On the Boundedness of Some Hardy-type Operators

By

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A DISSERTATION SUBMITTED IN THE PARTIAL FULFILLMENT

OF THE REQUIREMENT FOR THE DEGREE OF

DOCTOR OF PHILOSOPHY

IN

MATHEMATICS

Supervised By

Dr. Amjad Hussain

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Quaid-i-Azam University Islamabad, Pakistan

2024

AUTHOR'S DECLARATION

I, **Samia Bashir** hereby declare that my PhD thesis entitled **"On the Boundedness of Some Hardy-type Operators"** is my own work and has not been submitted previously by me for taking any degree from Quaid-I-Azam University, Islamabad, Pakistan or anywhere else in the country/world.

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On the Boundedness of Some Hardy-type Operators

By

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CERTIFICATE

A DISSERTATION SUBMITTED IN THE PARTIAL FULFILLMENT OF THE REQUIREMENTS FOR THE DEGREE OF THE

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Abstract

In this work, we contribute to the theory of Hardy-type operators in a number of ways on both $\mathbb{R}^n$ and $\mathbb{Q}_p^n$. Firstly, we characterize the central BMO spaces with variable exponent via the boundedness of commutators of Hardy-type operators on variable exponent Lebesgue and central Morrey spaces. Some boundedness results for the Hardy operator and its adjoint operator are also demonstrated on variable exponent Lebesgue and central Morrey spaces. Furthermore, we obtaine the boundedness of variable-order fractional Hardy-type operators from grand Herz spaces to weighted spaces, subject to appropriate weight conditions. Secondly, in the framework of variable exponent, we introduce some new p-adic function spaces. The fractional p-adic Hardy-type operators on the p-adic Lebesgue and central Morrey spaces with variable exponents are shown to be bounded. We characterize some varaible p-adic function spaces by proving the boundedness of commutators formed by p-adic Hardy-type integral operators and p-adic variable exponent λ-central BMO functions on the aforementioned spaces. Furthermore, the continuity of theses operators on p-adic variable exponent Herz-type spaces is discussed as well.

Dedications

"Put Allah first and everything will work out, may be not the way you planned but, just how it's meant to be".

(Anonymous)

This dissertation is dedicated to the sustainer and the best planner of the Universe **"The Almighty Allah"** who always has a greater plan for me. I always pray for the directions to follow it, patience to wait on it and knowledge to know when it comes. Without the hardships I would not have valued ease.

Preface

The aim of this thesis is to study Hardy-type integral operators on variable exponent function spaces. Our main results include the characterization of some function spaces via commutators of Hardy-type operators and the boundedness of these operators on function spaces defined different underlying spaces.

In Chapter 1, we give some basic definitions along with some necessary lemmas to be used in the subsequent chapters of this thesis. In addition, we define some function spaces and give introduction to the Hardy-type operators with $\mathbb{R}^n$ and $\mathbb{Q}_p^n$ as underlying spaces.

In Chapter 2, we come up with the characterization of variable exponent central-BMO spaces via commutators of Hardy-type operator on Lebesgue and central Morrey spaces. The continuity of Hardy-type operators on aforementioned spaces is established as well. The contents of this Chapter has been published in [50].

In Chapter 3, we investigate the boundedness of variable order Hardy-type operators on the variable exponent grand Herz-Morrey space. In this Chapter, we mainly proved the Soboleve-type theorem for Hardy-type operators on the variable exponent grand Herz-Morrey space. The contents of this Chapter has been published in [51].

In Chapter 4, we obtain the characterization of p-adic variable exponent central-BMO space via commutators of p-adic Hardy-type operators on p-adic variable exponent Lebesgue space. The boundedness of these operators is also made possible on the other hand. The contents of this Chapter are ready to submit for publication in well reputed journal of mathematics.

In Chapter 5, we consider continuity properties of Hardy-type operators defined on p-adic field on p-adic variable exponent central Morrey spaces. We also define and characterize the p-adic variable exponent λ-central BMO space by showing the boundedness of Hardy-type operator on these spaces. The contents of this Chapter are ready to submit for publication in well reputed journal of mathematics.

Our study in Chapter 6 adds to and extends the results of Chapter 5 in two ways. Firstly, we prove the boundedness of p-adic fractional Hardy-type operators on the

p-adic variable Herz spaces. Secondly, the similar results are proved true on p-adic variable exponent Herz-Morrey spaces. The contents of this Chapter are ready to submit for publication in well reputed journal of mathematics.

Samia Bashir
Islamabad, Pakistan
February 27, 2024

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All the praises, thanks and acknowledgements are for The Creator **Almighty Allah**, The most Beneficent, The Most Merciful, Who gave me strength and enabled me to undertake and execute this research task. Countless salutations upon the Holy Prophet **Hazrat Muhammad (S.A.W)**, source of knowledge for enlightening with the essence of faith in Allah and guiding the mankind, the true path of life. In accordance of Almighty Allah's order his creature must also be acknowledged.

Though only my name appears on the cover of this dissertation, in reality, there are a number of great people who have contributed to the execution of this task. I am indebted to thank my supervisor **Dr. Amjad Hussain** for providing a research-oriented environment and the best facilities within confined resources. I have a great exhilaration for my supervisor for his valuable guidance, constructive criticism and encouragement throughout my research work. I offer my sincere thanks for his scientific and personal support over the past seven years. I express my thanks to **Prof. Dr. Tariq Shah**, Chairman, Department of Mathematics, Quaid-I-Azam University Islamabad, for providing research facilities. Besides my supervisor, I owe my deepest gratitude to all my teachers especially **Dr. Muhammad Shabbir, Dr. Muhammad Aslam, Dr. Zia Bashir,** and **Dr. Maria Samreen** for their precious guidance and moral support.

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In the end, I would like to thank my institute **Quaid-I-Azam University Islamabad (QAU)** for making my Ph.D possible. May QAU progress by leaps and bounds in every field of Science and Technology. Ameen.

Samia Bashir

Contents

Chapter 1
Introduction to some Function Spaces and Operators

1.1 Introduction

Averaging operators are considered well-known fascinating mathematical objects in harmonic analysis due to their frequent appearance in both analysis and applications. One of the most celebrated averaging operators is the Hardy-Littlewood maximal operator. It, for example, controls the boundedness of a variety of integral operators and has several implications in the theory of partial differential equations (PDEs). Likewise, in analysis, theory and applications, the Hardy-type operators are popular averaging operators extensively studied over the past hundred years and subjected to many generalizations in various settings. There are many others, but we will limit ourselves to these two, for these are the main focus of our objective.

On the other hand continuity of these averaging operators on various function spaces is also a well-developed area in analysis and needs further attention concerning new trends in generalizing function spaces. It needs results in developing new methods, solving existing problems, and applications to the theory of PDEs. There is a need to focus on the new trends in operator theory and the theory of function spaces, including variable exponent spaces defined on different underlying spaces such as the Euclidean space $\mathbb{R}^n$ and the p-adic space $\mathbb{Q}_p^n$. With recent advancements in PDE theory, their modeling, and the emergence of new issues, the objective of boundedness of averaging operators on function spaces becomes more effective and fascinating.

In recent years, there has been a growing interest in the study of function spaces equipped with variable exponents, leading to the development of a new framework known as variable exponent analysis. These spaces provide a powerful tool for analyzing functions with variable growth or decay rates and have found applications in various areas of mathematics, including partial differential equations, harmonic analysis, and image processing. One can better understand the heterogeneity and complexity inherent in many real-world phenomena by taking into consideration the theory of variable exponent function spaces. The introduction of variable exponents extends the traditional framework of function spaces that opens up new avenues for mathematical analysis, numerical methods, and applications in various scientific and engineering fields [1, 2, 3]. These spaces offer a versatile framework for analyzing and modeling phenomena with non-standard regularity, anisotropy, or localized features. The ability to adapt the exponent to the local properties of the functions provides a powerful tool for capturing the behavior of complex systems.

The purpose of this thesis is twofold. Firstly, we investigate the boundedness of Hardy-type operators along with their commutators on function spaces with variable exponents defined over $\mathbb{R}^n$. Also, in some cases, we provide necessary and sufficient conditions for such boundedness results. Secondly, based on [4, 5], we define some new function spaces with variable exponents on p-adic field with $\mathbb{Q}_p^n$ as the underlying space. We then discuss the continuity of p-adic Hardy-type operators along with their commutators on these spaces.

1.2 Introduction to Variable Exponent Function Spaces

The concept of variable exponent function spaces was initially introduced by Orlicz [6] in the 1930s, who developed the theory of Orlicz spaces based on the growth function of the exponent. However, it was not until the 1990s that the general theory of variable exponent function spaces began to be systematically studied. Function spaces with variable exponent started significant progress when some of their essential features were provided by Kováčik and Rákosník [7]. The seminal works of Diening, Harjulehto, Hästö, and Růžička [1, 8, 9, 10] have laid the foundation for this research area, providing fundamental results, characterizations, and functional analytic tools for studying variable exponent function spaces. For a detailed history and recent developments in the theory of variable exponent function spaces, we refer the interested readers to the books [8, 9]. These spaces have a wide range of applications, including electrorheological fluid modeling [1], image processing [2], and differential equations with nonstandard growth [3]. Variable exponent λ-central BMO spaces, Morrey type spaces, and associated function spaces, on the other hand, have fascinating applications in analyzing the boundedness of integral operators; see, for instance, [11, 12, 13, 14, 15, 16].

All over remaining of this thesis, the constant $C > 0$ may vary from step to another and do not dependent on main parameters involved. The sign "$\approx$" between two function f and g implies that there exist constants c^1 and c^2 such that $c^1 f \leq g \leq c^2 f$.

1.2.1 Variable Exponent Function Spaces on $\mathbb{R}^n$

This section serves to define the variable exponent function spaces with $\mathbb{R}^n$ as underlying space. For this section we refer the reader to some standard references [7, 8, 17, 18, 19] from the literature.

Definition 1.2.1 *Consider a measurable function $q(\cdot)$: $D \to [1, \infty)$ with $D \subset \mathbb{R}^n$.*

(i) Denote by $L^{q(\cdot)}(D)$ the variable exponent Lebesgue space:

$$L^{q(\cdot)}(D) = \left\{ g \text{ measurable} : \int_D \left(\frac{|g(z)|}{\xi} \right)^{q(z)} dz < \infty \text{ where } \xi \text{ is a constant} \right\}.$$

in which we define the norm:

$$\|g\|_{L^{q(\cdot)}(D)} = \inf \left\{ \xi > 0 : \int_D \left(\frac{|g(z)|}{\xi} \right)^{q(z)} dz \leq 1 \right\}.$$

(ii) The local version of $L^{q(\cdot)}(D)$ is given by

$$L^{q(\cdot)}_{\text{loc}}(D) := \left\{ g : g \in L^{q(\cdot)}(F) \text{ for all compact subsets } F \subset D \right\}.$$

This thesis use the following notation for remaining discussion:

(a) Let $g \in L^1_{\text{loc}}(D)$, then the Hardy-Littlewood maximal function $\mathcal{M}$ is defined as

$$\mathcal{M}g(z) := \sup_{t>0} t^{-n} \int_{B(z,t)} |g(x)| dx \quad (z \in D),$$

where

$$B(z,t) := \{z \in D : |z - s| < t\}.$$

(b) We denote by $\mathfrak{P}(D)$ the set of all functions $r(\cdot)$ which are measurable and satisfy:

$$r_- := \operatorname{ess\,inf}_{\zeta \in D} r(\zeta) > 1, \quad r_+ := \operatorname{ess\,sup}_{\zeta \in D} r(\zeta) < \infty. \tag{1.2.1}$$

(c) We denote by $\mathfrak{P}^{\log} = \mathfrak{P}^{\log}(D)$ the set of all functions $r \in \mathfrak{P}(D)$ which are measurable and satisfy (1.2.1) along with log condition given as below:

$$|r(\xi) - r(\zeta)| \leq \frac{C(r)}{-\ln|\xi - \zeta|}, \quad |\xi - \zeta| \leq \frac{1}{2}, \quad \xi, \zeta \in D. \tag{1.2.2}$$

(d) If D is unbounded, then $\mathfrak{P}_{0,\infty}(D)$ and $\mathfrak{P}_{\infty}(D)$ become the subsets of $\mathfrak{P}(D)$. Functions belonging to $\mathfrak{P}_{0,\infty}(D)$ and $\mathfrak{P}_{\infty}(D)$ satisfy:

$$|r(d) - r_{\infty}| \leq \frac{C}{\ln(e + |d|)}, \tag{1.2.3}$$

where $r_{\infty} \in (1, \infty)$.

$$|r(d) - r_0| \leq \frac{C}{\ln|d|}, |d| \leq \frac{1}{2}, \tag{1.2.4}$$

respectively.

(e) Finally, $D_m = D(0, 2^m) = \{y \in \mathbb{R}^n : |y| < 2^m\}$, $A_m = D_m \setminus D_{m-1}$, for all $m \in \mathbb{Z}$, and $\chi_m = \chi_{A_m}$.

Definition 1.2.2 *[12, 13] A function $g \in L^{u(\cdot)}_{loc}(\mathbb{R}^n)$, for $u(\cdot) \in \mathfrak{P}(\mathbb{R}^n)$, is said to be in variable CBMO (central bounded mean oscillation) space if*

$$\|g\|_{CBMO^{u(\cdot)}(\mathbb{R}^n)} =: \sup_{r>0} \frac{\|(g - g_{B(0,r)})\chi_{B(0,r)}\|_{L^{u(\cdot)}(\mathbb{R}^n)}}{\|\chi_{B(0,r)}\|_{L^{u(\cdot)}(\mathbb{R}^n)}} < \infty.$$

If $u(x) = u$ is a constant, then $CBMO^{u(\cdot)}(\mathbb{R}^n)$ equals $CBMO^{u}(\mathbb{R}^n)$. We write $C^{u(\cdot)} =: CBMO^{u(\cdot)}(\mathbb{R}^n)$ simply here and in the following.

Definition 1.2.3 *[13] Let $\mu \in \mathbb{R}$, and $u(\cdot) \in \mathfrak{P}(\mathbb{R}^n)$. The variable exponent central Morrey space $\dot{B}^{u(\cdot),\mu}(\mathbb{R}^n)$ is given by*

$$\dot{B}^{u(\cdot),\mu}(\mathbb{R}^n) = \{g \in L^{u(\cdot)}_{loc}(\mathbb{R}^n) : \|g\|_{\dot{B}^{u(\cdot),\mu}(\mathbb{R}^n)} < \infty\},$$

where

$$\|g\|_{\dot{B}^{u(\cdot),\mu}(\mathbb{R}^n)} = \sup_{r>0} \frac{\|g\chi_{B(0,r)}\|_{L^{u(\cdot)}(\mathbb{R}^n)}}{|B(0,r)|^{\mu}\|\chi_{B(0,r)}\|_{L^{u(\cdot)}(\mathbb{R}^n)}}.$$

Definition 1.2.4 *[13] Let $\mu < 1/n$ and $u(\cdot) \in \mathfrak{P}(\mathbb{R}^n)$, then the variable exponent μ-central BMO space $CBMO^{u(\cdot),\mu}(\mathbb{R}^n)$ is given by*

$$CBMO^{u(\cdot),\mu}(\mathbb{R}^n) = \{g \in L^{u(\cdot)}_{loc}(\mathbb{R}^n) : \|g\|_{CBMO^{u(\cdot),\mu}(\mathbb{R}^n)} < \infty\},$$

where

$$\|g\|_{CBMO^{u(\cdot),\mu}(\mathbb{R}^n)} = \sup_{r>0} \frac{\|\chi_{B(0,r)}(g - g_{B(0,r)})\|_{L^{u(\cdot)}(\mathbb{R}^n)}}{\|\chi_{B(0,r)}\|_{L^{u(\cdot)}(\mathbb{R}^n)}|B(0,r)|^{\mu}}.$$

Remark 1.2.5 *An equivalent form of the definition given above can be written as:*

$$\|g\|_{CBMO^{u(\cdot),\mu}(\mathbb{R}^n)} = \sup_{\gamma\in\mathbb{Z}} \inf_{c\in\mathbb{C}} \frac{\|(g-c)\chi_{B_\gamma}\|_{L^{u(\cdot)}(\mathbb{R}^n)}}{|B_\gamma|^\mu \|\chi_{B_\gamma}\|_{L^{u(\cdot)}(\mathbb{R}^n)}}.$$

Note $B^{u(\cdot),\mu}(\mathbb{R}^n)$ and $CMO^{u(\cdot),\mu}(\mathbb{R}^n)$ inhomogeneous version of the variable exponent central Morrey space and the μ-central BMO space can be obtained respectively by taking the supremum on $R \geq 1$ in Definitions 1.2.3 and 1.2.4 in place of $R > 0$ here.

The results of this thesis apply to an inhomogeneous version of μ-central BMO space and a central Morrey space with variable exponents.

Definition 1.2.6 *Suppose $1 \leq u, v < \infty$, $\zeta \in \mathbb{R}$, then the homogeneous and inhomogeneous Herz spaces (classical version) are defined as:*

$$\|f\|_{K_v^{\zeta,u}(\mathbb{R}^n)} = \|f\|_{L^v(D(0,1))} + \left\{\sum_{\ell\in\mathbb{N}} 2^{\ell\zeta u}\|f\chi_\ell\|_{L^v(\mathbb{R}^n)}^u\right\}^{\frac{1}{u}}, \tag{1.2.5}$$

$$\|f\|_{\dot{K}_v^{\zeta,u}(\mathbb{R}^n)} = \left\{\sum_{\ell\in\mathbb{Z}} 2^{\ell\zeta u}\|f\chi_\ell\|_{L^v(\mathbb{R}^n)}^u\right\}^{\frac{1}{u}}, \tag{1.2.6}$$

respectively.

Definition 1.2.7 *Let $u \in [1,\infty)$, $v(\cdot) \in \mathfrak{P}(\mathbb{R}^n)$ and $\zeta \in \mathbb{R}$. $\dot{K}_{v(\cdot)}^{\zeta,u}(\mathbb{R}^n)$ is the homogenous version of Herz space and its norm is given as*

$$\dot{K}_{v(\cdot)}^{\zeta,u}(\mathbb{R}^n) = \left\{g \in L_{\text{loc}}^{v(\cdot)}(\mathbb{R}^n\setminus\{0\}) : \|g\|_{\dot{K}_{v(\cdot)}^{\zeta,u}(\mathbb{R}^n)} < \infty\right\}, \tag{1.2.7}$$

where

$$\|g\|_{\dot{K}_{v(\cdot)}^{\zeta,u}(\mathbb{R}^n)} = \left(\sum_{\ell=-\infty}^{\ell=\infty} \|2^{\ell\zeta} g\chi_\ell\|_{L^{v(\cdot)}}^u\right)^{\frac{1}{u}}.$$

Definition 1.2.8 *For $1 \leq u < \infty$, $v(\cdot) \in \mathfrak{P}(\mathbb{R}^n)$ and $\zeta \in \mathbb{R}$. The inhomogenous Herz space $K_{v(\cdot)}^{\zeta,u}(\mathbb{R}^n)$ is given by*

$$K_{v(\cdot)}^{\zeta,u}(\mathbb{R}^n) = \left\{h \in L_{\text{loc}}^{v(\cdot)}(\mathbb{R}^n\setminus\{0\}) : \|h\|_{K_{v(\cdot)}^{\zeta,u}(\mathbb{R}^n)} < \infty\right\}, \tag{1.2.8}$$

where

$$\|h\|_{K_{v(\cdot)}^{\zeta,u}(\mathbb{R}^n)} = \|h\|_{L^{v(\cdot)}(D(0,1))} + \left(\sum_{\ell=-\infty}^{\ell=\infty} \|2^{\ell\zeta} h\chi_\ell\|_{L^{v(\cdot)}}^u\right)^{\frac{1}{u}}.$$

Definition 1.2.9 *For $a(\cdot)$: $\mathbb{R}^n \to \mathbb{R}$, $0 < u < \infty$, $v(\cdot) \in \mathfrak{P}(\mathbb{R}^n)$ and $0 \leq \beta < \infty$. A variable Herz-Morrey spaces $M\dot{K}^{a(\cdot),\beta}_{u,v(\cdot)}(\mathbb{R}^n)$ is defined by*

$$M\dot{K}^{a(\cdot),\beta}_{u,v(\cdot)}(\mathbb{R}^n) = \left\{ g \in L^{v(\cdot)}_{\text{loc}}(\mathbb{R}^n \setminus \{0\}) : \|g\|_{M\dot{K}^{a(\cdot),\beta}_{u,v(\cdot)}(\mathbb{R}^n)} < \infty \right\},$$

where

$$\|g\|_{M\dot{K}^{a(\cdot),\beta}_{u,v(\cdot)}(\mathbb{R}^n)} = \sup_{k_0 \in \mathbb{Z}} 2^{-k_0\beta} \left(\sum_{\ell=-\infty}^{k_0} 2^{\ell a(\cdot) u} \|g\chi_\ell\|^u_{L^{v(\cdot)}(\mathbb{R}^n)} \right)^{\frac{1}{u}}.$$

Next, we define variable exponent grand Herz spaces.

Definition 1.2.10 *Let $a(\cdot) \in L^\infty(\mathbb{R}^n)$, $u \in [1, \infty)$, v: $\mathbb{R}^n \to [1, \infty)$, $\theta > 0$. A grand Herz spaces with variable exponent $\dot{K}^{a(\cdot),u),\theta}_{v(\cdot)}$ is defined by*

$$\dot{K}^{a(\cdot),u),\theta}_{v(\cdot)} = \left\{ g \in L^{v(\cdot)}_{\text{loc}}(\mathbb{R}^n \setminus \{0\}) : \|g\|_{\dot{K}^{a(\cdot),u),\theta}_{v(\cdot)}} < \infty \right\},$$

where

$$\begin{aligned} \|g\|_{\dot{K}^{a(\cdot),u),\theta}_{v(\cdot)}} &= \sup_{\phi > 0} \left(\phi^\theta \sum_{\ell \in \mathbb{Z}} 2^{\ell a(\cdot) u(1+\phi)} \|g\chi_\ell\|^{u(1+\phi)}_{L^{v(\cdot)}} \right)^{\frac{1}{u(1+\phi)}} \\ &= \sup_{\phi > 0} \phi^{\frac{\theta}{u(1+\phi)}} \|g\|_{\dot{K}^{a(\cdot),u(1+\phi)}_{v(\cdot)}}. \end{aligned}$$

1.2.2 Variable Exponent Function Spaces on $\mathbb{Q}_p^n$

The p-adic analysis finds its applications in different areas of mathematical sciences including mathematical physics, quantum mechanics, probability theory, and dynamical systems [21, 22]. Function spaces with variable exponents defined on p-adic fields with $\mathbb{Q}_p^n$ as underlying space are well known to pique interest not only in real and harmonic analysis, but also in applied mathematics. Variable exponent function spaces defined on $\mathbb{Q}_p^n$ attract less attention in the past and need careful consideration in the future. Unlike classical p-adic function spaces with fixed exponents, such as Lebesgue or Sobolev spaces, p-adic variable exponent function spaces allow the exponent to vary across the domain, providing a more flexible framework for studying functions with diverse characteristics. In order to introduce these spaces, a brief introduction of p-adic number is pre-requisite which is as below.

Let the field of rational numbers is denoted by $\mathbb{Q}$. The absolute value $|x|$ of $x \in \mathbb{Q}$ satisfies the below properties:
(i) $|x| \geq 0$, $|x| = 0$ iff $x = 0$,

(ii) $|xy| = |x||y|$,
(iii) $|x + y| \leq |x| + |y|$.
Therefore, the function $| \cdot | : \mathbb{Q} \to \mathbb{R}$ is termed as norm. Alternatively, for a prime a number p, the field of p-adic numbers $\mathbb{Q}_p$ is defined as the completion of the rational number field $\mathbb{Q}$ with respect to the p-adic norm $| \cdot |_p$, which is defined as

$$|x|_p =: \begin{cases} 0, & \text{if } x = 0, \\ p^{-\gamma}, & \text{if } x = p^{\gamma}\frac{a}{b}, \end{cases} \tag{1.2.9}$$

where a and b are integers that are co-prime to p. The p-adic order of x is denoted by the integer $\gamma =: ord(x)$ $(ord(0) =: +\infty)$. We extend this norm to $\mathbb{Q}_p^n$ as follows:

$$\|x\|_p =: \max_{1\leq i\leq n} |x_i|_p \quad for\ \ x = (x_1, ..., x_n) \in \mathbb{Q}_p^n, \tag{1.2.10}$$

and satisfies the 'strong triangular inequality'

$$\|x + y\|_p \leq \max\{\, \|x\|_p, \|y\|_p\}, \tag{1.2.11}$$

when $\|x\|_p \neq \|y\|_p$, there is equality. If $ord(x) =: \min_{1\leq i\leq n}\{ord(x_i)\}$, then $\|x\|_p =: p^{-ord(x)}$. The set $(\mathbb{Q}_p^n, \|.\|_p)$ is a complete ultrametric space, and $\mathbb{Q}_p$ is homeomorphic to a subset of the real line that is Cantor-like as a topological space. $x \neq 0$ is a p-adic number having a one of a kind series expansion, precisely,

$$x =: p^{ord(x)} \sum_{l=0}^{\infty} x_l p^l, \tag{1.2.12}$$

where $x_l \in \{0, 1, 2, ..., p-1\}$ and $x_0 \neq 0$ are used. We use $\gamma \in \mathbb{Z}$ to represent

$$B_\gamma(a) = \{\, x \in \mathbb{Q}_p^n : \|x - a\|_p \leq p^\gamma\}, \tag{1.2.13}$$

the ball with a radius p^γ and a center at $a = (a_1, ..., a_n) \in \mathbb{Q}_p^n$ and by

$$S_\gamma(a) = \{\, x \in \mathbb{Q}_p^n : \|x - a\|_p = p^\gamma\} := B_\gamma(a) \setminus B_{\gamma-1}(a), \tag{1.2.14}$$

the sphere that corresponds to the ball $B_\gamma(a)$. We use

$$B_\gamma(0) =: B_\gamma, \quad S_\gamma(0) =: S_\gamma, \tag{1.2.15}$$

and take note of the fact that

$$\mathbb{Q}_p^n \setminus \{0\} =: \bigcup_{\gamma\in\mathbb{Z}} S_\gamma. \tag{1.2.16}$$

The one-dimensional ball is represented by $B_\gamma(a_i) =: \{ x \in \mathbb{Q}_p : |x - a_i|_p \leq p^\gamma\}$, where $B_\gamma(a) = B_\gamma(a_1) \times ... \times B_\gamma(a_n)$. With regard to addition, $\mathbb{Q}_p^n$ is a locally compact commutative group with the additive Haar measure $d^n x =: dx$ (by $|F|$, we designate the Haar measure of the set F). We get a unique measure by normalizing the measure dx by $\int_{B_0} dx = 1$. From this point forward, we will use the normalized Haar measure; as a result,

$$p^{\gamma n} = |B_\gamma(a)|, \quad p^{\gamma n}(1 - p^{-n}) = |S_\gamma(a)|, \tag{1.2.17}$$

for any $a \in \mathbb{Q}_p^n$.

A partition of $\mathbb{Q}_p^n$ is made up of all disjoint balls with the same radius, γ, because inequality (1.2.11) states that, in $\mathbb{Q}_p^n$, if we have any two balls with same radius then either one contains the other or they are disjoint.

For any $x \in \mathbb{Q}_p^n$, the function $\hbar : \mathbb{Q}_p^n \to \mathbb{C}$ is referred to as a local constant if an integer $m(z) \in \mathbb{Z}$ exists in such a way that

$$\hbar(z + z') = \hbar(z) \quad for \;\; z' \in B_{m(z)}. \tag{1.2.18}$$

If $\hbar : \mathbb{Q}_p^n \longrightarrow \mathbb{C}$ is locally constant with the compact support, then it is called a test function (or a Schwartz-Bruhat function). Here, $S(\mathbb{Q}_p^n) =: S$ denotes the $\mathbb{C}$-vector space of such test functions.

Let $f : \mathbb{Q}_p^n \longrightarrow \mathbb{C}$ be measurable, then it is a member of the Lebesgue space $L^u(\mathbb{Q}_p^n)$, $1 \leq u < \infty$, when

$$\|f\|^u_{L^u(\mathbb{Q}_p^n)} =: \int_{\mathbb{Q}_p^n} |f(x)|^u dx < \infty, \tag{1.2.19}$$

where

$$\int_{\mathbb{Q}_p^n} |f(x)|^u dx =: \lim_{\gamma \to \infty} \int_{B_\gamma(0)} |f(x)|^u dx, \tag{1.2.20}$$

if the limit exists.

This section introduces the idea of p-adic Lebesgue spaces with variable exponents and lists some of their necessary features. The proofs are contained in citation [4].

If $u : \mathbb{Q}_p^n \longrightarrow [1, \infty)$ is a measurable function. The set of all measurable functions $u(\cdot)$ satisfying $u^- > 1$ and $u^+ < \infty$ is denoted by $\aleph(\mathbb{Q}_p^n)$, where $u^+ =: \operatorname{esssup}_{x \in \mathbb{Q}_p^n} u(x)$ and $u^- =: \operatorname{essinf}_{x \in \mathbb{Q}_p^n} u(x)$.

