# Generalized Fractional Mathematical Modelling and Simulation for Dynamical Systems

**By**

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March 2025

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Submitted in partial fulfillment of the requirements
for the degree of

**Doctor of Philosophy
in Mathematics**

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# Dedication

*Dedicated to my supervisor Prof. Dr. Tayab Kamran, my father Muhammad Razi Jafri, my husband Muhammad Azhar Rasheed, and my little world Husnain, Ayesha, Hassan and Farham. I am eternally grateful for their presence in my life.*

**Prof. Dr. Tayab Kamran**
**(My Supervisor)**

**Muhammad Razi Jafri**
**(My Father)**

**Muhammad Azhar Rasheed**
**(My Husband)**

**Husnain, Ayesha, Hassan and Farham**
**(My World)**

# DECLARATION BY THE SCHOLAR

I certify that:

- The work contained in this thesis is original and has been done by me under the guidance of my supervisor.
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- I have followed the guidelines provided by the Institute in preparing the thesis.
- I have complied with the standards and guidelines given in the Ethics Code of Conduct of the Institute.
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Nausheen Razi

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March, 2025

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This is to certify that the research work presented in this thesis entitled **Generalized Fractional Mathematical Modelling and Simulation for Dynamical Systems** was conducted by **Nausheen Razi** under the kind supervision of **Prof. Dr. Tayyab Kamran**. No part of this thesis has been submitted anywhere else for any other degree. This thesis is submitted to the Department of Mathematics, Quaid-i-Azam University, Islamabad in partial fulfillment of the requirements for the degree of Doctor of Philosophy in the field of Mathematics from the Department of Mathematics, Quaid-i-Azam University Islamabad, Pakistan.

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## CERTIFICATE

A THESIS SUBMITTED IN THE PARTIAL FULFILLMENT OF THE REQUIREMENT FOR A DEGREE OF THE

**DOCTOR OF PHILOSOPHY IN MATHEMATICS**

We accept this thesis as conforming to the required standard. This is to certify that the thesis entitled:**Generalized Fractional Mathematical Modelling and Simulation for Dynamical Systems.**

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Place: Quaid-I-Azam University Islamabad                                        Nausheen Razi

Date: March 17, 2025

# Abstract

Fractional calculus is an extension of classical calculus, that provides a more nuanced and flexible framework for modeling complex phenomena. Its concept is known since 1695 when L'Hospital asked Leibnitz about the fractional power of derivative. By fractional calculus, we may get more accurate results in many physical problems. The notion of fractional operators had not been much worthy for modeling the complex problems of real world. These complex real world problems can be based on those physical occurrences that show fractal behavior. To handle this type of problems, fractal-fractional theory plays a vital role.

Malware is a generic issue and many authors have discussed different mathematical models to explain its extremities. Due to its complex features involving chaotic behavior, heterogeneities and memory effect, some authors tried to solve it using the concept of fractional calculus and in advance form of fractal fractional theory. Till now we have seen the models which have a simple nature. So we decided to investigate a more complex mathematical model. This model has a variable infection rate which gives a deep insight of the behavior of malware. Moreover, infection rate is defined as a nonlinear function of infected nodes. To better understand the behavior of such type of malware and develop antivirus software to overcome the malware, we decided to deal this model by converting it into fractal fractional mathematical models. We also tried to find the impacts of different parameters on malware propagation for integer and non integer orders. We were interested in examining the impact of memory effects in this dynamical system in the sense of fractal fractional (FF) derivatives with three kernels known as Powerlaw, Exponential Decay and Mittag-Leffler. Initially the models were examined theoretically. Conditions for existence (Leray Schauder criteria), uniqueness (Lipschitz property) and stability (Ulam-Hyers and Ulam-Hyers-Rassias theorems) of the fractal fractional models were examined using concepts of fixed point theory. Secondly, numerical schemes were developed using Lagrange interpolation using two point formula and simulations were performed using Matlab codes on R2016a to verify the accuracy of theoretical results. Sensitivity analysis of different parameters such as initial infection rate, variable adjustment to sensitivity of infected nodes, immune rate of antivirus strategies and loss rate of immunity of removed nodes is investigated under FF model and is compared with classical. We have compared four different mathematical models (classical, fractional, fractal, fractal-fractional) so that in different forms of malware antivirus strategies could be developed accordingly. Moreover, constant and variable fractional and fractal orders have been compared by graphs. On investigation, we find that FF model describes the effects of memory on nodes in detail. Antivirus software can be developed considering the effect of FF orders and parameters to reduce persistence and eradication of infection. Small changes cause significant perturbation in infected nodes and malware can be driven into passive mode by understanding its propagation by FF derivatives and may take necessary actions to prevent the disaster caused by cyber attackers.

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**List of Notations**

- $FF$.........................Fractal-Fractional
- $FFP$.........................Fractal-fractional model with Power law kernel
- $FFE$........................Fractal-fractional model with Exponential Decay kernel
- $FFM$........................Fractal-fractional model with Mittag-Leffler kernel
- $Classical\,Model$...........Fractal-fractional model with $\mathfrak{p} = \mathfrak{q} = 1$
- $Fractional\,Model$...........Fractal-fractional model with $\mathfrak{p} = 1$
- $Fractal\,Model$............ Fractal-fractional model with $\mathfrak{q} = 1$
- $Fractional\,Model$.........Fractal-fractional model with $\mathfrak{p} = \mathfrak{q} = 1$

# Chapter 1

# Introduction and Preliminaries

In this chapter, first we give introduction and describe some definitions and theorems, on basis of which we develop our results in this thesis. Then, we explain classical model which we want to observe under fractal fractional theory and convert it into its fractal-fractional form.

## 1.1 Introduction

With the passage of time, everything is going to be changed. Now, in our daily life, we are fully dependent on technology. We store everything from pictures to every document in our computers and mobiles. Everything has its positive and adverse effects. Although the modern technologies have made our lives convenient, on the other hand it has also introduced many problems in our lives. Many crimes usually known as cyber crimes are due to some destructive softwares on internet. In this perspective, a malicious software known as "Malware" is commonly heard.

Malware is a program or a file that is designed intentionally to harm, interrupt or damage a computer network. It is also used for stealing the information about the users without their knowledge. It is used by cyber criminals. They use it to get information about the individual and its activities. They use it to track activities on the Internet, and get sensitive information about accounts as well. Malware software has its own defense system.

It can be hidden from antivirus programs [1]. It is used in the form of a malicious code in general. It exists in different forms. Sometimes it is a combination of two or more types. Most known types of malware are: Virus, Worms, Adware, Spyware, Trojan, Rootkit, Backdoors, Keyloggers, Ransomware, Cookies, Sniffers, Botnet, Spam, Mobile malware [1, 2, 3].

Different types of malware propagate differently either by self-propagation or through user interactions or by internet, Wifi, Bluetooth etc.[3]. Many articles have been written on some commonly used types of malware, their behaviors and about prevention tools. A well-known type of malware is Virus that can be spread through USB, unknown emails or corrupt links. Different types of Viruses are discussed [4] and the financial losses are a lot.

Most harmful type of malware commonly used in the business community is ransomware but now a days another common form of malware is adware. In the present era, marketing is essential. Now a days internet and smart-phones are usually used for marketing. When we use a mobile application or visit a website, we see many advertisements. There is a lot of chance of adware attack using it. Cyber criminals can get information whether you use an app or not [5]. The software has been developed to deal with such programs or malware according to its nature. The software is usually called antivirus. These antivirus softwares work as vaccination for the malware. It is very necessary to detect the malware as it may be in camouflage form. The technique used by malware programmers to make the malware difficult to read and understand is known as Obfuscation [2]. Rabia [2] discussed different techniques of obfuscation and corresponding detective techniques of malware. Antivirus software can detect and kill malware. Moreover, it may be installed in computers to act as shield against propagation of malware [4, 6]. To develop effective antivirus software against some particular malware, one should know how the malware propagates and works in computers. To analyze the malware prevalence and for its prevention, mathematical techniques are very useful. We can easily simulate the data to discuss its propagation and corresponding solution. Many scholars worked on mathematical modeling of malware propagation based on epidemic modeling [7, 8]. Malware propagation has a resemblance with the transmission of infectious disease found in human and other living bodies. It can be considered as Mathematical Epidemiology in which most commonly used model is $SIR$ model as [9] and the references in it. As the malicious programs behave like an infection in humans, it can be treated as epidemiology. Specially the mathematical features of an infectious disease and the role of a computer virus produced in a computer network can be related to each other [10].

We can see that formulating mathematical models is of great importance for accurate prediction of malicious propagation over network [11] but as the malware is spreading epidemically, it should be treated as the other epidemics are considered. As the time passed, due to the increase in the spread of virus and its different aspects we have to study the mathematical models in fractional calculus instead of classical. A lot of work has been done in the classical calculus on it. Then, the researches started to investigate about it in fractional calculus.

Fractional calculus is the generalization of the ordinary derivative and integral concept [12, 13, 14]. The notion of fractional operators had not been much worthy for modeling the complex problems of real world. These complex real world problems can be based on those physical occurrences that show fractal behavior. To handle this type of problems, a nonstandard derivative was introduced [15] known as fractal derivative which scales independent variable. Atangana [15] introduced new concept of differentiation. The term fractal-fractional (FF) derivative was used in the paper where combination of two concepts fractal derivative and fractional derivative was developed. He derived fractal fractional derivatives in Caputo sense and in Riemann-Liouville sense with three different forms of kernels (Powerlaw, Exponential Decay, Mittag-Leffler). He presented the new definition of Fractal Laplace transform and then used it to solve the fractal differential equations and found corresponding integrals. Initially introduced Riemann-Liouville and Caputo fractional derivatives have some difficulties with the kernels. So to overcome these difficulties, Caputo and Fabrizio fractional derivative was introduced in 2015 with the kernel in the form of exponential function, and Atangana and Baleanu replaced it in 2016 with the Mittag-Leffler function [16, 17]. It attracted many researchers in different fields of science, technology, engineering etc. Many articles have been written on mathematical modeling in the form of fractal-fractional [18]. Different models have been constructed on different diseases and their solutions are to be found using concepts of pure mathematics [19, 20, 21, 22, 23, 24, 25] and fractal theory [26, 27] along with numerical simulations.

Malware constitute a chaotic behavior in the real world and it is random and unpredictable in a non-linear rule. Phenomena of malware relates to data heterogeneities that cannot be well-defined using other forms of derivatives [28]. Due to its complex features involving chaotic behavior, heterogeneities and memory effect, some authors tried to solve it using the concept of fractional calculus and in advance form of fractal fractional theory. To see the memory effect, earlier fractional derivatives of variable order [29, 30] were used but recently fractal fractional orders are used to check memory effect [31]. Inspired by above theory, we also tried to work on malware with the help of fractal fractional theory. Till now, we have seen the models which have a simple nature. So, we decided to investigate a more complex mathematical model as presented by Feng et al. [32]. The authors described a model along with a different aspect. This model has a variable infection rate which gives a deep insight of the behavior of malware. Moreover, infection rate is defined as a non-linear function of infected nodes. To better understand the behavior of such type of malware and develop antivirus software to overcome the malware, we decided to deal this model by converting it into fractal fractional mathematical models with the kernels defined above. We also tried to find the impacts of different parameters on malware propagation for integer and non-integer orders. The aim of Feng et al. was to develop a real model that can be used for predicting malware propagation in computer networks. Our goal is to investigate the behavior of this model under the change of fractal orders and fractional orders with three kernels (Powerlaw, Exponential Decay, Mittag-Leffler). We want to seek a solution which can give us better estimate of parameters to prevent the malware propagation. We want to investigate about its physical significance too. This model is different in the sense that it involves a non-linear function to describe undetermined dynamical parameter which varies due to the sensitivity of infection rate.

We organized our thesis as: in chapter 1, some basic definitions and theorems are described that are useful for our work. In chapter 2, fractal fractional mathematical model is formulated with Powerlaw kernel. In chapter 3, this model has been discussed with Exponential Decay kernel. Then in chapter 4, we investigated the behavior of mathematical model with Mittag-Leffler kernel. In chapter 5, we compared three kernels and mathematical models (classical, fractional, fractal and fractal-fractional) by graphs and concluded our findings.

## 1.2 Preliminaries

Now we state some definitions and theorems from classical and fractional calculus which were helpful in our work.

### 1.2.1 Classical Calculus

This section has some definitions and results from fixed point theory which are needed in the sequel. We use the following results [19]:
Let $\Psi$ displays a subclass of non-decreasing operators $\psi\colon [0,\infty) \to [0,\infty)$ such that

$$\sum_{j=1}^{\infty} \psi^j(\varkappa) < \infty$$

for all $\varkappa > 0$, where $\psi^j$ is $j^{th}$ iterate of $\psi$, then following lemma holds:

**Lemma 1.2.1.** [19] Every function $\psi\colon [0,\infty) \to [0,\infty)$ satisfies the following condition: if $\psi \in \Psi$ is non-decreasing, then for each $\varkappa > 0$,
$\lim_{j\to\infty} \psi^j(\varkappa) = 0 \implies \psi(\varkappa) < \varkappa$.

**Definition 1.2.2.** [19] Let $\mathfrak{X}$ be a normed space and $\mathfrak{F}\colon \mathfrak{X} \to \mathfrak{X}$ with $\psi\colon [0,\infty) \to [0,\infty)$ and $\phi\colon \mathfrak{X}^2 \to [0,\infty)$, then $\mathfrak{F}$ is a *$\phi$-$\psi$-contraction* if for $\mathfrak{w}_1, \mathfrak{w}_2 \in \mathfrak{X}$,
$\phi(\mathfrak{w}_1, \mathfrak{w}_2) \cdot d(\mathfrak{F}_{\mathfrak{w}_1}, \mathfrak{F}_{\mathfrak{w}_2}) \leq \psi(d(\mathfrak{w}_1, \mathfrak{w}_2))$.

**Definition 1.2.3.** [19] If $\mathfrak{F}\colon \mathfrak{X} \to \mathfrak{X}$ and $\phi\colon \mathfrak{X}^2 \to [0,\infty)$ then, $\mathfrak{F}$ is *$\phi$-admissible* if for $\mathfrak{w}_1, \mathfrak{w}_2 \in \mathfrak{X}$,
$\phi(\mathfrak{w}_1, \mathfrak{w}_2) \geq 1 \implies \phi(\mathfrak{F}_{\mathfrak{w}_1}, \mathfrak{F}_{\mathfrak{w}_2}) \geq 1$.

With the help of above Definitions 1.2.2 and 1.2.3, the authors derived the following results for existence of fixed point.

**Theorem 1.2.4.** [19] Let $(\mathfrak{X}, d)$ be a complete metric space and $\mathfrak{F}\colon \mathfrak{X} \to \mathfrak{X}$ be a *$\phi$-$\psi$-contraction* mapping with the conditions:

1. $\mathfrak{F}$ is *$\phi$-admissible*;

2. $\exists\ \mathfrak{x}_0 \in \mathfrak{X}$ with the condition $\phi(\mathfrak{x}_0, \mathfrak{F}\mathfrak{x}_0) \geq 1$;

3. if $\{\mathfrak{x}_n\}$ is a sequence in $\mathfrak{X}$ such that $\phi(\mathfrak{x}_n, \mathfrak{x}_{n+1}) \geq 1$ for all $n$ and $\mathfrak{x}_n \to \mathfrak{x} \in \mathfrak{X}$ as $n \to \infty$, implies $\phi(\mathfrak{x}_n, \mathfrak{x}) \geq 1$ for all $n \in \mathbb{N}$.

   Then, $\mathfrak{F}$ has a fixed point.

For existence of solution in the support of Theorem 1.2.4, Leray Schauder criteria is also used which is defined as:

**Theorem 1.2.5.** [21] Let $\mathfrak{X}$ be a Banach space and $\mathfrak{E}$ be a bounded, closed set in $\mathfrak{X}$ such that $\mathfrak{E}$ is convex and $\mathfrak{U}$ be an open set in $\mathfrak{E}$ with the property $0 \in \mathfrak{U}$, then a compact and continuous operator $\mathfrak{G} \colon \bar{\mathfrak{U}} \to \mathfrak{E}$, shows either

(a) $\mathfrak{G}$ has a fixed point in $\bar{\mathfrak{U}}$,

or

(b) $\exists\, \mathfrak{x} \in \partial\mathfrak{U}$ and $\eta \in (0,1)$ s.t. $\mathfrak{x} = \eta\, \mathfrak{G}(\mathfrak{x})$.

Moreover, for compactness of operator, Arzela-Ascoli's theorem is used which is defined as:

**Theorem 1.2.6.** [22] Let $\mathfrak{V} \subset \mathbb{R}^n$, $\mathfrak{W} \subset C(\mathfrak{V}, \mathbb{R}^m)$. Then $\mathfrak{W}$ is compact $\Leftrightarrow$ $\mathfrak{W}$ is closed, bounded and equicontinuous.

### 1.2.2 Fractional Calculus

Fractional calculus as a generalized form of classical calculus plays an important role in dealing with complex dynamical systems. We study it in two parts which are given below.

#### Constant order Fractal Fractional Derivatives

Now-a-days fractal theory along with fractional order derivative is most widely used to understand the behavior of variables. To study the behavior of a given variable with respect to a scaled variable, Chen et al. [26] defined:

**Definition 1.2.7.** Fractal derivative (earlier defined as Hausdorff derivative [27] ) of a function $\mathfrak{f}(\varkappa)$ with respect to a fractal order $\mathfrak{p} \in (0,1)$ is defined as:

$$\frac{d\mathfrak{f}(\varkappa)}{d\varkappa^{\mathfrak{p}}} = \lim_{\varkappa \to \varkappa_1} \frac{\mathfrak{f}(\varkappa) - \mathfrak{f}(\varkappa_1)}{\varkappa^{\mathfrak{p}} - \varkappa_1^{\mathfrak{p}}}.$$

Combining the concepts of fractal differentiation and fractal derivative, Atangana introduced a new concept of differentiation and with the same operator, he constructed the fractal-fractional integral associated to the fractal-fractional derivatives with different forms of kernels [15] as:

**Definition 1.2.8.** Let $\mathfrak{G}(\varkappa)$ be continuous on $(a,b)$ and if $\mathfrak{G}$ is fractal differentiable on this interval having order $\mathfrak{p}$, then the **Fractal Fractional derivative** of $\mathfrak{G}$ of order $\mathfrak{q}$ in Riemann-liouville sense in terms of **power law kernel** is defined as:

$$^{FFP}D_{a,\varkappa}^{\mathfrak{q},\mathfrak{p}}\mathfrak{G}(\varkappa) = \frac{1}{\Gamma(n-\mathfrak{q})}\frac{d}{d\varkappa^{\mathfrak{p}}}\int_a^{\varkappa}(\varkappa-\mathfrak{u})^{n-\mathfrak{q}-1}\,\mathfrak{G}(\mathfrak{u})d\mathfrak{u},$$

where $(n-1<\mathfrak{p},\mathfrak{q}\leq n),\ n\in\mathbf{N}$.

**Definition 1.2.9.** Let $\mathfrak{G}(\varkappa)$ be continuous on $(a,b)$, then **Fractal fractional integral** of $\mathfrak{G}$ with **power law kernel** having order $\mathfrak{q}$ and taking $n=1$ is:

$$^{FFP}I_{a,\varkappa}^{\mathfrak{q},\mathfrak{p}}\mathfrak{G}(\varkappa) = \frac{\mathfrak{p}}{\Gamma(\mathfrak{q})}\int_a^{\varkappa}\mathfrak{u}^{(\mathfrak{p}-1)}\,(\varkappa-\mathfrak{u})^{\mathfrak{q}-1}\,\mathfrak{G}(\mathfrak{u})d\mathfrak{u}.$$

**Definition 1.2.10.** Let $\mathfrak{G}(\varkappa)$ be continuous on $(a,b)$ and if $\mathfrak{G}$ is fractal differentiable on this interval having order $\mathfrak{p}$ , then **Fractal Fractional derivative** of $\mathfrak{G}$ having order $\mathfrak{q}$ in Riemann Liouville sense in terms of **Exponential decay kernel** is defined as:

$$^{FFE}D_{a,\varkappa}^{\mathfrak{q},\mathfrak{p}}\mathfrak{G}(\varkappa) = \frac{\mathfrak{M}(\mathfrak{q})}{\Gamma(1-\mathfrak{q})}\frac{d}{d\varkappa^{\mathfrak{p}}}\int_a^{\varkappa}exp[\frac{-\mathfrak{q}}{1-\mathfrak{q}}\,(\varkappa-\mathfrak{u})]\,\mathfrak{G}(\mathfrak{u})d\mathfrak{u},$$

where $(0<\mathfrak{p},\mathfrak{q}\leq n),\ n\in\mathbf{N}$ and $\mathfrak{M}(0)=\mathfrak{M}(1)=1$.

**Definition 1.2.11.** Let $\mathfrak{G}(\varkappa)$ be continuous on $(a,b)$, then **Fractal fractional integral** of $\mathfrak{G}$ with **exponential decay kernel** having order $\mathfrak{q}$ and taking $n=1$ is:

$$^{FFE}I_{a,\varkappa}^{\mathfrak{q},\mathfrak{p}}\mathfrak{G}(\varkappa) = \frac{\mathfrak{p}\,(1-\mathfrak{q})\,\varkappa^{(\mathfrak{p}-1)}\,\mathfrak{G}(\varkappa)}{\mathfrak{M}(\mathfrak{q})} + \frac{\mathfrak{p}\,\mathfrak{q}}{\mathfrak{M}(\mathfrak{q})}\int_a^{\varkappa}\mathfrak{u}^{\mathfrak{p}-1}\,\mathfrak{G}(\mathfrak{u})d\mathfrak{u}.$$

**Definition 1.2.12.** Let $\mathfrak{G}(\varkappa)$ be continuous on $(a,b)$ and if $\mathfrak{G}$ is fractal differentiable on this interval having order $\mathfrak{p}$, then **Fractal Fractional derivative** of $\mathfrak{G}$ of order $\mathfrak{q}$ in Riemann Liouville sense in terms of **Mittag-Leffler kernel** is defined as:

$$^{FFM}D_{a,\varkappa}^{\mathfrak{q},\mathfrak{p}}\mathfrak{G}(\varkappa) = \frac{\mathfrak{AB}(\mathfrak{q})}{1-\mathfrak{q}}\frac{d}{d\varkappa^{\mathfrak{p}}}\int_a^{\varkappa}E_{\mathfrak{q}}[-\frac{\mathfrak{q}}{1-\mathfrak{q}}\,(\varkappa-\mathfrak{u})^{\mathfrak{q}}]\,\mathfrak{G}(\mathfrak{u})d\mathfrak{u},$$

where $\mathfrak{AB}(\mathfrak{q})=1-\mathfrak{q}+\frac{\mathfrak{q}}{\Gamma(\mathfrak{q})}$ and $(n-1<\mathfrak{p},\mathfrak{q}\leq n),\ n\in\mathbf{N}$.

**Definition 1.2.13.** Let $\mathfrak{G}(\varkappa)$ be continuous on $(a,b)$, then **Fractal fractional integral** of $\mathfrak{G}$ with **Mittag-Leffler kernel** having order $\mathfrak{q}$ is:

$$^{FFM}I_{a,\varkappa}^{\mathfrak{q},\mathfrak{p}}\mathfrak{G}(\varkappa) = \frac{\mathfrak{p}\,(1-\mathfrak{q})\,\varkappa^{\mathfrak{p}-1}\,\mathfrak{G}(\varkappa)}{\mathfrak{AB}(\mathfrak{q})} + \frac{\mathfrak{p}\,\mathfrak{q}}{\mathfrak{AB}(\mathfrak{q})\,\Gamma(\mathfrak{q})}\int_a^{\varkappa} u^{\mathfrak{p}-1}\,(\varkappa-\mathfrak{u})^{\mathfrak{q}-1}\,\mathfrak{G}(\mathfrak{u})d\mathfrak{u}.$$

**Variable Order Fractional Derivative**

Samko and Ross used extension of constant order Riemann Liouville integral to variable order in 1993 [33].

**Definition 1.2.14.** The new form of Riemann Liouville integral with variable order is

$$_aI_{\varkappa}^{\mathfrak{q}(\varkappa)}(\mathfrak{G}(\varkappa)) = \frac{1}{\Gamma(\mathfrak{q}(\varkappa))}\int_a^{\mathfrak{x}}(\mathfrak{x}-\mathfrak{u})^{\mathfrak{q}(\varkappa)-1}\mathfrak{G}(\mathfrak{u})d\mathfrak{u}.$$

**Definition 1.2.15.** Variable order Riemann Liouville fractional derivative for $\mathfrak{q}(\varkappa) \in (0,1)$ is

$$_aD_{\varkappa}^{\mathfrak{q}(\varkappa)}(\mathfrak{G}(\varkappa)) = \frac{1}{\Gamma(1-\mathfrak{q}(\varkappa))}\frac{d}{d\varkappa}\int_a^{\mathfrak{x}}(\mathfrak{x}-\mathfrak{u})^{-\mathfrak{q}(\varkappa)}\mathfrak{G}(\mathfrak{u})d\mathfrak{u}.$$

**Definition 1.2.16.** Let $\mathfrak{G}(\varkappa)$ be continuous on $(a,b)$ and if $\mathfrak{G}$ is fractal differentiable on this interval having constant order $\mathfrak{p}$, then **Fractal Fractional derivative** of $\mathfrak{G}$ of variable order $\mathfrak{q}(\varkappa)$ in Riemann-liouville sense in terms of **power law kernel** is defined as:

$$^{FFP}D_{a,\varkappa}^{\mathfrak{q}(\varkappa),\mathfrak{p}}\mathfrak{G}(\varkappa) = \frac{1}{\Gamma(1-\mathfrak{q}(\varkappa))}\frac{d}{d\varkappa^{\mathfrak{p}}}\int_a^{\varkappa}(\varkappa-\mathfrak{u})^{-\mathfrak{q}(\varkappa)}\,\mathfrak{G}(\mathfrak{u})d\mathfrak{u},$$

where $(0<\mathfrak{p},\mathfrak{q}(\varkappa)\leq 1)$.

**Definition 1.2.17.** Let $\mathfrak{G}(\varkappa)$ be continuous on $(a,b)$, then **Fractal fractional integral** of $\mathfrak{G}$ with **power law kernel** having variable order $\mathfrak{q}(\varkappa)$ and taking $n=1$ is:

$$^{FFP}I_{a,\varkappa}^{\mathfrak{q}(\varkappa),\mathfrak{p}}\mathfrak{G}(\varkappa) = \frac{\mathfrak{p}}{\Gamma(\mathfrak{q}(\varkappa))}\int_a^{\varkappa}\mathfrak{u}^{(\mathfrak{p}-1)}\,(\varkappa-\mathfrak{u})^{\mathfrak{q}(\varkappa)-1}\,\mathfrak{G}(\mathfrak{u})d\mathfrak{u}.$$

**Definition 1.2.18.** Let $\mathfrak{G}(\varkappa)$ be continuous on $(a,b)$ and if $\mathfrak{G}$ is fractal differentiable on this interval having constant order $\mathfrak{p}$, then **Fractal Fractional derivative** of $\mathfrak{G}$ having variable order $\mathfrak{q}(\varkappa)$ in Riemann Liouville sense in terms of **exponential decay kernel** is defined as:

$$^{FFE}D_{a,\varkappa}^{\mathfrak{q}(\varkappa),\mathfrak{p}}\mathfrak{G}(\varkappa) = \frac{\mathfrak{M}(\mathfrak{q}(\varkappa))}{\Gamma(1-\mathfrak{q}(\varkappa))}\frac{d}{d\varkappa^{\mathfrak{p}}}\int_a^{\varkappa} exp[\frac{-\mathfrak{q}(\varkappa)}{1-\mathfrak{q}(\varkappa)}\,(\varkappa-\mathfrak{u})]\,\mathfrak{G}(\mathfrak{u})d\mathfrak{u},$$

where $(0<\mathfrak{p},\mathfrak{q}(\varkappa)\leq 1)$ and $\mathfrak{M}(0)=\mathfrak{M}(1)=1$.

**Definition 1.2.19.** Let $\mathfrak{G}(\varkappa)$ be continuous on $(a,b)$, then **Fractal fractional integral** of $\mathfrak{G}$ with **exponential decay kernel** having variable order $\mathfrak{q}(\varkappa)$ and taking $n=1$ is:

$$ {}^{FFE}I_{a,\varkappa}^{\mathfrak{q}(\varkappa),\mathfrak{p}}\mathfrak{G}(\varkappa) = \frac{\mathfrak{p}\,(1-\mathfrak{q}(\varkappa))\,\varkappa^{(\mathfrak{p}-1)}\,\mathfrak{G}(\varkappa)}{\mathfrak{M}(\mathfrak{q}(\varkappa))} + \frac{\mathfrak{p}\,\mathfrak{q}(\varkappa)}{\mathfrak{M}(\mathfrak{q}(\varkappa))}\int_a^{\varkappa} \mathfrak{u}^{\mathfrak{p}-1}\,\mathfrak{G}(\mathfrak{u})d\mathfrak{u}. $$

**Definition 1.2.20.** Let $\mathfrak{G}(\varkappa)$ be continuous on $(a,b)$ and if $\mathfrak{G}$ is fractal differentiable on this interval having constant order $\mathfrak{p}$, then **Fractal Fractional derivative** of $\mathfrak{G}$ of variable order $\mathfrak{q}(\varkappa)$ in Riemann Liouville sense in terms of **Mittag-Leffler kernel** is defined as:

$$ {}^{FFM}D_{a,\varkappa}^{\mathfrak{q}(\varkappa),\mathfrak{p}}\mathfrak{G}(\varkappa) = \frac{\mathfrak{AB}(\mathfrak{q}(\varkappa))}{1-\mathfrak{q}(\varkappa)}\frac{d}{d\varkappa^{\mathfrak{p}}}\int_a^{\varkappa} E_{\mathfrak{q}}(\varkappa)[-\frac{\mathfrak{q}}{1-\mathfrak{q}(\varkappa)}\,(\varkappa-\mathfrak{u})^{\mathfrak{q}(\varkappa)}]\,\mathfrak{G}(\mathfrak{u})d\mathfrak{u}, $$

where $\mathfrak{AB}(\mathfrak{q}(\varkappa)) = 1-\mathfrak{q}(\varkappa)+\frac{\mathfrak{q}(\varkappa)}{\Gamma(\mathfrak{q}(\varkappa))}$ and $(0<\mathfrak{p},\mathfrak{q}(\varkappa)\leq 1)$.

**Definition 1.2.21.** Let $\mathfrak{G}(\varkappa)$ be continuous on $(a,b)$, then **Fractal fractional integral** of $\mathfrak{G}$ with **Mittag-Leffler kernel** having variable order $\mathfrak{q}(\varkappa)$ is:

$$ {}^{FFM}I_{a,\varkappa}^{\mathfrak{q}(\varkappa),\mathfrak{p}}\mathfrak{G}(\varkappa) = \frac{\mathfrak{p}\,(1-\mathfrak{q}(\varkappa))\,\varkappa^{\mathfrak{p}-1}\,\mathfrak{G}(\varkappa)}{\mathfrak{AB}(\mathfrak{q}(\varkappa))} + \frac{\mathfrak{p}\,\mathfrak{q}(\varkappa)}{\mathfrak{AB}(\mathfrak{q}(\varkappa))\,\Gamma(\mathfrak{q}(\varkappa))}\int_a^{\varkappa} u^{\mathfrak{p}-1}\,(\varkappa-\mathfrak{u})^{\mathfrak{q}(\varkappa)-1}\,\mathfrak{G}(\mathfrak{u})d\mathfrak{u}. $$

**Variable Order Fractal Derivative**

Najat et al. defined Riemann Liouville FF derivative with variable fractal order [12]:

**Definition 1.2.22.** Let $\mathfrak{G}(\varkappa)$ be a differential function. Let $\mathfrak{q}$ be a constant fractional order s.t. $0<\mathfrak{q}\leq 1$ and $0<\mathfrak{p}(\varkappa)<1$ be continuous function, then fractal fractional derivative of $\mathfrak{G}$ having order $\mathfrak{q}$ and fractal variable dimension $\mathfrak{p}(\varkappa)$ in Riemann-Liouville sense in terms of **power law kernel** is defined as:

$$ {}^{FFP}D_{a,\varkappa}^{\mathfrak{q},\mathfrak{p}(\varkappa)}\mathfrak{G}(\varkappa) = \frac{1}{\Gamma(1-\mathfrak{q})}\frac{d}{d\varkappa^{(\mathfrak{p}(\varkappa))}}\int_a^{\varkappa} (\varkappa-\mathfrak{u})^{\mathfrak{q}-1}\,\mathfrak{G}(\mathfrak{u})d\mathfrak{u}, $$

where $(0<\mathfrak{p}(\varkappa),\mathfrak{q}\leq 1)$.

**Definition 1.2.23.** The fractal fractional integral of $\mathfrak{G}$ having order $\mathfrak{q}$ and fractal variable dimension $\mathfrak{p}(\varkappa)$ in Riemann-Liouville sense in terms of **power law kernel** is defined as

$$ {}^{FFP}I_{a,\varkappa}^{\mathfrak{q},\mathfrak{p}(\varkappa)}\mathfrak{G}(\varkappa) = \frac{1}{\Gamma(\mathfrak{q})}\int_a^{\varkappa} (\varkappa-\mathfrak{u})^{\mathfrak{q}-1}\,\mathfrak{G}(\mathfrak{u})\,[\mathfrak{p}'(\mathfrak{u})\ln(\mathfrak{u})+\frac{\mathfrak{p}(\mathfrak{u})}{\mathfrak{u}}]\,\mathfrak{u}^{\mathfrak{p}(\mathfrak{u})}d\mathfrak{u}. $$

**Definition 1.2.24.** Let $\mathfrak{G}(\varkappa)$ be a differential function. Let $\mathfrak{q}$ be a constant fractional order s.t. $0 < \mathfrak{q} \leq 1$ and $0 < \mathfrak{p}(\varkappa) < 1$ be continuous function, then FF derivative of $\mathfrak{G}$ having order $\mathfrak{q}$ and fractal variable dimension $\mathfrak{p}(\varkappa)$ in Riemann Liouville sense in terms of **exponential decay kernel** is defined as:

$$^{FFE}D_{a,\varkappa}^{\mathfrak{q},\mathfrak{p}(\varkappa)}\mathfrak{G}(\varkappa) = \frac{\mathfrak{M}(\mathfrak{q})}{\Gamma(1-\mathfrak{q})}\frac{d}{d\varkappa^{\mathfrak{p}(\varkappa)}}\int_a^{\varkappa} exp[\frac{-\mathfrak{q}}{1-\mathfrak{q}}(\varkappa-\mathfrak{u})]\,\mathfrak{G}(\mathfrak{u})d\mathfrak{u},$$

where $(0 < \mathfrak{p}(\varkappa), \mathfrak{q} \leq 1)$ and $\mathfrak{M}(0) = \mathfrak{M}(1) = 1$.

**Definition 1.2.25.** The FF integral of $\mathfrak{G}$ having order $\mathfrak{q}$ and fractal variable dimension $\mathfrak{p}(\varkappa)$ in Riemann-Liouville sense in terms of **exponential decay kernel** having order $\mathfrak{q}$ and taking $n = 1$ is:

$$^{FFE}I_{a,\varkappa}^{\mathfrak{q},\mathfrak{p}(\varkappa)}\mathfrak{G}(\varkappa) = \frac{\mathfrak{p}\,(1-\mathfrak{q})\,\varkappa^{(\mathfrak{p}-1)}\,\mathfrak{G}(\varkappa)}{\mathfrak{M}(\mathfrak{q})} + \frac{\mathfrak{p}\,\mathfrak{q}}{\mathfrak{M}(\mathfrak{q})}\int_a^{\varkappa}\mathfrak{u}^{\mathfrak{p}-1}\,\mathfrak{G}(\mathfrak{u})\,[\mathfrak{p}'(\mathfrak{u})\ln(\mathfrak{u}) + \frac{\mathfrak{p}(\mathfrak{u})}{\mathfrak{u}}]\,\mathfrak{u}^{\mathfrak{p}(\mathfrak{u})}d\mathfrak{u}.$$

**Definition 1.2.26.** Let $\mathfrak{G}(\varkappa)$ be a differential function. Let $\mathfrak{q}$ be a constant fractional order s.t. $0 < \mathfrak{q} \leq 1$ and $0 < \mathfrak{p}(\varkappa) < 1$ be continuous function, then FF derivative of $\mathfrak{G}$ having order $\mathfrak{q}$ and fractal variable dimension $\mathfrak{p}(\varkappa)$ in Riemann Liouville sense in terms of **Mittag-Leffler kernel** is defined as:

$$^{FFM}D_{a,\varkappa}^{\mathfrak{q},\mathfrak{p}(\varkappa)}\mathfrak{G}(\varkappa) = \frac{\mathfrak{AB}(\mathfrak{q})}{1-\mathfrak{q}}\frac{d}{d\varkappa^{\mathfrak{p}(\varkappa)}}\int_a^{\varkappa} E_{\mathfrak{q}}[-\frac{\mathfrak{q}}{1-\mathfrak{q}}(\varkappa-\mathfrak{u})^{\mathfrak{q}}]\,\mathfrak{G}(\mathfrak{u})d\mathfrak{u},$$

where $\mathfrak{AB}(\mathfrak{q}) = 1 - \mathfrak{q} + \frac{\mathfrak{q}}{\Gamma(\mathfrak{q})}$ and $(0 < \mathfrak{p}(\varkappa), \mathfrak{q} \leq 1)$.

**Definition 1.2.27.** The fractal fractional integral of $\mathfrak{G}$ having order $\mathfrak{q}$ and fractal variable dimension $\mathfrak{p}(\varkappa)$ in Riemann-liouville sense in terms of **Mittag-Leffler kernel** having order $\mathfrak{q}$ is:

$$^{FFM}I_{a,\varkappa}^{\mathfrak{q},\mathfrak{p}(\varkappa)}\mathfrak{G}(\varkappa) = \frac{\mathfrak{p}\,(1-\mathfrak{q})\,\varkappa^{\mathfrak{p}-1}\,\mathfrak{G}(\varkappa)}{\mathfrak{AB}(\mathfrak{q})} + \frac{\mathfrak{p}\,\mathfrak{q}}{\mathfrak{AB}(\mathfrak{q})\,\Gamma(\mathfrak{q})}\int_a^{\varkappa}\mathfrak{u}^{\mathfrak{p}-1}\,(\varkappa-\mathfrak{u})^{\mathfrak{q}-1}\,\mathfrak{G}(\mathfrak{u})\,[\mathfrak{p}'(\mathfrak{u})\ln(\mathfrak{u}) + \frac{\mathfrak{p}(\mathfrak{u})}{\mathfrak{u}}]\,\mathfrak{u}^{\mathfrak{p}(\mathfrak{u})}d\mathfrak{u}.$$

## 1.3 Description of classical mathematical model

To develop an effective antivirus against some particular malware, one should know how the malware propagates and works in computers. For this purpose, Feng et al. [32] presented a model of propagation of malware through internet via three states: susceptible, infected and removed. They assumed that the total no. of nodes in the network at time $\varkappa$

is $N(\varkappa)$ along with this assumption that each node changes with respect to time in three states. They described that susceptible state($\Delta$) represents a node which has a weakness that the malware can easily exploit it, infected state($\aleph$) shows that when it is infected, it can infect its neighboring nodes and still infectious and removed state($\Theta$) represents that a detection tool has been installed which helps in identifying and removing a malware. Considering these definitions of states, they represented the model in the form of ODES as:

$$\begin{aligned}\frac{d\Delta}{d\varkappa} &= \Pi\theta - \beta(\varkappa)\aleph(\varkappa)\Delta(\varkappa) - (\mu+\nu)\Delta(\varkappa) + \zeta\Theta(\varkappa-\tau),\\ \frac{d\aleph}{d\varkappa} &= \beta(\varkappa)\aleph(\varkappa)\Delta(\varkappa) - (\mu+\kappa)\aleph(\varkappa),\\ \frac{d\Theta}{d\varkappa} &= (1-\Pi)\theta + \nu\Delta(\varkappa) + \kappa\aleph(\varkappa) - \zeta\Theta(\varkappa-\tau) - \mu\Theta(\varkappa);\end{aligned} \tag{1.3.1}$$

where $\Pi$ shows the susceptible rate of new nodes, $\theta$ shows the number of new nodes, $\zeta$ is the loss rate of immunity of the recovered nodes, $\mu$ is the replacement rate, $\nu$ is the real time immune rate of antivirus strategies, $\kappa$ is the recovered rate of infected nodes, $\tau$ is the change in time and $\beta(\varkappa)$ is the infection rate at time $\varkappa$. As $\beta(\varkappa)$ is infection rate and it depends on many factors discussed in the paper, so the authors defined $\beta(\varkappa) = \beta_0\mathfrak{f}_1(\aleph(\varkappa))$ where $\mathfrak{f}_1$ is a nonlinear function of $\aleph$ and $\beta_0$ is the initial infection rate. Again assuming $\mathfrak{f}(\aleph(\varkappa)) = \mathfrak{f}_1(\aleph(\varkappa))\aleph(\varkappa)$ which is undetermined dynamical operator. To determine this, the authors defined this as: $\mathfrak{f}(\aleph(\varkappa)) = \frac{\aleph(\varkappa)}{1+\alpha\aleph(\varkappa)}$ where $\alpha$ is used to adjust the sensitivity of the infection rate to the number of infected nodes $\aleph(\varkappa)$. We have a system as:

$$\begin{aligned}\frac{d\Delta}{d\varkappa} &= \Pi\theta - \beta_0\mathfrak{f}(\aleph(\varkappa))\Delta(\varkappa) - (\mu+\nu)\Delta(\varkappa) + \zeta\Theta(\varkappa-\tau),\\ \frac{d\aleph}{d\varkappa} &= \beta_0\mathfrak{f}(\aleph(\varkappa))\Delta(\varkappa) - (\mu+\kappa)\aleph(\varkappa),\\ \frac{d\Theta}{d\varkappa} &= (1-\Pi)\theta + \nu\Delta(\varkappa) + \kappa\aleph(\varkappa) - \zeta\Theta(\varkappa-\tau) - \mu\Theta(\varkappa).\end{aligned} \tag{1.3.2}$$

The initial conditions are defined as: $\Delta(0) = \Delta_0 \geq 0, \aleph(0) = \aleph_0 \geq 0$ and $\Theta(0) = \Theta_0 \geq 0$. Moreover, $N(\varkappa) = \Delta(\varkappa) + \aleph(\varkappa) + \Theta(\varkappa)$ as given above.

Threshold of system is defined as:

$$\mathfrak{R}_o = \frac{\beta_0\theta(\Pi\mu+\zeta)\mathfrak{f}'(0)}{\mu(\mu+\kappa)(\mu+\zeta+\nu)}.$$

The disease free equilibrium $(DFE)$ point for the deterministic system is:

$$\mathbf{E} = (\frac{(\Pi\mu+\zeta)\theta}{\mu(\mu+\zeta+\nu)}, 0, \frac{((1-\Pi)\mu+\nu)\theta}{\mu(\mu+\zeta+\nu)}).$$

## 1.4 Fractal-Fractional Mathematical Model

Influenced by concept of fractal fractional calculus, we convert the model (1.3.2) in terms of fractal fractional derivatives as:

$$\begin{aligned}
{}^{FF}D^{\mathfrak{q},\mathfrak{p}}_{0,\varkappa}\Delta(\varkappa) &= \Pi\theta - \beta_0\mathfrak{f}(\aleph(\varkappa))\Delta(\varkappa) - (\mu+\nu)\Delta(\varkappa) + \zeta\Theta(\varkappa-\tau),\\
{}^{FF}D^{\mathfrak{q},\mathfrak{p}}_{0,\varkappa}\aleph(\varkappa) &= \beta_0\mathfrak{f}(\aleph(\varkappa))\Delta(\varkappa) - (\mu+\kappa)\aleph(\varkappa),\\
{}^{FF}D^{\mathfrak{q},\mathfrak{p}}_{0,\varkappa}\Theta(\varkappa) &= (1-\Pi)\theta + \nu\Delta(\varkappa) + \kappa\aleph(\varkappa) - \zeta\Theta(\varkappa-\tau) - \mu\Theta(\varkappa),
\end{aligned} \tag{1.4.1}$$

with $\Delta(0) = \Delta_0 \geq 0, \aleph(0) = \aleph_0 \geq 0, \Theta(0) = \Theta_0 \geq 0$ and $N(\varkappa) = \Delta(\varkappa) + \aleph(\varkappa) + \Theta(\varkappa)$, for $\varkappa \in \mathrm{J} = [0, \mathrm{T}],\ \mathrm{T} > 0$. Also $p$, $q \in (0, 1]$ and all parameters are to be taken non-negative.

# Chapter 2

# Fractal Fractional Mathematical Model with Power Law Kernel

In this chapter, first we convert FF model with powerlaw kernel and then in fixed point problem. We apply results from fixed point theory to establish existence, uniqueness and convergence of solution of our proposed model. Moreover, we check the stability of our model. Furthermore, we generated Matlab code for our fractal fractional model to simulate the results. In the last, we analyze the results and conclude them.

## 2.1 Conversion of Classical Mathematical Model to Fractal-Fractional Mathematical Model with Powerlaw kernel

First, we discuss fractal-fractional mathematical model in terms of power law kernel. So, our required model is

$$\begin{aligned}
{}^{FFP}D_{0,\varkappa}^{\mathfrak{q},\mathfrak{p}}\Delta(\varkappa) &= \Pi\theta - \beta_0 f(\aleph(\varkappa))\Delta(\varkappa) - (\mu+\nu)\Delta(\varkappa) + \zeta\Theta(\varkappa-\tau), \\
{}^{FFP}D_{0,\varkappa}^{\mathfrak{q},\mathfrak{p}}\aleph(\varkappa) &= \beta_0 f(\aleph(\varkappa))\Delta(\varkappa) - (\mu+\kappa)\aleph(\varkappa), \\
{}^{FFP}D_{0,\varkappa}^{\mathfrak{q},\mathfrak{p}}\Theta(\varkappa) &= (1-\Pi)\theta + \nu\Delta(\varkappa) + \kappa\aleph(\varkappa) - \zeta\Theta(\varkappa-\tau) - \mu\Theta(\varkappa),
\end{aligned} \tag{2.1.1}$$

with the initial conditions $\Delta(0) = \Delta_0 \geq 0, \aleph(0) = \aleph_0 \geq 0$, $\Theta(0) = \Theta_0 \geq 0$, and $N(\varkappa) = \Delta(\varkappa) + \aleph(\varkappa) + \Theta(\varkappa)$, for $\varkappa \in \mathrm{J} = [0, \mathrm{T}], \mathrm{T} > 0$. Also $\mathfrak{p}$, $\mathfrak{q} \in (0, 1]$ and all parameters are to be taken non-negative.

