DISPERSION RELATIONS IN THE NON-RELATIVISTIC THEORY

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CONTENTS

CHAPTER I : INTRODUCTION AND THE KRAMERS-KRONIG DISPERSION RELATION.

Section I: Introduction:

Stationary collision theory, deals with elements of the scattering and transition matrices for particular values of energy. Useful general results can be obtained from the assumed symmetry and unitarity of the S.Matrix. General results of a different kind are obtained if the S and T matrices are considered to be functions of complex, instead of real physical parameters like energy, scattering angle, angular momentum. In other words the conventional S and T matrices are analytically continued into the complex plane. The regions of regularity, location of singularities, their nature and characterestics, if now, are determined, then, cauchys theorem may be used to express in a compact form the analytic properties of the continued S-matrix. This mathematical expression is then called a Dispersion Relation.

Thus a dispersion relation can be derived only after the analytic behaviour of the S or T matrix element has been established or conjectured, so that locations of poles and branch cuts and the asymptotic dependence are believed to be known. Originally as derived by Kronig and Kramers in optics, the analytic behaviour underlying the dispersion relation was inferred from causality, which is the statement that a light singnal has limiting speed c, so that the scattered e.m. radiation cannot outrun the incident wave. Causality connects events that occur at different times and hence relates Fourier comps of the e.m. field that correspond to different frequencies. But in nonrelativistic quantum mechanics there is no limiting speed and hence no causality of this kind, so that it might first be thought that there are no dispersion relations. However, solutions of the schroedinger equation that correspond to different values of the energy, or the angular momentum are in fact connected by the assumption that the potential energy is independent of these parameters (or in more complicated situations has a specified dependence on them).

The full scope of the causality requirement in establishing dispersion relations becomes evident only in quantum field theory or more generally in the relativistic theory of the elementary particles.

Section II The Kramer's-Kronig dispersion relation.

The name "dispersion relation" is historically derived from the long-known relation between the real part (index of refraction and the imaginary part (absorption co-efficient) of the "complex index of refraction" in optics. The complex index of refraction is defined as

$$n(w) \equiv \mathrm{Re.}n(w) + i\,\mathrm{Im.}n(w) \equiv n_r(w) + i n_i(w) \quad \ldots\ldots\ldots\ldots (1)$$

where

n_r = usual index of refraction at the angular freq. w of the incident light, while n_i is proportional to the absorption coeff α of the medium

$$n_i(w) = \frac{c}{2w} \alpha(w) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (2)$$

Classical e.m. theory relates n_r and $\alpha(\omega)$ by the simple formula

$$n_r(w) = 1 + \frac{c}{\pi} P \int_0^{\infty} \frac{\alpha(w')}{w'^2 - w^2} dw' \ldots\ldots\ldots\ldots\ldots (3)$$

where P denotes the Cauchy principal value. n(w) is defined only for positive frequencies, but it is perfectly in order and very convenient to formally define (for proof See e.g.muirheads particle physics, page 418.)

$$n(-w) = n^*(w) \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (4)$$

The above is known as <u>crossing relation</u>. It links the physical and unphysical regions Eq.(3) can be written as;

$$\mathrm{Re}\left[n(w) - 1\right] = \frac{P}{\pi} \int_{-\infty}^{+\infty} \frac{\mathrm{Im}\left[n(w') - 1\right]}{w' - w} dw' \ldots\ldots\ldots\ldots (5)$$

Because: $\alpha(w) = n_i(w) \frac{2w}{c}$

$$\alpha(-w) = -n_i(-w) \frac{2w}{c}$$

$$\alpha(-w) = n_i(w) \frac{2w}{c} = \alpha(w)$$

Thus (w) is an even fn. consequently $\int_0^{\infty} \frac{\alpha(w')}{w'^2 - w^2} dw'$ can be replaced by $\frac{1}{2}\int_{-\infty}^{+\infty} \frac{\alpha(w')}{w'^2 - w^2} dw'$

$$\text{or} \quad n_r(w) - 1 = \frac{c}{\pi} P \int_{-\infty}^{\infty} dw' \frac{\alpha(w')/2w'}{w' - w}$$

or $$\mathrm{Re}\left[n(w)-1\right] = \frac{P}{\pi}\int_{-\infty}^{\infty} \frac{\mathrm{Im}\left[n(w')-1\right]}{w'-w}\,dw'$$

This is the real part of the equation $n(w)-1 = \frac{1}{i\pi} P\int_{-\infty}^{\infty} dw' \frac{\left[n(w')-1\right]}{w'-w}$

or as the limit as $\epsilon \rightarrow +0$ of

$$n(w)-1 = \frac{1}{2\pi i}\int_{-\infty}^{\infty} dw' \, \frac{n(w')-1}{w'-(w+i\epsilon)} \quad \ldots\ldots\ldots\ldots \quad (6)$$

Because: $$P\int_{-\infty}^{\infty} \frac{F(x)}{x-x_0}\,dx = \frac{1}{2} \mathop{\mathrm{Lt}}_{\epsilon\rightarrow+0} \int_{-\infty}^{\infty} \frac{F(x)}{x-x_0-i\epsilon}\,dx$$

where F(x)is a fn. which can be analytically continued into the upper half-plane, is everywhere regular and vanishes asymptotically for $|z|\rightarrow\infty$ (See [illegible] quantum mechanics page 718) The eq.(6) is a relation which may be considered as a direct consequence of the analytic behaviour of n(w) where n(w) is analytically continued to complex values of w;n(w) is regular in the upper half-plane and n(w)-1 tends to zero as $|w|\rightarrow\infty$. Then taking the contour integral of

$$\frac{n(w')-1}{w'-w}$$

along the real axis closed by an infinite semicircle in the upper-half plane, Cauchy's integral formula

$$f(z) = \frac{1}{2\pi i}\int \frac{f(z')}{z'-z}\,dz'$$

leads immediately to equation (6).

The question is whether such an anglytic behaviour of n(w) can be postulated or inferred on physical grounds. Actually, Kramer's has shown that the necessary analytic behaviour is physically equivalent to wave propagation through a material medium so that no signals are transmetted with a velocity greater than that of light in vacuum. In fact it can be shown that

$$n(w) - 1 = 4\pi\int_{-\infty}^{\infty} P(t)e^{iwt}\, dt$$

where P(t) is the polarization of the medium at a time t if the light signal reached the medium at t=0 now causality demands that P(t) = 0 for t<0 and then by Titchmarsh's theorem the desired analytic behaviour of n(w)-1 follows immediately. Titchmarsh Theorem. If a fn.F(w) has any one of the following properties.

(1) satisfies Hilbert Transforms (dispersion rels.)

(2) has a Fourier Transform which vanishes for t < 0

(3) is analytic in the upper half plane; automatically has the other two properties.

The above argument can be reversed. Starting with the "causality condition" and the very general structure of light propagation eq.(6) can be derived and hence the dispersion relations (3) or (5) which relates two such different observables as the index of refraction and the absorption coefficient. No detailed knowledge of the electromagnetic scattering theory is required to establish this relation.

According to the classical light-despersion theory of Lorentz. [For proof see Goldbergers* review article on dispersion relation in "Elementary particles and dispersion relations".]

* John Wiley and Sons Inc. New York (1960)

$$n(w) = 1 + \frac{2\pi c^2}{w^2} Nf(w,0) \ldots\ldots\ldots\ldots \quad (7)$$

where,

N= number of scattering centers per cm^3

f(w,0)= forward scattering amplitude of light at frequency w

Eq.(7) shows that Imf (w,0) is odd.

Hence eq.(5) can be written as:

$$Ref(w,0) = \frac{w^2}{\pi} P\int_{-\infty}^{\infty} \frac{Imf(w',0)}{w'^2(w'-w)} dw' = \frac{2w^2}{\pi} P\int_{0}^{\infty} \frac{Imf(w',0)}{w'(w'^2-w^2)} dw'$$

Because

$$Re(n(w)-1) = \frac{2\pi c^2 N}{w^2} Ref(w,0)$$

and from (5),

$$\frac{2\pi c^2 N}{w^2} Ref(w,0) = 2\frac{P}{\pi}\int_{-\infty}^{\infty} \frac{2\pi c^2 N}{w'^2} \frac{Imf(w',0)}{(w'-w)} dw'$$

from which (8) follows.

The optical theorem relates the total scattering c.s. and forward amplitude

$$\sigma(w) = \frac{4\pi c}{w} Imf(w,0)$$

Hence (8) gives

$$Ref(w,0) = \frac{w^2}{2\pi^2 c} P\int_{0}^{\infty} \frac{\sigma(w')}{w'^2-w^2} dw' \quad \ldots\ldots\ldots\ldots\ldots \quad (9)$$

Thus the knowledge of the total scattering cross section at all frequencies allows us to calculate the forward scattering amplitude at at any frequency.

By setting $w=0$, $Ref(0,0)=0$ is obtained which is not true for free electrons, since they can respond to zero frequency. Thus eq.(9) holds only for the case when the bound electrons cause the scattering. The mathematical reason for this discrepancy is that for free electrons the assumed asymptotic behaviour $n(w)-1 \rightarrow 0$ for $|w| \rightarrow \infty$ is not satisified. Due to eq.(7) $f(w,0)$ cannot tend to zero when $|w| \longrightarrow \infty$, instead $f(\infty,0)$ is infinite when $|w| \longrightarrow \infty$ and both sides of eq.(8) become infinite. This difficulty of insufficient asymptotic behaviour of the amplitude is a typical difficulty frequently encountred in dispersion relations.

The situation can be remedied by means of the following subtraction procedure. Take Lt as $|w| \longrightarrow \infty$ of eq(9) to obtain

$$Ref(\infty,0) = -\frac{1}{2\pi^2 c} P\int_0^\infty \sigma(w')\,dw' \quad \ldots\ldots\ldots\ldots\ldots\ldots (10)$$

where in eq (9) w'^2 is neglected compared to w^2 as $|w|^2 \rightarrow 0$.

Substract eq.(10) from eq.(9) obtaining

$$Ref(w,0) - Ref(\infty,0) = \frac{w^2}{2\pi^2 c} P\int_0^\infty \frac{\sigma(w')}{w'^2-w^2}\,dw' + \frac{1}{2\pi^2 c} P\int_0^\infty \sigma(w')\,dw'$$

$$= \frac{1}{2\pi^2 c} P\int_0^\infty dw'\sigma(w') \frac{w^2+w'^2-w^2}{w'^2-w^2}$$

$$Ref(w,0)-Ref(\infty,0) = \frac{1}{2\pi^2 c} P\int_0^\infty \frac{w'^2\sigma(w')}{w'^2-w^2}\,dw' \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (11)$$

This subtracted dispersion relation is strictly valid for all volues of w, even for free electrons. f $(\infty,0)$ is convenient whereas f(0,0) can be calculated from e.m. theory. Put w=0 in equation (11) to obtain

$$\mathrm{Re}f(\infty,0) = \mathrm{Re}f(0,0) - \frac{1}{2\pi^2 c} P\int_0^\infty \frac{w'^2\sigma(w')}{w'^2}\, dw'$$

Inserting this in eq. (11), we get

$$\mathrm{Re}f(w,0) - \mathrm{Re}f(0,0) = \frac{w^2}{2\pi^2 c} P\int_0^\infty \frac{\sigma(w')}{w'^2 - w^2}\, dw'$$

This is the Kramers-Kronig despersion relation.

REMARKS.

The Kramers-Kronig relation is unsatisfactory in at least two respects first, the Kramers Krong approach was based on classical electrodynamics and used quite a circuitous procedure to introduce the scattering amplitude Through eq.(7) . It would be preferable to discuss directly and quantum-theoretically the whole elementary scattering process. Secondly, the quantum mechanical approaches to such a direct study of scattering were comfined to treatments of problems with a finite range enteraction, and this interaction radus played a vital role. The first attempt at a complete formulation of the causality condition and of dispersion relations within the framework of quantum field theory was made in 1954* by Gell-Mann, Goldberger, and Thirring. The statement of causality used in this work is as follows, the measurement of two observable quantities should not interfereif the points of measurement have a space-like separation. In

*M. Gell Mann, M. L. Goldberger, and W. Thirring 1954 Phys. Rev. 95, 1612 terms of the Heesenberg field operators describing the particles, we demand that the commutator (or anticommutator for Fermi particles) of two such operators taken at space-like points vanish.

CHAPTER II. ONE DIMENSIONAL REPRESENTATION.

ction I : The analytic properties of the scattering amplitude.

Schroedinger equation for scattering is

$$[\nabla^2+k^2-U(r)]\psi_{\underline{k}}(r) = 0$$

d the exact scattering amplitude can be written as

$$f(\underline{k}, 0) = -\frac{1}{4\pi}\int e^{-i\hat{\underline{k}}\underline{r}}\ U(r)\psi^{+}_{\underline{k}}(\underline{r})d^3(\underline{r}) \ldots\ldots\ldots\ldots\ldots\ldots (1)$$

ere ψ^{+}_{k} is the outgoing solution of the scattering problem.

$\hat{\underline{k}}$ = final momentum vector $|\hat{\underline{k}}| = k$

k_o = incident momentum vector $|\underline{k}_o| = k$

efine

omentum transfer vector $\underline{\Delta} = \underline{k} - \underline{k}_o$ .... (2-a)

$\Delta = \underline{k} - \underline{k}_o = 2k\ \text{Sin}\ \theta/2$ .... (2-b)

hus f may be considered a fn. of the magnitudes of the momentum and omentum transfervector. $f = f\ (k,\Delta)$

To study analytic behaveour of $f(k,\Delta)$ it is convenient o express ψ^{+} in (1) as

$$\psi^{+}_{k}(r) = e^{i\underline{k}_o\underline{r}} + \int G(\underline{r},\underline{r}';k)U(r')e^{i\underline{k}_o\underline{r}'}d^3\underline{r}' \ldots\ldots\ldots\ldots\ldots (3\text{-a})$$

ere $G(\underline{r},r',k)$ is the full green's function satisfying

$$\left\{\nabla_r^2+k^2-U(r)\right\} G(\underline{r},\underline{r}',k)=\delta(\underline{r}-\underline{r}') \quad \text{................(3-b)}$$

uivalently, G is the solution of the integral equation

$$-G(\underline{r},\underline{r}';k) = + G_o^+(\underline{r},r',k) + \int G_o^+(\underline{r},\underline{r}'',k)U(r'')G(\underline{r}'',\underline{r}',k)d^3\underline{r}'' \quad \text{..........................(3-c)}$$

ere

$$G_o^+(\underline{r},\underline{r}'',k)= \frac{1}{4\pi} \exp \frac{(ik|r-\underline{r}'|)}{|\underline{r}-\underline{r}'|} \quad \text{........................(3-d)}$$

the outgoing unperturbed Green function.

om $\left\{\nabla_r^2+k^2-U(r)\right\} G(\underline{r},r',k) = \delta(\underline{r}-\underline{r}')$ it follows that

$$G(\underline{r},r';k)=G(r';r;k) \quad \text{(4-a)}$$

om $G_o^+(\underline{r},\underline{r}',k) = +\frac{1}{4\pi} \exp \frac{(i|\underline{k}|\,|\underline{r}-\underline{r}'|)}{|\underline{r}-\underline{r}'|}$ it follows that

$$\left[G_o^+(\underline{r},\underline{r}',k)\right]^*=G_o^+(\underline{r}',\underline{r}-k)$$

ake complex conyugate of (3-c)

$$-G^*(\underline{r},\underline{r}',k)=G_o^+(\underline{r}',r-k)+\int G_o^+(\underline{r},\underline{r}'',-k)U(r'')G^*(\underline{r}'',\underline{r}',k)d^3\underline{r}''$$

hus $G^*(\underline{r},\underline{r}'-k)$ satisfied the same relation as $G(\underline{r}';r,k)$. [Here the fact at $U(r)$ is a central potential is necessary.]

ice, $G(\underline{r},\underline{r}',k)=G(\underline{r}',r,k)$; Thus,

$$G(\underline{r},\underline{r}',k)=G^{*}(\underline{r},\underline{r}',-k) \quad \ldots\ldots\ldots\ldots \quad (5)$$

bstitute for ψ_k^{+} its source representation (3-a).

to (1) to get

$$f(k,\Delta) = -\frac{1}{4\pi}\int \exp -i(\hat{\underline{k}} - \underline{k}_o)\underline{r}\; U(r)d^3\underline{r}$$

$$-\frac{1}{4\pi}\iint \exp(-i\hat{\underline{k}}.\underline{r})\exp(i\underline{k}_o\underline{r}')U(r)G(\underline{r},\underline{r}',k)$$

$$U(r')d^3\underline{r}\; d^3\underline{r}' \quad \ldots\ldots\ldots\ldots \quad (6)$$

fine ausciliary vector q as

$$q = \underline{n}\, q = \frac{1}{2}\,(\hat{\underline{k}} + \underline{k}_o)$$

$$q\,\underline{q} = q^2 = \frac{1}{2}(\underline{k} + \underline{k}_o)\frac{1}{2}\,(\hat{\underline{k}} + \underline{k}_o) = \frac{1}{4}\,(k^2+k^2+2k^2\cos\theta)$$

$$= \frac{1}{4}(k^2+k^2+2k^2(1-2\sin^2\frac{\theta}{2})$$

$$= \frac{1}{4}(4k^2-4k^2\sin^2\theta/_2)$$

$$q = (k^2 - \frac{1}{4}\Delta^2)^{\frac{1}{2}} \quad \ldots\ldots\ldots\ldots \quad (7\text{-a})$$

so, $q\;\_ = \frac{1}{2}\,(\hat{\underline{k}} + \underline{k}_o)\,(\hat{\underline{k}} - k_o) = \frac{1}{2}\,(k^2-k^2)=0$

,

$$q \cdot \underline{\Delta} = 0 \quad \ldots\ldots \quad (7\text{-b})$$

d $\quad \frac{1}{2}(\hat{\underline{k}} - \underline{k}_o)(\underline{r} + \underline{r}') + q\hat{\underline{n}}(\underline{r} - \underline{r}')$

$$= \tfrac{1}{2}\hat{\underline{k}}\cdot\underline{r} + \tfrac{1}{2}\hat{\underline{k}}\cdot\underline{r}' - \tfrac{1}{2}\underline{k}_o\,\underline{r} + \tfrac{1}{2}(\hat{\underline{k}} + \underline{k}_o)(\underline{r} - \underline{r}')$$

$$= \tfrac{1}{2}\hat{k}\cdot\underline{r} + \tfrac{1}{2}\hat{\underline{k}}\cdot\underline{r}' - \tfrac{1}{2}\underline{k}_o\,\underline{r}' - \underline{k}_o\,\underline{r} + \tfrac{1}{2}\hat{\underline{k}}\cdot\underline{r} - \tfrac{1}{2}\hat{\underline{k}}\cdot\underline{r}' + \tfrac{1}{2}\underline{k}_o\,\underline{r} - \tfrac{1}{2}\underline{k}_o\,\underline{r}'$$

$$= \hat{\underline{k}}\cdot\underline{r} - \underline{k}_o\cdot\underline{r}'$$

nce

$$\hat{\underline{k}}\cdot\underline{r} - \underline{k}_o\cdot\underline{r}' = \tfrac{1}{2}\underline{\Delta}\cdot(\underline{r} + \underline{r}') + q\hat{\underline{n}}(\underline{r} - \underline{r}') \quad \ldots\ldots \quad (7\text{-c})$$

.(6) becomes;

$$f(k,\Delta) = -\frac{1}{4\pi}\int \exp(-i\ \_\cdot\underline{r})U(r)\,d^3\underline{r} - \frac{1}{4\pi}\iint \exp\Big[-i/_2\ \_\ (\underline{r} + \underline{r}') - qn(\underline{r} - \underline{r})\Big]U(r)G(\underline{rr}',k) \times U(r')\,d^3\underline{r}\,d^3\underline{r}\,.$$

$$k,\Delta) = f_B(\Delta) - \frac{1}{4}\iint \exp\Big[-i/_2\,\underline{\Delta}\,(\underline{r} + \underline{r}')\Big]\exp\Big[i(k^2 - \tfrac{1}{4}\Delta^2)^{1/2}\,\underline{n}(\underline{r}' - \underline{r})\Big] \times U(r)G(\underline{r},\underline{r}',k)U(r')\,d^3\underline{r}\,d^3\underline{r}' \quad \ldots\ldots \quad (8)$$

here f_B is the first Born approximation

$$f_B(\Delta) = -\frac{1}{4\pi}\int \exp(-i\,\underline{\Delta}\cdot\underline{r})U(r)\,d^3\underline{r}$$

ich depends only on the magnitude of in case of a central potential.

.(8) represents the physical scattering amplitude for positive real
romenta k

menta k Extension of $f(k,\Delta)$ to arbitrary complexvalues of k (keeping

fixed)

Consider first, the extension to negative real k-values.

Using $G^*(\underline{r},\underline{r}', k) = G(\underline{r}',\underline{r}, -k)$

is seen from equation (8) that

$$^*(k,\Delta) = f_B(\Delta) - \frac{1}{4\pi}\iint \exp\left[i/_2\,(+\,\underline{\Delta}\,.(\underline{r}+\underline{r}')\right] \exp\left[-i\left(k^2 - \frac{\Delta^2}{4}\right)^{1/2} \hat{\underline{n}}.(\underline{r}'-\underline{r})\right] U(r)G^*(\underline{r},\underline{r}',k)U(r')d^3r\ d^3r$$

$$= f_B(\Delta) - \frac{1}{4\pi}\iint \exp\left[-i/_2\,(-\,\underline{\Delta}\,).(\underline{r}+\underline{r}')\right]\exp\left[i(k^2 - \Delta^2)^{1/2}\,\hat{\underline{n}}.(\underline{r}-\underline{r}')\right] U(r')G(\underline{r}'\underline{r};-k)U(r)d^3\underline{r}\ d^3\underline{r}'$$

$^*(k,\Delta) = f\ \ f(-k,\Delta)$ ....(9)

But this is only a formal relation. For "physical angles" $|\cos\theta| \leqslant 1$, the momentum k for any specified is restricted to

$$k \geqslant \frac{1}{2}\Delta$$

To extend $f(k,\Delta)$ to the portion of real k-axis where $|k| \leqslant \frac{1}{2}\Delta$, it must be ensured that the integral defining f exists.

Contrary to appearances, there are no branch pts at

$k = \frac{1}{2}\Delta$

Because

$$f(k,\Delta) = \frac{1}{2}\left[f(k,\Delta) + f^*(-k,\Delta)\right]$$

$$f(k,\Delta) = \frac{1}{2}\Big\{f_B(\Delta) - \frac{1}{4\pi}\iint \exp\left[-i/_2\ \underline{\Delta}\cdot(\underline{r}+\underline{r}')\right]\exp\left[i\left(k^2-\frac{\Delta^2}{4}\right)^{1/2}\hat{\underline{n}}\cdot(\underline{r}'-r)\right]$$

$$U(r)G(\underline{r},\underline{r}',k)U(r')\ d^3r\ d^3r'$$

$$+f_B(\Delta) - \frac{1}{4\pi}\iint \exp\left[i/_2\ (-\underline{\Delta})\cdot(\underline{r}+\underline{r}')\right]\exp\left[-i\left(k^2-\frac{\Delta^2}{4}\right)^{1/2}\hat{\underline{n}}\cdot(\underline{r}'-r)\right]$$

$$U(r)G^*(r,r',-k)\ U(r')\ d^3\underline{r}\ d^3\underline{r}'\Big\}$$

$$= \frac{1}{2}\Big\{2f_B(\Delta) - \frac{1}{4\pi}\iint \exp\left[-i/_2\ \Delta\cdot(\underline{r}+\underline{r}')\right]\exp\left[i\left(k - \frac{1}{4}\Delta^2\right)^{1/2}\hat{\underline{n}}\cdot(\underline{r}'-\underline{r})\right.$$

$$+\exp -i\left(k^2-\frac{\Delta^2}{4}\right)^{1/2}\hat{\underline{n}}\cdot(\underline{r}'-\underline{r})$$

$$\times\ U(r)G(r,r',-k)U(r')d^3\underline{r}\ d^3\underline{r}'$$

$$= \text{--------} \times \left(\mathrm{Cos}\left[\left(k^2-\frac{1}{4}\ \ {}^2\right)^{1/2} n\cdot(\underline{r}'-\underline{r})\right]\right) \times \text{-------}$$

Because Cos (....) is even consequently there will be no branch pt at $|\underline{k}| = \frac{1}{4}\Delta$; ~~$(k^2-\frac{1}{4}\Delta^2)^{1/2}$ has two branches~~ $+(k^2-\frac{1}{4}\Delta^2)^{1/2}$ ~~and~~ has two branches $+(k^2-\frac{1}{4}\Delta^2)^{1/2}$ and $-(k^2-\frac{\Delta^2}{4})^{1/2}$. But Cos $(-\theta)$=Cos θ; Or, Cos $\left[(k^2-\frac{1}{4}\Delta^2)^{1/2}\ \hat{\underline{n}}\cdot(\underline{r}'-\underline{r})\right]$ has no branch pt.