Let $g : \mathbb{Q}_p^n \longrightarrow \mathbb{R}$ is measurable then $L^{u(\cdot)}(\mathbb{Q}_p^n)$ denotes the space of all $u \in \aleph(\mathbb{Q}_p^n)$ given by

$$\|g\|_{L^{u(\cdot)}(\mathbb{Q}_p^n)} =: \inf \left\{ \sigma > 0 : \wp_{u(\cdot)} \left(\frac{g}{\sigma} \right) \leq 1 \right\} < \infty, \tag{1.2.21}$$

where $\wp_{u(\cdot)}(g) =: \int_{\mathbb{Q}_p^n} |g(y)|^{u(y)} dy$.

For the Lebesgue space with a variable exponent, we now have

$$\|g\|_{L^{u(\cdot)}(\mathbb{Q}_p^n)} \leq \wp_{u(\cdot)}(g) + 1, \tag{1.2.22}$$

$$\wp_{u(\cdot)}(g) \leq \left(1 + \|g\|_{L^{u(\cdot)}(\mathbb{Q}_p^n)}\right)^{u^+}, \tag{1.2.23}$$

$$\|g\|_{L^{u(\cdot)}(\mathbb{Q}_p^n)} = \||g|^s\|^{\frac{1}{s}}_{L^{u(\cdot)/s}(\mathbb{Q}_p^n)}, \quad s \in (0, u^-]. \tag{1.2.24}$$

The Hölder's inequality holds true for variable exponent Lebesgue spaces i.e.

$$\int_{\mathbb{Q}_p^n} |g(\xi)h(\xi)|d\xi \leq C\|g\|_{L^{u(\cdot)}(\mathbb{Q}_p^n)}\|h\|_{L^{u'(\cdot)}(\mathbb{Q}_p^n)}, \tag{1.2.25}$$

here, u and u' are conjugate exponents.

For $u \in \aleph(\mathbb{Q}_p^n)$, we say that $u \in W_0(\mathbb{Q}_p^n)$, If

$$\gamma\left(u^-(B_\gamma(\xi)) - u^+(B_\gamma(\xi))\right) \leq C, \tag{1.2.26}$$

for any $C > 0$, $\xi \in \mathbb{Q}_p^n$ and all $\gamma \in \mathbb{Z}$. Similarly, for $C > 0$ and any $\xi, \eta \in \mathbb{Q}_p^n$, we say that $u \in W^\infty(\mathbb{Q}_p^n)$ if

$$|u(\xi) - u(\eta)| \leq C\frac{1}{\log_p(p + \min\{\|\eta\|_p, \|\xi\|_p\})}. \tag{1.2.27}$$

Class $W_0^\infty(\mathbb{Q}_p^n)$ is described as $W_0^\infty(\mathbb{Q}_p^n) =: W^\infty(\mathbb{Q}_p^n) \cap W_0(\mathbb{Q}_p^n)$.

Definition 1.2.11 *A function $f \in L^{u(\cdot)}_{loc}(\mathbb{Q}_p^n)$ for $u(\cdot) \in \aleph(\mathbb{Q}_p^n)$ is in p-adic $CMO^{u(\cdot)}(\mathbb{Q}_p^n)$ with variable exponent if*

$$\|f\|_{CMO^{u(\cdot)}(\mathbb{Q}_p^n)} =: \sup_{\gamma\in\mathbb{Z}} \|\chi_{B_\gamma}\|^{-1}_{L^{u(\cdot)}(\mathbb{Q}_p^n)}\|(f - f_{B_\gamma})\|_{L^{u(\cdot)}(\mathbb{Q}_p^n)} < \infty,$$

where

$$f_{B_\gamma} = \frac{1}{|B_\gamma|}\int_{B_\gamma} f(x)dx.$$

If $u(x) = u$ is a constant, then $CMO^{u(\cdot)}(\mathbb{Q}_p^n)$ equals $CMO^u(\mathbb{Q}_p^n)$. We write $C^{u(\cdot)} =: CMO^{u(\cdot)}(\mathbb{Q}_p^n)$ simply here and in the following.

Definition 1.2.12 *Let $u(\cdot) \in \aleph(\mathbb{Q}_p^n)$ and $1/n > \mu$. The p-adic μ-central BMO space with variable exponent $CBMO^{u(\cdot),\mu}(\mathbb{Q}_p^n)$ can be defined as*

$$CBMO^{u(\cdot),\mu}(\mathbb{Q}_p^n) = \{g \in L^{u(\cdot)}_{loc}(\mathbb{Q}_p^n) : \|g\|_{CBMO^{u(\cdot),\mu}(\mathbb{Q}_p^n)} < \infty\},$$

where

$$\|g\|_{CBMO^{u(\cdot),\mu}(\mathbb{Q}_p^n)} = \sup_{\gamma\in\mathbb{Z}} \frac{\|(g - g_{B_\gamma})\chi_{B_\gamma}\|_{L^{u(\cdot)}(\mathbb{Q}_p^n)}}{|B_\gamma|^\mu\|\chi_{B_\gamma}\|_{L^{u(\cdot)}(\mathbb{Q}_p^n)}}.$$

Here and in what follows, we write $C^{u(\cdot),\mu} =: CBMO^{u(\cdot),\mu}(\mathbb{Q}_p^n)$ for simplicity.

Remark 1.2.13 *An equivalent form of the definition given above can be written as:*

$$\|g\|_{CBMO^{u(\cdot),\mu}(\mathbb{Q}_p^n)} = \sup_{\gamma\in\mathbb{Z}} \inf_{c\in\mathbb{C}} \frac{\|(g-c)\chi_{B_\gamma}\|_{L^{u(\cdot)}(\mathbb{Q}_p^n)}}{|B_\gamma|^\mu \|\chi_{B_\gamma}\|_{L^{u(\cdot)}(\mathbb{Q}_p^n)}}.$$

Definition 1.2.14 *Let $u(\cdot) \in \aleph(\mathbb{Q}_p^n)$ and $\mu \in \mathbb{R}$. The p-adic variable exponent central Morrey space $\dot{B}^{u(\cdot),\mu}(\mathbb{Q}_p^n)$ can be defined as*

$$\dot{B}^{u(\cdot),\mu}(\mathbb{Q}_p^n) = \{g \in L_{loc}^{u(\cdot)}(\mathbb{Q}_p^n) : \|g\|_{\dot{B}^{u(\cdot),\mu}(\mathbb{Q}_p^n)} < \infty\},$$

where

$$\|g\|_{\dot{B}^{u(\cdot),\mu}(\mathbb{Q}_p^n)} = \sup_{\gamma\in\mathbb{Z}} \frac{\|g\chi_{B_\gamma}\|_{L^{u(\cdot)}(\mathbb{Q}_p^n)}}{|B_\gamma|^\mu \|\chi_{B_\gamma}\|_{L^{u(\cdot)}(\mathbb{Q}_p^n)}}.$$

Definition 1.2.15 *Let $\beta \in \mathbb{R}$, $0 < m < \infty$, and $u(\cdot) \in \aleph(\mathbb{Q}_p^n)$. $\dot{K}_{u(\cdot)}^{\beta,m}(\mathbb{Q}_p^n)$ is the homogeneous version of p-adic Herz space and its norm is given by*

$$\dot{K}_{u(\cdot)}^{\beta,m}(\mathbb{Q}_p^n) = \left\{g \in L_{\text{loc}}^{u(\cdot)}(\mathbb{Q}_p^n) : \|g\|_{\dot{K}_{u(\cdot)}^{\beta,m}(\mathbb{Q}_p^n)} < \infty\right\},$$

where

$$\|g\|_{\dot{K}_{u(\cdot)}^{\beta,m}(\mathbb{Q}_p^n)} = \left(\sum_{\ell=-\infty}^{\infty} \|p^{\ell\beta} g\chi_\ell\|_{L^{u(\cdot)}(\mathbb{Q}_p^n)}^m\right)^{\frac{1}{m}}.$$

Definition 1.2.16 *Suppose $\beta \in \mathbb{R}$, $0 < m < \infty$, $\lambda \in [0,\infty)$ and $u(\cdot) \in \aleph(\mathbb{Q}_p^n)$. $M\dot{K}_{m,u(\cdot)}^{\beta,\lambda}(\mathbb{Q}_p^n)$ is the homogeneous version of p-adic Herz-Morrey space and its norm is given by*

$$M\dot{K}_{m,u(\cdot)}^{\beta,\lambda}(\mathbb{Q}_p^n) = \left\{g \in L_{\text{loc}}^{u(\cdot)}(\mathbb{Q}_p^n) : \|g\|_{M\dot{K}_{m,u(\cdot)}^{\beta,\lambda}(\mathbb{Q}_p^n)} < \infty\right\},$$

where

$$\|g\|_{M\dot{K}_{m,u(\cdot)}^{\beta,\lambda}(\mathbb{Q}_p^n)} = \sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda} \left(\sum_{\ell=-\infty}^{k_0} \|p^{\ell\beta} g\chi_\ell\|_{L^{u(\cdot)}(\mathbb{Q}_p^n)}^m\right)^{\frac{1}{m}}.$$

Having finishing the streak of definitions we are now going to introduce some integral operators of our interest.

1.3 Introduction to Some Integral Operators

The Hardy operator, the fractional Hardy operator, the p-adic Hardy operator, and the fractional p-adic Hardy operator are all integral operators of our interest in this thesis. The following subsections provide a quick overview of the operators that were defined on different underlying spaces.

1.3.1 Hardy-type Operators on $\mathbb{R}^n$

The Hardy operator was firstly introduced by Hardy [23] in the early 20th century. The Hardy operators and their associated Hardy inequalities are widely studied in a variety of function spaces. Various stages of development of the classical Hardy inequality like for example its extensions and early weighted generalization were studied by Kufner [24]. They cover both the discrete and continuous forms of the Hardy inequality. These operators find their applications in the theory of partial differential equations [25, 26, 27] as well as in characterizing the function spaces [28, 29, 30].

The extension of Hardy operators was first introduced by Samko [31] in 1993 in his work of fractional integration and differentiation. He defined the concept of fractional integral operators with variable order and established their basic properties.

Hardy introduced the operator

$$hg(z) = \frac{1}{z}\int_0^z g(\eta)d\eta, \quad z > 0, \tag{1.3.1}$$

in [23] which satisfy:

$$\|hg\|_{L^u(\mathbb{R}^+)} \leq \frac{u}{u-1}\|g\|_{L^u(\mathbb{R}^+)}, \quad 1 < u < \infty. \tag{1.3.2}$$

Additionally, it was demonstrated that the constant $u/(u-1)$ found in (1.3.2) is the best one. In [32] Faris and in [33] Christ and his coauthor introduced an extension of (1.3.1) and (1.3.2) to the high-dimensional Euclidian space $\mathbb{R}^n$, the equivalent forms of which are:

$$Hg(\xi) = \frac{1}{|\xi|^n}\int_{|\eta|\leq|\xi|} g(\eta)d\eta, \quad H^*g(\xi) = \int_{|\eta|>|\xi|} \frac{g(\eta)}{|\eta|^n}d\eta, \ \xi \in \mathbb{R}^n \setminus \{\mathbf{0}\}. \tag{1.3.3}$$

Additionally, in [33], the operator norms of H and H^* were computed and matched with those of the corresponding one-dimensional Hardy operators. Usefulness of Hardy integral inequalities in analysis and their applications have garnered considerable attention. With regards to their generalizations, variants, and applications there are numerous papers out there, for instance see [28, 32, 34, 35] and the references cited therein.

Now , we turn towards the definition of fractional Hardy operators [28]

$$\mathcal{H}_\alpha g(\xi) =: \frac{1}{|\xi|^{n-\alpha}}\int_{|\eta|\leq|\xi|} g(\eta)d\eta, \ \mathcal{H}^*_\alpha g(\xi) =: \int_{|\eta|>|\xi|} \frac{g(\eta)}{|\eta|^{n-\alpha}}d\eta, \ \xi \in \mathbb{R}^n \setminus \{0\}, \tag{1.3.4}$$

as well as their commutators

$$\mathcal{H}_{\alpha,b}f = b\mathcal{H}_\alpha f - \mathcal{H}_\alpha(bf), \ \mathcal{H}^*_{\alpha,b}f = b\mathcal{H}^*_\alpha f - \mathcal{H}^*_\alpha(bf). \tag{1.3.5}$$

It is worth noting that if we take $\alpha = 0$ in (1.3.4), we obtain (1.3.3). Also, if we select $\alpha = 0$ and $n = 1$, (1.3.4) reduces to (1.3.1). The following definition serves to define the variable-order fractional Hardy operator.

Definition 1.3.1 *Let $g \in L^1_{\text{loc}}(\mathbb{R}^n)$, and $0 \leq \zeta(\xi) < n$, then the variable order high-dimensional fractional Hardy-type operators are defined by*

$$\mathcal{H}g(\xi) := \frac{1}{|\xi|^{n-\zeta(\xi)}} \int_{|\eta|<|\xi|} g(\eta)d\eta, \quad \mathcal{H}^* g(\xi) := \int_{|\eta|\geq|\xi|} \frac{g(\eta)}{|\eta|^{n-\zeta(\xi)}} d\eta, \quad \xi \in \mathbb{R}^n \setminus 0.$$

Since Hardy-type operators gain well deserved attention as compared to the variable-order fractional Hardy-type operators. So, in third chapter, we decided to establish the boundedness of variable-order Hardy-type operators in variable exponents grand Herz-Morrey space.

1.3.2 Hardy-type Operators on $\mathbb{Q}_p^n$

The p-adic integral operators has a deep connections to p-adic Fourier analysis, p-adic differential equations, and p-adic dynamical systems [21, 22, 36, 37, 38, 39, 40]. Also, they have great importance in wavelet theory, pseudo-differential equations and harmonic analysis, among other fields [41, 42, 43, 44, 45, 46]. In the book [22], published in 1989, Vladimirov et al. introduced a formulation of p-adic quantum mechanics and introduced p-adic pseudo-differential operators, p-adic stochastic processes, and p-adic quantum theory. Most importantly, this book contains a thorough re-creation of the acclaimed Schwartz theory of distributions over p-adic fields. Since the publication of this monograph, many researchers took interest to study the harmonic analysis on p-adic fields, resulting in numerious generalizations in operator theory and function spaces.

The study of Hardy-type operators on p-adic field provide an insights into the behavior of functions defined on p-adic field and associated function spaces. Fu et al. introduced the p-adic Hardy-type operators H^p and $H^{p,*}$ for the first time in [47] and computed their sharp bounds on on Lebesgue space defined on Q_p^n. The authors in [46] established the boundedness of p-adic Hardy operators along with their commutators in central Morrey spaces. Optimal bounds for the p-adic Hardy operator and associated adjoint operator on higher dimensional product spaces are computed in [48]. Finally, the definitions and study of high-dimensional fractional p-adic Hardy-type operators were given in [48] and are as below:

$$\mathcal{H}^p_\alpha g(\xi) = \frac{1}{|\xi|_p^{n-\alpha}} \int_{|\eta|_p\leq|\xi|_p} g(\eta)d\eta, \ \ \mathcal{H}^{p,*}_\alpha g(\xi) \ = \int_{|\eta|_p>|\xi|_p} \frac{g(\eta)}{|\eta|_p^{n-\alpha}} d\eta, \ \xi \in \mathbb{Q}_p^n \setminus \{\mathbf{0}\}, \tag{1.3.6}$$

where g is a locally integrable function and $\alpha \in [0, \infty)$. The commutator operator of p-adic Hardy-type operators are given by

$$\mathcal{H}^{p}_{\alpha,b}g = b\mathcal{H}^{p}_{\alpha}g - \mathcal{H}^{p}_{\alpha}(bg), \;\; \mathcal{H}^{p,*}_{\alpha,b}g = b\mathcal{H}^{p,*}_{\alpha}g - \mathcal{H}^{p,*}_{\alpha}(bg). \tag{1.3.7}$$

Taking $\alpha = 0$, we get the commutator operators of p-adic Hardy-type operators. For detailed information, see [49].

1.4 Our Contribution

We contribute to the theory of Hardy-type operators in a number of ways on both $\mathbb{R}^n$ and $\mathbb{Q}^n_p$. Firstly, we characterize the variable exponent central BMO spaces via the boundedness of commutators of Hardy-type operators on variable Lebesgue and central Morrey spaces. Some boundedness results for the Hardy operator and corresponding adjoint operator are also demonstrated on variable exponent Lebesgue and central Morrey spaces. These results have been published and are online [50]. Furthermore, we also obtained the boundedness of variable-order fractional Hardy-type operators from grand Herz spaces to weighted spaces, subject to appropriate weight conditions. The results of this chapters have also been published in [51].

In the framework of variable exponent, we introduce some new p-adic function spaces. The fractional p-adic Hardy-type operators on the p-adic Lebesgue and central Morrey spaces with variable exponents are shown to be bounded. We characterize some p-adic function spaces as well by proving the boundedness of commutators generated by p-adic Hardy-type operators and p-adic variable exponent λ-central BMO functions on the aforementioned spaces. Furthermore, the continuity of theses operators on p-adic variable exponent Herz-type spaces is discussed as well. Future submissions of these findings to appropriate scientific journal are planned.

Chapter 2
Characterization of Variable Exponent Central BMO Space Via Commutators of Fractional Hardy Operator

2.1 Introduction

Characterization of function spaces via commutators of integral operators is an interesting issue in harmonic analysis mainly because of its numerous applications in the theory of partial differential equations. Many authors worked on such problems by defining commutators of these integral operators. Among many others, the high dimensional fractional Hardy-type operators are also used in the characterization of function spaces via their commutators. For example, in [28] Fu et al. gave the characterization of central BMO space via commutators of Hardy operator on Herz spaces. Later on, Zhao and Lu [29] characterized the λ-central BMO space for $\lambda \geq 0$ via the commutators of the same operator. The difficulty caused by taking $\lambda < 0$ for the same characterization was tackled in [30]. Since the Hardy operator is center-symmetric, using this property the authors characterized the central BMO space through the commutators of rough fractional Hardy operator in [52]. A characterization of weighted central Campanato spaces using the commutators of Hardy operator was given in [53]. Recently, in [54], Wei introduced the mixed central bounded mean oscillation space and characterized it using the boundedness of the commutators of the high dimensional Hardy operator on mixed Herz spaces. Finally, two new characterizations of central BMO space via the commutators of the rough Hardy operator were reported in the literature in [55].

Although the Hardy operator is a celebrated operator in analysis but there are only few publication discussing this operator on variable exponent function spaces.

To mention a few we cite here some references considering the continuity of Hardy-type operators along with their commutators in variable exponents function spaces [56, 68, 57, 58, 59, 60]. In this paper, we characterize the variable exponent central BMO space via the commutators of Hardy-type operators on variable exponent Lebesgue and central Morrey spaces. In addition, we discuss the continuity of Hardy-type operators on variable exponents Lebesgue and central Morrey spaces.

We divide this chapter into the following sections. Next Section gives some basic propositions and lemmas regarding the function spaces with variable exponents. Section 3 of this Chapter investigates the boundedness of the fractional Hardy operators in the context of variable exponent Lebesgue space along with the characterization of variable central BMO space via its commutators. Finally, similar boundedness of Hardy operator and characterization of central BMO space via its commutators on the variable exponent central Morrey space is done in the last Section.

2.2 Preliminaries

If $u'(\cdot) = u(\cdot)/(u(\cdot)-1)$ then we give the following inequality known as the generalized Hölder inequality:

Lemma 2.2.1 ([7]) Let $u(\cdot) \in \mathfrak{P}(H)$, where H is an open subset of $\mathbb{R}^n$. If $g \in L^{u(\cdot)}(H)$ and $h \in L^{u'(\cdot)}(H)$, then we have

$$\int_H |g(\eta)h(\eta)|d\eta \leq r_u \|g\|_{L^{u(\cdot)}(H)} \|h\|_{L^{u'(\cdot)}(H)}, \tag{2.2.1}$$

where $r_u = 1 + \frac{1}{u_-} - \frac{1}{u_+}$.

Lemma 2.2.2 ([61]) Suppose $u(\cdot) \in \mathfrak{B}(\mathbb{R}^n)$. Then, for every ball $B \subset \mathbb{R}^n$, there exists a $C > 0$ such that

$$\frac{1}{|B|} \|\chi_B\|_{L^{u(\cdot)}(\mathbb{R}^n)} \|\chi_B\|_{L^{u'(\cdot)}(\mathbb{R}^n)} \leq C.$$

Lemma 2.2.3 ([62]) If $u(\cdot) \in \mathfrak{B}(\mathbb{R}^n)$ then, for every ball B in $\mathbb{R}^n$ and all measurable subsets $S \subset B$, there exists constants $C > 0$ and $\delta, \delta_1 \in (0,1)$ such that

$$\frac{\|\chi_B\|_{L^{u(\cdot)}(\mathbb{R}^n)}}{\|\chi_S\|_{L^{u(\cdot)}(\mathbb{R}^n)}} \leq C\frac{|B|}{|S|}, \quad \frac{\|\chi_S\|_{L^{u(\cdot)}(\mathbb{R}^n)}}{\|\chi_B\|_{L^{u(\cdot)}(\mathbb{R}^n)}} \leq C\left(\frac{|S|}{|B|}\right)^{\delta}, \quad \frac{\|\chi_S\|_{L^{u'(\cdot)}(\mathbb{R}^n)}}{\|\chi_B\|_{L^{u'(\cdot)}(\mathbb{R}^n)}} \leq C\left(\frac{|S|}{|B|}\right)^{\delta_1}. \tag{2.2.2}$$

Lemma 2.2.4 ([68]) Let $u(\cdot) \in \mathfrak{B}(\mathbb{R}^n)$, $0 < \alpha < \frac{n}{u_+}$ and define $v(\cdot)$ by

$$\frac{1}{v(\cdot)} = \frac{1}{u(\cdot)} - \frac{\alpha}{n}. \tag{2.2.3}$$

Then

$$\|\chi_{B_j}\|_{L^{v(\cdot)}(\mathbb{R}^n)} \leq C2^{-j\alpha}\|\chi_{B_j}\|_{L^{u(\cdot)}(\mathbb{R}^n)}. \tag{2.2.4}$$

Lemma 2.2.5 ([9]) Let $u(\cdot) \in \mathfrak{B}(\mathbb{R}^n)$ satisfying conditions (1.2.2) and (1.2.3) of Chapter 1, then

$$\|\chi_Q\|_{L^{u(\cdot)}(\mathbb{R}^n)} \approx \begin{cases} |Q|^{\frac{1}{u(x)}}, & \text{if } |Q| \leq 2^n \text{and } x \in Q, \\ |Q|^{\frac{1}{u(\infty)}}, & \text{if } |Q| \geq 1, \end{cases}$$

for all $Q \subset \mathbb{R}^n$, where $u(\infty) = \lim_{x\to\infty} u(x)$.

Following propositions will be helpful in proving results on central Morrey spaces.

Proposition 2.2.6 ([12]) If $u(\cdot) \in \mathfrak{B}(\mathbb{R}^n)$, then $f \in C^{u(\cdot)}$ if and only if there exists a collection of numbers $\{c_B\}_B$ in such a way that

$$\|g\|_{C_*^{u(\cdot)}} =: \sup_{r>0} \frac{\|(g - c_{B(0,r)})\chi_{B(0,r)}\|_{L^{u(\cdot)}(\mathbb{R}^n)}}{\|\chi_{B(0,r)}\|_{L^{u(\cdot)}(\mathbb{R}^n)}} < \infty.$$

Proposition 2.2.7 ([12]) If $u(\cdot) \in \mathfrak{B}(\mathbb{R}^n)$, then $f \in C^{u(\cdot)}$ if and only if

$$\|f\|_{C_{**}^{u(\cdot)}} =: \sup_{r>0} \inf_{c} \|\chi_{B(0,r)}\|^{-1}_{L^{u(\cdot)}(\mathbb{R}^n)} \|(f - c)\chi_{B(0,r)}\|_{L^{u(\cdot)}(\mathbb{R}^n)} < \infty.$$

2.3 Characterization of Central BMO Via Commutators on Lebesgue Space

2.3.1 Main Results

The first result of this section gives us the boundedness of the operator defined in (1.3.4) on the variable exponent Lebesgue spaces.

Theorem 2.3.1 Let $u(\cdot) \in \mathfrak{B}(\mathbb{R}^n)$ satisfying conditions (1.2.2) and (1.2.3) of Chapter 1. Also, let $0 < \alpha < \min\{\frac{n}{u_+}, \frac{n}{v'_+}\}$ and define $v(\cdot)$ by

$$\frac{1}{v(\cdot)} = \frac{1}{u(\cdot)} - \frac{\alpha}{n}, \tag{2.3.1}$$

then both H_α and H^*_α map $L^{u(\cdot)}(\mathbb{R}^n)$ into $L^{v(\cdot)}(\mathbb{R}^n)$ and $L^{v'(\cdot)}(\mathbb{R}^n)$ into $L^{u'(\cdot)}(\mathbb{R}^n)$.

If $\alpha = 0$ in the above theorem then we have the following corollary:

Corollary 2.3.2 Let $u(\cdot) \in \mathfrak{B}(\mathbb{R}^n)$ satisfying conditions (1.2.2) and (1.2.3) of Chapter 1, then both H and H^* map $L^{u(\cdot)}(\mathbb{R}^n)$ into $L^{u(\cdot)}(\mathbb{R}^n)$ and $L^{u'(\cdot)}(\mathbb{R}^n)$ into $L^{u'(\cdot)}(\mathbb{R}^n)$.

The second result of this section gives the characterization of $C^{u(\cdot)}(\mathbb{R}^n)$ via the commutators operators defined in (1.3.5) on Variable Lebesgue space.

Theorem 2.3.3 Let $u(\cdot) \in \mathfrak{B}(\mathbb{R}^n)$ satisfying conditions (1.2.2) and (1.2.3) of Chapter 1. Also, let $0 < \alpha < \min\{\frac{n}{u_+}, \frac{n}{v'_+}\}$ and define $v(\cdot)$ by

$$\frac{1}{v(\cdot)} = \frac{1}{u(\cdot)} - \frac{\alpha}{n}, \tag{2.3.2}$$

then the following statements are equivalent:

(1) $b \in C^{v(\cdot)} \cap C^{u'(\cdot)}$.
(2) Both $H_{\alpha,b}$ and $H^*_{\alpha,b}$ map $L^{u(\cdot)}(\mathbb{R}^n)$ into $L^{v(\cdot)}(\mathbb{R}^n)$ and $L^{v'(\cdot)}(\mathbb{R}^n)$ into $L^{u'(\cdot)}(\mathbb{R}^n)$.

If $\alpha = 0$ in the above Theorem we obtain the following corollary which is Theorem 4.1 in [12].

Corollary 2.3.4 Let $u(\cdot) \in \mathfrak{B}(\mathbb{R}^n)$ satisfying conditions (1.2.2) and (1.2.3) of Chapter 1, then the following statements are equivalent:

(1) $b \in C^{u(\cdot)} \cap C^{u'(\cdot)}$.
(2) Both H_b and H^*_b are bounded on $L^{u(\cdot)}(\mathbb{R}^n)$ and $L^{u'(\cdot)}(\mathbb{R}^n)$.

The next Lemma is an extension of Lemma 2.6 in [28] to the variable exponent central BMO space.

Lemma 2.3.5 Let $g \in C^{u(\cdot)}$ and $l, m \in \mathbb{Z}$, then

$$|g(x) - g_{B_l}| \leq |g(x) - g_{B_m}| + C|m - l|\|g\|_{C^{u(\cdot)}}. \tag{2.3.3}$$

Proof. Let $i \in \mathbb{Z}$, then using inequality (2.2.1) we have

$$\begin{aligned}
|g_{B_i} - g_{B_{i+1}}| &= \frac{1}{|B_i|}\int_{B_i} |g(y) - g_{B_{i+1}}|dy \\
&\leq C\frac{1}{|B_{i+1}|}\|(g - g_{B_{i+1}})\chi_{B_{i+1}}\|_{L^{u(\cdot)}}\|\chi_{B_{i+1}}\|_{L^{u'(\cdot)}} \\
&\leq C\|g\|_{C^{u(\cdot)}},
\end{aligned}$$

where in the last inequality, we made use of Lemma 2.2.2 to obtain the desired output. Next, if $l < m$, then

$$|g(x) - g_{B_l}| \leq |g(x) - g_{B_m}| + C\sum_{i=l}^{m-1} |g_{B_i} - g_{B_{i+1}}| \leq |g(x) - g_{B_m}| + C(m - l)\|g\|_{C^{u(\cdot)}}. \tag{2.3.4}$$

Similarly, if $m < l$, then

$$|g(x) - g_{B_l}| \leq |g(x) - g_{B_m}| + C\sum_{i=m}^{l-1} |g_{B_i} - g_{B_{i+1}}| \leq |g(x) - g_{B_m}| + C(l-m)\|g\|_{C^{u(\cdot)}}. \tag{2.3.5}$$

The inequalities (2.3.4) and (2.3.5) yield the inequality (2.3.3).

2.3.2 Proof of the Main Results

The proof of theorem 2.3.1 is essentially same as that of the proof of Theorem 2.3.3 with proper adjustment of the function b. So, we only give the proof of Theorem 2.3.3.

