Study of Ag-water and Cu-water nanofluids

By

Zahid Ahmed

Department of Mathematics
Quaid-i-Azam University
Islamabad, Pakistan

2015

Study of Ag-water and Cu-water nanofluids

By

Zahid Ahmed

Supervised By

Dr. Sohail Nadeem

Department of Mathematics
Quaid-i-Azam University
Islamabad, Pakistan

2015

Study of Ag-water and Cu-water nanofluids

By

Zahid Ahmed

A DISSERTATION SUBMITTED IN THE PARTIAL FULFILLMENT OF THE REQUIREMENT FOR THE DEGREE OF

MASTER OF PHILOSOPHY

IN

MATHEMATICS

Supervised By

Dr. Sohail Nadeem

Department of Mathematics
Quaid-i-Azam University
Islamabad, Pakistan
2015

Preface

Nanofluids attain significant attention in recent years due to its wide range of applications in industrial as well as socioeconomic domain such as automotive industry, medical arena, nuclear power plant cooling system as well as electronic cooling systems. Nanofluids are proved to be capable to handle some of the very important problems of emerged industrial growth most significant among which is the problem to enhance the heat transfer ability of fluids and thermal management. The idea of nanofluids was first given by Choi [1] in 2001. Nanofluids are now expected to play its role nearly in each and every industry. One of the main reasons for its wide spread implication is that its effective physical as well as thermophysical properties are adjustable according to need. Many authors have presented different kind of nanofluids for various flow geometries. Mention may be made to the interesting work of [2-11]. The geometry, deformation and intrinsic motion such as rotation and spin motion of individual fluid element may affect the motion of the fluid and its heat transfer characteristics. Classical Naiver Stokes model take into account the fluid motion as a whole but it does not discusses the behavior of individual fluid element when undergoes with spin inertia and micro-rotational inertia. The concept of micropolar fluid first proposed by Eringin [12] can explain these intrinsic behaviors at very best both theoretically and practically. Micropolar fluid model supports couple stress and body torque. Unlike the ordinary fluid models micropolar fluid model possesses the asymmetric stress tensor [13]. Micropolar fluid theory is developed to be a generalized case of Naiver Stokes model in fact the micro-rotation parameter that appear in the momentum equation shows the deviation of micropolar fluid model from that of classical Naiver Stokes model. Due to its tremendous applications and uses many researchers have examined various aspects of this useful theory. Some are quoted in the studies [14-22]. This dissertation is the motivation by aforementioned research. It consists of three chapters. Important definitions and relevant equations are presented in chapter one.

It has been observed that some physical properties of nanofluids are sensible to temperature. In the heat transfer fluids the internal friction or heat generation or absorption phenomena can influence the fluid viscosity that is viscosity may change with temperature. Since the rise in temperature fasten the heat flow it is very necessary to study the effects of temperature dependent viscosity of the base-fluid on the heat transfer process. Chapter two therefor embraces the same theme. The subject matter of this chapter presents the meticulous review of the paper by Vajravelu Kuppalapalle [23].

In chapter three micropolar fluid theory is incorporated on nanofluids. The fluid viscosities (dynamic viscosity, spin gradient viscosity, micro-inertia density) are taken as inverse function of temperature. The flow is analyzed with different values of physical and thermophysical parameters such as solid-particle volume fraction, viscosity parameter, and micro-inertia viscosity parameter. The effect of these parameters on the fluid flow and on the heat transfer activity is determined numerically.

Contents

Chapter 1

Introduction

This chapter deals with the basic definitions and derivation of some governing equations.

1.1 Viscous and Inviscid Fluids

Two types of forces acted upon fluid elements namely body forces and surface forces. The body force is proportional to the mass of the fluids on which it acts and the surface force is enhanced by the surface area of the fluid element under consideration. The normal force that acts upon the unit area of the fluid elements are regarded as normal stress where as the tangential force on any unit area is said to be shearing stress.

The fluid is said to be viscous if normal as well as shearing stress is present, on the other hand fluid falls in the inviscid category if it is lake of shearing stress. Water and Air are examples of inviscid fluids where as syrups and heavy oils are usually taken as viscous fluids.

1.2 Viscosity

Due to shearing stress a viscous fluid produces resistance to the body moving through it as well as between the fluid elements, that property of the fluid which controls the rate of flow is called viscosity of the fluid. Mathematically, the coefficient of dynamic viscosity μ can be written as

$$T_{xy} = \mu(\frac{\partial u}{\partial y})$$

T_{xy} is shearing stress along xy-direction and u be the x-component of the velocity vector.

1.3 Compressible and Incompressible Fluids

The fluids that requires very large variation in pressure to produce some appreciable change in the density are regarded as incompressible fluids. The remaining fluids are compressible fluids. The hydrodynamics often deals with the incompressible fluids.

1.4 Rotational and Irrotational flows

In the fluid flow if the fluid particle rotate about their own axis then the flow is said to be rotational otherwise flow is irrotational.

1.5 Thermal conductivity

Transfer of heat per change in temperature is regarded as thermal conductivity. It is a thermophysical property of fluids . Fourier law of heat conduction states that the heat flux is proportional to the change in temperature. Mathematically,

$$Q \propto -\Delta T$$

$$Q = -K\Delta T$$

Q is the heat flux, ΔT is the change in temperature where K represients the thermal conductivity of the fluid. The negative sign shows that the direction of heat transfer is from higher temperature point to lower temperature point.

1.6 Specific heat

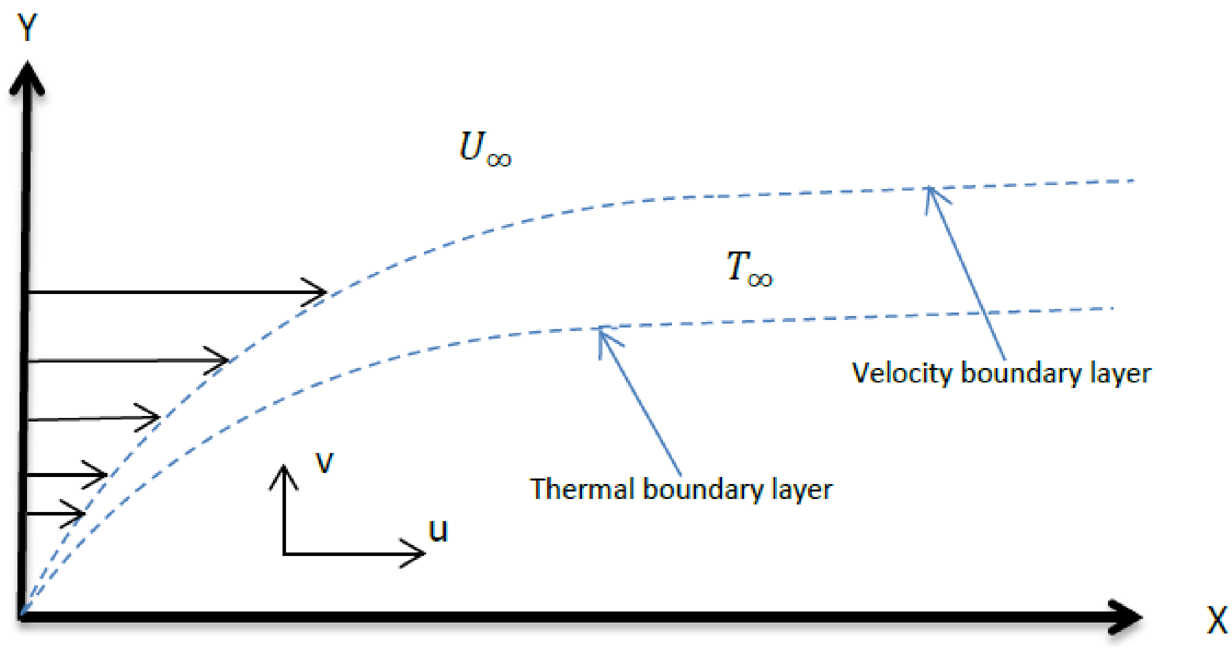
The amount of heat required to raise the temperature by one degree of a unit mass of a fluid is referred to as specific heat of that fluid. Thus

$$C = \frac{\Delta Q}{\Delta T}$$

where ΔQ is the quantity of heat added to increase the temperature by ΔT. Specific heat of a fluid can be determined mainly by two processes that are specific heat at constant volume and specific heat at constant pressure.

1.7 Boundary layer

A boundary layer is a very thin layer in the neighborhood of the plate in which the velocity gradient normal to the wall is very large. In the boundary layer the viscous stress $\mu(\frac{\partial u}{\partial y})$ becomes very important even when μ is very small.



The viscous and inertial forces are of the same order with in the boundary layer. Outside the boundary layer $\frac{\partial u}{\partial y}$ is very small so viscous forces may be ignored and thus boundary layer flow may be regarded as inviscid there. Most of the heat transfer to and from the fluid body takes place with in the boundary layer. The slip effect near the solid wall of the fluid couses mass deficit in the boundary layer as compared to the inviscid flow outside the boundary layer. The velocity boundary layer thickness is the distance from the solid boundaries to where the velocity is 99% of the free stream velocity or where the flow behaves as it is inviscid. Similarly thermal boundary layer can be defined as the distance from the solid boundaries to where the temperature is 99% of that of inviscid flow.

Naiver Stokes equations for general viscous flows are very difficult to solve. These equations therefor do not contribute in the field of aerodynamics, ship-hulls and viscous fluid dynamics for several decades until the concept of boundary layer approximation were introduced. Through boundary layer approximations the Naiver stokes equations are analyzed and terms that do not contribute significantly are dropped. This process evolves the model equations to rather simple and reasonable shape.

1.8 Dynamic Similarity

Two flows are dynamically similar if with similar geometrical boundaries the velocity field is similar that is both flows have similar stream lines. This is so if at all the similar geometries the forces acting on each point has some fixed ratio at every instant. Osborne Reynolds was the first who consider the laws of similar flows.

When the fluid flows are analyzed different types of forces are given importance at different situations and hence various laws are discovered to determine the dynamic similarity of different flows.

1.9 Some useful dimensionless numbers

1.9.1 Reynolds number

The dimensionless number that measures the ratio of inertial forces (force that resist motion) to viscous forces (heavy and gluey).

Mathematically,

$$\begin{aligned} \mathrm{Re} &= \frac{\text{inertial forces}}{\text{viscous forces}}, \\ &= \frac{VL}{\nu}, \end{aligned}$$

where L ,V and ν are length velocity and kinematic viscosity respectively.

1.9.2 Prandtle number

The ratio of viscous forces to thermal forces is represented by a dimensionless number called Prandtle number.

Mathematically,

$$\mathrm{Pr} = \frac{\mu g C_p}{k}$$

C_p is specific heat at constant pressure, k thermal conductivity of the fluid, μ the coefficient of dynamic viscosity and g is the gravitational acceleration. Prandtle number gives reasonable measure about the relative thickness of velocity boundary layer and thermal boundary layer. If Prandtle number is less than 1 than thermal boundary layer is thicker than the velocity boundary layer. If it is greater than 1 then thermal boundary layer is thinner than the velocity boundary layer. If Prandtle number is 1 then both boundary layers coincides.

1.9.3 Eckert number

Eckert number is a dimensionless number that measures the ratio of kinetic energy and change in thermal energy. Mathematically it can be written as

$$Ec = \frac{V^2}{C_p \Delta T}$$

V is the free stream velocity of the fluid, ΔT is the change in temperature of the fluid.

1.10 Nanofluids

Fluids that contains nanometer sized particle dispersed in it is called nanofluids. These nanoparticles are typically made of metals, oxides, carbides or carbon nanotubes. The base fluid commonly used are water, ethylene glycol and oil. The idea of nanofluid was first proposed by Choi [1] with the aim to enhance the thermal conductivity of the fluid. The eminent characteristic of nanofluid is that its physical as well as thermophysical properties are adjustable by setting the volume fraction of solid particles in base fluid. In nanofluids transform properties like effective density, effective dynamic viscosity and effective thermal conductivity are thus involve in the fluid flow and heat transfer phenomena.

1.11 Micropolar fluids

Micropolar fluid may be define as the fluid consist of rigid, randomly oriented particles suspended in a viscous medium. These fluid particles possess rotational as well as spin inertia. The individual fluid element in micropolar fluid may under goes into rotation and can shrink or expand independently. Animal blood and liquid crystals are good examples of micropolar fluids.

Micropolar fluid model was first proposed by Eringin [12] . This model is a generalization of classical Naiver Stokes fluid model. Unlike the Naiver Stokes model the stress tensor in micropolar fluid model is not symmetric. This model can support couple stress and body torque.

1.12 Basic equations

1.12.1 Continuity equation

In fluid mechanics, continuity equation is an equation that represent the law of conservation of mass which states that the rate at which the mass enters into the system is equal to the rate at which mass leaves the system.

Mathematically

$$\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla}.(\rho \mathbf{V}) = 0,$$

ρ is the density of the fluid, $\mathbf{V}$ is the flow velocity of the fluid. In three dimensional flow $\mathbf{V} = \mathbf{V}(u, v, w)$, $\boldsymbol{\nabla}$ is an operator given by

$$\boldsymbol{\nabla} = (i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}),$$

i, j and k are unit vectors.

For an incompressible fluid flow the continuity equation reduces to the following form

$$\boldsymbol{\nabla}.\mathbf{V} = 0.$$

1.12.2 Momentum equation

The law of conservation of linear momentum says that the rate of change of linear momentum is balanced by the force applied on the system. The equation that aims to represent the conservation of linear momentum in fluid dynamics can be written as

$$\rho\frac{d\mathbf{V}}{dt} = \boldsymbol{\nabla}.\mathbf{S} + \rho\mathbf{B},$$

where B is the body force, $\mathbf{V} = \mathbf{V}(u, v, w)$ be the velocity vector in three dimensional space, $\mathbf{S}$ the Cauchy stress tensor given as

$$\mathbf{S} = -p\mathbf{I} + \mu\mathbf{A}_1,$$

p the pressure on the fluid, $\mathbf{I}$ the identity tensor, $\mathbf{A}_1$ the first Rivlin Erickson tensor(a kinematic tensor) defined by

$$\mathbf{A}_1 = \text{grad}\,\mathbf{V} + (\text{grad}\,\mathbf{V})^t,$$

where

$$\text{grad}\,\mathbf{V} = \begin{bmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{bmatrix}$$

$\frac{d}{dt}$ is the material derivative which can be written as

$$\frac{d}{dt} = \frac{\partial}{\partial t} + (\mathbf{V}.\boldsymbol{\nabla}).$$

1.12.3 Angular momentum equation

The general equation for the law of conservation of angular momentum of micropolar fluid model can be written as mentioned below

$$\rho j \frac{d\mathbf{w}}{dt} = \rho\mathbf{g} + \boldsymbol{\nabla}.\mathbf{C} + \mathbf{S}_x$$

where $\mathbf{S}_x$ is the vector $\boldsymbol{\epsilon}_{ijk}\mathbf{S}_{ij}$, $\boldsymbol{\epsilon}_{ijk}$ is the alternating tensor of Levi-civita, $\mathbf{S}_{ij}$ is the stress tensor . In the case of micropolar fluid stress tensor can be written as[3]

$$\mathbf{S}_{ij} = (-p + \lambda v_{k,k})\delta_{ij} + \mu(v_{i,j} + v_{j,i}) + \kappa(v_{j,i} - \epsilon_{ijm}w_m) - 2\kappa\epsilon_{mij}w_m)$$

This stress tensor is asymmetric unless the micro-rotational viscosity κ vanishes.

The components of couple stress can be written as

$$\mathbf{C}_{ij} = c_0 w_{k,k}\delta_{ij} + c_d(w_{i,j} + w_{j,i}) + c_a(w_{j,i} - w_{i,j})$$

c_0, c_a and c$_d$ are coefficients of angular viscosities.

1.12.4 Energy equation

The energy equation in fluid dynamics represents the law of conservation of energy. This equation is derived from the first law of thermodynamics. Mathematically, the general equation of the conservation of energy for viscous fluid can be written as

$$\boldsymbol{\nabla}.(k\nabla T) + \varphi = \rho\frac{d}{dt}(C_pT) - \frac{dp}{dt}$$

k is the thermal conductivity of the fluid, ρ is the density of the fluid, C_p is the specific heat of the fluid , T is the temperature of the fluid, and φ represents the diffusibility terms in the equation which can be written as

$$\varphi = \mu[2 * (\in_{xx}^2 + \in_{yy}^2 + \in_{zz}^2) - \frac{2}{3} * (\boldsymbol{\nabla}.\mathbf{V}) * (\in_{xx} + \in_{yy} + \in_{zz}) + (\boldsymbol{\gamma}_{xy}^2 + \boldsymbol{\gamma}_{yz}^2 + \boldsymbol{\gamma}_{zx}^2)$$

where

$$\in_{xx} = \frac{\partial u}{\partial x}, \quad \in_{yy} = \frac{\partial v}{\partial y}, \quad \in_{zz} = \frac{\partial w}{\partial z}$$

$$\boldsymbol{\gamma}_{xy} = (\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}), \ \boldsymbol{\gamma}_{yz} = (\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}), \quad \boldsymbol{\gamma}_{zx} = (\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z})$$

For incompressible fluids the terms involving $(\boldsymbol{\nabla}.\mathbf{V})$ vanishes. This energy equation works until heat is added to the fluid body only by conduction. For convective, radiative or other heat transfers additional terms appear in the main equation.

1.13 Research methodology

The physical problem is modelled with the help of boundary layer approximation. Suitable similarity transformations are used to non-linear boundary value problem. The transformed non-linear ordinary differential equations are then solved by an implicit finite difference scheme known as Keller box method. This scheme consists of finite difference method, Newton's method and block elimination method. It is an implicit scheme with second order accuracy and is unconditionally stable.

Keller box scheme is much faster, more efficient and flexible to apply on non-linear boundary value problems. This scheme works effectively on the problem having complex physical situations and can be easily implemented to accommodate the variable physical and thermophysical properties of the fluid. The numerical solutions are obtained in four steps. At the very first the system of ordinary differential equations are converted into the system of first order ordinary differential equations which are then discritize by using central difference scheme. In the third step the discritized equations are linearized by applying Newton's linearization method and write the equations into vector matrix form. At very last these linearized system of equations are solved by block elimination technique. The detailed demonstration of this method is given in the next chapters when incorporated to modeled problems.

Chapter 2

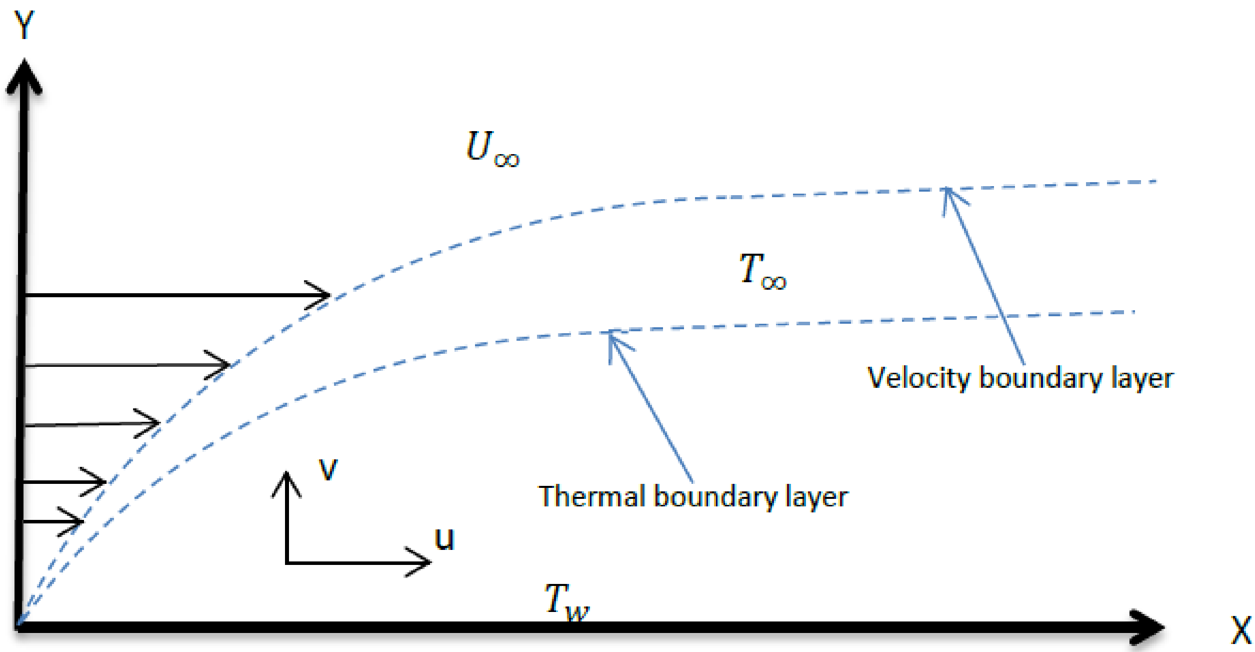
The study of Ag-water and Cu-water nanofluids with variable viscosity

2.1 Introduction

In this chapter we have examined the steady flow of incompressible nanofluid with variable temperature dependent viscosity for two dimensional flow. The effects of different physical parameters on the Ag-water and Cu-water nanofluid phenomena are considered. The governing equations of motion and energy are given in the presence of dissipation effects. The coupled partial differential equations are simplified with the help of existing suitable similarity transformations. The reduced equations are solved numerically. The subject matter of this chapter presents the meticulous review of the paper by Vajravelu Kuppalapalle [23].