For $k \leqslant \frac{1}{2}\Delta$, the integral (8) will exist if the potential

put under the restriction as

$$\int_0^\infty |U(r)|\, r e^{Kr}\, dr \text{ exists}$$

d further

$$\Delta < 2K$$

ere K^{-1} defines the "range" of the force.

fine,

$$T(z,\Delta) = -\frac{1}{4\pi} \exp\left[-i/_2\ \underline{\Delta}.(\underline{r} + \underline{r}')\right) \exp\left[i(z^2 - \frac{\Delta^2}{4})^{1/2}\ \underline{\hat{n}}.(\underline{r}'-r)\right]$$
$$\times U(r) G(\underline{r}, r'; z) U(r')$$

ere $z=k+i\beta$ is the complex momentum. As upper half complex plane is under nsideration, $\mathrm{Im}z=\beta > 0$ is considered. Thus

$$f(z,\Delta) = f_B(\Delta) + \iint T(z,\Delta)\, d^3\underline{r}\, d^3\underline{r}'$$

that

$$f(k,\Delta) = \underset{\beta \to +0}{\mathrm{Lt}}\ f(z,\Delta)$$

ere $f(k,\Delta)$ is the limit as approached from aboves of a function of compex argument. The analytic behaviour of T is determined by that of $(\underline{r},\underline{r}';z)$:

$G(\underline{r},\underline{r}',z)$ can be represented by the bilinear formula over all genfunctions.

$$G(r,r';z)=\sum_{b=1}^{N} \frac{\phi_b(\underline{r})\phi_b^*(\underline{r}')}{z^2 - z_b^2} + \int \frac{\psi_{k'}(\underline{r})\psi_{k'}^*(\underline{r}')\, d^3 k'}{z^2 - k'^2}$$

re ϕ_b represents discrete bound-state eigenfunctions belonging to the genvalue $z_b{}^2$ and $\psi_{k'}$ is in the continuum. The discrete eigen values E_b, ich correspond to bound states are negative, so that the complex pts.of ngularity in the upper half of the z-plane are given by

$$z_b = i(|E_b|)^{1/2}$$

e all poles lie on the imaginary axis. Thus $G(\underline{r},\underline{r}',z)$ is analytic in the per half complex k plane except for discrete poles at the position of und states The cut, parallel to and just below the real axis, extending om minus infinity to plus infinity, corresponding to the positive energy ntinuum is excluded, because we are considering $Im z > 0$.

The valid application of the Cauchy's theorem demands that ere be only a finite number of poles. It has been shown [V.Bargmann, roceedings of the National Academy of Science.U.S.38,961 (1952)] that for e class of potentials.

$$\int_0^\infty r\,|U(r)|\,dr \text{ exist}$$

ere can be only a finite number of poles.

Consider the integral

$$I = \frac{1}{2\pi i}\oint_C z\frac{'T(z',\triangle)}{z'^2 - z^2}$$

here the closed contowr consists part of the real axis and a semicercle in the upper half plane such that it encompasses all poles.

$I = 2\pi i$ [sum of Residues at poles enclosed by C]. Poles of I enclosed by C are all bound state poles lying on the positive imaginary axis and a pole at $z' = z$ [and not $z' = -z$]

Residue at bound state poles $= \sum_{b=1}^{N} T_b(z_b) \underset{z'\to z_b}{\mathrm{Lt}} \frac{z'}{z'^2 - z^2}$

where $T_b(z_b)$ = Residue of $'T(z',\triangle)$ at z_b

Residue at $z' = z$ is

$$\underset{z'\to z}{\mathrm{Lt}} (z' - z) \frac{z'\,T(z',\triangle)}{(z'+z)(z'-z)}$$

$$= \frac{z\,T(z,\triangle)}{2z} = \frac{1}{2}T(z,\triangle)$$

Hence

$$\frac{1}{2\pi i}\int_C \frac{z'T(z',\triangle)}{z'^2 - z^2} = \frac{1}{2}T(z,\triangle) + \sum_{b=1}^{N}\frac{z_b\,T_b(z_b)}{z_b^2 - z^2} \quad \ldots\ldots\ldots \quad (12)$$

The contribution to $T(z',\triangle)$ from the continuum vanishes because we are considering $\mathrm{Im} z > 0$ Residue of $T(z',\triangle)$ at $z'=z_b$ is

$$T_b(z_b) = \underset{z'\to z_b}{Lt}\,(z' - z_b)T\,(z',\triangle)$$

$$= \underset{z\to z_b}{Lt}\,(z'-z_b)\left\{-\frac{1}{4\pi}\exp\left[-i/_2\,\underline{\triangle}\cdot(\underline{r}+\underline{r}')\right)\exp\left[i(z'^2-\frac{\triangle^2}{4})^{\frac{1}{2}}\underline{n}.(\underline{r}'-\underline{r})\right]\right.$$

$$\left.\times\,\lambda U(r)\sum_{b=1}^{N}\frac{\phi_b(\underline{r})\phi_b^*(\underline{r}')U(r')}{z'^2-z_b^2}\right\}$$

$$= -\frac{1}{4\pi}\exp\left[-\frac{i}{2}\,\underline{\triangle}\cdot(\underline{r}+\underline{r}')\right)\exp\left[i(z_b-\frac{1}{4}\triangle^2)^{\frac{1}{2}}\underline{n}\,(\underline{r}'-\underline{r})\right]$$

$$\times\,\lambda U(r)\,\frac{\phi_b(r)\,\phi_b^*(r')U(r')}{2z_b}$$

$$= \frac{F_b(z_b)}{2z_b}\quad\text{say,}$$

hen

$$\frac{1}{2}T\,(z,\triangle) + \sum_{b=1}^{N}\frac{z_b\,F_b(z_b)}{2z_b\,(z_b^2-z^2)} = \frac{1}{2\pi i}\int_C\frac{z'T(z',\triangle)}{z'^2-z^2}\,dz$$

r $$T(z,\triangle) = \frac{F_b(z_b)}{z^2-z_b^2} + \frac{1}{i\pi}\int_C\frac{z'T\,(z',\quad)}{z'^2-z^2}\,dz' \qquad\text{.................}\quad(13)$$

Integrating this with respect to $\underline{r}$ and r' and substituting

n $f(z,\triangle)= f_B(z,\triangle)+\iint T(z,\triangle)\,d^3\underline{r}\,d^3\underline{r}$ we get,

$$f(z,\triangle)=f_B(\triangle) + \sum_{b=1}^{N}\iint\frac{F_b(z_b)}{z^2-z_b^2}d^3\underline{r}d^3\underline{r}' + \frac{1}{i\pi}\iint d^3\underline{r}\,d^3\underline{r}'\,\frac{z'T(z',\triangle)}{z'^2-z^2}\,dz.$$

itting

$$R_b(z_b, \triangle) = \iint F_b(z_b)\, d^3\underline{r}\, d^3\underline{r}'$$

$$= \iint -\frac{1}{4\pi} \exp\left[-i/_2\ \triangle\ (\underline{r} + \underline{r}')\right] \exp\left[i(z_b{}^2 - \frac{\triangle^2}{4})^{1/2} \hat{\underline{n}}(\underline{r}' - \underline{r})\right]$$

$$\times\ U(r)U(r')\ \phi_b(\underline{r})\phi_b^*(\underline{r}')\, d^3\underline{r}\ d^3\underline{r}' \quad (14)$$

hus

$$f(z, \triangle) = f_B(\triangle) + \sum_{b=1}^{N} \frac{R_b(z_b, \triangle)}{z^2 - zb^2} + \frac{1}{i\pi}\iint d^3\underline{r}\ d^3\underline{r}'\ \frac{z'T(z', \triangle)}{z'^2 - z^2}\ dz' \quad (15)$$

Using the asymptotic behaviour $\phi_b(r) = e^{-|E_b|^{1/2} r}$ of bound states,

t can be shown that R_b is <u>real</u> and it is <u>finite</u> as long as $\triangle < 2\ K$

rovided

a) $\int_0^\infty |U(r)|\ r\ e^{Kr} dr$ exist

b) $\int_0^\infty r\ |U(r)|\ dr$ exist

are valid.

Two assertions are assumed here.

<u>ssertion I</u>. The contour integration in eq (15) may be interchanged with t he two space integrations.

<u>ssertion II</u>. If the contour recedes to infinity, the contribution from the semicircle vanishes, so that in (15) the contour integration may be taken along the real axis.

The first assertion can be proved by showing that $T\, d^3r\, d^3r'$ converges uniformly on the semicircle with respect to z. e.second assertion demands a study of the asymptotic behaviour of G th respect to z. To retain continuity, the proof of the above two sertions will be postponed till the following section. Assuming the o assertions we have from equation (15)

$$\frac{1}{i\pi}\iint d^3r\, d^3r' \int_C \frac{z'T(z',\Delta)}{z'^2 - z^2}\, dz' = \frac{1}{i\pi}\iiint_C d^3r\, d^3r' \frac{z'T(z',\Delta)}{z'^2 - z^2}\, dz'$$

$$= \frac{1}{i\pi}\int_C \frac{z'}{z'^2 - z^2}\, dz' \left\{ f(z',\Delta) - f_B(\Delta) \right\}$$

Since for the contour receding to infinity, the contrib- ion from the semicircle vonishes so that we can integrate along the al axis.

nce,

$$\frac{1}{i\pi}\iint d^3\underline{r}\, d^3\underline{r}' \int_C z' \frac{T(z',\Delta)}{z'^2 - z^2}\, dz'$$

$$= \frac{1}{i\pi}\int_{-\infty}^{+\infty} f_B(\Delta) \frac{k'}{k'^2 - z^2}\, dk' + \frac{1}{i\pi}\int_{-\infty}^{\infty} \frac{k'}{k'^2 - z^2} f(k',\Delta)\, dk'$$

$$= \frac{f_B(\Delta)}{i\pi}\int_{-\infty}^{+\infty} \frac{k'}{k'^2 - z^2}\, dk' + \frac{1}{i\pi}\int_{-\infty}^{+\infty} \frac{k'\, f(k'\Delta)}{k'^2 - z^2}$$

$$= \frac{f_B(\Delta)}{i\pi}\left\{ \int_{-\infty}^{0} \frac{k'}{k'^2 - z^2}\, dk' + \int_{0}^{\infty} \frac{k'}{k'^2 - z^2}\, dk' \right\} + \frac{1}{i\pi}\int_{-\infty}^{+\infty} - - - -$$

$$= f_B \frac{(\triangle)}{i\pi}\int_0^{\infty} \frac{-(-k')d(-k')}{(-k')^2-z^2} + \int_0^{\infty}\frac{k'}{k'^2-z^2}\,dk' + \frac{1}{i\pi}\int_{-\infty}^{\infty} k'\frac{f(k'\triangle)}{k'^2-z^2}$$

$$= \frac{1}{i\pi}\int_{-\infty}^{\infty} k'\frac{f\ (k'\triangle)}{k'^2-z^2}$$

e.

$$\frac{1}{i\pi}\iint d^3\underline{r}\ d^3\underline{r}'\int_C z'\ \frac{T(z',\triangle)}{z'^2-z^2}\,dz' = \frac{1}{i\pi}\int_{-\infty}^{\infty} k'\frac{f(k'\triangle)}{k'^2-z^2}$$

nce,

$$f(z,\triangle) = f_B\ (\triangle) + \sum_{b=1}^{N}\frac{R_b(z_b)}{z^2-z_b^2} + \frac{1}{i\pi}\int_{-\infty}^{\infty}\frac{k'\ f(k',\triangle)}{k'^2-z^2}\,dk' \quad \text{for } \mathrm{Im}\, z>0$$

for Imz 0

s $f(-k,\triangle) = f^*(k,\triangle)$

$$\int_{-\infty}^{\infty}\frac{k'\ f(k',\triangle)}{k'^2-z^2}\,dk'$$

$$= \int_{-\infty}^{0}\frac{k'\ f(k'\triangle)\ dk'}{k'^2-z^2} + \int_0^{\infty}\frac{k'\ f(k',\triangle)}{k'^2-z^2}\,dk'$$

$$= -\int_0^{\infty}\frac{(-k')f(-k'\triangle)(-dk')}{(-k')^2-z^2} + \int_0^{\infty}\frac{k'\ f(k',\triangle)}{k'^2-z^2}\,dk'$$

$$= \int_0^{\infty}\frac{k'\left[f\ (k'\triangle) - f\ (-k'\triangle)\right]\ dk'}{k'^2-z^2}$$

$$\int_0^{\infty} k' \frac{[f(k'\Delta) - f^{*}(k'\Delta)]}{k'^2 - z^2} dk'$$

$$2i \int_0^{\infty} \frac{k' \operatorname{Im} f(k'\Delta)}{k'^2 - z^2} dk'$$

ence

$$f(z, \Delta) = f_B(\Delta) + \sum_{b=1}^{N} \frac{R_b(z_b)}{z^2 - z_b^2} + \frac{2}{\pi} \int_0^{\infty} k' \frac{\operatorname{Im} f(k', \Delta)}{k'^2 - z^2} dk' \tag{17}$$

f z is taken real (z=k) then as $z = k + i\beta$ Lt f(z) f (k',) =f ( )

$$,\Delta) + \sum_{b=1}^{N} \frac{R_b(z_b)}{k^2 - z_b^2} + \frac{2}{\pi} \operatorname{Lt}_{\beta \to +0} \int_0^{\infty} \frac{k' \operatorname{Im} f(k', \Delta)}{k'^2 - (k + i\beta)^2} dk' \quad \ldots\ldots\ldots\ldots\ldots\ldots \tag{18}$$

In this case $-z_b^2 = |E_b|$

Instead of momentum, as the dispersed variable, at times, t is more convenient to introduce

$$z^2 = s$$

so that if it is real, $s = k;^2$ i.e apart from a factor $\frac{\hbar^2}{2m}$, s represents he energy.

It is also customary to introduce, instead of Δ, a new variable t defined as the sq. of the momentum transfer

$$\Delta^2 = 4z^2 \sin^2 \theta/_2 = 2s\ (1-\cos\theta) \equiv t$$

is considered a fn. of s and t Eq.(17) becomes.

$$f(s,t)= f_B\ (t) + \sum_{b=1}^{N} \frac{R_b(i\,|E_b|,\ t\ )}{s + |E_b|} + \frac{1}{\pi}\int_0^{\infty} \frac{Imf(s',t)}{s'-s}\ ds'$$

where $Ims \neq 0\ (z^2=s;\ s'=k^2)$

To allow for real values of s,

$$f(s,t) = f_B(t) + \sum_{b=1}^{N} \frac{R_b(i\,|E_b|,t)}{s + |E_b|} + \frac{1}{\pi} \underset{\eta\to 0}{Lt} \int_0^{\infty} \frac{Imf(s',t)}{s'-s-i\eta}\ ds' \quad \text{---}(19)$$

This is valid for all values of s, on the entire first sheet f the complex s-plane. However, this is valid only as long as the limit-g condition on the momentum transfer is satisfied i.e $t < 4K^2$ where $^{-1}$ is the range of the force. Since, in the symbolic operator form

$$P\ \frac{1}{x-xo} + i\pi\,\delta(x-xo) = \frac{1}{x-xo-i\epsilon}$$

hich must be applied onto a fn. regular in the neighborhood of the real xis.

$$f(s,t) = f_B(t) + \sum_{b=1}^{N} \frac{R_b(i\,|E_b|\,,t)}{s + |E_b|} + \frac{1}{\pi} P\int_0^{\infty} \frac{Imf(s',t)}{s' - s}\ ds'$$

$$+ i\ \frac{\pi}{\pi}\int_0^{\infty} Imf(s',t)\,\delta(s'-s)\ ds'$$

$$f(s,t)= f_B(t) + \sum_{b=1}^{N} \frac{R_b(i\,|E_b|\,,t)}{s + |E_b|} + \frac{1}{\pi}\ P\int_0^{\infty} \frac{Imf(s',t)}{s' - s}\ ds'$$

+ i Imf(s t)

erefore, since $f_B(t)$ & R_b is real

$$\mathrm{Re}f(s,t)=f_B+\sum_{b=1}^{N}\frac{R_b(i|E_b|,t)}{s+|E_b|}+\frac{1}{\pi}P\int_0^\infty\frac{\mathrm{Im}f(s',t)}{s'-s}\,ds'$$

ıd, Imf = Imf

We have thus obtained an unsubtracted dispersion relation
The fact that no subtractions were necessary implies that $f(s,t)-f_B(t)$
or $f(s,t) - f_B(t)$ tends to zero when $|s|\to\infty$. This shows that for the lass of potentials considered the first Born approximation becomes eliable for high energies even if the whole Born series expansion ould not converge.

ction II. Convergence of the Born's series for the green's nction and allied topics. In this section it will be shown that for a neral class of potentials, the Born series for the green's function ways converges for sufficiently high energies, and in the high energy mit, the series is approximated by its leading term. And, the same is e case for wave functions and the scattering amplitudes.

In the integral formulation of the scattering problem, a ave function $\psi(\underline{r})$ is sought which satisfies

$$\Psi = \Psi_o + G_o V \Psi \qquad \text{......... (1)}$$

$$= \Psi_o + G V \Psi_o \qquad \text{......... (2)}$$

ere the (outgoing wave) green's function $G(\underline{r},\underline{r}')$ obeys

$$G = G_o + G_o V G \qquad \text{.......... (3)}$$

The free particle functions are given by

$$\Psi_o(\underline{r}) = \exp\left[i\ \underline{k}.\underline{r}\right]$$

$$G_o(\underline{r}\ \underline{r}') = -\frac{1}{4\pi}\ \frac{\exp\left[ik\ \underline{r} - \underline{r}'\right]}{|\underline{r} - \underline{r}'|}$$

where sq. of wave no. $k^2 = \frac{2m\ E}{\hbar^2}$

For $\hbar = 1$, k represents particle momentum

From (1) we get

$$\Psi = \Psi_o + G_o V\{\Psi_o + G_o V \Psi\} = \Psi_o + G_o V \Psi_o + G_o V G_o V \Psi$$

$$= \Psi_o + G_o V \Psi_o + G_o V G_o V \Psi_o + G_o V G_o V G_o V \Psi$$

f $\quad \Psi = \sum_{n=0}^{\infty} \Psi_n$

hen $\Psi_{n+1}(r) = \int G_o(\underline{r},\underline{s}) V(\underline{s}) \Psi_n(\underline{s})\ d\underline{s} \qquad \text{............ (4)}$

nd $G = G_o + G_o V G_o + G_o V G_o V G_o + G_o V G_o V G_o V G_o + G_o V G_o V G_o V G_o V G_o + G_o V G_o V G_o V G_o V G_o V G$.

$$G = \sum_{n=o}^{\infty} G_n \qquad \ldots\ldots\ldots\ldots\ldots \text{(4-a)}$$

en

$$G_{n+1}(r,\underline{r}') = \int G_o(\underline{r},s)\, V(s) G_n(\underline{s},\underline{r}')\, ds$$

$$G_{m+n+1}(r,r') = \int G_m(\underline{r},\underline{s}) V(\underline{s}) G_n(\underline{s},\underline{r}')\, d\underline{s} \quad \ldots\ldots\ldots (5)$$

It is useful to introduce an associated set of functions:

$$go\ (r,r') = 1$$

$$g_n\ (\underline{r},r') = G_n(\underline{r},\underline{r}')/G_o(\underline{r},\underline{r}')$$

$$g(\underline{r},\underline{r}') = G(\underline{r},\underline{r}')/G_o(\underline{r},\underline{r}')$$

n place of (4-a) consider the series

$$\frac{G}{G_o} = \frac{G_o}{G_o} + \frac{G_1}{G_o} + \frac{G_2}{G_o}$$

$$g(|\underline{r}|,|\underline{r}'|) = \sum_{n=o}^{\infty} g_n(|\underline{r}|,|r'|)$$

With the notation

$$D_{r,r'}(s) = \frac{1}{4\pi} \frac{|\underline{r} - \underline{r}'|}{|r-s||\underline{r}'-\underline{s}|} \exp\left[ik(|\underline{r} - \underline{s}| + |\underline{r}'-\underline{s}| - |\underline{r} - \underline{r}'|)\right] \ldots\ldots (6)$$

$$= \frac{1}{4\pi} \frac{\exp\left[ik|r-s|\right]}{|r-s|} \frac{\exp\left[ik|\underline{r}' - \underline{s}|\right]}{|r'-s|} \left[\frac{\exp ik|r-r'|}{|r-r'|}\right]^{-1}$$

$$= \frac{G_o(\underline{r},s) G_o(\underline{r}',\underline{s})}{G_o(r\ r')}$$

om (5) we get

$$G(r,r)/G_o(r\ r') = \int \frac{G_n(r,s)V(s)G_m(s,r')}{G_o(r,r')}\, d\underline{s}$$

$$g_{n+m+1}(r,r') = \int \frac{G_n(\underline{r},\underline{s})}{G_o(r,r')} \frac{G_o(\underline{r},\underline{s})G_o(\underline{r}',\underline{s})}{G_o(\underline{r},\underline{s})G_o(\underline{r}',\underline{s})} G_m(\underline{s},r')\, d\underline{s}$$

$$g_{n+m+1}(r,r') = \int \frac{G_n(\underline{r},\underline{s})}{G_o(\underline{r},\underline{s})} \frac{G_o(\underline{r},\underline{s})G_o(\underline{r}',s)}{G_o(\underline{r},\underline{r}')} \frac{G_m(s,r')}{G_o(s,r')}\, d\underline{s}$$

$$= \int g_n(r,s)D_{rr}(s)V(s)g_m(s,r')\, ds. \quad \ldots\ldots\ldots\ldots (7\text{-a})$$

norm" may be assigned to the g functions as follows:

$$\|g_i\| = \operatorname*{Max}_{r,r'} |g_i\ (\underline{r},\underline{r}')| \quad \ldots\ldots\ldots\ldots\ldots\ldots (7\text{-b})$$

e the norm $\|g_i\|$ is the manimum numerical value attained by $g_i(\underline{r},\underline{r}')$ $\underline{r}$ and $\underline{r}'$ vary. The norm is well defined for any fn. of two variables d is either a non-negative real number or infinity.

A corresponding numerical measure of the potential may be xpressed in terms of a dimensionless quantity $I_v(r)$

$$I_v(\underline{r}) = \int \frac{V(s)}{|\underline{r}-\underline{s}|}\, d\underline{s}$$

Where corresponding to (7) we define the "strength" V of ne potential V as

$$\|V\| = \frac{1}{2\pi} \max_r I_v(r)$$

Simple Potentials. Consider first the Born Series for simple potentials. potential V(s) will be termed "simple" if both V and its gradient exist

r all s, are bounded in magnitude, and vanish when $|(\underline{s})|$ exceeds some
nite magnitude S. The strength $\|V\|$ of a simple potential is certainly
nite and one may distinguish between weak and strong potentials accor-
ng to whether $\|V\|<1$ or $\|V\|\geqslant 1$.