Proof of Theorem 2.3.3. We give proof of this theorem in two steps:
(1) $\Rightarrow$ (2) Let $b \in C^{v(\cdot)} \cap C^{u'(\cdot)}$, then

$$\begin{aligned}
\|H_{\alpha,b}f\|_{L^{v(\cdot)}(\mathbb{R}^n)} &= \sum_{k=-\infty}^{\infty} \|\chi_k H_{\alpha,b}(f)\|_{L^{v(\cdot)}(\mathbb{R}^n)}, \\
&= \sum_{k=-\infty}^{\infty} \left\|\chi_k(\cdot)\left(\frac{1}{|\cdot|^{n-\alpha}}\int_{|t|\leq|\cdot|}(f(t))(b(\cdot)-b(t))dt\right)\right\|_{L^{v(\cdot)}(\mathbb{R}^n)}, \\
&\leq \sum_{k=-\infty}^{\infty} \left\|\chi_k(\cdot)|\cdot|^{\alpha-n}\sum_{j=-\infty}^{k}\int_{A_j}(f(t)(b(\cdot)-b(t)))dt\right\|_{L^{v(\cdot)}(\mathbb{R}^n)}. \qquad (2.3.6)
\end{aligned}$$

Let us consider the inner integral first which can be decomposed as:

$$\int_{A_j} |b(x) - b(t)||f(t)|dt \leq \int_{A_j} |b(x) - b_{B_k}||f(t)|dt + \int_{A_j} |b_{B_k} - b(t)||f(t)|dt. \tag{2.3.7}$$

In view of the generalized Hölders inequality (2.3.6), the fisrt component of the above inequality (2.3.7) implies

$$\int_{B_j} |b_{B_k} - b(x)||f(t)|dt \leq C|b_{B_k} - b(x)|\|f_j\|_{L^{u(\cdot)}(\mathbb{R}^n)}\|\chi_{B_j}\|_{L^{u'(\cdot)}(\mathbb{R}^n)}. \tag{2.3.8}$$

The second component of the inequality (2.3.7), by virtue of Lemma 2.3.5 and the Hölder inequality, gives us:

$$\begin{aligned}\int_{A_j}|f(t)||b_{B_k}-b(t)|dt &\le \int_{A_j}|b_{B_j}-b(t)||f(t)|dt + C(k-j)\|b\|_{C^{u'(\cdot)}}\int_{A_j}|f(t)|dt\\ &\le C\|(b-b_{B_j})\chi_{B_j}\|_{L^{u'(\cdot)}(\mathbb{R}^n)}\|f_j\|_{L^{u(\cdot)}(\mathbb{R}^n)}\\ &+ C\|\chi_{B_j}\|_{L^{u'(\cdot)}(\mathbb{R}^n)}\|b\|_{C^{u'(\cdot)}}(k-j)\|f_j\|_{L^{u(\cdot)}(\mathbb{R}^n)}\\ &\le C\chi_{B_j}\|_{L^{u'(\cdot)}(\mathbb{R}^n)}\|b\|_{C^{u'(\cdot)}}\|\|f_j\|_{L^{u(\cdot)}(\mathbb{R}^n)}\\ &+ C\|\chi_{B_j}\|_{L^{u'(\cdot)}(\mathbb{R}^n)}(k-j)\|b\|_{C^{u'(\cdot)}}\|f_j\|_{L^{u(\cdot)}(\mathbb{R}^n)}\\ &\le C\|\chi_{B_j}\|_{L^{u'(\cdot)}(\mathbb{R}^n)}(k-j)\|b\|_{C^{u'(\cdot)}}\|f_j\|_{L^{u(\cdot)}(\mathbb{R}^n)}.\end{aligned} \tag{2.3.9}$$

We infer from (2.3.6)-(2.3.9) that:

$$\begin{aligned}&\|H_{\alpha,b}f\|_{L^{v(\cdot)}(\mathbb{R}^n)}\\ &\le \sum_{k=-\infty}^{\infty}\left\|\chi_k(\cdot)|\cdot|^{\alpha-n}\sum_{j=-\infty}^{k}\int_{A_j}|f(t)||b(t)-b(\cdot)|dt\right\|_{L^{v(\cdot)}}\\ &\le C\sum_{k=-\infty}^{\infty}\sum_{j=-\infty}^{k}2^{-k(n-\alpha)}\|(b_{B_k}-b)\chi_{B_k}\|_{L^{v(\cdot)}(\mathbb{R}^n)}\|\chi_{B_j}\|_{L^{u'(\cdot)}(\mathbb{R}^n)}\|f_j\|_{L^{u(\cdot)}(\mathbb{R}^n)}\\ &+ C\sum_{k=-\infty}^{\infty}\sum_{j=-\infty}^{k}2^{-k(n-\alpha)}(k-j)\|b\|_{C^{u'(\cdot)}}\|\chi_{B_k}\|_{L^{v(\cdot)}(\mathbb{R}^n)}\|\chi_{B_j}\|_{L^{u'(\cdot)}(\mathbb{R}^n)}\|f_j\|_{L^{u(\cdot)}(\mathbb{R}^n)}\\ &\le C\sum_{k=-\infty}^{\infty}\sum_{j=-\infty}^{k}2^{-k(n-\alpha)}(k-j)\|\chi_{B_k}\|_{L^{v(\cdot)}(\mathbb{R}^n)}\|\chi_{B_j}\|_{L^{u'(\cdot)}(\mathbb{R}^n)}\|f_j\|_{L^{u(\cdot)}(\mathbb{R}^n)}.\end{aligned} \tag{2.3.10}$$

Next, our objective is to sort out the product term $\|\chi_{B_k}\|_{L^{v(\cdot)}(\mathbb{R}^n)}\|\chi_{B_j}\|_{L^{u'(\cdot)}(\mathbb{R}^n)}$. A use of Lemma 2.2.2 and the Lemma 2.2.3 help us to write:

$$\begin{aligned}\|\chi_{B_k}\|_{L^{v(\cdot)}(\mathbb{R}^n)}\|\chi_{B_j}\|_{L^{u'(\cdot)}(\mathbb{R}^n)} &\le 2^{kn}\|\chi_{B_k}\|^{-1}_{L^{v'(\cdot)}(\mathbb{R}^n)}\|\chi_{B_j}\|_{L^{u'(\cdot)}(\mathbb{R}^n)}\\ &\le C2^{kn}2^{(j-k)n\delta_1}\|\chi_{B_k}\|^{-1}_{L^{v'(\cdot)}(\mathbb{R}^n)}\|\chi_{B_k}\|_{L^{u'(\cdot)}(\mathbb{R}^n)}.\end{aligned}$$

Since $\frac{1}{u'(\cdot)}=\frac{1}{v'(\cdot)}-\frac{\alpha}{n}$, so by virtue of Lemma 2.2.4, we obtain

$$\|\chi_{B_k}\|_{L^{v(\cdot)}(\mathbb{R}^n)}\|\chi_{B_j}\|_{L^{u'(\cdot)}(\mathbb{R}^n)} \le 2^{k(n-\alpha)}2^{(j-k)n\delta_1}. \tag{2.3.11}$$

Hence, (2.3.10) and (2.3.11) together yield

$$\begin{aligned}\|H_{\alpha,b}f\|_{L^{v(\cdot)}(\mathbb{R}^n)} &\leq C\sum_{k=-\infty}^{\infty}\sum_{j=-\infty}^{k}(k-j)2^{-(k-j)n\delta_1}\|f_j\|_{L^{u(\cdot)}(\mathbb{R}^n)}\\ &\leq C\sum_{j=-\infty}^{\infty}\|f_j\|_{L^{u(\cdot)}(\mathbb{R}^n)}\sum_{k=j}^{\infty}(k-j)2^{-(k-j)n\delta_1}\\ &\leq C\|f\|_{L^{u(\cdot)}(\mathbb{R}^n)}.\end{aligned}$$

In a similar fashion it is easy to prove that

$$\|H^*_{\alpha,b}f\|_{L^{v(\cdot)}(\mathbb{R}^n)} \leq C\|f\|_{L^{u(\cdot)}(\mathbb{R}^n)}.$$

A similar procedure as used in the $(L^{u(\cdot)}(\mathbb{R}^n), L^{v(\cdot)}(\mathbb{R}^n))$ boundedness of $H_{\alpha,b}$ and $H^*_{\alpha,b}$ and the fact that $\frac{1}{u'(\cdot)} = \frac{1}{v'(\cdot)} - \frac{\alpha}{n}$, implies that both $H_{\alpha,b}$ and $H^*_{\alpha,b}$ map $L^{v'(\cdot)}(\mathbb{R}^n)$ into $L^{u'(\cdot)}(\mathbb{R}^n)$.

(2) $\Rightarrow$ (1) Using the fact that both $H_{\alpha,b}$ and $H^*_{\alpha,b}$ map $L^{u(\cdot)}(\mathbb{R}^n)$ into $L^{v(\cdot)}(\mathbb{R}^n)$ and $L^{v'(\cdot)}(\mathbb{R}^n)$ into $L^{u'(\cdot)}(\mathbb{R}^n)$, we have to show that $b \in C^{v(\cdot)} \cap C^{u'(\cdot)}$.
For any ball $B =: B(0,r)$ and $x \in B$, we obtain

$$\begin{aligned}|b(x)-b_B| &=: \left|\frac{1}{|B|}\int_B (b(x)-b(t))dt\right|,\\ &\leq C\left|\frac{|B|^{-\frac{\alpha}{n}}}{|x|^{n-\alpha}}\int_{|t|\leq|x|}(b(x)-b(t))\chi_B(t)dt\right|\\ &+ C\left|\int_{|t|>|x|}\frac{(b(x)-b(t))\chi_B(t)|B|^{-1}|t|^{n-\alpha}}{|t|^{n-\alpha}}dt\right|,\\ &\leq C|B|^{-\frac{\alpha}{n}}\left|H_{\alpha,b}\chi_B(x)\right| + C|B|^{-1}\left|H^*_{\alpha,b}f_0(x)\right|,\end{aligned}$$

where $f_0(x) = |x|^{n-\alpha}\chi_B(x)$. Hence

$$\|(b-b_B)\chi_B\|_{L^{s(\cdot)}(\mathbb{R}^n)} \leq C|B|^{-\frac{\alpha}{n}}\|H_{\alpha,b}(\chi_B)\|_{L^{s(\cdot)}(\mathbb{R}^n)} + C|B|^{-1}\|H^*_{\alpha,b}(f_0)\|_{L^{s(\cdot)}(\mathbb{R}^n)}. \tag{2.3.12}$$

In order to arrive at our claim, we split the problem into the following two cases:
Case 1: $s(\cdot) = v(\cdot)$. In this case using the $(L^{u(\cdot)}(\mathbb{R}^n), L^{v(\cdot)}(\mathbb{R}^n))$ boundedness of $H_{\alpha,b}$ and $H^*_{\alpha,b}$, one has

$$\begin{aligned}\|(b-b_B)\chi_B\|_{L^{v(\cdot)}(\mathbb{R}^n)} &\leq C|B|^{-\frac{\alpha}{n}}\|H_{\alpha,b}(\chi_B)\|_{L^{v(\cdot)}(\mathbb{R}^n)} + C|B|^{-1}\|H^*_{\alpha,b}(f_0)\|_{L^{v(\cdot)}(\mathbb{R}^n)}\\ &\leq C|B|^{-\frac{\alpha}{n}}\|\chi_B\|_{L^{u(\cdot)}(\mathbb{R}^n)} + C|B|^{-1}\|f_0\|_{L^{u(\cdot)}(\mathbb{R}^n)}\\ &\leq C|B|^{-\frac{\alpha}{n}}\|\chi_B\|_{L^{u(\cdot)}(\mathbb{R}^n)}.\end{aligned}$$

Using Lemma 2.2.5 and the condition $\frac{1}{v(x)} = \frac{1}{u(x)} - \frac{\alpha}{n}$, we have

$$\|\chi_B\|_{L^{u(\cdot)}(\mathbb{R}^n)} \approx |B|^{\frac{1}{u(x)}} \approx \|\chi_B\|_{L^{v(\cdot)}(\mathbb{R}^n)} |B|^{\frac{\alpha}{n}}.$$

Therefore

$$\|(b - b_B)\chi_B\|_{L^{v(\cdot)}(\mathbb{R}^n)} \leq C\|\chi_B\|_{L^{v(\cdot)}(\mathbb{R}^n)}.$$

Case 2: $s(\cdot) = u'(\cdot)$. In this case using the $(L^{v'(\cdot)}(\mathbb{R}^n), L^{u'(\cdot)}(\mathbb{R}^n))$ boundedness of $H_{\alpha,b}$ and $H^*_{\alpha,b}$, we have

$$\begin{aligned}\|(b - b_B)\chi_B\|_{L^{u'(\cdot)}(\mathbb{R}^n)} &\leq C|B|^{-\frac{\alpha}{n}} \|H_{\alpha,b}(\chi_B)\|_{L^{u'(\cdot)}(\mathbb{R}^n)} + C|B|^{-1} \|H^*_{\alpha,b}(f_0)\|_{L^{u'(\cdot)}(\mathbb{R}^n)} \\ &\leq C|B|^{-\frac{\alpha}{n}} \|\chi_B\|_{L^{v'(\cdot)}(\mathbb{R}^n)} + C|B|^{-1} \|f_0\|_{L^{v'(\cdot)}(\mathbb{R}^n)} \\ &\leq C|B|^{-\frac{\alpha}{n}} \|\chi_B\|_{L^{v'(\cdot)}(\mathbb{R}^n)}.\end{aligned}$$

Using Lemma 2.2.5 and the condition $\frac{1}{v'(x)} = \frac{1}{u'(x)} + \frac{\alpha}{n}$, we have

$$\|\chi_B\|_{L^{v'(\cdot)}(\mathbb{R}^n)} \approx |B|^{\frac{1}{v'(x)}} \approx \|\chi_B\|_{L^{u'(\cdot)}(\mathbb{R}^n)} |B|^{\frac{\alpha}{n}}.$$

Therefore

$$\|(b - b_B)\chi_B\|_{L^{u'(\cdot)}(\mathbb{R}^n)} \leq C\|\chi_B\|_{L^{u'(\cdot)}(\mathbb{R}^n)}.$$

We thus conclude from these cases that $b \in C^{v(\cdot)} \cap C^{u'(\cdot)}$. Thus the proof is completed.

2.4 Characterization of Central BMO Via Commutators on Central Morrey Space

2.4.1 Main Results

The first result for this section gives the continuity properties of H_α and H^*_α the variable exponent on central Morrey space.

Theorem 2.4.1 Let $u(\cdot) \in \mathfrak{B}(\mathbb{R}^n)$ satisfying conditions (1.2.2) and (1.2.3) of Chapter 1. Also, let $0 < \alpha < \min\{\frac{n}{u_+}, \frac{n}{v'_+}\}$ and define $v(\cdot)$ by

$$\frac{1}{v(\cdot)} = \frac{1}{u(\cdot)} - \frac{\alpha}{n}, \tag{2.4.1}$$

then for $\gamma = \beta + \frac{\alpha}{n}$ with $\gamma < 0$, both H_α and H^*_α map $\dot{M}^{u(\cdot),\beta}(\mathbb{R}^n)$ into $\dot{M}^{v(\cdot),\gamma}(\mathbb{R}^n)$ and $\dot{M}^{v'(\cdot),\beta}(\mathbb{R}^n)$ into $\dot{M}^{u'(\cdot),\gamma}(\mathbb{R}^n)$.

Remark 2.4.2 *Theorem 2.4.1 improves and unifies Theorem 3.1 and Theorem 4.1 in [60] by removing extra parameters involved in the stated conditions for these theorems. Also, it uses a different methodology for its proof.*

If $\alpha = 0$ in Theorem 2.4.1 then it yields the following corollary:

Corollary 2.4.3 Let $u(\cdot) \in \mathfrak{B}(\mathbb{R}^n)$ satisfying conditions (1.2.2) and (1.2.3) of Chapter 1, then for $\beta < 0$, both H and H^* map $\dot{M}^{u(\cdot),\beta}(\mathbb{R}^n)$ on $\dot{M}^{u(\cdot),\beta}(\mathbb{R}^n)$.

The next Theorem characterizes the variable central BMO space via the commutators of Hardy operators on central Morrey space.

Theorem 2.4.4 Let $u(\cdot) \in \mathfrak{B}(\mathbb{R}^n)$ satisfying conditions (1.2.2) and (1.2.3) of Chapter 1. Also, let $0 < \alpha < \min\{\frac{n}{u_+}, \frac{n}{v'_+}\}$ and define $v(\cdot)$ by

$$\frac{1}{v(\cdot)} = \frac{1}{u(\cdot)} - \frac{\alpha}{n}, \tag{2.4.2}$$

then for $\gamma = \beta + \frac{\alpha}{n}$ with $\gamma < 0$, the following statements are equivalent:

(1) $b \in C^{v(\cdot)} \cap C^{u'(\cdot)}$.
(2) Both $H_{\alpha,b}$ and $H^*_{\alpha,b}$ map $\dot{M}^{u(\cdot),\beta}(\mathbb{R}^n)$ into $\dot{M}^{v(\cdot),\gamma}(\mathbb{R}^n)$ and $\dot{M}^{v'(\cdot),\beta}(\mathbb{R}^n)$ into $\dot{M}^{u'(\cdot),\gamma}(\mathbb{R}^n)$.

The last Theorem has the following corollary:

Corollary 2.4.5 Let $u(\cdot) \in \mathfrak{B}(\mathbb{R}^n)$ satisfying conditions (1.2.2) and (1.2.3) of Chapter 1, then for $\beta < 0$, the following statements are equivalent:

(1) $b \in C^{u(\cdot)} \cap C^{u'(\cdot)}$.
(2) Both H_b and H^*_b are bounded on $\dot{M}^{u(\cdot),\beta}(\mathbb{R}^n)$ and $\dot{M}^{u'(\cdot),\beta}(\mathbb{R}^n)$.

2.4.2 Proofs of Main Results

We first present the proof of Theorem 2.4.1.

Proof of Theorem 2.4.1. Without loss of generality we may assume $B_{k_0} = B(0, R)$ with $k_0 \in \mathbb{Z}$ for a fixed ball $B(0, R) \subset \mathbb{R}^n$. Let $f = f_1 + f_2$ where $f_1 = f_{\chi_{2B_{k_0}}}$ and $f_2 = f_{\chi_{(2B_{k_0})^c}}$ then

$$\begin{aligned} \|(H_\alpha f)\chi_{B_{k_0}}\|_{L^{v(\cdot)}(\mathbb{R}^n)} &\le \|(H_\alpha f_1)\chi_{B_{k_0}}\|_{L^{v(\cdot)}(\mathbb{R}^n)} + \|(H_\alpha f_2)\chi_{B_{k_0}}\|_{L^{v(\cdot)}(\mathbb{R}^n)} \\ &=: I_1 + I_2. \end{aligned}$$

In view of Theorem 2.3.1, we estimate I_1 as below:

$$\begin{aligned} I_1 =: &\|(H_\alpha f_1)\chi_{B_{k_0}}\|_{L^{v(\cdot)}(\mathbb{R}^n)} \\ &\leq \|\big(H_\alpha f\big)\chi_{2B_{k_0}}\|_{L^{v(\cdot)}(\mathbb{R}^n)} \\ &\leq C\|f\chi_{2B_{k_0}}\|_{L^{u(\cdot)}(\mathbb{R}^n)} \\ &\leq C\|f\|_{\dot{M}^{u(\cdot),\beta}}\|\chi_{2B_{k_0}}\|_{L^{u(\cdot)}(\mathbb{R}^n)}|2B_{k_0}|^{\beta} \\ &\leq C\|f\|_{\dot{M}^{u(\cdot),\beta}}\|\|\chi_{B_{k_0}}\|_{L^{u(\cdot)}(\mathbb{R}^n)}|B_{k_0}|^{\beta}. \end{aligned}$$

Using Lemma 2.2.5 and the condition $\frac{1}{v(\cdot)} = \frac{1}{u(\cdot)} - \frac{\alpha}{n}$ we get

$$\|\chi_{B_{k_0}}\|_{L^{u(\cdot)}(\mathbb{R}^n)} \approx |B_{k_0}|^{\frac{1}{u(x)}} \approx \|\chi_{B_{k_0}}\|_{L^{v(\cdot)}(\mathbb{R}^n)}|B_{k_0}|^{\frac{\alpha}{n}}. \tag{2.4.3}$$

Therefore, by virtue of $\gamma - \beta = \frac{\alpha}{n}$, we get

$$I_1 =: C\|f\|_{\dot{M}^{u(\cdot),\beta}}\|\|\chi_{B_{k_0}}\|_{L^{v(\cdot)}(\mathbb{R}^n)}|B_{k_0}|^{\gamma}.$$

Now in order to estimate I_2, we proceed as below:

$$\begin{aligned} |H_\alpha f_2(x)| &= \left| |x|^{\alpha-n}\int_{|y|\leq|x|} f_2(y)dy \right| \\ &\leq C\sum_{k=2k_0}^{\infty}\frac{1}{|B_k|^{1-\frac{\alpha}{n}}}\int_{C_k}|f(y)|dy \\ &\leq C\sum_{k=2k_0}^{\infty}\frac{1}{|B_k|^{1-\frac{\alpha}{n}}}\|f\chi_{B_k}\|_{L^{u(\cdot)}(\mathbb{R}^n)}\|\chi_{B_k}\|_{L^{u'(\cdot)}}. \end{aligned}$$

Using Lemma 2.2.2, it is easy to see that

$$\begin{aligned} |H_\alpha f_2(x)| &\leq C\sum_{k=2k_0}^{\infty}|B_k|^{\frac{\alpha}{n}+\beta}\|f\|_{\dot{M}^{u(\cdot),\beta}} \\ &\leq C\|f\|_{\dot{M}^{u(\cdot),\beta}}\sum_{k=2k_0}^{\infty}|B_k|^{\gamma}, \end{aligned}$$

where we have used the condition $\gamma = \frac{\alpha}{n} + \beta$. Since $\beta < -\frac{\alpha}{n}$, so that we have

$$|H_\alpha f_2(x)| \leq C\|f\|_{\dot{M}^{u(\cdot),\beta}}|B_{k_0}|^{\gamma}.$$

Finally, we get

$$I_2 =: \|(H_\alpha f_2)\chi_{B_{k_0}}\|_{L^{v(\cdot)}(\mathbb{R}^n)} \leq C\|f\|_{\dot{M}^{u(\cdot),\beta}}\|\chi_{B_{k_0}}\|_{L^{v(\cdot)}(\mathbb{R}^n)}|B_{k_0}|^{\gamma}.$$

Combining the estimates for I_1 and I_2, one has

$$\|H_\alpha f\|_{\dot{M}^{v(\cdot),\gamma}(\mathbb{R}^n)} \leq C\|f\|_{\dot{M}^{u(\cdot),\beta}(\mathbb{R}^n)}.$$

Similarly, the following inequalities can be established as well:

$$\begin{aligned}\|H_\alpha^* f\|_{\dot{M}^{v(\cdot),\gamma}(\mathbb{R}^n)} &\leq C\|f\|_{\dot{M}^{u(\cdot),\beta}(\mathbb{R}^n)},\\ \|H_\alpha f\|_{\dot{M}^{u'(\cdot),\gamma}(\mathbb{R}^n)} &\leq C\|f\|_{\dot{M}^{v'(\cdot),\beta}(\mathbb{R}^n)},\\ \|H_\alpha^* f\|_{\dot{M}^{u'(\cdot),\gamma}(\mathbb{R}^n)} &\leq C\|f\|_{\dot{M}^{v'(\cdot),\beta}(\mathbb{R}^n)}.\end{aligned}$$

We thus finish the proof of Theorem 2.4.1.

Proof of Theorem 2.4.4. For (1) → (2), following the proof of Theorem 2.4.1, we write

$$\begin{aligned}\|\big(H_{\alpha,b}f\big)\chi_{B_{k_0}}\|_{L^{v(\cdot)}(\mathbb{R}^n)} &\leq \|\big(H_{\alpha,b}f_1\big)\chi_{B_{k_0}}\|_{L^{v(\cdot)}(\mathbb{R}^n)} + \|\big(H_{\alpha,b}f_2\big)\chi_{B_{k_0}}\|_{L^{v(\cdot)}(\mathbb{R}^n)}\\ &=: J_1 + J_2.\end{aligned}$$

Making use of the Theorem 2.3.3 for estimation of J_1, we get

$$\begin{aligned}J_1 &=: \|(H_{\alpha,b}f_1)\chi_{B_{k_0}}\|_{L^{v(\cdot)}(\mathbb{R}^n)}\\ &\leq \|\big(H_{\alpha,b}f\big)\chi_{2B_{k_0}}\|_{L^{v(\cdot)}(\mathbb{R}^n)}\\ &\leq C\|f\chi_{2B_{k_0}}\|_{L^{u(\cdot)}(\mathbb{R}^n)}\\ &\leq C\|f\|_{\dot{M}^{u(\cdot),\beta}}\|\chi_{2B_{k_0}}\|_{L^{u(\cdot)}(\mathbb{R}^n)}|2B_{k_0}|^{\beta}\\ &\leq C\|f\|_{\dot{M}^{u(\cdot),\beta}}\|\|\chi_{B_{k_0}}\|_{L^{u(\cdot)}(\mathbb{R}^n)}|B_{k_0}|^{\beta}.\end{aligned}$$

The relation (2.4.3) and the condition $\gamma = \beta + \frac{\alpha}{n}$, help us to write

$$J_1 =: C\|f\|_{\dot{M}^{u(\cdot),\beta}}\|\|\chi_{B_{k_0}}\|_{L^{v(\cdot)}(\mathbb{R}^n)}|B_{k_0}|^{\gamma}.$$

Next, to estimate J_2 we need to the decomposition:

$$\begin{aligned}|H_{\alpha,b}f_2(x)| &= \left|\frac{1}{|x|^{n-\alpha}}\int_{|y|\leq|x|}(b(x)-b(y))f_2(y)dy\right|\\ &\leq C\left|\sum_{k=2k_0}^{\infty}\frac{1}{|B_k|^{1-\frac{\alpha}{n}}}\int_{C_k}(b(x)-b(y))f(y)dy\right|\\ &\leq C\left|\sum_{k=2k_0}^{\infty}\frac{1}{|B_k|^{1-\frac{\alpha}{n}}}\int_{C_k}(b(x)-c)f(y)dy\right|\\ &+ C\left|\sum_{k=2k_0}^{\infty}\frac{1}{|B_k|^{1-\frac{\alpha}{n}}}\int_{C_k}(b(y)-c)f(y)dy\right|\\ &=: J_{21} + J_{22}.\end{aligned}$$

Using the generalized Hölder inequality, J_{21} assumes the following form:

$$\begin{aligned} J_{21} &= C\left|\sum_{k=2k_0}^{\infty}|B_k|^{\frac{\alpha}{n}-1}(b(x)-c)\int_{C_k}f(y)dy\right| \\ &\leq C\sum_{k=2k_0}^{\infty}|B_k|^{\frac{\alpha}{n}-1}|b(x)-c|\|f\chi_{B_k}\|_{L^{u(\cdot)}(\mathbb{R}^n)}\|\chi_{B_k}\|_{L^{u'(\cdot)}}, \end{aligned}$$

which in view of Lemma 2.2.2 gives us:

$$\begin{aligned} J_{21} &\leq C|b(x)-c|\sum_{k=2k_0}^{\infty}|B_k|^{\frac{\alpha}{n}+\beta}\|f\|_{\dot{M}^{u(\cdot),\beta}} \\ &\leq C|b(x)-c|\|f\|_{\dot{M}^{u(\cdot),\beta}}\sum_{k=2k_0}^{\infty}|B_k|^{\gamma} \\ &\leq C|b(x)-c|\|f\|_{\dot{M}^{u(\cdot),\beta}}|B_{k_0}|^{\gamma}, \end{aligned}$$

where series in the second last step converges due to the fact that $\gamma < 0$.

