Study of Whistler Waves in Nonthermal Magnetized Plasma

By

Abul Hasan

DEPARTMENT OF PHYSICS
QUAID-I-AZAM UIVERSITY
ISLAMABAD

2022

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The thesis is submitted in partial fulfillment of the requirements for the award of degree of

MASTER OF PHILOSOPHY
IN
PHYSICS

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2022

Declaration

This is to certify that the work presented in the thesis titled **"Study of Whistler Waves in Nonthermal Magnetized Plasma"** was carried out by **Mr. Abul Hasan** under our supervision and is accepted in its present form by the Department of Physics, Quaid-i-Azam University, Islamabad as satisfying the dissertation requirement of the degree of

MASTER OF PHILOSOPHY IN PHYSICS

Supervisor

Prof. Dr. Arshad M. Mirza(S.I)
Department of Physics
Quaid-i-Azam University
Islamabad, Pakistan.

Submitted through

Prof. Dr. Nawazish Ali Khan
Chairman
Department of Physics
Quaid-i-Azam University
Islamabad, Pakistan.

Dedicated TO My beloved
AMMI, ABU,BROTHER and SISTERS
Whose affection is reason of every success in my life

Acknowledgement

All the acclamations and glories are for ALMIGHTY ALLAH, the Creator of all things and only worthy to be worshiped. Countless salaams for the Holy Prophet MUHAMMAD (PEACE BE UPON HIM), whose teaching has guided us through darkness.

My immense gratitude goes to my honorable supervisor Prof. Dr. Arshad M. Mirza for their expert guidance, ideas, inspiration, motivation and keen interest in my research work. Furthermore I thank to Prof. Dr. Nawazish Ali Khan (Chairman Department of Physics) for providing the facilities for research work. I wish to record here with immense pleasure, my gratitude to my Ammi and Abu whose prayers, sacrifices and love has provided me inspiration to complete this thesis. I really have no words to thanks all my family members, my brother Abu Sufyan and my sisters Kausar, Zubaida, Marhama and Tayyaba.

I thank to my dear friend M.Qasim sardar(my roommate) for his frequent support in every way he can. I also thank to my friend and roommate Hasnain with whom I have spent a memorable time. I am also thankful to my friends Ali Kazmi, Asad Joya, Sareer Hassan, Asad Munir , M.Mehak M.Adnan, Zeeshan , Ismail khan , Shamsher, Khawar, M.irfan and Aqil zada with whom I enjoyed a lot. I am also thankful to my Ph.d colleagues Dr.Muhammad Usman Malik, Dr.Gul-e-Ali, Dr.Usman Hasan, M.Irfan Bajwa and Dr.Abdul Rehman(khucha) for their intellectual company and expert guidance.

I am also thankful to the staff of the Department of Physics, Quaid-i-Azam University, Islamabad, especially Mr. Muhammad Sadiq, Mr. Ijaz Ahmad, Mr.Ibrahim , Mr.Yaseen, Mr.Akhlaq Ahmad, Mr.Faisal, Mr.Muhammad Kamran and Mr.Abdul Ghaffar for their continuous cooperation all the time during my research.

ABUL HASAN

Abstract

In this study, we used the kinetic theory of plasmas to analyse Whistler waves propagating parallel to the background magnetic field with electron temperature anisotropy. The Bi-Maxwellian and Cairns distributions were used to determine the perturbed distribution function. It was used to derive a dispersion relation for Oblique whistler waves. These distributions have been found to greatly alter the instability condition for the growth of oblique whistler instability. The dispersion characteristics and growth rate were also compared to the Maxwellian distribution.

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Chapter 1

Introduction

1.0.1 what is plasma?

As we are fimiliary with the states of matter i-e solid liquid and gas. We are also fimiliary with the properties of these states like the atoms of solid states are tightly bonded and therefor have a fixed shape and the atom of solid can't easily move. Due to fixed arrangement of atoms position, the atoms only vibrate(to and fro motion) which is know as simple harmonic motion.

While in liquid state, atoms or molecules in a liquid state can freely move because they have a significant amount of energy that can overcome potential energies due to mutual interactions. However, molecules can not leave the surface of liquid because at the surface they experience a barrier or large potential energy which they can not overcome. Similarly the third form of matter is gaseous state. In gaseous state the atoms/molecules are at large distance from each other as compared to the liquid state. These atoms/molecules have sufficient amount of kinetic energy and can move freely. That's why we need a container or vessel for gaseous state.

These states are interconvertable i-e solid state changes into liquid by increasing the temperature of solid at certain temperature(at melting point) convert into liquid. Similarly liquid state changes into gaseous state by increasing the the temperature of liquid at certain temperature(at boiling point) convert into gas.

Now the question arises that "what will happen when we heat or increase the temperature of gaseous state?". So the answer to this question is when we heat the gaseous state of matter at certain temperature ($10,000K$ or above) the atoms of gaseous state get ionized by producing ions and free electrons. So at this temperature or above it we get a new state of matter which we called a fourth state of matter that is Plasma.

So, by heating method we can change the solid state into liquid, liquid into gas and gas into plasma. So if we want a Plasma state a huge amount of heat is required to a gas so that the atoms of gaseous state get sufficient mount of kinetic energy to get ionized.

In plasma physics we study charged particles which are present in sufficient number so that the long range coulombic force determine its statistical properties. The density is low enough that coulombic force dominates the force due to neighborhood particles. Plasma physics is related to study of low density ionized gases. While studying oscillations in electric discharges, in 1929, Langmuir and Tonks used the term “Plasma” as collection of charged particles [1].

1.1 Definition of plasma

Plasma can be define as

> ”*Plasma* is quasineutral gas with charged particles and neutral particles and it shows collective behaviour”.

In this definition the term quasineutral means it is neutral enough so that we can take plasma density of electron and that of ions as equal. On the average, plasma looks quite neutral from outside because the electric charge fields due to randomly distributed particles cancel each others affect. Whereas the collective behaviour means that its not only dependent on local distribution but also on plasma state in remote regions.

1.2 Conditions of plasma

There are some conditions for any ionized gas to be consider as plasma. These conditions are

a)Debye length

b)Plasma pararmeter

c)Plasma frequency

Debye length

Thermal particle energy tends to perturb the electrical neutrality and due to any charge separation the electrostatic potential energy tends to restore charge neutrality. And debye length is the length for which a balance is obtained between these two. Debye length is equal to [2]

$$\lambda_D = \left(\frac{\epsilon_o k_B T_e}{n_e e^2}\right)^{\frac{1}{2}}$$

For a plasma to be quasi neutral

$$\lambda_D \ll L$$

Plasma parameter

The shielding effect is caused by collective behaviour within a debye sphere with radius λ_D. It is necessary that debye sphere contains enough particles. The number of particles inside a debye sphere is $\frac{4\pi}{3}n_e\lambda_D^3$. The term $n_e\lambda_D^3$ is called plasma parameter. Second criterion for plasma is

$$n_e\lambda_D^3 \gg 1$$

Plasma frequency

For plasmas which are not fully ionized, they have substantial amount of neutral particles. If charged particles hit with neutrals constantly, electrons will be forced into equilibrium with neutrals, and plasma will cease to exist. For electrons to be remain unaffected by collisions with neutrals, average time between two electron-neutral collisions must be larger than reciprocal of plasma frequency.

$$w_{pe}\tau > 1$$

This is the third plasma criteria.

1.3 Micro and Macro treatment of Plasma

There are two common approaches to studying plasma properties and processes. One is macroscopical approach, it includes thermodynamic and fluid behaviour and other is microscopical approach and that includes kinetic and statistical descriptions. Microscopic approach deals mainly with quantities like temperature and average velocities as function of time and position. These quantities are emphasized in measuring plasma properties. On the other hand, microscopic approach deals with correlations between plasma particles, and also with the microfields that these particles produce and velocity-space distributions of these particles. It is difficult to directly measure microscopic properties but they play a pivotal role in determining macroscopic properties for a plasma. The distinction between micro and macro is convenient for beginning of general discussion about unstable plasma modes.

1.4 Kinetic theory

Kinetic theory of plasma describes and predicts the condition of plasma from microscopic interactions and motions of its constituents. It provides an essential basis for an introductory course on plasma physics as well as for advanced kinetic theory [3].

Plasma oscillations and waves are frequently treated ineffectively using fluid. The velocity distribution functions $f(v)$ are taken into account for this, and the treatment is called as kinetic theory. In fluids there are only four independent variables. This is because the anticipated velocity distribution for each specie is a Maxwellian distribution, which can only be determined by temperature. When the thermal velocity of charged particles approaches the phase velocity of waves, however, the wave particle interaction differs from that of fluid treatment. As a result, the description for this interaction must be based on the dynamics of the particle's phase space distribution function, in which particle position and velocity are independent variables. Kinetic theory equations for a plasma with self-consistent fields provide this kind of description.

1.5 Plasma distributions

The location and velocity of each plasma particle as a function of time are given in a detailed description of a plasma. This description of a real plasma can only be obtained in a few recent studies involving the use of modern computers to track the position and velocity of a large number of plasma particles. Therefore different distribution functions are used to describe a plasma. A distribution function is equal to number of particles per unit volume in 6-dimensional velocity configuration phase space. For example, Maxwellian velocity distribution is

$$\hat{n} f(v) = \hat{n} \left(\frac{m}{2\pi k_B T} \right)^{\frac{3}{2}} \exp \left[-\frac{mv^2}{2k_B T} \right]$$

Despite the fact that this distribution cannot be achieved in laboratory plasma experiments, it is used in a number of theoretical treatments. Because ions and electrons act differently on a time scale less than ion cyclotron relaxation times, it's common to design separate distribution functions for each charge specie.

1.6 Anisotropic distributions

All velocity distributions are not simple like isotropic Maxwellian distribution. The presence of magnetic fields in plasma is responsible for anisotropy due to the fact that it leads to different particle velocities which can be parallel and perpendicular to the magnetic field. In this approach, gyrating particles are an interesting example. In this situation, the velocity distribution is determined by $v_\perp$ and $v_\parallel$ rather than the angle of gyration. These two velocity components are independent so equilibrium distribution can be modeled as product of two Maxwellian like

$$f\left(v_\perp, v_\parallel\right) = \frac{n}{\left(\pi^3 \langle v_\perp \rangle^2 \langle v_\parallel \rangle^{\frac{1}{2}}\right)} \exp \left[-\frac{v_\perp^2}{\langle v_\perp \rangle^2} - \frac{v_\parallel^2}{\langle v_\parallel \rangle^2} \right]$$

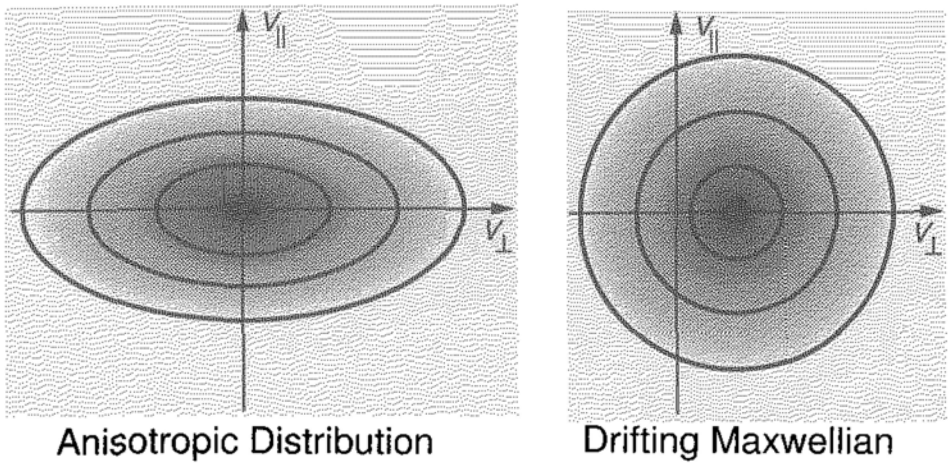

Figure 1.1: Contours of constant f for Maxwellian and anisotropic Bimaxwellian velocity distribution

This resulting distribution function is called Bi-Maxwellian distribution. It accounts explicitly for difference in two mean velocities $\langle v_{\perp} \rangle$ and $\langle v_{||} \rangle$ which are parallel and perpendicular to the magnetic field direction.

The most of these velocity distributions are found in space plasmas. These types of distributions are two-dimensional and gyrotropic, which means they are not affected by the phase angle of gyro motion. The left hand side of above figure shows sketch of Bi-Maxwellian with $T_{\perp} > T_{||}$, which means average thermal energy of particles perpendicular to magnetic field is greater than the average thermal energy parallel to magnetic field. The Bi-Maxwellian contours are deformed into an elliptical shape while isotropic Maxwellian contours are circular.

1.7 Plasma waves

Mostly people are quite familiar with the word waves like the waves that propagate on the surface of lakes , sea and oceans and break on beaches. The disturbances in the atmosphere also create waves but a lot of people conn't recognize it properly, although they have proper understanding of it. In the behaviour of plasmas, wave phenomena are extremely essential. In fact, One of the three conditions for the presence of a plasma is that the particle-particle collision rate is lower than the plasma oscillation frequency. As a result, the collective interactions that control the plasma gas are as dependent on electric and magnetic field effects as much as, or more so than, simple collisions [4].

The near-Earth space environment is in a dynamic state as a result of multiple forces working at the same time; the combination of these forces never allows a proper thermodynamic equilibrium to develop. However, quasi steady states can exist, which are often close to marginal states where only a small amount of free energy is required to produce unstable waves. As a result, the space environment is awash in naturally occurring waves. Even the early probes in almost every region of space demonstrated this. Waves can be found in partially ionized ionospheric layers, the collisiondominated neutral atmosphere, and the fully ionized magnetosphere at high altitudes.

Waves are indicators of local conditions as well as plays an important role for determining spatio-temporal evolution of the medium. They are also important in accelerating and heating particles, causing anomalous plasma resistivity, energy transport and in modifying particle velocity distributions. Therefore, wave study is crucial in understanding of geospace environment [5].

1.8 Whistler wave modes

These whistler modes are found in geospace plasmas and also in other planet's magnetospheres as well. In the frequency range above the ion cyclotron frequency and below the lower of plasma and electron cyclotron frequencies, electronic whistler oscillations are prominent wave modes [6].

Various types of waves such as upper-hybrid waves, whistler waves, electrostatic solitary waves, electron cyclotron waves, and Langmuir waves have been observed inside/outside the diffusion region using space measurements in current sheets in the Earth's magnetopause and magnetotail, including Magnetospheric Multiscale mission (MMS) observations [11]. one of them is Whistler waves which can be excited during reconnection by electron-scale kinetic physics. Whistler waves, for example, are linked to a temperature anisotropy instability caused by electron perpendicular heating or a loss cone instability, according to space observations. Particle in cell (PIC) simulations and space observations, on the other hand, show that electron beams can also produce whistler waves. Further studies are necessary to understand the roles of these waves in reconnection.

There are several studies of whistler wave propagation in the presence of density nonuniformities, there are significantly fewer investigations in the presence of magnetic nonuniformities, particularly strongly nonuniform fields with gradient scale lengths smaller than the wavelength. Whistler modes can be trapped or scattered by density nonuniformities in the ionosphere, such as ducts or striations. Whistler modes can be focused, guided, and reflected by magnetic field gradients, however there are few situations where these effects have been proved experimentally.

There are two solutions for the dispersion relation in the cold plasma approximation of electromagnetic (EM) wave propagation along background magnetic field with electric field perpendicular to Bo, corresponding to right(R) and left(L) hand circularly polarized modes. The R hand circularly polarized mode corresponds to parallel propagating whistler wave in this frequency range. Its dispersion relationship is as follows:

$$\frac{c^2k^2}{w^2} = 1 - \frac{\frac{w_{pe}^2}{w^2}}{1 - \frac{w_{ce}}{w}}$$

The spectrum components of the electron whistler mode can be observed in all ranges, from the EM (whistler) to the electrostatic (lower hybrid) limits. Typical spectrogram of VLF his E field data from ionosphere shows that its spectrum is sharply bounded below by lower hybrid resonant frequency

$$w_{LH} = \left[\frac{w_{pi}^2 w_{ce}^2}{w_{pe}^2 + w_{ce}^2} \right]^{\frac{1}{2}}$$

At lower hybrid resonance, the fall off in wave electric (E) field power spectrum is precise enough that the observed cut off frequency can be used to estimate local plasma density. There is no such boundary in the magnetic spectrum at lower hybrid resonance, indicating that oscillations towards the low frequency limit are electrostatic [7]. Although electrons carry most of the waves along this branch, there is a branch of low frequency oscillations known as hydromagnetic or ion whistlers that exists at ion cyclotron frequencies [8].

Lightning generates a wide range of electromagnetic radiations. These rays are trapped in a wave guide produced by the lower ionosphere and the ground. The spectrum of lightning between frequency and time for whistler and sferics is shown in figure.

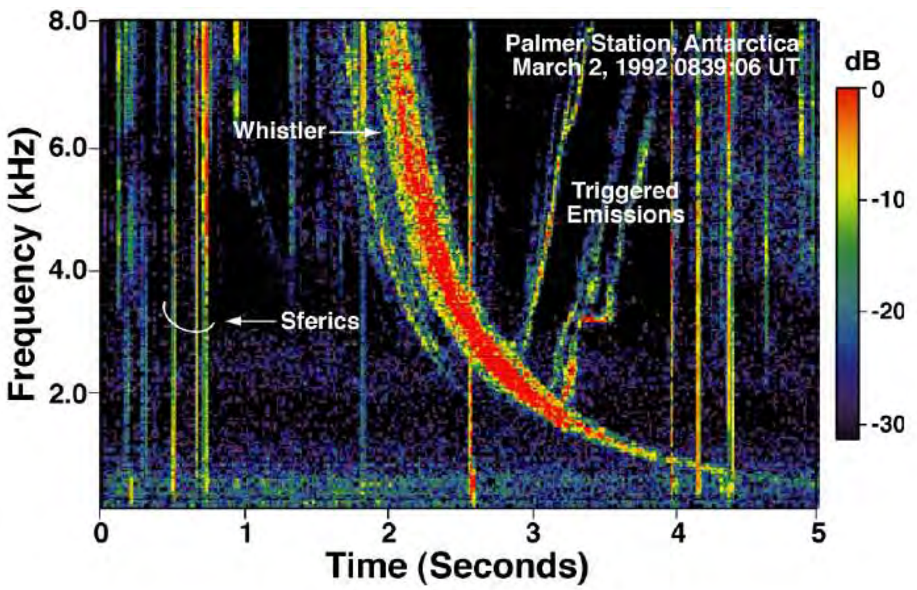

Figure 1.2: A frequency time-spectrograph of lightening spherics and resulting whistlers

Here red colour shows most intense and black shows least intense electrical component of EM waves. Whistlers are EM waves with a continuous tone that begin at higher frequencies of around 10kHz, rapidly decline in frequency, and end at low frequencies of a few hundred Hz in a couple of moments, as shown in the figure. It can be observed from the cold plasma dispersion relation that higher frequency wave components travel rapidly than lower frequency wave components, which results in the falling tone of whistlers caused by lightning. Whistlers are described as descending tones produced by atmospherics propagating through the ducts (field aligned plasmas)of the magnetosphere. The propagation of these whistlers is observed in northern and southern hemisphere's magnetically conjugate regions. Whistlers lasts from fraction of second to several seconds. Whistlers frequency range is from 30kHz to less than 1kHz and the one which can be heard by using simple equipments lying between 1kHz to 9kHz.

Whistlers and similar phenomena have been seen in space plasma, laboratory plasma, and solid state physics. Whistlers are described in different ways in space and in laboratory plasmas. Depending on their noises and spectrum, they are given different titles such as hiss, chorus, roar triggered emissions, and so on. In laboratory plasma, every wave in whistler mode is called a whistler, while in solid state physics, these waves are called helicons. Observations from ground and space and difficulties due to lack of parameter control leads to the desire of studying these waves in controlled laboratory plasma. Many contributions were made on whistler waves in laboratory plasma physics.

Work on whistlers grow rapidly in recent times due to spacecraft's data and computer models which can calculate whistler rays path in plasmasphere [9].

Two unique types of whistlers emerge from the analysis of whistlers using these methods which are named as ducted and non-ducted propagation. The whistler radiations in plasmasphere are consist of EM waves which have upper cut off frequency either local plasma frequency (f_p) or gyrofrequency (f_g) [10]. As the plasmasphere has a high cold plasma density, f_p is larger than f_g, allowing whistler mode emission in the VLF frequency range.

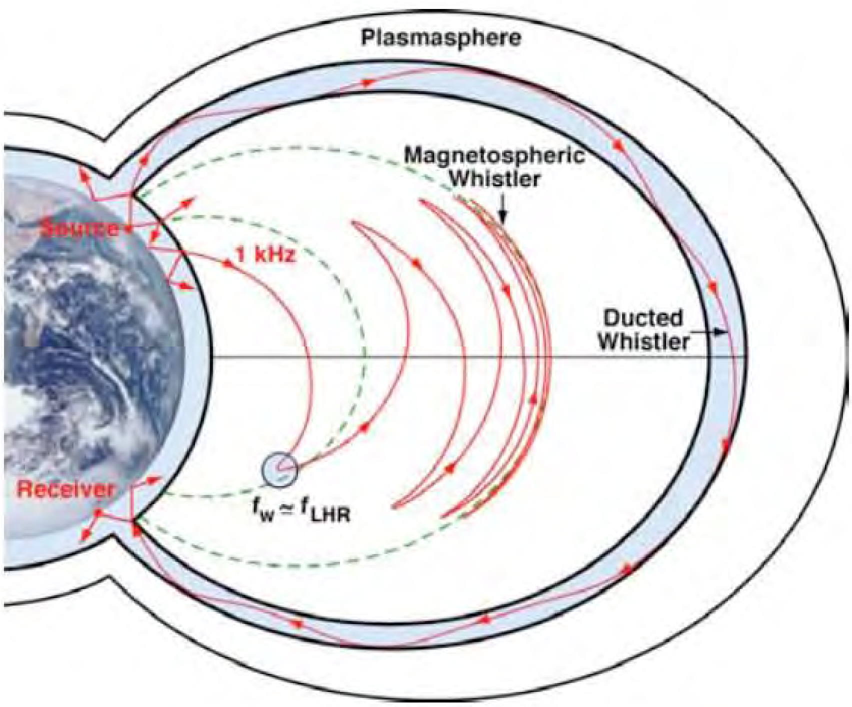

Figure 1.3: The prorogation paths of ducted and non-ducted whistlers

Ducted whistlers follow the earth's magnetic field lines, are refracted within ionospheric plasma, and then diffuse out into the plasmasphere along these magnetic field lines. These ducts make it easier for VLF lightning to travel through the ionosphere and consequently reaches at the other hemisphere, as shown in above figure. A single lightning strike can illuminate more than one duct, as a result numerous whistlers can be seen in opposite hemispheres at separate moments. These may reflect in the opposite hemisphere in some situations, allowing them to reach the source region and can be seen on the ground. At the source's end, there is only a little chance of being received. When there are more flashes, non-ducted whistlers are produced, which propagate into the plasmasphere. This sort of whistler does not follow the earth's magnetic field lines; instead, its route is determined by the index of refraction of the plasmasphere and their own frequencies.

1.9 History of Whistlers

Whistlers were initially discovered in 1919 by Barkhausen, who was listening in on Allied phone calls during World War I. They may actually have been observed as early as 1888 by J Fuchs of the Sonnblick High Altitude Observatory in Austria. In 1928, Eckersley of the Marconi Company discovered a link between whistlers and numerous atmospheric events, as well as a positive correlation between their frequency and solar activity [12]. Eckersley also reported an observation by Tremellen indicating an association of whistlers with visible lightning. Burton and Boardman performed the first quantitative measurement of the frequency time relationship of a whistler in 1933, using a whistler recorded in Ireland. In 1935, Eckersley developed a dispersion rule that explained how whistler frequency changed over time and demonstrated that Burton and Boardman's published analysis fitted the new theory closely. For about 15 years, interest in whistlers was then lagged until Storey began his doctoral studies at Cambridge in 1951. He made a detailed experimental study of whistlers and related atmospheric signals and identified short whistlers which were not associated with loud clicks.

RA Helliwell began simultaneous studies of whistlers at spaced stations at Stanford and Seattle in 1951. Around 25% of the whistlers observed were coincident at the two stations separated by 1120 km, corresponded to the area coverage predicted by Storey. Helliwell proved his prediction of a relationship between causal atmospherics in the Southern Hemisphere and short whistlers in the Northern Hemisphere in 1954. Morgan and Allcock obtained simultaneous observations of whistler echo trains around the geomagnetic conjugate point at Unalaska and Wellington and Dunedin, New Zealand, in 1955. Their results fully confirm the predicted behavior. In 1955, Koster and Storey confirmed the predicted absence of whistlers on the geomagnetic equator.

Many first laboratory experiments were driven in order to use whistler and lower hybrid waves as means of heating in fusion plasmas. Gallet *et al.* [13] has done some earliest investigations on whistlers, he considered possibilty of using these waves as diagnostic of magnetic field in toroidal plasma.

Gould and Fisher [14] performed experimental and theoretical investigation of radiation pattern produced by small dipole antenna in magnetized plasma. Hooke and Bernabei [15] showed that plasma resonance with short perpendicular exists at frequencies near lower hybrid resonance. Then Sugai [16] showed that these ducts formed even when antennas are driven at frequencies at which whistlers didn't propagate.

A study of whistler sources was based on the impulses produced by nuclear explosions. From 1953 to 1962 Stanford made broadband very low frequency recordings during detonations of a large number of nuclear devices. At one or more receiving stations, a bombexcited whistler was detected in five cases. It was discovered that nuclear sources produce whistlers that are identical to those produced by natural lightning in every measurable way, and that a whistler can be excited by a nuclear explosion in the hemisphere opposite that of the whistler path's entrance. It was also discovered that nuclear explosions which take place above the lower edge of the ionosphere produce signals similar to those arising from subionospheric explosions. This meant that the source of the electromagnetic impulse was in fact between the earth and the ionosphere and not at the location of the bomb itself.

1.10 Applications Of whistler

One of the first uses of whistler occurred in 1959, when the Soviet Union deployed two satellites to the moon to detect ion density along their journey. They confirmed that higher density plasma can be detected just beyond the ionosphere, but that this density drops rapidly near 10,000 km altitude [17].

The potential for remote diagnostics of the ionosphere has been clear since the theoretical approach for whistlers was revealed. The dispersion of low-frequency whistlers, or more precisely, "nose whistlers," can be used to calculate the average electron density. Electron temperature, large-scale electric fields, ion composition, and fractional densities have been measure. Magnetic fields in solar and laser plasmas have been calculated, and whistler emissions have been used to infer the presence of non-maxwellian electron distributions. Whistlers were used to determine the electron density profiles of the equator. They were used to measure electric fields in the east west. Due to these classical methods along with many others, we are able to find out cold plasma densities in magnetosphere.

Ground-based experiments can be used to generate VLF signals in a controlled manner. Because these signals may travel deep into the earth, they are employed to communicate with submarines and to image subterranean structures.Whistler waves have also been used in laboratory plasma. Helicon plasma sources, for example, are employed in the production of microelectronic devices used in computers [18]. They're also used in toroidal fusion devices to drive currents.

Whistlers' precipitation of energetic electrons is a key aspect of energy transfer from the magnetosphere to the atmosphere. Controlled precipitation using VLF injection could be useful in communication and aeronautics. Experiments with active whistler wave injection from space are deserving of additional consideration. To assess spatial wave properties, free-flying or tethered diagnostic packages are needed.

1.11 Oblique Whistler Wave Mode

Whistler mode waves are electromagnetic waves that occur at frequencies lower than the electron gyrofrequency in magnetised plasmas. Non-maxwellian electron distributions, such as beams, loss cones, rings, and temperature anisotropies, cause various plasma instabilities, which result in whistler frequency emissions [19]. They are given various names based on their frequency-time spectral properties and sound as heard through a loudspeaker, such as plasmaspheric hiss, lion roars, chorus, tweeks, and so on.

Whistler mode waves often propagate parallel or quasi-parallel ($\theta_{kB_o} < 30°$) to the ambient field B_0, according to satellite measurements and theoretical research. Here θ_{kB_o} is the angle between wave vector k and ambient field B_0. In space plasma zones, however, highly oblique whistler mode waves have been observed. When waves travel parallel to the ambient magnetic field direction, Whistler mode wave fields are understood to be right-hand circularly polarized. The magnetic field components of whistler mode chorus waves observed in the magnetosphere are circularly polarized regardless of propagation angles, according to Goldstein & Tsurutani and Tsurutani et al. The polarisation plane is perpendicular to the wave propagation direction k in this case. Following previous observations, Verkhoglyadova et al. and Bellan demonstrated that the wave magnetic fields are certainly circularly polarized (to first order) at all angles of propagation in the limit of cold plasma ($k^2\rho_e^2 \ll 1$) with a low frequency ($\omega \ll |\Omega_e| \cos\theta_{kB_o}$) approximation. Here ρ_e is the electron gyroradius. The wave electric field components are elliptically polarized, according to Verkhoglyadova et al., since their polarization plane is not orthogonal to the wave vector k. While the total wave electric field is elliptically polarized, Bellan demonstrated that the electric field components transverse to the propagation vector k are circularly polarized for all propagation angles.