It can be shown that

$$\lim_{k\to\infty} \|g_k\| = 0 \qquad \ldots\ldots\ (8)$$

[Page 1085, Nuovo Cimento, Vol.10, No.6, 1958]

From the Inequality $|z_1|+|z_2|\geqslant|z_1+z_2|$

with $z_1=|\underline{r}-\underline{s}|$ & $z_2=|\underline{s}-\underline{r}'|$ we obtain

$$|\underline{r}-\underline{s}|+|\underline{s}-\underline{r}'|\geqslant|\underline{r}-\underline{r}'|$$

us from (6)

$$\left|D_{rr'}(s)\right| = \frac{1}{4\pi}\frac{|\underline{r}-\underline{r}'|}{|\underline{r}-\underline{s}||\underline{r}'-\underline{s}|} \qquad \frac{1}{4\pi}\frac{|\underline{r}-\underline{s}|+|\underline{s}-\underline{r}'|}{|\underline{r}-\underline{s}||\underline{s}-\underline{r}'|}$$

$$\left|D_{rr'}(s)\right|\leqslant\frac{1}{4\pi}|s-r'|^{-1}+\frac{1}{4\pi}|r-s|^{-1}=\frac{1}{4\pi}|\underline{r}'-\underline{s}|^{-1}+\frac{1}{4\pi}|\underline{r}-\underline{s}|^{-1}$$

By (7-a) and (7-b)

$$\|g_{n+m+1}\|=\max_{r,r'}|g_{n+m+1}(r,r')|=\max_{r,r'}\left|\int g_n(\underline{r},\underline{s})D_{rr'}(s)V(s)\, g_m(\underline{s},\underline{r}')\,d\underline{s}\right|$$

$$\leqslant\max_{r,r'}\int|g_n(\underline{r},\underline{s})||D_{rr'}(s)||V(s)||g_m(s,r')|\,d\underline{s}$$

$$\leqslant\max_{rr'}\int|g_n(r,s)|\frac{1}{4\pi}\left[|\underline{r}-\underline{s}|^{-1}+|\underline{r}'-\underline{s}|^{-1}\right]|gm(s,r')||V(s)|ds$$

$$=\|gn\|\|gm\|\max\frac{1}{4\pi}\int\left[|\underline{r}-\underline{s}|^{-1}+|\underline{r}'-\underline{s}|^{-1}\right]|V(s)|\,ds$$

$$=\|gn\|\|gm\|\max_{r'}\frac{1}{4\pi}\int\frac{|V(\underline{s})|ds}{|r'-s|}+\max_{r}\frac{1}{4\pi}\int\frac{|V(s)|\,ds}{|r-s|}$$

But $$= \frac{1}{2\pi} \max_{r'} \int \frac{|V(\underline{s})|}{|\underline{r}'-\underline{s}|}\, d\underline{s} = \|V\|$$

Hence

$$\|g_{n+m+1}\| \leq \|g_n\|\,\|g_m\| \left[\frac{\|V\|}{2} + \frac{\|V\|}{2}\right] = \|g_n\|\,\|g_m\|\,\|V\|$$

$$\|g_{n+m+1}\| \leq \|g_n\|\,\|g_m\|\,\|V\| \qquad \text{.......... (10)}$$

The equation $g(rr') = \sum_{n=0}^{\infty} g_n(rr')$ may now be treated by

mparison with a geometric series; Because from (10)

$$\|g_n\| \leq \|g_{n-1}\|\,\|g_0\|\,\|V\| = \|g_{n-1}\|\,\|V\| \leq \|g_{n-2}\|\,\|g_0\|\,\|V\|\,\|V\| = \|g_{n-2}\|\,\|V\|^2$$
$$\leq \|g_0\|\,\|V\|^n = \|V\|^n$$

nce, $\|g_n\| \leq \|V\|^n$

nce, $\|g\| = \sum_{n=0}^{\infty} g_n$

$$\|g\| \leq \sum_n \|V\|^n = (1-\|V\|)^{-1}$$

Which forms a convergent series for $\|V\|<1$ Hence, $\|g\|$ nverges uniformly and absolutely in r, r' if $\|V\|<1$. The absolute and uniform nvergence of $G = \sum_{n=0}^{\infty} G_n$ is also implied for all real values of k, except a spatiol neighborhood of the singular points $\underline{r} = \underline{r}'$.

<u>Potentials of arbitrary strength</u>. Now even if $V \nless 1$ equation) shows that $\|g_1\|\,\|V\|<1$ for large k. Let $\|g_{2n}\| = \|g_x\|$ where x is even,

Using equation (10) we have

$$\|g_x\| = \|g_{x-2+1-1}\| \leq \|g_{x-2}\|\,\|g_1\|\,\|V\|$$

$$\leq \|g_{x-4}\|\,\|g_1\|^2\,\|V\|^2$$

$$\|g_{2n}\| \qquad \leq \|g_{x-(x-2)}\|\,\|g_1\|^{(x-2)/2}\,\|V\|^{(x-2)/2}$$

$$\|g_2\|\,\|g_1\|^{(2n-2)/2}\,{}_V{}^{(2n-2)/2}\,\|V\|$$

$$\leqslant \|g_1\| \, \|V\| \, \|g_1\|^{n-1} \, \|V\|^{n-1}$$

$$\|g_{2n}\| \leqslant (\|g_1\| \, \|V\|)^{n}$$

t

$$\|g_{2n+1}\| = \|g_y\| \text{ where y is odd.}$$

us

$$\|g_y\| \leqslant \|g_{y-2}\| \, \|g_1\| \, \|V\|$$

$$\leqslant \|g_{y-4}\| \, \|g_1\|^2 \, \|V\|^2$$

$$\leqslant \|g_{y-(y-1)}\| \, \|g_1\|^{\frac{y-1}{2}} \, \|V\|^{\frac{y-1}{2}}$$

$$= \|g_1\| \, \|g_1\|^{\frac{2n+1-1}{2}} \, \|V\|^{\frac{2n+1-1}{2}}$$

$$\|g_{2n+1}\| \leqslant (\|g_1\| \, \|V\|)^{n} \, \|g_1\| \qquad \ldots\ldots\ldots\ldots\ldots \quad (13)$$

Thus from 12 and (13) it is concluded that for sufficiently arge k, $(\|g_1\|\|V\| < 1)$, the Born series

$$g(\underline{r},\underline{r}') = \sum_{n=o}^{\infty} g_{2n}(\underline{r},\underline{r}') + \sum_{n=o}^{\infty} g_{2n+1}(\underline{r},r') \quad \ldots\ldots \quad (13\text{-a})$$

Converges uniformly and absolutely and that

$$\|g\| \leqslant \sum_n (\|g_1\|\|V\|)^n + \|g_1\| \sum_n (\|g_1\| \, \|V\|)^n$$

$$= \left[1 - \|g\| \, \|V\|\right]^{-1} + \|g_1\| (1 - g_1 \, V)^{-1}$$

$$\|g\| \leqslant (1 + \|g_1\|)(1 - \|g_1\|\|V\|)^{-1}$$

From (13-A) It is obvious that

$G(r,r') \rightarrow G_o(rr')$ for large k becomes $\mathrm{Lt}_{k\to\infty}$

$\max_{rr'} |g_n(r,r',)| = o$ for $n \geqslant 1$

<u>Convergence of the wave fn. and the scattering amplitude:</u>

for real positive values of k)

$$\Psi_{n+1}(\underline{r}) = \int G_n(\underline{r},\underline{s}) \, V(s) \, _o(\underline{s}) \, d\underline{s} \quad \ldots\ldots\ldots\ldots \quad (14\text{-A})$$

$$= -\frac{1}{4\pi} \int \exp \frac{i\,k|\underline{r}-\underline{s}|}{|\underline{r}-\underline{s}|} \, \frac{G_n(\underline{r},\underline{s})}{G_o(\underline{r},\underline{s})} \, V(\underline{s}) \, _o(\underline{s}) \, d\underline{s}$$

$$\left|\Psi_{n+1}(\underline{r})\right| \leqslant \frac{1}{4\pi}\int \frac{1}{|\underline{r}-\underline{s}|}\left|gn(\underline{r},s)\right|\left|V(\underline{s})\right|\left|\Psi_o(\underline{s})\right| \quad d\underline{s} \qquad {}_o(s)=e^{iks}$$

$$\leqslant \max_{r,s}\frac{1}{4\pi}\int\frac{1}{|\underline{r}-\underline{s}|}\left|g_n(r,s)\right|\left|V(\underline{s})\right| \; ds$$

$$= \|g_n\| \max_{r,s}\frac{1}{4\pi}\int\frac{1}{|\underline{r}-\underline{s}|}\left|V(\underline{s})\right| \; ds$$

$$= \frac{1}{2\pi}\|g_n\| \max_{s}\|V\|$$

$$= \frac{1}{2\pi}\|g_n\|\,\|V\|$$

$$\left|\Psi_{n+1}(\underline{r})\right| \leqslant \frac{1}{2\pi}\|gn\|\,\|V\| < \|g_n\|\,\|V\|$$

Thus $\Psi = \sum_{n=o}^{\infty}\Psi_n$ converges whenever $\sum_n g_n$ converges.

also,

$$\underset{k\to\infty}{Lt}\ \|g_n\| = o \quad \text{for } n \geqslant 1,$$

Thus for sufficiently high energy $\Psi \to \Psi_o$

Because of (14-A) the convergent Born series satisfies the integral equation

$$\Psi = \Psi_o + G_oV = \Psi_o + GV\Psi_\theta \qquad \text{.......................} \quad (15\text{-}A)$$

and for larger

$$\Psi(r) = \Psi_o(r) + \frac{\exp(ikr)}{r}\, f(\theta,\phi) \qquad \text{.......................} \quad (15\text{-}B)$$

where

$$f(\theta,\phi) = \sum_{n=o}^{\infty} f_n(\theta,\phi) \quad \text{...............................} \quad (16)$$

with

$$f(\theta,\phi) = \underset{r\to\infty}{Lt}\ r\exp\left[-ikr\right]\int G_o(\underline{r},\underline{s})\ V(s)\Psi_n(\underline{s})\ d\underline{s}$$

$|\underline{r}-\underline{s}| = r - s\cos\alpha$; $i|k'|\,|\underline{r}-\underline{s}| = ikr - i\underline{k}'\underline{s}$ where $|k'| = k$

$$G_o(r,s) = -(4\pi|r-s|)^{-1}\exp(ik|\underline{r}-\underline{s}|) \approx -(4\pi r)^{-1}\exp(ikr)\exp(-i\underline{k}'.\underline{s})$$

$$f_n(\theta,\phi) = r\exp\left[-ikr\right]\left(-\frac{1}{4\pi}\right)\int\frac{1}{r}\exp(ikr)\exp(-i\underline{k}'.\underline{s})\ V(s)\Psi_n(s)\ d\underline{s}$$

$$= -\frac{1}{4\pi}\int\exp(-i\underline{k}'.\underline{s})\ V(\underline{s})\Psi_n(\underline{s})\ d\underline{s} \quad \text{.......................} \quad (17)$$

θ, and ϕ are polar angles of $\underline{k}'$ with respect to $\underline{k}$

so

$$f_n(\theta,\phi) = r\exp(-ikr)\, G_n(r,\underline{s})\, V(\underline{s})\ {}_{o}(\underline{s})\, d\underline{s}$$

$$f_n(\theta,\phi) = r\exp(-ikr)\ -\frac{1}{4\pi}\int \frac{\exp(ik|\underline{r}-\underline{s}|}{|\underline{r}-\underline{s}|}\frac{G_n(\underline{r},\underline{s})}{G_o(\underline{r},\underline{s})}V(\underline{s})\Psi_\theta(\underline{s})\, d\underline{s}$$

$$f_n(\theta,\phi) \leqslant \int \frac{r}{|r-s|}\left|\gamma_n(r,s)\right|\left|V(\underline{s})\right| d\underline{s} \leqslant \max_{r,s}\int \frac{r}{|r-s|}\left|\gamma_n(r,s)\right|\left|V(s)\right| ds$$

$$f_n(\theta,\phi) \leqslant \int \|\gamma n\| \max_{r}\int \frac{r}{|r-s|}\left|V(s)\right| d\underline{s}$$

Thus $f(\theta,\phi) = \sum_{n=0}^{\infty} f_n(\theta,\phi)$ converges, because for sufficie-

ly large s the integral is both finite and bounded Ihms

$$\operatorname*{Lt}_{k\to\infty} f(\theta,\phi) = f_o(\theta,\phi)$$

l results so far derived are valid for finite potentials only.

eneral potentials.

ippose $\quad I_v(r) = \int \frac{V(s)}{|\underline{r}-\underline{s}|}\, ds$

atisfies

$I_v(r) < \infty$ for all r .... (A)

$I_v(r)$ is continuous in r .... (B)

$I_v(r) \sim 0\left(\frac{1}{r}\right)$ as $r \to \infty$ .... (C)

hence,

$$\|v\| < \infty$$

nd $\|v\|$ can be approximated by a simple potential in the sense that

iven $\epsilon < 0$, a simple potential U can be constructed

$$\int \frac{|V(\underline{s}) - U(\underline{s})|}{|\ \underline{r} - \underline{s}\ |}\, d\underline{s}$$

[E.C. Titchmarch = Theory of functions, Chapter X section

2.2]

Let $g_1(r,r')$ and $\bar{g}_1(\underline{r},\underline{r}')$ denote the first order g

unctions for V and U respectively i.e by (7-a)

$$g_1(rr') = \int g_o(r,s)\, D_{rr'}(s)\, V(s)\, g_o(s,r')\, ds$$

r

$$g_1(rr') = \int D_{rr'}(s)\, V(s)\, ds.$$

nd

$$\bar{g}_1(r,r') = \int D_{rr'}(s) U(s)\, ds$$

r

$$g_1(rr') - \bar{g}_1(rr') = \int D_{rr'}(s)\left[V(s) - U(s)\right] d\underline{s}$$

$$\left| g_1(rr') - \bar{g}_1(rr') \right| \leqslant \int \left| D_{rr'}(s) \right| \left| (V(s) - U(s) \right| d\underline{s}$$

ut,

$$\left| D_{rr'}(s) \right| \leqslant \frac{1}{4\pi}\left[\frac{1}{r-s} + \frac{1}{r'-s} \right]$$

hus,

$$\left| g_1(rr') - \bar{g}_1(rr') \right| \leqslant \frac{1}{4\pi}\int \left[|\underline{r}-\underline{s}|^{-1} + |\underline{r}' - \underline{s}|^{-1} \right] |V(s) - U(s)|\, d\underline{s}$$

$$\left| g_1(r,r') - \bar{g}_1(r,r') \right| \leqslant \frac{1}{4\pi}\left[\int \frac{|V(s) - U(s)|}{|r-s|} d\underline{s} + \int \frac{|V(s) - U(s)|}{|r'-s|} d\underline{s} \right]$$

$$\leqslant \frac{1}{4\pi}\,(\epsilon + \epsilon) = \frac{\epsilon}{2\pi}$$

ence,

$$\left| g_1(r,r') - \bar{g}_1(r,r') \right| \leqslant \frac{\epsilon}{2\pi}$$

herefore,

$\mathop{Lt}\limits_{k\to\infty} \left\| g_1(r,r') - \bar{g}_1(r,r') \right\|$ is less than any positive number and ust be zero.

$$\mathop{Lt}\limits_{k\to\infty} \left\| g_1(r,r') - \bar{g}_1(r,r') \right\| = o$$

It is thus concluded that (A), (B) and (C) may be offered s sufficient conds. On any potential to ensure that the non-relativistic orn series converges at high energies. Moreover, in the high energy limit nly the leading term of each series contributes.

:tension to complex energy.

The convergence of the Born series is uniform also in the ›per half z-plane with $z=k+i\beta$: $\beta > 0$

$$D_z(r,r'|s) = \frac{1}{4\pi}\frac{|\underline{r}-\underline{r}'|}{|r-s||r'-s|}\exp\left[i(k+i\beta)\left(|r-s|+|r'-s|-|r-r'|\right)\right]$$

$$D_z(r,r'|s) = \frac{1}{4\pi}\frac{|r-r'|}{|r-s||r'-s|}\left|\exp\left[-\beta\left(|\underline{r}-\underline{s}|+|\underline{r}'-\underline{s}|-|\underline{r}-\underline{r}'|\right)\right]\right|$$

$$= |D_k(r,r'|s)|\ \exp\left[-\beta\left(|\underline{r}'-\underline{s}|+|\underline{r}'-\underline{s}|-|r-\underline{r}'|\right)\right]$$

ut,

$$|\underline{r}-\underline{s}|+|\underline{r}'-\underline{s}| \geq |\underline{r}-\underline{s}-r'+s| = |r-r'|$$

ence

$$|D_z(r,r'\ s)| \leq |D_k(r,r'|\ s)|\left|\exp\left[-\beta\left(|r-r'|-|r-r'|\right)\right]\right|$$

$$= |D_k(r,r'|\ s)|$$

$$|D_z(r,r'|\ s)| \leq |D_k\ (r,r'\ s)|$$

Thus the Born series will converge uniformly for complex values.

The preceding results may be stated as

$$\operatorname*{Lt}_{\substack{z\to\infty\\ \beta\geq 0}}\ \max_{rr'}\left|g_1(r,r',z)\right| = 0$$

It has been shown that for $|z| > R$, $\beta \geq 0$ the Born series for :(r,r',z) and consequently $G(r,r',z)$ converges uniformly in z. Ultimately he series the represented by its leading term with arbitrary precision .e

$$\left|G(rr',z) - G_o(rr',z)\right| \leq \delta(z)\frac{e^{-\beta|\underline{r}-\underline{r}'|}}{|r-r'|} \quad \ldots\ldots\ldots\ldots\ (23)$$

here $\delta(z) \to 0$ uniformly as $|z|\to\infty$ in any nammer for $\beta \geq 0$.

For the validity of the above statements

$$I_v = \int\frac{|V|}{|r-r'|}d^3r' \quad \text{satisfies}$$

$$I_v(r) < \infty \qquad \text{for all } \underline{r} \quad \dots\dots\dots\dots\dots\dots \text{(A)}$$

$$I_v(\underline{r}) \text{ is continuous in } \underline{r} \quad \dots\dots\dots\dots\dots\dots \text{(B)}$$

nd $$I_v(\underline{r}) \sim 0\left(\frac{1}{r}\right) \text{ as } r \to \infty \quad \dots\dots\dots\dots\dots\dots \text{(C)}$$

A) is equivalent to

$$\max_{r'} \frac{1}{2\pi}\int \frac{|V(\underline{r})|}{|\underline{r} - \underline{r}'|} \, d^3r \leq M \leq \infty \quad \dots\dots\dots\dots\dots \text{(D)}$$

B) is equivalent to

$$\frac{1}{4\pi}\int e^{\kappa r}\, V(\underline{r}) \, d^3r \leq L_\alpha < \infty \quad \dots \kappa > 0 \dots\dots\dots \text{(E)}$$

κ^{-1} = range of the pot.

D) and (E) implies

$$\max_{r'} \frac{1}{4\pi}\int e^{\kappa r} \frac{|V(\underline{r})|}{|\underline{r}-\underline{r}'|} \, d^3r \leq N_\alpha \leq \infty \quad \dots\dots\dots\dots\dots\dots \text{(F)}$$

ection III: The convergence of the partial wave amplitude.

The dispersion relation is

$$f(s,t) = f_B(t) + \sum_{b=1}^{N} \frac{R_b(i|E_b|,t)}{s+E_b} + \frac{1}{\pi} \underset{\eta\to 0}{Lt} \int_0^{\infty} \frac{Imf(s',t)}{s'-s-i\eta} ds'$$

hich is valid for all values of s on the entire complex s-plane subject the condition

$$t < 4K^2$$

For nonforward scattering $t \neq o$ the evaluation of the integal demands the knowledge of $Imf(s',t)$ doun to $s'=0$ which is practically impossible because experiments can give us values of $Imf(s',t)$ for $s' \geqslant \frac{1}{4} t$ [as $4 \sin^2 \theta/_2 = t$, and for physical angles $|\sin\theta| \leqslant 1$] . In other ords, the dispersion integral is extended not only over the physical egion but also covers the unphysical region

$$0 \leqslant s' \leqslant \frac{1}{4} t$$

Therefore, some devise for computing the contributions rom the unphysical region is required, i.e we need the analytic contiuation of $Imf(s',t)$ into the unphysical region. Consider the partial ave expansion

$$f(s,t) = \frac{1}{\sqrt{\Delta}} \sum_{\ell=o}^{\infty} (2\ell+1)e^{i\delta_\ell(s)} \sin\delta_\ell(s) P_\ell(\cos\theta = 1-\frac{t}{2s}) \quad (22)$$

or s real. $\delta_\ell(s)$ is the (real) phase shift induced by the potential.

The analytic continuation of eq.(22) with respect to s for fixed down to s=0 will now be justified to hold for $t < 4K^2$. In other ords, it will be shown that for $z=\cos\theta$ restricted to real values (not ecessarily $|z| < 1$) the imaginary part of (22) converges uniformly with espect to s as long as

$$|z| < 1 + \left(\frac{2K^2}{s}\right) \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \qquad (24)$$

r potentials satisfying

$$\frac{1}{4\pi}\int e^{Kr}\,|U(r)|\,d^3r < \infty \qquad K>0$$

$$\max(r')\,\frac{1}{2\pi}\int\frac{|U(\underline{r})|\,d^3r}{|\underline{r}-\underline{r}'|} \ll \infty$$

As the integral involves Imf(s't) it is sufficient to

onsider only

$$f_1(s,t) = f(s,t) - f_B \quad \text{(because } f_B \text{ is real)}$$

where, $f_1(k,\Delta) = -\frac{1}{4\pi}\iint \exp\left[-\frac{1}{2}\,i\,\underline{\Delta}\cdot(\underline{r}+\underline{r}')\right]\exp\left[i(k^2-\frac{1}{4}\Delta^2)^{1/2}\right.$

$$\left.\underline{n}\cdot(\underline{r}-\underline{r}')\right]U(r')G(\underline{r},\underline{r}'k)U(r)\,d^3r'\,d^3r \qquad \ldots\ldots\ldots\ldots\ldots\ldots (25)$$

Introduce the angle (ν,ϕ), $(\nu'\,\phi)$ define by the

quations.

$$\underline{\Delta}\cdot\underline{r} = r\cos\nu, \qquad \underline{\Delta}\cdot\underline{r}' = r'\cos\nu'$$

$$\underline{n}\cdot\underline{r} = r\sin\nu\cos\varphi \qquad \underline{n}\cdot\underline{r}' = r'\sin\nu'\cos\varphi'$$

ubstituting in (25) we get

$$f_1(k,\Delta) = -\frac{1}{4\pi}\iint\exp\left[-\frac{i\Delta}{2}r\cos\nu \pm i(k^2-\frac{\Delta^2}{4})^{1/2}r\sin\nu\cos\varphi\right]\times U(r)$$

$$\times\exp\left[-\frac{i\Delta}{2}r'\cos\nu' \pm i(k^2-\frac{\Delta^2}{4})^{1/2}r'\sin\nu'\cos\varphi'\right]U(r')\ g(\underline{r}\underline{r}',k)$$

$$\times\, G(r,r';k)\,d^3r\,d^3r' \qquad \ldots\ldots\ldots (26)$$

$$f_1(k,\Delta) \leq \frac{1}{4\pi}\iint\left|\exp\left[-\frac{i\Delta}{2}r\cos\nu \pm i(k^2-\frac{\Delta^2}{4})^{1/2}r\sin\nu\cos\varphi\right]\right|\,|U(r)|$$

$$\left|\exp\left[-i\frac{\Delta}{2}r'\cos\nu' \pm i(k^2-\frac{\Delta^2}{4})r'\cos\varphi'\sin\nu'\right]\right|U(r')|\,|G(rr'k)|\,d^3r\,d^3r'$$

Thus eq (26) will represent an analytic function of Δ, for

fixed k, for those values of Δ for which.