## 2.2 Formulation of Model as Fixed Point Problem

In this section, we convert FF model (2.1.1) in fixed point problem. We apply results of fixed point theory on model (2.1.1). Consider $\Xi = \mathfrak{Y}^3$, a Banach space and $\mathfrak{Y} = C(\mathrm{J}, \mathbb{R})$ represents the class of all continuous functions with the norm defined by

$$|||F||_{\Xi} = ||(\Delta, \aleph, \Theta)||_{\Xi} = max\{|\Delta(\varkappa)| + |\aleph(\varkappa)| + |\Theta(\varkappa)| : \varkappa \in \mathrm{J}\}.$$

First, rewrite given model (2.1.1) as:

$$\begin{aligned}
&\Upsilon_1(\varkappa, \Delta(\varkappa), \aleph(\varkappa), \Theta(\varkappa)) = \Pi\theta - \beta_0 f(\aleph(\varkappa))\Delta(\varkappa) - (\mu + \nu)\Delta(\varkappa) + \zeta\Theta(\varkappa - \tau), \\
&\Upsilon_2(\varkappa, \Delta(\varkappa), \aleph(\varkappa), \Theta(\varkappa)) = \beta_0 f(\aleph(\varkappa))\Delta(\varkappa) - (\mu + \kappa)\aleph(\varkappa), \qquad (2.2.1)\\
&\Upsilon_3(\varkappa, \Delta(\varkappa), \aleph(\varkappa), \Theta(\varkappa)) = (1 - \Pi)\theta + \nu\Delta(\varkappa) + \kappa I(\varkappa) - \zeta\Theta(\varkappa - \tau) - \mu\Theta(\varkappa).
\end{aligned}$$

Comparing models (2.1.1) and (2.2.1), we have

$$\begin{aligned}
{}^{FFP}D^{\mathfrak{q},\mathfrak{p}}_{0,\varkappa}\Delta(\varkappa) &= \Upsilon_1(\varkappa, \Delta(\varkappa), \aleph(\varkappa), \Theta(\varkappa)), \\
{}^{FFP}D^{\mathfrak{q},\mathfrak{p}}_{0,\varkappa}\aleph(\varkappa) &= \Upsilon_2(\varkappa, \Delta(\varkappa), \aleph(\varkappa), \Theta(\varkappa)), \qquad (2.2.2)\\
{}^{FFP}D^{\mathfrak{q},\mathfrak{p}}_{0,\varkappa}\Theta(\varkappa) &= \Upsilon_3(\varkappa, \Delta(\varkappa), \aleph(\varkappa), \Theta(\varkappa)).
\end{aligned}$$

Since

$$\begin{aligned}
{}^{FFP}D^{\mathfrak{q},\mathfrak{p}}_{0,\varkappa}g(\varkappa) &= \frac{1}{\Gamma(1-\mathfrak{q})}\frac{d}{d\varkappa^{\mathfrak{p}}}\int_0^{\varkappa}(\varkappa - \mathfrak{u})^{-\mathfrak{q}}g(\mathfrak{u})d\mathfrak{u}. \\
{}^{FFP}D^{\mathfrak{q},\mathfrak{p}}_{0,\varkappa}g(\varkappa) &= \frac{1}{\Gamma(1-\mathfrak{q})}\frac{1}{\mathfrak{p}\varkappa^{\mathfrak{p}-1}}\frac{d}{d\varkappa}\int_0^{\varkappa}(\varkappa - \mathfrak{u})^{-\mathfrak{q}}g(\mathfrak{u})d\mathfrak{u}. \\
{}^{FFP}D^{\mathfrak{q},\mathfrak{p}}_{0,\varkappa}g(\varkappa) &= (\frac{1}{\mathfrak{p}\varkappa^{\mathfrak{p}-1}})\frac{1}{\Gamma(1-\mathfrak{q})}\frac{d}{d\varkappa}\int_0^{\varkappa}(\varkappa - \mathfrak{u})^{-\mathfrak{q}}g(\mathfrak{u})d\mathfrak{u}. \\
{}^{FFP}D^{\mathfrak{q},\mathfrak{p}}_{0,\varkappa}g(\varkappa) &= (\frac{1}{\mathfrak{p}\varkappa^{\mathfrak{p}-1}})\,{}^{RL}D^{\mathfrak{q}}_{0,\varkappa}g(\varkappa),
\end{aligned}$$

where from ([34]) for $n = 1$, we have

$$\frac{1}{\Gamma(1-\mathfrak{q})}\frac{d}{d\varkappa}\int_0^{\varkappa}(\varkappa - \mathfrak{u})^{-\mathfrak{q}}g(\mathfrak{u})d\mathfrak{u} = {}^{RL}D^{\mathfrak{q}}_{0,\varkappa}g(\varkappa).$$

So, model (2.2.2) can be written as

$$
\begin{aligned}
(\frac{1}{\mathfrak{p}\varkappa^{\mathfrak{p}-1}})^{RL}D^{\mathfrak{q}}_{0,\varkappa}\Delta(\varkappa) &= \Upsilon_1(\varkappa,\Delta(\varkappa),\aleph(\varkappa),\Theta(\varkappa)),\\
(\frac{1}{\mathfrak{p}\varkappa^{\mathfrak{p}-1}})^{RL}D^{\mathfrak{q}}_{0,\varkappa}\aleph(\varkappa) &= \Upsilon_2(\varkappa,\Delta(\varkappa),\aleph(\varkappa),\Theta(\varkappa)),\\
(\frac{1}{\mathfrak{p}\varkappa^{\mathfrak{p}-1}})^{RL}D^{\mathfrak{q}}_{0,\varkappa}\Theta(\varkappa) &= \Upsilon_3(\varkappa,\Delta(\varkappa),\aleph(\varkappa),\Theta(\varkappa)).
\end{aligned}
\tag{2.2.3}
$$

Hence, we get

$$
\begin{aligned}
{}^{RL}D^{\mathfrak{q}}_{0,\varkappa}\Delta(\varkappa) &= \mathfrak{p}\varkappa^{\mathfrak{p}-1}\Upsilon_1(\varkappa,\Delta(\varkappa),\aleph(\varkappa),\Theta(\varkappa)),\\
{}^{RL}D^{\mathfrak{q}}_{0,\varkappa}\aleph(\varkappa) &= \mathfrak{p}\varkappa^{\mathfrak{p}-1}\Upsilon_2(\varkappa,\Delta(\varkappa),\aleph(\varkappa),\Theta(\varkappa)),\\
{}^{RL}D^{\mathfrak{q}}_{0,\varkappa}\Theta(\varkappa) &= \mathfrak{p}\varkappa^{\mathfrak{p}-1}\Upsilon_3(\varkappa,\Delta(\varkappa),\aleph(\varkappa),\Theta(\varkappa)).
\end{aligned}
\tag{2.2.4}
$$

In general, we can write model (2.2.4) as

$$
\begin{aligned}
{}^{RL}D^{\mathfrak{q}}_{0,\varkappa}\mathcal{F}(\varkappa) &= \mathfrak{p}\varkappa^{\mathfrak{p}-1}\Upsilon(\varkappa,\mathcal{F}(\varkappa)),\\
\mathcal{F}(0) &= \mathcal{F}_0,
\end{aligned}
\tag{2.2.5}
$$

where

$$
\begin{aligned}
(\mathfrak{p},\mathfrak{q}) &\in (0,1],\\
\varkappa &\in \mathrm{J},\\
\mathcal{F}(\varkappa) &= (\Delta(\varkappa),\aleph(\varkappa),\Theta(\varkappa))^{\top},\\
\mathcal{F}_0 &= (\Delta_0,\aleph_0,\Theta_0)^{\top}.
\end{aligned}
$$

Applying fractal-fractional integral on model (2.2.5), using the result in ([15]), we get

$$
\mathcal{F}(\varkappa)-\mathcal{F}(0) = \frac{\mathfrak{p}}{\Gamma(\mathfrak{q})}\int_0^{\varkappa}\mathfrak{u}^{(\mathfrak{p}-1)}\,(\varkappa-\mathfrak{u})^{\mathfrak{q}-1}\,\Upsilon(\mathfrak{u},\mathcal{F}(\mathfrak{u}))d\mathfrak{u}. \tag{2.2.6}
$$

Hence, we can write

$$
\begin{aligned}
\Delta(\varkappa) &= \Delta(0)+\frac{\mathfrak{p}}{\Gamma(\mathfrak{q})}\int_0^{\varkappa}\mathfrak{u}^{(\mathfrak{p}-1)}\,(\varkappa-\mathfrak{u})^{\mathfrak{q}-1}\,\Upsilon_1(\mathfrak{u},\Delta(\mathfrak{u}),\aleph(\mathfrak{u}),\Theta(\mathfrak{u}))d\mathfrak{u},\\
\aleph(\varkappa) &= \aleph(0)+\frac{\mathfrak{p}}{\Gamma(\mathfrak{q})}\int_0^{\varkappa}\mathfrak{u}^{(\mathfrak{p}-1)}\,(\varkappa-\mathfrak{u})^{\mathfrak{q}-1}\,\Upsilon_2(\mathfrak{u},\Delta(\mathfrak{u}),\aleph(\mathfrak{u}),\Theta(\mathfrak{u}))d\mathfrak{u},\\
\Theta(\varkappa) &= \Theta(0)+\frac{\mathfrak{p}}{\Gamma(\mathfrak{q})}\int_0^{\varkappa}\mathfrak{u}^{(\mathfrak{p}-1)}\,(\varkappa-\mathfrak{u})^{\mathfrak{q}-1}\,\Upsilon_3(\mathfrak{u},\Delta(\mathfrak{u}),\aleph(\mathfrak{u}),\Theta(\mathfrak{u}))d\mathfrak{u}.
\end{aligned}
\tag{2.2.7}
$$

So, now we can transform (2.1.1) into a fixed point problem.

Define an operator $F\colon \Xi \to \Xi$ by

$$
F(\mathcal{F}(\varkappa)) = \mathcal{F}(0)+\frac{\mathfrak{p}}{\Gamma(\mathfrak{q})}\int_0^{\varkappa}\mathfrak{u}^{(\mathfrak{p}-1)}\,(\varkappa-\mathfrak{u})^{\mathfrak{q}-1}\,\Upsilon(\mathfrak{u},\mathcal{F}(\mathfrak{u}))d\mathfrak{u}. \tag{2.2.8}
$$

## 2.3 Existence of Solution

For existence, we prove a theorem on the basis of Theorem 1.2.4 as in ([35]).

**Theorem 2.3.1.** Suppose that $\exists\ V\colon \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}$ , $\psi \in \Psi$ and $\Upsilon \in C(\mathrm{J} \times \Xi, \Xi)$ satisfying the following conditions:

$(\beta_1):$ $\forall \mathcal{F}_1, \mathcal{F}_2 \in \Xi$ and $\varkappa \in \mathrm{J}$,

$$|\Upsilon(\varkappa, \mathcal{F}_1(\varkappa)) - \Upsilon(\varkappa, \mathcal{F}_2(\varkappa))| \leq \ell\, \psi(|\mathcal{F}_1(\varkappa) - \mathcal{F}_2(\varkappa)|),$$

with $V(\mathcal{F}_1(\varkappa), \mathcal{F}_2(\varkappa)) \geq 0$ and $\ell = \frac{\Gamma(p+q)}{\mathfrak{p}\ T^{(\mathfrak{p}+\mathfrak{q}-1)}\,\Gamma(\mathfrak{p})}$;

$(\beta_2)$: $\exists\ \mathcal{F}_0 \in \Xi$ such that $\forall\, \varkappa \in \mathrm{J}$,

$$V(\mathcal{F}_0(\varkappa), F(\mathcal{F}_0(\varkappa))) \geq 0,$$

and

$$V(\mathcal{F}_1(\varkappa), \mathcal{F}_2(\varkappa)) \geq 0$$

gives

$$V(F(\mathcal{F}_1(\varkappa)), F(\mathcal{F}_2(\varkappa))) \geq 0\ ;$$

$(\beta_3)$: $\forall \{\mathcal{F}_n\}_{n\geq 1} \subseteq \Xi$ with $\mathcal{F}_n \to \mathcal{F}$,

$$V(\mathcal{F}_n(\varkappa), \mathcal{F}_{n+1}(\varkappa)) \geq 0 \implies V(\mathcal{F}_n(\varkappa), \mathcal{F}(\varkappa)) \geq 0, \tag{2.3.1}$$

for every $n$ and $\varkappa$.

Hence, we say that $F$ has a fixed point. So a solution of malware propagation model exists.

*Proof.* Take $\mathcal{F}_1, \mathcal{F}_2 \in \Xi$ so that

$$V(\mathcal{F}_1(\varkappa), \mathcal{F}_2(\varkappa)) \geq 0,$$

for every $\varkappa \in \mathrm{J}$.

Now, we take

$$\begin{aligned}
|F(\mathcal{F}_1(\varkappa)) - F(\mathcal{F}_2(\varkappa))| &= \Big|\frac{\mathfrak{p}}{\Gamma(\mathfrak{q})} \int_0^{\varkappa} \mathfrak{u}^{(\mathfrak{p}-1)}\, (\varkappa - \mathfrak{u})^{\mathfrak{q}-1} \left(\Upsilon(\mathfrak{u}, \mathcal{F}_1(\mathfrak{u})) - \Upsilon(\mathfrak{u}, \mathcal{F}_2(\mathfrak{u}))\right) d\mathfrak{u}\Big|, \\
&\leq \frac{\mathfrak{p}}{\Gamma(\mathfrak{q})} \int_0^{\varkappa} \mathfrak{u}^{(\mathfrak{p}-1)}\, (\varkappa - \mathfrak{u})^{\mathfrak{q}-1} \left|\Upsilon(\mathfrak{u}, \mathcal{F}_1(\mathfrak{u})) - \Upsilon(\mathfrak{u}, \mathcal{F}_2(\mathfrak{u}))\right| d\mathfrak{u}.
\end{aligned}$$

By using $(\beta_1)$, we deduce

$$|F(\digamma_1(\varkappa)) - F(\digamma_2(\varkappa))| \;\leq\; \frac{\mathfrak{p}}{\Gamma(\mathfrak{q})} \int_0^{\varkappa} \mathfrak{u}^{(\mathfrak{p}-1)}\,(\varkappa - \mathfrak{u})^{\mathfrak{q}-1}\,\ell\,\psi(|\digamma_1(\mathfrak{u}) - \digamma_2(\mathfrak{u})|)\,d\mathfrak{u}.$$

Now, by using the definition of norm

$$|F(\digamma_1(\varkappa)) - F(\digamma_2(\varkappa))| \;\leq\; \frac{\mathfrak{p}}{\Gamma(\mathfrak{q})} \int_0^{\varkappa} \mathfrak{u}^{(\mathfrak{p}-1)}\,(\varkappa - \mathfrak{u})^{\mathfrak{q}-1}\,\ell\,\psi(||\digamma_1 - \digamma_2||_{\Xi})\,d\mathfrak{u}.$$

After doing some computations and using the definition of beta function and using definition of $\ell$, we get

$$||F(\digamma_1(\varkappa)) - F(\digamma_2(\varkappa))||_{\Xi} \leq \psi(||\digamma_1 - \digamma_2||_{\Xi}). \tag{2.3.2}$$

Moreover, if we define a function $\phi\colon \Xi^2 \to [0,\infty)$ such that $\phi(\digamma_1, \digamma_2) = 1$ for $V(\digamma_1(\varkappa), \digamma_2(\varkappa)) \geq 0$, and zero otherwise, then for each $\digamma_1, \digamma_2 \in \Xi$ equation (2.3.2) can be written as:

$$\phi(\digamma_1, \digamma_2)\, d(F(\digamma_1), F(\digamma_2)) \leq \psi(d(\digamma_1, \digamma_2)).$$

This shows that $F$ is a *$\phi$-$\psi$-contraction*.

Now, suppose that $\digamma_1, \digamma_2 \in \Xi$ with the property that $\phi(\digamma_1, \digamma_2) \geq 1$. By the definition of $\phi$, we deduce

$$V(\digamma_1(\varkappa), \digamma_2(\varkappa)) \geq 0,$$

and by $(\beta_2)$, we get

$V(\digamma_0(\varkappa), F(\digamma_0(\varkappa))) \geq 0$ and $V(\digamma_1(\varkappa), \digamma_2(\varkappa)) \geq 0$.

$\Longrightarrow V(F(\digamma_1(\varkappa)), F(\digamma_2(\varkappa))) \geq 0$.

So, by applying definition of $\phi$, we have

$$\phi(F(\digamma_1), F(\digamma_2)) \geq 1.$$

Hence, $F$ is *$\phi$-admissible*. (*)

Moreover, by $(\beta_2)$, it can be seen that for some $\digamma_0$ in $\Xi$, $\forall \varkappa \in \mathrm{J}$, we have

$V(\digamma_0(\varkappa), F(\digamma_0(\varkappa))) \geq 0 \implies \phi(\digamma_0, F(\digamma_0)) \geq 1$. (**)

Now, consider $\{\digamma_n\}_{n\geq 1} \subseteq \Xi$ with $\digamma_n \to \digamma$ and for all $n$ and $\phi(\digamma_n, \digamma_{n+1}) \geq 1$.

By definition of $\phi$ this implies $V(\digamma_n(\varkappa), \digamma_{n+1}(\varkappa)) \geq 0$.

Thus, by $(\beta_3)$ this implies $V(\digamma_n(\varkappa), \digamma(\varkappa)) \geq 0$.

Hence, $\phi(\digamma_n, \digamma) \geq 1$ for all $n$. (***)
So (*), (**), (***) show the conditions of Theorem 1.2.4 are satisfied, so we can say that there exists some $\digamma^* \in \Xi$ such that $F(\digamma^*) = \digamma^*$.
Hence, $\digamma^* = (\Delta^*, \aleph^*, \Theta^*)^\top$ is a solution of our model. $\square$

Theorem 1.2.5 also establishes that solution of model exists and on basis of this model we also define the following theorem as:

**Theorem 2.3.2.** Let $\Xi$ be a Banach space, $\mathfrak{N}_\epsilon$ be a bounded and closed set in $\Xi$ and $A$ be an open in $\mathfrak{N}_\epsilon$ with $0 \in A$, then there exists a compact and continuous operator $F$ with the conditions $(\beta_4)$ and $(\beta_5)$ from $\overline{A} \to \mathfrak{N}_\epsilon$ which satisfies one of the two conditions,
(a) $F$ has a fixed point in $\overline{A}$,
or
(b) there exists $\digamma \in \partial A$ and $\omega \in (0,1)$ s.t $\digamma = \omega\, F(\digamma)$;
where
$(\beta_4)$: Suppose $\Upsilon \in C(\mathrm{J}\times\Xi, \Xi)$ and there exists $\phi \in L^1(\mathrm{J}, [0,\infty))$ and $B \in C([0,\infty),[0,\infty))$ where $B$ is an increasing function satisfying the condition $|\mathfrak{F}(\varkappa, \digamma(\varkappa))| \leq \phi(\varkappa)\, B(|\digamma(\varkappa)|)$ for all $\varkappa \in \mathrm{J}$ and $\digamma \in \Xi$;
$(\beta_5)$: If $\phi^* = sup_{\varkappa\in \mathrm{J}}|\phi(\varkappa)|$ then $\exists$ a number $r$ s.t $\frac{r}{\digamma_0 + \lambda\, \phi^*\, B(r)} > 1$ where $\lambda = \frac{\mathfrak{p}\, \mathfrak{T}^{\mathfrak{p}+\mathfrak{q}-1}\, \Gamma(\mathfrak{p})}{\Gamma(\mathfrak{p}+\mathfrak{q})}$.

*Proof.* Consider $F\colon \Xi \to \Xi$ as

$$F(\digamma(\varkappa)) = \digamma(0) + \frac{\mathfrak{p}}{\Gamma(\mathfrak{q})}\int_0^{\varkappa} \mathfrak{u}^{(\mathfrak{p}-1)}\,(\varkappa-\mathfrak{u})^{\mathfrak{q}-1}\,\Upsilon(\mathfrak{u}, (\digamma\mathfrak{u}))d\mathfrak{u},$$

and $\mathfrak{N}_\epsilon = \{\digamma \in \Xi : ||\digamma||_\Xi \leq \epsilon\}$ for some positive $\epsilon$.
We show that $F$ is compact on $\mathfrak{N}_\epsilon$. For this, we prove that $F$ is uniformly bounded and equicontinuous.
Since $\Upsilon$ is continuous, this implies $F$ is continuous.
Now for $\digamma$ in $\mathfrak{N}_\epsilon$, we obtain

$$|F(\digamma(\varkappa))| \leq |\digamma(0)| + \frac{\mathfrak{p}}{\Gamma(\mathfrak{q})}\int_0^{\varkappa} \mathfrak{u}^{(\mathfrak{p}-1)}\,(\varkappa-\mathfrak{u})^{\mathfrak{q}-1}\,|\Upsilon(\mathfrak{u}, \digamma(\mathfrak{u}))|d\mathfrak{u}$$

and from $(\beta_4)$, we have

$$\begin{aligned}|F(\digamma(\varkappa))| &\leq \digamma_0 + \frac{\mathfrak{p}}{\Gamma(\mathfrak{q})}\int_0^{\varkappa} \mathfrak{u}^{(\mathfrak{p}-1)}\,(\varkappa-\mathfrak{u})^{\mathfrak{q}-1}\,\phi(\mathfrak{u})\,B(|\digamma(\mathfrak{u})|)d\mathfrak{u}\\ &\leq \digamma_0 + \frac{\mathfrak{p}}{\Gamma(\mathfrak{q})}\int_0^{\varkappa} \mathfrak{u}^{(\mathfrak{p}-1)}\,(\varkappa-\mathfrak{u})^{\mathfrak{q}-1}\,\phi^*\,B(||\digamma||_{\Xi})d\mathfrak{u}\\ &\leq \digamma_0 + \frac{\mathfrak{p}}{\Gamma(\mathfrak{q})}\,\phi^*\,B(||\digamma||_{\Xi})\int_0^{\varkappa} \mathfrak{u}^{(\mathfrak{p}-1)}\,(\varkappa-\mathfrak{u})^{\mathfrak{q}-1}\,d\mathfrak{u},\end{aligned}$$

after simplification of the integral, we get the beta function. So applying value of beta function and $\lambda$, we get

$$|F(\digamma(\varkappa))| \leq \digamma_0 + \lambda\,\phi^*\,B(\epsilon).$$

Hence, by applying norm, we have

$$||F(\digamma(\varkappa))|| \leq \digamma_0 + \lambda\,\phi^*\,B(\epsilon) < \infty. \tag{2.3.3}$$

This implies $F$ is uniformly bounded.

Now, take $\varkappa, \varkappa^* \in$ J such that $\varkappa < \varkappa^*$ and $\digamma \in \mathfrak{N}_\epsilon$ arbitrarily.

If we suppose $\Upsilon^* = sup\,|\Upsilon(\varkappa, \digamma(\varkappa))|$, then

$$\begin{aligned}|F(\digamma(\varkappa^*)) - F(\digamma(\varkappa))| &= |\frac{\mathfrak{p}}{\Gamma(\mathfrak{q})}\int_0^{\varkappa^*} \mathfrak{u}^{(\mathfrak{p}-1)}\,(\varkappa^*-\mathfrak{u})^{\mathfrak{q}-1}\,\Upsilon(\mathfrak{u}, \digamma(\mathfrak{u}))\,d\mathfrak{u}\\ &- \frac{\mathfrak{p}}{\Gamma(\mathfrak{q})}\int_0^{\varkappa} \mathfrak{u}^{(\mathfrak{p}-1)}\,(\varkappa-\mathfrak{u})^{\mathfrak{q}-1}\,\Upsilon(\mathfrak{u}, \digamma(\mathfrak{u}))\,d\mathfrak{u}|\\ &\leq \frac{\mathfrak{p}}{\Gamma(\mathfrak{q})}|\int_0^{\varkappa^*} \mathfrak{u}^{(\mathfrak{p}-1)}\,(\varkappa^*-\mathfrak{u})^{\mathfrak{q}-1}\,d\mathfrak{u}\\ &- \int_0^{\varkappa} \mathfrak{u}^{(\mathfrak{p}-1)}\,(\varkappa-\mathfrak{u})^{\mathfrak{q}-1}\,d\mathfrak{u}\,|\,\cdot|\Upsilon(\mathfrak{u}, \digamma(\mathfrak{u})|\\ &\leq \frac{\mathfrak{p}}{\Gamma(\mathfrak{q})}|(\varkappa^*)^{(\mathfrak{p}+\mathfrak{q}-1)}\,\beta(\mathfrak{p},\mathfrak{q}) - \varkappa^{(\mathfrak{p}+\mathfrak{q}-1)}\,\beta(\mathfrak{p},\mathfrak{q})|\,\Upsilon^*\\ &\leq \frac{\mathfrak{p}}{\Gamma(\mathfrak{p}+\mathfrak{q})}\,\Upsilon^*\,[(\varkappa^*)^{(\mathfrak{p}+\mathfrak{q}-1)} - \varkappa^{(\mathfrak{p}+\mathfrak{q}-1)}],\end{aligned}$$

that is independent from $\digamma$. When $\varkappa^* \to \varkappa$ its value becomes zero. Hence $||F(\digamma(\varkappa^*)) - F(\digamma(\varkappa))||_{\Xi} \to 0$. Thus proved that $F$ is equicontinuous. So $F$ is compact. As $F$ satisfies the conditions of Theorem 2.3.2, we say that $F$ will satisfy either one or the other condition mentioned in Theorem 2.3.2. For this using $(\beta_5)$, we construct $A = \{\digamma \in \Xi : ||\digamma||_{\Xi} < r\}$, where $r > 0$ is defined above. Hence, we can write

$$||F(\digamma(\varkappa))|| \leq \digamma_0 + \lambda\,\phi^*\,B(r). \tag{2.3.4}$$

Assume, there exists $\digamma \in \partial A$ and $\omega \in (0,1)$ where $\digamma = \omega F(\digamma)$.

For $\digamma$, $\omega$ and using $(\beta_5)$, we get

$$\begin{aligned} r &= ||\digamma||_{\Xi} \\ &= \omega ||F(\digamma)||_{\Xi} \\ &< ||F(\digamma)||_{\Xi} \\ &< \digamma_0 + \lambda\,\phi^*\,B(||\digamma||_{\Xi}) \\ &< \digamma_0 + \lambda\,\phi^*\,B(r). \end{aligned}$$

This gives us $r < r$, which is impossible. Thus, condition $(b)$ is not satisfied. Hence, by condition $(a)$, $F$ possesses a fixed point in $\overline{A}$. □

## 2.4 Uniqueness

Now, we will prove uniqueness with the help of theorems using lipschitz condition [33] along with some other conditions.

**Theorem 2.4.1.** Let $\Delta, \aleph, \Theta, \Delta_1, \aleph_1, \Theta_1 \in \mathfrak{Y} = C(\mathrm{J}, \mathbb{R})$ and we assume that

**(Condition)** : $||\Delta|| \leq \mu_1$, $||\aleph|| \leq \mu_2$, $\alpha \in (0, \infty)$,

$||f(\aleph(\varkappa))|| = ||\frac{\aleph(\varkappa)}{1+\alpha\,\aleph(\varkappa)}|| \leq \frac{||\aleph(\varkappa)||}{||1+\alpha\,\aleph(\varkappa)||} \leq \mu_3$ (where $\mu_3 = \frac{1}{\alpha}$),

$||\Theta|| \leq \mu_4$ for some $\mu_i > 0\,, i = 1, 2, 3, 4$.

Moreover, $||\frac{1}{1+\alpha\,\aleph(\varkappa)}|| \leq b_1, ||\frac{1}{1+\alpha\,\aleph_1(\varkappa)}|| \leq b_2$, where $b_1 = \frac{1}{\alpha||\aleph(\varkappa)||}$, $b_2 = \frac{1}{\alpha||\aleph_1(\varkappa)||}$, $b_1 > 0, b_2 > 0$ and $b = b_1 \cdot b_2$, then $\Upsilon_1, \Upsilon_2, \Upsilon_3$ defined in model (2.1.1) are lipschitz functions with the following values

$w_1 = (\beta_0\,\mu_3 + \mu + \nu), w_2 = (\beta_0\,\mu_1\,b + \mu + \gamma), w_3 = (\zeta + \mu)$, where $0 < w_j < 1, j = 1, 2, 3$.

*Proof.* Considering $\Upsilon_1$ for each $\Delta, \Delta_1 \in \mathfrak{Y}$, we take

$$\begin{aligned}
|| \quad & \Upsilon_1(\varkappa, \Delta(\varkappa), \aleph(\varkappa), \Theta(\varkappa)) - \Upsilon_1(\varkappa, \Delta_1(\varkappa), \aleph(\varkappa), \Theta(\varkappa))|| \\
= \quad & ||(\Pi\theta - \beta_0 f(\aleph(\varkappa))\Delta(\varkappa) - (\mu+\nu)\Delta(\varkappa) + \zeta\,\Theta(\varkappa - \tau)) \\
- \quad & (\Pi\theta - \beta_0 f(\aleph(\varkappa))\Delta_1(\varkappa) - (\mu+\nu)\Delta_1(\varkappa) + \zeta\Theta(\varkappa - \tau))|| \\
= \quad & || - \beta_0 f(\aleph(\varkappa))(\Delta(\varkappa) - \Delta_1(\varkappa)) - (\mu+\nu)(\Delta(\varkappa) - \Delta_1(\varkappa))|| \\
= \quad & ||(-\beta_0 f(\aleph(\varkappa)) - (\mu+\nu))(\Delta(\varkappa) - \Delta_1(\varkappa))|| \\
= \quad & ||(-(\beta_0 f(\aleph(\varkappa)) + (\mu+\nu)))(\Delta(\varkappa) - \Delta_1(\varkappa))|| \\
= \quad & ||(\beta_0 f(\aleph(\varkappa)) + (\mu+\nu))(\Delta(\varkappa) - \Delta_1(\varkappa))|| \\
\leq \quad & (||(\beta_0 f(\aleph(\varkappa)) + (\mu+\nu))||)\,||\Delta(\varkappa) - \Delta_1(\varkappa)|| \\
\leq \quad & (||(\beta_0 f(\aleph(\varkappa))|| + ||(\mu+\nu)||)\,||\Delta(\varkappa) - \Delta_1(\varkappa)|| \\
\leq \quad & (\beta_0\,\mu_3 + \mu + \nu)\,||\Delta(\varkappa) - \Delta_1(\varkappa)|| \\
\leq \quad & w_1\,||\Delta(\varkappa) - \Delta_1(\varkappa)||.
\end{aligned}$$

Hence, $\Upsilon_1$ is Lipschitz with respect to $\Delta$ with $w_1 > 0$.

Consider $\Upsilon_2$ for each $\aleph, \aleph_1 \in \mathfrak{Y}$, we take

$$\begin{aligned}
|| \quad & \Upsilon_2(\varkappa, \Delta(\varkappa), \aleph(\varkappa), \Theta(\varkappa)) - \Upsilon_2(\varkappa, \Delta(\varkappa), \aleph_1(\varkappa), \Theta(\varkappa))\,|| \\
= \quad & ||(\beta_0 f(\aleph(\varkappa))\Delta(\varkappa) - (\mu+\gamma)\aleph(\varkappa)) - (\beta_0 f(\aleph_1(\varkappa))\Delta(\varkappa) - (\mu+\gamma)\aleph_1(\varkappa))|| \\
= \quad & ||(\beta_0\Delta(\varkappa)(f(\aleph(\varkappa)) - f(\aleph_1(\varkappa))) + (\mu+\gamma)(-\aleph(\varkappa) + \aleph_1(\varkappa))|| \\
\leq \quad & ||(\beta_0\Delta(\varkappa)(f(\aleph(\varkappa)) - f(\aleph_1(\varkappa)))|| + ||(\mu+\gamma)(-\aleph(\varkappa) + \aleph_1(\varkappa))|| \\
\leq \quad & |\beta_0|\,||\Delta(\varkappa)||\,||(f(\aleph(\varkappa)) - f(\aleph_1(\varkappa)))|| + |(\mu+\gamma)|\,||(\aleph(\varkappa) - \aleph_1(\varkappa))|| \\
\leq \quad & \beta_0\,\mu_1\,||\frac{\aleph(\varkappa)}{1+\alpha\,\aleph(\varkappa)} - \frac{\aleph_1(\varkappa)}{1+\alpha\,\aleph_1(\varkappa)}|| + (\mu+\gamma)||\aleph(\varkappa) - \aleph_1(\varkappa))|| \\
\leq \quad & \beta_0\,\mu_1\,||\aleph(\varkappa) - \aleph_1(\varkappa)||\,\frac{1}{||(1+\alpha\,\aleph(\varkappa))\,(1+\alpha\,\aleph_1(\varkappa))||} + (\mu+\gamma)||\aleph(\varkappa) - \aleph_1(\varkappa)|| \\
\leq \quad & \beta_0\,\mu_1\,b\,||\aleph(\varkappa) - \aleph_1(\varkappa)|| + (\mu+\gamma)||\aleph(\varkappa) - \aleph_1(\varkappa)|| \\
\leq \quad & (\beta_0\,\mu_1\,b + \mu + \gamma)||\aleph(\varkappa) - \aleph_1(\varkappa)|| \\
\leq \quad & w_2||\aleph(\varkappa) - \aleph_1(\varkappa)||.
\end{aligned}$$

Hence, $\Upsilon_2$ is Lipschitz w.r.t $\aleph$ with $w_2 > 0$.

Considering $\Upsilon_3$ for each $\Theta, \Theta_1 \in \mathfrak{Y}$, we take

$$\begin{aligned}
|| \quad & \Upsilon_3(\varkappa, \Delta(\varkappa), \aleph(\varkappa), \Theta(\varkappa)) - \Upsilon_3(\varkappa, \Delta(\varkappa), \aleph_(\varkappa), \Theta_1(\varkappa))|| \\
= \quad & ||((1-\Pi)\theta + \nu\Delta(\varkappa) + \kappa\aleph(\varkappa) - \zeta\Theta(\varkappa - \tau) - \mu\Theta(\varkappa)) \\
- \quad & ((1-\Pi)\theta + \nu\Delta(\varkappa) + \kappa\aleph(\varkappa) - \zeta\Theta_1(\varkappa - \tau) - \mu\Theta_1(\varkappa))|| \\
= \quad & ||\zeta(\Theta_1(\varkappa - \tau) - \Theta(\varkappa - \tau)) + \mu(\Theta_1(\varkappa) - \Theta(\varkappa))|| \\
\leq \quad & ||\zeta(\Theta_1(\varkappa - \tau) - \Theta(\varkappa - \tau))|| + ||\mu(\Theta_1(\varkappa) - \Theta(\varkappa))|| \\
\leq \quad & |\zeta| \, ||\Theta(\varkappa - \tau) - \Theta_1(\varkappa - \tau)|| + |\mu|||(\Theta(\varkappa) - \Theta_1(\varkappa))||
\end{aligned}$$

For $\varkappa \in J$ and for $\tau \geq 0$, if $(\varkappa - \tau) \in J$, taking $\varkappa^* = max(\varkappa, \varkappa - \tau)$, we have

$$\begin{aligned}
\leq \quad & \zeta||\Theta(\varkappa^*) - \Theta_1(\varkappa^*)|| + \mu||\Theta(\varkappa^*) - \Theta_1(\varkappa^*)|| \\
\leq \quad & (\zeta + \mu)||\Theta(\varkappa^*) - \Theta_1(\varkappa^*)|| \\
\leq \quad & w_3||\Theta(\varkappa^*) - \Theta_1(\varkappa^*)||.
\end{aligned}$$

Hence, $\Upsilon_3$ is Lipschitz with respect to $\Theta$ with $w_3 > 0$. □

Moreover, we see the uniqueness of solution in Theorem 2.4.2 under the condition defined in Theorem 2.4.1.

**Theorem 2.4.2.** If $||\Delta|| \leq \mu_1, ||\aleph|| \leq \mu_2, ||\Theta|| \leq \mu_4$ for some $\mu_i > 0, i = 1, 2, 3, 4$ and $w_1 = (\beta_0 \, \mu_3 + \mu + \nu), w_2 = (\beta_0 \, \mu_1 \, b + \mu + \gamma), w_3 = (\zeta + \mu)$, where $0 < w_j < 1, \; j = 1, 2, 3$; then our model has a unique solution if $\lambda \, w_j < 1$ for $j = 1, 2, 3$.

*Proof.* Suppose the model has two solutions $(\Delta(\varkappa), \aleph(\varkappa), \Theta(\varkappa))$ and $(\Delta^*(\varkappa), \aleph^*(\varkappa), \Theta^*(\varkappa))$ with initial conditions defined above. So, we can write

$$\begin{aligned}
\Delta(\varkappa) &= \Delta(0) + \frac{\mathfrak{p}}{\Gamma(\mathfrak{q})} \int_0^{\varkappa} \mathfrak{u}^{(\mathfrak{p}-1)} \, (\varkappa - \mathfrak{u})^{\mathfrak{q}-1} \, \Upsilon_1(\mathfrak{u}, \Delta(\mathfrak{u}), \aleph(\mathfrak{u}), \Theta(\mathfrak{u})) d\mathfrak{u}, \\
\Delta^*(\varkappa) &= \Delta(0) + \frac{\mathfrak{p}}{\Gamma(\mathfrak{q})} \int_0^{\varkappa} \mathfrak{u}^{(\mathfrak{p}-1)} \, (\varkappa - \mathfrak{u})^{\mathfrak{q}-1} \, \Upsilon_1(\mathfrak{u}, \Delta^*(\mathfrak{u}), \aleph^*(\mathfrak{u}), \Theta^*(\mathfrak{u})) d\mathfrak{u}.
\end{aligned}$$

Take

$$\begin{aligned}
||\Delta(\varkappa) - \Delta^*(\varkappa)|| \quad &= \quad ||\frac{\mathfrak{p}}{\Gamma(\mathfrak{q})} \int_0^{\varkappa} \mathfrak{u}^{(\mathfrak{p}-1)} (\varkappa - \mathfrak{u})^{\mathfrak{q}-1} (\Upsilon_1(\mathfrak{u}, \Delta(\mathfrak{u}), \aleph(\mathfrak{u}), \Theta(\mathfrak{u})) \\
&- \quad \Upsilon_1(\mathfrak{u}, \Delta^*(\mathfrak{u}), \aleph^*(\mathfrak{u}), \Theta^*(\mathfrak{u}))) d\mathfrak{u}|| \\
&\leq \quad \frac{\mathfrak{p}}{\Gamma(\mathfrak{q})} \int_0^{\varkappa} \mathfrak{u}^{(\mathfrak{p}-1)} \, (\varkappa - \mathfrak{u})^{\mathfrak{q}-1} \, ||\Upsilon_1(\mathfrak{u}, \Delta(\mathfrak{u}), \aleph(\mathfrak{u}), \Theta(\mathfrak{u})) \\
&- \quad \Upsilon_1(\mathfrak{u}, \Delta^*(\mathfrak{u}), \aleph^*(\mathfrak{u}), \Theta^*(\mathfrak{u}))|| d\mathfrak{u},
\end{aligned}$$

since $\Upsilon_1$ is considered with respect to $\Delta$ and $\Delta^*$ so by Theorem 2.4.1 and definition of Beta function, we get

$$\begin{aligned}
||\Delta(\varkappa)-\Delta^*(\varkappa)|| &\leq \frac{\mathfrak{p}}{\Gamma(\mathfrak{q})}\int_0^{\varkappa} \mathfrak{u}^{(\mathfrak{p}-1)}\,(\varkappa-\mathfrak{u})^{\mathfrak{q}-1}\,||\Upsilon_1(\Delta)-\Upsilon_1(\Delta^*)||d\mathfrak{u}\\
&\leq \frac{\mathfrak{p}}{\Gamma(\mathfrak{q})}\,\mathrm{T}^{(\mathfrak{p}+\mathfrak{q}-1)}\,\beta(\mathfrak{p},\mathfrak{q})\,||\Upsilon_1(\Delta)-\Upsilon_1(\Delta^*)||\\
&\leq \frac{\mathfrak{p}\,\mathrm{T}^{(\mathfrak{p}+\mathfrak{q}-1)}\,\Gamma(\mathfrak{p})}{\Gamma(\mathfrak{p}+\mathfrak{q})}\,w_1||\Delta(\varkappa)-\Delta^*(\varkappa)||.
\end{aligned}$$

Hence,

$$||\Delta(\varkappa)-\Delta^*(\varkappa)|| \leq \lambda\,w_1\,||\Delta(\varkappa)-\Delta^*(\varkappa)||.$$

This implies that $(1-\lambda\,w_1)\,||\Delta(\varkappa)-\Delta^*(\varkappa)|| \leq 0$.
As $\lambda w_1 < 1$, so this is possible when $||\Delta(\varkappa)-\Delta^*(\varkappa)|| = 0$. Thus, $\Delta(\varkappa) = \Delta^*(\varkappa)$.
Similarly, we have

$$\begin{aligned}
\aleph(\varkappa) &= \aleph(0)+\frac{\mathfrak{p}}{\Gamma(\mathfrak{q})}\int_0^{\varkappa} \mathfrak{u}^{(\mathfrak{p}-1)}\,(\varkappa-\mathfrak{u})^{\mathfrak{p}-1}\,\Upsilon_2(\mathfrak{u},\Delta(\mathfrak{u}),\aleph(\mathfrak{u}),\Theta(\mathfrak{u}))d\mathfrak{u},\\
\aleph^*(\varkappa) &= \aleph(0)+\frac{\mathfrak{p}}{\Gamma(\mathfrak{q})}\int_0^{\varkappa} \mathfrak{u}^{(\mathfrak{p}-1)}\,(\varkappa-\mathfrak{u})^{\mathfrak{q}-1}\,\Upsilon_2(\mathfrak{u},\Delta^*(\mathfrak{u}),\aleph^*(\mathfrak{u}),\Theta^*(\mathfrak{u}))d\mathfrak{u}.
\end{aligned}$$

Take

$$\begin{aligned}
||\aleph(\varkappa)-\aleph^*(\varkappa)|| &= ||\frac{\mathfrak{p}}{\Gamma(\mathfrak{q})}\int_0^{\varkappa} \mathfrak{u}^{(\mathfrak{p}-1)}\,(\varkappa-\mathfrak{u})^{\mathfrak{q}-1}\,(\Upsilon_2(\mathfrak{u},\Delta(\mathfrak{u}),\aleph(\mathfrak{u}),\Theta(\mathfrak{u}))\\
&- \Upsilon_2(\mathfrak{u},\Delta^*(\mathfrak{u}),\aleph^*(\mathfrak{u}),\Theta^*(\mathfrak{u})))d\mathfrak{u}||\\
&\leq \frac{\mathfrak{p}}{\Gamma(\mathfrak{q})}\int_0^{\varkappa} \mathfrak{u}^{(\mathfrak{p}-1)}\,(\varkappa-\mathfrak{u})^{\mathfrak{q}-1}\,||\Upsilon_2(\mathfrak{u},\Delta(\mathfrak{u}),\aleph(\mathfrak{u}),\Theta(\mathfrak{u}))\\
&- \Upsilon_2(\mathfrak{u},\Delta^*(\mathfrak{u}),\aleph^*(\mathfrak{u}),\Theta^*(\mathfrak{u}))||d\mathfrak{u},
\end{aligned}$$

since $\Upsilon_2$ is considered with respect to $\aleph$ and $\aleph^*$ so by using previous result and definition of Beta function, we have

$$\begin{aligned}
||\aleph(\varkappa)-\aleph^*(\varkappa)|| &\leq \frac{\mathfrak{p}}{\Gamma(\mathfrak{q})}\int_0^{\varkappa} \mathfrak{u}^{(\mathfrak{p}-1)}\,(\varkappa-\mathfrak{u})^{\mathfrak{q}-1}\,||\Upsilon_2(\aleph)-\Upsilon_2(\aleph^*)||d\mathfrak{u}\\
&\leq \frac{\mathfrak{p}}{\Gamma(\mathfrak{q})}\,\mathrm{T}^{(\mathfrak{p}+\mathfrak{q}-1)}\,\beta(\mathfrak{p},\mathfrak{q})\,||\Upsilon_2(\aleph)-\Upsilon_2(\aleph^*)||\\
&\leq \frac{\mathfrak{p}\,\mathrm{T}^{(\mathfrak{p}+\mathfrak{q}-1)}\,\Gamma(\mathfrak{p})}{\Gamma(\mathfrak{p}+\mathfrak{q})}\,w_2||\aleph(\varkappa)-\aleph^*(\varkappa)||.
\end{aligned}$$

Hence, we obtain

$$||\aleph(\varkappa)-\aleph^*(\varkappa)|| \leq \lambda\,w_2\,||\aleph(\varkappa)-\aleph^*(\varkappa)||.$$

This implies that

$$(1-\lambda\, w_2)\,||\aleph(\varkappa)-\aleph^*(\varkappa)|| \leq 0.$$

As $\lambda w_2 < 1$, this is possible when $||\aleph(\varkappa)-\aleph^*(\varkappa)|| = 0$. Thus, $\aleph(\varkappa) = \aleph^*(\varkappa)$. Also, we have

$$\begin{aligned}
\Theta(\varkappa) &= \Theta(0) + \frac{\mathfrak{p}}{\Gamma(\mathfrak{q})}\int_0^{\varkappa} \mathfrak{u}^{(\mathfrak{p}-1)}\,(\varkappa-\mathfrak{u})^{\mathfrak{q}-1}\,\Upsilon_3(\mathfrak{u},\Delta(\mathfrak{u}),\aleph(\mathfrak{u}),\Theta(\mathfrak{u}))d\mathfrak{u},\\
\Theta^*(\varkappa) &= \Theta(0) + \frac{\mathfrak{p}}{\Gamma(\mathfrak{q})}\int_0^{\varkappa} \mathfrak{u}^{(\mathfrak{p}-1)}\,(\varkappa-\mathfrak{u})^{\mathfrak{q}-1}\,\Upsilon_3(\mathfrak{u},\Delta^*(\mathfrak{u}),\aleph^*(\mathfrak{u}),\Theta^*(\mathfrak{u}))d\mathfrak{u}.
\end{aligned}$$

Take

$$\begin{aligned}
||\Theta(\varkappa)-\Theta^*(\varkappa)|| &= ||\frac{\mathfrak{p}}{\Gamma(\mathfrak{q})}\int_0^{\varkappa} \mathfrak{u}^{(\mathfrak{p}-1)}\,(\varkappa-\mathfrak{u})^{\mathfrak{q}-1}\,(\Upsilon_3(\mathfrak{u},\Delta(\mathfrak{u}),\aleph(\mathfrak{u}),\Theta(\mathfrak{u}))\\
&- \Upsilon_3(\mathfrak{u},\Delta^*(\mathfrak{u}),\aleph^*(\mathfrak{u}),\Theta^*(\mathfrak{u})))d\mathfrak{u}||\\
&\leq \frac{\mathfrak{p}}{\Gamma(\mathfrak{q})}\int_0^{\varkappa} \mathfrak{u}^{(\mathfrak{p}-1)}\,(\varkappa-\mathfrak{u})^{\mathfrak{q}-1}\,||\Upsilon_3(\mathfrak{u},\Delta(\mathfrak{u}),\aleph(\mathfrak{u}),\Theta(\mathfrak{u}))\\
&- \Upsilon_3(\mathfrak{u},\Delta^*(\mathfrak{u}),\aleph^*(\mathfrak{u}),\Theta^*(\mathfrak{u}))||d\mathfrak{u},
\end{aligned}$$

since $\Upsilon_3$ is considered w.r.t $\Theta$ and $\Theta^*$ so by using previous result and definition of Beta function, we have

$$\begin{aligned}
||\Theta(\varkappa)-\Theta^*(\varkappa)|| &\leq \frac{\mathfrak{p}}{\Gamma(\mathfrak{q})}\int_0^{\varkappa} \mathfrak{u}^{(\mathfrak{p}-1)}\,(\varkappa-\mathfrak{u})^{\mathfrak{q}-1}\,||\Upsilon_3(\Theta)-\Upsilon_3(\Theta^*)||d\mathfrak{u}\\
&\leq \frac{\mathfrak{p}}{\Gamma(\mathfrak{q})}\,\mathrm{T}^{(\mathfrak{p}+\mathfrak{q}-1)}\,\beta(\mathfrak{p},\mathfrak{q})\,||\Upsilon_3(\Theta)-\Upsilon_3(\Theta^*)||\\
&\leq \frac{\mathfrak{p}\,\mathrm{T}^{(\mathfrak{p}+\mathfrak{q}-1)}\,\Gamma(\mathfrak{p})}{\Gamma(\mathfrak{p}+\mathfrak{q})}\,w_3||\Theta(\varkappa)-\Theta^*(\varkappa)||.
\end{aligned}$$

Hence, $||\Theta(\varkappa)-\Theta^*(\varkappa)|| \leq \lambda\, w_3\,||\Theta(\varkappa)-\Theta^*(\varkappa)||$.
$\implies (1-\lambda\, w_3)\,||\Theta(\varkappa)-\Theta^*(\varkappa)|| \leq 0.$
As $\lambda w_3 < 1$, this is possible when $||\Theta(\varkappa)-\Theta^*(\varkappa)|| = 0$. Thus $\Theta(\varkappa) = \Theta^*(\varkappa)$. That is, $(\Delta(\varkappa),\aleph(\varkappa),\Theta(\varkappa)) = (\Delta^*(\varkappa),\aleph^*(\varkappa),\Theta^*(\varkappa))$. Hence, solution is unique. □

## 2.5 Stability

In this section, we check stability of solution. We use Ulam−Hyers and Ulam−Hayers−Rassias theorems to check it. First, we define the following theorems for our model.

**Definition 2.5.1.** Model (2.2.1) is Ulam-Hyers stable [23], if for all $\epsilon_i > 0$, there exist $M_i > 0 \in \mathbb{R}$, which depend on $\Upsilon_i$, $i = 1,2,3$ respectively, and for all $(\Delta^*, \aleph^*, \Theta^*)$ satisfying the inequalities

$$\begin{aligned} |^{FFP}D_{0,\varkappa}^{\mathfrak{q},\mathfrak{p}}\Delta^*(\varkappa) - \Upsilon_1(\varkappa, \Delta^*(\varkappa), \aleph^*(\varkappa), \Theta^*(\varkappa))| &\le \epsilon_1, \\ |^{FFP}D_{0,\varkappa}^{\mathfrak{q},\mathfrak{p}}\aleph^*(\varkappa) - \Upsilon_2(\varkappa, \Delta^*(\varkappa), \aleph^*(\varkappa), \Theta^*(\varkappa))| &\le \epsilon_2, \\ |^{FFP}D_{0,\varkappa}^{\mathfrak{q},\mathfrak{p}}\Theta^*(\varkappa) - \Upsilon_3(\varkappa, \Delta^*(\varkappa), \aleph^*(\varkappa), \Theta^*(\varkappa))| &\le \epsilon_3, \end{aligned} \tag{2.5.1}$$

there exists $(\Delta, \aleph, \Theta) \in \Xi$ satisfying model (2.2.1) with the condition

$$\begin{aligned} |\Delta^*(\varkappa) - \Delta(\varkappa)| &\le M_1\, \epsilon_1, \\ |\aleph^*(\varkappa) - \aleph(\varkappa)| &\le M_2\, \epsilon_2, \\ |\Theta^*(\varkappa) - \Theta(\varkappa)| &\le M_3\, \epsilon_3. \end{aligned} \tag{2.5.2}$$

**Remark 2.5.2.** $(\Delta^*, \aleph^*, \Theta^*) \in \Xi$ is a solution of model (2.5.1) iff $\exists\, \eta_i \in C([0,\mathrm{T}], [0,\infty))$ such that for all $\varkappa \in \mathrm{J}$,

(i) $|\,\eta_i(\varkappa)\,| < \epsilon_i$,

(ii)

$$\begin{aligned} {}^{FFP}D_{0,\varkappa}^{\mathfrak{q},\mathfrak{p}}\Delta^*(\varkappa) &= \Upsilon_1(\varkappa, \Delta^*(\varkappa), \aleph^*(\varkappa), \Theta^*(\varkappa)) + \eta_1(\varkappa), \\ {}^{FFP}D_{0,\varkappa}^{\mathfrak{q},\mathfrak{p}}\aleph^*(\varkappa) &= \Upsilon_2(\varkappa, \Delta^*(\varkappa), \aleph^*(\varkappa), \Theta^*(\varkappa)) + \eta_2(\varkappa), \\ {}^{FFP}D_{0,\varkappa}^{\mathfrak{q},\mathfrak{p}}\Theta^*(\varkappa) &= \Upsilon_3(\varkappa, \Delta^*(\varkappa), \aleph^*(\varkappa), \Theta^*(\varkappa)) + \eta_3(\varkappa). \end{aligned} \tag{2.5.3}$$

**Theorem 2.5.3.** The fractal fraction model (2.1.1) is Ulam−Hayers stable on J such that $\lambda\, w_i < 1$, where $w_i$ and $\lambda$ are defined as above.

