A Comparative Analysis of Mean Charts for Skewed Distributions

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A THESIS SUBMITTED IN THE PARTIAL FULFILLMENT OF THE REQUIREMENTS FOR THE DEGREE OF MASTER OF PHILOSOPHY IN STATISTICS

Supervised By

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Faculty of Natural Sciences

Quaid-i-Azam University, Islamabad

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Declaration

I "Asad Raza" hereby solemnly declare that this thesis titled, "A Comparative Analysis of Mean Charts for Skewed Distributions"

- is done wholly in candidature for a degree of M.Phil Statistics at this University,
- where I got help from the published work of others, this is always clearly stated.
- where I have quoted from the work of others, the source is always mentioned. Except of such quotations, this thesis is entirely my own research work,
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Dated:______________________ Signature:______________________

CERTIFICATE

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(Reg. No. 02222211015)

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STATISTICS

We accept this thesis as conforming to the required standards

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DEPARTMENT OF STATISTICS
QUAID-I-AZAM UNIVERSITY
ISLAMABAD, PAKISTAN
2023

Dedication

I am feeling great honor and pleasure to dedicate this research work to

My affectionate Parents and My Uncle

Whose endless affection, prayers and wishes have been a great source of comfort for me during my whole education period and my life. I also dedicate this work to my mentors and teachers, who have imparted knowledge and wisdom, shaping my intellectual growth.

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Abstract

This study employs extensive simulation approaches to evaluate how well Shewhart control charts monitor mean of non-normally distributed data. The main goals is to assess the accuracy of the average run length (ARL) under normal approximation and validate it as a theoretical measure of control chart efficacy. We conducted a systematic analysis of three different non-normal distributions: the Weibull, exponential, and generalised exponential. Different settings of parameters are considered to evaluate the effectiveness of the Shewhart chart. To determine how sensitive control chart behaviour is, a variety of scenarios are included in the study, including varying numbers of simulations and observations. The outcomes showed significant differences between the generalised exponential and Weibull distributions' performances. In particular, the generalised exponential distribution demonstrated its stability and made control chart design easier by consistently achieving the desired ARL of 370 with a set threshold value across a wide range of parameter combinations. On the other hand, the Weibull distribution required different threshold values depending on shape parameters, which might make control tactics more difficult. These findings highlight the significance of comprehending distribution characteristics and their impact on control chart performance, and they offer useful insights into optimising control chart configurations for efficiently monitoring non-normally distributed processes.

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List of Abbreviations

SPC	Statistical Process Control
IC	In-Control
OOC	Out-of-Control
ARL	Average Run Length
CUSUM	Cumulative Sum
EWMA	Exponentially Weighted Moving Average
ARL_0	In-Control Average Run Length
ARL_1	Out-of-Control Average Run Length
UCL	Upper Control Limit
LCL	Lower Control Limit
NSPC	Non Parametric Statistical Process Control
SDRL	Standard Deviation Run Length

Chapter 1

Introduction

Quality control is an important aspect of modern production. Manufacturers must fulfil client criteria for product performance, as well as regulatory agency standards, whether the product is electronics, vehicle components, or food. This is an example of a definition of quality. To achieve these criteria and standards, a product must be manufactured consistently; hence, the manufacturing process must be supervised to guarantee that this aim is fulfilled. Various quality criteria are specified to monitor a manufacturing process (Montgomery, 2009). A quality characteristic represents a metric that assesses the quality of a process or product. For instance, in manufacturing, certain processes may require specific temperature levels or specific durations. These process attributes necessitate continuous monitoring throughout the product's production cycle. Some quality attributes are quantifiable attributes of the product itself, such as its dimensions, strength, capacity, density, or weight. For example, critical product dimensions like wire thickness or the strength of a plastic component are considered quality attributes. The practice of monitoring production involves taking samples from a process or a product and applying statistical techniques to determine if predefined standards are met. This practice is commonly referred to as the statistical process control (SPC). The purpose of SPC is to track the variation of a selected quality characteristic for the product of a given process and to alert when the variation deviates from "normal variation."There will always be random or normal fluctuation in any process. This is the process' inherent fluctuation, or background noise, brought on by several little, inescapable reasons. When a process merely experiences random variations, it is considered to be under statistical control. The results of a procedure could potentially contain other types of variability. Assignable causes of variation refer to the sources of variation that do not fall under the category of random causes of variation. If assignable causes of variation exist, the process is said to be out of statistical control according to basic SPC. Control charts are a useful tool for minimizing process variability and for identifying when a process might be out of statistical control. Control charts can be used to monitor a variety of quantities. An individual measurement is the most straightforward, In situations where each sample consists of a single observation, such as monitoring patient survival rates post-procedure, this approach proves beneficial. However, when a process yields a high output volume, it may be more practical to take samples of size "n" at specific time intervals "t." These samples' mean, range, and standard deviation can then be used to construct a control chart. A chart comprises a center line, representing the long-term average quality parameter under in-control conditions, like the mean of the underlying process distribution or the target quality statistic. The upper and lower control limits, typically set at a distance "L" from the center line in standard deviation units, serve as thresholds.

The amount and frequency of data available in businesses has grown dramatically during the last few decades. Concurrently, accessible computing capacity has grown tremendously, allowing for substantial advances in artificial intelligence and machine learning. The improvements are altering the needs for statistical process monitoring. Small sample sizes are less prevalent, and more frequent data collecting necessitates more flexibility in monitoring processes.

When there is a high possibility of an undesirable consequence, process monitoring tracks process data streams and signals. In statistical process monitoring (SPM), this translates to distinguishing between special-cause and common-cause variance (Montgomery, 2009). Process fluctuation inherent in the process's design is common-cause variation. A procedure that is solely impacted by common-cause variation generates steady results. A special-cause variation modifies the output distribution temporary. A damaged gasoline pipe that raises a car's basic fuel consumption, a traumatic experience that affects a child's school performance, and sudden unemployment that affects an individual's mental health are all examples of special-cause variation. Control charts are used in the SPM to monitor the variations of a process.

This study investigates the Shewhart control chart, a key tool of SPM, and its performance characteristics under the assumption of asymmetrical, non-normally distributed sample data. After using simulation to validate one theoretical measure of performance, the average run length (ARL), is used to investigate the influence of non-normally distributed data on the ARL. In today's digital era, companies deal with an abundance of large datasets that serve as the foundation for critical decision-making processes. However, when it comes to the distribution of measurement data, these datasets typically challenge the usual notion of normality. Such deviations from the normal distribution features are prevalent in practice, necessitating a reevaluation of the usefulness and adaptability of conventional statistical approaches. This work conducts a thorough investigation into the complexities of constructing schemes for the $\overline{x}$ control chart that are especially designed for non-normal data. We cast a wide net in our investigation, covering a variety of non-normal distributions that regularly arise in actual contexts. We concentrate on the well-known non-normality representations Weibull, exponential, and generalized exponential distributions. The influence of distinct m samples of n sizes on the performance of the $\overline{x}$ control chart when used to non-normal data is a major aspect of our research.

1.1 Basic Control Charts

The SPM offers real-time strategies for monitoring processes. Among these strategies, the control chart stands out as a tool for distinguishing between common-cause and special-cause variations within a process. A process that exhibits only common-cause variation is considered to be in-control (IC), while special-cause variation indicates an out-of-control process. Control charts are instrumental in detecting such out-of-control (OOC) events. Control charts of different types have been devised, with the Shewhart (Shewhart, 1926), cumulative sum (CUSUM) (**?**), and exponentially weighted moving average (EWMA) (Roberts, 2000) control charts being the most often used in practice. The Shewhart chart is simple to comprehend and use, and it is capable of quickly identifying major changes in the process's quality indicator mean. The CUSUM and EWMA charts, on the other hand, are more sophisticated since they integrate past data into their test results. In general, these two charts do better at detecting small changes in the mean (Vera do Carmo et al., 2004).

Control charts can be evaluated using performance metrics such as the average run length (ARL) and striking probability. A control chart's ARL represents the average number of samples taken before the chart detects an OOC signal in the process. The ARL is assessed in two scenarios: the in-control ARL, denoted as ARL_0, measures the average number of samples needed for a false OOC signal when the process is under control. Conversely, the OOC ARL, ARL_1, quantifies the average number of samples required for the chart to signal an OOC state once the process has deviated from control. Additionally, a control chart's striking probability signifies the likelihood that the chart will generate an OOC signal within a defined number of samples (Gandy and Kvaløy, 2013).

1.1.1 Shewhart Charts

The $\overline{x}$ and R charts are the most frequent Shewhart charts. The $\overline{x}$ chart, also known as the control chart for the mean, tracks the mean or average value of the selected quality feature. The sample mean, $\overline{x_i}$, is shown on the control chart at time i. The sample mean is computed by

$$\overline{x}_i = \frac{x_{i1} + x_{i2} + \ldots + x_{in}}{n}, i = 1, 2, 3, ..., n \tag{1.1}$$

where x_{ij} represents the j^{th} observation within the i^{th} sample, and j ranges from 1 to n, denoting the sample size. The process is IC if the obtained sample mean falls inside the control limits. If a sample plots beyond the control bounds, it is considered out of control, and the source of the variability must be examined (Saleh et al., 2015).

The midline of the $\overline{x}$ chart indicates the IC mean of the monitored quality characteristic, marked as μ_0. The upper control limit (UCL) and the lower control limit (LCL) are

$$UCL = \mu_0 + L \cdot \frac{\sigma_0}{\sqrt{n}} \tag{1.2}$$

$$LCL = \mu_0 - L \cdot \frac{\sigma_0}{\sqrt{n}} \tag{1.3}$$

where μ_0 represents the in-control mean, σ_0 represents the in-control standard deviation, and 'L,' known as the critical value, is often selected to indicate substantial process changes (Montgomery, 2009). Slight changes in the value of L or an increase in the sample size will affect the ARL and the likelihood of detection for the Shewhart chart.

The run lengths for the Shewhart Control Chart with three sigma control limits follow a geometric distribution (Koutras et al., 2007) with a probability of 0.002699796 if the process mean is on the target. Therefore, the ARL is just the expected value of such a distribution, which is 370.4.

1.2 Research Objectives

The objectives of the study are to

1. utilize simulation techniques to validate one theoretical measure of control chart performance, specifically the ARL. Verify its accuracy and reliability under normal approximation,

2. investigate the effect of non-normal data on the Shewhart control chart performance,

3. understand how departures from normality impact the efficacy of control charts,

4. examine the performance of the $\overline{x}$ control chart when applied to non-normal data using various configurations of simulations m and number of observation n,

5. and recognize how sensitive control chart behavior is to sample size and setup.

Under the premise of asymmetrical, regularly distributed sample data, this study explores the performance characteristics of the Shewhart control chart. The prevalence of deviations from normalcy in data distributions challenges established statistical approaches in an era characterised by the expansion of huge datasets and digitalization. To address this worry, the study uses simulation approaches to validate under idealized settings one theoretical measure of control chart performance, the ARL. Furthermore, this study expands its investigation to investigate the impact of non-normal data distributions on the ARL and, as a result, the performance of the Shewhart control chart. It investigates the consequences of data deviations from normality, including Weibull, exponential, and generalized exponential distributions. In addition to these findings, the study also investigates the performance of the control chart when applied to non-normal data using various configurations of m samples of n sizes. This broad study seeks to assist practitioners in quality control and process monitoring with practical insights and actionable recommendations. It aims to provide decision-makers in digitalized sectors with the information they need to effectively manage the problems provided by non-normal data distributions. Finally, this study adds to our understanding of control chart behavior, allowing for better decision-making in quality management and process control. The remainder of the thesis is organized as follows. Chapter 2 provides a comprehensive review of the related literature. In Chapter 3, the simulation procedure for determining ARL values from Weibull, exponential, and generalized exponential distributions is discussed, along with an exploration of the shift procedure. Chapter 4 presents tables showcasing ARL values, accompanied by an interpretation of the results. Finally, Chapter 5 concludes the thesis.

Chapter 2

Literature Review

Cryer and Ryan (1990) delved into the estimation of σ for X chart, emphasizing the conventional reliance on moving ranges. However, it revealed the inefficiency of this method compared to employ the sample standard deviation, particularly when dealing with correlated data. The review underscores the frequent occurrence of data correlation and the possibility of processes maintaining control while generating correlated data. Does and Schriever (1992) explored the use of control charts in statistical quality control for detecting control problems like outliers, level shifts, and excess variability. The study introduced a comprehensive framework for control charts and outlines tests for identifying special causes. Additionally, it contrasts the suggested control chart limits with conventional ones and assesses the test performance through run length distributions.

Padgett et al. (1992) investigated $\alpha - risks$ associated with Shewhart control charts in quality control. The authors assessed how these control charts perform in various scenarios and present findings based on simulated results. Roes et al. (1993) examined Shewhart-type control charts for individual observations in the SPC. The study investigated various standard deviation estimators and proposed a two-stage approach for retrospective testing. Furthermore, the review assesses the effectiveness of different estimators and the utility of moving ranges.

Quesenberry (1993) examined the impact of sample size on estimated limits for $\overline{x}$ and X control charts. The author investigated the run length distribution for control charts with both known and estimated control limits and offered guidance on choosing appropriate sample sizes for reliable control limit establishment. The author recommended starting with Q charts when parameters are assumed unknown and transitioning to X charts once sufficient data is available for accurate control limit determination. Alwan and Roberts (1995) discussed the issue of improperly positioned control limits in charts. The authors provided an overview of this problem and its consequences for data analysis. Additionally, they delved into different types of control charts and offered examples illustrating violations of control chart theory. Chou et al. (1998) addressed the challenge of dealing with non-normal data in the SPC and suggested a solution to transform such data into a normal distribution using the Johnson system of distributions. The authors outlined a process for estimating the most suitable Johnson distribution and validate its efficacy through simulation studies and practical case examples.

Woodall and Montgomery (1999) discussed SPC research questions and ideas. They explored many techniques and approaches for managing and overseeing processes, including control charts and associated tools. The authors highlighted the continuous need of improving SPC procedures and suggest directions for additional study and development in this area. Chakraborti et al. (2001) provided an overview of nonparametric control charts made for data with a single variable. They highlighted the advantages of these charts over standard distribution-based control charts. The authors explored many approaches and strategies used in nonparametric control charts, such as CUSUM-type charts and change-point detection techniques, and offer insightful information for future research. Rao et al. (2001) focused on the convergence and uniqueness of the ARL integral equations' solutions for control charts. According to the authors, under certain circumstances, the ARL is the only answer to the integral equation. They discussed the effectiveness of quadrature-based numerical approximations for calculating ARL and apply their findings to other control techniques, such as the Shewhart chart with correlated observations. Fu et al. (2002) presented a unified technique for calculating the run length distribution and the ARL for quality control systems based on finite Markov chains. This method also provides information on the variance or standard deviation of run lengths. The research has numerical results for well-known control schemes such as the CUSUM, EWMA, and Shewhart control charts.

