Some studies of Rough Pythagorean Fuzzy sets and Rough q-Rung Orthopair Fuzzy sets based on Soft Binary Relations, and their Applications

By

Muhammad Asim Bilal

Department of Mathematics

Quaid-i-Azam University,
Islamabad

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Supervised By

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بسم الله الرحمن الرحيم

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By

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A THESIS SUBMITTED IN THE PARTIAL FULFILLMENT OF THE REQUIREMENTS FOR THE DEGREE OF THE

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Introduction

Fuzzy Set (F_zS), Rough Set (R_fS), and Soft Set (S_fS) methodologies are invaluable tools for examining uncertainty and incompleteness within information systems. These frameworks offer distinct advantages and find wide-ranging applications across artificial intelligence, real-world problem-solving, and computer science domains.

In the realms of both natural and social sciences, a plethora of Fuzzy concepts abound. Zadeh [70] introduced Fuzzy Set (F_zS) theory in 1965, providing a mathematical framework to represent fuzziness. By adeptly portraying fuzziness using formal mathematical language, Zadeh ushered in a groundbreaking approach to processing Fuzzy and uncertain information. The structured and well-formulated mathematical methods inherent in F_zS theory have empowered humanity to grapple with data and information residing within uncertain boundaries. Notably, F_zS theory aligns seamlessly with the cognitive processes of human perception and reasoning, thereby enriching our understanding of complex phenomena.

Molodstove [40] introduced the concept of Soft Set (S_fS), distinguished by its innovative parameterization technique, which distinguishes it from traditional methods. This novel approach has fostered a broad spectrum of applications across various disciplines, including Riemann Integration, operational research, game theory, probability theory, and measure theory. Building upon Molodstove's foundational work, Maji et al. [42, 43] further enhanced the theoretical underpinnings of S_fSs by introducing various operations, thus enriching the methodological toolkit available for S_fS analysis.

Subsequent advancements by Ali et al. [3] refined and extended the operations proposed by Maji et al. [43], demonstrating the validity of De-Morgan's laws within the context of S_fS theory. Maji et al. [41] embarked on a pioneering effort to amalgamate the structural framework of S_fSs with that of Fuzzy Sets (F_zSs), giving rise to the innovative concept of Fuzzy Soft Set (F_zS_fS). This fusion of methodologies opened new avenues for tackling decision-making problems, as demonstrated by Feng et al. [23].

Further explorations into S_fS theory, as documented in [6, 44], have elucidated additional operations, thereby refining our understanding and expanding the applicability of S_fS methodologies. Ali and Shabir [5] contributed to the field by defining logic connectives tailored specifically for S_fSs and F_zS_fSs, enhancing the logical coherence of these methodologies. Additionally, in their subsequent work [4], Ali and Shabir presented improvements to the operations of F_zS_fS as defined in [41], further refining the methodological toolkit available for F_zS_fS analysis.

Pawlak [46] introduced Rough Set (R_fS) theory as a mathematical framework for addressing vagueness and uncertainty. Central to R_fS theory is the fundamental assumption that each object in a universe possesses some associated knowledge. For instance, in the context of citizens of a country like Pakistan, the information system might be structured around National Identity Cards, allowing for the classification of objects as indiscernible based on their similarity. The indiscernibility relation forms the mathematical foundation of R_fS theory, generating equivalence classes that represent concepts within the knowledge base. Every idea is characterized by a couple of ideas known as the lower estimation and the upper guess, working with extracting significant experiences from unsure information.

R_fS theory has garnered significant interest across various disciplines, including artificial intelligence, cognitive sciences, and machine learning. Intelligent systems, pattern recognition, and decision analysis are just a few of the many fields in which it can be used. This expansiveness features its flexibility and highlights its pertinence in different fields. R_fS theory broadens traditional set theory and offers unmistakable benefits. Dissimilar to statistical methods, it does not need extra information, for example, probability measures, or M_mD_gs as in F_zSs.

The combination of R_fSs with other mathematical structures, such as F_zSs, has further broadened its utility, enabling the description of attribute sets with ease. Various extensions and refinements of R_fS theory have been proposed in the literature, including covering-based

R_fSs, variable precision R_fSs, and R_fSs based on Binary Relations [66]. Dubois and Prade [20] integrated R_fSs and F_zSs based on Pawlak approximation space, enhancing the analytical capabilities of both methodologies.

Additionally, R_fS models have found applications in diverse problem domains, as evidenced by their utilization in various research endeavors [12, 13, 14, 15, 18, 20, 30, 50, 67, 73]. Further advancements in R_fS theory include the exploration of roughness in groups and subgroups [16], the development of generalized R_fS based on Binary Relations [72], and the investigation of roughness in semigroups (S_mG) [34] and order S_mGs based on pseudo-order [53]. Besides, mathematicians have examined the idea of rough Ideal (R_fI_d) of rings [18]. In addition, efforts have been made to integrate R_fSs, S_fSs, and F_zSs to define soft approximation spaces, [24].

In decision-making scenarios, diverse experts often yield varying evaluation results, necessitating a comprehensive framework beyond the conventional membership degree (M_mD_g) offered by Fuzzy Sets (F_zSs). In light of this need, Atanassov [7] presented the idea of the Intuitionistic Fuzzy Set (I_tF_zS), where I_tF_zS incorporates both the M_mD_g as well as the non-membership degree ($N_nM_mD_g$). Each element in an I_tF_zS U is characterized by its M_mD_g ($U_Y(\mathfrak{k})$) and $N_nM_mD_g$ ($U_N(\mathfrak{k})$). The relationship $U_Y(\mathfrak{k}) + U_N(\mathfrak{k}) \leq 1$ characterizes the level of certainty and non-trust in the element's classification inside the universal set ξ.

The novelty of I_tF_zS theory has spurred extensive research endeavors aimed at exploring its theoretical foundations and practical applications. Feng et al. [22] gave the concept of Intuitionistic Fuzzy Soft Set ($I_tF_zS_fS$), expanding upon $I_tF_zS_fS$ theory by considering various operations on it.

However, the constraint $U_Y(\mathfrak{k})+U_N(\mathfrak{k}) \leq 1$ confines the feasible selection of $U_Y(\mathfrak{k})$ and $U_N(\mathfrak{k})$ to form a triangular region in the first quadrant, as illustrated in Figure 1. This limitation poses challenges in scenarios where $U_Y(\mathfrak{k}) + U_N(\mathfrak{k}) > 1$, within the permissible range $U_Y(\mathfrak{k}), U_N(\mathfrak{k}) \in [0, 1]$. To address this issue, Yager [61] proposed the concept of Pythagorean Fuzzy Set (P_yF_zS), where the M_mD_g $U_Y(\mathfrak{k})$ and $N_nM_mD_g$ $U_N(\mathfrak{k})$ adhere to the relation

$U_Y^2(\mathfrak{k}) + U_N^2(\mathfrak{k}) \leq 1$. This relation delineates a unit quarter circle in the first quadrant, as depicted in Figure 2, providing a broader spectrum for expressing the M_mD_g and the $N_nM_mD_g$ of a Pythagorean Fuzzy Set.

By making use of P_yF_zS, we get an expanded set of membership and $N_nM_mD_g$s, effectively capturing the uncertainty associated with decision-making processes. Compared to I_tF_zSs, P_yF_zSs encompass a larger set of points, as illustrated in Figure 3, thereby offering a more comprehensive representation of uncertain information and decision-making scenarios. While

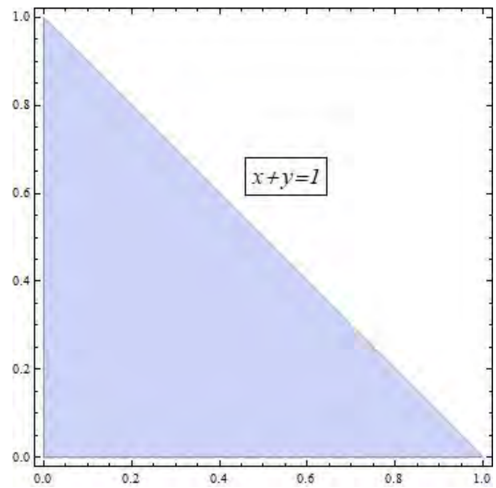

Figure 1

I_tF_zSs and P_yF_zS offer valuable frameworks for handling incomplete information, certain limitations persist. For instance, consider a scenario where a decision maker provides $U_Y(\mathfrak{k}) = 0.7$ and $U_N(\mathfrak{k}) = 0.9$. In this case, $U_Y^2(\mathfrak{k}) + U_N^2(\mathfrak{k}) = 1.30 > 1$, highlighting a discrepancy that I_tF_zSs and P_yF_zSs cannot accommodate.

To this end, Yager [62] introduced the novel idea of a q-Rung Orthopair Fuzzy Set (${}^qROF_zS$), which extends the capabilities of both I_tF_zSs and P_yF_zSs. In ${}^qROF_zS$s, the M_mD_g $U_Y(\mathfrak{k})$ and $N_nM_mD_g$ $U_N(\mathfrak{k})$ adhere to the relation $U_Y^q(\mathfrak{k}) + U_N^q(\mathfrak{k}) \leq 1$, where $q \geq 1$. For instance, the pair $(0.5, 0.4)$ represents an intuitionistic M_mD_g since $0.5 + 0.4 \leq 1$. Conversely, if the $N_nM_mD_g$ is 0.6, then $0.5+0.6 \geq 1$, indicating a Pythagorean M_mD_g which is not an intuitionistic M_mD_g. However, scenarios such as $(0.5, 0.9)$ cannot be adequately described by either

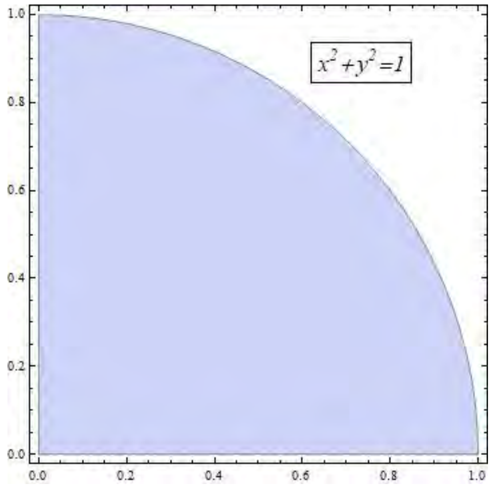

Figure 2

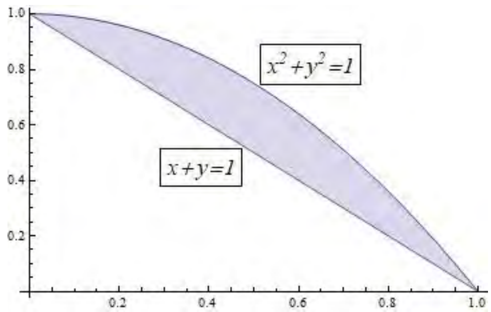

Figure 3

I_tF_zSs or P_yF_zSs. In this case, with $0.5 + 0.9 \geq 1$ and $0.5^2 + 0.9^2 \geq 1$, $(0.5, 0.9)$ represents a q-Rung Orthopair M_mD_g $(q \geq 3)$, making ${}^qROF_zS$ suitable for resolving decision-making dilemmas. Notably, for $q = 1$, ${}^qROF_zS$ reduces to an I_tF_zS, while for $q = 2$, it aligns with a P_yF_zS, illustrating the generalization of both models. ${}^qROF_zS$s offer a broader spectrum of Fuzzy information expression, providing increased flexibility and suitability for navigating uncertain environments.

In [69], Yager and Alajlan proposed the fundamental features of ${}^qROF_zS$, which play a crucial role in knowledge representation. Ali [8], leveraging the concept of orbits, offered an alternative interpretation of ${}^qROF_zS$.

Meanwhile, Peng et al. [51] scrutinized the exponential and aggregate operators of ${}^qROF_zS$s, endorsing various scoring functions and applying them to the selection of education management systems. Shaheen et al. [55] devised an alternative approach supporting ${}^{q}ROF_zS$ for deriving score functions.

Hussain et al. [28] established covering-based rough ${}^{q}ROF_zS$ and presented a methodology for resolving decision-making problems. Additionally, Bilal et al. [10] formulated a generalized rough ${}^{q}ROF_zS$ based on the Binary Relation of the dual universe, utilizing this framework to elucidate and determine certain fundamental concepts.

The ${}^{q}ROF_z$ aggregation operators for averaging information were subsequently introduced by Liu and Wang [35]. This foundational work has spurred further research into ${}^{q}ROF_z$ theory, with contributions from various scholars [8, 19, 37, 38, 39, 59, 68]. This ongoing exploration holds the promise of enhancing our capacity to manage uncertainty and incomplete information in decision-making processes.

In summary, this dissertation aims to advance the study of extended Fuzzy sets by introducing novel frameworks, algorithms, and applications. The theoretical contributions made in this work provide a strong foundation for addressing real-world problems in decision-making, pattern recognition, and more.

Motivation

The exploration of Pythagorean Fuzzy Sets (P_yF_zSs) has been extensive, with numerous researchers delving into its theory and applications across various fields. Peng et al. [47] laid down the foundations by defining Pythagorean Fuzzy Soft Sets ($P_yF_zS_fS$), elucidating basic operations, and showcasing applications. Zhang et al. [71] proposed an application of TOPSIS over P_yF_zS, expanding its utility in decision-making contexts. Additionally, Hussain et al. [28] introduced Pythagorean Fuzzy soft rough sets ($P_yF_zS_fR_fS$s), further enriching the theoretical framework. Olgun et al. [45] contributed to the advancement of P_yF_zS theory by defining Pythagorean Fuzzy Topological Spaces ($P_yF_zT_pS$s) and exploring Pythagorean Fuzzy

continuity between two $P_yF_zT_pS$s. Mean operators for P_yF_zSs have also been developed by several authors, as documented in [58, 60].
Recent research has seen a surge in the study of q-Rung Orthopair Fuzzy Sets (${}^qROF_zS$s). Peng et al. [52] introduced an algorithm for emergency decision-making using ${}^qROF_zS$s, demonstrating their effectiveness in critical scenarios. Hussain et al. [29] innovatively combined ${}^qROF_zS$ with Soft Sets, presenting aggregation operators that enhance their applicability. Furthermore, in [36, 51], researchers defined various mean operators tailored for ${}^qROF_zS$, such as ${}^qROF_z$ Bonferroni mean operators and exponential operators, contributing to the versatility of the model.

Chapter-wise Study

Chapter one serves as an introduction to foundational concepts essential for the subsequent chapters.
In the second chapter, inspired by the pioneering work of Yager [61], we extend the framework presented by Kanwal and Shabir [32] to the realm of Pythagorean Fuzzy Sets (P_yF_zSs). We delve into lower and upper approximations of P_yF_zSs using soft Binary Relations concerning F_rS and A_fSs, establishing their properties. Additionally, we introduce two types of Pythagorean Fuzzy Topological Spaces ($P_yF_zT_pS$s) derived from soft Binary Relation (S_fB_nR) and explore similarity relations between P_yF_zSs. The notion of roughness and accuracy for Pythagorean M_mD_gs with respect to F_rS and A_fSs is also introduced. Furthermore, an algorithm for decision-making using P_yF_zSs is provided, accompanied by an illustrative example demonstrating its application in real-world scenarios.
The third chapter focuses on the lower approximation (L_oA_p) and the upper approximation (U_pA_p) of a q-Rung Orthopair Fuzzy Set (${}^qROF_zS$) utilizing Crisp Binary Relations (C_rB_nR) with respect to F_rS and A_fSs, along with their associated properties. We delve into two types of q-Rung Orthopair Topological Spaces (${}^qROF_zT_pS$s) induced by C_rB_nRs and explore simi-

larity relations between ${}^{q}ROF_zS$s based on C_rB_nRs. Additionally, we introduce the concept of roughness and accuracy for q-Rung Orthopair M_mD_gs with respect to F_rS and A_fSs. An algorithm for decision-making using ${}^{q}ROF_zS$s is presented, along with a practical example illustrating its efficacy in decision-making contexts.

In the fourth chapter, we extend the approach outlined by Kanwal and Shabir [31] and Bilal et al. [9, 10] to the realm of ${}^{q}ROF_zS$ utilizing S_fB_nRs on dual universes, following Yager's pioneering idea. The chapter explores the properties of lower and upper approximations of ${}^{q}ROF_zS$s based on S_fB_nRs concerning F_rS and A_fSs. We discuss various types of q-Rung Orthopair Fuzzy Topologies (${}^{q}ROF_zT_p$s) constructed using soft Reflexive Relations (S_fR_fR) and introduce similarity relations between ${}^{q}ROF_zS$s based on S_fB_nRs. Moreover, we present a graphical solution for decision-making problems using ${}^{q}ROF_zS$s, accompanied by an illustrative example to demonstrate its application.

The fifth chapter delves into the approximation of P_yF_zSs in terms of S_fR_fRs. We discuss approximations of P_yF_zSs by F_rS and A_fSs, resulting in upper and lower $P_yF_zS_fS$s. Furthermore, we explore the approximation of P_yF_z subS_mGs (S_bS_mGs), Pythaorean Fuzzy Left Ideals ($P_yF_zL_fId$s), Pythagorean Fuzzy Right Ideals ($P_yF_zR_iI_d$s), Pythagorean Fuzzy Interior Ideals ($P_yF_zI_tI_d$s), and Pythagorean Fuzzy Bi-Ideals ($P_yF_zB_iI_d$s) of S_mGs, accompanied by illustrative examples.

In the sixth chapter, we extend our discussion to the approximation of ${}^{q}ROF_zS$s in terms of S_fR_fRs. Similar to the previous chapter, we explore approximations of ${}^{q}ROF_zS$s sets by F_rS and A_fSs, yielding upper and lower ${}^{q}ROF_zS_fS$s. Additionally, we delve into the approximation of ${}^{q}ROF_zS_bS_mG$s, ${}^{q}ROF_z$ left (right) Ideals, ${}^{q}ROF_zI_tI_d$s, and ${}^{q}ROF_zB_iI_d$s of S_mGs, complemented by relevant examples.

Chapter 1

Basic Concepts

This chapter lays the foundation with fundamental concepts related to Fuzzy Set (F_zS), Intuitionistic Fuzzy Set (I_tF_zS), Pythagorean Fuzzy Set (P_yF_zS) and its properties, q-Rung Orthopair Fuzzy Set (${}^qROF_zS$) and its properties, Crisp Binary Relation (C_rB_nR), Rough Set (R_fS), Rough Pythagorean Fuzzy Set ($R_fP_yF_zS$, Soft Set (S_fS), Soft Binary Relation (S_fB_nR), Pythagorean Fuzzy Soft Set ($P_yF_zS_fS$), S_mG (S_mG), and Pythagorean Fuzzy Ideals ($P_yF_zI_d$s), which are indispensable for subsequent chapters.

Throughout our discussion, ξ, ξ_1 and ξ_2 denote non-empty finite sets unless otherwise specified.

1.1 Fuzzy sets and their generalizations

Fuzzy Set (F_zS) theory, introduced by Zadeh [70], offers a powerful approach to handling vagueness and uncertainty in various systems. Fuzzy Set (F_zS) has become a valuable tool in both scientific and mathematical domains, especially for describing complex or ambiguous systems.

Definition 1.1.1. *[70] A F_zS U in ξ is a map $U : \xi \to [0, 1]$. A F_zS is non-empty if $U(\Bbbk) \neq 0$, for some $\Bbbk \in \xi$.*

Here, $U(ƙ)$ represents the M_mD_g of the object ƙ in the Fuzzy set, and the mapping U is termed the membership function (M_mf_n) on ξ.
The families of all F_zSs in ξ are denoted by $F_z(\xi)$.
In addition to considering M_mD_gs in F_zSs, situations in real life often necessitate the consideration of $N_nM_mD_g$s. To address this need, Atanassov [1] introduced the concept of Intuitionistic Fuzzy Set (I_tF_zS).