*Proof.* Let $\epsilon_1 > 0$ and $\Delta^* \in \mathfrak{Y}$ s.t

$$|^{FFP}D_{0,\varkappa}^{\mathfrak{q},\mathfrak{p}}\Delta^*(\varkappa) - \Upsilon_1(\varkappa, \Delta^*(\varkappa), \aleph^*(\varkappa), \Theta^*(\varkappa))| \le \epsilon_1,$$

by above remark (2.5.2), we have

$$\begin{aligned} \Delta^*(\varkappa) &= \Delta(0) + \frac{\mathfrak{p}}{\Gamma(\mathfrak{q})} \int_0^{\varkappa} \mathfrak{u}^{(\mathfrak{p}-1)}\, (\varkappa - \mathfrak{u})^{\mathfrak{q}-1}\, \Upsilon_1(\mathfrak{u}, \Delta^*(\mathfrak{u}), \aleph^*(\mathfrak{u}), \Theta^*(\mathfrak{u}))d\mathfrak{u} \\ &+ \frac{\mathfrak{p}}{\Gamma(\mathfrak{q})} \int_0^{\varkappa} \mathfrak{u}^{(\mathfrak{p}-1)}\, (\varkappa - \mathfrak{u})^{\mathfrak{q}-1}\, \eta_1(\mathfrak{u})\, d\mathfrak{u}. \end{aligned} \tag{2.5.4}$$

As $\Delta \in \mathfrak{Y}$ is the unique solution, then

$$\Delta(\varkappa) = \Delta(0) + \frac{\mathfrak{p}}{\Gamma(\mathfrak{q})} \int_0^{\varkappa} \mathfrak{u}^{(\mathfrak{p}-1)}\, (\varkappa - \mathfrak{u})^{\mathfrak{q}-1}\, \Upsilon_1(\mathfrak{u}, \Delta(\mathfrak{u}), \aleph(\mathfrak{u}), \Theta(\mathfrak{u}))d\mathfrak{u}.$$

That is

$$
\begin{aligned}
&|\Delta^*(\varkappa) - \Delta(\varkappa)| \\
&\quad = \; |\frac{\mathfrak{p}}{\Gamma(\mathfrak{q})} \int_0^{\varkappa} \mathfrak{u}^{(\mathfrak{p}-1)} \, (\varkappa - \mathfrak{u})^{\mathfrak{q}-1} \, \eta_1(\mathfrak{u}) \, d\mathfrak{u} \\
&\quad + \; \frac{\mathfrak{p}}{\Gamma(\mathfrak{q})} \int_0^{\varkappa} \mathfrak{u}^{(\mathfrak{p}-1)} \, (\varkappa - \mathfrak{u})^{\mathfrak{q}-1} [\Upsilon_1(\mathfrak{u}, \Delta^*(\mathfrak{u}), \aleph^*(\mathfrak{u}), \Theta^*(\mathfrak{u})) \\
&\quad - \; \Upsilon_1(\mathfrak{u}, \Delta(\mathfrak{u}), \aleph(\mathfrak{u}), \Theta(\mathfrak{u})) \, ] \, d\mathfrak{u}| \\
&\quad \leq \; \frac{\mathfrak{p}}{\Gamma(\mathfrak{q})} \int_0^{\varkappa} \mathfrak{u}^{(\mathfrak{p}-1)} \, (\varkappa - \mathfrak{u})^{\mathfrak{q}-1} \, |\eta_1(\mathfrak{u})| d\mathfrak{u} \\
&\quad + \; \frac{\mathfrak{p}}{\Gamma(\mathfrak{q})} \int_0^{\varkappa} \mathfrak{u}^{(\mathfrak{p}-1)} \, (\varkappa - \mathfrak{u})^{\mathfrak{q}-1} || \Upsilon_1(\mathfrak{u}, \Delta(\mathfrak{u}), \aleph(\mathfrak{u}), \Theta(\mathfrak{u})) \\
&\quad - \; \Upsilon_1(\mathfrak{u}, \Delta^*(\mathfrak{u}), \aleph^*(\mathfrak{u}), \Theta^*(\mathfrak{u})) \, || \, d\mathfrak{u} \\
&\quad \leq \; \frac{\mathfrak{p}}{\Gamma(\mathfrak{q})} \, \beta(\mathfrak{p}, \mathfrak{q}) \, T^{(\mathfrak{p}+\mathfrak{q}-1)} \, |\eta_1(u)| \\
&\quad + \; \frac{\mathfrak{p}}{\Gamma(\mathfrak{q})} \, \beta(\mathfrak{p}, \mathfrak{q}) \, T^{(\mathfrak{p}+\mathfrak{q}-1)} \, w_1 \, ||\Delta^* - \Delta|| \\
&\quad \leq \; \frac{\mathfrak{p}}{\Gamma(\mathfrak{q})} \, \beta(\mathfrak{p}, \mathfrak{q}) \, T^{(\mathfrak{p}+\mathfrak{q}-1)} \, \epsilon_1 \\
&\quad + \; \frac{\mathfrak{p}}{\Gamma(\mathfrak{q})} \, \beta(\mathfrak{p}, \mathfrak{q}) \, T^{(\mathfrak{p}+\mathfrak{q}-1)} \, w_1 \, ||\Delta^* - \Delta|| \\
&\quad \leq \; \lambda \, \epsilon_1 + \lambda \, w_1 \, ||\Delta^* - \Delta||.
\end{aligned}
$$

Hence, we have

$$
\begin{aligned}
||\Delta^* - \Delta|| \quad &\leq \quad \lambda \, \epsilon_1 + \, \lambda \, w_1 \, ||\Delta^* - \Delta|| \\
(1 - \lambda \, w_1) \, ||\Delta^* - \Delta|| \quad &\leq \quad \lambda \, \epsilon_1 \\
||\Delta^* - \Delta|| \quad &\leq \quad \frac{\lambda \, \epsilon_1}{(1 - \lambda \, w_1)}.
\end{aligned}
$$

If $\frac{\lambda}{(1-\lambda \, w_1)} = M_1$, then $||\Delta^* - \Delta|| \, \leq \, M_1 \, \epsilon_1$.
Similarly, we can prove that $||\aleph^* - \aleph|| \, \leq \, M_2 \, \epsilon_2$, and $||\Theta^* - \Theta|| \, \leq \, M_3 \, \epsilon_3$. □

Thus Ulam−Hayers stability criteria is fulfilled by our $FF$ model.

**Definition 2.5.4.** We define the Ulam−Hayers−Rassias stability criteria for our fractal−fractional model ([25]). Model (2.2.1) is Ulam-Hyers−Rassias stable with respect to the functions $\psi_i$, if for all $\epsilon_i > 0$, there exist $M_i > 0 \, \in [0, \infty)$, which depend on

$\Upsilon_i \, and \, \psi_i$, $i = 1, 2, 3$ respectively and for all $(\Delta^*, \aleph^*, \Theta^*)$ satisfying the inequalities:

$$
\begin{aligned}
|^{FFP}D^{\mathfrak{q},\mathfrak{p}}_{0,\varkappa}\Delta^*(\varkappa) - \Upsilon_1(\varkappa, \Delta^*(\varkappa), \aleph^*(\varkappa), \Theta^*(\varkappa))| &\leq \epsilon_1 \, \psi_1(\varkappa), \\
|^{FFP}D^{\mathfrak{q},\mathfrak{p}}_{0,\varkappa}\aleph^*(\varkappa) - \Upsilon_2(\varkappa, \Delta^*(\varkappa), \aleph^*(\varkappa), \Theta^*(\varkappa))| &\leq \epsilon_2 \, \psi_2(\varkappa), \\
|^{FFP}D^{\mathfrak{q},\mathfrak{p}}_{0,\varkappa}\Theta^*(\varkappa) - \Upsilon_3(\varkappa, \Delta^*(\varkappa), \aleph^*(\varkappa), \Theta^*(\varkappa))| &\leq \epsilon_3 \, \psi_3(\varkappa),
\end{aligned} \tag{2.5.5}
$$

there exists $(\Delta, \aleph, \Theta) \in \Xi$ satisfying model (2.2.1) with the conditions:

$$
\begin{aligned}
|\Delta^*(\varkappa) - \Delta(\varkappa)| &\leq M_1 \, \epsilon_1 \, \psi_1(\varkappa), \\
|\aleph^*(\varkappa) - \aleph(\varkappa)| &\leq M_2 \, \epsilon_2 \, \psi_2(\varkappa), \\
|\Theta^*(\varkappa) - \Theta(\varkappa)| &\leq M_3 \, \epsilon_3 \, \psi_3(\varkappa).
\end{aligned} \tag{2.5.6}
$$

**Remark 2.5.5.** $(\Delta^*, \aleph^*, \Theta^*) \in \Xi$ is a solution iff $\exists \, \eta_i \in C(\mathrm{J}, [0, \infty))$ such that for all $\varkappa \in \mathrm{J}$,

(i) $|\, \eta_i(\varkappa) \,| < \epsilon_i \, \psi_i(\varkappa)$,

(ii)

$$
\begin{aligned}
{}^{FFP}D^{\mathfrak{q},\mathfrak{p}}_{0,\varkappa}\Delta^*(\varkappa) &= \Upsilon_1(\varkappa, \Delta^*(\varkappa), \aleph^*(\varkappa), \Theta^*(\varkappa)) \, + \, \eta_1(\varkappa), \\
{}^{FFP}D^{\mathfrak{q},\mathfrak{p}}_{0,\varkappa}\aleph^*(\varkappa) &= \Upsilon_2(\varkappa, \Delta^*(\varkappa), \aleph^*(\varkappa), \Theta^*(\varkappa)) \, + \, \eta_2(\varkappa), \\
{}^{FFP}D^{\mathfrak{q},\mathfrak{p}}_{0,\varkappa}\Theta^*(\varkappa) &= \Upsilon_3(\varkappa, \Delta^*(\varkappa), \aleph^*(\varkappa), \Theta^*(\varkappa)) \, + \, \eta_3(\varkappa).
\end{aligned} \tag{2.5.7}
$$

**Theorem 2.5.6.** The fractal−fractional model (2.2.1) is Ulam−Hayers−Rassias stable when the following conditions are satisfied: for all $\varkappa \in \mathrm{J}$, there exist nondecreasing mappings $\psi_i \in C(\mathrm{J}, [0, \infty))$ and $\xi_i > 0$ depending upon $\psi_i$ such that ${}^{FFP}I^{\mathfrak{q},\mathfrak{p}}_{0,\varkappa}\psi_i(\varkappa) < \xi_i(\psi_i) \, \psi_i(\varkappa)$.

*Proof.* Let $\epsilon_1 > 0$ and $\Delta^* \in \mathfrak{Y}$ such that

$$
|^{FFP}D^{\mathfrak{q},\mathfrak{p}}_{0,\varkappa}\Delta^*(\varkappa) - \Upsilon_1(\varkappa, \Delta^*(\varkappa), \aleph^*(\varkappa), \Theta^*(\varkappa))| \leq \epsilon_1 \, \psi_1(\varkappa),
$$

then, by the conditions of remark 2.5.5, we consider

$$
\begin{aligned}
\Delta^*(\varkappa) = \Delta(0) &+ \frac{\mathfrak{p}}{\Gamma(\mathfrak{q})} \int_0^{\varkappa} \mathfrak{u}^{(\mathfrak{p}-1)} \, (\varkappa - \mathfrak{u})^{\mathfrak{q}-1} \, \Upsilon_1(\mathfrak{u}, \Delta^*(\mathfrak{u}), \aleph^*(\mathfrak{u}), \Theta^*(\mathfrak{u})) d\mathfrak{u} \\
&+ \frac{\mathfrak{p}}{\Gamma(\mathfrak{q})} \int_0^{\varkappa} \mathfrak{u}^{(\mathfrak{p}-1)} \, (\varkappa - \mathfrak{u})^{\mathfrak{q}-1} \, \eta_1(\mathfrak{u}) \, d\mathfrak{u}.
\end{aligned}
$$

As $\Delta \in \mathfrak{Y}$ is the unique solution, then

$$
\Delta(\varkappa) = \Delta(0) + \frac{\mathfrak{p}}{\Gamma(\mathfrak{q})} \int_0^{\varkappa} \mathfrak{u}^{(\mathfrak{p}-1)} \, (\varkappa - \mathfrak{u})^{\mathfrak{q}-1} \, \Upsilon_1(\mathfrak{u}, \Delta(\mathfrak{u}), \aleph(\mathfrak{u}), \Theta(\mathfrak{u})) d\mathfrak{u}.
$$

Therefore, we get

$$
\begin{aligned}
&|\Delta^*(\varkappa) - \Delta(\varkappa)| \\
&= |\frac{\mathfrak{p}}{\Gamma(\mathfrak{q})} \int_0^{\varkappa} \mathfrak{u}^{(\mathfrak{p}-1)} (\varkappa - \mathfrak{u})^{\mathfrak{q}-1} \eta_1(\mathfrak{u}) \, d\mathfrak{u} \\
&+ \frac{\mathfrak{p}}{\Gamma(\mathfrak{q})} \int_0^{\varkappa} \mathfrak{u}^{(\mathfrak{p}-1)} (\varkappa - \mathfrak{u})^{\mathfrak{q}-1} \\
&[ \quad \Upsilon_1(\mathfrak{u}, \Delta^*(\mathfrak{u}), \aleph^*(\mathfrak{u}), \Theta^*(\mathfrak{u})) - \Upsilon_1(\mathfrak{u}, \Delta(\mathfrak{u}), \aleph(\mathfrak{u}), \Theta(\mathfrak{u})) \,] \, d\mathfrak{u}| \\
&\leq \frac{\mathfrak{p}}{\Gamma(\mathfrak{q})} \int_0^{\varkappa} \mathfrak{u}^{(\mathfrak{p}-1)} (\varkappa - \mathfrak{u})^{\mathfrak{q}-1} |\eta_1(\mathfrak{u})| d\mathfrak{u} \\
&+ \frac{\mathfrak{p}}{\Gamma(\mathfrak{q})} \int_0^{\varkappa} \mathfrak{u}^{(\mathfrak{p}-1)} (\varkappa - \mathfrak{u})^{\mathfrak{q}-1} \\
&|| \quad \Upsilon_1(\mathfrak{u}, \Delta(\mathfrak{u}), \aleph(\mathfrak{u}), \Theta(\mathfrak{u})) - \Upsilon_1(\mathfrak{u}, \Delta^*(\mathfrak{u}), \aleph^*(\mathfrak{u}), \Theta^*(\mathfrak{u})) \,|| \, d\mathfrak{u} \\
&\leq \frac{\mathfrak{p}}{\Gamma(\mathfrak{q})} \int_0^{\varkappa} \mathfrak{u}^{(\mathfrak{p}-1)} (\varkappa - \mathfrak{u})^{\mathfrak{q}-1} \epsilon_1 \, \psi_1(\mathfrak{u}) \, d\mathfrak{u} \\
&+ \frac{\mathfrak{p}}{\Gamma(\mathfrak{q})} \beta(\mathfrak{p}, \mathfrak{q}) \, T^{(\mathfrak{p}+\mathfrak{q}-1)} \, w_1 \, ||\Delta^* - \Delta|| \\
&\leq \epsilon_1 \, \xi_1(\psi_1) \, \psi_1(\varkappa) + \lambda \, w_1 \, ||\Delta^* - \Delta||.
\end{aligned}
$$

Hence, we have

$$
\begin{aligned}
||\Delta^* - \Delta|| &\leq \epsilon_1 \, \xi_1(\psi_1) \, \psi_1(\varkappa) + \lambda \, w_1 \, ||\Delta^* - \Delta|| \\
(1 - \lambda \, w_1) \, ||\Delta^* - \Delta|| &\leq \epsilon_1 \, \xi_1(\psi_1) \, \psi_1(\varkappa) \\
||\Delta^* - \Delta|| &\leq \frac{\epsilon_1 \, \xi_1(\psi_1) \, \psi_1(\varkappa)}{(1 - \lambda \, w_1)}.
\end{aligned}
$$

If $\frac{\xi_1(\psi_1)}{(1-\lambda \, w_1)} = M_1(\Upsilon_1, \psi_1)$ then, we get

$$
||\Delta^* - \Delta|| \leq \epsilon_1 \, \psi_1(\varkappa) M_1(\Upsilon_1, \psi_1).
$$

Similarly, we can prove that

$$
||\aleph^* - \aleph|| \leq \epsilon_2 \, \psi_2(\varkappa) M_2(\Upsilon_2, \psi_2),
$$

$$
||\Theta^* - \Theta|| \leq \epsilon_3 \, \psi_3(\varkappa) M_3(\Upsilon_3, \psi_3).
$$

Thus, Ulam−Hayers−Rassias stability criteria is fulfilled by our fractal−fractional model.

□

## 2.6 Numerical Algorithm

Now, we make a numerical scheme using two−point Lagrangian interpolation formula [38] for our FF model. The difference between our scheme and others is that in our model $\Upsilon_1$ and $\Upsilon_3$ depend on $\varkappa$ and $(\varkappa - \tau)$, so we deal it differently in the end.

First, we take $\varkappa = \varkappa_{n+1}$ and $\mathfrak{u}^{\mathfrak{p}-1}\,\Upsilon_i(\mathfrak{u}, \Delta(\mathfrak{u}), \aleph(\mathfrak{u}), \Theta(\mathfrak{u})) = \digamma_i(\mathfrak{u})$ and get

$$\begin{aligned}
\Delta(\varkappa_{n+1}) &= \Delta(0) + \frac{\mathfrak{p}}{\Gamma(\mathfrak{q})}\int_0^{\varkappa_{n+1}} (\varkappa_{n+1} - \mathfrak{u})^{\mathfrak{q}-1}\,\digamma_1(\mathfrak{u})d\mathfrak{u},\\
\aleph(\varkappa_{n+1}) &= \aleph(0) + \frac{\mathfrak{p}}{\Gamma(\mathfrak{q})}\int_0^{\varkappa_{n+1}} (\varkappa_{n+1} - \mathfrak{u})^{\mathfrak{q}-1}\,\digamma_2(\mathfrak{u})d\mathfrak{u}, \qquad (2.6.1)\\
\Theta(\varkappa_{n+1}) &= \Theta(0) + \frac{\mathfrak{p}}{\Gamma(\mathfrak{q})}\int_0^{\varkappa_{n+1}} (\varkappa_{n+1} - \mathfrak{u})^{\mathfrak{q}-1}\,\digamma_3(\mathfrak{u})d\mathfrak{u}.
\end{aligned}$$

Approximating integral as the sum of integrals on sub intervals, we have

$$\begin{aligned}
\Delta(\varkappa_{n+1}) &= \Delta_0 + \frac{\mathfrak{p}}{\Gamma(\mathfrak{q})}\sum_{j=0}^{n}\int_{\varkappa_j}^{\varkappa_{j+1}} (\varkappa_{n+1} - \mathfrak{u})^{\mathfrak{q}-1}\digamma_1(\mathfrak{u})d\mathfrak{u},\\
\aleph(\varkappa_{n+1}) &= \aleph_0 + \frac{\mathfrak{p}}{\Gamma(\mathfrak{q})}\sum_{j=0}^{n}\int_{\varkappa_j}^{\varkappa_{j+1}} (\varkappa_{n+1} - \mathfrak{u})^{\mathfrak{q}-1}\digamma_2(\mathfrak{u})d\mathfrak{u}, \qquad (2.6.2)\\
\Theta(\varkappa_{n+1}) &= \Theta_0 + \frac{\mathfrak{p}}{\Gamma(\mathfrak{q})}\sum_{j=0}^{n}\int_{\varkappa_j}^{\varkappa_{j+1}} (\varkappa_{n+1} - \mathfrak{u})^{\mathfrak{q}-1}\digamma_3(\mathfrak{u})d\mathfrak{u}.
\end{aligned}$$

Now, we approximate the functions $\digamma_i(\mathfrak{u})$ by two points Lagrange interpolation polynomials on the interval $[\varkappa_j, \varkappa_{j+1}]$. We can write

$$\begin{aligned}
\digamma_1^*(\mathfrak{u}) &= \frac{\mathfrak{u} - \varkappa_{j-1}}{\varkappa_j - \varkappa_{j-1}}\,\varkappa_j^{\mathfrak{p}-1}\,\Upsilon_1(\mathfrak{u}_j, \Delta_j(\mathfrak{u}), \aleph_j(\mathfrak{u}), \Theta_j(\mathfrak{u}))\\
&- \frac{\mathfrak{u} - \varkappa_j}{\varkappa_j - \varkappa_{j-1}}\,\varkappa_{j-1}^{\mathfrak{p}-1}\,\Upsilon_1(\mathfrak{u}_{j-1}, \Delta_{j-1}(\mathfrak{u}), \aleph_{j-1}(\mathfrak{u}), \Theta_{j-1}(\mathfrak{u})),\\
\digamma_2^*(\mathfrak{u}) &= \frac{\mathfrak{u} - \varkappa_{j-1}}{\varkappa_j - \varkappa_{j-1}}\,\varkappa_j^{\mathfrak{p}-1}\,\Upsilon_2(\mathfrak{u}_j, \Delta_j(\mathfrak{u}), \aleph_j(\mathfrak{u}), \Theta_j(\mathfrak{u}))\\
&- \frac{\mathfrak{u} - \varkappa_j}{\varkappa_j - \varkappa_{j-1}}\,\varkappa_{j-1}^{\mathfrak{p}-1}\,\Upsilon_2(\mathfrak{u}_{j-1}, \Delta_{j-1}(\mathfrak{u}), \aleph_{j-1}(\mathfrak{u}), \Theta_{j-1}(\mathfrak{u})),\\
\digamma_3^*(\mathfrak{u}) &= \frac{\mathfrak{u} - \varkappa_{j-1}}{\varkappa_j - \varkappa_{j-1}}\,\varkappa_j^{\mathfrak{p}-1}\,\Upsilon_3(\mathfrak{u}_j, \Delta_j(\mathfrak{u}), \aleph_j(\mathfrak{u}), \Theta_j(\mathfrak{u}))\\
&- \frac{\mathfrak{u} - \varkappa_j}{\varkappa_j - \varkappa_{j-1}}\,\varkappa_{j-1}^{\mathfrak{p}-1}\,\Upsilon_3(\mathfrak{u}_{j-1}, \Delta_{j-1}(\mathfrak{u}), \aleph_{j-1}(\mathfrak{u}), \Theta_{j-1}(\mathfrak{u})).
\end{aligned}$$

Thus, we have

$$\begin{aligned}
\Delta(\varkappa_{n+1}) &= \Delta_0 + \frac{\mathfrak{p}}{\Gamma(\mathfrak{q})}\sum_{j=0}^{n}\int_{\varkappa_j}^{\varkappa_{j+1}} (\varkappa_{n+1}-\mathfrak{u})^{\mathfrak{q}-1} F_1^*(\mathfrak{u})d\mathfrak{u},\\
\aleph(\varkappa_{n+1}) &= \aleph_0 + \frac{\mathfrak{p}}{\Gamma(\mathfrak{q})}\sum_{j=0}^{n}\int_{\varkappa_j}^{\varkappa_{j+1}} (\varkappa_{n+1}-\mathfrak{u})^{\mathfrak{q}-1} F_2^*(\mathfrak{u})d\mathfrak{u},\\
\Theta(\varkappa_{n+1}) &= \Theta_0 + \frac{\mathfrak{p}}{\Gamma(\mathfrak{q})}\sum_{j=0}^{n}\int_{\varkappa_j}^{\varkappa_{j+1}} (\varkappa_{n+1}-\mathfrak{u})^{\mathfrak{q}-1} F_3^*(\mathfrak{u})d\mathfrak{u}.
\end{aligned} \tag{2.6.3}$$

Using values of $F_i^*(\mathfrak{u})$, we integrate the above integrals according to limits and taking $\varkappa_j - \varkappa_{j-1} = h$, we get the final results.

$$\begin{aligned}
\Delta(n+1) &= \Delta_0 + \frac{\mathfrak{p}\, h^{\mathfrak{q}}}{\Gamma(\mathfrak{q}+2)}\sum_{j=0}^{n}[\varkappa_j^{\mathfrak{p}-1}\Upsilon_1(\mathfrak{u}_j,\Delta_j,\aleph_j,\Theta_j)Z_1\\
&- \varkappa_{j-1}^{\mathfrak{p}-1}\,\Upsilon_1(\mathfrak{u}_{j-1},\Delta_{j-1},\aleph_{j-1},\Theta_{j-1})Z_2],\\
\aleph(n+1) &= \aleph_0 + \frac{\mathfrak{p}\, h^{\mathfrak{q}}}{\Gamma(\mathfrak{q}+2)}\sum_{j=0}^{n}[\varkappa_j^{\mathfrak{p}-1}\Upsilon_2(\mathfrak{u}_j,\Delta_j,\aleph_j,\Theta_j)\,Z_1\\
&- \varkappa_{j-1}^{\mathfrak{p}-1}\,\Upsilon_2(\mathfrak{u}_{j-1},\Delta_{j-1},\aleph_{j-1},\Theta_{j-1})\,Z_2],\\
\Theta(n+1) &= \Theta_0 + \frac{\mathfrak{p}\, h^{\mathfrak{q}}}{\Gamma(\mathfrak{q}+2)}\sum_{j=0}^{n}[\varkappa_j^{\mathfrak{p}-1}\Upsilon_3(\mathfrak{u}_j,\Delta_j,\aleph_j,\Theta_j)\,Z_1\\
&- \varkappa_{j-1}^{\mathfrak{p}-1}\,\Upsilon_3(\mathfrak{u}_{j-1},\Delta_{j-1},\aleph_{j-1},\Theta_{j-1})\,Z_2],
\end{aligned}$$

where

$Z_1 = (n+1-j)^{\mathfrak{q}}\,(n-j+2+\mathfrak{q}) - (n-j)^{\mathfrak{q}}\,(n-j+2+2\mathfrak{q}),$

$Z_2 = (n+1-j)^{\mathfrak{q}+1} - (n-j)^{\mathfrak{q}}\,(n-j+1+\mathfrak{q}).$

Since in the original model (1.3.1) for $\Upsilon_1$ and $\Upsilon_3$, $\Theta$ depends on $\varkappa$ and $(\varkappa-\tau) = \varkappa_1(say)$, so we write

$\Upsilon_1 = U_1(\varkappa_j,\Delta_j,\aleph_j,\Theta_j) + U_3(\varkappa_{1j},\Theta_j)$ and $\Upsilon_3 = U_2(\varkappa_j,\Delta_j,\aleph_j,\Theta_j) - U_3(\varkappa_{1j},\Theta_j)$ where

$U_1(\varkappa_j,\Delta_j,\aleph_j,\Theta_j) = \Pi\theta - \beta_0 f(\aleph(\varkappa))\Delta(\varkappa) - (\mu+\nu)\Delta(\varkappa),$

$U_2(\varkappa_j,\Delta_j,\aleph_j,\Theta_j) = (1-\Pi)\theta + \nu\Delta(\varkappa) + \kappa\aleph(\varkappa) - \mu\Theta(\varkappa),$

$U_3(\varkappa_{1j},\Theta_j) = \zeta\Theta(\varkappa-\tau).$

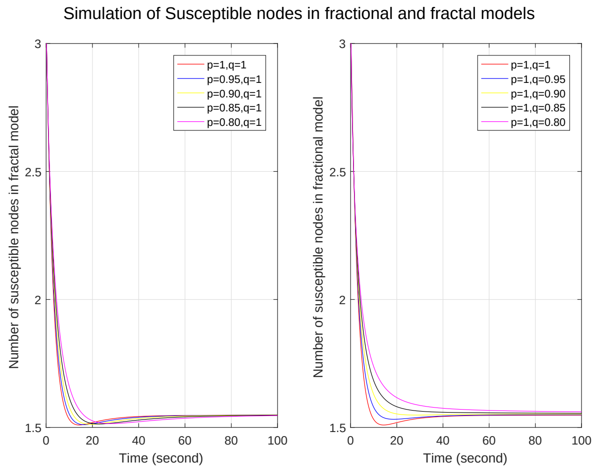


Figure 2.1: Trajectories of $\Delta(\varkappa)$ for different fractal orders $\mathfrak{p}$ when $\mathfrak{q} = 1$ and different fractional orders $\mathfrak{q}$ when $\mathfrak{p} = 1$.

Hence, our numerical scheme is

$$\begin{aligned}\Delta(n+1) = \Delta_0 + \frac{\mathfrak{p}\, h^{\mathfrak{q}}}{\Gamma(\mathfrak{q}+2)} \sum_{j=0}^{n} [\varkappa_j^{\mathfrak{p}-1}(U_1(\varkappa_j, \Delta_j, \aleph_j, \Theta_j) + U_3(\varkappa_{1j}, \Theta_j))\, Z_1 \\ - \varkappa_{j-1}^{\mathfrak{p}-1}\,(U_1(\varkappa_{j-1}, \Delta_{j-1}, \aleph_{j-1}, \Theta_{j-1}) + U_3(\varkappa_{1j}, \Theta_j))\, Z_2], \\ \aleph(n+1) = \aleph_0 + \frac{\mathfrak{p}\, h^{\mathfrak{q}}}{\Gamma(\mathfrak{q}+2)} \sum_{j=0}^{n} [\varkappa_j^{\mathfrak{p}-1}\Upsilon_2(\mathfrak{u}_j, \Delta_j, \aleph_j, \Theta_j)\, Z_1 \\ - \varkappa_{j-1}^{\mathfrak{p}-1}\,\Upsilon_2(\mathfrak{u}_{j-1}, \Delta_{j-1}, \aleph_{j-1}, \Theta_{j-1})\, Z_2], \\ \Theta(n+1) = \Theta_0 + \frac{\mathfrak{p}\, h^{\mathfrak{q}}}{\Gamma(\mathfrak{q}+2)} \sum_{j=0}^{n} [\varkappa_j^{\mathfrak{p}-1}(U_2(\varkappa_j, \Delta_j, \aleph_j, \Theta_j) - U_3(\varkappa_{1j}, \Theta_j))\, Z_1 \\ - \varkappa_{j-1}^{\mathfrak{p}-1}\,(U_2(\varkappa_{j-1}, \Delta_{j-1}, \aleph_{j-1}, \Theta_{j-1}) - U_3(\varkappa_{1j}, \Theta_j))\, Z_2].\end{aligned}$$

## 2.7 Discussion through Simulations based on Numerical algorithm

In this section, we see the simulation of $\Delta$, $\aleph$ and $\Theta$ under the effect of several fractal-fractional orders and also their behavior with respect to some parameters and compare the results of $FF$ model to the ordinary differential model. We take parameters as taken for figure2 in [32], $\Pi = 0.5, \theta = 0.8, \beta_0 = 0.02, \mu = 0.1, \nu = 0.2, \zeta = 0.01, \kappa = 0.2, \tau = 7.3, \alpha = 1$ and some estimated initial conditions $\Delta(0) = 3, \aleph(0) = 1, \Theta(0) = 0.1$.

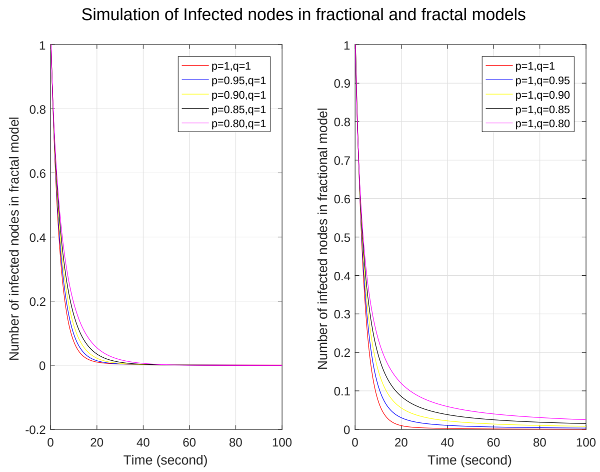


Figure 2.2: Trajectories of $\aleph(\varkappa)$ for different fractal orders $\mathfrak{p}$ when $\mathfrak{q} = 1$ and different fractional orders $\mathfrak{q}$ when $\mathfrak{p} = 1$.

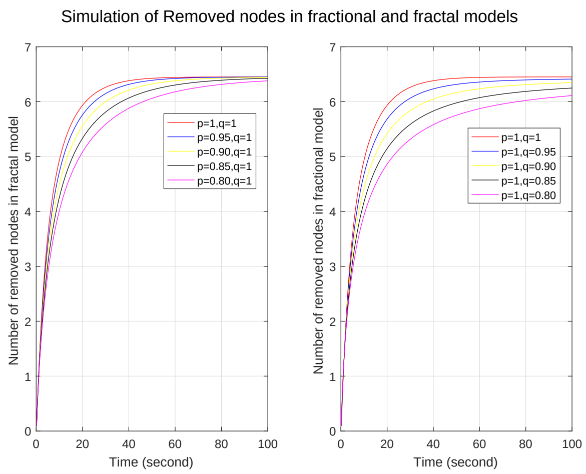


Figure 2.3: Trajectories of $\Theta(\varkappa)$ for different fractal orders $\mathfrak{p}$ when $\mathfrak{q} = 1$ and different fractional orders $\mathfrak{q}$ when $\mathfrak{p} = 1$.

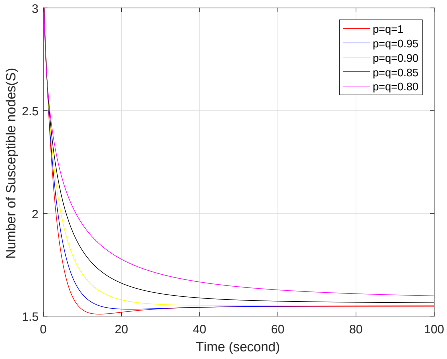


Figure 2.4: Trajectories of $\Delta(\varkappa)$ for different orders of $\mathfrak{p} = \mathfrak{q}$.

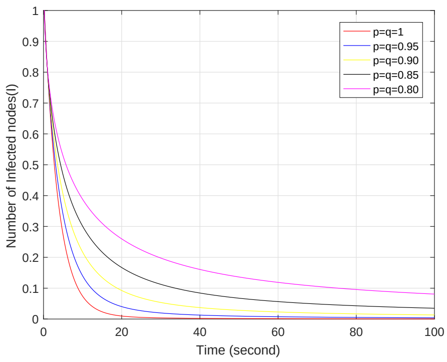


Figure 2.5: Trajectories of $\aleph(\varkappa)$ for different orders of $\mathfrak{p} = \mathfrak{q}$ .

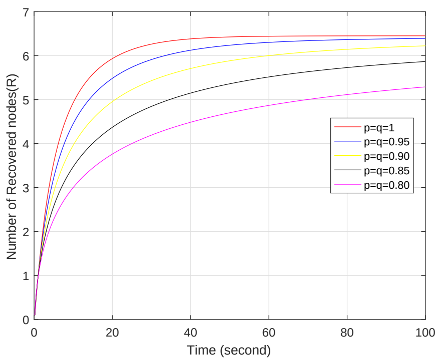


Figure 2.6: Trajectories of $\Theta(\varkappa)$ for different orders of $\mathfrak{p} = \mathfrak{q}$.

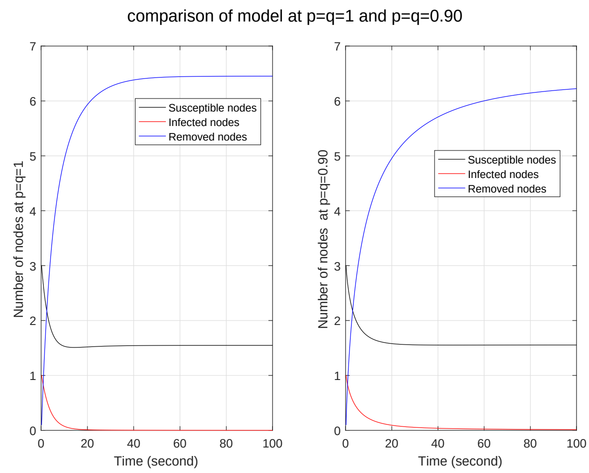


Figure 2.7: Comparison of $\Delta, \aleph,$ and $\Theta$ model for classical and FF model at level 0.90.

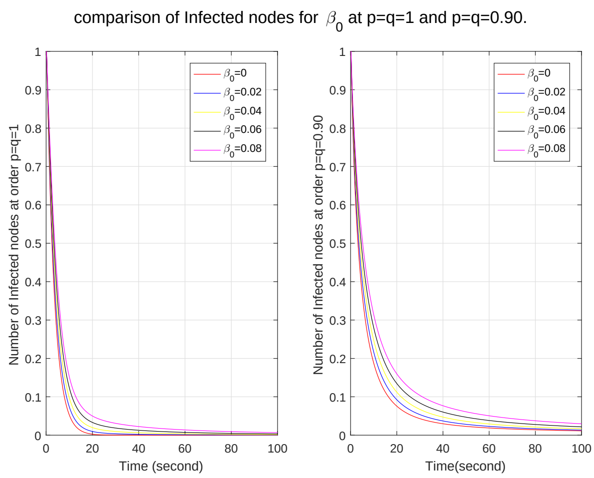


Figure 2.8: The effect of varying initial infection rate $\beta_0$ on infected nodes for classical and FF model at level 0.90.

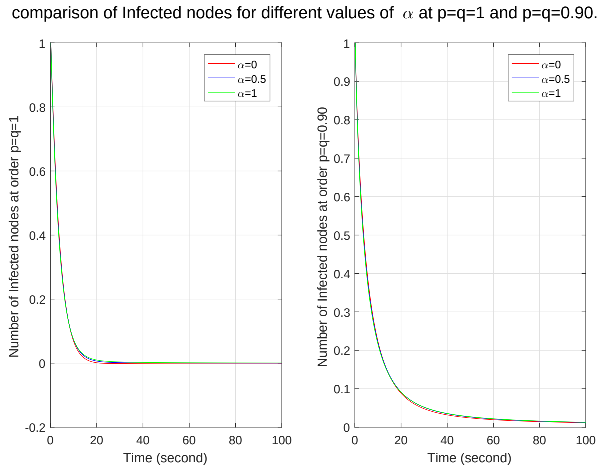


Figure 2.9: The effect of varying variable to adjust the infection rate sensitivity $\alpha$ on infected nodes for classical and FF model at level 0.90.

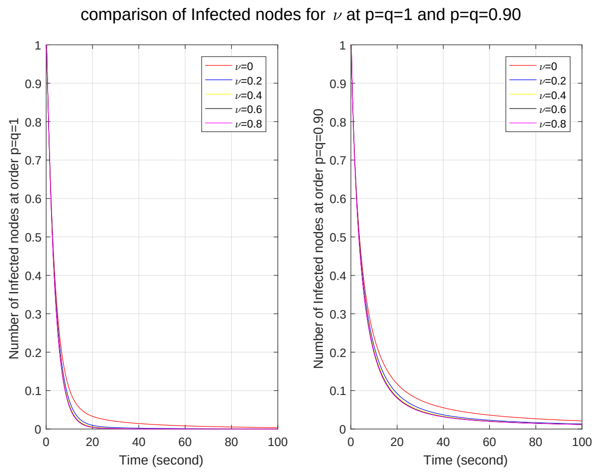


Figure 2.10: The effect of varying real-time immune rate $\nu$ on infected nodes for classical and FF model at level 0.90.

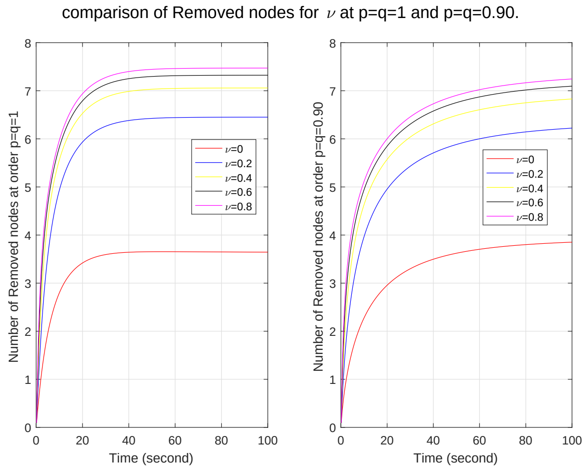


Figure 2.11: The effect of varying real-time immune rate $\nu$ on removed nodes for classical and FF model at level 0.90.

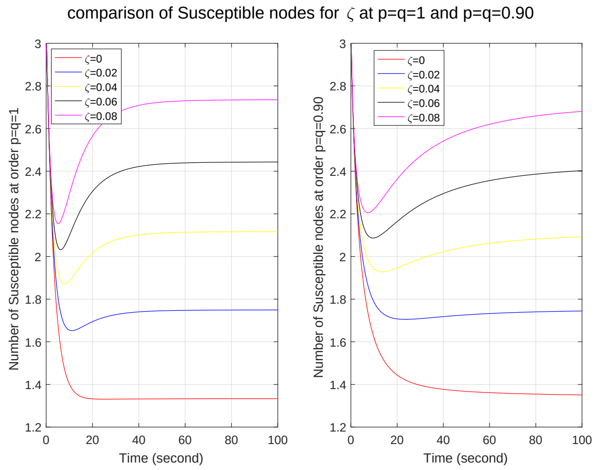


Figure 2.12: The effect of varying loss rate of immunity $\zeta$ on susceptible nodes for classical and FF model at level 0.90.

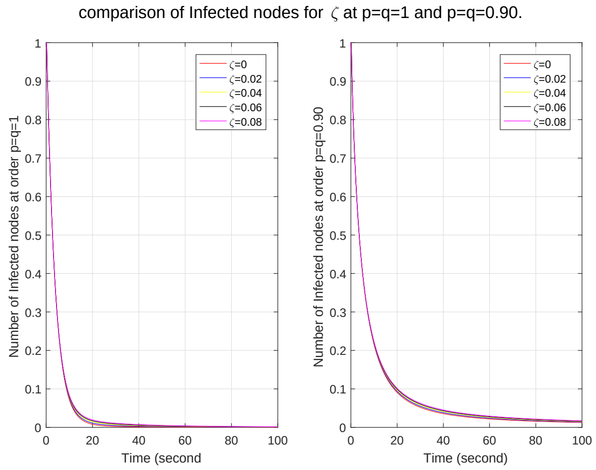


Figure 2.13: The effect of varying loss rate of immunity $\zeta$ on infected nodes for classical and FF model at level 0.90.

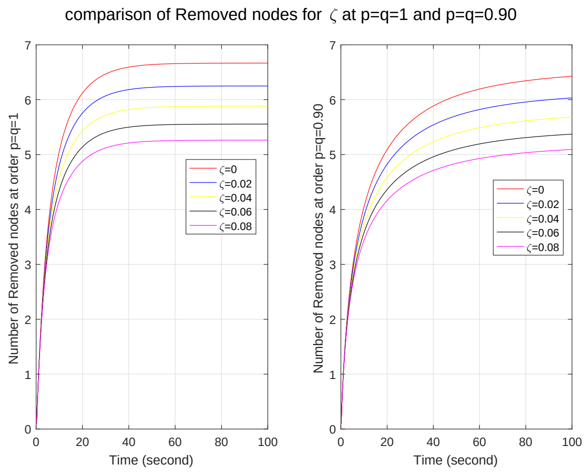


Figure 2.14: The effect of varying loss rate of immunity $\zeta$ on removed nodes for classical and FF model at level 0.90.

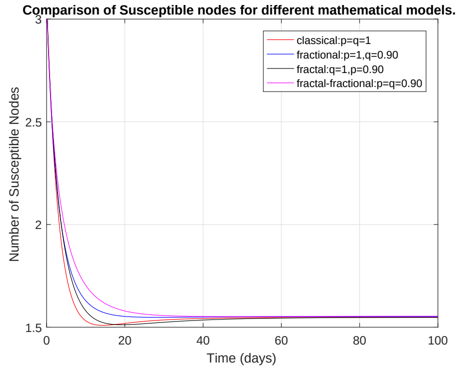


Figure 2.15: Trajectories of $\Delta$ showing comparison of different mathematical models.

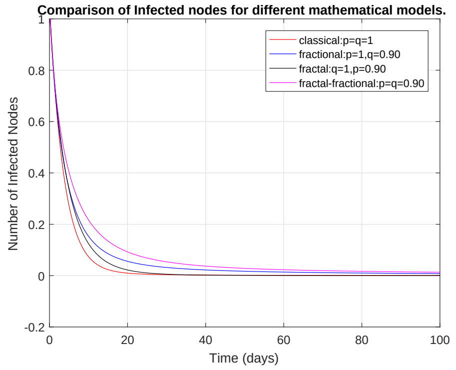


Figure 2.16: Trajectories of $\aleph$ showing comparison of different mathematical models.

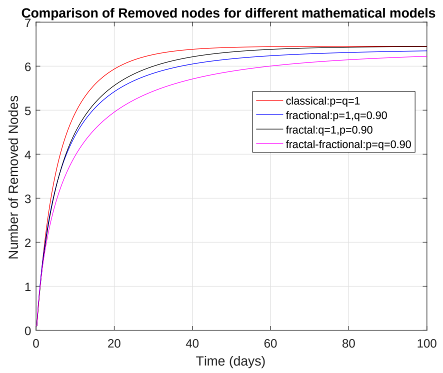


Figure 2.17: Trajectories of $\Theta$ showing comparison of different mathematical models.

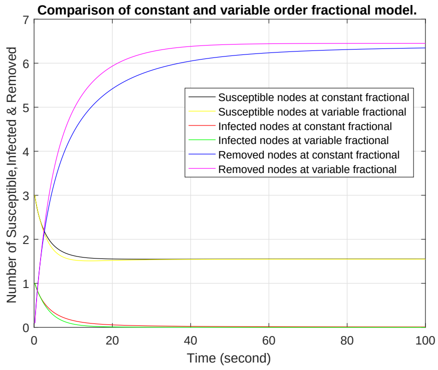


Figure 2.18: Trajectories of $\Delta$, $\aleph$ and $\Theta$ showing comparison of constant and variable fractional mathematical models.

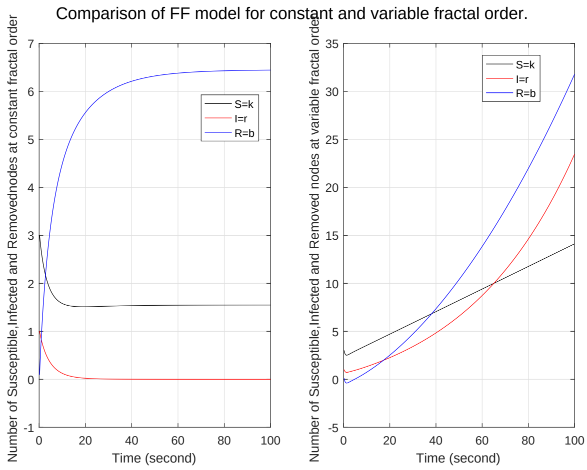


Figure 2.19: Trajectories of $\Delta$, $\aleph$ and $\Theta$ showing comparison of constant and variable fractal order mathematical models.

Figures 1-3 show the simulation of $\Delta$, $\aleph$ and $\Theta$ for different fractal and fractional orders separately. We can see in Fig.1, the simulation of $\Delta$ in one figure under the different fractional orders keeping fractal order one and in the other figure under different fractal orders by keeping fractional order one. We see that initially number of susceptible nodes is more at lower fractional and fractal orders. In fractal model, after 20s, this number becomes less at lower fractal order and then converges after 80s. On the other hand, in fractional model the no. of susceptible nodes is more at lower fractional orders throughout the time period and it converges slowly as compared to fractal model. So, we can see that in fractal model nodes converge more quickly that means no. of susceptible nodes is less affected by previous nodes. So we can predict that it has less memory effect, less sensitivity and more stability. Similarly Fig.2 is for $\aleph$. It represents higher number of nodes of infected nodes at lower fractal or fractional orders in both models. However, the convergence in fractal model is rapid, so we can conclude that it has less memory effect, less sensitivity and more stability. Fig.3 is for $\Theta$. In this figure, in both models the number of removed nodes is less at lower fractional or fractal orders but in fractal model the number of nodes is more than the number of nodes in fractional model at different orders. The nodes become stable more quickly as compared to fractional model. The fractal model has less memory effect, low sensitivity to initial conditions and limited information retention. It shows the uniqueness of the solution and existence of fixed point. Figs.4-6 show the simulation of $\Delta$, $\aleph$ and $\Theta$ under the combined effect of arbitrary fractal and fractional orders. Fig.4 shows that as $FF$ orders decrease, the number of susceptible nodes is higher at lower $FF$ orders and after some time it becomes stable rapidly and converges to the same limit except at order 0.80. This behavior depicts that these nodes have a higher risk of infection or perturbation. They are more sensitive to external influences, have strong memory effect and rapid stabilization. Fig.5 represents the behavior of infected nodes. The number of infected nodes first decreases then becomes stable at all fractal fractional orders. It depicts that the system has resilience to adapt infection where some nodes are still resistant and preventing further infection. Also, the number of nodes becomes zero at fractal fractional level one which describes that infection has been eradicated while at lower level of fractal fraction when no. of nodes goes on decreases but does not become zero show that the system has not been fully eradicated but also has the effect of memory which contributes to the persistence of infected nodes. In Fig.6, we see that number of removed nodes reduces as we reduce the fractal-fractional orders which are represented by $\mathfrak{p}$ and $\mathfrak{q}$. It describes that the memory effect is less in these nodes and these nodes have high stability and fast convergence. It depicts that at lower level of fractal fraction the persistence and containment of infection is high.