$$\frac{1}{4\pi}\int\exp\left\{(\mathrm{Im}\,\frac{\Delta}{2})(\cos\nu')\,r + \left|\mathrm{Im}\,(k^2-\frac{\Delta^2}{4})^{1/2}\right|r\sin\nu\cos\varphi\right\}|U(r)|\,d^3r < \infty$$

$$U(r)\,d^3r$$

or because of the condition

$$\frac{1}{4\pi}\int e^{Kr} \ |U(r)| \ d^3r$$

$$\max(\nu\varphi) \ \left\{ \left|\mathrm{Im}\ \frac{\Delta}{2}\right|\mathrm{Cos}\,\nu + \left|\mathrm{Im}(k^2 - \frac{\Delta^2}{4})^{1/2}\right|\mathrm{Sin}\,\nu\ \mathrm{Cos}\,\varphi \right\} < K \ldots\ldots \quad (27)$$

ere max($\nu\varphi$) means maximum with respect to the variation of ν and φ. ere is exactly similar condition for the r' integration involving $\nu'\varphi'$). The maximum value of (27) is obtained for $\mathrm{Cos}\,\varphi = 1$ and differentiating (27) w.r.t. ν and equating to zero

$$-\left|\mathrm{Im}\ \frac{\Delta}{2}\right| \mathrm{Sin}\,\nu + \left|\mathrm{Im}\ (k^2 - \frac{\Delta}{4}^2)^{1/2}\right|\mathrm{Cos}\,\nu = 0$$

$$\text{or } \tan\nu = \frac{\left|\mathrm{Im}\ (k^2 - \frac{\Delta}{4}^2)^{1/2}\right|}{\left|\mathrm{Im}\ \frac{\Delta}{2}\right|} = \frac{B}{A} \text{ say, } \ldots\ldots\ldots\ldots\ldots\ldots \quad (28)$$

ne maximum value of (27) is obtained if we substitute (28) in (27). 27) can be written as

$$\left\{A^2\mathrm{Cos}^2\nu + B^2\sin^2\nu + 2AB\ \mathrm{Sin}\,\nu\ \mathrm{Cos}\,\nu\right\}^{1/2}$$

$$\left\{A^2(1-\mathrm{Sin}^2\nu) + B^2\sin^2\nu + AB\ \mathrm{Sin}\ 2\nu\right\}^{1/2}$$

$$\mathrm{Cot}^2\nu = \frac{A^2}{B^2} = \mathrm{Cosec}^2\nu - 1;$$

$$\mathrm{Cosec}^2\nu = 1 + \frac{A^2}{B^2} = \frac{A^2+B^2}{B^2}$$

r

$$\mathrm{Sin}^2\nu = \frac{B^2}{A^2+B^2}; \text{ also } \mathrm{Sin}\ 2\nu = \frac{2\tan\nu}{1+\tan^2\nu} = \frac{2\ B/A}{1+\frac{B^2}{A^2}} = \frac{2AB}{A^2+B^2}$$

hus (27) becomes

$$\left\{A^2\frac{(A^2)}{A^2+B^2} + \frac{B^2(B^2)}{A^2+B^2} + 2\frac{A^2B^2}{A^2+B^2}\right\}^{1/2} < K$$

r

$$\left\{\frac{(A^2+B^2)^2}{(A^2+B^2)}\right\}^{1/2} < K$$

r

$$(A^2 + B^2)^{1/2} < K$$

$$\left\{\left|\text{Im}\ \frac{\Delta}{2}\right|^{2} + \left|\text{Im}\ (k^2 - \frac{\Delta^2}{4})^{1/2}\right|^{2}\right\}^{1/2} < K \ldots\ldots\ldots\ldots\ (29)$$

get an idea of (29) continue the eq. $\Delta = 2k\ \text{Sin}\ \theta/_2$ into the complex
ne.

$$\Delta = 2k\text{Sin}\ \Phi/_2$$

re

$$\Phi = \theta + i\psi$$

equality (29) becomes

$$\left\{\left|\text{Im}\ \text{Sin}\ \frac{\Phi}{2}\right|^{2} + \left|\text{Im}\ \text{Cos}\ \Phi/_2\right|^{2}\right\}^{1/2} < \frac{K}{k} \ldots\ldots\ldots\ (30)$$

$$\sin \Phi/_2 = \text{Sin}\ \frac{1}{2}(\theta + i\psi) = \sin\theta/_2\ \text{Cos}\ \frac{i\psi}{2} + \text{Cos}\ \frac{\theta}{2}\ \text{Sin}\ \frac{i\psi}{2}$$

$$\sin\frac{\Phi}{2} = \sin\theta/_2\ \text{Cosh}\ \frac{\psi}{2} + i\ \text{Cos}\ \theta/_2\ \text{Sinh}\ \frac{\psi}{2} \ldots\ldots(30\text{-a})$$

$$\text{Cos}\ \frac{1}{2}\Phi = \text{Cos}\ \frac{1}{2}\ \theta\text{Cosh}\frac{1}{2}\psi\ i\ \text{Sin}\ \theta/_2\ \text{Sinh}\ \frac{1}{2}\psi \ldots\ldots\ldots\ (30\text{-b})$$

us (30) becomes

$$\left\{\left|\text{Cos}\ \theta/_2\ \text{Sinh}\ \frac{\psi}{2}\right|^{2} + \left|-\text{Sin}\ 0/_2\ \text{Sinh}\ \frac{\psi}{2}\right|^{2}\right\}^{1/2} < \frac{K}{k}$$

$$\left\{\sin^2\theta/_2 + \text{Cos}^2\theta/_2)\ \text{Sinh}^2\frac{1}{2}\psi\right\}^{1/2} < \frac{K}{k}$$

$$\left|\sinh \frac{1}{2}\psi\right| < \frac{K}{k} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ (31)$$

The significance of (31) can be sun by expanding

$$f_1(k, \Delta) = F_1(k, \sin\Phi/2) = F_1(k, Z)$$

a power series

$$F_1(k, Z) = \sum_{p=o}^{\infty} A_p(k)\ Z^{2p} \ldots\ldots\ldots\ldots\ldots\ldots(31\text{-a})$$

ere only even powers occur. From (30-a) with $\text{Sin}\frac{1}{2}\Phi = Z = X + iY$, we have

$$X = \sin\frac{1}{2}\,\theta\ \text{Cosh}\ \frac{1}{2}\psi$$

$$Y = \text{Cos}\frac{1}{2}\,\theta\ \text{Sinh}\ \frac{1}{2}\psi \qquad \dots\dots\dots\dots\dots\dots (32)$$

or const ψ & $0 \leqslant \theta \leqslant 4\pi$, we have from (30)

$$\frac{X^2}{\text{Cosh}^2\ \frac{1}{2}\psi} + \frac{Y^2}{\text{Sinh}^2\ \frac{1}{2}\psi} = \text{Cos}^2\frac{1}{2}\,\theta + \text{Sin}^2\frac{1}{2}\,\theta = 1$$

.e (32) is an equation of an ellipse, in the Z plane, with focus $\pm$ 1, emimajor axis Cosh $\frac{1}{2}\psi$ semiminor axis Sinch $\frac{1}{2}$

According to $|\text{Sinh}\ \frac{1}{2}\psi| < \frac{K}{k}$

he series $F_1(k,Z) = \sum_{p=o}^{\infty} A_p(k)\ Z^{2p}$ converges within a circle of radius $\frac{K}{k}$ angent along the y-axis to a maximum ellipse defined by $|\sinh\ \frac{1}{2}\psi| < \frac{K}{k}$

The equation of this circle is

$$Z = \left(\frac{K}{k}\right)\ \exp\left(\frac{1}{2}\ i\theta\right)$$

nd $F_1(k,Z) = \sum_{p=o}^{\infty} A_p(k)\ Z^{2p}$ converges for $|Z| < \frac{K}{k}$. For the physical egion $Z = \text{Sin}\ \theta/2$ [actually $Z = \text{Sin}\frac{1}{2}\ (\theta + i\psi)$] $F_1(k,Z)$ represents a series hich is absolutily and uniformly convergent in $\sin\ \frac{1}{2}\ \theta$ for $\sin\ \frac{1}{2}\ \theta < \frac{K}{k}$

The following additional consequences can also be drawn rom the above considerations. (1) For fixed O, there is some $k = k_m$ hich is the maximum allowable. For $k < k_m$ the convergence of (31-a) is niform w.r.to k. This can be shown by straight forward application of he ratio and comparesion tests.

(2) If we reexpress (31-a) as a series of Legendre lynomials $P_\ell(Z^2)$, the resulting series will converge inside the lipse detirmined by eq.(31). This is because of a theorem which ates:

If a fn. of θ is analytic in the complex cos θ plane, with- an ellipse with focus at +1,-1, then function can be expressed in a gendre series over the entire inside of that ellipse".

Put $\triangle = 2\,Zk$,

The Series $F_1(k,\triangle) = \sum_{p=0}^{\infty} b_p(k)\,\triangle^{2p}$ .... (33)

Converges within a circle of radus $\triangle < 2K$ uniformly in k. experiments are analyzed by the series (33) then the continuation of ch a series would form a satisfactory basis for continuation of F_1 to the unphysical region. But, the above is completely equivalent to e partial wave expansion.

Introduce the variable

$$z = \cos\phi = 1-2\,Z^2$$

ith $z = x+iy$, from (30-a) & (30-b) e get,

$$x = \cos\theta \cosh\psi$$

$$y = \sin\theta \sinh\psi$$

hich again represent for constant ψ a family of ellipses in the z-plane, ith focus at +1,-1, semimajor axis $\cosh\psi$, semiminor axis $\sinh\psi$. The onvergence condition (31) becomes.

$$|\sinh\psi| = 2\left|\sinh\tfrac{1}{2}\psi \cosh\tfrac{1}{2}\psi\right| \leqslant \frac{2K}{k}\left[1+\frac{K^2}{k^2}\right]^{1/2}$$

$$|\cosh\psi| \leqslant 1 + \left(\frac{2K^2}{k^2}\right)$$

A power series expansion in z will therefore converge
hin a circle

$$|z| < 2\left(\frac{K}{k}\right)\left[\ 1+ \quad \left(\frac{K^2}{k^2}\right)\right]^{\frac{1}{2}}$$

reas the partial wave expansion in $P_\ell(z)$ will converge within the
ire ellipse of semimajor axis $\left(1 + 2\ \frac{K^2}{k^2}\right)$. Returning to real values
z, we obtain the condition

$$|z| < 1 + \frac{2K^2}{k^2}$$

In particular as $k \to 0$, the partial wave expansion conver-
for all real z if $K > 0$. If $K = 0$ the region of convergence shrinks
real values of cos θ.

It is to be noted that the region of convergence of the
ire amplitude $f(k,\Delta)$ are obtained from the above by the replacement
$\to \frac{1}{2} K$. This shrink age is entirely due to the properties of the
n approximation, i.e. the potential.

tion IV: Completion of the proof of the dispersion relations

Two assertions were made in section II. These two assertions
l be proved here.

To justify the interchange of the orders of integration it must
shown that $f(z,\Delta)$ defined by equation ( ) section II converges
formly with respect to z for z on the semicircle. Since it has been
en as granted the behaviour on the real part of the semicircle, it is
y necessary to establish uniform convergence for $|z| = R$, Im $z > 0$, for
ufficiently large. Assertion II is then permitted as soon as it can

demonstrated that

$$\lim_{z\to\infty\ \beta\geq 0} \left| f(z,\Delta) - f_B(\Delta)\right| = 0 \quad \ldots\ldots (23\text{-A})$$

ere it should be noticed that the real axis is included in the sertion. If it can be established that the limit in (23-A) is attained iformly with respect to arg z, then by a standard test for uniform nvergence of integrals both assertions can be justified. Towards this d a detailed knowledge of the behaviour of $G(\underline{r},r';z)$ for large z is quired as studied in section II preceding section.

To prove (23-A) consider,

$$f_1(z,\Delta) = f(z,\Delta) - f_B(\Delta) + \iint T(z,\Delta)\, d^3r\, d^3r'$$

ere, $T(z,\Delta) = \left(\frac{1}{4\pi}\right)\exp\left[-(i/2)\,\underline{\Delta}.(\underline{r}+\underline{r}')\right]\exp\left[i(z^2-\frac{1}{4}\Delta^2)^{1/2}\right.$

$$\left.\underline{n}.(\underline{r}'-\underline{r})\right]U(r)G(\underline{r},r';z)U(r')$$

In accordance with (23) split G as

$$G = G_o + (G - G_o)$$

with corresponding decomposition

$$f_1 = f_{11} + f_{12}$$

ere, $f_{11} = \iint -(1/4\pi)\exp\left[-(i/2)\,\underline{\Delta}.(\underline{r}+\underline{r}')\right]\exp\left[i\left(z^2-(1/4)\Delta^2\right)^{1/2}\underline{n}(\underline{r}'-\underline{r})\right]U(r)$

$$G_oU(r')\, d^3r\, d^3r' \ldots\ldots\ldots\ldots (24)$$

d, $f_{12} = -(1/4\pi)\exp\left[-(i/2)\,\underline{\Delta}.(\underline{r}+\underline{r}')\right]\exp\left[i\left(z-(1/4)\Delta^2\right)^{1/2}\underline{n}\,(\underline{r}'-\underline{r})\right]U(r)$

$$\left[G(r,r';z) - G_o(r,r';z)\right]U(r')\, d^3r\, d^3r'$$

equation (23)

$$_2 \leq (1/4\pi)\iint \exp\left\{-\mathrm{Im}\left[z^2-(1/4)\Delta^2\right]^{1/2}\underline{n}.(\underline{r}'-\underline{r})\right\}|\underline{r}'-\underline{r}|^{-1}\delta(z)\exp\left[(-\beta|\underline{r}'-\underline{r}|)\right]U(r)$$

$$U(r')\, d^3r\, d^3r' \ \ldots\ldots\ldots \quad (25)$$

r sufficiently large z

$$\left[z^2-(1/4)\Delta^2\right]^{1/2} = \mathrm{Im}\left[z\left(1-(\Delta^2/4z^2)\right)^{1/2}\right] = \mathrm{Im}\left\{z\left[1-\Delta^2/8z^2-\Delta^4/64z^4\ldots\ldots\right]\right\}$$

$$\ldots z\left[1-(\Delta^2/8z^2)\right] = \mathrm{Im}\,(k+i\beta)\left(1-\frac{(z^2)^*\,\Delta^2}{8z^2(z^2)^*}\right) = \mathrm{Im}(k+i\beta)\left(1-\frac{(k^2-\beta^2-2i\beta k)\Delta^2}{8|z|^4}\right)$$

$$= \beta - \frac{1}{8}\frac{\left[(k^2-\beta^2)\beta-(2\beta k)k\right]\Delta^2}{|z|^4} = \beta - \frac{1}{8}\frac{\left[\beta(k^2-\beta^2-2k^2)\right]\Delta^2}{|z|^4}$$

$$= \beta - \frac{1}{8}\left[-\beta\frac{z^2}{|z|^4}\right]\Delta^2 = \beta\left(1+\frac{1}{8}\frac{\Delta^2}{|z|^2}\right)$$

Therefore, for sufficiently large $|z|$

$$\mathrm{Im}\left(z^2-(1/4)\Delta^2\right)^{1/2} = \mathrm{Im}\,z\left[1-(\Delta^2/8z^2)\right] < \beta\left[1+(\Delta^2/8z^2)\right] \ldots\ldots\ldots\ldots (26)$$

…so, $\exp\left\{-\mathrm{Im}\left[z^2-(1/4)\Delta^2\right]^{1/2}\hat{\underline{n}}.(\underline{r}'\underline{r}) - \beta|\underline{r}'-\underline{r}|\right\} = \exp\left\{\mathrm{Im}\left[z^2-(1/4)\Delta^2\right]^{1/2}\underline{n}.(\underline{r}-\underline{r}')\right\}\exp\left[-\beta|r'-r|\right]$

$$= \exp\left\{\mathrm{Im}\left[z^2-(1/4)\Delta^2\right]^{1/2}|\underline{r}-\underline{r}'|\,\mathrm{Cos}\,\alpha\right\}\exp\left[-\beta|\underline{r}'-\underline{r}|\right]$$

…t $|\underline{r}-\underline{r}'|\mathrm{Cos}\,\alpha \leq |\underline{r}-\underline{r}'| \leq |\underline{r}'| + |\underline{r}|$; and, $|\underline{r}'-\underline{r}| \geq |\underline{r}'| - |r|$

…nce, $\exp\left\{-\mathrm{Im}\left[z^2-(1/4)\Delta^2\right]^{1/2}\underline{n}.(\underline{r}'-\underline{r}) - \beta|\underline{r}'-\underline{r}|\right\} \leq \exp\left\{\beta\left[1+(\Delta^2/8z^2)\right](|\underline{r}'|+|\underline{r}|) - \beta(|\underline{r}'|-|\underline{r}|)\right\}$

$$= \exp\left\{2\beta|r| + (\beta\Delta^2/8|z|^2)(|\underline{r}'|+|\underline{r}|)\right\}$$

…erefore, $\exp\left\{-\mathrm{Im}\left[z^2-(\Delta^2/4)\right]^{1/2}\hat{\underline{n}}.(\underline{r}'-\underline{r}) - \beta|\underline{r}'-\underline{r}|\right\} \leq \exp\left\{(\beta\Delta^2/8z^2)(|\underline{r}'|+|\underline{r}|)\right\}$

$$= \exp\left\{K(|\underline{r}'|+|\underline{r}|)\right\} \ldots\ldots\ldots\ldots\ldots (26\text{-A})$$

…us (25) becomes, $|f_{12}| \leq (1/4\pi)\delta(z)\left[\int\exp(Kr)|U(r)|\,d^3r\right]$

$$\times\left[\exp(Kr')|U(r')||\underline{r}'-\underline{r}|d^3r'\right]$$

…ing (E) and (F)

$$|f_{12}| \leq \delta(z)4\pi L_\alpha N_\alpha < \epsilon$$

… discuss f_{11} consider two cases:

…) $z = k_o + i\beta$, k_o fixed, $\beta\to\infty$; $z = k+i\beta_o$, β_o fixed, $k\to\infty$

(a). Let $y = (\underline{r}-\underline{r}')$; for β large from eq.(26)

$$\exp\left\{\mathrm{Im}\left[z^2-\frac{\Delta^2}{4}\right]^{1/2}\underline{n}.\underline{y}\right\} < \exp\left[\beta\,\underline{n}.y\right]\exp\left[\frac{\beta\Delta^2}{8|z|^2}\underline{y}.\underline{n}\right]$$

$$= \exp\left[\beta\,\underline{n}.y\right]\exp\left[K\,\underline{n}.\underline{y}\right]$$

Let $\underline{n}.\underline{y} = y\,\mathrm{Cos}\,\alpha$;

$$\exp\left\{\mathrm{Im}\left[z^2-\frac{\Delta}{4}^2\right]^{1/2}\hat{\underline{n}}.\underline{y}\right\} < \exp Ky\,\exp\left[\beta y\,\mathrm{Cos}\,\alpha\right]$$

…r eq.(24)

$$_{11} = -\frac{1}{4\pi}\iint \exp\left[i/_2\ \underline{\Delta}\cdot(\underline{r}+\underline{r}')\right] \exp\left[i\left(z^2-\frac{\Delta^2}{4}\right)^{1/2} \hat{\underline{n}}\cdot(\underline{r}'-\underline{r})\right]$$

$$\times\ U(\underline{r})\left(-\frac{1}{4\pi}\right) \exp\left[\frac{i(k+i\beta)|\underline{r}'-\underline{r}|}{|r'-r|}\right] U(r')\,d^3r\ d^3r'$$

$$_{11}(k_o+i\beta,\Delta)| \leqslant \frac{1}{16\pi^2}\iint \exp[Ky]\exp[\beta y\ \mathrm{Cos}\,\alpha]\ \exp\left[-\frac{\beta y}{y}\right]$$

$$|U(r')|\ |U(|y+\underline{r}'|)|\ d^3y\ d^3r'$$

$$_{11}(k_o+i\beta,\Delta)| \leqslant \frac{1}{16\pi^2}\iint \exp(Ky)\exp\left[-\beta(1-\mathrm{Cos}\,\alpha)y\right] |U(r')|\ |U(|\underline{y}+\underline{r}'|)\ d^3y\ d^3r' \quad (30$$

The above integral converges because:

$$\int e^{Ky}\ y^{-1}\ |U(|\underline{y}+\underline{r}'|)|\ d^3y = e^{K|\underline{r}-\underline{r}'|}\ |r-r'|^{-1}\ |U(r)|\ d^3r$$

$$\leqslant e^{Kr'}\int e^{Kr}\ |\underline{r}-\underline{r}'|^{-1}\ |U(r)|\,d^3r \leqslant e^{Kr'}\ 4\pi\ N_\alpha$$

Divide the 3-dim y space into 3 regions $y<a$; $y\geqslant a$, with $\alpha<\alpha_o$ nd the remaining region. Because of the convergence of (30)a and α_o ay be chosen to be independent of β and $\underline{r}'$ and so small that the total ontribution from the first two regions is $<\frac{1}{2}\epsilon$. The remaining contribu-ion is then less than $\exp\left\{-\beta(1-\mathrm{Cos}\,\alpha_o)\ a\right\}$ times a convergent integral. herefore, by choosing β large enough, and completily consistent with revious conditions on β, this contribution also becomes bounded by $\frac{1}{2}\epsilon$

The above results may be summarized as

$|f_{11}(z,\Delta)|<\epsilon$ for $\beta>R_1$ .... (32)

nd $|f_{12}(z,\Delta)|<\epsilon$ for $|z|>R$ .... (33)

b) $z = k+i\beta_o$, β_o fixed, $k\to\infty$

It shall be shown that for $k>R_{re}$, $o\leq\beta\leqslant R_{im}$,

rom (24) $f_1(z,\Delta)<\epsilon$ for $z>R=(R_{re}^2+R_{im}^2)^{1/2}$, ... (33-A)

$$f_{11}(k+i\beta_o,\Delta) = \frac{1}{16\pi^2}\iint \exp\left[-\frac{i}{2}\ \underline{\Delta}\cdot(\underline{r}'+\underline{r})\right]\exp\left\{ i\left[(k+i\beta_o)^2-\frac{\Delta^2}{4}\right]^{1/2}\right.$$

$$\left.\times\ \hat{\underline{n}}\cdot(\underline{r}'-\underline{r})\right\}\exp\left[i(k+i\beta_o)|\underline{r}'-\underline{r}|\right]|r'-r|^{-1}U(r)U(r')\,d^3r\,d^3r'$$

$$\text{-----}\ (34)$$

cause of (26) and (26-A), (34) is again an absolutely convergent tegral for k sufficiently large. The identity, $\mathrm{Im}z^2 = \mathrm{Im}[z^2-(\Delta^2/4)+(\Delta^2/4)]$ $^2/_4)$ gives $\mathrm{Im}z^2 = \mathrm{Im}\{[z^2-(\Delta^2/_4)]^{\frac{1}{2}}\}^2 = 2\mathrm{Re}[(z^2-(\Delta^2/_4))^{\frac{1}{2}}]\,\mathrm{Im}\{[z^2-(\Delta^2/_4)]^{\frac{1}{2}}\}$
$= \mathrm{Re}\{[z^2-(\Delta^2/_4)]^{\frac{1}{2}}\}\,\mathrm{Im}\{[z^2-(\Delta^2/_4)]^{\frac{1}{2}}\}$

om equation (26) for $|k|$ sufficiently large

$$\mathrm{Im}[z^2-(\Delta^2/_4)]^{\frac{1}{2}} > \beta;\ \text{or}\ k\beta > \mathrm{Re}\{[z^2-(\Delta^2/_4)]^{\frac{1}{2}}\}\beta$$

$$k > \mathrm{Re}\{[z^2-(\Delta^2/_4)]^{\frac{1}{2}}\},\ \text{whence}\ |k| \geq |\mathrm{Re}\,[z^2-(\Delta^2/_4)]^{\frac{1}{2}}|$$

in the previous case, the y-region is subdivided into three parts by choice of a and α_0. The only modification of the proof compared to that se is that for the third region, the Rieman-Lebesque lemma is applied to e radial y integration since there is an oscillatory factor,

$$+\exp\{iky - i\,\mathrm{Re}[z^2-(\Delta^2/4)]^{\frac{1}{2}}\, y\, \mathrm{Cos}\alpha\}$$

ich contains all the k dependence, which moreover cannot vanish in this gion. Thus by a suitable choice of R_r, the inequality (33-A) will be iform in β for $0 \leq \beta \leq R_{im}$ as long as $|k| > R_r$

CHAPTER-III TWO-DIMENSIONAL REPRESENTATION

Section-I Introduction: In the previous chapter it was seen that the scattering amplitude satisfies for a large class of potentials, a dispersion relation of the form given by equation (19) section I, Ch. II provided, momentum transfer is restricted by the condition $t < 4K^2$. It is natural to question the possibility of extending the above mentioned dispersion relation to larger values of t. This needs the study of the analytic properties of f(s,t) with respect to the momentum transfer variable t, which shall be done in the subsequent section. The analytic properties of f(s,t), considered as a function of two complex variables, will then be established in the topological product of the s and t planes. These analytic properties will then be exhibited by a dispersion relation in two variables. This relation is called the Mandelstam representation for potential scattering. In the final section the dispersion relation for the partial wave amplitudes will be studied.

Section-II Restrictions on the potentials: The potential of the form (the single yukawa potential)

$$\lambda V(r) = \lambda \exp(-\mu r)/r \qquad \ldots\ldots(1)$$

will be dealt with exclusively. Extension to a suitable linear combination of the type (1) i.e.

$$rV(r) = \int_0^\infty d\mu\, \sigma(\mu) \exp(-\mu r) \qquad \ldots\ldots(2)$$

can easely be effected. In fact (1) is a special form of (2) when $\sigma(\mu)$ $\sigma(\mu) = \delta(\mu)$.