Definition 1.1.2. *[1] An Intuitionistic Fuzzy Set* (I_tF_zS) U *in* ξ *takes the form:*

$$U = \{\langle ƙ, U_Y(ƙ), U_N(ƙ)\rangle : ƙ \in \xi\}$$

Here, $U_Y : \xi \to [0,1]$ *and* $U_N : \xi \to [0,1]$*, satisfying the constraint* $U_Y(ƙ) + U_N(ƙ) \leq 1$. *Each* $(U_Y(ƙ), U_N(ƙ))$ *pair, for any* $ƙ \in \xi$*, is termed an Intuitionistic Fuzzy Number.*

Definition 1.1.3. *[61] A Pythagorean Fuzzy Set* (P_yF_zS) U *in the universe* ξ *takes the form*

$$U = \{\langle ƙ, U_Y(ƙ), U_N(ƙ)\rangle : ƙ \in \xi\}$$

where $U_Y : \xi \to [0,1]$ *and* $U_N : \xi \to [0,1]$ *which satisfy the Pythagorean condition* $U_Y^2(ƙ) + U_N^2(ƙ) = r^2(ƙ)$ *for all* $ƙ \in \xi$*, where* $r : \xi \to [0,1]$. *Here, the* M_mD_g *of* ƙ *is represented by* $U_Y(ƙ)$*,* $U_N(ƙ)$ *represents the* $N_nM_mD_g$ *of* ƙ*, and* $r(ƙ)$ *denotes the strength of commitment at point* $ƙ \in \xi$. *The pair* $(U_Y(ƙ), U_N(ƙ))$ *for any* $ƙ \in \xi$ *is termed a Pythagorean Fuzzy Number.*

Definition 1.1.4. *[61] Let* $(r(ƙ), \theta(ƙ))$ *denotes polar coordinates of the Pythagorean Fuzzy Number* $(U_Y(ƙ), U_N(ƙ))$*, such that*

$$U_Y(ƙ) = r(ƙ)\cos(\theta(ƙ)) \qquad \text{and} \qquad U_N(ƙ) = r(ƙ)\sin(\theta(ƙ))$$

for $ƙ \in \xi$. *Then, the function* $d : \xi \to [0,1]$ *defined by* $d(ƙ) = (1 - \frac{2\theta(ƙ)}{\pi})$ *corresponds to the direction of commitment at* ƙ.

If $\theta(\Bbbk) = \frac{\pi}{2}$, *then* $U_Y(\Bbbk) = 0$, $U_N(\Bbbk) = r(\Bbbk)$, *indicating* $d(\Bbbk) = 0$. *Conversely, if* $\theta(\Bbbk) = 0$, *then* $U_Y(\Bbbk) = r(\Bbbk)$, $U_N(\Bbbk) = 0$, *resulting in* $d(\Bbbk) = 1$.

Here, $\pi_U(\Bbbk) = \sqrt{1 - U_Y^2(\Bbbk) - U_N^2(\Bbbk)}$ represents the indeterminacy of an object $\Bbbk \in \xi$. It is evident that P_yF_zS generalizes both I_tF_zS and F_zS. In decision-making problems, the P_yF_zS provides a larger membership space than the I_tF_zS. Therefore, a P_yF_zS exhibits greater capability than an I_tF_zS in modeling vagueness in real-life decision-making problems. All Pythagorean Fuzzy Sets in ξ are denoted as $P_yF_zS(\xi)$.

Definition 1.1.5. *Consider* $U = \{\langle \Bbbk, U_Y(\Bbbk), U_N(\Bbbk)\rangle : \Bbbk \in \xi\}$ *and* $V = \{\langle \Bbbk, V_Y(\Bbbk), V_N(\Bbbk)\rangle : \Bbbk \in \xi\}$ *be two* $P_yF_zS_fSs$ *in* ξ. *The basic operations on* $P_yF_zS(\xi)$, *defined by Yager[61], are as follows:*

i) $U \cup V = \{\langle \Bbbk, U_Y(\Bbbk) \vee V_Y(\Bbbk), U_N(\Bbbk) \wedge V_N(\Bbbk)\rangle : \Bbbk \in \xi\}$

ii) $U \cap V = \{\langle \Bbbk, U_Y(\Bbbk) \wedge V_Y(\Bbbk), U_Y(\Bbbk) \vee V_N(\Bbbk)\rangle : \Bbbk \in \xi\}$

iii) $U \subseteq V$ *if and only if* $U_N(\Bbbk) \geq V_N(\Bbbk)$ *and* $U_Y(\Bbbk) \leq V_Y(\Bbbk)$, *for all* $\Bbbk \in \xi$

iv) $U = V$ *if and only if* $U_Y(\Bbbk) = V_Y(\Bbbk)$ *and* $U_N(\Bbbk) = V_N(\Bbbk)$, *for all* $\Bbbk \in \xi$

v) $U^c = \{\langle \Bbbk, U_N(\Bbbk), U_Y(\Bbbk)\rangle : \Bbbk \in \xi\}$.

The P_yF_zS $1_U = \langle 1, 0\rangle$ *and* P_yF_zS $0_U = \langle 0, 1\rangle$, *where* $1(\Bbbk) = 1$ *and* $0(\Bbbk) = 0$, *for all* $\Bbbk \in \xi$.

In [61], Yager introduced a scoring function for comparing and ranking two P_yF_zNs based on their scores.

Definition 1.1.6. *Let* $(U_Y(\Bbbk), U_N(\Bbbk))$ *represent a* P_yF_zN *for* $\Bbbk \in \xi$. *Yager[61] defined the scoring function* f *of* P_yF_zN *as:*

$$f(r_U(\Bbbk), \theta_U(\Bbbk)) = \frac{1}{2} + r_U(\Bbbk)\left(\frac{1}{2} - \frac{2\theta_U(\Bbbk)}{\pi}\right)$$

where $r_U^2(\Bbbk) = U_Y^2(\Bbbk) + U_N^2(\Bbbk)$ *and* $\cos(\theta_U(\Bbbk)) = \frac{U_Y(\Bbbk)}{r_U(\Bbbk)}$.

Definition 1.1.7. *[61] Let $U(\Bbbk)$ and $V(\Bbbk)$ for $\Bbbk \in \xi$ be two P_yF_zNs, and $f(r_U(\Bbbk), \theta_U(\Bbbk))$ and $f(r_V(\Bbbk), \theta_V(\Bbbk))$ be the scores of $U(\Bbbk)$ and $V(\Bbbk)$ respectively. Then*

i) *$U > V$ if $f(r_U(\Bbbk), \theta_U(\Bbbk)) > f(r_V(\Bbbk), \theta_V(\Bbbk))$*

ii) *$U < V$ if $f(r_U(\Bbbk), \theta_U(\Bbbk)) < f(r_V(\Bbbk), \theta_V(\Bbbk))$*

iii) *$U = V$ if $f(r_U(\Bbbk), \theta_U(\Bbbk)) = f(r_V(\Bbbk), \theta_V(\Bbbk))$.*

Yager [62] proposed q-Rung Orthopair Fuzzy Set (${}^qROF_zS$), which enlarges the range of membership functions. In the following, a brief introduction of ${}^qROF_zS$s is given.

Definition 1.1.8. *[62] A q-Rung Orthopair Fuzzy Set (${}^qROF_zS$) U in ξ is defined by:*

$$U = \{\langle \Bbbk, U_Y(\Bbbk), U_N(\Bbbk)\rangle : \Bbbk \in \xi, q \geq 1\}$$

where $U_Y : \xi \to [0,1]$ and $U_N : \xi \to [0,1]$ satisfy $U_Y^q(\Bbbk) + U_N^q(\Bbbk) \leq 1$, for all $\Bbbk \in \xi$. Here, the M_mD_g of $\Bbbk$ is denoted by
$U_Y(\Bbbk)$, while $U_N(\Bbbk)$ indicates the $N_nM_mD_g$ of $\Bbbk$. The indeterminacy or hesitancy of $\Bbbk \in \xi$ is given by $\pi_U(\Bbbk) = 1 - (U_Y^q + U_N^q)^{\frac{1}{q}}$. The pair $(U_Y(\Bbbk), U_N(\Bbbk))$ for any $\Bbbk \in \xi$ is referred to as a q-Rung Orthopair Fuzzy Number (${}^qROF_zN$).

${}^qROF_z(\xi)$ denotes all ${}^qROF_zS$s in ξ.
It's worth noting that for $q = 1$, $(U_Y(\Bbbk), U_N(\Bbbk))$ represents an I_tF_zN, and for $q = 2$, it corresponds to a P_yF_zN. From Figure 1.1, it's apparent that ${}^qROF_zS$s cover a wide range of membership and $N_nM_mD_g$s, making them more versatile than P_yF_zSs and I_tF_zSs.
The ${}^qROF_zS$ $1_U = \langle 1, 0\rangle$ and ${}^qROF_zS$ $0_U = \langle 0, 1\rangle$, where $1(\Bbbk) = 1$ and $0(\Bbbk) = 0$, for all $\Bbbk \in \xi$.

Definition 1.1.9. *[35] Let $U = \{\langle \Bbbk, U_Y(\Bbbk), U_N(\Bbbk)\rangle : \Bbbk \in \xi, q \geq 1\}$ and $V = \{\langle \Bbbk, V_Y(\Bbbk), V_N(\Bbbk)\rangle : \Bbbk \in \xi, q \geq 1\}$ be two ${}^qROF_zSs$ in ξ. Lie and Wang [35] defined the basic operations on ${}^qROF_zS(\xi)$ as follows:*

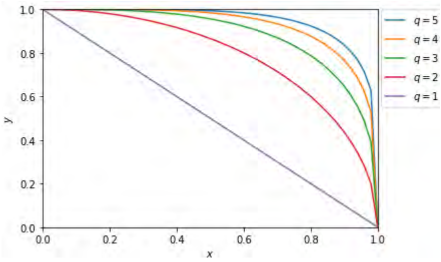

Figure 1.1

i) $U \cup V = \{\langle \acute{k}, U_Y(\acute{k}) \vee V_Y(\acute{k}), U_N(\acute{k}) \wedge V_N(\acute{k})\rangle : \acute{k} \in \xi, q \geq 1\}$

ii) $U \cap V = \{\langle \acute{k}, U_Y(\acute{k}) \wedge V_Y(\acute{k}), V_Y(\acute{k}) \vee V_N(\acute{k})\rangle : \acute{k} \in \xi, q \geq 1\}$

iii) $U \subseteq V$ *if and only if* $U_N(\acute{k}) \geq V_N(\acute{k})$ *and* $U_Y(\acute{k}) \leq V_Y(\acute{k})$*, for all* $\acute{k} \in \xi$

iv) $U = V$ *if and only if* $U_Y(\acute{k}) = V_Y(\acute{k})$ *and* $U_N(\acute{k}) = V_N(\acute{k})$*, for all* $\acute{k} \in \xi$

v) $U^c = \{\langle \acute{k}, U_N(\acute{k}), U_Y(\acute{k})\rangle : \acute{k} \in \xi\}$.

vi) $U \oplus V = \{\langle \acute{k}, [U_Y^q(\acute{k}) + V_Y^q(\acute{k}) - U_Y(\acute{k}).V_Y(\acute{k})]^{\frac{1}{q}}, U_N(\acute{k}).V_N(\acute{k})\rangle : \acute{k} \in \xi, q \geq 1\}$

vii) $U \otimes V = \{\langle \acute{k}, U_Y(\acute{k}).V_Y(\acute{k}), [U_N^q(\acute{k}) + V_N^q(\acute{k}) - U_N(\acute{k}).V_N(\acute{k})]^{\frac{1}{q}}\rangle : \acute{k} \in \xi, q \geq 1\}$

viii) $U^t = \{\langle \acute{k}, U_Y^t(\acute{k}), (1 - (1 - U_N^q(\acute{k}))^t)^{\frac{1}{q}}\rangle : \acute{k} \in \xi, q \geq 1, t \geq 0\}$

ix) $tU = \{\langle \acute{k}, (1 - (1 - U_Y^q(\acute{k}))^t)^{\frac{1}{q}}, U_N^t(\acute{k})\rangle : \acute{k} \in \xi, q \geq 1, t \geq 0\}$.

Definition 1.1.10. *[57] For any* ${}^qROF_zS$ $U = (U_Y(\acute{k}), U_N(\acute{k}))$*,* $\acute{k} \in \xi$*, we define the score function as*

$$S(U) = \frac{1}{2}(1 + U_Y^q(\acute{k}) - U_N^q(\acute{k})),$$

where $q \geq 1$*. A higher score value indicates a better Orthopair.*

1.2 Rough sets: Definitions and examples

Here, we present a couple of ideas linked with Rough Set (R_fS) theory. Moreover, some examples are added to demonstrate these concepts.

Definition 1.2.1. *A Crisp Binary Relation (C_rB_nR) J from ξ_1 to ξ_2 is a subset of $\xi_1 \times \xi_2$ and a Binary Relation (B_nR) on ξ_1 is a subset of $\xi_1 \times \xi_1$.*
If J is a B_nR on ξ_1, then:

1) *J is Reflexive if $(\Bbbk, \Bbbk) \in J$ for all $\Bbbk \in \xi_1$.*

2) *J is symmetric if $(\Bbbk_1, \Bbbk_2) \in J$ implies $(\Bbbk_2, \Bbbk_1) \in J$ for all $\Bbbk_1, \Bbbk_2 \in \xi_1$.*

3) *J is transitive if $(\Bbbk_1, \Bbbk_2), (\Bbbk_2, \Bbbk_3) \in J$ implies $(\Bbbk_1, \Bbbk_3) \in J$ for all $\Bbbk_1, \Bbbk_2, \Bbbk_3 \in \xi_1$.*

If J satisfies the above three conditions, then it is called an equivalence relation (E_qR).

In Pawlak's work [46], the theory of R_fS was originally introduced to manage imprecision and incompleteness within data frameworks.
If $\xi \neq \emptyset$ represents a finite set and J serves as an E_qR on ξ, then (ξ, J) is denoted as an approximation space (A_pS). A subset $U \subseteq \xi$, which is composed of a combination of some equivalence classes of ξ, is defined as definable. On the contrary, if U is not definable, it cannot be expressed as a union of equivalence classes.
When U is not definable, it can be estimated using the upper approximation (U_pA_p) and lower approximation (L_oA_p). These approximations are two definable subsets of ξ and are defined as follows:
$\underline{J}(U) = \cup\{[\Bbbk] : [\Bbbk]_J \subseteq U\}$ and $\overline{J}(U) = \cup\{[\Bbbk]_J : [\Bbbk]_J \cap U \neq \emptyset\}$. The pair $(\underline{J}(U), \overline{J}(U))$ constitutes a R_fS. The set $H = \overline{J}(U) - \underline{J}(U)$ represents the boundary region. If $\underline{J}(U)$ is equal to $\overline{J}(U)$, then U is definable, and $\overline{J}(U) - \underline{J}(U)$ results in the empty set.

Definition 1.2.2. *[46] A subset U of ξ represents a Crisp Set if $\underline{J}(U) = \overline{J}(U)$.*

The set ξ can be divided into three separate parts using the L_oA_p and U_pA_p of a subset $U \subseteq \xi$.

Theorem 1.2.3. *[46] Consider J be E_qR defined on a set ξ. If ξ has subsets U and V, then:*

i) $\underline{J}(U) \subseteq U \subseteq \overline{J}(U)$

ii) $U \subseteq V$ *implies* $\underline{J}(U) \subseteq \underline{J}(V)$

iii) $U \subseteq V$ *implies* $\overline{J}(U) \supseteq \overline{J}(V)$

iv) $\underline{J}(U \cap V) = \underline{J}(U) \cap \underline{J}(V)$

v) $\underline{J}(U) \cup \underline{J}(V) \subseteq \underline{J}(U \cup V)$

vi) $\overline{J}(U \cup V) = \overline{J}(U) \cup \overline{J}(V)$

vii) $\overline{J}(U \cap V) \subseteq \overline{J}(U) \cap \overline{J}(V)$.

Example 1.2.4. *Let (ξ, J) be an approximation space (A_pS), where $\xi = \{\Bbbk_1, \Bbbk_2, \Bbbk_3, \Bbbk_4, \Bbbk_5\}$ and J is an E_qR. Consider the following equivalence classes:*

$$[\Bbbk_1] = \{\Bbbk_1, \Bbbk_2, \Bbbk_3\},$$

$$[\Bbbk_3] = \{\Bbbk_4, \Bbbk_5\}.$$

Let $U = \{\Bbbk_1, \Bbbk_3\}$ and $V = \{\Bbbk_2, \Bbbk_4\}$. Then $\underline{J}(U) = \{\Bbbk_1, \Bbbk_3\}$ and $\overline{J}(U) = \{\Bbbk_1, \Bbbk_2, \Bbbk_3\}$, $\underline{J}(V) = \{\Bbbk_2, \Bbbk_4\}$ and $\overline{J}(V) = \{\Bbbk_1, \Bbbk_2, \Bbbk_3, \Bbbk_4\}$.
So $J(U) = (\{\Bbbk_1, \Bbbk_3\}, \{\Bbbk_1, \Bbbk_2, \Bbbk_3\})$ is a R_fS and $J(V) = (\{\Bbbk_2, \Bbbk_4\}, \{\Bbbk_1, \Bbbk_2, \Bbbk_3, \Bbbk_4\})$ is a Crisp Set.

1.3 Soft Sets and Soft Substructures

Molodtsov [40] introduced the theory of Soft Sets (S_fS) as a fundamental notion for handling uncertainty. S_fSs offer various operations that are useful in dealing with different types of situations. This theory suggests that every collection of objects in the universe ξ can be associated with a subset E of attributes (characteristics or parameters) for ξ.

$P(\xi)$ denotes the set of all Crisp subsets of ξ. When D is a subset of E, with E being the universal set of parameters, we explore fundamental definitions associated with S_fSs.

Definition 1.3.1. *Consider $\mathscr{S} : D \to P(\xi)$. Then $(\mathscr{S}, D)$ is termed as a Soft Set over ξ.*

Molodtsov provided several concrete examples to illustrate Soft Sets. One such example is presented below.

Example 1.3.2. *Suppose we have a universe ξ consisting of eight cars: $k_1, k_2, k_3, k_4, k_5, k_6, k_7$, and k_8. We want to evaluate these cars based on certain attributes to determine their suitability for a potential buyer. Let D be the set of attributes describing these cars:*

$$D = \{e_1 = \textit{luxury}, e_2 = \textit{fuel-efficient}, e_3 = \textit{spacious}, e_4 = \textit{reliable}, e_5 = \textit{affordable}\}$$

Now, let's define a S_fS $(\mathscr{S}, D)$ representing the preferences of the buyer:

$$\mathscr{S}(e_1) = \{k_1, k_2, k_5\}$$

$$\mathscr{S}(e_2) = \{k_2, k_3, k_4, k_7\}$$

$$\mathscr{S}(e_3) = \{k_4, k_6, k_8\}$$

$$\mathscr{S}(e_4) = \{k_1, k_4, k_6\}$$

$$\mathscr{S}(e_5) = \{k_3, k_5, k_7, k_8\}$$

This S_fS indicates which cars are preferred based on each attribute. For example, $\mathscr{S}(e_1)$ represents the subset of luxury cars, including k_1, k_2, and k_5. Similarly, $\mathscr{S}(e_2)$ represents the subset of fuel-efficient cars, and so on. The representation of S_fS is shown in the Table below:

Table 1.1: S_fS $(\mathscr{S}, D)$

$(\mathscr{S}, D)$	e_1	e_2	e_3	e_4	e_5
$\mathfrak{k}_1$	1	0	0	1	0
$\mathfrak{k}_2$	1	1	0	0	0
$\mathfrak{k}_3$	0	1	0	0	1
$\mathfrak{k}_4$	0	1	1	1	0
$\mathfrak{k}_5$	1	0	0	0	1
$\mathfrak{k}_6$	0	0	1	1	0
$\mathfrak{k}_7$	0	1	0	0	1
$\mathfrak{k}_8$	0	0	1	0	1

Each row corresponds to a car and each column corresponds to an attribute. The entries in the table indicate whether a particular car possesses the corresponding attribute (1) or not (0).