Moreover, in Fig.7 we compare $\Delta, \aleph$ and $\Theta$ model for fractal-fractional orders $\mathfrak{p}, \mathfrak{q}$ as $\mathfrak{p} = \mathfrak{q} = 1$ and $\mathfrak{p} = \mathfrak{q} = 0.90$ which shows behavior of three nodes in one figure. We can see from the comparison of nodes that at lower level of fractal fractional, memory effect is stronger and has more influence of initial conditions and previous nodes.

Since in our original model, infection rate depends on initial infection rate $\beta_0$ and also on a function of $\aleph$ which depends on another variable called $\alpha$. Therefore we see the impact of both variables on our fractal-fractional model too. Fig.8, first we see that in both models no. of infected nodes goes on increasing for higher initial infection rate i.e. as initial infection rate increases, no. of infected nodes also increases. It describes that outbreak is escalating and infected nodes are increasing rapidly. On the other hand, when we compare both models it shows that at lower fractal fractional order the malware spreads faster and has stronger memory effect than the integer order model.

Also in the original model, $\alpha$ is used to adjust the infection rate sensitivity to $\aleph$. Here $\alpha = 0$ means constant infection rate. According to the authors in [32], at $\alpha = 1$, the scale of malware spreading is smaller than the rate at $\alpha = 0$. From fig.9 we see that this condition is satisfied in both models. Also we observe that no. of infected nodes goes on decreasing more quickly as time passes in FF model as compared to integer order model. As a result, we can see that FF model has weaker memory effect and more stability and resilience to outbreaks in the nodes.

In Fig.10, we see the effect of real-time immune rate on $\aleph$ for classical and FF model. We see as we increase the immune rate, the number of infected nodes goes on decreasing in both models. As a comparison of both models, we see that no. of infected nodes is greater in FF model, which tells that in FF model nodes have stronger memory effect that enhances its sensitivity to network structure. Similarly Fig.11 shows that as the immune rate increases, no. of recovered nodes also increases in both models but in FF model this number is less than in classical model. It describes weak memory effect and decreased sensitivity in FF model.

Moreover, the recovered nodes lose their immunity after some time, so to see this impact we check the graphs of $\Delta, \aleph$ and $\Theta$. We see in Fig.12 that in both models as the loss rate of immunity is increasing, no. of susceptible nodes is also increasing that describes weaker memory effect and higher sensitivity but in comparison FF model has stronger memory effect than classical one except at zero value.

In Fig.13, we see the effect of loss rate of immunity on $\aleph$ for both models. We see as loss rate of immunity increases, the number of infected nodes goes on decreasing in both models which indicates stronger memory effect. As a comparison of both models, no. of infected nodes is greater in FF model, which tells that in FF model nodes have weaker memory effect that decrease its resilience to network structure. In Fig.14 as loss rate of immunity increases, the no. of removed nodes is also increases, it indicates that immune response is effective and has strong memory effect. When we compare both models, we see that no of removed nodes is less in FF model. Hence, we say that these nodes have weaker memory effect in FF model as compared to classical.

In Figs.15-17, we compare four mathematical models named classical, fractional, fractal and fractal-fractional. From Figs.15,16, we see that no. of susceptible and infected nodes is highest in FF model, then in fractional, fractal and classical simultaneously. It shows that fractal-fractional is more effective for expressing the complexity of malware propagation and fractional model may also be used for some types of networks. On the other hand, fractal and classical methods are not suitable for complex systems. The higher no. of nodes represents deeper memory effect and strong correlation between nodes. Moreover, convergence indicates the system is stable. Similarly in Fig 17, no. of removed nodes in fractal fractional model is lowest that show deep memory effect and strong correlation and convergence shows stability of the system. Fig.18 represents the comparison of constant order and variable order fractional mathematical models. We take variable fractional order as $\mathfrak{q} = 0.90 + 0.1/(1 + exp(-\varkappa))$. The behavior of nodes represents that constant order fractional model is more complex and probably more accurate in modeling malware propagation. Fig.19 represents the comparison of constant order and variable order fractal mathematical model. We take variable fractal order as $\mathfrak{p} = 0.1/0.9 + exp(-\varkappa)$. We can see that in variable fractal order model, the nodes do not converge and goes on increasing that means no stability in this case.

## 2.8 Conclusion

In this chapter, a deterministic mathematical model on malware propagation has been discussed in the sense of fractal fractional derivative. At first stage, the classical mathematical model given in [32] has been converted in fractal fractional model with power law kernel. Initially this model was examined theoretically. Conditions for existence (Leray Schauder criteria), uniqueness (Lipschitz property) and stability (Ulam-Hyers and Ulam-Hyers-Rassias theorems) of the fractal fractional model were examined using concepts of fixed point theory. Then, numerical scheme was developed and simulations were performed to verify the accuracy of theoretical results.

At second stage, FF model was examined under fractal dimensions and fractional orders separately. Then combined effect of fractal dimensions and fractional orders was discussed. We observed that at lower FF orders, the number of susceptible and infected nodes was higher. It demonstrates the sensitivity to external influences, resilience to adapt infection and strong memory effects. Under combined effect, we found out that removed nodes have higher containment of infection and persistence at lower level of FF orders.

At the next stage, we compared model for classical (FF order at one) and fractal fractional model for order $\mathfrak{p} = \mathfrak{q} = 0.90$. We examined the impact of different parameters such as initial infection rate, variable adjustment to sensitivity of infected nodes, immune rate of antivirus strategies and loss rate of immunity of recovered nodes of mathematical model [32] under $\mathfrak{p} = \mathfrak{q} = 1$ and $\mathfrak{p} = \mathfrak{q} = 0.90$.

We also compared four mathematical models named classical, fractional, fractal and fractal-fractional. Also we observed the behavior of nodes for constant order and variable order fractional and fractal mathematical models. Through the graphs, we find out the effect of memory on different types of nodes in system. We explored sensitivity, convergence, and stability of susceptible, infected, and removed nodes under fractal fractional model.

# Chapter 3

# Fractal Fractional Mathematical Model With Exponential Decay Kernel

In this chapter, we are converting classical model into Exponential Decay kernel FF mathematical model and then in fixed point problem. We check existence, uniqueness and stability by theorems of fixed point theory. Then, we make graphs on Matlab using code made by two-point Lagrange interpolation formula and simulate the data. After analysis we conclude the results.

## 3.1 Conversion in FF Mathematical Model With Exponential Decay Kernel

Now, we convert the model (1.3.2) in terms of fractal fractional derivatives with Exponential decay kernel as:

$$\begin{aligned}
{}^{FFE}D_{0,\varkappa}^{\mathfrak{q},\mathfrak{p}}\Delta(\varkappa) &= \Pi\theta - \beta_0 f(\aleph(\varkappa))\Delta(\varkappa) - (\mu+\nu)\Delta(\varkappa) + \zeta\Theta(\varkappa-\tau),\\
{}^{FFE}D_{0,\varkappa}^{\mathfrak{q},\mathfrak{p}}\aleph(\varkappa) &= \beta_0 f(\aleph(\varkappa))\Delta(\varkappa) - (\mu+\kappa)\aleph(\varkappa),\\
{}^{FFE}D_{0,\varkappa}^{\mathfrak{q},\mathfrak{p}}\Theta(\varkappa) &= (1-\Pi)\theta + \nu\Delta(\varkappa) + \kappa\aleph(\varkappa) - \zeta\Theta(\varkappa-\tau) - \mu\Theta(\varkappa),
\end{aligned} \tag{3.1.1}$$

with $\Delta(0) = \Delta_0 \geq 0, \aleph(0) = \aleph_0 \geq 0, \Theta(0) = \Theta_0 \geq 0$ and $N(\varkappa) = \Delta(\varkappa)+\aleph(\varkappa)+\Theta(\varkappa)$, for $\varkappa \in \mathrm{J} = [0,\mathrm{T}], \mathrm{T} > 0$. Also $\mathfrak{p}$ , $\mathfrak{q} \in (0,1]$ and all parameters are to be taken non-negative.

## 3.2 Fractal Fractional Model as Fixed Point Problem

Now, we convert mathematical model (3.1.1) in fixed point problem. We apply results of fixed point theory on model (3.1.1).
Consider $\Xi = \mathfrak{Y}^3$, a Banach space and $\mathfrak{Y} = C(\mathrm{J}, \mathbb{R})$ represents the class of all continuous functions with the norm defined as:

$$\|F\|_{\Xi} = \|(\Delta, \aleph, \Theta)\|_{\Xi} = max\{|\Delta(\varkappa)| + |\aleph(\varkappa)| + |\Theta(\varkappa)| : \varkappa \in \mathrm{J}\}.$$

First, we rewrite model (1.3.1) as:

$$\begin{aligned}
&\Upsilon_1(\varkappa, \Delta(\varkappa), \aleph(\varkappa), \Theta(\varkappa)) = \Pi\theta - \beta_0 f(\aleph(\varkappa))\Delta(\varkappa) - (\mu + \nu)\Delta(\varkappa) + \zeta\Theta(\varkappa - \tau),\\
&\Upsilon_2(\varkappa, \Delta(\varkappa), \aleph(\varkappa), \Theta(\varkappa)) = \beta_0 f(\aleph(\varkappa))\Delta(\varkappa) - (\mu + \kappa)\aleph(\varkappa),\\
&\Upsilon_3(\varkappa, \Delta(\varkappa), \aleph(\varkappa), \Theta(\varkappa)) = (1 - \Pi)\theta + \nu\Delta(\varkappa) + \kappa\aleph(\varkappa) - \zeta\Theta(\varkappa - \tau) - \mu\Theta(\varkappa).
\end{aligned} \tag{3.2.1}$$

Comparing models (1.3.1) and (3.2.1), we have

$$\begin{aligned}
{}^{FFE}D_{0,\varkappa}^{\mathfrak{q},\mathfrak{p}}\Delta(\varkappa) &= \Upsilon_1(\varkappa, \Delta(\varkappa), \aleph(\varkappa), \Theta(\varkappa)),\\
{}^{FFE}D_{0,\varkappa}^{\mathfrak{q},\mathfrak{p}}\aleph(\varkappa) &= \Upsilon_2(\varkappa, \Delta(\varkappa), \aleph(\varkappa), \Theta(\varkappa)),\\
{}^{FFE}D_{0,\varkappa}^{\mathfrak{q},\mathfrak{p}}\Theta(\varkappa) &= \Upsilon_3(\varkappa, \Delta(\varkappa), \aleph(\varkappa), \Theta(\varkappa)).
\end{aligned} \tag{3.2.2}$$

Since

$$\begin{aligned}
{}^{FFE}D_{a,\varkappa}^{\mathfrak{q},\mathfrak{p}}g(\varkappa) &= \frac{M(\mathfrak{q})}{\Gamma(1-\mathfrak{q})}\frac{d}{d\varkappa^{\mathfrak{p}}}\int_a^{\varkappa} exp[\frac{-\mathfrak{q}}{1-\mathfrak{q}}(\varkappa - u)]\, g(u)du.\\
{}^{FFE}D_{a,\varkappa}^{\mathfrak{q},\mathfrak{p}}g(\varkappa) &= \frac{M(\mathfrak{q})}{\Gamma(1-\mathfrak{q})}\frac{1}{\mathfrak{p}\varkappa^{\mathfrak{p}-1}}\frac{d}{d\varkappa}\int_a^{\varkappa} exp[\frac{-\mathfrak{q}}{1-\mathfrak{q}}(\varkappa - u)]\, g(u)du.\\
{}^{FFE}D_{a,\varkappa}^{\mathfrak{q},\mathfrak{p}}g(\varkappa) &= \frac{1}{\mathfrak{p}\varkappa^{\mathfrak{p}-1}}\frac{M(\mathfrak{q})}{\Gamma(1-\mathfrak{q})}\frac{d}{d\varkappa}\int_a^{\varkappa} exp[\frac{-\mathfrak{q}}{1-\mathfrak{q}}(\varkappa - u)]\, g(u)du.
\end{aligned}$$

Now, we can write

$$\frac{M(\mathfrak{q})}{\Gamma(1-\mathfrak{q})}\frac{d}{d\varkappa}\int_a^{\varkappa} exp[\frac{-\mathfrak{q}}{1-\mathfrak{q}}(\varkappa - u)]\, g(u)du$$

as Riemann Liouville fractional derivative with exponential decay kernel.
Therefore, we get

$${}^{FFE}D_{0,\varkappa}^{\mathfrak{q},\mathfrak{p}}\Upsilon(\varkappa) = (\frac{1}{\mathfrak{p}\varkappa^{\mathfrak{p}-1}})^{RL}D_{0,\varkappa}^{\mathfrak{q}}\Upsilon(\varkappa)$$

So model (3.2.2) can be written as:

$$
\begin{aligned}
(\frac{1}{\mathfrak{p}\varkappa^{\mathfrak{p}-1}})^{RL}D^{\mathfrak{q}}_{0,\varkappa}\Delta(\varkappa) &= \Upsilon_1(\varkappa,\Delta(\varkappa),\aleph(\varkappa),\Theta(\varkappa)),\\
(\frac{1}{\mathfrak{p}\varkappa^{\mathfrak{p}-1}})^{RL}D^{\mathfrak{q}}_{0,\varkappa}\aleph(\varkappa) &= \Upsilon_2(\varkappa,\Delta(\varkappa),\aleph(\varkappa),\Theta(\varkappa)),\\
(\frac{1}{\mathfrak{p}\varkappa^{\mathfrak{p}-1}})^{RL}D^{\mathfrak{q}}_{0,\varkappa}R(\varkappa) &= \Upsilon_3(\varkappa,\Delta(\varkappa),\aleph(\varkappa),\Theta(\varkappa)).
\end{aligned} \tag{3.2.3}
$$

Hence, we get

$$
\begin{aligned}
{}^{RL}D^{\mathfrak{q}}_{0,\varkappa}\Delta(\varkappa) &= \mathfrak{p}\varkappa^{\mathfrak{p}-1}\Upsilon_1(\varkappa,\Delta(\varkappa),\aleph(\varkappa),\Theta(\varkappa)),\\
{}^{RL}D^{\mathfrak{q}}_{0,\varkappa}\aleph(\varkappa) &= \mathfrak{p}\varkappa^{\mathfrak{p}-1}\Upsilon_2(\varkappa,\Delta(\varkappa),\aleph(\varkappa),\Theta(\varkappa)),\\
{}^{RL}D^{\mathfrak{q}}_{0,\varkappa}R(\varkappa) &= \mathfrak{p}\varkappa^{\mathfrak{p}-1}\Upsilon_3(\varkappa,\Delta(\varkappa),\aleph(\varkappa),\Theta(\varkappa)).
\end{aligned} \tag{3.2.4}
$$

In general, we can write model (3.2.4) as:

$$
\begin{aligned}
{}^{RL}D^{\mathfrak{q}}_{0,\varkappa}F(\varkappa) &= \mathfrak{p}\varkappa^{\mathfrak{p}-1}\Upsilon(\varkappa,F(\varkappa)),\\
F(0) &= F_0,
\end{aligned} \tag{3.2.5}
$$

where

$$
\begin{aligned}
&(\mathfrak{p},\mathfrak{q}) \in (0,1],\\
&\varkappa \in \mathrm{J},\\
F(\varkappa) &= (\Delta(\varkappa),\aleph(\varkappa),\Theta(\varkappa))^{\top},\\
F_0 &= (\Delta_0,\aleph_0,\Theta_0)^{\top}.
\end{aligned}
$$

Applying Fractal−Fractional integral on Model (3.2.5), using the result in [15], we have

$$
F(\varkappa) = F(0) + \frac{\mathfrak{p}\,(1-\mathfrak{q})\,\varkappa^{(\mathfrak{p}-1)}\,\Upsilon(\varkappa,F(\varkappa))}{M(\mathfrak{q})} + \frac{\mathfrak{p}\,\mathfrak{q}}{M(\mathfrak{q})}\int_0^{\varkappa} u^{\mathfrak{p}-1}\,\Upsilon(u,F(u))du, \tag{3.2.6}
$$

where $M(0) = M(1) = 1$.

Hence, we can write

$$
\begin{aligned}
\Delta(\varkappa) &= \Delta(0) + \frac{\mathfrak{p}\,(1-\mathfrak{q})\,\varkappa^{(\mathfrak{p}-1)}\,\Upsilon_1(\varkappa,\Delta(\varkappa),\aleph(\varkappa),\Theta(\varkappa))}{M(\mathfrak{q})} + \frac{\mathfrak{p}\,\mathfrak{q}}{M(\mathfrak{q})}\int_0^{\varkappa} u^{\mathfrak{p}-1}\,\Upsilon_1(u,F(u))du,\\
\aleph(\varkappa) &= \aleph(0) + \frac{\mathfrak{p}\,(1-\mathfrak{q})\,\varkappa^{(\mathfrak{p}-1)}\,\Upsilon_2(\varkappa,\Delta(\varkappa),\aleph(\varkappa),\Theta(\varkappa))}{M(\mathfrak{q})} + \frac{\mathfrak{p}\,\mathfrak{q}}{M(\mathfrak{q})}\int_0^{\varkappa} u^{\mathfrak{p}-1}\,\Upsilon_2(u,F(u))du,\\
\Theta(\varkappa) &= \Theta(0) + \frac{\mathfrak{p}\,(1-\mathfrak{q})\,\varkappa^{(\mathfrak{p}-1)}\,\Upsilon_3(\varkappa,\Delta(\varkappa),\aleph(\varkappa),\Theta(\varkappa))}{M(\mathfrak{q})} + \frac{\mathfrak{p}\,\mathfrak{q}}{M(\mathfrak{q})}\int_0^{\varkappa} u^{\mathfrak{p}-1}\,\Upsilon_3(u,F(u))du.
\end{aligned} \tag{3.2.7}
$$

So, now we can transform model (3.1.1) into a fixed point problem.

Define an operator $F\colon \Xi \to \Xi$ as

$$F(\digamma(\varkappa)) = \digamma(0) + \frac{\mathfrak{p}\,(1-\mathfrak{q})\,\varkappa^{(\mathfrak{p}-1)}\,\Upsilon(\varkappa)}{M(\mathfrak{q})} + \frac{\mathfrak{p}\,\mathfrak{q}}{M(\mathfrak{q})}\int_0^{\varkappa} u^{\mathfrak{p}-1}\,\Upsilon(u,\digamma(u))du. \qquad (3.2.8)$$

## 3.3 Existence of Solution

For existence, we prove the following theorem on the basis of Theorem 1.2.4 as in [35].

**Theorem 3.3.1.** Suppose that $\exists\, V_1\colon \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R},\ \psi \in \Psi$ and $\Upsilon \in C(\mathrm{J}\times\Xi, \Xi)$ satisfying the following three conditions:

$(\beta_6)$: $\forall \digamma_1, \digamma_2 \in \Xi$ and $\varkappa \in \mathrm{J}$,

$|\Upsilon(\varkappa, \digamma_1(\varkappa)) - \Upsilon(\varkappa, \digamma_2)(\varkappa)| \leq \ell_1\,\psi(|\digamma_1(\varkappa) - \digamma_2(\varkappa)|)$, with $V_1(\digamma_1(\varkappa), \digamma_2(\varkappa)) \geq 0$ and $\ell_1 = \frac{M(\mathfrak{q})}{p\,(1-\mathfrak{q})\,\varkappa^{p-1} + \mathfrak{q}\,\varkappa^{p}}$.

$(\beta_7)$: $\exists\, \digamma_0 \in \Xi$ and $\forall\, \varkappa \in \mathrm{J}$,

$V_1(\digamma_0(\varkappa), F(\digamma_0(\varkappa)) \geq 0$ and $V_1(\digamma_1(\varkappa), \digamma_2(\varkappa)) \geq 0$.

$\implies V_1(F(\digamma_1(\varkappa)), F(\digamma_2(\varkappa)) \geq 0$;

$(\beta_8)$: $\forall \{\digamma_n\}_{n\geq 1} \subseteq \Xi$ with $\digamma_n \to \digamma$,

$V_1(\digamma_n(\varkappa), \digamma_{n+1}(\varkappa)) \geq 0 \implies V_1(\digamma_n(\varkappa)), \digamma(\varkappa)) \geq 0$, for every $n$ , $\varkappa \in \mathrm{J}$.

Then, we say that $F$ has a fixed point. So, there exists a solution of the model of malware propagation.

*Proof.* Take $\digamma_1, \digamma_2 \in \Xi$ so that

$$V_1(\digamma_1(\varkappa), \digamma_2(\varkappa)) \geq 0,$$

for each $\varkappa \in \mathrm{J}$.

Now, we take

$$\begin{aligned} |F(\digamma_1(\varkappa)) - F(\digamma_2(\varkappa))| &= |\frac{\mathfrak{p}\,(1-\mathfrak{q})\,\varkappa^{\mathfrak{p}-1}\,[\Upsilon(u, \digamma_1(u)) - \Upsilon(u, \digamma_2(u))]}{M(\mathfrak{q})} \\ &+ \frac{\mathfrak{p}\,\mathfrak{q}}{M(\mathfrak{q})}\int_0^{\varkappa} u^{(\mathfrak{p}-1)}\,[\Upsilon(u, \digamma_1(u)) - \Upsilon(u, \digamma_2(u))]\,du| \end{aligned}$$

$$\begin{aligned} |F(\digamma_1(\varkappa)) - F(\digamma_2(\varkappa))| &\leq \frac{\mathfrak{p}\,(1-\mathfrak{q})\,\varkappa^{\mathfrak{p}-1}\,|\Upsilon(u, \digamma_1(u)) - \Upsilon(u, \digamma_2(u))|}{M(\mathfrak{q})} \\ &+ \frac{\mathfrak{p}\,\mathfrak{q}}{M(\mathfrak{q})}\int_0^{t} u^{(\mathfrak{p}-1)}\,|\Upsilon(u, \digamma_1(u)) - \Upsilon(u, \digamma_2(u))|\,du. \end{aligned}$$

Utilizing ($\beta_6$), we deduce

$$\begin{aligned} |F(\digamma_1(\varkappa)) - F(\digamma_2(\varkappa))| &\leq \frac{\mathfrak{p}\,(1-\mathfrak{q})\,\varkappa^{\mathfrak{p}-1}}{M(\mathfrak{q})}\,\ell_1\psi(|\digamma_1(u) - \digamma_2(u)|) \\ &+ \frac{p\,\mathfrak{q}}{M(\mathfrak{q})}\int_0^{\varkappa} u^{(\mathfrak{p}-1)}\,\ell_1\psi(|\digamma_1(u) - \digamma_2(u)|)\,du. \end{aligned}$$

Now, using definition of norm

$$\begin{aligned} |F(\digamma_1(\varkappa)) - F(\digamma_2(\varkappa))| &\leq \frac{\mathfrak{p}\,(1-\mathfrak{q})\,\varkappa^{\mathfrak{p}-1}}{M(\mathfrak{q})}\,\ell_1\psi(||\digamma_1 - \digamma_2||_{\Xi}) \\ &+ \frac{\mathfrak{p}\,\mathfrak{q}}{M(\mathfrak{q})}\int_0^{\varkappa} u^{(\mathfrak{p}-1)}\,\ell_1\psi(||\digamma_1 - \digamma_2||_{\Xi})\,du. \end{aligned}$$

After doing some computations and using the definition of beta function and $\ell_1$, we get

$$|F(\digamma_1(\varkappa)) - F(\digamma_2(\varkappa))| \leq \psi(||\digamma_1 - \digamma_2||_{\Xi}). \tag{3.3.1}$$

We can write it as:

$$d(F(\digamma_1), F(\digamma_2)) \leq \psi(d(\digamma_1, \digamma_2)). \tag{3.3.2}$$

Moreover, if we define a function $\phi$: $\Xi^2 \to [0, \infty)$ such that
$\phi(\digamma_1, \digamma_2) = 1$ for $V(\digamma_1(\varkappa), \digamma_2(\varkappa)) \geq 0$, and zero otherwise;
then, for each $\digamma_1, \digamma_2 \in \Xi$ equation (3.3.2) can be written as:

$$\phi(\digamma_1, \digamma_2)\, d(F(\digamma_1), F(\digamma_2)) \leq \psi(d(\digamma_1, \digamma_2)). \tag{3.3.3}$$

This shows that $F$ is a *$\phi$-$\psi$-contraction*.
Now, suppose that $\digamma_1, \digamma_2 \in \Xi$ with the property that $\phi(\digamma_1, \digamma_2) \geq 1$.
By the definition of $\phi$, we deduce

$$V_1(\digamma_1(\varkappa), \digamma_2(\varkappa)) \geq 0, \tag{3.3.4}$$

and by ($\beta_7$)
$V_1(\digamma_0(\varkappa), F(\digamma_0(\varkappa))) \geq 0$ and $V_1(\digamma_1(\varkappa), \digamma_2(\varkappa)) \geq 0$.
$\Longrightarrow\ V_1(F(\digamma_1(\varkappa)), F(\digamma_2(\varkappa))) \geq 0$.
So, by applying definition of $\phi$, we have

$$\phi(F(\digamma_1), F(\digamma_2)) \geq 1. \tag{3.3.5}$$

Hence, $F$ is $\phi$-*admissible.* (*)
Moreover, by $(\beta_7)$, it can be seen that for some $\digamma_0$ in $\Xi, \forall \varkappa \in$ J, we have
$V_1(\digamma_0(\varkappa), F(\digamma_0(\varkappa))) \geq 0 \implies \phi(\digamma_0, F(\digamma_0)) \geq 1.$ (**)
Now, consider $\{\digamma_n\}_{n\geq 1} \subseteq \Xi$ with $\digamma_n \to \digamma$ and for all $n$ and $\phi(\digamma_n, \digamma_{n+1}) \geq 1$.
By definition of $\phi$ this implies $V_1(\digamma_n(\varkappa), \digamma_{n+1}(\varkappa)) \geq 0$.
Thus, by $(\beta_8)$ this implies $V_1(\digamma_n(\varkappa), \digamma(\varkappa)) \geq 0$.
Hence, $\phi(\digamma_n, \digamma) \geq 1$ for all $n$. (***)
Now (*),(**),(***) show that the conditions of Theorem 1.2.4 are satisfied, so we can say that there exists some $\digamma^* \in \Xi$ such that $F(\digamma^*) = \digamma^*$.
So $\digamma^* = (\Delta^*, \aleph^*, \Theta^*)^\top$ is a solution of our model. □

Theorem 1.2.5 also establishes that solution of model exists and on basis of this model we also define the following theorem as:

**Theorem 3.3.2.** Let $\Xi$ be a Banach space, $\mathfrak{N}_{1\epsilon}$ be a bounded and closed set in $\Xi$ and $A_1$ be an open in $\mathfrak{N}_{1\epsilon}$ with $0 \in A_1$, then there exists a compact and continuous operator $F$ with the conditions $(\beta_9)$ and $(\beta_{10})$ from $\overline{A_1} \to \mathfrak{N}_{1\epsilon}$ which satisfies one of the two conditions,
(a) $F$ has a fixed point in $\overline{A_1}$,
or
(b) there exists $\digamma \in \partial A_1$ and $\omega_1 \in (0, 1)$ s.t $\digamma = \omega_1 F(\digamma)$;
where
$(\beta_9)$: Suppose $\Upsilon \in C(\mathrm{J}\times\Xi, \Xi)$ and there exists $\phi \in L^1(\mathrm{J}, [0, \infty))$ and $B_1 \in C([0, \infty), [0, \infty))$ where $B_1$ is an increasing function satisfying the condition $|\mathfrak{F}(\varkappa, \digamma(\varkappa))| \leq \phi(\varkappa)\, B_1(|\digamma(\varkappa)|)$, for all $\varkappa \in$ J and $\digamma \in \Xi$;
$(\beta_5)$: If $\phi^* = sup_{\varkappa\in J}|\phi(\varkappa)|$ then $\exists$ a number $r_1$ s.t $\frac{r_1}{\digamma_0+\lambda_1\, \phi^*\, B_1(r_1)} > 1$ where $\lambda_1 = \frac{\mathfrak{p}\,(1-\mathfrak{q})\, \varkappa^{\mathfrak{p}-1} + \mathfrak{q}\, \varkappa^{\mathfrak{p}}}{M(\mathfrak{q})}$.
If above conditions hold, then a solution exists for our model.

*Proof.* Consider $F\colon \Xi \to \Xi$ as:

$$F(\digamma(\varkappa)) = \digamma(0) + \frac{\mathfrak{p}\,(1-\mathfrak{q})\, \varkappa^{(\mathfrak{p}-1)}\, \Upsilon(\varkappa, \digamma(\varkappa))}{M(\mathfrak{q})} + \frac{\mathfrak{p}\,\mathfrak{q}}{M(\mathfrak{q})} \int_0^{\varkappa} u^{\mathfrak{p}-1}\, \Upsilon(u, \digamma(u))du,$$

and $\mathfrak{N}_{1\epsilon} = \{\digamma \in \Xi : ||\digamma||_\Xi \leq \epsilon\}$ for some positive $\epsilon$.
We show that $F$ is compact on $(\mathfrak{N}_1)_\epsilon$. For this, we prove that $F$ is uniformly bounded

and equicontinuous.

Since $\Upsilon$ is continuous this implies $F$ is continuous.

Now for $\digamma$ in $(\mathfrak{N}_1)_\epsilon$, we obtain

$$|F(\digamma(\varkappa))| \leq |\digamma(0)| + \frac{\mathfrak{p}\,(1-\mathfrak{q})\,\varkappa^{(\mathfrak{p}-1)}\,|\Upsilon(\varkappa,\digamma(\varkappa))|}{M(\mathfrak{q})} + \frac{\mathfrak{p}\,\mathfrak{q}}{M(\mathfrak{q})}\int_0^{\varkappa} u^{\mathfrak{p}-1}\,|\Upsilon(u,\digamma(u))|du,$$

and from $(\beta_9)$, we have

$$\begin{aligned} |F(\digamma(\varkappa))| &\leq \digamma_0 + \frac{\mathfrak{p}\,(1-\mathfrak{q})\,\varkappa^{(\mathfrak{p}-1)}\,\phi(\varkappa)\,B_1(|\digamma(\varkappa)|)}{M(\mathfrak{q})} + \frac{\mathfrak{p}\,\mathfrak{q}}{M(\mathfrak{q})}\int_0^{\varkappa} u^{\mathfrak{p}-1}\,\phi(u)\,B_1(|\digamma(u)|)du \\ &\leq \digamma_0 + \frac{\mathfrak{p}\,(1-\mathfrak{q})\,\varkappa^{(\mathfrak{p}-1)}\,\phi^*\,B_1(||\digamma||_\Xi)}{M(\mathfrak{q})} + \frac{\mathfrak{p}\,\mathfrak{q}}{M(\mathfrak{q})}\int_0^{\varkappa} u^{\mathfrak{p}-1}\phi^*\,B_1(||\digamma||_\Xi)du \\ &\leq \digamma_0 + \phi^*\,B(||\digamma||_\Xi)[\frac{\mathfrak{p}\,(1-\mathfrak{q})\,\varkappa^{(\mathfrak{p}-1)}}{M(\mathfrak{q})} + \frac{\mathfrak{p}\,\mathfrak{q}}{M(\mathfrak{q})}\int_0^{\varkappa} u^{\mathfrak{p}-1}du]. \end{aligned}$$

After simplification of the integral and applying value of $\lambda_1$, we get

$$|F(\digamma(\varkappa))| \leq \digamma_0 + \lambda_1\,\phi^*\,B_1(\epsilon). \tag{3.3.6}$$

Hence, by applying norm, we have

$$||F(\digamma(\varkappa))|| \leq \digamma_0 + \lambda_1\,\phi^*\,B_1(\epsilon) < \infty. \tag{3.3.7}$$

This implies $F$ is uniformly bounded.

Now, we take $\varkappa, \varkappa_1 \in$ J such that $\varkappa < \varkappa_1$ and $\digamma \in (\mathfrak{N}_1)_\epsilon$ arbitrarily. If we suppose $\Upsilon^* = sup\,|\Upsilon(\varkappa,\digamma(\varkappa))|$, then

$$\begin{aligned} |F(\digamma(\varkappa_1)) - F(\digamma(\varkappa))| &= |\frac{\mathfrak{p}\,(1-\mathfrak{q})\,\varkappa_1^{(\mathfrak{p}-1)}\,\Upsilon(\varkappa_1,\digamma(\varkappa_1))}{M(\mathfrak{q})} + \frac{\mathfrak{p}\mathfrak{q}}{M(\mathfrak{q})}\int_0^{\varkappa_1} u^{(\mathfrak{p}-1)}\,\Upsilon(u,\digamma(u))\,du \\ &- \frac{\mathfrak{p}\,(1-\mathfrak{q})\,\varkappa^{(\mathfrak{p}-1)}\,\Upsilon(\varkappa,\digamma(\varkappa))}{M(\mathfrak{q})} - \frac{\mathfrak{p}\mathfrak{q}}{M(\mathfrak{q})}\int_0^{\varkappa} u^{(\mathfrak{p}-1)}\,\Upsilon(u,\digamma(u))\,du| \\ &\leq \frac{\mathfrak{p}\,(1-\mathfrak{q})\,[\varkappa_1^{(\mathfrak{p}-1)}\,\Upsilon(\varkappa_1,\digamma(\varkappa_1) - \varkappa^{(\mathfrak{p}-1)}\,\Upsilon(\varkappa,\digamma_(\varkappa))]}{M(\mathfrak{q})} \\ &+ \frac{\mathfrak{p}\mathfrak{q}}{M(\mathfrak{q})}|\int_0^{\varkappa_1} u^{(\mathfrak{p}-1)}\,du - \int_0^{\varkappa} u^{(\mathfrak{p}-1)}\,du\,|\cdot|\Upsilon(u,\digamma(u)| \\ &\leq \frac{\mathfrak{p}\,(1-\mathfrak{q})\,[\varkappa_1^{(\mathfrak{p}-1)} - \varkappa^{(\mathfrak{p}-1)}]\,\Upsilon^*}{M(\mathfrak{q})} + \frac{\mathfrak{p}\mathfrak{q}}{M(\mathfrak{q})}[\varkappa_1^{\mathfrak{p}} - \varkappa^{\mathfrak{p}}]\,\Upsilon^* \end{aligned}$$

that is independent of $\digamma$. When $\varkappa_1 \to \varkappa$, its value becomes zero.

Hence $||F(\digamma(\varkappa_1)) - F(\digamma(\varkappa))||_\Xi \to 0$.

This proved that $F$ is equicontinuous. So $F$ is compact. As $F$ satisfies the conditions of

Theorem 3.3.2, we say that $F$ will satisfy either one or the other condition mentioned in Theorem 3.3.2. For this, using $(\beta_{10})$, we construct $A_1 = \{\mathcal{F} \in \Xi : ||\mathcal{F}||_\Xi < r_1\}$, where $r_1 > 0$ is defined above. Hence, we can write

$$||F(\mathcal{F}(\varkappa))|| \leq \mathcal{F}_0 + \lambda_1 \, \phi^* \, B_1(r_1). \tag{3.3.8}$$

Assume $\exists \mathcal{F} \in \partial A_1$ and $\omega_1 \in (0,1)$ where $\mathcal{F} = \omega_1 F(\mathcal{F})$. For $\mathcal{F}$ and $\omega_1$, we get

$$\begin{aligned} r_1 &= ||\mathcal{F}||_\Xi \\ &= \omega_1 ||F(\mathcal{F})||_\Xi \\ &< ||F(\mathcal{F})||_\Xi \\ &< \mathcal{F}_0 + \lambda_1 \, \phi^* \, B_1(||\mathcal{F}||_\Xi) \\ &< \mathcal{F}_0 + \lambda_1 \, \phi^* \, B_1(r_1). \end{aligned}$$

This gives us $r_1 < r_1$, which is impossible. Thus, the second condition is not satisfied. Hence, by first condition $F$ possesses a fixed point in $\overline{A_1}$. □

## 3.4 Uniqueness

We will prove uniqueness with the help of theorem using lipschitz condition proved in Theorem (2.4.1) along with some other condition which is described in Theorem (3.4.1).

**Theorem 3.4.1.** If $||\Delta|| \leq \mu_1, ||\aleph|| \leq \mu_2, ||\Theta|| \leq \mu_4$ for some $\mu_i > 0, i = 1, 2, 3, 4$ and $w_1 = (\beta_0 \, \mu_3 + \mu + \nu), w_2 = (\beta_0 \, \mu_1 \, b + \mu + \gamma), w_3 = (\zeta + \mu)$, where $0 < w_i < 1, i = 1, 2, 3$; then our model has a unique solution if $\lambda_1 \, w_i < 1$, for $i = 1, 2, 3$.

*Proof.* : Suppose the model has two solutions $(\Delta(\varkappa), \aleph(\varkappa), \Theta(\varkappa))$ and $(\Delta^*(\varkappa), \aleph^*(\varkappa), \Theta^*(\varkappa))$ with initial conditions defined above.Then, we can write

$$\begin{aligned} \Delta(\varkappa) &= \Delta(0) + \frac{\mathfrak{p}\,(1-\mathfrak{q})\,\varkappa^{(\mathfrak{p}-1)}\,\Upsilon_1(\varkappa, \mathcal{F}(\varkappa))}{M(\mathfrak{q})} + \frac{\mathfrak{p}\,\mathfrak{q}}{M(\mathfrak{q})} \int_0^\varkappa u^{\mathfrak{p}-1}\,\Upsilon_1(u, \mathcal{F}(u))du, \\ \Delta^*(\varkappa) &= \Delta(0) + \frac{\mathfrak{p}\,(1-\mathfrak{q})\,\varkappa^{(\mathfrak{p}-1)}\,\Upsilon_1(\varkappa, \mathcal{F}^*(\varkappa))}{M(\mathfrak{q})} + \frac{\mathfrak{p}\,\mathfrak{q}}{M(\mathfrak{q})} \int_0^\varkappa u^{\mathfrak{p}-1}\Upsilon_1(u, \mathcal{F}^*(u))du. \end{aligned}$$

Take

$$\begin{aligned}
||\Delta(\varkappa)-\Delta^*(\varkappa)|| &= ||\frac{\mathfrak{p}\,(1-\mathfrak{q})\,\varkappa^{(\mathfrak{p}-1)}\,[\Upsilon_1(\varkappa,F(\varkappa))-\Upsilon_1(\varkappa,F^*(\varkappa))]}{M(\mathfrak{q})}\\
&+ \frac{\mathfrak{p}\mathfrak{q}}{M(\mathfrak{q})}\int_0^{\varkappa} u^{(\mathfrak{p}-1)}\,(\Upsilon_1(u,F(u))-\Upsilon_1(u,F^*(u)))du||\\
&\leq \frac{\mathfrak{p}\,(1-\mathfrak{q})\,\varkappa^{(\mathfrak{p}-1)}\,||\Upsilon_1(\varkappa,F(\varkappa))-\Upsilon_1(\varkappa,F^*(\varkappa))||}{M(\mathfrak{q})}\\
&+ \frac{\mathfrak{p}\mathfrak{q}}{M(\mathfrak{q})}\int_0^{\varkappa} u^{(\mathfrak{p}-1)}\,||\Upsilon_1(u,F(u))-\Upsilon_1(u,F^*(u))||du,
\end{aligned}$$

since $\Upsilon_1$ is considered w.r.t $\Delta$ and $\Delta^*$, so by using integration and simplification

$$||\Delta(\varkappa)-\Delta^*(\varkappa)|| \leq \frac{\mathfrak{p}\,(1-\mathfrak{q})\,\varkappa^{\mathfrak{p}-1}+\varkappa^{\mathfrak{p}}\,\mathfrak{q}}{M(\mathfrak{q})}||\Upsilon_1(\Delta)-\Upsilon_1(\Delta^*)||.$$

Hence, using previous results

$$||\Delta(\varkappa)-\Delta^*(\varkappa)|| \leq \lambda_1\,w_1\,||\Delta(\varkappa)-\Delta^*(\varkappa)||.$$

$$(1-\lambda_1\,w_1)\,||\Delta(\varkappa)-\Delta^*(\varkappa)|| \leq 0.$$

As $\lambda_1 w_1 < 1$, so this is possible when $||\Delta(\varkappa)-\Delta^*(\varkappa)||=0$. Thus $\Delta(\varkappa)=\Delta^*(\varkappa)$. Similarly, we have

$$\begin{aligned}
\aleph(\varkappa) &= \aleph(0)+\frac{\mathfrak{p}\,(1-\mathfrak{q})\,\varkappa^{(\mathfrak{p}-1)}\,\Upsilon_2(\varkappa,F(\varkappa))}{M(\mathfrak{q})}+\frac{\mathfrak{p}\,\mathfrak{q}}{M(\mathfrak{q})}\int_0^{\varkappa} u^{\mathfrak{p}-1}\,\Upsilon_2(u,F(u))du,\\
\aleph^*(\varkappa) &= \aleph(0)+\frac{\mathfrak{p}\,(1\mathfrak{q})\,\varkappa^{(\mathfrak{p}-1)}\,\Upsilon_2(\varkappa,F^*(\varkappa))}{M(\mathfrak{q})}+\frac{\mathfrak{p}\,\mathfrak{q}}{M(\mathfrak{q})}\int_0^{\varkappa} u^{\mathfrak{p}-1}\Upsilon_2(u,F^*(u))du.
\end{aligned}$$

$$\begin{aligned}
||\aleph(\varkappa)-\aleph^*(\varkappa)|| &= ||\frac{\mathfrak{p}\,(1-\mathfrak{q})\,\varkappa^{(\mathfrak{p}-1)}\,[\Upsilon_2(\varkappa,F(\varkappa))-\Upsilon_2(\varkappa,F^*(\varkappa))]}{M(\mathfrak{q})}\\
&+ \frac{\mathfrak{p}\mathfrak{q}}{M(\mathfrak{q})}\int_0^{\varkappa} u^{(\mathfrak{p}-1)}\,(\Upsilon_2(u,F(u))-\Upsilon_2(u,F^*(u)))du||\\
&\leq \frac{\mathfrak{p}\,(1-\mathfrak{q})\,\varkappa^{(\mathfrak{p}-1)}\,||\Upsilon_2(\varkappa,F(\varkappa))-\Upsilon_2(\varkappa,F^*(\varkappa))||}{M(\mathfrak{q})}\\
&+ \frac{\mathfrak{p}\mathfrak{q}}{M(\mathfrak{q})}\int_0^{\varkappa} u^{(\mathfrak{p}-1)}\,||\Upsilon_2(u,F(u))-\Upsilon_2(u,F^*(u))||du,
\end{aligned}$$

since $\Upsilon_2$ is considered w.r.t $\aleph$ and $\aleph^*$, so after integration and simplification,

$$||\aleph(\varkappa)-\aleph^*(\varkappa)|| \leq \frac{\mathfrak{p}\,(1-\mathfrak{q})\,\varkappa^{\mathfrak{p}-1}+\varkappa^{\mathfrak{p}}\,\mathfrak{q}}{M(\mathfrak{q})}||\Upsilon_2(\aleph)-\Upsilon_2(\aleph^*)||.$$

Hence, we obtain

$$||\aleph(\varkappa) - \aleph^*(\varkappa)|| \leq \lambda_1 \, w_2 \, ||\aleph(\varkappa) - \aleph^*(\varkappa)||.$$

$$\implies (1 - \lambda_1 \, w_2) \, ||\aleph(\varkappa) - \aleph^*(\varkappa)|| \leq 0.$$

As $\lambda_1 w_2 < 1$, so this is possible when $||\aleph(\varkappa) - \aleph^*(\varkappa)|| = 0$. Thus $\aleph(\varkappa) = \aleph^*(\varkappa)$.
Also, we have

$$\begin{aligned}
\Theta(\varkappa) &= \Theta(0) + \frac{\mathfrak{p}\,(1-\mathfrak{q})\,\varkappa^{(\mathfrak{p}-1)}\,\Upsilon_3(\varkappa, F(\varkappa))}{M(\mathfrak{q})} + \frac{\mathfrak{p}\,\mathfrak{q}}{M(\mathfrak{q})} \int_0^{\varkappa} u^{\mathfrak{p}-1}\,\Upsilon_3(u, F(u))du, \\
\Theta^*(\varkappa) &= \Theta(0) + \frac{\mathfrak{p}\,(1\mathfrak{q})\,\varkappa^{(\mathfrak{p}-1)}\,\Upsilon_3(\varkappa, F^*(\varkappa))}{M(\mathfrak{q})} + \frac{\mathfrak{p}\,\mathfrak{q}}{M(\mathfrak{q})} \int_0^{\varkappa} u^{\mathfrak{p}-1}\Upsilon_3(u, F^*(u))du.
\end{aligned}$$

Take

$$\begin{aligned}
||\Theta(\varkappa) - \Theta^*(\varkappa)|| &= ||\frac{\mathfrak{p}\,(1-\mathfrak{q})\,\varkappa^{(\mathfrak{p}-1)}\,[\Upsilon_3(\varkappa, F(\varkappa)) - \Upsilon_3(\varkappa, F^*(\varkappa))]}{M(\mathfrak{q})} \\
&+ \frac{\mathfrak{p}\mathfrak{q}}{M(\mathfrak{q})} \int_0^{\varkappa} u^{(\mathfrak{p}-1)}\,(\Upsilon_3(u, F(u)) - \Upsilon_3(u, F^*(u)))du|| \\
&\leq \frac{\mathfrak{p}\,(1-\mathfrak{q})\,\varkappa^{(\mathfrak{p}-1)}\,||\Upsilon_3(\varkappa, F(\varkappa)) - \Upsilon_3(\varkappa, F^*(\varkappa))||}{M(\mathfrak{q})} \\
&+ \frac{\mathfrak{p}\mathfrak{q}}{M(\mathfrak{q})} \int_0^{\varkappa} u^{(\mathfrak{p}-1)}\,||\Upsilon_3(u, F(u)) - \Upsilon_3(u, F^*(u))||du,
\end{aligned}$$

since $\Upsilon_3$ is considered w.r.t $\Theta$ and $\Theta^*$, so by using integration and simplification, we have

$$||\Theta(\varkappa) - \Theta^*(\varkappa)|| \leq \frac{\mathfrak{p}\,(1-\mathfrak{q})\,\varkappa^{\mathfrak{p}-1} + \varkappa^{\mathfrak{p}}\,\mathfrak{q}}{M(\mathfrak{q})} ||\Upsilon_3(\Theta) - \Upsilon_3(\Theta^*)||.$$

By using previous results

$$||\Theta(\varkappa) - \Theta^*(\varkappa)|| \leq \lambda_1 \, w_3 \, ||\Theta(\varkappa) - \Theta^*(\varkappa)||.$$

$$\implies (1 - \lambda_1 \, w_3) \, ||\Theta(\varkappa) - \Theta^*(\varkappa)|| \leq 0.$$

As $\lambda_1 w_3 < 1$, so this is possible when $||\Theta(\varkappa) - \Theta^*(\varkappa)|| = 0$. Thus $\Theta(\varkappa) = \Theta^*(\varkappa)$. That is, $(\Delta(\varkappa), \aleph(\varkappa), \Theta(\varkappa)) = (\Delta^*(\varkappa), \aleph^*(\varkappa), \Theta^*(\varkappa))$. Hence, the solution is unique. □

## 3.5 Stability

In this section, we have to check the stability of the solution. We use Ulam−Hyers and Ulam−Hayers−Rassias theorems to check it. First, we define theorems for our model along with the definition and remark defined below.