The conditions on rV(r) which permits the representation (2) are, (assuming $\sigma(\mu)$ to be bounded almost everywhere except for δ-function singularities).

(i) rV(r) has derivatives of all orders $0 < r < \infty$

(ii) $|d^k (rV)/dr^k| < (\text{const})\, k!/r^{k+1}$ for $k = 0, 1, 2, \ldots\ldots (0 < r < \infty)$

also, the previous restrictions on the potential, (see chapter-II,) i.e.

$$|V(r)| < M/r^2$$

$$\int_{M',M''}^{\infty} r^2 dr\, |V(r)| < \infty$$

$$\int_0 r dr\, |V(r)| < \infty \text{ ; } M, M', M'' \text{ are positive numbers}$$

imply that, $\sigma(\mu)/\mu \to 0$ as $\mu \to 0$

and and, $\sigma(\mu) \to 0$ as $\mu \to \infty$

Section-III The analytic properties of the scattering amplitude in the momentum transfer variable t. Units in which $h^2/2m=1$, Energy$=k^2=s$ shall be used.

The integral representation of the exact scattering amplitude is

$$f(s,t)=f^0(t)-\frac{\lambda^2}{4\pi}\int \exp(-i\underline{k}'.\underline{r}')V(r')G(\underline{r}',\underline{r},s)V(r)\exp(i\underline{k}.\underline{r})d^3r'd^3r \text{......(3)}$$

where, $f^{(0)}(t)=-\frac{\lambda}{4\pi}\int \exp\left[-i(\underline{k}'-\underline{k}).\underline{r}'\right] V(r')d^3r'$

Let

$$\exp\left[-i(\underline{k}'-\underline{k}).\underline{r}'\right] = \exp\left[-i\,|\underline{k}'-\underline{k}|\,r'\cos\Theta\right]$$

and $d^3r = r'^2 \sin\Theta\, dr'\, d\Theta\, d\phi$

or, $f^{(0)}(t)=-\frac{\lambda}{4\pi}\int_0^{\infty}\int_0^{2\pi}\int_0^{\pi} \exp\left[-i\,|\underline{k}'-\underline{k}|\,r'\cos\Theta\right]\sin\Theta\, d\Theta\, r'^2 V(r')\,d\phi\, dr$

Let $\Delta = (\underline{k}'-\underline{k})$

Carrying out the integrations one obtains

$$f^{(0)}(t) = -\frac{\lambda}{\Delta}\int^{\infty} r'V(r')\, \sin\Delta r' dr'$$

Putting $V(r') = \lambda \exp\left[-\mu r'\right]/r'$

$$f^{(0)}(t) = -\frac{\lambda}{\Delta} \exp\left[-\mu r'\right] \sin\Delta r' dr'$$

$$= -\frac{\lambda}{2i\Delta}\int^{\infty} \exp\left[-\mu r'+i\Delta r'\right]dr' + \frac{1}{2i\Delta}\int_0^{\infty}\exp\left[-\mu r'-i\Delta r'\right]dr'$$

which by the use of gamma function becomes

$$f^{(0)}(t) = -\frac{\lambda}{2i\Delta}\left[\frac{-1}{-\mu+i\Delta} - \frac{1}{\mu+i\Delta}\right] = -\lambda\left[\mu^2+\Delta^2\right]$$

$$f^{(0)}(t) = -\lambda\left[\mu^2+t\right]^{-1}$$

also, $G(\underline{r}\ \underline{r}';s) = \langle \underline{r}'\left[s+i\epsilon-(-\nabla^2+\lambda V)\right]^{-1}|\underline{r}\rangle$

$$= \int \exp\, i\underline{q}'.\underline{r}')G(\underline{q}',\underline{q}',s)\exp(-i\underline{q}.r)d^3q'd^3q$$

Eq.(3) becomes with $V(r') = \exp(-\mu r)/r$

$$f(s,t) = f^{(0)}(t) - \frac{\lambda^2}{4\pi}\int \exp\left[-i(\underline{k}'-q').\underline{r}\right]V(r')G(q'q;s)\exp -i(\underline{k}-\underline{q}).\underline{r}\, V(r)\, d^3r'd^3q'd^3q$$

Elementary integration over the angles give (as shown eviously)

$$\int \exp\left[i(\underline{k}'-q').\underline{r}'\right]V(r')d^3r' = 4\pi\left|k'-q'\right|^{-1}\int_0^{\infty} r\, V(r')\,\mathrm{Sin}\,(|\underline{k}'-q'|r')dr'$$

$$= 4\pi\left[\mu^2 + (\underline{k}^1-q^1)^2\right]^{-1} \quad \ldots\ldots\ldots\ldots\ldots\ldots(5)$$

nce,

$$s,t) = f^o(t) - \lambda^2\, 4\pi\int\left[\mu + (\underline{k}'-q')^2\right]^{-1} G(q'q,s)^2 \times \left[\mu^2 + (\underline{k}-q)^2\right]^{-1} d^3q'd^3q \quad \ldots.(6)$$

The full green's function satisfies the (symbolic) integral uation

$$G = G_o + G_oVG = G_o + GVG_o \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(7)$$

so, the free particle green's function is given by

$$G_o(\underline{r}',\underline{r};k) = -\exp\left[ik|\underline{r}'-\underline{r}| \,/4\pi|\underline{r}'-\underline{r}|\right]$$

Eq.(6) will be used to show that for real (s, $f(st)-f^o(t)$ s analytic for $|t| < 4\mu^2$ and that the first singularity actually accur t= $4\mu^2$. For this, the possible zeros of afactor such as $\mu^2+(\underline{k}-q)^2$ s studied.

Choose Co-ord system in which

$$\underline{k} = k\,(\mathrm{Cos}\,\tfrac{1}{2}\,\theta,\ -\sin\tfrac{1}{2}\,\theta,\ 0)$$

$$q = (q_1, q_2, q_3) = q\,(\sin\gamma\,\sin\phi, \mathrm{Cos}\,\gamma\,\sin\phi, \mathrm{Cos}\phi)$$

here θ is the scattering angle.

herefore,

$$^2+ (\underline{k}-q)^2 = \mu^2+k^2+q^2+2kq\left[\mathrm{Cos}\,\tfrac{1}{2}\,\theta\,\mathrm{Sin}\,\gamma\,\mathrm{Sin}\phi - \mathrm{Sin}\,\tfrac{1}{2}\,\theta\,\mathrm{Cos}\gamma\,\sin\phi\right]$$

$$= \mu^2+k^2+q^2+2kq\left[(\sin\phi)\,\mathrm{Cos}\,\tfrac{1}{2}\,\theta\,\mathrm{Sin}\gamma - \mathrm{Sin}\,\tfrac{1\theta}{2}\mathrm{Cos}\,\gamma\right]$$

$$= \mu^2+k^2+q^2+2kq\,\sin\phi\,\sin(\theta/2-\gamma)$$

uch no sot exists for real θ,put $\theta=\theta_1+i\theta_2$ and hence the condition for he zeros of $\mu^2+(\underline{k}-q)^2$ is

$$\mu^2+k^2+q^2-2kq\,\mathrm{Sin}\,\phi\,\sin\left[\gamma - \tfrac{1}{2}(\theta_1+i\theta_2)\right] = 0$$

$$+k^2+q^2-2kq\ \operatorname{Sin}\phi\left[\sin(\gamma-\tfrac{1}{2}\theta_1)\operatorname{Cos}\frac{i\theta_2}{2}-\operatorname{Cos}(\gamma-\tfrac{1}{2}\theta_1)\operatorname{Sin}\frac{i\theta_2}{2}\right]=0$$

$$\mu^2+k^2+q^2-2kq\ \operatorname{Sin}\phi\ \sin(\gamma-\tfrac{1}{2}\theta_1)\operatorname{Cosh}\frac{\theta_2}{2}+i\ 2kq\ \sin\phi\operatorname{Cos}(\gamma-\tfrac{1}{2}\theta_1)\ \operatorname{Sinh}\frac{\theta_2}{2}=0$$

Equating real and imaginary parts, we get,

$$2kq\ \sin\phi\operatorname{Cos}(\gamma-\tfrac{1}{2}\theta_1)\ \operatorname{Sinh}\frac{\theta_2}{2}=0$$

$$\operatorname{Cos}(\gamma-\tfrac{1}{2}\theta_1)=0 \qquad \ldots\ldots\ldots\ldots\ldots\ldots (8)$$

d $\operatorname{Cosh}\theta_2/2=(\mu^2+k^2+q^2)/2kq\ \operatorname{Sin}\phi\ \sin(\gamma-\tfrac{1}{2}\theta_1)$

since $\sin(\gamma-1/2\ \theta_1)=1$, we have

$$\operatorname{Cosh}\frac{\theta_2}{2}=(\mu^2+k^2+q^2)/2kq\ \sin\phi \qquad \ldots\ldots\ldots\ldots\ldots\ldots (9)$$

so, $2k\operatorname{Sin}\tfrac{1}{2}\theta=2k\operatorname{Sin}\tfrac{1}{2}(\theta_1+i\theta_2)$

$$=2k\left[\sin\frac{\theta_1}{2}\operatorname{Cos}\frac{i\theta_2}{2}+\operatorname{Cos}\frac{\theta_1}{2}+\operatorname{Sin}\frac{i\theta_2}{2}\right]$$

$$=2k\left[\sin\frac{\theta_1}{2}\operatorname{Cosh}\frac{\theta_2}{2}+i\operatorname{Cos}\frac{\theta_1}{2}\operatorname{Sinh}\frac{\theta_2}{2}\right]$$

s, $\operatorname{Cos}(\gamma-\tfrac{1}{2}\theta_1)=0$, & $q_1=q\operatorname{Sin}\gamma\operatorname{Sin}\phi$, $q_2=q\operatorname{Cos}\gamma\operatorname{Sin}\phi$

e have $\operatorname{Cos}\gamma\operatorname{Cos}\tfrac{1}{2}\theta_1=-\operatorname{Sin}\gamma\sin\tfrac{1}{2}\theta_1$

$$q_2\operatorname{Cos}\tfrac{1}{2}\theta_1=-q_1\operatorname{Sin}\tfrac{1}{2}\theta_1$$

$$k\operatorname{Sin}\tfrac{1}{2}\theta=(q^2\operatorname{Sin}^2\phi)^{-1}\left[-q_2(\mu^2+k^2+q^2)+iq_1\left\{(\mu^2+k^2+q^2)^2-4k^2q^2\operatorname{Sin}^2\phi\right\}^{1/2}\right] (10)$$

$$|t|=\left|2k\operatorname{Sin}\tfrac{1}{2}\theta\right|^2$$

$$|=\left|2k\sin\tfrac{1}{2}\theta\right|^2=(q^2\sin^2\phi)^{-2}\left[q_2^2(\mu^2+k^2+q^2)^2+q_1^2\left\{(\mu^2+k^2+q^2)^2-4k^2q^2\operatorname{Sin}^2\phi\right\}\right]$$

s $q^2=q_1^2+q_2^2+q^2\operatorname{Cos}^2\phi$ or $q_2^2=q^2\operatorname{Sin}^2\phi-q_1^2$

$$t|=(q^2\sin^2\phi)^{-2}\left[\left\{(q^2\operatorname{Sin}^2\phi-q_1^2)+q_1^2\right\}(\mu^2+k^2+q^2)^2-4q_1k^2q^2\operatorname{Sin}^2\phi\right]$$

$$t|=(q^2\operatorname{Sin}^2\phi)^{-1}\left[(\mu^2+k^2+q^2)^2-4q_1^2k^2\right] \qquad \ldots\ldots\ldots\ldots\ldots\ldots (11)$$

This expression and (10) must be varied with respect to q_1, and sin to find that solution with the minimum value of $|t|$.

The solution is obviously sin = 1 . By straight forward f, $q_2 = o$, $q_1^2 = k^2 + \mu^2$. This yields from (11) and (10)

$$|t| \text{ (minimum)} = 4\mu^2, \; t^{1/2} = \pm 2\mu$$

Thus as long as $|t| < 4\mu$, the integral (3) as well as its der- ative w.r. to t are gauranteed to exist.

It will now be shown how the argument may be extended to yield re complete information on the analytic properties of f(s,t) in t.

Eq (3) may be symbolically written as

$$\begin{aligned}
f &= f^{(o)} - \lambda^2 VGV \\
&= f^{(o)} - \lambda^2 V(G + \lambda G_o VG)V \\
&= f^{(o)} - \lambda^2 VG_o V - \lambda^3 VG_o V(G_o + \lambda GVG_o)V \\
&= f^{(o)} - \lambda^2 VG_o V - \lambda^3 (VG_o)^2 V - \lambda^4 VG_o V(G_o + \lambda G_o VG)VG_o V \\
&= f^{(o)} - \lambda^2 VG_o V - \lambda^3 (VG_o)^2 V - \lambda^4 (VG_o)^3 V - \lambda^5 VG_o VG_o V(G_o + \lambda GVG_o) VG_o V \\
&= f^{(o)} - \lambda^2 VG_o V - \lambda^3 (VG_o)^2 V - \lambda^4 (VG_o)^3 V - \lambda^5 (VG_o)^4 V - \lambda^6 (VG_o)^2 VGV(G_o V
\end{aligned}$$

t

$$\begin{aligned}
f &= \sum_{i=o}^{2n} f^{(i)} + R_{2n+1} \\
&= f_{2n+1} + R_{2n+1}
\end{aligned}$$

ere

$$\begin{aligned}
f^{(i)} &= -(\lambda)^{i+1} (VG\,)^i V \\
f_{2n+1} &= \sum_{i=o}^{2n} f^{(i)} \\
R_{2n+1}(s,t) &= -(\lambda)^{2n+2} (VG_o)^n VGV(GoV)^n
\end{aligned}$$

Consider $VG_o V$

By definition,

$$G_oV = \int \exp\left[-i\underline{k}'\cdot\underline{r}'_1\right] V(r'_1) G_o(\underline{r}'_1,\underline{r}'_2,s) \exp\left[iq'_2\cdot\underline{r}'_2\right] V(r'_2)\, d^3r'_1\, d^3r'_2$$

ut $G_o(\underline{r}'_1,\underline{r}'_2,s) = \int (2\pi)^{-3} \exp\left[iq'_1\cdot\underline{r}'_1\right] \dfrac{\exp\left[-iq'_1\cdot\underline{r}'_2\right] dq'^3}{s+i\eta-\underline{q}'^2_1}$

erefore,

$$_oV = (2\pi)^{-3} \int \frac{dq^{3}_1{}'}{s+i\eta-\underline{q}'^2_1} \int \exp\left[-i(\underline{k}'-q'_1)\cdot\underline{r}'_1\right] V(r'_1)\, d^3r'_1 \int \exp -i(q'_1-q'_2)\cdot\underline{r}'_2\; V(r_2)\, d^3r'_2$$

milar to eq(5)

$$\int \exp\left[-i(\underline{k}'-q'_1)-\underline{r}'_1\right] V(r'\ )\, d^3r' = 4\pi\left[\mu^2+(\underline{k}'-q'\ )^2\right]^{-1}$$

nally,

$$R_{2n+1}(s,t) = -\lambda^{2n+2}\, (4\pi)^{2n+2} \int \frac{1}{\mu^2+(\underline{k}'-q'_1)^2}\, \frac{(2\pi)^{-3}}{s+i\epsilon-q'^2_1}\, \frac{1}{\mu^2+(q_1'-q'_2)^2}$$

$$\times \frac{(2\pi)^{-3}}{s+i\epsilon-(q'_2)^2} \cdots\cdots \frac{1}{\mu^2+(q_n'-q')^2}\; G(q',q;s)\; \frac{1}{\mu^2+(q_n'-q')^2}$$

$$\times \frac{(2\ \ )^{-3}}{s+i\epsilon-q_n^2} \cdots\cdots \frac{1}{\mu^2+(q_1-\underline{k})^2}$$

$$\times\, d^3q' \ldots d^3q_n'\, d^3q_n \ldots d^3q\, d^3q'd^3q \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (14)$$

It shall now be proved that $R_{2n+1}(s,t)$ is, for real s, analytic t for $|t| < (2n+2)^2\mu^2$, whereas, $f^{(i)}(s,t)$, $i\neq 0$ is analytic for all t xcept for a cut from $t = -(i+1)^2\mu^2$ to $-\infty$.

Consider the angular integrations

$$\int \frac{1}{\mu^2+(\underline{k}-q_1)^2}\, d\Omega_1,\; \frac{1}{\mu^2+(q_1-q_2)^2}\, d\Omega_2$$

$$\frac{1}{\mu^2+(q_{n-1}-q_n)^2}\, d\Omega_n\; \frac{1}{\mu^2+(q_n-q\ )^2} \quad \ldots\ldots\ldots (15)$$

This can be solved by the successive application of the rmula

$$\int d\Omega' \quad \frac{1}{a_1-\underline{\hat{n}}_1\cdot\underline{\hat{n}}'} \quad \frac{1}{a_2-\underline{\hat{n}}_2\cdot\underline{\hat{n}}'} = -4\pi\int_{-1}^{+1}\frac{dy}{(1-y^2)^{2}(a_{12}-\hat{n}_1\cdot\hat{n}_2)} \quad \ldots\ldots (16)$$

ere

$$_{12} =)1-y^2)^{-1}\left[\frac{1}{2}\left\{a_1(1+y)+a_2^2(1-y)\right\}^2-(1+y)\right]\ldots\ldots\ldots\ldots\ldots\ldots\ldots(17)$$

nsider the last integration of eq.(15) integrated over the direction

$$\frac{1}{\mu^2+(q_{n-1}-q_n)^2} \quad d\Omega_n \quad \frac{1}{\mu^2+(q_n-q)^2}$$

mparing with (16)

$$a_2 = (\mu^2+q^2+q_n^2)/2qq_n$$

$$a_1 = a_1\{q_1,q_2,\ldots\ q_{n-1},\ y\ \ldots y_{n-1},\ q_n,k)$$

Choose a co-ordinate system in which

$$q_{n-1} = q_{n-1}\ (\mathrm{Cos}\ \tfrac{1}{2}\ \theta,\ -\ \mathrm{Sin}\ \tfrac{1}{2}\ \theta,\ 0)$$

$$q = q(\sin\gamma\ \mathrm{Sin}\phi\ ,\mathrm{Cos}\gamma\ \mathrm{Sin}\phi\ ,\mathrm{Cos}\phi\ )$$

Consider the zeros of $a_{12}-\underline{\hat{n}}_1\cdot\hat{n}_2$

r

$$a_{12} - \sin\phi\left[\mathrm{Cos}\ \tfrac{1}{2}\ \theta\ \mathrm{Sin}\gamma\ -\mathrm{Sin}\ \tfrac{1}{2}\ \theta\mathrm{Cos}\ \gamma\right] = 0$$

$$a_{12} - \sin\phi\ \sin\ (\gamma-\tfrac{1}{2}\ \theta) = 0$$

solution does not exist for real θ.

ence, put

$$\theta = \theta_1+i\theta_2$$

r

$$a_{12}\ -\sin\phi\ \sin\left[\gamma - \tfrac{1}{2}\ (\theta_1+i\theta_2\right] = 0$$

$$a_{12}\ -\sin\phi\left[\sin(\gamma-\tfrac{1}{2}\ \theta_1)\mathrm{Cos}(-\tfrac{i}{2}\ \theta_2)-\mathrm{Cos}(\gamma-\tfrac{1}{2}\ \theta_1)\mathrm{Sin}(-\tfrac{i}{2}\ \theta_2)\right] = 0$$

Equating real and imaginary parts,

$$-\sin\phi \operatorname{Cos}(\gamma - \tfrac{1}{2}\theta_1)\sinh\left(\frac{\theta_2}{2}\right) = 0$$

Hence $\operatorname{Cos}(\gamma - \frac{1}{2}\theta_1) = 0$ .... (18-a)

And $\operatorname{Sin}(\gamma - \frac{1}{2}\theta_1) = 1$ .... (18-b)

also,

$$\sin\phi \sin(\gamma - \tfrac{1}{2}\theta_1) \operatorname{Cosh}\frac{\theta_2}{2} = a_{12} \quad \text{........ (18-c)}$$

$$\operatorname{Sin}\tfrac{1}{2}\theta = \operatorname{Sin}\tfrac{1}{2}\theta_1 \operatorname{Cos}\tfrac{i}{2}\theta_2 + \operatorname{Sin}\tfrac{i}{2}\theta_2 \operatorname{Cos}\frac{\theta_1}{2}$$

$$= \operatorname{Cosh}\frac{\theta_2}{2}\operatorname{Sin}\frac{\theta_1}{2} + i\sinh\frac{\theta_2}{2}\operatorname{Cos}\frac{\theta_1}{2} \quad \text{........ (19)}$$

ecause $\operatorname{Cos}(\gamma - \frac{1}{2}\theta_1) = 0$ or $\operatorname{Cos}\gamma\operatorname{Cos}\frac{\theta_1}{2} = -\operatorname{Sin}\gamma\operatorname{Sin}\frac{\theta_1}{2}$

d $\operatorname{Sin}(\gamma - \frac{1}{2}\theta_1) = 1$ or $\sin\gamma\cos\frac{1}{2}\theta_1 - \operatorname{Cos}\gamma\sin(\theta_1/2)^2 = 1$

$$\operatorname{Cos}\gamma\,\frac{\left[(1+\operatorname{Cos}\gamma\operatorname{Sin}(\theta_1/2)\right]}{\operatorname{Sin}\gamma} = -\operatorname{Sin}\gamma\operatorname{Sin}(\theta_1/2)$$

$$\text{or } \operatorname{Cos}\gamma + \sin(\theta_1/2)\left[\operatorname{Cos}^2\gamma + \operatorname{Sin}^2\gamma\right] = 0$$

$$\sin\theta_1 = -\operatorname{Cos}\gamma$$

$$\text{and } \operatorname{Cos}(\theta_1/2) = \operatorname{Sin}\gamma$$

Putting in (19) one obtains

$$\operatorname{Sin}\tfrac{1}{2}\theta = -\operatorname{Cos}\gamma\operatorname{Cosh}\tfrac{1}{2}\theta_2 + i\operatorname{Sin}\gamma\left(\operatorname{Cosh}^2\frac{\theta_2}{2} - 1\right)^{1/2} \quad \text{.............. (20)}$$

lso, with $\sin\phi = 1$, (18-c) gives

$$\operatorname{Cosh}\tfrac{1}{2}\theta_2 = \left[(1-y^2)\right]^{-1}\left[\tfrac{1}{2}\left\{a_1(1+y) + a_2(1-y)\right\}^2 - (1+y^2)\right]$$

$$= a_{12}/q$$

Equation (20) means

$$|t| = 4k^2\left[\operatorname{Cosh}^2\tfrac{1}{2}\theta_2 - \operatorname{Sin}^2\gamma\right] \quad \text{................ (22)}$$

Equation (22) will now be minimized for this, the minimisation f a_1 is required with respect to all variables, except q_n and k, i.e. wit

espect to the ones. On which it alone depends. Assume that this mini-
um is given by (it shall be proved imminently)

$$a_1 = (n\mu)^2 + q_n^2 + k^2/2kq_n \qquad (23)$$

inimize a_{12} with respect to q_n and y, in that order, for fixed q .This
ields straightforwardly

$$a_{12} = \left[(n+1)^2\mu^2+q^2+k^2\right]/2kq \qquad (24)$$

nd incidentally proves(23) if one gives a little thought to the structure
f a_1. This sequence of minimisations has now reduced (22) to the form

$$|t| = q^{-2}\left[\left\{(n+1)^2\mu^2+k^2+q^2\right\}^2-4k^2q^2\sin^2\gamma\right] \qquad (25)$$

But, this is of exactly the same structure as equation (11) ith μ replaced by $(n+1)\mu$. It has thus been proved that the minimus value f t for which $R_{(2n+1)}$ may possibly have a singularity is $(2n+2)^2\mu^2$.