Albers and Kallenberg (2005) provided simple corrections to improve the performance of typical control charts. The authors investigated how the performance of control charts is affected by the estimation of the underlying mean and standard deviation. They proposed one-sided and two-sided limits with corrections. The new proposals make a clear link to the actual performance characteristics of the chart. Tsai et al. (2005) presented a new approach for constructing control limits for $\overline{x}$ control charts when the number of subgroups is small. The proposed method uses student's t-distribution and ensures that the control limits are close to the true limits, even when only a few initial subgroups are available. The performance of the approach is studied through Monte Carlo simulation, which shows that the proposed control limits perform similarly to the true limits. Jensen et al. (2006) investigated the effect of parameter estimate on attribute control chart. The work emphasizes the importance of larger sample sizes in Phase I in order to attain comparable performance with known-parameters scenarios. In addition, the study underlines the significance of graphical depiction of the empirical run length distribution for comparison reasons.

Chakraborti (2006) discussed parameter estimation and design considerations in prospective applications of the $\overline{x}$ chart. The work provides different cases for when the mean and variance are known, when only the mean is known, and when both the mean and variance are unknown. The paper also presented simulations to support the proposed methods. Albers and Kallenberg (2007) discussed two aspects of standard control charts: the effect of estimating parameters and the use of group statistics. They explored the behavior of control charts based on small groups and compared the performance of different group statistics. The paper also introduced a combined control chart that takes into account the tail behavior of the data. Schoonhoven et al. (2009) explored the design of the $\overline{x}$ control chart under normality, focusing on the choice of standard deviation estimators and factors for accurate control limits. They investigated the performance of the control chart through simulations, analyzing characteristics of the run length distribution in both IC and OOC situations. Schoonhoven and Does (2010) investigated the non-normality design strategies for the $\overline{x}$ control chart. The authors analysed numerous standard deviation estimators and examined their impact on control chart performance when non-normality is present. The results revealed that whether the control limits are normal or compensated for non-normality determined the optimality of estimator. The study provided insights on the application of the *overlinex* control chart in unexpected situations, which might be useful for quality control practitioners. Schoonhoven and Does (2012) investigated how Shewhart-type control charts are used in statistical process control to track individual observations. The study examined various approaches to estimate standard deviation and suggested a two-stage procedure for retrospective testing. They also evaluated the efficiency of various estimators and emphasises the value of utilising a chart for shifting ranges.

Yang et al. (2012) studied the effectiveness of the X control chart and the 3-CUSUM chart for spotting changes in process mean and variation. The study concluded that when it comes to shift detection, the X chart is typically more reliable and convenient. The authors also discussed the 3-CUSUM chart's shortcomings and suggested combining the X chart with other improvements for improved performance. Psarakis et al. (2014) investigated the effects of parameter estimate on the properties of control charts. They emphasised how crucial accurate parameter estimation is to the success of control charts. The analysis also suggested other metrics for assessing the performance and dispersion of control charts, including the median run length, and interquartile range. Jones-Farmer et al. (2014) provided a thorough overview of Phase I analysis for process monitoring and improvement.

They investigated the use of graphical methods, such as multivariate charts, to investigate the causes of process variability. The study emphasized the crucial role of Phase I analysis in furnishing valuable insights about the process and directing resources toward improvement efforts.

Faraz et al. (2015) investigated the effectiveness of the S^2 control chart with estimated parameters. They study revealed that relying solely on the ARL as an indicator of the IC performance can be deceptive, given the typically large standard deviation of the run length (SDRL) values. To mitigate this, the authors proposed a control limit adjustment method based on bootstrapping. This approach ensured the desired IC performance with a specified probability while minimally affecting the chart's OOC performance. Chakraborti et al. (2015) focused on nonparametric control charts. The authors underscored the significance of nonparametric charts in addressing issues within SPC and pointed out the limited adoption of nonparametric techniques in real-world applications. In conclusion, the study expressed hope for the future of this direction and suggested ways in which it may improve SPC procedures. Keefe et al. (2015) used a method for producing and simulating self-starting charts that track the means of each individual observation. The performance of self-starting Shewhart and CUSUM charts is examined by the authors using data from a simulated study. The practical ramifications of these findings are discussed in the conclusion, along with suggestions for applying self-starting charts to real-world situations. Qiu (2018) delved into nonparametric approaches for SPC to detect changes in process mean and variance. Additionally, they introduced an multivariate nonparametric Shewhart chart using component-wise signs. The authors supported their methods with practical examples and comparative analyses.

The correction parameters needed to get the necessary unconditional ARL for Shewhart control charts in the work of Goedhart et al. (2016). It offered a thorough investigation of the effects various adjustment factors have on the effectiveness of these charts. The study also covered the usage of these adjustment variables in various quality control processes. Goedhart et al. (2017a) discussed a method for modifying the control limits in Shewhart control charts to take process variability into consideration. These changes are designed to ensure a specific likelihood and a minimum level of IC performance. The authors offered analytical derivations for these adjusted limits and delve into the trade-off between maintaining IC performance and handling OOC situations. The paper also provided practical guidelines for practitioners on how to strike a balance between these aspects in practice. Goedhart et al. (2017b) discussed the utilization of Shewhart X and $\overline{x}$ charts for the process mean monitoring. In the presence of estimating control limits, considering the variability in Phase I data, which can result in inconsistent

Phase II performance. A comparison of the suggested control chart with tolerance intervals and self-starting charts for process monitoring is also given. Goedhart et al. (2018) presented two methods for determining adjusted control limits for the Shewhart X and $\overline{x}$ control charts to ensure a specified IC performance. The first method is based on the bootstrap approach, while the second method is an analytical approach. This study simplifies the analytical expressions using the theory on tolerance intervals for individual observations.

The central limit theorem is used in Huberts et al. (2018) to evaluate the performance of a Shewhart control chart for large non-normally distributed datasets for various distributions and sample sizes. The authors discussed the practical applicability of these findings for big data enterprises in the SPM. Durtka (2018) investigated the influence of non-normally distributed data on the ARL in the SPC. The author investigated the influence of non-normality on ARLs through simulation studies and advised that quality managers should ensure if the data they are sampling is from normally distributed data. Capizzi and Masarotto (2020) introduced a cautious parameter learning approach to improve the performance of control charts. The authors suggested ready-to-use versions of three popular control charts and discuss directions for further research. The proposed approach offers improved performance in detecting OOC situations. Non-parametric control approaches for SPM that do not rely on particular assumptions about the underlying distribution are discussed in Goedhart et al. (2020). In a normal distribution, the parameters μ_0 and σ_0 are often unknown and must be calculated using a Phase I reference sample.

Chapter 3

Control charts for skewed distributions

This study is centered on utilizing Shewhart control charts to monitor non-normally distributed data while using simulation techniques to achieve the ARL. It delves into the impact of non-normal data on the efficacy of Shewhart control charts and examines how departures from the normal distribution effect their performance. Additionally, the study delves into the performance of the $\overline{x}$ control chart when used the non-normally distributed data sets, particularly considering various sample sizes "n" and the number of simulations "m". The research explores three non-normal distributions: Weibull, exponential, and generalized exponential. For the Weibull distribution. The analysis covers three different shape (γ) parameter combinations (0.8, 1, 1.5) with a constant rate (λ) parameter of one. Similarly, for the generalized exponential distribution, it examines various shape parameters (0.8, 1, 1.5) with a fixed rate parameter of 1. The research methodology involves conducting simulations to achieve an IC ARL target value set at 370, acting as a reference for assessing control chart performance. The simulations encompass diverse scenarios, including different numbers of simulations m = (30, 50, 100, 200, 1000, 5000, 10000) and multiple number of observation n = (5, 30, 50, 100, 250, 1000), providing insights into how sample size and setup effect the control chart performance. In the study, the Shewhart control chart utilizes control limits set at k times the standard deviation, following a geometric distribution that results in an expected ARL of 370.4 when the process mean is on target and within control limits. To evaluate the impact of shifts in process means, shifts represented as d values in terms of multiples of the standard error are introduced, ranging from 0.0 to 3.0 standard errors, indicating various degrees of deviation from the target mean. To achieve the desired ARL value of 370, adjustments are made to parameters such as number of observations and control chart threshold (k) values.

Specifically, for the generalized exponential distributions with γ and λ parameter combinations (0.8, 1), (1, 1), and (1.5, 1), the control chart threshold (k) is set at 3, approximating an IC ARL of 370 as the number of observation (n) approaches 1000. However, for the Weibull exponential distribution with different γ and λ parameters, distinct k values are employed to achieve the desired ARL. Specifically, for $\gamma = 0.5$ and $\lambda = 1$, k is set at 3.6, while for $\gamma = 1$ and $\lambda = 1$, k is set at 3, and for $\gamma = 1.5$ and $\lambda = 1$, k is set at 2.33.

3.1 Weibull Distribution

The Weibull distribution is a continuous probability distribution employed for the analysis of lifespan data, modeling the occurrence of failures over time, and assessing the reliability of products. Its versatility extends to accommodating a wide spectrum of data from various domains, including economics, hydrology, biology, and engineering sciences (Hallinan Jr, 1993). Essentially, it serves as an essential tool for modeling extreme values in probability distributions and is commonly applied to represent the reliability of systems, survival probabilities, wind speed patterns, and diverse datasets from a range of fields. The probability density function of a Weibull distribution is

$$f(x;\gamma,\lambda) = \begin{cases} \frac{\gamma}{\lambda}\left(\frac{x}{\lambda}\right)^{\gamma-1} e^{-(x/\lambda)^{\gamma}}, & x \geq 0 \\ 0, & x < 0 \end{cases} \tag{3.1}$$

1. When the shape parameter γ is less than 1, it signifies that the failure rate decreases over time. This can be likened to the concept of the Lindy effect, which is often associated with Pareto distributions rather than Weibull distributions. In this scenario, there is a prevalence of "infant mortality," where defective items tend to fail early, leading to a decreasing failure rate as these flawed items are gradually removed from the population.

2. When the shape parameter γ is equal to 1, it indicates a constant failure rate over time. This suggests that failures occur randomly due to external events. In essence, the Weibull distribution transforms into an exponential distribution under this condition.

3. A value of the shape parameter γ greater than 1 implies that the failure rate increases with time. This phenomenon can be attributed to an "aging" process or the presence of components that become more prone to failure as time progresses.

The mean of Weibull distribution with γ and λ is

$$\mu = \lambda \cdot \Gamma\left(1 + \frac{1}{\gamma}\right) \tag{3.2}$$

while the variance is

$$\sigma^2 = \lambda^2 \left[\Gamma\left(1 + \frac{2}{\gamma}\right) - \left(\Gamma\left(1 + \frac{1}{\gamma}\right)\right)^2\right] \tag{3.3}$$

3.2 Generalized Exponential Distribution

Gupta and Kundu (2007) discussed two-parameter generalized exponential distribution for analyzing positive lifetime data, serving as a suitable alternative to the two-parameter gamma or Weibull distribution. Specifically, when the shape parameter γ is set to one, it coincides with the one-parameter exponential distribution, making it an extension or generalization of the one-parameter exponential distribution. The generalized exponential distribution also has practical interpretations. For instance, in a parallel system composed of n components, where the system operates as long as at least one component functions, the system's lifetime distribution can be expressed as

$$F(x; n, \lambda) = (1 - e^{-\lambda x})^n, \quad x > 0 \tag{3.4}$$

The density function of the generalized exponential distribution is given by

$$f(x; \alpha, \lambda) = \alpha\lambda(1 - e^{-\lambda x})^{\alpha-1} e^{-\lambda x}, \quad x > 0 \tag{3.5}$$

where both α and λ are greater than zero. The density function exhibit various shapes. When α is less than or equal to one, the function decreases, and for α greater than one, it forms a unimodal, skewed, right-tailed distribution similar to the Weibull or gamma density function. It is worth noting that even for very large values of the shape parameter, the distribution remains asymmetric.

The expected value of a generalized exponential distribution is

$$E(X) = \frac{1}{\lambda}[\psi(\alpha + 1) - \psi(1)] \tag{3.6}$$

while the variance is

$$V(X) = \frac{1}{\lambda^2}[\psi'(1) - \psi'(\alpha + 1)] \tag{3.7}$$

where $\psi(.)$ represents the digamma function, and $\psi'(.)$ is the trigamma function, which are first and second derivatives of the gamma function, respectively. Importantly, as λ increases, the mean of a generalized exponential distribution also increases to infinity, assuming a fixed λ. Similarly, for a fixed λ, the variance grows and approaches to $\frac{\pi^2}{6\lambda}$.

3.3 Guaranteed ARL

One kind of control chart design that seeks to accomplish a guaranteed ARL performance is the guaranteed control chart. The control limits are determined using the actual process parameters in a standard control chart design, and the ARL performance is predicated on the assumption of normally distributed data. But in reality, the data might be non-normal, and it is frequently uncertain what the real process parameters are. Despite non-normality and uncertainty in parameter estimate, the guaranteed control chart design considers these aspects and seeks to get a given ARL value with a given probability (Weiß et al., 2018).

In this study, we set different threshold values based on parameter combinations to achieved the target ARL when the number of observations (n) approaches to 1000. In particular, the control chart threshold (k) is set at 3 for all parameter combinations of the generalized exponential distributions and exponential distribution with γ and λ parameter (1,1), to obtain the appropriate IC ARL of 370. However, alternative values of k are used for the Weibull distribution with γ and λ parameters. In particular, k is set at 3.6 for $\gamma = 0.5$ and $\lambda = 1$, at 3 for $\gamma = 1$ and $\lambda = 1$, and at 2.33 for $\gamma = 1.5$ and $\lambda = 1$ to get the target ARL values of 370.

3.4 Shift Procedure

The shift in terms of multiples of the standard error is represented as:

$$\delta = \frac{\sqrt{n}\,|\mu - \mu_0|}{\sigma} \tag{3.8}$$

where

- n is the sample size.
- σ is the process standard deviation.
- μ is the current process mean from which a sample is taken.
- μ_0 is the target process mean.