2.2 Model Problem

Consider a two dimensional, steady, boundary layer laminar flow over a semi-infinite impermeable flat surface. The flow takes place over positive xy-plane with the surface at $y = 0$. The lower surface is fixed at constant temperature T_w and T_∞ is the temperature of the fluid far away from the surface. Fluid near the surface is at rest and attains free stream velocity U_∞ far away from the surface as shown in the $Fig(2.1)$.



Fig(2.1) Schematic diagram of boundary layer flow

The assumed fluid is water based nanofluid with nano-solid particles of either copper(Cu) or silver(Ag). The working fluid is assumed to be incompressible with the property of having no chemical reaction in between. Internal heat generation or absorption is negligible. In addition radiative heat transfer is ignored. The viscosity of the base fluid is taken as the inverse function of temperature. In the presence of viscous dissipation the governing equations for conservation of mass, momentum and energy are

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{2.1}$$

$$\rho_{nf}[u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}] = 2\frac{\partial}{\partial x}[\mu_{nf}(\frac{\partial u}{\partial x})] + \frac{\partial}{\partial y}[\mu_{nf}(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x})] \tag{2.2}$$

$$\rho_{nf}[u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y}] = 2\frac{\partial}{\partial y}[\mu_{nf}(\frac{\partial v}{\partial x})] + \frac{\partial}{\partial x}[\mu_{nf}(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x})] \tag{2.3}$$

$$u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} = \frac{K_{nf}}{(\rho C_p)_{nf}}[\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2}] + \frac{2\mu_{nf}}{(\rho C_p)_{nf}}[(\frac{\partial u}{\partial x})^2 + (\frac{\partial v}{\partial y})^2] + \frac{\mu_{nf}}{(\rho C_p)_{nf}}(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x})^2] \quad (2.4)$$

where u and v are the fluid velocity components in the stream direction and cross stream direction respectively, T is the temperature of the fluid. The effective density of the nanofluid $(\rho)_{nf} = (1-\phi)\rho_{f\infty} + \phi\rho_s$ where ρ_s the density of the solid particles, $\rho_{f\infty}$ is the density of the base fluid and ϕ denotes the volume fraction of the nano-solid particles in the fluid, μ_{nf} is the effective dynamic viscosity of the nanofluid which is given as [24] $\mu_{nf} = \frac{\mu_f}{(1-\phi)^{2.5}}$, here μ_f is the coefficient of dynamic viscosity which vary as an inverse function of temperature i.e. $\frac{1}{\mu_f} = a(T - T_r)$, where $a = \frac{\delta}{\mu_{f\infty}}$, $T_r = T_\infty - \frac{1}{\delta}$, δ and a are constants where $a > 0$ for liquids, K_{nf} is the thermal conductivity of the nanofluid is given as [26] $K_{nf} = K_{f\infty}[\frac{K_s+2K_{f\infty}-2\phi(K_{f\infty}-K_s)}{K_s+2K_{f\infty}+\phi(K_{f\infty}-K_s)}]$. The effective heat capacitance of the of the nanofluid can be written as $(\rho C_p)_{nf} = (1-\phi)(\rho C_p)_{f\infty} + \phi(\rho C_p)_s$, the suffixes nf , f_∞ and s indicates the thermophysical properties of nanofluid, base fluid and solid particles respectively.

We now incorporate boundary layer approximation on Eqs(2.1-2.4) with the aim to eliminate terms in the governing equations that do not contribute in the solution of the problem significantly. Let δ_v and δ_t be respectively the velocity and thermal boundary layer thickness. Let $O(u)$, $O(\frac{\partial}{\partial x})$, $O(\frac{\partial^2}{\partial x^2})$ and $O(T)$ be 1, $O(\frac{\partial}{\partial y}) = \frac{1}{\delta_v}$, $O(\frac{\partial^2}{\partial y^2}) = \frac{1}{\delta_v^2}$, We check the order of each terms in the governing equations and drop terms of order δ_v, δ_t and smaller. Since the viscous and inertial forces with in the boundary layer are of the same order only if $O(\frac{\mu_{nf}}{\rho_{nf}}) = \delta_v^2$, similarly the equivalence of conduction and convection terms in energy equations implies that $O(\frac{\mu_{nf}}{\rho_{nf}}) = \delta_v^2$. The reduced boundary value problem is

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \quad (2.5)$$

$$\rho_{nf}[u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}] = \frac{\partial}{\partial y}(\mu_{nf}\frac{\partial u}{\partial y}) \quad (2.6)$$

$$u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial y}=\alpha_{nf}\frac{\partial^2 T}{\partial y^2}+\frac{\mu_{nf}}{(\rho C_p)_{nf}}(\frac{\partial u}{\partial y})^2] \tag{2.7}$$

The appropriate boundary conditions for the proposed problem are

$$u(x,y)=0, v(x,y)=0, T(x,y)=T_w \text{ at } y=0$$
$$u(x,y)\rightarrow U_\infty, T(x,y)\rightarrow T_\infty \text{ as } y\rightarrow\infty$$

$\alpha_{nf}=\frac{K_{nf}}{(\rho C_p)_{nf}}$ be the thermal diffusivity of the nanofluid.

2.3 Similarity transformation

The non-linear partial differential equations (2.5) to (2.7) are transformed into a set of non-linear ordinary differential equations by choosing specific forms of velocity and temperature, of the following form

$$\eta=(\frac{U_\infty}{2\nu_\infty x})^{\frac{1}{2}}y\ ,\ \psi=(2\nu_\infty U_\infty x)^{\frac{1}{2}}f(\eta),\ \theta(\eta)=\frac{T-T_\infty}{T_w-T\infty},\ u=\frac{\partial\psi}{\partial y},\ v=-\frac{\partial\psi}{\partial x}$$

where η is the similarity variable, f and θ are the dimensionless velocity and temperature respectively. The velocity components automatically satisfy the continuity equation. The resulting non-linear ordinary differential equations are given as

$$\frac{f'''}{\left(1-\frac{\theta}{\theta_r}\right)}+\frac{f''\theta'}{\theta_r\left(1-\frac{\theta}{\theta_r}\right)^2}+(1-\phi)^{2.5}[1-\phi+\phi(\frac{\rho_s}{\rho_{f\infty}})]ff''=0 \tag{2.8}$$

$$\theta''+\Pr\frac{K_{f\infty}}{K_{nf}}[1-\phi+\phi\frac{(\rho C_p)_s}{(\rho C_p)_{f\infty}}]f\theta'+\Pr\frac{K_{f\infty}}{K_{nf}}\frac{Ec}{(1-\phi)^{2.5}(1-\frac{\theta}{\theta_r})}(f'')^2=0 \tag{2.9}$$

The transformed boundary conditions are

$$f(\eta)=0,\ \theta(\eta)=1,\ f\ '(\eta)=0\ \ at\ \eta=0$$
$$f\ '(\eta)\rightarrow 1,\ \theta(\eta)\rightarrow 0\ as\ \eta\rightarrow\infty$$

where $\theta_r = -\frac{1}{\delta(T_w - T\infty)}$, is the fluid viscosity parameter, $\Pr = \frac{\nu_{f\infty}}{\alpha_{f\infty}}$, is the Prandtle number, $Ec = \frac{U_\infty^2}{[(T_w - T_\infty)C_{pf\infty}]}$, is the Eckert number, and the term θ_r can be defined by fluid viscosity and the operating temperature difference. If θ_r is very large that is if $(T_w - T\infty)$ is so small than the variation of viscosity due to change in temperature is as small as it can be ignore. But if θ_r is smaller the effect of variable viscosity on fluid flow and heat transfer is supposed to be consider in the case. Also it should be noted that θ_r is negative for liquids and positive for gases.

2.4 Methods of solution

2.4.1 System of first order equations

We now have momentum equation of third order and energy equation of second order. These two equations are converted into the system of first order equations by introducing new variables u, v and w that is

$$f' = u \tag{2.10}$$

$$u' = v \tag{2.11}$$

$$\theta' = w \tag{2.12}$$

$$v'(1 - \frac{\theta}{\theta_r}) + \frac{1}{\theta_r}vw + (1-\phi)^{2.5}[1 - \phi + \phi(\frac{\rho_s}{\rho_{f\infty}})][fv - \frac{2}{\theta_r}fv\theta + \frac{1}{\theta_r^2}fv\theta^2] = 0 \tag{2.13}$$

$$w'(1 - \frac{\theta}{\theta_r}) + \Pr\frac{K_{f\infty}}{K_{nf}}[1 - \phi + \phi\frac{(\rho C_p)_s}{(\rho C_p)_{f\infty}}][fw - \frac{1}{\theta_r}fw\theta] + \Pr\frac{K_{f\infty}}{K_{nf}}\frac{Ec}{(1-\phi)^{2.5}}v^2 = 0 \tag{2.14}$$

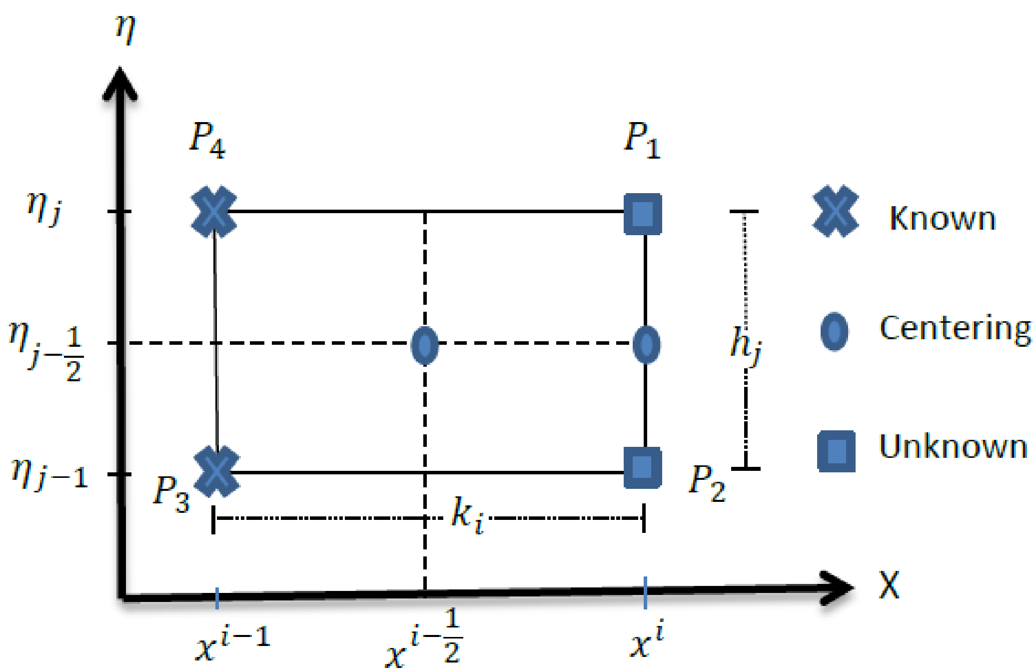
Boundary conditions yield

$$f(\eta) = 0,\ \theta(\eta) = 1,\ u(\eta) = 0 \quad at\ \eta = 0$$
$$u(\eta) \rightarrow 1,\ \theta(\eta) \rightarrow 0\ as\ \eta \rightarrow \infty$$

2.4.2 Discretization

Consider the net rectangle in the (x-η) plane shown in the $Fig(2.2)$. The net points are defined as

$$x^0 = 0, \quad x^i = x^{i-1} + k_i\ , \qquad i = 1, 2...I$$
$$\eta_0 = 0, \quad \eta_j = \eta_{j-1} + h_j, \qquad j = 1, 2...J$$



Fig(2.2) Schematic diagram of Mesh points

where 'k_i' is the Δx-spacing and 'h_j' is the $\Delta\eta$-spacing. Here i and j are just sequence of numbers that indicate the coordinate location. The finite difference form for any point is manipulated as

$$(\ )^i_{j-\frac{1}{2}} = \frac{1}{2}[(\ )^i_{j-1} + (\ )^i_j]$$

$$(\ )^{i-\frac{1}{2}}_j = \frac{1}{2}[(\ )^{i-1}_j + (\ )^i_j]$$

$$(\frac{\partial u}{\partial x})^{i-\frac{1}{2}}_{j-\frac{1}{2}} = \frac{u^i_{j-\frac{1}{2}} - u^{i-1}_{j-\frac{1}{2}}}{k_i}$$

$$(\frac{\partial u}{\partial \eta})^{i-\frac{1}{2}}_{j-\frac{1}{2}} = \frac{u^{i-\frac{1}{2}}_j - u^{i-\frac{1}{2}}_{j-1}}{h_j}$$

We now apply the finite difference scheme on the system of first order ordinary differential equations by centring at $(x^i, \eta_{j-\frac{1}{2}})$ on the first three equations. The resulting discritized equations are

$$f^i_j = f^i_{j-1} + \frac{h_j}{2}(u^i_{j-1} + u^i_j) \tag{2.15}$$

$$u^i_j = u^i_{j-1} + \frac{h_j}{2}(v^i_{j-1} + v^i_j) \tag{2.16}$$

$$\theta^i_j = \theta^i_{j-1} + \frac{h_j}{2}(w^i_{j-1} + w^i_j) \tag{2.17}$$

The last two equations are discritized by centring at $(x^{i-\frac{1}{2}}, \eta_{j-\frac{1}{2}})$, we get

$$
\begin{aligned}
&(v_j^i - v_{j-1}^i) - \frac{1}{2\theta_r}[(v_j^i - v_{j-1}^i)(\theta_j^i + \theta_{j-1}^i)] + \frac{h_j}{\theta_r}[\frac{(v_j^i + v_{j-1}^i)}{2}\frac{(w_j^i + w_{j-1}^i)}{2}] + h_j(1-\phi)^{2.5} \\
&[1-\phi+\phi(\frac{\rho_s}{\rho_{f\infty}})][\frac{(f_j^i + f_{j-1}^i)}{2}\frac{(v_j^i + v_{j-1}^i)}{2}] + \frac{h_j}{\theta_r}[\frac{(f_j^i + f_{j-1}^i)}{2}\frac{(v_j^i + v_{j-1}^i)}{2}\frac{(\theta_j^i + \theta_{j-1}^i)^2}{4}] \\
&-\frac{2h_j}{\theta_r}(1-\phi)^{2.5}[1-\phi+\phi(\frac{\rho_s}{\rho_{f\infty}})]h_j(1-\phi)^{2.5}[1-\phi+\phi(\frac{\rho_s}{\rho_{f\infty}})][\frac{(f_j^i + f_{j-1}^i)}{2}\frac{(v_j^i + v_{j-1}^i)}{2}] \\
&+\frac{h_j}{\theta_r}[\frac{(f_j^i + f_{j-1}^i)}{2}\frac{(v_j^i + v_{j-1}^i)}{2}\frac{(\theta_j^i + \theta_{j-1}^i)^2}{4}] - \frac{2h_j}{\theta_r}(1-\phi)^{2.5}[1-\phi+\phi(\frac{\rho_s}{\rho_{f\infty}})] \\
&[\frac{(f_j^i + f_{j-1}^i)}{2}\frac{(v_j^i + v_{j-1}^i)}{2}\frac{(\theta_j^i + \theta_{j-1}^i)}{2}] = M_{j-1}
\end{aligned}
$$

$$
\begin{aligned}
&(w_j^i - w_{j-1}^i) - \frac{1}{2\theta_r}[(w_j^i - w_{j-1}^i)(\theta_j^i + \theta_{j-1}^i)] + \Pr\frac{K_{f\infty}}{K_{nf}}[1-\phi+\phi\frac{(\rho C_p)_s}{(\rho C_p)_{f\infty}}]h_j[\frac{(f_j^i + f_{j-1}^i)}{2} \\
&\frac{(w_j^i + w_{j-1}^i)}{2}] - \Pr\frac{h_j}{\theta_r}\frac{K_{f\infty}}{K_{nf}}[1-\phi+\phi\frac{(\rho C_p)_s}{(\rho C_p)_{f\infty}}][\frac{(f_j^i + f_{j-1}^i)}{2}\frac{(w_j^i + w_{j-1}^i)}{2}\frac{(\theta_j^i + \theta_{j-1}^i)}{2}] \\
&+\Pr\frac{K_{f\infty}}{K_{nf}}\frac{Ec}{(1-\phi)^{2.5}}h_j\frac{(v_j^i + v_{j-1}^i)^2}{4} \\
= \;& N_{j-1}
\end{aligned}
$$

where

$$
\begin{aligned}
M_{j-\frac{1}{2}} \;=\; & -(v_j^{i-1} - v_{j-1}^{i-1}) + \frac{h_j}{\theta_r}(v'\theta)_{j-\frac{1}{2}}^{i-1} - \frac{h_j}{\theta_r}(vw)_{j-\frac{1}{2}}^{i-1} - h_j(1-\phi)^{2.5}[1-\phi+\phi(\frac{\rho_s}{\rho_{f\infty}})] \\
& -\frac{h_j}{\theta_r^2}(fv\theta^2)_{j-\frac{1}{2}}^{i-1} + \frac{2h_j}{\theta_r}(1-\phi)^{2.5}[1-\phi+\phi(\frac{\rho_s}{\rho_{f\infty}})](fv\theta)_{j-\frac{1}{2}}^{i-1} \qquad (2.18)
\end{aligned}
$$

$$
\begin{aligned}
N_{j-\frac{1}{2}} \;=\; & -(w_j^{i-1} - w_{j-1}^{i-1}) + \frac{h_j}{\theta_r}(w'\theta)_{j-\frac{1}{2}}^{i-1} - \Pr\frac{K_{f\infty}}{K_{nf}}[1-\phi+\phi\frac{(\rho C_p)_s}{(\rho C_p)_{f\infty}}](fw)_{j-\frac{1}{2}}^{i-1} \\
& +\frac{\Pr}{\theta_r}\frac{K_{f\infty}}{K_{nf}}[1-\phi+\phi\frac{(\rho C_p)_s}{(\rho C_p)_{f\infty}}]h_j(fw\theta)_{j-\frac{1}{2}}^{i-1} - \Pr\frac{K_{f\infty}}{K_{nf}}\frac{Ec}{(1-\phi)^{2.5}}h_j(v^2)_{j-\frac{1}{2}}^{i-1} \qquad (2.19)
\end{aligned}
$$

If we assume f_j^{i-1}, u_j^{i-1}, v_j^{i-1}, $\theta_j^{i-1} and\ w_j^{i-1}$ to be known for $0 \le j \le J$ then $M_{j-\frac{1}{2}}$, $N_{j-\frac{1}{2}}$ contains only known quantities and we have to obtain the solution of unknowns ($f_j^i, u_j^i, v_j^i, \theta_j^i$,

w_j^i) for $0 \leq j \leq J$. We write the unknowns at $ith-level$ as $(f_j, u_j, v_j, \theta_j, w_j)$ that is to ignore the superscripts for the sake of simplicity.