Similarly, Lemma 2.2.1 and Proposition 2.2.7 are used, respectively, in establishing the below inequality for J_{22}.

$$\begin{aligned} J_{22} &= C\left|\sum_{k=2k_0}^{\infty}|B_k|^{\frac{\alpha}{n}-1}\int_{C_k}(b(y)-c)f(y)dy\right| \\ &\leq C\sum_{k=2k_0}^{\infty}|B_k|^{\frac{\alpha}{n}-1}\|f\chi_{B_k}\|_{L^{u(\cdot)}(\mathbb{R}^n)}\|(b(y)-c)\chi_{B_k}\|_{L^{u'(\cdot)}} \\ &\leq C\|b\|_{C^{u'(\cdot)}}\sum_{k=2k_0}^{\infty}|B_k|^{\frac{\alpha}{n}-1}\|f\chi_{B_k}\|_{L^{u(\cdot)}(\mathbb{R}^n)}\|\chi_{B_k}\|_{L^{u'(\cdot)}}, \end{aligned}$$

which, by virtue of Lemma 2.2.2 and the condition $\gamma < 0$, yields

$$J_{22} \leq C\|f\|_{\dot{M}^{u(\cdot),\beta}}\sum_{k=2k_0}^{\infty}|B_k|^{\gamma} \leq C\|f\|_{\dot{M}^{u(\cdot),\beta}}|B_{k_0}|^{\gamma}.$$

Hence, we have

$$J_2 \leq C\|(b(x)-c)\chi_{B_{k_0}}\|_{L^{v(\cdot)}}\|f\|_{\dot{M}^{u(\cdot),\beta}}|B_{k_0}|^{\gamma} + C\|f\|_{\dot{M}^{u(\cdot),\beta}}\|\chi_{B_{k_0}}\|_{L^{v(\cdot)}(\mathbb{R}^n)}|B_{k_0}|^{\gamma},$$

which on making use of Proposition 2.2.7 results in the following inequality:

$$J_2 \leq C\|f\|_{\dot{M}^{u(\cdot),\beta}}\|\chi_{B_{k_0}}\|_{L^{v(\cdot)}(\mathbb{R}^n)}|B_{k_0}|^{\gamma}.$$

Combining the estimates J_1 and J_2, we get

$$\|H_{\alpha,b}f\|_{\dot{M}^{v(\cdot),\gamma}(\mathbb{R}^n)} \leq C\|f\|_{\dot{M}^{u(\cdot),\beta}(\mathbb{R}^n)}.$$

Similarly, the following inequalities can be established as well:

$$\|H^*_{\alpha,b}f\|_{\dot{M}^{v(\cdot),\gamma}(\mathbb{R}^n)} \leq C\|f\|_{\dot{M}^{u(\cdot),\beta}(\mathbb{R}^n)},$$
$$\|H_{\alpha,b}f\|_{\dot{M}^{u'(\cdot),\gamma}(\mathbb{R}^n)} \leq C\|f\|_{\dot{M}^{v'(\cdot),\beta}(\mathbb{R}^n)},$$
$$\|H^*_{\alpha,b}f\|_{\dot{M}^{u'(\cdot),\gamma}(\mathbb{R}^n)} \leq C\|f\|_{\dot{M}^{v'(\cdot),\beta}(\mathbb{R}^n)}.$$

Thus the proof of the case (1) $\rightarrow$ (2) is complete.

(2) $\Rightarrow$ (1) Using the fact that both $H_{\alpha,b}$ and $H^*_{\alpha,b}$ map $\dot{M}^{u(\cdot),\beta}(\mathbb{R}^n)$ into $\dot{M}^{v(\cdot),\gamma}(\mathbb{R}^n)$ and $\dot{M}^{v'(\cdot),\beta}(\mathbb{R}^n)$ into $\dot{M}^{u'(\cdot),\gamma}(\mathbb{R}^n)$, we have to show that $b \in C^{v(\cdot)} \cap C^{u'(\cdot)}$.
For $f_0(x) = |x|^{n-\alpha}\chi_B(x)$, here we rewrite (2.3.12):

$$\|(b-b_B)\chi_B\|_{L^{s(\cdot)}(\mathbb{R}^n)} \leq C|B|^{-\frac{\alpha}{n}}\|H_{\alpha,b}(\chi_B)\|_{L^{s(\cdot)}(\mathbb{R}^n)} + C|B|^{-1}\|H^*_{\alpha,b}(f_0)\|_{L^{s(\cdot)}(\mathbb{R}^n)}.$$

Next, we split the problem into the following two cases:
Case 1: $s(\cdot) = v(\cdot)$. In this case using the $(\dot{M}^{u(\cdot),\beta}, \dot{M}^{v(\cdot),\gamma})$ boundedness of $H_{\alpha,b}$ and $H^*_{\alpha,b}$, one has

$$\begin{aligned}
\|(b-b_B)\chi_B\|_{L^{v(\cdot)}(\mathbb{R}^n)} &\leq C|B|^{\gamma-\frac{\alpha}{n}}\|H_{\alpha,b}(\chi_B)\|_{\dot{M}^{v(\cdot),\gamma}}\|\chi_B\|_{L^{v(\cdot)}(\mathbb{R}^n)} \\
&\quad + C|B|^{\gamma-1}\|H^*_{\alpha,b}(f_0)\|_{\dot{M}^{v(\cdot),\gamma}}\|\chi_B\|_{L^{v(\cdot)}(\mathbb{R}^n)} \\
&\leq C|B|^{\gamma-\frac{\alpha}{n}}\|\chi_B\|_{\dot{M}^{u(\cdot),\beta}}\|\chi_B\|_{L^{v(\cdot)}(\mathbb{R}^n)} \\
&\quad + C|B|^{\gamma-1}\|f_0\|_{\dot{M}^{u(\cdot),\beta}}\|\chi_B\|_{L^{v(\cdot)}(\mathbb{R}^n)} \\
&\leq C\|\chi_B\|_{L^{v(\cdot)}(\mathbb{R}^n)}.
\end{aligned}$$

Case 2: $s(\cdot) = u'(\cdot)$. In this case using the $(\dot{M}^{v'(\cdot),\beta}, \dot{M}^{u'(\cdot),\gamma})$ boundedness of $H_{\alpha,b}$ and $H^*_{\alpha,b}$, one has

$$\begin{aligned}
\|(b-b_B)\chi_B\|_{L^{u'(\cdot)}(\mathbb{R}^n)} &\leq C|B|^{\gamma-\frac{\alpha}{n}}\|H_{\alpha,b}(\chi_B)\|_{\dot{M}^{u'(\cdot),\gamma}}\|\chi_B\|_{L^{u'(\cdot)}(\mathbb{R}^n)} \\
&\quad + C|B|^{\gamma-1}\|H^*_{\alpha,b}(f_0)\|_{\dot{M}^{u'(\cdot),\gamma}}\|\chi_B\|_{L^{u'(\cdot)}(\mathbb{R}^n)} \\
&\leq C|B|^{\gamma-\frac{\alpha}{n}}\|\chi_B\|_{\dot{M}^{v'(\cdot),\beta}}\|\chi_B\|_{L^{u'(\cdot)}(\mathbb{R}^n)} \\
&\quad + C|B|^{\gamma-1}\|f_0\|_{\dot{M}^{v'(\cdot),\beta}}\|\chi_B\|_{L^{u'(\cdot)}(\mathbb{R}^n)} \\
&\leq C\|\chi_B\|_{L^{u'(\cdot)}(\mathbb{R}^n)}.
\end{aligned}$$

From these cases, we conclude that $b \in C^{v(\cdot)} \cap C^{u'(\cdot)}$. We thus complete the proof.

2.5 Conclusion

In this Chapter we showed that the Hardy-type operators are bounded on variable exponent Lebesgue and central Morrey spaces. In addition, we gave a characterization of central Bounded Mean Oscillation (BMO) space via the commutators of these operators on variable exponents Lebesgue and central Morrey spaces.

Chapter 3
Continuity of Hardy Operators on Grand Herz Spaces With Variable Exponent

3.1 Introduction

The investigation of boundedness properties of operators in variable exponent spaces is a challenging and active area of research, for reference see [58, 59, 60]. The behavior of operators in these spaces often differs significantly from that in classical Lebesgue or Sobolev spaces. Consequently, the study of boundedness properties of fractional Hardy operators in variable exponent spaces requires new techniques and approaches.

The concept of Variable exponent grand Herz spaces was introduced in [63] and opens new dimensions in the specified function spaces. Sultan et al. [64, 65] introduced the idea of grand variable Herz-Morrey spaces and proved boundedness for Riesz potential operator in these spaces. Grand weighted Herz spaces and grand weighted Herz-Morrey spaces was introduced by Sultan et al. in [66, 67] respectively.

Inspired by above cited work, in this chapter, we consider the boundedness of fractional Hardy-type operators of variable order from grand Herz spaces to weighted space under some proper assumptions on weight functions.

In this chapter, the content has been divided into four main sections. In addition to the introduction, the next Section is dedicated to exploring fundamental lemmas and propositions. The third Section focuses on the Sobolev-type theorems for fractional Hardy-type operators with variable orders in grand Herz spaces. The last Section includes concluding remarks.

3.2 Preliminaries

We are assuming that order of the fractional integral operator is:

$$I^{\zeta(\xi)}h(\xi) = \int_{\mathbb{R}^n} \frac{h(\xi)}{|\eta - \xi|^{n-\zeta(\xi)}} d\eta, \qquad 0 < \zeta(\xi) < n. \tag{3.2.1}$$

$\zeta(\xi)$ is not continuous rather we are assuming that it is a measurable function in $\mathbb{R}^n$ satisfying:

(1) $\zeta_0 := \operatorname{ess\,inf}_{y\in\mathbb{R}^n} \zeta(y) > 0$,

(2) $\operatorname{ess\,sup}_{y\in\mathbb{R}^n} p(y)\zeta(y) < n$,

(3) $\operatorname{ess\,sup}_{y\in\mathbb{R}^n} p(\infty)\zeta(y) < n$,

where $p(\infty) = \lim\limits_{|x|\to\infty} p(x)$.

The following proposition is one of the main requirement to prove our main results. This proposition was proved in [20] and commonly known as Sobolev theorem for Riesz potential operator in Lebesgue spaces under the some necessary assumptions on exponent.

Proposition 3.2.1 *Suppose that*

$$p(\cdot) \in \mathfrak{B}^{\log}(\mathbb{R}^n) \cap \mathfrak{B}_{0,\infty}(\mathbb{R}^n) \cap \mathfrak{B}(\mathbb{R}^n)$$

and assume

$$1 < p(\infty) \leq p(x) \leq p_+ < \infty.$$

Let $\zeta(x)$ satisfy the above conditions (1)–(3). Then, we have following weighted Sobolev-type estimate for the fractional operator $I^{\zeta(z)}$

$$\|(1+|z|)^{-\lambda(z)} I^{\zeta(z)}(f)\|_{L^{q(\cdot)}(\mathbb{R}^n)} \leq C\|f\|_{L^{p(\cdot)}(\mathbb{R}^n)},$$

where

$$\frac{1}{q(z)} = \frac{1}{p(z)} - \frac{\zeta(z)}{n}$$

is the Sobolev exponent.

$$\lambda(z) = C\zeta(z)\left(1 - \frac{\zeta(z)}{n}\right) \leq \frac{n}{4}C,$$

with C is being the Dini-Lipschitz constant from the inequality (1.2.3) *in which $a(\cdot)$ replaced by $p(\cdot)$.*

Lemma 3.2.2 *If the indices appearing in the statement of Proposition 3.2.1 satisfy the condition given at there then the following inequality holds:*

$$\|(1+|z_1|)^{-\lambda(z_1)}|z_1|^{\zeta(z_1)}.\chi_{D_\ell}(z_1)\|_{L^q(\mathbb{R}^n)} \leq C\|\chi_{D_\ell}\|_{L^p(\mathbb{R}^n)}.$$

Proof: Differing slightly from the procedure given in [68], we have

$$\begin{aligned} I^{\zeta(z_1)}(\chi_{D_\ell})(z_1) \geq I^{\zeta(z_1)}(\chi_{D_\ell})(z_1).(\chi_{D_\ell})(z_1) &= \int_{D_\ell} \frac{1}{|z_1-z_2|^{\zeta(z_1)-n}} dy.\chi_{D_\ell}(z_1) \\ &\geq C|z_1|^{\zeta(z_1)}.\chi_{D_\ell}(z_1). \end{aligned}$$

Multiplying both sides with $(1+|z_1|)^{-\lambda(z_1)}$, the above inequality can be written as:

$$(1+|z_1|)^{-\lambda(z_1)}|z_1|^{\zeta(z_1)}.\chi_{D_\ell}(z_1) \leq C(1+|z_1|)^{-\lambda(z_1)} I^{\zeta(z_1)}(\chi_{D_\ell})(z_1).$$

Applying $L^q(\mathbb{R}^n)$ norm on both sides and using Proposition 3.2.1, we obtain

$$\begin{aligned} \|(1+|z_1|)^{-\lambda(z_1)}|z_1|^{\zeta(z_1)}.\chi_{D_\ell}\|_{L^q(\mathbb{R}^n)} &\leq C\|(1+|z_1|)^{-\lambda(z_1)} I^{\zeta(z_1)}(\chi_{D_\ell})\|_{L^q(\mathbb{R}^n)} \\ &\leq C\|\chi_{D_\ell}\|_{L^p(\mathbb{R}^n)}. \end{aligned}$$

Thus we finish the proof of Lemma 3.2.2.

Remark 3.2.3 *(i) If $\zeta(z)$ is satisfying the condition* (1.2.3)*:*

$$|\zeta(z)-\zeta_\infty| \leq \frac{C_\infty}{\ln(e+\|z|)}$$

for $x \in \mathbb{R}^n$. Then, $(1+|z|)^{-\lambda(z)}$ is equivalent to the weight $(1+|z|)^{-\lambda_\infty}$.
(ii) We can replace the variable order of Riesz potential operator $\zeta(x)$ by $\zeta(y)$ if we consider potentials over bounded domain, these potentials vary unessentially if the function $\zeta(x)$ is satisfying the logarithmic smoothness condition given in (1.2.2) *because:*

$$C_1|z_1-z_2|^{n-\zeta(z_2)}| \leq |z_1-z_2|^{n-\zeta(z_1)} \leq C_2|z_1-z_2|^{n-\zeta(z_2)}.$$

Lemma 3.2.4 *[19] Let $1 < A$ and $p \in \mathfrak{P}_{0,\infty}(\mathbb{R}^n)$. Then,*

$$\frac{1}{t_0} r^{\frac{n}{p(0)}} \leq \|\chi_{B(0,Ar)\setminus B(0,r)}\|_{p(\cdot)} \leq t_0 r^{\frac{n}{p(0)}} \tag{3.2.2}$$

for $0 < r \leq 1$ and

$$\frac{1}{t_\infty} r^{\frac{n}{p_\infty}} \leq \|\chi_{B(0,Ar)\setminus B(0,r)}\|_{p(\cdot)} \leq t_\infty r^{\frac{n}{p_\infty}} \tag{3.2.3}$$

for $r \geq 1$, where $1 \leq t_0$ and $1 \leq t_\infty$ depend on A but not on r.

Lemma 3.2.5 *[8] Consider a measurable subset D such that $D \subseteq \mathbb{R}^n$ and $p_-(D) \geq 1$, $\leq p_+(D) \leq \infty$. Then,*

$$\|gh\|_{L^{r(\cdot)}(D)} \leq \|g\|_{L^{p(\cdot)}(D)} \|h\|_{L^{q(\cdot)}(D)}$$

holds, where $g \in L^{p(\cdot)}(D)$, $h \in L^{q(\cdot)}(D)$ and

$$\frac{1}{r(t)} = \frac{1}{p(t)} + \frac{1}{q(t)}$$

for every $t \in H$.

3.3 Sobolev-type Theorem for Hardy-type Operators in Grand Herz Spaces

The main results of our chapter along with their proofs are given below.

Theorem 3.3.1 *Let $1 < u < \infty$,*

$$1/q_1(z_1) - 1/q_2(z_1) = \zeta(\cdot)/n,$$

$0 < \zeta(\cdot) < n$ and $a, q_2 \in \mathfrak{P}_{0,\infty}(\mathbb{R}^n)$, such that

$$\frac{-n}{q_{1\infty}} < a_\infty < \frac{n}{q'_{1\infty}}, \quad \frac{-n}{q_1(0)} < a(0) < \frac{n}{q'_1(0)},$$

where $q_{1\infty} = \lim_{x\to\infty} q_1(x)$ and $q'_{1\infty} = \lim_{x\to\infty} q'_1(x)$. Then,

$$\|(1+|z_1|)^{-\lambda(z_1)}\mathcal{H}(f)\|_{\dot{K}^{a(\cdot),u),\theta}_{q_2(\cdot)}(\mathbb{R}^n)} \leq C\|f\|_{\dot{K}^{a(\cdot),u),\theta}_{q_1(\cdot)}(\mathbb{R}^n)}.$$

Proof. Let $f \in \dot{K}^{a(\cdot),u),\theta}_{q_2(\cdot)}(\mathbb{R}^n)$ and

$$f(z_1) = \sum_{j=-\infty}^{\infty} f(z_1)\chi_j(z_1) = \sum_{j=-\infty}^{\infty} f_j(z_1),$$

we have

$$\begin{aligned}|\mathcal{H}(f)(z_1).\chi_\ell(z_1)| &\leq \frac{1}{|z_1|^{n-\zeta(z_1)}} \int_{D_\ell} |f(x)|dx.\chi_\ell(z_1) \\ &\leq C2^{-\ell n} \sum_{j=-\infty}^{\ell} \|f_j\|_{q_1(\cdot)} \|\chi_j\|_{q'_1(\cdot)}.|z_1|^{\zeta(z_1)}\chi_\ell(z_1).\end{aligned}$$

Consequently, we have

$$\begin{aligned}
&\|\chi_\ell(1+|z_1|)^{-\lambda(z_1)}\mathcal{H}(f)\|_{q_2(\cdot)}\\
&\leq C2^{-\ell n}\|f_j\|_{L^{q_1(\cdot)}}\|\chi_j\|_{q_1'(\cdot)}\|(1+|z_1|)^{-\lambda(z_1)}|z_1|^{\zeta(z_1)}\chi_\ell(z_1)\|_{q_2(\cdot)}\\
&\leq C2^{-\ell n}\sum_{j=-\infty}^{\ell}\|f_j\|_{q_1(\cdot)}\|\chi_j\|_{q_1'(\cdot)}\|(1+|z_1|)^{-\lambda(z_1)}|z_1|^{\zeta(z_1)}\chi_{D_\ell}(z_1)\|_{q_2(\cdot)}\\
&\leq C2^{-\ell n}\sum_{j=-\infty}^{\ell}\|f_j\|_{q_1(\cdot)}\|\chi_j\|_{q_1'(\cdot)}\|\chi_{D_\ell}\|_{q_1(\cdot)}\\
&\leq C\sum_{j=-\infty}^{\ell}2^{-\ell n}\|f_j\|_{q_1(\cdot)}\|\chi_j\|_{q_1'(\cdot)}\|\chi_{D_\ell}\|_{q_1(\cdot)},
\end{aligned}$$

where we used Lemma 3.2.2 in the second last step of the above inequality.

Next by definition of Herz-Morrey spaces, we have

$$\begin{aligned}
&\|(1+|z_1|)^{-\lambda(z_1)}\mathcal{H}(f)\|_{\dot{K}^{a(\cdot),v),\theta}_{q_2(\cdot)}(\mathbb{R}^n)}\\
&=\sup_{\phi>0}\left(\phi^\theta\sum_{\ell\in\mathbb{Z}}2^{\ell a(\cdot)v(1+\phi)}\|\chi_\ell(1+|z_1|)^{-\lambda(z_1)}\mathcal{H}(f)\|_{q_2(\cdot)}^{v(1+\phi)}\right)^{\frac{1}{v(1+\phi)}}\\
&\leq\sup_{\phi>0}\left(\phi^\theta\sum_{\ell\in\mathbb{Z}}2^{\ell a(\cdot)v(1+\phi)}\left(\sum_{j=-\infty}^{\ell}2^{-\ell n}\|f_j\|_{q_1(\cdot)}\|\chi_j\|_{q_1'(\cdot)}\|\chi_{D_\ell}\|_{q_1(\cdot)}\right)^{v(1+\phi)}\right)^{\frac{1}{v(1+\phi)}}\\
&\leq\sup_{\phi>0}\left(\phi^\theta\sum_{\ell=-\infty}^{-1}2^{\ell a(\cdot)v(1+\phi)}\left(\sum_{j=-\infty}^{\ell}2^{-\ell n}\|f_j\|_{q_1(\cdot)}\|\chi_j\|_{q_1'(\cdot)}\|\chi_{D_\ell}\|_{q_1(\cdot)}\right)^{v(1+\phi)}\right)^{\frac{1}{v(1+\phi)}}\\
&+\sup_{\phi>0}\left(\phi^\theta\sum_{\ell=0}^{\infty}2^{\ell a(\cdot)v(1+\phi)}\left(\sum_{j=-\infty}^{\ell}2^{-\ell n}\|f_j\|_{q_1(\cdot)}\|\chi_j\|_{q_1'(\cdot)}\|\chi_{D_\ell}\|_{q_1(\cdot)}\right)^{v(1+\phi)}\right)^{\frac{1}{v(1+\phi)}}\\
&=: E_1+E_2.
\end{aligned}$$

Now, we will find the estimate for E_1. By the Lemma (3.2.4)

$$2^{-\ell n}\|\chi_j\|_{q_1'(\cdot)}\|\chi_{D_\ell}\|_{q_1(\cdot)}\leq C2^{-\ell n}2^{\frac{kn}{q_1(0)}}2^{\frac{jn}{q_1'(0)}}\leq C2^{\frac{(j-\ell)n}{q_1'(0)}}. \tag{3.3.1}$$

Applying above results to E_1 to get

$$E_1 \leq \sup_{\phi>0}\left(\phi^{\theta}\sum_{\ell=-\infty}^{-1}2^{\ell a(\cdot)v(1+\phi)}\left(\sum_{j=-\infty}^{\ell}2^{-\ell n}\|f_j\|_{q_1(\cdot)}\|\chi_j\|_{q_1'(\cdot)}\|\chi_{D_\ell}\|_{q_1(\cdot)}\right)^{v(1+\phi)}\right)^{\frac{1}{v(1+\phi)}}$$

$$\leq C\sup_{\phi>0}\left(\phi^{\theta}\sum_{\ell=-\infty}^{-1}2^{\ell a(0)v(1+\phi)}\left(\sum_{j=-\infty}^{\ell}2^{\frac{(j-\ell)n}{q_1'(0)}}\|f_j\|_{q_1(\cdot)}\right)^{v(1+\phi)}\right)^{\frac{1}{v(1+\phi)}}.$$

Let $b := \frac{n}{q_1'(0)} - a(0)$. Applying the fact $2^{-v(1+\phi)} < 2^{-v}$, the Hölder's inequality and Fubini's theorem to get,

$$E_1 \leq C\sup_{\phi>0}\left(\phi^{\theta}\sum_{\ell=-\infty}^{-1}\left(\sum_{j=-\infty}^{\ell}2^{a(0)j}\|f_j\|_{q_1(\cdot)}2^{b(j-\ell)}\right)^{v(1+\phi)}\right)^{\frac{1}{v(1+\phi)}}$$

$$\leq C\sup_{\phi>0}\left(\phi^{\theta}\sum_{\ell=-\infty}^{-1}\left(\sum_{j=-\infty}^{\ell}2^{a(0)j}\|f_j\|_{q_1(\cdot)}2^{b(j-\ell)}\right)^{v(1+\phi)}\right)^{\frac{1}{v(1+\phi)}}$$

$$\leq C\sup_{\phi>0}\left[\phi^{\theta}\sum_{\ell=-\infty}^{-1}\left(\sum_{j=-\infty}^{\ell}2^{a(0)v(1+\phi)j}\|f_j\|_{q_1(\cdot)}^{v(1+\phi)}2^{bu(1+\phi)(j-\ell)/2}\right)\right.$$

$$\left.\times\left(\sum_{j=-\infty}^{\ell}2^{b(v(1+\phi))'(j-\ell)/2}\right)^{\frac{v(1+\phi)}{(v(1+\phi))'}}\right]^{\frac{1}{v(1+\phi)}}$$

$$\leq C\sup_{\phi>0}\left[\phi^{\theta}\sum_{\ell=-\infty}^{-1}\sum_{j=-\infty}^{\ell}2^{a(0)v(1+\phi)j}\|f_j\|_{q_1(\cdot)}^{v(1+\phi)}2^{bu(1+\phi)(j-\ell)/2}\right]^{\frac{1}{v(1+\phi)}}$$

$$\leq C\sup_{\phi>0}\left[\phi^{\theta}\sum_{j=-\infty}^{-1}2^{a(\cdot)v(1+\phi)j}\|f_j\|_{q_1(\cdot)}^{v(1+\phi)}\sum_{\ell=j}^{-1}2^{bu(1+\phi)(j-\ell)/2}\right]^{\frac{1}{v(1+\phi)}}$$

$$\leq C\sup_{\phi>0}\left(\phi^{\theta}\sum_{j=-\infty}^{-1}2^{a(0)v(1+\phi)j}\|f_j\|_{q_1(\cdot)}^{v(1+\phi)}\sum_{\ell=j}^{-1}2^{bu(1+\phi)(j-\ell)/2}\right)^{\frac{1}{v(1+\phi)}}$$

$$\leq C\sup_{\phi>0}\left(\phi^{\theta}\sum_{l=-\infty}^{-1}2^{a(0)v(1+\phi)j}\|f_j\|_{q_1(\cdot)}^{v(1+\phi)}\right)^{\frac{1}{v(1+\phi)}}$$

$$= C\sup_{\phi>0}\left(\phi^{\theta}\sum_{j\in\mathbb{Z}}2^{a(\cdot)v(1+\phi)j}\|f_j\|_{q_1(\cdot)}^{v(1+\phi)}\right)^{\frac{1}{v(1+\phi)}}$$

$$\leq C\|f\|_{\dot{K}_{q_1(\cdot)}^{a(\cdot),v),\theta}(\mathbb{R}^n)}.$$

Now for E_2, using Minkowski's inequality we have

$$
\begin{aligned}
E_2 &\le \sup_{\phi>0}\left(\phi^{\theta}\sum_{\ell=0}^{\infty}2^{\ell a(\cdot)v(1+\phi)}\left(\sum_{j=-\infty}^{\ell}\|\chi_\ell(1+|z_1|)^{-\lambda(z_1)}\mathcal{H}(f_j)\|_{q_2(\cdot)}\right)^{v(1+\phi)}\right)^{\frac{1}{v(1+\phi)}}\\
&\le \sup_{\phi>0}\left(\phi^{\theta}\sum_{\ell=0}^{\infty}2^{\ell a(\cdot)v(1+\phi)}\left(\sum_{j=-\infty}^{-1}\|\chi_\ell(1+|z_1|)^{-\lambda(z_1)}\mathcal{H}(f_j)\|_{q_2(\cdot)}\right)^{v(1+\phi)}\right)^{\frac{1}{v(1+\phi)}}\\
&+ \sup_{\phi>0}\left(\phi^{\theta}\sum_{\ell=0}^{\infty}2^{\ell a(\cdot)v(1+\phi)}\left(\sum_{j=0}^{\ell}\|\chi_\ell(1+|z_1|)^{-\lambda(z_1)}\mathcal{H}(f_j)\|_{q_2(\cdot)}\right)^{v(1+\phi)}\right)^{\frac{1}{v(1+\phi)}}\\
&:= A_1 + A_2.
\end{aligned}
$$

We can easily find the approximation for A_2 in a way similar to E_1. We will replace $q_1'(0)$ with $q_{1\infty}'$ and by virtue of the fact $b := \frac{n}{q_{1\infty}'} - a_\infty > 0$ to get our desired results. For A_1, we have

$$
2^{-\ell n}\|\chi_{D_\ell}\|_{q_1(\cdot)}\|\chi_j\|_{q_1'(\cdot)} \le C2^{-\ell n}2^{\frac{\ell n}{q_{1\infty}}}2^{\frac{jn}{q_1'(0)}} \le C2^{\frac{-\ell n}{q_{1\infty}}}2^{\frac{jn}{q_1'(0)}}. \tag{3.3.2}
$$

As $a_\infty - \frac{n}{q_{1\infty}'} < 0$ we have

$$
\begin{aligned}
A_1 &\le C\sup_{\phi>0}\left(\phi^{\theta}\sum_{\ell=0}^{\infty}2^{\ell a_\infty v(1+\phi)}\left(\sum_{j=-\infty}^{-1}\|\chi_\ell(1+|z_1|)^{-\lambda(z_1)}\mathcal{H}(f_j)\|_{q_2(\cdot)}\right)^{v(1+\phi)}\right)^{\frac{1}{v(1+\phi)}}\\
&\le C\sup_{\phi>0}\left[\phi^{\theta}\sum_{\ell=0}^{\infty}2^{\ell a_\infty v(1+\phi)}\left(\sum_{j=-\infty}^{-1}2^{-\ell n}2^{\frac{\ell n}{q_{1\infty}}}2^{\frac{jn}{q_1'(0)}}\|f_j\|_{q_1(\cdot)}\right)^{v(1+\phi)}\right]^{\frac{1}{v(1+\phi)}}\\
&\le C\sup_{\phi>0}\left[\phi^{\theta}\sum_{\ell=0}^{\infty}2^{\frac{\ell a_\infty-\ell n}{q_{1\infty}'}v(1+\phi)}\left(\sum_{j=-\infty}^{-1}2^{\frac{jn}{q_1'(0)}}\|f_j\|_{q_1(\cdot)}\right)^{v(1+\phi)}\right]^{\frac{1}{v(1+\phi)}}\\
&\le C\sup_{\phi>0}\left(\phi^{\theta}\left(\sum_{j=-\infty}^{-1}2^{\frac{jn}{q_1'(0)}}\|f_j\|_{q_1(\cdot)}\right)^{v(1+\phi)}\right)^{\frac{1}{v(1+\phi)}}\\
&\le C\sup_{\phi>0}\left(\phi^{\theta}\left(\sum_{j=-\infty}^{-1}2^{a(0)j}\|f_j\|_{q_1(\cdot)}2^{\frac{jn}{q_1'(0)}-a(0)j}\right)^{v(1+\phi)}\right)^{\frac{1}{v(1+\phi)}}.
\end{aligned}
$$