**Definition 3.5.1.** Model(3.1.1) is Ulam-Hyers stable as in [23] if, for all $\epsilon_i > 0$, there exist $M_i > 0 \in [0, \infty)$, which depend on $\Upsilon_i$ respectively, $i = 1, 2, 3$ and for all $(\Delta^*, \aleph^*, \Theta^*)$ satisfying the inequalities

$$\begin{aligned} |{}^{FFE}D_{0,\varkappa}^{\mathfrak{q},\mathfrak{p}}\Delta^*(\varkappa) - \Upsilon_1(\varkappa, F^*(\varkappa))| &\leq \epsilon_1, \\ |{}^{FFE}D_{0,\varkappa}^{\mathfrak{q},\mathfrak{p}}\aleph^*(\varkappa) - \Upsilon_2(\varkappa, F^*(\varkappa))| &\leq \epsilon_2, \\ |{}^{FFE}D_{0,\varkappa}^{\mathfrak{q},\mathfrak{p}}\Theta^*(\varkappa) - \Upsilon_3(\varkappa, F^*(\varkappa))| &\leq \epsilon_3, \end{aligned} \tag{3.5.1}$$

then, there exists $(\Delta, \aleph, \Theta) \in \Xi$ satisfying the model (3.1.1) with the condition

$$\begin{aligned} |\Delta^*(\varkappa) - \Delta(\varkappa)| &\leq M_1\,\epsilon_1, \\ |\aleph^*(\varkappa) - \aleph(\varkappa)| &\leq M_2\,\epsilon_2, \\ |\Theta^*(\varkappa) - \Theta(\varkappa)| &\leq M_3\,\epsilon_3. \end{aligned} \tag{3.5.2}$$

**Remark 3.5.2.** $(\Delta^*, \aleph^*, \Theta^*) \in \Xi$ is a solution of model (3.2.2) iff $\exists\, \eta_i \in C([0, \mathrm{T}], [0, \infty))$ such that for all $\varkappa \in \mathrm{J}$,

(i) $|\,\eta_i(\varkappa)\,| < \epsilon_i$,

(ii)

$$\begin{aligned} {}^{FFE}D_{0,\varkappa}^{\mathfrak{q},\mathfrak{p}}\Delta^*(\varkappa) &= \Upsilon_1(\varkappa, F^*(\varkappa)) + \eta_1(\varkappa), \\ {}^{FFE}D_{0,\varkappa}^{\mathfrak{q},\mathfrak{p}}\aleph^*(\varkappa) &= \Upsilon_2(\varkappa, F^*(\varkappa)) + \eta_2(\varkappa), \\ {}^{FFE}D_{0,\varkappa}^{\mathfrak{q},\mathfrak{p}}\Theta^*(\varkappa) &= \Upsilon_3(\varkappa, F^*(\varkappa)) + \eta_3(\varkappa). \end{aligned} \tag{3.5.3}$$

**Theorem 3.5.3.** The fractal fraction model (3.1.1) is Ulam−Hayers stable on J s.t. $\lambda_1\, w_i < 1$, where $w_i$ and $\lambda_1$ are defined with the conditions as above.

*Proof.* :Let $\epsilon_1 > 0$ and $\Delta^* \in \mathfrak{Y}$ such that

$$|{}^{FFE}D_{0,\varkappa}^{\mathfrak{q},\mathfrak{p}}\Delta^*(\varkappa) - \Upsilon_1(\varkappa, F^*(\varkappa))| \leq \epsilon_1,$$

by remark 3.5.2, we have

$$\begin{aligned}\Delta^*(\varkappa) &= \Delta(0) + \frac{\mathfrak{p}\,(1-\mathfrak{q})\,\varkappa^{\mathfrak{p}-1}\,\Upsilon_1(\varkappa, F^*(\varkappa))}{M(\mathfrak{q})} + \frac{\mathfrak{p}\mathfrak{q}}{M(\mathfrak{q})}\int_0^{\varkappa} u^{(\mathfrak{p}-1)}\,\Upsilon_1(u, F^*(u))du \\ &+ \frac{\mathfrak{p}\mathfrak{q}}{M(\mathfrak{q})}\int_0^{\varkappa} u^{(\mathfrak{p}-1)}\,\eta_1(u)\,du.\end{aligned}$$

As $\Delta \in \mathfrak{Y}$ is the unique solution, then

$$\Delta(\varkappa) = \Delta(0) + \frac{\mathfrak{p}\,(1-\mathfrak{q})\,\varkappa^{(\mathfrak{p}-1)}\,\Upsilon_1(\varkappa, F(\varkappa))}{M(\mathfrak{q})}\frac{\mathfrak{p}\,\mathfrak{q}}{M(\mathfrak{q})}\int_0^{\varkappa} u^{\mathfrak{p}-1}\,\Upsilon_1(u, F(u))du.$$

That is

$$\begin{aligned}&|\Delta^*(\varkappa) - \Delta(\varkappa)| \\ &= |\frac{\mathfrak{p}\,(1-\mathfrak{q})\,\varkappa^{(\mathfrak{p}-1)}\,[\Upsilon_1(\varkappa, F^*(\varkappa)) - \Upsilon_1(\varkappa, F(\varkappa))]}{M(\mathfrak{q})} \\ &+ \frac{\mathfrak{p}\mathfrak{q}}{M(\mathfrak{q})}\int_0^{\varkappa} u^{(\mathfrak{p}-1)}\,\eta_1(u)\,du \\ &+ \frac{\mathfrak{p}\mathfrak{q}}{M(\mathfrak{q})}\int_0^{\varkappa} u^{(\mathfrak{p}-1)}\,[\Upsilon_1(u, F^*(u)) - \Upsilon_1(u, F(u))\,]\,du| \\ &\leq \frac{\mathfrak{p}\,(1-\mathfrak{q})\,\varkappa^{(\mathfrak{p}-1)}\,||\Upsilon_1(\varkappa, F^*(\varkappa)) - \Upsilon_1(\varkappa, F(\varkappa))||}{M(\mathfrak{q})} \\ &+ \frac{\mathfrak{p}\mathfrak{q}}{M(\mathfrak{q})}\int_0^{\varkappa} u^{(\mathfrak{p}-1)}\,|\eta_1(u)|du \\ &+ \frac{\mathfrak{p}\mathfrak{q}}{M(\mathfrak{q})}\int_0^{\varkappa} u^{(\mathfrak{p}-1)}\,||\Upsilon_1(u, F^*(u)) - \Upsilon_1(u, F(u))\,||\,du \\ &\leq \frac{\mathfrak{p}\,(1-\mathfrak{q})\,\varkappa^{(\mathfrak{p}-1)}\,w_1\,||\Delta^* - \Delta||}{M(\mathfrak{q})} + \frac{\mathfrak{q}\,\varkappa^{\mathfrak{p}}}{M(\mathfrak{q})}\,|\eta_1| + \frac{\mathfrak{p}\mathfrak{q}}{M(\mathfrak{q})}\int_0^{\varkappa} u^{(\mathfrak{p}-1)}\,w_1\,||\Delta^* - \Delta||\,du \\ &\leq [\frac{\mathfrak{p}\,(1-\mathfrak{q})\,\varkappa^{(\mathfrak{p}-1)} + \mathfrak{q}\,\varkappa^{\mathfrak{p}}}{M(\mathfrak{q})}]\,w_1\,||\Delta^* - \Delta|| + \frac{\mathfrak{q}\,\varkappa^{\mathfrak{p}}\,\epsilon_1}{M(\mathfrak{q})} \\ &\leq \lambda_1\,w_1\,||\Delta^* - \Delta|| + \frac{\mathfrak{q}\,\varkappa^{\mathfrak{p}}\,\epsilon_1}{M(\mathfrak{q})}.\end{aligned}$$

Hence, we have

$$\begin{aligned}||\Delta^* - \Delta|| &\leq \lambda_1\,w_1\,||\Delta^* - \Delta|| + \frac{\mathfrak{q}\,\varkappa^{\mathfrak{p}}\,\epsilon_1}{M(\mathfrak{q})} \\ (1-\lambda_1\,w_1)\,||\Delta^* - \Delta|| &\leq \frac{\mathfrak{q}\,\varkappa^{\mathfrak{p}}\,\epsilon_1}{M(\mathfrak{q})} \\ ||\Delta^* - \Delta|| &\leq \frac{\frac{\mathfrak{q}\,\varkappa^{\mathfrak{p}}\,\epsilon_1}{M(\mathfrak{q})}}{(1-\lambda_1\,w_1)}.\end{aligned}$$

If $\frac{\mathfrak{q}\,\varkappa^{\mathfrak{p}}}{M(\mathfrak{q})\,(1-\lambda_1\,w_1)} = M_1$, then $||\Delta^* - \Delta|| \leq M_1\,\epsilon_1$.

Similarly, we can prove that $||\aleph^* - \aleph|| \leq M_2\,\epsilon_2$, and $||\Theta^* - \Theta|| \leq M_3\,\epsilon_3$ .

Thus Ulam−Hayers stability criteria is fulfilled by our fractal−fractional model. □

Now we define the Ulam−Hayers−Rassias stability criteria for our fractal−fractional model (3.1.1).

**Definition 3.5.4.** [25] Model (3.1.1) is Ulam-Hyers−Rassias stable w.r.t the functions $\psi_i$, if for all $\epsilon_i > 0$, there exist $M_i > 0 \in [0, \infty)$, which depend on $\Upsilon_i \, and \, \psi_i$, respectively $i = 1, 2, 3$ and for all $(\Delta^*, \aleph^*, \Theta^*)$ satisfying the inequalities:

$$\begin{aligned} |^{FFE}D^{\mathfrak{q},\mathfrak{p}}_{0,\varkappa}\Delta^*(\varkappa) - \Upsilon_1(\varkappa, F^*(\varkappa))| &\leq \epsilon_1 \, \psi_1(\varkappa), \\ |^{FFE}D^{\mathfrak{q},\mathfrak{p}}_{0,\varkappa}\aleph^*(\varkappa) - \Upsilon_2(\varkappa, F^*(\varkappa))| &\leq \epsilon_2 \, \psi_2(\varkappa), \\ |^{FFE}D^{\mathfrak{q},\mathfrak{p}}_{0,\varkappa}\Theta^*(\varkappa) - \Upsilon_3(\varkappa, F^*(\varkappa))| &\leq \epsilon_3 \, \psi_3(\varkappa), \end{aligned} \tag{3.5.4}$$

there exists $(\Delta, \aleph, \Theta) \in \Xi$ satisfying model with the condition

$$\begin{aligned} |\Delta^*(\varkappa) - \Delta(\varkappa)| &\leq M_1 \, \epsilon_1 \, \psi_1(\varkappa), \\ |\aleph^*(\varkappa) - \aleph(\varkappa)| &\leq M_2 \, \epsilon_2 \, \psi_2(\varkappa), \\ |\Theta^*(\varkappa) - \Theta(\varkappa)| &\leq M_3 \, \epsilon_3 \, \psi_3(\varkappa). \end{aligned} \tag{3.5.5}$$

**Remark 3.5.5.** $(\Delta^*, \aleph^*, \Theta^*) \in \Xi$ is a solution iff $\exists \, \eta_i \in C([0, \mathrm{T}], [0, \infty))$ s.t for all $\varkappa \in \mathrm{J}$

(i) $|\, \eta_i(\varkappa) \,| < \epsilon_i \, \psi_i(\varkappa)$,

(ii)

$$\begin{aligned} {}^{FFE}D^{\mathfrak{q},\mathfrak{p}}_{0,\varkappa}\Delta^*(\varkappa) &= \Upsilon_1(\varkappa, F^*(\varkappa)) + \eta_1(\varkappa), \\ {}^{FFE}D^{\mathfrak{q},\mathfrak{p}}_{0,\varkappa}\aleph^*(\varkappa) &= \Upsilon_2(\varkappa, F^*(\varkappa)) + \eta_2(\varkappa), \\ {}^{FFE}D^{\mathfrak{q},\mathfrak{p}}_{0,\varkappa}\Theta^*(\varkappa) &= \Upsilon_3(\varkappa, F^*(\varkappa)) + \eta_3(\varkappa). \end{aligned} \tag{3.5.6}$$

**Theorem 3.5.6.** The fractal−fractional model (3.1.1) is Ulam−Hayers−Rassias stable when the following conditions are satisfied, for all $\varkappa \in \mathrm{J}$ there exists nondecreasing mappings $\psi_i \in C([0, \mathrm{T}], [0, \infty))$ and $\xi_i > 0$ depending upon $\psi_i$ such that ${}^{FFE}I^{\mathfrak{q},\mathfrak{p}}_{0,\varkappa}\psi_i(\varkappa) < \xi_i(\psi_i) \, \psi_i(\varkappa)$ and $\lambda_1 > 0, w_i > 0$, where $w_i$ and $\lambda_1$ are defined as before.

*Proof.* Let $\epsilon_1 > 0$ and $\Delta^* \in Y$ such that

$$|^{FFE}D^{\mathfrak{q},\mathfrak{p}}_{0,\varkappa}\Delta^*(\varkappa) - \Upsilon_1(\varkappa, \Delta^*(\varkappa), \aleph^*(\varkappa), \Theta^*(\varkappa))| \leq \epsilon_1 \, \psi_1(\varkappa),$$

then, by the conditions of remark 3.5.5, we consider

$$\begin{aligned} \Delta^*(\varkappa) &= \Delta(0) + \frac{\mathfrak{p}\,(1-\mathfrak{q})\,\varkappa^{\mathfrak{p}-1}\,\Upsilon_1(\varkappa, F^*(\varkappa))}{M(\mathfrak{q})} + \frac{\mathfrak{p}\mathfrak{q}}{M(\mathfrak{q})} \int_0^{\varkappa} u^{(\mathfrak{p}-1)} \, \Upsilon_1(u, F^*(u)) du \\ &+ \frac{\mathfrak{p}\mathfrak{q}}{M(\mathfrak{q})} \int_0^{\varkappa} u^{(\mathfrak{p}-1)} \, \eta_1(u) \, du. \end{aligned}$$

As $\Delta \in \mathfrak{Y}$ is the unique solution, then

$$\Delta(\varkappa) = \Delta(0) + \frac{\mathfrak{p}\,(1-\mathfrak{q})\,\varkappa^{(\mathfrak{p}-1)}\,\Upsilon_1(\varkappa, F(\varkappa))}{M(\mathfrak{q})} + \frac{\mathfrak{p}\,\mathfrak{q}}{M(\mathfrak{q})}\int_0^{\varkappa} u^{\mathfrak{p}-1}\,\Upsilon_1(u, F(u))du.$$

Therefore, we have

$$\begin{aligned}
&|\Delta^*(\varkappa) - \Delta(\varkappa)| \\
&\quad = |\frac{\mathfrak{p}\,(1-\mathfrak{q})\,\varkappa^{(\mathfrak{p}-1)}\,[\Upsilon_1(\varkappa, F^*(\varkappa)) - \Upsilon_1(\varkappa, F(\varkappa))]}{M(\mathfrak{q})} \\
&\quad + \frac{\mathfrak{p}\mathfrak{q}}{M(\mathfrak{q})}\int_0^{\varkappa} u^{(\mathfrak{p}-1)}\,\eta_1(u)\,du \\
&\quad + \frac{\mathfrak{p}\mathfrak{q}}{M(\mathfrak{q})}\int_0^{\varkappa} u^{(\mathfrak{p}-1)}\,[\Upsilon_1(u, F^*(u)) - \Upsilon_1(u, F(u))\,]\,du| \\
&\quad \leq \frac{\mathfrak{p}\,(1-\mathfrak{q})\,\varkappa^{(\mathfrak{p}-1)}\,||\Upsilon_1(\varkappa, F^*(\varkappa)) - \Upsilon_1(\varkappa, F(\varkappa))||}{M(\mathfrak{q})} \\
&\quad + \frac{\mathfrak{p}\mathfrak{q}}{M(\mathfrak{q})}\int_0^{\varkappa} u^{(\mathfrak{p}-1)}\,|\eta_1(u)|du \\
&\quad + \frac{\mathfrak{p}\mathfrak{q}}{M(\mathfrak{q})}\int_0^{\varkappa} u^{(\mathfrak{p}-1)}\,||\Upsilon_1(u, F^*(u)) - \Upsilon_1(u, F(u))\,||\,du \\
&\quad \leq \frac{\mathfrak{p}\,(1-\mathfrak{q})\,\varkappa^{(\mathfrak{p}-1)}\,w_1\,||\Delta^* - \Delta||}{M(\mathfrak{q})} + \frac{\mathfrak{q}\,\varkappa^{\mathfrak{p}}}{M(\mathfrak{q})}\,|\eta_1| + \frac{\mathfrak{p}\mathfrak{q}}{M(\mathfrak{q})}\int_0^{\varkappa} u^{(\mathfrak{p}-1)}\,w_1\,||\Delta^* - \Delta||\,du \\
&\quad \leq [\frac{\mathfrak{p}\,(1-\mathfrak{q})\,\varkappa^{(\mathfrak{p}-1)} + \mathfrak{q}\,\varkappa^{\mathfrak{p}}}{M(\mathfrak{q})}]\,w_1\,||\Delta^* - \Delta|| + \frac{\mathfrak{q}\,\varkappa^{\mathfrak{p}}\,\epsilon_1\psi_1(\varkappa)}{M(\mathfrak{q})} \\
&\quad \leq \lambda_1\,w_1\,||\Delta^* - \Delta|| + \frac{\mathfrak{q}\,\varkappa^{\mathfrak{p}}\,\epsilon_1\,\xi_1(\psi_1)\,\psi_1(\varkappa)}{M(\mathfrak{q})}.
\end{aligned}$$

Hence, we get

$$\begin{aligned}
||\Delta^* - \Delta|| &\leq \frac{\mathfrak{q}\,\varkappa^{\mathfrak{p}}\,\epsilon_1\,\xi_1(\psi_1)\,\psi_1(\varkappa)}{M(\mathfrak{q})} + \lambda_1\,w_1\,||\Delta^* - \Delta||, \\
(1-\lambda_1\,w_1)\,||\Delta^* - \Delta|| &\leq \frac{\mathfrak{q}\,\varkappa^{\mathfrak{p}}\,\epsilon_1\,\xi_1(\psi_1)\,\psi_1(\varkappa)}{M(\mathfrak{q})}, \\
||\Delta^* - \Delta|| &\leq \frac{\frac{\mathfrak{q}\,\varkappa^{\mathfrak{p}}\,\epsilon_1\,\xi_1(\psi_1)\,\psi_1(\varkappa)}{M(\mathfrak{q})}}{(1-\lambda_1\,w_1)}.
\end{aligned}$$

If $\frac{\mathfrak{q}\,\varkappa^{\mathfrak{p}}\,\xi_1(\psi_1)}{M(\mathfrak{q})(1-\lambda_1\,w_1)} = M_1(\Upsilon_1, \psi_1)$ then, we get

$$||\Delta^* - \Delta|| \leq \epsilon_1\,\psi_1(\varkappa)M_1(\Upsilon_1, \psi_1).$$

Similarly, we can prove that

$$||\aleph^* - \aleph|| \leq \epsilon_2\,\psi_2(\varkappa)M_2(\Upsilon_2, \psi_2),$$

$$||\Theta^* - \Theta|| \le \epsilon_3 \, \psi_3(\varkappa) M_3(\Upsilon_3, \psi_3).$$

Thus Ulam−Hayers−Rassias stability criteria is fulfilled by our fractal−fractional model.

□

## 3.6 Numerical Algorithm

For numerical scheme of our fractal−fractional model, we proceed as already many authors did [37, 36]. As in our model $\Upsilon_1$ and $\Upsilon_3$ depends on $\varkappa$ and $(\varkappa - \tau)$, so we deal it differently in the end. First, we take $\varkappa = \varkappa_{n+1}$ and $u^{p-1}\,\Upsilon_i(u, \Delta(u), \aleph(u), \Theta(u)) = H_i(u)$, $i = 1, 2, 3$; in model (3.2.7) and get

$$\begin{aligned}
\Delta(n+1) &= \Delta(0) + \frac{\mathfrak{p}\,(1-\mathfrak{q})\,\varkappa_n^{\mathfrak{p}-1}\,\Upsilon_1(\varkappa, \Delta^n(\varkappa), \aleph^n(\varkappa), \Theta^n(\varkappa))}{M(\mathfrak{q})} + \frac{\mathfrak{p}\,\mathfrak{q}}{M(\mathfrak{q})}\int_0^{\varkappa_{n+1}} F_1(u)du,\\
\aleph(n+1) &= \aleph(0) + \frac{\mathfrak{p}\,(1-\mathfrak{q})\,\varkappa_n^{\mathfrak{p}-1}\,\Upsilon_2(\varkappa, \Delta^n(\varkappa), \aleph^n(\varkappa), \Theta^n(\varkappa))}{M(\mathfrak{q})} + \frac{\mathfrak{p}\,\mathfrak{q}}{M(\mathfrak{q})}\int_0^{\varkappa_{n+1}} F_2(u)du,\\
\Theta(n+1) &= \Theta(0) + \frac{\mathfrak{p}\,(1-\mathfrak{q})\,\varkappa_n^{\mathfrak{p}-1}\,\Upsilon_3(\varkappa, \Delta^n(\varkappa), \aleph^n(\varkappa), \Theta^n(\varkappa))}{M(\mathfrak{q})} + \frac{\mathfrak{p}\,\mathfrak{q}}{M(\mathfrak{q})}\int_0^{\varkappa_{n+1}} F_3(u)du.
\end{aligned} \tag{3.6.1}$$

Taking difference between consecutive terms, we get

$$\begin{aligned}
\Delta(n+1) &= \Delta(n) + \frac{\mathfrak{p}\,(1-\mathfrak{q})\,\varkappa_n^{\mathfrak{p}-1}\,\Upsilon_1(\varkappa, F^n(\varkappa))}{M(\mathfrak{q})} - \frac{\mathfrak{p}\,(1-\mathfrak{q})\,\varkappa_{n-1}^{\mathfrak{p}-1}\,\Upsilon_1(\varkappa, F^{n-1}(\varkappa))}{M(\mathfrak{q})}\\
&+ \frac{\mathfrak{p}\,\mathfrak{q}}{M(\mathfrak{q})}\int_{\varkappa_n}^{\varkappa_{n+1}} F_1(u)du,\\
\aleph(n+1) &= \aleph(n) + \frac{\mathfrak{p}\,(1-\mathfrak{q})\,\varkappa_n^{\mathfrak{p}-1}\,\Upsilon_2(\varkappa, F^n(\varkappa))}{M(\mathfrak{q})} - \frac{\mathfrak{p}\,(1-\mathfrak{q})\,\varkappa_{n-1}^{\mathfrak{p}-1}\,\Upsilon_2(\varkappa, F^{n-1}(\varkappa))}{M(\mathfrak{q})}\\
&+ \frac{\mathfrak{p}\,\mathfrak{q}}{M(\mathfrak{q})}\int_{\varkappa_n}^{\varkappa_{n+1}} F_2(u)du,\\
\Theta(n+1) &= \Theta(n) + \frac{\mathfrak{p}\,(1-\mathfrak{q})\,\varkappa_n^{\mathfrak{p}-1}\,\Upsilon_3(\varkappa, F^n(\varkappa))}{M(\mathfrak{q})} - \frac{\mathfrak{p}\,(1-\mathfrak{q})\,\varkappa_{n-1}^{\mathfrak{p}-1}\,\Upsilon_3(\varkappa, F^{n-1}(\varkappa))}{M(\mathfrak{q})}\\
&+ \frac{\mathfrak{p}\,\mathfrak{q}}{M(\mathfrak{q})}\int_{\varkappa_n}^{\varkappa_{n+1}} F_3(u)du.
\end{aligned}$$

Approximating the functions $\Upsilon_i(u)$ by two points Lagrange interpolation polynomials on the interval $[\varkappa_j, \varkappa_{j+1}]$, we can write

$$\begin{aligned}
H_1^*(u,\Delta(u),\aleph(u),\Theta(u)) &= \frac{u-\varkappa_{j-1}}{\varkappa_j-\varkappa_{j-1}}\,\Upsilon_1(u_j,\Delta_j(u),\aleph_j(u),\Theta_j(u)) \\
&- \frac{u-\varkappa_j}{\varkappa_j-\varkappa_{j-1}}\,\Upsilon_1(u_{j-1},\Delta_{j-1}(u),\aleph_{j-1}(u),\Theta_{j-1}(u)), \\
H_2^*(u,\Delta(u),\aleph(u),\Theta(u)) &= \frac{u-\varkappa_{j-1}}{\varkappa_j-\varkappa_{j-1}}\,\Upsilon_2(u_j,\Delta_j(u),\aleph_j(u),\Theta_j(u)) \\
&- \frac{u-\varkappa_j}{\varkappa_j-\varkappa_{j-1}}\,\Upsilon_2(u_{j-1},\Delta_{j-1}(u),\aleph_{j-1}(u),\Theta_{j-1}(u)), \\
H_3^*(u,\Delta(u),\aleph(u),\Theta(u)) &= \frac{u-\varkappa_{j-1}}{\varkappa_j-\varkappa_{j-1}}\,\Upsilon_3(u_j,\Delta_j(u),\aleph_j(u),\Theta_j(u)) \\
&- \frac{u-\varkappa_j}{\varkappa_j-\varkappa_{j-1}}\,\Upsilon_3(u_{j-1},\Delta_{j-1}(u),\aleph_{j-1}(u),\Theta_{j-1}(u)).
\end{aligned}$$

Thus, we have

$$\begin{aligned}
\Delta(n+1) &= \Delta(n) + \frac{\mathfrak{p}\,(1-\mathfrak{q})\,\varkappa_n^{\mathfrak{p}-1}\,\Upsilon_1(\varkappa,F^n(\varkappa))}{M(\mathfrak{q})} \\
&- \frac{\mathfrak{p}\,(1-\mathfrak{q})\,\varkappa_{n-1}^{\mathfrak{p}-1}\,\Upsilon_1(\varkappa,F^{n-1}(\varkappa))}{M(\mathfrak{q})} \\
&+ \frac{\mathfrak{p}\,\mathfrak{q}}{M(\mathfrak{q})}\int_{\varkappa_n}^{\varkappa_{n+1}} F_1^*(u)du, \\
\aleph(n+1) &= \aleph(n) + \frac{\mathfrak{p}\,(1-\mathfrak{q})\,\varkappa_n^{\mathfrak{p}-1}\,\Upsilon_2(\varkappa,F^n(\varkappa))}{M(\mathfrak{q})} \\
&- \frac{\mathfrak{p}\,(1-\mathfrak{q})\,\varkappa_{n-1}^{\mathfrak{p}-1}\,\Upsilon_2(\varkappa,F^{n-1}(\varkappa))}{M(\mathfrak{q})} \\
&+ \frac{\mathfrak{p}\,\mathfrak{q}}{M(\mathfrak{q})}\int_{\varkappa_n}^{\varkappa_{n+1}} F_2^*(u)du, \\
\Theta(n+1) &= \Theta(n) + \frac{\mathfrak{p}\,(1-\mathfrak{q})\,\varkappa_n^{\mathfrak{p}-1}\,\Upsilon_3(\varkappa,F^n(\varkappa))}{M(\mathfrak{q})} \\
&- \frac{\mathfrak{p}\,(1-\mathfrak{q})\,\varkappa_{n-1}^{\mathfrak{p}-1}\,\Upsilon_3(\varkappa,F^{n-1}(\varkappa))}{M(\mathfrak{q})} \\
&+ \frac{\mathfrak{p}\,\mathfrak{q}}{M(\mathfrak{q})}\int_{\varkappa_n}^{\varkappa_{n+1}} F_3^*(u)du.
\end{aligned}$$

By integrating the above integrals according to limits and taking $\varkappa_j - \varkappa_{j-1} = h$, we get the final results.

$$
\begin{aligned}
\Delta(n+1) &= \Delta(n) + \frac{\mathfrak{p}\,(1-\mathfrak{q})\,\varkappa_n^{\mathfrak{p}-1}\,\Upsilon_1(\varkappa, F^n(\varkappa))}{M(\mathfrak{q})} - \frac{\mathfrak{p}\,(1-\mathfrak{q})\,\varkappa_{n-1}^{\mathfrak{p}-1}\,\Upsilon_1(\varkappa, F^{n-1}(\varkappa))}{M(\mathfrak{q})} \\
&+ \frac{\mathfrak{p}\,\mathfrak{q}}{M(\mathfrak{q})}[\frac{3}{2}\,h\,\varkappa_n^{\mathfrak{p}-1}\,\Upsilon_1(\varkappa, F^n(\varkappa)) - \frac{1}{2}\,h\,\varkappa_{n-1}^{\mathfrak{p}-1}\,\Upsilon_1(\varkappa, F^{n-1}(\varkappa)), \\
\aleph(n+1) &= \aleph(n) + \frac{\mathfrak{p}\,(1-\mathfrak{q})\,\varkappa_n^{\mathfrak{p}-1}\,\Upsilon_2(\varkappa, F^n(\varkappa))}{M(\mathfrak{q})} - \frac{\mathfrak{p}\,(1-\mathfrak{q})\,\varkappa_{n-1}^{\mathfrak{p}-1}\,\Upsilon_2(\varkappa, F^{n-1}(\varkappa))}{M(\mathfrak{q})} \\
&+ \frac{\mathfrak{p}\,\mathfrak{q}}{M(\mathfrak{q})}[\frac{3}{2}\,h\,\varkappa_n^{\mathfrak{p}-1}\,\Upsilon_2(\varkappa, F^n(\varkappa)) - \frac{1}{2}\,h\,\varkappa_{n-1}^{\mathfrak{p}-1}\,\Upsilon_2(\varkappa, F^{n-1}(\varkappa)), \\
\Theta(n+1) &= \Theta(n) + \frac{\mathfrak{p}\,(1-\mathfrak{q})\,\varkappa_n^{\mathfrak{p}-1}\,\Upsilon_3(\varkappa, F^n(\varkappa))}{M(\mathfrak{q})} - \frac{\mathfrak{p}\,(1-\mathfrak{q})\,\varkappa_{n-1}^{\mathfrak{p}-1}\,\Upsilon_3(\varkappa, F^{n-1}(\varkappa))}{M(\mathfrak{q})} \\
&+ \frac{\mathfrak{p}\,\mathfrak{q}}{M(\mathfrak{q})}[\frac{3}{2}\,h\,\varkappa_n^{\mathfrak{p}-1}\,\Upsilon_3(\varkappa, F^n(\varkappa)) - \frac{1}{2}\,h\,\varkappa_{n-1}^{\mathfrak{p}-1}\,\Upsilon_3(\varkappa, F^{n-1}(\varkappa)).
\end{aligned}
$$

Since in the original model in $\Upsilon_1$ and $\Upsilon_3$ , $\Theta$ depends on $\varkappa$ and $(\varkappa - tau) = \varkappa_1(say)$, so we write

$\Upsilon_1 = U_1(\varkappa^n, \Delta^n, \aleph^n, \Theta^n) + U_3((\varkappa_1)^n, \Theta^n)$, and $\Upsilon_3 = U_2(\varkappa^n, \Delta^n, \aleph^n, \Theta^n) - U_3((\varkappa_1)^n, \Theta^n)$, where

$U_1(\varkappa^n, \Delta^n, \aleph^n, \Theta^n) = \Pi\,\theta - \beta_0 f(\aleph(\varkappa))\Delta(\varkappa) - (\mu + \nu)\Delta(\varkappa)$,

$U_2(\varkappa^n, \Delta^n, \aleph^n, \Theta^n) = (1 - \Pi)\theta + \nu\Delta(\varkappa) + \kappa\,\aleph(\varkappa) - \mu\Theta(\varkappa)$,

$U_3((\varkappa_1)^n, \Theta^n) = \zeta\Theta(\varkappa - \tau)$.

Hence our numerical scheme is :

$$
\begin{aligned}
\Delta(n+1) &= \Delta(n)+\frac{\mathfrak{p}\,(1-\mathfrak{q})\,\varkappa_n^{\mathfrak{p}-1}\,[U_1(\varkappa^n,\Delta^n,\aleph^n,\Theta^n)+U_3((\varkappa_1)^n,\Theta^n)]}{M(\mathfrak{q})}\\
&- \frac{\mathfrak{p}\,(1-\mathfrak{q})\,\varkappa_{n-1}^{\mathfrak{p}-1}\,[U_1(\varkappa^{n-1},\Delta^{n-1},\aleph^{n-1},\Theta^{n-1})+U_3((\varkappa_1)^{n-1},\Theta^{n-1})}{M(\mathfrak{q})}\\
&+ \frac{\mathfrak{p}\,\mathfrak{q}}{M(\mathfrak{q})}[\frac{3}{2}\,h\,\varkappa_n^{\mathfrak{p}-1}\,(U_1(\varkappa^n,\Delta^n,\aleph^n,\Theta^n)+U_3((\varkappa_1)^n,\Theta^n))\\
&- \frac{1}{2}\,h\,\varkappa_{n-1}^{\mathfrak{p}-1}\,(U_1(\varkappa^{n-1},\Delta^{n-1}(\varkappa),\aleph^{n-1}(\varkappa),\Theta^{n-1}(\varkappa))+U_3((\varkappa_1)^{n-1},\Theta^{n-1})),\\
\aleph(n+1) &= \aleph(n)+\frac{\mathfrak{p}\,(1-\mathfrak{q})\,\varkappa_n^{\mathfrak{p}-1}\,\Upsilon_2(\varkappa^n,\Delta^n(\varkappa),\aleph^n(\varkappa),\Theta^n(\varkappa))}{M(\mathfrak{q})}\\
&- \frac{\mathfrak{p}\,(1-\mathfrak{q})\,\varkappa_{n-1}^{\mathfrak{p}-1}\,\Upsilon_2(\varkappa^{n-1},\Delta^{n-1}(\varkappa),\aleph^{n-1}(\varkappa),\Theta^{n-1}(\varkappa))}{M(\mathfrak{q})}\\
&+ \frac{\mathfrak{p}\,\mathfrak{q}}{M(\mathfrak{q})}[\frac{3}{2}\,h\,\varkappa_n^{\mathfrak{p}-1}\,\Upsilon_2(\varkappa^n,\Delta^n(\varkappa),\aleph^n(\varkappa),\Theta^n(\varkappa))\\
&- \frac{1}{2}\,h\,\varkappa_{n-1}^{\mathfrak{p}-1}\,\Upsilon_2(\varkappa^{n-1},\Delta^{n-1}(\varkappa),\aleph^{n-1}(\varkappa),\Theta^{n-1}(\varkappa)),\\
\Theta(n+1) &= \Theta(n)+\frac{\mathfrak{p}\,(1-\mathfrak{q})\,\varkappa_n^{\mathfrak{p}-1}\,(U_1(\varkappa^n,\Delta^n,\aleph^n,\Theta^n)-U_3((\varkappa_1)^n,\Theta^n))}{M(\mathfrak{q})}\\
&- \frac{\mathfrak{p}\,(1-\mathfrak{q})\,\varkappa_{n-1}^{\mathfrak{p}-1}\,(U_3(\varkappa^{n-1},\Delta^{n-1}(\varkappa),\aleph^{n-1}(\varkappa),\Theta^{n-1}(\varkappa))-U_3((\varkappa_1)^{n-1},\Theta^{n-1}))}{M(\mathfrak{q})}\\
&+ \frac{\mathfrak{p}\,\mathfrak{q}}{M(\mathfrak{q})}[\frac{3}{2}\,h\,\varkappa_n^{\mathfrak{p}-1}\,(U_1(\varkappa^n,\Delta^n,\aleph^n,\Theta^n)-U_3((\varkappa_1)^n,\Theta^n))\\
&- \frac{1}{2}\,h\,\varkappa_{n-1}^{\mathfrak{p}-1}\,(U_1(\varkappa^{n-1},\Delta^{n-1}(\varkappa),\aleph^{n-1}(\varkappa),\Theta^{n-1}(\varkappa))-U_3((\varkappa_1)^{n-1},\Theta^{n-1})).
\end{aligned}
$$

## 3.7 Simulations based on Numerical algorithm

Now, we see the effect of different fractal and fractional orders, change in infection rate, loss rate of immunity of recovered nodes and real time immune rate of antivirus strategies using Matlab R2016a under parameters $\Pi = 0.5, \theta = 0.8, \beta_0 = 0.02, \mu = 0.1, \nu = 0.2, \zeta = 0.01, \kappa = 0.2, \tau = 7.3, \alpha = 1$ with initial conditions $\Delta(0) = 3, \aleph(0) = 1, \Theta(0) = 0.1$, on the deterministic model given by system of ODES.

Figs.1-3 show the simulation of $\Delta$, $\aleph$ and $\Theta$ for different fractal and fractional orders separately that means in one figure under the different fractional orders keeping fractal order one (fractional model) and in the other figure under different fractal orders by keeping fractional order one (fractal model). We observe that graphs for no. of nodes

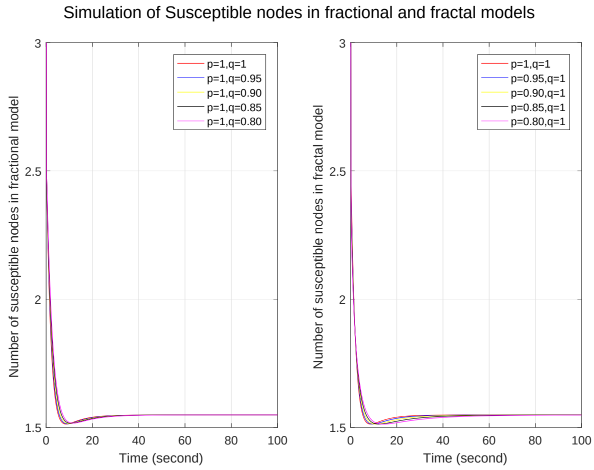


Figure 3.1: Trajectories of $\Delta(\varkappa)$ for different fractal orders $\mathfrak{p}$ when $\mathfrak{q} = 1$ and different fractional orders $\mathfrak{q}$ when $\mathfrak{p} = 1$.

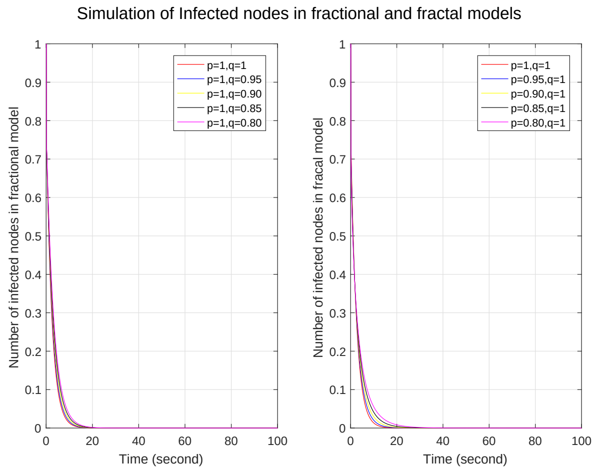


Figure 3.2: Trajectories of $\aleph(\varkappa)$ for different fractal orders $\mathfrak{p}$ when $\mathfrak{q} = 1$ and different fractional orders $\mathfrak{q}$ when $\mathfrak{p} = 1$.

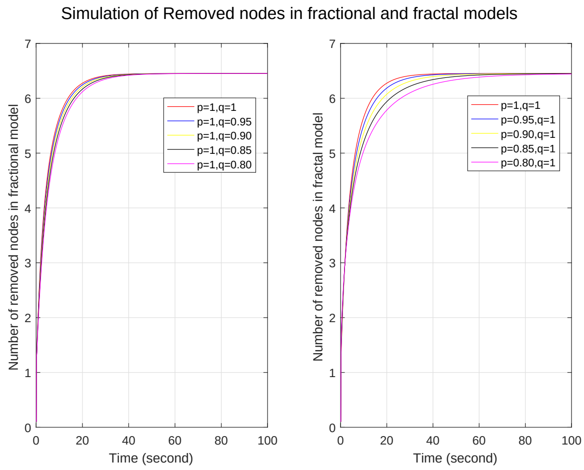


Figure 3.3: Trajectories of $\Theta(\varkappa)$ for different fractal orders $\mathfrak{p}$ when $\mathfrak{q} = 1$ and different fractional orders $\mathfrak{q}$ when $\mathfrak{p} = 1$.

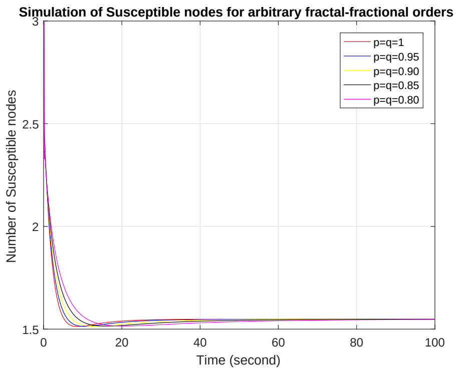


Figure 3.4: Fractal and fractional trajectories of $\Delta(\varkappa)$ with different orders of $\mathfrak{p} = \mathfrak{q}$.

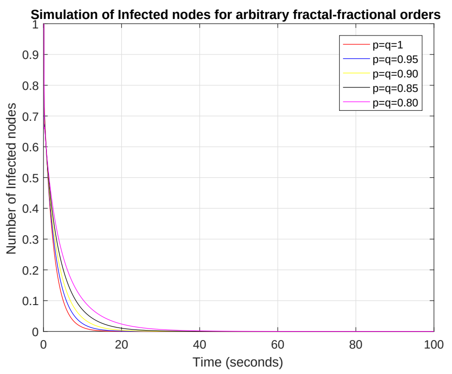


Figure 3.5: Fractal and fractional trajectories of $\aleph(\varkappa)$ with different orders of $\mathfrak{p} = \mathfrak{q}$.

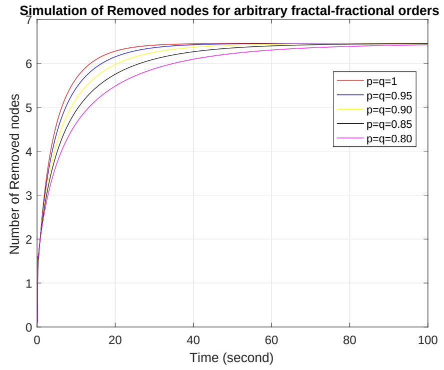


Figure 3.6: Fractal and fractional trajectories of $\Theta(\varkappa)$ with different orders of $\mathfrak{p} = \mathfrak{q}$.

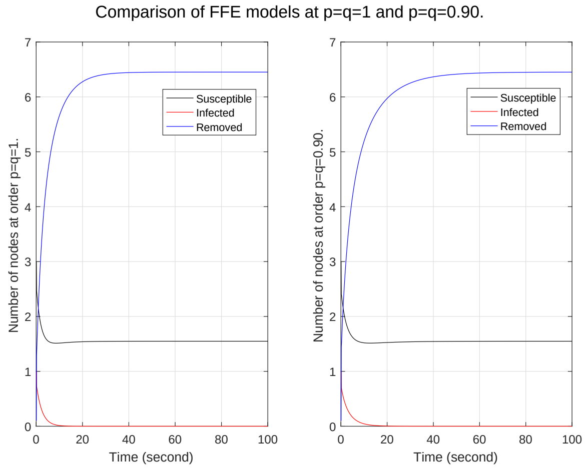


Figure 3.7: Comparison of $\Delta\aleph\Theta$ model at $\mathfrak{p} = \mathfrak{q} = 1$ and $\mathfrak{p} = \mathfrak{q} = 0.90$.

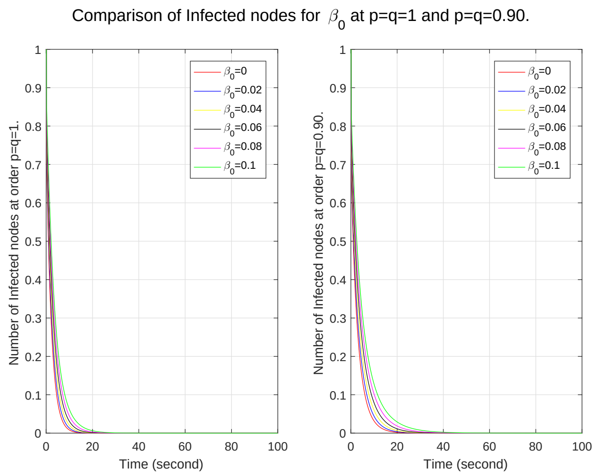


Figure 3.8: The effect of varying initial infection rate $\beta_0$ on infected nodes when $\mathfrak{p} = \mathfrak{q} = 1$ and $\mathfrak{p} = \mathfrak{q} = 0.90$.

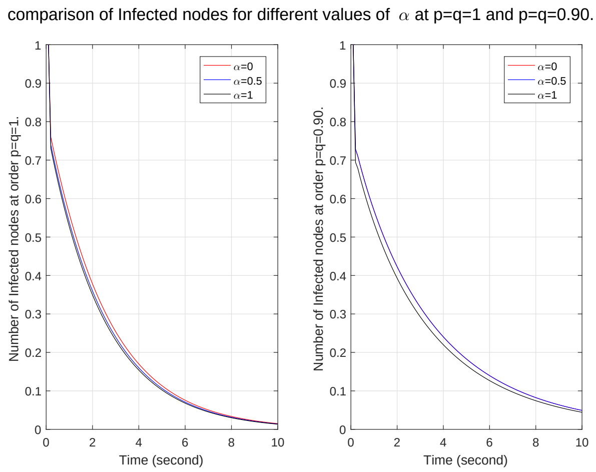


Figure 3.9: The effect of $\alpha$ (varying variable to adjust the infection rate sensitivity) on infected nodes when $\mathfrak{p} = \mathfrak{q} = 1$ and $\mathfrak{p} = \mathfrak{q} = 0.90$.

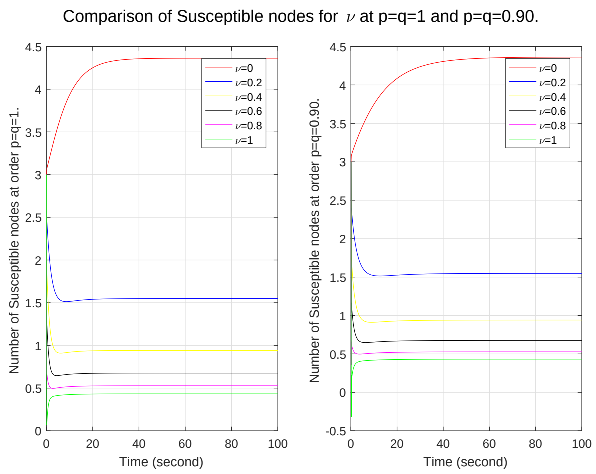


Figure 3.10: The effect of varying real-time immune rate $\nu$ on susceptible nodes when $\mathfrak{p} = \mathfrak{q} = 1$ and $\mathfrak{p} = \mathfrak{q} = 0.90$.

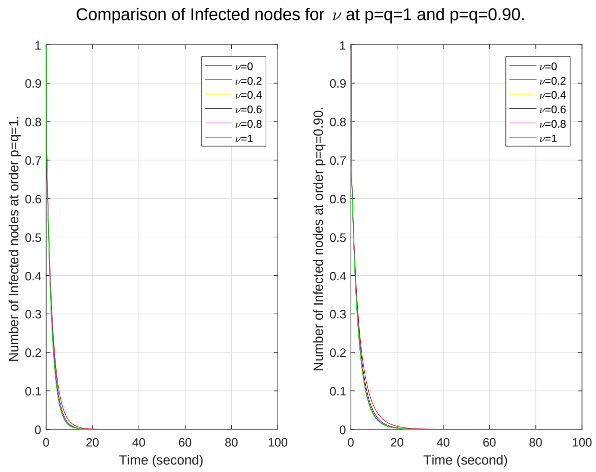


Figure 3.11: The effect of varying real-time immune rate $\nu$ on infected nodes when $\mathfrak{p} = \mathfrak{q} = 1$ and $\mathfrak{p} = \mathfrak{q} = 0.90$.

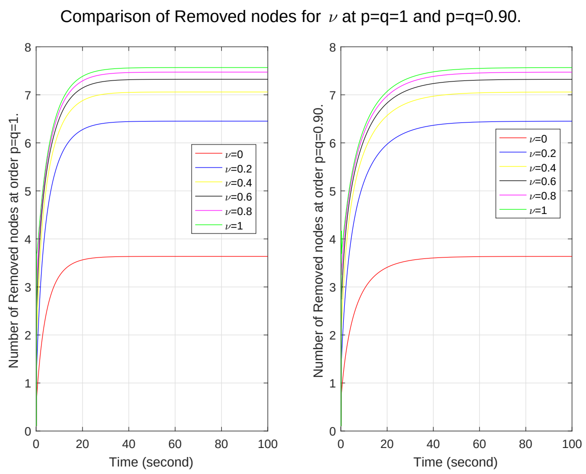


Figure 3.12: The effect of varying real-time immune rate $\nu$ on removed nodes when $\mathfrak{p} = \mathfrak{q} = 1$ and $\mathfrak{p} = \mathfrak{q} = 0.90$.

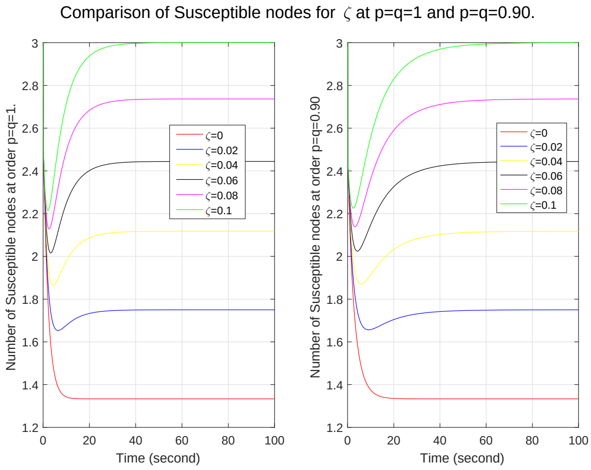


Figure 3.13: The effect of varying loss rate of immunity $\zeta$ on susceptible nodes when $\mathfrak{p} = \mathfrak{q} = 1$ and $\mathfrak{p} = \mathfrak{q} = 0.90$.