Example 1.3.2 demonstrates how a S_fS can be used to evaluate objects based on multiple attributes, providing flexibility in decision-making processes. Now, let's discuss some essential operations of S_fSs.

Definition 1.3.3. *A S_fB_nR $(\mathscr{S}, D)$ can be defined as a S_fS from ξ_1 to ξ_2, denoted as $\mathscr{S} : D \to P(\xi_1 \times \xi_2)$.*

The aforementioned definition suggests that $(\mathscr{S}, D)$ denotes a set of B_nRs from ξ_1 to ξ_2 that are parameterized. Each parameter e in D corresponds to a Binary Relation $\mathscr{S}(e)$ from ξ_1 to ξ_2.
Maji et al. [41] combined the structures of S_fS and F_zS and introduced the novel concept of the Fuzzy Soft Set (F_zS_fS), which provides a parameterized collection of F_zSs on ξ. Peng et al. [47] proposed the idea of a Pythagorean Fuzzy Soft Set $(P_yF_zS_fS)$.

Definition 1.3.4. *[47] A pair $(\mathscr{S}, D)$ is termed as a Pythagorean Fuzzy Soft Set $(P_yF_zS_fS)$ over ξ if $\mathscr{S} : D \to P_yF_zS(\xi)$ such that $\mathscr{S}(e)$ is a P_yF_zS in ξ for each $e \in D$. Therefore, a*

Pythagorean Soft Set ($P_yF_zS_fS$) over ξ is a parameterized collection of P_yF_zSs in ξ.

Definition 1.3.5. *[47] The subset relationship between two $P_yF_zS_fSs$ $(\mathscr{S}_1, D_1)$ and $(\mathscr{S}_2, D_2)$ over a finite set ξ is defined as follows: $(\mathscr{S}_1, D_1)$ is a $P_yF_zS_f$ subset of $(\mathscr{S}_2, D_2)$ if $D_1 \subseteq D_2$ and $\mathscr{S}_1(e) \subseteq \mathscr{S}_2(e)$ for all $e \in D_1$.*
Two $P_yF_zS_fSs$ $(\mathscr{S}_2, D_2)$ and $(\mathscr{S}_1, D_1)$ are considered to be equal if $(\mathscr{S}_1, D_1) \subseteq (\mathscr{S}_2, D_2)$ and $(\mathscr{S}_2, D_2) \subseteq (\mathscr{S}_1, D_1)$.

Definition 1.3.6. *[47] The union and intersection of $P_yF_zS_fSs$ $(\mathscr{S}_1, D)$, $(\mathscr{S}_2, D)$ over the common universal set ξ are the $P_yF_zS_fSs$ $(\mathscr{S}_3, D)$ and $(\mathscr{S}_4, D)$, respectively, where $\mathscr{S}_3(e) = \mathscr{S}_1(e) \cup \mathscr{S}_2(e)$ and $\mathscr{S}_4(e) = \mathscr{S}_1(e) \cap \mathscr{S}_2(e)$ for all $e \in D$.*

Molodtsov's idea of an S_fS has been expanded and combined with different frameworks. The concept of q-Rung Orthopair Fuzzy Soft Set (${}^qROF_zS_fS$) and a q-Rung Orthopair Soft Topology (${}^qROF_zS_fT$) was introduced by Hamid et al. [27], together with an MCGDM approach.

Definition 1.3.7. *[27] Let D denote a subset of set of parameters, and ξ a finite universe. If $\mathscr{S} : D \to {}^qROF_zS(\xi)$ is defined such that $\mathscr{S}(e) \in {}^qROF_zS(\xi)$ for each $e \in D$, then the pair $(\mathscr{S}, D)$ is referred to as ${}^qROF_zS_fS$ over ξ. Consequently, a ${}^qROF_zS_fS$ in ξ provides a parameterization of ${}^qROF_zSs$ in ξ.*

Definition 1.3.8. *[27] For two ${}^qROF_zS_fSs$ $(\mathscr{S}_1, D_1)$, $(\mathscr{S}_2, D_2)$ over a common universal set ξ, we define $(\mathscr{S}_1, D_1)$ as a ${}^qROF_zS_f$ subset of $(\mathscr{S}_2, D_2)$ if $D_1 \subseteq D_2$ and $\mathscr{S}_1(e) \subseteq \mathscr{S}_2(e)$ for all $e \in D_1$. Additionally, two ${}^qROF_zS_fSs$ $(\mathscr{S}_1, D_1)$, $(\mathscr{S}_2, D_2)$ are considered ${}^qROF_zS_fS$ equal if $(\mathscr{S}_1, D_1) \subseteq (\mathscr{S}_2, D_2)$ and $(\mathscr{S}_2, D_2) \subseteq (\mathscr{S}_1, D_1)$.*

Finally, let's define some substructures of Soft Sets with respect to a S_mG M.

1.4 Semigroups: Definitions and Examples

In this section, we delve into the intricacies of S_mGs, presenting definitions and examples.

Definition 1.4.1. *A S_mG is defined as a non-empty set M with a binary operation $\cdot$, where the operation satisfies the associative property. Here are some key concepts related to S_mGs:*

1) *The product UV of two subsets U and V of S_mG M consists of all products mn, where $m \in U$ and $n \in V$.*

2) *The S_mGs M and V can be combined to form a new S_mG $M \times V$ through the Cartesian product. In this new S_mG, the operation is defined as $(m, n)(m', n') = (mm', nn')$ for all $n, n' \in V$ and $m, m' \in M$.*

3) *A subsemigroup (S_bS_mG) of a S_mG M is a non-empty subset U such that for all $m, n \in U$, the product $mn \in U$.*

4) *If a non-empty subset $U \subseteq M$ is such that $MU \subseteq U$ ($UM \subseteq U$), then it is referred to as a Left (Right) Ideal of the S_mG M.*

5) *The set $U \subseteq M$ that is non-empty and satisfies $MUM \subseteq U$ is called an Interior Ideal (I_tI_d) of M.*

6) *Every I_tI_d is an Ideal, but it is not necessarily the case that every Ideal is an I_{tId}.*

7) *If $UMU \subseteq U$, then a S_bS_mG U of a S_mG M is called a Bi-Ideal (B_iI_d) of M.*

8) *Every B_iI_d is not necessarily a one-sided Ideal, although every one-sided Ideal is a B_iI_d.*

In this thesis, we represent a subsemigroup, Right Ideal, Left Ideal, Bi-Ideal, and Interior Ideal as S_bS_mG, R_iI_d, L_fI_d, B_iI_d, and I_tI_d, respectively.

Definition 1.4.2. *[26]*

- *A* P_yF_zS $V = \{\langle m, V_Y(m), V_N(m)\rangle : m \in M\}$ *in* M *is a* P_yF_z S_bS_mG *of the* S_mG M *if it satisfies the following:*
$V_Y(mn) \geq V_Y(m) \wedge V_Y(n)$ *and* $V_N(mn) \leq V_N(m) \vee V_N(n)$*, for all* $m, n \in M$*.*

- *A Pythagorean Fuzzy Left Ideal (*$P_yF_zL_fI_d$*) is denoted as* V *and is characterized by the conditions,* $V_Y(mn) \geq V_Y(n)$ *and* $V_N(mn) \leq V_Y(n)$*, for all* $m, n \in M$*.*

- *A Pythagorean Fuzzy* R_iI_d *(*$P_yF_zR_iI_d$*) of* M *is represented by* V *and is defined by the criteria,* $V_Y(mn) \geq V_Y(m)$ *and* $V_N(mn) \leq V_Y(m)$*, for all* $m, n \in M$*.*

- *If a set* V *satisfies both the conditions of being a* $P_yF_zL_fI_d$ *and a* $P_yF_zR_iI_d$*, then it is termed as a Pythagorean Fuzzy Ideal (*$P_yF_zI_d$*).*

- *A subset* V *of a* S_mG M*, which is a Pythagorean Fuzzy* S_bS_mG *(*$P_yF_zS_bS_mG$*), is considered a Pythagorean Fuzzy* I_tI_d *(*$P_yF_zI_tI_d$*) of* M *if it fulfills the condition, for all* $a, m, n \in M$*:*
$V_Y(nam) \geq V_Y(a)$ *and* $V_N(nam) \leq V_N(a)$*.*

- *A subset* V *of a* S_mG M*, which is a Pythagorean Fuzzy* S_bS_mG *(*$P_yF_zS_bS_mG$*), is termed as a Pythagorean Fuzzy* B_iI_d *(*$P_yF_zB_iI_d$*) of* M *if it meets the following criteria, for all* $a, m, n \in M$*:*
$V_Y(nam) \geq V_Y(n) \wedge V_Y(m)$ *and* $V_N(nam) \leq V_N(n) \vee V_N(m)$*.*

Definition 1.4.3. *Consider a* S_fS $(\mathscr{S}, D)$ *over a* S_mG M*.*

1) *If for every element* $e \in D$ *with* $\mathscr{S}(e) \neq \emptyset$*,* $\mathscr{S}(e)$ *forms a* S_bS_mG *in* M*, then* $(\mathscr{S}, D)$ *is a Soft* S_bS_mG *(*$S_fS_bS_mG$*) over* M*.*

2) *For every element* $e \in D$ *with* $\mathscr{S}(e) \neq \emptyset$*, if* $\mathscr{S}(e)$ *is an Ideal of* M*, then* $(\mathscr{S}, D)$ *over* M *is considered to be a Soft* I_d *(*S_fI_d*) over* M*.*

3) *If for all* $e \in D$ *with* $\mathscr{S}(e) \neq \emptyset$*,* $\mathscr{S}(e)$ *is a* B_iI_d *of* M*, then the pair* $(\mathscr{S}, D)$ *is a Soft* B_iI_d *(*$S_fB_iI_d$*) over* M*.*

4) *For every element e in D such that $\mathscr{S}(e)$ is not empty, if $\mathscr{S}(e)$ is an Interior Ideal of M, then $(\mathscr{S}, D)$ is a Soft Interior Ideal ($S_fI_tI_d$) over M.*

Definition 1.4.4. *A S_fB_nR $(\mathscr{S}, D)$ from a S_mG M_1 to a S_mG M_2 is considered as a soft compatible (S_fC_m) if, for any $i, k \in M_1$ and $j, l \in M_2$,*
$(i, j), (k, l) \in \mathscr{S}(e)$ implies $(ij, kl) \in \mathscr{S}(e)$.
The relation $(\mathscr{S}, D)$ is a soft compatible relation ($S_fC_mR_l$) over M if $i\mathscr{S}(e) \cdot j\mathscr{S}(e) \subseteq (ij)\mathscr{S}(e)$. When $m \in i\mathscr{S}(e)$ and $n \in j\mathscr{S}(e)$, it implies that $(i, m) \in \mathscr{S}(e)$ and $(j, n) \in \mathscr{S}(e)$ in accordance with the compatibility of $(\mathscr{S}, D)$.
From this compatibility, we can conclude that $(ij, mn) \in \mathscr{S}(e)$, which indicates that $mn \in (ij)\mathscr{S}(e)$. Furthermore, $\mathscr{S}(e)i \cdot \mathscr{S}(e)j \subseteq \mathscr{S}(e)(ij)$.

Definition 1.4.5. *A soft compatible relation ($S_fC_mR_l$) $(\mathscr{S}, D)$ from a S_mG M_1 to a S_mG M_2 is referred to as being soft complete relation ($S_fC_{mp}R_l$) if, for any elements $m, n \in M_1$ and $e \in D$, it holds that $m\mathscr{S}(e) \cdot n\mathscr{S}(e) = mn\mathscr{S}(e)$. In the same way, it is termed as a $S_fC_{mp}R_l$ if, for all $m', n' \in M_2$ and $e \in D$, the equation $= \mathscr{S}(e)m'n' = \mathscr{S}(e)m' \cdot \mathscr{S}(e)n'$ is satisfied.*

In this chapter, the foundational concepts essential to this research were discussed in detail, including Fuzzy Sets, Rough Sets, and their extensions such as Pythagorean Fuzzy Sets and q-Rung Orthopair Fuzzy Sets. The mathematical frameworks and notations introduced here form the basis for the methods and algorithms developed in subsequent chapters. By establishing a clear understanding of these fundamental ideas, this chapter provides the groundwork for exploring advanced topics in approximation theory and decision-making applications in the later parts of this thesis.

Chapter 2

Approximations of Pythagorean Fuzzy Sets over dual universes by Soft Binary Relations

In this chapter, Section 2.1 explores the lower approximation (L_oA_p) and upper approximation (U_pA_p) of Pythagorean Fuzzy Sets ($R_fP_yF_zS$) using S_fB_nRs in the context of foresets (F_rS) and aftersets (A_fSs), along with a presentation of their properties. Section 2.2 introduces two kinds of Pythagorean Fuzzy Topological Spaces ($^qROF_zT_pS$s) derived from S_fB_nRs. Following this, Section 2.3 discusses similarity relations among P_yF_zSs based on S_fB_nRs. In Section 2.4, the concepts of roughness degree ($R_fN_sD_g$) and accuracy degree ($A_cR_cD_g$) for Pythagorean M_mD_gs with respect to F_rS and A_fSs are presented. Finally, Section 2.5 outlines an algorithm designed for addressing decision-making problems using P_yF_zSs, accompanied by an illustrative example demonstrating its application in practical scenarios.

2.1 Approximating Pythagorean Fuzzy Sets with Soft Binary Relations

In this section, we explore the utilization of S_fB_nRs from ξ_1 to ξ_2 to approximate a P_yF_zS over U by employing F_rS, resulting in two Pythagorean Fuzzy Soft Sets ($P_yF_zS_fS$s) over ξ_2. Similarly, we approximate a P_yF_zS over ξ_2 using A_fSs, leading to two $P_yF_zS_fS$s over ξ_1. Furthermore, we discuss some of their properties.

Definition 2.1.1. *Let $(\mathscr{S}, D)$ be a S_fB_nR from ξ_1 to ξ_2 and $U = \{\langle t, U_Y(t), U_N(t)\rangle : t \in \xi_2\}$ be a P_yF_zS in ξ_2. Then we define the lower approximation (L_oA_p) $\underline{\mathscr{S}}^U = (\underline{\mathscr{S}}^{U_Y}, \underline{\mathscr{S}}^{U_N})$ and the upper approximation (U_pA_p) $\overline{\mathscr{S}}^U = (\overline{\mathscr{S}}^{U_Y}, \overline{\mathscr{S}}^{U_N})$ of $U = \{\langle t, U_Y(t), U_N(t)\rangle : t \in \xi_2\}$ with respect to A_fSs as follows:*

$$\underline{\mathscr{S}}^U(e)(k) = \begin{cases} \left(\bigwedge_{t \in k\mathscr{S}(e)} U_Y(t), \bigvee_{t \in k\mathscr{S}(e)} U_N(t)\right) & if \quad k\mathscr{S}(e) \neq \emptyset; \\ (1, 0) & otherwise. \end{cases}$$

and

$$\overline{\mathscr{S}}^U(e)(k) = \begin{cases} \left(\bigvee_{t \in k\mathscr{S}(e)} U_Y(t), \bigwedge_{t \in k\mathscr{S}(e)} U_N(t)\right) & if \quad k\mathscr{S}(e) \neq \emptyset; \\ (0, 1) & otherwise. \end{cases}$$

where $k\mathscr{S}(e) = \{t \in \xi_2 : (k, t) \in \mathscr{S}(e)\}$, which is referred to as the A_fS of k for all $k \in \xi_1$ and $e \in D$.

1) $\underline{\mathscr{S}}^{U_Y}(e)(k)$ *indicates the degree to which k definitely possesses the property e.*

2) $\underline{\mathscr{S}}^{U_N}(e)(k)$ *indicates the degree to which k probably does not possess the property e.*

3) $\overline{\mathscr{S}}^{U_Y}(e)(k)$ *indicates the degree to which k probably possesses the property e.*

4) $\overline{\mathscr{S}}^{U_N}(e)(k)$ *indicates the degree to which k definitely does not possess the property e.*

Definition 2.1.2. *Let* $(\mathscr{S}, D)$ *be a* S_fB_nR *from* ξ_1 *to* ξ_2 *and* $U = \{\langle \Bbbk, U_Y(\Bbbk), U_N(\Bbbk)\rangle : \Bbbk \in \xi_1\}$ *be a* P_yF_zS *in* ξ_1. *Then we define* L_oA_p ${}^U\underline{\mathscr{S}} = ({}^{U_Y}\underline{\mathscr{S}}, {}^{U_N}\underline{\mathscr{S}})$ *and* U_pA_p ${}^U\overline{\mathscr{S}} = ({}^{U_Y}\overline{\mathscr{S}}, {}^{U_N}\overline{\mathscr{S}})$ *of* $U = \{\langle \Bbbk, U_Y(\Bbbk), U_N(\Bbbk)\rangle : \Bbbk \in \xi_1\}$ *with respect to* F_rS *as follows:*

$$^U\underline{\mathscr{S}}(e)(t) = \begin{cases} \left(\bigwedge_{\Bbbk\in\mathscr{S}(e)t} U_Y(\Bbbk), \bigvee_{\Bbbk\in\mathscr{S}(e)t} U_N(\Bbbk)\right) & if \quad \mathscr{S}(e)t \neq \emptyset; \\ (1,0) & otherwise. \end{cases}$$

and

$$^U\overline{\mathscr{S}}(e)(t) = \begin{cases} \left(\bigvee_{\Bbbk\in\mathscr{S}(e)t} U_Y(\Bbbk), \bigwedge_{\Bbbk\in\mathscr{S}(e)t} U_N(\Bbbk)\right) & if \quad \mathscr{S}(e)t \neq \emptyset; \\ (0,1) & otherwise. \end{cases}$$

where $\mathscr{S}(e)t = \{\Bbbk \in \xi_1 : (\Bbbk, t) \in \mathscr{S}(e)\}$, *which is referred to as the* F_rS *of* t *for all* $t \in \xi_2$ *and* $e \in D$.

1) $\underline{\mathscr{S}}^{U_Y}(e)(t)$ *represents the degree to which* t *definitely possesses the property* e.

2) $\underline{\mathscr{S}}^{U_N}(e)(t)$ *represents the degree to which* t *probably does not possess the property* e.

3) $\overline{\mathscr{S}}^{U_Y}(e)(t)$ *represents the degree to which* t *probably possesses the property* e.

4) $\overline{\mathscr{S}}^{U_N}(e)(t)$ *represents the degree to which* t *definitely does not possess the property* e.

Here, we have $\underline{\mathscr{S}}^U : D \to P_yF_zS(\xi_1)$, $\overline{\mathscr{S}}^U : D \to P_yF_zS(\xi_1)$, ${}^U\underline{\mathscr{S}} : D \to P_yF_zS(\xi_2)$ and ${}^U\overline{\mathscr{S}} : D \to P_yF_zS(\xi_2)$.

The example below demonstrates these concepts.

