and Γ_5 are estimated same as L_4 and L_5.

Putting all the estimates in (5.8) results in

$$\begin{aligned}&\frac{1}{4}\frac{d}{dt}(\|\mathcal{U}\|^4_{L^4}+\|\mathcal{W}\|^4_{L^4}+\|\mathcal{V}\|^4_{L^4})+\||\nabla\mathcal{U}||\mathcal{U}|\|^2_{L^2}+\frac{1}{2}\|\nabla|\mathcal{U}|^2\|^2_{L^2}\\&+\||\nabla\mathcal{W}||\mathcal{W}|\|^2_{L^2}+\frac{1}{2}\|\nabla|\mathcal{W}|^2\|^2_{L^2}+\|div\ \mathcal{W}\|^2_{L^2}+2\|\mathcal{W}\|^4_{L^4}+\||\nabla\mathcal{V}||\mathcal{V}|\|^2_{L^2}\\&+2\|\nabla|\mathcal{V}||\mathcal{V}|\|^2_{L^2}\\&\leq\frac{1}{4}\|\nabla|\mathcal{U}|^2\|^2_{L^2}+\frac{1}{4}\||\mathcal{U}|\nabla\mathcal{U}\|^2_{L^2}+C\Bigg\|\Big\|\big\|\nabla\Psi\big\|_{L^{l,\infty}_{x_1}}\Big\|_{L^{m,\infty}_{x_2}}\Bigg\|^{\frac{2}{3-(\frac{1}{l}+\frac{1}{m}+\frac{1}{n})}}_{L^{n,\infty}_{x_3}}\|\mathcal{U}\|^4_{L^4}+\frac{1}{2}\||\mathcal{U}||\nabla\mathcal{U}|\|^2_{L^2}\\&+\frac{1}{2}\||\mathcal{W}||\nabla\mathcal{W}|\|^2_{L^2}+C\Big(\|\mathcal{U}\|^4_{L^4}+\|\mathcal{W}\|^4_{L^4}\Big)+C\||\mathcal{V}|\nabla|\mathcal{V}|\|^2_{L^2}+\frac{1}{4}\|\nabla|\mathcal{U}|^2\|^2_{L^2}\\&+C\||\mathcal{V}|\nabla|\mathcal{V}|\|^2_{L^2}.\end{aligned}\tag{5.21}$$

Simplification yields

$$\begin{aligned}&\frac{1}{4}\frac{d}{dt}(\|\mathcal{U}\|^4_{L^4}+\|\mathcal{W}\|^4_{L^4}+\|\mathcal{V}\|^4_{L^4})+\||\nabla\mathcal{U}||\mathcal{U}|\|^2_{L^2}+\frac{1}{2}\|\nabla|\mathcal{U}|^2\|^2_{L^2}\\&+\||\nabla\mathcal{W}||\mathcal{W}|\|^2_{L^2}+\frac{1}{2}\|\nabla|\mathcal{W}|^2\|^2_{L^2}+\|div\ \mathcal{W}\|^2_{L^2}+2\|\mathcal{W}\|^4_{L^4}+\||\nabla\mathcal{V}||\mathcal{V}|\|^2_{L^2}\\&+2\|\nabla|\mathcal{V}||\mathcal{V}|\|^2_{L^2}\\&\leq C\Bigg(1+\Bigg\|\Big\|\big\|\nabla\Psi\big\|_{L^{l,\infty}_{x_1}}\Big\|_{L^{m,\infty}_{x_2}}\Bigg\|^{\frac{2}{3-(\frac{1}{l}+\frac{1}{m}+\frac{1}{n})}}_{L^{n,\infty}_{x_3}}\Bigg)\Big(\|\mathcal{U}\|^4_{L^4}+\|\mathcal{W}\|^4_{L^4}+\|\mathcal{V}\|^4_{L^4}\Big).\end{aligned}\tag{5.22}$$

By Gronwall's inequality

$$\sup_{0\leq t\leq T}(\|\mathcal{U}\|^4_{L^4}+\|\mathcal{W}\|^4_{L^4}+\|\mathcal{V}\|^4_{L^4})\leq\exp\Big\{\int_0^T C\Bigg(1+\Bigg\|\Big\|\big\|\nabla\Psi\big\|_{L^{l,\infty}_{x_1}}\Big\|_{L^{m,\infty}_{x_2}}\Bigg\|^{\frac{2}{3-(\frac{1}{l}+\frac{1}{m}+\frac{1}{n})}}_{L^{n,\infty}_{x_3}}\Bigg)d\tau\Big\}\tag{5.23}$$

$$\sup_{0\le t\le T}(\|\mathcal{U}\|_{L^4}^4+\|\mathcal{W}\|_{L^4}^4+\|\mathcal{V}\|_{L^4}^4)<\infty. \tag{5.24}$$

Hence proved.

5.4 Logarithmic improvement

Theorem 5.4.1 Considering same assumptions as for Theorem 5.2.1 and Theorem 5.3.1 The sufficient conditions

$$\int_0^T \frac{\left\|\left\|\left\|\Psi\right\|_{L_{x_1}^{l,\infty}}\right\|_{L_{x_2}^{m,\infty}}\right\|_{L_{x_3}^{n,\infty}}^{\frac{2}{2-(\frac{1}{l}+\frac{1}{m}+\frac{1}{n})}}}{1+\ln(1+\|\Psi\|_{L^2}^2)}dt<\infty, \tag{5.25}$$

and

$$\int_0^T \frac{\left\|\left\|\left\|\nabla\Psi\right\|_{L_{x_1}^{l,\infty}}\right\|_{L_{x_2}^{m,\infty}}\right\|_{L_{x_3}^{n,\infty}}^{\frac{2}{3-(\frac{1}{l}+\frac{1}{m}+\frac{1}{n})}}}{1+\ln(1+\|\Psi\|_{L^2}^2)}dt<\infty, \tag{5.26}$$

are the logarithmic imporvements of the conditions (5.7) and (5.17).

Proof. we can prove above condition by Continuing from inequality (5.15)

$$\text{As } 1+\ln(1+\|\Psi\|_{L^2}^2)\le 1+\ln(\Pi(t))$$

$$\text{Where } \Pi(t)=e+\|\mathcal{U}\|_{L^4}^4+\|\mathcal{W}\|_{L^4}^4+\|\mathcal{V}\|_{L^4}^4$$

$$\frac{d}{dt}\Pi(t)\le C\left(\frac{1+\left\|\left\|\left\|\Psi\right\|_{L_{x_1}^{l,\infty}}\right\|_{L_{x_2}^{m,\infty}}\right\|_{L_{x_3}^{n,\infty}}^{\frac{2}{2-(\frac{1}{l}+\frac{1}{m}+\frac{1}{n})}}}{1+\ln(e+\|\Psi\|_{L^2}^2)}\right)\Big(\|\mathcal{U}\|_{L^4}^4+\|\mathcal{W}\|_{L^4}^4+\|\mathcal{V}\|_{L^4}^4\Big)(1+\ln(e+\|\Psi\|_{L^2}^2)$$

$$\frac{d}{dt}\Pi(t)\le C\left(\frac{1+\left\|\left\|\left\|\Psi\right\|_{L_{x_1}^{l,\infty}}\right\|_{L_{x_2}^{m,\infty}}\right\|_{L_{x_3}^{n,\infty}}^{\frac{2}{2-(\frac{1}{l}+\frac{1}{m}+\frac{1}{n})}}}{1+\ln(e+\|\Psi\|_{L^2}^2)}\right)(\Pi(t))(1+\ln(\Pi(t))$$

$$\Longrightarrow \frac{d}{dt}(1+\ln\Pi(t))\le C\left(\frac{1+\left\|\left\|\left\|\Psi\right\|_{L_{x_1}^{l,\infty}}\right\|_{L_{x_2}^{m,\infty}}\right\|_{L_{x_3}^{n,\infty}}^{\frac{2}{2-(\frac{1}{l}+\frac{1}{m}+\frac{1}{n})}}}{1+\ln(1+\|\Psi\|_{L^2}^2)}\right)(1+\ln\Pi(t)).$$

Gronwall's Lemma results in

$$\sup_{0\le t\le T}\ln\Pi(t)\le(1+\ln\Pi(0))\exp\Big\{C\left(\frac{1+\left\|\left\|\left\|\Psi\right\|_{L_{x_1}^{l,\infty}}\right\|_{L_{x_2}^{m,\infty}}\right\|_{L_{x_3}^{n,\infty}}^{\frac{2}{2-(\frac{1}{l}+\frac{1}{m}+\frac{1}{n})}}}{1+\ln(1+\|\Psi\|_{L^2}^2)}\right)d\tau\Big\} \tag{5.27}$$

$$\Longrightarrow \sup_{0\leq t\leq T}\left(\|\mathcal{U}\|_{L^4}^4+\|\mathcal{W}\|_{L^4}^4+\|\mathcal{V}\|_{L^4}^4\right)\leq C, \tag{5.28}$$

which proves our theorem on interval $(0,T]$.

Following on the same steps as for (5.25), condition (5.26) can be proved. These bounds ensure the regularity of weak solutions on the interval $(0,T]$.