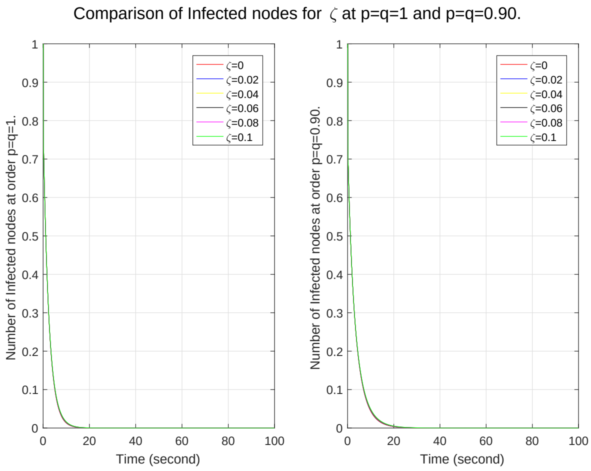


Figure 3.14: The effect of varying loss rate of immunity $\zeta$ on infected nodes when $\mathfrak{p} = \mathfrak{q} = 1$ and $\mathfrak{p} = \mathfrak{q} = 0.90$.

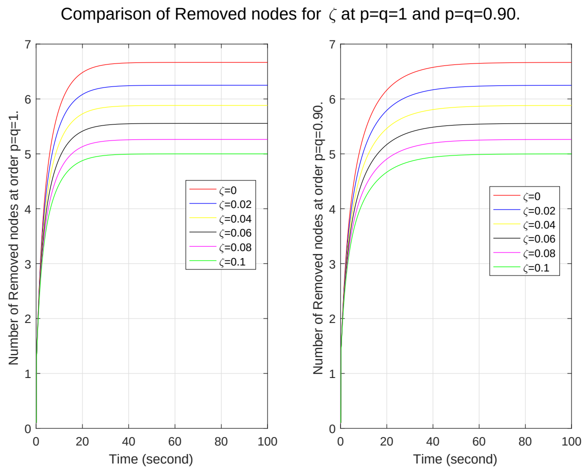


Figure 3.15: The effect of varying loss rate of immunity $\zeta$ on removed nodes when $\mathfrak{p} = \mathfrak{q} = 1$ and $\mathfrak{p} = \mathfrak{q} = 0.90$.

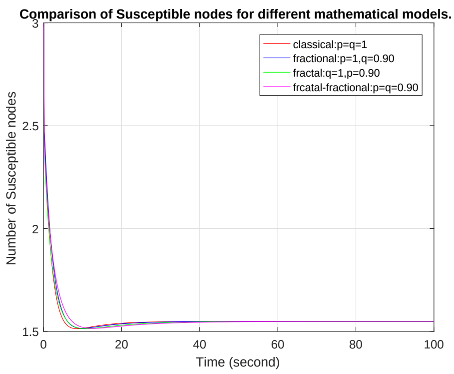


Figure 3.16: Trajectories of $\Delta$ for different mathematical models.

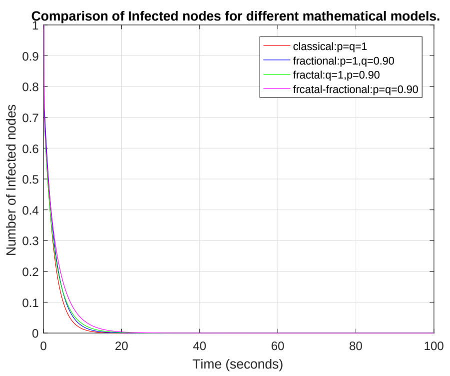


Figure 3.17: Trajectories of $\aleph$ for different mathematical models.

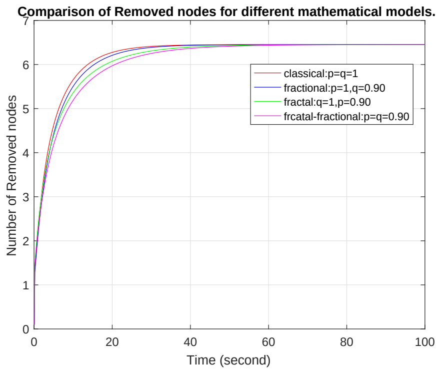


Figure 3.18: Trajectories of $\Theta$ for different mathematical models.

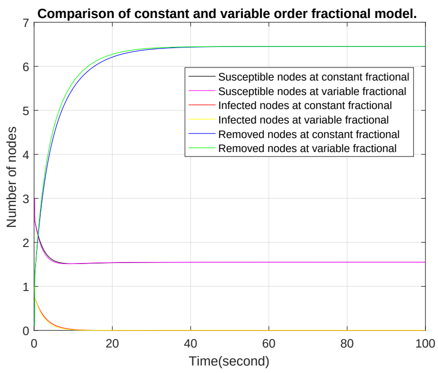


Figure 3.19: Trajectories of $\Delta$, $\aleph$ and $\Theta$ showing comparison of constant and variable fractional mathematical models.

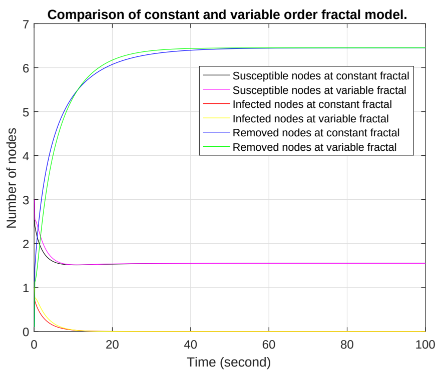


Figure 3.20: Trajectories of $\Delta$, $\aleph$ and $\Theta$ showing comparison of constant and variable fractal mathematical models.

in fractional model are very close to each other which show that nodes are strongly connected and indicate a more homogeneous structure with fewer variation while distance between nodes at different orders represents long-range corrections and depicts a more heterogenous and complex structure. For fractal models, in fig.1 initially at lower fractal orders, no. of nodes is greater then it goes to decrease which shows that initially system has strong memory effect and as time passes, the memory effect decays. In figs.2-3, no. of infected nodes goes on decreasing and no. of removed nodes goes on decreasing for lower fractal orders. It also represents that in fractal model system has strong memory effect.

Figs.4-6 show the simulation of $\Delta$, $\aleph$ and $\Theta$ under the combined effect of different fractal and fractional orders. We see that in fig.4, as FF order decreases, no. of susceptible nodes is greater initially, then decreases and finally converges. It means initially there is a higher degree of connectivity and vulnerability to infection, decrease shows a reduction in both connectivity and vulnerability probably due to increased immunity and convergence shows that the system became stable. The effect of FF orders shows that the memory effect is more initially, then it decays with passage of time. In fig.5, the greater no. of nodes at lower FF orders indicates a stronger memory effect and a higher potential for epidemic spread. We also observe that no. of infected nodes become zero early at higher ff order, it tells us about increased resilience and improved immunity of the system. It also indicates about lower transmission rates of infection and increased isolation. In fig.6, no. of removed nodes is less at lower ff orders, it shows a longer persistence and increased prevalence of infection, also a longer and stronger memory effect. In Fig.7, we compare classical model $\mathfrak{p} = \mathfrak{q} = 1$ and fractal-fractional model at $\mathfrak{p} = \mathfrak{q} = 0.90$ which show behavior of all nodes in one look.

Since in our original model, infection rate depends on initial infection rate $\beta_0$ and also on a function of $\aleph$ which depends on another variable called $\alpha$. Therefore, we see the impact of both variables on our fractal-fractional model too. Now we examine the graphs in two ways. First, we examine the behavior of needs for different ff orders in each model. Secondly, we compare the behavior of these nodes in both models. In fig.8, as the initial infection rate increases, no. of nodes also increases, it shows that the system may be more susceptible to infection, may exhibit a faster spread of infection due to a larger pool of infected nodes, may shows non-linear dynamics where small changes in $\beta_0$ lead to large changes in no. of infected nodes. It also represents that system has a stronger memory effect. As we compare, we see that in lower ff model the system has more flexibility to become infectious and has a stronger memory effect. In the original model, $\alpha$ is used to adjust the infection rate sensitivity to $\aleph$ and $\alpha = 0$ means constant infection rate. In fig.9 we see the effect of this varying variable. Although there seems a very slight difference but it plays a role in the dynamics of system in coordination of other parameters.

In figs.10-12, we see the effect of real-time immune rate $\nu$ on $\Delta$, $\aleph$ and $\Theta$ for $\mathfrak{p} = \mathfrak{q} = 1$ and $\mathfrak{p} = \mathfrak{q} = 0.90$. Fig.10 describes that as immune rate increases, no. of susceptible nodes become less in each model. We can say that it shows a strong immune response, decreased risk of infection, more resilience to infection and it may eradicate infection entirely. Moreover, as we compare both models, we see that that at lower ff model, no. of nodes is smaller. It shows increased complexity, slower spread of infection, increased clustering, improved resilience and enhanced robustness. In fig.11, no. of infected nodes are very close to each other for different values of $\nu$ in each model. It depicts that the system may have reached a saturation point and it may exhibit diminishing returns. Also for ff model at 0.90, no.of nodes are greater, which represents increased complexity, faster speed of infections, increased vulnerability and reduced resilience. Similarly, in fig.12 the no. of removed nodes is greater for higher real time immune rate in each model. It shows an effective immune response which is capable of eliminating infected nodes efficiently, faster clearance rate, increased resilience in the system and enhanced robustness of system. As we compare both models, the no. of nodes is less in ff model at level 0.90 which shows strong memory effect and increased complexity of the system.

As we know that the recovered nodes lose their immunity after some time, so to see this impact, we check the graphs of $\Delta, \aleph, \text{and} \Theta$. Figs.13-15, show the effect of $\zeta$ (loss rate of immunity). From fig.13 we see that no. of susceptible nodes goes on increasing as lost rate of immunity gets higher in each model. Also, ff model at 0.90 exhibits robustness to immunity loss and may introduce unique effects that mitigate the impact of immunity loss and show effective immune response with increased resilience. By fig.14 no. of infected nodes remain same at different rates of immunity loss in each model represents that the immune response may have reached a saturation point and system has reached at equilibrium state. It also shows robustness and resilience of system. Moreover, the no. of infected nodes approaches to zero earlier in classical model. It describes that ff model at 0.90 shows a delayed eradication of infection, slower immune response, less efficiency and increased vulnerability in the system. Similarly, no. of removed nodes goes on decreasing at higher loss of immunity in each model in fig.15. It shows reduced immune efficiency, longer persistence, increase in vulnerability and reduced resilience of the system. In comparison of models,ff model at 0.90 shows that no. of removed nodes is less than classical which indicates impaired immune function, persistence of infection, increase in vulnerability and decrease in system's resilience.

In Figs.16-18, we compare four mathematical models named classical, fractional, fractal and fractal-fractional. From Figs.16,17 no. of susceptible and infected nodes is highest in FF model, then in fractional, fractal and classical simultaneously. It shows that fractal-fractional is more effective for expressing the complexity of malware propagation and fractional model may also be used in some cases. On the other hand, fractal and classical methods are not suitable for complex systems. The higher no. of nodes represents deeper memory effect and strong correlation between nodes. Moreover, convergence indicates that the system is stable. Similarly in Fig.18, no. of removed nodes in fractal fractional model is lowest that show deep memory effect and strong correlation. Convergence shows stability of the system. In Fig.19, we see the difference between constant fractional order and variable fractional order. Similarly, Fig.20 shows the comparison of constant and variable fractal order. We take variable fractional order as $\mathfrak{q}(\varkappa) = 0.90+0.1/(1+exp(-\varkappa))$ and variable fractal order as $\mathfrak{p}(\varkappa) = 0.1/0.9 + exp(-\varkappa)$. We see that susceptible and infected nodes merge earlier after some time, i.e variable order becomes constant. The no. of nodes in variable order is greater than constant order depicts that variable order has more advantage of removing nodes that lead to more effective epidemic control, the system is more adaptive and variable fractional and fractal order represent more effective control strategies.

## 3.8 Conclusion

In this chapter, we have examined fractal fractional model with exponential decay kernel theoretically. Conditions for existence (Leray Schauder criteria), uniqueness (Lipschitz property) and stability (Ulam-Hyers and Ulam-Hyers-Rassias theorems) of the fractal fractional model were examined using concepts of fixed point theory. In second stage, numerical scheme was developed and simulations were performed to verify the accuracy of theoretical results. Our FF model was examined under fractal dimensions and fractional orders separately and combined effect of fractal dimensions and fractional orders. We observed that at lower FF orders, the number of susceptible and infected nodes was higher while no. of removed nodes is lower. It demonstrates the sensitivity to external influences, resilience to adapt infection and strong memory effects. We found out that removed nodes have higher containment of infection and persistence at lower level of FF orders. At the next stage, we compared model for classical (FF order at one) and fractal fractional model for orders $p = q = 0.90$. We examined the impact of different parameters such as initial infection rate, variable adjustment to sensitivity of infected nodes, immune rate of antivirus strategies and loss rate of immunity of recovered nodes of mathematical model [6] under $p = q = 1$ and $p = q = 0.90$. Through the graphs we find out the effect of memory on different types of nodes in system. We explored sensitivity, convergence, and stability of susceptible, infected, and removed nodes under fractal fractional model. It will help us to predict about the vulnerabilities in computer systems. Antivirus strategies can be made by developing software that may help in containment and eradication of infection in the nodes by keeping an eye on the behavior of nodes. The graphs gave a clear insight that by choosing appropriate variable infection rate, the prevalence of malware can be controlled. Continuing this process, we investigated the impacts of other parameters too on malware model. We also compared four methods (classical, fractional, fractal, fractal-fractional). We discussed the cases when these models may be more suitably used. Moreover, we tried to see the impact of variable order fractional derivative and variable order fractal derivative. Although sometimes we see a very small difference, but it may play a role in malware propagation as small changes may cause large perturbations.

# Chapter 4

# Fractal Fractional Mathematical Model with Mittag-Leffler kernel

In this chapter, we will convert classical mathematical model in Fractal-Fractional with Mittag Leffler kernel. After converting in fixed point problem, we treat it under fixed point theory and find results about existence, uniqueness and stability. Numerical simulation is done, analysis is performed by coding in Matlab and results are concluded.

## 4.1 Conversion in Fractal-Fractional Mathematical Model with Mittag-Leffler Kernel

We discuss the model (1.3.2) in terms of fractal fractional derivatives with Mittag leffler kernel as:

$$\begin{aligned} {}^{FFM}D^{\mathfrak{q},\mathfrak{p}}_{0,\varkappa}\Delta(\varkappa) &= \Pi\Theta - \beta_0 f(\aleph(\varkappa))\Delta(\varkappa) - (\mu+\nu)\Delta(\varkappa) + \zeta\Theta(\varkappa-\tau), \\ {}^{FFM}D^{\mathfrak{q},\mathfrak{p}}_{0,\varkappa}\aleph(\varkappa) &= \beta_0 f(\aleph(\varkappa))\Delta(\varkappa) - (\mu+\kappa)\aleph(\varkappa), \\ {}^{FFM}D^{\mathfrak{q},\mathfrak{p}}_{0,\varkappa}\Theta(\varkappa) &= (1-\Pi)\Theta + \nu\Delta(\varkappa) + \kappa\aleph(\varkappa) - \zeta\Theta(\varkappa-\tau) - \mu\Theta(\varkappa), \end{aligned} \tag{4.1.1}$$

with $\Delta(0) = \Delta_0 \geq 0, \aleph(0) = \aleph_0 \geq 0, \Theta(0) = \Theta_0 \geq 0$ and $N(\varkappa) = \Delta(\varkappa)+\aleph(\varkappa)+\Theta(\varkappa)$, for $\varkappa \in \mathrm{J} = [0,\mathrm{T}], \mathrm{T} > 0$ . Also $\mathfrak{p}$ , $\mathfrak{q} \in (0,1]$ and all parameters are to be taken non-negative.

## 4.2 Formulation of Model as Fixed Point Problem

In this section, we convert fractal fractional model as a problem of fixed point theory. Consider $\Xi = \mathfrak{Y}^3$, a Banach space and $\mathfrak{Y} = C(\mathrm{J}, \mathbb{R})$ represents the class of all continuous functions with the norm defined as:

$$|||F||_{\Xi} = ||(\Delta, \aleph, \Theta)||_{\Xi} = max\{|\Delta(\varkappa)| + |\aleph(\varkappa)| + |\Theta(\varkappa)| : \varkappa \in \mathrm{J}\}.$$

Firstly, we rewrite Model 1.3.2 as:

$$\begin{aligned}
&\Upsilon_1(\varkappa, \Delta(\varkappa), \aleph(\varkappa), \Theta(\varkappa)) = \Pi\Theta - \beta_0 f(\aleph(\varkappa))\Delta(\varkappa) - (\mu + \nu)\Delta(\varkappa) + \zeta\Theta(\varkappa - \tau),\\
&\Upsilon_2(\varkappa, \Delta(\varkappa), \aleph(\varkappa), \Theta(\varkappa)) = \beta_0 f(\aleph(\varkappa))\Delta(\varkappa) - (\mu + \kappa)\aleph(\varkappa), \qquad (4.2.1)\\
&\Upsilon_3(\varkappa, \Delta(\varkappa), \aleph(\varkappa), \Theta(\varkappa)) = (1 - \Pi)\Theta + \nu\Delta(\varkappa) + \kappa\aleph(\varkappa) - \zeta\Theta(\varkappa - \tau) - \mu\Theta(\varkappa).
\end{aligned}$$

Comparing Models 1.3.2 and 4.1.1 we have,

$$\begin{aligned}
{}^{FFM}D^{\mathfrak{q},\mathfrak{p}}_{0,\varkappa}\Delta(\varkappa) &= \Upsilon_1(\varkappa, \Delta(\varkappa), \aleph(\varkappa), \Theta(\varkappa)),\\
{}^{FFM}D^{\mathfrak{q},\mathfrak{p}}_{0,\varkappa}\aleph(\varkappa) &= \Upsilon_2(\varkappa, \Delta(\varkappa), \aleph(\varkappa), \Theta(\varkappa)), \qquad (4.2.2)\\
{}^{FFM}D^{\mathfrak{q},\mathfrak{p}}_{0,\varkappa}\Theta(\varkappa) &= \Upsilon_3(\varkappa, \Delta(\varkappa), \aleph(\varkappa), \Theta(\varkappa)).
\end{aligned}$$

Since

$$\begin{aligned}
{}^{FFM}D^{\mathfrak{q},\mathfrak{p}}_{0,\varkappa}g(\varkappa) &= \frac{AB(\mathfrak{q})}{1-\mathfrak{q}}\frac{d}{d\varkappa^{\mathfrak{p}}}\int_0^{\varkappa} E_{\mathfrak{q}}[-\frac{\mathfrak{q}}{1-\mathfrak{q}}\,(\varkappa - u)^{\mathfrak{q}}]\,g(u)du.\\
{}^{FFM}D^{\mathfrak{q},\mathfrak{p}}_{0,\varkappa}g(\varkappa) &= \frac{AB(\mathfrak{q})}{1-\mathfrak{q}}\frac{1}{\mathfrak{p}\varkappa^{\mathfrak{p}-1}}\frac{d}{d\varkappa}\int_0^{\varkappa} E_{\mathfrak{q}}[-\frac{\mathfrak{q}}{1-\mathfrak{q}}\,(\varkappa - u)^{\mathfrak{q}}]\,g(u)du.\\
{}^{FFM}D^{\mathfrak{q},\mathfrak{p}}_{0,\varkappa}g(\varkappa) &= \frac{1}{\mathfrak{p}\varkappa^{\mathfrak{p}-1}}\frac{AB(\mathfrak{q})}{1-\mathfrak{q}}\frac{d}{d\varkappa}\int_0^{\varkappa} E_{\mathfrak{q}}[-\frac{\mathfrak{q}}{1-\mathfrak{q}}\,(\varkappa - u)^{\mathfrak{q}}]\,g(u)du.
\end{aligned}$$

Now, we can write

$$\frac{AB(\mathfrak{q})}{1-\mathfrak{q}}\frac{d}{d\varkappa}\int_0^{\varkappa} E_{\mathfrak{q}}[-\frac{\mathfrak{q}}{1-\mathfrak{q}}\,(\varkappa - u)^{\mathfrak{q}}]\,g(u)du,$$

as Riemann Liouville fractional derivative with Mittag Laffler kernel.

Therefore, we get

$${}^{FFM}D^{\mathfrak{q},\mathfrak{p}}_{0,\varkappa}g(\varkappa) = (\frac{1}{\mathfrak{p}\varkappa^{\mathfrak{p}-1}})\,{}^{RL}D^{\mathfrak{q}}_{0,\varkappa}g(\varkappa)$$

Hence, Model 4.2.2 can be written as:

$$\begin{aligned}
(\frac{1}{\mathfrak{p}\varkappa^{\mathfrak{p}-1}})\,{}^{RL}D^{\mathfrak{q}}_{0,\varkappa}\Delta(\varkappa) &= \Upsilon_1(\varkappa, \Delta(\varkappa), \aleph(\varkappa), \Theta(\varkappa)),\\
(\frac{1}{\mathfrak{p}\varkappa^{\mathfrak{p}-1}})\,{}^{RL}D^{\mathfrak{q}}_{0,\varkappa}\aleph(\varkappa) &= \Upsilon_2(\varkappa, \Delta(\varkappa), \aleph(\varkappa), \Theta(\varkappa)), \qquad (4.2.3)\\
(\frac{1}{\mathfrak{p}\varkappa^{\mathfrak{p}-1}})\,{}^{RL}D^{\mathfrak{q}}_{0,\varkappa}\Theta(\varkappa) &= \Upsilon_3(\varkappa, \Delta(\varkappa), \aleph(\varkappa), \Theta(\varkappa)).
\end{aligned}$$

Moreover, we have

$$\begin{aligned}
{}^{RL}D^{\mathfrak{q}}_{0,\varkappa}\Delta(\varkappa) &= \mathfrak{p}\varkappa^{\mathfrak{p}-1}\Upsilon_1(\varkappa,\Delta(\varkappa),\aleph(\varkappa),\Theta(\varkappa)),\\
{}^{RL}D^{\mathfrak{q}}_{0,\varkappa}\aleph(\varkappa) &= \mathfrak{p}\varkappa^{\mathfrak{p}-1}\Upsilon_2(\varkappa,\Delta(\varkappa),\aleph(\varkappa),\Theta(\varkappa)),\\
{}^{RL}D^{\mathfrak{q}}_{0,\varkappa}R(\varkappa) &= \mathfrak{p}\varkappa^{\mathfrak{p}-1}\Upsilon_3(\varkappa,\Delta(\varkappa),\aleph(\varkappa),\Theta(\varkappa)).
\end{aligned} \tag{4.2.4}$$

In general, we can rewrite Model 4.2.4 as:

$$\begin{aligned}
{}^{RL}D^{\mathfrak{q}}_{0,\varkappa}\digamma(\varkappa) &= \mathfrak{p}\varkappa^{\mathfrak{p}-1}\Upsilon(\varkappa,\digamma(\varkappa)),\\
\digamma(0) &= \digamma_0,
\end{aligned} \tag{4.2.5}$$

where for $(\mathfrak{p},\mathfrak{q}) \in (0,1]$ and for $\varkappa \in \mathrm{J}$, we have

$$\begin{aligned}
\digamma(\varkappa) &= (\Delta(\varkappa),\aleph(\varkappa),\Theta(\varkappa))^\top,\\
\digamma_0 &= (\Delta_0,\aleph_0,\Theta_0)^\top.
\end{aligned} \tag{4.2.6}$$

Applying Fractal−Fractional integral on Model (4.2.5), using the result [15], we have

$$\digamma(\varkappa) = \digamma(0) + \frac{\mathfrak{p}\,(1-\mathfrak{q})\,\varkappa^{\mathfrak{p}-1}\,g(\varkappa)}{AB(\mathfrak{q})} + \frac{\mathfrak{p}\,\mathfrak{q}}{AB(\mathfrak{q})\,\Gamma(\mathfrak{q})}\int_0^{\varkappa} u^{\mathfrak{p}-1}\,(\varkappa-u)^{\mathfrak{q}-1}\,g(u)du, \tag{4.2.7}$$

where $AB(\mathfrak{q}) = 1-\mathfrak{q}+\frac{\mathfrak{q}}{\Gamma(\mathfrak{q})}$.

Hence, we can write

$$\begin{aligned}
\Delta(\varkappa) &= \Delta(0) + \frac{\mathfrak{p}\,(1-\mathfrak{q})\,\varkappa^{\mathfrak{p}-1}\,\Upsilon_1(\varkappa,\digamma(\varkappa))}{AB(\mathfrak{q})} + \frac{\mathfrak{p}\,\mathfrak{q}}{AB(\mathfrak{q})\,\Gamma(\mathfrak{q})}\int_0^{\varkappa} u^{\mathfrak{p}-1}\,(\varkappa-u)^{\mathfrak{q}-1}\,\Upsilon_1(u,\digamma(u))du,\\
\aleph(\varkappa) &= \aleph(0) + \frac{\mathfrak{p}\,(1-\mathfrak{q})\,\varkappa^{\mathfrak{p}-1}\,\Upsilon_2(\varkappa,\digamma(\varkappa))}{AB(\mathfrak{q})} + \frac{\mathfrak{p}\,\mathfrak{q}}{AB(\mathfrak{q})\,\Gamma(\mathfrak{q})}\int_0^{\varkappa} u^{\mathfrak{p}-1}\,(\varkappa-u)^{\mathfrak{q}-1}\,\Upsilon_2(u,\digamma(u))du,\\
\Theta(\varkappa) &= \Theta(0) + \frac{\mathfrak{p}\,(1-\mathfrak{q})\,\varkappa^{\mathfrak{p}-1}\,\Upsilon_3(\varkappa,\digamma(\varkappa))}{AB(\mathfrak{q})} + \frac{\mathfrak{p}\,\mathfrak{q}}{AB(\mathfrak{q})\,\Gamma(\mathfrak{q})}\int_0^{\varkappa} u^{\mathfrak{p}-1}\,(\varkappa-u)^{\mathfrak{q}-1}\,\Upsilon_3(u,\digamma(u))du.
\end{aligned} \tag{4.2.8}$$

So, now we can transform Model (4.2.2) into a fixed point problem.

Define an operator $F\colon \Xi \to \Xi$ as

$$F(\digamma(\varkappa)) = \digamma(0) + \frac{\mathfrak{p}\,(1-\mathfrak{q})\,\varkappa^{\mathfrak{p}-1}\,\Upsilon(\varkappa,\digamma(\varkappa))}{AB(\mathfrak{q})} + \frac{\mathfrak{p}\,\mathfrak{q}}{AB(\mathfrak{q})\,\Gamma(\mathfrak{q})}\int_0^{\varkappa} u^{\mathfrak{p}-1}\,(\varkappa-u)^{\mathfrak{q}-1}\,\Upsilon(u,\digamma(u))du. \tag{4.2.9}$$

## 4.3 Existence of Solution

For existence, we prove the following theorem on the basis of Theorem 1.2.4 as in [35].

**Theorem 4.3.1.** Suppose that $\exists\, V_2 \colon \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R},\ \psi \in \Psi$ and $\Upsilon \in C(\mathrm{J} \times \Xi, \Xi)$ satisfying the following three conditions :

$(\beta_{11})\colon\ \forall \digamma_1, \digamma_2 \in \Xi$ and $\varkappa \in \mathrm{J}$,

$$|\Upsilon(\varkappa, \digamma_1(\varkappa)) - \Upsilon(\varkappa, \digamma_2)(\varkappa)| \leq \ell_2\, \psi(|\digamma_1(\varkappa) - \digamma_2(\varkappa)|),$$

with $V(\digamma_1(\varkappa), \digamma_2(\varkappa)) \geq 0$ and $\ell_2 = \frac{1}{[\frac{\mathfrak{p}(1-\mathfrak{q})}{AB(\mathfrak{q})}\, \varkappa^{\mathfrak{p}-1} + \frac{\mathfrak{q}\,\mathfrak{p}\,\Gamma(\mathfrak{p})\, \varkappa^{\mathfrak{p}+\mathfrak{q}-1}}{AB(\mathfrak{q})\,\Gamma(\mathfrak{p}+\mathfrak{q})}]}$.

$(\beta_{12})\colon\ \exists \digamma_0 \in \Xi$ and $\forall\, \varkappa \in \mathrm{J}$,

$V_2(\digamma_0(\varkappa), F(\digamma_0(\varkappa)) \geq 0$ and $V_2(\digamma_1(\varkappa), \digamma_2(\varkappa)) \geq 0 \implies V_2(F(\digamma_1(\varkappa)), F(\digamma_2(\varkappa)) \geq 0;$

$(\beta_{13})$: $\forall \{\digamma_n\}_{n \geq 1} \subseteq \Xi$ with $\digamma_n \to \digamma$,

$V_2(\digamma_n(\varkappa), \digamma_{n+1}(\varkappa)) \geq 0 \implies V_2(\digamma_n(\varkappa)), \digamma(\varkappa)) \geq 0$, for every $n$ , $\varkappa \in \mathrm{J}$.

Then we say that $F$ has a fixed point. So, there exists a solution of the model of malware propagation.

*Proof.* Take $\digamma_1, \digamma_2 \in \Xi$ so that

$$V_2(\digamma_1(\varkappa), \digamma_2(\varkappa)) \geq 0, \tag{4.3.1}$$

for each $\varkappa \in \mathrm{J}$.

Now we take

$$\begin{aligned} |F(\digamma_1(\varkappa)) - F(\digamma_2(\varkappa))| &= |\frac{\mathfrak{p}(1-\mathfrak{q}) \cdot \varkappa^{\mathfrak{p}-1}}{AB(\mathfrak{q})} [\Upsilon(\varkappa, \digamma_1(\varkappa)) - \Upsilon(\varkappa, \digamma_2(\varkappa)] \\ &+ \frac{\mathfrak{p}\mathfrak{q}}{AB(\mathfrak{q})\,\Gamma(\mathfrak{q})} \int_0^{\varkappa} u^{(\mathfrak{p}-1)}\, (\varkappa - u)^{\mathfrak{q}-1}\, [\Upsilon(u, \digamma_1(u)) - \Upsilon(u, \digamma_2(u))]\, du| \\ &\leq \frac{\mathfrak{p}(1-\mathfrak{q}) \cdot \varkappa^{\mathfrak{p}-1}}{AB(\mathfrak{q})} |\Upsilon(\varkappa, \digamma_1(\varkappa)) - \Upsilon(\varkappa, \digamma_2(\varkappa)| \\ &+ \frac{\mathfrak{p}\mathfrak{q}}{AB(\mathfrak{q})\,\Gamma(\mathfrak{q})} \int_0^{\varkappa} u^{(\mathfrak{p}-1)}\, (\varkappa - u)^{\mathfrak{q}-1}\, |\Upsilon(u, \digamma_1(u)) - \Upsilon(u, \digamma_2(u))|\, du, \end{aligned}$$

using$(\beta_{11})$

$$\begin{aligned} |F(\digamma_1(\varkappa)) - F(\digamma_2(\varkappa))| &\leq \frac{\mathfrak{p}(1-\mathfrak{q}) \cdot \varkappa^{\mathfrak{q}-1}}{AB(\mathfrak{q})}\, \ell_2\, \psi(|\digamma_1(\varkappa) - \digamma_2(\varkappa)| \\ &+ \frac{\mathfrak{p}\mathfrak{q}}{AB(\mathfrak{q})\,\Gamma(\mathfrak{q})} \int_0^{\varkappa} u^{\mathfrak{p}-1}\, (\varkappa - u)^{\mathfrak{q}-1}\, \ell_2 \psi(|\digamma_1(u) - \digamma_2(u)|\, du, \end{aligned}$$

by using the definition of norm

$$\begin{aligned} |F(\digamma_1(\varkappa)) - F(\digamma_2(\varkappa))| &\leq \frac{\mathfrak{p}(1-\mathfrak{q}) \cdot \varkappa^{\mathfrak{p}-1}}{AB(\mathfrak{q})}\, \ell_2\, \psi(||\digamma_1 - \digamma_2|| \\ &+ \frac{\mathfrak{p}\mathfrak{q}}{AB(\mathfrak{q})\,\Gamma(\mathfrak{q})} \int_0^{\varkappa} u^{(\mathfrak{p}-1)}\, (\varkappa - u)^{\mathfrak{q}-1}\, \ell_2\, \psi(||\digamma_1 - \digamma_2||_{\Xi}\, du. \end{aligned}$$

After doing some computations and using the definition of beta function, we get

$$\begin{aligned}
|F(\digamma_1(\varkappa)) - F(\digamma_2(\varkappa))| &\leq \frac{\mathfrak{p}(1-\mathfrak{q})\cdot \varkappa^{\mathfrak{p}-1}}{AB(\mathfrak{q})}\, \ell_2\, \psi(||\digamma_1 - \digamma_2||_{\Xi}) \\
&+ \frac{\mathfrak{p}\,\mathfrak{q}\,\Gamma(\mathfrak{p})\,\varkappa^{\mathfrak{p}+\mathfrak{q}-1}}{AB(\mathfrak{q})\,\Gamma(\mathfrak{p}+\mathfrak{q})}\, \ell_2\, \psi(||\digamma_1 - \digamma_2||_{\Xi}) \\
&\leq \Big[\frac{\mathfrak{p}(1-\mathfrak{q})\cdot \varkappa^{\mathfrak{p}-1}}{AB(\mathfrak{q})} + \frac{\mathfrak{p}\,\mathfrak{q}\,\Gamma(\mathfrak{p})\,\varkappa^{\mathfrak{p}+\mathfrak{q}-1}}{AB(\mathfrak{q})\,\Gamma(\mathfrak{p}+\mathfrak{q})}\Big]\, \ell_2\, \psi(||\digamma_1 - \digamma_2||_{\Xi}).
\end{aligned}$$

Using value of $\ell_2$, we get

$$|F(\digamma_1(\varkappa)) - F(\digamma_2(\varkappa))| \leq \psi(||\digamma_1 - \digamma_2||_{\Xi}). \tag{4.3.2}$$

Moreover, if we define a function $\phi\colon \Xi^2 \to [0,\infty)$ such that $\phi(\digamma_1, \digamma_2) = 1$ for $V(\digamma_1(\varkappa), \digamma_2(\varkappa)) \geq 0$, and zero otherwise, then for each $\digamma_1, \digamma_2 \in \Xi$ equation, (4.3.2) can be written as

$$\phi(\digamma_1, \digamma_2)\, d(F(\digamma_1), F(\digamma_2)) \leq \psi(d(\digamma_1, \digamma_2)). \tag{4.3.3}$$

This shows that $F$ is a *$\phi$-$\psi$-contraction.*

Now suppose that $\digamma_1, \digamma_2 \in \Xi$ with the property that $\phi(\digamma_1, \digamma_2) \geq 1$.

By the definition of $\phi$, we deduce

$$V_2(\digamma_1(\varkappa), \digamma_2(\varkappa)) \geq 0, \tag{4.3.4}$$

and by $(\beta_{12})$

$V_2(\digamma_0(\varkappa), F(\digamma_0(\varkappa)) \geq 0$ and $V_2(\digamma_1(\varkappa), \digamma_2(\varkappa)) \geq 0$.

$\implies V_2(F(\digamma_1(\varkappa)), F(\digamma_2(\varkappa)) \geq 0$.

So, by applying definition of $\phi$, we have

$$\phi(F(\digamma_1), F(\digamma_2) \geq 1. \tag{4.3.5}$$

Hence $F$ is *$\phi$-admissible.* (*)

Moreover, by $(\beta_{12})$, it can be seen that for some $\digamma_0$ in $\Xi$, $\forall \varkappa \in$ J, we have

$V_2(\digamma_0(\varkappa), F(\digamma_0(\varkappa))) \geq 0 \implies \phi(\digamma_0, F(\digamma_0)) \geq 1.$ (**)

Now, consider $\{\digamma_n\}_{n\geq 1} \subseteq \Xi$ with $\digamma_n \to \digamma$ and for all $n$ and $\phi(\digamma_n, \digamma_{n+1}) \geq 1$.

By definition of $\phi$ this implies $V_2(\digamma_n(\varkappa), \digamma_{n+1}(\varkappa)) \geq 0$.

Thus by $(\beta_{13})$ this implies $V_2(\digamma_n(\varkappa), \digamma(\varkappa)) \geq 0$.

Hence $\phi(\digamma_n, \digamma) \geq 1$ for all $n$. (***)

So (*),(**),(***) show the conditions of Theorem 1.2.4 are satisfied, so we can say that there exists some $\digamma^* \in X$such that $F(\digamma^*) = \digamma^*$.
Hence $\digamma^* = (\Delta^*, \aleph^*, \Theta^*)^\top$ is a solution of our model. □

Our next theorem on the basis of theorem(1.2.5) also establishes that the solution of the model exists. For this, we have to show that $F$ is compact.

**Theorem 4.3.2.** Let $\Xi$ be a Banach space, $(\mathfrak{N}_2)_\epsilon$ be a bounded and closed set in $\Xi$ and $A_2$ be an open in $(\mathfrak{N}_2)_\epsilon$ with $0 \in A_2$, then there exists a compact and continuous operator $F$ with the conditions $(\beta_{14})$ and $(\beta_{15})$ from $\overline{A_2} \to (\mathfrak{N}_2)_\epsilon$ which satisfies one of the two conditions,
(a) $F$ has a fixed point in $\overline{A_2}$,
or
(b) there exists $\digamma \in \partial A_2$ and $\omega_2 \in (0,1)$ s.t $\digamma = \omega_2 F(\digamma)$;
where
$(\beta_{14})$: There exists $\phi \in L^1(J, [0,\infty))$ and $B_2 \in C([0,\infty), [0,\infty))$ where $B_2$ is an increasing function satisfying the condition $|\Upsilon(\varkappa, \digamma(\varkappa))| \leq \phi(\varkappa)\, B_2(|\digamma(\varkappa)|)$ for all $\varkappa \in$ J and $\digamma \in \Xi$;
$(\beta_{15})$: If $\phi^* = sup_{\varkappa \in J}|\phi(\varkappa)|$, then $\exists$ a number $r_2$ s.t $\frac{r_2}{\digamma_0 + \lambda_2\, \phi^*\, B_2(r_2)} > 1$ where $\lambda_2 = [\frac{\mathfrak{p}(1-\mathfrak{q})}{AB(\mathfrak{q})}\,\varkappa^{\mathfrak{p}-1} + \frac{\mathfrak{q}\,\mathfrak{p}\,\Gamma(\mathfrak{p})\,\varkappa^{\mathfrak{p}+\mathfrak{q}-1}}{AB(\mathfrak{q})\,\Gamma(\mathfrak{p}+\mathfrak{q})}]$.
If above conditions holds then a solution exists for our model.

*Proof.* Consider $F\colon \Xi \to \Xi$ as

$$F(\digamma(\varkappa)) = \digamma(0) + \frac{\mathfrak{p}\,(1-\mathfrak{q})\,\varkappa^{\mathfrak{p}-1}\,\Upsilon(\varkappa)}{AB(\mathfrak{q})} + \frac{\mathfrak{p}\,\mathfrak{q}}{AB(\mathfrak{q})\,\Gamma(\mathfrak{q})}\int_0^{\varkappa} u^{(\mathfrak{p}-1)}\,(\varkappa-u)^{\mathfrak{q}-1}\,\Upsilon(u, \digamma(u))du,$$

and $(\mathfrak{N}_2)_\epsilon = \{\digamma \in \Xi : ||\digamma||_\Xi \leq \epsilon\}$ for some positive $\epsilon$.
We show that $F$ is compact on $(\mathfrak{N}_2)_\epsilon$. For this, we prove that $F$ is uniformly bounded and equicontinuous.
Since $\Upsilon$ is continuous, this implies $F$ is continuous.
Now for $\digamma$ in $(\mathfrak{N}_2)_\epsilon$ :

$$|F(\digamma(\varkappa))| \leq |\digamma(0)| + \frac{\mathfrak{p}\,(1-\mathfrak{q})\,\varkappa^{\mathfrak{p}-1}}{AB(\mathfrak{q})}\,|\Upsilon(\varkappa)| + \frac{\mathfrak{p}\,\mathfrak{q}}{AB(\mathfrak{q})\,\Gamma(\mathfrak{q})}\int_0^{\varkappa} u^{(\mathfrak{p}-1)}\,(\varkappa-u)^{\mathfrak{q}-1}\,|\Upsilon(u, \digamma(u))|du,$$

and from $(\beta_{14})$:

$$\begin{aligned}|F(\digamma(\varkappa))| &\leq \digamma_0 + \frac{\mathfrak{p}\,(1-\mathfrak{q})\,\varkappa^{\mathfrak{p}-1}}{AB(\mathfrak{q})}\,\phi(\varkappa)\,B_2(|\digamma(\varkappa)|)\\ &+ \frac{\mathfrak{p}\,\mathfrak{q}}{AB(\mathfrak{q})\,\Gamma(\mathfrak{q})}\int_0^{\varkappa} u^{(\mathfrak{p}-1)}\,(\varkappa-u)^{\mathfrak{q}-1}\,\phi(u)\,B_2(|\digamma(u)|)du\\ &\leq \digamma_0 + \frac{\mathfrak{p}\,(1-\mathfrak{q})\,\varkappa^{\mathfrak{p}-1}}{AB(\mathfrak{q})}\,\phi^*\,B_2(||\digamma||_\Xi) + \frac{\mathfrak{p}\,\mathfrak{q}}{AB(\mathfrak{q})\,\Gamma(\mathfrak{q})}\int_0^{\varkappa} u^{(\mathfrak{p}-1)}\,(\varkappa-u)^{\mathfrak{q}-1}\,\phi^*\,B_2(||\digamma||_\Xi)du\\ &\leq \digamma_0 + \frac{\mathfrak{p}\,(1-\mathfrak{q})\,\varkappa^{\mathfrak{p}-1}}{AB(\mathfrak{q})}\,\phi^*\,B_2(||\digamma||_\Xi) + \frac{\mathfrak{p}\,\mathfrak{q}}{AB(\mathfrak{q})\,\Gamma(\mathfrak{q})}\,\phi^*\,B_2(||\digamma||_\Xi)\int_0^{\varkappa} u^{(\mathfrak{p}-1)}\,(\varkappa-u)^{\mathfrak{q}-1}\,du.\end{aligned}$$

After simplification of the integral we get the beta function. So applying values of beta function, we get

$$\begin{aligned}|F(\digamma(\varkappa))| &\leq \digamma_0 + \frac{\mathfrak{p}\,(1-\mathfrak{q})\,\varkappa^{\mathfrak{p}-1}}{AB(\mathfrak{q})}\,\phi^*\,B_2(||\digamma||_\Xi) + \frac{\mathfrak{p}\,\mathfrak{q}}{AB(\mathfrak{q})\,\Gamma(\mathfrak{q})}\,\phi^*\,B_2(||\digamma||_\Xi)\,\frac{\mathfrak{p}\,\mathfrak{q}\,\varkappa^{\mathfrak{p}+\mathfrak{q}-1}\,\Gamma(\mathfrak{p})}{AB(\mathfrak{q})\,\Gamma(\mathfrak{p}+\mathfrak{q})}\\ &\leq \digamma_0 + [\frac{\mathfrak{p}\,(1-\mathfrak{q})\,\varkappa^{\mathfrak{p}-1}}{AB(\mathfrak{q})} + \frac{\mathfrak{p}\,\mathfrak{q}\,\varkappa^{\mathfrak{p}+\mathfrak{q}-1}\,\Gamma(\mathfrak{p})}{AB(\mathfrak{q})\,\Gamma(\mathfrak{p}+\mathfrak{q})}]\,\phi^*\,B_2(||\digamma||_\Xi).\end{aligned}$$

Applying the value of $\lambda_2$, we have

$$|F(\digamma(\varkappa))| \leq \digamma_0 + \lambda_2\,\phi^*\,B_2(\epsilon).$$

Hence applying norm, we have

$$||F(\digamma(\varkappa))|| \leq \digamma_0 + \lambda_2\,\phi^*\,B_2(\epsilon) < \infty. \tag{4.3.6}$$

This implies $F$ is uniformly bounded.

Now, take $\varkappa, \varkappa^* \in$ J such that $\varkappa < \varkappa^*$ and $\digamma \in N_\epsilon$ arbitrarily. If we suppose that

$\Upsilon^* = sup|\Upsilon(\varkappa, F(\varkappa))|$, then

$$\begin{aligned}
|F(F(\varkappa^*)) - F(F(\varkappa))| &= |\frac{\mathfrak{p}(1-\mathfrak{q})}{AB(\mathfrak{q})}[(\varkappa^*)^{\mathfrak{p}-1}\Upsilon(\varkappa^*) - \varkappa^{\mathfrak{p}-1}\Upsilon(\varkappa)] \\
&+ \frac{\mathfrak{p}\,\mathfrak{q}}{AB(\mathfrak{q})\,\Gamma(\mathfrak{q})}[\int_0^{\varkappa^*} u^{\mathfrak{p}-1}(\varkappa^*-u)^{\mathfrak{q}-1}\Upsilon(u, F(u))\,du \\
&- \int_0^{\varkappa} u^{(\mathfrak{p}-1)}(\varkappa-u)^{\mathfrak{q}-1}\Upsilon(u, F(u))\,du]| \\
&\leq \frac{\mathfrak{p}(1-\mathfrak{q})}{AB(\mathfrak{q})}|(\varkappa^*)^{\mathfrak{p}-1}\Upsilon(\varkappa^*) - \varkappa^{\mathfrak{p}-1}\Upsilon(\varkappa)| \\
&+ \frac{\mathfrak{p}\,\mathfrak{q}}{AB(\mathfrak{q})\,\Gamma(\mathfrak{q})}|\int_0^{\varkappa^*} u^{\mathfrak{p}-1}(\varkappa^*-u)^{\mathfrak{q}-1}\,du \\
&- \int_0^{\varkappa} u^{(\mathfrak{p}-1)}(\varkappa-u)^{\mathfrak{q}-1}\,du| \cdot |\Upsilon(u, F(u)| \\
&\leq \frac{\mathfrak{p}(1-\mathfrak{q})}{AB(\mathfrak{q})}[(\varkappa^*)^{\mathfrak{p}-1} - \varkappa^{\mathfrak{p}-1}]\,\Upsilon^* \\
&+ \frac{\mathfrak{p}\,\mathfrak{q}}{AB(\mathfrak{q})\,\Gamma(\mathfrak{q})}[\int_0^{\varkappa^*} u^{(\mathfrak{p}-1)}(\varkappa^*-u)^{\mathfrak{q}-1}\,du \\
&- \int_0^{\varkappa} u^{(\mathfrak{p}-1)}(\varkappa-u)^{\mathfrak{q}-1}\,du]\,\Upsilon^* \\
&\leq \frac{\mathfrak{p}(1-\mathfrak{q})}{AB(\mathfrak{q})}[(\varkappa^*)^{\mathfrak{p}-1} - \varkappa^{\mathfrak{p}-1}]\,\Upsilon^* \\
&+ \frac{\mathfrak{p}\,\mathfrak{q}}{AB(\mathfrak{q})\,\Gamma(\mathfrak{q})}|(\varkappa^*)^{(\mathfrak{p}+\mathfrak{q}-1)}\beta(\mathfrak{p},\mathfrak{q}) - \varkappa^{(\mathfrak{p}+\mathfrak{q}-1)}\beta(\mathfrak{p},\mathfrak{q})|\,\Upsilon^* \\
&\leq \frac{\mathfrak{p}(1-\mathfrak{q})}{AB(\mathfrak{q})}[(\varkappa^*)^{\mathfrak{p}-1} - \varkappa^{\mathfrak{p}-1}]\,\Upsilon^* \\
&+ \frac{\mathfrak{p}\,\mathfrak{q}}{AB(\mathfrak{q})\,\Gamma(\mathfrak{p}+\mathfrak{q})}\,\Upsilon^*\,[(\varkappa^*)^{(\mathfrak{p}+\mathfrak{q}-1)} - \varkappa^{(\mathfrak{p}+\mathfrak{q}-1)}],
\end{aligned}$$

that is independent of $F$. When $\varkappa^* \to \varkappa$ its value becomes zero. Hence $||F(F(\varkappa^*)) - F(F(\varkappa))||_{\Xi} \to 0$.

This proved that $F$ is equicontinuous. So $F$ is compact. As $F$ satisfies the conditions of Theorem 4.3.2, we say that $F$ will satisfy either one or the other condition mentioned in Theorem 4.3.2. For this using($\beta_{15}$), we construct $A_2 = \{F \in \Xi : ||F||_{\Xi} < r_2\}$, where $r_2 > 0$ is defined above. Hence, we can write

$$||F(F(\varkappa))|| \leq F_0 + \lambda_2\,\phi^*\,B_2(r_2). \tag{4.3.7}$$

Assume, $\exists F \in \partial A_2$ and $\omega_2 \in (0,1)$ where $F = \omega_2 F(F)$.

For this $\digamma, w_2$ and using($\beta_{15}$), we get

$$\begin{aligned} r_2 &= ||\digamma||_\Xi \\ &= w||F(\digamma)||_\Xi \\ &< ||F(\digamma)||_\Xi \\ &< \digamma_0 + \lambda_2\, \phi^*\, B_2(||\digamma||_\Xi) \\ &< \digamma_0 + \lambda_2\, \phi^*\, B_2(r_2). \end{aligned}$$

This gives $r_2 < r_2$, which is impossible. Thus second condition is not satisfied. Hence, by first condition $F$ possesses a fixed point in $\overline{A_2}$. □

## 4.4 Uniqueness

In this section, we will prove uniqueness with the help of Theorem 4.4.1 using lipschitz condition proved in Theorem(2.4.1).