Now, by using the condition that $a(0) < \frac{n}{q_1'(0)}$ and Hölder's inequality we have

$$
\begin{aligned}
A_1 &\leq C \sup_{\phi>0} \left[\phi^{\theta} \left(\sum_{j=-\infty}^{-1} 2^{a(0)jv(1+\phi)} \|f_j\|_{q_1(\cdot)}^{v(1+\phi)} \right) \right. \\
&\quad \times \left. \left(\sum_{j=-\infty}^{-1} 2^{(\frac{jn}{q_1'(0)} - a(0)j)(v(1+\phi))'} \right)^{\frac{v(1+\phi)}{(v(1+\phi))'}} \right]^{\frac{1}{v(1+\phi)}} \\
&\leq C \sup_{\phi>0} \left(\phi^{\theta} \left(\sum_{j\in\mathbb{Z}} 2^{a(0)jv(1+\phi)} \|f_j\|_{q_1(\cdot)}^{v(1+\phi)} \right) \right)^{\frac{1}{v(1+\phi)}} \\
&\leq C \|f\|_{\dot{K}_{q_1(\cdot)}^{a(\cdot),v),\theta}(\mathbb{R}^n)}.
\end{aligned}
$$

Combining these estimates we get

$$
\|(1+|z_1|)^{-\lambda(z_1)} \mathcal{H}(f)\|_{\dot{K}_{q_2(\cdot)}^{a(\cdot),v),\theta}(\mathbb{R}^n)} \leq C \|f\|_{\dot{K}_{q_1(\cdot)}^{a(\cdot),v),\theta}(\mathbb{R}^n)}.
$$

Theorem 3.3.2 *Let* $1 < v < \infty$,

$$1/q_1(z_1) - 1/q_2(z_1) = \zeta(\cdot)/n,$$

$0 < \zeta(\cdot) < n$, *and* $a, q_2 \in \mathfrak{P}_{0,\infty}(\mathbb{R}^n)$ *such that*

$$\frac{-n}{q_{2\infty}} < a_\infty < \frac{n}{q'_{2\infty}}, \quad \frac{-n}{q_2(0)} < a(0) < \frac{n}{q'_2(0)}.$$

Then,

$$\|(1+|z_1|)^{-\lambda(z_1)}\mathcal{H}^*(f)\|_{\dot{K}^{a(\cdot),v),\theta}_{q_2(\cdot)}(\mathbb{R}^n)} \leq C\|f\|_{\dot{K}^{a(\cdot),v),\theta}_{q_1(\cdot)}(\mathbb{R}^n)}.$$

Proof. Let $f \in \dot{K}^{a(\cdot),u),\theta}_{q_2(\cdot)}(\mathbb{R}^n)$ and

$$f(z_1) = \sum_{j=-\infty}^{\infty} f(z_1)\chi_j(z_1) = \sum_{j=-\infty}^{\infty} f_j(z_1).$$

We have

$$\begin{aligned}
&|(1+|z_1|)^{-\lambda(z_1)}\mathcal{H}^*(f)(z_1).\chi_\ell(z_1)| \\
&\leq \int_{\mathbb{R}^n\setminus\ell} \frac{1}{|z_1|^{n-\zeta(z_1)}}|f(x)|dx.(1+|z_1|)^{-\lambda(z_1)}\chi_\ell(z_1) \\
&\leq C\sum_{j=\ell+1}^{\infty} \|f_j\|_{q_1(\cdot)}\|(1+|z_1|)^{-\lambda(z_1)}|\cdot|^{\zeta(z_1)-n}\chi_j\|_{q'_1(\cdot)}\chi_\ell(z_1).
\end{aligned}$$

It is known, see e.g. [56] that

$$\begin{aligned}
I^{\zeta(\cdot)}(\chi_{D_j})(z_1) &\geq I^{\zeta(\cdot)}(\chi_{D_j})(z_1).(\chi_{D_j})(z_1) \\
&= \int_{D_j} \frac{1}{|z_1 - z_2|^{\zeta(z_1)-n}}dy.\chi_{D_j}(z_1) \\
&\geq C|z_1|^{\zeta(z_1)}.\chi_{D_j}(z_1) \\
&\geq C|z_1|^{\zeta(z_1)}.\chi_j(z_1).
\end{aligned}$$

Consequently, we have

$$\begin{aligned}
&\|\chi_\ell(1+|z_1|)^{-\lambda(z_1)}\mathcal{H}^*(f)\|_{q_2(\cdot)} \\
&\leq C\sum_{j=\ell+1}^{\infty} \|f_j\|_{q_1(\cdot)}\|(1+|z_1|)^{-\lambda(z_1)}|\cdot|^{\zeta(z_1)-n}\chi_j\|_{L^{q'_1(\cdot)}(\tau)}\|\chi_\ell\|_{q_2(\cdot)} \\
&\leq C\sum_{j=\ell+1}^{\infty} 2^{-jn}\|f_j\|_{q_1(\cdot)}\|(1+|z_1|)^{-\lambda(z_1)}|z_1|^{\zeta(z_1)}\chi_j\|_{q'_1(\cdot)}\|\chi_\ell\|_{q_2(\cdot)} \\
&\leq C2^{-jn}\sum_{j=\ell+1}^{\infty} \|f_j\|_{L^{q_1(\cdot)}}\|\chi_j\|_{q'_2(\cdot)}\|\chi_\ell\|_{q_2(\cdot)}.
\end{aligned}$$

$$
\begin{aligned}
&\|(1+|z_1|)^{-\lambda(z_1)}\mathcal{H}^*(f)\|_{\dot{K}^{a(\cdot),v),\theta}_{q_2(\cdot)}(\mathbb{R}^n)} \\
&= \sup_{\phi>0}\left(\phi^\theta \sum_{\ell\in\mathbb{Z}} 2^{\ell a(\cdot)v(1+\phi)}\|\chi_\ell(1+|z_1|)^{-\lambda(z_1)}\mathcal{H}^*(f)\|_{q_2(\cdot)}^{v(1+\phi)}\right)^{\frac{1}{v(1+\phi)}} \\
&\le \sup_{\phi>0}\left(\phi^\theta \sum_{\ell\in\mathbb{Z}} 2^{\ell a(\cdot)v(1+\phi)}\left(\sum_{j=\ell+1}^{\infty} 2^{-jn}\|f_j\|_{q_1(\cdot)}\|\chi_j\|_{q_2'(\cdot)}\|\chi_{D_\ell}\|_{q_2(\cdot)}\right)^{v(1+\phi)}\right)^{\frac{1}{v(1+\phi)}} \\
&\le \sup_{\phi>0}\left(\phi^\theta \sum_{\ell=-\infty}^{-1} 2^{\ell a(\cdot)v(1+\phi)}\left(\sum_{j=\ell+1}^{\infty} 2^{-jn}\|f_j\|_{q_1(\cdot)}\|\chi_j\|_{q_2'(\cdot)}\|\chi_{D_\ell}\|_{q_2(\cdot)}\right)^{v(1+\phi)}\right)^{\frac{1}{v(1+\phi)}} \\
&+ \sup_{\phi>0}\left(\phi^\theta \sum_{\ell=0}^{\infty} 2^{\ell a(\cdot)v(1+\phi)}\left(\sum_{j=\ell+1}^{\infty} 2^{-jn}\|f_j\|_{q_1(\cdot)}\|\chi_j\|_{q_2'(\cdot)}\|\chi_{D_\ell}\|_{q_2(\cdot)}\right)^{v(1+\phi)}\right)^{\frac{1}{v(1+\phi)}} \\
&=: E_1 + E_2.
\end{aligned}
$$

We will first estimate E_2. For this we have

$$
2^{-jn}\|\chi_\ell\|_{q_2(\cdot)}\|\chi_j\|_{q_2'(\cdot)} \le C2^{-jn}2^{\frac{\ell n}{q_{2\infty}}}2^{\frac{jn}{q_{2\infty}'}} \le C2^{\frac{(\ell-j)n}{q_{2\infty}}}. \tag{3.3.3}
$$

$$
\begin{aligned}
E_2 &\le C\sup_{\phi>0}\left(\phi^\theta \sum_{\ell=0}^{\infty} 2^{\ell a(\cdot)v(1+\phi)}\left(\sum_{j=\ell+1}^{\infty} \|\chi_\ell(1+|z_1|)^{-\lambda(z_1)}\mathcal{H}^*(f_j)\|_{q_2(\cdot)}\right)^{v(1+\phi)}\right)^{\frac{1}{v(1+\phi)}} \\
&\le C\sup_{\phi>0}\left(\phi^\theta \sum_{\ell=0}^{\infty}\left(\sum_{l=\ell+1}^{\infty} 2^{a_\infty j}\|f_j\|_{q_1(\cdot)}2^{d(\ell-j)}\right)^{v(1+\phi)}\right)^{\frac{1}{v(1+\phi)}},
\end{aligned}
$$

where

$$
d = \frac{n}{q_{2\infty}} + a_\infty > 0.
$$

Then, by virtue of the well known Hölder's theorem for series and $2^{-v(1+\phi)} < 2^{-v}$ yields

$$E_2 \leq C \sup_{\phi>0} \left[\phi^\theta \sum_{\ell=0}^{\infty} \left(\sum_{j=\ell+1}^{\infty} 2^{a_\infty v(1+\phi)j} \|f_j\|_{q_1(\cdot)}^{v(1+\phi)} 2^{du(1+\phi)(\ell-j)/2} \right) \right.$$
$$\left. \times \left(\sum_{j=\ell+1}^{\infty} 2^{d(v(1+\phi))'(\ell-j)/2} \right)^{\frac{v(1+\phi)}{(v(1+\phi))'}} \right]^{\frac{1}{v(1+\phi)}}$$
$$\leq C \sup_{\phi>0} \left[\phi^\theta \sum_{\ell=0}^{\infty} \sum_{j=\ell+1}^{\infty} 2^{a_\infty v(1+\phi)j} \|f_j\|_{q_1(\cdot)}^{v(1+\phi)} 2^{du(1+\phi)(\ell-j)/2} \right]^{\frac{1}{v(1+\phi)}}$$
$$\leq C \sup_{\phi>0} \left(\phi^\theta \sum_{j=0}^{\infty} 2^{a_\infty v(1+\phi)j} \|f_j\|_{q_1(\cdot)}^{v(1+\phi)} \sum_{\ell=0}^{j-1} 2^{du(1+\phi)(\ell-j)/2} \right)^{\frac{1}{v(1+\phi)}}$$
$$< C \sup_{\phi>0} \left(\phi^\theta \sum_{j\in\mathbb{Z}} 2^{a_\infty v(1+\phi)j} \|f_j\|_{q_1(\cdot)}^{v(1+\phi)} \sum_{\ell=-\infty}^{j-1} 2^{du(1+\phi)(\ell-j)/2} \right)^{\frac{1}{v(1+\phi)}}$$
$$= C \sup_{\phi>0} \left(\phi^\theta \sum_{j\in\mathbb{Z}} 2^{a(\cdot)v(1+\phi)j} \|f_j\|_{q_1(\cdot)}^{v(1+\phi)} \right)^{\frac{1}{v(1+\phi)}}$$
$$\leq C\|f\|_{\dot{K}_{q_1(\cdot)}^{a(\cdot),v),\theta}(\mathbb{R}^n)}.$$

For E_1, by using Minkowski's inequality

$$E_1 \leq \sup_{\phi>0} \left(\phi^\theta \sum_{\ell=-\infty}^{-1} 2^{\ell a(\cdot)v(1+\phi)} \left(\sum_{j=\ell+1}^{\infty} \|\chi_\ell (1+|z_1|)^{-\lambda(z_1)} \mathcal{H}^*(f_j)\|_{q_2(\cdot)} \right)^{v(1+\phi)} \right)^{\frac{1}{v(1+\phi)}}$$
$$\leq \sup_{\phi>0} \left(\phi^\theta \sum_{\ell=-\infty}^{-1} 2^{\ell a(\cdot)v(1+\phi)} \left(\sum_{j=\ell+1}^{-1} \|\chi_\ell (1+|z_1|)^{-\lambda(z_1)} \mathcal{H}^*(f_j)\|_{q_2(\cdot)} \right)^{v(1+\phi)} \right)^{\frac{1}{v(1+\phi)}}$$
$$+ \sup_{\phi>0} \left(\phi^\theta \sum_{\ell=-\infty}^{-1} 2^{\ell a(\cdot)v(1+\phi)} \left(\sum_{j=0}^{\infty} \|\chi_\ell (1+|z_1|)^{-\lambda(z_1)} \mathcal{H}^*(f_j)\|_{q_2(\cdot)} \right)^{v(1+\phi)} \right)^{\frac{1}{v(1+\phi)}}$$
$$:= D_1 + D_2.$$

The estimate of D_1 is attained in a way similar to E_2 by replacing $q_{2\infty}$ with $q_2(0)$ and using the inequality

$$\frac{n}{q_2(0)} + a(0) > 0$$

and

$$0 < \frac{n}{q_{2\infty}} + a_\infty.$$

For D_2 we have

$$2^{-jn}\|\chi_{D_\ell}\|_{q_2(\cdot)}\|\chi_j\|_{q_2'(\cdot)} \leq C2^{-jn}2^{\frac{\ell n}{q_2(0)}}2^{\frac{jn}{q'2\infty}} \leq C2^{\frac{\ell n}{q_2(0)}}2^{\frac{-jn}{q_{2\infty}}}, \tag{3.3.4}$$

$$
\begin{aligned}
D_2 &\le C \sup_{\phi>0} \left(\phi^\theta \sum_{\ell=-\infty}^{-1} 2^{\ell a(0) v(1+\phi)} \left(\sum_{j=0}^{\infty} \|\chi_\ell (1+|z_1|)^{-\lambda(z_1)} \mathcal{H}^*(f_j)\|_{q_2(\cdot)} \right)^{v(1+\phi)} \right)^{\frac{1}{v(1+\phi)}} \\
&\le C \sup_{\phi>0} \left(\phi^\theta \sum_{\ell=-\infty}^{-1} 2^{\ell a(0) v(1+\phi)} \left(\sum_{j=0}^{\infty} 2^{-jn} 2^{\frac{\ell n}{q_2(0)}} 2^{\frac{jn}{q'_{2\infty}}} \|f_j\|_{q_1(\cdot)} \right)^{v(1+\phi)} \right)^{\frac{1}{v(1+\phi)}} \\
&\le C \sup_{\phi>0} \left(\phi^\theta \sum_{\ell=-\infty}^{-1} 2^{\ell a(0) v(1+\phi)} \left(\sum_{j=0}^{\infty} 2^{\frac{\ell n}{q_2(0)}} 2^{\frac{-jn}{q_{2\infty}}} \|f_j\|_{q_1(\cdot)} \right)^{v(1+\phi)} \right)^{\frac{1}{v(1+\phi)}} \\
&\le C \sup_{\phi>0} \left(\phi^\theta \sum_{\ell=-\infty}^{-1} 2^{\ell(a(0)+n)/q_2(0) v(1+\phi)} \left(\sum_{j=0}^{\infty} 2^{\frac{-jn}{q_{2\infty}}} \|f_j\|_{q_1(\cdot)} \right)^{v(1+\phi)} \right)^{\frac{1}{v(1+\phi)}} \\
&\le C \sup_{\phi>0} \left(\phi^\theta \left(\sum_{j=0}^{\infty} 2^{a_\infty j} \|f_j\|_{q_1(\cdot)} 2^{j(nq_{2\infty}+a_\infty)} \right)^{v(1+\phi)} \right)^{\frac{1}{v(1+\phi)}} \\
&\le C \sup_{\phi>0} \left(\phi^\theta \left(\sum_{j=0}^{\infty} 2^{a_\infty j u(1+\phi)} \|f_j\|_{q_1(\cdot)}^{v(1+\phi)} \right)^{v(1+\phi)} \left(\sum_{j=0}^{\infty} 2^{j(nq_{2\infty}+a_\infty) v(1+\phi)} \right)^{\frac{v(1+\phi)}{(v(1+\phi))'}} \right)^{\frac{1}{v(1+\phi)}} \\
&\le C \sup_{\phi>0} \left(\phi^\theta \left(\sum_{j\in\mathbb{Z}} 2^{a(\cdot) j v(1+\phi)} \|f_j\|_{q_1(\cdot)}^{v(1+\phi)} \right) \right)^{\frac{1}{v(1+\phi)}} \\
&\le C \|f\|_{\dot{K}_{q_1(\cdot)}^{a(\cdot),v),\theta}(\mathbb{R}^n)}.
\end{aligned}
$$

The estimates for E_1 and E_2 in combined form yields

$$
\|(1+|z_1|)^{-\lambda(z_1)} \mathcal{H}^*(f)\|_{\dot{K}_{q_2(\cdot)}^{a(\cdot),v),\theta}(\mathbb{R}^n)} \le C \|f\|_{\dot{K}_{q_1(\cdot)}^{a(\cdot),v),\theta}(\mathbb{R}^n)}.
$$

3.4 Conclusions

Our primary goal of determining the sufficient conditions for ensuring the boundedness of fractional Hardy operators in Grad Herz spaces with variable exponents has been successfully accomplished. We investigated the interplay between the variable order and exponent, exploring how different combinations of these parameters affect the boundedness properties of the operators.

Chapter 4
p-adic Hardy-type Operator and Commutators on Variable p-adic Lebesgue space

4.1 Introduction

An essential component of harmonic analysis is to discuss the characterization of function spaces and regularity theory of partial differential equations, both of which make use of the boundedness properties of commutator operators. The definition and discussion of the commutators of the p-adic Hardy-type operators H^p and H^p_α with the locally integrable function b can be found in the citations [46, 47], and [69]. With regard to the boundedness of H^p and H^p_α and their commutators on variable exponent p-adic function spaces, there exists a clear gap in the literature which we want to fill here in the remaining of this thesis.

In this Chapter, we investigate the boundedness of fractional p-adic Hardy operators and commutators on the variable exponent lebesgue spaces. For this purpose, we use the concept of the Sobolev limitting exponent given in [4,5] to prove our boundedness results.

The next Section contains some fundamental lemmas to be used in this chapter and subsequent chapters.

4.2 Preliminaries

Lemma 4.2.1 *[5] Assume that $u \in \aleph(\mathbb{Q}_p^n)$ is an L-Lipschitz function for a value of $L \geq 0$, then $u \in W_0(\mathbb{Q}_p^n)$.*

Lemma 4.2.2 *[5] Suppose $u \in W_0(\Omega_p^n)$, where $\Omega_p^n \in \mathbb{Q}_p^n$ is a bounded set, then there arise an extension function $\tilde{u} \in W_0^\infty(\mathbb{Q}_p^n)$ which is constant outside of some fixed ball.*

Lemma 4.2.3 *[5] Let $u(\cdot) \in W_0^\infty(\mathbb{Q}_p^n)$. Then,*

$$\|\chi_{B_k}\|_{L^{u(\cdot)}(\mathbb{Q}_p^n)} \le Cp^{kn/u(x,k)},$$

where

$$u(x,k) =: \begin{cases} u(x), & k < 0, \\ u(\infty), & k \ge 0. \end{cases}$$

Lemma 2.2 in [48] is extended to the p-adic variable exponent central BMO space in the following Lemma.

Lemma 4.2.4 Let $g \in C^{u(\cdot)}$ and $m, l \in \mathbb{Z}$, then

$$|g(x) - g_{B_m}| \le |g(x) - g_{B_l}| + p^n |l - m| \|g\|_{C^{u(\cdot)}}. \tag{4.2.1}$$

Proof. Let $i \in \mathbb{Z}$, then using inequality (1.2.25) we have

$$\begin{aligned} |g_{B_i} - g_{B_{i+1}}| &\le \frac{1}{|B_i|} \int_{B_i} |g(y) - g_{B_{i+1}}| dy \\ &\le p^n \frac{1}{|B_{i+1}|} \|(g - g_{B_{i+1}})\chi_{B_{i+1}}\|_{L^{u(\cdot)}} \|\chi_{B_{i+1}}\|_{L^{u'(\cdot)}} \\ &\le p^n \|g\|_{C^{u(\cdot)}}, \end{aligned}$$

where in the last inequality, we made use of Lemma 4.2.3 to obtain the desired outcome. Next, if $m < l$, then

$$|g(x) - g_{B_m}| \le |g(x) - g_{B_l}| + \sum_{i=m}^{l-1} |g_{B_i} - g_{B_{i+1}}| \le |g(x) - g_{B_l}| + p^n (l - m) \|g\|_{C^{u(\cdot)}}. \tag{4.2.2}$$

Similarly, if $l < m$, then

$$|g(x) - g_{B_m}| \le |g(x) - g_{B_l}| + \sum_{i=l}^{m-1} |g_{B_i} - g_{B_{i+1}}| \le |g(x) - g_{B_l}| + p^n (m - l) \|g\|_{C^{u(\cdot)}}. \tag{4.2.3}$$

The inequalities (4.2.2) and (4.2.3) yield the inequality (4.2.1).

4.3 p-adic Hardy Operators on Variable Exponent p-adic Lebesgue Spaces

Theorem 4.3.1 *Let* $0 < \alpha < \min\{\frac{n}{u_+}, \frac{n}{v'_+}\}$. *Suppose, moreover, that* $u \in \aleph(\mathbb{Q}_p^n)$, *where* v *is the Sobolev limiting exponent define as*

$$\frac{1}{v(x)} = \frac{1}{u(x)} - \frac{\alpha}{n}. \tag{4.3.1}$$

Then, the operator $H_\alpha^p, H_\alpha^{p,*} : L^{u(\cdot)}(\mathbb{Q}_p^n) \longrightarrow L^{v(\cdot)}(\mathbb{Q}_p^n)$ *and* $L^{v'(\cdot)}(\mathbb{Q}_p^n) \longrightarrow L^{u'(\cdot)}(\mathbb{Q}_p^n)$ *are bounded.*

The following corollary results from the above theorem if $\alpha = 0$:

Corollary 4.3.2 Let $u \in \aleph(\mathbb{Q}_p^n)$, then both H^p and $H^{p,*}$ map $L^{u(\cdot)}(\mathbb{Q}_p^n)$ into $L^{u(\cdot)}(\mathbb{Q}_p^n)$ and $L^{u'(\cdot)}(\mathbb{Q}_p^n)$ into $L^{u'(\cdot)}(\mathbb{Q}_p^n)$.

Proof of Theorem 4.3.1 Repeated use of Holder's inequality gives

$$\begin{aligned}
\|H_\alpha^p f\|_{L^{v(\cdot)}(\mathbb{Q}_p^n)} &= \sum_{k=-\infty}^{\infty} \|\chi_k H_\alpha^p(f)\|_{L^{v(\cdot)}(\mathbb{Q}_p^n)}, \\
&= \sum_{k=-\infty}^{\infty} \left\| \frac{\chi_k(\cdot)}{|\cdot|_p^{n-\alpha}} \int_{B(0,|\cdot|_p)} |f(t)|dt \right\|_{L^{v(\cdot)}(\mathbb{Q}_p^n)}, \\
&\lesssim \sum_{k=-\infty}^{\infty} \left\| \chi_k(\cdot) \frac{1}{|\cdot|_p^{n-\alpha}} \sum_{j=-\infty}^{k} \int_{S_j} |f(t)|dt \right\|_{L^{v(\cdot)}(\mathbb{Q}_p^n)}.
\end{aligned}$$

Inequality (1.2.25) implies

$$\int_{S_j} |f(t)|dt \lesssim \|f_j\|_{L^{u(\cdot)}(\mathbb{Q}_p^n)} \|\chi_j\|_{L^{u'(\cdot)}(\mathbb{Q}_p^n)}.$$

As a result of Lemma 4.2.3 and equality (4.3.1)

$$
\begin{aligned}
\|H_\alpha^p f\|_{L^{v(\cdot)}(\mathbb{Q}_p^n)} &\lesssim \sum_{k=-\infty}^{\infty}\sum_{j=-\infty}^{k} p^{-k(n-\alpha)}\|f_j\|_{L^{u(\cdot)}(\mathbb{Q}_p^n)}\|\chi_j\|_{L^{u'(\cdot)}(\mathbb{Q}_p^n)}\|\chi_k\|_{L^{v(\cdot)}(\mathbb{Q}_p^n)},\\
&\lesssim \sum_{k=-\infty}^{\infty}\sum_{j=-\infty}^{k} \|f_j\|_{L^{u(\cdot)}(\mathbb{Q}_p^n)}\|\chi_{B_j}\|_{L^{u'(\cdot)}(\mathbb{Q}_p^n)}\|\chi_{B_k}\|^{-1}_{L^{u'(\cdot)}(\mathbb{Q}_p^n)},\\
&\lesssim \sum_{k=-\infty}^{\infty}\sum_{j=-\infty}^{k} p^{(j-k)n/u'(\cdot)}\|f_j\|_{L^{u(\cdot)}(\mathbb{Q}_p^n)},\\
&\lesssim \sum_{j=-\infty}^{\infty}\|f_j\|_{L^{u(\cdot)}(\mathbb{Q}_p^n)}\sum_{k=j}^{\infty} p^{(j-k)n/u'(\cdot)},\\
&\lesssim \|f\|_{L^{u(\cdot)}(\mathbb{Q}_p^n)}.
\end{aligned}
$$

Likewise, the following inequalities are obvious;

$$
\begin{aligned}
\|H_\alpha^{p,*} f\|_{L^{v(\cdot)}(\mathbb{Q}_p^n)} &\lesssim \|f\|_{L^{u(\cdot)}(\mathbb{Q}_p^n)},\\
\|H_\alpha^{p} f\|_{L^{u'(\cdot)}(\mathbb{Q}_p^n)} &\lesssim \|f\|_{L^{v'(\cdot)}(\mathbb{Q}_p^n)},\\
\|H_\alpha^{p,*} f\|_{L^{u'(\cdot)}(\mathbb{Q}_p^n)} &\lesssim \|f\|_{L^{v'(\cdot)}(\mathbb{Q}_p^n)}.
\end{aligned}
$$

This conclusion completes Theorem 4.3.1.

4.4 Necessary and Sufficient condition for the Boundedness of Commutator of H_α^p

This section presents the boundedness analysis of operators defined in (1.3.6) and (1.3.7) on p-adic variable Lebesgue spaces.

Theorem 4.4.1 *Let $0 < \alpha < \min\{\frac{n}{u_+}, \frac{n}{v'_+}\}$. Suppose, moreover, that $u \in \aleph(\mathbb{Q}_p^n)$, where v is the Sobolev limiting exponent (4.3.1). The following are equivalent statements:*

1) $b \in C^{v(\cdot)} \cap C^{u'(\cdot)}$.
2) $H_{\alpha,b}^p$ and $H_{\alpha,b}^{p,*}$ from $L^{u(\cdot)}(\mathbb{Q}_p^n)$ to $L^{v(\cdot)}(\mathbb{Q}_p^n)$ and $L^{v'(\cdot)}(\mathbb{Q}_p^n)$ to $L^{u'(\cdot)}(\mathbb{Q}_p^n)$.

If $\alpha = 0$ in the preceding theorem, then the following corollary holds:

Corollary 4.4.2 Let $u \in \aleph(\mathbb{Q}_p^n)$, then the subsequent claims are equivalent:

(1) $b \in C^{u(\cdot)} \cap C^{u'(\cdot)}$.
(2) Both H_b^p and $H_b^{p,*}$ are bounded on $L^{u(\cdot)}(\mathbb{Q}_p^n)$ and $L^{u'(\cdot)}(\mathbb{Q}_p^n)$.

Proof of Theorem 4.4.1.