Chapter 6

Double logarithmic pressure regularity

6.1 Introduction

We propose and prove two-logarithmic regularity conditions for a variety of fluid dynamics models on an unbounded Euclidean 3D domain. These integrable constraints guarantee the smoothness of weak solutions. We specifically obtain new double-log regularity results for the pressure terms of the magneto-micropolar system and the Navier-Stokes-Nernst-Planck system in the time interval of regular points $[0, T]$.

To avoid notational verbosity, we rewrite systems (4.1) and (4.2) in the new symbols that also match our online published copy on ResearchGate with DOI: 10.13140/RG.2.2.18887.37284.

We prove a new result for the following magneto-micropolar system

$$\begin{cases} \frac{\partial \nu}{\partial t} + \nu \cdot \nabla \nu - \triangle \nu + \nabla \Psi - \nabla \times m - \beta \cdot \nabla \beta = 0, \\ \frac{\partial m}{\partial t} - \triangle m + \nu \cdot \nabla m - \nabla \times \nu + 2m - \nabla div\ m = 0, \\ \frac{\partial \beta}{\partial t} - \triangle \beta + \nu \cdot \nabla \beta - \beta \cdot \nabla \nu = 0, \\ \nabla \cdot \nu = 0, \quad \nabla \cdot \beta = 0, \\ (\nu, m, \beta)|_{t=0} = (\nu_0, m_0, \beta_0), \end{cases} \tag{6.1}$$

where $\nu(x,t)$, $\beta(x,t)$, $m(x,t)$ and $\Psi(x,t)$ represent the velocity field, magnetic field, micro-rotational velocity, and pressure, in that order, for the system (6.1).

The second system we analyze is the following Navier-Stokes-Nernst-Planck system

$$\begin{cases} \frac{\partial \vartheta}{\partial t} + \vartheta \cdot \nabla \vartheta - \triangle \vartheta + \nabla \Psi - \Delta \psi \nabla \psi = 0, \\ \nabla \cdot \vartheta = 0, \\ \frac{\partial q^+}{\partial t} + \vartheta \cdot \nabla q^+ - \nabla \cdot (\nabla q^+ + q^+ \nabla \psi) = 0, \\ \frac{\partial q^-}{\partial t} + \vartheta \cdot \nabla q^- - \nabla \cdot (\nabla q^- - q^- \nabla \psi) = 0, \\ \Delta \psi = q^- - q^+, \quad \text{in} \ \ \mathbb{R}^3 \times \mathbb{R}_+, \\ (\vartheta, q^+, q^-)|_{t=0} = (\vartheta_0, q_0^+, q_0^-), \end{cases} \tag{6.2}$$

where $\vartheta(x,t)$ is a velocity field vector, $\Psi(x,t)$, and ψ are scalars describing pressure and electrostatic potential of the flow with $q^-(x,t)$ and $q^+(x,t)$ are the densities of the negative and positive charged particles.

Our problem to refine regularity of weak solutions immensely effect physicality of solutions. Similar to other sub-tri-dimensional fluid models, the regularity problem in terms of pressure is being studied by Hua and Bao-quan [85] for (6.1) and presented criteria:

$$\int_0^T \|\Psi\|_{L^l}^n dt < \infty, \quad \text{with } \frac{2}{n} + \frac{3}{l} \leq 2, \ \ \frac{3}{2} < l \leq \infty. \tag{6.3}$$

Above criteria was improved by Li and Chen [83] given as:

$$\int_0^T \|\Psi\|_{\dot{B}^0_{\infty,\infty}} dt < \infty. \tag{6.4}$$

Recently Tang et al. [73] proved an improved regularity criteria given as:

$$\int_0^T \frac{\|\Psi\|^2_{\dot{B}^{-1}_{\infty,\infty}}}{1 + \ln(e + \|\Psi\|_{\dot{B}^{-1}_{\infty,\infty}})} dt < \infty. \tag{6.5}$$

Our new regularity criteria for (6.1)) in the critical Besov space is better and improve the results (6.3), (6.4) and (6.5) in terms of pressure as proved in Theorem 6.2.1. For the system (6.2), Zhao [86] presented the regularity criteria in terms of $\Omega = \nabla \times \mathcal{U}$ given as:

$$\int_0^T \frac{\|\Omega\|^{\frac{2}{2-\kappa_1}}_{\dot{B}^{-\kappa_1}_{\infty,\infty}}}{1 + \ln(e + \|\Omega\|_{\dot{B}^{-\kappa_1}_{\infty,\infty}})} dt < \infty, \quad \text{with } \ 0 < \kappa_1 < 2.$$

Recently, Zhao [87] proved an improved regularity conditions for (6.2) in terms of pressure given as:

$$\int_0^T \frac{\|\Psi\|^2_{L^3}}{1 + \ln(e + \|\Psi\|_{L^3})} dt < \infty, \tag{6.6}$$

$$\int_0^T \frac{\|\Psi\|_{\dot{B}^0_{\infty,\infty}}}{\sqrt{1 + \ln(e + \|\Psi\|_{\dot{B}^0_{\infty,\infty}})}} dt < \infty, \tag{6.7}$$

$$\int_0^T \frac{\|\Psi\|^2_{\dot{B}^{-1}_{\infty,\infty}}}{1 + \ln(e + \|\Psi\|_{\dot{B}^{-1}_{\infty,\infty}})} dt < \infty, \tag{6.8}$$

$$\int_0^T \frac{\|\nabla\Psi\|^{\frac{2}{3}}_{\dot{B}^0_{\infty,\infty}}}{(1 + \ln(e + \|\nabla\Psi\|_{\dot{B}^0_{\infty,\infty}}))^{\frac{2}{3}}} dt < \infty.$$

The above analysis clear motivates us to present Theorem 6.3.1 for (6.2).

6.2 Regularity criteria for the magneto-micropolar system

In this section, by employing energy methods we will proof our main result.

Theorem 6.2.1. Let $(\nu_0, m_0, \beta_0) \in H^1(\mathbb{R}^3)$ with $\nabla \cdot \nu_0 = \nabla \cdot \beta_0 = 0$ in the sense of distribution. Let $T > 0$ and (ν, m, β) is a weak solution of system (6.1) on the interval $(0, T]$. If the pressure Ψ satisfies the following condition

$$\int_0^T \frac{\|\Psi\|^2_{\dot{B}^{-1}_{\infty,\infty}}}{\left(e + \ln\left(e + \|\Psi\|_{\dot{B}^{-1}_{\infty,\infty}}\right) \ln\left(e + \ln\left(e + \|\Psi\|_{\dot{B}^{-1}_{\infty,\infty}}\right)\right)\right.} dt < \infty, \tag{6.9}$$

then (ν, m, β) is a regular solution in $\mathbb{R}^3 \times (0, T]$.

Proof. In order to find L^4 estimates, multiply $(6.1)_1$ with $|\nu|^2\nu$, integrating by parts, using divergence free condition and its relations for such evaluations

$$\begin{aligned} &\frac{1}{4}\frac{d}{dt}\int_{\mathbb{R}^3} |\nu|^4 dx + \int_{\mathbb{R}^3} |\nabla \nu|^2 |\nu|^2 dx + \frac{1}{2}\int_{\mathbb{R}^3} |\nabla |\nu|^2|^2 dx \\ \leq &\int_{\mathbb{R}^3} |\Psi||\nu|^2|\nabla \nu| dx + \int_{\mathbb{R}^3} |m||\nu|^2|\nabla \nu| dx - \int_{\mathbb{R}^3} |\beta||\nabla(|\nu|^2\nu)||\beta| dx. \end{aligned} \tag{6.10}$$

Similarly, evaluating L^4 estimates for $(6.1)_2$

$$\begin{aligned} \frac{1}{4}\frac{d}{dt}\int_{\mathbb{R}^3} |m|^4 dx + \int_{\mathbb{R}^3} |\nabla m|^2 |m|^2 dx + \frac{1}{2}\int_{\mathbb{R}^3} |\nabla |m|^2|^2 dx + \int_{\mathbb{R}^3} |div\ m|^2 \\ +2\int_{\mathbb{R}^3} |m|^4 \\ \leq 3\int_{\mathbb{R}^3} |\nu||m|^2|\nabla m| dx. \end{aligned} \tag{6.11}$$

In case of $(6.1)_3$

$$\begin{aligned} \frac{1}{4}\frac{d}{dt}\int_{\mathbb{R}^3} |\beta|^4 dx + \int_{\mathbb{R}^3} |\nabla \beta|^2 |\beta|^2 dx + 2\int_{\mathbb{R}^3} |\nabla |\beta||\beta||^2 dx \\ \leq \int_{\mathbb{R}^3} |\beta||\nabla(|\beta|^2\beta)|\nu|| dx. \end{aligned} \tag{6.12}$$