Absolute values are used because a shift in the process mean above or below the target results has the same probabilities when the measures being sampled are normally distributed. A shift of $\delta = 1$ indicates that the process mean has shifted one standard error away from the target mean.

The probability that a sample mean falls outside of a control limit if the true process mean has shifted from the target mean by one standard error is calculated as follows:

$$
\begin{aligned}
P(\bar{x} \notin [LCL, UCL]) &= P\left(\bar{x} \geq \text{UCL} \,|\, \mu\right) + P\left(\bar{x} \leq \text{LCL} \,|\, \mu\right) \\
&= P\left(\frac{\bar{x} - \mu}{\sigma/\sqrt{n}} \geq \frac{\mu_0 + 3(\sigma/\sqrt{n}) - (\mu_0 + \delta(\sigma/\sqrt{n}))}{\sigma/\sqrt{n}}\right) + \\
& P\left(\frac{\bar{x} - \mu}{\sigma/\sqrt{n}} \leq \frac{\mu_0 - 3(\sigma/\sqrt{n}) - (\mu_0 + \delta(\sigma/\sqrt{n}))}{\sigma/\sqrt{n}}\right) \\
&= P\left(z \geq \sigma/\sqrt{n}\frac{(3 - \delta)}{\sigma/\sqrt{n}}\right) + P\left(z \leq \sigma/\sqrt{n}\frac{(-3 - \delta)}{\sigma/\sqrt{n}}\right) \\
&= P\left(z \geq 3 - \delta\right) + P\left(z \leq -3 - \delta\right) = P(z \geq 2) + P(z \leq -4) \\
&= 0.02275013 + 0.00003167 = 0.02278180 \qquad (3.9)
\end{aligned}
$$

3.5 Simulation Design

A simulation process is carried out in the R programming language, involving the generation of random samples based on specified distribution and parameters. This entailed calculating the true mean and standard deviation of the process and subsequently computing the standard error of the sample mean by dividing the standard deviation by the square root of the sample size. To establish control chart limits, the conventional K-sigma rule is applied to determine the UCL and LCL. A vector of potential shift values was created to introduce shifts to the process mean, with a subsequent check to determine if the sample mean fell outside the control limits. In the cases where the sample mean exceeded these limits, a signal indicator was set to 1, indicating an OOC signal. The run length is stored in a designated vector, and the ARL is calculated by taking the mean of the run lengths obtained from all simulations. The steps are given below.

1. Generate the data from Weibull or Generalized exponential distribution with number of observation n and simulations m, where the μ_0 is computed from 3.2 and 3.6 and σ_0 is computed by taking the square root of 3.3 and 3.7, respectively.

2. The UCL and LCL are determined using 1.2 and 1.3.

3. After that, the run length (RL) for each data set is computed by recording the OOC signals by $RL = (\overline{x} > \text{UCL} \,||\, \overline{x} < \text{LCL})$, where $\overline{x}$ is the mean of

the generated data set. Then, the ARL is calculated by taking average of the run length values.

Chapter 4

Results and Discussion

Table 4.1: ARL values for different combinations of n, m, and δ for Weibull distribution with $\gamma = 0.8$ and $\lambda = 1$

		Weibull distribution ($\gamma = 0.8, \lambda = 1, k = 3.6$)						
	m	30	50	100	200	1000	5000	10000
n	δ	ARL	ARL	ARL	ARL	ARL	ARL	ARL
5	0.0	88.67	84.36	84.44	78.73	81.09	79.03	79.67
	0.2	45.97	40.52	45.69	48.29	43.35	44.74	45.26
	0.4	25.33	23.08	22.59	27.75	27.06	27.14	27.86
	0.6	19.30	17.14	17.54	19.13	19.61	18.72	18.84
	0.8	18.13	15.04	14.42	11.69	14.00	13.36	13.50
	1.0	10.13	11.34	10.44	9.40	10.26	10.24	10.00
	1.2	7.73	9.32	5.89	7.94	8.10	8.22	7.94
	1.4	6.13	5.94	7.71	7.51	6.77	6.44	6.57
	1.6	4.40	5.74	4.95	5.22	5.39	5.50	5.56
	1.8	5.33	3.94	4.25	4.25	4.99	4.66	4.73
	2.0	3.60	3.92	3.83	4.19	4.07	4.19	4.19
	2.2	4.43	3.44	3.50	3.86	3.59	3.64	3.63
	2.4	3.57	2.92	3.02	3.16	3.20	3.26	3.31
	2.6	2.90	3.12	2.81	3.10	2.90	3.03	2.98
	2.8	3.67	2.86	2.55	2.68	2.76	2.74	2.73
	3.0	2.93	2.28	2.11	2.59	2.61	2.56	2.53
30	0.0	268.30	169.90	209.39	175.38	181.17	184.22	179.93
	0.2	108.77	85.66	107.89	101.88	99.65	101.51	102.90
	0.4	55.70	72.30	58.49	57.06	61.38	59.47	60.89
	0.6	37.07	42.32	45.29	39.47	37.53	38.26	38.46

	0.8	28.63	23.02	21.17	26.30	24.59	25.13	25.40
	1.0	19.60	17.68	18.36	17.97	18.67	17.69	17.61
	1.2	13.07	9.50	11.97	14.11	12.88	12.79	12.56
	1.4	9.63	12.00	9.57	8.79	9.83	9.62	9.63
	1.6	8.97	7.24	9.12	7.14	7.44	7.57	7.43
	1.8	6.80	5.70	6.21	5.68	5.79	5.79	5.99
	2.0	4.83	4.30	5.22	4.84	4.90	4.85	4.85
	2.2	4.33	4.44	4.50	4.12	4.06	4.03	4.02
	2.4	3.97	3.60	3.38	3.42	3.38	3.42	3.42
	2.6	2.93	2.88	2.95	3.02	3.00	2.99	2.98
	2.8	2.80	2.56	2.04	2.46	2.80	2.60	2.61
	3.0	2.57	2.72	2.11	2.14	2.43	2.35	2.33
50	0.0	202.37	232.56	213.67	246.02	222.58	226.64	223.62
	0.2	148.50	117.58	132.77	146.56	125.74	126.35	126.46
	0.4	101.03	87.32	71.01	73.71	73.21	72.43	74.31
	0.6	58.20	46.24	48.94	47.77	46.08	46.09	46.60
	0.8	27.57	31.00	30.66	25.80	29.47	29.90	30.60
	1.0	24.23	22.64	20.25	20.70	19.96	20.32	20.82
	1.2	15.77	14.68	14.81	14.66	15.37	14.66	14.56
	1.4	11.83	12.64	9.57	10.36	10.56	10.86	10.80
	1.6	7.77	9.54	8.11	8.65	7.93	8.27	8.09
	1.8	5.23	6.62	5.75	6.88	6.27	6.44	6.38
	2.0	6.07	4.78	5.12	4.57	5.40	5.10	5.14
	2.2	4.30	4.52	3.84	4.39	4.37	4.20	4.20
	2.4	3.73	4.46	3.63	3.82	3.46	3.51	3.53
	2.6	3.33	2.48	3.19	2.89	3.11	2.96	3.03
	2.8	2.43	2.96	2.77	2.63	2.62	2.64	2.63
	3.0	2.60	2.26	2.43	2.46	2.29	2.34	2.31
100	0.00	247.37	271.74	262.52	264.29	281.68	282.66	280.26
	0.2	194.77	138.44	165.20	173.34	161.21	165.63	160.08
	0.4	99.17	121.22	90.30	94.75	100.23	94.32	94.28
	0.6	72.30	60.58	73.24	55.08	63.89	58.90	58.49
	0.8	34.20	35.92	36.72	44.02	38.07	37.51	36.90
	1.0	32.70	17.70	26.88	24.91	25.09	25.16	25.05
	1.2	13.17	18.62	18.73	17.96	17.40	17.55	17.21
	1.4	9.87	13.70	12.54	13.19	12.40	12.74	12.49
	1.6	9.70	10.22	8.93	8.89	8.93	9.34	9.19
	1.8	5.43	6.96	7.02	7.05	7.22	7.02	7.14
	2.0	6.90	5.92	5.79	5.77	5.46	5.53	5.56

	2.2	5.03	5.02	3.84	4.39	4.49	4.49	4.42
	2.4	4.00	3.52	4.16	3.72	3.82	3.69	3.69
	2.6	2.67	3.48	3.55	3.03	3.15	3.11	3.10
	2.8	2.63	2.86	2.89	2.68	2.62	2.63	2.63
	3.0	2.03	2.20	2.32	2.13	2.30	2.33	2.31
250	0.0	287.93	299.74	348.01	355.79	351.34	329.32	336.47
	0.2	215.23	215.78	189.58	183.11	198.44	212.68	205.78
	0.4	129.37	117.98	122.08	144.18	127.15	124.09	125.38
	0.6	87.83	82.76	77.70	80.57	77.39	76.21	75.56
	0.8	40.30	58.42	47.24	44.51	46.76	47.46	47.54
	1.0	28.53	40.00	26.49	32.55	29.92	30.83	31.50
	1.2	19.93	19.04	19.02	20.95	20.16	21.19	20.85
	1.4	18.10	14.24	14.13	13.83	14.97	14.72	15.05
	1.6	10.47	11.48	13.08	10.75	10.68	10.65	10.58
	1.8	8.03	7.60	7.20	8.43	8.15	7.73	7.89
	2.0	5.30	4.76	6.50	6.34	5.78	6.14	5.99
	2.2	5.43	5.16	4.13	4.64	4.64	4.69	4.75
	2.4	4.07	4.52	3.72	4.41	3.74	3.84	3.87
	2.6	2.83	2.94	3.37	3.26	3.07	3.10	3.17
	2.8	2.40	2.24	3.10	2.89	2.68	2.68	2.72
	3.0	2.33	2.34	2.15	2.44	2.32	2.26	2.31
1000	0.00	371.53	366.42	369.37	367.26	363.96	361.54	371.71
	0.2	235.07	255.20	259.48	226.98	255.13	257.07	264.42
	0.4	148.50	152.84	185.83	165.32	166.13	162.99	162.65
	0.6	108.00	82.24	95.65	93.18	93.76	96.06	98.48
	0.8	61.20	46.28	74.64	60.50	59.94	60.79	60.56
	1.0	37.60	42.26	36.42	36.87	38.03	38.95	38.52
	1.2	25.23	26.56	26.99	23.86	24.57	25.65	25.80
	1.4	18.40	19.36	17.36	17.01	17.52	17.66	17.54
	1.6	10.60	14.58	13.36	10.89	12.51	12.28	12.23
	1.8	8.03	10.26	8.81	10.14	9.23	8.70	9.02
	2.00	5.80	8.50	7.08	6.51	6.67	6.51	6.80
	2.2	5.53	5.60	5.02	5.64	4.89	5.13	5.14
	2.4	4.10	3.98	4.21	3.89	4.16	4.02	4.01
	2.6	3.43	3.24	3.09	3.51	3.43	3.32	3.28
	2.8	2.57	2.78	2.99	2.87	2.79	2.74	2.72
	3.0	2.60	2.74	2.37	2.42	2.29	2.28	2.30

Table 4.2: ARL values for different combinations of n, m, and δ for Weibull distribution with $\gamma = 1$ and $\lambda = 1$

		Weibull distribution ($\gamma = 1, \lambda = 1, k = 3$)						
	m	30	50	100	200	1000	5000	10000
n	δ	ARL	ARL	ARL	ARL	ARL	ARL	ARL
5	0.0	132.63	114.12	103.83	110.97	102.96	108.09	108.25
	0.2	60.57	60.72	49.85	56.31	56.80	55.33	55.92
	0.4	40.63	33.70	39.93	33.94	33.95	32.69	33.10
	0.6	13.83	19.72	19.86	19.48	20.95	20.96	21.00
	0.8	15.23	15.16	14.98	13.74	14.24	14.47	14.45
	1.0	10.00	11.24	11.85	9.47	10.58	10.74	10.55
	1.2	9.13	7.92	6.83	6.71	8.24	8.19	8.05
	1.4	7.97	6.44	6.87	6.63	6.45	6.48	6.40
	1.6	5.03	5.46	5.26	5.43	5.28	5.20	5.32
	1.8	3.90	4.98	4.92	3.95	4.32	4.53	4.42
	2.0	4.57	3.82	4.13	4.32	3.87	3.90	3.79
	2.2	3.20	3.14	3.18	3.31	3.33	3.32	3.34
	2.4	2.67	2.46	2.82	3.08	3.06	3.00	2.95
	2.6	3.57	2.50	2.69	2.63	2.73	2.70	2.68
	2.8	2.10	2.06	2.43	2.40	2.42	2.50	2.50
	3.0	2.37	2.42	2.18	2.32	2.25	2.24	2.26
30	0.0	303.60	267.32	265.16	249.05	240.63	230.01	234.87
	0.2	109.47	129.30	126.36	145.44	116.45	123.66	125.50
	0.4	73.80	84.72	71.21	61.03	74.12	70.81	71.16
	0.6	30.43	41.70	41.10	46.05	39.24	42.39	42.63
	0.8	20.80	29.10	30.26	27.61	27.23	27.89	27.05
	1.0	16.23	21.00	17.69	18.46	17.87	18.36	18.26
	1.2	10.50	14.10	12.47	15.04	12.48	12.65	13.07
	1.4	6.03	9.06	8.75	9.38	9.64	9.26	9.49
	1.6	7.47	7.24	7.97	7.18	7.02	7.16	7.19
	1.8	6.13	5.38	6.22	5.45	5.52	5.54	5.67
	2.0	3.57	4.28	4.37	4.80	4.67	4.43	4.47
	2.2	4.23	3.92	3.43	3.88	3.71	3.69	3.73
	2.4	2.73	3.22	3.28	2.95	3.11	3.14	3.17
	2.6	3.47	2.76	2.85	2.78	2.73	2.69	2.66
	2.8	2.00	2.22	2.42	2.36	2.31	2.40	2.36