In the Earth's dawnside outer radiation belt, the STEREO observation by Cattell et al. [20] [2008] reported the finding of obliquely propagating whistler waves with amplitudes as high as $\sim 240mV/m$. The propagation angle is reported to be in the range of $45° - 70°$, the ambient magnetic field intensity (derived from measured electric field intensity) is $300nT$ to $350nT$, and the observed background electron number density is in the range of $25cm^{-3}$, resulting in a local plasma to electron gyrofrequency ratio of roughly 3 or so. The peak frequency detected is $0.2fce$, which is within the whistler range.

Yoon [2011] examined the nonlinear wave properties of obliquely propagating whistler waves using an arbitrarily high wave amplitude. The claimed goal of such a prescription was to accelerate numerical computations so that long-term nonlinear dynamical development could be studied within a reasonable computing time scale. Because oblique whistlers are compressional waves, He discovered that they will gradually steepen.. However, even though the steepening of oblique whistler waves is a general consequence of a compressional wave, by re-examining the problem.

Chapter 2

Model Of Oblique Whistler Modes

2.1 Fluid Model

When the phase velocity of a wave excited in a plasma exceeds than the thermal velocity, the main body of the plasma particles are out of resonance with the wave and the number of the resonant particles which exchange the energy with the wave is small. Plasma can be treated as a fluid in this situation. The electron and ion fluids are expected to be interpenetrating each other in the fluid plasma. Through the electromagnetic field, they interact and exchange momentum and energy through collisions. Although transport coefficients based on collisional processes are given by the kinetic theory, transport coefficients associated with the waves excited in the plasma are determined in the frame work of the fluid theory.

So, The fluid model in plasma is one that considers the entire flow of fluid constituents. The Ordinary collisions between particles in fluid maintain all of the fluid's constituents moving together. Fluid model works when the collisions are incredibly high. In this model plasma is described the macroscopic quantities such as density, flux, average velocity, pressure, temperature or heat flux. Ions and electrons can be treated independently as two fluid plasmas in a general description. Fluid models are accurate for high collisions in order to keep velocity distributions close to Maxwell-Boltzmann distribution. This model can not resolve wave particle affects due to the fact that in fluid model plasma is taken as single flow at certain temperature and at each spatial location.

2.2 Kinetic Model

The velocity distributions in kinetic models are not considered to be purely Maxwell-Boltzmann. For collisionless plasma, a kinetic model was considered. This model is more intensive computationally. The dynamics of a system of charged particles interacting with an electromagnetic field are described by the Vlasov equation.

2.3 Kinetic Theory

A plasma contains a huge number of interacting ions and electrons, it is best to analyze it using a statistical approach. The fluid description is the most basic and can be used to describe a variety of phenomena. We investigate velocity distributions for each species for those phenomena where fluid treatment is inappropriate, and this treatment is known as kinetic theory.

The dependent variables in fluid theory are functions of only four independent variables : x, y, z, and t. This is possible because the velocity distribution of each species is assumed to be Maxwellian everywhere and can therefore be uniquely specified by only one number, the temperature T. The wave-particle interaction is significantly different from that described by the fluid equations if the velocity of a significant number of charged particles (usually the thermal velocity) is close to the phase velocity of waves. A proper description of this interaction must be based upon the dynamics of the particles' phase space distribution function in which the particles' velocity and position are independent variables. Such a description is provided by the kinetic theory equations for a plasma with self-consistent fields.

Due to the rarity of collisions in high-temperature plasmas, deviations from thermal equilibrium can be maintained for longer periods of time. Consider two velocity distributions $f_1(v_x)$ and $f_2(v_x)$ in a one-dimensional system; if the area under the curve is the same for both, we cannot distinguish between them using the fluid model. In realistic case where electric and magnetic fields are present, the distribution function is not symmetric there. Let if we perturb the system then number of particles will be different from equilibrium state.

$$dN_\alpha(t+\triangle t, x+\triangle x, u_s + a\triangle t)$$

We can write the distribution function as

$$[h_\alpha(t+\triangle t, x+\triangle x, u_s+a\triangle t) - h_\alpha(t,x,u_s)]\, d^3x d^3u_s = (\frac{\partial h_\alpha}{\partial t})_{\text{col.}} d^3x d^3u_s \triangle t$$

$\left(\frac{\partial h_\alpha}{\partial t}\right)_{col}$ is rate of change of h_α due to collisions
Expanding the first term

$$\left[h_\alpha(t,x,u_s) + \frac{\partial h_\alpha}{\partial t}\triangle t + \frac{\partial h_\alpha}{\partial x}\triangle x + \frac{\partial h_\alpha}{\partial u_s}\triangle u_s + ... - h_\alpha(t,x,u_s)\right] d^3x d^3u_s = (\frac{\partial h_\alpha}{\partial t})_{\text{col.}} d^3x d^3u_s \triangle t$$

$$\left[\frac{\partial h_\alpha}{\partial t} + u_s.\nabla h_\alpha + a.\nabla h_\alpha\right] d^3x d^3u_s \triangle t = (\frac{\partial h_\alpha}{\partial t})_{\text{col.}} d^3x d^3u_s \triangle t$$

Let $\triangle t \to 0$,so

$$\frac{\partial h_\alpha}{\partial t} + u_s.\nabla h_\alpha + a.\nabla h_\alpha = \alpha(\frac{\partial h_\alpha}{\partial t})_{\text{col.}}$$

To find acceleration in third term

$$F = q_\alpha \left(E + u \times B\right)$$

$$a = \frac{q_\alpha}{m_\alpha} \left(E + u_s \times B\right)$$

Putting this in the above equation we have,

$$\frac{\partial h_\alpha}{\partial t} + u_s.\nabla h_\alpha + \frac{q_\alpha}{m_\alpha} \left(E + u_s \times B\right).\nabla h_\alpha = (\frac{\partial h_\alpha}{\partial t})_{\text{col.}}$$

This equation is Boltzmann Vlasov equation. For collisionless plasma it can be written as

$$\frac{\partial h_\alpha}{\partial t} + u_s.\nabla h_\alpha + \frac{q_\alpha}{m_\alpha} \left(E + u_s \times B\right).\nabla h_\alpha = 0$$

2.4 Electromagnetic Dispersion Relation

Vlasov equation For collisionless plasma is

$$\frac{\partial h_\alpha(t,x,u_s)}{\partial t} + u_s.\nabla h_\alpha\left(t,x,u_s\right) + \left(\frac{q_\alpha}{m_\alpha}E + \frac{q_\alpha}{m_\alpha}u_{s\alpha} \times B\right).\nabla h_\alpha\left(t,x,u_s\right) = 0$$

As we know Maxwell equations are

$$\nabla.E = \frac{1}{\epsilon_o}\sum_\alpha q_\alpha^{'} \int h_\alpha d^3v \qquad \nabla \times E = -\frac{\partial B}{\partial t}$$

$$\frac{1}{\mu_o}\nabla \times B = \epsilon_o \frac{\partial E}{\partial t} +_\alpha q_\alpha \int v h_\alpha d^3 v \qquad \nabla.B = 0$$

If we perturb the system slightly then

$$h_\alpha\left(r,u_s,t\right) = h_{\alpha o}\left(r,u_s\right) + h_{\alpha 1}\left(r,u_s,t\right)$$

$$B = B_o\left(r\right) + B_1$$

$$E = E_o + E_1 = 0 + E_1$$

Here $h_1\left(r,u_s,t\right)$, B_1 and E_1 depend upon $e^{i(k.r-wt)}$. Taking first order terms

$$\frac{\partial h_{\alpha 1}}{\partial t} + u_{\alpha s}.\nabla h_{\alpha 1} + \frac{q_\alpha}{m_\alpha}\left(u_{\alpha s} \times B\right).\nabla h_{\alpha 1} = -\frac{q_\alpha}{m_\alpha}\left(E_1 + u_{\alpha s} \times B_1\right).\nabla h_{\alpha o}$$

$$ik.E_1 = \frac{1}{\epsilon_o}\sum_\alpha n_\alpha q_\alpha \int h_{\alpha o} d^3v$$

$$\frac{1}{\mu_o}\vec{k}\times\vec{B}_1 = -\omega\left(\epsilon_o\vec{E}_1 + \frac{i}{\omega}\sum_\alpha n_\alpha q_\alpha \int \vec{u}_s\ h_{\alpha 1}\ d^3v\right) \tag{2.1}$$

$$\vec{B}_1 = \frac{1}{\omega}\left(\vec{k}\times\vec{E}_1\right)$$

From above equation we have

$$\frac{Dh_{\alpha 1}\left(\vec{r}\left(t\right),\vec{u}_s\left(t\right),\vec{t}\right)}{Dt} = \frac{\partial h_{\alpha 1}}{\partial t} + \frac{\partial h_{\alpha 1}}{\partial \vec{r}}\cdot\frac{\partial \vec{r}}{\partial t} + \frac{\partial h_{\alpha 1}}{\partial \vec{u}}\cdot\frac{\partial \vec{u}}{\partial t} = -\frac{q_\alpha}{m_\alpha}\left(\vec{E}_1 + \vec{u}_s\times\vec{B}_1\right)\cdot\vec{\nabla}_v\ h_{\alpha 0}$$

$$h_{\alpha 1} = -\frac{q_\alpha}{m_\alpha}{}_{-\infty}^{t} d\acute{t}\ \left(\vec{E}_1 + \vec{u}_s\times\vec{B}_1\right)\cdot\vec{\nabla}_v\ h_{\alpha 0}$$

Using equation (2.1) we have

$$\begin{aligned} h_{\alpha 1} &= -\frac{q_\alpha}{m_\alpha}{}_{-\infty}^{t} dt^{'}\ \left[\vec{E}_1\left(\vec{r}^{'},t^{'}\right) + \vec{u}_s^{'}\left(t^{'}\right)\times\frac{1}{\omega}\left(\vec{k}\times\vec{E}_1\left(\vec{r}^{'},t^{'}\right)\right)\right]\cdot\vec{\nabla}_{v'}\ h_{\alpha 0} \\ &= {}_{-\infty}^{t} X dt^{'} \end{aligned} \tag{2.2}$$

As X is defined as

$$X = -\frac{q_\alpha}{m_\alpha}\ \left[\vec{E}_1\left(\vec{r}',t'\right) + \vec{u}_s'\left(t'\right)\times\frac{1}{\omega}\left(\vec{k}\times\vec{E}_1\left(\vec{r}',t'\right)\right)\right]\cdot\vec{\nabla}_{v'}\ h_{\alpha 0}$$

Taking into account wave propagation perpendicular to the uniform applied magnetic field $\overrightarrow{B}_{o.}$. And B_o is in the direction of z axis, so
$\overrightarrow{B}_{o.} = B_o\hat{\mathbf{z}}$. Propagation Vector $\overrightarrow{k}$ is normal to $\overrightarrow{B}_{o.}$ and along the

$$\overrightarrow{k} = k_x\hat{x} + k_z\hat{z} \tag{2.3}$$

Charged particles motion in a uniform magnetic field $\overrightarrow{B}_{o.}$ can be described as
Body Math

$$\frac{d\vec{r}'}{dt'} = \vec{u}_s$$

$$\frac{d\vec{u}_s'}{dt'} = \frac{q}{m}(\vec{u}_s'\times B_o)$$

At $t = t'$, we can write

$$\vec{r}' = \vec{r} \qquad , \qquad \tilde{\mathbf{u}}_s = (u_{s\perp}Cos\theta, u_{s\perp}Sin\theta, u_{s\|}) \tag{2.4}$$

Here we take the cylindrical coordinates

$$\frac{du'_{sx}}{dt'} = \frac{q}{m} u'_{sy} B_o \qquad \frac{du'_{sy}}{dt'} = -\frac{q}{m} u'_{sx} B_o$$

As $\omega_c = \dfrac{qB_o}{m}$ so,

$$\frac{du'_{sx}}{dt'} = \omega_c u'_{sy} \qquad \frac{du'_{sy}}{dt'} = -\omega_c u'_{sx}$$

$$\frac{d^2 u'_{sx}}{dt^{2'}} = \omega_c \frac{du'_{sy}}{dt} \qquad \frac{d^2 u'_{sx}}{dt^{2'}} = -\omega_c^2 u'_{sx}$$

So, the solution of above equations will be

$$\begin{aligned} u'_{sx}(t') &= u_{s\perp} Cos \left[\theta - \omega_c(t' - t)\right] \\ u'_{sy}(t') &= u_{s\perp} Sin \left[\theta - \omega_c(t' - t)\right] \\ u'_{sz}(t') &= u_{s\parallel} \end{aligned} \tag{2.5}$$

Now

$$\frac{dx'}{dt'} = u'_{sx}(t')$$

Body Math Integrating from t to t'

$$\int_t^{t'} \frac{dx'}{dt'} dt' = \int u_\perp \cos\left[\theta - \omega_c(t' - t)\right] dt'$$

After integration we have

$$\begin{aligned} x'(t') &= x + \frac{u_{s\perp}}{\omega_c}\left[\sin\theta - \sin\left\{\theta - \omega_c(t' - t)\right\}\right] \\ y'(t') &= y - \frac{u_{s\perp}}{\omega_c}\left[\cos\theta - \cos\left\{\theta - \omega_c(t' - t)\right\}\right] \\ z'(t') &= z + u_{sz}(t' - t) \end{aligned} \tag{2.6}$$

Using the mathematical identity for triple cross product i-e

$$\vec{A} \times \left(\vec{B} \times \vec{C}\right) = \left(\vec{A}.\vec{C}\right)\vec{B} - \left(\vec{A}.\vec{B}\right)\vec{C}$$

So,

$$\vec{u}_s \times \left(\vec{k} \times \vec{E}_1\right) = \left(\vec{u}_s \cdot \vec{E}_1\right)\vec{k} - \left(\vec{k} \cdot \vec{u}_s\right)\vec{E}_1$$

Here X becomes,

$$X = -\frac{q_\alpha}{m_\alpha}\left[\left(1-\frac{\vec{k}\cdot\vec{u}_s'(t')}{\omega}\right)\vec{E}_1(\vec{r}',t')+\frac{1}{\omega}\left(\vec{u}_s'(t')\cdot\vec{E}_1(\vec{r}',t')\right)\vec{k}\right]\cdot\vec{\nabla}_{v'}\ h_{\alpha 0}$$

Now equation (2.2) becomes

$$h_{\alpha 1} = -\frac{q_\alpha}{m_\alpha}\int_{-\infty}^{t} dt'\left[\left(1-\frac{\vec{k}\cdot\vec{u}_s'(t')}{\omega}\right)\vec{E}_1(\vec{r}',t')+\frac{1}{\omega}\left(\vec{u}_s'(t')\cdot\vec{E}_1(\vec{r}',t')\right)\vec{k}\right]\cdot\vec{\nabla}_{v'}\ h_{\alpha 0} \qquad (2.7)$$

We have$h_o(r,u_s) = h_o\left(u_{s\perp},u_{s\parallel}\right)$

$$g_o(r,u) = g_o\left(u_\perp,u_\parallel\right)$$

And

$$u_{s\perp}^2 = u_{sx}^2 + u_{sy}^2$$

$$\vec{E}_1(\vec{r}',t') = \vec{E}\exp\left[i\left(\vec{k}\cdot(\vec{r}'-\ \vec{r})-\omega(t'-\ t)\right)\right] \qquad (2.8)$$

The exponential term becomes

$$\exp\left[i\left(\vec{k}\cdot(\vec{r}'-\ \vec{r})-\omega(t'-\ t)\right)\right] = \exp\left[i\left(k_x(x'-\ x)+k_z(z'-\ z)-\omega(t'-\ t)\right)\right]$$

Where $k_y = 0$. Now using equations (2.6) we have

$$= \exp\left[i\left(\frac{k_x u_\perp}{\omega_c}\left[\sin\theta-\sin\left\{\theta-\omega_c(t^{'}-t)\right\}\right]+(k_z u_z-\omega)(t'-\ t)\right)\right]$$

So (2.8) becomes

$$\vec{E}_1(\vec{r}',t') = \vec{E}\exp\left[i\left(\frac{k_x u_{s\perp}}{\omega_c}\left[\sin\theta-\sin\left\{\theta-\omega_c(t^{'}-t)\right\}\right]+(k_z u_{sz}-\omega)(t'-\ t)\right)\right]$$

Putting this in above equation (2.7) we have

$$h_1 = -\frac{q}{m}\int_{-\infty}^{t} dt'\left\{\begin{array}{c}\left[\left(1-\frac{\vec{k}\cdot\vec{u}_s'(t')}{\omega}\right)\vec{E}+\frac{1}{\omega}\left(\vec{u}_s'(t')\cdot\vec{E}\right)\vec{k}\right]\\ \exp i\left(\frac{k_x u_{s\perp}}{\omega_c}\left[\sin\theta-\sin\left\{\theta-\omega_c(t^{'}-t)\right\}\right]+(k_z u_{sz}-\omega)(t'-\ t)\right)\end{array}\right\}\cdot\vec{\nabla}_{v'}\ h_0$$

Taking

$$\tau = t^{'}-t \qquad d\tau = dt'$$

$$t' \rightarrow t \quad \rightarrow \ \tau \rightarrow 0 \qquad\qquad t' \rightarrow -\infty \quad \tau \rightarrow -\infty$$

So

$$g_1 = -\frac{q}{m}\int_{-\infty}^{0} d\tau \left\{ \begin{array}{c} \left[\left(1 - \frac{\vec{k}\cdot\vec{u}'(t')}{\omega}\right)\vec{E} + \frac{1}{\omega}\left(\vec{u}'(t')\cdot\vec{E}\right)\vec{k}\right] \\ \exp i\left(\frac{k_x u_\perp}{\omega_c}\left[\sin\theta - \sin\{\theta - \omega_c\tau\}\right] + (k_z u_z - \omega)\tau\right) \end{array} \right\} \cdot \vec{\nabla}_{v'}\ g_0 \tag{2.9}$$

From the generating function of bessel function we have

$$e^{ix\sin\theta} = \sum_{n=-\infty}^{\infty} J_n(x)\, e^{in\theta} \tag{2.10}$$

Now taking

$$X_1 = \ \exp i\left(\frac{k_x u_{s\perp}}{\omega_c}\left[\sin\theta - \sin(\theta - \omega_c\tau)\right] + (k_z u_{sz} - \omega)\tau\right)$$

From equation (2.9) and using (2.10) we have

$$X_1 = \sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty} J_n\left(\frac{k_x u_{s\perp}}{\omega_c}\right) J_m\left(\frac{k_x u_{s\perp}}{\omega_c}\right) e^{-in(\theta-\omega_c\tau)} e^{im\theta} e^{i(k_z u_{sz}-\omega)\tau}$$

Putting value of X_1 in equation (2.9) we have

$$\begin{aligned} h_1 \ &= \ -\frac{q}{m}\int_{-\infty}^{0} d\tau \left[\left(1 - \frac{\vec{k}\cdot\vec{u}'_s(t')}{\omega}\right)\vec{E} + \frac{1}{\omega}\left(\vec{u}'_s(t')\cdot\vec{E}\right)\vec{k}\right]\cdot\vec{\nabla}_{v'}\ h_0 \\ &\quad \sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty} J_n\left(\tfrac{k_x u_{s\perp}}{\omega_c}\right) J_m\left(\tfrac{k_x u_{s\perp}}{\omega_c}\right) e^{-in(\theta\ -\ \omega_c\tau)} e^{im\theta} e^{i(k_z u_{sz} -\ \omega)\tau} \end{aligned} \tag{2.11}$$

Now taking the term

$$X_2 = \ \left[\left(1 - \frac{\vec{k}\cdot\vec{u}'_s(t')}{\omega}\right)\vec{E} + \frac{1}{\omega}\left(\vec{u}'_s(t')\cdot\vec{E}\right)\vec{k}\right]\cdot\vec{\nabla}_{v'} h_0$$

From (2.11) it simplifies as

$$\begin{aligned} X_2 \ &= \ \frac{\partial h_o}{\partial u'_{sx}}\left[\left(1 - \frac{k_z u'_{sz}}{\omega}\right)E_x + \frac{k_x}{\omega}\left(u'_{sy}E_y + u'_{sz}E_z\right)\right] + \frac{\partial h_o}{\partial u'_{sy}}\left(1 - \frac{k_x u'_{sx} + k_z u'_{sz}}{\omega}\right)E_y \\ &\quad + \frac{\partial h_o}{\partial u'_{sz}}\left[\left(1 - \frac{k_x u'_{sx}}{\omega}\right)E_z + \frac{k_z}{\omega}\left(u'_{sx}E_x + u'_{sy}E_y\right)\right] \end{aligned} \tag{2.12}$$

From chain rule

$$\frac{\partial h_o}{\partial u'_{sx}} = \frac{\partial h_o}{\partial u_{s\perp}}\frac{\partial u_\perp}{\partial u'_{sx}}$$

Where as $\quad u_{s\perp}^2 = \acute{u}_{sx}'^2 + \acute{u}_{sy}^2$

$$\frac{\partial u_{s\perp}}{\partial u'_{sx}} = \frac{u'_{sx}}{u_{s\perp}}$$

So

$$\frac{\partial h_o}{\partial u'_{sx}} = \frac{u'_{sx}}{u_{s\perp}}\frac{\partial h_o}{\partial u_{s\perp}}$$

Similarly

$$\frac{\partial h_o}{\partial u'_{sy}} = \frac{u'_{sy}}{u_{s\perp}}\frac{\partial h_o}{\partial u_{s\perp}} \qquad \frac{\partial h_o}{\partial u'_{sz}} = \frac{\partial h_o}{\partial u_{sz}} = \frac{\partial h_o}{\partial u_{s\parallel}}$$

Where as $u'_{sz} = u_{sz}$

Now using above equation (2.12) we have,

$$\begin{aligned} X_2 &= \left[\frac{\partial h_o}{\partial u_{s\perp}}\left(1 - \frac{k_z u_{sz}}{\omega}\right) + \frac{\partial h_o}{\partial u_{sz}}\frac{k_z u_{s\perp}}{\omega}\right]\left(E_x\frac{u'_{sx}}{u_{s\perp}} + E_y\frac{u'_{sy}}{u_{s\perp}}\right) \\ &+ \left[\frac{\partial h_o}{\partial u_{s\perp}}\frac{k_z u_{sz}}{\omega} - \frac{\partial h_o}{\partial u_{sz}}\frac{k_x u_{s\perp}}{\omega}\right]\left(E_z\frac{u'_{sx}}{u_{s\perp}}\right) + \frac{\partial h_o}{\partial u_{sz}}E_z \end{aligned}$$

As

$$\begin{aligned} \frac{u'_{sx}}{u_{s\perp}} &= \cos(\theta - \omega_c\tau) = \frac{e^{i(\theta-\omega_c\tau)} + e^{-i(\theta-\omega_c\tau)}}{2} \\ \frac{u'_{sy}}{u_{s\perp}} &= \sin(\theta - \omega_c\tau) = \frac{e^{i(\theta-\omega_c\tau)} - e^{-i(\theta-\omega_c\tau)}}{2i} \end{aligned}$$

Using above equations we have

$$\begin{aligned} X_2 &= \left[\left(1 - \frac{k_z u_{sz}}{\omega}\right)\frac{\partial h_o}{\partial u_{s\perp}} + \frac{k_z u_{s\perp}}{\omega}\frac{\partial h_o}{\partial u_{sz}}\right]\left(E_x\frac{e^{i(\theta-\omega_c\tau)} + e^{-i(\theta-\omega_c\tau)}}{2} + E_y\frac{e^{i(\theta-\omega_c\tau)} - e^{-i(\theta-\omega_c\tau)}}{2i}\right) \\ &+ \left[\frac{k_x u_{sz}}{\omega}\frac{\partial h_o}{\partial u_{s\perp}} - \frac{k_x u_{s\perp}}{\omega}\frac{\partial h_o}{\partial u_{sz}}\right]\left(\frac{e^{i(\theta-\omega_c\tau)} + e^{-i(\theta-\omega_c\tau)}}{2}\right)E_z + \frac{\partial h_o}{\partial u_{sz}}E_z \end{aligned}$$

$$U = \left[\left(1 - \frac{k_z u_{sz}}{\omega}\right)\frac{\partial h_o}{\partial u_{s\perp}} + \frac{k_z u_{s\perp}}{\omega}\frac{\partial h_o}{\partial u_{sz}}\right] \qquad V = \left[\frac{k_x u_{sz}}{\omega}\frac{\partial h_o}{\partial u_{s\perp}} - \frac{k_x u_{s\perp}}{\omega}\frac{\partial h_o}{\partial u_z}\right]$$

$$a = \frac{k_x u_{s\perp}}{\omega_c} \qquad \omega_c = \frac{qB_o}{m}$$

So now

$$X_2 = \frac{UE_x}{2}\left[e^{i(\theta-\omega_c\tau)}+e^{-i(\theta-\omega_c\tau)}\right]-i\frac{UE_y}{2}\left[e^{i(\theta-\omega_c\tau)}-e^{-i(\theta-\omega_c\tau)}\right]+ \frac{I_z}{2}\left[e^{i(\theta-\omega_c\tau)}+e^{-i(\theta-\omega_c\tau)}\right]+\frac{\partial h_o}{\partial u_{sz}}E_z$$

Putting X_2 in equation (2.11) we have

$$h_1 = -\frac{q}{m}\sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty}\int_{-\infty}^{0} d\tau \times \left\{\begin{array}{c} \frac{UE_x}{2}\left[\begin{array}{c} e^{-i(n-1)\theta}e^{[i(k_z u_{sz}-\omega-(n-1)\omega_c)\tau]}J_n(a) \\ +e^{-i(n+1)\theta}e^{[i(k_z u_{sz}-\omega-(n+1)\omega_c)\tau]}J_n(a) \end{array}\right] \\ -i\frac{UE_y}{2}\left[\begin{array}{c} e^{-i(n-1)\theta}e^{[i(k_z u_{sz}-\omega-(n-1)\omega_c)\tau]}J_n(a) \\ -e^{-i(n+1)\theta}e^{[i(k_z u_{sz}-\omega-(n+1)\omega_c)\tau]}J_n(a) \end{array}\right] \\ +\frac{VE_z}{2}\left[\begin{array}{c} e^{-i(n-1)\theta}e^{[i(k_z u_{sz}-\omega-(n-1)\omega_c)\tau]}J_n(a) \\ +e^{-i(n+1)\theta}e^{[i(k_z u_{sz}-\omega-(n+1)\omega_c)\tau]}J_n(a) \end{array}\right] \\ +\frac{\partial h_o}{\partial u_{sz}}E_z e^{-in\theta}e^{i(k_z u_{sz}-\omega-n\omega_c)\tau}J_n(a) \end{array}\right\} J_m(a)\, e^{im\theta}$$

Where as

$$n+1\to n \qquad J_n(a)\to J_{n-1}(a) \qquad ; \qquad n-1\to n \qquad J_n(a)\to J_{n+1}(a)$$

$$h_1 = -\frac{q}{m}\sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty}\int_{-\infty}^{0} d\tau \times \left\{\begin{array}{c} \frac{UE_x}{2}\left[J_{n-1}(a)+J_{n+1}(a)\right] \\ +i\frac{UE_y}{2}\left[J_{n-1}(a)-J_{n+1}(a)\right] \\ +\frac{VE_z}{2}\left[J_{n-1}(a)+J_{n+1}(a)\right] \\ +\frac{\partial h_o}{\partial u_{sz}}E_z J_n(a) \end{array}\right\} J_m(a)\, e^{-in\theta}e^{i(k_z u_{sz}-\omega-n\omega_c)\tau}e^{im\theta}$$

As Recurrence relations of bessel function is

$$\frac{J_{n-1}(a)+J_{n+1}(a)}{2}=\frac{n}{a}J_n(a) \qquad ; \qquad \frac{J_{n-1}(a)-J_{n+1}(a)}{2}=\frac{d}{da}J_n(a)=J_n'(a)$$

So h_1 becomes

$$h_1 = -\frac{q}{m}\sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty}\left\{\begin{array}{c} U\frac{n}{a}J_n(a)\,E_x + iUJ_n'(a)\,E_y \\ +\left[V\frac{n}{a}J_n(a) + \frac{\partial h_o}{\partial u_{sz}}J_n(a)\right]E_z \end{array}\right\}\frac{J_m(a)\,e^{i(m-n)\theta}}{i\,(k_z u_{sz} - \omega - n\omega_c)} \tag{2.13}$$