Example 2.1.3. *Suppose a student wants to buy new shoes.*

Let $\xi_1 = \{$*the set of available shoe designs*$\} = \{\Bbbk_1, \Bbbk_2, \Bbbk_3, \Bbbk_4\}$, $\xi_2 = \{$*the colors of all designs*$\} = \{t_1, t_2, t_3, t_4\}$, *and let the set of attributes be* $D = \{$*the set of stores near his house*$\} = \{e_1, e_2\}$. *Take a* S_fB_nR $\mathscr{S} : D \to P(\xi_1 \times \xi_2)$ *by*

$$\mathscr{S}(e_1) = \begin{array}{c} \\ t_1 \\ t_2 \\ t_3 \\ t_4 \end{array}\!\!\begin{array}{c} \begin{array}{cccc} \Bbbk_1 & \Bbbk_2 & \Bbbk_3 & \Bbbk_4 \end{array} \\ \begin{pmatrix} 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix} \end{array}, \quad \mathscr{S}(e_2) = \begin{array}{c} \\ t_1 \\ t_2 \\ t_3 \\ t_4 \end{array}\!\!\begin{array}{c} \begin{array}{cccc} \Bbbk_1 & \Bbbk_2 & \Bbbk_3 & \Bbbk_4 \end{array} \\ \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 \end{pmatrix} \end{array}$$

represent the relation between colors and designs available at stores e_1, e_2, respectively. Now let two P_yF_zSs U and V on ξ_2 and ξ_1, respectively, where U represents the preference of the colors and V represents the preference of the designs given by the student and defined by:

$$U = \{\langle t_1, 0.9, 0\rangle, \langle t_2, 0.8, 0.3\rangle, \langle t_3, 0.4, 0.7\rangle, \langle t_4, 0, 1\rangle\},$$

$$V = \{\langle \Bbbk_1, 1, 0\rangle, \langle \Bbbk_2, 0.7, 0.2\rangle, \langle \Bbbk_3, 0.5, 0.6\rangle, \langle \Bbbk_4, 0.1, 0.8\rangle\}.$$

Table 2.1 shows that L_oA_p and U_pA_p $\underline{\mathscr{S}}^U$, $\overline{\mathscr{S}}^U$ of P_yF_zS U with respect to A_fSs $\Bbbk_i\mathscr{S}(e_j)$ are two P_yF_zSs on ξ_1.

Table 2.1: A_pS of P_yF_zS with respect to A_fSs

	$\overline{\mathscr{S}}^{\mathbf{U}}(e_1)(\Bbbk_i)$	$\underline{\mathscr{S}}^{\mathbf{U}}(e_1)(\Bbbk_i)$	$\overline{\mathscr{S}}^{\mathbf{U}}(e_2)(\Bbbk_i)$	$\underline{\mathscr{S}}^{\mathbf{U}}(e_2)(\Bbbk_i)$
$\Bbbk_{\mathbf{1}}$	$(0.9, 0)$	$(0.8, 0.3)$	$(0.4, 0.7)$	$(0, 1)$
$\Bbbk_{\mathbf{2}}$	$(0.4, 0.7)$	$(0.4, 0.7)$	$(0.9, 0)$	$(0, 1)$
$\Bbbk_{\mathbf{3}}$	$(0.9, 0)$	$(0, 1)$	$(0.4, 0.7)$	$(0.4, 0.7)$
$\Bbbk_{\mathbf{4}}$	$(0.4, 0.7)$	$(0, 1)$	$(0, 1)$	$(1, 0)$

Similarly Table 2.2 shows that L_oA_p and U_pA_ps ${}^V\underline{\mathscr{S}}$, ${}^V\overline{\mathscr{S}}$ of P_yF_zS V with respect to F_rS $\mathscr{S}(e_j)\ell_i$ are two P_yF_zSs on ξ_2, where $j = 1, 2$ and $i = 1, 2, 3, 4$.

Table 2.2: Approximations of P_yF_zS with respect to F_rS

	${}^{\mathbf{V}}\overline{\mathscr{S}}(e_1)(\ell_i)$	${}^{\mathbf{V}}\underline{\mathscr{S}}(e_1)(\ell_i)$	${}^{\mathbf{V}}\overline{\mathscr{S}}(e_2)(\ell_i)$	${}^{\mathbf{V}}\underline{\mathscr{S}}(e_2)(\ell_i)$
ℓ_1	$(1, 0)$	$(0.5, 0.6)$	$(0.7, 0.2)$	$(0.7, 0.2)$
ℓ_2	$(1, 0)$	$(0.5, 0.6)$	$(0.7, 0.2)$	$(0.7, 0.2)$
ℓ_3	$(0.7, 0.2)$	$(0.1, 0.8)$	$(1, 0)$	$(0.5, 0.6)$
ℓ_4	$(0.5, 0.6)$	$(0.1, 0.8)$	$(1, 0)$	$(0.7, 0.2)$

Theorem 2.1.4. *Let $(\mathscr{S}, D)$ be a S_fB_nR from ξ_1 to ξ_2, that is, $\mathscr{S} : D \to P(\xi_1 \times \xi_2)$. For any three P_yF_zSs $U = \{\langle \ell, U_Y(\ell), U_N(\ell)\rangle : \ell \in \xi_2\}$, $U_1 = \{\langle \ell, U_{1_Y}(\ell), U_{1_N}(\ell)\rangle : \ell \in \xi_2\}$, and $U_2 = \{\langle \ell, U_{2_Y}(\ell), U_{2_N}(\ell)\rangle : \ell \in \xi_2\}$ of ξ_2, we have the following:*

i) $U_1 \subseteq U_2$ *implies* $\underline{\mathscr{S}}^{U_1} \subseteq \underline{\mathscr{S}}^{U_2}$

ii) $U_1 \subseteq U_2$ *implies* $\overline{\mathscr{S}}^{U_1} \subseteq \overline{\mathscr{S}}^{U_2}$

iii) $\underline{\mathscr{S}}^{U_1 \cap U_2} = \underline{\mathscr{S}}^{U_1} \cap \underline{\mathscr{S}}^{U_2}$

iv) $\overline{\mathscr{S}}^{U_1 \cap U_2} \subseteq \overline{\mathscr{S}}^{U_1} \cap \overline{\mathscr{S}}^{U_2}$

v) $\underline{\mathscr{S}}^{U_1} \cup \underline{\mathscr{S}}^{U_2} \subseteq \underline{\mathscr{S}}^{U_1 \cup U_2}$

vi) $\overline{\mathscr{S}}^{U_1} \cup \overline{\mathscr{S}}^{U_2} = \overline{\mathscr{S}}^{U_1 \cup U_2}$

vii) $\overline{\mathscr{S}}^{1_{\xi_2}} = \underline{\mathscr{S}}^{1_{\xi_2}} = 1_{\xi_1}$, *if* $\ell\mathscr{S}(e) \neq \emptyset$

viii) $\underline{\mathscr{S}}^{U} = (\overline{\mathscr{S}}^{U^c})^c$ *and* $\overline{\mathscr{S}}^{U} = (\underline{\mathscr{S}}^{U^c})^c$, *if* $\ell\mathscr{S}(e) \neq \emptyset$

ix) $\underline{\mathscr{S}}^{0_{\xi_2}} = 0_{\xi_1} = \overline{\mathscr{S}}^{0_{\xi_2}}$.

Proof.

i) Let $U_1 \subseteq U_2$, that is, for all $\mathfrak{k} \in \xi_2$, $U_{1_Y}(\mathfrak{k}) \leq U_{2_Y}(\mathfrak{k})$, and $U_{1_N}(\mathfrak{k}) \geq U_{2_N}(\mathfrak{k})$.
If $\mathfrak{K}\mathscr{S}(e) = \emptyset$, then $\underline{\mathscr{L}}^{U_1} = (1,0) = \underline{\mathscr{L}}^{U_2}$.
If $\mathfrak{K}\mathscr{S}(e) \neq \emptyset$, then
$\underline{\mathscr{L}}^{U_{1Y}}(e)(\mathfrak{K}) = \bigwedge_{\mathfrak{k}\in\mathfrak{K}\mathscr{S}(e)} U_{1_Y}(\mathfrak{k}) \leq \bigwedge_{\mathfrak{k}\in\mathfrak{K}\mathscr{S}(e)} U_{2_Y}(\mathfrak{k}) = \underline{\mathscr{L}}^{U_{2Y}}(e)(\mathfrak{K})$
and $\underline{\mathscr{L}}^{U_{1N}}(e)(\mathfrak{K}) = \bigvee_{\mathfrak{k}\in\mathfrak{K}\mathscr{S}(e)} U_{1_N}(\mathfrak{k}) \geq \bigvee_{\mathfrak{k}\in\mathfrak{K}\mathscr{S}(e)} U_{2_N}(\mathfrak{k}) = \underline{\mathscr{L}}^{U_{2N}}(e)(\mathfrak{K})$.
Thus, $\underline{\mathscr{L}}^{U_{1Y}}(e)(\mathfrak{K}) \leq \underline{\mathscr{L}}^{U_{2Y}}(e)(\mathfrak{K})$ and $\underline{\mathscr{L}}^{U_{1N}}(e)(\mathfrak{K}) \geq \underline{\mathscr{L}}^{U_{2N}}(e)(\mathfrak{K})$. Hence, $\underline{\mathscr{L}}^{U_1} \subseteq \underline{\mathscr{L}}^{U_2}$.

ii) Let $U_1 \subseteq U_2$, that is, for all $\mathfrak{k} \in \xi_2$, $U_{1_Y}(\mathfrak{k}) \leq U_{2_Y}(\mathfrak{k})$, and $U_{1_N}(\mathfrak{k}) \geq U_{2_N}(\mathfrak{k})$.
If $\mathfrak{K}\mathscr{S}(e) = \emptyset$, then $\overline{\mathscr{S}}^{U_1} = (0,1) = \overline{\mathscr{S}}^{U_2}$.
If $\mathfrak{K}\mathscr{S}(e) \neq \emptyset$, then
$\overline{\mathscr{S}}^{U_{1Y}}(e)(\mathfrak{K}) = \bigvee_{\mathfrak{k}\in\mathfrak{K}\mathscr{S}(e)} U_{1_Y}(\mathfrak{k}) \leq \bigvee_{\mathfrak{k}\in\mathfrak{K}\mathscr{S}(e)} U_{2_Y}(\mathfrak{k}) = \overline{\mathscr{S}}^{U_{2Y}}(e)(\mathfrak{K})$
and $\overline{\mathscr{S}}^{U_{1N}}(e)(\mathfrak{K}) = \bigwedge_{\mathfrak{k}\in\mathfrak{K}\mathscr{S}(e)} U_{1_N}(\mathfrak{k}) \geq \bigwedge_{\mathfrak{k}\in\mathfrak{K}\mathscr{S}(e)} U_{2_N}(\mathfrak{k}) = \overline{\mathscr{S}}^{U_{2N}}(e)(\mathfrak{K})$.
Thus, $\overline{\mathscr{S}}^{U_{1Y}}(e)(\mathfrak{K}) \leq \overline{\mathscr{S}}^{U_{2Y}}(e)(\mathfrak{K})$ and $\overline{\mathscr{S}}^{U_{1N}}(e)(\mathfrak{K}) \geq \overline{\mathscr{S}}^{U_{2N}}(e)(\mathfrak{K})$. Hence, $\overline{\mathscr{S}}^{U_1} \subseteq \overline{\mathscr{S}}^{U_2}$.

iii) Consider $(\underline{\mathscr{L}}^{U_{1Y}} \cap \underline{\mathscr{L}}^{U_{2Y}})(e)(\mathfrak{K}) = \underline{\mathscr{L}}^{U_{1Y}}(e)(\mathfrak{K}) \wedge \underline{\mathscr{L}}^{U_{2Y}}(e)(\mathfrak{K}) = (\bigwedge_{\mathfrak{k}\in\mathfrak{K}\mathscr{S}(e)} U_{1_Y}(\mathfrak{k})) \wedge (\bigwedge_{\mathfrak{k}\in\mathfrak{K}\mathscr{S}(e)} U_{2_Y}(\mathfrak{k})) = \bigwedge_{\mathfrak{k}\in\mathfrak{K}\mathscr{S}(e)}(U_{1_Y}(\mathfrak{k}) \wedge U_{2_Y}(\mathfrak{k})) = \underline{\mathscr{L}}^{U_1\cap U_2}(e)(\mathfrak{K})$, and
$(\underline{\mathscr{L}}^{U_{1N}} \cup \underline{\mathscr{L}}^{U_{2N}})(e)(\mathfrak{K}) = \underline{\mathscr{L}}^{U_{1N}}(e)(\mathfrak{K}) \vee \underline{\mathscr{L}}^{U_{2N}}(e)(\mathfrak{K}) = (\bigvee_{\mathfrak{k}\in\mathfrak{K}\mathscr{S}(e)} U_{1_N}(\mathfrak{k})) \vee (\bigvee_{\mathfrak{k}\in\mathfrak{K}\mathscr{S}(e)} U_{2_N}(\mathfrak{k}))$
$= \bigvee_{\mathfrak{k}\in\mathfrak{K}\mathscr{S}(e)}(U_{1_N}(\mathfrak{k}) \vee U_{2_N}(\mathfrak{k})) = \underline{\mathscr{L}}^{U_1\cup U_2}(e)(\mathfrak{K})$. Thus, $\underline{\mathscr{L}}^{U_1\cap U_2} = \underline{\mathscr{L}}^{U_1} \cap \underline{\mathscr{L}}^{U_2}$.

iv) Given that $U_1 \cap U_2 \subseteq U_1$ and $U_1 \cap U_2 \subseteq U_2$, it follows from part (ii) that $\overline{\mathscr{S}}^{U_1\cap U_2} \subseteq \overline{\mathscr{S}}^{U_1}$ and $\overline{\mathscr{S}}^{U_1\cap U_2} \subseteq \overline{\mathscr{S}}^{U_2}$. Therefore, we conclude that $\overline{\mathscr{S}}^{U_1\cap U_2} \subseteq \overline{\mathscr{S}}^{U_1} \cap \overline{\mathscr{S}}^{U_2}$.

v) Given that $U_1 \subseteq U_1 \cup U_2$ and $U_2 \subseteq U_1 \cup U_2$, it follows from part (i) that $\underline{\mathscr{L}}^{U_1} \subseteq \underline{\mathscr{L}}^{U_1\cup U_2}$ and $\underline{\mathscr{L}}^{U_2} \subseteq \underline{\mathscr{L}}^{U_1\cup U_2}$. Therefore, we conclude that $\underline{\mathscr{L}}^{U_1} \cup \underline{\mathscr{L}}^{U_2} \subseteq \underline{\mathscr{L}}^{U_1\cup U_2}$.

vi) Consider $(\overline{\mathscr{S}}^{U_{1Y}} \cup \overline{\mathscr{S}}^{U_{2Y}})(e)(\mathfrak{K}) = \overline{\mathscr{S}}^{U_{1Y}}(e)(\mathfrak{K}) \vee \overline{\mathscr{S}}^{U_{2Y}}(e)(\mathfrak{K}) = (\bigvee_{\mathfrak{k}\in\mathfrak{K}\mathscr{S}(e)} U_{1_Y}(\mathfrak{k})) \vee (\bigvee_{\mathfrak{k}\in\mathfrak{K}\mathscr{S}(e)} U_{2_Y}(\mathfrak{k})) = \bigvee_{\mathfrak{k}\in\mathfrak{K}\mathscr{S}(e)}(U_{1_Y}(\mathfrak{k}) \vee U_{2_Y}(\mathfrak{k})) = \overline{\mathscr{S}}^{U_1\cup U_2}(e)(\mathfrak{K})$ and
$(\overline{\mathscr{S}}^{U_{1N}} \cap \overline{\mathscr{S}}^{U_{2N}})(e)(\mathfrak{K}) = \overline{\mathscr{S}}^{U_{1N}}(e)(\mathfrak{K}) \wedge \overline{\mathscr{S}}^{U_{2N}}(e)(\mathfrak{K}) = (\bigwedge_{\mathfrak{k}\in\mathfrak{K}\mathscr{S}(e)} U_{1_N}(\mathfrak{k})) \wedge (\bigwedge_{\mathfrak{k}\in\mathfrak{K}\mathscr{S}(e)} U_{2_N}(\mathfrak{k}))$
$= \bigwedge_{\mathfrak{k}\in\mathfrak{K}\mathscr{S}(e)}(U_{1_N}(\mathfrak{k}) \wedge U_{2_N}(\mathfrak{k})) = \overline{\mathscr{S}}^{U_1\cap U_2}(e)(\mathfrak{K})$. Thus, $\overline{\mathscr{S}}^{U_1\cup U_2} = \overline{\mathscr{S}}^{U_1} \cup \overline{\mathscr{S}}^{U_2}$.

vii) Since $\underline{\mathscr{L}}^{1_{\xi_2}}(e)(\mathfrak{k}) = \bigwedge_{\mathfrak{t}\in\mathfrak{k}\mathscr{S}(e)} 1(\mathfrak{t}) = \bigwedge_{\mathfrak{t}\in\mathfrak{k}\mathscr{S}(e)} 1 = 1$ and $\underline{\mathscr{L}}^{0}(e)(\mathfrak{k}) = \bigwedge_{\mathfrak{t}\in\mathfrak{k}\mathscr{S}(e)} 0(\mathfrak{t}) = \bigwedge_{\mathfrak{t}\in\mathfrak{k}\mathscr{S}(e)} 0 = 0$. Thus, $\underline{\mathscr{L}}^{1_{\xi_2}} = 1_{\xi_1}$.

Similarly, we can prove that $\overline{\mathscr{S}}^{1_{\xi_2}} = 1_{\xi_1}$.

viii) Consider

$\overline{\mathscr{S}}^{U_Y^c}(e)(\mathfrak{k}) = \bigvee_{\mathfrak{t}\in\mathfrak{k}\mathscr{S}(e)} U_Y^c(\mathfrak{t}) = \bigvee_{\mathfrak{t}\in\mathfrak{k}\mathscr{S}(e)} U_N(\mathfrak{t}) = \underline{\mathscr{L}}^{U_N}(e)(\mathfrak{k}) = (\underline{\mathscr{L}}^{U_Y}(e)(\mathfrak{k}))^c$ and

$\overline{\mathscr{S}}^{U_N^c}(e)(\mathfrak{k}) = \bigwedge_{\mathfrak{t}\in\mathfrak{k}\mathscr{S}(e)} U_N^c(\mathfrak{t}) = \bigwedge_{\mathfrak{t}\in\mathfrak{k}\mathscr{S}(e)} U_Y(\mathfrak{t}) = \underline{\mathscr{L}}^{U_Y}(e)(\mathfrak{k}) = (\underline{\mathscr{L}}^{U_N}(e)(\mathfrak{k}))^c$.

Thus, $\overline{\mathscr{S}}^{U^c} = (\overline{\mathscr{S}}^{U_Y^c}, \overline{\mathscr{S}}^{U_N^c}) = ((\underline{\mathscr{L}}^{U_Y})^c, (\underline{\mathscr{L}}^{U_N})^c) = (\underline{\mathscr{L}}^{U_Y}, \underline{\mathscr{L}}^{U_N})^c = (\underline{\mathscr{L}}^{U})^c$. Which gives that $(\overline{\mathscr{S}}^{U^c})^c = \underline{\mathscr{L}}^U$. Similarly, $\overline{\mathscr{S}}^{U} = (\underline{\mathscr{L}}^{U^c})^c$.

ix) The proof is straightforward.

□

Theorem 2.1.5. *Let $(\mathscr{S}, D)$ be a S_fB_nR from ξ_1 to ξ_2, that is, $\mathscr{S} : D \to P(\xi_1 \times \xi_2)$. For any three P_yF_zSs $U = \{\langle k, U_Y(k), U_N(k)\rangle : k \in \xi_1\}$, $U_1 = \{\langle k, U_{1_Y}(k), U_{1_N}(k)\rangle : k \in \xi_1\}$, and $U_2 = \{\langle k, U_{2_Y}(k), U_{2_N}(k)\rangle : k \in \xi_1\}$ of ξ_1, we have the following:*

i) $U_1 \subseteq U_2$ *implies* ${}^{U_1}\underline{\mathscr{L}} \subseteq {}^{U_2}\underline{\mathscr{L}}$

ii) $U_1 \subseteq U_2$ *implies* ${}^{U_1}\overline{\mathscr{S}} \subseteq {}^{U_2}\overline{\mathscr{S}}$

iii) ${}^{U_1\cap U_2}\underline{\mathscr{L}} = {}^{U_1}\underline{\mathscr{L}} \cap {}^{U_2}\underline{\mathscr{L}}$

iv) ${}^{U_1\cap U_2}\overline{\mathscr{S}} \subseteq {}^{U_1}\overline{\mathscr{S}} \cap {}^{U_2}\overline{\mathscr{S}}$

v) ${}^{U_1}\underline{\mathscr{L}} \cup {}^{U_2}\underline{\mathscr{L}} \subseteq {}^{U_1\cup U_2}\underline{\mathscr{L}}$

vi) ${}^{U_1}\overline{\mathscr{S}} \cup {}^{U_2}\overline{\mathscr{S}} = {}^{U_1\cup U_2}\overline{\mathscr{S}}$

vii) ${}^{1_{\xi_1}}\overline{\mathscr{S}} = {}^{1_{\xi_1}}\underline{\mathscr{L}} = 1_{\xi_2}$, *if* $\mathscr{S}(e)t \neq \emptyset$

viii) ${}^{U}\underline{\mathscr{L}} = ({}^{U^c}\overline{\mathscr{S}})^c$, *and* ${}^{U}\overline{\mathscr{S}} = ({}^{U^c}\underline{\mathscr{L}})^c$ *if* $\mathscr{S}(e)t \neq \emptyset$

ix) ${}^{0_{\xi_1}}\underline{\mathscr{L}} = 0_{\xi_2} = {}^{0_{\xi_1}}\overline{\mathscr{S}}$.