Adding (6.10), (6.11) and (6.12) implies

$$\frac{1}{4}\frac{d}{dt}(\|\nu\|^4_{L^4} + \|m\|^4_{L^4} + \|\beta\|^4_{L^4}) + \||\nabla \nu||\nu|\|^2_{L^2} + \frac{1}{2}\|\nabla |\nu|^2\|^2_{L^2}$$

$$+\||\nabla m||m|\|_{L^2}^2+\frac{1}{2}\|\nabla|m|^2\|_{L^2}^2+\|div\ m\|_{L^2}^2+2\|m\|_{L^4}^4+\||\nabla\beta||\beta|\|_{L^2}^2$$

$$+2\|\nabla|\beta||\beta|\|_{L^2}^2$$

$$\leq 2\int_{\mathbb{R}^3}|\Psi||\nu|^2|\nabla\nu|dx+\int_{\mathbb{R}^3}|m||\nu|^2|\nabla\nu|dx+\int_{\mathbb{R}^3}|\nu||m|^2|\nabla m|dx$$

$$-\int_{\mathbb{R}^3}|\beta||\nabla(|\nu|^2\nu)||\beta|dx+\int_{\mathbb{R}^3}|\beta||\nabla(|\beta|^2\beta)|\nu||dx$$

$$=N_1+N_2+N_3+N_4+N_5. \tag{6.13}$$

Estimating N_1 by Holder's and Young's inequality

$$N_1\leq\frac{1}{4}\|\nabla|\nu|^2\|_{L^2}^2+C\int_{\mathbb{R}^3}|\Psi|^2|\nu|^2dx.$$

By Calderon Zygmund inequality

$$C\int_{\mathbb{R}^3}|\Psi|^2|\nu|^2dx\leq C\|\nu\|_{L^4}^2\|\Psi\|_{L^4}^2\leq C\|\nabla\Psi\|_{L^2}\|\Psi\|_{\dot{B}^{-1}_{\infty,\infty}}\|\nu\|_{L^4}^2$$

$$\leq C\||\nu||\nabla\nu|\|_{L^2}\|\Psi\|_{\dot{B}^{-1}_{\infty,\infty}}\|\nu\|_{L^4}^2.$$

By Young's inequality

$$\leq\frac{1}{2}(\||\nu||\nabla\nu|\|_{L^2}^2)+C\|\Psi\|_{\dot{B}^{-1}_{\infty,\infty}}^2\|\nu\|_{L^4}^4\leq\||\nu||\nabla\nu|\|_{L^2}^2+C\|\Psi\|_{\dot{B}^{-1}_{\infty,\infty}}^2\|\nu\|_{L^4}^4.$$

Finally, getting estimate

$$N_1\leq\||\nu||\nabla\nu|\|_{L^2}^2+C(1+\|\Psi\|_{\dot{B}^{-1}_{\infty,\infty}}^2)(\|\nu\|_{L^4}^4+\|m\|_{L^4}^4+\|\beta\|_{L^4}^4).$$

Now estimating N_2, N_3,

$$N_2\leq\frac{1}{2}\||\nabla\nu||\nu|\|_{L^2}^2+C\big(\|\nu\|_{L^4}^4+\|m\|_{L^4}^4+\|\beta\|_{L^4}^4\big).$$

$$N_3\leq\frac{1}{2}\||\nabla m||m|\|_{L^2}^2+C\big(\|\nu\|_{L^4}^4+\|m\|_{L^4}^4+\|\beta\|_{L^4}^4\big).$$

Estimating N_4 and N_5 by applying Young's inequaltiy, Sobolev embedding theorem and Gagliardo-Nirenberg inequality

$$N_4\leq C\||\nu||\beta|^2\|_{L^2}\|\nabla|\nu|^2\|_{L^2}\leq C\||\nu||\beta|^2\|_{L^2}^2+\frac{1}{2}\|\nabla|\nu|^2\|_{L^2}^2$$

$$\leq C\||\beta|^2\|_{L^6}^2\||\nu|\|_{L^3}^2+\frac{1}{4}\|\nabla|\nu|^2\|_{L^2}^2\leq C\||\nabla\beta|^2\|_{L^2}^2\||\nu|\|_{L^2}\||\nabla\nu|\|_{L^2}+\frac{1}{4}\|\nabla|\nu|^2\|_{L^2}^2$$

$$\leq C\||\beta|\nabla|\beta|\|_{L^2}^2+\frac{1}{4}\|\nabla|\nu|^2\|_{L^2}^2.$$

$$N_5 \le C\||\nu||\beta|^2\|_{L^2}^2 + \frac{1}{8}\|\nabla|\beta|^2\|_{L^2} \le C\||\beta|\nabla|\beta|\|_{L^2}^2.$$

Adding all estimates and putting in (6.13), by Gronwalls's inequality

$$\|\nu\|_{L^4}^4 + \|m\|_{L^4}^4 + \|\beta\|_{L^4}^4 \le \exp\Big(C\int_0^t (1+\|\Psi\|_{\dot{B}^{-1}_{\infty,\infty}}^2)dt\Big)(\|\nu_0\|_{L^4}^4 + \|m_0\|_{L^4}^4 + \|\beta_0\|_{L^4}^4).$$

Proceeding by multiplying $(6.1)_1$ with $(-\Delta\nu)$, $(6.1)_2$ with $(-\Delta m)$ and $(6.1)_3$ with $(-\Delta\beta)$ in $L^2(\mathbb{R}^3)$. Adding all the resulting equations, using integration by parts and Gagliardo-Nirenberg inequalities

$$\|\nabla\nu\|_{L^4} \le C\|\nu\|_{L^4}^{\frac{1}{5}}\|\Delta\nu\|_{L^2}^{\frac{4}{5}} \quad \text{and} \quad \|\nabla\nu\|_{L^3} \le C\|\nu\|_{L^4}^{\frac{2}{5}}\|\Delta\nu\|_{L^2}^{\frac{3}{5}}.$$

We get

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\Big(\|\nabla\nu\|_{L^2}^2 + \|\nabla m\|_{L^2}^2 + \|\nabla\beta\|_{L^2}^2\Big) + \|\Delta\nu\|_{L^2}^2 + \|\Delta m\|_{L^2}^2 + \|\Delta\beta\|_{L^2}^2 \\
&\qquad + \|\nabla \text{div } m\|_{L^2}^2 + 2\|\nabla m\|_{L^2}^2 \\
&= \int_{\mathbb{R}^3}(\nu\cdot\nabla)\nu\cdot\Delta\nu dx + \int_{\mathbb{R}^3}(\nu\cdot\nabla)m\cdot\Delta m dx + \int_{\mathbb{R}^3}(\beta\cdot\nabla)\beta\cdot\Delta\nu dx - \int_{\mathbb{R}^3}(\nabla\times m)\Delta\nu dx \\
&\qquad - \int_{\mathbb{R}^3}(\nabla\times\nu)\Delta m dx + \int_{\mathbb{R}^3}(\nu\cdot\nabla\beta)\Delta\beta dx - \int_{\mathbb{R}^3}(\beta\cdot\nabla\nu)\Delta\beta dx \\
&\le \|\nu\|_{L^4}\|\nabla\nu\|_{L^4}\|\Delta\nu\|_{L^2} + \|\nu\|_{L^4}\|\nabla m\|_{L^4}\|\Delta m\|_{L^2} + \|\nabla\nu\|_{L^3}\|\nabla\beta\|_{L^3}^2 + \|\nabla m\|_{L^2}\|\Delta\nu\|_{L^2} \\
&\qquad + \|\nabla\nu\|_{L^2}\|\Delta m\|_{L^2} + \|\nabla\nu\|_{L^3}\|\nabla\beta\|_{L^3}^2 + \|\nabla\nu\|_{L^3}\|\nabla\beta\|_{L^3}^2 \\
&\le C\|\nu\|_{L^4}^{\frac{6}{5}}\|\Delta\nu\|_{L^2}^{\frac{9}{5}} + \|m\|_{L^2}^{\frac{1}{2}}\|\Delta m\|_{L^2}^{\frac{1}{2}}\|\Delta\nu\|_{L^2} + \|\nu\|_{L^4}\|m\|_{L^4}^{\frac{1}{5}}\|\Delta m\|_{L^2}^{\frac{9}{5}} + \\
&\qquad + \|\nu\|_{L^2}^{\frac{1}{2}}\|\Delta\nu\|_{L^2}^{\frac{1}{2}}\|\Delta m\|_{L^2} + 3\|\nu\|_{L^4}^{\frac{2}{5}}\|\Delta m\|_{L^2}^{\frac{3}{5}}\|\beta\|_{L^4}^{\frac{4}{5}}\|\Delta\beta\|_{L^2}^{\frac{6}{5}} \\
&\le C\|\nu\|_{L^4}^{12} + C\|m\|_{L^2}^2 + C(\|\nu\|_{L^4}^{12} + \|m\|_{L^4}^{12}) + \frac{1}{2}(\|\Delta\nu\|_{L^2}^2 + \|\Delta m\|_{L^2}^2) + C\|\nu\|_{L^2}^2 \\
&\qquad + C(\|\nu\|_{L^4}^{12} + \|\beta\|_{L^2}^2) + C(\|\Delta\nu\|_{L^2}^2 + \|\Delta\beta\|_{L^2}^2) \\
&\le C\Big(1 + \|\nu\|_{L^4}^{12} + \|\beta\|_{L^4}^{12} + \|m\|_{L^4}^{12}\Big) + \frac{1}{2}\Big(\|\Delta\nu\|_{L^2}^2 + \|\Delta\beta\|_{L^2}^2 + \|\Delta m\|_{L^2}^2\Big)
\end{aligned}$$