Now from (**??**) and (2.1)we have

$$\vec{k}\times\left(\vec{k}\times\vec{E}_1\right) = -\frac{\omega^2}{c^2}\left(\vec{E}_1 + \frac{i}{\omega\epsilon_0}\sum_\alpha n_\alpha q_\alpha\int\vec{u}_s\ h_{\alpha 1}\ d^3v\right)$$

$$-\overrightarrow{k}\times\left(\overrightarrow{k}\times\overrightarrow{E}_1\right) = \frac{\omega^2}{c^2}\overrightarrow{E}_1 + i\frac{\omega}{c^2\epsilon_0}\sum_\alpha n_\alpha q_\alpha\int\vec{u}_s h_{\alpha 1}d\mathbf{v} \tag{2.14}$$

Separating the equation in component we have

$$\begin{aligned}
\epsilon_{xx}E_x + \epsilon_{xy}E_y + \epsilon_{xz}E_z &= 0\\
\epsilon_{yx}E_x + \epsilon_{yy}E_y + \epsilon_{yz}E_z &= 0\\
\epsilon_{zx}E_x + \epsilon_{zy}E_y + \epsilon_{zz}E_z &= 0
\end{aligned}$$

It can be written as

$$\overleftrightarrow{\epsilon}\cdot\overrightarrow{E} = 0$$

Where

$$\overleftrightarrow{\epsilon} = \begin{bmatrix} \epsilon_{xx} & \epsilon_{xy} & \epsilon_{xz} \\ \epsilon_{yx} & \epsilon_{yy} & \epsilon_{yz} \\ \epsilon_{zx} & \epsilon_{zy} & \epsilon_{zz} \end{bmatrix}$$

Now using equation(2.13) , equation (2.14) becomes

Body Math

$$\begin{aligned}
L.H.S &= -\overrightarrow{k}\times\left(\overrightarrow{k}\times\overrightarrow{E}_1\right) = -\left(k_\perp\hat{x} + k_\parallel\hat{z}\right)\times\left[\left(k_\perp\hat{x} + k_\parallel\hat{z}\right)\times\left(E_x\hat{x} + E_y\hat{y} + E_z\hat{z}\right)\right]\\
&= -\left(k_\perp\hat{x} + k_\parallel\hat{z}\right)\times\left\{-k_\parallel E_y\hat{x} + \left(-k_\perp E_z + k_\parallel E_x\right)\hat{y} + k_\perp E_y\hat{z}\right\}\\
&= -\left\{-\left(-k_\perp k_\parallel E_z + k_\parallel^2 E_x\right)\hat{x} + \left(-k_\perp^2 E_y - k_\parallel^2 E_y\right)\hat{y} + \left(-k_\perp^2 E_z + k_\perp k_\parallel E_x\right)\hat{z}\right\}
\end{aligned}$$

$$
\begin{aligned}
&\left(-k_\perp k_\| E_z + k_\|^2 E_x\right)\hat{x} + \left(k_\perp^2 E_y + k_\|^2 E_y\right)\hat{y} + \left(k_\perp^2 E_z - k_\perp k_\| E_x\right)\hat{z} \\
= \; &\frac{\omega^2}{c^2}(E_x\hat{x}+E_y\hat{y}+E_z\hat{z}) + i\frac{\omega}{c^2\epsilon_o}\sum_\alpha n_\alpha q_\alpha \int (u_{s\perp}\cos\theta\hat{x}+u_{s\perp}\sin\theta\hat{y}+u_{sz}\hat{z}) \times \\
&\left[-\frac{q}{m}\sum_{n,m}\left[\begin{array}{c} U\left\{\frac{nJ_n(a)}{a}\right\}E_x - iUJ_n'(a)E_y \\ +\left\{V\frac{nJ_n(a)}{a}+\frac{\partial h_o}{\partial u_{sz}}J_n(a)\right\}E_z \end{array}\right]\left\{\frac{J_m(a)\exp\{-i(m-n)\theta\}}{i(k_z u_{sz}-\omega+n\Omega)}\right\}\right]d\mathbf{v} \qquad (2.15)
\end{aligned}
$$

Know calculating $\int \vec{u}_s\ h_{\alpha 1}\ d^3v$ we have

$(\mathbf{A})$ $\quad u_{s\perp}Cos\theta\hat{x}$

$$
\begin{aligned}
\int d^3v h_1 u_{s\perp}\cos\theta\hat{x} &= \int_0^\infty u_{s\perp}du_{s\perp}\int_{-\infty}^\infty du_{s\|}\int_0^{2\pi} d\theta\ h_1 u_{s\perp}Cos\theta\hat{x} \\
&= -\frac{q}{m}\sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty}\int_0^\infty u_{s\perp}^2 du_{s\perp}\int_{-\infty}^\infty du_{s\|}\int_0^{2\pi}\cos\theta e^{i(m-n)\theta}d\theta \\
&\quad \left\{\begin{array}{c} U\frac{n}{a}J_n(a)E_x + iUJ_n'(a)E_y \\ +\left[V\frac{n}{a}J_n(a)+\frac{\partial h_o}{\partial u_{sz}}J_n(a)\right]E_z \end{array}\right\}\frac{J_m(a)}{i(k_z u_{sz}-\omega-n\omega_c)}\hat{x} \qquad (2.16)
\end{aligned}
$$

$\int_0^{2\pi}\cos\theta e^{i(m-n)\theta}d\theta = \int_0^{2\pi}\cos(m-n)\theta\cos\theta + i\sin(m-n)\theta\sin\theta d\theta$

As $\quad \int_0^{2\pi}\cos ax\cos bx\ dx = \left\{\begin{array}{cc} \pi\delta_{ab} & a\neq 0 \\ 0 & for\ all\ others \end{array}\right\}$

$\int_0^{2\pi}\sin ax\cos bx\ dx = \{0 \qquad for\ all\ integral\ a\ and\ b\}$

So $\quad \int_0^{2\pi}\cos(m-n)\theta\cos\theta\ d\theta = \pi \qquad if\ \ and\ only\ if\ m-n=\pm 1$

$$\int_0^{2\pi}\cos\theta e^{\pm i\theta}d\theta = \pi$$

Integral exists only if $m = n+1 \qquad m = n-1$ so we have

$$
\begin{aligned}
{}_{m=-\infty}^{\infty}\int_0^{2\pi}\cos\theta e^{i(m-n)\theta}d\theta J_m(a) &= \int_0^{2\pi}\cos\theta e^{i\theta}d\theta J_{n+1}(a) + \int_0^{2\pi}\cos\theta e^{-i\theta}d\theta J_{n-1}(a) \\
&= 2\pi\left(\frac{J_{n-1}(a)+J_{n+1}(a)}{2}\right) = 2\pi\left(\frac{n}{a}\right)J_n(a) \\
&= \int_0^{2\pi}d\theta\left(\frac{n}{a}\right)J_n(a)
\end{aligned}
$$

So equation (2.16) becomes

$$\int d^3u_s h_1 u_{s\perp}\cos\theta\hat{x} = -\frac{q}{m}\sum_{n=-\infty}^{\infty}\int d^3u_s u_{s\perp}\left\{\begin{array}{c} U\frac{n}{a}J_n(a)E_x + iUJ_n'(a)E_y \\ +\left[V\frac{n}{a}J_n(a)+\frac{\partial h_o}{\partial u_{sz}}J_n(a)\right]E_z\end{array}\right\}\frac{\left(\frac{n}{a}\right)J_n(a)}{i(k_zu_{sz}-\omega+n\omega_c)}\hat{x} \quad (2.17)$$

Similarly,

(**B**) $u_{s\perp}\sin\theta\hat{y}$

$$\int d^3u_s h_1 u_{s\perp}\sin\theta\hat{y} = -\frac{q}{m}\sum_{n=-\infty}^{\infty}\int d^3u_s u_{s\perp}\left\{\begin{array}{c} U\frac{n}{a}J_n(a)E_x + iUJ_n'(a)E_y \\ +\left[V\frac{n}{a}J_n(a)+\frac{\partial h_o}{\partial u_{sz}}J_n(a)\right]E_z\end{array}\right\}\frac{-J_n'(a)}{(k_zu_{sz}-\omega-n\omega_c)}\hat{y} \quad (2.18)$$

(**C**) $v_{\|}\ \hat{z}$

$$\int d^3u_s h_1 u_{s\|}\hat{z} = -\frac{q}{m}\sum_{n=-\infty}^{\infty}\int d^3u_s u_{s\|}\left\{\begin{array}{c} U\frac{n}{a}J_n(a)E_x + iUJ_n'(a)E_y \\ +\left[W\frac{n}{a}J_n(a)+\frac{\partial h_o}{\partial u_{sz}}J_n(a)\right]E_z\end{array}\right\}\frac{J_n(a)}{i(k_zu_{sz}-\omega-n\omega_c)}\hat{z} \quad (2.19)$$

Now putting(2.17) (2.18) (2.19) in (2.15) we have

$$\left(\frac{-k_\perp k_\| E_z + k_\|^2 E_x}{\omega^2}c^2 - E_x\right)\hat{x} + \left(\frac{k_\perp^2 E_y + k_\|^2 E_y}{\omega^2}c^2 - E_y\right)\hat{y} + \left(\frac{k_\perp^2 E_z - k_\perp k_\| E_x}{\omega^2}c^2 - E_z\right)\hat{z}$$

$$= -\sum_\alpha \frac{n_\alpha q_\alpha^2}{m\omega\epsilon_0}\sum_{n=-\infty}^{\infty}\int d^3u_s$$

$$\left[\begin{array}{l} \left\{\begin{array}{c} U\frac{n}{a}J_n(a)E + iUJ_n'(a)E_y \\ +\left[V\frac{n}{a}J_n(a)+\frac{\partial h_o}{\partial u_{sz}}J_n(a)\right]E_z\end{array}\right\}\frac{u_{s\perp}\left(\frac{n}{a}\right)J_n(a)}{(k_zu_{sz}-\omega-n\omega_c)}\hat{x}+ \\ \left\{\begin{array}{c} U\frac{n}{a}J_n(a)E_x + iUJ_n'(a)E_y \\ +\left[V\frac{n}{a}J_n(a)+\frac{\partial h_o}{\partial u_{sz}}J_n(a)\right]E_z\end{array}\right\}\frac{-iu_{s\perp}J_n'(a)}{(k_zu_{sz}-\omega-n\omega_c)}\hat{y}+ \\ \left\{\begin{array}{c} U\frac{n}{a}J_n(a)E_x + iUJ_n'(a)E_y \\ +\left[V\frac{n}{a}J_n(a)+\frac{\partial h_o}{\partial u_{sz}}J_n(a)\right]E_z\end{array}\right\}\frac{u_{s\|}J_n(a)}{(k_zu_{sz}-\omega-n\omega_c)}\hat{z}\end{array}\right]$$

Now separating the components with E_x, E_y and E_z term and comparing it we have

(1) x-component

$$\left(1-\frac{k_{\parallel}^{2}c^{2}}{\omega^{2}}-\sum_{\alpha}\frac{n_{\alpha}q_{\alpha}^{2}}{m\omega\epsilon_{0}}\sum_{n=-\infty}^{\infty}\int d^{3}u_{s}\left\{U\frac{n}{a}J_{n}\left(a\right)\right\}\frac{u_{s\perp}\left(\frac{n}{a}\right)J_{n}\left(a\right)}{\left(k_{z}u_{sz}-\omega-n\omega_{c}\right)}\right)E_{x}+$$
$$\left(-\sum_{\alpha}\frac{n_{\alpha}q_{\alpha}^{2}}{m\omega\epsilon_{0}}\sum_{n=-\infty}^{\infty}\int d^{3}u_{s}\left\{iUJ_{n}'\left(a\right)\right\}\frac{u_{s\perp}\left(\frac{n}{a}\right)J_{n}\left(a\right)}{\left(k_{z}u_{sz}-\omega-n\omega_{c}\right)}\right)E_{y}+$$
$$\left(\frac{k_{\perp}k_{\parallel}c^{2}}{\omega^{2}}-\sum_{\alpha}\frac{n_{\alpha}q_{\alpha}^{2}}{m\omega\epsilon_{0}}\sum_{n=-\infty}^{\infty}\int d^{3}u_{s}\left\{V\frac{n}{a}J_{n}\left(a\right)+\frac{\partial h_{o}}{\partial u_{sz}}J_{n}\left(a\right)\right\}\frac{u_{s\perp}\left(\frac{n}{a}\right)J_{n}\left(a\right)}{\left(k_{z}u_{sz}-\omega-n\omega_{c}\right)}\right)E_{z}$$
$$=\ 0$$

(2) y-component

$$\left(-\sum_{\alpha}\frac{n_{\alpha}q_{\alpha}^{2}}{m\omega\epsilon_{0}}\sum_{n=-\infty}^{\infty}\int d^{3}u_{s}\left\{U\frac{n}{a}J_{n}\left(a\right)\right\}\frac{-iu_{s\perp}J_{n}'\left(a\right)}{\left(k_{z}u_{sz}-\omega-n\omega_{c}\right)}\right)E_{x}+$$
$$\left(1-\frac{\left(k_{\perp}^{2}+k_{\parallel}^{2}\right)c^{2}}{\omega^{2}}-\sum_{\alpha}\frac{n_{\alpha}q_{\alpha}^{2}}{m\omega\epsilon_{0}}\sum_{n=-\infty}^{\infty}\int d^{3}u_{s}\left\{iUJ_{n}'\left(a\right)\right\}\frac{-iu_{s\perp}J_{n}'\left(a\right)}{\left(k_{z}u_{sz}-\omega-n\omega_{c}\right)}\right)E_{y}+$$
$$\left(-\sum_{\alpha}\frac{n_{\alpha}q_{\alpha}^{2}}{m\omega\epsilon_{0}}\sum_{n=-\infty}^{\infty}\int d^{3}u_{s}\left\{V\frac{n}{a}J_{n}\left(a\right)+\frac{\partial h_{o}}{\partial u_{sz}}J_{n}\left(a\right)\right\}\frac{-iu_{s\perp}J_{n}'\left(a\right)}{\left(k_{z}u_{sz}-\omega-n\omega_{c}\right)}\right)E_{z}$$
$$=\ 0$$

(3) z-component

$$\left(1+\frac{k_{\perp}k_{\parallel}c^{2}}{\omega^{2}}-\sum_{\alpha}\frac{n_{\alpha}q_{\alpha}^{2}}{m\omega\epsilon_{0}}\sum_{n=-\infty}^{\infty}\int d^{3}u_{s}\left\{U\frac{n}{a}J_{n}\left(a\right)\right\}\frac{u_{s\parallel}J_{n}\left(a\right)}{\left(k_{z}u_{sz}-\omega-n\omega_{c}\right)}\right)E_{x}+$$
$$\left(-\sum_{\alpha}\frac{n_{\alpha}q_{\alpha}^{2}}{m\omega\epsilon_{0}}{}_{n=-\infty}^{\infty}\int d^{3}u_{s}\left\{iUJ_{n}'\left(a\right)\right\}\frac{u_{s\parallel}J_{n}\left(a\right)}{\left(k_{z}u_{sz}-\omega-n\omega_{c}\right)}\right)E_{y}+$$
$$\left(1-\frac{k_{\perp}^{2}c^{2}}{\omega^{2}}-\sum_{\alpha}\frac{n_{\alpha}q_{\alpha}^{2}}{m\omega\epsilon_{0}}{}_{n=-\infty}^{\infty}\int d^{3}u_{s}\left\{V\frac{n}{a}J_{n}\left(a\right)+\frac{\partial h_{o}}{\partial u_{sz}}J_{n}\left(a\right)\right\}\frac{u_{s\parallel}J_{n}\left(a\right)}{\left(k_{z}u_{sz}-\omega-n\omega_{c}\right)}\right)E_{z}$$
$$=\ 0$$

Now by comparing $E_x, E_{y,} E_z$ terms from the above equations we get

$$
\begin{aligned}
\epsilon_{xx} &= 1-\frac{k_\parallel^2 c^2}{\omega^2}-\sum_\alpha\sum_{n=-\infty}^{\infty}\frac{n_\alpha q_\alpha^2}{m\omega\epsilon_0}\int d^3u_s\ u_{s\perp}U\frac{\left[\left(\frac{n}{a}\right)J_n(a)\right]^2}{(k_z u_{sz}-\omega-n\omega_c)}\\
\epsilon_{xy} &= -\sum_\alpha\sum_{n=-\infty}^{\infty}\frac{n_\alpha q_\alpha^2}{m_\alpha\omega\epsilon_o}\int d^3u_s\ u_{s\perp}\left\{iUJ_n'(a)\right\}\frac{\left(\frac{n}{a}\right)J_n(a)}{(k_z u_{sz}-\omega-n\omega_c)}\\
\epsilon_{xz} &= \frac{k_\perp k_\parallel c^2}{\omega^2}-\sum_\alpha\sum_{n=-\infty}^{\infty}\frac{n_\alpha q_\alpha^2}{m_\alpha\omega\epsilon_o}\int d^3u_s\ u_{s\perp}\left\{V\frac{n}{a}J_n(a)+\frac{\partial h_o}{\partial u_{sz}}J_n(a)\right\}\frac{\left(\frac{n}{a}\right)J_n(a)}{(k_z u_{sz}-\omega-n\omega_c)}\\
\epsilon_{yx} &= -\sum_\alpha\sum_{n=-\infty}^{\infty}\frac{n_\alpha q_\alpha^2}{m_\alpha\omega\epsilon_o}\int d^3u_s u_{s\perp}\left\{U\frac{n}{a}J_n(a)\right\}\frac{-iJ_n'(a)}{(k_z u_{sz}-\omega-n\omega_c)}\\
\epsilon_{yy} &= 1-\frac{\left(k_\perp^2+k_\parallel^2\right)c^2}{\omega^2}-\sum_\alpha\sum_{n=-\infty}^{\infty}\frac{n_\alpha q_\alpha^2}{m_\alpha\omega\epsilon_o}\int d^3u_s u_{s\perp}\left\{UJ_n'(a)\right\}\frac{J_n'(a)}{(k_z u_{sz}-\omega-n\omega_c)}\\
\epsilon_{yz} &= -\sum_\alpha\sum_{n=-\infty}^{\infty}\frac{n_\alpha q_\alpha^2}{m_\alpha\omega\epsilon_o}\int d^3u_s u_{s\perp}\left\{V\frac{n}{a}J_n(a)+\frac{\partial h_o}{\partial u_{sz}}J_n(a)\right\}\frac{-iJ_n'(a)}{(k_z u_{sz}-\omega-n\omega_c)}\\
\epsilon_{zx} &= 1+\frac{k_\perp k_\parallel c^2}{\omega^2}-\sum_\alpha\sum_{n=-\infty}^{\infty}\frac{n_\alpha q_\alpha^2}{m_\alpha\omega\epsilon_o}\int d^3u_s\ u_{s\parallel}\left\{U\frac{n}{a}J_n(a)\right\}\frac{J_n(a)}{(k_z u_{sz}-\omega-n\omega_c)}\\
\epsilon_{zy} &= -\sum_\alpha\sum_{n=-\infty}^{\infty}\frac{n_\alpha q_\alpha^2}{m_\alpha\omega\epsilon_o}\int d^3u_s\ u_{s\parallel}\left\{iUJ_n'(a)\right\}\frac{J_n(a)}{(k_z u_{sz}-\omega-n\omega_c)}\\
\epsilon_{zz} &= 1-\frac{k_\perp^2 c^2}{\omega^2}-\sum_\alpha\sum_{n=-\infty}^{\infty}\frac{n_\alpha q_\alpha^2}{m_\alpha\omega\epsilon_o}\int d^3u_s\ u_{s\parallel}\left\{V\frac{n}{a}J_n(a)+\frac{\partial h_o}{\partial u_{sz}}J_n(a)\right\}\frac{J_n(a)}{(k_z u_{sz}-\omega-n\omega_c)}
\end{aligned}
$$

$$
\overleftrightarrow{\epsilon}=\overleftrightarrow{A}-\sum_\alpha\sum_{n=-\infty}^{\infty}\frac{\omega_{p\alpha}^2}{\omega}\int d^3u_s\frac{S_{ij}}{(k_z u_{sz}-\omega-n\omega_c)} \tag{2.20}
$$

$$
S_{ij}=\left\{\begin{matrix}
u_{s\perp}\left(\frac{n}{a}\right)^2J_n^2(a)U & iu_{s\perp}\left(\frac{n}{a}\right)J_n(a)J_n'(a)U & u_{s\perp}\left(\frac{n}{a}\right)J_n^2(a)\left\{V\left(\frac{n}{a}\right)+\frac{\partial h_o}{\partial u_{s\parallel}}\right\}\\
-iu_{s\perp}\left(\frac{n}{a}\right)J_n(a)J_n'(a)U & u_{s\perp}UJ_n'^{\,2}(a) & -iu_{s\perp}J_n(a)J_n'(a)\left\{V\left(\frac{n}{a}\right)+\frac{\partial h_o}{\partial u_{s\parallel}}\right\}\\
u_{s\parallel}\left(\frac{n}{a}\right)J_n^2(a)U & iu_{s\parallel}J_n(a)J_n'(a)U & u_{s\parallel}J_n^2(a)\left\{V\left(\frac{n}{a}\right)+\frac{\partial h_o}{\partial u_{s\parallel}}\right\}
\end{matrix}\right\} \tag{2.21}
$$

$$
\overleftrightarrow{A}=\begin{bmatrix}
1-\frac{k_\parallel^2c^2}{\omega^2} & 0 & \frac{k_\perp k_\parallel c^2}{\omega^2}\\
0 & 1-\frac{\left(k_\perp^2+k_\parallel^2\right)c^2}{\omega^2} & 0\\
1+\frac{k_\perp k_\parallel c^2}{\omega^2} & 0 & 1-\frac{k_\perp^2c^2}{\omega^2}
\end{bmatrix} \tag{2.22}
$$

$$
U=\left[\left(1-\frac{k_z u_{sz}}{\omega}\right)\frac{\partial h_o}{\partial u_{s\perp}}+\frac{k_z u_{s\perp}}{\omega}\frac{\partial h_o}{\partial u_{s\parallel}}\right] \qquad V=\left[\frac{k_x u_{sz}}{\omega}\frac{\partial h_o}{\partial u_{s\perp}}-\frac{k_x u_{s\perp}}{\omega}\frac{\partial h_o}{\partial u_{s\parallel}}\right]
$$

$$a = \frac{k_x u_{s\perp}}{\omega} \quad \omega_c = \frac{qB_o}{m}$$

Where as $\quad u_z = u_\parallel$

From above components ϵ_{yy} is taken as kinetic relation for whistler modes, which is

$$\epsilon_{yy} = 1 - \frac{\left(k_\perp^2 + k_\parallel^2\right) c^2}{\omega^2} - \sum_\alpha \sum_{n=-\infty}^{\infty} \frac{n_\alpha q_\alpha^2}{m_\alpha \omega \epsilon_o} \int d^3 u_s u_{s\perp} \left\{U J_n{}'\left(a\right)\right\} \frac{J_n{}'\left(a\right)}{\left(k_z u_{sz} - \omega - n\omega_c\right)}$$

This is generalized kinetic dispersion relation for oblique whistler modes.

Chapter 3

Oblique whistler waves mode with Bi-Maxwellian distribution

3.1 Bi-Maxwellian distribution function

The Bi-maxwellian distribution is an anisotropic distribution that isn't the same as the simple Maxwellian distribution. Anisotropy in plasma caused by the presence of a magnetic field causes different particle velocities that can be parallel or perpendicular to the magnetic field. The velocity distribution of gyrating particles is determined by the independent variables $v_\perp$ and $v_\parallel$. In this case, equilibrium distribution is modelled as product of two Maxwellian and resulting distribution is Bi-Maxwellian which is equal to

$$f\left(v_\perp, v_\parallel\right) = \frac{n}{\left(\pi^3 \langle v_\perp \rangle^2 \langle v_\parallel \rangle^{\frac{1}{2}}\right)} Exp\left[-\frac{v_\perp^2}{\langle v_\perp \rangle^2} - \frac{v_\parallel^2}{\langle v_\parallel \rangle^2}\right]$$

Now we will use this distribution in kinetic treatment of oblique whistler waves.

3.2 Oblique Whistler treatment with Bi-Maxwellian distribution

For Bi-Maxwellian distribution starting from relation propagation

$$1 - \frac{\left(k_\perp^2 + k_\parallel^2\right) c^2}{\omega^2} - \sum_\alpha \frac{\omega_{p\alpha}^2}{\omega} \int d^3v \frac{v_\perp \left(J_n'\left(a\right)\right)^2}{\left(k_\parallel v_\parallel - \omega - n\omega_c\right)} \left(\left(1 - \frac{k_\parallel v_\parallel}{\omega}\right) \frac{\partial f_o}{\partial v_\perp} + \frac{k_\parallel v_\perp}{\omega} \frac{\partial f_o}{\partial v_\parallel}\right) = 0 \quad (3.1)$$

As Bi-Maxwellian distribution is,

$$f_o = A\exp\left(-\frac{v_\perp^2}{v_{t\perp}^2}-\frac{v_\parallel^2}{v_{t\parallel}^2}\right)$$

where as,

$$A = \frac{1}{(\pi)^{3/2}\, v_{t\perp}^2 v_{t\parallel}}$$

Taking the derivative of the perpendicular and parallel component of velocity respectively we have,

$$\frac{\partial f_o}{\partial v_\perp} = -f_o\left(\frac{2v_\perp}{v_{t\perp}^2}\right)$$

$$\frac{\partial f_o}{\partial v_\parallel} = -f_o\left(\frac{2v_\parallel}{v_{t\parallel}^2}\right)$$

Putting these values in Eq (3.1) , we have,

$$1-\frac{\left(k_\perp^2+k_\parallel^2\right)c^2}{\omega^2}+\sum_\alpha\frac{\omega_{p\alpha}^2}{\omega}\int d^3v\frac{v_\perp\left(J_n'(a)\right)^2 f_0}{\left(k_\parallel v_\parallel-\omega-n\omega_c\right)}\left(\left(1-\frac{k_\parallel v_\parallel}{\omega}\right)\left(\frac{2v_\perp}{v_{t\perp}^2}\right)+\frac{k_\parallel v_\perp}{\omega}\left(\frac{2v_\parallel}{v_{t\parallel}^2}\right)\right)=0$$

After simplifying the terms we have,

$$1-\frac{\left(k_\perp^2+k_\parallel^2\right)c^2}{\omega^2}+\sum_\alpha\frac{2\omega_{p\alpha}^2}{\omega v_{t\perp}^2}A\int d^3v\frac{v_\perp^2\left(J_n'(a)\right)^2\exp\left(-\frac{v_\perp^2}{v_{t\perp}^2}-\frac{v_\parallel^2}{v_{t\parallel}^2}\right)}{\left(k_\parallel v_\parallel-\omega-n\omega_c\right)}\left(1+\frac{k_\parallel v_\parallel}{\omega}A_T\right)=0$$

Since,

$$A_T = \frac{v_{t\perp}^2}{v_{t\parallel}^2}-1=\frac{T_\perp}{T_\parallel}-1$$

Here $\int d^3v = \int v_\perp dv_\perp dv_\parallel d\theta$, and their limits are taken as $dv_\perp$ is from $0\to\infty$, $dv_\parallel$ is from $-\infty\to\infty$ and for $d\theta$, $0\to 2\pi$, so

$$1-\frac{\left(k_\perp^2+k_\parallel^2\right)c^2}{\omega^2}+\sum_\alpha\frac{2\omega_{p\alpha}^2}{\omega v_{t\perp}^2}A\int 2\pi dv_\perp dv_\parallel\frac{v_\perp^3\left(J_n'(a)\right)^2\exp\left(-\frac{v_\perp^2}{v_{t\perp}^2}-\frac{v_\parallel^2}{v_{t\parallel}^2}\right)}{\left(k_\parallel v_\parallel-\omega-n\omega_c\right)}\left(1+\frac{k_\parallel v_\parallel}{\omega}A_T\right)=0 \quad (3.2)$$

Now integrating the perpendicular component of velocity we have,

$$\int_0^\infty dv_\perp \left(J_n'(a)\right)^2 v_\perp^3 \exp\left(-\frac{v_\perp^2}{v_{t\perp}^2}\right)$$

Defining the values of x, dx *and* b we have,

$$x = \frac{k_\perp v_\perp}{\omega_c} \quad , \quad dx = \frac{k_\perp}{\omega_c} dv_\perp$$

$$b = \frac{k_\perp^2 v_{t\perp}^2}{2\omega_c^2}$$

So integrating the perpendicular component of velocity will become,

$$\left(\frac{\omega_c}{k_\perp}\right)^4 \int_0^\infty dx \left(J_n'(x)\right)^2 x^3 \exp\left(-\frac{x^2}{2b}\right) = \left(\frac{\omega_c}{k_\perp}\right)^4 e^{-b} \left(\begin{array}{c} b\left(2b^2 - 2bn + n^2\right) I_n(b) - \\ 2b^3 I_{n+1}(b) \end{array}\right)$$