Proof.

The proof follows a similar approach to the proof of Theorem 2.1.4. □

The following example demonstrates that equality does not hold in parts (iv) and (v) of Theorem 2.1.4.

Example 2.1.6. *Utilizing the information given in Example 2.1.3, define two P_yF_zSs U_1, U_2 on ξ_2 by:*

$U_1 = \{\langle t_1, 0.3, 0.7\rangle, \langle t_2, 0.2, 0.9\rangle, \langle t_3, 0.9, 0.4\rangle, \langle t_4, 0.6, 0.5\rangle\}$,

$U_2 = \{\langle t_1, 0.4, 0.6\rangle, \langle t_2, 0.4, 0.5\rangle, \langle t_3, 0.3, 0.8\rangle, \langle t_4, 0.1, 0.9\rangle\}$.

Then, $U_1 \cap U_2 = \{\langle t_1, 0.3, 0.7\rangle, \langle t_2, 0.2, 0.9\rangle, \langle t_3, 0.3, 0.8\rangle, \langle t_4, 0.1, 0.9\rangle\}$, *and*

$U_1 \cup U_2 = \{\langle t_1, 0.4, 0.6\rangle, \langle t_2, 0.4, 0.5\rangle, \langle t_3, 0.9, 0.4\rangle, \langle t_4, 0.6, 0.5\rangle\}$.

Table 2.3: Union of L_oA_ps and L_oA_ps of union of two P_yF_zSs

	$(\underline{\mathscr{S}}^{U_1} \cup \underline{\mathscr{S}}^{U_2})(e_1)(k_i)$	$(\underline{\mathscr{S}}^{U_1} \cup \underline{\mathscr{S}}^{U_2})(e_2)(k_i)$	$\underline{\mathscr{S}}^{\mathbf{U_1 \cup U_2}}(e_1)(k_i)$	$\underline{\mathscr{S}}^{\mathbf{U_1 \cup U_2}}(e_2)(k_i)$
k_1	$(0.4, 0.6)$	$(0.6, 0.5)$	$(0.4, 0.6)$	$(0.6, 0.5)$
k_2	$(0.9, 0.4)$	$(0.2, 0.9)$	$(0.9, 0.4)$	$(0.4, 0.6)$
k_3	$(0.2, 0.9)$	$(0.9, 0.4)$	$(0.4, 0.6)$	$(0.9, 0.4)$
k_4	$(0.6, 0.5)$	$(1, 0)$	$(0.6, 0.5)$	$(1, 0)$

Table 2.4: Intersection of U_pA_ps and U_pA_ps of intersection of two P_yF_zSs

	$(\overline{\mathscr{S}}^{U_1} \cap \overline{\mathscr{S}}^{U_2})(e_1)(k_i)$	$(\overline{\mathscr{S}}^{U_1} \cap \overline{\mathscr{S}}^{U_2})(e_2)(k_i)$	$\overline{\mathscr{S}}^{\mathbf{U_1 \cap U_2}}(e_1)(k_i)$	$\overline{\mathscr{S}}^{\mathbf{U_1 \cap U_2}}(e_2)(k_i)$
k_1	$(0.3, 0.7)$	$(0.3, 0.8)$	$(0.3, 0.7)$	$(0.3, 0.8)$
k_2	$(0.3, 0.8)$	$(0.4, 0.5)$	$(0.3, 0.8)$	$(0.3, 0.7)$
k_3	$(0.4, 0.5)$	$(0.3, 0.8)$	$(0.3, 0.7)$	$(0.3, 0.8)$
k_4	$(0.3, 0.8)$	$(0, 1)$	$(0.3, 0.8)$	$(0, 1)$

Now observing Table 2.3, we can easily see that $\underline{\mathscr{S}}^{U_1\cup U_2}(e_1)(\Bbbk_3) \neq (\underline{\mathscr{S}}^{U_1} \cup \underline{\mathscr{S}}^{U_2})(e_1)(\Bbbk_3)$ *and* $(\underline{\mathscr{S}}^{U_1} \cup \underline{\mathscr{S}}^{U_2})(e_2)(\Bbbk_2) \neq \underline{\mathscr{S}}^{U_1\cup U_2}(e_2)(\Bbbk_2)$*. Thus, union of* L_oA_p*s of two* P_yF_zS*s is not equal to the* L_oA_p *of the union of two* P_yF_zS*s, that is,* $\underline{\mathscr{S}}^{U_1} \cup \underline{\mathscr{S}}^{U_2} \neq \underline{\mathscr{S}}^{U_1\cup U_2}$.
Similarly, according to Table 2.4, the intersection of U_pA_p*s of two* P_yF_zS*s is not equal to the* U_pA_p *of the intersection of two* P_yF_zS*s, specifically,* $\overline{\mathscr{S}}^{U_1} \cap \overline{\mathscr{S}}^{U_2} \neq \overline{\mathscr{S}}^{U_1\cap U_2}$*. Therefore, equality does not hold in parts (iv) and (v) of Theorem 2.1.4.*

Theorem 2.1.7. *Let* $(\mathscr{S}_1, D)$ *and* $(\mathscr{S}_2, D)$ *be two* S_fB_nR*s from* ξ_1 *to* ξ_2 *such that* $(\mathscr{S}_1, D) \subseteq (\mathscr{S}_2, D)$*, that is,* $\mathscr{S}_1(e) \subseteq \mathscr{S}_2(e)$*, for all* $e \in D$*. Then, for any* $U \in P_yF_zS(\xi_2)$*,* $\underline{\mathscr{S}_2}^U \subseteq \underline{\mathscr{S}_1}^U$ *and* $\overline{\mathscr{S}_1}^U \subseteq \overline{\mathscr{S}_2}^U$.

Proof.
If $\Bbbk\mathscr{S}_1(e) = \emptyset$, then $\underline{\mathscr{S}_2}^{U_Y}(e)(\Bbbk) \leq 1 = \underline{\mathscr{S}_1}^{U_Y}(e)(\Bbbk)$, and $\underline{\mathscr{S}_1}^{U_N}(e)(\Bbbk) = 0 \leq \underline{\mathscr{S}_2}^{U_N}(e)(\Bbbk)$. This implies that $\underline{\mathscr{S}_2}^U \subseteq \underline{\mathscr{S}_1}^U$.
If $\Bbbk\mathscr{S}_1(e) \neq \emptyset$, then $\underline{\mathscr{S}_1}^{U_Y}(e)(\Bbbk) = \bigwedge_{\mathfrak{k}\in\Bbbk\mathscr{S}_1(e)} U_Y(\mathfrak{k}) \geq \bigwedge_{\mathfrak{k}\in\Bbbk\mathscr{S}_2(e)} U_Y(\mathfrak{k}) = \underline{\mathscr{S}_2}^{U_Y}(e)(\Bbbk)$, and $\underline{\mathscr{S}_1}^{U_N}(e)(\Bbbk) = \bigvee_{\mathfrak{k}\in\Bbbk\mathscr{S}_1(e)} U_N(\mathfrak{k}) \leq \bigvee_{\mathfrak{k}\in\Bbbk\mathscr{S}_2(e)} U_N(\mathfrak{k}) = \underline{\mathscr{S}_2}^{U_N}(e)(\Bbbk)$. Thus, $\underline{\mathscr{S}_2}^U \subseteq \underline{\mathscr{S}_1}^U$. Similarly, $\overline{\mathscr{S}_1}^U \subseteq \overline{\mathscr{S}_2}^U$. □

Theorem 2.1.8. *Let* $(\mathscr{S}_1, D)$ *and* $(\mathscr{S}_2, D)$ *be two* S_fB_nR*s from* ξ_1 *to* ξ_2 *such that* $(\mathscr{S}_1, D) \subseteq (\mathscr{S}_2, D)$*, that is,* $\mathscr{S}_1(e) \subseteq \mathscr{S}_2(e)$*, for all* $e \in D$*. Then, for any* $U \in P_yF_zS(\xi_1)$*,* ${}^U\underline{\mathscr{S}}_2 \subseteq {}^U\underline{\mathscr{S}}_1$ *and* ${}^U\overline{\mathscr{S}_1} \subseteq {}^U\overline{\mathscr{S}_2}$.

Proof.
The proof can be derived using the same approach as in Theorem. 2.1.7. □

Theorem 2.1.9. *Let* $(\mathscr{S}_1, D)$ *and* $(\mathscr{S}_2, D)$ *be two* S_fB_nR*s from* ξ_1 *to* ξ_2*. Then, for any* $U \in P_yF_zS(\xi_2)$*, the following are true:*

i) $\underline{\mathscr{S}_1}^U \subseteq \underline{(\mathscr{S}_1 \cap \mathscr{S}_2)}^U$ *and* $\underline{\mathscr{S}_2}^U \subseteq \underline{(\mathscr{S}_1 \cap \mathscr{S}_2)}^U$.

ii) $\overline{(\mathscr{S}_1 \cap \mathscr{S}_2)}^U \subseteq \overline{\mathscr{S}_1}^U$ *and* $\overline{(\mathscr{S}_1 \cap \mathscr{S}_2)}^U \subseteq \overline{\mathscr{S}_2}^U$.

Proof.

The proof is a direct consequence of Theorem 2.1.7. □

Similarly, we have the following.

Theorem 2.1.10. *Let $(\mathscr{S}_1, D)$ and $(\mathscr{S}_2, D)$ be two S_fB_nRs from ξ_1 to ξ_2. Then, for any $U \in P_yF_zS(\xi_1)$, the following are true:*

i) ${}^U\underline{\mathscr{S}}_1 \subseteq {}^U(\underline{\mathscr{S}_1 \cap \mathscr{S}_2})$, *and* ${}^U\underline{\mathscr{S}_2} \subseteq {}^U(\underline{\mathscr{S}_1 \cap \mathscr{S}_2})$.

ii) ${}^U\overline{(\mathscr{S}_1 \cap \mathscr{S}_2)} \subseteq {}^U\overline{\mathscr{S}_1}$ *and* ${}^U\overline{(\mathscr{S}_1 \cap \mathscr{S}_2)} \subseteq {}^U\overline{\mathscr{S}_2}$.

Theorem 2.1.11. *Let $(\mathscr{S}, D)$ be a S_fB_nR from ξ_1 to ξ_2 and $\{U_i : i \in I\}$ be a family of P_yF_zSs defined on ξ_2. Then the following hold:*

i) $\underline{\mathscr{S}}^{(\bigcap_{i\in I} U_i)} = \bigcap_{i\in I} \underline{\mathscr{S}}^{U_i}$

ii) $\bigcup_{i\in I} \underline{\mathscr{S}}^{U_i} \subseteq \underline{\mathscr{S}}^{(\bigcup_{i\in I} U_i)}$

iii) $\overline{\mathscr{S}}^{(\bigcup_{i\in I} U_i)} = \bigcup_{i\in I} \overline{\mathscr{S}}^{U_i}$

iv) $\overline{\mathscr{S}}^{(\bigcap_{i\in I} U_i)} \subseteq \bigcap_{i\in I} \overline{\mathscr{S}}^{U_i}$.

Proof.

i) Let $U_i \in P_yF_zS(\xi_2)$, for $i \in I$. Then
$\underline{\mathscr{S}}^{(\bigcap_{i\in I} U_{i_Y})}(e)(\mathfrak{k}) = \bigwedge_{\mathfrak{k}\in\mathfrak{k}\mathscr{S}(e)}(\wedge_{i\in I} U_{i_Y}(\mathfrak{k})) = \bigwedge_{i\in I}(\wedge_{\mathfrak{k}\in\mathfrak{k}\mathscr{S}(e)} U_{i_Y}(\mathfrak{k})) = \bigcap_{i\in I} \underline{\mathscr{S}}^{U_{i_Y}}(e)(\mathfrak{k})$
and
$\underline{\mathscr{S}}^{(\bigcup_{i\in I} U_{i_N})}(e)(\mathfrak{k}) = \bigvee_{\mathfrak{k}\in\mathfrak{k}\mathscr{S}(e)}(\vee_{i\in I} U_{i_N}(\mathfrak{k})) = \bigvee_{i\in I}(\vee_{\mathfrak{k}\in\mathfrak{k}\mathscr{S}(e)} U_{i_N}(\mathfrak{k})) = \bigcup_{i\in I} \underline{\mathscr{S}}^{U_{i_N}}(e)(\mathfrak{k})$.
Thus, $\underline{\mathscr{S}}^{(\bigcap_{i\in I} U_i)} = \bigcap_{i\in I} \underline{\mathscr{S}}^{U_i}$.

ii) Given that $U_i \subseteq \bigcup_{i\in I} U_i$ for each $i \in I$, it follows that $\underline{\mathscr{S}}^{U_i} \subseteq \underline{\mathscr{S}}^{(\bigcup_{i\in I} U_i)}$. Consequently, $\bigcup_{i\in I} \underline{\mathscr{S}}^{U_i} \subseteq \underline{\mathscr{S}}^{(\bigcup_{i\in I} U_i)}$.

iii) The proof follows in a similar manner to part (i).

iv) The proof follows in a similar manner to part (ii).

□

Theorem 2.1.12. *Let $(\mathscr{S}, D)$ be an S_fB_nR from ξ_1 to ξ_2, and let $\{U_i : i \in I\}$ be a family of P_yF_zSs defined on ξ_1. The following properties hold:*

i) $^{(\bigcap_{i\in I} U_i)}\underline{\mathscr{S}} = \bigcap_{i\in I} {}^{U_i}\underline{\mathscr{S}}$

ii) $\bigcup_{i\in I} {}^{U_i}\underline{\mathscr{S}} \subseteq {}^{(\bigcup_{i\in I} U_i)}\underline{\mathscr{S}}$

iii) $^{(\bigcup_{i\in I} U_i)}\overline{\mathscr{S}} = \bigcup_{i\in I} {}^{U_i}\overline{\mathscr{S}}$

iv) $^{(\bigcap_{i\in I} U_i)}\overline{\mathscr{S}} \subseteq \bigcap_{i\in I} {}^{U_i}\overline{\mathscr{S}}$

Proof.

The proof can be derived using the same approach as in the Theorem 2.1.11. □

Theorem 2.1.13. *Let $(\mathscr{S}, D)$ be a soft Reflexive Relation (S_fR_lR) over ξ. For any $U \in P_yF_zS(\xi)$, the following properties hold for L_oA_p and U_pA_p with respect to A_fS:*

i) For all $e \in D$, $\underline{\mathscr{S}}^U(e) \leq U \leq \overline{\mathscr{S}}^U(e)$

ii) For all $e \in D$, it holds that $\underline{\mathscr{S}}^U(e) \leq \overline{\mathscr{S}}^U(e)$.

Proof.

For $\mathfrak{k} \in \xi$

i) Consider $\underline{\mathscr{S}}^{U_Y}(e)(\mathfrak{k}) = \bigwedge_{\mathfrak{t}\in\mathfrak{k}\mathscr{S}(e)} U_Y(\mathfrak{t}) \leq U_Y(\mathfrak{k})$, since $\mathfrak{k} \in \mathfrak{k}\mathscr{S}(e)$, and $\underline{\mathscr{S}}^{U_N}(e)(\mathfrak{k}) = \bigvee_{\mathfrak{t}\in\mathfrak{k}\mathscr{S}(e)} U_N(\mathfrak{t}) \geq U_N(\mathfrak{k})$, since $\mathfrak{k} \in \mathfrak{k}\mathscr{S}(e)$. Thus, $\underline{\mathscr{S}}^U(e) \leq U$.
Also, $\overline{\mathscr{S}}^{U_Y}(e)(\mathfrak{k}) = \bigvee_{\mathfrak{t}\in\mathfrak{k}\mathscr{S}(e)} U_Y(\mathfrak{t}) \geq U_Y(\mathfrak{k})$, since $\mathfrak{k} \in \mathfrak{k}\mathscr{S}(e)$, and $\overline{\mathscr{S}}^{U_N}(e)(\mathfrak{k}) = \bigwedge_{\mathfrak{t}\in\mathfrak{k}\mathscr{S}(e)} U_N(\mathfrak{t}) \leq U_N(\mathfrak{k})$, since $\mathfrak{k} \in \mathfrak{k}\mathscr{S}(e)$. Thus, $\overline{\mathscr{S}}^U(e) \geq U$.

ii) From part (i) we get that $\underline{\mathscr{S}}^U(e) \leq U \leq \overline{\mathscr{S}}^U(e)$ which implies that $\underline{\mathscr{S}}^U(e) \leq \overline{\mathscr{S}}^U(e)$.

□

Theorem 2.1.14. *Let $(\mathscr{S}, D)$ be a S_fR_fR over ξ. For any $U \in P_yF_zS(\xi)$, the following properties for L_oA_p and U_pA_ps with respect to F_rS hold, for all $e \in D$:*

i) $^U\underline{\mathscr{L}}(e) \leq U \leq {}^U\overline{\mathscr{S}}(e)$.

ii) $^U\underline{\mathscr{L}}(e) \leq {}^U\overline{\mathscr{S}}(e)$.

Proof.
The proof can be derived using the same approach as in Theorem. 2.1.13. □

2.2 Pythagorean Fuzzy Topologies induced by Soft Binary Reflexive Relations

Cheng [17] introduced the concept of a Fuzzy Topological Space and extended several fundamental notions of Topology. Olgun [45] developed the concept of Pythagorean Fuzzy Topological Spaces ($P_yF_zT_pS$) and examined the continuity between two $P_yF_zT_pS$ spaces.
In this context, we propose two types of Pythagorean Fuzzy Topologies that are derived from a soft Reflexive Relation (S_fR_lR).

Definition 2.2.1. *[45] A family $\mathfrak{A} \subseteq P_yF_zS(\xi)$ of P_yF_zSs on ξ is termed a Pythagorean Fuzzy Topology ($P_yF_zT_p$) on ξ if it meets the following:*

1) $0, 1 \in \mathfrak{A}$

2) $U_1 \cap U_2 \in \mathfrak{A}$*, for all* $U_1, U_2 \in \mathfrak{A}$

3) $\bigcup_{i\in I} U_i \in \mathfrak{A}$*, for all* $U_i \in \mathfrak{A}$*,* $i \in I$.

If $\mathfrak{A}$ is a $P_yF_zT_p$ on ξ, then the pair $(\xi, \mathfrak{A})$ is called a Pythagorean Fuzzy Topological Space ($P_yF_zT_pS$). The elements of $\mathfrak{A}$ are referred to as P_yF_z open sets.

Theorem 2.2.2. *If $(\mathscr{S}, D)$ is a S_fR_fR on ξ, then*

$$\mathfrak{T}_e = \{U \in P_yF_zS(\xi) : \underline{\mathscr{S}}^U(e) = U\}$$

is a $P_yF_zT_p$ on ξ for each $e \in D$.