	3.0	1.77	2.14	2.03	2.15	2.05	2.11	2.08
50	0.0	266.20	163.36	262.63	284.09	280.12	279.15	268.58
	0.2	163.93	157.96	133.88	156.48	152.89	148.19	151.04
	0.4	81.77	82.28	84.05	84.84	85.81	86.62	85.55
	0.6	49.50	61.48	49.86	48.80	53.49	50.84	50.68
	0.8	24.30	25.80	30.21	33.30	31.29	32.63	31.77
	1.0	17.47	21.46	21.27	21.47	21.47	20.49	21.02
	1.2	19.03	15.72	12.22	14.24	15.50	14.66	14.56
	1.4	10.67	13.64	8.27	10.96	10.04	10.40	10.45
	1.6	4.80	8.14	6.98	7.36	7.61	7.87	7.65
	1.8	5.87	5.58	5.66	5.74	5.72	5.91	5.96
	2.0	3.10	4.52	4.64	4.84	4.91	4.76	4.65
	2.2	4.17	4.02	3.45	3.76	3.60	3.84	3.83
	2.4	3.03	3.12	2.93	3.12	3.10	3.18	3.22
	2.6	2.27	2.58	2.61	2.53	2.76	2.74	2.72
	2.8	2.43	2.50	2.98	2.19	2.30	2.34	2.35
	3.0	2.17	1.98	2.11	2.10	2.05	2.09	2.08
100	0.0	332.23	362.00	298.90	294.69	309.96	308.55	312.92
	0.2	209.33	217.86	186.38	176.99	188.18	185.25	181.64
	0.4	98.30	103.94	104.16	120.39	100.08	104.55	105.82
	0.6	73.43	61.02	58.49	61.12	62.55	62.56	62.41
	0.8	29.60	44.64	40.34	38.96	39.02	38.23	39.35
	1.0	19.80	20.88	23.63	25.38	24.69	25.41	25.01
	1.2	13.63	17.64	19.03	18.00	16.92	16.77	17.02
	1.4	10.07	10.52	11.45	11.59	11.67	12.02	12.04
	1.6	8.33	8.68	8.77	8.50	8.55	8.49	8.51
	1.8	6.20	6.78	6.46	6.83	6.53	6.38	6.50
	2.0	4.40	5.26	5.81	5.22	5.04	4.97	5.05
	2.2	4.50	4.08	3.91	3.91	3.92	4.03	3.96
	2.4	4.20	3.34	3.48	3.38	3.46	3.25	3.26
	2.6	2.23	3.10	2.66	2.57	2.74	2.76	2.71
	2.8	2.27	2.12	2.54	2.15	2.44	2.38	2.36
	3.0	2.57	2.40	2.10	2.03	1.96	2.03	2.07
250	0.0	330.40	334.36	268.43	320.10	354.04	338.76	350.51
	0.2	273.93	194.96	246.35	199.92	223.67	223.42	223.57
	0.4	150.57	136.34	126.06	136.69	132.24	130.97	131.05
	0.6	83.17	88.44	68.24	74.96	79.73	78.28	78.00
	0.8	43.77	35.32	48.65	44.21	48.58	46.74	46.25
	1.0	38.90	22.86	30.80	30.85	31.09	30.29	30.13

	1.2	22.70	19.68	17.33	20.78	19.95	19.59	19.50
	1.4	11.00	13.40	13.16	13.65	13.07	13.56	13.65
	1.6	10.57	8.32	9.83	10.35	9.54	9.80	9.66
	1.8	8.60	9.10	8.03	9.11	7.31	6.96	6.97
	2.0	5.37	5.16	6.32	6.01	5.30	5.21	5.32
	2.2	4.17	4.54	3.80	4.22	4.38	4.31	4.15
	2.4	2.90	3.54	3.36	3.16	3.37	3.35	3.36
	2.6	3.43	2.76	2.86	2.63	2.73	2.74	2.80
	2.8	2.03	2.00	2.20	2.17	2.39	2.38	2.37
	3.0	2.40	2.32	1.97	2.01	2.09	2.03	2.01
1000	0.0	342.77	361.94	370.43	362.74	363.97	368.97	369.75
	0.2	293.03	249.20	258.90	251.59	252.15	257.70	258.93
	0.4	88.43	161.26	157.19	164.66	155.44	155.66	158.18
	0.6	76.63	92.26	95.63	88.50	90.56	94.37	96.62
	0.8	49.17	63.80	53.51	51.72	58.21	56.99	57.49
	1.0	39.57	35.00	36.14	36.76	34.45	35.24	35.53
	1.2	22.30	32.00	21.58	23.22	23.94	22.95	23.12
	1.4	16.87	14.06	15.42	16.04	16.14	15.52	15.44
	1.6	8.43	8.48	12.27	9.64	10.40	10.88	10.75
	1.8	7.53	7.22	7.36	7.42	7.74	7.68	7.61
	2.0	8.40	4.76	7.03	5.17	5.65	5.82	5.64
	2.2	4.37	4.24	4.45	4.68	4.37	4.43	4.39
	2.4	4.67	4.20	3.06	3.36	3.53	3.45	3.49
	2.6	2.60	2.82	2.79	3.14	2.71	2.82	2.84
	2.8	2.50	2.46	2.33	2.30	2.43	2.37	2.36
	3.0	1.87	1.74	2.02	2.01	2.06	1.99	2.01

Table 4.3: ARL values for different combinations of n, m, and δ for Weibull distribution with $\gamma = 1.5$ and $\lambda = 1$

		Weibull distribution ($\gamma = 1.5, \lambda = 1, k = 2.33$)						
	m	30	50	100	200	1000	5000	10000
n	δ	ARL	ARL	ARL	ARL	ARL	ARL	ARL
5	0.0	193.90	220.02	190.77	202.37	199.73	189.40	197.50
	0.2	96.13	87.30	90.19	69.68	84.83	86.73	85.44

	0.4	31.93	45.56	40.24	41.84	41.79	42.15	43.44
	0.6	21.87	24.58	23.36	22.03	24.52	24.61	24.66
	0.8	14.13	14.54	13.65	14.43	14.91	15.42	15.43
	1.0	9.70	11.30	10.19	9.99	10.96	10.32	10.54
	1.2	9.87	6.38	7.26	6.75	7.74	7.60	7.57
	1.4	5.80	5.20	5.84	6.06	5.70	5.68	5.68
	1.6	5.03	4.10	4.26	4.13	4.65	4.54	4.53
	1.8	3.30	3.24	3.72	3.64	3.68	3.68	3.69
	2.0	3.43	2.70	3.34	3.39	3.06	3.08	3.09
	2.2	2.70	2.68	2.19	2.88	2.59	2.72	2.67
	2.4	2.23	2.78	2.26	2.41	2.38	2.37	2.40
	2.6	1.70	2.00	1.90	2.18	2.13	2.15	2.13
	2.8	1.60	1.70	1.66	1.89	1.92	1.92	1.94
	3.0	1.83	1.52	1.90	1.66	1.81	1.81	1.79
30	0.0	214.13	267.22	241.56	320.85	309.70	316.73	315.80
	0.2	160.57	139.98	168.76	146.06	161.38	158.45	162.57
	0.4	64.87	97.70	102.84	86.59	83.45	84.07	82.69
	0.6	34.53	38.76	40.56	46.07	46.35	46.00	46.21
	0.8	20.70	25.72	25.08	26.09	27.36	26.41	26.51
	1.0	16.73	17.58	14.53	16.18	17.37	16.39	16.95
	1.2	11.47	9.56	11.49	10.17	11.48	11.31	11.01
	1.4	7.33	10.04	7.88	8.62	8.26	7.96	7.81
	1.6	5.77	5.78	4.96	6.20	5.88	5.82	5.78
	1.8	6.63	3.44	4.48	4.09	4.54	4.37	4.36
	2.0	3.70	2.78	3.44	3.81	3.38	3.44	3.40
	2.2	2.37	2.64	2.65	2.80	2.75	2.86	2.78
	2.4	2.77	2.42	2.53	2.42	2.46	2.38	2.36
	2.6	2.03	2.12	1.94	2.00	2.04	2.08	2.06
	2.8	1.73	1.76	1.75	1.69	1.79	1.80	1.81
	3.0	1.77	1.40	1.59	1.62	1.67	1.61	1.61
50	0.0	252.33	367.98	314.41	312.59	329.13	325.07	329.89
	0.2	148.43	173.36	172.95	164.02	172.30	183.72	179.82
	0.4	87.50	87.14	95.39	98.37	93.32	98.08	94.99
	0.6	51.67	51.58	56.38	54.41	54.22	53.15	51.73
	0.8	30.63	31.76	29.26	29.16	30.50	30.11	30.52
	1.0	17.17	20.38	20.39	19.89	18.15	18.60	18.88
	1.2	12.03	10.66	11.90	12.55	12.37	12.39	12.26
	1.4	6.83	7.50	7.64	8.56	8.68	8.36	8.49
	1.6	7.07	5.40	6.24	6.18	5.99	6.00	5.94

	1.8	3.90	4.46	4.34	4.34	4.33	4.56	4.52
	2.0	4.27	3.26	4.00	3.41	3.69	3.53	3.58
	2.2	3.30	2.66	2.66	2.70	2.92	2.84	2.85
	2.4	1.93	2.34	2.21	2.18	2.36	2.35	2.36
	2.6	1.77	2.22	1.90	2.15	1.97	2.06	2.02
	2.8	1.73	1.86	1.69	1.83	1.72	1.78	1.80
	3.0	1.37	1.80	1.56	1.54	1.58	1.57	1.60
100	0.0	252.53	285.76	343.36	288.67	349.56	342.89	340.05
	0.2	214.23	197.04	204.31	189.63	201.71	201.65	204.39
	0.4	106.90	77.72	111.47	109.63	112.71	111.04	108.51
	0.6	48.50	65.42	58.22	55.10	58.81	59.07	60.97
	0.8	32.43	35.82	35.65	33.92	36.51	35.03	34.88
	1.0	19.07	17.48	18.11	19.48	20.51	20.96	21.08
	1.2	10.43	13.82	11.45	13.19	13.94	13.64	13.37
	1.4	7.57	7.98	9.99	8.25	9.28	9.14	9.05
	1.6	6.43	6.22	6.51	5.97	6.45	6.52	6.41
	1.8	5.87	4.78	4.65	4.49	4.68	4.74	4.77
	2.0	3.80	3.98	3.63	3.63	3.65	3.70	3.66
	2.2	2.20	2.66	3.34	2.62	2.80	2.93	2.91
	2.4	2.73	2.34	2.28	2.51	2.39	2.35	2.37
	2.6	1.77	1.92	2.03	2.03	2.11	2.03	2.02
	2.8	1.97	1.64	1.61	1.62	1.70	1.76	1.75
	3.0	1.53	1.32	1.51	1.51	1.54	1.52	1.55
250	0.0	368.30	342.90	351.17	323.11	347.79	351.14	349.57
	0.2	302.23	201.72	225.86	235.58	230.69	228.18	232.14
	0.4	103.83	118.00	121.18	123.20	129.02	128.83	127.03
	0.6	70.67	68.60	57.41	66.44	69.52	67.85	70.62
	0.8	41.27	42.74	36.09	35.30	38.54	39.72	39.98
	1.0	30.90	21.16	22.04	23.92	23.60	23.80	23.63
	1.2	11.13	16.46	12.82	13.96	15.00	15.00	15.32
	1.4	10.80	10.58	9.31	9.37	9.98	9.92	9.99
	1.6	6.47	5.80	6.44	6.06	7.01	6.96	6.87
	1.8	4.73	4.76	4.88	5.46	4.99	5.05	5.00
	2.0	3.33	3.38	3.60	3.58	3.72	3.74	3.79
	2.2	2.90	2.52	2.79	2.99	2.86	2.93	2.97
	2.4	2.37	2.30	2.27	2.41	2.52	2.42	2.41
	2.6	1.80	1.86	1.86	2.04	1.97	2.03	2.01
	2.8	1.60	1.50	1.76	1.72	1.71	1.72	1.72
	3.0	1.37	1.42	1.64	1.51	1.58	1.51	1.52

1000	0.0	328.17	321.28	371.30	370.37	373.70	369.65	366.93
	0.2	350.60	216.02	269.37	253.75	256.90	267.14	262.44
	0.4	122.07	121.30	127.53	131.48	147.11	148.59	150.04
	0.6	107.77	81.86	66.60	81.69	76.43	80.23	82.08
	0.8	33.77	39.68	39.94	47.05	45.32	46.74	46.88
	1.0	26.07	30.54	24.69	24.20	29.12	27.34	27.32
	1.2	12.90	14.84	15.67	15.54	16.31	16.39	16.99
	1.4	9.47	9.36	10.55	11.63	10.47	11.07	10.99
	1.6	7.50	7.62	7.38	7.55	7.35	7.55	7.61
	1.8	4.70	4.86	4.99	5.16	5.30	5.35	5.27
	2.0	2.87	3.96	3.52	3.92	4.05	4.02	4.00
	2.2	2.93	2.30	2.92	3.06	3.07	3.07	3.04
	2.4	1.93	2.26	2.14	2.39	2.45	2.44	2.46
	2.6	1.93	2.20	1.85	1.99	1.96	2.01	2.02
	2.8	1.67	1.78	1.61	1.67	1.66	1.72	1.72
	3.0	1.33	1.56	1.50	1.42	1.51	1.50	1.52

Table 4.4: ARL values for different combinations of n, m, and δ for Generalized exponential distribution with $\alpha = 0.8$ and $\lambda = 1$

		Gen exponential distribution ($\alpha = 0.8, \lambda = 1, k = 3$)						
	m	30	50	100	200	1000	5000	10000
n	δ	ARL	ARL	ARL	ARL	ARL	ARL	ARL
5	0.0	127.30	91.50	101.39	99.13	102.44	113.80	107.00
	0.2	235.30	187.18	206.15	196.56	182.33	189.60	180.00
	0.4	369.50	299.20	342.93	326.58	384.49	398.00	375.80
	0.6	669.47	588.48	579.00	684.76	561.25	587.60	562.80
	0.8	1566.27	1183.74	1370.68	1499.90	1462.60	1437.40	1409.60
	1.0	2595.90	2871.10	3004.12	3229.75	3263.41	3223.40	3793.20
	1.2	6346.30	6982.08	6416.53	6394.15	6115.85	6805.80	6406.60
	1.4	12668.10	14868.60	10215.07	14055.10	14042.48	14908.00	17141.00
	1.6	17766.27	29411.68	23696.53	26612.09	32839.28	24482.60	29342.20
	1.8	54792.27	62777.94	51652.22	43319.53	60225.39	58598.00	50346.20
	2.0	146420.20	94236.56	92452.39	111469.18	116354.40	134365.00	116242.40
	2.2	369197.50	191918.82	218638.53	249208.64	670151.06	400106.40	482933.80
	2.4	454837.03	486256.24	467445.20	481810.33	473494.20	259974.80	763030.20
	2.6	783706.60	1126099.24	1063845.69	1216383.12	1055399.00	1395642.60	1611826.00
	2.8	2594347.83	2349584.72	2139946.50	2766024.91	6945702.60	5173965.00	1743176.80
	3.0	5052089.70	6013067.86	6210624.70	5128014.37	6611963.00	6503289.00	6700874.00
30	0.0	222.27	296.92	243.10	240.27	228.60	222.32	220.05