It is now easily seen that the above reasoning has also estab-
ished the analytic properties in t of $f^{(n)}$, the (n+1)st term in the Born
eries. Eor, its dependence on t is given essentially by an integral of
he form(15) with q replaced by k'. In the integral (16), then, if this
s again the result of the last angular integration, $\hat{\underline{n}} . \hat{\underline{n}} = \cos\theta$. Since
$_{12}$ is now a real positive quantity, it is deduced that $f^{(n)}$ is an
nalytic function of CosO except for a cut along the positive real axis
trting at the minimum value of a_{12}. According to (25) with $q = |k'| = k$
he cut starts at

$$\cos\theta = 1 + \left[(n+1)^2\mu^2/2k^2\right] = 1 - (t/2k^2) \qquad (26)$$

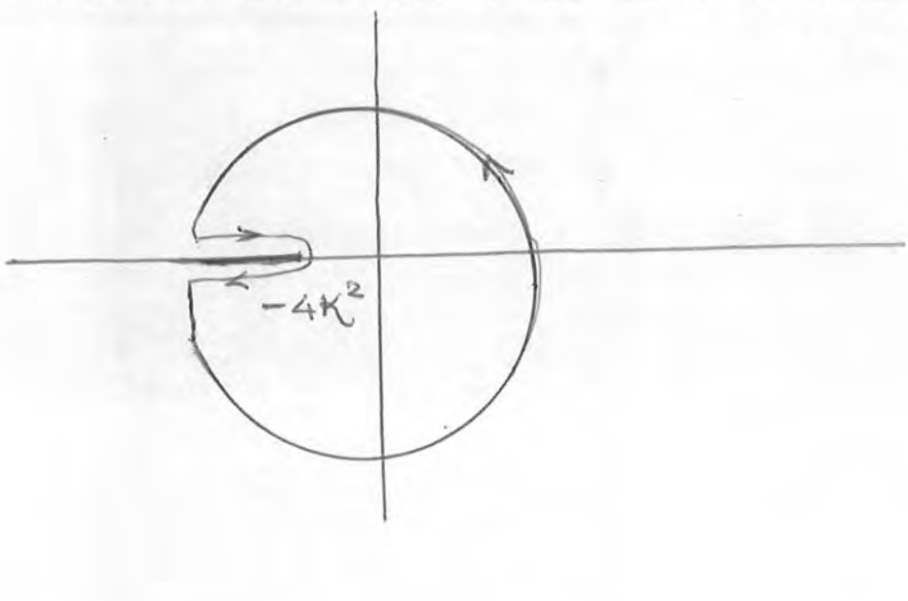
r $t = -(n+1)^2\mu^2$ as asserted previously. On combining this result with
he previous one for $R_{(2n+1)}$, it may be concluded that f(s,t) is for
eal s analytic in t except along the indicated cuts.

ction IV : Explicit construction of the Mandelstam representation:

From the above discussion it is seem that $f(s,t) - f_B(t)$ is analy-
c in t plane cut from $t= -4K^2$ to $t= -\infty$. Consider the dispersion
lation given by equation (19) section II Chapter II. The first Born
proximation $f_B(t)$ is real. Therefore, $Imf(s,t)$ is analytic in the
t t-plane. Also, if there are a finite number of bound states, R_b
a finite polynomial in t, and hence it is certainly regular for
l finite t-values. Since, $Imf(s,t)$ is analytic in the cut t-plane
or $s' \geqslant 0$) and also analytic in the cut s-plane, it is obvious that
$(s,t) - f_B(t)$ can be analytically extended into the entire s,t plane,
ith the exception of bound state poles). The analytic function obtain-
l by extending the right hand side of equation (19) will be identical
th the actual analytic function $f - f_B$ because of the common region
analyticily, $0<t<4K^2$, arbitrary s. Analytic extension of the right
nd side of the eq(19) i.e. of the integral term in eq (19) can be
presented using cauchy's theorem. $Imf(s',t)$ is analytic in the cut
-plane ($s'\geqslant 0$), the cut extending from $t= -4K^2$ to $t=-\infty$. Assume that,

$$Imf(s,t) \longrightarrow 0 \quad \text{for} \quad |t| \longrightarrow \infty \quad \ldots\ldots\ldots\ldots \quad (1)$$

sing Cauchy's theorem, for any complex t, not on the cut, around the
ontour



ne obtains,

$$\frac{1}{2\pi i}\oint \frac{\operatorname{Im}f(s',t')}{t'-t}\,dt' = \operatorname{Im}f(s',t)$$

$$\operatorname{Im}f(s',t) = \lim_{t'\to\infty}\frac{1}{2\pi i}\oint_C \frac{\operatorname{Im}f(s',t')}{t'-t}\,dt' + \lim_{\epsilon\to 0}\frac{1}{2\pi i}\int_{-\infty}^{-4K^2}\frac{\operatorname{Im}f(s',t'+i\epsilon)-\operatorname{Im}f(s't-i\epsilon)}{t'-t}$$

fine, $u(s',t') = (1/2i)\lim_{\epsilon\to 0}\left[\operatorname{Im}f\,(s',t'+i\epsilon) - \operatorname{Im}f(s',t'-i\epsilon)\right]$

$$\operatorname{Im}f(s',t') = \frac{1}{\pi}\int_{-\infty}^{-4K^2}\frac{u(s',t')}{t'-t}\,dt' = -\frac{1}{\pi}\int_{4K^2}^{\infty}\frac{u(s',-t')(-dt')}{-t'-t}$$

, $$\operatorname{Im}f(s',t') = \frac{1}{\pi}\int_{4K^2}^{\infty}\frac{\rho(s',t')dt'}{t'+t}$$

ere $\rho(s',t') = -u(s',-t')$; (s',t' are real and positive). $\rho(s',t')$ is the ~~the~~ discontinuity across the cut. It should be noted that by introduc-g for convenience, the notation $\rho(s,t)$ instead of $-u(s,-t)$ the argument of ρ is the negative of the actual (physical or unphysical) values of t. no bound states are present and no subtractions are necessary, then,

$$f(s,t) = f_B(t) + \frac{1}{\pi^2}\lim_{\eta\to 0}\int_0^{\infty}\int_{4K^2}^{\infty}\frac{\rho(s',t')\,dt'ds'}{(s'-s-i\eta)(t'+t)} \quad \ldots\ldots\ldots\ldots\ldots\ldots(2)$$

sumption (1) actually does not hold. If (n+1) subtractions are necessary, d subtractions are made at t=0, then,

$$\operatorname{Im}f(s',t) = \sum_{j=0}^{n}\frac{t^j}{j!}g_j(s') + (-1)^{n+1}\frac{t^{n+1}}{\pi}\int_{4K^2}^{\infty}\frac{\rho(s',t)\,dt'}{(t'+t)t'^{n+1}} \quad \ldots\ldots(3)$$

Here $g_j(s')$ is the j^{th} derivative of $\operatorname{Im}f(s',t)$ taken at t=0 order to represent f(s,t) in the entire (s,t) complex domain, substitute q.(3) in

$$(s,t) = f_B(t) + \sum_{b=1}^{N}\frac{R_b(i\,|E_b|\,t)}{s+|E_b|} + \frac{1}{\pi}\lim_{\eta\to 0}\int_0^{\infty}\frac{\operatorname{Im}f(s',t)ds'}{s'-s-i\eta}$$

otaining,

$$s,t) = f_B(t) + \sum_{b=1}^{N} \frac{R_b(i\,|E_b|\,t)}{s + |E_b|} + (-1)^{n+1}\, t^{n+1}\, \frac{1}{\pi^2} \lim_{\eta \to 0}$$

$$\lim_{\to 0} \int_0^{\infty}\!\!\int_{4k^2}^{\infty} \frac{\rho(s',t')\, dt'ds'}{(t'+t)t'^{\,n+1}(s'-s-i\eta)} + \sum_{j=1}^{n} \frac{t^j}{j!} \frac{1}{\pi} \lim_{\eta \to 0} \int_0^{\infty} \frac{g_j(s')\, ds'}{s'-s-i\eta} \quad \ldots\ldots\ldots (4)$$

is "double dispersion relation" is valid for every s and t which do t lie on the respective cuts.

mments: The Mandelstam fepresentation, apart from possible unknown bstraction constants g_j, determines the scattering amplitude completely st by its singularties i.e. the locations and strengths (residua) of s poles and the discontinuity across the branch cut. In analogy with assical potential theory, one may consider the poles as "point charges" d the cut as a "line charge" which then fully determine the function $-f_B$. If the Mandetslam rep. is considered as a given statement about e full analytic properties of the amplitude, one can go backward and duce the one variable dispersion rels.

lculation of weight function $\rho(s,t)$

The Unitarity Relation is :

$$\mathrm{Im}f(k,\theta) = \frac{k}{4\pi} \int f^*(k,\theta')\, f(k,\Theta)\, d\Omega'$$

ere $d\Omega' = \sin\theta' d\theta' d\varphi'$

Θ = angle between $(\theta,0)$ and (θ',φ') devection, that is,

$$\cos\Theta = \cos\theta\cos\theta' + \sin\theta\sin\theta'\cos\varphi'$$

n terms of (s,t)this becomes,

$$\mathrm{Im}f(s,t) = \frac{\sqrt{s}}{4\pi} \int f^*(s,t_1) f(s,t_2)\, d\Omega' \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (5)$$

ere $t_1 = (\hat{\underline{k}}-\underline{k})^2$, $t_2 = (\underline{k}-\underline{k}_o)^2$; $t = (\hat{\underline{k}}-\underline{k}_o)^2$

here $k_o^2 = \hat{k}^2 = k^2 = s$; $d\Omega' = \sin\theta' d\theta' d\varphi'$

e. $d\Omega'$ is the surface element of the unit sphere described by $\underline{k}/k$ there are no bound states and assuming that no subtractions are cessary, then from (3) and (5)

$$\mathrm{Imf}(s,t) = \frac{(s)^{1/2}}{4\pi}\int d\Omega'\left[f_B(t_1) + \frac{1}{\pi^2}\iint\frac{\rho(s',t')dt'ds'}{(s'-s+i\eta)(t'+t_1)}\right]$$

$$\times \left[f_B(t_2) + \frac{1}{\pi^2}\iint\frac{\rho(s'',t'')dt''ds''}{(s''-s-i\eta)(t''+t_2)} \right. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(6)}$$

$$\mathrm{Imf}(s,t) = \frac{(s)^{1/2}}{4\pi}\int d\Omega'\left[\frac{\lambda^2}{(t_1+K^2)(t_2+K^2)} - \frac{\lambda}{\pi^2}\iint\frac{\rho(s',t')dt'ds'}{(s'-s+i\eta)(t_1+t')(t_2+K^2)}\right.$$

$$- \frac{\lambda}{\pi^2}\iint\frac{\rho(s'',t'')dt''ds''}{(s''-s-i\eta)(t_2+t'')(t_1+K^2)} + \frac{1}{\pi^4}\iiiint\frac{\rho(s',t')\rho(s'',t'')dt'ds'dt''ds''}{(s'-s+i\eta)(s''-s-i\eta)(t_1+t')(t_2+t'')}$$

$$\ldots\ldots\ldots \text{(7)}$$

r convenience consider a single Yukawa potential.

$$U(r) = \lambda\frac{e^{-Kr}}{r}$$

i.e $\sigma(\mu) = \delta(\mu-K)$

For a Yukawa potential

$$f_B(t) = -\frac{\lambda}{K^2+t}$$

ile working out (6) the following angular integrations are encountered.

$$I_1 = \int d\Omega'\frac{1}{(t_1+K^2)(t_2+K^2)}, \quad I_3 = \int d\Omega'\frac{1}{(t_1+K^2)(t_2+t'')}$$

$$I_2 = \int d\Omega'\frac{1}{(t_1+t')(t_2+K^2)} \quad I_4 = \int d\Omega'\frac{1}{(t_1+t')(t_2+t'')}$$

They are all of the type

$$I_n = \int d\Omega'\frac{1}{(t_1+\alpha)(t_2+\beta)} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(8)}$$

ubstituting in (8)

$$\tau_1 = 1+\frac{\alpha}{2s}, \quad \tau_2 = 1+\frac{\beta}{2s}$$

$t_1 = (\hat{\underline{k}}-\underline{k})^2$, $t_2=(\underline{k}-\underline{k}_o)^2$ and $\hat{k}^2=k_o^2=k^2=s$

e obtains

$$I_n = \int d\Omega' \frac{1}{\left(2k^2-2\hat{\underline{k}}.\underline{k}+(\tau_1-1)2s\right)\left(2k^2-2\underline{k}.\underline{k}_o+(\tau_2-1)2s\right)}$$

$$I_n = \frac{1}{4s^2}\int d\Omega' \frac{1}{(\tau_1-\hat{\underline{k}}.\underline{k}/k^2)(\tau_2-\underline{k}.\underline{k}_o/k^2)}$$

This integral can be solved by Feynman's method.(see, for xample, M.L Goldberger's article in Relations de dispersion it partic-ies elementaires.p.63)

The result is

$$I_n = \frac{4\pi}{4s^2}\int_{a_n}^{\infty} \frac{1}{A_n(\xi)} \frac{1}{(\xi-\hat{\underline{k}}.\underline{k}_o/k^2)} d\xi$$

here

$$A_n(\xi) = \left[(\tau_1\tau_2-\xi)^2-(\tau_1^2-1)(\tau_2^2-1)\right]^{1/2}$$

nd

$$a_n = \tau_1\tau_2 + (\tau_1^2-1)^{1/2}(\tau_2^2-1)^{1/2}$$

Introducing a new variable ϑ by setting

$$\xi = \left(1+\frac{\vartheta}{2s}\right)$$

and noting that

$$\frac{\hat{\underline{k}}.\underline{k}_o}{k^2} = 1-\frac{t}{2s}$$

One obtains

$$I_n = \frac{\pi}{s^2}\int_{b_n}^{\infty} \frac{1}{(\vartheta+t)} \frac{1}{A_n(1+\vartheta/2s)} d\vartheta$$

here $b_n = 2s\left\{\tau_1\tau_2-1+\left[(\tau_1^2-1)(\tau_2^2-1)\right]^{1/2}\right.$

ecause $\vartheta = 2s(\xi-1)$

or $b_n = 2s(a_n-1)=2s\left\{\tau_1\tau_2-1+\left[(\tau_1^2-1)(\tau^2-1)\right]^{1/2}\right.$

Instead of b_n, the lower limit can be taken as zero by serting in the arguments the factor $\Theta(\vartheta - b_n)$, where,

$$\Theta(\vartheta - b_n) = \begin{cases} 1 \text{ if } \vartheta > b_n \\ 0 \text{ if } \vartheta < b_n \end{cases}$$

the "step function".

$$I_n = \frac{\pi}{s^2} \int_0^\infty \frac{\Theta(\vartheta - b_n)}{(\vartheta + t)} \frac{1}{A_n(1 + \vartheta/2s)} d\vartheta$$

om (7)

$$f(s,t) = \frac{\sqrt{s}}{4\pi} \int d\Omega' \Big[\lambda^2 I_1 - \frac{\lambda}{\pi^2} \iint \frac{\rho(s',t)dt'ds'}{(s'-s+i\eta)} . I_2 - \frac{\lambda}{\pi^2} \iint \frac{\rho(s'',t'')dt''ds''}{(s''-s-i\eta)} I_3$$

$$+ \frac{1}{\pi^4} \iiiint \frac{\rho(s',t')\rho(s'',t'')dt'ds'dt''ds''I_4}{(s'-s+i\eta)(s''-s-i\eta)}$$

nally,

$$\text{Im} f(s,t) = \frac{1}{4s^{3/2}} \int_0^\infty \frac{d\vartheta}{\vartheta + t} \times \Big[\frac{\lambda^2 \Theta(\vartheta - b_1)}{A_1(1-\vartheta/2s)} - \frac{\lambda}{\pi^2} \iint \frac{\Theta(v-b_2)}{A_2(1-\vartheta/2s)} \frac{\rho(s',t')dt'ds'}{(s'-s+i\eta)}$$

$$- \frac{\lambda}{4\pi^2} \iint \frac{\Theta(\;-b_3)}{A_3(1-\vartheta/2s)} \frac{\rho(s'',t'')dt''ds''}{(s''-s-i\eta)}$$

$$+ \frac{}{\pi^4} \iiiint \frac{\Theta(\vartheta - b_4)}{A_4(1-\vartheta/2s)} \frac{\rho(s',t')\rho(s''t'')dt'ds'dt''ds''}{(s'-s+i\eta)(s''-s-i\eta)}$$

The A_n fns. depend, besides on , on t',t" or on both through heir respective τ's (except A_1 which depends only on K^2). The same is rue for the b_n's. From the above equation, the discontinuity across the ut $\rho(s,t)$, can be calculated with the help of

$$\rho(s',t') = \frac{1}{2i} \operatorname*{Lt}_{\epsilon \to 0} \Big[\text{Imf}(s',-t'-i\epsilon) - \text{Imf}(s',-t'+i\epsilon) \Big]$$

nd

$$\operatorname*{Lt}_{\eta \to 0} \frac{\eta}{x^2 + \eta^2} = \pi\delta(x)$$

sing

$$\tau_1 = 1 + (\alpha/2s), \quad \tau_2 = 1 + (\beta/2s)$$

$$A_n(\xi) = \Big[(\tau_1\tau_2 - \xi)^2 - (\tau_1^2 - 1)(\tau_2^2 - 1) \Big]^{1/2}$$

$$b_n = 2s\Big\{ \tau_1\tau_2 - 1 + \big[(\tau_1^2 - 1)(\tau_2^2 - 1) \big]^{1/2} \Big\}$$

ne eventually obtains,

$$\rho(s,t) = \lambda^2 K(s,t;K^2,K^2) - 2\lambda \frac{P}{\pi^2}\int_{4K^2}^{\infty}\int_{4K^2}^{\infty} \frac{K(s,t;t',K^2)}{s'-s}\,\rho(s',t')dt'ds' + \frac{1}{\pi^4}\lim_{\eta\to 0}\int_{4K^2}^{\infty}\int_{0}^{\infty}\int_{4K^2}^{\infty}\int_{0}^{\infty} \frac{K(s,t;t',t'')}{(s'-s+i\eta)(s''-s-i\eta)} \times \rho(s',t')\rho(s'',t'')dt''ds''dt'ds' \quad \ldots\ldots (10)$$

Hence,

$$K(s,t;a,b) = \frac{\pi}{2}\,\Theta\left\{t-a-b-\frac{ab}{2s}-\frac{\sqrt{ab}}{2s}\left[16s^2+4s(a+b)+ab\right]^{1/2}\right\} \times\left\{s\left[t-(\sqrt{a}+\sqrt{b})^2\right]\left[t-(\sqrt{a}-\sqrt{b})^2\right]-tab\right\}^{-1/2} \quad \ldots\ldots(11)$$

$a,b = t', t'', K^2$

n (11) put $a=b=K^2$ obtaining

$$K(s,t;K^2,K^2) = \frac{\pi}{2}\,\Theta\left\{t-K^2-K^2-\frac{K^2K^2}{2s}-\frac{(K^2K^2)^{1/2}}{2s}\left[16s^2+4s(K^2+K^2)+K^2K^2\right]^{1/2}\right\} \times\left\{s\left[t-(K+K)^2\right]\left[t-(K-K)^2\right] -t\,KK\right\}^{-1/2}$$

$$K(s,t;K^2,K^2) = \frac{\pi}{2}\,\Theta\left(t-4K^2-\frac{K^4}{s}\right) \times \left[s(t-4K^2)\ t-tK^4\right]^{-1/2}$$

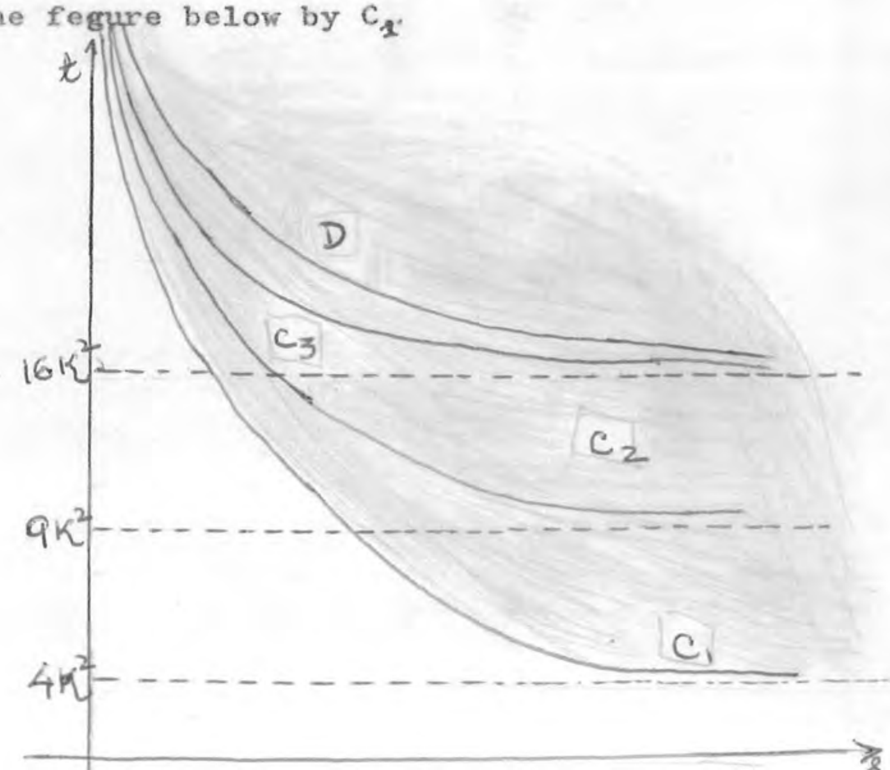
$K(s,t;K^2,K^2)$ fails to vanish only if (because of the Θ function)

$$t > 4K^2 + \frac{K^4}{s} \quad \ldots\ldots (12)$$

i.e the region in the st-plane where the first term of q(10) is different from zero is bounded from below by the curve

$$st - 4K^2 s - K^4 = 0 \quad \ldots\ldots (13)$$

his curve is denoted in the fegure below by C_1.



In the general case

$$\Theta\left\{t-a-b-\frac{ab}{2s}-\frac{\sqrt{ab}}{2s}\left[16s^2+4s(a+b)+ab\right]^{1/2}\right\}$$

will becomes for small s

$$\Theta\left[t-\frac{ab}{2s}-\frac{ab}{2s}(ab)^{1/2}\right]$$

Hence for small s,

K(s,t;a,b) is different from zero only in a region bounded from below by the curve $t \approx \frac{ab}{s}$

For large s,

$$\Theta\left\{t-a-b-\frac{ab}{2s}-\frac{\sqrt{ab}}{2s}\left[16s^2+4s(a+b)+ab\right]^{1/2}\right\} \rightarrow \Theta\left[t-a-b\ \frac{ab}{2s}(16s^2)^{1/2}\right]$$

Therefore, for large s K(s,t,ab) is different from zero in a region which is bounded from below by the horizontal straight line t $(\sqrt{a}+\sqrt{b})^2$

Consider the second term of equation (10). The lower limit of t' is $4K^2$. This term will contribute to $\rho(s,t)$ only in the region where $K(s,t,4K^2,K^2)\neq 0$. From (11) $K(s,t;4K^2,K^2)\neq 0$, if $\Theta\left\{t-4K^2-K^2-\frac{4K^2K^2}{2s}-\frac{(4K^2K^2)^{1/2}}{2s}\ 16s^2+4s(4K^2+K^2)+4K^2K^2\right\}^{1/2}\neq 0$

Or second term of (10) will not vanish in the region which is bounded from below by the curve C_2 given by

$$s(t-9K^2)(t-K^2) - 4K\,t = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (15)$$

In the third term the lower limit of t', t" is $4K^2$

Hence the third term will contribute to $\rho(s,t)$ in the region where $K(s,t;4K^2,4K^2)\neq 0$. Again using (11)

$$t-4K^2-4K^2\frac{16K^4}{2s}-\frac{4K^2}{2s}\left[16s^2+4s(8K^2)+16K^4\right]^{1/2}=0$$

will give the curve C_3 which is the lower bound of the region in which the third term contributes

$$s(t-16K^2)-16K^4=0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (16)$$

Thus the region where the $\rho(s,t)$ fails to vanish is bounded from below by curve C_1. This region is entirely outside the physical region.