$$f_j = f_{j-1} + \frac{h_j}{2}(u_{j-1} + u_j) \tag{2.20}$$

$$u_j = u_{j-1} + \frac{h_j}{2}(v_{j-1} + v_j) \tag{2.21}$$

$$\theta_j = \theta_{j-1} + \frac{h_j}{2}(w_{j-1} + w_j) \tag{2.22}$$

$$\begin{aligned}
&(v_j - v_{j-1}) - \frac{1}{2\theta_r}[(v_j - v_{j-1})(\theta_j + \theta_{j-1})] + \frac{h_j}{\theta_r}[\frac{(v_j + v_{j-1})}{2}\frac{(w_j + w_{j-1})}{2}] + h_j(1-\phi)^{2.5} \\
&[1 - \phi + \phi(\frac{\rho_s}{\rho_{f\infty}})][\frac{(f_j + f_{j-1})}{2}\frac{(v_j + v_{j-1})}{2}] + \frac{h_j}{\theta_r}[\frac{(f_j + f_{j-1})}{2}\frac{(v_j + v_{j-1})}{2}\frac{(\theta_j + \theta_{j-1})^2}{4}] \\
&-\frac{2h_j}{\theta_r}(1-\phi)^{2.5}[1 - \phi + \phi(\frac{\rho_s}{\rho_{f\infty}})][\frac{(f_j + f_{j-1})}{2}\frac{(v_j + v_{j-1})}{2}\frac{(\theta_j + \theta_{j-1})}{2}] = M_{j-1}
\end{aligned} \tag{2.23}$$

$$\begin{aligned}
&(w_j - w_{j-1}) - \frac{1}{2\theta_r}[(w_j - w_{j-1})(\theta_j + \theta_{j-1})] + \Pr\frac{K_{f\infty}}{K_{nf}}[1 - \phi + \phi\frac{(\rho C_p)_s}{(\rho C_p)_{f\infty}}]h_j \\
&[\frac{(f_j + f_{j-1})}{2}\frac{(w_j + w_{j-1})}{2}] - \Pr\frac{h_j}{\theta_r}\frac{K_{f\infty}}{K_{nf}}[1 - \phi + \phi\frac{(\rho C_p)_s}{(\rho C_p)_{f\infty}}] \\
&[\frac{(f_j + f_{j-1})}{2}\frac{(w_j + w_{j-1})}{2}\frac{(\theta_j + \theta_{j-1})}{2}] + \Pr\frac{K_{f\infty}}{K_{nf}}\frac{Ec}{(1-\phi)^{2.5}}h_j\frac{(v_j + v_{j-1})^2}{4} \\
= \;& N_{j-1}
\end{aligned} \tag{2.24}$$

The boundary conditions become

$$f_0 = 0, \quad u_0 = 0, u_J = 1, \theta_0 = 1, \theta_J = 0.$$

2.4.3 Newton's Linearization

The above finite difference equations are linearized by using Newton's linearization technique.

$$f^{(k+1)} = f^{(k)} + \delta f^{(k)}, \quad u^{(k+1)} = u^{(k)} + \delta u^{(k)}$$

$$v^{(k+1)} = v^{(k)} + \delta v^{(k)}, \quad \theta^{(k+1)} = \theta^{(k)} + \delta\theta^{(k)}$$

$$w^{(k+1)} = w^{(k)} + \delta w^{(k)}$$

We substitute the above expressions in the equations and ignoring the quadratic and higher order terms in δ. The resulting equations then take the following form

$$\delta f_j - \delta f_{j-1} - \frac{h_j}{2}(\delta u_j + \delta u_{j-1}) = (r_1)_j \tag{2.25}$$

$$\delta u_j - \delta u_{j-1} - \frac{h_j}{2}(\delta v_j + \delta v_{j-1}) = (r_2)_j \tag{2.26}$$

$$\delta\theta_j - \delta\theta_{j-1} - \frac{h_j}{2}(\delta w_j + \delta w_{j-1}) = (r_3)_j \tag{2.27}$$

$$\begin{aligned} & (a_1)_j\delta v_j + (a_2)_j\delta v_{j-1} + (a_3)_j\delta u_j + (a_4)_j\delta u_{j-1} + (a_5)_j\delta f_j \\ & +(a_6)_j\delta f_{j-1} + (a_7)_j\delta w_j + (a_8)_j\delta w_{j-1} + (a_9)_j\delta\theta_j + (a_{10})\delta\theta_{j-1} \\ = \quad & (r_4)_j \end{aligned}$$

$$\begin{aligned} & (b_1)_j\delta v_j + (b_2)_j\delta v_{j-1} + (b_3)_j\delta u_j + (b_4)_j\delta u_{j-1} + (b_5)_j\delta f_j \\ & +(b_6)_j\delta f_{j-1} + (b_7)_j\delta w_j + (b_8)_j\delta w_{j-1} + (b_9)_j\delta\theta_j + (b_{10})\delta\theta_{j-1} \\ = \quad & (r_5)_j \end{aligned}$$

$$\begin{aligned} (a_1)_j \quad = \quad & 1 - \frac{1}{\theta_r}\theta_{j-\frac{1}{2}} + \frac{h_j}{2\theta_r}w_{j-\frac{1}{2}} + \frac{h_j}{2}(1-\phi)^{2.5}[1-\phi+\phi(\frac{\rho_s}{\rho_{f\infty}})]f_{j-\frac{1}{2}} \\ & -\frac{h_j}{\theta_r}(1-\phi)^{2.5}[1-\phi+\phi(\frac{\rho_s}{\rho_{f\infty}})]f_{j-\frac{1}{2}}\theta_{j-\frac{1}{2}} \\ & +\frac{h_j}{2\theta_r^2}(1-\phi)^{2.5}[1-\phi+\phi(\frac{\rho_s}{\rho_{f\infty}})]f_{j-\frac{1}{2}}\theta_{j-\frac{1}{2}}\theta_{j-\frac{1}{2}} \end{aligned}$$

$$
\begin{aligned}
(a_2)_j &= -1+\frac{1}{\theta_r}\theta_{j-\frac{1}{2}}+\frac{h_j}{2\theta_r}w_{j-\frac{1}{2}}+\frac{h_j}{2}(1-\phi)^{2.5}[1-\phi+\phi(\frac{\rho_s}{\rho_{f\infty}})]f_{j-\frac{1}{2}} \\
&\quad -\frac{h_j}{\theta_r}(1-\phi)^{2.5}[1-\phi+\phi(\frac{\rho_s}{\rho_{f\infty}})]f_{j-\frac{1}{2}}\theta_{j-\frac{1}{2}} \\
&\quad +\frac{h_j}{2\theta_r^2}(1-\phi)^{2.5}[1-\phi+\phi(\frac{\rho_s}{\rho_{f\infty}})]f_{j-\frac{1}{2}}\theta_{j-\frac{1}{2}}\theta_{j-\frac{1}{2}}
\end{aligned}
$$

$$(a_3)_j=(a_4)_j=0$$

$$
\begin{aligned}
(a_5)_j &= \frac{h_j}{2}(1-\phi)^{2.5}[1-\phi+\phi(\frac{\rho_s}{\rho_{f\infty}})]v_{j-\frac{1}{2}}-\frac{h_j}{\theta_r}(1-\phi)^{2.5}[1-\phi+\phi(\frac{\rho_s}{\rho_{f\infty}})]v_{j-\frac{1}{2}}\theta_{j-\frac{1}{2}} \\
&\quad +\frac{h_j}{2\theta_r^2}v_{j-\frac{1}{2}}\theta_{j-\frac{1}{2}}\theta_{j-\frac{1}{2}}
\end{aligned}
$$

$$(a_6)_j=(a_5)_j$$

$$(a_7)_j=\frac{h_j}{2\theta_r}v_{j-\frac{1}{2}}$$

$$(a_8)_j=(a_7)_j$$

$$(a_9)_j=-\frac{1}{2\theta_r}(v_j-v_{j-1})-\frac{h_j}{\theta_r}(1-\phi)^{2.5}[1-\phi+\phi(\frac{\rho_s}{\rho_{f\infty}})]v_{j-\frac{1}{2}}f_{j-\frac{1}{2}}+\frac{h_j}{\theta_r^2}v_{j-\frac{1}{2}}f_{j-\frac{1}{2}}\theta_{j-\frac{1}{2}}$$

$$(a_{10})_j=(a_9)_j$$

$$(b_1)_j=\Pr\frac{K_{f\infty}}{K_{nf}}Ah_jv_{j-\frac{1}{2}}$$

$$(b_2)_j=(b_1)_j$$

$$(b_3)_j=(b_4)_j=0$$

$$(b_5)_j=\Pr\frac{K_{f\infty}}{K_{nf}}[1-\phi+\phi\frac{(\rho C_p)_s}{(\rho C_p)_{f\infty}}]\frac{h_j}{2}w_{j-\frac{1}{2}}-\frac{1}{\theta_r}\Pr\frac{K_{f\infty}}{K_{nf}}[1-\phi+\phi\frac{(\rho C_p)_s}{(\rho C_p)_{f\infty}}]\frac{h_j}{2}w_{j-\frac{1}{2}}\theta_{j-\frac{1}{2}}$$

$$(b_6)_j=(b_5)_j$$

$$(b_7)_j=1-\frac{1}{2\theta_r}\theta_{j-\frac{1}{2}}+\Pr\frac{K_{f\infty}}{K_{nf}}[1-\phi+\phi\frac{(\rho C_p)_s}{(\rho C_p)_{f\infty}}]\frac{h_j}{2}f_{j-\frac{1}{2}}-\frac{1}{\theta_r}\Pr\frac{K_{f\infty}}{K_{nf}}[1-\phi+\phi\frac{(\rho C_p)_s}{(\rho C_p)_{f\infty}}]\frac{h_j}{2}f_{j-\frac{1}{2}}\theta_{j-\frac{1}{2}}$$

$$(b_8)_j=-1+\frac{1}{2\theta_r}\theta_{j-\frac{1}{2}}+\Pr\frac{K_{f\infty}}{K_{nf}}[1-\phi+\phi\frac{(\rho C_p)_s}{(\rho C_p)_{f\infty}}]\frac{h_j}{2}f_{j-\frac{1}{2}}-\frac{1}{\theta_r}\Pr\frac{K_{f\infty}}{K_{nf}}[1-\phi+\phi\frac{(\rho C_p)_s}{(\rho C_p)_{f\infty}}]\frac{h_j}{2}f_{j-\frac{1}{2}}\theta_{j-\frac{1}{2}}$$

$$(b_9)_j=-\frac{1}{\theta_r}(w_j-w_{j-1})-\frac{h_j}{2\theta_r}\Pr\frac{K_{f\infty}}{K_{nf}}[1-\phi+\phi\frac{(\rho C_p)_s}{(\rho C_p)_{f\infty}}]w_{j-\frac{1}{2}}f_{j-\frac{1}{2}}$$

$$(b_{10})_j = (b_9)_j$$

$$(r_1)_j = f_{j-1} - f_j + h_j u_{j-\frac{1}{2}}$$

$$(r_2)_j = u_{j-1} - u_j + h_j v_{j-\frac{1}{2}}$$

$$(r_3)_j = \theta_{j-1} - \theta_j + h_j w_{j-\frac{1}{2}}$$

$$\begin{aligned}(r_4)_j &= (\frac{1}{\theta_r}\theta_{j-1} - 1)(v_j - v_{j-1}) - \frac{h_j}{\theta_r} w_{j-\frac{1}{2}} v_{j-\frac{1}{2}} - h_j(1-\phi)^{2.5}[1 - \phi + \phi(\frac{\rho_s}{\rho_{f\infty}})] v_{j-\frac{1}{2}} f_{j-\frac{1}{2}} \\ &+ \frac{2h_j}{\theta_r}(1-\phi)^{2.5}[1 - \phi + \phi(\frac{\rho_s}{\rho_{f\infty}})] f_{j-\frac{1}{2}} v_{j-\frac{1}{2}} \theta_{j-\frac{1}{2}} - \frac{h_j}{\theta_r^2} f_{j-\frac{1}{2}} v_{j-\frac{1}{2}} \theta_{j-\frac{1}{2}} + M_{j-\frac{1}{2}}\end{aligned}$$

$$\begin{aligned}(r_5)_j &= (\frac{1}{\theta_r}\theta_{j-\frac{1}{2}} - 1)(w_j - w_{j-1}) - \Pr \frac{K_{f\infty}}{K_{nf}}[1 - \phi + \phi \frac{(\rho C_p)_s}{(\rho C_p)_{f\infty}}] h_j w_{j-\frac{1}{2}} f_{j-\frac{1}{2}} \\ &+ \frac{1}{\theta_r} \Pr \frac{K_{f\infty}}{K_{nf}}[1 - \phi + \phi \frac{(\rho C_p)_s}{(\rho C_p)_{f\infty}}] h_j w_{j-\frac{1}{2}} f_{j-\frac{1}{2}} \theta_{j-\frac{1}{2}} - \Pr \frac{K_{f\infty}}{K_{nf}} A h_j v_{j-\frac{1}{2}} v_{j-\frac{1}{2}} + N_{j-\frac{1}{2}}\end{aligned}$$

We now write this system of equations into matrix-vector form.

$$A\varphi = R \tag{2.28}$$

where

$$A = \begin{bmatrix} [A_1] & [C_1] & & & & & \\ [B_1] & [A_2] & . & & & & \\ & . & . & . & & & \\ & & . & . & . & & \\ & & & . & . & . & \\ & & & & [B_{J-1}] & [A_{J-1}] & [C_{J-1}] \\ & & & & & [B_J] & [A_J] \end{bmatrix}$$

$$\varphi = \begin{bmatrix} [\delta_1] \\ [\delta_2] \\ . \\ . \\ [\delta_J] \end{bmatrix}, \quad R = \begin{bmatrix} [R_1] \\ [R_2] \\ . \\ . \\ [R_J] \end{bmatrix}, \quad [R_j] = \begin{bmatrix} (r_1)_j \\ (r_2)_j \\ (r_3)_j \\ (r_4)_j \\ (r_5)_j \end{bmatrix}$$

$$[\delta_1] = \begin{bmatrix} v_0 \\ w_0 \\ f_1 \\ v_1 \\ w_1 \end{bmatrix}, \quad [\delta_j] = \begin{bmatrix} u_{j-1} \\ \theta_{j-1} \\ f_j \\ v_j \\ w_j \end{bmatrix}, \quad \text{For } 2 \leq j \leq J$$

$$[A_1] = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ -\frac{h_j}{2} & 0 & 0 & -\frac{h_j}{2} & 0 \\ 0 & -\frac{h_j}{2} & 0 & 0 & -\frac{h_j}{2} \\ (a_2)_1 & (a_{10})_1 & (a_3)_1 & (a_1)_1 & (a_9)_1 \\ (b_2)_1 & (b_{10})_1 & (b_3)_1 & (b_1)_1 & (b_9)_1 \end{bmatrix}$$

$$[A_j] = \begin{bmatrix} -\frac{h_j}{2} & 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & -\frac{h_j}{2} & 0 \\ 0 & -1 & 0 & 0 & -\frac{h_j}{2} \\ 0 & (a_8)_j & (a_3)_j & (a_1)_j & (a_9)_j \\ 0 & (b_8)_j & (b_3)_j & (b_1)_j & (b_9)_j \end{bmatrix} \quad \text{For } \ 2 \leq j \leq J$$

$$[B_j] = \begin{bmatrix} 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -\frac{h_j}{2} & 0 \\ 0 & 0 & 0 & 0 & -\frac{hj}{2} \\ 0 & 0 & (a_4)_j & (a_2)_j & (a_{10})_j \\ 0 & 0 & (b_4)_j & (b_2)_j & (b_{10})_j \end{bmatrix}$$

$$[C_j] = \begin{bmatrix} -\frac{h_j}{2} & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & (a_7)_j & 0 & 0 & 0 \\ 0 & (b_7)_j & 0 & 0 & 0 \end{bmatrix}$$

2.4.4 Block elimination technique

The above system of matrix-vectors can be solved by block elimination method as discussed by Keller[27](1978). The very first step is to decompose the system into two factors that is

$$A = LU \tag{2.29}$$

L be the lower triangular matrix system

$$L = \begin{bmatrix} [\alpha_1] & & & & & & \\ [B_2] & [\alpha_2] & & & & & \\ & [B_3] & [\alpha_3] & & & & \\ & & . & . & & & \\ & & & . & . & & \\ & & & & [B_{J-1}] & [\alpha_{J-1}] & \\ & & & & & [B_J] & [\alpha_J] \end{bmatrix}$$

U be the upper triangular matrix system

$$U = \begin{bmatrix} [I] & [\Gamma_1] & & & & & \\ & [I_2] & [\Gamma_2] & & & & \\ & & . & . & & & \\ & & & . & . & & \\ & & & & . & . & \\ & & & & & [I_{J-1}] & [\Gamma_{J-1}] \\ & & & & & & [I_J] \end{bmatrix}$$

The above systems of matrix-vectors results in the following equations

$$[\alpha_1] = [A_1] \tag{2.30}$$

$$[\alpha_j] = [A_j] - [B_j][\Gamma_{j-1}] \qquad \text{for} \quad j = 2, 3...J \tag{2.31}$$

$$[\alpha_j][\Gamma_j] = [C_j] \qquad \text{for} \quad j = 1, 2....J-1 \tag{2.32}$$

Using above equations we can calculate the matrices $[\alpha_j]$ and $[\Gamma_j]$.

Now using Eqs (2.28) and (2.29), we get

$$LU\varphi = R \tag{2.33}$$

Let

$$U\varphi = W \tag{2.34}$$

where

$$W = \begin{bmatrix} [W_1] \\ [W_2] \\ [W_3] \\ [W_4] \\ [W_5] \end{bmatrix}$$

Thus

$$LW = R$$

which gives the following relations

$$[\alpha_1][W_1] = [R_1] \tag{2.35}$$

$$[\alpha_j][W_j] = [R_j] - [B_j][W_{j-1}], \qquad \text{for } 2 \leq j \leq J \tag{2.36}$$

Since $[\alpha_j]$ and $[R_j]$ are known matrices thus $[W_j]$ can be calculated using above equations. The step computing $[\alpha_j]$, $[\Gamma_j]$ and $[w_j]$ is known as forward sweep. At the end we use Eq(2.38) to calculate the components of $[\delta_j]$. This step is called backward sweep, where $[\delta_j]$ is computed by using the following equations

$$[\delta_J] = [W_J] \tag{2.37}$$

$$[\delta_j] = [W_J] - [\Gamma_j][\delta_{j+1}], \text{ for } \ 1 \leq j \leq J-1 \tag{2.38}$$

These calculations are repeated until the convergence criterion is achieved. This last step is done with the help of Matlab programming. Matlab 7.9 version is used for the algorithm. $\Delta\eta = 0.01$ step size is taken for all of the computations. Error tolerance is fixed at 10^{-6} for all of the calculations. The edge of the boundary layer is fixed at $\eta_\infty = 5$, throughout the computations. Initial solutions are guess as under

$$f_0 = \frac{1}{2}\frac{\eta^2}{\eta_\infty},\ u_0 = \frac{\eta}{\eta_0},\ v_0 = \frac{1}{\eta_\infty},\ \theta_0 = 1 - \frac{\eta}{\eta_\infty},\ w_0 = -\frac{1}{\eta_\infty}$$

2.5 Comparison Method

Temperature gradient is calculated with different values of parameters for the case when there is no viscous dissipation. This is done with the help of Maple software. A finite difference method which is called midrich method is incorporated for this matter. Error tolerance is set at 10^{-6}, maximum mesh point is taken as 501, step size is taken as 0.01. Continuation approach is implemented to solve the problem. The results are in very good agreement with the solutions obtained by Keller box method.

2.6 Numerical results

All of the thermophysical properties of assumed fluid is taken as constant except the viscosity of the fluid which is considered as the inverse function of temperature. $Table(2.1)$ shows the constant thermophysical properties of Copper, Silver and base fluid(water). $Table(2.2)$ presents the computed numerical values of temperature gradient near the wall for Cu-water nanofluid. $Table(2.3)$ contains numerical results for temperature gradient near the wall for Ag-water nanofluid. Flow is analyzed in the presence of fluid viscosity parameter, the Eckert number and the nano-particle volume fraction parameter.