Proof.
1) According to Theorem 2.1.4, for each $e \in D$, we have $\underline{\mathscr{S}}^0(e) = 0$ and $\underline{\mathscr{S}}^1(e) = 1$, which implies that $0, 1 \in \mathfrak{T}_e$.
2) $U_1, U_2 \in \mathfrak{T}_e$ implies $\underline{\mathscr{S}}^{U_1}(e) = U_1$ and $\underline{\mathscr{S}}^{U_2}(e) = U_2$. According to the Theorem 2.1.4, $\underline{\mathscr{S}}^{U_1 \cap U_2}(e) = (\underline{\mathscr{S}}^{U_1} \cap \underline{\mathscr{S}}^{U_2})(e) = U_1 \cap U_2$.
This implies that $U_1 \cap U_2 \in \mathfrak{T}_e$.
3) If $U_i \in \mathfrak{T}_e$, then $\underline{\mathscr{S}}^{U_i} = U_i$ for each $i \in I$. Since the relation is soft Reflexive, Theorem 2.1.13 gives us:

$$\underline{\mathscr{S}}^{(\bigcup_{i \in I} U_i)}(e) \leq \bigcup_{i \in I} U_i. \tag{2.2.1}$$

Also, because $U_i \leq \bigcup_{i \in I} U_i$, we have $\underline{\mathscr{S}}^{U_i}(e) \leq \underline{\mathscr{S}}^{(\bigcup_{i \in I} U_i)}(e)$. This implies:

$$\bigcup_{i \in I} \underline{\mathscr{S}}^{U_i}(e) \leq \underline{\mathscr{S}}^{(\bigcup_{i \in I} U_i)}(e). \tag{2.2.2}$$

Thus:

$$\bigcup_{i \in I} U_i \leq \underline{\mathscr{S}}^{(\bigcup_{i \in I} U_i)}(e). \tag{2.2.3}$$

From Equations (2.2.1) and (2.2.3), we get $\underline{\mathscr{S}}^{(\bigcup_{i \in I} U_i)}(e) = \bigcup_{i \in I} U_i$.
Therefore, $\mathfrak{T}_e$ is a $P_yF_zT_p$ on ξ. □

Theorem 2.2.3. *If $(\mathscr{S}, D)$ is a S_fR_fR on ξ, then*

$$\mathfrak{T}'_e = \{U \in P_yF_zS(\xi) : {}^U\underline{\mathscr{S}}(e) = U\}$$

is a $P_yF_zT_p$ *on* ξ *for each* $e \in D$.

Proof.

The proof can be derived using the same approach as in Theorem. 2.1.14. □

2.3 Similarity Relations Associated with Soft Binary Relations

Here, we discuss rough approximations based B_nRs between P_yF_zSs and associated properties.

Definition 2.3.1. *Let* $(\mathscr{S}, D)$ *be a* S_fR_fR *over* ξ. *For* $U_1, U_2 \in P_yF_zS(\xi)$, *we define*

$U_1\tilde{R}U_2$ *if and only if* $\overline{\mathscr{S}}^{U_1} = \overline{\mathscr{S}}^{U_2}$

$U_1\underline{R}U_2$ *if and only if* $\underline{\mathscr{S}}^{U_1} = \underline{\mathscr{S}}^{U_2}$

U_1RU_2 *if and only if* $\underline{\mathscr{S}}^{U_1} = \underline{\mathscr{S}}^{U_2}$ *and* $\overline{\mathscr{S}}^{U_1} = \overline{\mathscr{S}}^{U_2}$.

Definition 2.3.2. *Let* $(\mathscr{S}, D)$ *be a* S_fR_fR *over* ξ. *For* $U_1, U_2 \in P_yF_zS(\xi)$, *we define*

$U_1\tilde{r}U_2$ *if and only if* ${}^{U_1}\overline{\mathscr{S}} = {}^{U_2}\overline{\mathscr{S}}$

$U_1\underline{r}U_2$ *if and only if* ${}^{U_1}\underline{\mathscr{S}} = {}^{U_2}\underline{\mathscr{S}}$

U_1rU_2 *if and only if* ${}^{U_1}\underline{\mathscr{S}} = {}^{U_2}\underline{\mathscr{S}}$ *and* ${}^{U_1}\overline{\mathscr{S}} = {}^{U_2}\overline{\mathscr{S}}$.

The aforementioned Binary Relations can be denoted as follows: the lower Pythagorean Fuzzy Similarity relation ($L_oP_yF_zS_mR$), the upper Pythagorean Fuzzy Similarity relation ($U_pP_yF_zS_mR$), and the Pythagorean Fuzzy Similarity relation ($P_yF_zS_mR$).

Proposition 2.3.3. *The Binary Relations* $\underline{R}$, $\tilde{R}$, R *are* E_qRs *on* $P_yF_zS(\xi)$.

Proof.

The proof is straightforward. □

Proposition 2.3.4. *The Binary Relations $\underline{r}$, $\tilde{r}$, r are E_qRs on $P_yF_zS(\xi)$.*

Proof.

The proof is straightforward. $\square$

Theorem 2.3.5. *Let $(\mathscr{S}, D)$ be a S_fR_fR on ξ and $U_1, U_2, U_3, U_4 \in P_yF_zS(\xi)$. Then:*

i) $U_1\tilde{R}U_2$ *if and only if* $U_1\tilde{R}(U_1 \cup U_2)\tilde{R}U_2$

ii) *If* $U_1\tilde{R}U_2$ *and* $U_3\tilde{R}U_4$, *then* $(U_1 \cup U_3)\tilde{R}(U_2 \cup U_4)$

iii) *If* $U_1 \leq U_2$ *and* $U_2\tilde{R}0$, *then* $U_1\tilde{R}0$

iv) $(U_1 \cup U_2)\tilde{R}0$ *if and only if* $U_1\tilde{R}0$ *and* $U_2\tilde{R}0$

v) *If* $U_1 \leq U_2$ *and* $U_1\tilde{R}1$, *then* $U_2\tilde{R}1$

vi) *If* $(U_1 \cap U_2)\tilde{R}1$, *then* $U_1\tilde{R}1$ *and* $U_2\tilde{R}1$.

Proof.

i) If $U_1\tilde{R}U_2$, then $\overline{\mathscr{S}}^{U_1} = \overline{\mathscr{S}}^{U_2}$. According to the Theorem 2.1.4, $\overline{\mathscr{S}}^{U_2} = \overline{\mathscr{S}}^{U_1} = \overline{\mathscr{S}}^{U_1} \cup \overline{\mathscr{S}}^{U_2} = \overline{\mathscr{S}}^{U_1 \cup U_2}$, so we have $U_1\tilde{R}(U_1 \cup U_2)\tilde{R}U_2$.
Conversely, if $U_1\tilde{R}(U_1 \cup U_2)\tilde{R}U_2$, then $U_1\tilde{R}(U_1 \cup U_2)$ and $(U_1 \cup U_2)\tilde{R}U_2$. Which implies that $\overline{\mathscr{S}}^{U_1} = \overline{\mathscr{S}}^{U_1 \cup U_2}$ and $\overline{\mathscr{S}}^{U_1 \cup U_2} = \overline{\mathscr{S}}^{U_2}$. Thus, $\overline{\mathscr{S}}^{U_1} = \overline{\mathscr{S}}^{U_2}$. Hence, $U_1\tilde{R}U_2$.

ii) If $U_1\tilde{R}U_2$ and $U_3\tilde{R}U_4$, which implies that $\overline{\mathscr{S}}^{U_1} = \overline{\mathscr{S}}^{U_2}$ and $\overline{\mathscr{S}}^{U_3} = \overline{\mathscr{S}}^{U_4}$. Now according to the Theorem 2.1.4, $\overline{\mathscr{S}}^{U_1 \cup U_3} = \overline{\mathscr{S}}^{U_1} \cup \overline{\mathscr{S}}^{U_3} = \overline{\mathscr{S}}^{U_2} \cup \overline{\mathscr{S}}^{U_4} = \overline{\mathscr{S}}^{U_2 \cup U_4}$. Thus, $(U_1 \cup U_3)\tilde{R}(U_2 \cup U_4)$.

iii) Let $U_1 \leq U_2$ and $U_2\tilde{R}0$. Then $\overline{\mathscr{S}}^{U_2} = \overline{\mathscr{S}}^{0}$. Also, since $U_1 \leq U_2$, so we have $\overline{\mathscr{S}}^{U_1} \subseteq \overline{\mathscr{S}}^{U_2} = \overline{\mathscr{S}}^{0}$. But $\overline{\mathscr{S}}^{0} \subseteq \overline{\mathscr{S}}^{U_1}$, so $\overline{\mathscr{S}}^{U_1} = \overline{\mathscr{S}}^{0}$. Hence, $U_1\tilde{R}0$.

iv) If $(U_1 \cup U_2)\tilde{R}0$, then $\overline{\mathscr{S}}^{U_1} \cup \overline{\mathscr{S}}^{U_2} = \overline{\mathscr{S}}^{U_1 \cup U_2} = \overline{\mathscr{S}}^0$. Since $\overline{\mathscr{S}}^{U_1} \subseteq \overline{\mathscr{S}}^{U_1} \cup \overline{\mathscr{S}}^{U_2} = \overline{\mathscr{S}}^0$, so we have $\overline{\mathscr{S}}^{U_1} = \overline{\mathscr{S}}^0$. Similarly, $\overline{\mathscr{S}}^{U_2} = \overline{\mathscr{S}}^0$. Hence, $U_1\tilde{R}0$ and $U_1\tilde{R}0$.

Conversely, if $U_1\tilde{R}0$ and $U_2\tilde{R}0$, then $\overline{\mathscr{S}}^{U_1} = \overline{\mathscr{S}}^0$ and $\overline{\mathscr{S}}^{U_2} = \overline{\mathscr{S}}^0$. According to the Theorem 2.1.4, $\overline{\mathscr{S}}^{(U_1 \cup U_2)} = \overline{\mathscr{S}}^{U_1} \cup \overline{\mathscr{S}}^{U_2} = \overline{\mathscr{S}}^0 \cup \overline{\mathscr{S}}^0 = \overline{\mathscr{S}}^0$. Hence, $(U_1 \cup U_2\tilde{R})0$.

v) If $U_1\tilde{R}1$, then $\overline{\mathscr{S}}^{U_1} = \overline{\mathscr{S}}^1$. Since $U_1 \leq U_2$, so $\overline{\mathscr{S}}^1 = \overline{\mathscr{S}}^{U_1} \subseteq \overline{\mathscr{S}}^{U_2}$. But $\overline{\mathscr{S}}^{U_2} \subseteq \overline{\mathscr{S}}^1$ so, $\overline{\mathscr{S}}^1 = \overline{\mathscr{S}}^{U_1}$. Hence, $U_2\tilde{R}1$.

vi) If $U_1 \cap U_2\tilde{R}1$, then $\overline{\mathscr{S}}^{U_1 \cap U_2} = \overline{\mathscr{S}}^1$. According to the Theorem 2.1.4, we have $\overline{\mathscr{S}}^{U_1} \cap \overline{\mathscr{S}}^{U_2} \supseteq \overline{\mathscr{S}}^{U_1 \cap U_2} = \overline{\mathscr{S}}^1$. Thus, $\overline{\mathscr{S}}^1 = \overline{\mathscr{S}}^{U_1}$ and $\overline{\mathscr{S}}^1 = \overline{\mathscr{S}}^{U_2}$.

Hence, $U_1\tilde{R}1$ and $U_2\tilde{R}1$.

□

Theorem 2.3.6. *Let $(\mathscr{S}, D)$ be a $S_f R_f R$ on ξ and $U_1, U_2, U_3, U_4 \in P_y F_z S(\xi)$. Then:*

i) $U_1\tilde{r}U_2$ if and only if $U_1\tilde{r}(U_1 \cup U_2)\tilde{r}U_2$

ii) If $U_1\tilde{r}U_2$ and $U_3\tilde{r}U_4$, then $(U_1 \cup U_3)\tilde{r}(U_2 \cup U_4)$

iii) If $U_1 \leq U_2$ and $U_2\tilde{r}0$, then $U_1\tilde{r}0$

iv) $(U_1 \cup U_2)\tilde{r}0$ if and only if $U_1\tilde{r}0$ and $U_2\tilde{r}0$

v) If $U_1 \leq U_2$ and $U_1\tilde{r}1$, then $U_2\tilde{r}1$

vi) If $(U_1 \cap U_2)\tilde{r}1$, then $U_1\tilde{r}1$ and $U_2\tilde{r}1$.

Proof.

The proof can be derived using the same approach as in Theorem. 2.3.5 □

Theorem 2.3.7. *Let $(\mathscr{S}, D)$ be a $S_f R_f R$ on ξ and $U_1, U_2, U_3, U_4 \in P_y F_z S(\xi)$. Then:*

i) $U_1\underline{R}U_2$ if and only if $U_1\underline{R}(U_1 \cap U_2)\underline{R}U_2$

ii) If $U_1\underline{R}U_2$ and $U_3\underline{R}U_4$, then $(U_1 \cap U_3)\underline{R}(U_2 \cap U_4)$

iii) If $U_1 \leq U_2$ and $U_2\underline{R}0$, then $U_1\underline{R}0$

iv) $(U_1 \cup U_2)\underline{R}0$ if and only if $U_1\underline{R}0$ and $U_2\underline{R}0$

v) If $U_1 \leq U_2$ and $U_1\underline{R}1$, then $U_2\underline{R}1$

vi) If $(U_1 \cap U_2)\underline{R}1$, then $U_1\underline{R}1$ and $U_2\underline{R}1$.

Proof.

The proof is straightforward. □

Theorem 2.3.8. *Let* $(\mathscr{S}, D)$ *be a* S_fR_fR *on* ξ *and* $U_1, U_2, U_3, U_4 \in P_yF_zS(\xi)$. *Then:*

i) $U_1\underline{r}U_2$ if and only if $U_1\underline{r}(U_1 \cap U_2)\underline{r}U_2$

ii) If $U_1\underline{r}U_2$ and $U_3\underline{r}U_4$, then $(U_1 \cap U_3)\underline{r}(U_2 \cap U_4)$

iii) If $U_1 \leq U_2$ and $U_2\underline{r}0$, then $U_1\underline{r}0$

iv) $(U_1 \cup U_2)\underline{r}0$ if and only if $U_1\underline{r}0$ and $U_2\underline{r}0$

v) If $U_1 \leq U_2$ and $U_1\underline{r}1$, then $U_2\underline{r}1$

vi) If $(U_1 \cap U_2)\underline{r}1$, then $U_1\underline{r}1$ and $U_2\underline{r}1$.

Proof.

The proof is straightforward. □

Theorem 2.3.9. *Let* $(\mathscr{S}, D)$ *be a* S_fR_fR *on* ξ *and* $U_1, U_2 \in P_yF_zS(\xi)$. *Then:*

i) U_1RU_2 if and only if $U_1\tilde{R}(U_1 \cup U_2)\tilde{R}U_2$ and $U_1\underline{R}(U_1 \cap U_2)\underline{R}U_2$

ii) If $U_1 \leq U_2$ and U_2R0, then U_1R0

iii) $(U_1 \cup U_2)R0$ if and only if U_1R0 and U_2R0

iv) If $(U_1 \cap U_2)R1$, then U_1R1 and U_2R1.

v) If $U_1 \leq U_2$ and U_1R1, then U_2R1

Proof.

Theorems 2.3.5 and 2.3.7 directly lead to this conclusion. □

Theorem 2.3.10. *Let $(\mathscr{S}, D)$ be a S_fR_fR on ξ and $U_1, U_2 \in P_yF_zS(\xi)$. Then:*

i) U_1rU_2 if and only if $U_1\tilde{r}(U_1 \cup U_2)\tilde{r}U_2$ and $U_1\underline{r}(U_1 \cap U_2)\underline{r}U_2$

ii) If $U_1 \leq U_2$ and U_2r0, then U_1r0

iii) $(U_1 \cup U_2)r0$ if and only if U_1r0 and U_2r0

iv) If $(U_1 \cap U_2)r1$, then U_1r1 and U_2r1.

v) If $U_1 \leq U_2$ and U_1r1, then U_2r1

Proof.

Theorems 2.3.6 and 2.3.8 directly lead to this conclusion. □

2.4 Accuracy Measure

The approximation of P_yF_zSs provides a novel method to evaluate the precision of M_mD_gs that describe objects. Hussain et al. [28] introduced the concept of the $(\mathscr{A}, \mathscr{B})$-level cut set of a P_yF_zS and explored its properties.

In this section, we introduce the notions of the roughness degree $(R_fN_sD_g)$ and $A_cR_cD_g$ concerning A_fSs and F_rS with respect to M_mD_gs of P_yF_zS.

Definition 2.4.1. *[28] Let $U \in P_yF_zS(\xi)$ and let $\mathscr{A}, \mathscr{B} \in [0,1]$ satisfy $\mathscr{A}^2 + \mathscr{B}^2 \leq 1$. The $(\mathscr{A}, \mathscr{B})$-level cut set of U is defined as:*

$$U_{\mathscr{A}}^{\mathscr{B}} = \{\Bbbk \in \xi : U_Y(\Bbbk) \geq \mathscr{A} \quad \text{and} \quad U_N(\Bbbk) \leq \mathscr{B}\}.$$

Hussain et al. [28] discussed several important properties of the $(\mathscr{A}, \mathscr{B})$-level cut set of $U \in P_y F_z S(\xi)$.

Lemma 2.4.2. *[28] Let $U, V \in P_y F_z S(\xi)$. Let $\mathscr{A}_1, \mathscr{A}_2, \mathscr{B}_1, \mathscr{B}_2 \in [0,1]$ be such that $\mathscr{A}_1^2 + \mathscr{B}_1^2 \leq 1$ and $\mathscr{A}_2^2 + \mathscr{B}_2^2 \leq 1$. Then the following hold:*

1) *$U \subseteq V$ implies $U_{\mathscr{A}_1}^{\mathscr{B}_1} \subseteq V_{\mathscr{A}_1}^{\mathscr{B}_1}$*

2) *If $\mathscr{A}_1 \geq \mathscr{A}_2$ and $\mathscr{B}_1 \leq \mathscr{B}_2$, then $U_{\mathscr{A}_1}^{\mathscr{B}_1} \subseteq U_{\mathscr{A}_2}^{\mathscr{B}_2}$.*

Note that if $(\mathscr{S}, D)$ is an $S_f B_n R$ over ξ, then $\underline{\mathscr{S}}^{U_{\mathscr{A}}^{\mathscr{B}}}$ represents the lower approximation $L_o A_p$ of the Crisp Set $U_{\mathscr{A}}^{\mathscr{B}}$, and $(\underline{\mathscr{S}}^U(e))_{\mathscr{A}}^{\mathscr{B}}$ denotes the $(\mathscr{A}, \mathscr{B})$-level cut of $\underline{\mathscr{S}}^U(e)$ with respect to the $A_f S$s. Therefore, for all $e \in D$,

$$\begin{aligned}(\underline{\mathscr{S}}^U(e))_{\mathscr{A}}^{\mathscr{B}} &= \{\mathfrak{k} \in \xi : \underline{\mathscr{S}}^{U_Y}(e)(\mathfrak{k}) \geq \mathscr{A} \quad \text{and} \quad \underline{\mathscr{S}}^{U_N}(e)(\mathfrak{k}) \leq \mathscr{B}\} \\ &= \{\mathfrak{k} \in \xi : \wedge_{\mathfrak{t} \in \mathfrak{k}\mathscr{S}(e)} U_Y(\mathfrak{t}) \geq \mathscr{A} \quad \text{and} \quad \vee_{\mathfrak{t} \in \mathfrak{k}\mathscr{S}(e)} U_N(\mathfrak{t}) \leq \mathscr{B}\}\end{aligned}$$

and

$$\begin{aligned}(\overline{\mathscr{S}}^U(e))_{\mathscr{A}}^{\mathscr{B}} &= \{\mathfrak{k} \in \xi : \overline{\mathscr{S}}^{U_Y}(e)(\mathfrak{k}) \geq \mathscr{A} \quad \text{and} \quad \overline{\mathscr{S}}^{U_N}(e)(\mathfrak{k}) \leq \mathscr{B}\} \\ &= \{\mathfrak{k} \in \xi : \vee_{\mathfrak{t} \in \mathfrak{k}\mathscr{S}(e)} U_Y(\mathfrak{t}) \geq \mathscr{A} \quad \text{and} \quad \wedge_{\mathfrak{t} \in \mathfrak{k}\mathscr{S}(e)} U_N(\mathfrak{t}) \leq \mathscr{B}\}.\end{aligned}$$

Similarly, for all $e \in D$,

$$\begin{aligned}({}^U\underline{\mathscr{S}}(e))_{\mathscr{A}}^{\mathscr{B}} &= \{\mathfrak{t} \in \xi : {}^{U_Y}\underline{\mathscr{S}}(e)(\mathfrak{t}) \geq \mathscr{A} \quad \text{and} \quad {}^{U_N}\underline{\mathscr{S}}(e)(\mathfrak{t}) \leq \mathscr{B}\} \\ &= \{\mathfrak{t} \in \xi : \wedge_{\mathfrak{k} \in \mathscr{S}(e)\mathfrak{t}} U_Y(\mathfrak{k}) \geq \mathscr{A} \quad \text{and} \quad \vee_{\mathfrak{k} \in \mathscr{S}(e)\mathfrak{t}} U_N(\mathfrak{k}) \leq \mathscr{B}\}\end{aligned}$$

and

$$\begin{aligned}({}^U\overline{\mathscr{S}}(e))_{\mathscr{A}}^{\mathscr{B}} &= \{\mathfrak{t} \in \xi : {}^{U_Y}\overline{\mathscr{S}}(e)(\mathfrak{t}) \geq \mathscr{A} \quad \text{and} \quad {}^{U_N}\overline{\mathscr{S}}(e)(\mathfrak{t}) \leq \mathscr{B}\} \\ &= \{\mathfrak{t} \in \xi : \vee_{\mathfrak{k} \in \mathscr{S}(e)\mathfrak{t}} U_Y(\mathfrak{k}) \geq \mathscr{A} \quad \text{and} \quad \wedge_{\mathfrak{k} \in \mathscr{S}(e)\mathfrak{t}} U_N(\mathfrak{k}) \leq \mathscr{B}\}\end{aligned}$$

with respect to F_rS.