$$\begin{aligned}
&\frac{d}{dt}\Big(\|\nabla\nu\|_{L^2}^2 + \|\nabla m\|_{L^2}^2 + \|\nabla\beta\|_{L^2}^2\Big) + \|\Delta\nu\|_{L^2}^2 + \|\Delta m\|_{L^2}^2 + \|\Delta\beta\|_{L^2}^2 + \|\nabla \text{div } m\|_{L^2}^2 + 2\|\nabla m\|_{L^2}^2 \\
&\qquad \le \|\nabla\nu\|_{L^2}^2 + \|\nabla\beta\|_{L^2}^2 + \|\nabla m\|_{L^2}^2 + C\Big(1 + \|\nu\|_{L^4}^{12} + \|\beta\|_{L^4}^{12} + \|m\|_{L^4}^{12}\Big)
\end{aligned}$$

$$\begin{aligned}
&\Big(\|\nabla\nu\|_{L^2}^2 + \|\nabla m\|_{L^2}^2 + \|\nabla\beta\|_{L^2}^2\Big) + \int_0^t \Big(\|\Delta\nu\|_{L^2}^2 + \|\Delta m\|_{L^2}^2 + \|\Delta\beta\|_{L^2}^2 + \|\nabla \text{div } m\|_{L^2}^2 + 2\|\nabla m\|_{L^2}^2\Big)dt \\
&\qquad \le \|\nabla\nu_0\|_{L^2}^2 + \|\nabla\beta_0\|_{L^2}^2 + \|\nabla m_0\|_{L^2}^2 + C\int_0^t \Big(1 + \|\nu\|_{L^4}^{12} + \|\beta\|_{L^4}^{12} + \|m\|_{L^4}^{12}\Big)dt.
\end{aligned}$$

By $\dot{H}^1(\mathbb{R}^3) \subset L^6(\mathbb{R}^3)$, we achieve

$$e + \|\Psi(\cdot,t)\|_{L^3} \leq e + C\|\nu\|_{L^6}^2 \leq e + C(\|\nabla\nu\|_{L^2}^2 + \|\nabla\beta\|_{L^2}^2 + \|\nabla m\|_{L^2}^2)$$

$$\leq e+C\big(\|\nabla\nu_0\|_{L^2}^2+\|\nabla\beta_0\|_{L^2}^2+\|\nabla m_0\|_{L^2}^2\big)+C\int_0^t \big(1+\|\nu(\cdot,t)\|_{L^4}^{12}+\|\beta(\cdot,t)\|_{L^4}^{12}+\|m(\cdot,t)\|_{L^4}^{12}\big)dt$$

$$\leq C\big(e+\|\nabla\nu_0\|_{L^2}^2+\|\nabla\beta_0\|_{L^2}^2+\|\nabla m_0\|_{L^2}^2\big)(e+t)\sup_{0\leq t\leq t}\big(1+\|\nu(\cdot,t)\|_{L^4}^{12}+\|\beta(\cdot,t)\|_{L^4}^{12}+\|m(\cdot,t)\|_{L^4}^{12}\big)$$

$$\leq C_0(e+t)\exp\Big(C\int_0^t(1+\|\Psi\|_{\dot{B}_{\infty,\infty}^{-1}}^2)dt\Big).$$

Where C_0 represents constants, Now using $L^3(\mathbb{R}^3) \subset \dot{B}_{\infty,\infty}^{-1}(\mathbb{R}^3)$

$$e + \|\Psi\|_{\dot{B}_{\infty,\infty}^{-1}} \leq C(e+t)\exp\Big(C\int_0^t(1+\|\Psi\|_{\dot{B}_{\infty,\infty}^{-1}}^2)dt\Big).$$

Now, applying ln on both sides

$$\ln\Big(e + \|\Psi\|_{\dot{B}_{\infty,\infty}^{-1}}\Big) \leq \ln(C(e+t)) + \Big(C\int_0^t(1+\|\Psi\|_{\dot{B}_{\infty,\infty}^{-1}}^2)dt\Big). \tag{6.14}$$

Now, we let

$\Lambda(t) = \ln(e + \|\Psi\|_{\dot{B}_{\infty,\infty}^{-1}})$.

$\Theta(t) = \ln(C(e+t)) + \Big(C\int_0^t(1+\|\Psi\|_{\dot{B}_{\infty,\infty}^{-1}}^2)dt\Big)$.

Inequality (6.14) implies

$0< \Lambda(t) \leq \Theta(t)$.

Now we get

$$(e+\Lambda(t))\ln(e+\Lambda(t)) \leq (e+\Theta(t)\ln(e+\Theta(t)).$$

Now, we have

$$\begin{aligned}\frac{d}{dt}\ln(e+\Theta(t)) &= \frac{1}{(e+\Theta(t))}\Big(\frac{1}{e+t}+C(1+\|\Psi\|^2_{\dot{B}^{-1}_{\infty,\infty}})\Big)\\ &\leq \frac{1}{e^2}+C\frac{1+\|\Psi\|^2_{\dot{B}^{-1}_{\infty,\infty}}}{e+\Theta(t)}\\ &= \frac{1}{e^2}+C\frac{1+\|\Psi\|^2_{\dot{B}^{-1}_{\infty.\infty}}}{e+\Theta(t)\ln(e+\Theta(t))}\ln(e+\Theta(t))\\ &= \frac{1}{e^2}+C\frac{1+\|\Psi\|^2_{\dot{B}^{-1}_{\infty,\infty}}}{(e+\Lambda(t))\ln(e+\Lambda(t))}\ln(e+\Theta(t)).\end{aligned}$$

Applying Gronwall's inequality

$$\ln(e+\Theta(t)) \leq \ln(e+\Theta(0))\exp\Big(\frac{T}{e^2}+C\int_0^T\frac{1+\|\Psi\|^2_{\dot{B}^{-1}_{\infty,\infty}}}{(e+\Lambda(t))\ln(e+\Lambda(t))}dt\Big). \tag{6.15}$$

Resulting as

$$(e+\Theta(t)) \leq (e+\Theta(0))^{\exp\Big(\frac{T}{e^2}+C\int_0^T\frac{1+\|\Psi\|^2_{\dot{B}^{-1}_{\infty,\infty}}}{(e+\Lambda(t))\ln(e+\Lambda(t))}dt\Big)}.$$

These bounds proof our criteria. □

6.3 Regularity criteria for the Navier-Stokes-Nernst-Planck system

Theorem 6.3.1 Suppose that $(\vartheta_0, q_0^-, q_0^+) \in H^1(\mathbb{R}^3)\cap H^2(\mathbb{R}^3)$ with $\nabla\cdot\vartheta_0 = 0$. Let $T>0$ and (ϑ, q^-, q^+), a weak solution to system (6.2) satisfies the condition (6.9) then it is a regular in $\mathbb{R}^3\times(0,T]$.

Proof. To find L^4 estimates, multiply $(6.2)_1$ with $|\vartheta|^2\vartheta$, integrating by parts, using divergence free condition and its relations for such evaluations

$$\begin{aligned}\frac{1}{4}\frac{d}{dt}\int_{\mathbb{R}^3}|\vartheta|^4dx+\int_{\mathbb{R}^3}|\nabla\vartheta|^2|\vartheta|^2dx+\frac{1}{2}\int_{\mathbb{R}^3}|\nabla|\vartheta|^2|^2dx\\ \leq -\int_{\mathbb{R}^3}\nabla\Psi\cdot\vartheta|\vartheta|^2dx+\int_{\mathbb{R}^3}\Delta\Psi\nabla\Psi\cdot\vartheta|\vartheta|^2dx\\ = M_1+M_2.\end{aligned} \tag{6.16}$$

Now, we will estimate M_1

$$M_1 \leq C\int_{\mathbb{R}^3}\Psi|\vartheta|^2\cdot|\nabla\vartheta|dx$$

$$M_1 \leq C\|\Psi\|_{L^4}\|\vartheta\|_{L^4}\||\vartheta||\nabla\vartheta|\|_{L^2}$$

$$M_1 \leq \frac{1}{4}\||\vartheta||\nabla\vartheta|\|_{L^2}^2 + C\|\Psi\|_{L^4}^2\|\vartheta\|_{L^4}^2$$

$$M_1 \leq \frac{1}{4}\||\vartheta||\nabla\vartheta|\|_{L^2}^2 + C\|\nabla\Psi\|_{L^2}\|\Psi\|_{\dot{B}^{-1}_{\infty,\infty}}\|\vartheta\|_{L^4}^2$$

$$\leq \frac{1}{4}\||\vartheta||\nabla\vartheta|\|_{L^2}^2 + C\|\Psi\|_{\dot{B}^{-1}_{\infty,\infty}}(\||\vartheta||\nabla\vartheta|\|_{L^2}^2 + \|(\nabla q^-, \nabla q^+)\|_{L^2}^{\frac{1}{2}})\|\vartheta\|_{L^4}^2$$

$$\leq \frac{1}{4}\||\vartheta||\nabla\vartheta|\|_{L^2}^2 + C(1 + \|\Psi\|_{\dot{B}^{-1}_{\infty,\infty}}^2 + \|(\nabla q^-, \nabla q^+)\|_{L^2}^2)(1 + \|\vartheta\|_{L^4}^4).$$