**Theorem 4.4.1.** If $||\Delta|| \leq \mu_1, ||\aleph|| \leq \mu_2, ||\Theta|| \leq \mu_4$ for some $\mu_i > 0, i = 1, 2, 4$ and $w_1 = (\beta_0\, \mu_3 + \mu + \nu), w_2 = (\beta_0\, \mu_1\, b + \mu + \gamma), w_3 = (\zeta + \mu)$, where $0 < w_i < 1, i = 1, 2, 3$; then our model has a unique solution if $\lambda_2\, w_i < 1$, for $i = 1, 2, 3$.

*Proof.* Suppose the model has two solutions $(\Delta(\varkappa), \aleph(\varkappa), \Theta(\varkappa))$ and $(\Delta^*(\varkappa), \aleph^*(\varkappa), \Theta^*(\varkappa))$ with initial conditions defined above. Then, we can write

$$\begin{aligned} \Delta(\varkappa) &= \Delta(0) + \frac{\mathfrak{p}(1-\mathfrak{q})\, \varkappa^{\mathfrak{p}-1}\Upsilon_1(\varkappa, \digamma(\varkappa))}{AB(\mathfrak{q})} \\ &+ \frac{\mathfrak{p}\,\mathfrak{q}}{AB(\mathfrak{q})\,\Gamma(\mathfrak{q})} \int_0^{\varkappa} u^{(\mathfrak{p}-1)}\,(\varkappa-u)^{\mathfrak{q}-1}\,\Upsilon_1(u, \digamma(u))du, \\ \Delta^*(\varkappa) &= \Delta(0) + \frac{\mathfrak{p}(1-\mathfrak{q})\, \varkappa^{\mathfrak{p}-1}\Upsilon_1(\varkappa, \digamma^*(\varkappa)}{AB(\mathfrak{q})} \\ &+ \frac{\mathfrak{p}\,\mathfrak{q}}{AB(\mathfrak{q})\,\Gamma(\mathfrak{q})} \int_0^{\varkappa} u^{(\mathfrak{p}-1)}\,(\varkappa-u)^{\mathfrak{q}-1}\,\Upsilon_1(u, \digamma^*(u))du. \end{aligned}$$

Take

$$|\Delta(\varkappa) - \Delta^*(\varkappa)| = ||\frac{\mathfrak{p}(1-\mathfrak{q})\, \varkappa^{\mathfrak{p}-1}[\Upsilon_1(\varkappa, \digamma(\varkappa)) - \Upsilon_1(\varkappa, \digamma^*(\varkappa))]}{AB(\mathfrak{q})}$$
$$+\frac{\mathfrak{p}\,\mathfrak{q}}{AB(\mathfrak{q})\,\Gamma(\mathfrak{q})} \int_0^{\varkappa} u^{(\mathfrak{p}-1)}\,(\varkappa-u)^{\mathfrak{q}-1}\,(\Upsilon_1(u, \digamma(u)) - \Upsilon_1(u, \digamma^*(u)))du||$$

$$|\Delta(\varkappa) - \Delta^*(\varkappa)| \leq \frac{\mathfrak{p}(1-\mathfrak{q})\,\varkappa^{\mathfrak{p}-1}||\Upsilon_1(\varkappa, F(\varkappa)) - \Upsilon_1(\varkappa, F^*(\varkappa))||}{AB(\mathfrak{q})}$$
$$+\frac{\mathfrak{p}\,\mathfrak{q}}{AB(\mathfrak{q})\,\Gamma(\mathfrak{q})}\int_0^{\varkappa} u^{(\mathfrak{p}-1)}\,(\varkappa-u)^{\mathfrak{q}-1}\,||(\Upsilon_1(u, F(u)) - \Upsilon_1(u, F^*(u)))||du.$$

Since $\Upsilon_1$ is considered w.r.t $\Delta$ and $\Delta^*$, so by using result in Theorem 2.4.1 and definition of Beta function, we get

$$\begin{aligned}
|\Delta(\varkappa) - \Delta^*(\varkappa)| &\leq \frac{\mathfrak{p}(1-\mathfrak{q})\,\varkappa^{\mathfrak{p}-1}}{AB(\mathfrak{q})}\,||\Upsilon_1(\Delta) - \Upsilon_1(\Delta^*)|| \\
&+ \frac{\mathfrak{p}\,q}{AB(\mathfrak{q})\,\Gamma(\mathfrak{q})}\int_0^{\varkappa} u^{(\mathfrak{p}-1)}\,(\varkappa-u)^{\mathfrak{q}-1}\,||(\Upsilon_1(\Delta) - \Upsilon_1(\Delta^*))||du \\
&\leq \frac{\mathfrak{p}(1-\mathfrak{q})\,\varkappa^{\mathfrak{p}-1}}{AB(\mathfrak{q})}\,||\Upsilon_1(\Delta) - \Upsilon_1(\Delta^*)|| \\
&+ \frac{\mathfrak{p}\,\mathfrak{q}}{AB(\mathfrak{q})\,\Gamma(\mathfrak{q})}\,\varkappa^{(\mathfrak{p}+\mathfrak{q}-1)}\,\beta(\mathfrak{p},\mathfrak{q})\,||\Upsilon_1(\Delta) - \Upsilon_1(\Delta^*)|| \\
&\leq \frac{\mathfrak{p}(1-\mathfrak{q})\,\varkappa^{\mathfrak{p}-1}}{AB(\mathfrak{q})}\,w_1||\Delta - \Delta^*|| + \frac{\mathfrak{p}\,\mathfrak{q}\,\varkappa^{(\mathfrak{p}+\mathfrak{q}-1)}\,\Gamma(\mathfrak{p})}{AB(\mathfrak{q})\,\Gamma(\mathfrak{p}+\mathfrak{q})}\,w_1||\Delta - \Delta^*|| \\
&\leq [\frac{\mathfrak{p}(1-\mathfrak{q})\,\varkappa^{\mathfrak{p}-1}}{AB(\mathfrak{q})} + \frac{\mathfrak{p}\,\mathfrak{q}\,\varkappa^{(\mathfrak{p}+\mathfrak{q}-1)}\,\Gamma(\mathfrak{p})}{AB(\mathfrak{q})\,\Gamma(\mathfrak{p}+\mathfrak{q})}]\,w_1||\Delta - \Delta^*||.
\end{aligned}$$

Hence

$$|\Delta(\varkappa) - \Delta^*(\varkappa)| \leq \lambda_2\,w_1\,||\Delta - \Delta^*||,$$

$$||\Delta - \Delta^*|| \leq \lambda_2\,w_1\,||\Delta - \Delta^*||,$$

This implies that $(1-\lambda_2\,w_1)\,||\Delta - \Delta^*|| \leq 0$.

As $\lambda_2 w_1 < 1$, so this is possible when $||\Delta - \Delta^*|| = 0$. Thus $\Delta = \Delta^*$.

Similarly, we have

$$\begin{aligned}
\aleph(\varkappa) &= \aleph(0) + \frac{\mathfrak{p}(1-\mathfrak{q})\,\varkappa^{\mathfrak{p}-1}\Upsilon_2(\varkappa, F(\varkappa))}{AB(\mathfrak{q})} \\
&+ \frac{\mathfrak{p}\,\mathfrak{q}}{AB(\mathfrak{q})\,\Gamma(\mathfrak{q})}\int_0^{\varkappa} u^{(\mathfrak{p}-1)}\,(\varkappa-u)^{\mathfrak{q}-1}\,\Upsilon_2(u, F(u))du, \\
\aleph^*(\varkappa) &= \aleph(0) + \frac{\mathfrak{p}(1-\mathfrak{q})\,\varkappa^{\mathfrak{p}-1}\Upsilon_2(\varkappa, F^*(\varkappa))}{AB(\mathfrak{q})} \\
&+ \frac{\mathfrak{p}\,\mathfrak{q}}{AB(\mathfrak{q})\,\Gamma(\mathfrak{q})}\int_0^{\varkappa} u^{(\mathfrak{p}-1)}\,(\varkappa-u)^{\mathfrak{q}-1}\,\Upsilon_2(u, F^*(u))du.
\end{aligned}$$

Take

$$|\aleph(\varkappa) - \aleph^*(\varkappa)| = ||\frac{\mathfrak{p}(1-\mathfrak{q})\,\varkappa^{\mathfrak{p}-1}[\Upsilon_2(\varkappa, F(\varkappa)) - \Upsilon_2(\varkappa, F^*(\varkappa))]}{AB(\mathfrak{q})}$$
$$+\frac{\mathfrak{p}\,\mathfrak{q}}{AB(\mathfrak{q})\,\Gamma(\mathfrak{q})}\int_0^{\varkappa} u^{(\mathfrak{p}-1)}\,(\varkappa-u)^{\mathfrak{q}-1}\,(\Upsilon_2(u, F(u)) - \Upsilon_2(u, F^*(u)))du||$$

$$|\aleph(\varkappa)-\aleph^*(\varkappa)| \leq \frac{\mathfrak{p}(1-\mathfrak{q})\,\varkappa^{\mathfrak{p}-1}||\Upsilon_2(\varkappa,F(\varkappa))-\Upsilon_2(\varkappa,F^*(\varkappa))||}{AB(\mathfrak{q})}$$
$$+\frac{\mathfrak{p}\,\mathfrak{q}}{AB(\mathfrak{q})\,\Gamma(\mathfrak{q})}\int_0^{\varkappa} u^{(\mathfrak{p}-1)}\,(\varkappa-u)^{\mathfrak{q}-1}\,||(\Upsilon_2(u,F(u))-\Upsilon_2(u,F^*(u)))||du.$$

Since $\Upsilon_2$ is considered w.r.t $\aleph$ and $\aleph^*$, so by using result in Theorem 2.4.1 and definition of Beta function, we get

$$\begin{aligned}
|\aleph(\varkappa)-\aleph^*(\varkappa)| &\leq \frac{\mathfrak{p}(1-\mathfrak{q})\,\varkappa^{\mathfrak{p}-1}}{AB(\mathfrak{q})}\,||\Upsilon_2(\aleph)-\Upsilon_2(\aleph^*)|| \\
&+ \frac{\mathfrak{p}\,\mathfrak{q}}{AB(\mathfrak{q})\,\Gamma(\mathfrak{q})}\int_0^{\varkappa} u^{(\mathfrak{p}-1)}\,(\varkappa-u)^{\mathfrak{q}-1}\,||(\Upsilon_2(\aleph)-\Upsilon_2(\aleph^*))||du \\
&\leq \frac{\mathfrak{p}(1-\mathfrak{q})\,\varkappa^{\mathfrak{p}-1}}{AB(\mathfrak{q})}\,||\Upsilon_2(\aleph)-\Upsilon_2(\aleph^*)|| \\
&+ \frac{\mathfrak{p}\,\mathfrak{q}}{AB(\mathfrak{q})\,\Gamma(\mathfrak{q})}\,\varkappa^{(\mathfrak{p}+\mathfrak{q}-1)}\,\beta(\mathfrak{p},\mathfrak{q})\,||\Upsilon_2(\aleph)-\Upsilon_2(\aleph^*)|| \\
&\leq \frac{\mathfrak{p}(1-\mathfrak{q})\,\varkappa^{\mathfrak{p}-1}}{AB(\mathfrak{q})}\,w_2||\aleph-\aleph^*|| + \frac{\mathfrak{p}\,\mathfrak{q}\,\varkappa^{(\mathfrak{p}+\mathfrak{q}-1)}\,\Gamma(\mathfrak{p})}{AB(\mathfrak{q})\,\Gamma(\mathfrak{p}+\mathfrak{q})}\,w_2||\aleph-\aleph^*|| \\
&\leq [\frac{\mathfrak{p}(1-\mathfrak{q})\,\varkappa^{\mathfrak{p}-1}}{AB(\mathfrak{q})} + \frac{\mathfrak{p}\,\mathfrak{q}\,\varkappa^{(\mathfrak{p}+\mathfrak{q}-1)}\,\Gamma(\mathfrak{p})}{AB(\mathfrak{q})\,\Gamma(\mathfrak{p}+\mathfrak{q})}]\,w_2||\aleph-\aleph^*||.
\end{aligned}$$

Hence

$$|\aleph(\varkappa)-\aleph^*(\varkappa)| \leq \lambda_2\,w_2\,||\aleph-\aleph^*||,$$

$$||\aleph-\aleph^*|| \leq \lambda_2\,w_2\,||\aleph-\aleph^*||,$$

This implies that $(1-\lambda_2\,w_2)\,||\aleph-\aleph^*|| \leq 0$.

As $\lambda_2 w_2 < 1$, so this is possible when $||\aleph-\aleph^*|| = 0$. Thus $\aleph = \aleph^*$. Also we have

$$\begin{aligned}
\Theta(\varkappa) &= \Theta(0) + \frac{\mathfrak{p}(1-\mathfrak{q})\,\varkappa^{\mathfrak{p}-1}\Upsilon_3(\varkappa,F(\varkappa))}{AB(\mathfrak{q})} \\
&+ \frac{\mathfrak{p}\,\mathfrak{q}}{AB(\mathfrak{q})\,\Gamma(\mathfrak{q})}\int_0^{\varkappa} u^{(\mathfrak{p}-1)}\,(\varkappa-u)^{\mathfrak{q}-1}\,\Upsilon_3(u,F(u))du, \\
\Theta^*(\varkappa) &= \Theta(0) + \frac{\mathfrak{p}(1-\mathfrak{q})\,\varkappa^{\mathfrak{p}-1}\Upsilon_3(\varkappa,F^*(\varkappa))}{AB(\mathfrak{q})} \\
&+ \frac{\mathfrak{p}\,\mathfrak{q}}{AB(\mathfrak{q})\,\Gamma(\mathfrak{q})}\int_0^{\varkappa} u^{(\mathfrak{p}-1)}\,(\varkappa-u)^{\mathfrak{q}-1}\,\Upsilon_3(u,F^*(u))du.
\end{aligned}$$

Take

$$|\Theta(\varkappa)-\Theta^*(\varkappa)| = ||\frac{\mathfrak{p}(1-\mathfrak{q})\,\varkappa^{\mathfrak{p}-1}[\Upsilon_3(\varkappa,F(\varkappa))-\Upsilon_3(\varkappa,F^*(\varkappa))]}{AB(\mathfrak{q})}$$
$$+\frac{\mathfrak{p}\,\mathfrak{q}}{AB(\mathfrak{q})\,\Gamma(\mathfrak{q})}\int_0^{\varkappa} u^{(\mathfrak{p}-1)}\,(\varkappa-u)^{\mathfrak{q}-1}\,(\Upsilon_3(u,F(u))-\Upsilon_3(u,F^*(u)))du||$$

$$|\Theta(\varkappa) - \Theta^*(\varkappa)| \leq \frac{\mathfrak{p}(1-\mathfrak{q})\,\varkappa^{\mathfrak{p}-1}||\Upsilon_3(\varkappa, F(\varkappa)) - \Upsilon_3(\varkappa, F^*(\varkappa))||}{AB(\mathfrak{q})}$$

$$+\frac{\mathfrak{p}\,\mathfrak{q}}{AB(\mathfrak{q})\,\Gamma(\mathfrak{q})}\int_0^{\varkappa} u^{(\mathfrak{p}-1)}\,(\varkappa-u)^{\mathfrak{q}-1}\,||(\Upsilon_3(u, F(u)) - \Upsilon_3(u, F^*(u)))||du.$$

As $\Upsilon_3$ is considered w.r.t $\Theta$ and $\Theta^*$, so by using result in Theorem 2.4.1 and definition of Beta function, we get

$$\begin{aligned}
|\Theta(\varkappa) - \Theta^*(\varkappa)| &\leq \frac{\mathfrak{p}(1-\mathfrak{q})\,\varkappa^{\mathfrak{p}-1}}{AB(\mathfrak{q})}\,||\Upsilon_3(\Theta) - \Upsilon_3(\Theta^*)|| \\
&+ \frac{\mathfrak{p}\,\mathfrak{q}}{AB(\mathfrak{q})\,\Gamma(\mathfrak{q})}\int_0^{\varkappa} u^{(\mathfrak{p}-1)}\,(\varkappa-u)^{\mathfrak{q}-1}\,||(\Upsilon_3(\Theta) - \Upsilon_3(\Theta^*))||du \\
&\leq \frac{\mathfrak{p}(1-\mathfrak{q})\,\varkappa^{\mathfrak{p}-1}}{AB(\mathfrak{q})}\,||\Upsilon_3(\Theta) - \Upsilon_3(\Theta^*)|| \\
&+ \frac{\mathfrak{p}\,\mathfrak{q}}{AB(\mathfrak{q})\,\Gamma(\mathfrak{q})}\,\varkappa^{(\mathfrak{p}+\mathfrak{q}-1)}\,\beta(\mathfrak{p},\mathfrak{q})\,||\Upsilon_3(\Theta) - \Upsilon_3(\Theta^*)|| \\
&\leq \frac{\mathfrak{p}(1-\mathfrak{q})\,\varkappa^{\mathfrak{p}-1}}{AB(\mathfrak{q})}\,w_3||\Theta - \Theta^*|| + \frac{\mathfrak{p}\,\mathfrak{q}\,\varkappa^{(\mathfrak{p}+\mathfrak{q}-1)}\,\Gamma(\mathfrak{p})}{AB(\mathfrak{q})\,\Gamma(\mathfrak{p}+\mathfrak{q})}\,w_3||\Theta - \Theta^*|| \\
&\leq [\frac{\mathfrak{p}(1-\mathfrak{q})\,\varkappa^{\mathfrak{p}-1}}{AB(\mathfrak{q})} + \frac{\mathfrak{p}\,\mathfrak{q}\,\varkappa^{(\mathfrak{p}+\mathfrak{q}-1)}\,\Gamma(\mathfrak{p})}{AB(\mathfrak{q})\,\Gamma(\mathfrak{p}+\mathfrak{q})}]\,w_3||\Theta - \Theta^*||.
\end{aligned}$$

Hence

$$|\Theta(\varkappa) - \Theta^*(\varkappa)| \leq \lambda_2\,w_3\,||\Theta - \Theta^*||,$$

$$||\Theta - \Theta^*|| \leq \lambda_2\,w_3\,||\Theta - \Theta^*||,$$

This implies that $(1-\lambda_2\,w_3)\,||\Theta - \Theta^*|| \leq 0$.
As $\lambda_2 w_3 < 1$, so this is possible when $||\Theta - \Theta^*|| = 0$. Thus $\Theta = \Theta^*$.
So $(\Delta(\varkappa), \aleph(\varkappa), \Theta(\varkappa)) = (\Delta^*(\varkappa), \aleph^*(\varkappa), \Theta^*(\varkappa))$. Hence, the solution is unique. □

## 4.5 Stability

We have to check the stability of the solution. We use Ulam−Hyers and Ulam−Hayers−Rassias theorems to check it. First we define these theorems for our model as:

**Definition 4.5.1.** Model 4.2.2 is Ulam-Hyers stable [23] if, for all $\epsilon_i > 0$, there exist $M_i > 0 \in [0, \infty)$, which depend on $\Upsilon_i$ respectively $i = 1, 2, 3$ and for all $(\Delta^*, \aleph^*, \Theta^*)$

satisfying the inequalities

$$\begin{aligned} |{}^{FFM}D_{0,\varkappa}^{\mathfrak{q},\mathfrak{p}}\Delta^*(\varkappa) - \Upsilon_1(\varkappa, F^*(\varkappa))| &\le \epsilon_1, \\ |{}^{FFM}D_{0,\varkappa}^{\mathfrak{q},\mathfrak{p}}\aleph^*(\varkappa) - \Upsilon_2(\varkappa, F^*(\varkappa))| &\le \epsilon_2, \\ |{}^{FFM}D_{0,\varkappa}^{\mathfrak{q},\mathfrak{p}}\Theta^*(\varkappa) - \Upsilon_3(\varkappa, F^*(\varkappa))| &\le \epsilon_3, \end{aligned} \tag{4.5.1}$$

then there exists $(\Delta, \aleph, \Theta) \in \Xi$ satisfying the Model 4.2.2 with the condition

$$\begin{aligned} |\Delta^*(\varkappa) - \Delta(\varkappa)| &\le M_1\,\epsilon_1, \\ |\aleph^*(\varkappa) - \aleph(\varkappa)| &\le M_2\,\epsilon_2, \\ |\Theta^*(\varkappa) - \Theta(\varkappa)| &\le M_3\,\epsilon_3. \end{aligned} \tag{4.5.2}$$

**Remark 4.5.2.** $(\Delta^*, \aleph^*, \Theta^*) \in \Xi$ is a solution of Model 4.2.2 iff $\exists\, \eta_i \in C([0, \mathrm{T}], [0, \infty))$ such that for all $\varkappa \in$ J,

(i) $|\,\eta_i\,| < \epsilon_i$,

(ii)

$$\begin{aligned} {}^{FFM}D_{0,\varkappa}^{\mathfrak{q},\mathfrak{p}}\Delta^*(\varkappa) &= \Upsilon_1(\varkappa, F^*(\varkappa)) + \eta_1(\varkappa), \\ {}^{FFM}D_{0,\varkappa}^{\mathfrak{q},\mathfrak{p}}\aleph^*(\varkappa) &= \Upsilon_2(\varkappa, F^*(\varkappa)) + \eta_2(\varkappa), \\ {}^{FFM}D_{0,\varkappa}^{\mathfrak{q},\mathfrak{p}}\Theta^*(\varkappa) &= \Upsilon_3(\varkappa, F^*(\varkappa)) + \eta_3(\varkappa). \end{aligned} \tag{4.5.3}$$

**Theorem 4.5.3.** The fractal fraction model 4.2.2 is Ulam−Hayers stable on J s.t. $\lambda_2\, w_i < 1$, where $w_i$ and $\lambda_2$ are defined with the conditions given above.

*Proof.* Let $\epsilon_1 > 0$ and $\Delta^* \in \mathfrak{Y}$

$$|{}^{FFM}D_{0,\varkappa}^{\mathfrak{q},\mathfrak{p}}\Delta^*(\varkappa) - \Upsilon_1(\varkappa, F^*(\varkappa))| \le \epsilon_1,$$

then by above remark 4.5.2, we have

$$\begin{aligned} \Delta^*(\varkappa) &= \Delta(0) + \frac{\mathfrak{p}\,(1-\mathfrak{q})\,\varkappa^{\mathfrak{p}-1}}{AB(\mathfrak{q})}\,\Upsilon_1(\varkappa, F^*(\varkappa)) \\ &+ \frac{\mathfrak{p}\,\mathfrak{q}}{AB(\mathfrak{q})\,\Gamma(\mathfrak{q})}\int_0^{\varkappa} u^{(\mathfrak{p}-1)}\,(\varkappa-u)^{\mathfrak{q}-1}\,\Upsilon_1(u, F^*(u))du \\ &+ \frac{\mathfrak{p}\,\mathfrak{q}}{AB(\mathfrak{q})\,\Gamma(\mathfrak{q})}\int_0^{\varkappa} u^{(\mathfrak{p}-1)}\,(\varkappa-u)^{\mathfrak{q}-1}\,\eta_1(u)\,du. \end{aligned}$$

As $\Delta \in \mathfrak{Y}$ is the unique solution, then

$$\begin{aligned}\Delta(\varkappa) &= \Delta(0) + \frac{\mathfrak{p}\,(1-\mathfrak{q})\,\varkappa^{\mathfrak{p}-1}}{AB(\mathfrak{q})}\,\Upsilon_1(\varkappa, F^*(\varkappa)) \\ &+ \frac{\mathfrak{p}\,\mathfrak{q}}{AB(\mathfrak{q})\Gamma(\mathfrak{q})}\int_0^{\varkappa} u^{(\mathfrak{p}-1)}\,(\varkappa-u)^{\mathfrak{q}-1}\,\Upsilon_1(u, F(u))du.\end{aligned}$$

So,

$$\begin{aligned}&|\Delta^*(\varkappa) - \Delta(\varkappa)| \\ &= |\frac{\mathfrak{p}\,(1-\mathfrak{q})\,\varkappa^{\mathfrak{p}-1}}{AB(\mathfrak{q})}\,[\Upsilon_1(\varkappa, F^*(\varkappa)) - \Upsilon_1(\varkappa, F(\varkappa))] \\ &+ \frac{\mathfrak{p}\,\mathfrak{q}}{AB(\mathfrak{q})\Gamma(\mathfrak{q})}\int_0^{\varkappa} u^{(\mathfrak{p}-1)}\,(\varkappa-u)^{\mathfrak{q}-1}\,\eta_1(u)\,du \\ &+ \frac{\mathfrak{p}\,\mathfrak{q}}{AB(\mathfrak{q})\,\Gamma(\mathfrak{q})}\int_0^{\varkappa} u^{(\mathfrak{p}-1)}\,(\varkappa-u)^{\mathfrak{q}-1} \\ &[\quad \Upsilon_1(u, F^*(u)) - \Upsilon_1(u, F(u))\,]\,du| \\ &\leq \frac{\mathfrak{p}\,(1-\mathfrak{q})\,\varkappa^{\mathfrak{p}-1}}{AB(\mathfrak{q})}\,||\Upsilon_1(\varkappa, F^*(\varkappa)) - \Upsilon_1(\varkappa, F(\varkappa))|| \\ &+ \frac{\mathfrak{p}\,\mathfrak{q}}{AB(\mathfrak{q})\,\Gamma(\mathfrak{q})}\int_0^{\varkappa} u^{(\mathfrak{p}-1)}\,(\varkappa-u)^{\mathfrak{q}-1}\,|\eta_1(u)|du \\ &+ \frac{\mathfrak{p}\,\mathfrak{q}}{AB(\mathfrak{q})\,\Gamma(\mathfrak{q})}\int_0^{\varkappa} u^{(\mathfrak{p}-1)}\,(\varkappa-u)^{\mathfrak{q}-1} \\ &||\quad \Upsilon_1(u, F^*(u)) - \Upsilon_1(u, F(u))\,||\,du \\ &\leq \frac{\mathfrak{p}(1-\mathfrak{q})\,\varkappa^{\mathfrak{p}-1}}{AB(\mathfrak{q})}\,w_1\,||\Delta^* - \Delta|| + \frac{\mathfrak{p}\,\mathfrak{q}}{AB(\mathfrak{q})\,\Gamma(\mathfrak{q})}\,\beta(\mathfrak{p},\mathfrak{q})\,\varkappa^{(\mathfrak{p}+\mathfrak{q}-1)}\,|\eta_1| \\ &+ \frac{\mathfrak{p}\,\mathfrak{q}}{AB(\mathfrak{q})\,\Gamma(\mathfrak{q})}\,\beta(\mathfrak{p},\mathfrak{q})\,\varkappa^{(\mathfrak{p}+\mathfrak{q}-1)}\,w_1\,||\Delta^* - \Delta|| \\ &\leq \frac{\mathfrak{p}\,\mathfrak{q}}{\Gamma(\mathfrak{q})}\,\beta(\mathfrak{p},\mathfrak{q})\,\varkappa^{(\mathfrak{p}+\mathfrak{q}-1)}\,\epsilon_1 \\ &+ [\frac{\mathfrak{p}(1-\mathfrak{q})\varkappa^{\mathfrak{p}-1}}{AB(\mathfrak{q})} + \frac{\mathfrak{p}\,\mathfrak{q}}{AB(\mathfrak{q})\,\Gamma(\mathfrak{q})}\,\beta(\mathfrak{p},\mathfrak{q})\,\varkappa^{(\mathfrak{p}+\mathfrak{q}-1)}]\,w_1\,||\Delta^* - \Delta|| \\ &\leq \frac{\mathfrak{p}\,\mathfrak{q}\,\varkappa^{\mathfrak{p}+\mathfrak{q}-1}\,\Gamma(\mathfrak{p})}{AB(\mathfrak{q})\,\Gamma(\mathfrak{p}+\mathfrak{q})}\epsilon_1 + \lambda_2\,w_1\,||\Delta^* - \Delta||.\end{aligned}$$

Hence, we have

$$\begin{aligned}||\Delta^* - \Delta|| &\leq \frac{\mathfrak{p}\,\mathfrak{q}\,\varkappa^{\mathfrak{p}+\mathfrak{q}-1}\,\Gamma(\mathfrak{p})}{AB(\mathfrak{q})\,\Gamma(\mathfrak{p}+\mathfrak{q})}\epsilon_1 + \lambda_2\,w_1\,||\Delta^* - \Delta||, \\ (1-\lambda_2\,w_1)\,||\Delta^* - \Delta|| &\leq \frac{\mathfrak{p}\,\mathfrak{q}\,\varkappa^{\mathfrak{p}+\mathfrak{q}-1}\,\Gamma(\mathfrak{p})}{AB(\mathfrak{q})\,\Gamma(\mathfrak{p}+\mathfrak{q})}\epsilon_1, \\ ||\Delta^* - \Delta|| &\leq \frac{\frac{\mathfrak{p}\,\mathfrak{q}\,\varkappa^{\mathfrak{p}+\mathfrak{q}-1}\,\Gamma(\mathfrak{p})}{AB(\mathfrak{q})\,\Gamma(\mathfrak{p}+\mathfrak{q})}\epsilon_1}{(1-\lambda_2\,w_1)}.\end{aligned}$$

If $\frac{\mathfrak{p}\,\mathfrak{q}\,\varkappa^{\mathfrak{p}+\mathfrak{q}-1}}{AB(\mathfrak{q})\,\Gamma(\mathfrak{q})\,(1-\lambda_2\,w_1)} = M_1$, then $||\Delta^* - \Delta|| \leq M_1\,\epsilon_1$.

Similarly, we can prove that

$||\aleph^* - \aleph|| \leq M_2\,\epsilon_2$ and $||\Theta^* - \Theta|| \leq M_3\,\epsilon_3$ .

Thus Ulam−Hayers stability criteria is fulfilled by our fractat−fractional model. □

**Definition 4.5.4.** We define the Ulam−Hayers−Rassias stability criteria for our fractal−fractional model as: Model 4.2.2 is Ulam-Hyers−Rassias stable [25] w.r.t the functions $\psi_i$, if for all $\epsilon_i, > 0$ , there exist $M_i > 0 \in [0, \infty)$ , which depend on $\Upsilon_i$ and $\psi_i$ where $i = 1, 2, 3$, and for all $(\Delta^*, \aleph^*, \Theta^*)$ satisfying the inequalities,

$$\begin{aligned}
|{}^{FFM}D_{0,\varkappa}^{\mathfrak{q},\mathfrak{p}}\Delta^*(\varkappa) - \Upsilon_1(\varkappa, F^*(\varkappa))| &\leq \epsilon_1\,\psi_1(\varkappa),\\
|{}^{FFM}D_{0,\varkappa}^{\mathfrak{q},\mathfrak{p}}\aleph^*(\varkappa) - \Upsilon_2(\varkappa, F^*(\varkappa))| &\leq \epsilon_2\,\psi_2(\varkappa),\\
|{}^{FFM}D_{0,\varkappa}^{\mathfrak{q},\mathfrak{p}}\Theta^*(\varkappa) - \Upsilon_3(\varkappa, F^*(\varkappa))| &\leq \epsilon_3\,\psi_3(\varkappa),
\end{aligned} \tag{4.5.4}$$

then there exists $(\Delta, \aleph, \Theta) \in \Xi$ satisfying the Model (4.2.2) with the condition

$$\begin{aligned}
|\Delta^*(\varkappa) - \Delta(\varkappa)| &\leq M_1\,\epsilon_1\,\psi_1(\varkappa),\\
|\aleph^*(\varkappa) - \aleph(\varkappa)| &\leq M_2\,\epsilon_2\,\psi_2(\varkappa),\\
|\Theta^*(\varkappa) - \Theta(\varkappa)| &\leq M_3\,\epsilon_3\,\psi_3(\varkappa).
\end{aligned} \tag{4.5.5}$$

**Remark 4.5.5.** $(\Delta^*, \aleph^*, \Theta^*) \in \Xi$ is a solution iff $\exists\,\eta_i \in C([0, \mathrm{T}], [0, \infty))$ such that for all $\varkappa \in \mathrm{J}$,

(i) $|\,\eta_i\,| < \epsilon_i\,\psi_i(\varkappa)$,

(ii)

$$\begin{aligned}
{}^{FFM}D_{0,\varkappa}^{\mathfrak{q},\mathfrak{p}}\Delta^*(\varkappa) &= \Upsilon_1(\varkappa, F^*(\varkappa)) + \eta_1(\varkappa),\\
{}^{FFM}D_{0,\varkappa}^{\mathfrak{q},\mathfrak{p}}\aleph^*(\varkappa) &= \Upsilon_2(\varkappa, F^*(\varkappa)) + \eta_2(\varkappa),\\
{}^{FFM}D_{0,\varkappa}^{\mathfrak{q},\mathfrak{p}}\Theta^*(\varkappa) &= \Upsilon_3(\varkappa, F^*(\varkappa)) + \eta_3(\varkappa).
\end{aligned} \tag{4.5.6}$$

**Theorem 4.5.6.** The fractal−fractional model 4.2.2 is Ulam−Hayers−Rassias stable when the following conditions are satisfied:

For all $\varkappa \in \mathrm{J}$ there exists nondecreasing mappings $\psi_i \in C([0, \mathrm{T}], [0, \infty)$ and $\xi_i > 0$ depending upon $\psi_i$ such that ${}^{FFM}I_{0,\varkappa}^{\mathfrak{q},\mathfrak{p}}\psi_i(\varkappa) < \xi_i\,\psi_i(\varkappa)$ and $\lambda_2 > 0, w_i > 0$ where $w_i$ and $\lambda_2$ are defined as before.

*Proof.* Let $\epsilon_1 > 0$ and $\Delta^* \in \mathfrak{Y}$ such that

$$|^{FFM}D^{\mathfrak{q},\mathfrak{p}}_{0,\varkappa}\Delta^*(\varkappa) - \Upsilon_1(\varkappa, F^*(\varkappa))| \leq \epsilon_1\, \psi_1(\varkappa),$$

then by the conditions of remark 4.5.5, we consider

$$\begin{aligned}
\Delta^*(\varkappa) &= \Delta(0) + \frac{\mathfrak{p}\,(1-\mathfrak{q})\,\varkappa^{\mathfrak{p}-1}}{AB(\mathfrak{q})}\,\Upsilon_1(\varkappa, F^*(\varkappa)) \\
&+ \frac{\mathfrak{p}\,\mathfrak{q}}{AB(\mathfrak{q})\,\Gamma(\mathfrak{q})}\int_0^{\varkappa} u^{(\mathfrak{p}-1)}\,(\varkappa-u)^{\mathfrak{q}-1}\,\Upsilon_1(u, F^*(u))du \\
&+ \frac{\mathfrak{p}\,\mathfrak{q}}{AB(\mathfrak{q})\,\Gamma(\mathfrak{q})}\int_0^{\varkappa} u^{(\mathfrak{p}-1)}\,(\varkappa-u)^{\mathfrak{q}-1}\,\eta_1(u)\,du.
\end{aligned}$$

As $\Delta \in \mathfrak{Y}$ is the unique solution, then

$$\Delta(\varkappa) = \Delta(0) + \frac{\mathfrak{p}\,(1-\mathfrak{q})\,\varkappa^{\mathfrak{p}-1}}{AB(\mathfrak{q})}\,\Upsilon_1(\varkappa, F^*(\varkappa))$$
$$+\frac{\mathfrak{p}\,\mathfrak{q}}{AB(\mathfrak{q})\Gamma(\mathfrak{q})}\int_0^{\varkappa} u^{(\mathfrak{p}-1)}\,(\varkappa-u)^{\mathfrak{q}-1}\,\Upsilon_1(u, F(u))du.$$

Now

$$\begin{aligned}
&|\Delta^*(\varkappa) - \Delta(\varkappa)| \\
&= |\frac{\mathfrak{p}\,(1-\mathfrak{q})\,\varkappa^{\mathfrak{p}-1}}{AB(\mathfrak{q})}\,[\Upsilon_1(\varkappa, F^*(\varkappa)) - \Upsilon_1(\varkappa, F(\varkappa))] \\
&+ \frac{\mathfrak{p}\,\mathfrak{q}}{AB(\mathfrak{q})\Gamma(\mathfrak{q})}\int_0^{\varkappa} u^{(\mathfrak{p}-1)}\,(\varkappa-u)^{\mathfrak{q}-1}\,\eta_1(u)\,du \\
&+ \frac{\mathfrak{p}\,\mathfrak{q}}{AB(\mathfrak{q})\,\Gamma(\mathfrak{q})}\int_0^{\varkappa} u^{(\mathfrak{p}-1)}\,(\varkappa-u)^{\mathfrak{q}-1} \\
&[\ \Upsilon_1(u, F^*(u)) - \Upsilon_1(u, F(u))\,]\,du| \\
&\leq \frac{\mathfrak{p}\,(1-\mathfrak{q})\,\varkappa^{\mathfrak{p}-1}}{AB(\mathfrak{q})}\,||\Upsilon_1(\varkappa, F^*(\varkappa)) - \Upsilon_1(\varkappa, F(\varkappa))|| \\
&+ \frac{\mathfrak{p}\,\mathfrak{q}}{AB(\mathfrak{q})\,\Gamma(\mathfrak{q})}\int_0^{\varkappa} u^{(\mathfrak{p}-1)}\,(\varkappa-u)^{\mathfrak{q}-1}\,|\eta_1(u)|du \\
&+ \frac{\mathfrak{p}\,\mathfrak{q}}{AB(\mathfrak{q})\,\Gamma(\mathfrak{q})}\int_0^{\varkappa} u^{(\mathfrak{p}-1)}\,(\varkappa-u)^{\mathfrak{q}-1} \\
&||\ \Upsilon_1(u, F^*(u)) - \Upsilon_1(u, F(u))\,||\,du \\
&\leq \frac{\mathfrak{p}(1-\mathfrak{q})\,\varkappa^{\mathfrak{p}-1}}{AB(\mathfrak{q})}\,w_1\,||\Delta^* - \Delta|| + \frac{\mathfrak{p}\,\mathfrak{q}}{AB(\mathfrak{q})\,\Gamma(\mathfrak{q})}\int_0^{\varkappa} u^{(\mathfrak{p}-1)}\,(\varkappa-u)^{\mathfrak{q}-1}\,\epsilon_1\,\psi_1(u)du \\
&+ \frac{\mathfrak{p}\,\mathfrak{q}}{AB(\mathfrak{q})\,\Gamma(\mathfrak{q})}\,\beta(\mathfrak{p},\mathfrak{q})\,\varkappa^{(\mathfrak{p}+\mathfrak{q}-1)}\,w_1\,||\Delta^* - \Delta|| \\
&\leq \epsilon_1\,\xi_1\,\psi_1(\varkappa) + [\frac{\mathfrak{p}(1-\mathfrak{q})\varkappa^{\mathfrak{p}-1}}{AB(\mathfrak{q})} + \frac{\mathfrak{p}\,\mathfrak{q}}{AB(\mathfrak{q})\,\Gamma(\mathfrak{q})}\,\beta(\mathfrak{p},\mathfrak{q})\,\varkappa^{(\mathfrak{p}+\mathfrak{q}-1)}]\,w_1\,||\Delta^* - \Delta|| \\
&\leq \epsilon_1\,\xi_1\,\psi_1(\varkappa) + \lambda_2\,w_1\,||\Delta^* - \Delta||.
\end{aligned}$$

Thus, we have

$$\begin{aligned} ||\Delta^* - \Delta|| &\le \epsilon_1\, \xi_1\, \psi_1(\varkappa) + \lambda_2\, w_1\, ||\Delta^* - \Delta||, \\ (1-\lambda_2\, w_1)\, ||\Delta^* - \Delta|| &\le \epsilon_1\, \xi_1\, \psi_1(\varkappa), \\ ||\Delta^* - \Delta|| &\le \frac{\epsilon_1\, \xi_1\, \psi_1(\varkappa)}{(1-\lambda_2\, w_1)}. \end{aligned}$$

If $\frac{\xi_1}{(1-\lambda_2\, w_1)} = M_1$, then $||\Delta^* - \Delta|| \le \epsilon_1\, \psi_1(\varkappa)\, M_1(\Upsilon_1, \psi_1)$.
Similarly, we can prove that
$||\aleph^* - \aleph|| \le M_2(\Upsilon_2, \psi_2)\, \epsilon_2\, \psi_2(\varkappa)$,
$||\Theta^* - \Theta|| \le M_3(\Upsilon_3, \psi_3)\, \epsilon_3\, \psi_3(\varkappa)$.
Hence Ulam−Hayers−Rassias stability criteria is fulfilled by our fractal−fractional model.

□

## 4.6 Numerical Algorithm

For numerical scheme of our FF model, we proceed as [37]. The difference between our scheme and others is that in our model $\Upsilon_1$ and $\Upsilon_3$ depends on $\varkappa$ and $(\varkappa - \tau)$, so we deal it differently. We know that

$$\begin{aligned} \Delta(\varkappa) &= \Delta(0) + \frac{\mathfrak{p}\,(1-\mathfrak{q})\,\varkappa^{\mathfrak{p}-1}}{AB(\mathfrak{q})}\,\Upsilon_1(\varkappa, F(\varkappa)) \\ &+ \frac{\mathfrak{p}\,\mathfrak{q}}{AB(\mathfrak{q})\Gamma(\mathfrak{q})}\int_0^{\varkappa} u^{(\mathfrak{p}-1)}\,(\varkappa-u)^{\mathfrak{q}-1}\,\Upsilon_1(u, F(u))du, \\ \aleph(\varkappa) &= \aleph(0) + \frac{\mathfrak{p}\,(1-\mathfrak{q})\,\varkappa^{\mathfrak{p}-1}}{AB(\mathfrak{q})}\,\Upsilon_2(\varkappa, F(\varkappa)) \\ &+ \frac{\mathfrak{p}\,\mathfrak{q}}{AB(\mathfrak{q})\Gamma(\mathfrak{q})}\int_0^{\varkappa} u^{(\mathfrak{p}-1)}\,(\varkappa-u)^{\mathfrak{q}-1}\,\Upsilon_2(u, F(u))du, \\ \Theta(\varkappa) &= \Theta(0) + \frac{\mathfrak{p}\,(1-\mathfrak{q})\,\varkappa^{\mathfrak{p}-1}}{AB(\mathfrak{q})}\,\Upsilon_3(\varkappa, F(\varkappa)) \\ &+ \frac{\mathfrak{p}\,\mathfrak{q}}{AB(\mathfrak{q})\Gamma(\mathfrak{q})}\int_0^{\varkappa} u^{(\mathfrak{p}-1)}\,(\varkappa-u)^{\mathfrak{q}-1}\,\Upsilon_3(u, F(u))du. \end{aligned}$$

First we take $\varkappa = \varkappa_{n+1}$ that means we take iterations and let $u^{\mathfrak{p}-1}\,\Upsilon_i(u, F(u)) = H_i(u)$, so

$$\Delta(\varkappa_{n+1}) = \Delta(0) + \frac{\mathfrak{p}\,(1-\mathfrak{q})\,{\varkappa_n}^{\mathfrak{p}-1}}{AB(\mathfrak{q})}\,\Upsilon_1(\varkappa(n), F(n)) + \frac{\mathfrak{p}\,\mathfrak{q}}{AB(\mathfrak{q})\Gamma(\mathfrak{q})}\int_0^{\varkappa_{n+1}} (\varkappa_{n+1}-u)^{\mathfrak{q}-1}\,H_1(u)du,$$

$$\aleph(\varkappa_{n+1}) = \aleph(0)+\frac{\mathfrak{p}\,(1-\mathfrak{q})\,{\varkappa_n}^{\mathfrak{p}-1}}{AB(\mathfrak{q})}\,\Upsilon_2(\varkappa(n),\digamma(n))+\frac{\mathfrak{p}\,\mathfrak{q}}{AB(\mathfrak{q})\Gamma(\mathfrak{q})}\int_0^{\varkappa_{n+1}}(\varkappa_{n+1}-u)^{\mathfrak{q}-1}\,H_2(u)du,$$

$$\Theta(\varkappa_{n+1}) = \Theta(0)+\frac{\mathfrak{p}\,(1-\mathfrak{q})\,{\varkappa_n}^{\mathfrak{p}-1}}{AB(\mathfrak{q})}\,\Upsilon_3(\varkappa(n),\digamma(n))+\frac{\mathfrak{p}\,\mathfrak{q}}{AB(\mathfrak{q})\Gamma(\mathfrak{q})}\int_0^{\varkappa_{n+1}}(\varkappa_{n+1}-u)^{\mathfrak{q}-1}\,H_3(u)du.$$

Approximating integral as the sum of integrals on sub intervals, we have

$$\Delta(\varkappa_{n+1}) = \Delta_0+\frac{\mathfrak{p}(1-\mathfrak{q}){\varkappa_n}^{\mathfrak{p}-1}}{AB(\mathfrak{q})}\Upsilon_1(\varkappa(n),\digamma(n))+\frac{\mathfrak{p}\mathfrak{q}}{AB(\mathfrak{q})\Gamma(\mathfrak{q})}\sum_{j=0}^{n}\int_{\varkappa_j}^{\varkappa_{j+1}}(\varkappa_{n+1}-u)^{\mathfrak{q}-1}H_1(u)du,$$

$$\aleph(\varkappa_{n+1}) = \aleph_0+\frac{\mathfrak{p}(1-\mathfrak{q}){\varkappa_n}^{\mathfrak{p}-1}}{AB(\mathfrak{q})}\Upsilon_2(\varkappa(n),\digamma(n))+\frac{\mathfrak{p}\mathfrak{q}}{AB(\mathfrak{q})\Gamma(\mathfrak{q})}\sum_{j=0}^{n}\int_{\varkappa_j}^{\varkappa_{j+1}}(\varkappa_{n+1}-u)^{\mathfrak{q}-1}H_2(u)du,$$

$$\Theta(\varkappa_{n+1}) = \Theta_0+\frac{\mathfrak{p}(1-\mathfrak{q}){\varkappa_n}^{\mathfrak{p}-1}}{AB(\mathfrak{q})}\Upsilon_3(\varkappa(n),\digamma(n))+\frac{\mathfrak{p}\mathfrak{q}}{AB(\mathfrak{q})\Gamma(\mathfrak{q})}\sum_{j=0}^{n}\int_{\varkappa_j}^{\varkappa_{j+1}}(\varkappa_{n+1}-u)^{\mathfrak{q}-1}H_3(u)du.$$

Now we approximate the functions $H_i(u)$ by two point Lagrange interpolation polynomials on the interval $[\varkappa_j,\varkappa_{j+1}]$ as:

$H_1^*(u) = \frac{u-\varkappa_{j-1}}{\varkappa_j-\varkappa_{j-1}}\,\varkappa_j^{\mathfrak{p}-1}\,\Upsilon_1(u_j,\Delta_j(u),\aleph_j(u),\Theta_j(u)) - \frac{u-\varkappa_j}{\varkappa_j-\varkappa_{j-1}}\,\varkappa_{j-1}^{\mathfrak{p}-1}\,\Upsilon_1(u_{j-1},\Delta_{j-1}(u),\aleph_{j-1}(u),\Theta_{j-1}(u)),$
$H_2^*(u) = \frac{u-\varkappa_{j-1}}{\varkappa_j-\varkappa_{j-1}}\,\varkappa_j^{\mathfrak{p}-1}\,\Upsilon_2(u_j,\Delta_j(u),\aleph_j(u),\Theta_j(u)) - \frac{u-\varkappa_j}{\varkappa_j-\varkappa_{j-1}}\,\varkappa_{j-1}^{\mathfrak{p}-1}\,\Upsilon_2(u_{j-1},\Delta_{j-1}(u),\aleph_{j-1}(u),\Theta_{j-1}(u)),$
$H_3^*(u) = \frac{u-\varkappa_{j-1}}{\varkappa_j-\varkappa_{j-1}}\,\varkappa_j^{\mathfrak{p}-1}\,\Upsilon_3(u_j,\Delta_j(u),\aleph_j(u),\Theta_j(u)) - \frac{u-\varkappa_j}{\varkappa_j-\varkappa_{j-1}}\,\varkappa_{j-1}^{\mathfrak{p}-1}\,\Upsilon_3(u_{j-1},\Delta_{j-1}(u),\aleph_{j-1}(u),\Theta_{j-1}(u)).$

Thus, we have

$$\begin{aligned}
\Delta(\varkappa_{n+1}) &= \Delta_0+\frac{\mathfrak{p}(1-\mathfrak{q}){\varkappa_n}^{\mathfrak{p}-1}}{AB(\mathfrak{q})}\Upsilon_1(\varkappa(n),\digamma(n))\\
&+\frac{\mathfrak{p}\mathfrak{q}}{AB(\mathfrak{q})\Gamma(\mathfrak{q})}\sum_{j=0}^{n}\int_{\varkappa_j}^{\varkappa_{j+1}}(\varkappa_{n+1}-u)^{\mathfrak{q}-1}H_1^*(u)du,\\
\aleph(\varkappa_{n+1}) &= \aleph_0+\frac{\mathfrak{p}(1-\mathfrak{q}){\varkappa_n}^{\mathfrak{p}-1}}{AB(\mathfrak{q})}\Upsilon_2(\varkappa(n),\digamma(n))\\
&+\frac{\mathfrak{p}\mathfrak{q}}{AB(\mathfrak{q})\,\Gamma(\mathfrak{q})}\sum_{j=0}^{n}\int_{\varkappa_j}^{\varkappa_{j+1}}(\varkappa_{n+1}-u)^{\mathfrak{q}-1}H_2^*(u)du,\\
\Theta(\varkappa_{n+1}) &= \Theta_0+\frac{\mathfrak{p}(1-\mathfrak{q}){\varkappa_n}^{\mathfrak{p}-1}}{AB(\mathfrak{q})}\Upsilon_3(\varkappa(n),\digamma(n))\\
&+\frac{\mathfrak{p}\mathfrak{q}}{AB(\mathfrak{q})\Gamma(\mathfrak{q})}\sum_{j=0}^{n}\int_{\varkappa_j}^{\varkappa_{j+1}}(\varkappa_{n+1}-u)^{\mathfrak{q}-1}H_3^*(u)du.
\end{aligned}$$

By integrating the above integrals according to limits and taking $\varkappa_j - \varkappa_{j-1} = h$, we get the final results

$$\begin{aligned}
\Delta(n+1) &= \Delta_0 + \frac{\mathfrak{p}\,(1-\mathfrak{q})\,\varkappa_n{}^{\mathfrak{p}-1}}{AB(\mathfrak{q})}\,\Upsilon_1(\varkappa(n), F(n)) \\
&+ \frac{\mathfrak{p}\,\mathfrak{q}\,h^{\mathfrak{q}}}{AB(\mathfrak{q})\,\Gamma(\mathfrak{q}+2)}\sum_{j=0}^{n}(\varkappa_j^{\mathfrak{p}-1}\Upsilon_1(u_j,\Delta_j,\aleph_j,\Theta_j)\,Z_1 - \varkappa_{j-1}^{\mathfrak{p}-1}\,\Upsilon_1(u_{j-1},\Delta_{j-1},\aleph_{j-1},\Theta_{j-1})\,Z_2), \\
\aleph(n+1) &= \aleph_0 + \frac{\mathfrak{p}\,(1-\mathfrak{q})\,\varkappa_n{}^{\mathfrak{p}-1}}{AB(\mathfrak{q})}\,\Upsilon_2(\varkappa(n), F(n)) \\
&+ \frac{\mathfrak{p}\,\mathfrak{q}\,h^{\mathfrak{q}}}{AB(\mathfrak{q})\,\Gamma(\mathfrak{q}+2)}\sum_{j=0}^{n}(\varkappa_j^{\mathfrak{p}-1}\Upsilon_2(u_j,\Delta_j,\aleph_j,\Theta_j)\,Z_1 - \varkappa_{j-1}^{\mathfrak{p}-1}\,\Upsilon_2(u_{j-1},\Delta_{j-1},\aleph_{j-1},\Theta_{j-1})\,Z_2), \\
\Theta(n+1) &= \Theta_0 + \frac{\mathfrak{p}\,(1-\mathfrak{q})\,\varkappa_n{}^{\mathfrak{p}-1}}{AB(\mathfrak{q})}\,\Upsilon_3(\varkappa(n), F(n)) \\
&+ \frac{\mathfrak{p}\,\mathfrak{q}\,h^{\mathfrak{q}}}{AB(\mathfrak{q})\,\Gamma(\mathfrak{q}+2)}\sum_{j=0}^{n}(\varkappa_j^{\mathfrak{p}-1}\Upsilon_3(u_j,\Delta_j,\aleph_j,\Theta_j)\,Z_1 - \varkappa_{j-1}^{\mathfrak{p}-1}\,\Upsilon_3(u_{j-1},\Delta_{j-1},\aleph_{j-1},\Theta_{j-1})\,Z_2),
\end{aligned}$$

where,

$Z_1 = (n+1-j)^{\mathfrak{q}}\,(n-j+2+\mathfrak{q}) - (n-j)^{\mathfrak{q}}\,(n-j+2+2\mathfrak{q})$,

$Z_2 = (n+1-j)^{\mathfrak{q}+1} - (n-j)^{\mathfrak{q}}\,(n-j+1+\mathfrak{q})$.