Lemma 2.4.3. *Let $(\mathscr{S}, D)$ be a S_fR_fR on a non-empty set ξ and $U \in P_yF_zS(\xi)$. Let $\mathscr{A}, \mathscr{B} \in [0,1]$ be such that $\mathscr{A}^2 + \mathscr{B}^2 \leq 1$. Then, for all $e \in D$,*

$$\overline{\mathscr{S}}^{U_{\mathscr{A}}^{\mathscr{B}}}(e) = (\overline{\mathscr{S}}^{U}(e))_{\mathscr{A}}^{\mathscr{B}} \qquad \text{and} \qquad \underline{\mathscr{S}}^{U_{\mathscr{A}}^{\mathscr{B}}}(e) = (\underline{\mathscr{S}}^{U}(e))_{\mathscr{A}}^{\mathscr{B}}.$$

Proof.
Let $\mathscr{A}, \mathscr{B} \in [0,1]$ be such that $\mathscr{A}^2 + \mathscr{B}^2 \leq 1$. Then

$$\begin{aligned}
(\underline{\mathscr{S}}^{U}(e))_{\mathscr{A}}^{\mathscr{B}} &= \{\mathfrak{k} \in \xi : \underline{\mathscr{S}}^{U_Y}(e)(\mathfrak{k}) \geq \mathscr{A} \quad \text{and} \quad \underline{\mathscr{S}}^{U_N}(e)(\mathfrak{k}) \leq \mathscr{B}\} \\
&= \{\mathfrak{k} \in \xi : \wedge_{\mathfrak{t} \in \mathfrak{k}\mathscr{S}(e)} U_Y(\mathfrak{t}) \geq \mathscr{A} \quad \text{and} \quad \vee_{\mathfrak{t} \in \mathfrak{k}\mathscr{S}(e)} U_N(\mathfrak{t}) \leq \mathscr{B}\} \\
&= \{\mathfrak{k} \in \xi : U_Y(\mathfrak{t}) \geq \mathscr{A} \quad \text{and} \quad U_N(\mathfrak{t}) \leq \mathscr{B}, \quad \text{for all} \quad \mathfrak{t} \in \mathfrak{k}\mathscr{S}(e)\} \\
&= \{\mathfrak{k} \in \xi : \mathfrak{k}\mathscr{S}(e) \subseteq U_{\mathscr{A}}^{\mathscr{B}}\} \\
&= \underline{\mathscr{S}}^{U_{\mathscr{A}}^{\mathscr{B}}}(e)(\mathfrak{k}).
\end{aligned}$$

Similarly, we can easily show that $\overline{\mathscr{S}}^{U_{\mathscr{A}}^{\mathscr{B}}}(e) = (\overline{\mathscr{S}}^{U}(e))_{\mathscr{A}}^{\mathscr{B}}$. □

Lemma 2.4.4. *Let $(\mathscr{S}, D)$ be a S_fR_fR on a non-empty set ξ and $U \in P_yF_zS(\xi)$. Let $\mathscr{A}, \mathscr{B} \in [0,1]$ be such that $\mathscr{A}^2 + \mathscr{B}^2 \leq 1$. Then, for all $e \in D$,*

$${}^{U_{\mathscr{A}}^{\mathscr{B}}}\overline{\mathscr{S}}(e) = ({}^{U}\overline{\mathscr{S}}(e))_{\mathscr{A}}^{\mathscr{B}} \qquad \text{and} \qquad {}^{U_{\mathscr{A}}^{\mathscr{B}}}\underline{\mathscr{S}}(e) = ({}^{U}\underline{\mathscr{S}}(e))_{\mathscr{A}}^{\mathscr{B}}.$$

Proof.
The proof can be derived using the same approach as in Lemma 2.4.3. □

The $A_cR_cD_g$ and $R_fN_sD_g$ of a P_yF_zS is defined below.

Definition 2.4.5. *Let $(\mathscr{S}, D)$ be a S_fR_lR over a non-empty set ξ. The $A_cR_cD_g$ for the membership of $U \in P_yF_zS(\xi)$, with respect to parameters $\mathscr{A}, \mathscr{B}, \mathscr{G}, \theta \in [0,1]$ satisfying $\mathscr{A} \leq$*

$\mathscr{G}$, $\mathscr{B} \geq \theta$, $\mathscr{A}^2 + \mathscr{B}^2 \leq 1$, and $\mathscr{G}^2 + \theta^2 \leq 1$, and with respect to A_fS, is defined as:

$$\delta^{(\mathscr{G},\theta)}_{(\mathscr{A},\mathscr{B})}(\mathscr{S}^U)(e) = \frac{|\underline{\mathscr{S}}^{U^\theta_\mathscr{G}}(e)|}{|\overline{\mathscr{S}}^{U^\mathscr{B}_\mathscr{A}}(e)|}$$

for all $e \in D$. The $R_fN_sD_g$ for the membership of $U \in P_yF_zS(\xi)$ is defined as:

$$\eta^{(\mathscr{G},\theta)}_{(\mathscr{A},\mathscr{B})}(\mathscr{S}^U)(e) = 1 - \delta^{(\mathscr{G},\theta)}_{(\mathscr{A},\mathscr{B})}(\mathscr{S}^U)(e)$$

for all $e \in D$. Similarly, the $A_cR_cD_g$ for the membership of $U \in P_yF_zS(\xi)$ with respect to F_rS is defined as:

$$\delta^{(\mathscr{G},\theta)}_{(\mathscr{A},\mathscr{B})}({}^UF)(e) = \frac{|\,{}^{U^\theta_\mathscr{G}}\underline{\mathscr{S}}(e)|}{|\,{}^{U^\mathscr{B}_\mathscr{A}}\overline{\mathscr{S}}(e)|}$$

for all $e \in D$. The $R_fN_sD_g$ for the membership of $U \in P_yF_zS(\xi)$ with respect to F_rS is defined as:

$$\eta^{(\mathscr{G},\theta)}_{(\mathscr{A},\mathscr{B})}({}^UF)(e) = 1 - \delta^{(\mathscr{G},\theta)}_{(\mathscr{A},\mathscr{B})}({}^UF)(e)$$

for all $e \in D$.

In the context of a Soft E_qR (S_fE_qR), the concepts of F_rS and A_fS coincide. Specifically, $\underline{\mathscr{S}}^{U^\theta_\mathscr{G}}(e)$ contains elements of ξ with $\mathscr{G}$ as the minimal definite M_mD_g and θ as the maximal definite $N_nM_mD_g$ in U. Conversely, $\overline{\mathscr{S}}^{U^\mathscr{B}_\mathscr{A}}(e)$ includes elements with $\mathscr{A}$ as the minimal possible M_mD_g and $\mathscr{B}$ as the maximal possible $N_nM_mD_g$ in U, for all $e \in D$.
In simpler terms, $\underline{\mathscr{S}}^{U^\theta_\mathscr{G}}(e)$ represents the union of soft equivalence classes in U's L_oA_p, where $\mathscr{G}$ signifies the lowest definite M_mD_g and θ represents the highest definite $N_nM_mD_g$. On the other hand, $\overline{\mathscr{S}}^{U^\mathscr{B}_\mathscr{A}}(e)$ represents the union in U's U_pA_p, where $\mathscr{A}$ denotes the lowest possible M_mD_g and $\mathscr{B}$ denotes the highest possible $N_nM_mD_g$.
Therefore, $(\mathscr{G},\theta)$ and $(\mathscr{A},\mathscr{B})$ act as thresholds defining the levels of certainty regarding the membership status of elements u in U. Consequently, $\delta^{(\mathscr{G},\theta)}_{(\mathscr{A},\mathscr{B})}(\mathscr{S}^U)(e)$ can be interpreted as the $A_cR_cD_g$ of U's membership with respect to the specified thresholds $(\mathscr{G},\theta)$ and $(\mathscr{A},\mathscr{B})$.

These concepts are further illustrated in the following example.

Example 2.4.6. *Let* $\xi = \{\Bbbk_1, \Bbbk_2, \Bbbk_3, \Bbbk_4, \Bbbk_5, \Bbbk_6, \Bbbk_7, \Bbbk_8, \Bbbk_9, \Bbbk_{10}, \Bbbk_{11}\}$ *be a collection of houses and* $D = \{Green\ Surroundings, Wooden\ House, Cheap, costly\} = \{e_1, e_2, e_3, e_4\}$ *be a parameters set.*

Define a S_fE_qR $\mathscr{S} : D \to P_yF_zS(\xi \times \xi)$ *by the following soft equivalence classes:*

For $\mathscr{S}(e_1)$*, the soft equivalence classes* $\Bbbk_i\mathscr{S}(e_1)$ *are* $\{\Bbbk_1, \Bbbk_9\}, \{\Bbbk_2, \Bbbk_4, \Bbbk_6, \Bbbk_7\}, \{\Bbbk_3, \Bbbk_5, \Bbbk_8, \Bbbk_{10}\}, \{\Bbbk_{11}\}$.

For $\mathscr{S}(e_2)$*, the soft equivalence classes* $\Bbbk_i\mathscr{S}(e_2)$ *are* $\{\Bbbk_1\}, \{\Bbbk_2, \Bbbk_3, \Bbbk_5, \Bbbk_9\}, \{\Bbbk_4, \Bbbk_7\}, \{\Bbbk_8, \Bbbk_{11}\}, \{\Bbbk_6, \Bbbk_{10}\}$.

For $\mathscr{S}(e_3)$*, the soft equivalence classes* $\Bbbk_i\mathscr{S}(e_3)$ *are* $\{\Bbbk_1\}, \{\Bbbk_2\}, \{\Bbbk_3, \Bbbk_4, \Bbbk_5, \Bbbk_7, \Bbbk_8, \Bbbk_9, \Bbbk_{10}\}, \{\Bbbk_6\}, \{\Bbbk_{11}\}$.

For $\mathscr{S}(e_4)$*, the soft equivalence classes* $\Bbbk_i\mathscr{S}(e_4)$ *are* $\{\Bbbk_1, \Bbbk_2, \Bbbk_3, \Bbbk_4, \Bbbk_5, \Bbbk_6, \Bbbk_8, \Bbbk_{11}\}, \{\Bbbk_9\}, \{\Bbbk_7\}\{\Bbbk_{10}\}$.

Define a P_yF_zS

$U : \xi \to [0,1]$ *by* $U = \{\langle \Bbbk_1, 0.9, 0.3\rangle, \langle \Bbbk_2, 0.6, 0.7\rangle, \langle \Bbbk_3, 0.3, 0.8\rangle, \langle \Bbbk_4, 0, 0.9\rangle, \langle \Bbbk_5, 0.2, 0.91\rangle, \langle \Bbbk_6, 0.4, 0.8\rangle, \langle \Bbbk_7, 0.6, 0.5\rangle, \langle \Bbbk_8, 0.8, 0.1\rangle, \langle \Bbbk_9, 1, 0\rangle, \langle \Bbbk_{10}, 0, 0.8\rangle, \langle \Bbbk_{11}, 0.99, 0.01\rangle\}$.

Take $(\mathscr{G}, \theta) = (0.7, 0.4)$ *and* $(\mathscr{A}, \mathscr{B}) = (0.5, 0.6)$ *then* $(\mathscr{G}, \theta)-$*level and* $(\mathscr{A}, \mathscr{B})-$*level cuts* $U_{0.7}^{0.4}$ *and* $U_{0.5}^{0.6}$ *are,* $U_{\mathscr{A}}^{\mathscr{B}} = U_{0.5}^{0.6} = \{\Bbbk : U_Y(\Bbbk) \geq 0.5, \quad U_N(\Bbbk) \leq 0.6\} = \{\Bbbk_1, \Bbbk_7, \Bbbk_8, \Bbbk_9, \Bbbk_{11}\}$, $U_{\mathscr{G}}^{\theta} = U_{0.7}^{0.4} = \{\Bbbk_1, \Bbbk_8, \Bbbk_9, \Bbbk_{11}\}$. *Then*

$\underline{\mathscr{S}}^{U_{\mathscr{G}}^{\theta}}(e_1) = \underline{\mathscr{S}}^{U_{0.7}^{0.4}}(e_1) = \{\Bbbk \in \xi : \Bbbk\mathscr{S}(e_1) \subseteq U_{\mathscr{G}}^{\theta}\} = \{\Bbbk_1, \Bbbk_9, \Bbbk_{11}\}$, $\underline{\mathscr{S}}^{U_{0.7}^{0.4}}(e_2) = \{\Bbbk_1, \Bbbk_8, \Bbbk_{11}\}$, $\underline{\mathscr{S}}^{U_{0.7}^{0.4}}(e_3) = \{\Bbbk_1, \Bbbk_{11}\}$, $\underline{\mathscr{S}}^{U_{0.7}^{0.4}}(e_4) = \{\Bbbk_9\}$, *and*

$\overline{\mathscr{S}}^{U_{\mathscr{A}}^{\mathscr{B}}}(e_1) = \overline{\mathscr{S}}^{U_{0.5}^{0.6}}(e_1) = \{\Bbbk \in \xi : \Bbbk\mathscr{S}(e_1) \cap U_{0.5}^{0.6} \neq \emptyset\}, = \{\Bbbk_1, \Bbbk_2, \Bbbk_3, \Bbbk_4, \Bbbk_5, \Bbbk_6, \Bbbk_7, \Bbbk_8, \Bbbk_9, \Bbbk_{10}, \Bbbk_{11}\}$,

$\overline{\mathscr{S}}^{U_{0.5}^{0.6}}(e_2) = \{\Bbbk_1, \Bbbk_2, \Bbbk_3, \Bbbk_4, \Bbbk_5, \Bbbk_7, \Bbbk_8, \Bbbk_9, \Bbbk_{11}\}$, $\overline{\mathscr{S}}^{U_{0.5}^{0.6}}(e_3) = \{\Bbbk_1, \Bbbk_3, \Bbbk_4, \Bbbk_5, \Bbbk_7, \Bbbk_8, \Bbbk_9, \Bbbk_{10}, \Bbbk_{11}\}$,

$\overline{\mathscr{S}}^{U_{0.5}^{0.6}}(e_4) = \{\Bbbk_1, \Bbbk_2, \Bbbk_3, \Bbbk_4, \Bbbk_5, \Bbbk_6, \Bbbk_7, \Bbbk_8, \Bbbk_9, \Bbbk_{11}\}$.

Thus $\delta_{(\mathscr{A},\mathscr{B})}^{(\mathscr{G},\theta)}\mathscr{S}^U(e_1) = \frac{|\underline{\mathscr{S}}^{U_{0.7}^{0.4}}(e_1)|}{|\overline{\mathscr{S}}^{U_{0.5}^{0.6}}(e_1)|} = \frac{3}{11}$, $\delta_{(\mathscr{A},\mathscr{B})}^{(\mathscr{G},\theta)}(\mathscr{S}^U)(e_2) = \frac{1}{3}$, $\delta_{(\mathscr{A},\mathscr{B})}^{(\mathscr{G},\theta)}(\mathscr{S}^U)(e_3) = \frac{2}{9}$, $\delta_{(\mathscr{A},\mathscr{B})}^{(\mathscr{G},\theta)}(\mathscr{S}^U)(e_4) = \frac{1}{10}$.

Theorem 2.4.7. *Let* $(\mathscr{S}, D)$ *be a soft Reflexive Relation* (S_fR_lR) *defined on a non-empty set* ξ*,* $U \in P_yF_zS(\xi)$*, and let* $\mathscr{A}, \mathscr{B}, \mathscr{G}, \theta \in [0,1]$ *satisfy* $\mathscr{A} \leq \mathscr{G}$*,* $\mathscr{B} \geq \theta$*,* $\mathscr{A}^2 + \mathscr{B}^2 \leq 1$*, and* $\mathscr{G}^2 + \theta^2 \leq 1$*. Then for all* $e \in D$*, with respect to* A_fS*, the* $A_cR_cD_g$ *of* U*'s membership is bounded as follows:*

$$0 \leq \delta_{(\mathscr{A},\mathscr{B})}^{(\mathscr{G},\theta)}(\mathscr{S}^U)(e) \leq 1.$$

Proof.

Let $U \in P_yF_zS(\xi)$ and let $\mathscr{A}, \mathscr{B}, \mathscr{G}, \theta \in [0,1]$ be such that $\mathscr{A} \leq \mathscr{G}$, $\mathscr{B} \geq \theta$, $\mathscr{A}^2 + \mathscr{B}^2 \leq 1$, and $\mathscr{G}^2 + \theta^2 \leq 1$. According to Lemma 2.4.2, we have $U_{\mathscr{G}}^{\theta} \subseteq U_{\mathscr{A}}^{\mathscr{B}}$. According to the Theorem 2.1.4, it follows that $\underline{\mathscr{L}}^{U_{\mathscr{G}}^{\theta}}(e) \subseteq \overline{\mathscr{F}}^{U_{\mathscr{G}}^{\theta}}(e) \subseteq \overline{\mathscr{F}}^{U_{\mathscr{A}}^{\mathscr{B}}}(e)$, which implies $\mid \underline{\mathscr{L}}^{U_{\mathscr{G}}^{\theta}}(e) \mid \leq \mid \overline{\mathscr{F}}^{U_{\mathscr{A}}^{\mathscr{B}}}(e) \mid$. Thus, $\frac{|\underline{\mathscr{L}}^{U_{\mathscr{G}}^{\theta}}(e)|}{|\overline{\mathscr{F}}^{U_{\mathscr{A}}^{\mathscr{B}}}(e)|} \leq 1$. Consequently, $0 \leq \delta_{(\mathscr{A},\mathscr{B})}^{(\mathscr{G},\theta)}(\mathscr{S}^U)(e) \leq 1$ for $e \in D$. □

Corollary 2.4.8. *Let $(\mathscr{S}, D)$ be a S_fR_lR on ξ, $U \in P_yF_zS(\xi)$, and let $\mathscr{A}, \mathscr{B}, \mathscr{G}, \theta \in [0,1]$ be such that $\mathscr{A} \leq \mathscr{G}$, $\mathscr{B} \geq \theta$, $\mathscr{A}^2 + \mathscr{B}^2 \leq 1$, and $\mathscr{G}^2 + \theta^2 \leq 1$. Then $0 \leq \eta_{(\mathscr{A},\mathscr{B})}^{(\mathscr{G},\theta)}(\mathscr{S}^U)(e) \leq 1$ for $e \in D$ with respect to the A_fSs.*

Proof.