Now, estimating M_2 we will get

$$M_2 \leq \int_{\mathbb{R}^3}(q^- - q^+)\nabla\Psi \cdot \vartheta|\vartheta|^2 dx$$

$$\leq \|(q^-, q^+)\|_{L^3}\|\nabla\Psi\|_{L^6}\|\vartheta|\vartheta|^2\|_{L^2}$$

$$\leq C\|(q^-, q^+)\|_{L^3}^{\frac{3}{2}}\|(\nabla q^-, \nabla q^+)\|_{L^2}^{\frac{1}{2}}\||\vartheta|^2\|_{L^2}^{\frac{3}{2}}$$

$$\leq C\|(\nabla q^-, \nabla q^+)\|_{L^2}^{\frac{1}{2}}\||\vartheta|^2\|_{L^2}^{\frac{3}{4}}\|\nabla|\vartheta|^2\|_{L^2}^{\frac{3}{4}}$$

$$\leq \frac{1}{4}\|\nabla|\vartheta|^2\|_{L^2}^2 + C\|(\nabla q^-, \nabla q^+)\|_{L^2}^{\frac{4}{5}}\|\vartheta\|_{L^4}^{\frac{12}{5}}$$

$$\leq \frac{1}{4}\|\nabla|\vartheta|^2\|_{L^2}^2 + C(1 + \|(\nabla q^-, \nabla q^+)\|_{L^2}^2)(1 + \|\vartheta\|_{L^4}^4).$$

Now, combining both estimates and putting in (6.16), we are down to

$$\frac{d}{dt}\|\vartheta\|_{L^4}^4 \leq C(1 + \|(\nabla q^-, \nabla q^+)\|_{L^2}^2 + \|\Psi\|_{\dot{B}^{-1}_{\infty,\infty}}^2)(1 + \|\vartheta\|_{L^4}^4).$$

Proceeding by multiplying $(6.2)_1$ with $(-\Delta\vartheta)$ in $L^2(\mathbb{R}^3)$. Adding all the resulting equations, we are down to

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^3}|\nabla\vartheta|^2 dx + \int_{\mathbb{R}^3}|\Delta\vartheta|^2 dx$$

$$= \int_{\mathbb{R}^3}(\vartheta \cdot \nabla)\vartheta \cdot \Delta - \int_{\mathbb{R}^3}\Delta\Psi\nabla\Psi \cdot \Delta$$

$$= O_1 + O_2. \tag{6.17}$$

Now, estimating O_1

$$|O_1| \leq \|\vartheta\|_{L^4}\|\nabla\vartheta\|_{L^4}\|\Delta\vartheta\|_{L^2}$$

$$\leq C\|\vartheta\|_{L^4}^{\frac{6}{5}}\|\Delta\vartheta\|_{L^2}^{\frac{9}{5}}$$

$$\leq C\|\vartheta\|_{L^4}^{12} + \epsilon\|\Delta\vartheta\|_{L^2}^2.$$

For O_2

$$|O_2| \leq \int_{\mathbb{R}^3} \Delta\Psi \nabla\Psi \cdot \Delta.$$

Using Holder's, Young's and interpolation inequality

$$\leq C(1 + \|(\nabla q^-, \nabla q^+)\|_{L^2}^2) + \epsilon \|\Delta\vartheta\|_{L^2}^2.$$

Adding both estimates and putting in (6.17), and applying Gronwall's inequality

$$\|\nabla\vartheta\|_{L^2}^2 + \int_0^T \|\Delta\vartheta\|_{L^2}^2 \leq \|\nabla\vartheta_0\|_{L^2}^2 + C\int_0^T (\|\vartheta\|_{L^4}^{12} + \|(\nabla q^-, \nabla q^+)\|_{L^2}^2 + 1)dt.$$

Which implies

$$\begin{aligned} e + \|\Psi(\cdot,t)\|_{L^3} &\leq e + C\|\vartheta\|_{L^6}^2 \leq e + C\|\nabla\vartheta\|_{L^2}^2 \\ &\leq e + C\|\nabla\vartheta_0\|_{L^2}^2 + C\int_0^t (\|\vartheta\|_{L^4}^{12} + \|(\nabla q^-, \nabla q^+)\|_{L^2}^2 + 1)dt \\ &\leq e + C\|\nabla\vartheta_0\|_{L^2}^2 + C(e+t)\Big(\sup_{0\leq t\leq t} \|\vartheta\|_{L^4}^{12} + \int_0^t \|(\nabla q^-, \nabla q^+)\|_{L^2}^2 + 1)dt\Big) \\ &\leq C_0(e+t)\exp\Big(C\int_0^t (1 + \|\Psi\|_{\dot{B}_{\infty,\infty}^{-1}}^2 + \|(\nabla q^-, \nabla q^+)\|_{L^2}^2\Big)dt. \end{aligned}$$

Where C_0 represents constants, Now using $L^3(\mathbb{R}^3) \subset \dot{B}_{\infty,\infty}^{-1}(\mathbb{R}^3)$

$$e + \|\Psi\|_{\dot{B}_{\infty,\infty}^{-1}} \leq e + \|\Psi(\cdot,t)\|_{L^3} \leq C(e+t)\exp\Big(C\int_0^t (1 + \|\Psi\|_{\dot{B}_{\infty,\infty}^{-1}}^2 + \|(\nabla q^-, \nabla q^+)\|_{L^2}^2)dt\Big).$$

Now, applying ln on both sides

$$\ln\Big(e + \|\Psi\|_{\dot{B}_{\infty,\infty}^{-1}}\Big) \leq \ln(C(e+t)) + \Big(C\int_0^t (1 + \|\Psi\|_{\dot{B}_{\infty,\infty}^{-1}}^2 + \|(\nabla q^-, \nabla q^+)\|_{L^2}^2)dt\Big).$$

Let

$\Lambda(t) = \ln(e + \|\Psi\|_{\dot{B}_{\infty,\infty}^{-1}})$. Following on the same steps as for (6.15), we get the final estimate

$$(e + \Theta(t)) \leq (e + \Theta(0))^{\exp\left(\frac{T}{e^2} + C\int_0^T \frac{1 + \|\Psi\|_{\dot{B}_{\infty,\infty}^{-1}}^2 + \|(\nabla q^-, \nabla q^+)\|_{L^2}^2}{(e+\Lambda(t))\ln(e+\Lambda(t))}dt\right)}.$$

Now to bound $\|(\nabla q^-, \nabla q^+)\|_{L^2}^2 \leq \infty$. Multiply third equation with Δq^- and fourth equation with Δq^+ for L^2 norm bounds

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^3} |\nabla q^-|^2 dx + \int_{\mathbb{R}^3} |\Delta q^-|^2 dx = \int_{\mathbb{R}^3} (\vartheta \cdot \nabla)q^- \cdot \Delta q^- dx - \int_{\mathbb{R}^3} \nabla \cdot (q^- \cdot \nabla\Psi)\Delta q^- dx$$

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^3}|\nabla q^+|^2dx+\int_{\mathbb{R}^3}|\Delta q^+|^2dx=\int_{\mathbb{R}^3}(\vartheta\cdot\nabla)q^+\cdot\Delta q^+dx-\int_{\mathbb{R}^3}\nabla\cdot(q^+\cdot\nabla\Psi)\Delta q^+dx.$$

Adding both equations for L^2 estimates

$$\frac{1}{2}\frac{d}{dt}(\|\nabla q^-\|_{L^2}^2+\|\nabla q^+\|_{L^2}^2)+(\|\Delta q^-\|_{L^2}^2+\|\Delta q^+\|_{L^2}^2)$$

$$=\int_{\mathbb{R}^3}(\vartheta\cdot\nabla)q^-\cdot\Delta q^-dx+\int_{\mathbb{R}^3}(\vartheta\cdot\nabla)q^+\cdot\Delta q^+dx-\int_{\mathbb{R}^3}\nabla\cdot(q^-\cdot\nabla\Psi)\Delta q^-dx-\int_{\mathbb{R}^3}\nabla\cdot(q^+\cdot\nabla\Psi)\Delta q^+dx$$

$$=X_1+X_2+X_3+X_4. \tag{6.18}$$

Now, we will bound X_1

$$|X_1|\leq\|(\vartheta\cdot\nabla)q^-\|_{L^2}\|\Delta q^-\|_{L^2}\leq C\|(\vartheta\cdot\nabla)q^-\|_{L^2}^2+\epsilon\|\Delta q^-\|_{L^2}^2.$$

By Holder's and Young's inequality, we get

$$\leq C\|\nabla\vartheta\|_{L^2}^2\|\nabla q^-\|_{L^2}^2\|\nabla q^-\|_{L^6}+\epsilon\|\Delta q^-\|_{L^2}^2\|\Delta q^-\|_{L^2}^2\leq C\|\nabla q^-\|_{L^2}^2+2\epsilon\|\Delta q^-\|_{L^2}^2.$$