Putting it back in Eq (3.2) and after some simplifications we have ,

$$\begin{aligned} & 1 - \frac{\left(k_\perp^2 + k_\parallel^2\right) c^2}{\omega^2} + \sum_\alpha \frac{v_{t\perp}^2 \omega_{p\alpha}^2}{2\omega} A \int 2\pi dv_\parallel \frac{\exp\left(-\frac{v_\parallel^2}{v_{t\parallel}^2}\right)}{\left(k_\parallel v_\parallel - \omega - n\omega_c\right)} \times \\ & \left(1 + \frac{k_\parallel v_\parallel}{\omega} A_T\right) \frac{e^{-b}}{b} \left(\begin{array}{c} \left(2b^2 - 2bn + n^2\right) I_n(b) - \\ 2b^2 I_{n+1}(b) \end{array}\right) \\ = \; & 0 \end{aligned}$$

Let defining some term in above equation i-e

$$I(nb) = \frac{e^{-b}}{b} \left(\begin{array}{c} \left(2b^2 - 2bn + n^2\right) I_n(b) - \\ 2b^2 I_{n+1}(b) \end{array}\right)$$

Putting this value in above equation we have,

$$1 - \frac{\left(k_\perp^2 + k_\parallel^2\right) c^2}{\omega^2} + \sum_\alpha \frac{v_{t\perp}^2 \omega_{p\alpha}^2}{2\omega} A \int 2\pi dv_\parallel \frac{\exp\left(-\frac{v_\parallel^2}{v_{t\parallel}^2}\right)}{\left(k_\parallel v_\parallel - \omega - n\omega_c\right)} \left(1 + \frac{k_\parallel v_\parallel}{\omega} A_T\right) I(nb) = 0 \tag{3.3}$$

Taking the parallel component of velocity for integration i-e,

$$\int_{-\infty}^{\infty} dv_\parallel \frac{\left(1+\frac{k_\parallel v_\parallel}{\omega}A_T\right)}{k_\parallel v_\parallel-(\omega\pm\omega_c)}\exp\left(-\frac{v_\parallel^2}{v_{t\parallel}^2}\right)$$

Let $S=\dfrac{v_\parallel}{v_{t\parallel}}$ and $dSv_{t\parallel}=dv_\parallel$

$$=\int_{-\infty}^{\infty} v_{t\parallel} dS \frac{\left(1+\frac{k_\parallel v_{t\parallel} S}{\omega}A_T\right)}{k_\parallel v_{t\parallel}S-(\omega\pm\omega_c)}\exp\left(-S^2\right)$$

$$=\frac{1}{k_\parallel}\int_{-\infty}^{\infty} dS\frac{\exp\left[-S^2\right]}{S-\frac{(\omega\pm\omega_c)}{k_\parallel v_{t\parallel}}}+\frac{1}{k_\parallel}\frac{k_\parallel v_{t\parallel}A_T}{\omega}\int_{-\infty}^{\infty} dS\frac{S\exp\left[-S^2\right]}{S-\frac{(\omega\pm\omega_c)}{k_\parallel v_{t\parallel}}}$$

Here, we defined $\xi=\frac{(\omega\pm\omega_c)}{k_\parallel v_{t\parallel}}$. Now simplifying parallel components we have,

$$=\frac{1}{k_\parallel}\left[\int_{-\infty}^{\infty} ds\frac{\exp\left[-s^2\right]}{s-\xi}+\frac{k_\parallel v_{t\parallel}A_T}{\omega}\int_{-\infty}^{\infty} ds\frac{s\exp\left[-s^2\right]}{s-\xi}\right]$$

The plasma dispersion function and its derivative is defined as,

$$Z(\xi)=\frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty} ds\frac{e^{-s^2}}{S-\xi}$$

$$\acute{Z}(\xi)=-\frac{2}{\sqrt{\pi}}\int_{-\infty}^{\infty} ds\frac{se^{-s^2}}{(s-\xi)}$$

Using the values of $Z(\xi)$ and $\acute{Z}(\xi)$ to get,

$$=\frac{\sqrt{\pi}}{k_\parallel}\left[Z(\xi)-\frac{k_\parallel v_{t\parallel}A_T}{2\omega}\acute{Z}(\xi)\right]$$

Putting this value in Eq (3.3) and simplifying equation we have,

$$1-\frac{\left(k_\perp^2+k_\parallel^2\right)c^2}{\omega^2}+\sum_\alpha \frac{\omega_{p\alpha}^2}{\omega}\ \frac{I(nb)}{k_\parallel v_{t\parallel}}\left[Z(\xi)-\frac{k_\parallel v_{t\parallel}A_T}{2\omega}\acute{Z}(\xi)\right]=0$$

Expanding the plasma dispersion function for large argument we have,

$$Z(\xi)=i\sqrt{\pi}e^{-\xi^2}-\frac{1}{\xi}\left(1+\frac{1}{2\xi^2}+\frac{3}{4\xi^4}+\frac{15}{8\xi^6}\right)........$$

The derivative of the plasma dispersion function has the following form,

$$\acute{Z}(\xi)=-2i\sqrt{\pi}\xi e^{-\xi^2}+\frac{1}{\xi^2}$$

Now we have,

$$1-\frac{\left(k_\perp^2+k_\parallel^2\right)c^2}{\omega^2}+\sum_\alpha \frac{\omega_{p\alpha}^2}{\omega}\frac{I\left(nb\right)}{k_\parallel v_{t\parallel}}\left[i\sqrt{\pi}e^{-\xi^2}-\frac{1}{\xi}\left(1+\frac{1}{2\xi^2}+....\right)-\frac{k_\parallel v_{t\parallel}A_T}{2\omega}\left(-2i\sqrt{\pi}\xi e^{-\xi^2}+\frac{1}{\xi^2}\right)\right]=0$$

Separating the Real and Imaginary parts we have,

$$D_r\left(\omega\right)=1-\frac{\left(k_\perp^2+k_\parallel^2\right)c^2}{\omega^2}+\sum_\alpha \frac{\omega_{p\alpha}^2}{\omega}\frac{I\left(nb\right)}{k_\parallel v_{t\parallel}}\left[-\frac{1}{\xi}\left(1+\frac{1}{2\xi^2}....\right)-\frac{k_\parallel v_{t\parallel}A_T}{2\omega}\left(\frac{1}{\xi^2}\right)\right]$$

$$D_i\left(\omega\right)=1-\frac{\left(k_\perp^2+k_\parallel^2\right)c^2}{\omega^2}+\sum_\alpha \frac{\omega_{p\alpha}^2}{\omega}\frac{I\left(nb\right)}{k_\parallel v_{t\parallel}}\left[\sqrt{\pi}e^{-\xi^2}-\frac{k_\parallel v_{t\parallel}A_T}{2\omega}\left(-2\sqrt{\pi}\xi e^{-\xi^2}\right)\right]$$

As,

$$D_r\left(\omega\right)=0$$

So, the real part of the dispersion relation is,

$$D_r\left(\omega\right)=1-\frac{\left(k_\perp^2+k_\parallel^2\right)c^2}{\omega^2}+\sum_\alpha \frac{\omega_{p\alpha}^2}{\omega}\frac{I\left(nb\right)}{k_\parallel v_{t\parallel}}\left[-\frac{1}{\xi}\left(1+\frac{1}{2\xi^2}....\right)-\frac{k_\parallel v_{t\parallel}A_T}{2\omega}\left(\frac{1}{\xi^2}\right)\right]=0$$

Taking the terms of plasma dispersion function upto $\frac{1}{\xi^2}$ we have,

$$1-\frac{\left(k_\perp^2+k_\parallel^2\right)c^2}{\omega^2}+\sum_\alpha \frac{\omega_{p\alpha}^2}{\omega}\frac{I\left(nb\right)}{k_\parallel v_{t\parallel}}\left[-\frac{1}{\xi}-\frac{k_\parallel v_{t\parallel}A_T}{2\omega}\frac{1}{\xi^2}\right]=0$$

Putting value of ξ where $\xi=\frac{\omega_r-|\omega_c|}{k_\parallel v_{t\parallel}}$ and as $\frac{\omega_r-|\omega_c|}{k_\parallel v_{t\parallel}}>>1$ so we have $|\omega_c|>>\omega_r$,

$$1-\frac{\left(k_\perp^2+k_\parallel^2\right)c^2}{\omega^2}+\sum_\alpha \frac{\omega_{p\alpha}^2}{\omega}I\left(nb\right)\left[\frac{1}{|\omega_c|}-\frac{A_T}{2\omega}\frac{\left(k_\parallel v_{t\parallel}\right)^2}{|\omega_c|^2}\right]=0$$

$$\frac{\omega}{|\omega_c|}=\frac{\left(k_\perp^2+k_\parallel^2\right)c^2}{\omega_{p\alpha}^2 I\left(nb\right)}\left(1+\frac{\omega_{p\alpha}^2 I\left(nb\right)}{\left(\frac{k_\perp^2}{k_\parallel^2}+1\right)}\frac{A_T}{2}\frac{\left(v_{t\parallel}\right)^2}{c^2|\omega_c|^2}\right)$$

Putting the values of $c^2=\frac{1}{\mu_o\epsilon_o}$, $\omega_{p\alpha}^2=\frac{ne^2}{m\epsilon_o}$, $v_{t\parallel}^2=\frac{T_\parallel}{m}$ and $\omega_c=\frac{eB}{m}$ in $\frac{\omega_{p\alpha}^2}{2}\frac{v_{t\parallel}^2}{c^2|\omega_c|^2}$ to get $\frac{nT_\parallel\mu_o}{2B^2}$. Now putting this value in $\frac{\omega}{|\omega_c|}$ expression we have,

$$\frac{\omega}{|\omega_c|}=\frac{\left(k_\perp^2+k_\parallel^2\right)c^2}{\omega_{p\alpha}^2 I\left(nb\right)}\left(1+\frac{A_T I\left(nb\right)}{\left(\frac{k_\perp^2}{k_\parallel^2}+1\right)}\frac{nT_\parallel\mu_o}{2B^2}\right)$$

Now further simplifying it by putting $\beta_{\|} = \frac{nT_{\|}\mu_o}{B^2}$ we have,

$$\frac{\omega}{|\omega_c|} = \frac{\left(k_{\perp}^2 + k_{\|}^2\right) c^2}{\omega_{p\alpha}^2 I\left(nb\right)} \left(1 + \frac{\beta_{\|} A_T I\left(nb\right)}{2\left(\frac{k_{\perp}^2}{k_{\|}^2} + 1\right)} \right)$$

Where speed of light is defined as $c^2 = \frac{1}{\mu_o \epsilon_o}$, the plasma frequency as $\omega_{p\alpha}^2 = \frac{ne^2}{m\epsilon_o}$, the parallel thermal velocity as $v_{t\|}^2 = \frac{T_{\|}}{m}$ and

the cyclotron frequency as $\omega_c = \frac{eB}{m}$.This is the required dispersion relation for oblique Whistler waves mode when we take the Bi-Maxwellian distribution function.

To find growth rate of this model using Bi-Maxwellian distribution function, consider

$$\gamma = -\frac{D_i\left(\omega\right)}{\frac{\partial D_r(\omega)}{\partial \omega_r}} \tag{3.4}$$

As the expression for $D_r\left(\omega\right)$ and $D_i\left(\omega\right)$ respectively are,

$$D_r\left(\omega\right) = 1 - \frac{\left(k_{\perp}^2 + k_{\|}^2\right) c^2}{\omega^2} + \sum_{\alpha} \frac{\omega_{p\alpha}^2}{\omega} \frac{I\left(nb\right)}{k_{\|} v_{t\|}} \left[-\frac{1}{\xi}\left(1 + \frac{1}{\xi^2}....\right) - \frac{k_{\|} v_{t\|} A_T}{2\omega}\left(\frac{1}{\xi^2}\right)\right]$$

$$D_i\left(\omega\right) = \sum_{\alpha} \frac{\omega_{p\alpha}^2}{\omega} \frac{I\left(nb\right)}{k_{\|} v_{t\|}} \left[\sqrt{\pi} e^{-\xi^2} - \frac{k_{\|} v_{t\|} A_T}{2\omega}\left(-2\sqrt{\pi}\xi e^{-\xi^2}\right)\right]$$

As $|\omega_c| >> \omega_r$ so we have $\omega - |\omega_c| = -|\omega_c|$ and Simplifying the $D_i\left(\omega\right)$ part we have,

$$D_i\left(\omega\right) = \sum_{\alpha} \sqrt{\pi} e^{-\left(\frac{\omega - |\omega_c|}{k_{\|} v_{t\|}}\right)^2} \frac{\omega_{p\alpha}^2}{\omega} \frac{I\left(nb\right)}{k_{\|} v_{t\|}} \left[1 - \left(\frac{k_{\|} v_{t\|} A_T}{\omega}\right)\left(\frac{-|\omega_c|}{k_{\|} v_{t\|}}\right)\right]$$

$$D_i\left(\omega\right) = \sum_{\alpha} \sqrt{\pi} e^{-\left(\frac{\omega - |\omega_c|}{k_{\|} v_{t\|}}\right)^2} \frac{\omega_{p\alpha}^2}{\omega^2} \frac{I\left(nb\right)}{k_{\|} v_{t\|}} \left(\omega + A_T\left(|\omega_c|\right)\right)$$

Now Simplifying $D_r\left(\omega\right)$ expression we have,

$$D_r\left(\omega\right) = 1 - \frac{\left(k_{\perp}^2 + k_{\|}^2\right) c^2}{\omega^2} + \sum_{\alpha} \frac{\omega_{p\alpha}^2}{\omega} \frac{I\left(nb\right)}{k_{\|} v_{t\|}} \left[-\frac{1}{\xi}\left(1 + \frac{1}{2\xi^2}....\right) - \frac{k_{\|} v_{t\|} A_T}{2\omega}\left(\frac{1}{\xi^2}\right)\right]$$

$$D_r\left(\omega\right) = 1 - \frac{\left(k_{\perp}^2 + k_{\|}^2\right) c^2}{\omega^2} + \sum_{\alpha} \frac{\omega_{p\alpha}^2}{\omega} I\left(nb\right) \left[-\frac{1}{\left(\omega_r - |\omega_c|\right)} - \frac{\left(k_{\|} v_{t\|}\right)^2}{2\left(\omega_r - |\omega_c|\right)^3} - \frac{A_T}{2\omega} \frac{\left(k_{\|} v_{t\|}\right)^2}{\left(\omega_r - |\omega_c|\right)^2}\right]$$

Taking the derivative of the real part we have,

$$\frac{\partial D_r(\omega)}{\partial \omega_r} = +2\frac{\left(k_\perp^2 + k_\parallel^2\right)c^2}{\omega^3} + \sum_\alpha \omega_{p\alpha}^2 \; I(nb)\left[\begin{array}{c} \frac{(2\omega - |\omega_c|)}{[\omega(\omega - |\omega_c|)]^2} + \frac{\left(k_\parallel v_{t\parallel}\right)^2\left(4\omega^3 - 9\omega^2|\omega_c| + 6\omega|\omega_c|^2 - |\omega_c|^3\right)}{2\left(\omega(\omega-|\omega_c|)^3\right)^2} + \\ \frac{A_T}{2}\frac{\left(k_\parallel v_{t\parallel}\right)^2\left(4\omega^3 - 6\omega^2|\omega_c| + 2\omega^1|\omega_c|^2\right)}{\left(\omega^2(\omega-|\omega_c|)^2\right)^2} \end{array}\right]$$

As $|\omega_c| >> \omega$, neglecting the higher power terms of ω, and using $(\omega_r - |\omega_c|) = -|\omega_c|$

$$\frac{\partial D_r(\omega)}{\partial \omega_r} = 2\frac{\left(k_\perp^2 + k_\parallel^2\right)c^2}{\omega^3} + \sum_\alpha \omega_{p\alpha}^2 \; I(nb)\left[-\frac{1}{\omega^2|\omega_c|} - \frac{\left(k_\parallel v_{t\parallel}\right)^2}{2\omega^2|\omega_c|^3} + A_T\frac{\left(k_\parallel v_{t\parallel}\right)^2}{\omega^3|\omega_c|^2}\right]$$

Now putting these values in Eq (3.4) we have,

$$\gamma = -\frac{\sum_\alpha \sqrt{\pi} I(nb)\,\omega_{p\alpha}^2 e^{-\left(\frac{\omega - |\omega_c|}{k_\parallel v_{t\parallel}}\right)^2}\left(\omega + A_T(|\omega_c|)\right)}{\omega^2 k_\parallel v_{t\parallel}\left(2\frac{\left(k_\perp^2+k_\parallel^2\right)c^2}{\omega^3} + \sum_\alpha \omega_{p\alpha}^2 \; I(nb)\left[-\frac{1}{\omega^2|\omega_c|} - \frac{\left(k_\parallel v_{t\parallel}\right)^2}{2\omega^2|\omega_c|^3} + A_T\frac{\left(k_\parallel v_{t\parallel}\right)^2}{\omega^3|\omega_c|^2}\right]\right)}$$

After some further simplification we have,

$$\gamma = -\frac{\sum_\alpha \sqrt{\pi} I(nb)\,\omega_{p\alpha}^2 e^{-\left(\frac{\omega - |\omega_c|}{k_\parallel v_{t\parallel}}\right)^2}\left(\omega + A_T(|\omega_c|)\right)}{k_\parallel v_{t\parallel}\left(2\frac{\left(k_\perp^2+k_\parallel^2\right)c^2}{\omega} - \; I(nb)\left[\frac{\omega_{p\alpha}^2}{|\omega_c|} + \frac{\omega_{p\alpha}^2\left(k_\parallel v_{t\parallel}\right)^2}{2|\omega_c|^3} - A_T\frac{\left(k_\parallel v_{t\parallel}\right)^2\omega_{p\alpha}^2}{\omega|\omega_c|^2}\right]\right)}$$

Putting the value of c^2, $\omega_{p\alpha}^2$, $v_{t\parallel}^2$ and ω_c in $\frac{k_\parallel v_{t\parallel}}{|\omega_c|}$ to get $\left(\frac{k_\parallel c}{\omega_{p\alpha}}\right)\sqrt{\beta_\parallel}$. Putting it in above equation and rearranging we have

$$\gamma = -\frac{\sum_\alpha |\omega_c|\sqrt{\pi} I(nb)\, e^{-\left(\frac{\frac{\omega}{|\omega_c|}-1}{\left(\frac{k_\parallel c}{\omega_{p\alpha}}\right)\sqrt{\beta_\parallel}}\right)^2}\left(\frac{\omega}{|\omega_c|} + A\left(\frac{\omega}{|\omega_c|} - 1\right)\right)}{\left(\frac{k_\parallel c}{\omega_{p\alpha}}\right)\sqrt{\beta_\parallel}\left(\frac{2|\omega_c|}{\omega}\left(\frac{k_\parallel c}{\omega_{p\alpha}}\right)^2\left(1 + \frac{k_\perp^2}{k_\parallel^2}\right) - I(nb)\left[1 + \frac{1}{2}\left(\frac{k_\parallel c}{\omega_{p\alpha}}\right)^2\beta_\parallel - \frac{A_T}{\left(\frac{\omega}{|\omega_c|}\right)}\left(\frac{k_\parallel c}{\omega_{p\alpha}}\right)^2\beta_\parallel\right]\right)}$$

For simplification let $x = \frac{k_\parallel c}{\omega_{p\alpha}}$ the $\frac{\gamma}{|\omega_c|}$ will be,

$$\frac{\gamma}{|\omega_c|} = -\frac{\sum_\alpha \sqrt{\pi} I(nb)\, e^{-\left(\frac{\frac{\omega}{|\omega_c|}-1}{x\sqrt{\beta_\parallel}}\right)^2}\left(\frac{\omega}{|\omega_c|} + A\left(\frac{\omega}{|\omega_c|} - 1\right)\right)}{x\sqrt{\beta_\parallel}\left(\frac{2|\omega_c|}{\omega}x^2\left(1 + \frac{k_\perp^2}{k_\parallel^2}\right) - I(nb)\left[1 + \frac{1}{2}x^2\beta_\parallel - \frac{A_T}{\left(\frac{\omega}{|\omega_c|}\right)}x^2\beta_\parallel\right]\right)}$$

Here, we have taken $A_T = \frac{T_\perp}{T_\parallel} - 1 = \frac{v_{t\perp}^2}{v_{t\parallel}^2} - 1$ and $\beta_\parallel = \frac{nT_\parallel \mu_\circ}{B^2}$. Real frequency is denoted by ω, $\omega_{p\alpha}$ is plasma frequency, ω_c is cyclotron frequency,

$k_\parallel$ is wave vector, c is speed of light, B is magnetic field and $T_\perp$ and $T_\parallel$are perpendicular and parallel electron temperatures.

So, this is the required growth rate formula of oblique Whistler mode using Bi-Maxwellian distribution.

3.3 Results and Discussion

Here we will discuss the results obtained for oblique whistler waves with electron temperature anisotropy for Bi-Maxwellian distributed electrons. We'll look at how the oblique whistler wave's dispersion properties and growth rate change when the electron temperature anisotropy, parallel electron beta, and $k_\perp/k_\parallel$ grow.

In figure 3.1(a) and 3.1(b) The dispersion relation and the growth rate of the oblique whistler waves with electron temperature anisotropy by increasing $k_\perp/k_\parallel$ leaving the other parameters of plasma unchanged. It is discovered that increasing the $k_\perp/k_\parallel$ population of electrons reduces the real frequency of oblique whistler waves, whereas increasing the $k_\perp/k_\parallel$ population increases the growth of the oblique whistler instability. In this figure we have choose the $A_T = 0.1$ and $\beta_\parallel = 0.504$.

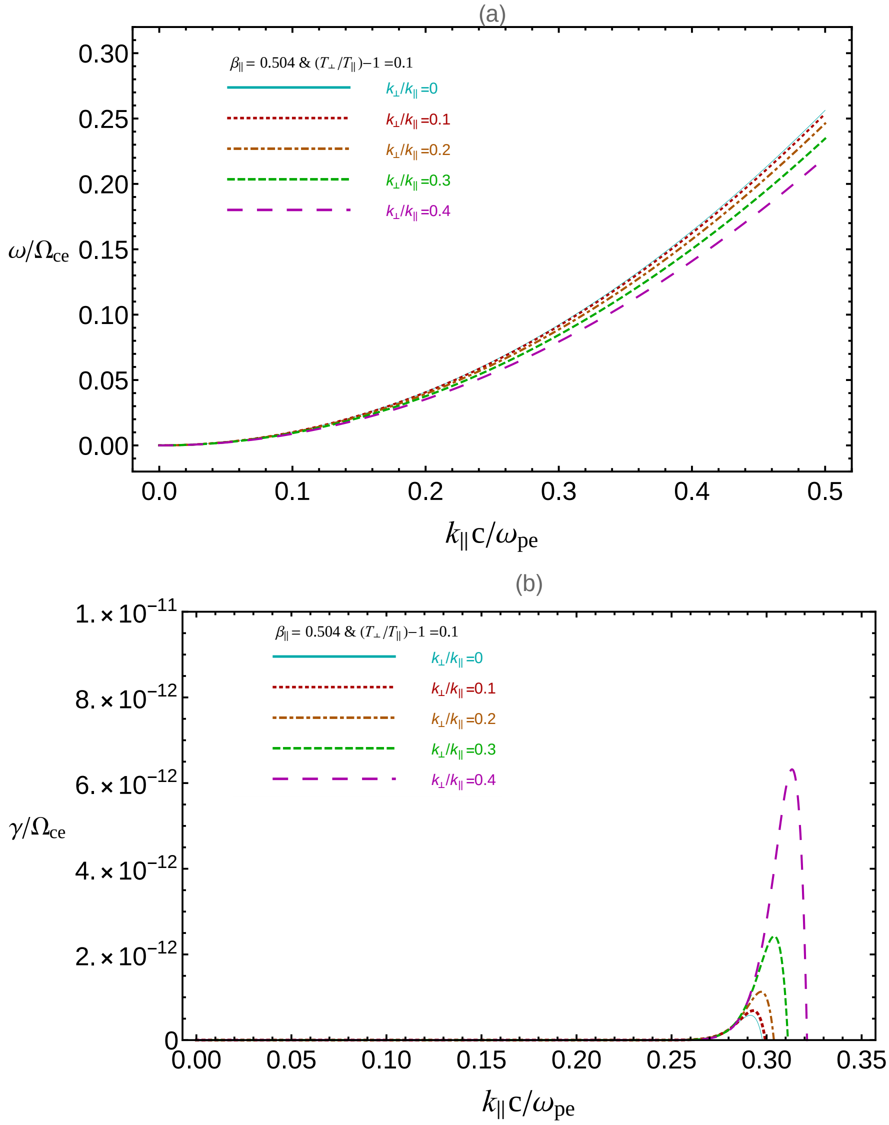

Figure 3.1: (a) Oblique whistler mode dispersion relation for $A_T = 0.1$, $\beta_\| = 0.504$ and for changing values of $k_\perp/k_\|$. (b) Oblique whistler mode growth rate for $A_T = 0.1$, $\beta_\| = 0.504$ and for changing values of $k_\perp/k_\|$. Here blue solid curve represents $k_\perp/k_\| = 0$, red dotted curve represents $k_\perp/k_\| = 0.1$, brown dot-dashed curve represents $k_\perp/k_\| = 0.2$, green small dashed curve represents $k_\perp/k_\| = 0.3$ and purple large dashed curve represents $k_\perp/k_\| = 0.4$

Figures 3.2(a) and 3.2(b) show how increasing the $k_\perp/k_\|$, while choosing a higher value of anisotropy, changes the dispersion relation and causes oblique whistler instability to grow. In this figure we have choose the $A_T = 1$ and $\beta_\| = 0.504$.

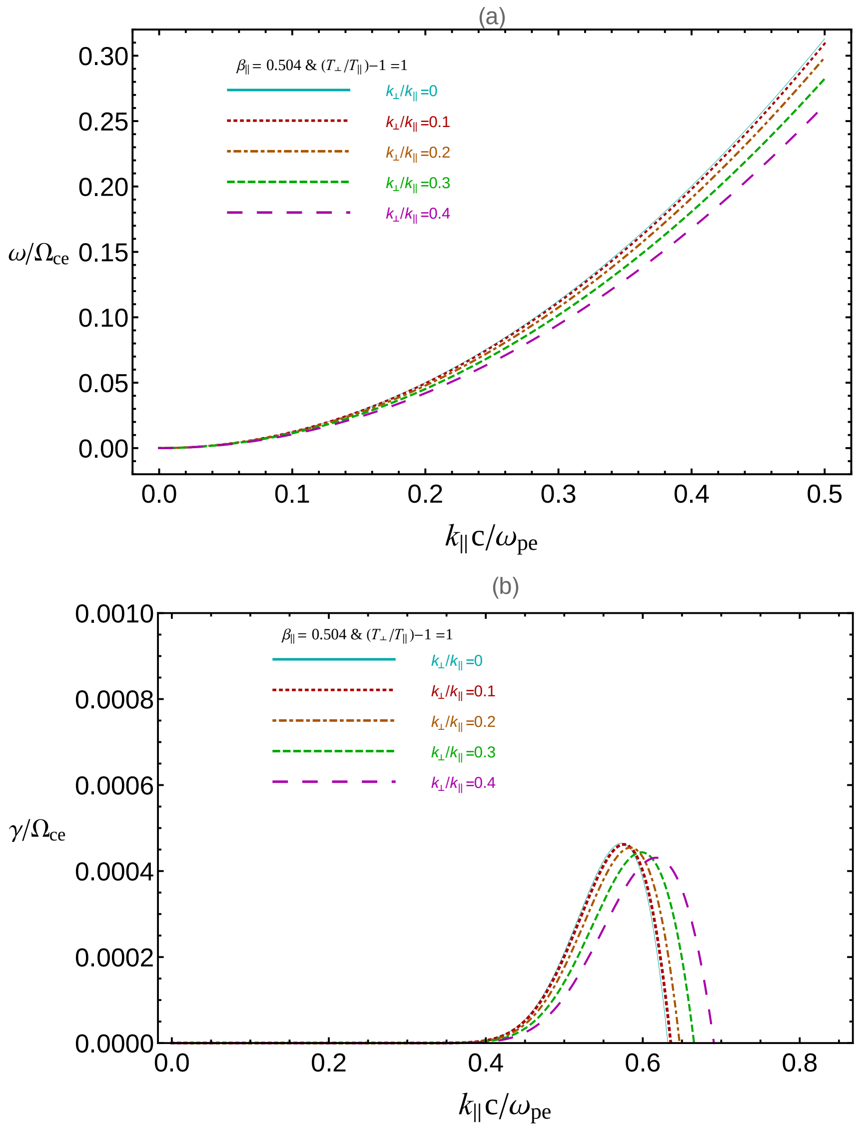

Figure 3.2: (a) Oblique whistler mode dispersion relation for $A_T = 1$, $\beta_\| = 0.504$ and for changing values of $k_\perp/k_\|$. (b) Oblique whistler mode growth rate for $A_T = 1$, $\beta_\| = 0.504$ and for changing values of $k_\perp/k_\|$. Here blue solid curve represents $k_\perp/k_\| = 0$, red dotted curve represents $k_\perp/k_\| = 0.1$, brown dot-dashed curve represents $k_\perp/k_\| = 0.2$, green small dashed curve represents $k_\perp/k_\| = 0.3$ and purple large dashed curve represents $k_\perp/k_\| = 0.4$

Figures 3.3(a) and 3.3(b) show that by taking same value for $\beta_\|$., choosing a higher value of anisotropy and increasing the $k_\perp/k_\|$, changes the dispersion relation and causes oblique whistler instability to grow. In this figure we have choose the $A_T = 1.5$ and $\beta_\| = 0.504$.