$Table(2.1)$

	Base fluid	Cu-nano-particles	Ag-nano-particles
$C_p(J/kgK)$	4179	385.0	235.0
$\rho(kg/m^3)$	997.1	8933.0	10500
$k(w/mK)$	0.613	400.0	429.0
$\alpha(10^{-7}m^2/s)$	1.470	1163.1	1738.6

Table.(2.2) Temperature gradient values of Cu-water nanofluid

	ϕ	$Ec = 0.0$		$Ec = 0.25$	$Ec = 0.50$
		Maple results	Matlab results		
$\theta_r = -1$	0.0	-1.041315	-1.041358	-0.728118	-0.375123
	0.05	-1.0157876	-1.015819	-0.658840	-0.259730
	0.10	-0.979076	-0.979102	-0.591917	-0.161907
	0.15	-0.936794	-0.936817	-0.527641	-0.075412
	0.20	-0.891570	-0.891590	-0.465633	0.003736
$\theta_r = -5$	0.0	-0.938238	-0.938288	-0.642745	-0.328257
	0.05	-0.919774	-0.919802	-0.574472	-0.206570
	0.10	-0.889914	-0.889937	-0.508089	-0.101004
	0.15	-0.854041	-0.854061	-0.444437	-0.007389
	0.20	-0.814749	-0.814766	-0.383314	0.077525
$\theta_r = -10$	0.0	-0.921660	-0.921710	-0.632423	-0.332490
	0.05	-0.904149	-0.904177	-0.565200	-0.213460
	0.10	-0.875276	-0.875210	-0.499672	-0.109612
	0.15	-0.840363	-0.840384	-0.436760	-0.017309
	0.20	-0.801986	-0.802003	-0.376398	0.066317
$\theta_r \to \infty$	0.0	-0.903934	-0.904173	-0.622895	-0.340774
	0.05	-0.887387	-0.887589	-0.557276	-0.226215
	0.10	-0.859533	-0.859720	-0.492970	-0.125570
	0.15	-0.825625	-0.825798	-0.431099	-0.035888
	0.20	-0.788211	-0.788372	-0.371716	0.045292

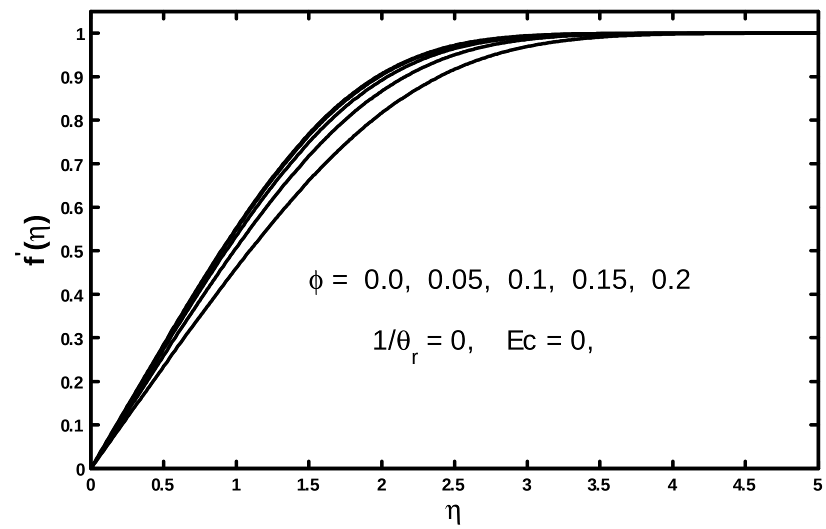
Table(2.3) Temperature gradient values of Ag-water nanofluid

	ϕ	$Ec = 0.0$		$Ec = 0.25$	$Ec = 0.50$
		Maple results	Matlab results		
$\theta_r = -1$	0.0	-1.041325	-1.041372	-0.728090	-0.375179
	0.05	-1.019483	-1.019517	-0.649850	-0.237143
	0.10	-0.982245	-0.982271	-0.574185	-0.122164
	0.15	-0.937760	-0.937781	-0.501809	-0.021946
	0.20	-0.889635	-0.8897	-0.4325	0.0686
$\theta_r = -5$	0.0	-0.938260	-0.938280	-0.646633	-0.328441
	0.05	-0.923993	-0.924019	-0.565022	-0.182172
	0.10	-0.894215	-0.894238	-0.488476	-0.056536
	0.15	-0.856706	-0.856724	-0.415667	0.053748
	0.20	-0.815010	-0.815026	-0.346199	0.153486
$\theta_r = -10$	0.0	-0.921685	-0.921710	-0.632607	-0.333261
	0.05	-0.908419	-0.908445	-0.555801	-0.190553
	0.10	-0.879711	-0.879734	-0.480512	-0.065220
	0.15	-0.843248	-0.843268	-0.408463	0.044228
	0.20	-0.802547	-0.802563	-0.339699	0.142693
$\theta_r \to \infty$	0.0	-0.903945	-0.903996	-0.622376	-0.340755
	0.05	-0.891684	-0.891727	-0.547414	-0.203102
	0.10	-0.864080	-0.864122	-0.473470	-0.082818
	0.15	-0.828711	-0.828764	-0.402517	0.023731
	0.20	-0.789060	-0.789121	-0.334752	0.119617

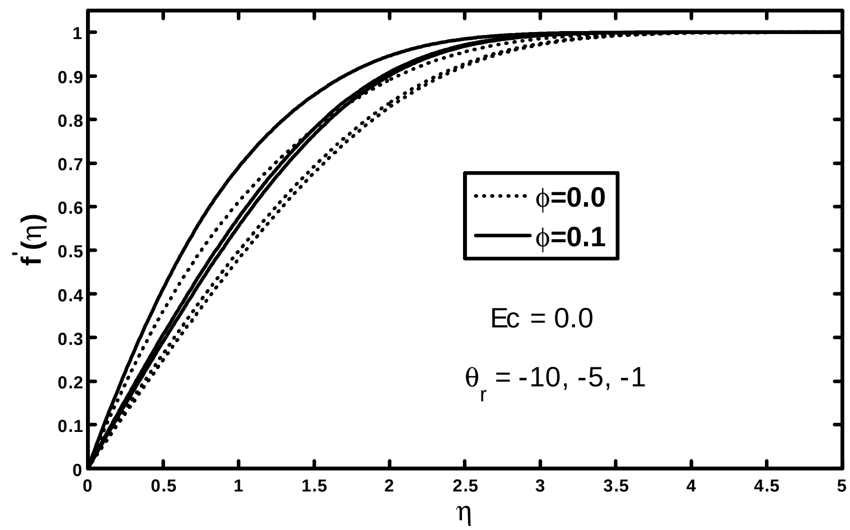
2.7 Graphical results and Discussion

Numerical as well as graphical results are obtained to study the impact of different physical properties of nanofluid on the flow and heat transfer phenomena. Most importantly the effect of viscosity parameter 'θ_r'on the flow and heat transfer is discussed. The functions $f(\eta)$ and $\theta(\eta)$ can be utilize to determine the skin friction coefficient $C_f = \frac{1}{(1-\phi)^{2.5}}(1 - \frac{1}{\theta_r})^{-1}(\mathrm{Re}_x)^{-\frac{1}{2}}f''(0)$, and the Nusselt number $Nu_x = -(\mathrm{Re}_x)^{\frac{1}{2}} \frac{K_{nf}}{K_{f\infty}} \theta'(0)$. Here $\mathrm{Re}_x = U_\infty x/\nu_{f\infty}$ is the local Reynolds number.

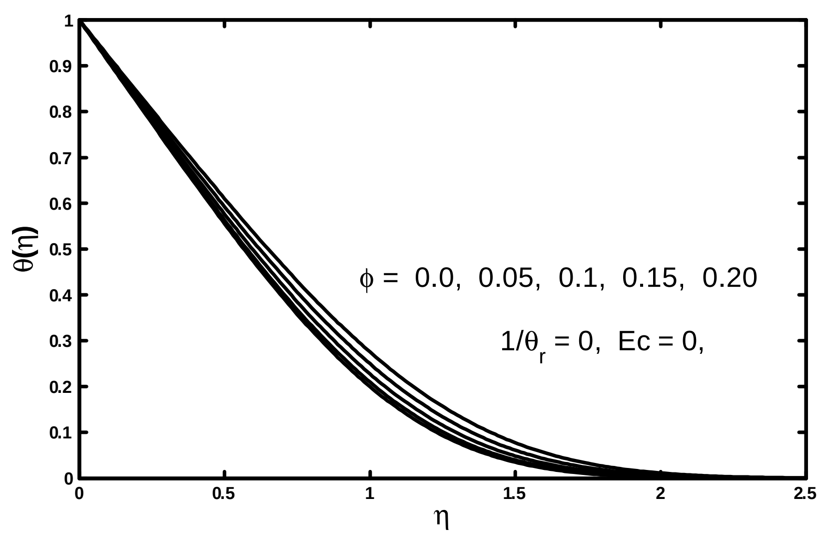
Numerical results of wall temperature gradient are mentioned in *Tables* (2.2) and (2.3) with Ag-water and Cu-water nanofluid respectively. The obtained results uncover some very intresting facts. Some of these observations are discussed meticulously with valid arguments. It is observed that the general behavier of Ag-water and Cu-water nanofluids are all same. The numerical results shows that the magnitude of wall temperature gradient decreases with increasing values of solid particle volume fraction parameter, hence Nusselt number decreases with increasing values of volume fraction parameter, this couses the thining of thermal boundary layer thickness. It is so because the decrease in Nusselt number depicts the decrease in the heat flux from the solid boundaries to the fluid across the boundary layer. Since the heat flux depends on the temperature difference, for decreasing heat flux the free stream temperature is nearer to achieve. This phenomena is even true for different fixed values of viscosity parameter. It is very clear that the magnitude of wall temperature gradient increases as $\theta_r \to 0$, hence the increase in the heat flux would bring the thickning effect in thermal boundary layer thickness. The magnitude of wall temperature gradient decreases with increase in the value of Eckert number. It is because heat flux decreases in the presence of viscous dissipation, since the presence of it rises the temperature of the fluid which results in the decrease of heat transfer and thus thining of thermal boundary layer thickness occure. *Figs.* (2.3) and (2.4) depicts the velocity profile. From the graphical results it is observed that the velocity is zero at the wall and tends to unity as the distence increases from the boundary. *Fig*(2.3) shows that the velocity profile rises with the increase in the solid particle volume fraction. It is so because with the increase in the volume fraction of solid particle the thickness of the momentum boundary layer decreases and free stream velocity is shortly achieved. This behavier is repeated for fixed values of viscosity parameter as well. *Fig*(2.4) shows that velocity profile increases as $\theta_r \to 0$. *Fig.* (2.5) to (2.8) presents the temperature profile, which starts from unity at the wall and tends to zero the distence increases from the boundary. From *Fig*(2.5) it is observed that the growth in volume fraction of solid particle results in the rise of temperature distribution, this is obvous since the increase of solid particle volume fraction increases the thermal conductivity of nanofluid. *Fig*(2.6) shows that the increase in the magnitude of viscosity parameter increases the temperature profile and hence causes thining of thermal boundary layer thickness. *Figs.* (2.7) and (2.8) are the temperature distribution representation in the presence of viscous dissipation with costant viscosity and variable viscosity respectively. It is observed that the increase in the value of Eckert number increases the temperature distribution, this is even true for negative value of viscosity parameter.



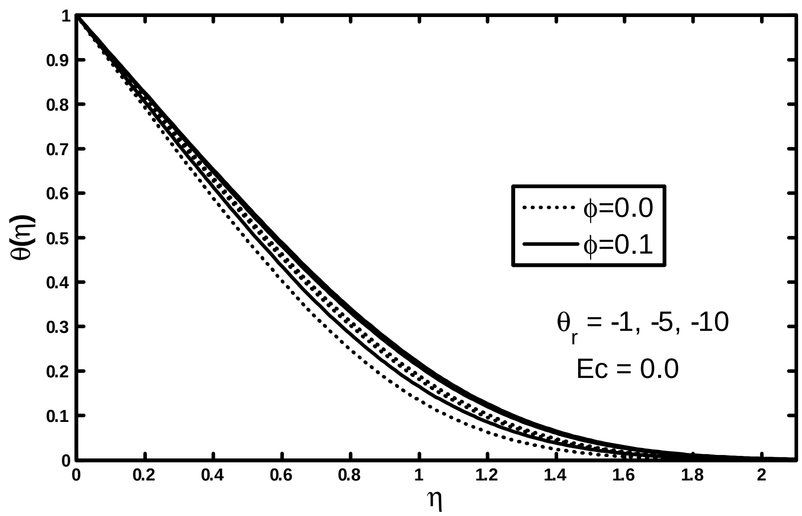
Fig(2.3). Variation of velocity with differnt values of ϕ



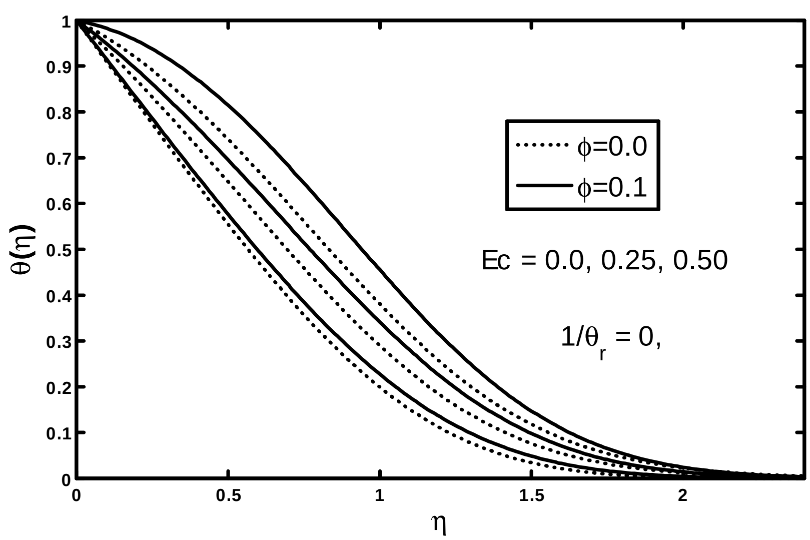
Fig(2.4). Variation of velocity with different values of viscosity parameter



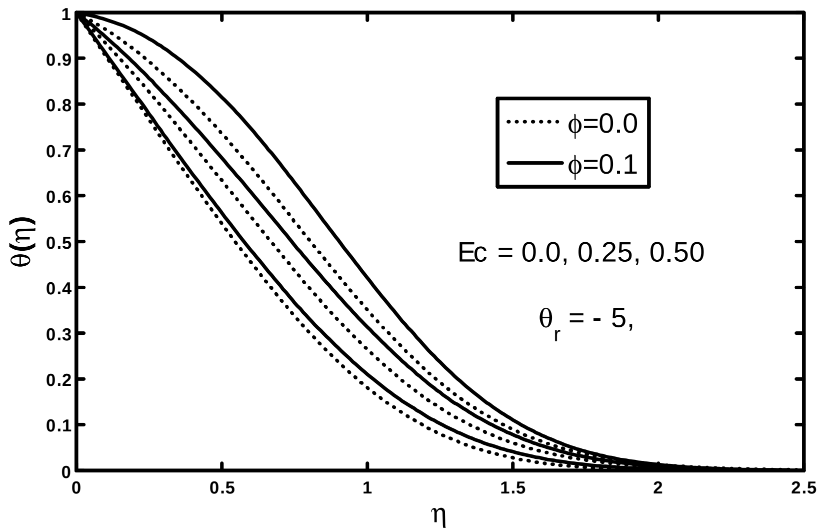
Fig(2.5). Variation of temperature with different values of ϕ



Fig(2.6). Variation of temperature with different values of viscosity parameter



Fig(2.7). Variation of temperature with different values of Ec



Fig(2.8). Variation of temperature with different values of Ec

Chapter 3

The study of Ag-water and Cu-water micropolar nanofluids with temperature dependant viscosities

3.1 Introduction

In this chapter micropolar fluid theory is incorporated on the steady flow of incompressible nanofluids. Two dimensional flow of Ag-water or Cu-water nanofluid is considered in the absence of dissipation effect. The fluid viscosities (dynamic viscosity, spin gradient viscosity, micro-inertia density) are taken as function of temperature. The governing partial differential equations are simplified with the help of suitable transformations. The reduced equations are then solved numerically by a second order finite difference scheme known as Keller box method. The flow is analyzed with different values of physical and thermophysical parameters such as solid-particle volume fraction, viscosity parameters θ_r, and micro-inertia viscosity parameter. The effect of these parameters on the fluid flow and on the heat transfer activity is determined.

3.2 Flow analysis

Consider a two dimensional, steady, boundary layer laminar flow of a micropolar fluid over a semi-infinite impermeable flat surface. The flow takes place in the positive xy-plane with the surface being at $y = 0$. The lower surface is fixed at constant temperature T_w where as the temperature of the fluid far away from the surface is taken to be T_∞. Fluid near the surface is at rest and attains free stream velocity U_∞ far away from the surface as shown in $Fig(2.1)$.

The assumed fluid is water based micropolar nanofluid with nano-solid particles of either copper(Cu) or silver(Ag). The working fluid is assumed to be incompressible, no-chemical reactions, negligible internal heat generation or absorption, and negligible radiative heat transfer. The viscosity of the nanofluid is considered variable and is taken as the inverse function of temperature. In the absence of viscous dissipation the usual boundary layer approximation leads the governing equations to the following form

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{3.1}$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \frac{1}{\rho_{nf}}\frac{\partial}{\partial y}\left(\mu_{nf} + \kappa\right)\frac{\partial u}{\partial y} + \frac{\kappa}{\rho_{nf}}\frac{\partial N}{\partial y} \tag{3.2}$$

$$u\frac{\partial N}{\partial x} + v\frac{\partial N}{\partial y} = \frac{1}{(\rho)_{nf}\, j}\frac{\partial}{\partial y}(\gamma^*\frac{\partial N}{\partial y}) - \frac{\kappa}{(\rho)_{nf}\, j}(2N + \frac{\partial u}{\partial y}) \tag{3.3}$$

$$u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} = \alpha_{nf}\frac{\partial^2 T}{\partial y^2} \tag{3.4}$$

Eqs. (3.1) to (3.4) represent the law of conservation of mass, momentum, angular momentum and energy respectively.

The appropriate boundary conditions are

$$u(x,y) = 0,\ v(x,y) = 0,\ T(x,y) = T_w,\ N = -N_0\tfrac{\partial u}{\partial y}\ at\ y = 0$$
$$u(x,y) \to U_\infty,\ T(x,y) \to T_\infty,\ N \to 0\ as\ y \to \infty$$

where u and v are the fluid velocity components in the stream direction and cross stream direction respectively, T is the temperature of the fluid, N is the micro-rotation or angular velocity of the fluid particle, κ is the vertex viscosity or micro-rotation viscosity. The effective density of the nanofluid is $(\rho)_{nf} = (1-\phi)\rho_{f\infty} + \phi\rho_s$, where ρ_s is the density of the solid particles, $\rho_{f\infty}$ is the density of the base fluid and ϕ denotes the volume fraction of the nano-solid particles in the fluid, μ_{nf} is the effective dynamic viscosity of the nanofluid which is given as $\mu_{nf} = \frac{\mu_f}{(1-\phi)^{2.5}}$, here μ_f is the coefficient of dynamic viscosity which vary as an inverse function of temperature i.e. $\frac{1}{\mu_f} = a(T - T_r)$, since $a = \frac{\delta}{\mu_{f\infty}}$, $T_r = T_\infty - \frac{1}{\delta}$, δ and a are constants, and $a > 0$ for liquids. $\gamma^*$is the spin gradient viscosity and is given as [25] $\gamma^* = \left(\mu_{nf} + \frac{\kappa}{2}\right) j$, where j is the micro inertia density. The thermal diffusibility of the nanofluid is $\alpha_{nf} = \frac{K_{nf}}{(\rho C_p)_{nf}}$. K_{nf} is the thermal conductivity of the nanofluid which is given as $K_{nf} = K_{f\infty}[\frac{K_s+2K_{f\infty}-2\phi(K_{f\infty}-K_s)}{K_s+2K_{f\infty}+\phi(K_{f\infty}-K_s)}]$. The effective heat capacitance of the of the nanofluid can be written as $(\rho C_p)_{nf} = (1-\phi)(\rho C_p)_{f\infty} + \phi(\rho C_p)_s$. The suffixes nf , f_∞ and s indicate the thermophysical properties of nanofluid, base fluid and solid particles respectively.