Similarly

$$|X_2|\leq C\|\nabla q^+\|_{L^2}^2+2\epsilon\|\Delta q^+\|_{L^2}^2$$

$$|X_3|\leq-\int_{\mathbb{R}^3}^{-}\nabla\Psi\cdot\Delta q^-dx-\int_{\mathbb{R}^3}q^-\Delta\Psi\cdot\Delta q^-dx$$

$$\leq C(\|\nabla q^-\|_{L^3}^2\|\nabla\Psi\|_{L^6}^2+\|(q^-,q^+)\|_{L^4}^4)+\epsilon\|\Delta q^-\|_{L^2}^2$$

$$\leq 3\epsilon\|(\Delta q^-,\Delta q^+)\|_{L^2}^2+C(1+\|(\nabla q^-,\nabla q^+)\|_{L^2}^2).$$

Similarly

$$|X_4|\leq 3\epsilon\|(\Delta q^-,\Delta q^+)\|_{L^2}^2+C(1+\|(\nabla q^-,\nabla q^+)\|_{L^2}^2).$$

Placing all the estimates in (6.18), and Gronwall's lemma yields

$$\|\nabla q^-\|_{L^2}^2+\|\nabla q^+\|_{L^2}^2+\int_0^T(\|\Delta q^-\|_{L^2}^2+\|\Delta q^+\|_{L^2}^2)dt\leq C(1+\|\nabla q_0^-\|_{L^2}^2+\|\nabla q_0^+\|_{L^2}^2)\leq\infty.$$

Hence, our result has been proved and improves the conditions (6.6), (6.7) and (6.8).

Chapter 7

Geometric constraints for the fractional system on the entire three-dimensional domain

7.1 Introduction

This chapter establishes two new geometric constraints, one on the vorticity for the Beale-Kato-Majda type result and the other on the gradient velocity, vital for controlling the blow-up of weak solutions to the 3D incompressible fractional magneto-micropolar system for a finite time.

We study the following Cauchy problem of a generalised 3D incompressible magneto-micropolar system in the whole spatial domain, i.e., $\mathbb{R}^3$, and in the finite time domain, i.e., [0,T]:

$$\begin{cases} \mathfrak{h}_t + \mathfrak{h}\cdot\nabla\mathfrak{h} + (-\Delta)^{\hat{\alpha_1}}\mathfrak{h} + \nabla\Psi - \mathfrak{B}\cdot\nabla\mathfrak{B} - 2(\nabla\times\mathfrak{D}) = 0, \\ \mathfrak{D}_t + \mathfrak{h}\cdot\nabla\mathfrak{D} + (-\Delta)^{\hat{\alpha_2}}\mathfrak{D} - \nabla\mathrm{div}\mathfrak{D} + 4\mathfrak{D} - 2(\nabla\times\mathfrak{h}) = 0, \\ \mathfrak{B}_t + \mathfrak{h}\cdot\nabla\mathfrak{B} - \mathfrak{B}\cdot\nabla\mathfrak{h} + (-\Delta)^{\hat{\alpha_3}}\mathfrak{B} = 0, \\ \nabla\cdot\mathfrak{h} = 0, \quad \nabla\cdot\mathfrak{B} = 0, \\ (\mathfrak{h},\mathfrak{D},\mathfrak{B})|_{t=0} = (\mathfrak{h}_0,\mathfrak{D}_0,\mathfrak{B}_0), \end{cases} \tag{7.1}$$

where $\mathfrak{h}$, $\mathfrak{D}$, $\mathfrak{B}$ are the velocity field, micro-rotational velocity, and magnetic field, respectively, while $\Psi(x,t)$ is the scalar pressure. Throughout the paper, $\Omega = \nabla\times\mathfrak{h}$ denotes the vorticity and parameters $\hat{\alpha_1}, \hat{\alpha_2}, \hat{\alpha_3} > 0$. The divergence free conditions are satisfied by velocity and magnetic field with initial conditions $\mathfrak{h}_0, \mathfrak{D}_0$, and $\mathfrak{B}_0$ given for velocity, micro-rotation and magnetic field. The Zygmund operator $\Pi = (-\Delta)^{\frac{1}{2}}$ is defined in terms of Fourier transform

$$\widehat{\Pi^{\hat{i}} f(\xi)} = |\xi|^{\hat{i}}\widehat{f(\xi)}, \quad \forall \ \hat{i} \geq 0.$$

As system (7.1) is coupled with the Navier-Stokes equations with fractional micro-rotation and magnetic diffusion that allows us to deal with more complex problems of the movement of liquid crystals, ferromagnetic particles, animal blood flow, diluted aqueous polymer solutions, etc. Recently, Deng and Shang [88] showed the global existence of system (7.1) using Lebesgue and Sobolev spaces. Fan and Zhong [89] showed the local existence and uniqueness of the system (7.1) and also proved the following geometric constraints for the velocity and gradient velocity that keep the solutions of the given system from blowing up in the given time interval

$$\mathfrak{h} \in L^{\frac{2\hat{s}}{\hat{s}-r}}(0,T;\dot{X}_{r,\hat{s}-1}),$$

$$\nabla\mathfrak{h} \in L^{\frac{2s}{\hat{s}-r}}(0,T;\dot{X}_{r}),$$

$$\nabla\mathfrak{h} \in L^{\frac{2\hat{s}}{\hat{s}-r}}(0,T;\dot{X}_{r,\hat{s}}),$$

$$\nabla\mathfrak{h} \in L^{1}(0,T;\dot{B}^{0}_{\infty,\infty}),$$

for $0 < r < \hat{s} \leq \frac{5}{4}$.

Although numerous well-posedness and regularity results are proven for other systems, the detailed study of the local and global regularity of the system (7.1) clearly lacks. We try to fill this void by proving more rigorous geometric constraints for vorticity and gradient velocity because the geometric constraints are used to control blow-up of any three-dimensional system in given time interval. These results are new and improved, proven in more general and regular Besov function spaces.

Remark 7.1.1 When $\mathfrak{B} = 0$ system (7.1) becomes fractional micropolar system and for $\mathfrak{D} = 0$ and $\mathfrak{B} = 0$, system (7.1) reduces to fractional NSE.

7.2 Beale-Kato-Majda type criteria

In this section, we establish geometric constraints of the Belae-Kato-Majda type by using weak formulations and divergence free properties.

Theorem 7.2.1 Let $(\mathfrak{h}_0, \mathfrak{D}_0, \mathfrak{B}_0) \in H^n(\mathbb{R}^3)$ with $n > \frac{5}{2}$ and $\nabla \cdot \mathfrak{h}_0 = 0$, $\nabla \cdot \mathfrak{B}_0 = 0$ in distributional sense. If a weak solution $(\mathfrak{h}, \mathfrak{D}, \mathfrak{B})$ of system (7.1) satisfies the following constraint for vorticity

$$\int_0^T \frac{\|\Omega\|_{\dot{B}^0_{\infty,\infty}}}{\sqrt{(1+\log(e+\|\Omega\|_{\dot{B}^0_{\infty,\infty}})}} dt < \infty, \tag{7.2}$$

then it remains its smoothness in $(0,T]$, until its blow-up at $T = T^*$.

Proof. Testing $(7.1)_1$ with $\Delta\mathfrak{h}$ and using the divergence-free properties of velocity and magnetic field, we have that

$$\int_{\mathbb{R}^3} \mathfrak{h}_t \cdot \Delta\mathfrak{h}\ dx + \int_{\mathbb{R}^3} \mathfrak{h} \cdot \nabla\mathfrak{h} \cdot \Delta\mathfrak{h}\ dx + \int_{\mathbb{R}^3} (-\Delta)^{\hat{\alpha_1}}\mathfrak{h} \cdot \Delta\mathfrak{h}\ dx + \int_{\mathbb{R}^3} \nabla\Psi \cdot \Delta\mathfrak{h}\ dx$$

$$-\int_{\mathbb{R}^3} \mathfrak{B}\cdot\nabla\mathfrak{B}\cdot\Delta\mathfrak{h}\; dx - \int_{\mathbb{R}^3} 2(\nabla\times\mathfrak{D})\cdot\Delta\mathfrak{h}\; dx = 0$$

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\nabla\mathfrak{h}\|_{L^2}^2 + \|\Pi^{1+\hat{\alpha_1}}\mathfrak{h}\|_{L^2}^2 &= \int_{\mathbb{R}^3} \mathfrak{B}\cdot\nabla\mathfrak{B}\cdot\Delta\mathfrak{h}\; dx - \int_{\mathbb{R}^3} \mathfrak{h}\cdot\nabla\mathfrak{h}\cdot\Delta\mathfrak{h}\; dx.\\ &= X_1 + X_2.\end{aligned} \tag{7.3}$$

Similarly, testing (7.1)$_2$ with $\Delta\mathfrak{D}$, we get that

$$\int_{\mathbb{R}^3} \mathfrak{D}_t\cdot\Delta\mathfrak{D}\; dx + \int_{\mathbb{R}^3} \mathfrak{h}\cdot\nabla\mathfrak{D}\cdot\Delta\mathfrak{D}\; dx + \int_{\mathbb{R}^3} (-\Delta)^{\beta}\mathfrak{D}\cdot\Delta\mathfrak{D}\; dx - \int_{\mathbb{R}^3} \nabla\mathrm{div}\mathfrak{D}\cdot\Delta\mathfrak{D}\; dx$$

$$+\int_{\mathbb{R}^3} 4\mathfrak{D}\cdot\Delta\mathfrak{D}\; dx - \int_{\mathbb{R}^3} 2(\nabla\times\mathfrak{h})\cdot\Delta\mathfrak{D}\; dx = 0$$