	0.2	370.20	351.36	297.58	354.58	417.11	403.95	397.32
	0.4	611.17	738.02	527.82	709.02	744.46	693.27	719.68
	0.6	1357.40	1116.38	1375.19	1206.75	1112.26	1142.97	1152.33
	0.8	1391.23	1613.40	1782.50	1597.19	1522.64	1496.93	1509.91
	1.0	1094.43	1467.04	1408.42	1718.51	1647.64	1507.20	1543.43
	1.2	1133.90	918.50	1161.02	1275.43	1277.73	1211.01	1248.49
	1.4	1012.00	987.04	863.09	907.42	869.89	900.17	897.15
	1.6	619.00	587.00	681.60	642.69	666.62	649.30	652.93
	1.8	406.30	387.02	495.84	451.00	469.89	475.11	459.80
	2.0	321.53	286.28	288.62	317.57	328.92	347.16	336.92
	2.2	220.73	231.02	238.39	261.86	240.99	251.01	248.20
	2.4	157.27	178.12	210.29	183.33	183.37	184.36	182.17
	2.6	127.30	151.60	134.96	145.74	138.82	137.01	139.91
	2.8	110.90	101.18	111.41	102.42	104.41	106.20	106.71
	3.0	88.73	63.38	70.72	82.21	80.41	80.82	82.75
30	0.0	301.43	244.26	239.77	249.45	280.51	264.56	265.04
	0.2	537.43	448.04	401.83	448.44	432.99	435.39	443.40
	0.4	523.50	649.26	624.76	691.79	649.95	655.88	663.83
	0.6	762.20	872.64	770.94	815.31	743.79	800.59	778.37
	0.8	665.37	744.02	651.60	743.31	685.70	707.13	700.28
	1.0	428.63	546.42	499.21	541.85	532.66	552.67	544.72
	1.2	433.63	385.20	380.24	370.93	392.77	386.74	384.68
	1.4	261.57	205.06	298.26	259.35	257.74	263.00	263.87
	1.6	179.93	223.40	181.98	188.21	177.01	181.25	185.05
	1.8	111.03	129.16	121.57	133.72	129.44	129.17	133.76
	2.0	88.80	103.70	73.28	90.06	92.49	97.57	95.63
	2.2	48.33	78.72	77.74	63.46	66.86	68.73	70.54
	2.4	81.80	48.80	51.90	48.80	53.79	51.09	53.32
	2.6	37.40	39.20	41.21	37.34	40.40	39.26	39.38
	2.8	29.77	29.78	31.79	36.42	30.87	30.88	30.82
	3.0	20.70	22.86	21.43	22.86	24.66	24.68	23.89
100	0.0	304.67	209.08	319.92	284.82	308.33	303.21	310.71
	0.2	441.13	492.60	447.66	445.16	458.22	441.37	448.11
	0.4	762.10	475.96	632.49	521.41	555.92	527.30	533.94
	0.6	576.80	537.54	469.26	481.80	465.83	471.56	462.48
	0.8	427.10	296.22	309.07	356.86	350.98	340.78	342.90
	1.0	266.07	227.46	261.55	246.47	224.45	233.26	234.32
	1.2	231.70	178.34	151.77	184.29	156.69	158.50	162.47
	1.4	127.90	73.34	109.55	107.24	107.67	107.96	108.39
	1.6	80.73	66.12	74.83	77.47	77.99	77.62	75.07
	1.8	50.23	44.20	68.57	45.65	53.30	54.19	54.43
	2.0	33.27	30.84	42.32	37.33	38.87	39.03	39.30
	2.2	30.33	28.92	37.97	29.33	30.00	28.60	28.84
	2.4	19.23	20.92	23.15	20.56	21.39	21.85	22.00
	2.6	16.90	15.70	19.21	17.32	16.60	16.43	16.88
	2.8	11.33	10.20	12.17	12.83	13.00	13.17	12.97
	3.0	10.73	11.58	11.02	10.68	10.78	10.27	10.37
250	0.0	246.93	391.92	338.04	343.72	348.62	342.45	345.23
	0.2	371.03	423.24	405.98	345.27	413.86	415.83	413.69
	0.4	464.13	481.88	412.56	407.66	389.18	384.27	386.37
	0.6	232.40	266.72	297.99	278.21	290.99	296.32	294.98

	0.8	220.67	190.40	199.50	198.98	212.63	203.38	196.70
	1.0	124.37	141.16	110.15	138.55	137.13	131.07	132.77
	1.2	97.97	83.38	91.41	73.86	88.53	88.63	87.81
	1.4	55.60	72.24	52.62	62.20	58.57	58.33	60.06
	1.6	43.13	44.92	42.65	36.82	42.51	41.60	41.42
	1.8	19.80	31.80	26.29	28.78	31.52	29.32	30.10
	2.0	23.50	21.94	22.68	22.14	22.05	22.02	21.24
	2.2	17.93	14.36	17.95	14.01	16.09	15.96	15.82
	2.4	13.07	10.66	12.20	12.93	11.99	12.24	11.95
	2.6	10.30	10.82	8.17	9.56	9.51	9.21	9.40
	2.8	7.63	7.82	6.69	6.45	7.33	7.19	7.37
	3.0	5.23	4.46	5.57	6.05	5.51	5.89	5.91
1000	0.0	365.53	375.72	381.88	373.29	365.34	368.00	361.50
	0.2	337.23	393.14	384.32	402.64	375.56	369.73	379.75
	0.4	269.93	279.26	289.09	287.84	304.60	296.98	295.43
	0.6	205.87	147.46	164.85	196.88	208.16	206.19	201.87
	0.8	144.37	110.06	112.47	126.02	140.81	132.53	134.65
	1.0	58.40	99.22	97.39	84.51	82.34	85.75	86.86
	1.2	71.17	56.84	48.23	54.26	58.25	58.84	57.64
	1.4	40.17	44.64	49.34	37.00	38.87	39.85	39.37
	1.6	26.77	30.78	29.71	26.21	28.29	27.18	27.26
	1.8	18.73	24.42	21.98	19.56	19.56	19.18	19.63
	2.0	12.70	12.00	14.90	14.84	14.65	14.42	14.06
	2.2	9.17	10.02	9.47	10.80	10.66	10.60	10.61
	2.4	7.63	8.78	7.25	8.37	7.98	8.15	8.09
	2.6	6.40	6.02	5.53	6.28	6.00	6.25	6.34
	2.8	8.17	4.56	4.80	4.76	4.85	4.92	4.98
	3.0	3.87	4.68	3.76	4.08	4.14	3.99	4.05

Table 4.5: ARL values for different combinations of n, m, and δ for Generalized exponential distribution with $\alpha = 1$ and $\lambda = 1$

		Gen exponential distribution ($\alpha = 1, \lambda = 1, k = 3$)						
	m	30	50	100	200	1000	5000	10000
n	δ	ARL	ARL	ARL	ARL	ARL	ARL	ARL
5	0.0	106.57	114.36	99.50	101.45	104.40	183.40	105.04
	0.2	252.77	204.48	234.11	239.34	262.20	211.20	190.68
	0.4	416.23	494.10	420.98	434.98	325.20	314.00	402.22
	0.6	1411.97	1148.10	962.57	977.38	448.60	747.40	725.42
	0.8	1406.13	2345.84	2440.41	2334.66	1339.80	1883.00	1338.10
	1.0	3868.73	5116.26	4701.47	4675.22	8132.20	5173.60	2767.29
	1.2	10302.07	9411.78	11940.35	12023.78	10086.20	12801.20	6122.35
	1.4	34693.77	36037.08	27685.56	29786.16	8904.80	17122.00	11903.13
	1.6	66113.30	50624.82	63620.34	65784.76	56874.80	98676.40	27405.14
	1.8	196352.80	138891.56	143794.33	143794.33	44547.40	103272.60	49914.78
	2.0	222300.90	351273.92	311976.14	311976.14	392269.40	331488.80	116016.70

	2.2	744464.20	957516.70	735435.20	735435.20	1460825.20	600358.00	236364.74
	2.4	1908072.00	2031631.64	2044075.20	2044075.20	2102233.00	3262944.00	520972.51
	2.6	4888776.70	4773259.74	4553745.85	4553745.85	5923970.80	8493413.80	1089265.64
	2.8	11249500.00	9037212.52	12034520.02	12034520.02	17932466.00	7077457.00	2355726.34
	3.0	26110320.00	26646192.98	33560596.57	33560596.57	22635280.00	42264171.20	52451679.28
30	0.0	183.97	245.96	236.74	218.13	241.43	239.29	234.33
	0.2	479.13	438.64	444.36	508.05	460.27	454.35	451.87
	0.4	877.00	885.66	740.36	803.16	848.61	802.67	790.43
	0.6	1195.17	1013.94	1207.38	1242.31	1136.53	1106.87	1095.77
	0.8	714.07	1142.94	1083.88	1047.18	1078.83	1027.00	1063.55
	1.0	897.43	1037.72	713.67	753.82	777.04	767.94	784.93
	1.2	618.67	416.18	531.92	485.01	499.84	539.00	536.82
	1.4	417.33	404.16	308.47	359.02	353.82	349.47	341.52
	1.6	191.37	225.44	215.65	255.28	240.90	234.50	236.16
	1.8	227.87	200.42	171.98	144.57	169.52	166.59	168.31
	2.0	105.57	113.12	106.69	124.18	113.78	114.22	116.14
	2.2	77.87	76.12	89.58	78.58	80.93	81.17	83.22
	2.4	54.60	64.28	62.51	62.31	58.47	61.34	60.96
	2.6	52.43	45.08	39.93	50.30	45.02	45.44	46.00
	2.8	34.00	30.02	39.39	30.13	35.61	34.53	35.18
	3.0	22.57	22.46	26.06	24.61	27.79	27.19	26.41
50	0.0	197.50	302.74	244.78	277.77	270.86	273.46	276.59
	0.2	504.07	424.28	523.02	421.95	481.12	472.88	478.28
	0.4	869.83	647.48	719.11	606.50	672.19	662.96	654.28
	0.6	533.07	658.38	611.92	659.90	636.63	622.55	628.32
	0.8	627.73	473.16	498.07	431.91	478.74	465.30	467.53
	1.0	344.30	295.20	312.09	286.49	295.33	304.52	306.60
	1.2	139.17	239.50	183.73	194.34	202.88	198.44	199.66
	1.4	147.23	166.02	124.33	149.71	127.50	133.03	132.55
	1.6	76.40	73.58	73.81	84.33	90.61	89.47	87.82
	1.8	69.67	59.34	55.18	65.37	64.66	60.37	60.41
	2.0	45.33	53.34	47.44	36.09	40.76	42.75	42.10
	2.2	28.47	24.84	35.30	27.23	30.96	31.35	30.93
	2.4	19.17	20.10	20.06	21.91	23.02	23.06	23.21
	2.6	18.10	20.22	17.57	17.83	17.48	17.68	17.30
	2.8	15.77	13.98	13.61	14.00	13.64	13.40	13.18
	3.0	8.20	8.76	9.81	10.03	10.37	10.31	10.33
100	0.0	283.93	330.18	317.70	326.19	317.06	310.55	319.33
	0.2	583.00	399.40	478.44	465.63	439.85	443.89	466.29
	0.4	492.33	392.20	400.48	447.01	468.17	468.68	477.78
	0.6	335.03	406.84	362.45	326.62	356.98	354.65	358.14
	0.8	262.37	194.40	224.44	226.85	226.07	240.21	233.22
	1.0	143.43	158.20	159.25	157.64	140.94	148.40	145.58
	1.2	45.47	91.34	98.48	95.63	95.15	94.63	91.91
	1.4	67.77	51.32	53.57	58.66	60.97	60.34	62.16
	1.6	41.37	39.22	45.03	43.24	41.81	42.47	41.61
	1.8	21.33	24.54	25.41	30.64	28.57	28.54	29.03
	2.0	24.23	19.16	23.19	23.15	20.62	20.40	20.65
	2.2	17.77	14.04	15.24	13.39	14.66	14.80	15.02
	2.4	14.10	9.16	12.52	12.05	11.41	11.42	11.27
	2.6	7.03	9.82	8.30	8.36	8.61	8.51	8.61

	2.8	6.97	6.88	7.64	7.19	6.58	6.79	6.78
	3.0	6.00	5.70	4.69	5.21	5.27	5.40	5.41
250	0.0	345.13	312.12	332.32	323.28	342.77	351.75	343.14
	0.2	504.77	543.72	355.69	436.68	394.27	419.82	418.17
	0.4	338.27	317.18	367.32	352.75	370.38	337.00	344.44
	0.6	212.57	223.56	198.51	212.60	236.66	226.62	228.12
	0.8	131.27	161.52	133.81	131.03	154.09	138.29	141.61
	1.0	90.80	94.48	89.19	84.76	84.06	88.28	86.38
	1.2	61.10	51.48	59.12	49.43	56.39	55.64	55.76
	1.4	36.73	29.96	38.98	39.43	36.00	36.95	36.24
	1.6	31.53	19.74	23.28	25.31	22.98	25.25	24.61
	1.8	18.50	16.76	15.04	17.90	17.89	17.39	17.42
	2.0	12.27	11.02	12.16	13.16	12.25	12.60	12.39
	2.2	9.60	8.00	8.60	8.40	9.10	9.25	9.13
	2.4	7.03	6.46	7.31	6.99	7.09	6.90	6.90
	2.6	5.87	6.16	5.54	5.86	5.23	5.38	5.41
	2.8	5.40	3.66	4.02	3.88	4.08	4.19	4.21
	3.0	3.13	3.58	3.90	3.15	3.36	3.49	3.47
1000	0.0	357.67	334.24	367.48	372.86	366.32	361.55	367.91
	0.2	350.70	291.84	348.62	336.60	358.08	368.06	358.96
	0.4	289.13	262.30	222.60	251.34	256.10	263.72	260.51
	0.6	120.20	138.26	138.77	173.83	155.61	163.00	159.93
	0.8	87.07	117.12	84.47	99.46	95.89	98.71	97.30
	1.0	67.20	60.48	60.13	60.19	60.66	59.60	60.47
	1.2	44.00	48.80	42.57	36.72	38.38	38.45	38.84
	1.4	28.53	23.66	24.69	26.41	24.38	25.20	25.37
	1.6	23.37	20.30	16.15	17.44	17.89	16.64	17.16
	1.8	7.97	10.42	11.96	10.84	12.22	12.09	11.91
	2.0	6.40	7.02	8.35	8.83	8.49	8.38	8.58
	2.2	6.70	5.40	6.50	6.49	6.38	6.45	6.36
	2.4	5.93	4.18	4.59	5.32	4.76	4.85	4.90
	2.6	4.13	3.14	4.16	3.87	4.01	3.85	3.83
	2.8	2.77	3.36	2.93	3.26	3.02	2.99	3.12
	3.0	2.67	2.66	2.51	2.63	2.53	2.55	2.57