3.3 Similarity transformations

Eqs. (3.1) to (3.4) are the non-linear partial differential equations which are converted into a set of non-linear ordinary differential equations by choosing a specific forms of the velocity, angular velocity and temperature.

$$\eta = (\frac{U_\infty}{2\nu_\infty x})^{\frac{1}{2}} y \ , \ \psi = (2\nu_\infty U_\infty x)^{\frac{1}{2}} f(\eta) \ , \ \theta(\eta) = \frac{T-T_\infty}{T_w - T\infty}$$
$$N = U_\infty(\frac{U_\infty}{2\nu_\infty x})^{\frac{1}{2}} g(\eta) \quad , u = \frac{\partial\psi}{\partial y}, \ v = -\frac{\partial\psi}{\partial x}$$
$$j = x^2 \,\mathrm{Re}_x^{-1} f^{\,2}(\eta)$$

where η is the similarity variable, f , N and θ are the dimensionless velocity, micro-rotation and temperature respectively. The velocity components automatically satisfy the continuity equation (3.1). Using the above similarity transformations the governing equations (3.2) to (3.4) are transformed as

$$\frac{f'''}{\left(1-\frac{\theta}{\theta_r}\right)} + \frac{f''\theta'}{\theta_r\left(1-\frac{\theta}{\theta_r}\right)^2} + (1-\phi)^{2.5}[1-\phi+\phi(\frac{\rho_s}{\rho_{f\infty}})]ff''$$
$$= \quad -K(1-\phi)^{2.5}f''' + K(1-\phi)^{2.5}g'$$

$$\frac{f^{\ 2}g''}{(1-\frac{\theta}{\theta_r})} + (1-\phi)^{2.5}[\frac{K}{2}f^{\ 2}g'' - K(f\ '' + 2g) + Kff\ 'g' + [1-\phi+\phi(\frac{\rho_s}{\rho_{f\infty}})](f^{\ 2}f\ 'g + f^{\ 3}g')]$$
$$= \quad -\frac{2ff\ 'g'}{(1-\frac{\theta}{\theta_r})} + \frac{f^{\ 2}g'\theta'}{\theta_r(1-\frac{\theta}{\theta_r})^2}$$

$$\theta'' + \Pr\frac{K_{f\infty}}{K_{nf}}[1-\phi+\phi\frac{(\rho C_p)_s}{(\rho C_p)_{f\infty}}]f\theta' = 0$$

The boundary conditions in transformed form reduce to

$$f(\eta) = 0,\ \theta(\eta) = 1,\ g(\eta) = -N_0 f\ ''(\eta),\ f\ '(\eta) = 0 \ \ at\ \eta = 0$$
$$f\ '(\eta) \to 1,\ \theta(\eta) \to 0,\ g(\eta) \to 0 \ \ as\ \eta \to \infty$$

where $K = \frac{\kappa}{\mu_{f\infty}}$, is the micropolar parameter, $\theta_r = -\frac{1}{\delta(T_w - T\infty)}$, represents the fluid viscosity parameter and $\Pr = \frac{\nu_{f\infty}}{\alpha_{f\infty}}$,show the Prandtle number of the base fluid.

3.4 Solution procedure

3.4.1 System of first order ODEs

According to the procedure discussed in chapter two, we once again write the following terms

$$f' = u \tag{3.5}$$

$$u' = v \tag{3.6}$$

$$\theta' = w \tag{3.7}$$

$$g' = m \tag{3.8}$$

$$(1+K\epsilon_3)v' - \frac{1}{\theta_r}(1+2K\epsilon_3)v'\theta + \frac{K\epsilon_3}{\theta_r^2}v'\theta^2 + \frac{1}{\theta_r}vw + \epsilon_1 fv + \frac{\epsilon_1}{\theta_r^2}fv\theta^2 - \frac{2\epsilon_1}{\theta_r}fv\theta + K\epsilon_3 m + \frac{K\epsilon_3}{\theta_r^2}m\theta^2 - \frac{2K\epsilon_3}{\theta_r}m\theta = 0 \tag{3.9}$$

$$w' - \frac{1}{\theta_r} w'\theta + \Pr \frac{K_{f\infty}}{K_{nf}} \in_2 fw - \frac{1}{\theta_r} \Pr \frac{K_{f\infty}}{K_{nf}} \in_2 fw\theta = 0 \tag{3.10}$$

$$(1 + \frac{K}{2}\epsilon_3) f^2 m' - \frac{1}{\theta_r}(1 + K\epsilon_3) f^2 m'\theta + \frac{K\epsilon_3}{2\theta_r^2} f^2 m'\theta^2 - K\epsilon_3 v$$
$$- \frac{1}{\theta_r^2} K\epsilon_3 v\theta^2 + \frac{2}{\theta_r} K\epsilon_3 v\theta - 2K\epsilon_3 g - \frac{2}{\theta_r^2} K\epsilon_3 \theta^2 g$$
$$+ \frac{4K}{\theta_r}\epsilon_3 \theta g + K\epsilon_3 fum + \frac{K}{\theta_r^2}\epsilon_3 fum\theta^2 - \frac{2K}{\theta_r}\epsilon_3 fum\theta$$

$$+\epsilon_1 f^2 ug + \frac{\epsilon_1}{\theta_r^2} f^2 ug\theta^2 - \frac{2\epsilon_1}{\theta_r} f^2 ug\theta + \epsilon_1 f^3 m + \frac{\epsilon_1}{\theta_r^2} f^3 m\theta^2 = 0$$

we substitute $\epsilon_1 = (1-\phi)^{2.5}[1 - \phi + \phi(\frac{\rho_s}{\rho_{f\infty}})]$, $\epsilon_2 = [1 - \phi + \phi\frac{(\rho C_p)_s}{(\rho C_p)_{f\infty}}]$, $\epsilon_3 = (1-\phi)^{2.5}$, to make the equations simple.

3.4.2 Discretization

Consider the net rectangle in the (x-η) plane shown in the Fig(2.2). The net points are defined as

$x^0 = 0, \quad x^i = x^{i-1} + k_i \;, \qquad i = 1, 2...I$

$\eta_0 = 0, \quad \eta_j = \eta_{j-1} + h_j, \qquad j = 1, 2...J$

where 'k_i' is the Δx-spacing and 'h_j' is the $\Delta\eta$-spacing. Here i and j are just sequence of numbers that indicate the coordinate location. The finite difference form for any point is manipulated as

$$(\;)^i_{j-\frac{1}{2}} = \frac{1}{2}[(\;)^i_{j-1} + (\;)^i_j]$$

$$(\;)^{i-\frac{1}{2}}_j = \frac{1}{2}[(\;)^{i-1}_j + (\;)^i_j]$$

$$(\frac{\partial u}{\partial x})^{i-\frac{1}{2}}_{j-\frac{1}{2}} = \frac{u^i_{j-\frac{1}{2}} - u^{i-1}_{j-\frac{1}{2}}}{k_i}$$

$$(\frac{\partial u}{\partial \eta})^{i-\frac{1}{2}}_{j-\frac{1}{2}} = \frac{u_j^{i-\frac{1}{2}} - u_{j-1}^{i-\frac{1}{2}}}{h_j}$$

We now apply the finite difference scheme on the system of first order ordinary differential equations by centring at $(x^i, \eta_{j-\frac{1}{2}})$ on the first three equations. The resulting discritized equations are

$$f_j^i = f_{j-1}^i + \frac{h_j}{2}(u_{j-1}^i + u_j^i) \tag{3.11}$$

$$u_j^i = u_{j-1}^i + \frac{h_j}{2}(v_{j-1}^i + v_j^i) \tag{3.12}$$

$$\theta_j^i = \theta_{j-1}^i + \frac{h_j}{2}(w_{j-1}^i + w_j^i) \tag{3.13}$$

$$g_j^i = g_{j-1}^i + \frac{h_j}{2}(m_{j-1}^i + m_j^i) \tag{3.14}$$

$$\begin{aligned} & (1+K\epsilon_3)(v_j^i - v_{j-1}^i) - \frac{1}{\theta_r}(1+2K\epsilon_3)(v_j^i - v_{j-1}^i)\theta_{j-\frac{1}{2}}^i + K\frac{\epsilon_3}{\theta_r^2}(v_j^i - v_{j-1}^i)(\theta_{j-\frac{1}{2}}^i)^2 \\ & + \frac{h_j}{\theta_r} v_{j-\frac{1}{2}}^i w_{j-\frac{1}{2}}^i + \epsilon_1 h_j f_{j-\frac{1}{2}}^i v_{j-\frac{1}{2}}^i + \frac{\epsilon_1 h_j}{\theta_r^2} f_{j-\frac{1}{2}}^i v_{j-\frac{1}{2}}^i (\theta_{j-\frac{1}{2}}^i)^2 - \frac{2\epsilon_1}{\theta_r} h_j f_{j-\frac{1}{2}}^i v_{j-\frac{1}{2}}^i \theta_{j-\frac{1}{2}}^i \\ & + K h_j \epsilon_3 m_{j-\frac{1}{2}}^i + K\frac{h_j}{\theta_r^2}\epsilon_3 m_{j-\frac{1}{2}}^i (\theta_{j-\frac{1}{2}}^i)^2 - \frac{2K\epsilon_3}{\theta_r} h_j m_{j-\frac{1}{2}}^i \theta_{j-\frac{1}{2}}^i \\ = \; & M_{j-\frac{1}{2}} \end{aligned} \tag{3.15}$$

$$(w_j^i - w_{j-1}^i) - \frac{1}{\theta_r}(w_j^i - w_{j-1}^i)\theta_{j-\frac{1}{2}}^i + \Pr\frac{K_{f\infty}}{K_{nf}}\epsilon_2 h_j f_{j-\frac{1}{2}}^i w_{j-\frac{1}{2}}^i$$

$$-\frac{h_j}{\theta_r}\Pr\frac{K_{f\infty}}{K_{nf}}\epsilon_2 f_{j-\frac{1}{2}}^i w_{j-\frac{1}{2}}^i \theta_{j-\frac{1}{2}}^i = N_{j-\frac{1}{2}} \tag{3.16}$$

$$
\begin{aligned}
&(1+\frac{K}{2}\epsilon_3)(f^i_{j-\frac{1}{2}})^2(m^i_j-m^i_{j-1})-\frac{1}{\theta_r}(1+K\epsilon_3)(f^i_{j-\frac{1}{2}})^2\theta^i_{j-\frac{1}{2}}(m^i_j-m^i_{j-1})\\
&+\frac{K\epsilon_3}{2\theta_r^2}(f^i_{j-\frac{1}{2}})^2(\theta^i_{j-\frac{1}{2}})^2(m^i_j-m^i_{j-1})-h_jK\epsilon_3v^i_{j-\frac{1}{2}}-\frac{h_j}{\theta_r^2}K\epsilon_3v^i_{j-\frac{1}{2}}(\theta^i_{j-\frac{1}{2}})^2\\
&+\frac{2K}{\theta_r}h_j\epsilon_3v^i_{j-\frac{1}{2}}\theta^i_{j-\frac{1}{2}}-2h_jK\epsilon_3g^i_{j-\frac{1}{2}}-\frac{2h_jK}{\theta_r^2}\epsilon_3(\theta^i_{j-\frac{1}{2}})^2g^i_{j-\frac{1}{2}}\\
&+\frac{4h_jK\epsilon_3}{\theta_r}\theta^i_{j-\frac{1}{2}}g^i_{j-\frac{1}{2}}+h_jK\epsilon_3f^i_{j-\frac{1}{2}}u^i_{j-\frac{1}{2}}m^i_{j-\frac{1}{2}}+\frac{h_jK}{\theta_r^2}\epsilon_3f^i_{j-\frac{1}{2}}u^i_{j-\frac{1}{2}}m^i_{j-\frac{1}{2}}(\theta^i_{j-\frac{1}{2}})^2\\
&-\frac{2Kh_j}{\theta_r}\epsilon_3f^i_{j-\frac{1}{2}}u^i_{j-\frac{1}{2}}m^i_{j-\frac{1}{2}}\theta^i_{j-\frac{1}{2}}+h_j\epsilon_1(f^i_{j-\frac{1}{2}})^2u^i_{j-\frac{1}{2}}g^i_{j-\frac{1}{2}}+\frac{h_j\epsilon_1}{\theta_r^2}(f^i_{j-\frac{1}{2}})^2u^i_{j-\frac{1}{2}}g^i_{j-\frac{1}{2}}(\theta^i_{j-\frac{1}{2}})^2\\
&-\frac{2h_j\epsilon_1}{\theta_r}(f^i_{j-\frac{1}{2}})^2u^i_{j-\frac{1}{2}}g^i_{j-\frac{1}{2}}\theta^i_{j-\frac{1}{2}}+h_j\epsilon_1(f^i_{j-\frac{1}{2}})^3m^i_{j-\frac{1}{2}}+\frac{h_j\epsilon_1}{\theta_r^2}(f^i_{j-\frac{1}{2}})^3m^i_{j-\frac{1}{2}}(\theta^i_{j-\frac{1}{2}})^2\\
&-2\frac{h_j}{\theta_r}\epsilon_1(f^i_{j-\frac{1}{2}})^3m^i_{j-\frac{1}{2}}\theta^i_{j-\frac{1}{2}}+2h_jf^i_{j-\frac{1}{2}}u^i_{j-\frac{1}{2}}m^i_{j-\frac{1}{2}}-2\frac{h_j}{\theta_r}f^i_{j-\frac{1}{2}}u^i_{j-\frac{1}{2}}m^i_{j-\frac{1}{2}}\theta^i_{j-\frac{1}{2}}\\
&\frac{h_j}{\theta_r}(f^i_{j-\frac{1}{2}})^2m^i_{j-\frac{1}{2}}w^i_{j-\frac{1}{2}}=P_{j-1} \qquad (3.17)
\end{aligned}
$$

where

$$
\begin{aligned}
M_{j-\frac{1}{2}} \;=\;& -h_j(1+K\epsilon_3)(v')^{i-1}_{j-\frac{1}{2}}+\frac{h_j}{\theta_r}(1+2K\epsilon_3)(v'\theta)^{i-1}_{j-\frac{1}{2}}-\frac{K\epsilon_3}{\theta_r^2}h_j(v'\theta^2)^{i-1}_{j-\frac{1}{2}}\\
&-\frac{h_j}{\theta_r}(vw)^{i-1}_{j-\frac{1}{2}}-\epsilon_1h_j(fv)^{i-1}_{j-\frac{1}{2}}-\frac{\epsilon_1h_j}{\theta_r^2}(fv\theta^2)^{i-1}_{j-\frac{1}{2}}+2\frac{\epsilon_1h_j}{\theta_r}(fv\theta)^{i-1}_{j-\frac{1}{2}}\\
&+K\epsilon_3h_jm^{i-1}_{j-\frac{1}{2}}+\frac{Kh_j}{\theta_r^2}\epsilon_3(m\theta^2)^{i-1}_{j-\frac{1}{2}}-2\frac{Kh_j}{\theta_r}\epsilon_3(m\theta)^{i-1}_{j-\frac{1}{2}}
\end{aligned}
$$

$$
N_{j-\frac{1}{2}}=-h_j(w')^{i-1}_{j-\frac{1}{2}}+\frac{h_j}{\theta_r}(w'\theta)^{i-1}_{j-\frac{1}{2}}-\Pr h_j\frac{K_{f\infty}}{K_{nf}}\epsilon_2(fw)^{i-1}_{j-\frac{1}{2}}+\frac{h_j}{\theta_r}\Pr\frac{K_{f\infty}}{K_{nf}}\epsilon_2(fw\theta)^{i-1}_{j-\frac{1}{2}}
$$

$$
\begin{aligned}
P_{j-\frac{1}{2}} \;=\;& -h_j(1+\frac{K}{2}\epsilon_3)(f^2m')^{i-1}_{j-\frac{1}{2}} + \frac{h_j}{\theta_r}(1+K\epsilon_3)(f^2\theta m')^{i-1}_{j-\frac{1}{2}} - \frac{K\epsilon_3}{2\theta_r^2}h_j(f^2\theta^2m')^{i-1}_{j-\frac{1}{2}} \\
& +h_jK\epsilon_3 v^{i-1}_{j-\frac{1}{2}} + \frac{h_j}{\theta_r^2}K\epsilon_3(v\theta^2)^{i-1}_{j-\frac{1}{2}} - \frac{2h_j}{\theta_r}K\epsilon_3(v\theta)^{i-1}_{j-\frac{1}{2}} + 2h_jK\epsilon_3 g^{i-1}_{j-\frac{1}{2}} \\
& +2\frac{h_jK}{\theta_r^2}\epsilon_3(\theta^2g)^{i-1}_{j-\frac{1}{2}} - 4\frac{h_jK}{\theta_r}\epsilon_3(\theta g)^{i-1}_{j-\frac{1}{2}} - h_jK\epsilon_3(fum)^{i-1}_{j-\frac{1}{2}} - \frac{h_jK}{\theta_r^2}\epsilon_3(fum\theta^2)^{i-1}_{j-\frac{1}{2}} \\
& +2\frac{h_jK}{\theta_r}\epsilon_3(fum\theta)^{i-1}_{j-\frac{1}{2}} - \epsilon_1h_j(f^2ug)^{i-1}_{j-\frac{1}{2}} - \frac{h_j\epsilon_1}{\theta_r^2}(f^2ug\theta^2)^{i-1}_{j-\frac{1}{2}} + 2\frac{h_j}{\theta_r}\epsilon_1(f^2ug\theta)^{i-1}_{j-\frac{1}{2}} \\
& -h_j\epsilon_1(f^3m)^{i-1}_{j-\frac{1}{2}} - \frac{h_j}{\theta_r^2}\epsilon_1(f^3m\theta^2)^{i-1}_{j-\frac{1}{2}} + 2\frac{h_j}{\theta_r}\epsilon_1(f^3m\theta)^{i-1}_{j-\frac{1}{2}} - 2h_j(fum)^{i-1}_{j-\frac{1}{2}} \\
& +2\frac{h_j}{\theta_r}(fum\theta)^{i-1}_{j-\frac{1}{2}} - \frac{h_j}{\theta_r}(f^2mw)^{i-1}_{j-\frac{1}{2}} \qquad (3.18)
\end{aligned}
$$

If we assume f_j^{i-1}, u_j^{i-1}, v_j^{i-1}, θ_j^{i-1}, w_j^{i-1}, $g_j^{i-1} and\ m_j^{i-1}$ to be known for $0 \le j \le J$ then $M_{j-\frac{1}{2}}$, $N_{j-\frac{1}{2}}$ and $P_{j-\frac{1}{2}}$contains only known quantities and we have to obtain the solution of unknowns ($f_j^i, u_j^i, v_j^i, \theta_j^i,\ w_j^i, g_j^i, m_j^i$) for $0 \le j \le J$. We write the unknowns at $ith-level$ as $(f_j, u_j, v_j, \theta_j, w_j, g_j, m_j)$ i.e ignore the superscripts for the sake of simplicity.