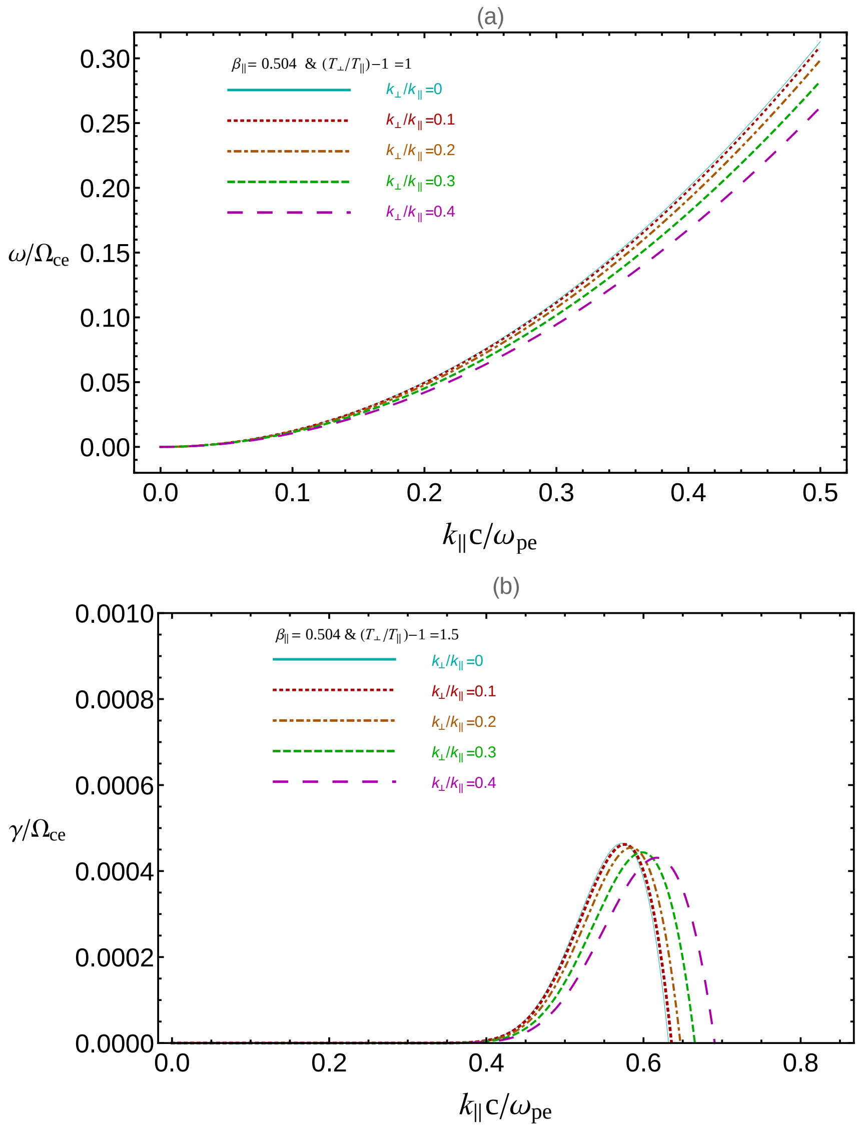

Figure 3.3: (a) Oblique whistler mode dispersion relation for $A_T = 1$, $\beta_{\parallel} = 0.504$ and for changing values of $k_{\perp}/k_{\parallel}$. (b) Oblique whistler mode growth rate for $A_T = 1.5$, $\beta_{\parallel} = 0.504$ and for changing values of $k_{\perp}/k_{\parallel}$. Here blue solid curve represents $k_{\perp}/k_{\parallel} = 0$, red dotted curve represents $k_{\perp}/k_{\parallel} = 0.1$, brown dot-dashed curve represents $k_{\perp}/k_{\parallel} = 0.2$, green small dashed curve represents $k_{\perp}/k_{\parallel} = 0.3$ and purple large dashed curve represents $k_{\perp}/k_{\parallel} = 0.4$

In Figures 3.4(a) and 3.4(b) even with the variation in the real frequency and the growth rate of oblique waves by increasing the $k_{\perp}/k_{\parallel}$ value, however only if the electron temperature anisotropy is set to a positive magnitude. In this figure we have choose the $A_T = 0.1$ and $\beta_{\parallel} = 1.872$

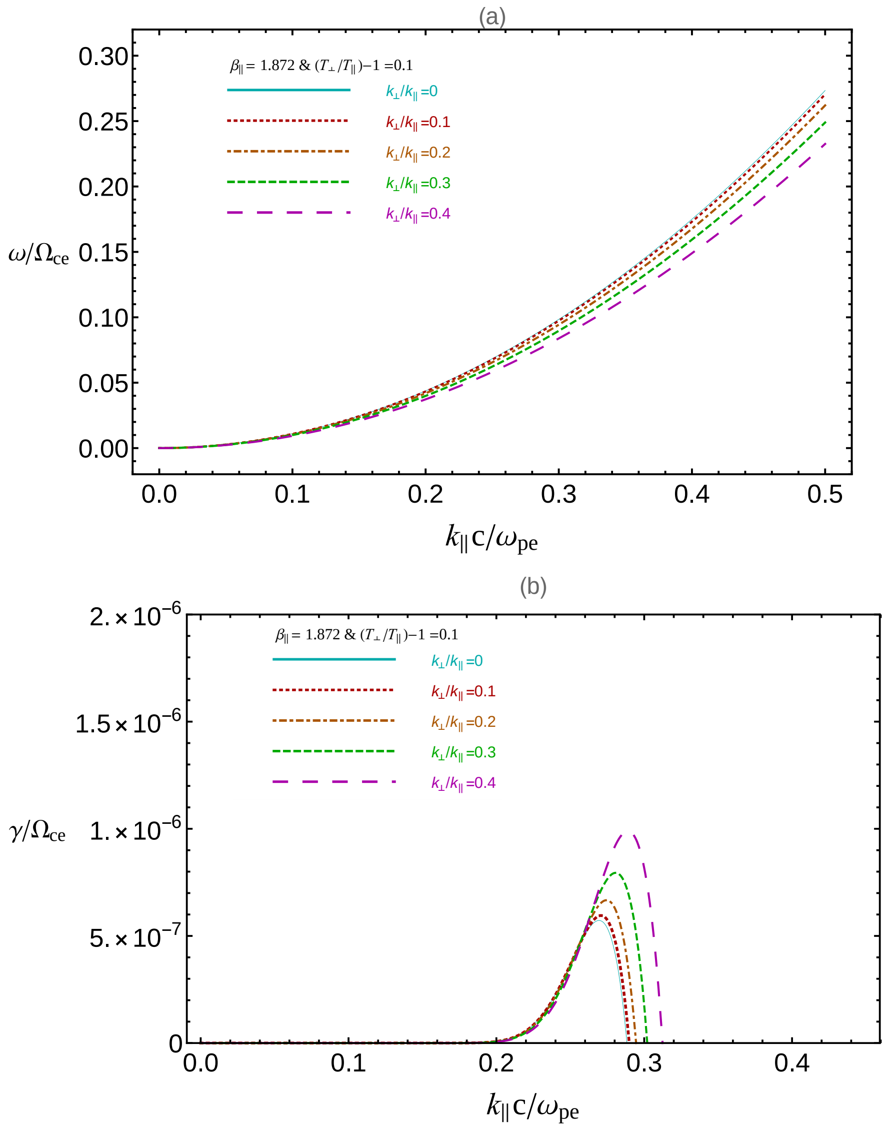

Figure 3.4: (a) Oblique whistler mode dispersion relation for $A_T = 0.1$, $\beta_{\parallel} = 1.872$ and for changing values of $k_{\perp}/k_{\parallel}$. (b) Oblique whistler mode growth rate for $A_T = 0.1$, $\beta_{\parallel} = 1.872$ and for changing values of $k_{\perp}/k_{\parallel}$. Here blue solid curve represents $k_{\perp}/k_{\parallel} = 0$, red dotted curve represents $k_{\perp}/k_{\parallel} = 0.1$, brown dot-dashed curve represents $k_{\perp}/k_{\parallel} = 0.2$, green small dashed curve represents $k_{\perp}/k_{\parallel} = 0.3$ and purple large dashed curve represents $k_{\perp}/k_{\parallel} = 0.4$

Figures 3.5(a) and 3.5(b) show that by taking same value for $\beta_{\parallel}$, choosing a higher value of anisotropy and increasing the $k_{\perp}/k_{\parallel}$, changes the dispersion relation and causes oblique whistler instability to grow. In this figure we have choose the $A_T = 1$ and $\beta_{\parallel} = 1.872$

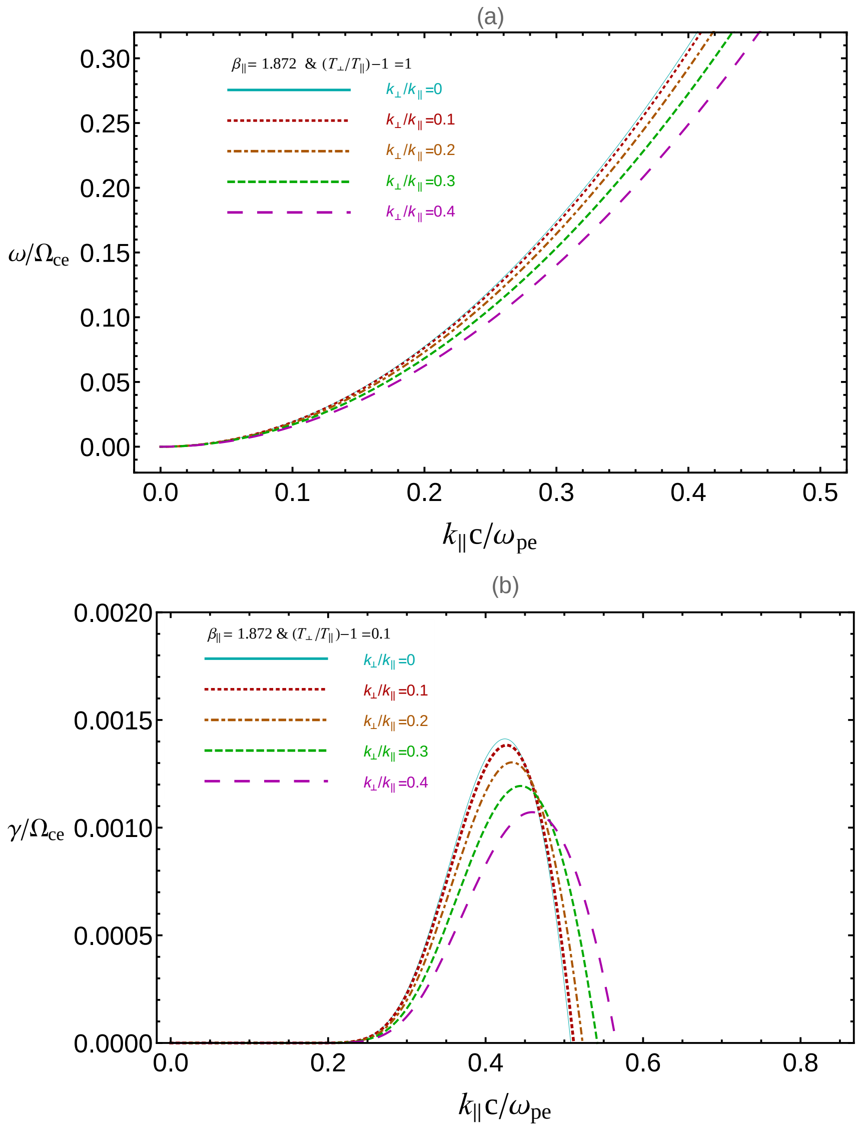

Figure 3.5: (a) Oblique whistler mode dispersion relation for $A_T = 1$, $\beta_\| = 1.872$ and for changing values of $k_\perp/k_\|$. (b) Oblique whistler mode growth rate for $A_T = 0.1$, $\beta_\| = 1.872$ and for changing values of $k_\perp/k_\|$. Here blue solid curve represents $k_\perp/k_\| = 0$, red dotted curve represents $k_\perp/k_\| = 0.1$, brown dot-dashed curve represents $k_\perp/k_\| = 0.2$, green small dashed curve represents $k_\perp/k_\| = 0.3$ and purple large dashed curve represents $k_\perp/k_\| = 0.4$

Figures 3.6(a) and 3.6(b) show the fluctuation in the real frequency and growth rate of whistler waves by increasing the $k_\perp/k_\|$, but this time by setting the parallel electron beta greater than 1. This shows us the importance of parallel electron beta in determining the dispersion characteristics of oblique whistler waves with electron temperature anisotropy. The change in parallel electron beta increases the growth rate. In this figure we have choose the $A_T = 1.5$ and $\beta_\| = 1.872$

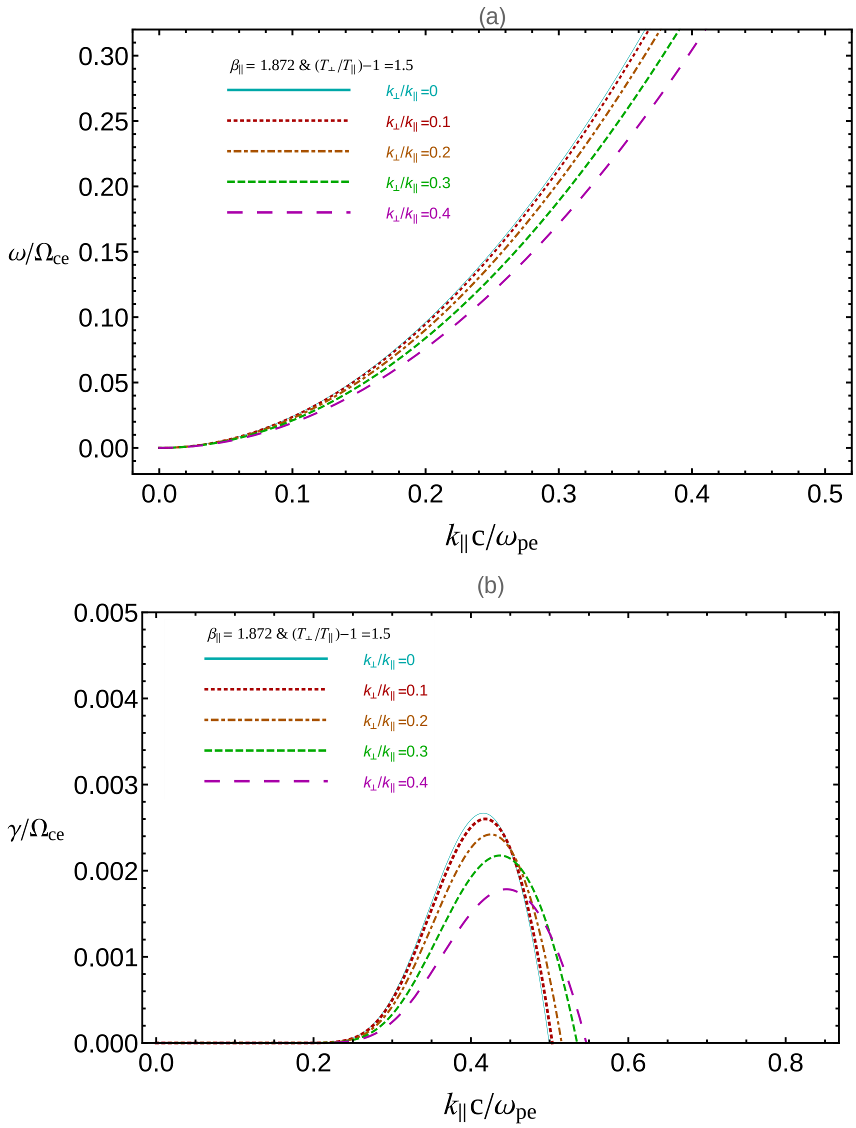

Figure 3.6: (a) Oblique whistler mode dispersion relation for $A_T = 1.5$, $\beta_{\|} = 1.872$ and for changing values of $k_\perp/k_\|$. (b) Oblique whistler mode growth rate for $A_T = 1.5$, $\beta_\| = 1.872$ and for changing values of $k_\perp/k_\|$. Here blue solid curve represents $k_\perp/k_\| = 0$, red dotted curve represents $k_\perp/k_\| = 0.1$, brown dot-dashed curve represents $k_\perp/k_\| = 0.2$, green small dashed curve represents $k_\perp/k_\| = 0.3$ and purple large dashed curve represents $k_\perp/k_\| = 0.4$

By comparing all the figures, we can say that oblique whistler treatment with Bi-Maxwellian distribution shows dependence of growth rate and dispersion relation upon electron temperature anisotropy. we see that the growth rate increases by increasing dispersion relation and electron temperature anisotropy.

Chapter 4

Oblique whistler waves mode with Cairns distribution

4.1 Anisotropic Cairns distribution function

The distribution function for plasma particles deviates from Maxwellian distribution if collisions between slower particles are more than collisions between high energy particles. These high-energy particles' mean free path is proportional to v^4, and the distribution function does not relax to a Maxwellian distribution. In the space and astrophysical environment, distribution functions with tails like a power law are observed. The presence of ion and electron populations which are not in thermodynamic equilibrium in space plasma observations led to model these effects by simplest analytical way. These non-thermal velocity distributions include a ring structure. One of these non-thermal distribution function was introduced by Cairns *et al.* to explain the reverse-polarity structures observed in space plasma. Although the Cairns distribution can be used to represent non-Maxwellian plasma, it is not used to fit velocity distribution data. The enlarged high energy tail of this nonthermal distribution is superimposed on a Maxwellian-like low energy component. Anisotropic cairns distribution function is

$$f_{cairns} = \frac{1}{\pi^{\frac{3}{2}}\left(3\Lambda+1\right)v_{t\alpha\perp}^2 v_{t\alpha\parallel}}\left[1+\Lambda\left\{\frac{v_\perp^4}{v_{t\perp}^4}+\frac{v_\parallel^4}{v_{t\parallel\,\perp}^4}\right\}\right] Exp\left\{-\left(\frac{v_\perp^2}{v_{t\alpha\perp}^2}+\frac{v_\parallel^2}{v_{t\alpha\parallel}^2}\right)\right\}$$

Here $v_{t\alpha\perp}$ and $v_{t\alpha\parallel}$ are perpendicular and parallel thermal velocities respectively. And they are equal to $v_{t\alpha\perp} = \frac{2k_BT_\perp}{m_\alpha}$ and $v_{t\alpha\parallel} = \frac{2k_BT_\parallel}{m_\alpha}$. Here Λ is nonthermality parameter.

4.2 Oblique Whistler waves with anisotropic cairns distribution

The kinetic relation for right and left handed circularly polarized Oblique whistler modes is

$$1-\frac{\left(k_\perp^2+k_\parallel^2\right)c^2}{\omega^2}-\sum_\alpha\frac{\omega_{p\alpha}^2}{\omega}\int d^3v\frac{v_\perp\left(J_n'(a)\right)^2}{\left(k_\parallel v_\parallel-\omega-n\omega_c\right)}\left(\left(1-\frac{k_\parallel v_\parallel}{\omega}\right)\frac{\partial f_o}{\partial v_\perp}+\frac{k_\parallel v_\perp}{\omega}\frac{\partial f_o}{\partial v_\parallel}\right)=0 \qquad (4.1)$$

As Cairns distribution function is,

$$f_o=A\left(1+\Lambda\left(\frac{v_\perp^4}{v_{t\perp}^4}+\frac{v_\parallel^4}{v_{t\parallel}^4}\right)\right)\exp\left(-\frac{v_\perp^2}{v_{t\perp}^2}-\frac{v_\parallel^2}{v_{t\parallel}^2}\right)$$

Where as,

$$A=\frac{1}{(\pi)^{3/2}\left(1+\frac{11}{4}\Lambda\right)v_{t\perp}^2v_{t\parallel}}$$

Now taking the derivative of f_o w.r.t $v_\perp$ as,

$$\frac{\partial f_o}{\partial v_\perp}=A\left[4\Lambda\frac{v_\perp^3}{v_t^4}-\left(1+\Lambda\left\{\frac{v_\perp^4}{v_{t\perp}^4}+\frac{v_\parallel^4}{v_{t\parallel}^4}\right\}\right)\left(2\frac{v_\perp}{v_{t\perp}^2}\right)\right]\exp\left(-\frac{v_\perp^2}{v_{t\perp}^2}-\frac{v_\parallel^2}{v_{t\parallel}^2}\right)$$

$$\frac{\partial f_o}{\partial v_\perp}=A\left[v_\perp\left(-\frac{2}{v_{t\perp}^2}-2\Lambda\frac{v_\parallel^4}{v_{t\perp}^2v_{t\parallel}^4}\right)+4\Lambda\frac{v_\perp^3}{v_{t\perp}^4}-2\Lambda\frac{v_\perp^5}{v_{t\perp}^6}\right]\exp\left(-\frac{v_\perp^2}{v_{t\perp}^2}-\frac{v_\parallel^2}{v_{t\parallel}^2}\right)$$

Similarly taking the derivative of f_o w.r.t $v_\parallel$ as,

$$\frac{\partial f_o}{\partial v_\parallel}=A\left[4\Lambda\frac{v_\parallel^3}{v_{t\parallel}^4}-2\frac{v_\parallel}{v_{t\parallel}^2}-2\Lambda\frac{v_\parallel v_\perp^4}{v_{t\parallel}^2v_{t\perp}^4}-2\Lambda\frac{v_\parallel^5}{v_{t\parallel}^6}\right]\exp\left(-\frac{v_\perp^2}{v_{t\perp}^2}-\frac{v_\parallel^2}{v_{t\parallel}^2}\right)$$

$$\frac{\partial f_o}{\partial v_\parallel}=A\left[v_\parallel\left(-\frac{2}{v_{t\parallel}^2}-2\Lambda\frac{v_\perp^4}{v_{t\parallel}^2v_{t\perp}^4}\right)+4\Lambda\frac{v_\parallel^3}{v_{t\parallel}^4}-2\Lambda\frac{v_\parallel^5}{v_{t\parallel}^6}\right]\exp\left(-\frac{v_\perp^2}{v_{t\perp}^2}-\frac{v_\parallel^2}{v_{t\parallel}^2}\right)$$

Now putting the values of $\frac{\partial f_o}{\partial v_\perp}$and $\frac{\partial f_o}{\partial v_\parallel}$ in Eq (4.1) we get,

$$1-\frac{\left(k_\perp^2+k_\parallel^2\right)c^2}{\omega^2}+\sum_\alpha\frac{\omega_{p\alpha}^2}{\omega}A\int d^3v\frac{v_\perp\left(J_n'(a)\right)^2\exp\left(-\frac{v_\perp^2}{v_{t\perp}^2}-\frac{v_\parallel^2}{v_{t\parallel}^2}\right)}{\left(k_\parallel v_\parallel-\omega-n\omega_c\right)}\times \tag{4.2}$$

$$\left(\begin{array}{c}\left(1-\frac{k_\parallel v_\parallel}{\omega}\right)\left[\begin{array}{c}v_\perp\left(-\frac{2}{v_{t\perp}^2}-2\Lambda\frac{v_\parallel^4}{v_{t\perp}^2v_{t\parallel}^4}\right)+\\ 4\Lambda\frac{v_\perp^3}{v_{t\perp}^4}-2\Lambda\frac{v_\perp^5}{v_{t\perp}^6}\end{array}\right]+\\ \frac{k_\parallel v_\perp}{\omega}\left[4\Lambda\frac{v_\parallel^3}{v_{t\parallel}^4}-2\frac{v_\parallel}{v_{t\parallel}^2}-2\Lambda\frac{v_\parallel v_\perp^4}{v_{t\parallel}^2v_{t\perp}^4}-2\Lambda\frac{v_\parallel^5}{v_{t\parallel}^6}\right]\end{array}\right)$$

$$=\quad 0$$

Simplifying the last terms of above equation and separating the $v_\perp$,$v_\perp^3$,$v_\perp^5$ terms we have,

$$\left(\begin{array}{c}\left(1-\frac{k_\parallel v_\parallel}{\omega}\right)\left[v_\perp\left(-\frac{2}{v_{t\perp}^2}-2\Lambda\frac{v_\parallel^4}{v_{t\perp}^2v_{t\parallel}^4}\right)+4\Lambda\frac{v_\perp^3}{v_{t\perp}^4}-2\Lambda\frac{v_\perp^5}{v_{t\perp}^6}\right]+\\ \frac{k_\parallel v_\perp}{\omega}\left[4\Lambda\frac{v_\parallel^3}{v_{t\parallel}^4}-2\frac{v_\parallel}{v_{t\parallel}^2}-2\Lambda\frac{v_\parallel v_\perp^4}{v_{t\parallel}^2v_{t\perp}^4}-2\Lambda\frac{v_\parallel^5}{v_{t\parallel}^6}\right]\end{array}\right)$$

$$\begin{aligned}=\quad & v_\perp\left[-\frac{2}{v_{t\perp}^2}-2\Lambda\frac{v_\parallel^4}{v_{t\perp}^2v_{t\parallel}^4}-\frac{k_\parallel v_\parallel}{\omega}\left(-\frac{2}{v_{t\perp}^2}-2\Lambda\frac{v_\parallel^4}{v_{t\perp}^2v_{t\parallel}^4}\right)+\frac{k_\parallel}{\omega}\left(4\Lambda\frac{v_\parallel^3}{v_{t\parallel}^4}-2\frac{v_\parallel}{v_{t\parallel}^2}-2\Lambda\frac{v_\parallel^5}{v_{t\parallel}^6}\right)\right]+\\ & v_\perp^3\left[4\Lambda\frac{1}{v_{t\perp}^4}-\frac{k_\parallel v_\parallel}{\omega}\left(4\Lambda\frac{1}{v_{t\perp}^4}\right)\right]+v_\perp^5\left[-2\Lambda\frac{1}{v_{t\perp}^6}+\frac{k_\parallel v_\parallel}{\omega}\left(2\Lambda\frac{1}{v_{t\perp}^6}-2\Lambda\frac{1}{v_{t\parallel}^2v_{t\perp}^4}\right)\right]\end{aligned}$$

Let defining some terms,

$$a_1=\left[-\frac{2}{v_{t\perp}^2}-2\Lambda\frac{v_\parallel^4}{v_{t\perp}^2v_{t\parallel}^4}-\frac{k_\parallel v_\parallel}{\omega}\left(-\frac{2}{v_{t\perp}^2}-2\Lambda\frac{v_\parallel^4}{v_{t\perp}^2v_{t\parallel}^4}\right)+\frac{k_\parallel}{\omega}\left(4\Lambda\frac{v_\parallel^3}{v_{t\parallel}^4}-2\frac{v_\parallel}{v_{t\parallel}^2}-2\Lambda\frac{v_\parallel^5}{v_{t\parallel}^6}\right)\right]$$

$$a_2=\left[4\Lambda\frac{1}{v_{t\perp}^4}-\frac{k_\parallel v_\parallel}{\omega}\left(4\Lambda\frac{1}{v_{t\perp}^4}\right)\right]$$

$$a_3=\left[-2\Lambda\frac{1}{v_{t\perp}^6}+\frac{k_\parallel v_\parallel}{\omega}\left(2\Lambda\frac{1}{v_{t\perp}^6}-2\Lambda\frac{1}{v_{t\parallel}^2v_{t\perp}^4}\right)\right]$$

So the above equation becomes,

$$=v_\perp a_1+v_\perp^3a_2+v_\perp^5a_3$$

Putting this in Eq (4.2) we have,

$$1-\frac{\left(k_\perp^2+k_\parallel^2\right)c^2}{\omega^2}+\sum_\alpha \frac{\omega_{p\alpha}^2}{\omega}A\int d^3v\frac{v_\perp\left(J_n'(a)\right)^2\exp\left(-\frac{v_\perp^2}{v_{t\perp}^2}-\frac{v_\parallel^2}{v_{t\parallel}^2}\right)}{\left(k_\parallel v_\parallel-\omega-n\omega_c\right)}\left(v_\perp c_1+v_\perp^3c_2+v_\perp^5c_3\right)=0$$