1) $\Rightarrow$ 2) We focus on the evidence of the boundedness of $H_{\alpha,b}^p$, because the arguments of $H_{\alpha,b}^{p,*}$ are comparable with the essential adjustments. We begin

$$\begin{aligned}
\|H_{\alpha,b}^p f\|_{L^{v(\cdot)}(\mathbb{Q}_p^n)} &= \sum_{k=-\infty}^{\infty} \|\chi_k H_{\alpha,b}^p(f)\|_{L^{v(\cdot)}(\mathbb{Q}_p^n)} \\
&= \sum_{k=-\infty}^{\infty} \left\| \chi_k(\cdot) \left(\frac{1}{|\cdot|_p^{n-\alpha}} \int_{B(0,|\cdot|_p)} (f(t)(b(\cdot)-b(t)))dt \right) \right\|_{L^{v(\cdot)}(\mathbb{Q}_p^n)} \\
&\lesssim \sum_{k=-\infty}^{\infty} \left\| \chi_k(\cdot) \frac{1}{|\cdot|_p^{n-\alpha}} \sum_{j=-\infty}^{k} \int_{S_j} |b(\cdot)-b(t)||f(t)|dt \right\|_{L^{v(\cdot)}(\mathbb{Q}_p^n)}.
\end{aligned}$$

It is simple to see

$$\int_{S_j} |b(x)-b(t)||f(t)|dt \lesssim \int_{S_j} |b(x)-b_{B_k}||f(t)|dt + \int_{S_j} |b_{B_k}-b(t)||f(t)|dt. \quad (4.4.1)$$

Inequality (1.2.25) implies

$$\int_{S_j} |b(x)-b_{B_k}||f(t)|dt \lesssim |b(x)-b_{B_k}|\|f_j\|_{L^{u(\cdot)}(\mathbb{Q}_p^n)}\|\chi_j\|_{L^{u'(\cdot)}(\mathbb{Q}_p^n)}. \quad (4.4.2)$$

By Lemma 4.2.4, we have

$$\begin{aligned}
\int_{S_j} |b(t)-b_{B_k}||f(t)|dt &\le \int_{B_j} |b(t)-b_{B_j}||f(t)|dt + C(k-j)\|b\|_{C^{u'(\cdot)}} \int_{B_j} |f(t)|dt \\
&\lesssim \|(b-b_{B_j})\chi_j\|_{L^{u'(\cdot)}(\mathbb{Q}_p^n)}\|f_j\|_{L^{u(\cdot)}(\mathbb{Q}_p^n)} \\
&\quad + (k-j)\|b\|_{C^{u'(\cdot)}}\|\chi_j\|_{L^{u'(\cdot)}(\mathbb{Q}_p^n)}\|f_j\|_{L^{u(\cdot)}(\mathbb{Q}_p^n)} \\
&\lesssim \|b\|_{C^{u'(\cdot)}}\|\chi_j\|_{L^{u'(\cdot)}(\mathbb{Q}_p^n)}\|f_j\|_{L^{u(\cdot)}(\mathbb{Q}_p^n)} \\
&\quad + (k-j)\|b\|_{C^{u'(\cdot)}}\|\chi_j\|_{L^{u'(\cdot)}(\mathbb{Q}_p^n)}\|f_j\|_{L^{u(\cdot)}(\mathbb{Q}_p^n)} \\
&\lesssim (k-j)\|b\|_{C^{u'(\cdot)}}\|\chi_j\|_{L^{u'(\cdot)}(\mathbb{Q}_p^n)}\|f_j\|_{L^{u(\cdot)}(\mathbb{Q}_p^n)}.
\end{aligned}$$

Next, (4.4.1), (4.4.2) and (4.4.3) leads to

$$\begin{aligned}
&\left\|\chi_k(\cdot)\frac{1}{|\cdot|_p^{n-\alpha}}\int_{B(0,|\cdot|_p)}((b(\cdot)-b(t))f(t))dt\right\|_{L^{v(\cdot)}}\\
&\lesssim p^{-k(n-\alpha)}\|(b-b_{B_k})\chi_j\|_{L^{v(\cdot)}(\mathbb{Q}_p^n)}\|\chi_j\|_{L^{u'(\cdot)}}\|f_j\|_{L^{u(\cdot)}(\mathbb{Q}_p^n)}\\
&+p^{-k(n-\alpha)}(k-j)\|b\|_{C^{u'(\cdot)}}\|\chi_k\|_{L^{v(\cdot)}(\mathbb{Q}_p^n)}\|\chi_j\|_{L^{u'(\cdot)}}\\
&\times\|f_j\|_{L^{u(\cdot)}(\mathbb{Q}_p^n)}\\
&\lesssim p^{-k(n-\alpha)}(k-j)\|b\|_{C^{v(\cdot)}\cap C^{u'(\cdot)}}\|\chi_{B_k}\|_{L^{v(\cdot)}(\mathbb{Q}_p^n)}\\
&\times\|\chi_{B_j}\|_{L^{u'(\cdot)}(\mathbb{Q}_p^n)}\|f_j\|_{L^{u(\cdot)}(\mathbb{Q}_p^n)}.
\end{aligned}$$

This clearly shows

$$\begin{aligned}
\|H_{\alpha,b}^p f\|_{L^{v(\cdot)}(\mathbb{Q}_p^n)} &\lesssim \|b\|_{C^{v(\cdot)}\cap C^{u'(\cdot)}}\sum_{k=-\infty}^{\infty}\sum_{j=-\infty}^{k}(k-j)p^{(j-k)n/u'(\cdot)}\|f_j\|_{L^{u(\cdot)}(\mathbb{Q}_p^n)}\\
&\lesssim \|b\|_{C^{v(\cdot)}\cap C^{u'(\cdot)}}\sum_{j=-\infty}^{\infty}\|f_j\|_{L^{u(\cdot)}(\mathbb{Q}_p^n)}\sum_{k=j}^{\infty}(k-j)p^{(j-k)n/u'(\cdot)}\\
&\lesssim \|b\|_{C^{v(\cdot)}\cap C^{u'(\cdot)}}\|f\|_{L^{u(\cdot)}(\mathbb{Q}_p^n)}
\end{aligned}$$

where (4.3.1) and Lemma 4.2.3 are used in above case. Similarly, the following can be easily achieved

$$\begin{aligned}
\|H_{\alpha,b}^{p,*}f\|_{L^{v(\cdot)}(\mathbb{Q}_p^n)} &\lesssim \|b\|_{C^{v(\cdot)}\cap C^{u'(\cdot)}}\|f\|_{L^{u(\cdot)}(\mathbb{Q}_p^n)}\\
\|H_{\alpha,b}^{p}f\|_{L^{u'(\cdot)}(\mathbb{Q}_p^n)} &\lesssim \|b\|_{C^{v(\cdot)}\cap C^{u'(\cdot)}}\|f\|_{L^{v'(\cdot)}(\mathbb{Q}_p^n)}\\
\|H_{\alpha,b}^{p,*}f\|_{L^{u'(\cdot)}(\mathbb{Q}_p^n)} &\lesssim \|b\|_{C^{v(\cdot)}\cap C^{u'(\cdot)}}\|f\|_{L^{v'(\cdot)}(\mathbb{Q}_p^n)},
\end{aligned}$$

Which is the desired result.

2) ⇒ 1) Condition $b\in C^{v(\cdot)}\cap C^{u'(\cdot)}$ turns out to be a prerequisite for the conclusion, which is that both $H_{\alpha,b}^p$ and $H_{\alpha,b}^{p,*}$ are bounded from $L^{u(\cdot)}(\mathbb{Q}_p^n)$ to $L^{v(\cdot)}(\mathbb{Q}_p^n)$ and $L^{v'(\cdot)}(\mathbb{Q}_p^n)$ to $L^{u'(\cdot)}(\mathbb{Q}_p^n)$.

For any ball $B =: B_\gamma$, we gain

$$\begin{aligned}|b(x) - b_B| &=: \left|\frac{1}{|B|}\int_B (b(x) - b(t))dt\right| \\ &\leq C\left|\frac{|B|^{-1}|x|_p^{n-\alpha}}{|x|_p^{n-\alpha}}\int_{B(0,|x|_p)}(b(x) - b(t))\chi_B(t)dt\right| \\ &+ C\left|\int_{\mathbb{Q}_p^n\setminus B(0,|x|_p)}\frac{(b(x) - b(t))\chi_B(t)|B|^{-1}|t|_p^{n-\alpha}}{|t|_p^{n-\alpha}}dt\right| \\ &\leq C\left||B|^{-1}|x|_p^{n-\alpha}(H^p_{\alpha,b}\chi_B(x))\right| + C\left||B|^{-1}|t|_p^{n-\alpha}(H^{p,*}_{\alpha,b}\chi_B(x))\right|.\end{aligned}$$

It follows from the boundedness of $H^p_{\alpha,b}$ and $H^{p,*}_{\alpha,b}$ that

$$\|(b - b_B)\chi_B\|_{L^{s(\cdot)}(\mathbb{Q}_p^n)} \leq C|B|^{-\frac{\alpha}{n}}\left(\|H^p_{\alpha,b}(\chi_B)\|_{L^{s(\cdot)}(\mathbb{Q}_p^n)} + \|H^{p,*}_{\alpha,b}(\chi_B)\|_{L^{s(\cdot)}(\mathbb{Q}_p^n)}\right).$$

In order to arrive at our estimations, we split the problem into two cases: $s = v(\cdot)$, $s = u'(\cdot)$.

Case 1: $s = v(\cdot)$

$$\begin{aligned}\|(b - b_B)\chi_B\|_{L^{v(\cdot)}(\mathbb{Q}_p^n)} &\leq C|B|^{-\frac{\alpha}{n}}\left(\|H^p_{\alpha,b}(\chi_B)\|_{L^{v(\cdot)}(\mathbb{Q}_p^n)} + \|H^{p,*}_{\alpha,b}(\chi_B)\|_{L^{v(\cdot)}(\mathbb{Q}_p^n)}\right) \\ &\leq C|B|^{-\frac{\alpha}{n}}\left(\|\chi_B\|_{L^{u(\cdot)}(\mathbb{Q}_p^n)} + \|\chi_B\|_{L^{u(\cdot)}(\mathbb{Q}_p^n)}\right) \\ &\leq C\|\chi_B\|_{L^{v(\cdot)}(\mathbb{Q}_p^n)}\end{aligned}$$

where we used limiting exponent $\frac{1}{u(x)} - \frac{1}{v(x)} =: \frac{\alpha}{n}$, , which implies $b \in C^{v(\cdot)}$.

Case 2: $s = u'(\cdot)$

We know that both $H^p_{\alpha,b}$ and $H^{p,*}_{\alpha,b}$ map $L^{v'(\cdot)}(\mathbb{Q}_p^n)$ to $L^{u'(\cdot)}(\mathbb{Q}_p^n)$. Therefore, by utilization of Sobolev limiting exponent

$$\begin{aligned}\|(b - b_B)\chi_B\|_{L^{u'(\cdot)}(\mathbb{Q}_p^n)} &\leq C|B|^{-\frac{\alpha}{n}}\left(\|H^p_{\alpha,b}(\chi_B)\|_{L^{u'(\cdot)}(\mathbb{Q}_p^n)} + \|H^{p,*}_{\alpha,b}(\chi_B)\|_{L^{u'(\cdot)}(\mathbb{Q}_p^n)}\right) \\ &\leq C|B|^{-\frac{\alpha}{n}}\left(\|\chi_B\|_{L^{u^{\star\prime}(\cdot)}(\mathbb{Q}_p^n)} + \|\chi_B\|_{L^{u^{\star\prime}(\cdot)}(\mathbb{Q}_p^n)}\right) \\ &\leq C\|\chi_B\|_{L^{u'(\cdot)}(\mathbb{Q}_p^n)}.\end{aligned}$$

Therefore, we obtain that b belongs to $C^{v(\cdot)} \cap C^{u'(\cdot)}$.

4.5 Conclusion

The boundedness of commutators of fractional p-adic Hardy operators on variable exponent p-adic Lebesgue spaces is used to characterise variable exponent p-adic central BMO spaces in this chapter. On variable exponent p-adic Lebesgue spaces, the boundedness of fractional p-adic Hardy operators is also obtained.

Chapter 5
p-adic Variable λ-central BMO Estimates for the Commutators of p-adic Fractional Hardy-type Operators

5.1 Introduction

In continuation to the previous Chapter, we introduce p-adic variable exponent λ-central BMO spaces and investigate the boundedness of commutators of fractional p-adic Hardy-type operators on central Morrey spaces. The boundedness of H_α^p and its dual on p-adic variable exponent central Morrey spaces are also established. To be more specific, in this Chapter, we primarily focus on characterizing p-adic variable exponent λ-central BMO spaces via commutators of fractional Hardy-type operator on central Morrey spaces.

To fulfill our assertion, we need following lemma:

Lemma 5.1.1 *Assume that* $b \in C^{u(\cdot),\lambda}(\mathbb{Q}_p^n)$ *and* $j, k \in \mathbb{Z}$, $\lambda \geq 0$. *Then,*

$$|b_{B_j} - b_{B_k}| \lesssim |j-k| C^{u(\cdot),\lambda}(\mathbb{Q}_p^n) \max\{|B_j|^\lambda, |B_k|^\lambda\}.$$

Proof. We can make the assumption that $k > j$ without losing generality. Bear in mind that $b_{B_i} = (1/|B_i|)\int_{B_i} b(x)dx$. Holder's inequality and Lemma 4.2.3 provide us

$$\begin{aligned}
|b_{B_j} - b_{B_k}| &\lesssim \frac{1}{|B_i|}\int_{B_i} |b(x) - b_{B_{i+1}}|dx \\
&\lesssim \frac{1}{|B_i|}\int_{B_{i+1}} |b(x) - b_{B_{i+1}}|dx \\
&\lesssim \frac{1}{|B_i|}\|(b - b_{B_{i+1}})\chi_{B_{i+1}}\|_{L^{u(\cdot)}(\mathbb{Q}_p^n)}\|\chi_{B_{i+1}}\|_{L^{u'(\cdot)}(\mathbb{Q}_p^n)} \\
&\lesssim \frac{1}{|B_i|}\|b\|_{C^{u(\cdot),\lambda}(\mathbb{Q}_p^n)}|B_{i+1}|^{\lambda}\|\chi_{B_{i+1}}\|_{L^{u(\cdot)}(\mathbb{Q}_p^n)}\|\chi_{B_{i+1}}\|_{L^{u'(\cdot)}(\mathbb{Q}_p^n)} \\
&\lesssim \|b\|_{C^{u(\cdot),\lambda}(\mathbb{Q}_p^n)}\frac{|B_{i+1}|^{\lambda+1}}{|B_i|} \\
&\lesssim p^n\|b\|_{C^{u(\cdot),\lambda}(\mathbb{Q}_p^n)}|B_{i+1}|^{\lambda}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
|b_{B_j} - b_{B_k}| &\leq \sum_{i=j}^{k-1} |b_{B_{i+1}} - b_{B_i}| \\
&\lesssim \|b\|_{C^{u(\cdot),\lambda}(\mathbb{Q}_p^n)}\sum_{i=j}^{k-1} |B_{i+1}|^{\lambda} \\
&\lesssim (k-j)\|b\|_{C^{u(\cdot),\lambda}(\mathbb{Q}_p^n)}|B_k|^{\lambda}.
\end{aligned}$$

5.2 Variable p-adic Centtral Morrey Space Estimates

Following Chapters 1 and 2, which covered the rudimentary theory of p-adic variable exponent function spaces and special findings relating to the p-adic fractional Hardy operators, we now present the following conclusion:

Theorem 5.2.1 *Suppose that $u \in \aleph(\mathbb{Q}_p^n)$, and v is defined by*

$$\frac{1}{v(x)} = \frac{1}{u(x)} - \frac{\alpha}{n}, \tag{5.2.1}$$

where $0 < \alpha < \min\{\frac{n}{u_+}, \frac{n}{v'_+}\}$. Let $\lambda = \lambda_1 + \frac{\alpha}{n}$ with $\lambda_1 > -1$, then, the operators $H_\alpha^p, H_\alpha^{p,*} : \dot{B}^{(\lambda_1,u(\cdot))}(\mathbb{Q}_p^n) \longrightarrow \dot{B}^{(\lambda,v(\cdot))}(\mathbb{Q}_p^n)$ and $\dot{B}^{(\lambda_1,v'(\cdot))}(\mathbb{Q}_p^n) \longrightarrow \dot{B}^{(\lambda,u'(\cdot))}(\mathbb{Q}_p^n)$ are bounded.

Theorem 5.2.1 yields the following corollary if $\alpha = 0$:

Corollary 5.2.2 *Let $u \in \aleph(\mathbb{Q}_p^n)$, then for $\lambda_1 > -1$, H^p and $H^{p,*}$ both map $\dot{B}^{(\lambda_1,u(\cdot))}(\mathbb{Q}_p^n)$ on $\dot{B}^{(\lambda_1,u(\cdot))}(\mathbb{Q}_p^n)$.*

For convenience, we write $\sum_{j=-\infty}^{\infty} f(x)\chi_j(x) = \sum_{j=-\infty}^{\infty} f_j(x)$.

Proof of Theorem 5.2.1. It is clear from the definition of the p-adic fractional Hardy operator and (1.2.25), that

$$\begin{aligned}
|H_\alpha^p f(x).\chi_k(x)| &\lesssim \frac{1}{|x|_p^{n-\alpha}} \int_{B(0,|x|_p)} |f(t)|dt.\chi_k(x) \\
&\lesssim p^{-k(n-\alpha)} \left(\sum_{j=-\infty}^{k} \int_{S_j} |f(t)|dt \right) \chi_k(x) \\
&\lesssim \|f\|_{\dot{B}^{(\lambda_1,u(\cdot))}(\mathbb{Q}_p^n)} p^{-k(n-\alpha)} \sum_{j=-\infty}^{k} |B_j|^{\lambda_1} \|\chi_j\|_{L^{u(\cdot)}(\mathbb{Q}_p^n)} \|\chi_j\|_{L^{u'(\cdot)}(\mathbb{Q}_p^n)} \chi_k(x).
\end{aligned}$$

Using (5.2.1), Lemma 4.2.3 and applying the norm to both sides of the aforementioned inequality, we get

$$\begin{aligned}
\|\chi_k H_\alpha^p(f)\|_{L^{v(\cdot)}(\mathbb{Q}_p^n)} &\lesssim \|f\|_{\dot{B}^{(\lambda_1,u(\cdot))}(\mathbb{Q}_p^n)} \sum_{j=-\infty}^{k} p^{-k(n-\alpha)} |B_j|^{\lambda_1} \|\chi_{B_j}\|_{L^{u(\cdot)}(\mathbb{Q}_p^n)} \|\chi_{B_j}\|_{L^{u'(\cdot)}(\mathbb{Q}_p^n)} \\
&\quad \times \|\chi_{B_k}(\cdot)\|_{L^{v(\cdot)}(\mathbb{Q}_p^n)} \\
&\lesssim \|f\|_{\dot{B}^{(\lambda_1,u(\cdot))}(\mathbb{Q}_p^n)} |B_k|^{\lambda} \|\chi_{B_k}\|_{L^{v(\cdot)}(\mathbb{Q}_p^n)} \sum_{j=-\infty}^{k} p^{n(\lambda_1+1)(j-k)} \\
&\lesssim \|f\|_{\dot{B}^{(\lambda_1,u(\cdot))}(\mathbb{Q}_p^n)} |B_k|^{\lambda} \|\chi_{B_k}\|_{L^{v(\cdot)}(\mathbb{Q}_p^n)}
\end{aligned}$$

where we used the fact $\lambda_1 + 1 > 0$. Therefore, the desired boundedness of H_α^p is obtained.

Similarly, the boundedness of $H_\alpha^{p,*}$ and the following can be easily established.

$$\begin{aligned}
\|H_\alpha^{p,*} f\|_{\dot{B}^{(\lambda,v(\cdot))}(\mathbb{Q}_p^n)} &\lesssim \|f\|_{\dot{B}^{(\lambda_1,u(\cdot))}(\mathbb{Q}_p^n)} \\
\|H_\alpha^{p} f\|_{\dot{B}^{(\lambda,u'(\cdot))}(\mathbb{Q}_p^n)} &\lesssim \|f\|_{\dot{B}^{(\lambda_1,v'(\cdot))}(\mathbb{Q}_p^n)} \\
\|H_\alpha^{p,*} f\|_{\dot{B}^{(\lambda,u'(\cdot))}(\mathbb{Q}_p^n)} &\lesssim \|f\|_{\dot{B}^{(\lambda_1,v'(\cdot))}(\mathbb{Q}_p^n)}.
\end{aligned}$$

This concludes our results.

5.3 p-adic Variable λ-central BMO Estimates for $H^p_{\alpha,b}$

The current section illustrates the necessary and sufficient condition for the boundedness of commutators of p-adic Hardy operators when the symbol functions belongs to $C^{(v(.),\lambda_1)} \cap C^{(u'(.),\lambda_1)}$.

Theorem 5.3.1 *Let $0 < \alpha < \min\{\frac{n}{u_+}, \frac{n}{v'_+}\}$. Suppose, that $u \in \aleph(\mathbb{Q}_p^n)$, where u is as defined in (5.2.1). If $\lambda = \lambda_1 + \lambda_2 + \frac{\alpha}{n}$ with $0 \leq \lambda_1 < 1/n$, $-1 < \lambda_2$. Then*
(i) $b \in C^{(v(.),\lambda_1)} \cap C^{(u'(.),\lambda_1)}$.
(ii) $H^p_{\alpha,b}$ and $H^{p,*}_{\alpha,b}$ from $\dot{B}^{(\lambda_2,u(.))}(\mathbb{Q}_p^n)$ to $\dot{B}^{(\lambda,v(.))}(\mathbb{Q}_p^n)$ and from $\dot{B}^{(\lambda_2,v'(.))}(\mathbb{Q}_p^n)$ to $\dot{B}^{(\lambda,u'(.))}(\mathbb{Q}_p^n)$.

The last Theorem has the following corollary:

Corollary 5.3.2 *Let $u \in \aleph(\mathbb{Q}_p^n)$, then for $\beta = \lambda_1 + \lambda_2 < 0$, the following statements are equivalent:*
(1) $b \in C^{(u(.),\lambda_1)} \cap C^{(u'(.),\lambda_1)}$.
(2) H^p_b and $H^{p,*}_b$ from $\dot{B}^{(\beta,u(.))}(\mathbb{Q}_p^n)$ to $\dot{B}^{(\beta,u'(.))}(\mathbb{Q}_p^n)$.

Proof of Theorem 5.3.1. It is not difficult to see that

$$\begin{aligned}
|H^p_{\alpha,b}f(y)\chi_k(y)| &\lesssim \frac{1}{|y|_p^{n-\alpha}} \int_{B(0,|y|_p)} |f(t)(b(y)-b(t))|dt.\chi_k(y) \\
&\lesssim \frac{1}{|y|_p^{n-\alpha}} \int_{B_k} |f(t)(b(y)-b(t))|dt.\chi_k(y) \\
&\lesssim p^{-k(n-\alpha)} \sum_{j=-\infty}^{k} \int_{S_j} |f(t)(b(y)-b_{B_k})|dt.\chi_k(y) \\
&+ p^{-k(n-\alpha)} \sum_{j=-\infty}^{k} \int_{S_j} |(b(t)-b_{B_k})f(t)|dt.\chi_k(y) \\
&= \mathcal{A}_1 + \mathcal{A}_2.
\end{aligned}$$

For $\mathcal{A}_1$, inequality (1.2.25) produces the following inequality;

$$\begin{aligned}
\mathcal{A}_1 &\lesssim p^{-k(n-\alpha)}|b(x)-b_{B_k}|\chi_k(x) \sum_{j=-\infty}^{k} \int_{S_j} |f(t)|dt \\
&\lesssim p^{-k(n-\alpha)}|b(x)-b_{B_k}|\chi_k(x) \sum_{j=-\infty}^{k} \|f_j\|_{L^{u(.)}(\mathbb{Q}_p^n)} \|\chi_j\|_{L^{u'(.)}(\mathbb{Q}_p^n)} \\
&\lesssim \|f\|_{\dot{B}^{(\lambda_2,u(.))}(\mathbb{Q}_p^n)} p^{-k(n-\alpha)}|b(x)-b_{B_k}|\chi_k(x) \sum_{j=-\infty}^{k} |B_j|^{\lambda_2} \|\chi_j\|_{L^{u(.)}(\mathbb{Q}_p^n)} \|\chi_j\|_{L^{u'(.)}(\mathbb{Q}_p^n)}.
\end{aligned}$$

Using Lemma 4.2.3, inequality (5.2.1) and the norm on both sides, we get

$$\begin{aligned}\|\mathcal{A}_1\|_{L^{v(\cdot)}(\mathbb{Q}_p^n)} &\lesssim \|f\|_{\dot{B}^{(\lambda_2,u(\cdot))}(\mathbb{Q}_p^n)}p^{-k(n-\alpha)}\|(b-b_{B_k})\chi_{B_k}\|_{L^{v(\cdot)}(\mathbb{Q}_p^n)}\\ &\times\sum_{j=-\infty}^{k}|B_j|^{\lambda_2}\|\chi_{B_j}\|_{L^{u(\cdot)}(\mathbb{Q}_p^n)}\|\chi_{B_j}\|_{L^{u'(\cdot)}(\mathbb{Q}_p^n)}\\ &\lesssim \|f\|_{\dot{B}^{(\lambda_2,u(\cdot))}(\mathbb{Q}_p^n)}\|b\|_{C^{v(\cdot),\lambda_1}}p^{-k(n-\alpha)}|B_k|^{\lambda_1}\|\chi_{B_k}\|_{L^{v(\cdot)}(\mathbb{Q}_p^n)}\\ &\times\sum_{j=-\infty}^{k}|B_j|^{\lambda_2}\|\chi_{B_j}\|_{L^{u(\cdot)}(\mathbb{Q}_p^n)}\|\chi_{B_j}\|_{L^{u'(\cdot)}(\mathbb{Q}_p^n)}\\ &\lesssim \|f\|_{\dot{B}^{(\lambda_2,u(\cdot))}(\mathbb{Q}_p^n)}\|b\|_{C^{v(\cdot),\lambda_1}}|B_k|^{\lambda}\|\chi_{B_k}\|_{L^{v(\cdot)}(\mathbb{Q}_p^n)}\sum_{j=-\infty}^{k}p^{n(j-k)(\lambda_2+1)}\\ &\lesssim \|f\|_{\dot{B}^{(\lambda_2,u(\cdot))}(\mathbb{Q}_p^n)}\|b\|_{C^{v(\cdot),\lambda_1}}|B_k|^{\lambda}\|\chi_{B_k}\|_{L^{v(\cdot)}(\mathbb{Q}_p^n)}.\end{aligned}$$

The fact $\lambda_2+1>0$ is used in above inequality. Now we will look at $\mathcal{A}_2$'s estimate. By Lemma 5.1.1, we have

$$\begin{aligned}\mathcal{A}_2 &\lesssim p^{-k(n-\alpha)}\chi_k(x)\sum_{j=-\infty}^{k}\int_{S_j}|(b(t)-b_{B_j})f(t)|dt\\ &+p^{-k(n-\alpha)}\chi_k(x)\sum_{j=-\infty}^{k}|b_{B_j}-b_{B_k}|\int_{S_j}|f(t)|dt\\ &=\mathcal{A}_{21}+\mathcal{A}_{22}.\end{aligned}$$

Where

$$\begin{aligned}\mathcal{A}_{21} &\lesssim p^{-k(n-\alpha)}\chi_k(x)\sum_{j=-\infty}^{k}\|b(t)-b_{B_j}\|_{L^{u'(\cdot)}(\mathbb{Q}_p^n)}\|f_j\|_{L^{u(\cdot)}(\mathbb{Q}_p^n)}\\ &\lesssim \|b\|_{C^{(u'(\cdot),\lambda_1)}}\|f\|_{\dot{B}^{(\lambda_2,u(\cdot))}(\mathbb{Q}_p^n)}p^{-k(n-\alpha)}\chi_k(x)\\ &\times\sum_{j=-\infty}^{k}|B_j|^{\lambda_1+\lambda_2}\|\chi_{B_j}\|_{L^{u(\cdot)}(\mathbb{Q}_p^n)}\|\chi_{B_j}\|_{L^{u'(\cdot)}(\mathbb{Q}_p^n)}.\end{aligned}$$

As a result, the $\mathcal{A}_{21}$ norm implies

$$\begin{aligned}\|\mathcal{A}_{21}\|_{L^{v(\cdot)}(\mathbb{Q}_p^n)} &\lesssim \|b\|_{C^{(u'(\cdot),\lambda_1)}}\|f\|_{\dot{B}^{(\lambda_2,u(\cdot))}(\mathbb{Q}_p^n)}p^{-k(n-\alpha)}\|\chi_k\|_{L^{v(\cdot)}(\mathbb{Q}_p^n)}\\ &\times\sum_{j=-\infty}^{k}|B_j|^{\lambda_1+\lambda_2}\|\chi_{B_j}\|_{L^{u(\cdot)}(\mathbb{Q}_p^n)}\|\chi_{B_j}\|_{L^{u'(\cdot)}(\mathbb{Q}_p^n)}\\ &\lesssim \|b\|_{C^{(u'(\cdot),\lambda_1)}}\|f\|_{\dot{B}^{(\lambda_2,u(\cdot))}(\mathbb{Q}_p^n)}|B_k|^{\lambda}\|\chi_{B_k}\|_{L^{v(\cdot)}(\mathbb{Q}_p^n)}\sum_{j=-\infty}^{k}p^{n(j-k)(\lambda_1+\lambda_2+1)}\\ &\lesssim \|b\|_{C^{(u'(\cdot),\lambda_1)}}\|f\|_{\dot{B}^{(\lambda_2,u(\cdot))}(\mathbb{Q}_p^n)}|B_k|^{\lambda}\|\chi_{B_k}\|_{L^{v(\cdot)}(\mathbb{Q}_p^n)}.\end{aligned}$$