$$f_j = f_{j-1} + \frac{h_j}{2}(u_{j-1} + u_j) \qquad (3.19)$$

$$u_j = u_{j-1} + \frac{h_j}{2}(v_{j-1} + v_j) \qquad (3.20)$$

$$\theta_j = \theta_{j-1} + \frac{h_j}{2}(w_{j-1} + w_j) \qquad (3.21)$$

$$g_j = g_{j-1} + \frac{h_j}{2}(m_{j-1} + m_j) \qquad (3.22)$$

$$
\begin{aligned}
& (1+K\epsilon_3)(v_j - v_{j-1}) - \frac{1}{\theta_r}(1+2K\epsilon_3)(v_j - v_{j-1})\theta_{j-\frac{1}{2}} + K\frac{\epsilon_3}{\theta_r^2}(v_j - v_{j-1})(\theta_{j-\frac{1}{2}})^2 \\
& + \frac{h_j}{\theta_r} v_{j-\frac{1}{2}} w_{j-\frac{1}{2}} + \epsilon_1 h_j f_{j-\frac{1}{2}} v_{j-\frac{1}{2}} + \frac{\epsilon_1 h_j}{\theta_r^2} f_{j-\frac{1}{2}} v_{j-\frac{1}{2}} (\theta_{j-\frac{1}{2}})^2 - \frac{2\epsilon_1}{\theta_r} h_j f_{j-\frac{1}{2}} v_{j-\frac{1}{2}} \theta_{j-\frac{1}{2}} \\
& + K h_j \epsilon_3 m_{j-\frac{1}{2}} + K\frac{h_j}{\theta_r^2}\epsilon_3 m_{j-\frac{1}{2}}(\theta_{j-\frac{1}{2}})^2 - \frac{2K\epsilon_3}{\theta_r} h_j m_{j-\frac{1}{2}} \theta_{j-\frac{1}{2}} \\
= \; & M_{j-\frac{1}{2}} \qquad (3.23)
\end{aligned}
$$

$$
(w_j - w_{j-1}) - \frac{1}{\theta_r}(w_j - w_{j-1})\theta_{j-\frac{1}{2}} + \Pr\frac{K_{f\infty}}{K_{nf}}\epsilon_2 h_j f_{j-\frac{1}{2}} w_{j-\frac{1}{2}} - \frac{h_j}{\theta_r}\Pr\frac{K_{f\infty}}{K_{nf}}\epsilon_2 f_{j-\frac{1}{2}} w_{j-\frac{1}{2}} \theta_{j-\frac{1}{2}} = N_{j-\frac{1}{2}} \qquad (3.24)
$$

$$
\begin{aligned}
& (1+\frac{K}{2}\epsilon_3)(f_{j-\frac{1}{2}})^2(m_j - m_{j-1}) - \frac{1}{\theta_r}(1+K\epsilon_3)(f_{j-\frac{1}{2}})^2\theta_{j-\frac{1}{2}}(m_j - m_{j-1}) \\
& + \frac{K\epsilon_3}{2\theta_r^2}(f_{j-\frac{1}{2}})^2(\theta_{j-\frac{1}{2}})^2(m_j - m_{j-1}) - h_j K\epsilon_3 v_{j-\frac{1}{2}} - \frac{h_j}{\theta_r^2}K\epsilon_3 v_{j-\frac{1}{2}}(\theta_{j-\frac{1}{2}})^2 \\
& + \frac{2K}{\theta_r} h_j \epsilon_3 v_{j-\frac{1}{2}}\theta_{j-\frac{1}{2}} - 2h_j K\epsilon_3 g_{j-\frac{1}{2}} - \frac{2h_j K}{\theta_r^2}\epsilon_3(\theta_{j-\frac{1}{2}})^2 g_{j-\frac{1}{2}} + \frac{4h_j K\epsilon_3}{\theta_r}\theta_{j-\frac{1}{2}} g_{j-\frac{1}{2}} \\
& + h_j K\epsilon_3 f_{j-\frac{1}{2}} u_{j-\frac{1}{2}} m_{j-\frac{1}{2}} + \frac{h_j K}{\theta_r^2}\epsilon_3 f_{j-\frac{1}{2}} u_{j-\frac{1}{2}} m_{j-\frac{1}{2}}(\theta_{j-\frac{1}{2}})^2 - \frac{2Kh_j}{\theta_r}\epsilon_3 f_{j-\frac{1}{2}} u_{j-\frac{1}{2}} m_{j-\frac{1}{2}}\theta_{j-\frac{1}{2}} \\
& + h_j\epsilon_1(f_{j-\frac{1}{2}})^2 u_{j-\frac{1}{2}} g_{j-\frac{1}{2}} + \frac{h_j\epsilon_1}{\theta_r^2}(f_{j-\frac{1}{2}})^2 ug(\theta)^2 - \frac{2h_j\epsilon_1}{\theta_r}(f_{j-\frac{1}{2}})^2 u_{j-\frac{1}{2}} g_{j-\frac{1}{2}}\theta_{j-\frac{1}{2}} \\
& + h_j\epsilon_1(f_{j-\frac{1}{2}})^3 m_{j-\frac{1}{2}} + \frac{h_j\epsilon_1}{\theta_r^2}(f_{j-\frac{1}{2}})^3 m_{j-\frac{1}{2}}(\theta_{j-\frac{1}{2}})^2 - 2\frac{h_j}{\theta_r}\epsilon_1(f_{j-\frac{1}{2}})^3 m_{j-\frac{1}{2}}\theta_{j-\frac{1}{2}} \\
& + 2h_j f_{j-\frac{1}{2}} u_{j-\frac{1}{2}} m_{j-\frac{1}{2}} - 2\frac{h_j}{\theta_r} f_{j-\frac{1}{2}} u_{j-\frac{1}{2}} m_{j-\frac{1}{2}}\theta_{j-\frac{1}{2}} + \frac{h_j}{\theta_r}(f_{j-\frac{1}{2}})^2 m_{j-\frac{1}{2}} w_{j-\frac{1}{2}} \\
= \; & P_{j-\frac{1}{2}} \qquad (3.25)
\end{aligned}
$$

The boundary conditions become

$$
f_0 = 0, \;\; u_0 = 0, \; u_J = 1, \; \theta_0 = 1, \; \theta_J = 0, \; g_0 = -N_0 v_0, \; g_J = 0.
$$

3.4.3 Newton linearization technique

The above finite difference equations are linearized by using Newton's linearization technique.

$$f^{(k+1)} = f^{(k)} + \delta f^{(k)}, \quad u^{(k+1)} = u^{(k)} + \delta u^{(k)}$$

$$v^{(k+1)} = v^{(k)} + \delta v^{(k)}, \quad \theta^{(k+1)} = \theta^{(k)} + \delta\theta^{(k)}$$

$$w^{(k+1)} = w^{(k)} + \delta w^{(k)}, \quad g^{(k+1)} = g^{(k)} + \delta g^{(k)}$$

$$m^{(k+1)} = m^{(k)} + \delta m^{(k)}$$

We substitute the above expressions in the equations and ignoring the quadratic terms in δ and higher orders. The resulting system of equations are given as under

$$\delta f_j - \delta f_{j-1} - \frac{h_j}{2}(\delta u_j + \delta u_{j-1}) = (r_1)_j \tag{3.26}$$

$$\delta u_j - \delta u_{j-1} - \frac{h_j}{2}(\delta v_j + \delta v_{j-1}) = (r_2)_j \tag{3.27}$$

$$\delta\theta_j - \delta\theta_{j-1} - \frac{h_j}{2}(\delta w_j + \delta w_{j-1}) = (r_3)_j \tag{3.28}$$

$$\delta m_j - \delta m_{j-1} - \frac{h_j}{2}(\delta m_j + \delta m_{j-1}) = (r_4)_j \tag{3.29}$$

$$\begin{aligned} & (a_1)_j\delta v_j + (a_2)_j\delta v_{j-1} + (a_3)_j\delta f_j + (a_4)_j\delta f_{j-1} + (a_5)_j\delta u_j + (a_6)_j\delta u_{j-1} + (a_7)_j\delta\theta_j \\ & +(a_8)_j\delta\theta_{j-1} + (a_9)_j\delta w_j + (a_{10})\delta w_{j-1} + (a_{11})_j\delta g_j + (a_{12})_j\delta g_{j-1} + (a_{13})_j\delta m_j + (a_{14})\delta m_{j-1} \\ = \; & (r_5)_j \end{aligned}$$

$$
\begin{aligned}
&(b_1)_j\delta v_j + (b_2)_j\delta v_{j-1} + (b_3)_j\delta f_j + (b_4)_j\delta f_{j-1} + (b_5)_j\delta u_j + (b_6)_j\delta u_{j-1} + (b_7)_j\delta\theta_j + \\
&(b_8)_j\delta\theta_{j-1} + (b_9)_j\delta w_j + (b_{10})\delta w_{j-1} + (b_{11})_j\delta g_j + (b_{12})_j\delta g_{j-1} + (b_{13})_j\delta m_j + (b_{14})\delta m_{j-1} \\
= \quad &(r_6)_j
\end{aligned}
$$

$$
\begin{aligned}
&(c_1)_j\delta v_j + (c_2)_j\delta v_{j-1} + (c_3)_j\delta f_j + (c_4)_j\delta f_{j-1} + (c_5)_j\delta u_j + (c_6)_j\delta u_{j-1} + (c_7)_j\delta\theta_j + \\
&(c_8)_j\delta\theta_{j-1} + (c_9)_j\delta w_j + (c_{10})\delta w_{j-1} + (c_{11})_j\delta g_j + (c_{12})_j\delta g_{j-1} + (c_{13})_j\delta m_j + (c_{14})\delta m_{j-1} \\
= \quad &(r_7)_j
\end{aligned}
$$

$$
\begin{aligned}
(a_1)_j \quad = \quad & (1+K\epsilon_3) - \frac{1}{\theta_r}(1+2K\epsilon_3)\theta_{j-\frac{1}{2}} + \frac{K\epsilon_3}{\theta_r^2}\theta^2_{j-\frac{1}{2}} + \frac{h_j}{2\theta_r}w_{j-\frac{1}{2}} \\
& + \frac{\epsilon_1 h_j}{2}f_{j-\frac{1}{2}} + \frac{\epsilon_1 h_j}{2\theta_r^2}f_{j-\frac{1}{2}}\theta^2_{j-\frac{1}{2}} - \frac{\epsilon_1 h_j}{\theta_r}f_{j-\frac{1}{2}}\theta_{j-\frac{1}{2}}
\end{aligned}
$$

$$
\begin{aligned}
(a_2)_j \quad = \quad & -(1+K\epsilon_3) + \frac{1}{\theta_r}(1+2K\epsilon_3)\theta_{j-\frac{1}{2}} - \frac{K\epsilon_3}{\theta_r^2}\theta^2_{j-\frac{1}{2}} + \frac{h_j}{2\theta_r}w_{j-\frac{1}{2}} \\
& + \frac{\epsilon_1 h_j}{2}f_{j-\frac{1}{2}} + \frac{\epsilon_1 h_j}{2\theta_r^2}f_{j-\frac{1}{2}}\theta^2_{j-\frac{1}{2}} - \frac{\epsilon_1 h_j}{\theta_r}f_{j-\frac{1}{2}}\theta_{j-\frac{1}{2}}
\end{aligned}
$$

$$
(a_3)_j = \frac{\epsilon_1 h_j}{2}v_{j-\frac{1}{2}} + \frac{\epsilon_1 h_j}{2\theta_r^2}v_{j-\frac{1}{2}}\theta^2_{j-\frac{1}{2}} - \frac{\epsilon_1 h_j}{\theta_r}v_{j-\frac{1}{2}}\theta_{j-\frac{1}{2}}
$$

$$
(a_4)_j = (a_3)_j
$$

$$
(a_5)_j = (a_6)_j = 0
$$

$$
\begin{aligned}
(a_7)_j &= -\frac{(1+2K\epsilon_3)}{2\theta_r}(v_j - v_{j-1}) + \frac{K\epsilon_3}{\theta_r^2}\theta_{j-\frac{1}{2}}(v_j - v_{j-1}) + \frac{\epsilon_1 h_j}{\theta_r^2} f_{j-\frac{1}{2}} v_{j-\frac{1}{2}} \theta_{j-\frac{1}{2}} \\
&\quad -\frac{\epsilon_1 h_j}{\theta_r} f_{j-\frac{1}{2}} v_{j-\frac{1}{2}} + \frac{K h_j}{\theta_r^2}\epsilon_3 m_{j-\frac{1}{2}} \theta_{j-\frac{1}{2}} - \frac{K h_j}{\theta_r}\epsilon_3 m_{j-\frac{1}{2}}
\end{aligned}
$$

$$(a_8)_j = (a_7)_j$$

$$(a_9)_j = \frac{h_j}{2\theta_r} v_{j-\frac{1}{2}}$$

$$(a_{10})_j = (a_9)_j$$

$$(a_{11})_j = (a_{12})_j = 0$$

$$(a_{13})_j = \frac{K h_j}{2}\epsilon_3 + \frac{K h_j}{2\theta_r^2}\epsilon_3 \theta_{j-\frac{1}{2}}^2 - \frac{K h_j}{\theta_r}\epsilon_3 \theta_{j-\frac{1}{2}}$$

$$(a_{14})_j = (a_{13})_j$$

$$(b_1)_j = (b_2)_j = 0$$

$$(b_3)_j = \frac{\Pr}{2} - \frac{h_j}{\theta_r}\epsilon_2 h_j w_{j-\frac{1}{2}} - \frac{h_j}{2\theta_r}\Pr\frac{K_{f\infty}}{K_{nf}}\epsilon_2 w_{j-\frac{1}{2}} \theta_{j-\frac{1}{2}}$$

$$(b_4)_j = (b_3)_j$$

$$(b_5)_j = (b_6)_j = 0$$

$$(b_7)_j = -\frac{h_j}{2\theta_r} \Pr \frac{K_{f\infty}}{K_{nf}} \epsilon_2 f_{j-\frac{1}{2}} w_{j-\frac{1}{2}}$$

$$(b_8)_j = (b_7)_j$$

$$(b_9)_j = 1 - \frac{1}{\theta_r}\theta_{j-\frac{1}{2}} + \Pr \frac{K_{f\infty}}{K_{nf}} \frac{\epsilon_2}{2} h_j f_{j-\frac{1}{2}} - \Pr \frac{K_{f\infty}}{K_{nf}} \frac{\epsilon_2}{2\theta_r} h_j f_{j-\frac{1}{2}} \theta_{j-\frac{1}{2}}$$

$$(b_{10})_j = -1 + \frac{1}{\theta_r}\theta_{j-\frac{1}{2}} + \Pr \frac{K_{f\infty}}{K_{nf}} \frac{\epsilon_2}{2} h_j f_{j-\frac{1}{2}} - \Pr \frac{K_{f\infty}}{K_{nf}} \frac{\epsilon_2}{2\theta_r} h_j f_{j-\frac{1}{2}} \theta_{j-\frac{1}{2}}$$

$$(b_{11})_j = (b_{12})_j = (b_{13})_j = (b_{14})_j = 0$$

$$(c_1)_j = -\frac{h_j K}{2}\epsilon_3 - \frac{h_j K}{2\theta_r^2}\epsilon_3 \theta^2_{j-\frac{1}{2}} + \frac{h_j K}{\theta_r}\epsilon_3 \theta_{j-\frac{1}{2}}$$

$$(c_2)_j = (c_1)_j$$

$$\begin{aligned}
(c_3)_j \quad = \quad & \frac{(1+K\epsilon_3)}{2}(m_j - m_{j-1}) f_{j-\frac{1}{2}} - \frac{(1+K\epsilon_3)}{\theta_r}(m_j - m_{j-1}) f_{j-\frac{1}{2}} \theta_{j-\frac{1}{2}} \\
& + \frac{K\epsilon_3}{2\theta_r^2}(m_j - m_{j-1}) f_{j-\frac{1}{2}} \theta^2_{j-\frac{1}{2}} + \frac{(2+K\epsilon_3)}{2} h_j u_{j-\frac{1}{2}} m_{j-\frac{1}{2}} + \frac{h_j K \epsilon_3}{2\theta_r^2} u_{j-\frac{1}{2}} m_{j-\frac{1}{2}} \theta^2_{j-\frac{1}{2}} \\
& - \frac{(1+K\epsilon_3)}{\theta_r} h_j u_{j-\frac{1}{2}} m_{j-\frac{1}{2}} \theta_{j-\frac{1}{2}} + \epsilon_1 h_j f_{j-\frac{1}{2}} u_{j-\frac{1}{2}} g_{j-\frac{1}{2}} + \frac{\epsilon_1 h_j}{\theta_r^2} f_{j-\frac{1}{2}} u_{j-\frac{1}{2}} g_{j-\frac{1}{2}} \theta^2_{j-\frac{1}{2}} \\
& - \frac{2h_j\epsilon_1}{\theta_r} f_{j-\frac{1}{2}} u_{j-\frac{1}{2}} g_{j-\frac{1}{2}} \theta_{j-\frac{1}{2}} + \frac{3\epsilon_1 h_j}{2} m_{j-\frac{1}{2}} f^2_{j-\frac{1}{2}} + \frac{3\epsilon_1}{2\theta_r^2} h_j m_{j-\frac{1}{2}} f^2_{j-\frac{1}{2}} \theta^2_{j-\frac{1}{2}} \\
& - \frac{3\epsilon_1 h_j}{\theta_r} m_{j-\frac{1}{2}} f^2_{j-\frac{1}{2}} \theta_{j-\frac{1}{2}} + \frac{h_j}{\theta_r} f_{j-\frac{1}{2}} m_{j-\frac{1}{2}} w_{j-\frac{1}{2}}
\end{aligned}$$

$$(c_4)_j = (c_3)_j$$

$$
\begin{aligned}
(c_5)_j &= \frac{(2+K\epsilon_3)}{2}h_j f_{j-\frac{1}{2}} m_{j-\frac{1}{2}} + \frac{h_j K}{2\theta_r^2}\epsilon_3 f_{j-\frac{1}{2}} m_{j-\frac{1}{2}} \theta_{j-\frac{1}{2}}^2 - \frac{(1+K\epsilon_3)}{\theta_r} h_j f_{j-\frac{1}{2}} m_{j-\frac{1}{2}} \theta_{j-\frac{1}{2}} \\
&\quad + \frac{\epsilon_1 h_j}{2} f_{j-\frac{1}{2}}^2 g_{j-\frac{1}{2}} + \frac{\epsilon_1}{2\theta_r^2} h_j f_{j-\frac{1}{2}}^2 g_{j-\frac{1}{2}} \theta_{j-\frac{1}{2}}^2 - \frac{\epsilon_1 h_j}{\theta_r} f_{j-\frac{1}{2}}^2 u_{j-\frac{1}{2}} g_{j-\frac{1}{2}} \theta_{j-\frac{1}{2}}
\end{aligned}
$$

$$(c_6)_j = (c_5)_j$$

$$
\begin{aligned}
(c_7)_j &= -\frac{(1+K\epsilon_3)}{2\theta_r}(m_j - m_{j-1}) f_{j-\frac{1}{2}}^2 + \frac{K\epsilon_3}{2\theta_r^2}(m_j - m_{j-1}) f_{j-\frac{1}{2}}^2 \theta_{j-\frac{1}{2}} - \frac{K\epsilon_3}{\theta_r^2} h_j v_{j-\frac{1}{2}} \theta_{j-\frac{1}{2}} \\
&\quad + \frac{K\epsilon_3}{\theta_r} v_{j-\frac{1}{2}} - 2\frac{K\epsilon_3}{\theta_r^2} h_j g_{j-\frac{1}{2}} \theta_{j-\frac{1}{2}} + 2\frac{K\epsilon_3}{\theta_r} h_j g_{j-\frac{1}{2}} + \frac{K\epsilon_3}{\theta_r^2} h_j f_{j-\frac{1}{2}} u_{j-\frac{1}{2}} m_{j-\frac{1}{2}} \theta_{j-\frac{1}{2}} \\
&\quad - \frac{(1+K\epsilon_3)}{\theta_r} h_j f_{j-\frac{1}{2}} u_{j-\frac{1}{2}} m_{j-\frac{1}{2}} + \frac{\epsilon_1}{\theta_r^2} h_j f_{j-\frac{1}{2}}^2 u_{j-\frac{1}{2}} g_{j-\frac{1}{2}} \theta_{j-\frac{1}{2}} - \frac{\epsilon_1}{\theta_r} h_j f_{j-\frac{1}{2}}^2 u_{j-\frac{1}{2}} g_{j-\frac{1}{2}} \\
&\quad + \frac{\epsilon_1 h_j}{\theta_r^2} f_{j-\frac{1}{2}}^3 m_{j-\frac{1}{2}} \theta_{j-\frac{1}{2}} - \frac{\epsilon_1}{\theta_r} h_j f_{j-\frac{1}{2}}^3 m_{j-\frac{1}{2}}
\end{aligned}
$$

$$(c_8)_j = (c_7)_j$$

$$(c_9)_j = \frac{h_j}{2\theta_r} f_{j-\frac{1}{2}}^2 m_{j-\frac{1}{2}}$$

$$(c_{10})_j = (c_9)_j$$

$$
\begin{aligned}
(c_{11})_j &= -h_j K\epsilon_3 - \frac{K\epsilon_3}{\theta_r^2} h_j \theta_{j-\frac{1}{2}}^2 + 2\frac{K\epsilon_3}{\theta_r} h_j \theta_{j-\frac{1}{2}} + \frac{\epsilon_1}{2} h_j f_{j-\frac{1}{2}}^2 u_{j-\frac{1}{2}} \\
&\quad + \frac{\epsilon_1}{2\theta_r^2} f_{j-\frac{1}{2}}^2 u_{j-\frac{1}{2}} \theta_{j-\frac{1}{2}}^2 - \frac{\epsilon_1}{\theta_r} h_j f_{j-\frac{1}{2}}^2 u_{j-\frac{1}{2}} \theta_{j-\frac{1}{2}}
\end{aligned}
$$