Here $\int d^3v = \int v_\perp dv_\perp dv_\parallel d\theta$, and their limits are taken as $dv_\perp$ is from $0\to\infty$, $dv_\parallel$ is from $-\infty\to\infty$ and for $d\theta$, $0\to 2\pi$, so

$$1-\frac{\left(k_\perp^2+k_\parallel^2\right)c^2}{\omega^2}+\sum_\alpha \frac{\omega_{p\alpha}^2}{\omega}A\int 2\pi v_\perp dv_\perp dv_\parallel\frac{v_\perp\left(J_n'(a)\right)^2\exp\left(-\frac{v_\perp^2}{v_{t\perp}^2}-\frac{v_\parallel^2}{v_{t\parallel}^2}\right)}{\left(k_\parallel v_\parallel-\omega-n\omega_c\right)}\left(v_\perp c_1+v_\perp^3c_2+v_\perp^5c_3\right)=0 \tag{4.3}$$

Let $x=\frac{k_\perp v_\perp}{\omega_c}\Rightarrow v_\perp=\frac{x\omega_c}{k_\perp}\Rightarrow dv_\perp=\frac{\omega_c}{k_\perp}\,dx,\;\; b=\frac{k_\perp^2v_{t\perp}^2}{2\omega_c^2}$ and $a=\frac{k_\perp v_\perp}{\omega_c}$

So the perpendicular component becomes

$$\int_0^\infty dv_\perp\left(J_n'(a)\right)^2v_\perp^3\exp\left(-\frac{v_\perp^2}{v_{t\perp}^2}\right)=\left(\frac{\omega_c}{k_\perp}\right)^4\int_0^\infty dx\left(J_n'(a)\right)^2x^3\exp\left(-\frac{x^2}{2b}\right)$$

$$\int_0^\infty dv_\perp\left(J_n'(a)\right)^2v_\perp^5\exp\left(-\frac{v_\perp^2}{v_{t\perp}^2}\right)=\left(\frac{\omega_c}{k_\perp}\right)^6\int_0^\infty dx\left(J_n'(a)\right)^2x^5\exp\left(-\frac{x^2}{2b}\right)$$

$$\int_0^\infty dv_\perp\left(J_n'(a)\right)^2v_\perp^7\exp\left(-\frac{v_\perp^2}{v_{t\perp}^2}\right)=\left(\frac{\omega_c}{k_\perp}\right)^8\int_0^\infty dx\left(J_n'(a)\right)^2x^7\exp\left(-\frac{x^2}{2b}\right)$$

Now integrating the perpendicular components we have,

1st component integration of perpendicular term,

$$\left(\frac{\omega_c}{k_\perp}\right)^4\int_0^\infty dx\left(J_n'(a)\right)^2x^3\exp\left(-\frac{x^2}{2b}\right)=\left(\frac{\omega_c}{k_\perp}\right)^4e^{-b}\begin{pmatrix} b\left(2b^2-2bn+n^2\right)I_n(b)- \\ 2b^3I_{n+1}(b)\end{pmatrix}$$

2nd component integration of perpendicular term,

$$\left(\frac{\omega_c}{k_\perp}\right)^6\int_0^\infty dx\left(J_n'(a)\right)^2x^5\exp\left(-\frac{x^2}{2b}\right)=\left(\frac{\omega_c}{k_\perp}\right)^6\left(-2b^2\right)e^{-b}\begin{pmatrix}\begin{pmatrix}4b^3-n^2(1+n)- \\ 2b^2(3+2n)+ \\ bn(4+3n)\end{pmatrix}I_n(b)- \\ b\left(4b^2-4b+n^2\right)I_{n+1}(b)\end{pmatrix}$$

And 3rd component integration of perpendicular term,

$$\left(\frac{\omega_c}{k_\perp}\right)^8 \int_0^\infty dx\,(J_n'(a))^2 x^7 \exp\left(-\frac{x^2}{2b}\right) = \left(\frac{\omega_c}{k_\perp}\right)^8 (4b^3)\, e^{-b} \left(\left(\begin{array}{c} 8b^4 - 2b^3(15+4n) + \\ n^2(2+3n+n^2) - \\ 2bn(6+7n+2n^2) + \\ b^2(24+26n+8n^2) \end{array} \right) I_n(b) + b \left(\begin{array}{c} 26b^2 - 8b^3 + 3n^2 - \\ 4b(3+n^2) \end{array} \right) I_{n+1}(b) \right)$$

Putting the values of integration of perpendicular components in Eq (4.3) we have,

$$1 - \frac{\left(k_\perp^2 + k_\parallel^2\right)c^2}{\omega^2} + \sum_\alpha \frac{\omega_{p\alpha}^2}{\omega} A \int 2\pi dv_\parallel \frac{\exp\left(-\frac{v_\parallel^2}{v_{t\parallel}^2}\right)}{\left(k_\parallel v_\parallel - \omega - n\omega_c\right)} \times \tag{4.4}$$

$$\left[\begin{array}{c} a_1 \left(\frac{\omega_c}{k_\perp}\right)^4 e^{-b} \left(\begin{array}{c} b(2b^2 - 2bn + n^2) I_n(b) - \\ 2b^3 I_{n+1}(b) \end{array} \right) + \\ a_2 \left(\frac{\omega_c}{k_\perp}\right)^6 (-2b^2)\, e^{-b} \left(\begin{array}{c} (4b^3 - n^2(1+n) - 2b^2(3+2n) + bn(4+3n)) I_n(b) - \\ b(4b^2 - 4b + n^2) I_{n+1}(b) \end{array} \right) + \\ a_3 \left(\frac{\omega_c}{k_\perp}\right)^8 (4b^3)\, e^{-b} \left(\begin{array}{c} \left(\begin{array}{c} 8b^4 - 2b^3(15+4n) + n^2(2+3n+n^2) - \\ 2bn(6+7n+2n^2) + b^2(24+26n+8n^2) \end{array} \right) I_n(b) + \\ b(26b^2 - 8b^3 + 3n^2 - 4b(3+n^2)) I_{n+1}(b) \end{array} \right) \end{array} \right] = 0$$

Simplifying the term we have,

$$\left[\begin{array}{c} a_1 \left(\frac{\omega_c}{k_\perp}\right)^4 e^{-b} \left(\begin{array}{c} b(2b^2 - 2bn + n^2) I_n(b) - \\ 2b^3 I_{n+1}(b) \end{array} \right) + \\ a_2 \left(\frac{\omega_c}{k_\perp}\right)^6 (-2b^2)\, e^{-b} \left(\begin{array}{c} (4b^3 - n^2(1+n) - 2b^2(3+2n) + bn(4+3n)) I_n(b) - \\ b(4b^2 - 4b + n^2) I_{n+1}(b) \end{array} \right) + \\ a_3 \left(\frac{\omega_c}{k_\perp}\right)^8 (4b^3)\, e^{-b} \left(\begin{array}{c} \left(\begin{array}{c} 8b^4 - 2b^3(15+4n) + n^2(2+3n+n^2) - \\ 2bn(6+7n+2n^2) + b^2(24+26n+8n^2) \end{array} \right) I_n(b) + \\ b(26b^2 - 8b^3 + 3n^2 - 4b(3+n^2)) I_{n+1}(b) \end{array} \right) \end{array} \right]$$

Now putting the values of a_1, a_2, a_3 we have,

$$\left[\begin{array}{c}\left(\frac{2}{v_{t\perp}^2}\left(\begin{array}{c}-1-\Lambda\frac{v_\parallel^4}{v_{t\parallel}^4}-\frac{k_\parallel v_\parallel}{\omega}\left(-1-\Lambda\frac{v_\parallel^4}{v_{t\parallel}^4}\right)+\\ \frac{k_\parallel v_{t\perp}^2}{\omega}\left(2\Lambda\frac{v_\parallel^3}{v_{t\parallel}^4}-\frac{v_\parallel}{v_{t\parallel}^2}-\Lambda\frac{v_\parallel^5}{v_{t\parallel}^6}\right)\end{array}\right)\right)\left(\frac{\omega_c}{k_\perp}\right)^4 be^{-b}\left(\begin{array}{c}\left(2b^2-2bn+n^2\right)I_n(b)-\\ 2b^2 I_{n+1}(b)\end{array}\right)+\\ \frac{4\Lambda}{v_{t\perp}^4}\left(1-\frac{k_\parallel v_\parallel}{\omega}\right)\left(\frac{\omega_c}{k_\perp}\right)^6\left(-2b^2\right)e^{-b}\left(\begin{array}{c}\left(\begin{array}{c}4b^3-n^2(1+n)-2b^2(3+2n)+\\ bn(4+3n)\end{array}\right)I_n(b)-\\ b\left(4b^2-4b+n^2\right)I_{n+1}(b)\end{array}\right)+\\ \frac{2}{v_{t\perp}^6}\Lambda\left(-1+\frac{k_\parallel v_\parallel}{\omega}\left(1-\frac{v_{t\perp}^2}{v_{t\parallel}^2}\right)\right)\left(\frac{\omega_c}{k_\perp}\right)^8\left(4b^3\right)e^{-b}\left(\begin{array}{c}\left(\begin{array}{c}8b^4-2b^3(15+4n)+\\ n^2\left(2+3n+n^2\right)-\\ 2bn\left(6+7n+2n^2\right)+\\ b^2\left(24+26n+8n^2\right)\end{array}\right)I_n(b)+\\ b\left(26b^2-8b^3+3n^2-4b\left(3+n^2\right)\right)I_{n+1}(b)\end{array}\right)\end{array}\right] \tag{4.5}$$

Simplifying,

$$\frac{2}{v_{t\perp}^2}\left(\frac{\omega_c}{k_\perp}\right)^4 b=\frac{2}{v_{t\perp}^2}\left(\frac{\omega_c}{k_\perp}\right)^4\frac{k_\perp^2 v_{t\perp}^2}{2\omega_c^2}=\frac{\omega_c^2}{k_\perp^2}=\frac{v_{t\perp}^2}{2b}$$

$$\frac{4}{v_{t\perp}^4}\left(\frac{\omega_c}{k_\perp}\right)^6\left(-2\left(\frac{k_\perp^2 v_{t\perp}^2}{2\omega_c^2}\right)^2\right)=\frac{4}{v_{t\perp}^4}\left(\frac{\omega_c}{k_\perp}\right)^6\left(-2\left(\frac{k_\perp^2 v_{t\perp}^2}{2\omega_c^2}\right)^2\right)=-2\frac{\omega_c^2}{k_\perp^2}=-\frac{v_{t\perp}^2}{b}$$

$$\frac{2}{v_{t\perp}^6}\left(\frac{\omega_c}{k_\perp}\right)^8\left(4b^3\right)=\frac{2}{v_{t\perp}^6}\left(\frac{\omega_c}{k_\perp}\right)^8\left(4\left(\frac{k_\perp^2 v_{t\perp}^2}{2\omega_c^2}\right)^3\right)=\frac{\omega_c^2}{k_\perp^2}=\frac{v_{t\perp}^2}{2b}$$

Putting it back in above Eq (4.5) we have,

$$\left[\begin{array}{c}\frac{v_{t\perp}^2}{2}\left(\begin{array}{c}-1-\Lambda\frac{v_\parallel^4}{v_{t\parallel}^4}-\frac{k_\parallel v_\parallel}{\omega}\left(-1-\Lambda\frac{v_\parallel^4}{v_{t\parallel}^4}\right)+\\ \frac{k_\parallel v_{t\perp}^2}{\omega}\left(2\Lambda\frac{v_\parallel^3}{v_{t\parallel}^4}-\frac{v_\parallel}{v_{t\parallel}^2}-\Lambda\frac{v_\parallel^5}{v_{t\parallel}^6}\right)\end{array}\right)\frac{e^{-b}}{b}\left(\begin{array}{c}\left(2b^2-2bn+n^2\right)I_n(b)-\\ 2b^2 I_{n+1}(b)\end{array}\right)+\\ \left[-v_{t\perp}^2\Lambda\left(1-\frac{k_\parallel v_\parallel}{\omega}\right)\frac{e^{-b}}{b}\left(\begin{array}{c}\left(4b^3-n^2(1+n)-2b^2(3+2n)+bn(4+3n)\right)I_n(b)-\\ b\left(4b^2-4b+n^2\right)I_{n+1}(b)\end{array}\right)\right]+\\ \frac{v_{t\perp}^2}{2}\Lambda\left(-1+\frac{k_\parallel v_\parallel}{\omega}\left(1-\frac{v_{t\perp}^2}{v_{t\parallel}^2}\right)\right)\frac{e^{-b}}{b}\left(\begin{array}{c}\left(\begin{array}{c}8b^4-2b^3(15+4n)+n^2\left(2+3n+n^2\right)-\\ 2bn\left(6+7n+2n^2\right)+b^2\left(24+26n+8n^2\right)\end{array}\right)I_n(b)+\\ b\left(26b^2-8b^3+3n^2-4b\left(3+n^2\right)\right)I_{n+1}(b)\end{array}\right)\end{array}\right]$$

Let further defining,

$$I_1(nb)=\frac{e^{-b}}{b}\left(\left(2b^2-2bn+n^2\right)I_n(b)-2b^2 I_{n+1}(b)\right)$$

$$I_2(nb) = \frac{e^{-b}}{b}\begin{pmatrix} \left(4b^3 - n^2(1+n) - 2b^2(3+2n) + bn(4+3n)\right) I_n(b) - \\ b\left(4b^2 - 4b + n^2\right) I_{n+1}(b) \end{pmatrix}$$

$$I_3(nb) = \frac{e^{-b}}{b}\begin{pmatrix} \begin{pmatrix} 8b^4 - 2b^3(15+4n) + n^2\left(2+3n+n^2\right) - \\ 2bn\left(6+7n+2n^2\right) + b^2\left(24+26n+8n^2\right) \end{pmatrix} I_n(b) + \\ b\left(26b^2 - 8b^3 + 3n^2 - 4b\left(3+n^2\right)\right) I_{n+1}(b) \end{pmatrix}$$

So the above equation becomes,

$$= -\frac{v_{t\perp}^2}{2}\begin{pmatrix} I_1(nb)\begin{pmatrix} 1 + \Lambda\frac{v_\parallel^4}{v_{t\parallel}^4} - \frac{k_\parallel v_\parallel}{\omega}\left(1 + \Lambda\frac{v_\parallel^4}{v_{t\parallel}^4}\right) - \\ \frac{k_\parallel v_{t\perp}^2}{\omega}\left(2\Lambda\frac{v_\parallel^3}{v_{t\parallel}^4} - \frac{v_\parallel}{v_{t\parallel}^2} - \Lambda\frac{v_\parallel^5}{v_{t\parallel}^6}\right) \end{pmatrix} \\ +I_2(nb)\left(2\Lambda\left(1 - \frac{k_\parallel v_\parallel}{\omega}\right)\right) - I_3(nb)\left(\Lambda\left(-1 + \frac{k_\parallel v_\parallel}{\omega}\left(1 - \frac{v_{t\perp}^2}{v_{t\parallel}^2}\right)\right)\right) \end{pmatrix}$$

Rearranging with powers of $v_\parallel$. Simplifying and put $A_T = \frac{v_{t\perp}^2}{v_{t\parallel}^2} - 1$ we have,

$$= -\frac{v_{t\perp}^2}{2}\begin{pmatrix} \left(I_1(nb) + 2\Lambda I_2(nb) + \Lambda I_3(nb)\right) \\ v_\parallel\left(\frac{k_\parallel}{\omega}\left\{A_T\left(I_1(nb) + \Lambda I_3(nb)\right) - 2\Lambda I_2(nb)\right\}\right) \\ v_\parallel^3\left(\left(-\frac{k_\parallel v_{t\perp}^2}{\omega}\left(2\Lambda\frac{1}{v_{t\parallel}^4}\right)\right) I_1(nb)\right) + v_\parallel^4\left(\Lambda\frac{1}{v_{t\parallel}^4}\right) I_1(nb) \\ v_\parallel^5\left(\Lambda\frac{k_\parallel}{\omega v_{t\parallel}^4} A_T I_1(nb)\right) \end{pmatrix} \tag{4.6}$$

Further defining some terms we have,

$$d_1 = \left(I_1(nb) + 2\Lambda I_2(nb) + \Lambda I_3(nb)\right)$$

$$d_2 = \frac{k_\parallel}{\omega}\left\{A_T\left(I_1(nb) + \Lambda I_3(nb)\right) - 2\Lambda I_2(nb)\right\}$$

$$d_3 = -\frac{k_\parallel v_{t\perp}^2}{\omega}\left(2\Lambda\frac{1}{v_{t\parallel}^4}\right) I_1(nb)$$

$$d_4 = \Lambda\frac{1}{v_{t\parallel}^4} I_1(nb)$$

$$d_5 = \Lambda\frac{k_\parallel}{\omega v_{t\parallel}^4} A_T I_1(nb)$$

Equation (4.6) becomes,

$$= -\frac{v_{t\perp}^2}{2}\left(d_1 + d_2 v_\parallel + d_3 v_\parallel^3 + d_4 v_\parallel^4 + d_5 v_\parallel^5\right)$$

Putting this value in Eq (4.4) we have,

$$1 - \frac{\left(k_\perp^2 + k_\parallel^2\right)c^2}{\omega^2} + \sum_\alpha \frac{\omega_{p\alpha}^2}{\omega} A \int 2\pi dv_\parallel \frac{\exp\left(-\frac{v_\parallel^2}{v_{t\parallel}^2}\right)}{\left(k_\parallel v_\parallel - \omega - n\omega_c\right)}\left(-\frac{v_{t\perp}^2}{2}\right) \times \left(d_1 + d_2 v_\parallel + d_3 v_\parallel^3 + d_4 v_\parallel^4 + d_5 v_\parallel^5\right) = 0$$

After Simplifying it we have,

$$1 - \frac{\left(k_\perp^2 + k_\parallel^2\right)c^2}{\omega^2} - \sum_\alpha \frac{\pi v_{t\perp}^2 \omega_{p\alpha}^2}{\omega} A \int dv_\parallel \frac{\exp\left(-\frac{v_\parallel^2}{v_{t\parallel}^2}\right)}{\left(k_\parallel v_\parallel - \omega - n\omega_c\right)} \times \left(d_1 + d_2 v_\parallel + d_3 v_\parallel^3 + d_4 v_\parallel^4 + d_5 v_\parallel^5\right) = 0 \tag{4.7}$$

Now we have to find the parallel component integration i-e

$$\int_{-\infty}^{\infty} dv_\parallel \frac{\left(d_1 + d_2 v_\parallel + d_3 v_\parallel^3 + d_4 v_\parallel^4 + d_5 v_\parallel^5\right)}{k_\parallel v_\parallel - (\omega \pm \omega_c)} \exp\left(-\frac{v_\parallel^2}{v_{t\parallel}^2}\right)$$

Let $s = \frac{v_\parallel}{v_{t\parallel}} \Rightarrow v_\parallel = s v_{t\parallel} \Rightarrow dv_\parallel = v_{t\parallel} ds$ and $\xi = \frac{\omega_r \pm \omega_c}{k_\parallel v_{t\parallel}}$. So Eq (4.7) becomes,

$$1 - \frac{\left(k_\perp^2 + k_\parallel^2\right)c^2}{\omega^2} - \sum_\alpha \frac{\pi v_{t\perp}^2 \omega_{p\alpha}^2}{\omega} A \frac{1}{k_\parallel} \int_{-\infty}^{\infty} ds \frac{\left(d_1 + d_2 v_{t\parallel} s + d_3 v_{t\parallel}^3 s^3 + d_4 v_{t\parallel}^4 s^4 + d_5 v_{t\parallel}^5 s^5\right)}{s - \xi} \exp\left(-s^2\right) \tag{4.8}$$

Expanding the plasma dispersion function for large argument we have,

$$Z(\xi) = i\sqrt{\pi} e^{-\xi^2} - \frac{1}{\xi}\left(1 + \frac{1}{2\xi^2} + \frac{3}{4\xi^4} + \frac{15}{8\xi^6}\right) \ldots\ldots\ldots$$

The derivative of the plasma dispersion function has the following form,

$$Z(\xi) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} dS \frac{e^{-S^2}}{S - \xi}$$

$$\acute{Z}(\xi) = -2i\sqrt{\pi}\xi e^{-\xi^2} + \frac{1}{\xi^2}$$

For n number of successive derivative

$$Z^{(n)}(\xi) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} dS \frac{\frac{d^{(n)}\left(e^{-S^2}\right)}{dS^{(n)}}}{(S - \xi)}$$

Using Eq (4.8) we have,

$$1-\frac{\left(k_\perp^2+k_\parallel^2\right)c^2}{\omega^2}-\sum_\alpha \frac{\pi v_{t\perp}^2\omega_{p\alpha}^2}{\omega}A\frac{\sqrt{\pi}}{k_\parallel}\times$$
$$\begin{pmatrix} Z(\xi)\,d_1-\frac{d_2}{2}v_{t\parallel}Z^{(1)}(\xi)+d_3v_{t\parallel}^3\left(-\frac{1}{8}\left(Z^{(3)}(\xi)+6Z^{(1)}(\xi)\right)\right)+\\ d_4v_{t\parallel}^4\frac{1}{16}\left(Z^{(4)}(\xi)+12Z(\xi)+12Z^{(2)}(\xi)\right)+\\ d_5v_{t\parallel}^5\frac{1}{32}\left(Z^{(5)}(\xi)+20Z^{(3)}(\xi)+60Z^{(1)}(\xi)\right)\end{pmatrix}$$

Where as $Z(\xi)$, $Z^{(1)}(\xi)$, $Z^{(2)}(\xi)$, $Z^{(3)}(\xi)$, $Z^{(4)}(\xi)$ and $Z^{(5)}(\xi)$ are the derivatives of the plasma dispersion function.

By simplifying the terms we get,

$$1-\frac{\left(k_\perp^2+k_\parallel^2\right)c^2}{\omega^2}-\sum_\alpha \frac{\pi v_{t\perp}^2\omega_{p\alpha}^2}{\omega}A\frac{\sqrt{\pi}}{k_\parallel}\times \tag{4.9}$$
$$\begin{pmatrix} \left(d_1+\frac{3}{4}d_4v_{t\parallel}^4\right)Z(\xi)+\left(-\frac{d_2}{2}v_{t\parallel}-\frac{3}{4}d_3v_{t\parallel}^3+\frac{15}{8}d_5v_{t\parallel}^5\right)Z^{(1)}(\xi)+\\ \left(\frac{3}{4}d_4v_{t\parallel}^4\right)Z^{(2)}(\xi)+\left(-\frac{1}{8}d_3v_{t\parallel}^3+\frac{5}{8}d_5v_{t\parallel}^5\right)Z^{(3)}(\xi)+\\ \left(\frac{1}{16}d_4v_{t\parallel}^4\right)Z^{(4)}(\xi)+\left(\frac{1}{32}d_5v_{t\parallel}^5\right)Z^{(5)}(\xi)\end{pmatrix}$$

Now expanding derivatives of dispersion function with respect to large argument, we have

$$Z(\xi)=i\sqrt{\pi}e^{-\xi^2}-\frac{1}{\xi}\left(1+\frac{1}{2\xi^2}+\frac{3}{4\xi^4}+\frac{15}{8\xi^6}..........\right)$$
$$Z^{(1)}(\xi)=-2i\sqrt{\pi}\xi e^{-\xi^2}-\left(-\frac{1}{\xi^2}-\frac{3}{2\xi^4}-\frac{15}{4\xi^6}-\frac{105}{8\xi^8}........\right)$$
$$Z^{(2)}(\xi)=-2i\sqrt{\pi}\left(1-2\xi^2\right)e^{-\xi^2}-\left(\frac{2}{\xi^3}+\frac{12}{2\xi^5}+\frac{90}{4\xi^7}+\frac{840}{8\xi^9}......\right)$$
$$Z^{(3)}(\xi)=-2i\sqrt{\pi}\left(4\xi^3-6\xi\right)e^{-\xi^2}-\left(-\frac{6}{\xi^4}-\frac{60}{2\xi^6}-\frac{630}{4\xi^8}-\frac{7560}{8\xi^{10}}......\right)$$
$$Z^{(4)}(\xi)=-2i\sqrt{\pi}\left[24\xi^2-6-8\xi^4\right]e^{-\xi^2}-\left(\frac{24}{\xi^5}+\frac{360}{2\xi^7}+\frac{5040}{4\xi^9}+\frac{75600}{8\xi^{11}}......\right)$$
$$Z^{(5)}(\xi)=-2i\sqrt{\pi}\left[60\xi-80\xi^3+16\xi^5\right]e^{-\xi^2}-\left(-\frac{120}{\xi^6}-\frac{2520}{2\xi^8}-\frac{45360}{4\xi^{10}}-\frac{831600}{8\xi^{12}}......\right)$$

Now putting the respective values in Eq (4.9) we have,

$$1-\frac{\left(k_\perp^2+k_\parallel^2\right)c^2}{\omega^2}-\sum_\alpha \frac{\pi v_{t\perp}^2\omega_{p\alpha}^2}{\omega}A\frac{\sqrt{\pi}}{k_\parallel}\times$$
$$\left(\begin{array}{c}\left(d_1+\frac{3}{4}d_4v_{t\parallel}^4\right)\left(i\sqrt{\pi}e^{-\xi^2}-\frac{1}{\xi}\left(1+\frac{1}{2\xi^2}+\frac{3}{4\xi^4}+\frac{15}{8\xi^6}..........\right)\right)+\\ \left(-\frac{d_2}{2}v_{t\parallel}-\frac{3}{4}d_3v_{t\parallel}^3+\frac{15}{8}d_5v_{t\parallel}^5\right)\left(-2i\sqrt{\pi}\xi e^{-\xi^2}-\left(-\frac{1}{\xi^2}-\frac{3}{2\xi^4}-\frac{15}{4\xi^6}-\frac{105}{8\xi^8}........\right)\right)+\\ \left(\frac{3}{4}d_4v_{t\parallel}^4\right)\left(-2i\sqrt{\pi}\left(1-2\xi^2\right)e^{-\xi^2}-\left(\frac{2}{\xi^3}+\frac{12}{2\xi^5}+\frac{90}{4\xi^7}+\frac{840}{8\xi^9}......\right)\right)+\\ \left(-\frac{1}{8}d_3v_{t\parallel}^3+\frac{5}{8}d_5v_{t\parallel}^5\right)\left(-2i\sqrt{\pi}\left(4\xi^3-6\xi\right)e^{-\xi^2}-\left(-\frac{6}{\xi^4}-\frac{60}{2\xi^6}-\frac{630}{4\xi^8}-\frac{7560}{8\xi^{10}}......\right)\right)+\\ \left(\frac{1}{16}d_4v_{t\parallel}^4\right)\left(-2i\sqrt{\pi}\left[24\xi^2-6-8\xi^4\right]e^{-\xi^2}-\left(\frac{24}{\xi^5}+\frac{360}{2\xi^7}+\frac{5040}{4\xi^9}+\frac{75600}{8\xi^{11}}......\right)\right)+\\ \left(\frac{1}{32}d_5v_{t\parallel}^5\right)\left(-2i\sqrt{\pi}\left[60\xi-80\xi^3+16\xi^5\right]e^{-\xi^2}-\left(-\frac{120}{\xi^6}-\frac{2520}{2\xi^8}-\frac{45360}{4\xi^{10}}-\frac{831600}{8\xi^{12}}......\right)\right)\end{array}\right)$$