Since in the original model in $\Upsilon_1$ and $\Upsilon_3$ , $\Theta$ depends on $\varkappa$ and $(\varkappa - tau) = \varkappa_1(say)$, so we write

$\Upsilon_1 = U_1(\varkappa_j, \Delta_j, \aleph_j, \Theta_j) + U_3((\varkappa_1)_j, \Theta_j)$,

and $\Upsilon_3 = U_2(\varkappa_j, \Delta_j, \aleph_j, \Theta_j) - U_3((\varkappa_1)_j, \Theta_j)$,

where

$U_1(\varkappa_j, \Delta_j, \aleph_j, \Theta_j) = \Pi\Theta - \beta_0 f(\aleph(\varkappa))\Delta(\varkappa) - (\mu+\nu)\Delta(\varkappa)$,

$U_2(\varkappa_j, \Delta_j, \aleph_j, \Theta_j) = (1-\Pi)\Theta + \nu\Delta(\varkappa) + \kappa\,\aleph(\varkappa) - \mu\Theta(\varkappa)$,

$U_3((\varkappa_1)_j, \Theta_j) = \zeta\Theta(\varkappa - \tau)$.

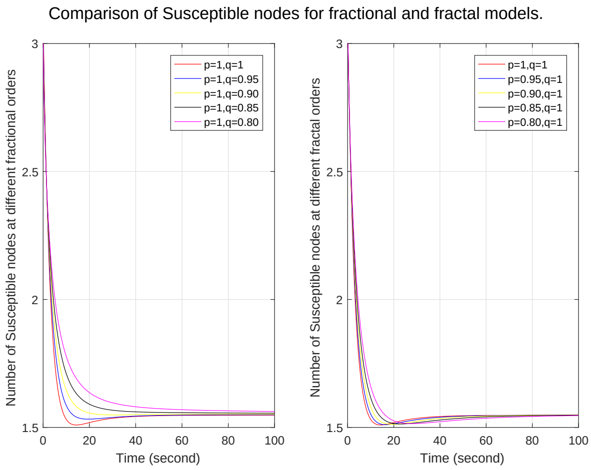


Figure 4.1: Trajectories of $\Delta(\varkappa)$ for different fractal orders $\mathfrak{p}$ when $\mathfrak{q}=1$ and different fractional orders $\mathfrak{q}$ when $\mathfrak{p}=1$.

Hence our numerical scheme is :

$$
\begin{aligned}
\Delta(n+1) &= \Delta_0 + \frac{\mathfrak{p}\,(1-\mathfrak{q})\,(\varkappa_n)^{\mathfrak{p}-1}}{AB(\mathfrak{q})}\,\Upsilon_1(\varkappa(n),\digamma(n)) \\
&+ \frac{\mathfrak{p}\mathfrak{q}h^{\mathfrak{q}}}{AB(\mathfrak{q})\Gamma(\mathfrak{q}+2)}\sum_{j=0}^{n}[\varkappa_j^{\mathfrak{p}-1}(U_1(\varkappa_j,\Delta_j,\aleph_j,\Theta_j)+U_3((\varkappa_1)_j,\Theta_j))\,Z_1 \\
&- \varkappa_{j-1}^{\mathfrak{p}-1}\,(U_1(\varkappa_{j-1},\Delta_{j-1},\aleph_{j-1},\Theta_{j-1})+U_3((\varkappa_1)_j,\Theta_j))\,Z_2], \\
\aleph(n+1) &= \aleph_0 + \frac{\mathfrak{p}\,(1-\mathfrak{q})\,{\varkappa_n}^{\mathfrak{p}-1}}{AB(\mathfrak{q})}\,\Upsilon_2(\varkappa(n),\digamma(n)) \\
&+ \frac{\mathfrak{p}\,\mathfrak{q}\,h^{\mathfrak{q}}}{AB(\mathfrak{q})\,\Gamma(\mathfrak{q}+2)}\sum_{j=0}^{n}[\varkappa_j^{\mathfrak{p}-1}\Upsilon_2(\varkappa_j,\Delta_j,\aleph_j,\Theta_j)\,Z_1 - \varkappa_{j-1}^{\mathfrak{p}-1}\,\Upsilon_2(\varkappa_{j-1},\Delta_{j-1},\aleph_{j-1},\Theta_{j-1})\,Z_2], \\
\Theta(n+1) &= \Theta_0 + \frac{\mathfrak{p}\,(1-\mathfrak{q})\,{\varkappa_n}^{\mathfrak{p}-1}}{AB(\mathfrak{q})}\,\Upsilon_3(\varkappa(n),\digamma(n)) \\
&+ \frac{\mathfrak{p}\mathfrak{q}h^{\mathfrak{q}}}{AB(\mathfrak{q})\Gamma(\mathfrak{q}+2)}\sum_{j=0}^{n}[\varkappa_j^{\mathfrak{p}-1}(U_1(\varkappa_j,\Delta_j,\aleph_j,\Theta_j)-U_3((\varkappa_1)_j,\Theta_j))\,Z_1 \\
&- \varkappa_{j-1}^{\mathfrak{p}-1}\,(U_1(\varkappa_{j-1},\Delta_{j-1},\aleph_{j-1},\Theta_{j-1})-U_3((\varkappa_1)_j,\Theta_j))\,Z_2].
\end{aligned}
$$

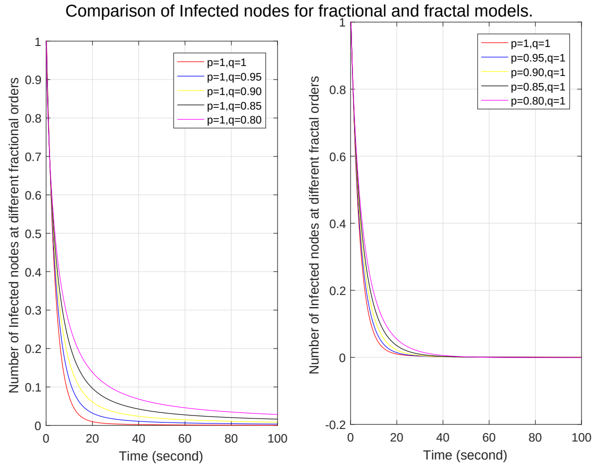


Figure 4.2: Trajectories of $\aleph(\varkappa)$ for different fractal orders $\mathfrak{p}$ when $\mathfrak{q} = 1$ and different fractional orders $\mathfrak{q}$ when $\mathfrak{p} = 1$.

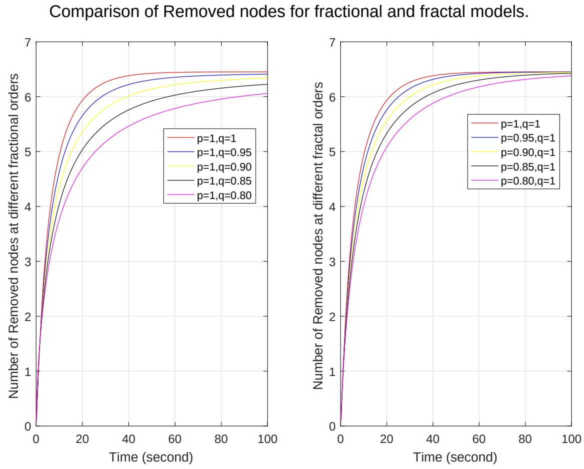


Figure 4.3: Trajectories of $\Theta(\varkappa)$ for different fractal orders $\mathfrak{p}$ when $\mathfrak{q} = 1$ and different fractional orders $\mathfrak{q}$ when $\mathfrak{p} = 1$.

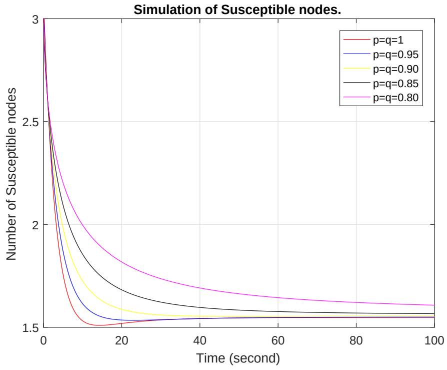


Figure 4.4: Fractal and fractional trajectories of $\Delta(\varkappa)$ with different orders of $\mathfrak{p} = \mathfrak{q}$.

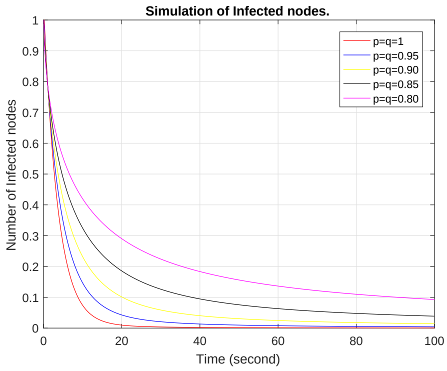


Figure 4.5: Fractal and fractional trajectories of $\aleph(\varkappa)$ with different orders of $\mathfrak{p} = \mathfrak{q}$.

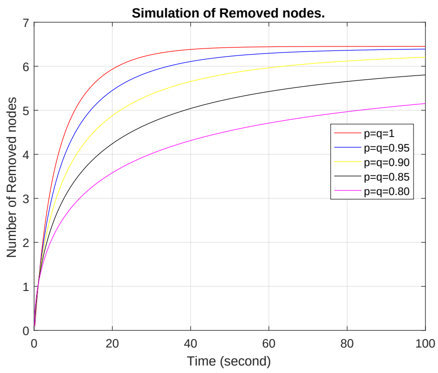


Figure 4.6: Fractal and fractional trajectories of $\Theta(\varkappa)$ with different orders of $\mathfrak{p} = \mathfrak{q}$.

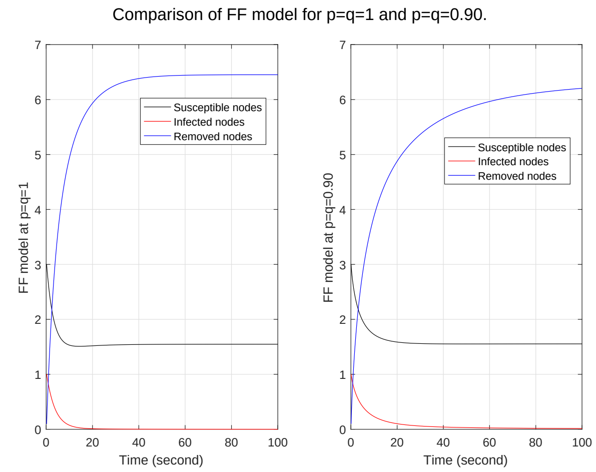


Figure 4.7: Comparison of $\Delta\aleph\Theta$ model at $\mathfrak{p} = \mathfrak{q} = 1$ and $\mathfrak{p} = \mathfrak{q} = 0.90$.

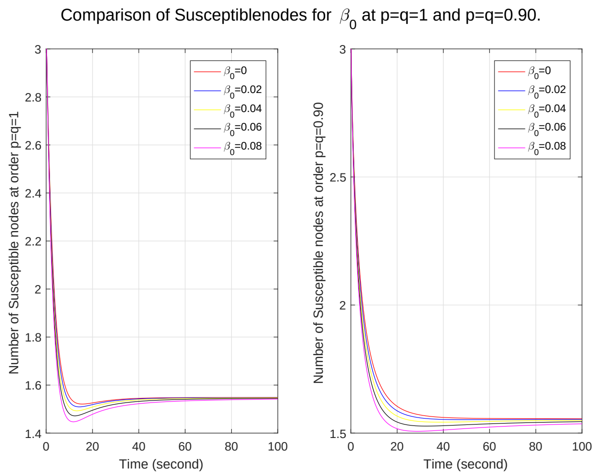


Figure 4.8: The effect of varying initial infection rate $\beta_0$ on susceptible nodes when $\mathfrak{p} = \mathfrak{q} = 1$ and $\mathfrak{p} = \mathfrak{q} = 0.90$.

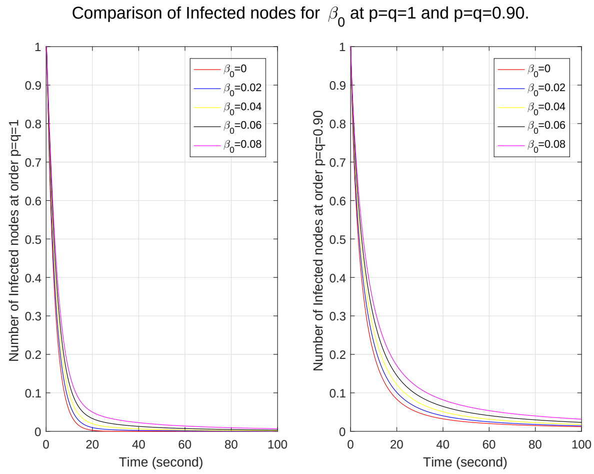


Figure 4.9: The effect of varying initial infection rate $\beta_0$ on infected nodes when $\mathfrak{p} = \mathfrak{q} = 1$ and $\mathfrak{p} = \mathfrak{q} = 0.90$.

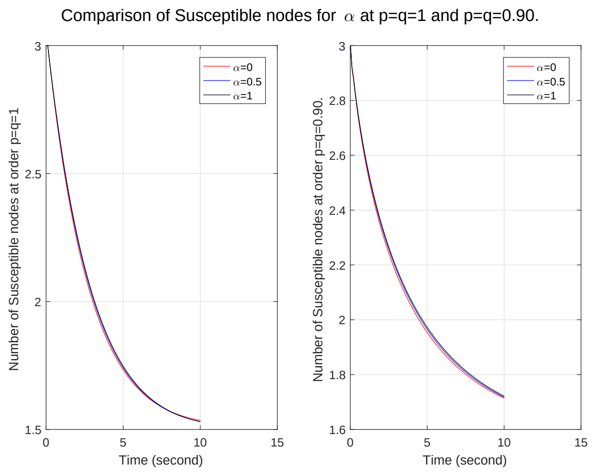


Figure 4.10: The effect of varying variable $'\alpha'$ to adjust the infection rate sensitivity on susceptible nodes when $\mathfrak{p} = \mathfrak{q} = 1$ and $\mathfrak{p} = \mathfrak{q} = 0.90$.

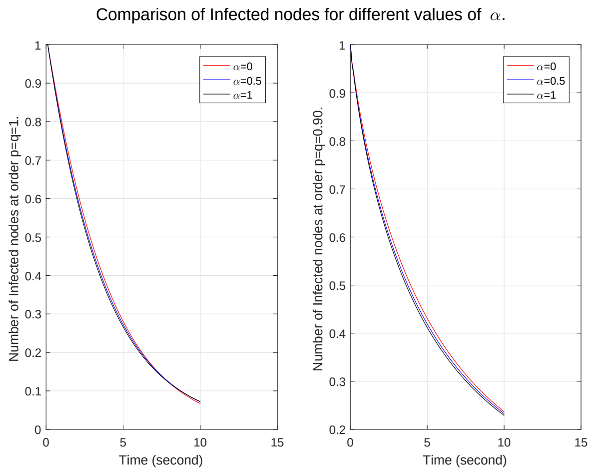


Figure 4.11: The effect of varying variable $'\alpha'$ to adjust the infection rate sensitivity on infected nodes when $\mathfrak{p} = \mathfrak{q} = 1$ and $\mathfrak{p} = \mathfrak{q} = 0.90$.

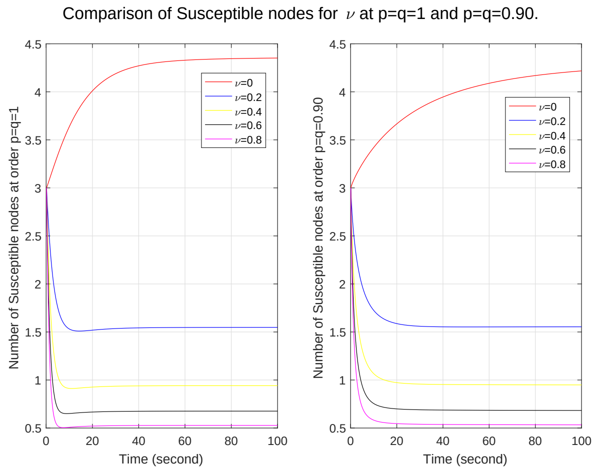


Figure 4.12: The effect of varying real-time immune rate $\nu$ on susceptible nodes when $\mathfrak{p} = \mathfrak{q} = 1$ and $\mathfrak{p} = \mathfrak{q} = 0.90$.

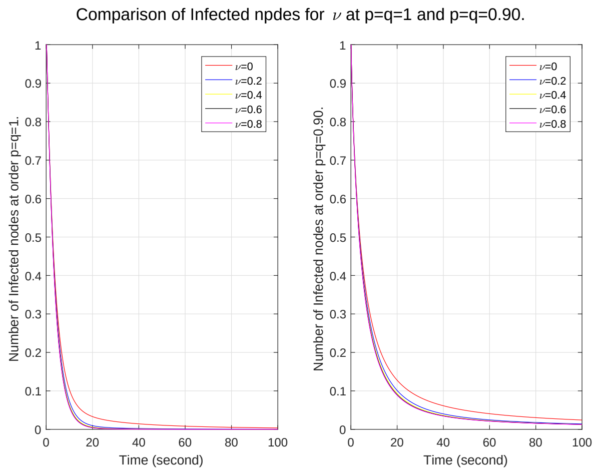


Figure 4.13: The effect of varying real-time immune rate $\nu$ on infected nodes when $\mathfrak{p} = \mathfrak{q} = 1$ and $\mathfrak{p} = \mathfrak{q} = 0.90$.

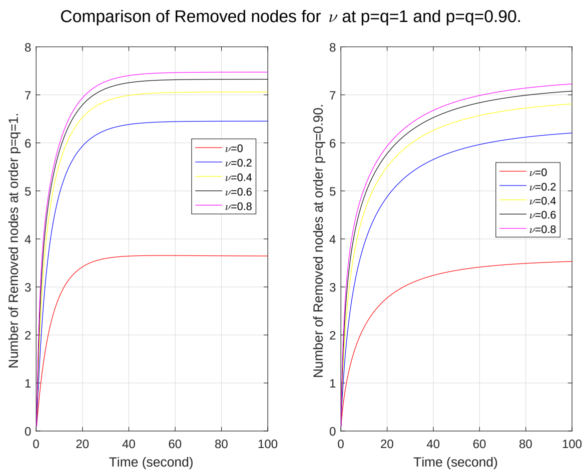


Figure 4.14: The effect of varying real-time immune rate $\nu$ on removed nodes when $\mathfrak{p} = \mathfrak{q} = 1$ and $\mathfrak{p} = \mathfrak{q} = 0.90$.

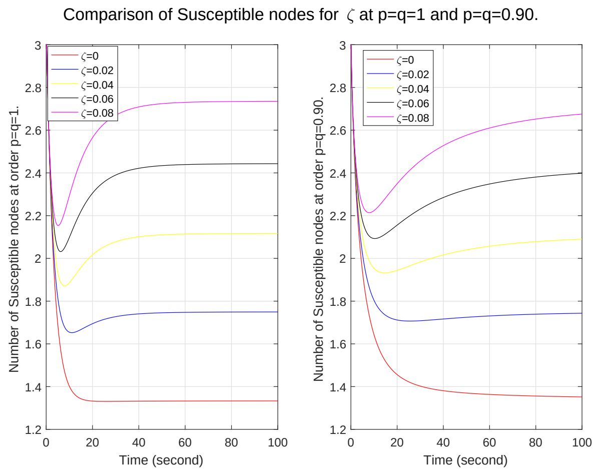


Figure 4.15: The effect of varying loss rate of immunity $\zeta$ on susceptible nodes when $\mathfrak{p} = \mathfrak{q} = 1$ and $\mathfrak{p} = \mathfrak{q} = 0.90$.

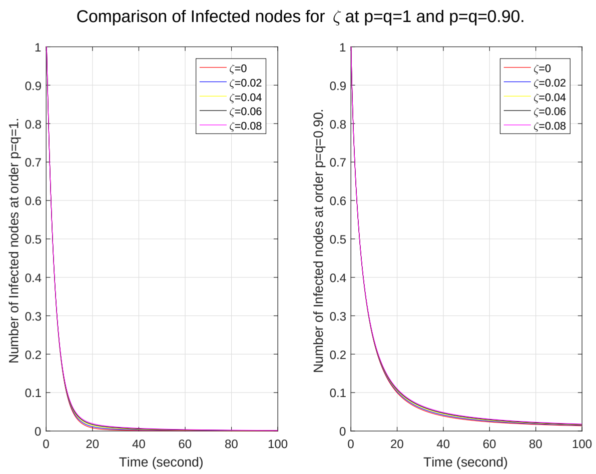


Figure 4.16: The effect of varying loss rate of immunity $\zeta$ on infected nodes when $\mathfrak{p} = \mathfrak{q} = 1$ and $\mathfrak{p} = \mathfrak{q} = 0.90$.

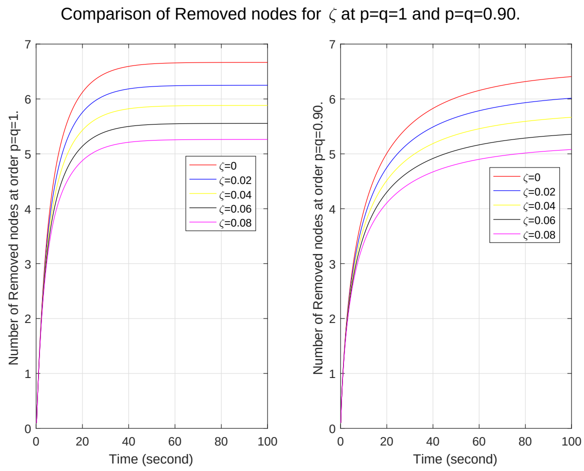


Figure 4.17: The effect of varying loss rate of immunity $\zeta$ on removed nodes when $\mathfrak{p} = \mathfrak{q} = 1$ and $\mathfrak{p} = \mathfrak{q} = 0.90$.

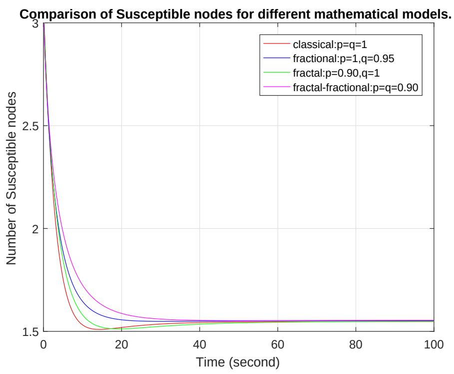


Figure 4.18: Comparison of $\Delta$ for different mathematical orders.

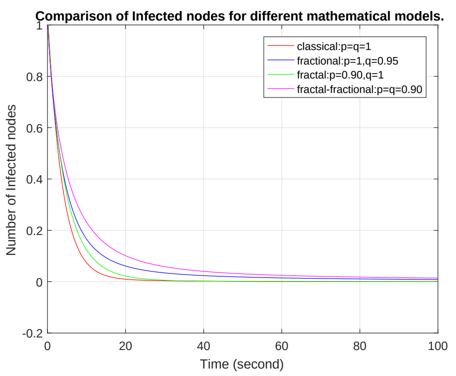


Figure 4.19: Comparison of $\aleph$ for different mathematical orders.

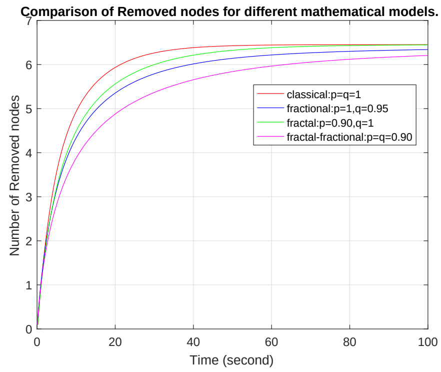


Figure 4.20: Comparison of Θ for different mathematical orders.

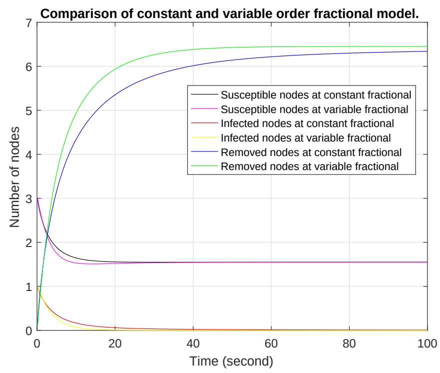


Figure 4.21: Trajectories of Δ, ℵ and Θ showing comparison of constant and variable fractional order mathematical models.

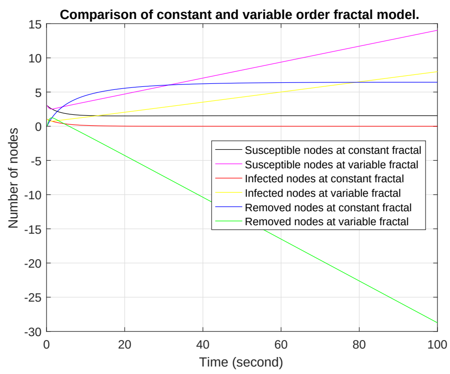


Figure 4.22: Trajectories of Δ, ℵ and Θ showing comparison of constant and variable fractal order mathematical models.

## 4.7 Discussion through Simulations based on Numerical algorithm

In this section, we see the simulation of $\Delta$, $\aleph$ and $\Theta$ under the effect of several fractal-fractional orders and also the behaviour of $\Delta, \aleph$ and $\Theta$ model with respect to some parameters and compare the results of fractal-fractional model to the ordinary differential model. We take the parameters as taken for figure 2 in [32], $\Pi = 0.5, \theta = 0.8, \beta_0 = 0.02, \mu = 0.1, \nu = 0.2, \zeta = 0.01, \kappa = 0.2, \tau = 7.3, \alpha = 1$ and some estimated initial conditions $\Delta(0) = 3, \aleph(0) = 1, \Theta(0) = 0.1$.

Figs.1-3 show the simulation of $\Delta$,$\aleph$ and $\Theta$ for different fractal and fractional orders separately. On left side of figure we take different fractional orders keeping fractal order one (fractional model) and on right side of figure, different fractal orders by keeping fractional order one (fractal model). We observe that graphs for no. of nodes in fractal model are very close to each other which show that nodes are strongly connected and indicate a more homogeneous structure with fewer variation while distance between nodes at different orders represents long-range corrections and depicts a more heterogenous and complex structure. For fractal models, in fig.1 initially at lower fractal orders, no. of nodes is greater then it goes to decrease which shows that initially system has strong memory effect and as time passes, it the memory effect decays. We see that in fractional and fractal models, no. of more susceptible nodes represents that system has higher risk of epidemic outbreaks, greater potential for infections, more connected network, increased clustering, increased vulnerability, reduced resilience, increased hereditary effects and strong memory effect. Moreover, when we compare both models we see that fractional model has stronger memory effect and is more sensitive to initial conditions. In figs.2,3 no. of infected nodes goes on increasing and no. of removed nodes goes on decreasing for lower orders. It confirms the above interpretations. When we compare the both models in fig.2, we see that fractal model shows more complex dynamics and behaviors and from fig.3 we see that the fractal model exhibits a more efficient removal mechanism, more sensitive and more fragile fractal system. It also represents that in fractional model system has strong memory effect.

Figs.4-6 show the simulation of $\Delta$, $\aleph$ and $\Theta$ under the combined effect of different fractal and fractional orders. We see that in fig.4, as fractal-fractional (ff) order decreases, no. of susceptible nodes is greater initially, then decreases and finally converges. It means initially there is a higher degree of connectivity and vulnerability to infection, decrease shows a reduction in both connectivity and vulnerability probably due to increased immunity and convergence shows that the system became stable. The effect of ff orders shows that the memory effect is more initially, then it decays with passage of time. In fig.5, the greater no. of nodes at lower ff orders indicates a stronger memory effect and a higher potential for epidemic spread. We also observe that no. of infected nodes become zero early at higher ff order, it tells us about increased resilience and improved immunity of the system. It also indicates about lower transmission rates of infection and increased isolation. In fig.6 no. of removed nodes is less at lower ff orders, it shows a longer persistence and increased prevalence of infection, robust network structure,more complex dynamics and a longer and stronger memory effect. In Fig.7, we compare classical model $\mathfrak{p} = \mathfrak{q} = 1$ and fractal-fractional model at $\mathfrak{p} = \mathfrak{q} = 0.90$ which show behavior of all nodes of the system.

Now as we see the impact of both variables, initial infection rate $\beta_0$ and $\alpha$ on our fractal-fractional model. We illustrate the graphs in two ways. First, we see the behavior of nodes for different ff orders in each model. Secondly, we compare the behavior of these nodes in both models. In fig.8, as initial birth rate increases, no. of susceptible nodes decreases that means system is more sensitive, nonlinear dynamics and increased complexity. In classical model, system has decreased epidemic risk and goes to a stable state when the risk is minimized. When we compare classical model with lower ff order model, system has more complex structure at $\mathfrak{p} = \mathfrak{q} = 0.90$. In fig.9, as the initial birth rate increases, no. of nodes also increases, it shows that the system may be more susceptible to infection, may exhibit a faster spread of infection due to a larger pool of infected nodes, may shows non-linear dynamics where small changes in $\beta_0$ lead to large changes in no. of infected nodes. It also represents that system has a stronger memory effect. As we compare, we see that in lower ff model the system has more flexibility to become infectious and has a stronger memory effect. In the original model, $\alpha$ is used to adjust the infection rate sensitivity to $\aleph$ and $\alpha = 0$ means constant infection rate. In figs.10,11 we see the effect of this varying variable. Although there seems a very slight difference at $\mathfrak{p} = \mathfrak{q} = 0.90$ but it plays a role in the dynamics of system in coordination of other parameters.

In figs.12-14, we see the effect of real-time immune rate $\nu$ on $\Delta$, $\aleph$ and $\Theta$ for $\mathfrak{p} = \mathfrak{q} = 1$ and $\mathfrak{p} = \mathfrak{q} = 0.90$. Fig.12 describes that as immune rate increases, no. of susceptible nodes become less in each model. We can say that it shows a strong immune response, decreased risk of infection, more resilience to infection and it may eradicate infection entirely. Moreover, as we compare both models, we see that at lower ff model, no. of nodes is greater. It shows increased complexity, increased spread of infection, increased clustering, improved resilience and enhanced robustness. In fig.13, no. of infected nodes are very close to each other for different values of $\nu$ in each model. It depicts that the system may have reached a saturation point and it may exhibit diminishing returns. Also for ff model at 0.90, no.of nodes are greater, which represents increased complexity, faster speed of infections, increased vulnerability and reduced resilience. Similarly, in fig.14 the no. of removed nodes is greater for higher real time immune rate in each model. It shows an effective immune response which is capable of eliminating infected nodes efficiently, faster clearance rate, increased resilience in the system and enhanced robustness of system. As we compare both models, the no. of nodes is less in ff model at level 0.90 which shows strong memory effect and increased complexity of the system.

As we know that the recovered nodes lose their immunity after some time, so to see this impact, we check the graphs of $\Delta, \aleph, \text{and} \Theta$. Figs.15-17, show the effect of $\zeta$ (loss rate of immunity). From fig.15 we see that no. of susceptible nodes goes on increasing as lost rate of immunity gets higher in each model. Also, ff model at 0.90 exhibits robustness to immunity loss and may introduce unique effects that mitigate the impact of immunity loss and show effective immune response with increased resilience. By fig.16 no. of infected nodes remains very close at different rates of immunity loss in each model represents that the immune response may have reached a saturation point and system has reached at equilibrium state. It also shows robustness and resilience of system. Moreover, the no. of infected nodes approaches to zero earlier in classical model. It describes that ff model at 0.90 shows a delayed eradication of infection, slower immune response, less efficiency and increased vulnerability in the system. Similarly, no. of removed nodes goes on decreasing at higher loss of immunity in each model in fig.17. It shows reduced

immune efficiency, longer persistence, increase in vulnerability and reduced resilience of the system. In comparison of models, ff model at 0.90 shows that no. of removed nodes is less than classical which indicates impaired immune function, persistence of infection, increase in vulnerability and decrease in system's resilience.
In figs.18-20, we compare four mathematical models (classical, fractional, fractal and fractal-fractional). From Figs.18,19 no. of susceptible and infected nodes is highest in FF model, then in fractional, fractal and classical simultaneously. It shows that fractal-fractional is more effective for expressing the complexity of malware propagation and fractional model may also be used in some cases. On the other hand, fractal and classical methods are not suitable for complex systems. The higher no. of nodes represents deeper memory effect and strong correlation between nodes. Moreover, convergence indicates that the system is stable. Similarly in Fig.20, no. of removed nodes in fractal fractional model is lowest that show deep memory effect and strong correlation. Convergence shows stability of the system. In fig.21 we see the difference between constant fractional order and variable fractional order. Similarly, fig.22 shows the comparison of constant and variable fractal order. We take variable fractional order as $\mathfrak{q}(\varkappa) = 0.90 + 0.1/(1 + exp(-\varkappa))$ and variable fractal order as $\mathfrak{p}(\varkappa) = 0.1/0.9 + exp(-\varkappa)$. In fig.21 we see that susceptible and infected nodes merge earlier after some time, i.e variable order becomes constant. The no. of removed nodes in variable order is greater than constant order depicts that variable order has more advantage of removing nodes that lead to more effective epidemic control, the system is more adaptive and variable fractional order represents more effective control strategies. From fig.22, we see that for fractal variable order, the system shows an irregular behavior. Negative no. of removed nodes shows rebound effect i.e epidemic is growing. It also shows that infection rate might be increased due to quick spread of virus than immunity of the system. It also indicates unstable dynamics of the system.

## 4.8 Conclusion

In this chapter, we have discussed FF model with Mittag-Leffler kernel considering as a fixed point problem. Conditions for existence (Leray Schauder criteria), uniqueness

(Lipschitz property) and stability (Ulam-Hyers and Ulam-Hyers-Rassias theorems) of the fractal fractional model were examined using concepts of fixed point theory. Numerical scheme was developed and simulations were performed to verify theoretical results. Our fractal fractional (FF) model was examined under fractal dimensions and fractional orders separately and combined effect of fractal dimensions and fractional orders. We observed that at lower FF orders, the number of susceptible and infected nodes was higher while no. of removed nodes is lower demonstrates the sensitivity to external influences, resilience to adapt infection and strong memory effects. It also showed that removed nodes have higher containment of infection and persistence at lower level of FF orders. We examined the impact of different parameters such as initial infection rate, variable adjustment to sensitivity of infected nodes, immune rate of antivirus strategies and loss rate of immunity of recovered nodes of mathematical model under $p = q = 1$ and $p = q = 0.90$. Through the graphs we find out the effect of memory on different types of nodes in system. We explored sensitivity, convergence, and stability of nodes under fractal fractional model. It will help us to predict about the vulnerabilities in computer systems. Antivirus strategies can be made by developing software that may help in containment and eradication of infection in the nodes by keeping an eye on the behavior of nodes. The graphs gave a clear insight that by choosing appropriate variable infection rate, the prevalence of malware can be controlled. Continuing this process, we investigated the impacts of other parameters too on malware model. We also compared four methods (classical, fractional, fractal, fractal-fractional). We discussed the cases when these models may be more suitably used. Moreover, we tried to see the impact of variable order fractional derivative and variable order fractal derivative about stability of the system.

# Chapter 5

# Comparison of three kernels and four forms of mathematical models

In this chapter, we compare the results of three kernels (Powerlaw kernel, Exponential Decay kernel, Mittag-leffler kernel) and four mathematical models (classical, fractional, fractal and fractal-fractional).

## 5.1 Comparison

In figs.1-3 we compare number of nodes for three kernels in four different models and figs.4-6 we compare four models for three kernels.
In fig.1, we see that under classical and fractal models, no. of susceptible nodes in $FFP$ and $FFM$ are same and in fractional and fractal-fractional values are approximately same. In classical and fractal models, model gets the same values in $FFP$ later than $FFE$ that means $FFP$ is slower than $FFE$. That means it is more sensitive to the initial conditions and exhibits more complexity. In fractional and fractal-fractional models, no. of susceptible nodes in $FFP$ and $FFM$ are more than $FFE$. It shows that $FFP$ and $FFM$ has increased risk of epidemic outbreak, higher rate of infection, less immune population, more vulnerable to the spread and potential for rapid epidemic spread. Moreover, convergence is rapid in classical, then fractal, fractional and fractal-fractional

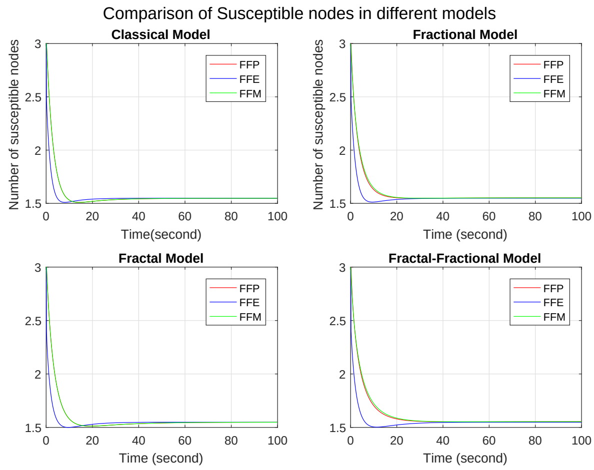


Figure 5.1: Comparison of Susceptible nodes for classical, fractional, fractal and fractal-fractional models

respectively. Similarly, we can compare the results in fig.2 and fig.3. $FFE$ may reduce the epidemic risk and system is more resilient to the spread of malware.

In figs.4-6 we see the comparison of classical, fractional, fractal and fractal-fractional mathematical models with respect to three kernels respectively. Observing the graphs, we see that the models constitute the sequence classical, fractal, fractional and fractal-fractional. By seeing the behavior of nodes, we conclude that FF model is less efficient in containing the spread of malware, exhibits an increased epidemic risk, capture the complex dynamics, exhibits non-linear dynamics and more sensitive to parameter uncertainty.

## 5.2 Conclusion

In this chapter, a deterministic mathematical model on malware propagation has been discussed in the sense of fractal fractional derivatives. At first stage, the classical mathematical model given in [32] has been converted in fractal fractional model with the kernels (Power law kernel, Exponential Decay kernel and Mittag-Leffler kernel). Initially the models were examined theoretically. For existence Leray Schauder criteria with Arzela Ascoli's theorem is used. Uniqueness is proved with the help of Lipschitz property and stability is checked by Ulam-Hyers and Ulam-Hyers-Rassias theorems. Secondly, nu-

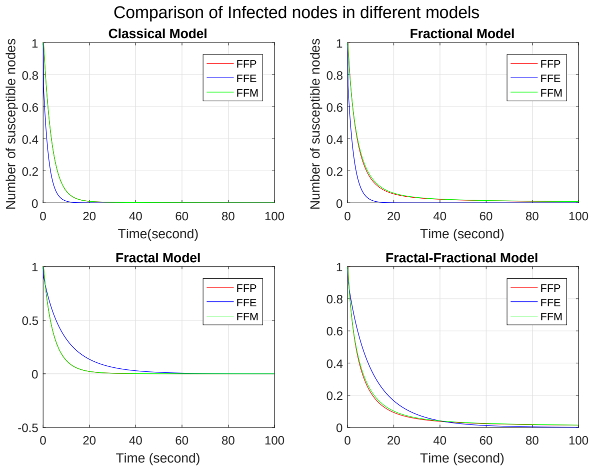


Figure 5.2: Comparison of Infected nodes for classical, fractional, fractal and fractal-fractional models

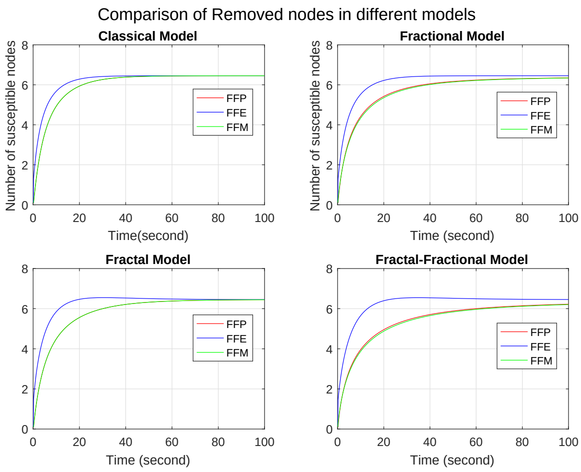


Figure 5.3: Comparison of Removed nodes for classical, fractional, fractal and fractal-fractional models

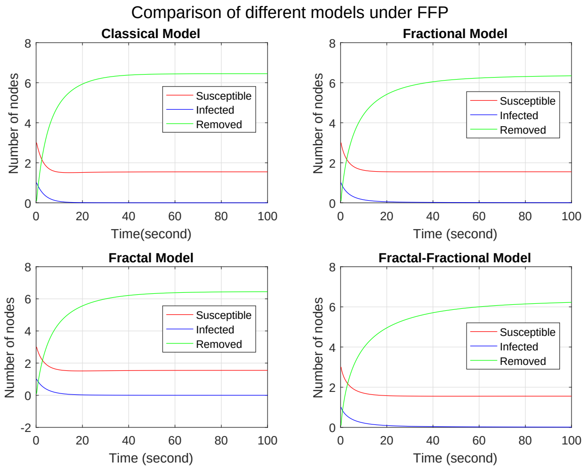


Figure 5.4: Comparison of classical, fractional, fractal, and fractal-fractional models under $FFP$

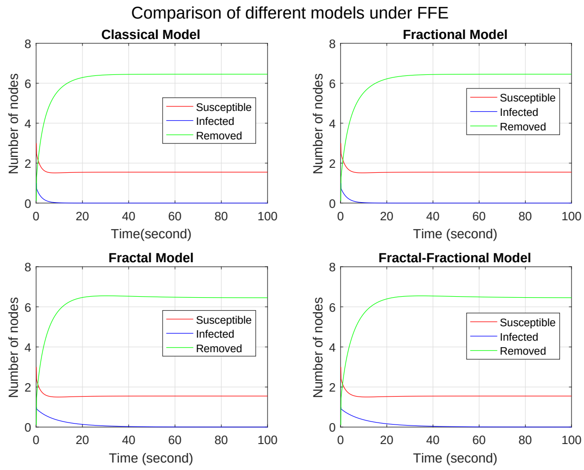


Figure 5.5: Comparison of classical, fractional, fractal, and fractal-fractional models under $FFE$

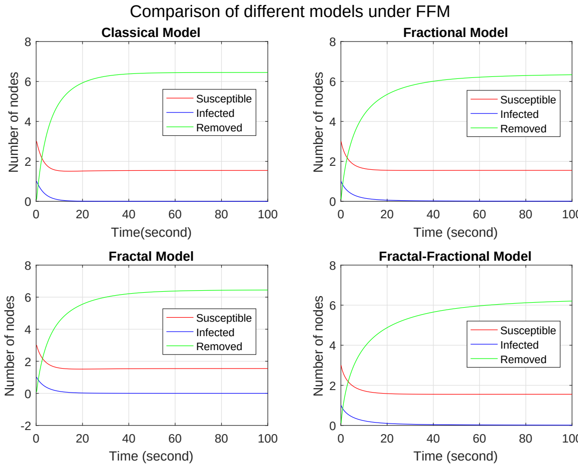


Figure 5.6: Comparison of classical, fractional, fractal, and fractal-fractional models under $FFM$

merical schemes were developed using Lagrange interpolation using two-points formula and simulations were performed using Matlab codes on R2016a to verify the accuracy of theoretical results.

During simulations, fractal fractional models were examined under fractal dimensions and fractional orders separately. Then combined effect of fractal dimensions and fractional orders was discussed. We also examined the impact of different parameters such as initial infection rate, variable adjustment to sensitivity of infected nodes, immune rate of antivirus strategies and loss rate of immunity of recovered nodes of mathematical model [32] under $\mathfrak{p} = \mathfrak{q} = 1$ and $\mathfrak{p} = \mathfrak{q} = 0.90$. Observing graphs we find out that higher number of susceptible and infected nodes and lower no. of removed nodes depicts that the system is more vulnerable to the spread of decrease, has reduced immunity, has non-linear dynamics, more sensitivity to uncertainty of parameters, potential for epidemic outbreaks, sensitivity to external influences, higher containment of infection and persistence, resilience to adapt infection and strong memory effects. When the nodes are closed to each other, it shows that the network is highly connected, can facilitate the spread of malware, can increase the risk of contagion and system may exhibit a clustered structure. Convergence shows stability of the system and convergence to one point shows that nodes have converged to a fixed point. Early convergence shows tat system becomes stable early and when no of infected nodes becomes zero, it means the disease has been eradicated and equilibrium state has been achieved. We also find out the effect of memory on different types of nodes in system. As we explore sensitivity, convergence, and stability of susceptible, infected, and removed nodes under fractal fractional model, it helps us to predict about the vulnerabilities in computer systems. Antivirus strategies can be made by developing software that may help in containment and eradication of infection in the nodes by keeping an eye on the behavior of nodes. In classical form, this model gave a clear insight that by choosing appropriate variable infection rate, the prevalence of malware can be controlled. Our FF model agrees with it. We also compared four methods named as classical, fractional, fractal, and fractal-fractional. We discussed the cases when these models may be more suitably used. Moreover, we tried to see the impact of variable order fractional derivative and variable order fractal derivative. Although sometimes we see a very small difference, but it may play a role in malware propagation as small changes may cause large perturbations. Our findings may be helpful in installing antivirus software in cyber security practice by keeping in view the past behaviors of previous nodes. This model is suitable for malware like red worms, Nimda, Slammer worms, and Wittyworms etc. That means the malware which depends on variable infection rate and time-delay factors, in that case our findings will help in developing antivirus strategies keeping in mind its cost factor.

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# Generalized Fractional Mathematical Modeling and Simulation for Dynamical Systems

*by* Nausheen Razi

# Generalized Fractional Mathematical Modeling and Simulation for Dynamical Systems

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