$$(c_{12})_j = (c_{11})_j$$

$$
\begin{aligned}
(c_{13})_j &= \frac{(2+K\epsilon_3)}{2} f_{j-\frac{1}{2}}^2 - \frac{1}{\theta_r}(1+K\epsilon_3) f_{j-\frac{1}{2}}^2 \theta_{j-\frac{1}{2}} + \frac{K\epsilon_3}{2\theta_r^2} f_{j-\frac{1}{2}}^2 \theta_{j-\frac{1}{2}}^2 \\
&+ \frac{(2+K\epsilon_3)}{2} h_j f_{j-\frac{1}{2}} u_{j-\frac{1}{2}} + \frac{K\epsilon_3}{2\theta_r^2} h_j f_{j-\frac{1}{2}} u_{j-\frac{1}{2}} \theta_{j-\frac{1}{2}}^2 \\
&- \frac{(1+K\epsilon_3)}{\theta_r} h_j f_{j-\frac{1}{2}} u_{j-\frac{1}{2}} \theta_{j-\frac{1}{2}} + \frac{\epsilon_1}{2} h_j f_{j-\frac{1}{2}}^3 - \frac{\epsilon_1}{\theta_r} f_{j-\frac{1}{2}}^3 \theta_{j-\frac{1}{2}} + \frac{h_j}{2\theta_r} f_{j-\frac{1}{2}}^2 w_{j-\frac{1}{2}}
\end{aligned}
$$

$$
\begin{aligned}
(c_{14})_j &= -\frac{(2+K\epsilon_3)}{2} f_{j-\frac{1}{2}}^2 + \frac{1}{\theta_r}(1+K\epsilon_3) f_{j-\frac{1}{2}}^2 \theta_{j-\frac{1}{2}} - \frac{K\epsilon_3}{2\theta_r^2} f_{j-\frac{1}{2}}^2 \theta_{j-\frac{1}{2}}^2 \\
&+ \frac{(2+K\epsilon_3)}{2} h_j f_{j-\frac{1}{2}} u_{j-\frac{1}{2}} + \frac{K\epsilon_3}{2\theta_r^2} h_j f_{j-\frac{1}{2}} u_{j-\frac{1}{2}} \theta_{j-\frac{1}{2}}^2 \\
&- \frac{(1+K\epsilon_3)}{\theta_r} h_j f_{j-\frac{1}{2}} u_{j-\frac{1}{2}} \theta_{j-\frac{1}{2}} + \frac{\epsilon_1}{2} h_j f_{j-\frac{1}{2}}^3 - \frac{\epsilon_1}{\theta_r} f_{j-\frac{1}{2}}^3 \theta_{j-\frac{1}{2}} + \frac{h_j}{2\theta_r} f_{j-\frac{1}{2}}^2 w_{j-\frac{1}{2}}
\end{aligned}
$$

$$(r_1)_j = f_{j-1} - f_j + h_j u_{j-\frac{1}{2}}$$

$$(r_2)_j = u_{j-1} - u_j + h_j v_{j-\frac{1}{2}}$$

$$(r_3)_j = \theta_{j-1} - \theta_j + h_j w_{j-\frac{1}{2}}$$

$$(r_4)_j = g_{j-1} - g_j + h_j m_{j-\frac{1}{2}}$$

$$
\begin{aligned}
(r_5)_j &= -(1+K\epsilon_3)(v_j - v_{j-1}) + \frac{1}{\theta_r}(1+2K\epsilon_3)(v_j - v_{j-1})\theta_{j-\frac{1}{2}} \\
&- \frac{K\epsilon_3}{\theta_r^2}(v_j - v_{j-1})\theta_{j-\frac{1}{2}}^2 - \frac{h_j}{\theta_r} w_{j-\frac{1}{2}} v_{j-\frac{1}{2}} - \epsilon_1 h_j f_{j-\frac{1}{2}} v_{j-\frac{1}{2}} - \frac{\epsilon_1 h_j}{\theta_r^2} f_{j-\frac{1}{2}} v_{j-\frac{1}{2}} \theta_{j-\frac{1}{2}}^2 \\
&+ 2\frac{\epsilon_1 h_j}{\theta_r} f_{j-\frac{1}{2}} v_{j-\frac{1}{2}} \theta_{j-\frac{1}{2}} - K h_j \epsilon_3 m_{j-\frac{1}{2}} - \frac{K\epsilon_3}{\theta_r^2} h_j m_{j-\frac{1}{2}} \theta_{j-\frac{1}{2}}^2 + 2\frac{K\epsilon_3}{\theta_r} h_j m_{j-\frac{1}{2}} \theta_{j-\frac{1}{2}} + M_{j-\frac{1}{2}}
\end{aligned}
$$

$$
\begin{aligned}
(r_6)_j &= -(w_j - w_{j-1}) + \frac{1}{\theta_r}(w_j - w_{j-1})\theta_{j-\frac{1}{2}} - \Pr\frac{K_{f\infty}}{K_{nf}}\epsilon_2 h_j f_{j-\frac{1}{2}} w_{j-\frac{1}{2}} \\
&\quad + \frac{h_j}{\theta_r}\Pr\frac{K_{f\infty}}{K_{nf}}\epsilon_2 f_{j-\frac{1}{2}} w_{j-\frac{1}{2}}\theta_{j-\frac{1}{2}} + N_{j-\frac{1}{2}}
\end{aligned}
$$

$$
\begin{aligned}
(r_7)_j &= -\frac{(1+K\epsilon_3)}{2}(m_j - m_{j-1})f_{j-\frac{1}{2}}^2 + \frac{(1+K\epsilon_3)}{\theta_r}(m_j - m_{j-1})f_{j-\frac{1}{2}}^2\theta_{j-\frac{1}{2}} \\
&\quad -\frac{K\epsilon_3}{2\theta_r^2}(m_j - m_{j-1})f_{j-\frac{1}{2}}^2\theta_{j-\frac{1}{2}}^2 + h_j K\epsilon_3 v_{j-\frac{1}{2}} + \frac{K\epsilon_3}{\theta_r^2}h_j v_{j-\frac{1}{2}}\theta_{j-\frac{1}{2}}^2 \\
&\quad -2\frac{K\epsilon_3}{\theta_r}h_j v_{j-\frac{1}{2}}\theta_{j-\frac{1}{2}} + 2h_j K\epsilon_3 g_{j-\frac{1}{2}} + 2\frac{K\epsilon_3}{\theta_r^2}h_j g_{j-\frac{1}{2}}\theta_{j-\frac{1}{2}}^2 - 4\frac{K\epsilon_3}{\theta_r}h_j\theta_{j-\frac{1}{2}}g_{j-\frac{1}{2}} \\
&\quad -(2+K\epsilon_3)h_j f_{j-\frac{1}{2}}u_{j-\frac{1}{2}}m_{j-\frac{1}{2}} - \frac{K\epsilon_3}{\theta_r^2}h_j f_{j-\frac{1}{2}}u_{j-\frac{1}{2}}m_{j-\frac{1}{2}}\theta_{j-\frac{1}{2}}^2 \\
&\quad +2\frac{(K\epsilon_3+1)}{\theta_r}h_j f_{j-\frac{1}{2}}u_{j-\frac{1}{2}}m_{j-\frac{1}{2}}\theta_{j-\frac{1}{2}} - \epsilon_1 h_j f_{j-\frac{1}{2}}^2 u_{j-\frac{1}{2}}g_{j-\frac{1}{2}} - \frac{\epsilon_1}{\theta_r^2}h_j f_{j-\frac{1}{2}}^2 u_{j-\frac{1}{2}}g_{j-\frac{1}{2}}\theta_{j-\frac{1}{2}}^2 \\
&\quad +2\frac{\epsilon_1}{\theta_r}h_j f_{j-\frac{1}{2}}^2 u_{j-\frac{1}{2}}g_{j-\frac{1}{2}}\theta_{j-\frac{1}{2}} - \epsilon_1 h_j f_{j-\frac{1}{2}}^3 m_{j-\frac{1}{2}} - \frac{\epsilon_1}{\theta_r^2}f_{j-\frac{1}{2}}^3 m_{j-\frac{1}{2}}\theta_{j-\frac{1}{2}}^2 \\
&\quad +2\frac{\epsilon_1}{\theta_r}h_j f_{j-\frac{1}{2}}^3 m_{j-\frac{1}{2}}\theta_{j-\frac{1}{2}} - \frac{h_j}{\theta_r}f_{j-\frac{1}{2}}^2 m_{j-\frac{1}{2}}w_{j-\frac{1}{2}} + P_{j-\frac{1}{2}}
\end{aligned}
$$

We now write this system of equations into matrix-vector form as.

$$
A\varphi = R \tag{3.30}
$$

where

$$
A = \begin{bmatrix}
[A_1] & [C_1] & & & & \\
[B_1] & [A_2] & . & & & \\
 & . & . & . & & \\
 & & . & . & . & \\
 & & & . & . & . \\
 & & & [B_{J-1}] & [A_{J-1}] & [C_{J-1}] \\
 & & & & [B_J] & [A_J]
\end{bmatrix}
$$

$$
\varphi = \begin{bmatrix} [\delta_1] \\ [\delta_2] \\ [\delta_3] \\ . \\ . \\ . \\ [\delta_J] \end{bmatrix}, \quad R = \begin{bmatrix} [R_1] \\ [R_2] \\ [R_3] \\ . \\ . \\ . \\ [R_J] \end{bmatrix}, \quad [R_j] = \begin{bmatrix} (r_1)_j \\ (r_2)_j \\ (r_3)_j \\ (r_4)_j \\ (r_5)_j \\ (r_6)_j \\ (r_7)_j \end{bmatrix}
$$

$$
[\delta_1] = \begin{bmatrix} v_0 \\ m_0 \\ w_0 \\ f_1 \\ v_1 \\ m_1 \\ w_1 \end{bmatrix}, \quad [\delta_j] = \begin{bmatrix} u_{j-1} \\ g_{j-1} \\ \theta_{j-1} \\ f_j \\ v_j \\ m_j \\ w_j \end{bmatrix}, \quad \text{For } 2 \leq j \leq J
$$

$$
[A_1] = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ -\frac{h_j}{2} & 0 & 0 & 0 & -\frac{h_j}{2} & 0 & 0 \\ 0 & 0 & -\frac{h_j}{2} & 0 & 0 & 0 & -\frac{h_j}{2} \\ 0 & -\frac{h_j}{2} & 0 & 0 & 0 & -\frac{h_j}{2} & 0 \\ (a_2)_1 & (a_{14})_1 & (a_{10})_1 & (a_3)_1 & (a_1)_1 & (a_{13})_1 & (a_9)_1 \\ (b_2)_1 & 0 & (b_{10})_1 & (b_3)_1 & (b_{21})_1 & 0 & (b_9)_1 \\ (b_2)_1 & (c_{14})_1 & (c_{10})_1 & (c_3)_1 & (c_1)_1 & (c_{13})_1 & (c_9)_1 \end{bmatrix}
$$

$$[A_j] = \begin{bmatrix} -\frac{h_j}{2} & 0 & 0 & 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & -\frac{h_j}{2} & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & -\frac{h_j}{2} \\ 0 & -1 & 0 & 0 & 0 & -\frac{h_j}{2} & 0 \\ 0 & 0 & (a_8)_j & (a_3)_j & (a_1)_j & (a_{13})_j & (a_9)_j \\ 0 & 0 & (b_8)_j & (b_3)_j & (b_1)_j & (b_{13})_j & (b_9)_j \\ (a_6)_j & (a_{12})_j & (c_8)_j & (c_3)_j & (c_1)_j & (c_{13})_j & (c_9)_j \end{bmatrix} \quad \text{For } \ 2 \le j \le J$$

$$[B_j] = \begin{bmatrix} 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\frac{h_j}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -\frac{h_j}{2} \\ 0 & 0 & 0 & 0 & 0 & -\frac{h_j}{2} & 0 \\ 0 & 0 & 0 & (a_4)_j & (a_2)_j & (a_{14})_j & (a_{10})_j \\ 0 & 0 & 0 & (b_4)_j & (b_2)_j & (b_{14})_j & (b_{10})_j \\ 0 & 0 & 0 & (c_4)_j & (c_2)_j & (c_{14})_j & (c_{10})_j \end{bmatrix}$$

$$[C_j] = \begin{bmatrix} -\frac{h_j}{2} & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & (a_7)_j & 0 & 0 & 0 & 0 \\ 0 & 0 & (b_7)_j & 0 & 0 & 0 & 0 \\ (c_5)_j & (c_{11})_j & (c_7)_j & 0 & 0 & 0 & 0 \end{bmatrix}$$

3.4.4 Block elimination technique

The very first step is to decompose the system into two factors that is

$$A = LU \tag{3.31}$$

L the lower triangular matrix system

$$L = \begin{bmatrix} [\alpha_1] & & & & & & \\ [B_2] & [\alpha_2] & & & & & \\ & [B_3] & [\alpha_3] & & & & \\ & & . & . & & & \\ & & & . & . & & \\ & & & & [B_{J-1}] & [\alpha_{J-1}] & \\ & & & & & [B_J] & [\alpha_J] \end{bmatrix}$$

U be the upper triangular matrix system

$$U = \begin{bmatrix} [I] & [\Gamma_1] & & & & \\ & [I_2] & [\Gamma_2] & & & \\ & & . & . & & \\ & & & . & . & \\ & & & & . & . \\ & & & & [I_{J-1}] & [\Gamma_{J-1}] \\ & & & & & [I_J] \end{bmatrix}$$

The above systems of matrix-vectors results in the following equations

$$[\alpha_1] = [A_1] \tag{3.32}$$

$$[\alpha_j] = [A_j] - [B_j][\Gamma_{j-1}] \qquad \text{for} \quad j = 2, 3...J \tag{3.33}$$

$$[\alpha_j][\Gamma_j] = [C_j] \qquad \text{for} \quad j = 1, 2....J - 1 \tag{3.34}$$

Using above equations we can calculate the matrices $[\alpha_j]$ and $[\Gamma_j]$.

Using Eqs. (3.30) and (3.31) we get

$$LU\varphi = R \tag{3.35}$$

Let

$$U\varphi = W \tag{3.36}$$

where

$$W = \begin{bmatrix} [W_1] \\ [W_2] \\ [W_3] \\ [W_4] \\ [W_5] \\ [W_6] \\ [W_7] \end{bmatrix}$$

$$LW = R$$

which gives the following relations

$$[\alpha_1][W_1] = [R_1] \tag{3.37}$$

$$[\alpha_j][W_j] = [R_j] - [B_j][W_{j-1}], \qquad \text{for } 2 \leq j \leq J \tag{3.38}$$

Since $[\alpha_j]$ and $[R_j]$ are known matrices thus $[W_j]$ can be calculated using above equations. The step computing $[\alpha_j]$, $[\Gamma_j]$ and $[w_j]$ is known as forward sweep. At the end we use Eq(3.42) to calculate the components of $[\delta_j]$. This step is called backward sweep, where $[\delta_j]$ is computed by using the following equations

$$[\delta_J] = [W_J] \tag{3.39}$$

$$[\delta_j] = [W_J] - [\Gamma_j][\delta_{j+1}], \quad \text{for} \quad 1 \leq j \leq J-1 \tag{3.40}$$

These calculations are repeated until the convergence criterion is achieved. This last step is done with the help of Matlab programming. Matlab 7.9 version is used for the algorithm, $\Delta\eta = 0.01$, step size is taken for all of the computations. Error tolerance is fixed at 10^{-6} for all of the calculations. The edge of the boundary layer is fixed at $\eta_\infty = 10$, throughout the computations.

3.5 Numerical results

Thermophysical properties are assumed to be constant except the dynamic viscosity which makes the microrotation viscosity and micro-inertia density variant as well. The viscosity of the nanofluid varies as inverse function of temperature.

$Table(3.1)$

	ϕ	$K = 0.5$		$K = 1$		$K = 2$	
		$\theta'(0)$	$f''(0)$	$\theta'(0)$	$f''(0)$	$\theta'(0)$	$f''(0)$
	0.0	−0.962254	0.483388	−0.994366	0.454148	−1.233808	0.502364
$\theta_r = -5$	0.05	−0.937341	0.546208	−0.963974	0.520707	−1.190608	0.588601
	0.1	−0.901948	0.588668	−0.916842	0.567407	−1.149227	0.668297
	0.15	−0.861304	0.614722	−0.877571	0.598027	−1.083047	0.721119
	0.20	−0.818569	0.626931	−0.831001	0.614291	−1.009486	0.753158
$\theta_r = -10$	0.0	−0.942205	0.461378	−0.967182	0.437371	−1.066112	0.435353
	0.05	−0.918766	0.520843	−0.940689	0.501396	−1.042839	0.513544
	0.10	−0.884556	0.560405	−0.903435	0.546223	−0.999765	0.576318
	0.15	−0.841158	0.584110	−0.860519	0.574646	−0.978345	0.620447
	0.20	−0.805721	0.594623	−0.814254	0.589104	−0.947964	0.710869
$\theta_r \to \infty$	0.0	−0.923294	0.436941	−0.946490	0.419082	−1.041574	0.418816
	0.05	−0.902081	0.492572	−0.921924	0.480041	−1.020184	0.493093
	0.10	−0.870297	0.529283	−0.886186	0.522139	−0.979681	0.553600
	0.15	−0.833266	0.550788	−0.858340	0.554977	−0.929622	0.596392
	0.20	−0.793493	0.559778	−0.832353	0.579887	−0.874869	0.620296

The numeric values used in computations for constant thermophysical properties are given in $Table(2.1)$. $Table(3.1)$ and $Table(3.2)$ shows the temperature gradient and velocity gradient values near the wall for Cu-water and Ag-water micropolar nanofluids respectively. These readings are obtained with respect to different values of variant parameters which are nano-particle volume fraction,micro-rotation parameter and dynamic viscosity parameter.