Table 4.6: ARL values for different combinations of n, m, and δ for Generalized exponential distribution with $\alpha = 1.5$ and $\lambda = 1$

		Gen exponential distribution ($\alpha = 1.5, \lambda = 1, k = 3$)						
	m	30	50	100	200	1000	5000	10000
n	δ	ARL	ARL	ARL	ARL	ARL	ARL	ARL
5	0.0	148.10	131.12	130.74	123.99	120.00	128.00	101.00
	0.2	250.83	310.06	263.90	251.92	286.20	206.20	256.00
	0.4	588.27	526.58	639.88	495.49	315.20	564.70	425.90
	0.6	1384.00	904.90	1026.81	1058.06	763.40	548.20	1018.00
	0.8	2259.63	2620.54	2072.91	2511.09	1254.00	665.50	2839.90

	1.0	4652.80	4760.82	5784.71	5147.64	4484.60	2941.70	3436.70
	1.2	8778.03	12175.88	11418.73	11576.09	17804.00	5149.30	5995.70
	1.4	19993.57	21770.10	19360.55	23350.65	26960.80	29397.50	22831.40
	1.6	28529.90	45140.90	48789.55	45955.65	61604.60	45653.20	37045.40
	1.8	117351.30	86712.74	116827.66	109482.86	156713.40	193534.30	88172.30
	2.0	252313.90	218589.34	266279.89	273353.74	420514.00	398315.00	318291.30
	2.2	736011.70	672610.80	609767.57	585457.73	913802.80	819659.70	1487524.10
	2.4	1590401.00	1408568.52	1392316.38	1374458.66	1711471.60	1083322.00	2270640.10
	2.6	2787383.00	2996861.64	3299257.55	3221497.26	4608160.40	4433888.00	6258963.30
	2.8	8472578.00	8448942.38	7295099.73	7505769.57	17949122.20	9779936.00	10109355.80
	3.0	18524710.00	16159501.42	19454961.11	18954374.11	41111504.60	15912400.00	29405318.00
30	0.0	224.90	212.94	213.83	259.15	256.93	253.21	256.58
	0.2	565.03	588.18	474.07	561.46	539.40	540.77	526.42
	0.4	747.97	860.48	751.64	727.44	755.90	754.96	780.11
	0.6	845.80	656.18	684.26	596.92	606.60	639.80	642.42
	0.8	287.40	376.92	397.47	364.01	387.68	394.44	385.45
	1.0	168.63	217.86	246.40	228.12	227.08	224.75	221.62
	1.2	121.17	152.92	136.32	136.75	127.45	133.16	129.74
	1.4	65.17	79.92	76.56	84.92	84.62	82.52	82.26
	1.6	51.17	57.94	55.54	55.53	50.98	51.83	51.16
	1.8	34.13	29.58	31.09	32.43	34.77	34.44	34.44
	2.0	25.33	25.10	23.55	24.77	26.08	23.66	23.49
	2.2	17.30	16.52	14.71	18.51	17.17	16.80	16.67
	2.4	12.93	9.30	11.54	11.28	11.69	12.31	12.21
	2.6	11.13	8.10	9.04	9.30	9.19	9.14	9.22
	2.8	7.43	6.54	6.90	7.74	7.14	7.11	7.14
	3.0	4.60	5.90	5.73	5.26	5.65	5.60	5.53
50	0.0	209.07	315.42	275.27	321.10	299.43	284.33	288.25
	0.2	475.83	445.44	491.10	580.73	509.26	510.86	506.63
	0.4	700.20	625.70	601.99	530.96	521.75	523.74	523.81
	0.6	404.53	372.78	326.03	293.05	336.00	344.15	341.15
	0.8	141.60	236.76	216.38	165.97	190.00	188.39	192.23
	1.0	109.77	102.40	103.67	116.65	115.81	109.73	111.74
	1.2	66.77	62.78	65.51	56.73	60.47	65.84	65.55
	1.4	42.10	41.42	37.21	44.03	40.38	39.17	41.13
	1.6	24.87	27.90	22.56	25.14	26.10	26.58	26.61
	1.8	14.43	14.24	20.36	18.05	17.71	17.40	17.49
	2.0	8.07	14.90	15.71	12.39	12.53	12.41	12.05
	2.2	10.73	8.38	9.83	8.67	9.02	8.94	8.75
	2.4	6.13	6.98	7.43	6.36	6.45	6.49	6.52
	2.6	4.23	4.60	5.25	5.40	5.00	4.98	5.04
	2.8	2.90	3.68	4.96	4.09	3.95	4.00	3.92
	3.0	3.53	3.32	3.59	3.06	3.29	3.25	3.28
100	0.0	325.10	300.06	339.86	328.37	331.49	319.90	325.79
	0.2	397.13	502.16	458.02	444.12	441.62	455.88	455.43
	0.4	299.90	269.12	325.84	407.95	360.19	356.57	355.52
	0.6	225.30	201.28	206.59	206.28	214.56	198.09	204.80
	0.8	111.60	113.02	105.07	107.33	110.86	108.34	111.94
	1.0	70.90	61.24	74.88	62.45	62.46	63.66	63.54
	1.2	42.87	32.62	40.24	35.85	36.21	37.53	38.16
	1.4	28.03	22.42	24.78	23.81	23.76	24.01	23.66

	1.6	17.07	15.62	17.61	15.78	15.12	15.07	15.09
	1.8	11.43	10.34	11.13	9.21	10.32	10.52	10.31
	2.0	9.90	6.80	8.68	7.16	7.29	7.48	7.16
	2.2	5.70	6.26	5.61	5.03	5.66	5.36	5.34
	2.4	4.07	4.72	3.57	3.45	3.99	4.07	4.08
	2.6	3.13	3.16	3.15	3.14	3.30	3.24	3.19
	2.8	2.87	2.96	2.66	2.30	2.55	2.58	2.58
	3.0	2.10	2.22	2.04	2.10	2.14	2.16	2.17
250	0.0	346.87	336.50	301.13	343.78	342.05	346.23	350.79
	0.2	392.27	392.56	401.14	362.58	403.92	390.93	383.80
	0.4	206.37	220.18	224.09	255.90	234.20	249.12	247.38
	0.6	163.37	150.62	121.99	125.80	135.81	137.02	137.71
	0.8	69.80	86.62	71.73	78.14	74.42	73.77	73.80
	1.0	32.30	47.86	45.81	44.08	39.80	42.28	41.72
	1.2	27.77	20.30	25.22	26.23	25.79	25.05	24.66
	1.4	19.37	17.44	14.82	15.58	15.20	15.92	15.30
	1.6	10.47	10.82	9.53	11.32	10.09	10.39	10.37
	1.8	7.27	7.10	7.65	7.32	6.72	6.86	6.99
	2.0	4.83	4.16	6.07	5.02	5.19	5.05	5.03
	2.2	3.10	3.56	3.59	4.08	3.78	3.75	3.71
	2.4	2.63	2.62	2.75	3.11	2.88	2.91	2.89
	2.6	2.57	2.16	2.58	2.29	2.29	2.33	2.35
	2.8	1.70	1.80	2.00	1.80	1.95	1.94	1.93
	3.0	1.37	1.68	1.59	1.67	1.66	1.68	1.68
1000	0.0	334.37	347.36	361.24	376.98	367.54	365.05	362.75
	0.2	333.33	249.40	301.07	366.13	307.70	328.20	325.07
	0.4	154.33	168.64	206.33	197.36	195.54	198.11	192.43
	0.6	81.10	89.62	106.84	112.84	105.54	100.84	102.27
	0.8	67.07	49.34	57.68	57.20	54.10	54.52	54.30
	1.0	20.93	32.90	29.13	29.98	30.82	31.41	31.00
	1.2	15.00	15.64	16.87	17.01	18.18	18.49	18.28
	1.4	10.00	10.18	12.51	9.54	11.50	11.44	11.40
	1.6	6.40	7.56	8.07	7.91	7.31	7.48	7.59
	1.8	4.80	4.98	5.40	5.15	5.44	5.28	5.25
	2.0	5.03	3.66	3.69	3.51	3.66	3.80	3.74
	2.2	3.63	3.24	2.90	2.77	2.94	2.90	2.87
	2.4	1.90	2.04	2.50	2.14	2.37	2.25	2.30
	2.6	1.80	2.04	2.17	1.85	1.84	1.89	1.88
	2.8	1.53	1.60	1.55	1.54	1.65	1.62	1.60
	3.0	1.80	1.50	1.33	1.45	1.45	1.42	1.41

4.1 Weibull with $\gamma = 0.8$, $\lambda = 1$, and $k = 3.6$

Table 4.1 lists the ARL values obtained from a Weibull distribution with $\gamma = 0.8$ and $\lambda = 1$ and threshold value $k = 3.6$, to explore different combinations of the number of observations (n) and number of simulations (m) under various δ

scenarios. The ARL values serve as a critical metric in assessing the performance of a control chart, particularly in its ability to detect shifts in the process mean. Ideally, for IC scenarios, the ARL should closely align with the target value of 370. Deviations from this target indicate the chart's sensitivity to changes in the process. Table 4.1 examines different number of simulations (30, 50, 100, 200, 1000, 5000, 10000) and δ values for a given number of observations $n = 5$. Under ideal conditions, $\delta = 0$, the ARL values for simulations 30, 50, 100, and 200 are 88.67, 84.36, 84.44, and 78.73, respectively, indicating the expected number of observations before a signal is generated. When the shift increases to 0.2, the ARL values decrease to 45.97, 40.52, 45.69, and 48.29, signifying increased sensitivity of the control chart to detect shifts in the process mean. With $\delta = 1.0$, the ARL values decrease further (10.13, 11.34, 10.44, 9.40). For a δ of 2.0, the ARL values remain relatively consistent (3.60, 3.92, 3.83, 4.19), indicating sustained sensitivity. A δ of 3.0 results in varied ARL reductions (2.93, 2.28, 2.11, 2.59), demonstrating the evolving ability to detect δ.

Considering $n = 30$ and $\delta = 0$, the ARL values for the respective number of simulations are 268.30, 169.90, 209.39, and 175.38. As δ increases to 0.2, the ARL values decrease to 108.77, 85.66, 107.89, and 101.88, indicating an increased sensitivity of the control chart to detect shifts in the process mean. A larger δ of 1.0 results in further ARL reductions (19.60, 17.68, 18.36, 17.97), underscoring heightened sensitivity. For δ 2.0, the ARL values exhibit variations (4.83, 4.30, 5.22, 4.84), signifying sustained sensitivity. A δ of 3.0 leads to additional ARL reductions (2.57, 2.72, 2.11, 2.14), highlighting the evolving ability to detect δ.

When the system is under control $\delta = 0$, the ARL values for number of simulations 30, 50, 100, and 200 with $n = 50$, are 202.37, 232.56, 213.67, and 246.02, respectively. As δ 0.2, the ARL values decrease to 148.50, 117.58, 132.77, and 146.56, suggesting an increased sensitivity of the control chart to identify δ in the process mean. With δ 1.0, the ARL values further decrease (24.23, 22.64, 20.25, 20.70), underscoring heightened sensitivity. A δ of 2.0 results in additional ARL reductions (6.07, 4.78, 5.12, 4.57), emphasizing sustained sensitivity. A δ of 3.0 leads to further ARL decreases (2.60, 2.26, 2.43, 2.46), highlighting the evolving capability to detect δ.

For $\delta = 0$, the ARL values for $n = 100$ and number of simulations 30, 50, 100, and 200 are 247.37, 271.74, 262.52, and 264.29, respectively. As the δ increases to 0.2, the ARL values decrease to 194.77, 138.44, 165.20, and 173.34, indicating an increased sensitivity of the control chart to identify δ in the process mean. With a large δ of 1.0, ARL values further decrease (32.70, 17.70, 26.88, 24.91). For a δ of 2.0, the ARL values continue to decrease (6.90, 5.92, 5.79, 5.77), highlighting sustained sensitivity. A δ of 3.0 leads to further ARL decreases (2.03, 2.20, 2.32, 2.13), emphasizing the evolving capability to detect δ. Considering $n = 250$ and $\delta = 0$, the ARL values for number of simulations 30, 50, 100, and 200 are 287.93, 299.74, 348.01, and 355.79, respectively. As the δ increases to 0.2, the ARL values decrease to 215.23, 215.78, 189.58, and 183.11, indicating an increased sensitivity of the control chart to detect δ in the process mean. With a larger δ of 1.0, the ARL values further decrease (28.53, 40.00, 26.49, 32.55), emphasizing increased sensitivity. For a δ of 2.0, the ARL values exhibit variations (5.30, 4.76, 6.50, 6.34), highlighting sustained sensitivity. A δ of 3.0 leads to further ARL decreases (2.33, 2.34, 2.15, 2.44).

Increasing the n to 1000 results in ARL values surpassing those observed with n=250 because as (n) to 1000 the behavior of the control chart tends to approximate normality. This is indicative of the central limit theorem, which states that as the number of simulations becomes sufficiently large, the distribution of sample means approaches a normal distribution. For $\delta = 0$, the ARL values for number of simulations 30, 50, 100, and 200 are 371.53, 366.42, 369.37, and 367.26, respectively. As δ increases to 0.2, the ARL values decrease to 235.07, 255.20, 259.48, and 226.98, indicating an increased sensitivity in the control chart's ability to detect δ in the process mean. With δ 1.0, the ARL values further decrease (37.60, 42.26, 36.42, 36.87). A δ of 2.0 introduces variations in ARL values (5.80, 8.50, 7.08, 6.51). A δ of 3.0 leads to further ARL decreases (2.60, 2.74, 2.37, 2.42). Notably, with $n = 1000$, ARL values approach the target of 370, suggesting that the larger number of simulations contributes to a more balanced control chart in terms of sensitivity and stability. As the δ magnitude increases, the ARL values consistently decrease, indicating an enhanced sensitivity of the control chart to out-of-control situations.