Now the estimate $\mathcal{A}_{22}$ by Lemma 5.1.1, Lemma 4.2.3 is

$$\begin{aligned}\mathcal{A}_{22} &\lesssim \|b\|_{C^{(u'(.),\lambda_1)}} p^{-k(n-\alpha)}|B_k|^{\lambda_1}\chi_k(x)\sum_{j=-\infty}^{k}(k-j)\|f_j\|_{L^{u(\cdot)}(\mathbb{Q}_p^n)}\|\chi_j\|_{L^{u'(\cdot)}(\mathbb{Q}_p^n)}\\ &\lesssim \|b\|_{C^{(u'(.),\lambda_1)}}\|f\|_{\dot{B}^{(\lambda_2,u(.))}(\mathbb{Q}_p^n)} p^{-k(n-\alpha)}|B_k|^{\lambda_1}\chi_k(x)\\ &\times\sum_{j=-\infty}^{k}(k-j)|B_j|^{\lambda_2}\|\chi_{B_j}\|_{L^{u(\cdot)}(\mathbb{Q}_p^n)}\|\chi_{B_j}\|_{L^{u'(\cdot)}(\mathbb{Q}_p^n)}\end{aligned}$$

and

$$\begin{aligned}\|\mathcal{A}_{22}\|_{L^{v(\cdot)}(\mathbb{Q}_p^n)} &\lesssim \|b\|_{C^{(u'(.),\lambda_1)}}\|f\|_{\dot{B}^{(\lambda_2,u(.))}(\mathbb{Q}_p^n)} p^{-k(n-\alpha)}|B_k|^{\lambda_1}\|\chi_{B_k}\|_{L^{v(\cdot)}(\mathbb{Q}_p^n)}\\ &\times\sum_{j=-\infty}^{k}(k-j)|B_j|^{\lambda_2}\|\chi_{B_j}\|_{L^{u(\cdot)}(\mathbb{Q}_p^n)}\|\chi_{B_j}\|_{L^{u'(\cdot)}(\mathbb{Q}_p^n)}\\ &\lesssim \|b\|_{C^{(u'(.),\lambda_1)}}\|f\|_{\dot{B}^{(\lambda_2,u(.))}(\mathbb{Q}_p^n)}|B_k|^{\lambda}\|\chi_{B_k}\|_{L^{v(\cdot)}(\mathbb{Q}_p^n)}\sum_{j=-\infty}^{k}p^{n(j-k)(\lambda_2+1)}\\ &\lesssim \|b\|_{C^{(u'(.),\lambda_1)}}\|f\|_{\dot{B}^{(\lambda_2,u(.))}(\mathbb{Q}_p^n)}|B_k|^{\lambda}\|\chi_{B_k}\|_{L^{v(\cdot)}(\mathbb{Q}_p^n)}.\end{aligned}$$

When we add all of the $\mathcal{A}_1$, $\mathcal{A}_2$, $\mathcal{A}_{21}$, and $\mathcal{A}_{22}$ approximations together, we get

$$\|H^p_{\alpha,b}f\|_{\dot{B}^{(\lambda,v(.))}(\mathbb{Q}_p^n)} \lesssim \|b\|_{C^{(v(.),\lambda_1)}\cap C^{(u'(.),\lambda_1)}}\|f\|_{\dot{B}^{(\lambda_2,u(.))}(\mathbb{Q}_p^n)}.$$

Similarly, the following inequalities can be established as well:

$$\begin{aligned}\|H^{p,*}_{\alpha,b}f\|_{\dot{B}^{(\lambda,v(.))}(\mathbb{Q}_p^n)} &\lesssim \|b\|_{C^{(v(.),\lambda_1)}\cap C^{(u'(.),\lambda_1)}}\|f\|_{\dot{B}^{(\lambda_2,u(.))}(\mathbb{Q}_p^n)}\\ \|H^{p}_{\alpha,b}f\|_{\dot{B}^{(u'(\cdot),\lambda)}(\mathbb{Q}_p^n)} &\lesssim \|b\|_{C^{(v(.),\lambda_1)}\cap C^{(u'(.),\lambda_1)}}\|f\|_{\dot{B}^{(v'(\cdot),\lambda_2)}(\mathbb{Q}_p^n)}\\ \|H^{p,*}_{\alpha,b}f\|_{\dot{B}^{(u'(\cdot),\lambda)}(\mathbb{Q}_p^n)} &\lesssim \|b\|_{C^{(v(.),\lambda_1)}\cap C^{(u'(.),\lambda_1)}}\|f\|_{\dot{B}^{(v'(\cdot),\lambda_2)}(\mathbb{Q}_p^n)}.\end{aligned}$$

Conversely, assuming that $H^p_{\alpha,b}$ and $H^{p,*}_{\alpha,b}$ are bounded from $\dot{B}^{(\lambda_2,u(.))}(\mathbb{Q}_p^n)$ to $\dot{B}^{(\lambda,v(.))}(\mathbb{Q}_p^n)$ and $\dot{B}^{(\lambda_2,v'(.))}(\mathbb{Q}_p^n)$ to $\dot{B}^{(\lambda,u'(.))}(\mathbb{Q}_p^n)$, we will show that $b\in C^{(v(.),\lambda_1)}\cap C^{(u'(.),\lambda_1)}$ for any ball $B=B_\gamma$, we gain;

$$\begin{aligned}|b(x)-b_B| &=: \left|\frac{1}{|B|}\int_B(b(x)-b(t))dt\right|\\ &\lesssim \left|\frac{|B|^{-1}|x|_p^{n-\alpha}}{|x|_p^{n-\alpha}}\int_{B(0,|x|_p)}(b(x)-b(t))\chi_B(t)dt\right|\\ &+\left|\int_{\mathbb{Q}_p^n\setminus B(0,|x|_p)}\frac{(b(x)-b(t))\chi_B(t)|B|^{-1}}{|t|_p^{n-\alpha}}dt\right|\\ &\lesssim \left||B|^{-1}|x|_p^{n-\alpha}(H^p_{\alpha,b}\chi_B(x))\right|+\left||B|^{-1}|x|_p^{n-\alpha}(H^{p,*}_{\alpha,b}\chi_B(x))\right|.\end{aligned}$$

The norm of above inequality implies

$$\begin{aligned}
&\|(b-b_B)\chi_B\|_{L^{s(.)}(\mathbb{Q}_p^n)}\\
&\lesssim |B|^{-\frac{\alpha}{n}}\left(\|H^p_{\alpha,b}(\chi_B)\|_{L^{s(.)}(\mathbb{Q}_p^n)}+\|H^{p,*}_{\alpha,b}(\chi_B)\|_{L^{s(.)}(\mathbb{Q}_p^n)}\right)\\
&\lesssim |B|^{-\frac{\alpha}{n}}|B|^{\lambda}\|\chi_B\|_{L^{s(.)}(\mathbb{Q}_p^n)}\left(\|H^p_{\alpha,b}(\chi_B)\|_{\dot{B}^{(\lambda,s(.))}(\mathbb{Q}_p^n)}+\|H^{p,*}_{\alpha,b}(\chi_B)\|_{\dot{B}^{(\lambda,s(.))}(\mathbb{Q}_p^n)}\right).
\end{aligned}$$

To arrive at our estimates, we divided the problem into two cases: $s=v(.)$,and $s=u'(.)$. Then by boundedness of $H^p_{\alpha,b}$ and $H^{p,*}_{\alpha,b}$.
<u>Case 1</u>: If $s=v(.)$,

$$\begin{aligned}
&\|(b-b_B)\chi_B\|_{L^{v(.)}(\mathbb{Q}_p^n)}\\
&\lesssim |B|^{\lambda-\frac{\alpha}{n}}\|\chi_B\|_{L^{v(.)}(\mathbb{Q}_p^n)}\left(\|H^p_{\alpha,b}(\chi_B)\|_{\dot{B}^{(\lambda,v(.))}(\mathbb{Q}_p^n)}+\|H^{p,*}_{\alpha,b}(\chi_B)\|_{\dot{B}^{(\lambda,v(.))}(\mathbb{Q}_p^n)}\right)\\
&\lesssim |B|^{\lambda-\frac{\alpha}{n}}\|\chi_B\|_{L^{v(.)}(\mathbb{Q}_p^n)}\left(\|\chi_B\|_{\dot{B}^{(\lambda_2,u(.))}(\mathbb{Q}_p^n)}+\|\chi_B\|_{\dot{B}^{(\lambda_2,u(.))}(\mathbb{Q}_p^n)}\right)\\
&\lesssim |B|^{\lambda-\frac{\alpha}{n}-\lambda_2}\|\chi_B\|_{L^{v(.)}(\mathbb{Q}_p^n)}\\
&\lesssim |B|^{\lambda_1}\|\chi_B\|_{L^{v(.)}(\mathbb{Q}_p^n)}
\end{aligned}$$

where equation (5.2.1) and $\lambda=\lambda_1+\lambda_2+\frac{\alpha}{n}$ are used. We obtain that b belongs to $C^{(v(.),\lambda_1)}$.

<u>Case 2</u>: If $s=u'(.)$
Since, both $H^p_{\alpha,b}$ and $H^{p,*}_{\alpha,b}$ map $\dot{B}^{(\lambda_2,v'(.))}(\mathbb{Q}_p^n)$ to $\dot{B}^{(\lambda,u'(.))}(\mathbb{Q}_p^n)$ due to duality. As a result, by employing equation (5.2.1) and $\lambda-\lambda_2-\frac{\alpha}{n}=\lambda_1$

$$\begin{aligned}
&\|(b-b_B)\chi_B\|_{L^{u'(.)}(\mathbb{Q}_p^n)}\\
&\lesssim |B|^{\lambda-\frac{\alpha}{n}}\|\chi_B\|_{L^{u'(.)}(\mathbb{Q}_p^n)}\left(\|H^p_{\alpha,b}(\chi_B)\|_{\dot{B}^{(\lambda,u'(.))}(\mathbb{Q}_p^n)}+\|H^{p,*}_{\alpha,b}(\chi_B)\|_{\dot{B}^{(\lambda,u'(.))}(\mathbb{Q}_p^n)}\right)\\
&\lesssim |B|^{\lambda-\frac{\alpha}{n}}\|\chi_B\|_{L^{u'(.)}(\mathbb{Q}_p^n)}\left(\|\chi_B\|_{\dot{B}^{(\lambda_2,v'(.))}(\mathbb{Q}_p^n)}+\|\chi_B\|_{\dot{B}^{(\lambda_2,v'(.))}(\mathbb{Q}_p^n)}\right)\\
&\lesssim |B|^{\lambda-\frac{\alpha}{n}-\lambda_2}\|\chi_B\|_{L^{u'(.)}(\mathbb{Q}_p^n)}\\
&\lesssim |B|^{\lambda_1}\|\chi_B\|_{L^{u'(.)}(\mathbb{Q}_p^n)}.
\end{aligned}$$

The proof is finished.

5.4 p-Adic Variable Central BMO Estimates for $H^p_{\alpha,b}$ on Variable Central Morrey Space

We introduce here certain findings concerning central Morrey spaces with p-adic variables. The subsequent theorem utilizes commutators of p-adic Hardy operators to define and characterize the p-adic variable central BMO space within the context of p-adic variable central Morrey spaces.

Theorem 5.4.1 Let $u \in \aleph(\mathbb{Q}_p^n)$. Also, let $0 < \alpha < \min\{\frac{n}{u_+}, \frac{n}{v'_+}\}$ and define $u(\cdot)$ by

$$\frac{1}{v(\cdot)} = \frac{1}{u(\cdot)} - \frac{\alpha}{n}, \tag{5.4.1}$$

then for $\lambda = \beta + \frac{\alpha}{n}$ with $\lambda < 0$, the following claims are interchangeable:

(1) $b \in C^{v(\cdot)} \cap C^{u'(\cdot)}$.
(2) Both $H^p_{\alpha,b}$ and $H^{p,*}_{\alpha,b}$ map $\dot{B}^{u(\cdot),\beta}(\mathbb{Q}_p^n)$ into $\dot{B}^{v(\cdot),\lambda}(\mathbb{Q}_p^n)$ and $\dot{B}^{v'(\cdot),\beta}(\mathbb{Q}_p^n)$ into $\dot{B}^{u'(\cdot),\lambda}(\mathbb{Q}_p^n)$.

The corollary following the aforementioned theorem is:

Corollary 5.4.2 Let $u \in \aleph(\mathbb{Q}_p^n)$, then for $\beta < 0$, the following statements are interchangeable:

(1) $b \in C^{u(\cdot)} \cap C^{u'(\cdot)}$.
(2) Both H^p_b and $H^{p,*}_b$ are bounded on $\dot{B}^{u(\cdot),\beta}(\mathbb{Q}_p^n)$ and $\dot{B}^{u'(\cdot),\beta}(\mathbb{Q}_p^n)$.

Proof of Theorem 5.4.1. For (1) $\to$ (2), While maintaining generalization, we can deduce that $B_\gamma = B_\gamma(0)$ by selecting $\gamma \in \mathbb{Z}$ to define a fix ball B_γ contained within $\mathbb{Q}_p^n$. Following the proof of Theorem 5.3.1, we write

$$\begin{aligned}\|\big(H^p_{\alpha,b}f\big)\chi_{B_\gamma}\|_{L^{v(\cdot)}(\mathbb{Q}_p^n)} &\leq \|\big(H^p_{\alpha,b}f_1\big)\chi_{B_\gamma}\|_{L^{v(\cdot)}(\mathbb{Q}_p^n)} + \|\big(H^p_{\alpha,b}f_2\big)\chi_{B_\gamma}\|_{L^{v(\cdot)}(\mathbb{Q}_p^n)} \\ &=: J_1 + J_2.\end{aligned}$$

Making use of the Theorem 4.4.1 for estimation of J_1, we get

$$\begin{aligned} J_1 &=: \|(H^p_{\alpha,b}f_1)\chi_{B_\gamma}\|_{L^{v(\cdot)}(\mathbb{Q}_p^n)} \\ &\leq \|\big(H^p_{\alpha,b}f\big)\chi_{2B_\gamma}\|_{L^{v(\cdot)}(\mathbb{Q}_p^n)} \\ &\lesssim \|f\chi_{2B_\gamma}\|_{L^{u(\cdot)}(\mathbb{Q}_p^n)} \\ &\lesssim \|f\|_{\dot{B}^{u(\cdot),\beta}}\|\chi_{2B_\gamma}\|_{L^{u(\cdot)}(\mathbb{Q}_p^n)}|2B_\gamma|^\beta \\ &\lesssim \|f\|_{\dot{B}^{u(\cdot),\beta}}\|\|\chi_{B_\gamma}\|_{L^{u(\cdot)}(\mathbb{Q}_p^n)}|B_\gamma|^\beta. \end{aligned}$$

Using Lemma 4.2.3 and the condition $\frac{1}{v(\cdot)}=\frac{1}{u(\cdot)}-\frac{\alpha}{n}$, we get

$$\|\chi_{B_\gamma}\|_{L^{u(\cdot)}(\mathbb{Q}_p^n)}\approx|B_\gamma|^{\frac{1}{u(x)}}\approx\|\chi_{B_\gamma}\|_{L^{v(\cdot)}(\mathbb{Q}_p^n)}|B_\gamma|^{\frac{\alpha}{n}}. \tag{5.4.2}$$

The relation (5.4.2) and the condition $\lambda=\beta+\frac{\alpha}{n}$, help us to write

$$J_1=:C\|f\|_{\dot{B}^{u(\cdot),\beta}}\|\|\chi_{B_\gamma}\|_{L^{v(\cdot)}(\mathbb{Q}_p^n)}|B_\gamma|^{\lambda}.$$

Next, we need the decomposition of J_2 for its estimate:

$$\begin{aligned}|H^p_{\alpha,b}f_2(x)|&=\left|\frac{1}{|x|_p^{n-\alpha}}\int_{|t|_p\le|x|_p}(b(x)-b(y))f_2(y)dy\right|\\&\lesssim\left|\sum_{k=2\gamma}^{\infty}\frac{1}{|B_k|^{1-\frac{\alpha}{n}}}\int_{S_k}(b(x)-b(y))f(y)dy\right|\\&\lesssim\left|\sum_{k=2\gamma}^{\infty}\frac{1}{|B_k|^{1-\frac{\alpha}{n}}}\int_{S_k}(b(x)-c)f(y)dy\right|\\&+\left|\sum_{k=2\gamma}^{\infty}\frac{1}{|B_k|^{1-\frac{\alpha}{n}}}\int_{S_k}(b(y)-c)f(y)dy\right|\\&=:J_{21}+J_{22}.\end{aligned}$$

Using the Hölder inequality, J_{21} reduces to the following form:

$$\begin{aligned}J_{21}&=C\left|\sum_{k=2\gamma}^{\infty}|B_k|^{\frac{\alpha}{n}-1}(b(x)-c)\int_{S_k}f(y)dy\right|\\&\lesssim\sum_{k=2\gamma}^{\infty}|B_k|^{\frac{\alpha}{n}-1}|b(x)-c|\|f\chi_{B_k}\|_{L^{u(\cdot)}(\mathbb{Q}_p^n)}\|\chi_{B_k}\|_{L^{u'(\cdot)}}\\&\lesssim|b(x)-c|\sum_{k=2\gamma}^{\infty}|B_k|^{\frac{\alpha}{n}+\beta}\|f\|_{\dot{B}^{u(\cdot),\beta}}\\&\lesssim|b(x)-c|\|f\|_{\dot{B}^{u(\cdot),\beta}}|B_\gamma|^{\lambda},\end{aligned}$$

where series in the second last step converges due to the fact that $\lambda<0$.

Similarly, (1.2.25) is used, in establishing the below inequality for J_{22}.

$$\begin{aligned}J_{22}&=C\left|\sum_{k=2\gamma}^{\infty}|B_k|^{\frac{\alpha}{n}-1}\int_{C_k}(b(y)-c)f(y)dy\right|\\&\lesssim\sum_{k=2\gamma}^{\infty}|B_k|^{\frac{\alpha}{n}-1}\|f\chi_{B_k}\|_{L^{u(\cdot)}(\mathbb{Q}_p^n)}\|(b(x)-c)\chi_{B_k}\|_{L^{u'(\cdot)}}\\&\lesssim\|b\|_{C^{u'(\cdot)}}\sum_{k=2\gamma}^{\infty}|B_k|^{\frac{\alpha}{n}-1}\|f\chi_{B_k}\|_{L^{u(\cdot)}(\mathbb{Q}_p^n)}\|\chi_{B_k}\|_{L^{u'(\cdot)}},\end{aligned}$$

the condition $\lambda < 0$, yields

$$J_{22} \lesssim \|f\|_{\dot{B}^{u(\cdot),\beta}} \sum_{k=2\gamma}^{\infty} |B_k|^{\lambda} \lesssim \|f\|_{\dot{B}^{u(\cdot),\beta}} |B_\gamma|^{\lambda}.$$

Hence, we have

$$\begin{aligned} J_2 &\lesssim \|(b(x)-c)\chi_{B_\gamma}\|_{L^{v(\cdot)}} \|f\|_{\dot{B}^{u(\cdot),\beta}} |B_\gamma|^{\lambda} + \|f\|_{\dot{B}^{u(\cdot),\beta}} \|\chi_{B_\gamma}\|_{L^{v(\cdot)}(\mathbb{Q}_p^n)} |B_\gamma|^{\lambda} \\ &\lesssim \|f\|_{\dot{B}^{u(\cdot),\beta}} \|\chi_{B_\gamma}\|_{L^{v(\cdot)}(\mathbb{Q}_p^n)} |B_\gamma|^{\lambda}. \end{aligned}$$

Combining the estimates of J_1 and J_2, we get

$$\|H^p_{\alpha,b} f\|_{\dot{B}^{v(\cdot),\lambda}(\mathbb{Q}_p^n)} \lesssim \|f\|_{\dot{B}^{u(\cdot),\beta}(\mathbb{Q}_p^n)}.$$

Similarly, the following inequalities can be obtained as well:

$$\begin{aligned} \|H^{p,*}_{\alpha,b} f\|_{\dot{B}^{v(\cdot),\lambda}(\mathbb{Q}_p^n)} &\lesssim \|f\|_{\dot{B}^{u(\cdot),\beta}(\mathbb{Q}_p^n)}, \\ \|H^p_{\alpha,b} f\|_{\dot{B}^{u'(\cdot),\lambda}(\mathbb{Q}_p^n)} &\lesssim \|f\|_{\dot{B}^{v'(\cdot),\beta}(\mathbb{Q}_p^n)} \\ \|H^{p,*}_{\alpha,b} f\|_{\dot{B}^{u'(\cdot),\lambda}(\mathbb{Q}_p^n)} &\lesssim \|f\|_{\dot{B}^{v'(\cdot),\beta}(\mathbb{Q}_p^n)}. \end{aligned}$$

Thus the proof of the case (1) $\rightarrow$ (2) is complete.

(2) $\Rightarrow$ (1) Using the fact that both $H^p_{\alpha,b}$ and $H^{p,*}_{\alpha,b}$ map $\dot{B}^{u(\cdot),\beta}(\mathbb{Q}_p^n)$ into $\dot{B}^{v(\cdot),\lambda}(\mathbb{Q}_p^n)$ and $\dot{B}^{v'(\cdot),\beta}(\mathbb{Q}_p^n)$ into $\dot{B}^{u'(\cdot),\lambda}(\mathbb{Q}_p^n)$, we have to show that $b \in C^{v(\cdot)} \cap C^{u'(\cdot)}$.
For setting $f_0(x) = |B|^{-1}|x|^{n-\alpha}\chi_B(x)$, the following result is obtained:

$$\|(b-b_B)\chi_B\|_{L^{s(\cdot)}(\mathbb{Q}_p^n)} \lesssim |B|^{-\frac{\alpha}{n}} \|H_{\alpha,b}(\chi_B)\|_{L^{s(\cdot)}(\mathbb{Q}_p^n)} + \|H^*_{\alpha,b}(f_0)\|_{L^{s(\cdot)}(\mathbb{Q}_p^n)}.$$

Next, we split the problem into the following two cases:
Case 1: $s(\cdot) = v(\cdot)$. Here using the $(\dot{B}^{u(\cdot),\beta}, \dot{B}^{v(\cdot),\lambda})$ boundedness of $H^p_{\alpha,b}$ and $H^{p,*}_{\alpha,b}$, one can have

$$\begin{aligned} &\|(b-b_B)\chi_B\|_{L^{v(\cdot)}(\mathbb{Q}_p^n)} \\ &\lesssim |B|^{\lambda-\frac{\alpha}{n}} \|H^p_{\alpha,b}(\chi_B)\|_{\dot{B}^{v(\cdot),\lambda}} \|\chi_B\|_{L^{v(\cdot)}(\mathbb{Q}_p^n)} + |B|^{\lambda} \|H^{p,*}_{\alpha,b}(f_0)\|_{\dot{B}^{v(\cdot),\lambda}} \|\chi_B\|_{L^{v(\cdot)}(\mathbb{Q}_p^n)} \\ &\lesssim |B|^{\lambda-\frac{\alpha}{n}} \|\chi_B\|_{\dot{B}^{u(\cdot),\beta}} \|\chi_B\|_{L^{v(\cdot)}(\mathbb{Q}_p^n)} + |B|^{\lambda} \|f_0\|_{\dot{B}^{u(\cdot),\beta}} \|\chi_B\|_{L^{v(\cdot)}(\mathbb{Q}_p^n)} \\ &\lesssim \|\chi_B\|_{L^{v(\cdot)}(\mathbb{Q}_p^n)}. \end{aligned}$$

Case 2: $s(\cdot) = u'(\cdot)$. In this case,using the $(\dot{B}^{v'(\cdot),\beta}, \dot{B}^{u'(\cdot),\lambda})$ boundedness of $H^p_{\alpha,b}$ and $H^{p,*}_{\alpha,b}$, one can get

$$
\begin{aligned}
&\|(b-b_B)\chi_B\|_{L^{u'(\cdot)}(\mathbb{Q}_p^n)} \\
&\lesssim |B|^{\lambda-\frac{\alpha}{n}}\|H^p_{\alpha,b}(\chi_B)\|_{\dot{B}^{u'(\cdot),\lambda}}\|\chi_B\|_{L^{u'(\cdot)}(\mathbb{Q}_p^n)} + |B|^{\lambda}\|H^{p,*}_{\alpha,b}(f_0)\|_{\dot{B}^{u'(\cdot),\lambda}}\|\chi_B\|_{L^{u'(\cdot)}} \\
&\lesssim |B|^{\lambda-\frac{\alpha}{n}}\|\chi_B\|_{\dot{B}^{v'(\cdot),\beta}}\|\chi_B\|_{L^{u'(\cdot)}(\mathbb{Q}_p^n)} + |B|^{\lambda}\|f_0\|_{\dot{B}^{v'(\cdot),\beta}}\|\chi_B\|_{L^{u'(\cdot)}(\mathbb{Q}_p^n)} \\
&\lesssim \|\chi_B\|_{L^{u'(\cdot)}(\mathbb{Q}_p^n)}.
\end{aligned}
$$

From these cases, we conclude that $b \in C^{v(\cdot)} \cap C^{u'(\cdot)}$. We thus complete the proof.

5.5 Conclusion

The main finding of this Chapter for characterizing p-adic variable exponent λ-central BMO spaces and central BMO spaces is the boundedness of commutators generated from p-adic fractional Hardy operators on p-adic variable exponent central Morrey spaces. Such outcomes in p-adic Hardy-type operators have never been attained before. Our work will pave the way for even more remarkable findings in p-adic function spaces. Also some estimates are accomplished for p-adic Hardy-type operators.

Chapter 6
Bounds for p-adic Hardy-type Operators and Commutator On p-adic Variable Herz-Morrey Spaces

6.1 Introduction

In this chapter, we use the definition of the p-adic variable Herz-type spaces and then prove some new results regarding the continuity of fractional p-adic Hardy-type operators along with their commutators on these spaces. The coming section gives the bounds for p-adic Hardy-type operators on p-adic variable exponent Herz space. Whereas the last section discusses the variable Herz-Morrey estimates for p-adic Hardy-type operators and commutators.

6.2 Variable p-adic Herz Space Estimates for Hardy Operators

The findings of this section present the continuity characteristics about H_α^p, $H_\alpha^{p,*}$, $H_{\alpha,b}^p$, and $H_{\alpha,}^{p,*}$, which are all associated with the variable exponent p-adic Herz space.

Theorem 6.2.1 *Let $0 < m_1 \leq m_2 < \infty$, $u(\cdot) \in \aleph(\mathbb{Q}_p^n)$, $0 < \alpha < \min\{\frac{n}{u_+}, \frac{n}{v'_+}\}$ and $-\frac{n}{v_+} < \beta < \frac{n}{u'_-}$. Defined $v(\cdot)$ by*

$$\frac{1}{v(\cdot)} = \frac{1}{u(\cdot)} - \frac{\alpha}{n}, \tag{6.2.1}$$

then both H_α^p and $H_\alpha^{p,*}$ map $\dot{K}_{v(\cdot)}^{\beta,m_2}(\mathbb{Q}_p^n)$ into $\dot{K}_{u(\cdot)}^{\beta,m_1}(\mathbb{Q}_p^n)$.

From the above theorem, if $\alpha = 0$, then the following result is true.

Corollary 6.2.2 *Let* $0 < m_1 \leq m_2 < \infty$, $u \in \aleph(\mathbb{Q}_p^n)$, *and* $-\frac{n}{u_+} < \beta < \frac{n}{u'_-}$. *Then both* H^p *and* $H^{p,*}$ *map* $\dot{K}_{u(\cdot)}^{\beta,m_2}(\mathbb{Q}_p^n)$ *into* $\dot{K}_{u(\cdot)}^{\beta,m_1}(\mathbb{Q}_p^n)$.