Using this property of ρ the integral eq. (10) can be solved in a series of infintely many but well defined steps. First of all in the region between the curves C_1 and C_2 ; $\rho(s,t)$ is completely given by the first ~~two~~ term~~s~~. From(11)

$$\rho(s,t) = \lambda^2 \frac{\pi}{2} \cdot \frac{1}{\{s[t-(\kappa+\kappa)^2](t)-t\kappa^4\}^{\frac{1}{2}}}$$

$$\rho(s,t) = \lambda^2 \frac{\pi}{2} \frac{1}{\{s(t-4\kappa^2)t-t\kappa^4\}^{1/2}}$$

This can be subtituted into the second term of eq. (10) and integrated over that region for which $\rho(s',t')$ is already known. The integrand contains $K(s,t;t';\kappa^2)$ and to evaluate the integral upto a certain t(keeping $s = s_0$ fixed), it is not needed to integrate upto t'-values bigger than a certain limit, because K does not vanish only when the envisaged $t > t_{crit}(t')$. Thus one can go upto a t such that $t \leq \max. t_{crit}(t')$. From (11) since $a = t'$; $b = \kappa^2$

$$t_{crit}(t') = t'+\kappa^2+(t'\kappa^2/2s_0) + [(t'\kappa^2)^{\frac{1}{2}}/2s_0][16s_0 + 4s_0(t'+\kappa^2)+t'\kappa^2]^{\frac{1}{2}}$$

If ρ is known in the first region then for a fixed s_0, ρ upto a t' is known such as given by the intersection of the vertical line $s=s_0$ with the curve C_2. This intersection is given by theequation,

$$s_0(t'-9\kappa^2)(t'-9\kappa^2) = 4\kappa^2 t'$$

Denoting the solution by $t'_m(s_0)$, the second term for any fixed $s = s_0$ can be computed upto

$$t \leq t'_m(s_0)+\kappa^2+(t'_m(s_0)\kappa^2)/2s_0 + \{[t'_m(s_0)\kappa^2]^{\frac{1}{2}}/2s_0\}\{16s_0^2+4s_0[t'_m(s_0^2)+\kappa^2](s_0) + t'_m(s_0)\kappa^2\}$$

Byworking this out explicitly, it will be found that the knowledge of ρ in the first region (i.e.between the curves C_1 and C_2) enablesto compute the second term in eq. (10) not only in the entire region between C_2 and C_3 (where the third term does not yet cotribute) but also slightly over this region, upto some curve D shown in the preceding figure. Thus ρ is determined for the range between C_1 and C_2, using only the first two terms. Beyond this region the third term also starts contributing and this region can be penetrated as discussed earlier by utilising the knowledge of ρ in the lower regions. Eventually the entire st-plane is covered.

This procedure enables one to represent ρ by a ~~by a~~ polynomial in the strength parameter of the potential, valid for any finite region In the region between C_1 and C_2, the function ρ contains only λ^2. In going upto curve C_3, there are terms in λ^2 and λ^3. In the next step(using the third term for the first time) a term in λ^4 is added and so on. In any finite region of the st-plane, ρ will be given exactly as a polynomial of finite order in λ. The farther the region that is considered, the higher the order. Thus ρ can be effectively approximated by a polynomials and the region where this approximation is valid is also known.

Once ρ is obtained in such a way, it can be fed back into the Mandelstam dispersion relation ( )in the following manner: Thedispersion relation over a previously defined region is computed and f(s,t) as a polynomial of λ is obtained. Larger and larger regions are covered in this way, integrating always over regions where

ρ is known. thus the sctttering ampltude $f(s,t)$ can be represented by a sequence of polynomials which is certain to converge. In other words, one can developfrom the Mandelstam representation an approximation, method different from the usall power series expansion.

The determination of $\rho(s,t)$ for no bound states and no subtractions have been discussed above. If subtractions are necessary, the same method can still be used. However, in that case the functions g_j must also be determined. For this it is more convenient to use a partial wave decomposition .

Section IV. Dispersion relations for partial wave amplitudes.

The partial wave decompositiom of the scattering amplitude is,

$$f(st) = \frac{1}{\sqrt{s}}\sum_{l=0}^{\infty}(2l + 1)\exp(i\delta_l)\sin\delta_l P_l(1 - \frac{t}{2s}) \qquad (1)$$

Introducing,

$$f_l(s) = \frac{1}{\sqrt{s}}\exp(i\,\delta_l(s))\sin\delta_l(s) \qquad (1\text{-a})$$

as the partial-wave amplitude, equation (1) can be written as,

$$f(st) = (2l + 1)f_l(s)P_l(1 - \frac{t}{2s})$$

writing, $x = 1 - \frac{t}{2s}$

$$f(s,x) = \sum_{l=0}^{\infty}(2l + 1)f_l(s)P_l(x) \qquad (2)$$

The Legendre formula states:

If, $F(x) = \sum_{l} c_l P_l(x)$

then, $c_l = \frac{2l+1}{2}\int_{-1}^{1}F(x)P_l(x)dx$

Multiplying both sides of (2) by $P_l(x)$ and integrating from -1 to +1 one obtains,

$$f_l(s) = \frac{1}{2}\int_{-1}^{+1}f(s,x)P_l(x)dx \qquad (3)$$

The analytic properties of f_l (as shall be seen) do not depend on the number of subtractions in the full representation. Therefore to be specific, it can be assumed that there is only one S-wave bound state and subtraction is necessary. In this case the Mandelstam representationwill be,

$$f(s,t) = f_B(t) - t\frac{1}{\pi^2}\int_{0}^{\infty}\int_{4K^2}^{\infty}\frac{(s',t')}{(t'+t)t'(s'-s-i\eta)}dt'ds' + \frac{R}{s+|E|} + \frac{1}{\pi}\int_{0}^{\infty}\frac{g(s')}{(s'-s-i\eta)}ds' \qquad (4)$$

Here $-|E|$ is the bound state enrgy, R is a constant (independent) of t) and

$$g(s) = Im.f(s,o)$$

The Born term is given explicitly by

$$f_B(t) = - \frac{8\lambda}{\kappa^2 + t} \qquad \text{(for the single Yukawa potential)}$$

Substitute (4) in the equation (3)

$$f_\ell(s) = \frac{1}{2}\int_{-1}^{+1} \frac{-\lambda P_\ell(x)\,dx}{\kappa^2 + 2s(1-x)} + \frac{1}{2}\int_{-1}^{+1}\frac{RP_\ell(x)dx}{(s + |E|)} - \frac{1}{2\pi^2}\int_{-1}^{+1} 2s(1-x)$$

$$\times \int_0^\infty \int_{4\kappa^2}^\infty \frac{\varrho(s',t')P_\ell(x)\,dt'\,ds'dx}{t' + 2s(1-x)\; t'(s' - s - i\eta)}$$

$$+ \frac{1}{2\pi}\int_{-1}^{+1} P_\ell(x) \int_0^\infty \frac{g(s')}{s' - s - i\eta}\,ds'dx \qquad \text{.......................(5)}$$

Hence there are integrals of the type

$$I_\ell = \frac{1}{2}\int_{-1}^{+1} \frac{P_\ell(x)}{a + 2s(1-x)}dx \qquad \text{..............................(6)}$$

where $a = \kappa^2, t'$. Such integrals have solutions,

$$\frac{1}{2}\int_{-1}^{+1} \frac{P_\ell(x)}{y - x}dx = Q_\ell(y) \qquad \text{..............................(7)}$$

where Q_ℓ is a Legendre polynomial of the second kind.

By the Rodrigues formula

$$P_\ell(x) = \frac{1}{2^n n!}\frac{d}{dx^n}(x^2 - 1)^n$$

$$Q_o(y) = \frac{1}{2}\int_{-1}^{+1} \frac{1}{y-x}dx = -\frac{1}{2}\left[\log(y-1) - \log(y+1)\right]$$

$$Q_o(y) = \frac{1}{2}\log\frac{(y+1)}{(y-1)} \qquad \text{..............................(8-a)}$$

$$Q_1(y) = -1 + \frac{1}{2}\, y \log\frac{y+1}{y-1} = \frac{1}{2}\int_{-1}^{+1}(-1)dx + \frac{1}{2}\int_{-1}^{+1}\frac{y}{y-x}dx$$

$$Q_1(y) = +\frac{1}{2}\int_{-1}^{+1} \frac{x}{y-x}dx = -\frac{1}{2}\int(-1)dx + \frac{1}{2}\int_{-1}\frac{y}{y-x}dx$$

$$= -\frac{1}{2}(2) + \frac{1}{2}y\log\frac{y+1}{y-1}$$

$$= -1 + \frac{1}{2}y\log\frac{(y+1)}{(y-1)} = -1 + yQ_o(y) \quad \text{.....(8-b)}$$

Because of recursion relation

$$(2n+1)xP_n(x) = (n+1)P_{n+1}(x) + nP_{n-1}(x)$$

or,

$$(2n-1)x\,P_{n-1}(x) = nP_n(x) + (n-1)P_{n-2}(x)$$

Eq. (7) becomes,

$$Q_n(y) = \frac{1}{2}\frac{(2n-1)xP_{n-1}(x)}{n(y-x)}dx - \frac{1}{2}\cdot\frac{(n-1)}{n}\int_{-1}\frac{P_{n-1}(x)}{y-x}dx$$

$$-\frac{(2n-1)}{2n}\int_{-1}^{+1}P_{n-1}(x)dx + \frac{(2n-1)}{2n}\int_{-1}^{+1}\frac{yP_{n-1}(x)dx}{y-x} - (n-1)\ P\ ($$

$$-\frac{(n-1)}{2n}\int_{-1}^{+1}\frac{P_{n-2}(x)dx}{y-x}$$

Here, $\int_{-1}^{+1}P_{n-1}(x)dx = o$, because $\int_{-1}^{+1}P\ (x) = \left[2/(2\ell+1)\right]\delta_{\ell 0}$

Therefore, $Q_n(y) = \frac{(2n-1)}{n}yQ_{n-1}(y) - \frac{(n-1)}{n}Q_{n-2}(y)$

Or,

$$(\ell+1)Q_\ell(y) = (2\ell+1)yQ_\ell(y) - \ell Q_{\ell-1}(y) \quad \text{..........(8-c)}$$

All Q_ℓ's can be expressed as combinations of Q_1 and Q_o where

$Q_o(y) = \frac{1}{2}\log(y+1)/(y-1)$ & $Q_1 = -1 + yQ_0(y)$.

Hence, the singularities are determined by $\frac{1}{2}\log(y+1)/(y-1)$

For $|y|>1$, $(y+1)/(y-1)$ is positve, $\log(y+1)/(y-1)$ is real

For $|y|<1$, $(y+1)/(y-1)$ is negative; consequently Q_ℓ is complex

Therefore by Schartz's reflection principle

$$\overset{*}{Q}_{\ell}(z) = Q_{\ell}(\overset{*}{z}) \qquad \text{.........................(8-d)}$$

Also, because of $\log(y + 1)/(y - 1)$, $y=1$, $y=-1$, are branch points and branch cut may conveniently be taken along the real axis to connect these points.

The discontinuity along the cut is given by,

$$D_{\ell}(y) = \frac{1}{2i\eta} \lim_{\eta \to 0} \left[Q_{\ell}(y + i\eta) - Q_{\ell}(y - i\eta) \right.$$

By equation (7)

$$Q_{\ell}(y + i\eta) = \frac{1}{2} \int_{-1}^{+1} \frac{P_{\ell}(x)dx}{y + i\eta - x};$$

$$Q_{\ell}(y - i\eta) = \frac{1}{2} \int_{-1}^{+1} \frac{P_{\ell}(x)dx}{y - i\eta - x}$$

Thus,

$$\frac{1}{2i} \lim_{\eta \to 0} Q_{\ell}(y + i\eta) - Q_{\ell}(y - i\eta) = \frac{1}{2i} \lim_{\eta \to 0} \frac{1}{2} \int_{-1}^{+1} \frac{-2i\ P_{\ell}(x)dx}{(y-x)^2 - \eta^2}$$

But, $\lim_{\eta \to 0} \frac{\eta}{(y-x)^2 \mp \eta^2} = \pi\,\delta(x-y)$

Hence, $D_{\ell}(y) = -\frac{\pi}{2} \int_{-1}^{+1} \delta(x-y) P_{\ell}(x)dx = -\frac{\pi}{2} P_{\ell}(y)$ for $|y| < 1$

Because of (8-d)

$$D_{\ell}(y) = \frac{1}{2i} \lim_{\eta \to 0} \left[Q_{\ell}(y + i\eta) - \overset{*}{Q}_{\ell}(y + i\eta) \right.$$

or, $\qquad D_{\ell}(y) = \mathrm{Im.}\,Q_{\ell}(y) = -\frac{\pi}{2} P_{\ell}(y)$ for $|y| < 1$

Eq. (6) may be written as,

$$I_{\ell} = \frac{1}{2} \int_{-1}^{+1} \frac{P\ (x)dx}{(a + 2s(1-x)} = \frac{1}{2s}\frac{1}{2} \int_{-1}^{+1} \frac{P_{\ell}(x)}{(a/2s)+1-x} dx$$

$$I_{\ell} = \frac{1}{2s} \int_{-1}^{+1} \frac{1}{2} \frac{P_{\ell}(x)dx}{y - x} = \frac{1}{2s} \cdot Q_{\ell}(y)$$

where $y = (a/2s) + 1$

Hence $I_{\ell} = \frac{1}{2s} Q_{\ell}\left[1 + (a/2s)\right]$ ....(10)

$Q_{\ell}(1 + (a/2s))$ is complex for $|1 + (a/2s)| <1$

Hence $I_\ell = \frac{1}{2} Q_\ell(1 + \frac{a}{2s})$ ....(10)

$Q_\ell(1 + (a/2s))$ is complex for $|1 + (a/2s)| < 1$

Since, $a > 0$ this means that $0 > \frac{a}{2s} > -2$

Or, I_ℓ is complex for $s > -a/4$

Similarly, I_ℓ is real for $s < -a/4$

Also, $$I_\ell^*(s) = \frac{1}{2s^*} Q_\ell^*(1 + (a/2s)) = \frac{1}{2s^*} Q_\ell(1 + \frac{a}{2s^*})$$

Hence, $$I_\ell^*(s) = I_\ell(s^*) \quad \text{for s coplex}$$

I_ℓ has branch points at $s = -a/4$ and $s = \infty$ and associted cut

along the negative real axis and a discontinuit y

$$\frac{1}{2i} \lim_{\eta \to 0} \left[I_\ell(s + i\eta) - I_\ell(s - i\eta) \right] = \frac{\pi}{2s\,2} P_\ell(1 + \frac{a}{2s}) \text{ for } s < -a/4$$

Further, $$\text{Im. } I_\ell(s) = \frac{1}{2s} \frac{\pi}{2} P_\ell(1 + \frac{a}{2s})$$

for s real $s < -a/4$. Note that for $s > -a/4$ the jump and I_ℓ are zero.

For $\ell = 0$, equation (5) may be written ass

$$f_0(s) = \frac{1}{2s} \cdot \frac{1}{2} \int_{-1}^{+1} \frac{-\lambda P_0(x)dx}{(K^2/2s) + 1 - x} + \frac{1}{2} R. \int_{-1}^{+1} \frac{P_0(x)dx}{s + |E|}$$

$$-\frac{1}{2\pi^2} \int_{-1}^{+1} \frac{2s(1-x)P_0(x)dx}{t' + 2s(1-x)} \int_0^\infty \int_{4K^2}^\infty \frac{(s';t';)dt'ds'}{t'\,t(s'-s-i\eta)} + \frac{1}{2\pi} \int_{-1}^{+1} P_0(x) \int_0^\infty \frac{g(s')ds'dx}{s'-s-i\eta}$$

Or, $$f_0(s) = -\frac{\lambda}{2s}Q_0\left(1 + \frac{a}{2s}\right) + R/(s+|E|)$$
$$-\frac{1}{2\pi^2}\int_{-1}^{+1}\left[1 - \frac{t'}{t'+2s(1-x)}\right]P_0(x)\,dx \int_0^{\infty}\int_{4\kappa^2}^{\infty}\frac{\rho(s';t')\,dt'\,ds'}{t'(s'-s-i\eta)}$$
$$+\frac{1}{2\pi}\int_{-1}^{+1}P_0(x)\int_0^{\infty}\frac{g(s')}{s'-s-i\eta}\,ds'\,dx$$

Hence, $$f_0(s) = -\frac{\lambda}{2s}Q_0(1 + \kappa^2/2s) + R/(s+|E|)$$
$$-\frac{1}{2\pi^2}\left[2 - \frac{t'}{2s}Q_0\right]\int_0^{\infty}\int_{4\kappa^2}^{\infty}\frac{\rho(s';t')\,dt'\,ds'}{t'(s'-s-i\eta)} + \frac{1}{2\pi}.2.\int_0^{\infty}\frac{g(s')\,ds'}{s'-s-i\eta}$$

$$f_0(s) = -\frac{\lambda}{s}Q_0(1 + \kappa^2/2s) + R/(s+E) +$$
$$+\frac{1}{\pi^2}\int_0^{\infty}\int_{4\kappa^2}^{\infty}\frac{\rho(s';t')}{(s'-s-i\eta)}\frac{1}{2s}\left[Q_0\left(1+\frac{t'}{2s}\right) - \frac{2s}{t'}\right]dt'\,ds' \qquad \text{..............(13)}$$

For $\ell \geq 1$, (from 12)

$$f_\ell(s) = -\frac{\lambda}{2s}Q_\ell\left(1 + \frac{\kappa^2}{2s}\right) + \frac{1}{\pi^2}\int_0^{\infty}\int_{4\kappa^2}^{\infty}\frac{\rho(s';t').1}{s'-s-i\eta}\,\frac{1}{2s}\,Q_\ell\left(1+\frac{t'}{2s}\right)dt'\,ds' \qquad \text{......(14)}$$

Here use has been made of fact that $P_\ell(x)dx = [2/(2l+1)]\,\delta_{\ell 0}$

Equations (13) and (14) are the Mandelstam representations for the partial wa ve amplitudes.

For $\ell \neq 0$, it is seen that a solution of the integral equation is not required, becuase, in principle at least, the weight function

is already known and f (s) can be determined by quadrature
For ℓ =0, a difficult nonlinear integral eqn is involved. Consider the
when s $>$ 0: $Q_0(1 + K^2/2s)$ and $Q_0(1 + t'/2s)$ are real, because $Q_0(1 + a/2s)$
is real for s $>$ -a/4. Taking imaginary part of (13) one obtains,

$$\mathrm{Im.}\ f_0(s) = \frac{1}{\pi^2}\mathrm{Im.}\int_0^\infty\int_{4K^2}^\infty \frac{\rho(s',t')}{s'-s-i\eta}\ \frac{1}{2s}\left[Q_0\left(1+\frac{t'}{2s}\right) - \frac{2s}{t'}\right]dt'ds' + \frac{1}{\pi}\mathrm{Im.}\int_0^\infty \frac{g(s')ds'}{s'-s-i\eta} \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(15)$$

Using the identity,

$$\frac{1}{s'-s-i\eta} = P\frac{1}{s'-s} + i\pi\delta(s'-s)$$

$$\frac{1}{\pi^2}\mathrm{Im.}\int_0^\infty\int_{4K^2}^\infty \frac{\rho(s',t')}{s'-s-i}\ \frac{1}{2s}\left[Q_0\left(1+\frac{t'}{2s}\right) - \frac{2s}{t'}\right]dt'ds' = \frac{1}{\pi}\int_{4K^2}^\infty \rho(s,t')\,\frac{1}{2s}\left[Q_0\left(1+\frac{t'}{2s}\right) - \frac{2s}{t'}\right]dt'$$

And, $$\frac{1}{\pi}\mathrm{Im.}\int_0^\infty \frac{g(s')\,ds'}{s'-s-i\eta} = g(s)$$

Hence, Eq. (15) becomes

$$g(s) = \mathrm{Im.}\,f_0(s) - \frac{1}{\pi}\int_{4K^2}^\infty \rho(s,t')\,\frac{1}{2s}\left[Q_0\left(1+\frac{t'}{2s}\right) - \frac{2s}{t'}\right]dt' \qquad \text{-----}\ (15\text{-A})$$

Substituteng this in equation (13) one obtains

$$f_0(s) = -\frac{\lambda}{2s}Q_0\left(1+\frac{K^2}{2s}\right) + \frac{R}{s+|E|} + \frac{1}{\pi}\int_0^\infty \frac{\mathrm{Im.}\,f_0(s')ds'}{s'-s-i\eta} \qquad \ldots\ldots\ldots\ldots\ldots(16)$$

which is an integral equation for $f_0(s)$. Itis nonliear because:

$$\mathrm{Im.}\,f_\ell(s) = \sqrt{s}\,\mathrm{Sin}\,\delta_\ell(s)\left[\exp i\delta_\ell(s) - \exp -i\delta_\ell(s)\right]/2i$$
$$= \sqrt{s}\,\mathrm{Sin}^2\ (s)$$

Also, from (1-a)

$$\left|f_\ell(s)\right|^2 = s^{-1}\sin^2\delta_\ell(s) \qquad \ldots\ldots\ldots\ldots\ldots(16)\text{- A}$$

Therefore,
$$Im.f_{\ell}(s) = \sqrt{s}|F_{\ell}(s)|^2$$

Once the nonlinear integral eq. is solved, eq. (15)-a, gives the solution of $g(s)$

The Mandelstam representation eq. (13) and (14) also exhibits the analytic properties ofthe partial wave amplitudes.The partial wave amplitudes $f_{\ell}(s)$ (for all ℓ) have acut along the real s-axis from $s = 0$ to $s = \infty$,and comes from the denominator $s'-s$. It is similar to that of the full amplitude. In addition, there is also a cut along the negative s-axis. The first Born terms $-(\lambda/2sQ_{\ell}(1+K^2/2s$ have a cut from $s = -K^2/4$ to $s = -\infty$ and the double integral terms have a cut from $s = -4K^2/4=-K^2$ to $s = -\infty$ .The cuts are represented below

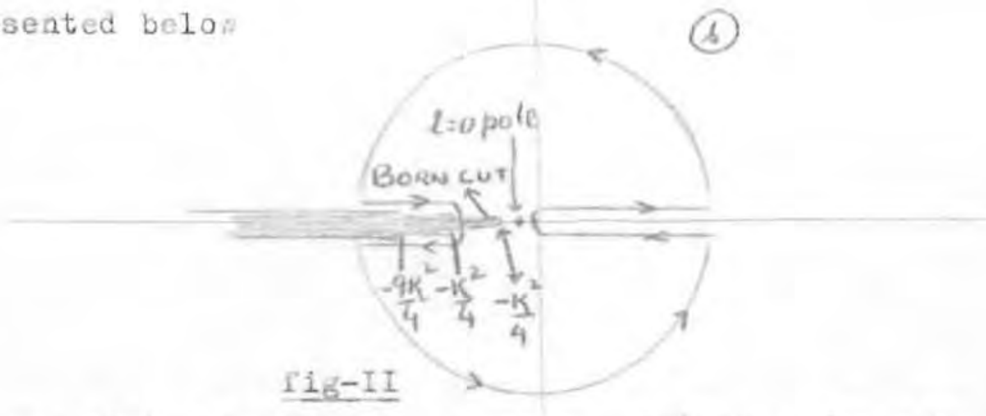


fig-II

further, $f(s)$ also has a pole at $s = -|E|$ Also, ρ can be expanded in a Born type series,

$$\rho(s',t') = \rho^{(1)} + \rho^{(2)} + \rho^{(3)} + \cdots$$

where ρ vanishes unless $t' \geq (n + 1)^2 K^2$.(seesection III). Hence for the nth term the effective lower limit of the t' integration is $(n + 1)^2 K^2$, and therefore, branch points occur at:

$$s = -\kappa^2,\ -\frac{9}{4}\kappa^2,\ -\frac{16}{4}\kappa^2 \ldots\ldots$$

The corresponding cuts overlap so htat the negative cut in $f_\ell(s)$ starts at $s = -\kappa^2/4$. Discounting the first Born approximation term the cut starts at $s = -\kappa^2$, of the full amplitude

The dispersion relations (13) and (14) may be formulated to exhibit explicitly the analytic properties of $f_\ell(s)$. This is done with the help of Cuachy's theorem. For this the knowledge of the brhaviour of $f_\ell(s)$ as $s \to \infty$ is necessary.

from (16-a), $|f_\ell(s)| < 1/\sqrt{s}$ so that $f_\ell(s) \to 0$, for $s \to \infty$. It can be shown that this holds true in any direction. Using the notation,

$$f_{\ell B}(s) = -(\lambda/2s)\,Q_\ell(1 + \kappa^2/2s) \text{ for all } \ell, \text{ or from (10)}$$

$f_{\ell B}(s) = -\lambda I_\ell(s)$ with $a = \kappa^2$.