4.2 Weibull with $\gamma = 1$, $\lambda = 1$, and $k = 3$

Table 4.2 displays the ARL results corresponding to a Weibull distribution with $\gamma = 1$, $\lambda = 1$, and a threshold value of $k = 3$. These ARL values are computed for diverse combinations of observation numbers n and number of simulations m across different shift (δ) scenarios. A δ of 0.00 signifies the ARL values IC, reflecting the system's performance under stable conditions.

The control chart exhibits the ARL values of 132.63, 114.12, 103.83, and 110.97 for simulations of 30, 50, 100, and 200, respectively, for $\delta = 0$ and n=5. As the shift magnitude increases beyond 0, the ARL values significantly decrease, indicating heightened sensitivity in detecting shifts in the process mean. For instance, with a δ of 1.00, the ARL drops to 10.00, 11.24, 11.85, and 9.47 across the respective sample sizes, signifying a quicker response to OCC conditions. Subsequent increases in δ magnitude ($\delta > 1.00$) further reduces the ARL values, reinforcing the control chart's ability to swiftly identify deviations from the IC state. This heightened sensitivity is evident from the ARL values of 4.57, 3.82, 4.13, and 4.32 for $\delta = 2.00$, and 2.37, 2.42, 2.18, and 2.32 for a $\delta = 3.00$. As observed previously, when δ is at 0, indicating the IC state, the ARL values stand at 303.60, 267.32, 265.16, and 249.05 for m equals to 30, 50, 100, and 200, respectively. These figures imply that, on average, a considerable number of samples would be needed before an OOC signal is triggered. By increasing δ magnitudes ($\delta > 0$), the ARL values decline, signifying heightened sensitivity of the control chart in detecting δ in the process mean. For instance, $\delta = 1.00$ results in the ARL values of 16.23, 21.00, 17.69, and 18.46 across the corresponding m values, indicating a quicker response to OOC conditions. Subsequent increases in δ magnitude ($\delta > 1.00$) lead to further reductions in the ARL values, underscoring the chart's adeptness in promptly identifying deviations from the IC state. This heightened sensitivity is evident from the ARL values of 3.57, 4.28, 4.37, and 4.80 for a δ of 2.00, and 1.77, 2.14, 2.03, and 2.15 for $\delta = 3.00$, aligned with the respective m values. Overall, the control chart with a sample size of 30 demonstrates effective performance in quickly signaling changes in the process mean as the δ magnitude increases, underscoring its utility in the process control.

For n=50, the IC ($\delta = 0$) is characterized by the ARL values of 266.20, 263.36, 262.63, and 284.09 for m values of 30, 50, 100, and 200, respectively, indicating a need for a substantial number of samples on average before triggering an OOC signal. As the shift magnitude increases ($\delta > 0$), the ARL values decrease, signifying an augmented sensitivity of the control chart in detecting δ in the process mean. For instance, with $\delta = 1.00$, the ARL values decrease to 17.47, 21.46, 21.27, and 21.47 across the corresponding m values, suggesting a swifter response to OOC conditions. Further increments in shift magnitude ($\delta > 1.00$) result in varied ARL values, emphasizing the chart's capacity to promptly identify deviations from the IC state. This enhanced sensitivity is evident from the ARL values of 3.10, 4.52, 4.64, and 4.84 for $\delta = 2.00$, and 2.17, 1.98, 2.11, and 2.10 for $\delta = 3.00$, aligned with the respective m values.

For $n = 100$, the IC case ($\delta = 0$) has the ARL values of 332.23, 362.00, 298.90, and 294.69 for $m = 30$, 50, 100, and 200, respectively, suggesting that, on the average, a substantial sample size is necessary before an OOC signal is triggered. By increasing shift magnitudes ($\delta > 0$), the ARL values decrease, indicating an increased sensitivity of the control chart to detect shifts in the process mean. For instance, with a δ of 1.00, the ARL values decrease to 19.80, 20.88, 23.63, and 25.38 across the corresponding m values, signifying a swifter response to OOC conditions. Subsequent increases in the shift magnitudes ($\delta > 1.00$) result in diverse ARL values, underscoring the chart's adeptness in promptly identifying deviations from the IC state. This heightened sensitivity is evident in the ARL values of 4.40, 5.26, 5.81, and 5.22 for a δ of 2.00, and 2.57, 2.40, 2.10, and 2.03 for a δ of 3.00, aligned with the respective m values.

Assuming $n = 250$ with $\delta = 0$ the ARL values are 330.40, 334.36, 268.43, and 320.10 for $m = 30$, 50, 100, and 200, respectively. These values suggest that, on the average, a substantial number of samples would be necessary before triggering an OOC signal when the process is IC. With increasing shift magnitudes ($\delta > 0$), the ARL values decrease, accentuating the heightened sensitivity of the control chart in detecting shifts in the process mean. For instance, with $\delta = 1.00$, the ARL values decrease to 38.90, 22.86, 30.80, and 30.85 across the corresponding m values, indicating a swifter response to OOC conditions. Subsequent increments in shift magnitude ($\delta > 1.00$) result in diverse ARL values, emphasizing the chart's proficiency in promptly identifying deviations from the IC state. This enhanced sensitivity is evident in the ARL values of 5.37, 5.16, 6.32, and 6.01 for $\delta = 2.00$, and 2.40, 2.32, 1.97, and 2.01 for $\delta = 3.00$.

A control chart with $\delta = 0$ and $n = 1000$ has the ARL values of 342.77, 361.94, 370.43, and 362.74 for $m = 30$, 50, 100, and 200, respectively, indicating the need for a substantial number of samples on average before an OOC signal is generated. The diminishing ARL values with increasing shift magnitudes ($\delta > 0$) underscore the heightened sensitivity of the control chart in detecting shifts in the process mean. With $\delta = 1.00$, the ARL values decrease to 39.57, 35.00, 36.14, and 36.76 across the corresponding m values, indicating a swifter response to OOC conditions. Further in shifts magnitude ($\delta > 1.00$) result in diverse ARL values, highlighting the chart's proficiency in promptly identifying deviations from the in-control state. As the number of observations n approaches 1000, the ARL values for the IC scenario converge to the target value of 370, emphasizing the chart's effectiveness in achieving the specified ARL target with $n = 1000$.

4.3 Weibull with $\gamma = 1.5$, $\lambda = 1$, $k = 2.33$

Table 4.3 also tabulates the results for the Weibull distribution with $\gamma = 1.5$ and $\lambda = 1$, delineate the inherent features of the process distribution. A threshold value of $k = 2.33$ indicates that the process is considered OOC if one consecutive points surpass the control limits. Different number of simulations (30, 50, 100, 200, 1000, 5000, 10000) are examined to observe their influence on the ARL values. In the absence of a shift ($\delta = 0$) and with $n = 5$, the ARL values for sample sizes 30, 50, 100, and 200 are 193.90, 220.02, 190.77, and 202.37, respectively. The presence of a shift ($\delta > 0$), such as a $\delta = 1.00$, results in significantly reduced ARL values (9.70, 11.30, 10.19, 9.99), indicating increased sensitivity in detecting shifts in the process mean. Exploring further, a $\delta = 2.00$ leads to continued ARL reduction (3.43, 2.70, 3.34, 3.39), highlighting the increased sensitivity of the control chart and for $\delta = 3.00$, the ARL values further decreased (1.83, 1.52, 1.90, 1.66), emphasizing the enhanced ability to detect shifts. The primary objective is to achieve a target ARL of 370, potentially requiring adjustments in sample sizes to maintain an optimal balance between sensitivity and stability.

When the process is under control ($\delta = 0$) with $n = 30$, the ARL values are 214.13, 267.22, 241.56, and 320.85 for sample sizes 30, 50, 100, and 200, respectively. As δ increases to 1.00, there is a significant decrease in the ARL values to 16.73, 17.58, 14.53, and 16.18, indicating increased sensitivity of the control chart to detect shifts in the process mean. For $\delta = 2.00$ the ARL values continued to decrease (3.70, 2.78, 3.44, 3.81), underscoring the heightened sensitivity of the control chart and for $\delta = 3.00$, there are further reductions in the ARL values (1.77, 1.40, 1.59, 1.62), emphasizing the enhanced ability to detect shifts. For $n = 50$ and $\delta = 0$, the ARL values for m 30, 50, 100, and 200 are 252.33, 367.98, 314.41, and 312.59, respectively. As the δ increases to 1.00, there is a noticeable decrease in the ARL values to 17.17, 20.38, 20.39, and 19.89, indicating an increased sensitivity of the control chart to detect shifts in the process mean. With a δ of 2.00, the ARL values continue to decrease (4.27, 3.26, 4.00, 3.41), emphasizing the heightened sensitivity of the control chart. A δ of 3.00 leads to further reductions in the ARL values (1.37, 1.80, 1.56, 1.54), showcasing the enhanced ability to detect shifts.

When the number of observations become 100 and there is no shift in the process (i.e., $\delta = 0$), the ARL values for number of simulations 30, 50, 100, and 200 are 252.53, 285.76, 343.36, and 288.67, respectively. As δ increases to 1.00, the ARL values decrease to 19.07, 17.48, 18.11, and 19.48, indicating an increased sensitivity of the control chart to detect shifts in the process mean. With $\delta = 2.00$, the ARL values continue to decrease (3.80, 3.98, 3.63, 3.63), highlighting the heightened sensitivity of the control chart. For $\delta = 3.00$ results in further reductions in the ARL values (1.53, 1.32, 1.51, 1.51), demonstrating the enhanced ability to detect shifts. The primary objective remains achieving a target ARL, by adjusting sample sizes necessary to maintain the desired balance between sensitivity and stability.

With $n = 250$ and $\delta = 0$, the ARL values for number of simulations 30, 50, 100, and 200 are 368.30, 342.90, 351.17, and 323.11, respectively. These values signify the anticipated number of observations before a signal is generated. As δ increases to 1.00, there is a decrease in the ARL values to 30.90, 21.16, 22.04, and 23.92, indicating an increased sensitivity of the control chart to detect shifts in the process mean. With $\delta = 2.00$, the ARL values continue to decrease (3.33, 3.38, 3.60, 3.58), underscoring the heightened sensitivity of the control chart. A δ of 3.00 leads to further reductions in the ARL values (1.37, 1.42, 1.64, 1.51), highlighting the enhanced ability to detect shifts. With $n = 1000$ and the process IC ($\delta = 0$), the ARL values for sample sizes 30, 50, 100, and 200 are 328.17, 321.28, 371.30, and 370.37, respectively. As δ increases to 1.00, there is a decrease in the ARL values to 26.07, 30.54, 24.69, and 24.20, indicating heightened sensitivity of the control chart to detect shifts in the process mean. With $\delta = 2.00$, the ARL values continue to decrease (2.87, 3.96, 3.52, 3.92), emphasizing the increased sensitivity of the control chart. A δ of 3.00 leads to further reductions in the ARL values (1.33, 1.56, 1.50, 1.42), showcasing the enhanced ability to detect shifts. Thus, as the number of observations approaches 1000, we observe that our primary goal of achieving the target IC ARL of 370 is accomplished.

4.4 Results for the generalized exponential distribution

Tables 4.4, 4.5 and 4.6 present the ARL values for the generalized exponential distribution with shape parameter values of 0.8, 1, 1.5, and a rate of 1. The combinations of number of simulations (m) and numbers of observations (n) are varied, and the ARL values are calculated for different shift scenarios. The goal is to achieve an IC ARL of 370, the threshold value (k) is set at 3 for all parameter combinations to maintain a consistent target ARL. The IC ARL at shift value zero acts as the baseline, showing the control system's performance while the process is operating within normal norms. The ARL values diverge from this baseline as the shift size rises, reflecting the system's ability to recognise and adapt to OOC situations. The results reveal that the number of simulations (m) influences the ARL values. Smaller number of simulations, such as $m = 30$, exhibit greater unpredictability, particularly in cases with larger shifts, reflecting a potential reduction in sensitivity to process changes. As the number of simulations to $m = 50$, there is a trend towards increased ARL values, indicating better detection of minor disturbances but potential concerns with larger shifts. When the number of simulations is increased to $m = 100$, the ARL values increase across the board, indicating a decreased sensitivity to shifts. A larger number of simulations of $m = 200$, on the other hand, yields lower ARL values, indicating better responsiveness to process changes, especially for larger adjustments.

By examining the results, it is evident that the IC ARL is influenced by the number of observations (n). For smaller values of n, the IC ARL tends to be lower, reflecting the system's sensitivity to detecting shifts. However, as n increases, the IC ARL tends to rise, suggesting that the system becomes less responsive to smaller shifts. When n is set to 5, the IC ARL is relatively low, but as n increases to

50, the ARL values also rise, indicating a reduced sensitivity to shifts. The trend continues as n further increases to 100 and 250, with the IC ARL values showing a gradual increase. However, using a large number of observations $n = 1000$, we successfully achieve our goal of $ARL = 370$. This means that the system becomes really good at keeping things in check and quickly spotting any differences from the normal state. Having a big dataset with $n = 1000$ gives the control system the ability to notice even small variations in the process, and as a result, the IC ARL closely matches our target of 370. It's like having a super sharp eye on the process, making sure everything stays on track and responding fast to any changes. The shift values in the table represent the magnitude of the deviation from the baseline, indicating the size of the disturbance introduced to the system. As the system grows more adept at identifying and responding to substantial changes in the process, larger shift values result in shorter ARLs. However, minor shift values, on the other hand, result in larger ARLs, implying a slower response to minor shocks.

To reach the desired IC ARL of 370, a compromise between simulations and number of observations must be struck. As indicated by the stabilising trend in ARL values, raising the number of observations (n) to 1000 appears to create a favourable situation for attaining the aim.

4.5 Comparison of results for Weibull and Generalized exponential distributions

The ARL values for the generalized exponential distribution with γ parameters of 0.8, 1, and 1.5 are examined at a λ of 1. Various number of simulations (m) and number of observation (n) combinations are examined, with the goal of achieving a consistent IC ARL of 370 with a fixed threshold value (k) set at 3 across all parameter combinations. Notably, when n was set to 1000, all parameter combinations with $k = 3$ consistently reached the goal ARL of 370.