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\nabla\mathfrak{D}\|_{L^2}^2 + \|\Pi^{1+\hat{\alpha_2}}\mathfrak{D}\|_{L^2}^2 + 4\|\nabla\mathfrak{D}\|_{L^2}^2 + \|\nabla\mathrm{div}\mathfrak{D}\|_{L^2}^2 &= -\int_{\mathbb{R}^3} \mathfrak{h}\cdot\nabla\mathfrak{D}\cdot\Delta\mathfrak{D}\; dx\\ &\quad + \int_{\mathbb{R}^3} 2(\nabla\times\mathfrak{h})\cdot\Delta\mathfrak{D}\; dx\\ &= X_3 + X_4.\end{aligned} \tag{7.4}$$

Finally, testing (7.1)$_3$ with $\Delta\mathfrak{B}$, we have

$$\int_{\mathbb{R}^3} \mathfrak{B}_t\cdot\Delta\mathfrak{B}\; dx + \int_{\mathbb{R}^3} \mathfrak{h}\cdot\nabla\mathfrak{B}\cdot\Delta\mathfrak{B}\; dx - \int_{\mathbb{R}^3} \mathfrak{B}\cdot\nabla\mathfrak{h}\cdot\Delta\mathfrak{B}\; dx + \int_{\mathbb{R}^3} (-\Delta)^{\hat{\alpha_3}}\mathfrak{B}\cdot\Delta\mathfrak{B}\; dx = 0$$

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\nabla\mathfrak{B}\|_{L^2}^2 + \|\Pi^{1+\hat{\alpha_3}}\mathfrak{B}\|_{L^2}^2 &= -\int_{\mathbb{R}^3} \mathfrak{h}\cdot\nabla\mathfrak{B}\cdot\Delta\mathfrak{B}\; dx + \int_{\mathbb{R}^3} \mathfrak{B}\cdot\nabla\mathfrak{h}\cdot\Delta\mathfrak{B}\; dx\\ &= X_5 + X_6.\end{aligned} \tag{7.5}$$

Now, adding (7.3), (7.4), and (7.5) to have

$$\frac{1}{2}\frac{d}{dt}\big(\|\nabla\mathfrak{h}\|_{L^2}^2 + \|\nabla\mathfrak{D}\|_{L^2}^2 + \|\nabla\mathfrak{B}\|_{L^2}^2\big) + \big(\|\Pi^{1+\hat{\alpha_1}}\mathfrak{h}\|_{L^2}^2 + \|\Pi^{1+\hat{\alpha_2}}\mathfrak{D}\|_{L^2}^2 + \|\Pi^{1+\hat{\alpha_3}}\mathfrak{B}\|_{L^2}^2\big)$$

$$+4\|\nabla\mathfrak{D}\|_{L^2}^2 + \|\nabla\mathrm{div}\mathfrak{D}\|_{L^2}^2 = \sum_{i=1}^{6} X_i. \tag{7.6}$$

Furthermore, we get an estimates for $X_i's$ in (7.6).For X_2, we derive

$$|X_2| \le \int_{\mathbb{R}^3} |\nabla\mathfrak{h}|^3 \le C\|\nabla\mathfrak{h}\|_{L^2}\|\nabla\mathfrak{h}\|_{L^4}^2.$$

Here, employing (1.1), we get that

$$\le C\|\nabla\mathfrak{h}\|_{BMO}\|\nabla\mathfrak{h}\|_{L^2}^2.$$

Using

$$\|\nabla \mathfrak{h}\|_{\dot{B}^0_{\infty,\infty}} \leq C\|\Omega\|_{\dot{B}^0_{\infty,\infty}},$$

and (1.2) yields

$$|X_2| \leq C\|\nabla \mathfrak{h}\|^2_{L^2}\Big(1 + \frac{\|\Omega\|_{\dot{B}^0_{\infty,\infty}}}{\sqrt{1+\log(e+\|\Omega\|_{\dot{B}^0_{\infty,\infty}})}}\Big)\log(e+\|\mathfrak{h}\|_{H^s})$$

$$\leq C\|\nabla \mathfrak{h}\|^2_{L^2}\Big(1 + \frac{\|\Omega\|_{\dot{B}^0_{\infty,\infty}}}{\sqrt{1+\log(e+\|\Psi\|_{\dot{B}^0_{\infty,\infty}})}}\Big)\log(e+\lambda(t)), \tag{7.7}$$

where $\lambda(t) := \sup_{T_*\leq s\leq t}\|\Pi^s \mathfrak{h}\|^2_{L^2} + \|\Pi^s \mathfrak{D}\|^2_{L^2} + \|\Pi^s V\|^2_{L^2}$. For the detailed reasoning and proven bounds on $s=3$, see [35].

Next, we get bounds for X_1

$$|X_1| \leq \int_{\mathbb{R}^3} |\nabla \mathfrak{B}|^2 \cdot \nabla \mathfrak{h}\ dx.$$

Now, using Holder's inequality with $\frac{1}{2}+\frac{5}{12}+\frac{1}{12}=1$, with Sobolev inequalities given

$$\leq C\|\nabla \mathfrak{h}\|_{L^2}\|\nabla \mathfrak{B}\|_{L^{12}}\|\nabla \mathfrak{B}\|_{L^{\frac{12}{5}}}$$

$$|X_1| \leq C\|\nabla \mathfrak{h}\|^2_{L^2} + C\|\Pi^{\hat{\alpha_3}}\mathfrak{B}\|^2_{L^2} + C\|\Pi^{\hat{\alpha_3}}\mathfrak{B}\|^2_{L^2}\|\nabla \mathfrak{h}\|^2_{L^2} + C\|\Pi^{1+\hat{\alpha_3}}\mathfrak{B}\|^2_{L^2} + C. \tag{7.8}$$

Following on the same steps as for X_1, the estimates for X_5, X_6, and X_3, are given as

$$|X_5| = |X_6| \leq C\|\nabla \mathfrak{h}\|^2_{L^2} + C\|\Pi^{\hat{\alpha_3}}\mathfrak{B}\|^2_{L^2} + C\|\Pi^{\hat{\alpha_3}}\mathfrak{B}\|^2_{L^2}\|\nabla \mathfrak{h}\|^2_{L^2} + C\|\Pi^{1+\hat{\alpha_3}}\mathfrak{B}\|^2_{L^2} + C \tag{7.9}$$

$$|X_3| \leq C\|\nabla \mathfrak{h}\|^2_{L^2} + C\|\Pi^{\hat{\alpha_2}}\mathfrak{D}\|^2_{L^2} + C\|\Pi^{\hat{\alpha_2}}\mathfrak{D}\|^2_{L^2}\|\nabla \mathfrak{h}\|^2_{L^2} + C\|\Pi^{1+\hat{\alpha_2}}\mathfrak{D}\|^2_{L^2} + C. \tag{7.10}$$

Lastly, for X_4, we obtain

$$|X_4| \leq C\|\nabla \mathfrak{h}\|_{L^2}\|\Delta \mathfrak{D}\|_{L^2}$$

$$\leq C\Big(\|\nabla \mathfrak{h}\|^2_{L^2} + \|\nabla \mathfrak{D}\|^2_{L^2} + \|\Pi^{1+\hat{\alpha_2}}\mathfrak{D}\|^2_{L^2}\Big). \tag{7.11}$$

Putting (7.7), (7.8), (7.9), (7.10), and (7.11) in (7.6), after arranging and conserving norms we have that

$$\frac{1}{2}\frac{d}{dt}\big(\|\nabla \mathfrak{h}\|^2_{L^2} + \|\nabla \mathfrak{D}\|^2_{L^2} + \|\nabla \mathfrak{B}\|^2_{L^2}\big) + \big(\|\Pi^{1+\hat{\alpha_1}}\mathfrak{h}\|^2_{L^2} + \|\Pi^{1+\hat{\alpha_2}}\mathfrak{D}\|^2_{L^2} + \|\Pi^{1+\hat{\alpha_3}}\mathfrak{B}\|^2_{L^2}\big)$$

$$+4\|\nabla \mathfrak{D}\|^2_{L^2} + \|\nabla \mathrm{div}\mathfrak{D}\|^2_{L^2}$$

$$\leq C\|\nabla\mathfrak{h}\|_{L^2}^2\Big(1+\frac{\|\Omega\|_{\dot{B}^0_{\infty,\infty}}}{\sqrt{1+\log(e+\|\Omega\|_{\dot{B}^0_{\infty,\infty}})}}\Big)\log(e+\lambda(t))+C\big(\|\nabla\mathfrak{h}\|_{L^2}^2+\|\nabla\mathfrak{D}\|_{L^2}^2\big)$$

$$+C\|\Pi^{\hat{\alpha_3}}\mathfrak{B}\|_{L^2}^2\|\nabla\mathfrak{h}\|_{L^2}^2+C\|\Pi^{\hat{\alpha_3}}\mathfrak{B}\|_{L^2}^2+C+C\|\Pi^{\hat{\alpha_2}}\mathfrak{D}\|_{L^2}^2+C\|\Pi^{\hat{\alpha_2}}\mathfrak{D}\|_{L^2}^2\|\nabla\mathfrak{h}\|_{L^2}^2+C.$$