Separating the Real and Imaginary part respectively we have,

$$D_r(\omega) = 1-\frac{\left(k_\perp^2+k_\parallel^2\right)c^2}{\omega^2}-\sum_\alpha \frac{\pi v_{t\perp}^2\omega_{p\alpha}^2}{\omega}A\frac{\sqrt{\pi}}{k_\parallel}\times$$
$$\left(\begin{array}{c}\left(d_1+\frac{3}{4}d_4v_{t\parallel}^4\right)\left(-\frac{1}{\xi}\left(1+\frac{1}{2\xi^2}+\frac{3}{4\xi^4}+\frac{15}{8\xi^6}..........\right)\right)+\\ \left(-\frac{d_2}{2}v_{t\parallel}-\frac{3}{4}d_3v_{t\parallel}^3+\frac{15}{8}d_5v_{t\parallel}^5\right)\left(-\left(-\frac{1}{\xi^2}-\frac{3}{2\xi^4}-\frac{15}{4\xi^6}-\frac{105}{8\xi^8}........\right)\right)+\\ \left(\frac{3}{4}d_4v_{t\parallel}^4\right)\left(-\left(\frac{2}{\xi^3}+\frac{12}{2\xi^5}+\frac{90}{4\xi^7}+\frac{840}{8\xi^9}......\right)\right)+\\ \left(-\frac{1}{8}d_3v_{t\parallel}^3+\frac{5}{8}d_5v_{t\parallel}^5\right)\left(-\left(-\frac{6}{\xi^4}-\frac{60}{2\xi^6}-\frac{630}{4\xi^8}-\frac{7560}{8\xi^{10}}......\right)\right)+\\ \left(\frac{1}{16}d_4v_{t\parallel}^4\right)\left(-\left(\frac{24}{\xi^5}+\frac{360}{2\xi^7}+\frac{5040}{4\xi^9}+\frac{75600}{8\xi^{11}}......\right)\right)+\\ \left(\frac{1}{32}d_5v_{t\parallel}^5\right)\left(-\left(-\frac{120}{\xi^6}-\frac{2520}{2\xi^8}-\frac{45360}{4\xi^{10}}-\frac{831600}{8\xi^{12}}......\right)\right)\end{array}\right)$$

$$D_i(\omega)=\frac{\sqrt{\pi}}{k_\parallel}\left(\begin{array}{c}\left(d_1+\frac{3}{4}d_4v_{t\parallel}^4\right)\left(i\sqrt{\pi}e^{-\xi^2}\right)+\\ \left(-\frac{d_2}{2}v_{t\parallel}-\frac{3}{4}d_3v_{t\parallel}^3+\frac{15}{8}d_5v_{t\parallel}^5\right)\left(-2i\sqrt{\pi}\xi e^{-\xi^2}\right)+\\ \left(\frac{3}{4}d_4v_{t\parallel}^4\right)\left(-2i\sqrt{\pi}\left(1-2\xi^2\right)e^{-\xi^2}\right)+\\ \left(-\frac{1}{8}d_3v_{t\parallel}^3+\frac{5}{8}d_5v_{t\parallel}^5\right)\left(-2i\sqrt{\pi}\left(4\xi^3-6\xi\right)e^{-\xi^2}\right)+\\ \left(\frac{1}{16}d_4v_{t\parallel}^4\right)\left(-2i\sqrt{\pi}\left[24\xi^2-6-8\xi^4\right]e^{-\xi^2}\right)+\\ \left(\frac{1}{32}d_5v_{t\parallel}^5\right)\left(-2i\sqrt{\pi}\left[60\xi-80\xi^3+16\xi^5\right]e^{-\xi^2}\right)\end{array}\right)$$

First simplifying the real part we have, and taking only $\frac{1}{\xi^2}$ up to terms we have,

$$D_r(\omega)=1-\frac{\left(k_\perp^2+k_\parallel^2\right)c^2}{\omega^2}-\sum_\alpha \frac{\pi v_{t\perp}^2\omega_{p\alpha}^2}{\omega}A\frac{\sqrt{\pi}}{k_\parallel}\left(\begin{array}{c}\left(d_1+\frac{3}{4}d_4v_{t\parallel}^4\right)\left(-\frac{1}{\xi}\right)+\\ \left(-\frac{d_2}{2}v_{t\parallel}-\frac{3}{4}d_3v_{t\parallel}^3+\frac{15}{8}d_5v_{t\parallel}^5\right)\left(-\left(-\frac{1}{\xi^2}\right)\right)+\end{array}\right)$$

Putting $\xi = \frac{\omega_r - \omega_c}{k_\parallel v_{t\parallel}}$ and Taking $\omega_r >> \omega_c$ as we have taken $\frac{\omega_r - \omega_c}{k_\parallel v_{t\parallel}} >> 1$ we have,

$$D_r(\omega) = 1 - \frac{\left(k_\perp^2 + k_\parallel^2\right)c^2}{\omega^2} - \sum_\alpha \frac{\pi v_{t\perp}^2 \omega_{p\alpha}^2}{\omega} A \frac{\sqrt{\pi}}{k_\parallel} \left(\begin{array}{c} \left(d_1 + \frac{3}{4} d_4 v_{t\parallel}^4\right)\left(\frac{k_\parallel v_{t\parallel}}{\omega_c}\right) + \\ \left(-\frac{d_2}{2} v_{t\parallel} - \frac{3}{4} d_3 v_{t\parallel}^3 + \frac{15}{8} d_5 v_{t\parallel}^5\right)\left(\frac{k_\parallel v_{t\parallel}}{\omega_c}\right)^2 \end{array} \right)$$

Now putting the values of d_1, d_2, d_3, d_4 and d_5 in above equation and Simplifying we have,

$$\begin{aligned} D_r(\omega) &= 1 - \frac{\left(k_\perp^2 + k_\parallel^2\right)c^2}{\omega^2} - \sum_\alpha \frac{\pi v_{t\perp}^2 \omega_{p\alpha}^2}{\omega} A \frac{\sqrt{\pi}}{k_\parallel} \left(\frac{k_\parallel v_{t\parallel}}{\omega_c}\right) \\ & \left(\begin{array}{c} \left(\left(I_1(nb) + 2\Lambda I_2(nb) + \Lambda I_3(nb)\right) + \frac{3}{4}\Lambda I_1(nb)\right) + \\ \left(\begin{array}{c} -\frac{1}{2}\frac{k_\parallel v_{t\parallel}}{\omega}\left\{A_T\left(I_1(nb) + \Lambda I_3(nb)\right) - 2\Lambda I_2(nb)\right\} - \\ \frac{3}{4}\left(-\frac{k_\parallel v_{t\parallel}}{\omega}\left(2\Lambda\frac{v_{t\perp}^2}{v_{t\parallel}^2}\right) I_1(nb)\right) + \frac{15}{8}\Lambda\frac{k_\parallel v_{t\parallel}}{\omega} A_T I_1(nb) \end{array} \right)\left(\frac{k_\parallel v_{t\parallel}}{\omega_c}\right) \end{array} \right) \end{aligned}$$

Putting $A_T = \frac{v_{t\perp}^2}{v_{t\parallel}^2} - 1$ we have

$$\begin{aligned} D_r(\omega) &= 1 - \frac{\left(k_\perp^2 + k_\parallel^2\right)c^2}{\omega^2} - \sum_\alpha \frac{\pi v_{t\perp}^2 \omega_{p\alpha}^2}{\omega} A \frac{\sqrt{\pi}}{k_\parallel} \left(\frac{k_\parallel v_{t\parallel}}{\omega_c}\right) \\ & \left(\begin{array}{c} \left(\left(I_1(nb) + 2\Lambda I_2(nb) + \Lambda I_3(nb)\right) + \frac{3}{4}\Lambda I_1(nb)\right) + \\ \left(\begin{array}{c} -\frac{1}{2}\left\{A_T\left(I_1(nb) + \Lambda I_3(nb)\right) - 2\Lambda I_2(nb)\right\} + \\ \frac{3}{2}\Lambda\left(A_T + 1\right) I_1(nb) + \frac{15}{8}\Lambda A_T I_1(nb) \end{array} \right)\left(\frac{k_\parallel v_{t\parallel}}{\omega_c}\right)\left(\frac{k_\parallel v_{t\parallel}}{\omega}\right) \end{array} \right) \end{aligned}$$

Separating the $I_1(nb)$,$I_2(nb)$,$I_3(nb)$ terms and simplifying we have,

$$\begin{aligned} D_r(\omega) &= 1 - \frac{\left(k_\perp^2 + k_\parallel^2\right)c^2}{\omega^2} - \sum_\alpha \frac{\pi v_{t\perp}^2 \omega_{p\alpha}^2}{\omega} A \frac{\sqrt{\pi}}{k_\parallel} \left(\frac{k_\parallel v_{t\parallel}}{\omega_c}\right) \\ & \left(\begin{array}{c} I_1(nb)\left(1 + \frac{3}{4}\Lambda + \frac{1}{2}\left(-A_T + 3\Lambda + \frac{27}{4}\Lambda A_T\right)\left(\frac{k_\parallel v_{t\parallel}}{\omega_c}\right)\left(\frac{k_\parallel v_{t\parallel}}{\omega}\right)\right) + \\ I_2(nb)\, 2\Lambda\left(1 + \frac{1}{2}\left(\frac{k_\parallel v_{t\parallel}}{\omega_c}\right)\left(\frac{k_\parallel v_{t\parallel}}{\omega}\right)\right) + \\ I_3(nb)\,\Lambda\left(1 - \frac{1}{2}A_T\left(\frac{k_\parallel v_{t\parallel}}{\omega_c}\right)\left(\frac{k_\parallel v_{t\parallel}}{\omega}\right)\right) \end{array} \right) \end{aligned}$$

Putting $A = \frac{1}{(\pi)^{3/2}\left(1 + \frac{11}{4}\Lambda\right) v_{t\perp}^2 v_{t\parallel}}$ in above equation we have,

$$
\begin{aligned}
& 1-\frac{\left(k_{\perp}^{2}+k_{\|}^{2}\right) c^{2}}{\omega^{2}}- \\
& \sum_{\alpha} \frac{\omega_{p\alpha}^{2}}{\omega \omega_{c}} \frac{1}{\left(1+\frac{11}{4} \Lambda\right)} \times\left(\begin{array}{c}
I_{1}(nb)\left(1+\frac{3}{4} \Lambda+\frac{1}{2}\left(-A_{T}+3 \Lambda+\frac{27}{4} \Lambda A_{T}\right)\left(\frac{k_{\|} v_{t\|}}{\omega_{c}}\right)\left(\frac{k_{\|} v_{t\|}}{\omega}\right)\right)+ \\
I_{2}(nb) 2 \Lambda\left(1+\frac{1}{2}\left(\frac{k_{\|} v_{t\|}}{\omega_{c}}\right)\left(\frac{k_{\|} v_{t\|}}{\omega}\right)\right)+ \\
I_{3}(nb) \Lambda\left(1-\frac{1}{2} A_{T}\left(\frac{k_{\|} v_{t\|}}{\omega_{c}}\right)\left(\frac{k_{\|} v_{t\|}}{\omega}\right)\right)
\end{array}\right) \\
= & \ 0
\end{aligned}
$$

Multiplying each term with ω^2 and Taking $\omega_r >> \omega_c$ as we have taken $\frac{\omega_r-\omega_c}{k_{\|} v_{t\|}} >> 1$ we have,

$$
\begin{aligned}
& -\left(k_{\perp}^{2}+k_{\|}^{2}\right) c^{2}- \\
& \sum_{\alpha} \frac{\omega \omega_{p\alpha}^{2}}{\omega_{c}} \frac{1}{\left(1+\frac{11}{4} \Lambda\right)} \times\left(\begin{array}{c}
I_{1}(nb)\left(1+\frac{3}{4} \Lambda+\frac{1}{2}\left(\begin{array}{c}
-A_{T}+3 \Lambda \\
+\frac{27}{4} \Lambda A_{T}
\end{array}\right)\left(\frac{k_{\|} v_{t\|}}{\omega_{c}}\right)\left(\frac{k_{\|} v_{t\|}}{\omega}\right)\right)+ \\
I_{2}(nb) 2 \Lambda\left(1+\frac{1}{2}\left(\frac{k_{\|} v_{t\|}}{\omega_{c}}\right)\left(\frac{k_{\|} v_{t\|}}{\omega}\right)\right)+ \\
I_{3}(nb) \Lambda\left(1-\frac{1}{2} A_{T}\left(\frac{k_{\|} v_{t\|}}{\omega_{c}}\right)\left(\frac{k_{\|} v_{t\|}}{\omega}\right)\right)
\end{array}\right) \\
= & \ 0
\end{aligned}
$$

Splitting the summation term we have,

$$
\begin{aligned}
& -\sum_{\alpha} \frac{\omega \omega_{p\alpha}^{2}}{\omega_{c}} \frac{1}{\left(1+\frac{11}{4} \Lambda\right)} \times\left(\begin{array}{c}
I_{1}(nb)\left(1+\frac{3}{4} \Lambda\right)+ \\
I_{2}(nb)(2 \Lambda)+ \\
I_{3}(nb)(\Lambda)
\end{array}\right)- \\
& \sum_{\alpha} \frac{\omega_{p\alpha}^{2}}{\left(1+\frac{11}{4} \Lambda\right)} \times\left(\begin{array}{c}
I_{1}(nb)\left(\frac{1}{2}\left(-A_{T}+3 \Lambda+\frac{27}{4} \Lambda A_{T}\right)\left(\frac{k_{\|} v_{t\|}}{\omega_{c}}\right)^{2}\right)+ \\
I_{2}(nb) 2 \Lambda\left(\frac{1}{2}\left(\frac{k_{\|} v_{t\|}}{\omega_{c}}\right)^{2}\right)+ \\
I_{3}(nb) \Lambda\left(-\frac{1}{2} A_{T}\left(\frac{k_{\|} v_{t\|}}{\omega_{c}}\right)^{2}\right)
\end{array}\right)
\end{aligned}
$$

Now putting the split terms into its original position,Simplifying and taking $\left(k_{\perp}^{2}+k_{\|}^{2}\right) c^{2}$ as common we have,

$$\left(k_\perp^2+k_\parallel^2\right)c^2\left[1-\sum_\alpha\frac{\omega_{p\alpha}^2}{\left(1+\frac{11}{4}\Lambda\right)\left(k_\perp^2+k_\parallel^2\right)c^2}\times\begin{pmatrix} I_1(nb)\left(\frac{1}{2}\begin{pmatrix}-A_T+3\Lambda+\\ \frac{27}{4}\Lambda A_T\end{pmatrix}\left(\frac{k_\parallel v_{t\parallel}}{\omega_c}\right)^2\right)+\\ I_2(nb)\,2\Lambda\left(\frac{1}{2}\left(\frac{k_\parallel v_{t\parallel}}{\omega_c}\right)^2\right)+\\ I_3(nb)\,\Lambda\left(-\frac{1}{2}A_T\left(\frac{k_\parallel v_{t\parallel}}{\omega_c}\right)^2\right)\end{pmatrix}\right]$$

$$=\sum_\alpha\frac{\omega\omega_{p\alpha}^2}{\omega_c}\frac{1}{\left(1+\frac{11}{4}\Lambda\right)}\times\begin{pmatrix} I_1(nb)\left(1+\frac{3}{4}\Lambda\right)+\\ I_2(nb)(2\Lambda)+\\ I_3(nb)(\Lambda)\end{pmatrix}$$

Dividing B.H.S by $\left(k_\perp^2+k_\parallel^2\right)c^2$ we have,

$$\left[1-\sum_\alpha\frac{\omega_{p\alpha}^2}{\left(1+\frac{11}{4}\Lambda\right)\left(k_\perp^2+k_\parallel^2\right)c^2}\times\begin{pmatrix} I_1(nb)\left(\frac{1}{2}\left(-A_T+3\Lambda+\frac{27}{4}\Lambda A_T\right)\left(\frac{k_\parallel v_{t\parallel}}{\omega_c}\right)^2\right)+\\ I_2(nb)\,2\Lambda\left(\frac{1}{2}\left(\frac{k_\parallel v_{t\parallel}}{\omega_c}\right)^2\right)+\\ I_3(nb)\,\Lambda\left(-\frac{1}{2}A_T\left(\frac{k_\parallel v_{t\parallel}}{\omega_c}\right)^2\right)\end{pmatrix}\right]$$

$$=\sum_\alpha\frac{\omega\omega_{p\alpha}^2}{\omega_c}\frac{1}{\left(1+\frac{11}{4}\Lambda\right)\left(k_\perp^2+k_\parallel^2\right)c^2}\times\begin{pmatrix} I_1(nb)\left(1+\frac{3}{4}\Lambda\right)+\\ I_2(nb)(2\Lambda)+\\ I_3(nb)(\Lambda)\end{pmatrix}$$

So we have,

$$\frac{\omega}{\omega_c}=\frac{\left(1+\frac{11}{4}\Lambda\right)\left(k_\perp^2+k_\parallel^2\right)c^2}{\omega_{p\alpha}^2\times\begin{pmatrix} I_1(nb)\left(1+\frac{3}{4}\Lambda\right)+\\ I_2(nb)(2\Lambda)+\\ I_3(nb)(\Lambda)\end{pmatrix}}\times\left[\begin{matrix}1-\sum_\alpha\frac{\omega_{p\alpha}^2}{\left(1+\frac{11}{4}\Lambda\right)\left(k_\perp^2+k_\parallel^2\right)c^2}\times\\ \begin{pmatrix} I_1(nb)\left(\frac{1}{2}\left(-A_T+3\Lambda+\frac{27}{4}\Lambda A_T\right)\left(\frac{k_\parallel v_{t\parallel}}{\omega_c}\right)^2\right)+\\ I_2(nb)\,2\Lambda\left(\frac{1}{2}\left(\frac{k_\parallel v_{t\parallel}}{\omega_c}\right)^2\right)+\\ I_3(nb)\,\Lambda\left(-\frac{1}{2}A_T\left(\frac{k_\parallel v_{t\parallel}}{\omega_c}\right)^2\right)\end{pmatrix}\end{matrix}\right]$$

This is the required dispersion relation for oblique Whistlers using cairns distribution function.

Now for the growth rate we have,

$$\gamma=-\frac{D_i(\omega)}{\frac{\partial D_r(\omega)}{\partial\omega_r}}\tag{4.10}$$

Putting the value of $A = \frac{1}{(\pi)^{3/2}\left(1+\frac{11}{4}\Lambda\right)v_{t\perp}^2 v_{t\|}}$ in $D_r(\omega)$ we have,

$$D_r(\omega) = 1 - \frac{\left(k_\perp^2 + k_\|^2\right)c^2}{\omega^2} - \sum_\alpha \frac{\omega_{p\alpha}^2}{\omega}\frac{1}{\left(1+\frac{11}{4}\Lambda\right)k_\| v_{t\|}}\left(\begin{array}{c}\left(d_1 + \frac{3}{4}d_4 v_{t\|}^4\right)\left(-\frac{1}{\xi}\left(1+\frac{1}{2\xi^2}\right)\right) + \\ \left(-\frac{d_2}{2}v_{t\|} - \frac{3}{4}d_3 v_{t\|}^3 + \frac{15}{8}d_5 v_{t\|}^5\right)\left(-\left(-\frac{1}{\xi^2}\right)\right) + \\ \left(\frac{3}{4}d_4 v_{t\|}^4\right)\left(-\left(\frac{2}{\xi^3}\right)\right)\end{array}\right)$$

Let defining some terms as,

$$\begin{aligned} e_1 &= \left(d_1 + \frac{3}{4}d_4 v_{t\|}^4\right) \\ e_2 &= \left(-\frac{d_2}{2}v_{t\|} - \frac{3}{4}d_3 v_{t\|}^3 + \frac{15}{8}d_5 v_{t\|}^5\right) \\ e_3 &= \left(\frac{3}{4}d_4 v_{t\|}^4\right) \end{aligned}$$

Than we have,

$$D_r(\omega) = 1 - \frac{\left(k_\perp^2 + k_\|^2\right)c^2}{\omega^2} - \sum_\alpha \frac{\omega_{p\alpha}^2}{\omega}\frac{1}{\left(1+\frac{11}{4}\Lambda\right)k_\| v_{t\|}}\left(-e_1\frac{1}{\xi} + e_2\frac{1}{\xi^2} - \left(2e_3 + \frac{1}{2}e_1\right)\frac{1}{\xi^3}\right)$$

Putting the value of ξ we have,

$$D_r(\omega) = 1 - \frac{\left(k_\perp^2 + k_\|^2\right)c^2}{\omega^2} - \sum_\alpha \frac{\omega_{p\alpha}^2}{\omega}\frac{1}{\left(1+\frac{11}{4}\Lambda\right)}\left(\begin{array}{c} -e_1\left(\frac{1}{\omega-|\omega_c|}\right) + e_2 k_\| v_{t\|}\left(\frac{1}{\omega-|\omega_c|}\right)^2 - \\ \left(2e_3 + \frac{1}{2}e_1\right)\left(k_\| v_{t\|}\right)^2\left(\frac{1}{\omega-|\omega_c|}\right)^3 \end{array}\right)$$

Take the derivative of $D_r(\omega)$ by extracting the $\frac{1}{\omega}$ term from d_2, d_3, d_4 which is present in e_2 .

$$\begin{aligned} \frac{\partial D_r(\omega)}{\partial \omega_r} &= +2\frac{\left(k_\perp^2 + k_\|^2\right)c^2}{\omega^3} - \\ &\sum_\alpha \frac{\omega_{p\alpha}^2}{\left(1+\frac{11}{4}\Lambda\right)}\left(\begin{array}{c} -e_1\left(\frac{(2\omega-|\omega_c|)}{(\omega(\omega-|\omega_c|))^2}\right) + e_2 k_\| v_{t\|}\frac{\left(4\omega^3 - 6\omega^2|\omega_c| + 2\omega|\omega_c|^2\right)}{\left(\omega^2(\omega-|\omega_c|)^2\right)^2} - \\ \left(2e_3 + \frac{1}{2}e_1\right)\left(k_\| v_{t\|}\right)^2\frac{\left(4\omega^3 - 9\omega^2|\omega_c| + 6\omega|\omega_c|^2 - |\omega_c|^3\right)}{\left(\omega(\omega-|\omega_c|)^3\right)^2} \end{array}\right) \end{aligned}$$

As we have $\frac{\omega_r - \omega_c}{k_\| v_{t\|}} >> 1$ therefore $\omega_r >> \omega_c$ and simplifying we have,

$$\frac{\partial D_r(\omega)}{\partial \omega_r} = +2\frac{\left(k_\perp^2+k_\parallel^2\right)c^2}{\omega^3} - \sum_\alpha \frac{\omega_{p\alpha}^2}{\left(1+\frac{11}{4}\Lambda\right)}\frac{1}{\left(\omega^2\left|\omega_c\right|\right)}\left(e_1+\frac{2e_2k_\parallel v_{t\parallel}}{\omega\left|\omega_c\right|}+\frac{\left(2e_3+\frac{1}{2}e_1\right)\left(k_\parallel v_{t\parallel}\right)^2}{\left|\omega_c\right|^2}\right)$$

Now putting the values of e_1,e_2,e_3 back we have,

$$\begin{aligned}\frac{\partial D_r(\omega)}{\partial \omega_r} &= +2\frac{\left(k_\perp^2+k_\parallel^2\right)c^2}{\omega^3} - \sum_\alpha \frac{\omega_{p\alpha}^2}{\left(1+\frac{11}{4}\Lambda\right)}\frac{1}{\left(\omega^2\left|\omega_c\right|\right)} \times \\ &\left(\begin{array}{c}\left(d_1+\frac{3}{4}d_4v_{t\parallel}^4\right)+ \\ \left(-\frac{d_2}{2}v_{t\parallel}-\frac{3}{4}d_3v_{t\parallel}^3+\frac{15}{8}d_5v_{t\parallel}^5\right)\frac{2k_\parallel v_{t\parallel}}{\omega\left|\omega_c\right|}+ \\ \frac{\left(2\left(\frac{3}{4}d_4v_{t\parallel}^4\right)+\frac{1}{2}\left(d_1+\frac{3}{4}d_4v_{t\parallel}^4\right)\right)\left(k_\parallel v_{t\parallel}\right)^2}{\left|\omega_c\right|^2}\end{array}\right)\end{aligned}$$

And now putting the values of d_1 ,d_2,d_3,d_4,d_5 we have,

$$\begin{aligned}\frac{\partial D_r(\omega)}{\partial \omega_r} &= +2\frac{\left(k_\perp^2+k_\parallel^2\right)c^2}{\omega^3} - \sum_\alpha \frac{\omega_{p\alpha}^2}{\left(1+\frac{11}{4}\Lambda\right)}\frac{1}{\left(\omega^2\left|\omega_c\right|\right)} \times \\ &\left(\begin{array}{c}\left(\left(I_1(nb)+2\Lambda I_2(nb)+\Lambda I_3(nb)\right)+\frac{3}{4}\Lambda\frac{1}{v_{t\parallel}^4}I_1(nb)\,v_{t\parallel}^4\right)+ \\ \left(\begin{array}{c}-\frac{k_\parallel}{2}\left\{A_T\left(I_1(nb)+\Lambda I_3(nb)\right)-2\Lambda I_2(nb)\right\}v_{t\parallel}- \\ \frac{3}{4}\left(-\frac{k_\parallel v_{t\perp}^2}{1}\left(2\Lambda\frac{1}{v_{t\parallel}^4}\right)I_1(nb)\right)v_{t\parallel}^3+ \\ \frac{15}{8}\Lambda\frac{k_\parallel}{v_{t\parallel}^4}A_TI_1(nb)\,v_{t\parallel}^5\end{array}\right)\frac{2k_\parallel v_{t\parallel}}{\omega\left|\omega_c\right|}+ \\ \frac{\left(\begin{array}{c}2\left(\frac{3}{4}\Lambda\frac{1}{v_{t\parallel}^4}I_1(nb)\,v_{t\parallel}^4\right)+ \\ \frac{1}{2}\left(\left(I_1(nb)+2\Lambda I_2(nb)+\Lambda I_3(nb)\right)+\frac{3}{4}\Lambda\frac{1}{v_{t\parallel}^4}I_1(nb)\,v_{t\parallel}^4\right)\end{array}\right)\left(k_\parallel v_{t\parallel}\right)^2}{\left|\omega_c\right|^2}\end{array}\right)\end{aligned}$$

Simplifying by putting $A_T=\frac{v_{t\perp}^2}{v_{t\parallel}^2}-1$ we have,

$$\begin{aligned}\frac{\partial D_r(\omega)}{\partial \omega_r} &= +2\frac{\left(k_\perp^2+k_\parallel^2\right)c^2}{\omega^3} - \sum_\alpha \frac{\omega_{p\alpha}^2}{\left(1+\frac{11}{4}\Lambda\right)}\frac{1}{\left(\omega^2\left|\omega_c\right|\right)} \times \\ &\left(\begin{array}{c}\left(\left(I_1(nb)+2\Lambda I_2(nb)+\Lambda I_3(nb)\right)+\frac{3}{4}\Lambda I_1(nb)\right)+ \\ \left(\begin{array}{c}-\left\{A_T\left(I_1(nb)+\Lambda I_3(nb)\right)-2\Lambda I_2(nb)\right\}+ \\ 3\Lambda\left(A_T+1\right)I_1(nb)+\frac{15}{4}\Lambda A_TI_1(nb)\end{array}\right)\frac{\left(k_\parallel v_{t\parallel}\right)^2}{\omega\left|\omega_c\right|}+ \\ \frac{\left(\frac{3}{2}\Lambda I_1(nb)+\frac{1}{2}\left(\left(I_1(nb)+2\Lambda I_2(nb)+\Lambda I_3(nb)\right)+\frac{3}{4}\Lambda I_1(nb)\right)\right)\left(k_\parallel v_{t\parallel}\right)^2}{\left|\omega_c\right|^2}\end{array}\right)\end{aligned}$$

Separating and simplifying the $I_1(nb)$, $I_2(nb)$ and $I_3(nb)$ term we have,

$$\frac{\partial D_r(\omega)}{\partial \omega_r} = +2\frac{\left(k_\perp^2 + k_\parallel^2\right)c^2}{\omega^3} - \sum_\alpha \frac{\omega_{p\alpha}^2}{\left(1+\frac{11}{4}\Lambda\right)} \frac{1}{\left(\omega^2 |\omega_c|\right)} \times \tag{4.11}$$
$$\left(\begin{array}{c} I_1(nb) \left(\begin{array}{c} 1 + \frac{3}{4}\Lambda + \\ \left(3\Lambda\left(1+\frac{9}{4}A_T\right) - A_T\right) \frac{\left(k_\parallel v_{t\parallel}\right)^2}{\omega|\omega_c|} + \\ \left(1+\frac{15}{4}\Lambda\right) \frac{\left(k_\parallel v_{t\parallel}\right)^2}{2|\omega_c|^2} \end{array} \right) + \\ I_2(nb)\, 2\Lambda \left(1 + \frac{\left(k_\parallel v_{t\parallel}\right)^2}{\omega|\omega_c|} + \frac{\frac{1}{2}\left(k_\parallel v_{t\parallel}\right)^2}{|\omega_c|^2}\right) + \\ I_3(nb) \left(\Lambda + \Lambda\frac{\left(k_\parallel v_{t\parallel}\right)^2}{\omega|\omega_c|} + \frac{\left(\frac{1}{2}(\Lambda)\right)\left(k_\parallel v_{t\parallel}\right)^2}{|\omega_c|^2}\right) \end{array} \right)$$

Now as the imaginary part is,

$$D_i(\omega) = -\sum_\alpha \frac{\omega_{p\alpha}^2}{\omega\left(1+\frac{11}{4}\Lambda\right)k_\parallel v_{t\parallel}} \left(\begin{array}{c} \left(d_1 + \frac{3}{4}d_4 v_{t\parallel}^4\right)\left(\sqrt{\pi}e^{-\xi^2}\right) + \\ \left(-\frac{d_2}{2}v_{t\parallel} - \frac{3}{4}d_3 v_{t\parallel}^3 + \frac{15}{8}d_5 v_{t\parallel}^5\right)\left(-2\sqrt{\pi}\xi e^{-\xi^2}\right) + \\ \left(\frac{3}{4}d_4 v_{t\parallel}^4\right)\left(-2\sqrt{\pi}\left(1-2\xi^2\right)e^{-\xi^2}\right) + \\ \left(-\frac{1}{8}d_3 v_{t\parallel}^3 + \frac{5}{8}d_5 v_{t\parallel}^5\right)\left(-2\sqrt{\pi}\left(4\xi^3 - 6\xi\right)e^{-\xi^2}\right) + \\ \left(\frac{1}{16}d_4 v_{t\parallel}^4\right)\left(-2\sqrt{\pi}\left[24\xi^2 - 6 - 8\xi^4\right]e^{-\xi^2}\right) + \\ \left(\frac{1}{32}d_5 v_{t\parallel}^5\right)\left(-2\sqrt{\pi}\left[60\xi - 80\xi^3 + 16\xi^5\right]e^{-\xi^2}\right) \end{array} \right)$$