A comparison investigation utilising the Weibull distribution with γ parameters of 0.8, 1, and 1.5 and a λ of 1 found that adjustments to the threshold value (k) were required for each Weibull distribution parameter combination to meet the same ARL target of 370. A threshold value of $k = 3.6$ was required for $\gamma = 0.8$ and $\lambda = 1$. When n was set to 1000, a threshold value of k=3 is found to be effective for $\gamma = 1$ and $\lambda = 1$, and a threshold value of $k = 2.33$ was found to be successful for $\gamma = 1.5$ and $\lambda = 1$.

This comparison illustrates an intriguing aspect regarding the two distributions' performance. When $n = 1000$, the generalized exponential distribution demonstrated robustness in meeting the desired ARL of 370 by maintaining a consistent $k = 3$. The Weibull distribution, on the other hand, needs multiple threshold values (k) to achieve the same ARL aim for different shape parameters. Although both distributions were capable of obtaining the desired ARL, the generalized exponential distribution performed more consistently as its convergence to centra limits theorem is independent of the shape parameter. This feature facilitates control system construction by maintaining a consistent threshold value. The Weibull distribution, on the other hand, while fulfilling the goal, required nuanced adjustments to the threshold value dependent on the individual shape parameter, potentially adding complexity to the management method. Figure 4.1 depicts the ARL values for the Weibull distribution with different shape parameters. From the figure, it is clear that as number of observations (n) increase the ARL values also approaches to the desired ARL value, especially when n approaches to 1000. Similarly, Figure 4.2 shows the ARL values for the generalized exponential distribution with different shape parameters and we reached the same conclusion as the case of Weibull distribution. Next, Figures 4.3 and 4.4 depict the ARL values for the Weibull and generalized exponential distributions with varying shifts (δ). Assuming $n = 1000$ with $ARL_0 = 370$, as δ increases the ARL values continuously decrease and we conclude that there is decreasing trend in the ARL values. Last two figures 4.5 and 4.6 are also shows the ARL values for the Weibull and generalized exponential distributions with varying shifts (δ) and varying simulations m. Assuming $n = 1000$ with $ARL_0 = 370$, as δ increases the ARL values continuously decrease.

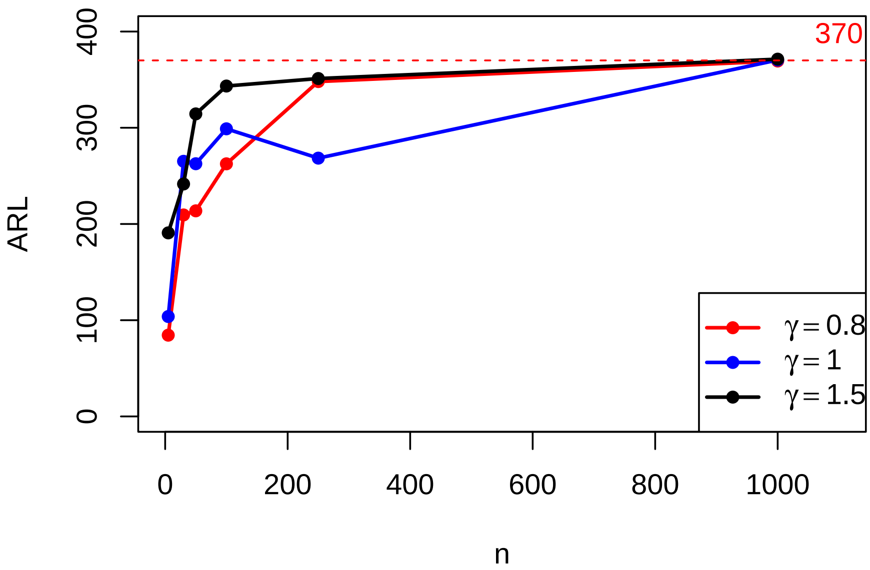

Figure 4.1: ARL values for Weibull distribution and there is increasing trend in ARL values. The target ARl is 370 and we obtain the targeted ARL when n approaches to 1000.

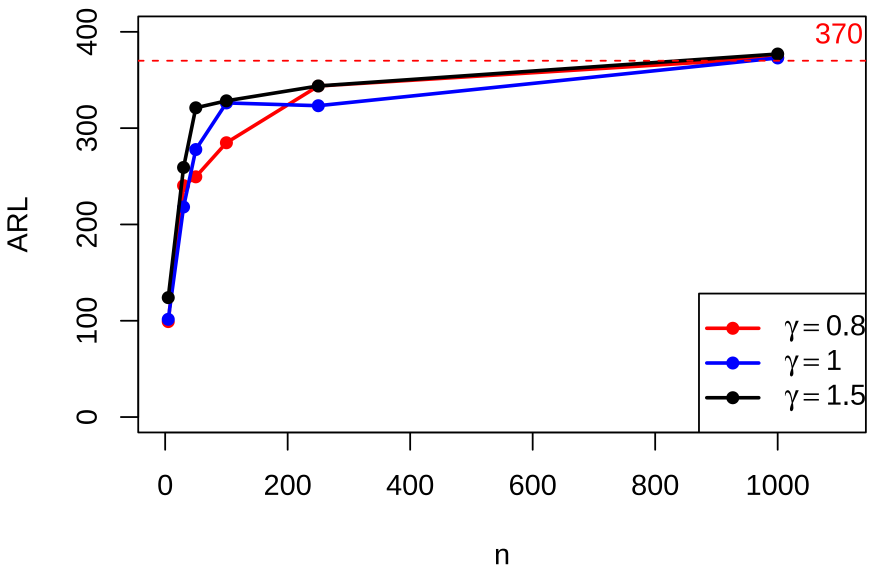

Figure 4.2: ARL values for the generalized exponential distribution and the desired ARL is achieved as n approaches to 1000.

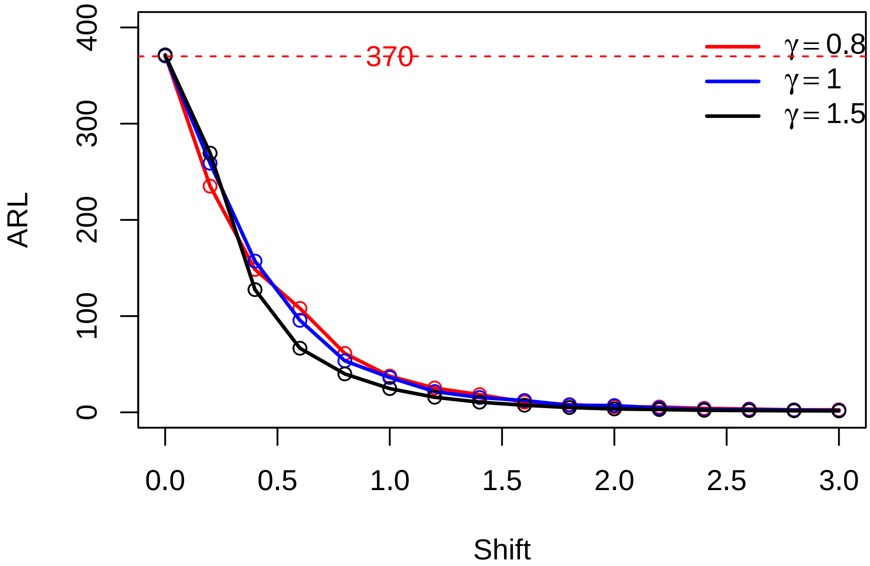

Figure 4.3: ARL for the Weibull distribution with varying shifts

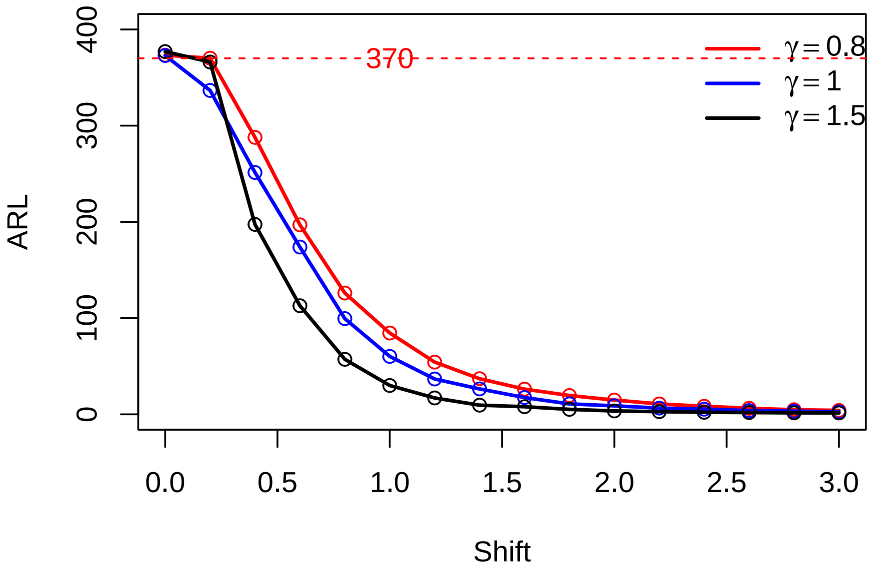

Figure 4.4: ARL for the generalized exponential distribution with varying shifts

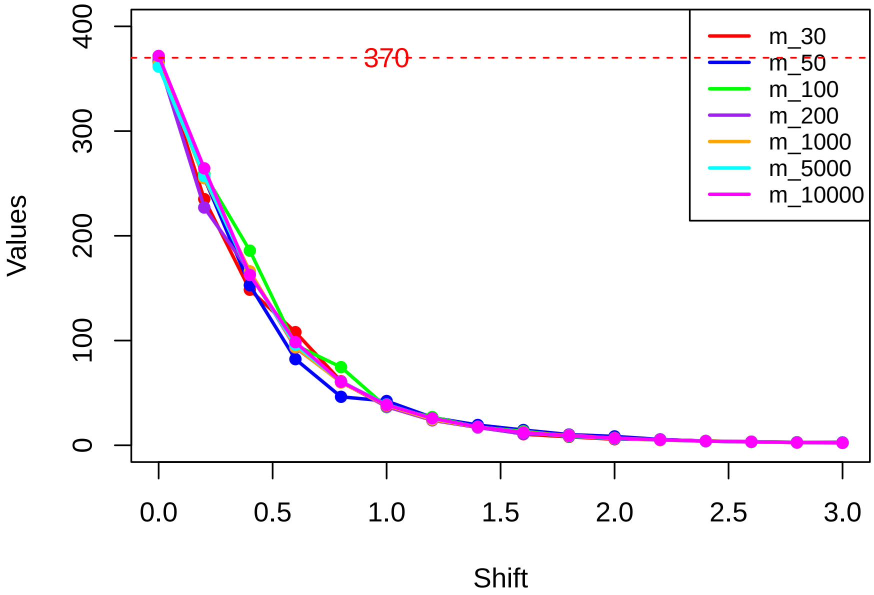

Figure 4.5: ARL for the Weibull distribution with varying shifts and number of simulations m

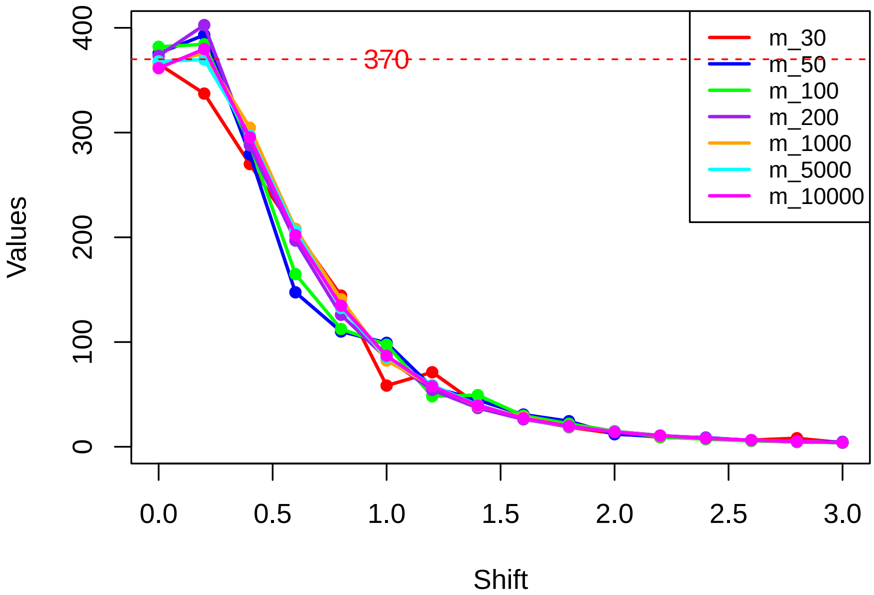

Figure 4.6: ARL for the generalized exponential distribution with varying shifts and number of simulations

Chapter 5

Conclusion and Recommendations

This thesis conducted a comprehensive exploration into the effectiveness of Shewhart control charts for monitoring mean when applied to non-normally distributed data with a specific focus on the ARL as a pivotal metric for evaluating control chart efficiency. Utilizing simulation techniques, the study validated the precision and reliability of ARL under normal approximation and investigated how non-normal data impacts the performance of Shewhart control charts.

The investigation particularly delved into the behavior of the $\overline{x}$ control chart when constructed with non-normal datasets, considering diverse setups of number of observation (n) and number of simulations (m). Three non-normal distributions Weibull, exponential, and generalized exponential are systematically analyzed, with parameters varied to assess their influence on control chart performance. A noteworthy observation from the study emphasized the robustness of the generalized exponential distribution, consistently achieving the target ARL of 370 with a fixed threshold value ($k = 3$) across various parameter combinations when n is set to 1000. In contrast, the Weibull distribution necessitated nuanced adjustments to the k to meet the same ARL target for different shape parameters. Despite both distributions demonstrating the ability to reach the target ARL, the generalized exponential distribution exhibited more uniform and simplified performance, enhancing ease of implementation in a control system. Hence, these findings underscore the susceptibility of control chart behavior to factors such as number of simulations, distribution type, and parameter values. The study underscoring the significance of understanding distribution characteristics and their potential impact on control chart performance. Ultimately, this research contributes to the advancement of statistical process control methodologies by providing guidance on optimizing control chart configurations for monitoring non-normally distributed processes.

Future efforts should focus on applying same technique on other non-normal distributions. Furthermore, more efforts should also focus on standardized control chart parameter, implement of this technique on real case studies, and emphasising continuous monitoring and adaptation. These recommendations seek to further the subject of statistical process control by facilitating more effective and efficient monitoring of non-normal distributed processes.

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