Cuachy's formula for $f_\ell - f_{\ell B} = c_\ell$(say), can be applied around the contour shown in fig.-2; because, c has a cut from $s = -\kappa^2$ to $s = -\infty$ and from $s = 0$ to $s = +\infty$. Also fro $c_\ell(s) \to 0$ for $s \to \infty$ thus (for $\ell \geq 1$)

$$c_\ell(s) = \frac{1}{2i\pi}\oint_C \frac{c(s)}{s'-s}\,ds'$$

for $s' \to \infty$

$$c_\ell(s) = \frac{1}{2i\pi}\int_\ell \frac{c(s')\,ds'}{s'-s} + \frac{1}{2i\pi}\lim_{\beta \to 0}\int_0^\infty \frac{c_\ell(s'+i\beta) - c_\ell(s'-i\beta)}{s'-s}\,ds'$$

$$+ \frac{1}{2i\pi}\lim_{\epsilon \to 0}\int_0^\infty \frac{c_\ell(s'+i\epsilon) - c_\ell(s'-i\epsilon)}{s'-s}\,ds'$$

where the first intengral is zero.

To take care of the case when s is real it is customary to write

$$f_\ell(s) - f_{\ell B}(s) = \int_0^\infty \frac{A_\ell(s')}{s'-s-i\eta}\, ds' + \int_{-\infty}^{-K^2} \frac{B_\ell(s')\, ds'}{s'-s-i\eta} \quad \text{for } \ell \geq 1 \quad \ldots\ldots\ldots\ldots (17)$$

where $\ell = 0$ $f_0(s)$ has a pole at $s = -|E|$ .Thus,

$$f_0(s) - f_{0B}(s) = \int_0^\infty \frac{A_0(s')ds'}{s'-s-i\eta} - \int_{-\infty}^{-K^2} \frac{B_0(s')\, ds'}{s'-s-i\eta} + \frac{R}{s+|E|} \quad \ldots\ldots\ldots\ldots (18)$$

The weight function A_ℓ and B_ℓ are real(for all ℓ), becaus e:

$c_\ell(s) = f_\ell - f_{\ell B} = f_{\ell B} + \lambda I_\ell$

discontinuity of $f_\ell(s')$ is real = $\mathrm{Im} f_\ell(s')$

The discontinuity of $I_\ell(s')$ is real = I m.$I_\ell(s')$ for s'real and $s' < -K^2/4$

For $s' > -K^2/4$, Im.$I_\ell(s')$ = 0.Taking imaginary part of (17) and (18)

one obtains for all ℓ.

$$\mathrm{Im}.f_\ell(s) = \mathrm{Im}.f_{\ell B}(s)\, ds' + \mathrm{Im}.\int_0^\infty \frac{A_\ell(s')ds'}{s'-s-i\eta} + \int_{-\infty}^{-K^2} \frac{B_\ell(s')ds'}{s'-s-i\eta}$$

Using, $(s'-s-i\eta)^{-1} = P\,(s'-s)^{-1} + i\pi\delta(s'-s)$

$\mathrm{Im}.f_\ell(s) - \mathrm{Im} f_{\ell B}(s) = \pi A_\ell(s) + 0$ for $s > 0$

$\mathrm{Im}[f(s) - f(s)] = 0 + \pi B_\ell(s)$ for $s < -K^2$

But for $s > 0$, $f_{\ell B}(s)$ is real,Hence, Im.$f_{\ell B}(s) = 0$; and for

$s < -K^2$, $f_{\ell B}(s)$ is ~~real~~ complex

Therefore,

$\mathrm{Im}.f_\ell(s) = \pi A_\ell(s)$, $\quad s > 0$

for all ℓ

$\mathrm{Im}.\ f_\ell(s) - \mathrm{Im}.f_{\ell B}(s) = \pi B_\ell(s)$, $s < -K^2$

Hence equation (17) and (18) can be written as,

$$f_0(s) = f_{0B}(s) + \frac{R}{s+|E|} + \frac{1}{\pi}\int_0^\infty \frac{\mathrm{Im}.f(s')ds'}{s'-s-i\eta} + \frac{1}{\pi}\int_{-\infty}^{-K^2} \frac{\mathrm{Im}[f_0(s') - f_{0B}(s')]}{s'-s-i\eta}\, ds'$$

$$f_\ell(s) = f_{\ell B}(s) + \frac{1}{\pi}\int_0^\infty \frac{Im.f_\ell(s')ds'}{s'-s-i\eta} + \frac{1}{\pi}\int_{-\infty}^{-K^2} \frac{Im[f_\ell(s') - f_{\ell B}(s)]ds'}{s'-s-i\eta} \quad \text{(19-b)}$$

where $\ell \geqslant 1$. If ρ is known the integrals over the part of the negative real axis is known. Take the imaginary part of (13) when $s < -K^2$ i.e.

$$Im[f_0(s) - f_{0B}(s)] = \frac{1}{\pi^2} Im\int_0^\infty\int_{4K^2}^\infty \frac{(s',t')}{s'-s-i\eta} \cdot \frac{1}{2s}[Q_0(1+\frac{t'}{2s}) - \frac{2s}{t'}]dt'ds' + \frac{1}{\pi} Im\int_0^\infty \frac{g(s')}{(s'-s-i\eta)}ds' \quad (20)$$

In the double integral $Q_0(1 + t'/2s)$ is complex if $s < -t'/4$ or as the lower limit of t' is $4K^2$, $Q_0(1 + t'/2s)$ iscomplex if $s < K^2$, Hence,

$$\frac{1}{\pi^2} Im\int_0^\infty\int_{4K^2}^\infty \frac{\rho(s',t')}{s'-s-i\eta} \cdot \frac{1}{2s} Q_0(1+t'/2s)\left(-\frac{2s}{t'}\right)dt'ds'$$

$$= \frac{1}{\pi^2}\int_0^\infty\int_{4K^2}^\infty \frac{\pi\delta(s'-s)\rho(s',t')}{s'-s-i\eta}\cdot\left(-\frac{1}{t'}\right)dt'ds'$$

$$+ \frac{P}{\pi^2}\int_0^\infty\int_{4K^2}^\infty \frac{(s',t')}{s'-s} Im.\frac{1}{2s}.Q_0(1+t'/2s) \quad + 0$$

Also, $Im\int_0^\infty \frac{g(s')}{s'-s'-i\eta}ds' = \int_0^\infty \pi\delta(s'-s)g(s')ds' = 0$ for $s < -K^2$

Using, $Im.I_\ell(s) = (2s)^{-1}(\pi/2)P_\ell(1 + t'/2s)$ for $s < -t'/4$

$= 0$ for $s > -t'/4$

or, equivalently, $Im.I_\ell(s) = (2s)^{-1}(\pi/2)P_\ell(1 + t'/2s)\theta(-s - t'/4)$

Eq. (20) becomes,

$$Im[f_0(s) - f_{0B}(s)] = \frac{P}{4\pi s}\int_0^\infty\int_{4K^2}^\infty \frac{\rho(s',t')}{s'-s} . P_0(1+\frac{t'}{2s})\theta(-s-\frac{t'}{4})dt'ds' \quad (21)$$

Equation (21) is also valid for all ℓ as can be seen by using eq.(14) instead of (13). Thus,

$$Im.[f_0(s) - f_{0B}(s)] = \frac{P}{4\pi s}\int_0^\infty\int_{4K^2}^\infty \frac{\rho(s',t')}{s'-s} . P_0(1+\frac{t'}{2s})\theta(-s-\frac{t'}{4})dt'ds' \quad (21)$$

is valid for all ℓ and $-\infty < s < -K^2$. When considering eq, (19-a,b) equation (21) should be taken under the integral(the last one). For $s > 0$, $Q_\ell(1 + K^2/2s)$ is real. Taking imaginary part of (14) one obtains

$$\mathrm{Im}.f_\ell(s) = \frac{1}{\pi^2}\mathrm{Im}\int_{4K^2}^{\infty}\int_0^{\infty} \frac{\rho(s',t')}{s' - s - i\eta} \cdot \frac{1}{2s} Q_\ell\left(1 + \frac{t'}{2s}\right) dt'\,ds'$$

$$= \frac{1}{\pi^2}\int_{4K^2}^{\infty} \pi\rho(s,t') \cdot \frac{1}{2s} Q_\ell(1 + t'/2s)\,dt'$$

$$= \frac{1}{2\pi s}\int_{4K^2}^{\infty} \rho(s,t') Q_\ell(1 + t'/2s)\,dt'; \quad \ell \geqslant 1;\ 0 < s < \infty$$

Hence the dispersiom relation 19-b, for $\ell \geqslant 1$ are not integral equation but are justquadratures. This is true for equation (14). For $\ell = 0$, one obtains a nonlinear integral equati on because with the substitution $\mathrm{Im}.f_0(s) = \sqrt{s}\,|f_0(s)|^2$ equation 19-a becomes,

$$f_0(s) = f_{0B}(s) + \frac{R}{s + |E|} + \frac{1}{\pi}\int_0^{\infty} \frac{\sqrt{s'}\,|f_0(s')|^2\,ds'}{s' - s - i\eta} + \frac{1}{\pi}\int_{-\infty}^{-K^2} \frac{\mathrm{Im}.[f_0(s') - f_0(s)]\,ds'}{s' - s - i\eta} \quad \ldots\ldots\ldots\ldots(22)$$

This nonlinear equation can be solved by a method known as the N/D method.

Assume that $f_0(s) = N(s)/D(s)$ ....(22-a)

where,

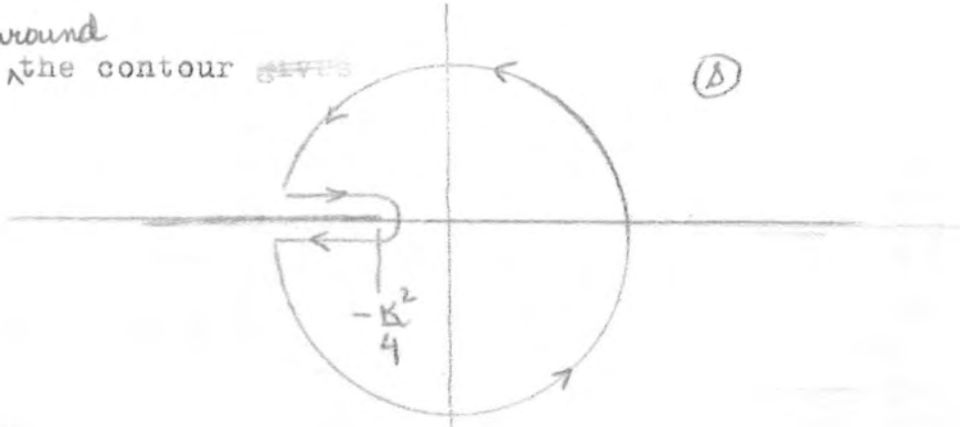
(a) N is real for $s > 0$, and has a branch cut alöng the negative real axis from $s = -K^2/4$ to $s = -\infty$

(b) D is real for $s < 0$ and has acut along the positive real axis from $s = 0$ to $s = -\infty$. Otherwise, both N and D are regular. It is alsö assumed that

(c) $D(s) \Rightarrow 1$ when $s \Rightarrow \infty$. Since $f_0(s) \to 0$, for $s \Rightarrow \infty$, therefore $N(s) \Rightarrow 0$, when $s \Rightarrow \infty$.

Because of the assumed properties of N, $N(s')/(s' - s)$

integrated around the contour ~~gives~~ (δ)



gives,

$$N(s) = \frac{1}{2\pi i}\oint \frac{N(s')}{s'-s}\,ds'$$

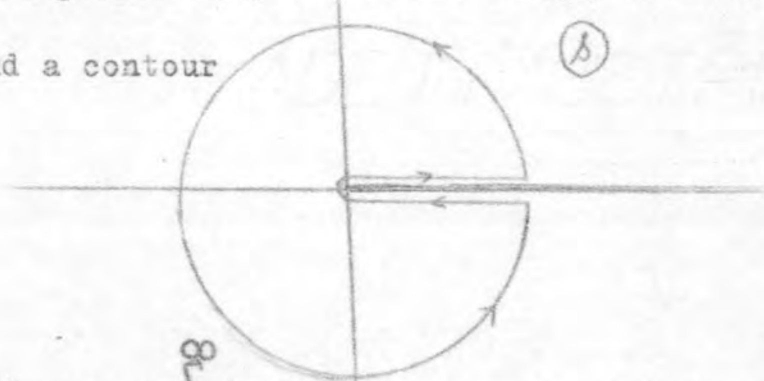
For $s' \longrightarrow \infty$, $N \longrightarrow 0$; Hence,

$$N(s) = \frac{1}{2\pi i}\int_C \frac{N(s')}{(s'-s)}\,ds' + \frac{1}{2\pi i}\lim_{\beta\to 0}\int_{-\infty}^{-K^2/4} \frac{N(s'+i\beta) - N(s'-i\beta)}{s'-s}\,ds'$$

But, $N^*(s') = N(s'^*)$ for s' complex. Hence,

$$N(s) = \frac{1}{\pi}\int_{-\infty}^{-K^2/4} \frac{Im.N(s')}{s'-s-i\eta}\,ds' \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(23)$$

similarly, remembering that $D(s) \longrightarrow 1$ for $s \longrightarrow \infty$, one obtains on integrating around a contour (δ)



$$D(s) = 1 + \frac{1}{\pi}\int_{0}^{\infty} \frac{Im.D(s')}{s'-s-i\eta}\,ds' \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(24)$$

Since $N(s)$ is real for $s>0$, equation (22-a) gives ~~imaginary~~

$$Im.D(s) = N(s)\,Im\left(1/f_0(s)\right) \text{ for } s>0.$$

Using $Im.\ f_\ell(s) = \sqrt{s}\,|f_\ell(s)|^2$, one obtains

$$\operatorname{Im}[f_0(s)]^{-1} = \operatorname{Im}\frac{f_0^*}{|f_0|^2} = -\frac{1}{|f_0|^2}\operatorname{Im}f_0 = -\sqrt{s})^{-1/2}$$

so that $\operatorname{Im}D(s) = -(s)^{1/2}N(s)$ (for $N(s)>0$) and eq. (24) may be written as

$$D(s) = 1 - \frac{1}{\pi}\int_0^\infty \frac{\sqrt{s'}N(s')}{s'-s-i\eta}ds' \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(25)$$

Also, $D(s)$ is real when $-\infty < s < -K^2/4$

Thus,

$$\operatorname{Im}N(s) = D(s)\operatorname{Im}f_0(s)$$

As,

$$f_{\ell B}(s) = -(\lambda/2s)Q_\ell(1 + K^2/2s)$$

And,

$$\operatorname{Im}I_\ell(s) = \frac{1}{2s}\frac{\pi}{2}P_\ell\left(1+\frac{K^2}{2s}\right)\theta\left(-s\ \frac{K^2}{4}\right)$$

$$\operatorname{Im}f_{0B} = -\lambda\operatorname{Im}I_\ell(s) = -\lambda\pi/4s \qquad \text{for } s<-K^2/4$$

From (21), as it appears applies in the region $-\infty < s < -K^2/4$

$$\operatorname{Im}.f_0 = -\frac{\lambda\pi}{4s} + \frac{1}{4\pi s}\int_0^\infty\int_{4K^2}^\infty \frac{\rho(s',t')\theta(-s-\frac{t'}{4})dt'ds'}{s'-s} = \alpha(s) \text{ say.}$$

Hence, $\operatorname{Im}.N(s) = D(s)\alpha(s)$, $\qquad s<-K^2/4$

Substituting in (23)

$$N(s) = \frac{1}{\pi}\int_{-\infty}^{-K^2/4}\frac{\alpha(s'')D(s'')}{s''-s-i\eta}ds'' \quad \ldots\ldots\ldots\ldots\ldots(26\_$$

where α is a known function. Equation (25) and (26) form a coupled integral eqution. Eliminate N from (25) to obtain,

$$D(s) = 1 - \frac{1}{\pi}\int_0^\infty \frac{\sqrt{s'}}{s'-s-i\eta}\ \frac{1}{\pi}\int_{-\infty}^{-K^2/4}\frac{\alpha(s'')D(s'')}{s''-s'-i\eta}ds''ds'$$

The integral $\displaystyle\int_0^\infty \frac{\sqrt{s'}\,ds'}{(s'-s-i\eta)(s''-s'-i\eta)} = \frac{\pi}{\sqrt{s''}+\sqrt{s}}$

Whence, $$D(s) = 1 - \frac{1}{\pi}\int_{-\infty}^{-K^2/4} \frac{(s'')D(s'')}{\sqrt{-s''}+\sqrt{s}}\,ds''$$

or,

$$D(s) = 1 + \frac{1}{\pi}\int_{K^2/4}^{\infty} \frac{\alpha(-s'')D(-s'')}{\sqrt{s''}+\sqrt{s}}\,ds'' \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(26)$$

Subsituting, $s'' = \frac{1}{x^2}$, for $s'' = K^2/4$, $x = 2/K$
for $s'' = \infty$, $x = 0$; $ds'' = -2x^{-3}$

Hence, $$D(s) = \underline{1} + \frac{2}{\pi}\int_0^{2/K} \frac{\alpha(-1/x^2)D(-x)\,dx}{x^2(1+x\sqrt{s})}$$

Put $(s)^{-1/2} = y$

Thus, $$D(-y) = 1 + \frac{2}{\pi}\int_0^{2/K} \frac{1}{x^2}\,\frac{(-1/x^2)}{1+x\sqrt{s}}\,D(-\cancel{x})\,dx$$

Solution of the D-eqution (26) is equivalent tocomputing theenergy binding-energy; because: the zeroes of D gives the poles of f_0 and since the poles of f_0, by the dispertion relations, are given by the bound state energy $s = -E$, therefore the solution of D id equivalent to computing the bindin g energy. The residue R is given by

$$R = \lim_{s \to -|E|} (s + |E|)\,\frac{N(s)}{D(s)}$$

An approximate method of solving the D-eq. suggested by Chew and Mandelstam, is to approximate the left hand cut by a pole. Let the ~~cut~~ pole replacing the ~~pole~~ cut be löcated at $s = -s_0 (s_0 > 0)$ and let its strength be characterised by a real parameter F. Thus setting, $$\mathrm{Im} f_0(s) = -\pi F \delta(s + s_0) \qquad \text{for } s < -K^2/4$$

But, for $s < -K^2/4$, $\mathrm{Im}\,N = D\,\mathrm{Im} f_0$; hence from (23) one obtains,

$$N(s) = \frac{1}{\pi}\int_{-\infty}^{-K^2/4} \frac{D(s')\,\mathrm{Im} f_0(s')\,ds'}{s' - s - i\eta} = \frac{1}{\pi}\int_{-\infty}^{-K^2/4} \frac{-\pi D(s) F \delta(s' + s_0)}{s' - s - i\eta}\,ds'$$

$$= FD(-s_0)/(s_0 + s) \quad \ldots\ldots\ldots\ldots(27)$$

Substituting in equation (25) one obtains,

$$D(s) = 1 - \frac{F}{\pi}\int_0^\infty \frac{(s')^{1/2} D(-s_0)ds'}{(s'-s-i\eta)(s_0+s')} = \frac{1- FD(-s_0)}{\sqrt{s_0}+\sqrt{-s}} \quad \ldots\ldots\ldots\ldots(28)$$

putting $s = - s_0$, one obtains,

$$D(-s_0)= 1 - \frac{FD(-s_0)}{2\sqrt{s_0}}$$

$$D(-s_0) = 2\sqrt{s_0}(F + 2\sqrt{s_0})^{-1}$$

(27) and (28) can be written as

$$N(s) = \frac{2F\sqrt{s_0}}{F + 2\sqrt{s_0}(s_0+ s)} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(28\text{-a})$$

$$D(s) = 1- \frac{2F\sqrt{s_0}}{(F+2\sqrt{s_0})\,(\sqrt{s_0}+\sqrt{-s})} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(28\text{-b})$$

$$fo(s) = \frac{N(s)}{D(s)} = \frac{2F\sqrt{s_0}}{(s + s_0)}\cdot\frac{(\sqrt{s_0}+ \sqrt{-s})}{(F+2\sqrt{s_0})(\sqrt{s_0}+\sqrt{-s}\ )-2F\sqrt{s_0}} \quad \ldots\ldots\ldots(29)$$

The effective range formula is,

$$\sqrt{s}\cot = - \frac{1}{\alpha} + \frac{1}{2}\,.r_{eff}s \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(30)$$

where,

r_{eff} = effective range,

α = scattering length defined by, $\alpha = -\lim_{k\to 0} \frac{\delta_0}{k}$

Becuase of, $f_\ell(s) = (s)^{-1/2}\exp.[i\delta_\ell(s)]\sin\delta_\ell(s)$

or,

$$Ref_\ell(s)= (s)^{-1/2}\cos\delta_\ell(s)\sin\delta_\ell(s)$$

$$Im.f_\ell(s) =(s)^{-1/2}\sin\delta_\ell(s)\sin\delta_\ell(s)$$

Therefore, $\sqrt{s}\cot\delta_0 = \sqrt{s}\,\frac{Re.f_0}{Im.f_0}$

Equation (29) can be written as ,

$$f_0(s) = \frac{2F\sqrt{s_0})}{(s_0+ s)}.\frac{(\sqrt{s_0} + \sqrt{-s})\left[\sqrt{-s}(F + 2\sqrt{s_0}) - \sqrt{s_0}(2\sqrt{s_0}- F)\right]}{\left[(-s\ )(F + 2\sqrt{s_0})^2 -\sqrt{s_0}(2\sqrt{s_0} -F)^2\right]}$$

bound state. For F negative, no bound state can exist. The reason for this is that s_0 must be somewhere "on the cut", i.e. $-s < -\kappa^2/4$ whereas the bound state if any, must be to the right of the cut, that is, $-s > -\kappa^2/4$. But if $F < 0$, the condition $\sqrt{-s} > 0$ (first sheet) cannot be satisfied unless $\sqrt{-s} > \sqrt{s_0}$ implying that $-s < -\kappa^2/4$. Thus irrespective of the magnitude of F and the value of s_0, there cannot be a bound state if $F < 0$. Therefore, $F > 0$ corresponds to an attractive potential and $F < 0$ to a repulsive potential; and $\sqrt{s_0}$ characterises its range.

If a bound state occurs, the binding energy is given by

$$|E| = \left(\frac{F - 2\sqrt{s_0}}{F + 2\sqrt{s_0}}\right) s_0$$

The accuracy of this approximation method can be improved by approximating the negative cut by several poles. This will bring in new parameters.

If instead of one S-bound state and one subtraction, there are several bound states several subtractions are necessary in the full Mandelstam representation, then, considering $(n+1)$ subtractio in the full representation i. e.

$$f(s,t) = f_B(t) + \sum_{b=1}^{N} \frac{R_b(i|E_b|, t)}{s + |E_b|} + (-1)^{n+1} t^{n+1} \frac{1}{\pi^2} \lim_{\eta \to 0} \int_0^\infty \int_{4\kappa^2}^\infty \frac{\rho(s',t')\, dt'\, ds'}{(t'-t)\, t'^{n+1} (s'-s-i\eta)}$$
$$+ \sum_{j=0}^{n} \frac{t^j}{j!} \frac{1}{\pi} \lim_{\eta \to 0} \int_0^\infty \frac{g_j(s')}{s' - s - i\eta} ds'$$

then, $f_\ell(s)/s^n \longrightarrow 0$ as $|s| \longrightarrow \infty$, that is, there will be at most n subtractions needed for each f_ℓ. Furthermore, all partial waves with $\ell \geqslant n+1$, will be determined by quadratures over the weight functio ρ, which is known in principle. However, for $\ell \leqslant n+1$, a nonlinear equation will have to be solved. An N/D method for each ℓ may be us i.e. $f_\ell = N_\ell/D_\ell$ and non-singular Fredholm-type equations for the functions are obtained. The kernel of these integral equations be given in tefms of the weight function ρ, but now subtrcted dispersion rels. may be needed for the N-equations. The subtraction constants remain undetermined.

Itis possible that the Schröedinger equation may be replaced by the Mandelstam representation and unitarity, thus forming a basis for a complete dynamical scheme, both for the scattring problem and the bound state problem. The interaction is charecterised by its range and its strength, and these features determine the location of the cuts and the discontinuity across the cuts. All partial wave amplitudes can be determined from these data. The poles of the amplitude then give the bound states. There is however one great problem: the exact number of necessary subtractions is not known and undetermined subtraction constanta may appear. This problem may partially be answered by considring analytic behaviour of the scattering amplitude as a funtion of complex variable angular momentum, an approach developed by Regge.*

* T. Regge, Nuovo Cimento 14, 951, (1959); 18, 947 (1960)
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All other relevant references are quoted within the text.