Continuing by the same methodology as above, by applying Gronwall's inequality, We achieved that

$$\|\nabla\mathfrak{h}\|_{L^2}^2+\|\nabla\mathfrak{D}\|_{L^2}^2+\|\nabla\mathfrak{B}\|_{L^2}^2+\int_0^T\|\Pi^{\hat{\alpha_1}}\mathfrak{h}\|_{L^2}^2+\|\Pi^{\hat{\alpha_2}}\mathfrak{D}\|_{L^2}^2+\|\Pi^{\hat{\alpha_3}}\mathfrak{B}\|_{L^2}^2+\|\Pi^{1+\hat{\alpha_1}}\mathfrak{h}\|_{L^2}^2$$

$$+\|\Pi^{1+\hat{\alpha_2}}\mathfrak{D}\|_{L^2}^2+\|\Pi^{1+\hat{\alpha_3}}\mathfrak{B}\|_{L^2}^2\,dt$$

$$\leq C_0\exp\Big(C\int_{T_*}^T\Big(1+\frac{\|\Omega\|_{\dot{B}^0_{\infty,\infty}}}{\sqrt{1+\log(e+\|\Omega\|_{\dot{B}^0_{\infty,\infty}})}}\Big)dt(1+\ln(e+\lambda(t))\Big)$$

$$\leq C_0(e+\lambda(t))^{C\epsilon}, \tag{7.12}$$

Where $C_0=\Big(\|\nabla\mathfrak{h}(\cdot,T_*)\|_{L^2}^2+\|\nabla\mathfrak{D}(\cdot,T_*)\|_{L^2}^2+\|\nabla\mathfrak{B}(\cdot,T_*)\|_{L^2}^2+1\Big)$ and for infinitesimally small constant $\epsilon>0$, $\exists\, T_*<T$, such that

$$\int_{T_*}^T\Big(1+\frac{\|\Omega\|_{\dot{B}^0_{\infty,\infty}}}{\sqrt{1+\log(e+\|\Omega\|_{\dot{B}^0_{\infty,\infty}})}}\Big)dt<\epsilon.$$

The bounds for $\lambda(t)$, and for $\Big(\|\Pi^{\hat{\alpha_1}}\mathfrak{h}\|_{L^2}^2+\|\Pi^{\hat{\alpha_2}}\mathfrak{D}\|_{L^2}^2+\|\Pi^{\hat{\alpha_3}}\mathfrak{B}\|_{L^2}^2+\|\Pi^{1+\hat{\alpha_1}}\mathfrak{h}\|_{L^2}^2+\|\Pi^{1+\hat{\alpha_2}}\mathfrak{D}\|_{L^2}^2+\|\Pi^{1+\hat{\alpha_3}}\mathfrak{B}\|_{L^2}^2\Big)$ are elaborately proved in [88].

For the L^2 bounds, taking inner product of $\mathfrak{h}$ with $(1)_1$, $\mathfrak{D}$ with $(1)_2$, and $\mathfrak{B}$ with $(1)_3$, estimating all the terms and adding, it is easy to see we get that

$$\|\mathfrak{h}\|_{L^2}^2+\|\mathfrak{D}\|_{L^2}^2+\|\mathfrak{B}\|_{L^2}^2+\int_0^T\big(\|\Pi^{\hat{\alpha_1}}\mathfrak{h}\|_{L^2}^2+\|\Pi^{\hat{\alpha_2}}\mathfrak{D}\|_{L^2}^2+\|\Pi^{\hat{\alpha_3}}\mathfrak{B}\|_{L^2}^2\big)dt\leq C. \tag{7.13}$$

Hence, (7.12) together with (1.5) and (7.13) proves our result

$$\|\nabla\big(\mathfrak{h},\mathfrak{D},\mathfrak{B}\big)\|_{L^2}^2\leq C.$$

By controlling one of the geometric constraints of vorticity in regular homogeneous Besov spaces we proved our solutions will keep their blow-up until reaching singular time.

7.3 Gradient velocity criteria

This section is concerned with demonstrating gradient velocity regularity in the critical Besov spaces.

Theorem 7.3.1 Suppose $(\mathfrak{h}_0, \mathfrak{D}_0, \mathfrak{B}_0) \in H^n(\mathbb{R}^3)$ with $n > \frac{5}{2}$ and $\nabla \cdot \mathfrak{h}_0 = 0$, $\nabla \cdot \mathfrak{B}_0 = 0$ in distributional sense. If a weak solution $(\mathfrak{h}, \mathfrak{D}, \mathfrak{B})$ of system (7.1) satisfies the following condition for gradient velocity

$$\int_0^T \frac{\|\nabla \mathfrak{h}\|^2_{\dot{B}^{-1}_{\infty,\infty}}}{1 + \log(e + \|\nabla \mathfrak{h}\|_{\dot{B}^{-1}_{\infty,\infty}})} dt < \infty, \tag{7.14}$$

then it remains its regularity in the interval $\mathbb{R}^3 \times [0, T]$.

Proof. To prove result (7.2) , we continue from equation (7.6).

Now, we get new estimates for X_2

$$\begin{aligned}
|X_2| &\leq C\|\nabla \mathfrak{h}\|_{L^2}\|\nabla \mathfrak{h}\|^2_{L^4} \\
&\leq C\|\nabla \mathfrak{h}\|^2_{L^2}\|\nabla \mathfrak{h}\|^2_{\dot{B}^{-1}_{\infty,\infty}} + \|\Delta \mathfrak{h}\|^2_{L^2}.
\end{aligned}$$

All the other estimates X_i's are evaluated same as for the previous result. Putting all the new estimates in (7.6) and following on the same steps as for (7.12), we have that

$$\begin{aligned}
\|\nabla \mathfrak{h}\|^2_{L^2} + \|\nabla \mathfrak{D}\|^2_{L^2} + \|\nabla \mathfrak{B}\|^2_{L^2} + \int_0^T \|\Pi^{\hat{\alpha_1}} \mathfrak{h}\|^2_{L^2} + \|\Pi^{\hat{\alpha_2}} \mathfrak{D}\|^2_{L^2} + \|\Pi^{\hat{\alpha_3}} \mathfrak{B}\|^2_{L^2} + \|\Pi^{1+\hat{\alpha_1}} \mathfrak{h}\|^2_{L^2} \\
+\|\Pi^{1+\hat{\alpha_2}} \mathfrak{D}\|^2_{L^2} + \|\Pi^{1+\hat{\alpha_3}} \mathfrak{B}\|^2_{L^2}\ dt \\
\leq C_0 \exp\left(C \int_{T_*}^T \left(1 + \frac{\|\nabla \mathfrak{h}\|^2_{\dot{B}^{-1}_{\infty,\infty}}}{1 + \log(e + \|\nabla \mathfrak{h}\|_{\dot{B}^{-1}_{\infty,\infty}})}\right) dt (1 + \ln(e + \lambda(t))\right) \\
\leq C \exp(C_0 \epsilon \ln(e + \lambda(t))) \leq C_0 (e + \lambda(t))^{C\epsilon}.
\end{aligned} \tag{7.15}$$

Whereas, for $\epsilon > 0$, $\exists\ T_* < T$, such that

$$\int_{T_*}^T \left(1 + \frac{\|\nabla \mathfrak{h}\|^2_{\dot{B}^{-1}_{\infty,\infty}}}{1 + \log(e + \|\nabla \mathfrak{h}\|_{\dot{B}^{-1}_{\infty,\infty}})}\right) dt < \epsilon.$$

Now, (7.15) together with (7.2) proves that for the finite time the solution remains regular. □

Corollary 7.3.2 The fact $\|\nabla \mathfrak{h}\|_{\dot{B}^{-1}_{\infty,\infty}} \approx \|\mathfrak{h}\|_{\dot{B}^{0}_{\infty,\infty}}$ implies the new improved result for (7.1) via velocity constraint

$$\int_0^T \frac{\|\mathfrak{h}\|^2_{\dot{B}^{0}_{\infty,\infty}}}{1 + \log(e + \|\mathfrak{h}\|_{\dot{B}^{0}_{\infty,\infty}})} dt < \infty. \tag{7.16}$$

7.4 Conclusions

This work employs the functional theoretical approach to the system (7.1) to prove two new geometric constraints. Considering its structural properties, these constraints are vital for analyzing the turbulence in a finite time interval for a possible blow-up. The results (7.2), (7.14), and (7.16) are proved in Besov spaces that are important due to their complexity and scale-invariant properties. Although NSE have been analyzed extensively for the finite time regularity or partial regularity (the regularity of the singular sets), the detailed regularity and partial regularity analyses of the well-posedness of system (7.1) is missing from the literature. The better results should be presented on this system.

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