$Table(3.2)$

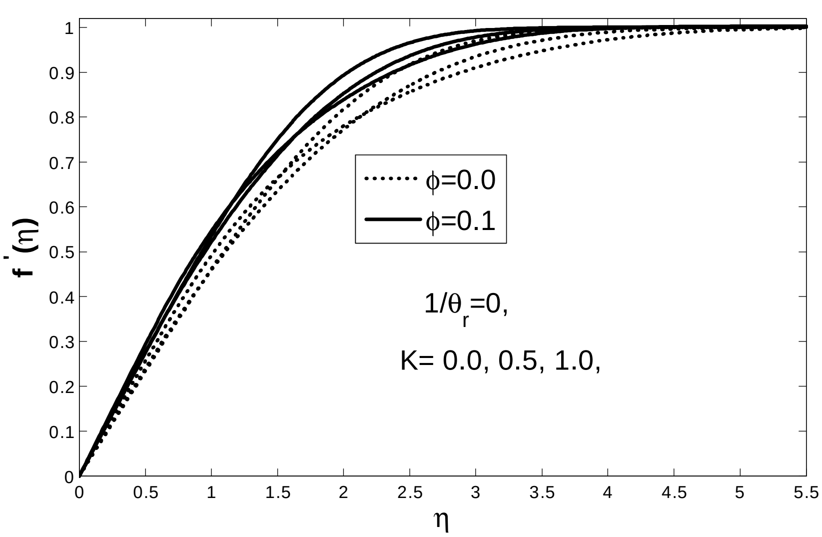
	ϕ	$K = 0.5$		$K = 1$		$K = 2$	
		$\theta'(0)$	$f''(0)$	$\theta'(0)$	$f''(0)$	$\theta'(0)$	$f''(0)$
$\theta_r = -5$	0.0	-0.962254	0.483388	-1.011872	0.457119	-1.260144	0.522138
	0.05	-0.941438	0.561971	-1.011133	0.541795	-1.215196	0.630988
	0.1	-0.905901	0.615241	-0.985668	0.631147	-1.158294	0.724400
	0.15	-0.862984	0.648823	-0.954471	0.678931	-1.090631	0.796842
	0.20	-0.818238	0.665781	-0.894400	0.706388	-1.0137132	0.844302
$\theta_r = -10$	0.0	-0.942205	0.461378	-0.967182	0.437371	-1.066112	0.435353
	0.05	-0.922845	0.536061	-0.944933	0.516938	-1.050301	0.531850
	0.10	-0.888463	0.585997	-0.907880	0.572765	-1.008217	0.610198
	0.15	-0.849246	0.616760	-0.862963	0.609042	-1.060851	0.737937
	0.20	-0.806141	0.631937	-0.826862	0.638123	-0.989569	0.781573
$\theta_r \to \infty$	0.0	-0.923294	0.436941	-0.963965	0.421252	-1.090451	0.435955
	0.05	-0.906372	0.507107	-0.943043	0.499438	-1.08090	0.529621
	0.10	-0.887393	0.558655	-0.906089	0.554331	-1.042381	0.609113
	0.15	-0.847187	0.588057	-0.872005	0.595542	-0.987293	0.669548
	0.20	-0.803495	0.601993	-0.863685	0.644665	-0.923841	0.706979

3.6 Graphical results and Discussions

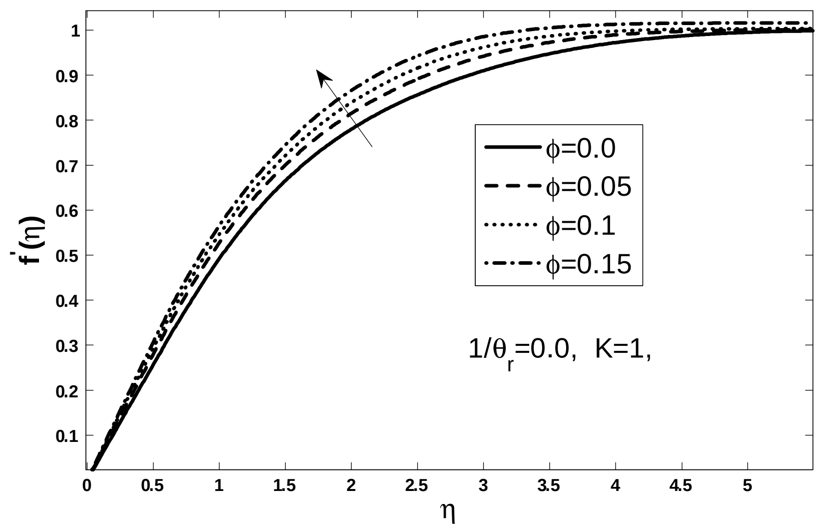
Numerical as well as graphical results are obtained to study the impact of different physical properties of nanofluid on the flow and heat transfer phenomena. Most importantly the effect of viscosity parameter 'θ_r'on the flow and heat transfer is discussed in the presence of rotational inertia. The functions $f(\eta)$ and $\theta(\eta)$ can be utilize to determine the skin friction coefficient $C_f = \frac{1}{(1-\phi)^{2.5}}(1 - \frac{1}{\theta_r})^{-1}(\mathrm{Re}_x)^{-\frac{1}{2}}f''(0)$, and the Nusselt number $Nu_x = -(\mathrm{Re}_x)^{\frac{1}{2}}\frac{K_{nf}}{K_{f\infty}}\theta'(0)$. Here

$\mathrm{Re}_x = U_\infty x/\nu_{f_\infty}$ is the local Reynolds number.

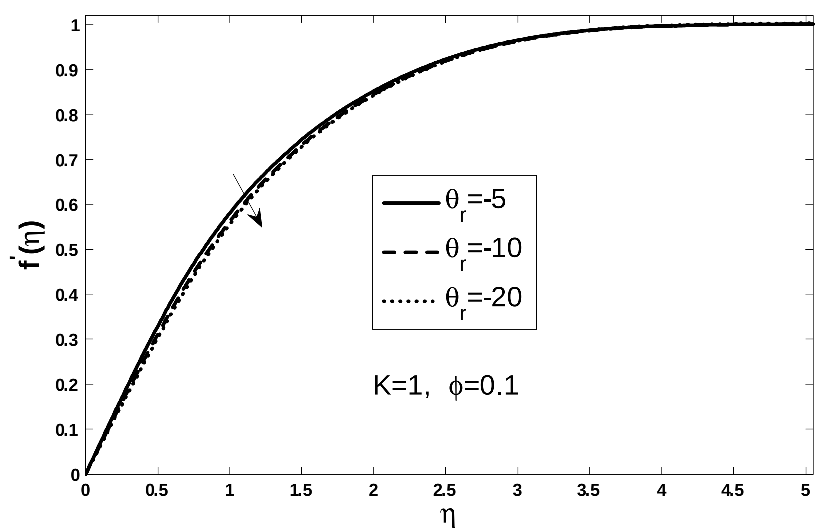
Numerical results of wall temperature gradient and velocity gradient are mentioned in $Tables$ (3.1) and (3.2) with Ag-water and Cu-water nanofluid respectively in the presence of micro-rotation parameter. The obtained results uncover some very interesting facts. Some of these observations are discussed meticulously with valid arguments. It is observed that the general behavior of Ag-water and Cu-water nanofluids are all same. The numerics depicts that for each fixed value of micropolar parameter and the viscosity parameter, the magnitude of wall temperature gradient decreases with increasing values of solid particle volume fraction, since with the decrease in the wall temperature gradient the heat flux from the solid boundaries to the fluid body decreases which causes the thinning of thermal boundary layer thickness. It is because with the low temperature potential between the two ends the thermal boundary layer is nearer to achieve. On the other hand it is very clear from the obtained data that with the fixed values of micropolar parameter and viscosity parameter the velocity gradient near the wall increases with increasing values of solid particle volume fraction parameter, this behavior of nanofluid causes the skin friction to rise and in result the momentum decreases which brings thinning effect in the momentum boundary layer. It can be noted from the numerical tables that for nanofluids with rotational inertia the magnitude of temperature gradient decreases with the increase in the magnitude of viscosity parameter where as the wall velocity gradient decreases as well. It is very obvious for heat flux since increasing viscosity parameter represents the decreasing difference of wall temperature and the temperature outside the thermal boundary layer, which ofcourse results in the fall of heat flux. Since the decrease in the wall velocity gradient indicates the decrease in the skin friction, it can be concluded that momentum boundary layer thickness increases with the increase in the magnitude of viscosity parameter. The obtained numerical results both for Ag-water and Cu-water nanofluids clearly show that with the rise in micro rotation parameter the Nusselt number increases and hence heat flux increases which causing the thickening effect on thermal boundary layer. $Figs.$ (3.1) to (3.4) presents the velocity profile with different parameters. It is observed that velocity is zero near the wall, it increases and tends to unity as the distance increases from the solid boundaries. Fig (3.2) shows that in the presence of rotational inertia the velocity profile increases with increasing value of solid particle volume fraction parameter. Fig (3.4) authenticated the same behavior when viscosity is considered as temperature dependent and a finite fixed value is given to the viscosity parameter. *Fig* (3.3) shows the decrease in velocity distribution with the increase in the magnitude of viscosity parameter. *Figs.* (3.5) to (3.8) presents the temperature profile in the presence of rotational inertia with the effect of different physical parameters. It is observed that the temperature distribution starts from unity near the wall and tends to vanish as the distance increases from the solid boundaries. *Fig* (3.5) indicates the increase in temperature distribution with the increase in the magnitude of viscosity parameter. It is because as the temperature distribution increases the heat flux decreases which results in the decrease of thermal potential difference between the two ends of fluid body, this is very obvious since viscosity parameter is reciprocal of this temperature difference. *Fig* (3.6) shows the increase in the temperature distribution with the rise in the volume fraction parameter. It is so because the rise in the volume fraction of solid particles increase the thermal conductivity of the nanofluid and hence increases the temperature distribution. It is very clear from the *Fig* (3.7) that the temperature profile decreases with the increasing values of micropolar parameter, the behavior is even true when viscosity is taken as function of temperature as shown in *Fig* (3.8).



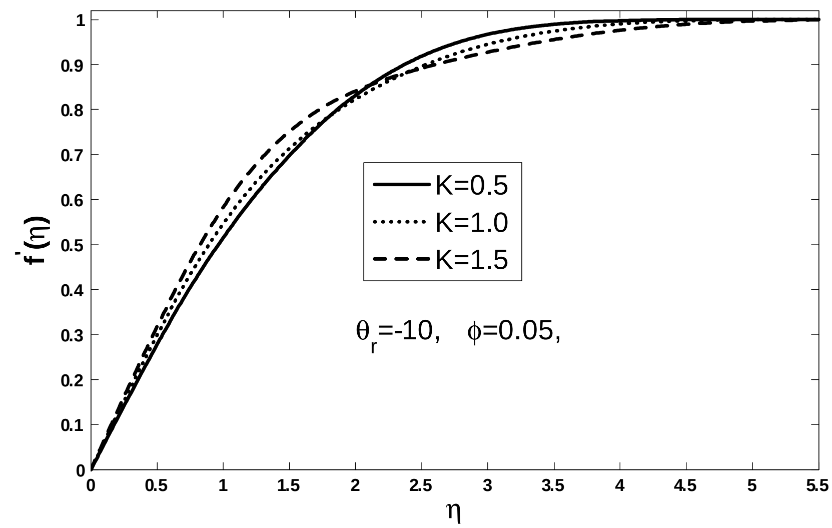
Fig(3.1) Variation of velocity with different values of micropolar parameter



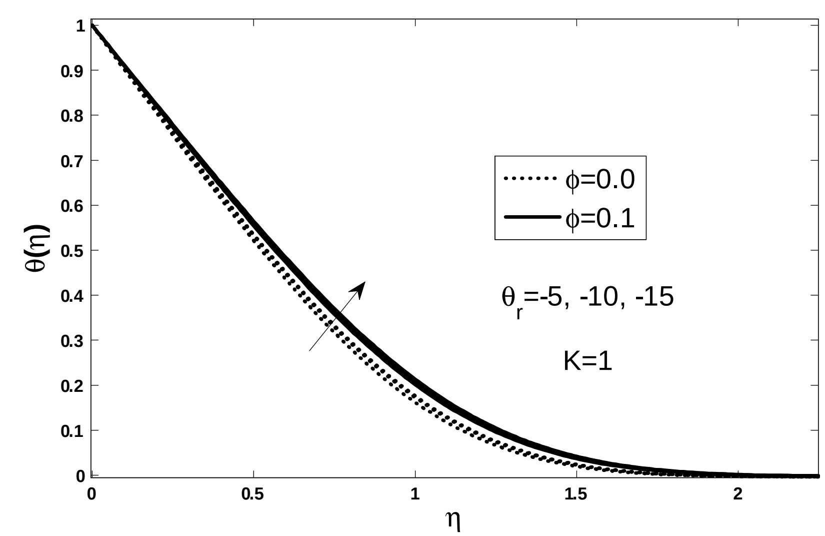
Fig(3.2). Variation of velocity with different values of ϕ



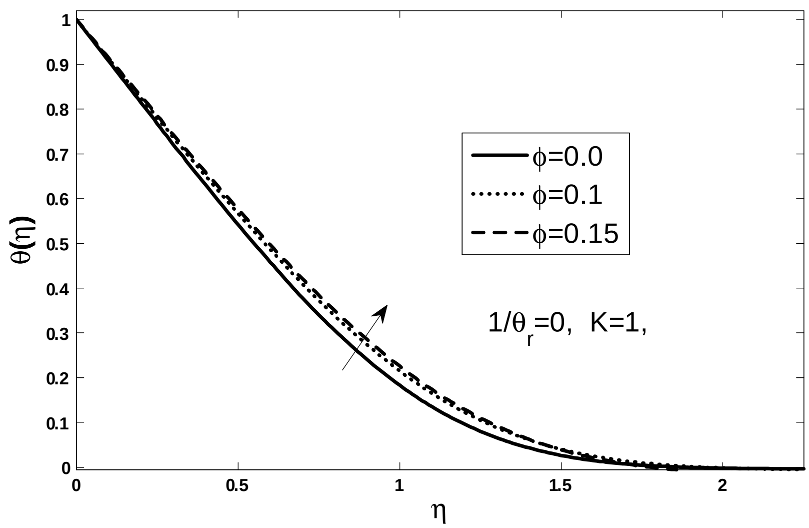
Fig(3.3). Variation of velocity with different values of viscosity parameter



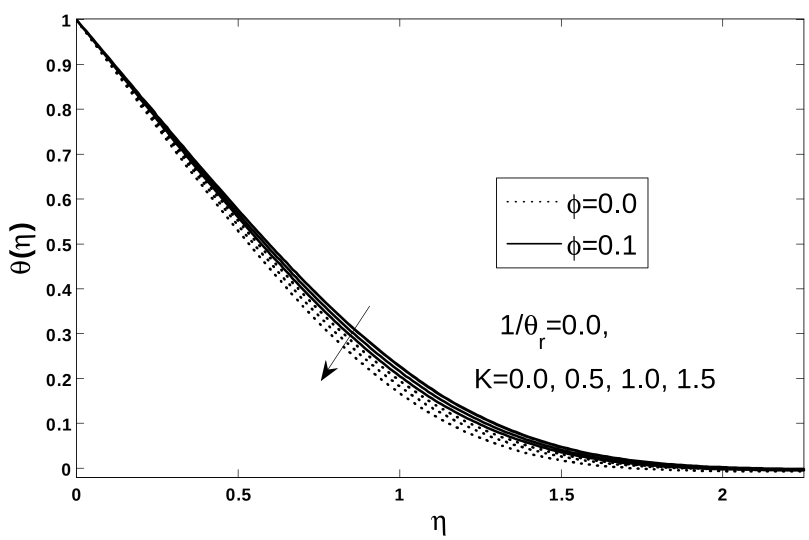
Fig(3.4). Variation of velocity with different values of micropolar parameter



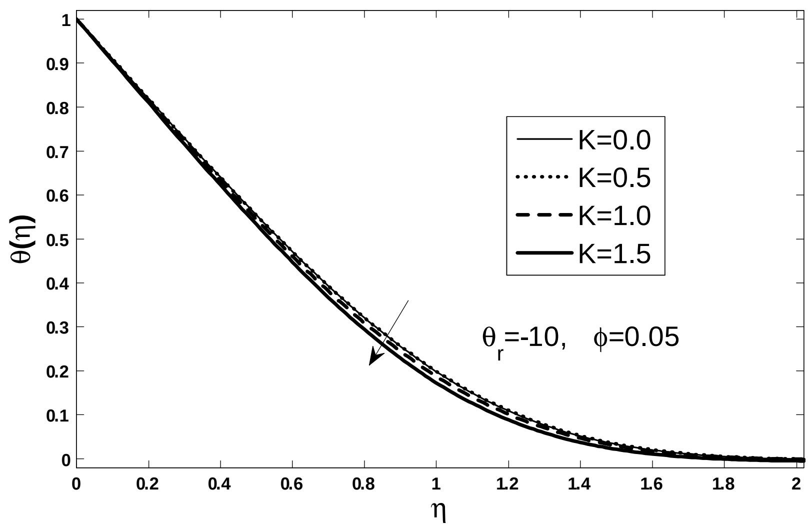
Fig(3.5). Variation of temperature with different values of viscosity parameter



Fig(3.6). Variation of temperature with different values of ϕ



Fig(3.7). Variation of temperature with different values of micropolar parameter



Fig(3.8). Variation of temperature with different values of micropolar parameter

Bibliography

[1] S. U. S. CHOI, Z. G. ZHANG, W. YU et al. Anomalously thermal conductivity enhancement in nanotube suspensions. *Applied Physics Letters*. 79 (2001): 2252-2254.

[2] S. Nadeem, R. U. Haq, Z.H. Khan. Numerical study of MHD boundary layer flow of a Maxwell fluid past a stretching sheet in the presence of nanoparticles. *Journal of the Taiwan Institute of Chemical Engineers*. 45 (2014): 121–126.

[3] A. Dib, A. Haiahem, B. Bou-said. Approximate analytical solution of squeezing unsteady nanofluid flow. *Powder Technology*. 269 (2015): 193–199

[4] S. Nadeem, S. Saleem. Unsteady mixed convection flow of nanofluid on a rotating cone with magnetic field. *Applied Nanoscience*. 4 (2014): 405–414.

[5] M. A. Sheremet, I. Pop. Conjugate natural convection in a square porous cavity filled by a nanofluid using Buongiorno's mathematical model. *International Journal of Heat and Mass Transfer*. 79 (2014): 137–145.

[6] M. Sheikholeslami, D.D. Ganji, M.Y. Javed, R. Ellahi. Effect of thermal radiation on magnetohydrodynamics nanofluid flow and heat transfer by means of two phase model. *Journal of Magnetism and Magnetic Materials*. 374 (2015): 36–43.

[7] S. Doganay, A. Turgut. Enhanced effectiveness of nanofluid based natural circulation mini loop. *Applied Thermal Engineering*. 75 (2015): 669-676.

[8] Y. Hu, Y. He, Cong Qi, B. Jiang, H. Inaki Schlaberg. Experimental and numerical study of natural convection in a square enclosure filled with nanofluid. *International Journal of Heat and Mass Transfer*. 78 (2014): 380–392.

[9] J. Zhang, Y. Diao, Y. Zhao, Y. Zhang. Experimental study of TiO2–water nanofluid flow and heat transfer characteristics in a multiport minichannel flat tube. *International Journal of Heat and Mass Transfer*. 79 (2014): 628–638.

[10] J. Bi, K. Vafai, D. M. Christopher. Heat transfer characteristics and CHF prediction in nanofluid boiling. *International Journal of Heat and Mass Transfer*. 80 (2015): 256–265.

[11] R. U. Haq, S. Nadeem, Z. H. Khan, N. S. Akbar. Thermal radiation and slip effects on MHD stagnation point flow of nanofluid over a stretching sheet. *Physica E*. 65 (2015): 17–23.

[12] A. C . Eringen. Theory of micropolar fluids. *Journal of Mathematics and Mechanics*. 16 (1966): 1–18.

[13] G. Lukaszewicz. Micropolar Fluids Theory and Applications, *Springer*. 1999.

[14] M. Hussainn, M. Ashraf, S. Nadeem, M. Khan. Radiation effects on the thermal boundary layer flow of a micropolar fluid towards a permeable stretching sheet. *Journal of the Franklin Institute*. 350 (2013): 194–210.

[15] M. Turkyilmazoglu. A note on micropolar fluid flow and heat transfer over a porous shrinking sheet. *International Journal of Heat and Mass Transfer*. 72 (2014): 388–391.

[16] N. S. Akbar, S. Nadeem. Peristaltic flow of a micropolar fluid with nano particles in small intestine. *Applied Nanoscience*. 3 (2013): 461–468.

[17] N. C. Roşca, I. Pop. Boundary layer flow past a permeable shrinking sheet in a micropolar fluid with a second order slip flow model. *European Journal of Mechanics B/Fluids*. 48 (2014): 115–122.

[18] A. Borrelli, G. Giantesio, M. C. Patria. Magnetoconvection of a micropolar fluid in a vertical channel. *International Journal of Heat and Mass Transfer*. 80 (2015): 614–625.

[19] G. C. Bourantas, V. C. Loukopoulos. MHD natural-convection flow in an inclined square enclosure filled with a micropolar nanofluid. *International Journal of Heat and Mass Transfer*. 79 (2014): 930–944.

[20] R. Mehmood, S. Nadeem, N. S. Akbar. Non-orthogonal stagnation point flow of a micropolar second grade fluid towards a stretching surface with heat transfer. *Journal of the Taiwan Institute of Chemical Engineers*. 44 (2013): 586–595.

[21] M. T. Darvishi, F. Khania, F. G. Awad, A. A. Khidir, P. Sibandab. Numerical investigation of the flow of a micropolar fluid through a porous channel with expanding or contracting walls. *Propulsion and Power Research*. 3 (2014): 133–142.

[22] M. S. Faltas, E. I. Saad. Slow motion of spherical droplet in a micropolar fluid flow perpendicular to a planar solid surface. *European Journal of Mechanics B/Fluids* 48 (2014) 266–276.

[23] V. Kuppalapalle, P. K. Vinayaka, NG Chiu-On. The effect of variable viscosity on the flow and heat transfer of a viscous Ag- water and Cu-water nanofluids. *Journal of Hydrodynamics*. 25 (2013): 1-9.

[24] H. C. Brinkman, The viscosity of concentrated suspensions and solution. *Journal of Chemical Physics*. 20 (1952): 571.

[25] C. Ahmadi. Self-similar solution of incompressible micropolar boundary layer flow over a semi-infinite plate. *International Journal of Engineering Science*. 14 (1976) :639–646

[26] J. A. Maxwell, Treatise on electricity and magnetism. *Oxford University Press*. 1904.

[27] H. B. KELLER , Numerical methods for two-point boundary value problems. New York, USA, *Dover*. 1992.