Proof of Theorem 6.2.1. Let $f_i = f(\chi_i)$ for any $i \in \mathbb{Z}$. Then $f = \sum_{k=-\infty}^{\infty} f_i$. So we have

$$\begin{aligned}
|H_\alpha^p f(x)\chi_j| &\lesssim \chi_j \frac{1}{|x|_p^{n-\alpha}} \int_{B(0,|x|_p)} |f(t)dt \\
&\lesssim \chi_j p^{j(\alpha-n)} \sum_{i=-\infty}^{j} \int_j f(t)dt \\
&\lesssim \chi_j p^{j(\alpha-n)} \sum_{i=-\infty}^{j} \|f_i\|_{L^{u(\cdot)}(\mathbb{Q}_p^n)} \|\chi_i\|_{L^{u'(\cdot)}(\mathbb{Q}_p^n)}. \qquad (6.2.2)
\end{aligned}$$

By taking the $L^{v(\cdot)}(\mathbb{Q}_p^n)$-norm applying Lemma 4.2.3, we have

$$\begin{aligned}
\|(H_\alpha^p f)\chi_j\|_{L^{v(\cdot)}(\mathbb{Q}_p^n)} &\lesssim p^{j(\alpha-n)} \sum_{i=-\infty}^{j} \|f_i\|_{L^{u(\cdot)}(\mathbb{Q}_p^n)} \|\chi_i\|_{L^{u'(\cdot)}(\mathbb{Q}_p^n)} \|\chi_j\|_{L^{v(\cdot)}(\mathbb{Q}_p^n)} \\
&\lesssim p^{j\alpha} \sum_{i=-\infty}^{j} \|f_i\|_{L^{u(\cdot)}(\mathbb{Q}_p^n)} \|\chi_i\|_{L^{u'(\cdot)}(\mathbb{Q}_p^n)} \|\chi_{B_i}\|_{L^{v'(\cdot)}(\mathbb{Q}_p^n)}^{-1} \frac{\|\chi_{B_i}\|_{L^{v'(\cdot)}(\mathbb{Q}_p^n)}}{\|\chi_{B_j}\|_{L^{v'(\cdot)}(\mathbb{Q}_p^n)}} \\
&\lesssim p^{j\alpha} \sum_{i=-\infty}^{j} p^{n\delta_2(i-j)} \|f_i\|_{L^{u(\cdot)}(\mathbb{Q}_p^n)} \|\chi_i\|_{L^{u'(\cdot)}(\mathbb{Q}_p^n)} \|\chi_{B_i}\|_{L^{v'(\cdot)}(\mathbb{Q}_p^n)}^{-1} \\
&\lesssim \sum_{i=-\infty}^{j} p^{(i-j)n/u'(\cdot)} \|f_i\|_{L^{u(\cdot)}(\mathbb{Q}_p^n)}. \qquad (6.2.3)
\end{aligned}$$

Let $f \in \dot{K}_{u(\cdot)}^{\beta,m_1}(\mathbb{Q}_p^n)$. Then by Jensen's inequality, we have

$$\begin{aligned}
\|H_\alpha^p f\|_{\dot{K}_{v(\cdot)}^{\beta,m_2}(\mathbb{Q}_p^n)}^{m_1} &= \left(\sum_{j=-\infty}^{\infty} p^{\beta m_2 j} \|(H_\alpha^p f)\chi_j\|_{L^{v(\cdot)}(\mathbb{Q}_p^n)}^{m_2} \right)^{m_1/m_2} \\
&\lesssim \sum_{j=-\infty}^{\infty} p^{\beta m_1 j} \|(H_\alpha^p f)\chi_j\|_{L^{v(\cdot)}(\mathbb{Q}_p^n)}^{m_1} \\
&\lesssim \sum_{j=-\infty}^{\infty} p^{\beta m_1 j} \left(\sum_{i=-\infty}^{j} p^{(i-j)n/u'(\cdot)} \|f_i\|_{L^{u(\cdot)}(\mathbb{Q}_p^n)} \right)^{m_1} \\
&\lesssim \sum_{j=-\infty}^{\infty} \left(\sum_{i=-\infty}^{j} p^{i\beta} p^{(n/u'(\cdot)-\beta)(i-j)} \|f_i\|_{L^{u(\cdot)}(\mathbb{Q}_p^n)} \right)^{m_1} \\
&= J. \qquad (6.2.4)
\end{aligned}$$

Note that $\beta < n/u'(\cdot)$. We are interested the the following two cases:
Case 1: If $1 < m_1 < \infty$, then Holder's inequality implies

$$\begin{aligned}
J &\lesssim \sum_{j=-\infty}^{\infty}\left(\sum_{i=-\infty}^{j} p^{i\beta m_1} p^{(n/u'(\cdot)-\beta)(i-j)m_1/2}\|f_i\|^{m_1}_{L^{u(\cdot)}(\mathbb{Q}_p^n)}\right)\left(\sum_{i=-\infty}^{j} p^{(n/u'(\cdot)-\beta)(i-j)m_1'/2}\right)^{\frac{m_1}{m_1'}}\\
&\lesssim \sum_{j=-\infty}^{\infty}\sum_{i=-\infty}^{j} p^{i\beta m_1} p^{(n/u'(\cdot)-\beta)(i-j)m_1/2}\|f_i\|^{m_1}_{L^{u(\cdot)}(\mathbb{Q}_p^n)}\\
&\lesssim \|f\|^{m_1}_{\dot{K}^{\beta,m_1}_{u(\cdot)}(\mathbb{Q}_p^n)}. \qquad (6.2.5)
\end{aligned}$$

Case 2: If $0 < m_1 \le 1$, then we obtain

$$\begin{aligned}
J &\lesssim \sum_{j=-\infty}^{\infty}\sum_{i=-\infty}^{j} p^{i\beta m_1} p^{(n/u'(\cdot)-\beta)(i-j)m_1}\|f_i\|^{m_1}_{L^{u(\cdot)}(\mathbb{Q}_p^n)}\\
&\lesssim \sum_{i=-\infty}^{\infty} p^{i\beta m_1}\|f_i\|^{m_1}_{L^{u(\cdot)}(\mathbb{Q}_p^n)}\sum_{j=i}^{\infty} p^{(n/u'(\cdot)-\beta)(i-j)m_1}\\
&\lesssim \|f\|^{m_1}_{\dot{K}^{\beta,m_1}_{u(\cdot)}(\mathbb{Q}_p^n)}. \qquad (6.2.6)
\end{aligned}$$

Then the required result for H^p_α follows from (6.2.4)-(6.2.6). Similarly, it is simple to demonstrate that

$$\|H^{p,*}_\alpha f\|_{\dot{K}^{\beta,m_2}_{v(\cdot)}(\mathbb{Q}_p^n)} \lesssim \|f\|_{\dot{K}^{\beta,m_1}_{u(\cdot)}(\mathbb{Q}_p^n)}.$$

Consequently we have proved the Theorem 6.2.1.

The next result gives the continuity of commutators of p-adic Hardy-type operators on p-adic variables exponent Herz space.

Theorem 6.2.3 *Let $0 < m_1 \leq m_2 < \infty$, $b \in C^{u'(\cdot)} \cap C^{v(\cdot)}$, $v(\cdot) \in W_0^{\infty}(\mathbb{Q}_p^n)$, $0 < \alpha < \min\{\frac{n}{u_+}, \frac{n}{v'_+}\}$ and $-\frac{n}{v_+} < \beta < \frac{n}{u'_-}$. Defined $v(\cdot)$ by*

$$\frac{1}{v(\cdot)} = \frac{1}{u(\cdot)} - \frac{\alpha}{n}, \tag{6.2.7}$$

then both $H^p_{\alpha,b}$ and $H^{p,*}_{\alpha,b}$ map $\dot{K}^{\beta,m_2}_{v(\cdot)}(\mathbb{Q}_p^n)$ into $\dot{K}^{\beta,m_1}_{u(\cdot)}(\mathbb{Q}_p^n)$.

The following corollary holds if $\alpha = 0$ in the preceding theorem.

Corollary 6.2.4 *Let $0 < m_1 \leq m_2 < \infty$, $b \in C^{u'(\cdot)} \cap C^{u(\cdot)}$, $u \in \aleph(\mathbb{Q}_p^n)$, and $-\frac{n}{u_+} < \beta < \frac{n}{u'_-}$. Then both H^p_b and $H^{p,*}_b$ map $\dot{K}^{\beta,m_2}_{u(\cdot)}(\mathbb{Q}_p^n)$ into $\dot{K}^{\beta,m_1}_{u'(\cdot)}(\mathbb{Q}_p^n)$.*

Proof of Theorem 6.2.3. By definition

$$\begin{aligned}
|H^p_{\alpha,b} f(x)\chi_j| &\lesssim p^{j(\alpha-n)} \sum_{l=-\infty}^{j} \int_{B_l} |(b(x)-b(t))f(t)|dt.\chi_j(x), \\
&\lesssim p^{j(\alpha-n)} \sum_{l=-\infty}^{j} \int_{B_l} |(b(x)-b_{B_j})f(t)|dt.\chi_j(x) \\
&+ p^{j(\alpha-n)} \sum_{l=-\infty}^{j} \int_{B_l} |(b(t)-b_{B_j})f(t)|dt.\chi_j(x), \\
&= U + U'.
\end{aligned}$$

Inequality (1.2.25) yields the following inequality for U.

$$U \lesssim p^{j(\alpha-n)} |b(x) - b_{B_j}|\chi_j(x) \sum_{l=-\infty}^{j} \|f_l\|_{L^{u(\cdot)}(\mathbb{Q}_p^n)} \|\chi_l\|_{L^{u'(\cdot)}(\mathbb{Q}_p^n)}.$$

Using Lemma (4.2.3), and $L^{v(\cdot)}(\mathbb{Q}_p^n)$ norm of above inequality reduces to

$$\begin{aligned}
\|U\|_{L^{v(\cdot)}(\mathbb{Q}_p^n)} &\lesssim p^{j(\alpha-n)} \|(b - b_{B_j})\chi_{B_j}\|_{L^{v(\cdot)}(\mathbb{Q}_p^n)} \sum_{l=-\infty}^{j} \|f_l\|_{L^{u(\cdot)}(\mathbb{Q}_p^n)} \|\chi_l\|_{L^{u'(\cdot)}(\mathbb{Q}_p^n)}, \\
&\lesssim p^{j(\alpha-n)} \|b\|_{C^{v(\cdot)}(\mathbb{Q}_p^n)} \|\chi_{B_j}\|_{L^{v(\cdot)}(\mathbb{Q}_p^n)} \sum_{l=-\infty}^{j} \|f_l\|_{L^{u(\cdot)}(\mathbb{Q}_p^n)} \|\chi_l\|_{L^{u'(\cdot)}(\mathbb{Q}_p^n)}, \\
&\lesssim \|b\|_{C^{v(\cdot)}(\mathbb{Q}_p^n)} \sum_{l=-\infty}^{j} p^{(l-j)n/u'(\cdot)} \|f_l\|_{L^{u(\cdot)}(\mathbb{Q}_p^n)}.
\end{aligned} \tag{6.2.8}$$

Next we estimate U' with the help of Lemma (4.2.3) and inequality (1.2.25) as

$$\begin{aligned}
U' &\lesssim p^{j(\alpha-n)} \sum_{l=-\infty}^{j} \int_{B_l} |b(t)-b_{B_l}||f(t)|dt.\chi_j(x) + \sum_{l=-\infty}^{j} |b_{B_j}-b_{B_l}| \int_{B_l} |f(t)|dt.\chi_j(x),\\
&\lesssim p^{j(\alpha-n)} \sum_{l=-\infty}^{j} \|f_l\|_{L^{u(\cdot)}(\mathbb{Q}_p^n)} \|(b-b_{B_l})\chi_{B_l}\|_{L^{u'(\cdot)}(\mathbb{Q}_p^n)}.\chi_j(x)\\
&+ p^{j(\alpha-n)} \sum_{l=-\infty}^{j} (j-l)\|f_l\|_{L^{u(\cdot)}(\mathbb{Q}_p^n)} \|b\|_{C^{u'(\cdot)}(\mathbb{Q}_p^n)} \|\chi_{B_l}\|_{L^{u'(\cdot)}(\mathbb{Q}_p^n)}.\chi_j(x),\\
&= U'_1 + U'_2.
\end{aligned}$$

Lemma (4.2.3) and norm of U'_1 becomes

$$\begin{aligned}
\|U'_1\|_{L^{v(\cdot)}(\mathbb{Q}_p^n)} &\lesssim \|b\|_{C^{u'(\cdot)}(\mathbb{Q}_p^n)} \sum_{l=-\infty}^{j} p^{j(\alpha-n)} \|f_l\|_{L^{u(\cdot)}(\mathbb{Q}_p^n)} \|\chi_{B_l}\|_{L^{u'(\cdot)}(\mathbb{Q}_p^n)} \|\chi_{B_j}\|_{L^{v(\cdot)}(\mathbb{Q}_p^n)},\\
&\lesssim \|b\|_{C^{u'(\cdot)}(\mathbb{Q}_p^n)} \sum_{l=-\infty}^{j} p^{(l-j)n/u'(\cdot)} \|f_l\|_{L^{u(\cdot)}(\mathbb{Q}_p^n)}. \qquad (6.2.9)
\end{aligned}$$

Similarly, norm of U'_2 gives

$$\begin{aligned}
\|U'_2\|_{L^{v(\cdot)}(\mathbb{Q}_p^n)} &\lesssim \|b\|_{C^{u'(\cdot)}(\mathbb{Q}_p^n)} \sum_{l=-\infty}^{j} (j-l) p^{j(\alpha-n)} \|f_l\|_{L^{u(\cdot)}(\mathbb{Q}_p^n)} \|\chi_{B_l}\|_{L^{u'(\cdot)}(\mathbb{Q}_p^n)} \|\chi_{B_j}\|_{L^{v(\cdot)}(\mathbb{Q}_p^n)},\\
&\lesssim \|b\|_{C^{u'(\cdot)}(\mathbb{Q}_p^n)} \sum_{l=-\infty}^{j} (j-l) p^{(l-j)n/u'(\cdot)} \|f_l\|_{L^{u(\cdot)}(\mathbb{Q}_p^n)}. \qquad (6.2.10)
\end{aligned}$$

Inequalities (6.2.8)-(6.2.10) imply that

$$\|(H^p_{\alpha,b}f)\chi_j\|_{L^{v(\cdot)}(\mathbb{Q}_p^n)} \lesssim \|b\|_{C^{u'(\cdot)}\cap C^{v(\cdot)}} \sum_{l=-\infty}^{j} (j-l) p^{(l-j)n/u'(\cdot)} \|f_l\|_{L^{u(\cdot)}(\mathbb{Q}_p^n)}. \qquad (6.2.11)$$

Let $f \in \dot{K}^{\beta,m_1}_{u(\cdot)}(\mathbb{Q}_p^n)$, Jensen inequality then gives us

$$\begin{aligned}
&\|H^p_{\alpha,b}f\|^{m_1}_{\dot{K}^{\beta,m_2}_{v(\cdot)}(\mathbb{Q}_p^n)} \qquad (6.2.12)\\
&= \left(\sum_{j=-\infty}^{\infty} p^{\beta m_2 j}\|(H^p_{\alpha,b}f)\chi_j\|^{m_2}_{L^{v(\cdot)}(\mathbb{Q}_p^n)}\right)^{m_1/m_2},\\
&\lesssim \sum_{j=-\infty}^{\infty} p^{\beta m_1 j}\|(H^p_{\alpha,b}f)\chi_j\|^{m_1}_{L^{v(\cdot)}(\mathbb{Q}_p^n)},\\
&\lesssim \|b\|^{m_1}_{C^{u'(\cdot)}\cap C^{v(\cdot)}} \sum_{j=-\infty}^{\infty} p^{\beta m_1 j}\left(\sum_{l=-\infty}^{j}(j-l)p^{(l-j)n/u'(\cdot)}\|f_l\|_{L^{u(\cdot)}(\mathbb{Q}_p^n)}\right)^{m_1},\\
&\lesssim \|b\|^{m_1}_{C^{u'(\cdot)}\cap C^{v(\cdot)}} \sum_{j=-\infty}^{\infty} \left(\sum_{l=-\infty}^{j}(j-l)p^{i\beta}p^{(n/u'(\cdot)-\beta)(l-j)}\|f_l\|_{L^{u(\cdot)}(\mathbb{Q}_p^n)}\right)^{m_1},\\
&= L. \qquad (6.2.13)
\end{aligned}$$

We look at two scenarios: $1 < m_1 < \infty$ and $0 < m_1 \leq 1$.

If $1 < m_1 < \infty$, then inequality (1) yields

$$\begin{aligned}
L &\lesssim \|b\|^{m_1}_{C^{u'(\cdot)}\cap C^{v(\cdot)}} \sum_{j=-\infty}^{\infty}\left(\sum_{l=-\infty}^{j} p^{l\beta m_1}p^{(n/u'(\cdot)-\beta)(l-j)m_1/2}\|f_l\|^{m_1}_{L^{u(\cdot)}(\mathbb{Q}_p^n)}\right)\\
&\quad\times\left(\sum_{l=-\infty}^{j}(j-l)^{m_1'}p^{(n/u'(\cdot)-\beta)(l-j)m_1'/2}\right)^{\frac{m_1}{m_1'}},\\
&\lesssim \|b\|^{m_1}_{C^{u'(\cdot)}\cap C^{v(\cdot)}} \sum_{j=-\infty}^{\infty}\sum_{l=-\infty}^{j} p^{l\beta m_1}p^{(n/u'(\cdot)-\beta)(l-j)m_1/2}\|f_l\|^{m_1}_{L^{u(\cdot)}(\mathbb{Q}_p^n)},\\
&\lesssim \|b\|^{m_1}_{C^{u'(\cdot)}\cap C^{v(\cdot)}}\|f\|^{m_1}_{\dot{K}^{\beta,m_1}_{u(\cdot)}(\mathbb{Q}_p^n)}.
\end{aligned}$$

For $0 < m_1 \leq 1$, we can use the Jensen inequality to get

$$\begin{aligned}
L &\lesssim \|b\|^{m_1}_{C^{u'(\cdot)}\cap C^{v(\cdot)}} \sum_{j=-\infty}^{\infty}\sum_{l=-\infty}^{j} p^{l\beta m_1}p^{(n/u'(\cdot)-\beta)(l-j)m_1}\|f_l\|^{m_1}_{L^{u(\cdot)}(\mathbb{Q}_p^n)},\\
&\lesssim \|b\|^{m_1}_{C^{u'(\cdot)}\cap C^{v(\cdot)}} \sum_{l=-\infty}^{\infty} p^{l\beta m_1}\|f_l\|^{m_1}_{L^{u(\cdot)}(\mathbb{Q}_p^n)}\sum_{j=l}^{\infty}(j-l)^{m_1}p^{(n/u'(\cdot)-\beta)(l-j)m_1},\\
&\lesssim \|b\|^{m_1}_{C^{u'(\cdot)}\cap C^{v(\cdot)}}\|f\|^{m_1}_{\dot{K}^{\beta,m_1}_{u(\cdot)}(\mathbb{Q}_p^n)}.
\end{aligned}$$

We achieve the desired outcome by combining the estimates from both cases.

Similar to the previous inequality, the following one can also be proven:

$$\|H^{p,*}_{\alpha,b}f\|_{\dot{K}^{\beta,m_2}_{v(\cdot)}(\mathbb{Q}^n_p)} \lesssim \|b\|_{C^{u'(\cdot)}\cap C^{v(\cdot)}}\|f\|_{\dot{K}^{\beta,m_1}_{u(\cdot)}(\mathbb{Q}^n_p)}.$$

The proof is now complete.

6.3 Variable Morrey-Herz Estimates for p-adic Hardy-type Operators and Commutators

This section proves the boundedness of H^p_α, $H^{p,*}_\alpha$, $H^p_{\alpha,b}$ and $H^{p,*}_{\alpha,b}$ on Morrey-Herz type spaces. Here $f_i = f(\chi_i)$ remains the same as used in previous section for any $i \in \mathbb{Z}$.

Theorem 6.3.1 *Let $0 < m_1 \leq m_2 < \infty$, $u(\cdot) \in \aleph(\mathbb{Q}^n_p)$, $0 < \alpha < \min\{\frac{n}{u_+}, \frac{n}{v'_+}\}$ and $\lambda - \frac{n}{v_-} < \beta < \frac{n}{u'_-} + \lambda$. Defined $v(\cdot)$ by*

$$\frac{1}{v(\cdot)} = \frac{1}{u(\cdot)} - \frac{\alpha}{n}, \tag{6.3.1}$$

then both H^p_α and $H^{p,*}_\alpha$ map $M\dot{K}^{\beta,\lambda}_{m_2,v(\cdot)}(\mathbb{Q}^n_p)$ into $M\dot{K}^{\beta,\lambda}_{m_1,u(\cdot)}(\mathbb{Q}^n_p)$.

If $\alpha = 0$, then the following is true:

Corollary 6.3.2 *Let $0 < m_1 \leq m_2 < \infty$, $u \in \aleph(\mathbb{Q}^n_p)$, and $-\frac{n}{u_+} < \beta < \frac{n}{u'_-}$. Then both H^p and $H^{p,*}$ map $M\dot{K}^{\beta,\lambda}_{m_2,u(\cdot)}(\mathbb{Q}^n_p)$ into $M\dot{K}^{\beta,\lambda}_{m_1,u(\cdot)}(\mathbb{Q}^n_p)$.*

Proof of Theorem 6.3.1. Consider

$$\begin{aligned}
\|f_i\|_{L^{u(\cdot)}(\mathbb{Q}^n_p)} &= p^{-i\beta}\left(p^{i\beta m_1}\|f_i\|^{m_1}_{L^{u(\cdot)}(\mathbb{Q}^n_p)}\right)^{\frac{1}{m_1}}, \\
&\lesssim p^{-i\beta}\left(\sum_{l=-\infty}^{i} p^{l\beta m_1}\|f_l\|^{m_1}_{L^{u(\cdot)}(\mathbb{Q}^n_p)}\right)^{\frac{1}{m_1}}, \\
&\lesssim p^{i(\lambda-\beta)}\|f\|_{M\dot{K}^{\beta,\lambda}_{m_1,u(\cdot)}(\mathbb{Q}^n_p)}.
\end{aligned} \tag{6.3.2}$$

By considering the inequalities (6.2.3), (6.3.2), and the Jensen inequality, we observe that

$$\begin{aligned}
&\|H^p_\alpha f\|^{m_1}_{M\dot{K}^{\beta,\lambda}_{m_2,v(\cdot)}(\mathbb{Q}^n_p)}\\
&=\sup_{j_0\in\mathbb{Z}} p^{-j_0\lambda m_1}\left(\sum_{j=-\infty}^{j_0} p^{j\beta m_2}\|(H^p_\alpha f)\chi_j\|^{m_2}_{L^{v(\cdot)}(\mathbb{Q}^n_p)}\right)^{m_1/m_2},\\
&\lesssim \sup_{j_0\in\mathbb{Z}} p^{-j_0\lambda m_1}\left(\sum_{j=-\infty}^{j_0} p^{j\beta m_1}\|(H^p_\alpha f)\chi_j\|^{m_1}_{L^{v(\cdot)}(\mathbb{Q}^n_p)}\right),\\
&\lesssim \sup_{j_0\in\mathbb{Z}} p^{-j_0\lambda m_1}\sum_{j=-\infty}^{j_0} p^{j\beta m_1}\left(\sum_{i=-\infty}^{j} p^{(i-j)n/u'(\cdot)}\|f_i\|_{L^{u(\cdot)}(\mathbb{Q}^n_p)}\right)^{m_1},\\
&\lesssim \sup_{j_0\in\mathbb{Z}} p^{-j_0\lambda m_1}\sum_{j=-\infty}^{j_0} p^{j\beta m_1}\left(\sum_{i=-\infty}^{j} p^{(i-j)n/u'(\cdot)}p^{i(\lambda-\beta)}\|f\|_{M\dot{K}^{\beta,\lambda}_{m_1,u(\cdot)}(\mathbb{Q}^n_p)}\right)^{m_1},\\
&\lesssim \|f\|^{m_1}_{M\dot{K}^{\beta,\lambda}_{m_1,u(\cdot)}(\mathbb{Q}^n_p)}\sup_{j_0\in\mathbb{Z}} p^{-j_0\lambda m_1}\sum_{j=-\infty}^{j_0} p^{j\lambda m_1}\left(\sum_{i=-\infty}^{j} p^{(\frac{n}{u'(\cdot)}+\lambda-\beta)(i-j)}\right)^{m_1},\\
&\lesssim \|f\|^{m_1}_{M\dot{K}^{\beta,\lambda}_{m_1,u(\cdot)}(\mathbb{Q}^n_p)}\sup_{j_0\in\mathbb{Z}} p^{-j_0\lambda m_1}\sum_{j=-\infty}^{j_0} p^{j\lambda m_1},\\
&\lesssim \|f\|^{m_1}_{M\dot{K}^{\beta,\lambda}_{m_1,u(\cdot)}(\mathbb{Q}^n_p)}.
\end{aligned}$$

Likewise, it is straightforward to show that

$$\|H^{p,*}_\alpha f\|_{M\dot{K}^{\beta,\lambda}_{m_2,v(\cdot)}(\mathbb{Q}^n_p)} \lesssim \|f\|_{M\dot{K}^{\beta,\lambda}_{m_1,u(\cdot)}(\mathbb{Q}^n_p)}.$$

Thus, we achieved the desired proofs.

Theorem 6.3.3 *Let* $0 < m_1 \le m_2 < \infty$, $b \in C^{u'(\cdot)} \cap C^{v(\cdot)}$, $u(\cdot) \in \aleph(\mathbb{Q}^n_p)$, $0 < \alpha < \min\{\frac{n}{u_+}, \frac{n}{v'_+}\}$ *and* $\lambda - \frac{n}{v_-} < \beta < \frac{n}{u'_-} + \lambda$. *Defined* $v(\cdot)$ *by*

$$\frac{1}{v(\cdot)} = \frac{1}{u(\cdot)} - \frac{\alpha}{n}, \tag{6.3.3}$$

then both $H^p_{\alpha,b}$ *and* $H^{p,*}_{\alpha,b}$ *map* $M\dot{K}^{\beta,\lambda}_{m_2,v(\cdot)}(\mathbb{Q}^n_p)$ *into* $M\dot{K}^{\beta,\lambda}_{m_1,u(\cdot)}(\mathbb{Q}^n_p)$.

The logical consequence of $\alpha = 0$ is as follows:

Corollary 6.3.4 *Let* $0 < m_1 \le m_2 < \infty$, $b \in C^{u'(\cdot)} \cap C^{u(\cdot)}$, $u(\cdot) \in \aleph(\mathbb{Q}^n_p)$, *and* $\lambda - \frac{n}{u_-} < \beta < \frac{n}{u'_-} + \lambda$. *Then both* H^p_b *and* $H^{p,*}_b$ *map* $M\dot{K}^{\beta,\lambda}_{m_2,u(\cdot)}(\mathbb{Q}^n_p)$ *into* $M\dot{K}^{\beta,\lambda}_{m_1,u'(\cdot)}(\mathbb{Q}^n_p)$.

Proof of Theorem 6.3.3. Considering (6.2.11), (6.3.2), and the Jensen inequality, we find that

$$\begin{aligned}
&\|H^p_{\alpha,b}f\|^{m_1}_{M\dot{K}^{\beta,\lambda}_{m_2,v(\cdot)}(\mathbb{Q}^n_p)}\\
&= \sup_{j_0\in\mathbb{Z}} p^{-j_0\lambda m_1}\left(\sum_{j=-\infty}^{j_0} p^{j\beta m_2}\|(H^p_{\alpha,b}f)\chi_j\|^{m_2}_{L^{v(\cdot)}(\mathbb{Q}^n_p)}\right)^{m_1/m_2},\\
&\lesssim \sup_{j_0\in\mathbb{Z}} p^{-j_0\lambda m_1}\left(\sum_{j=-\infty}^{j_0} p^{j\beta m_1}\|(H^p_{\alpha,b}f)\chi_j\|^{m_1}_{L^{v(\cdot)}(\mathbb{Q}^n_p)}\right),\\
&\lesssim \|b\|^{m_1}_{C^{u'(\cdot)}\cap C^{v(\cdot)}} \sup_{j_0\in\mathbb{Z}} p^{-j_0\lambda m_1}\sum_{j=-\infty}^{j_0} p^{j\beta m_1}\left(\sum_{i=-\infty}^{j} p^{(\frac{n(i-j)}{u'(\cdot)})}\|f_i\|_{L^{u(\cdot)}(\mathbb{Q}^n_p)}\right)^{m_1},\\
&\lesssim \|b\|^{m_1}_{C^{u'(\cdot)}\cap C^{v(\cdot)}} \sup_{j_0\in\mathbb{Z}} p^{-j_0\lambda m_1}\sum_{j=-\infty}^{j_0} p^{j\beta m_1}\\
&\times\left(\sum_{i=-\infty}^{j} p^{(\frac{n}{u'(\cdot)})(i-j)}p^{i(\lambda-\beta)}\|f\|_{M\dot{K}^{\beta,\lambda}_{m_1,u(\cdot)}(\mathbb{Q}^n_p)}\right)^{m_1},\\
&\lesssim \|b\|^{m_1}_{C^{u'(\cdot)}\cap C^{v(\cdot)}}\|f\|^{m_1}_{M\dot{K}^{\beta,\lambda}_{m_1,u(\cdot)}(\mathbb{Q}^n_p)} \sup_{j_0\in\mathbb{Z}} p^{-j_0\lambda m_1}\sum_{j=-\infty}^{j_0} p^{j\lambda m_1}\\
&\times\left(\sum_{i=-\infty}^{j} p^{(\frac{n}{u'(\cdot)}+\lambda-\beta)(i-j)}\right)^{m_1},\\
&\lesssim \|b\|^{m_1}_{C^{u'(\cdot)}\cap C^{v(\cdot)}}\|f\|^{m_1}_{M\dot{K}^{\beta,\lambda}_{m_1,u(\cdot)}(\mathbb{Q}^n_p)} \sup_{j_0\in\mathbb{Z}} p^{-j_0\lambda m_1}\sum_{j=-\infty}^{j_0} p^{j\lambda m_1},\\
&\lesssim \|b\|^{m_1}_{C^{u'(\cdot)}\cap C^{v(\cdot)}}\|f\|^{m_1}_{M\dot{K}^{\beta,\lambda}_{m_1,u(\cdot)}(\mathbb{Q}^n_p)}.
\end{aligned}$$

Noticing that $\frac{n}{u'(\cdot)}+\lambda-\beta>0$. Just like the above inequality, the next one can also be shown:

$$\|H^{p,*}_{\alpha,b}f\|_{M\dot{K}^{\beta,\lambda}_{m_2,v(\cdot)}(\mathbb{Q}^n_p)} \lesssim \|f\|_{M\dot{K}^{\beta,\lambda}_{m_1,u(\cdot)}(\mathbb{Q}^n_p)}.$$

This concludes the proof.

6.4 Conclusion

In this Chapter, as a first attempt, we proved the boundedness of p-adic Hardy-type operators on p-adic Herz-type spaces with variable exponents. This work will open ups new dimensions for research in this direction.

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