Extracting the ξ, ξ^2, ξ^3, ξ^4 and ξ^5 and Simplifying it we have,

$$D_i(\omega) = -\sum_\alpha \frac{\omega_{p\alpha}^2/\omega\sqrt{\pi}e^{-\xi^2}}{\left(1+\frac{11}{4}\Lambda\right)k_\parallel v_{t\parallel}} \left(\begin{array}{c} d_1 + \left(d_2 v_{t\parallel}\right)\xi + \\ \left(d_3 v_{t\parallel}^3\right)\xi^3 + \left(d_4 v_{t\parallel}^4\right)\xi^4 - \left(d_5 v_{t\parallel}^5\right)\xi^5 \end{array} \right)$$

Putting the value of ξ and also putting $(\omega_r - |\omega_c|) = -|\omega_c|$ as $\omega_r >> \omega_c$ we have,

$$D_i(\omega) = -\sum_\alpha \frac{\omega_{p\alpha}^2/\omega\sqrt{\pi}e^{-\xi^2}}{\left(1+\frac{11}{4}\Lambda\right)k_\parallel v_{t\parallel}} \left(\begin{array}{c} d_1 - \left(d_2 v_{t\parallel}\right)\left(\frac{|\omega_c|}{k_\parallel v_{t\parallel}}\right) - \left(d_3 v_{t\parallel}^3\right)\left(\frac{|\omega_c|}{k_\parallel v_{t\parallel}}\right)^3 + \\ \left(d_4 v_{t\parallel}^4\right)\left(\frac{|\omega_c|}{k_\parallel v_{t\parallel}}\right)^4 + \left(d_5 v_{t\parallel}^5\right)\left(\frac{|\omega_c|}{k_\parallel v_{t\parallel}}\right)^5 \end{array} \right)$$

Putting the values of d_1 , d_2, d_3, d_4, d_5 and also putting $A_T = \frac{v_{t\perp}^2}{v_{t\parallel}^2} - 1$ we have,

$$D_i(\omega) = -\sum_\alpha \frac{\omega_{p\alpha}^2/\omega\sqrt{\pi}e^{-\xi^2}}{\left(1+\frac{11}{4}\Lambda\right)k_\parallel v_{t\parallel}} \left(\begin{array}{c} (I_1(nb) + 2\Lambda I_2(nb) + \Lambda I_3(nb)) - \\ (\{A_T(I_1(nb) + \Lambda I_3(nb)) - 2\Lambda I_2(nb)\})\left(\frac{|\omega_c|}{\omega}\right) - \\ (-(A_T+1)(2\Lambda)I_1(nb))\left(\frac{|\omega_c|}{k_\parallel v_{t\parallel}}\right)^3\left(\frac{k_\parallel v_{t\parallel}}{\omega}\right) + \\ (\Lambda I_1(nb))\left(\frac{|\omega_c|}{k_\parallel v_{t\parallel}}\right)^4 + (\Lambda A_T I_1(nb))\frac{k_\parallel v_{t\parallel}}{\omega}\left(\frac{|\omega_c|}{k_\parallel v_{t\parallel}}\right)^5 \end{array} \right)$$

Simplifying and separating the $I_1(nb)$, $I_2(nb)$ and $I_3(nb)$ terms we have,

$$D_i(\omega) = -\frac{\omega_{p\alpha}^2/\omega\sqrt{\pi}e^{-\xi^2}}{\left(1+\frac{11}{4}\Lambda\right)k_\parallel v_{t\parallel}} \left(\begin{array}{c} I_1(nb)\left(\begin{array}{c} 1 - A_T\left(\frac{|\omega_c|}{\omega}\right) + \\ (2\Lambda)(A_T+1)\left(\frac{|\omega_c|}{k_\parallel v_{t\parallel}}\right)^3\left(\frac{k_\parallel v_{t\parallel}}{\omega}\right) + \\ \Lambda\left(\frac{|\omega_c|}{k_\parallel v_{t\parallel}}\right)^4 + \Lambda A_T\left(\frac{k_\parallel v_{t\parallel}}{\omega}\right)\left(\frac{|\omega_c|}{k_\parallel v_{t\parallel}}\right)^5 \end{array} \right) \\ I_2(nb)\left(2\Lambda\left(1+\left(\frac{|\omega_c|}{\omega}\right)\right)\right) + \\ I_3(nb)\left(\Lambda\left(1 - A_T\left(\frac{|\omega_c|}{\omega}\right)\right)\right) \end{array} \right) \tag{4.12}$$

Putting the values of Eq (4.11) and Eq (4.12) in Eq(4.10) we have,

$$\gamma = -\frac{ -\frac{\omega_{p\alpha}^2\sqrt{\pi}e^{-\xi^2}}{\omega\left(1+\frac{11}{4}\Lambda\right)k_\parallel v_{t\parallel}} \left(\begin{array}{c} I_1(nb)\left(\begin{array}{c} 1 - A_T\left(\frac{|\omega_c|}{\omega}\right) + \\ (2\Lambda)(A_T+1)\left(\frac{|\omega_c|}{k_\parallel v_{t\parallel}}\right)^3\left(\frac{k_\parallel v_{t\parallel}}{\omega}\right) + \\ \Lambda\left(\frac{|\omega_c|}{k_\parallel v_{t\parallel}}\right)^4 + \Lambda A_T\left(\frac{k_\parallel v_{t\parallel}}{\omega}\right)\left(\frac{|\omega_c|}{k_\parallel v_{t\parallel}}\right)^5 \end{array} \right) \\ I_2(nb)\left(2\Lambda\left(1+\left(\frac{|\omega_c|}{\omega}\right)\right)\right) + I_3(nb)\left(\Lambda\left(1 - A_T\left(\frac{|\omega_c|}{\omega}\right)\right)\right) \end{array} \right) }{ +2\frac{\left(k_\perp^2+k_\parallel^2\right)c^2}{\omega^3} - \sum_\alpha \frac{\omega_{p\alpha}^2}{\left(1+\frac{11}{4}\Lambda\right)}\frac{1}{(\omega^2|\omega_c|)} \times \left(\begin{array}{c} I_1(nb)\left(\begin{array}{c} 1 + \frac{3}{4}\Lambda + \\ \left(3\Lambda\left(1+\frac{9}{4}A_T\right) - A_T\right)\frac{\left(k_\parallel v_{t\parallel}\right)^2}{\omega|\omega_c|} + \\ \left(1+\frac{15}{4}\Lambda\right)\frac{\left(k_\parallel v_{t\parallel}\right)^2}{2|\omega_c|^2} \end{array} \right) + \\ I_2(nb)\,2\Lambda\left(1 + \frac{\left(k_\parallel v_{t\parallel}\right)^2}{\omega|\omega_c|} + \frac{\frac{1}{2}\left(k_\parallel v_{t\parallel}\right)^2}{|\omega_c|^2}\right) + \\ I_3(nb)\left(\Lambda + \Lambda\frac{\left(k_\parallel v_{t\parallel}\right)^2}{\omega|\omega_c|} + \frac{\left(\frac{1}{2}(\Lambda)\right)\left(k_\parallel v_{t\parallel}\right)^2}{|\omega_c|^2}\right) \end{array} \right) }$$

Putting the value of ξ and simplifying it to get $\frac{\gamma}{|\omega_c|}$ form we have,

$$\frac{\gamma}{|\omega_c|} = \frac{\omega_{p\alpha}^2 \sqrt{\pi} e^{-\left(\frac{\left(\frac{\omega}{|\omega_c|}-1\right)}{\left(\frac{k_\parallel c}{\omega_{p\alpha}}\right)\sqrt{\beta_\parallel}}\right)^2} \left(\begin{array}{c} I_1(nb) \left(\begin{array}{c} \frac{1}{|\omega_c|} - A_T \left(\frac{|\omega_c|}{\omega}\right) + \\ (2\Lambda)(A_T+1)\left(\frac{|\omega_c|}{k_\parallel v_{t\parallel}}\right)^2 \left(\frac{1}{\omega}\right) + \\ \Lambda |\omega_c|^3 \left(\frac{1}{k_\parallel v_{t\parallel}}\right)^4 + \\ \Lambda A_T \left(\frac{1}{\omega}\right)\left(\frac{|\omega_c|}{k_\parallel v_{t\parallel}}\right)^4 \end{array} \right) + \\ I_2(nb)\left(2\Lambda\left(\frac{1}{|\omega_c|} + \left(\frac{1}{\omega}\right)\right)\right) + \\ I_3(nb)\left(\Lambda\left(\frac{1}{|\omega_c|} - A_T\left(\frac{1}{\omega}\right)\right)\right) \end{array} \right)}{\left(1+\frac{11}{4}\Lambda\right) k_\parallel v_{t\parallel} \left[\begin{array}{c} 2\frac{\left(k_\perp^2+k_\parallel^2\right)c^2}{\omega^2} - \sum_\alpha \frac{\omega_{p\alpha}^2}{\left(1+\frac{11}{4}\Lambda\right)} \frac{1}{(\omega|\omega_c|)} \times \\ \left(\begin{array}{c} I_1(nb) \left(\begin{array}{c} 1+\frac{3}{4}\Lambda + \\ \left(\begin{array}{c} 3\Lambda\left(1+\frac{9}{4}A_T\right) - \\ A_T \end{array} \right) \frac{\left(k_\parallel v_{t\parallel}\right)^2}{\omega|\omega_c|} + \\ \left(1+\frac{15}{4}\Lambda\right)\frac{\left(k_\parallel v_{t\parallel}\right)^2}{2|\omega_c|^2} \end{array} \right) + \\ I_2(nb)\, 2\Lambda \left(\begin{array}{c} 1 + \frac{\left(k_\parallel v_{t\parallel}\right)^2}{\omega|\omega_c|} + \\ \frac{\frac{1}{2}\left(k_\parallel v_{t\parallel}\right)^2}{|\omega_c|^2} \end{array} \right) + \\ I_3(nb) \left(\begin{array}{c} \Lambda + \Lambda\frac{\left(k_\parallel v_{t\parallel}\right)^2}{\omega|\omega_c|} + \\ \frac{\left(\frac{1}{2}(\Lambda)\right)\left(k_\parallel v_{t\parallel}\right)^2}{|\omega_c|^2} \end{array} \right) \end{array} \right) \end{array} \right]}$$

Further defying some terms i-e

$$M_1 = \left(\begin{array}{c} \frac{1}{|\omega_c|} - A_T\left(\frac{|\omega_c|}{\omega}\right) + (2\Lambda)(A_T+1)\left(\frac{|\omega_c|}{k_\parallel v_{t\parallel}}\right)^2\left(\frac{1}{\omega}\right) + \\ \Lambda|\omega_c|^3\left(\frac{1}{k_\parallel v_{t\parallel}}\right)^4 + \Lambda A_T\left(\frac{1}{\omega}\right)\left(\frac{|\omega_c|}{k_\parallel v_{t\parallel}}\right)^4 \end{array} \right)$$

$$M_2 = 2\Lambda\left(\frac{1}{|\omega_c|} + \left(\frac{1}{\omega}\right)\right)$$

$$M_3 = \Lambda\left(\frac{1}{|\omega_c|} - A_T\left(\frac{1}{\omega}\right)\right)$$

And

$$N_1 = \left(\begin{array}{c} 1+\frac{3}{4}\Lambda + \left(\begin{array}{c} 3\Lambda\left(1+\frac{9}{4}A_T\right) - \\ A_T \end{array} \right) \frac{\left(k_\parallel v_{t\parallel}\right)^2}{\omega|\omega_c|} + \\ \left(1+\frac{15}{4}\Lambda\right)\frac{\left(k_\parallel v_{t\parallel}\right)^2}{2|\omega_c|^2} \end{array} \right)$$

$$N_2 = 2\Lambda\left(1 + \frac{\left(k_\parallel v_{t\parallel}\right)^2}{\omega|\omega_c|} + \frac{\frac{1}{2}\left(k_\parallel v_{t\parallel}\right)^2}{|\omega_c|^2}\right)$$

$$N_3 = \left(\Lambda + \Lambda \frac{\left(k_{\|} v_{t\|}\right)^2}{\omega \left|\omega_c\right|} + \frac{\left(\frac{1}{2}\left(\Lambda\right)\right)\left(k_{\|} v_{t\|}\right)^2}{\left|\omega_c\right|^2} \right)$$

So we have,

$$\frac{\gamma}{|\omega_c|} = \frac{\omega_{p\alpha}^2 \sqrt{\pi} e^{-\left(\frac{\left(\frac{\omega}{|\omega_c|} - 1\right)}{\left(\frac{k_{\|} c}{\omega_{p\alpha}}\right)\sqrt{\beta_{\|}}} \right)^2} \left(I_1(nb) M_1 + I_2(nb) M_2 + I_3(nb) M_3\right)}{\left(1 + \frac{11}{4}\Lambda\right) k_{\|} v_{t\|} \left[\begin{array}{c} 2\frac{\left(k_\perp^2 + k_{\|}^2\right)c^2}{\omega^2} - \sum_\alpha \frac{\omega_{p\alpha}^2}{\left(1+\frac{11}{4}\Lambda\right)} \frac{1}{(\omega|\omega_c|)} \times \\ \left(I_1(nb) N_1 + I_2(nb) N_2 + I_3(nb) N_3\right) \end{array} \right]}$$

This is the required growth rate for oblique whistler modes, using cairns distribution. Where Λ is non-thermality parameter. In the limit, that Λ approaches 0, we get expression for growth rate of Bi-Maxwellian distribution.

Bibliography

[1] Krall, Nicholas A., Alvin W. Trivelpiece, and K. R. Symon. "Principles of plasma physics." IEEE Transactions on Plasma Science 2.3 (1974): 196-196.

[2] Chen, Francis F. Introduction to plasma physics and controlled fusion. Vol. 1. New York: Plenum press, 1984.

[3] FROM THE EDITED VOLUME , Kinetic Theory Edited by George Z. Kyzas and Athanasios C. Mitropoulos.

[4] Kelley, Michael C. , Potter, Simeon and Liley, Bruce Sween. "plasma"

[5] Amatucci, W. E. "A review of laboratory investigations of space plasma waves." URSI Radio Science Bulletin 2006.319 (2006): 32-66

[6] Kurth, W. S., and Donald A. Gurnett. "Plasma waves in planetary magnetospheres." Journal of Geophysical Research: Space Physics 96.S01 (1991): 18977-18991.

[7] Storey, L. R. O. "An investigation of whistling atmospherics." Philosophical Transactions of the Royal Society of London. Series A, Mathematical and Physical Sciences 246.908 (1953): 113-141.

[8] Russell, C. T., R. L. McPherron, and P. J. Coleman. "Fluctuating magnetic fields in the magnetosphere." Space Science Reviews 12.6 (1972): 810-856.

[9] Kimura, I., 1966. Effects of ions on whistler-mode ray tracing, Radio Sci., 1 (New Series), 269-283

[10] Stix, T. H., 1992. Waves in Plasmas, AIP Press, New York, p. 27-28.

[11] Choi, Seung, et al. "Whistler waves generated by nongyrotropic and gyrotropic electron beams during asymmetric guide field reconnection." Physics of Plasmas 29.1 (2022): 012903.

[12] United States. Air Force. Office of Scientific Research.

Air Force Office of Scientific Research, Office of Aerospace Research, USAF, 1967.

[13] Gallet, R. M., et al. "Microwave whistler mode propagation in a dense laboratory plasma." Physical Review Letters 4.7 (1960): 347.

[14] Fisher, R. K., and R. W. Gould. "Resonance cones in the field pattern of a short antenna in an anisotropic plasma." Physical Review Letters 22.21 (1969): 1093.

[15] Hooke, W. M., and S. Bernabei. "Direct observation of waves propagating near the lower-hybrid-resonance frequency." Physical Review Letters 28.7 (1972): 407.

[16] Sugai, H., et al. "Whistler wave ducting caused by antenna actions." The Physics of Fluids 21.4 (1978): 690-694.

[17] Gringauz, K. I., V. G. Kurt, V. I. Moroz, and I. S. Shklovsky, 1960. Results of observations of charged particles up to R=100,000 km with the aid of charged particletraps on Soviet cosmic rockets, Soviet Astronomy A. J., 4, 680-688

[18] Stenzel, R. L. "Whistler waves in space and laboratory plasmas." Journal of Geophysical Research: Space Physics 104, no. A7 (1999): 14379-14395.

[19] Remya, B., Lee, K. H., Lee, L. C., & Tsurutani, B. T. (2016). Polarization of obliquely propagating whistler mode waves based on linear dispersion theory. Physics of Plasmas, 23(12), 122120.

[20] Yoon, Peter H., Vinay S. Pandey, and Dong-Hun Lee. "Oblique nonlinear whistler wave." Journal of Geophysical Research: Space Physics 119, no. 3 (2014): 1851-1862.

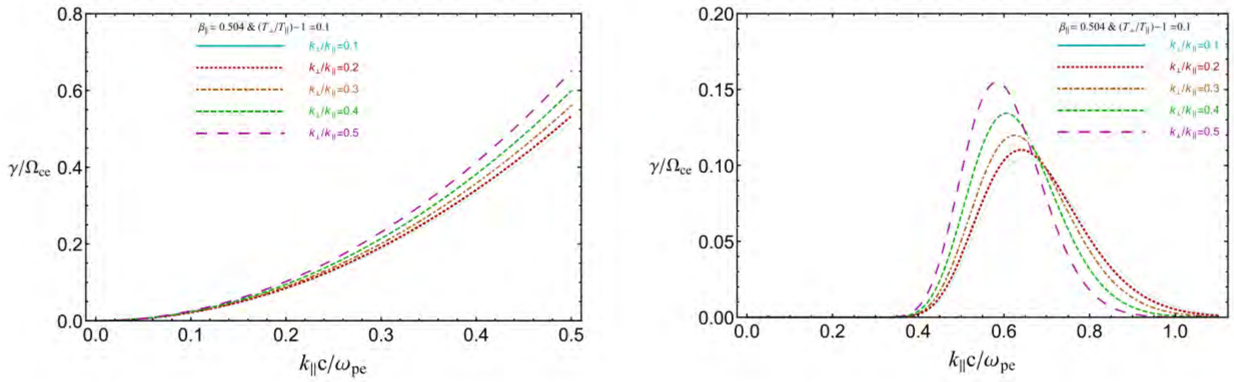

Figure 4.1: :(a) Oblique Whistler mode dispersion relation for $A_T = 0.1, \beta_{\parallel} = 0.504$, $\Lambda = 0.2$ and for changing values of $k_{\perp}/k_{\parallel}$. (b) Oblique Whistler mode growth rate for $A_T = 0.1, \beta_{\parallel} = 0.504$, $\Lambda = 0.2$ and for changing values of $k_{\perp}/k_{\parallel}$. Here blue solid curve represents $k_{\perp}/k_{\parallel} = 0.1$, red dotted curve represents$k_{\perp}/k_{\parallel} = 0.2$, brown dot-dashed curve represents $k_{\perp}/k_{\parallel} = 0.3$, green small dashed curve represents $k_{\perp}/k_{\parallel} = 0.4$ and purple large dashed curve represents $k_{\perp}/k_{\parallel} = 0.5$

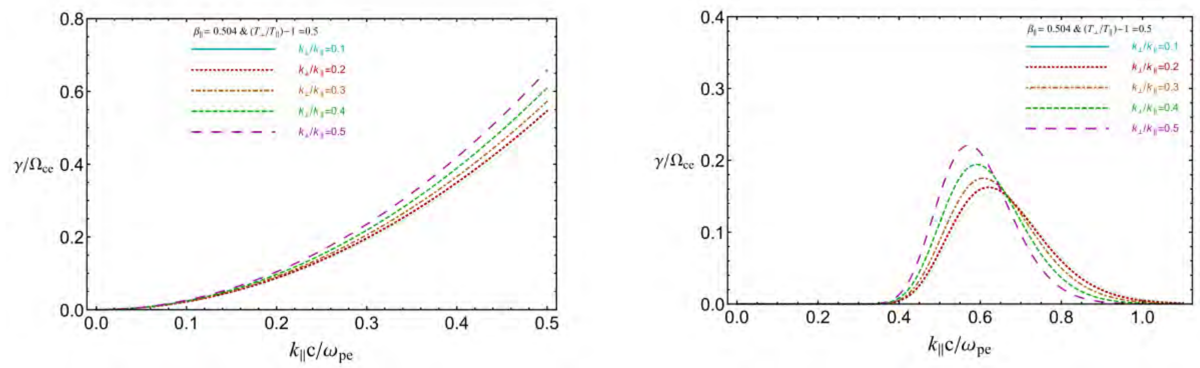

Figure 4.2: :(a) Oblique Whistler mode dispersion relation for $A_T = 0.5, \beta_{\parallel} = 0.504$, $\Lambda = 0.2$ and for changing values of $k_{\perp}/k_{\parallel}$. (b) Oblique Whistler mode growth rate for $A_T = 0.5, \beta_{\parallel} = 0.504$, $\Lambda = 0.2$ and for changing values of $k_{\perp}/k_{\parallel}$. Here blue solid curve represents $k_{\perp}/k_{\parallel} = 0.1$, red dotted curve represents$k_{\perp}/k_{\parallel} = 0.2$, brown dot-dashed curve represents $k_{\perp}/k_{\parallel} = 0.3$, green small dashed curve represents $k_{\perp}/k_{\parallel} = 0.4$ and purple large dashed curve represents $k_{\perp}/k_{\parallel} = 0.5$

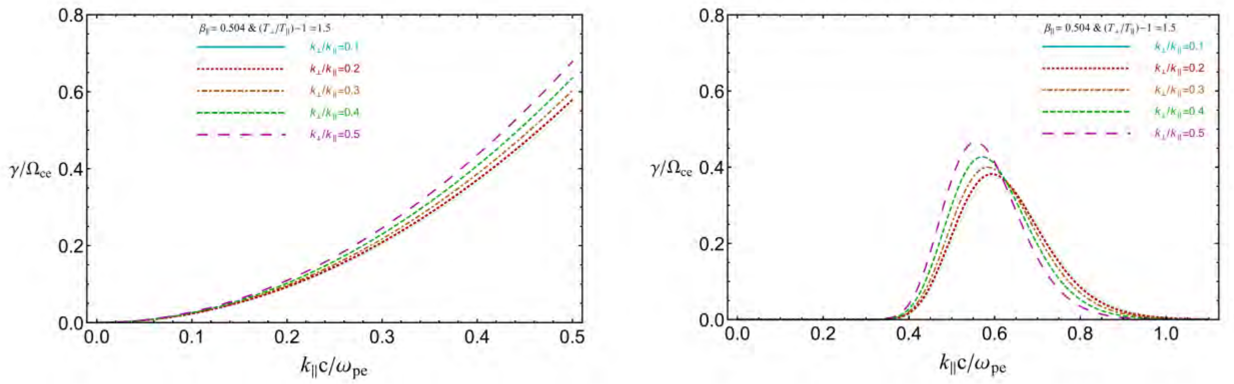

Figure 4.3: :(a) Oblique Whistler mode dispersion relation for $A_T = 1.5, \beta_{\|} = 0.504$, $\Lambda = 0.2$ and for changing values of $k_{\perp}/k_{\|}$. (b) Oblique Whistler mode growth rate for $A_T = 1.5, \beta_{\|} = 0.504$, $\Lambda = 0.2$ and for changing values of $k_{\perp}/k_{\|}$. Here blue solid curve represents $k_{\perp}/k_{\|} = 0.1$, red dotted curve represents$k_{\perp}/k_{\|} = 0.2$, brown dot-dashed curve represents $k_{\perp}/k_{\|} = 0.3$, green small dashed curve represents $k_{\perp}/k_{\|} = 0.4$ and purple large dashed curve represents $k_{\perp}/k_{\|} = 0.5$

4.3 Discussion and Conclusion

We treat oblique whistler waves with different anisotropic velocity distributions. First, we treat oblique whistler waves with anisotropic Bi-Maxwellian distribution and find out the dispersion relation and growth rate of oblique whistler waves. It was found that the growth rate of oblique whistler waves increases with increasing electron temperature anisotropy. Next, we treat the oblique whistler waves with Cairns distribution and find that the Non-Thermlity parameter Λ, electron temperature anisotropy, plasma density, and magnetic field are all observed to influence the actual frequency of Cairns scattered electrons. We investigated the plasma dispersion relation and growth rates for fixed values of $\beta_{\parallel}$ and Λ while varying the values of $K_{\perp}/K_{\parallel}$ and A_T (electron temperature anisotropy) in this case. The growth rate and dispersion relationship were found to be more sensitive to electron temperature anisotropy. The dispersion properties and growth rates of Cairns-distributed electrons were compared to those of Maxwellian-distributed electrons, and it was discovered that Cairns-distributed electrons had a higher growth rate than their Maxwellian counterparts. The actual frequency with cairns scattered electrons is also less than that obtained with Bi-Maxwellian distribution for $\beta_{\parallel} < 1$.

Bibliography

[1] Krall, Nicholas A., Alvin W. Trivelpiece, and K. R. Symon. "Principles of plasma physics." IEEE Transactions on Plasma Science 2.3 (1974): 196-196.

[2] Chen, Francis F. Introduction to plasma physics and controlled fusion. Vol. 1. New York: Plenum press, 1984.

[3] FROM THE EDITED VOLUME , Kinetic Theory Edited by George Z. Kyzas and Athanasios C. Mitropoulos.

[4] Kelley, Michael C. , Potter, Simeon and Liley, Bruce Sween. "plasma"

[5] Amatucci, W. E. "A review of laboratory investigations of space plasma waves." URSI Radio Science Bulletin 2006.319 (2006): 32-66

[6] Kurth, W. S., and Donald A. Gurnett. "Plasma waves in planetary magnetospheres." Journal of Geophysical Research: Space Physics 96.S01 (1991): 18977-18991.

[7] Storey, L. R. O. "An investigation of whistling atmospherics." Philosophical Transactions of the Royal Society of London. Series A, Mathematical and Physical Sciences 246.908 (1953): 113-141.

[8] Russell, C. T., R. L. McPherron, and P. J. Coleman. "Fluctuating magnetic fields in the magnetosphere." Space Science Reviews 12.6 (1972): 810-856.

[9] Kimura, I., 1966. Effects of ions on whistler-mode ray tracing, Radio Sci., 1 (New Series), 269-283

[10] Stix, T. H., 1992. Waves in Plasmas, AIP Press, New York, p. 27-28.

[11] Choi, Seung, et al. "Whistler waves generated by nongyrotropic and gyrotropic electron beams during asymmetric guide field reconnection." Physics of Plasmas 29.1 (2022): 012903.

[12] United States. Air Force. Office of Scientific Research.

Air Force Office of Scientific Research, Office of Aerospace Research, USAF, 1967.

[13] Gallet, R. M., et al. ”Microwave whistler mode propagation in a dense laboratory plasma.” Physical Review Letters 4.7 (1960): 347.

[14] Fisher, R. K., and R. W. Gould. ”Resonance cones in the field pattern of a short antenna in an anisotropic plasma.” Physical Review Letters 22.21 (1969): 1093.

[15] Hooke, W. M., and S. Bernabei. ”Direct observation of waves propagating near the lower-hybrid-resonance frequency.” Physical Review Letters 28.7 (1972): 407.

[16] Sugai, H., et al. ”Whistler wave ducting caused by antenna actions.” The Physics of Fluids 21.4 (1978): 690-694.

[17] Gringauz, K. I., V. G. Kurt, V. I. Moroz, and I. S. Shklovsky, 1960. Results of observations of charged particles up to R=100,000 km with the aid of charged particletraps on Soviet cosmic rockets, Soviet Astronomy A. J., 4, 680-688

[18] Stenzel, R. L. ”Whistler waves in space and laboratory plasmas.” Journal of Geophysical Research: Space Physics 104, no. A7 (1999): 14379-14395.

[19] Remya, B., Lee, K. H., Lee, L. C., & Tsurutani, B. T. (2016). Polarization of obliquely propagating whistler mode waves based on linear dispersion theory. Physics of Plasmas, 23(12), 122120.

[20] Yoon, Peter H., Vinay S. Pandey, and Dong-Hun Lee. ”Oblique nonlinear whistler wave.” Journal of Geophysical Research: Space Physics 119, no. 3 (2014): 1851-1862.