Definition 2.4.5 and Theorem 2.4.7 directly lead to this conclusion. □

Theorem 2.4.9. *Let $(\mathscr{S}, D)$ be a S_fR_lR on ξ, $U, V \in P_yF_zS(\xi)$, and $\mathscr{A}, \mathscr{B}, \mathscr{G}, \theta \in [0,1]$ be such that $\mathscr{A} \leq \mathscr{G}$, $\mathscr{B} \geq \theta$, $\mathscr{A}^2 + \mathscr{B}^2 \leq 1$, and $\mathscr{G}^2 + \theta^2 \leq 1$. If $U \leq V$, then the following assertions hold, for all $e \in D$:*

i) $\delta_{(\mathscr{A},\mathscr{B})}^{(\mathscr{G},\theta)}(\mathscr{S}^U)(e) \leq \delta_{(\mathscr{A},\mathscr{B})}^{(\mathscr{G},\theta)}(\mathscr{S}^V)(e)$, *whenever* $\overline{\mathscr{F}}^{U_{\mathscr{A}}^{\mathscr{B}}}(e) = \overline{\mathscr{F}}^{V_{\mathscr{A}}^{\mathscr{B}}}(e)$

ii) $\delta_{(\mathscr{A},\mathscr{B})}^{(\mathscr{G},\theta)}(\mathscr{S}^U)(e) \geq \delta_{(\mathscr{A},\mathscr{B})}^{(\mathscr{G},\theta)}(\mathscr{S}^V)(e)$ *whenever* $\underline{\mathscr{L}}^{U_{\mathscr{A}}^{\mathscr{B}}}(e) = \underline{\mathscr{L}}^{V_{\mathscr{A}}^{\mathscr{B}}}(e)$.

Proof.

i) Let $\mathscr{A}, \mathscr{B}, \mathscr{G}, \theta \in [0,1]$ be such that $\mathscr{A} \leq \mathscr{G}$, $\mathscr{B} \geq \theta$, and $\mathscr{A}^2 + \mathscr{B}^2 \leq 1$, $\mathscr{G}^2 + \theta^2 \leq 1$. Let $U, V \in P_yF_zS(\xi)$ be such that $U \leq V$ which implies $U_{\mathscr{G}}^{\theta} \subseteq V_{\mathscr{G}}^{\theta}$. Then According to the Theorem 2.1.4, $\underline{\mathscr{L}}^{U_{\mathscr{G}}^{\theta}}(e) \leq \underline{\mathscr{L}}^{V_{\mathscr{G}}^{\theta}}(e)$, this implies that $\frac{|\underline{\mathscr{L}}^{U_{\mathscr{G}}^{\theta}}(e)|}{|\overline{\mathscr{F}}^{U_{\mathscr{A}}^{\mathscr{B}}}(e)|} \leq \frac{|\underline{\mathscr{L}}^{V_{\mathscr{G}}^{\theta}}(e)|}{|\overline{\mathscr{F}}^{V_{\mathscr{G}}^{\theta}}(e)|}$, whenever $\overline{\mathscr{F}}^{U_{\mathscr{A}}^{\mathscr{B}}}(e) = \overline{\mathscr{F}}^{V_{\mathscr{A}}^{\mathscr{B}}}(e)$. Thus, $\delta_{(\mathscr{A},\mathscr{B})}^{(\mathscr{G},\theta)}(\mathscr{S}^U)(e) \leq \delta_{(\mathscr{A},\mathscr{B})}^{(\mathscr{G},\theta)}(\mathscr{S}^V)(e)$.

ii) The proof can be derived using the same approach as in part (i).

□

Corollary 2.4.10. *Let $(\mathscr{S}, D)$ denote a S_fR_fR defined on a non-empty set ξ. Suppose $U, V \in P_yF_zS(\xi)$ and let $\mathscr{A}, \mathscr{B}, \mathscr{G}, \theta \in [0,1]$ satisfy $\mathscr{A} \leq \mathscr{G}$, $\mathscr{B} \geq \theta$, $\mathscr{A}^2 + \mathscr{B}^2 \leq 1$, and $\mathscr{G}^2 + \theta^2 \leq 1$. If $U \leq V$, then the following assertions hold for all $e \in D$, with respect to the A_fSs:*

i) $\eta^{(\mathscr{G},\theta)}_{(\mathscr{A},\mathscr{B})}(\mathscr{S}^U)(e) \leq \eta^{(\mathscr{G},\theta)}_{(\mathscr{A},\mathscr{B})}(\mathscr{S}^V)(e)$, *whenever* $\overline{\mathscr{S}}^{U^{\mathscr{B}}_{\mathscr{A}}}(e) = \overline{\mathscr{S}}^{V^{\mathscr{B}}_{\mathscr{A}}}(e)$

ii) $\eta^{(\mathscr{G},\theta)}_{(\mathscr{A},\mathscr{B})}(\mathscr{S}^U)(e) \geq \eta^{(\mathscr{G},\theta)}_{(\mathscr{A},\mathscr{B})}(\mathscr{S}^V)(e)$, *whenever* $\underline{\mathscr{S}}^{U^{\mathscr{B}}_{\mathscr{A}}}(e) = \underline{\mathscr{S}}^{V^{\mathscr{B}}_{\mathscr{A}}}(e)$.

Proof.
The proof can be directly inferred from Theorem 2.4.9. □

Theorem 2.4.11. *Let $(\mathscr{S}, D)$ be a S_fR_fR on a non-empty set ξ, $U \in P_yF_zS(\xi)$, and $\mathscr{A}, \mathscr{B}, \mathscr{G}, \theta \in [0,1]$ such that $\mathscr{A} \leq \mathscr{G}$, $\mathscr{B} \geq \theta$, and $\mathscr{A}^2 + \mathscr{B}^2 \leq 1$, $\mathscr{G}^2 + \theta^2 \leq 1$. Suppose $(\mathscr{S}_1, D)$ is a S_fE_qR on ξ such that $\mathscr{S}(e) \subseteq \mathscr{S}_1(e)$ for all $e \in D$. Then with respect to the A_fSs, we have:*

$$\delta^{(\mathscr{G},\theta)}_{(\mathscr{A},\mathscr{B})}(\mathscr{S}^U)(e) \geq \delta^{(\mathscr{G},\theta)}_{(\mathscr{A},\mathscr{B})}({\mathscr{S}_1}^V)(e)$$

for all $e \in D$.

Proof.
Let $U \in P_yF_zS(\xi)$ and consider $(\mathscr{S}, D)$ and $(\mathscr{S}_1, D)$ as two S_fE_qRs on ξ such that $\mathscr{S}(e) \subseteq \mathscr{S}_1(e)$ for all $e \in D$. According to Theorem 2.1.4, we have $\underline{\mathscr{S}}^U(e) \geq \underline{\mathscr{S}_1}^U(e)$ and $\overline{\mathscr{S}}^U(e) \leq \overline{\mathscr{S}_1}^U(e)$. Using Lemma 2.4.2, it follows that $\underline{\mathscr{S}}^{U^{\theta}_{\mathscr{G}}}(e) \supseteq \underline{\mathscr{S}_1}^{U^{\theta}_{\mathscr{G}}}(e)$ and $\overline{\mathscr{S}}^{U^{\mathscr{B}}_{\mathscr{A}}}(e) \subseteq \overline{\mathscr{S}_1}^{U^{\mathscr{B}}_{\mathscr{A}}}(e)$, which implies $|\underline{\mathscr{S}}^{U^{\theta}_{\mathscr{G}}}(e)| \geq |\underline{\mathscr{S}_1}^{U^{\theta}_{\mathscr{G}}}(e)|$ and $|\overline{\mathscr{S}}^{U^{\mathscr{B}}_{\mathscr{A}}}(e)| \leq |\overline{\mathscr{S}_1}^{U^{\mathscr{B}}_{\mathscr{A}}}(e)|$. Dividing these inequalities, we obtain $\frac{|\underline{\mathscr{S}}^{U^{\theta}_{\mathscr{G}}}(e)|}{|\overline{\mathscr{S}}^{U^{\mathscr{B}}_{\mathscr{A}}}(e)|} \geq \frac{|\underline{\mathscr{S}_1}^{U^{\theta}_{\mathscr{G}}}(e)|}{|\overline{\mathscr{S}_1}^{U^{\mathscr{B}}_{\mathscr{A}}}(e)|}$.

Therefore, $\delta^{(\mathscr{G},\theta)}_{(\mathscr{A},\mathscr{B})}(\mathscr{S}^U)(e) \geq \delta^{(\mathscr{G},\theta)}_{(\mathscr{A},\mathscr{B})}({\mathscr{S}_1}^U)(e)$, for all $e \in D$. □

Corollary 2.4.12. *Let $(\mathscr{S}, D)$ be a S_fR_fR on a non-empty set ξ, $U \in P_yF_zS(\xi)$, and $\mathscr{A}, \mathscr{B}, \mathscr{G}, \theta \in [0,1]$ such that $\mathscr{A} \leq \mathscr{G}$, $\mathscr{B} \geq \theta$, and $\mathscr{A}^2 + \mathscr{B}^2 \leq 1$, $\mathscr{G}^2 + \theta^2 \leq 1$. Suppose $(\mathscr{S}_1, D)$ is a S_fE_qR on ξ such that $\mathscr{S}(e) \subseteq \mathscr{S}_1(e)$ for all $e \in D$. Then $\eta^{(\mathscr{G},\theta)}_{(\mathscr{A},\mathscr{B})}(\mathscr{S}^U)(e) \geq \eta^{(\mathscr{G},\theta)}_{(\mathscr{A},\mathscr{B})}({\mathscr{S}_1}^U)(e)$ with respect to the A_fSs.*

Proof.

The proof can be derived using the same approach as in Theorem 2.4.11. □

2.5 Application of proposed approach in Decision-Making

In decision-making problems, different experts have produced different evaluation results. Yager [61] introduced the P_yF_zS and described some of its operations. So far, many researchers have accomplished numerous works in P_yF_zS Theory and many applications have appeared in different aspects. Peng et al. [47] presented the idea of a $P_yF_zS_fS$, basic operations, and provided its application. Kanwal and Shabir [31] introduced the concepts of L_oA_p and U_pA_p of a F_zS in a S_mG using S_fB_nRs (S_fB_nR), applying them to practical problems. Hussain et al. [25] extended this framework by introducing $P_yF_zS_fR_fS$ and $S_fR_fP_yF_zS$, offering decision-making methods based on S_fB_nR.

In S_fB_nR theory, parameterized families of B_nRs on a universe are versatile for decision-making approaches, extending beyond traditional Binary Relations on sets. Rough approximations in the context of S_fB_nRs can handle diverse B_nRs, whereas Pawlak's R_fS theory deals with single B_nR.

We propose an alternative approach for decision-making problems using $P_yF_zS_fR_fS$ theory with S_fB_nRs, building upon the methodologies of Kanwal and Shabir [31] and Hussain et al. [25]. This approach utilizes only the information inherent in the decision problem data, avoiding the need for additional subjective inputs from decision-makers or other sources. Consequently, it minimizes the impact of subjective information on decision outcomes, ensuring greater objectivity and reducing the possibility of contradictory results arising from different experts tackling the same decision problem.

The rough L_oA_p and U_pA_p approximations are crucial as they closely estimate the set of the universe. Therefore, we derive the nearest values $\underline{\mathscr{S}}^U(e)(\Bbbk_i)$ and $\overline{\mathscr{S}}^U(e)(\Bbbk_i)$ with respect to the A_fSs for each decision alternative $\Bbbk_i \in \xi$ of the universe ξ, using the P_yF_zS's L_oA_p and

U_pA_p approximations of U. So, we define the choice-value δ_i for the decision alternative $\Bbbk_i$ on the universe ξ with respect to the A_fSs as follows:

$$\delta_i = \Sigma_{j=1}^{n} \underline{f}(\underline{r}^U(e_j)(\Bbbk_i), \underline{\theta}^U(e_j)(\Bbbk_i)) + \Sigma_{j=1}^{n} \overline{f}(\overline{r}^U(e_j)(\Bbbk_i), \overline{\theta}^U(e_j)(\Bbbk_i))$$

where

$$\begin{aligned}
\underline{f}(\underline{r}^U(e_j)(\Bbbk_i), \underline{\theta}^U(e_j)(\Bbbk_i)) &= \frac{1}{2} + \underline{r}^U(e_j)(\Bbbk_i)\left(\frac{1}{2} - \frac{2\underline{\theta}^U(e_j)(\Bbbk_i)}{\pi}\right) \\
\cos(\underline{\theta}^U(e_j)(\Bbbk_i)) &= \frac{\underline{\mathscr{S}}^{U_Y}(e_j)(\Bbbk_i)}{\underline{r}^U(e_j)(\Bbbk_i)} \\
(\underline{r}^U(e_j)(\Bbbk_i))^2 &= (\underline{\mathscr{S}}^{U_Y}(e_j)(\Bbbk_i))^2 + (\underline{\mathscr{S}}^{U_N}(e_j)(\Bbbk_i))^2 \\
\text{and} \qquad \overline{f}(\overline{r}^U(e_j)(\Bbbk_i), \overline{\theta}^U(e_j)(\Bbbk_i)) &= \frac{1}{2} + \overline{r}^U(e_j)(\Bbbk_i)\left(\frac{1}{2} - \frac{2\overline{\theta}^U(e_j)(\Bbbk_i)}{\pi}\right) \\
\cos(\overline{\theta}^U(e_j)(\Bbbk_i)) &= \frac{\overline{\mathscr{S}}^{U_Y}(e_j)(\Bbbk_i)}{\overline{r}^U(e_j)(\Bbbk_i)} \\
(\overline{r}^U(e_j)(\Bbbk_i))^2 &= (\overline{\mathscr{S}}^{U_Y}(e_j)(\Bbbk_i))^2 + (\overline{\mathscr{S}}^{U_N}(e_j)(\Bbbk_i))^2
\end{aligned}$$

The object $\Bbbk_i \in \xi$ with the maximum choice-value δ_i is considered the optimal decision for the given decision-making problem, while the object $\Bbbk_i \in \xi$ with the minimum value of δ_i is regarded as the worst decision. If there are multiple objects $\Bbbk_i \in \xi$ with the same maximum (or minimum) choice-value δ_i, one of them is chosen randomly as the optimal (or worst) decision for the problem at hand.

Algorithm 1

1: Compute the upper $P_yF_zS_fS$ approximation $\overline{\mathscr{S}}^U$ and lower $P_yF_zS_fS$ approximation $\underline{\mathscr{S}}^U$ of a P_yF_zS U with respect to the A_fSs.
2: Compute lower score function $\underline{f}(\underline{r}^U(e_j)(\Bbbk_i), \underline{\theta}^U(e_j)(\Bbbk_i))$ and upper score function $\overline{f}(\overline{r}^U(e_j)(\Bbbk_i), \overline{\theta}^U(e_j)(\Bbbk_i))$.
3: Compute the choice value

$$\delta_i = \Sigma_{j=1}^{|D|}\underline{f}(\underline{r}^U(e_j)(\Bbbk_i), \underline{\theta}^U(e_j)(\Bbbk_i)) + \Sigma_{j=1}^{|D|}\overline{f}(\overline{r}^U(e_j)(\Bbbk_i), \overline{\theta}^U(e_j)(\Bbbk_i))$$

4: The best decision is $\Bbbk_m \in \xi$ if $\delta_m = max_i\delta_i$, $i = 1, 2, 3, ... \mid \xi \mid$.
5: The bad decision is $\Bbbk_m \in \xi$ if $\delta_m = min_i\delta_i$, $i = 1, 2, 3, ... \mid \xi \mid$.
6: If m has multiple values, select any $\Bbbk_m$ as the optimal or least favorable alternative.

Algorithm 2

1: Compute the upper $P_yF_zS_fS$ approximation ${}^U\overline{\mathscr{S}}$ and lower $P_yF_zS_fS$ approximation ${}^U\underline{\mathscr{S}}$ of a P_yF_zS U with respect to the F_rS.
2: Compute lower score function $\underline{f}({}^U\underline{r}(e_j)(\mathfrak{k}_i), {}^U\underline{\theta}(e_j)(\mathfrak{k}_i))$ and upper score function $\overline{f}({}^U\overline{r}(e_j)(\mathfrak{k}_i), {}^U\overline{\theta}(e_j)(\mathfrak{k}_i))$.
3: Compute the choice value

$$\delta_i = \Sigma_{j=1}^{|D|}\underline{f}({}^U\underline{r}(e_j)(\mathfrak{k}_i), {}^U\underline{\theta}(e_j)(\mathfrak{k}_i)) + \Sigma_{j=1}^{|D|}\overline{f}({}^U\overline{r}(e_j)(\mathfrak{k}_i), {}^U\overline{\theta}(e_j)(\mathfrak{k}_i))$$

4: The best decision is $\mathfrak{k}_m \in \xi$ if $\delta_m = max_i\delta_i$, $i = 1, 2, 3, ... \mid \xi \mid$.
5: The bad decision is $\mathfrak{k}_m \in \xi$ if $\delta_m = min_i\delta_i$, $i = 1, 2, 3, ... \mid \xi \mid$.
6: If m has multiple values, select any $\Bbbk_m$ as the optimal or least favorable alternative.

2.5.1 An Application of the Decision-Making Approach:

Here, we demonstrate the steps of the decision-making methodology using an example to choose a car.

Example 2.5.1. *Ministry of Foreign Affairs, Pakistan needs to purchase a car for the reception of international guests. Foreign Minister is being provided with the following list of models and colors of cars available in three different showrooms in Islamabad, Pakistan.*
Let $\xi_1 = \{\textit{the set of available car models}\} = \{k_1, k_2, k_3, k_4, k_5, k_6\}$, $\xi_2 = \{\textit{the set of available colors of car}\}$ $\{t_1, t_2, t_3, t_4\}$, *and let the set of parameters* $D = \{\textit{showrooms}\} = \{e_1, e_2, e_3\}$.
Define a $S_f B_n R$ $\mathscr{S} : D \to P(\xi_1 \times \xi_2)$ *by*

$$\mathscr{S}(e_1) = \begin{array}{c} \\ t_1 \\ t_2 \\ t_3 \\ t_4 \end{array}\!\!\begin{array}{c} \begin{array}{cccccc} k_1 & k_2 & k_3 & k_4 & k_5 & k_6 \end{array} \\ \begin{pmatrix} 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \end{pmatrix} \end{array}, \quad \mathscr{S}(e_2) = \begin{array}{c} \\ t_1 \\ t_2 \\ t_3 \\ t_4 \end{array}\!\!\begin{array}{c} \begin{array}{cccccc} k_1 & k_2 & k_3 & k_4 & k_5 & k_6 \end{array} \\ \begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 \end{pmatrix} \end{array}$$

$$\textit{and} \quad \mathscr{S}(e_3) = \begin{array}{c} \\ t_1 \\ t_2 \\ t_3 \\ t_4 \end{array}\!\!\begin{array}{c} \begin{array}{cccccc} k_1 & k_2 & k_3 & k_4 & k_5 & k_6 \end{array} \\ \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 1 & 1 \end{pmatrix} \end{array}$$

represent the relation between models and colors available at showrooms e_1, e_2, e_3, *respectively.*
Minister gives preference for the models and colours in the form of two $P_y F_z S s$.
Let U *and* V *be two* $P_y F_z S s$ *in* ξ_2 *and* ξ_1, *respectively, where* U *represents the preference of colors and* V *represents the preference of models by the respected Minister, such that:*

$$U = \{\langle t_1, 0.9, 0.2\rangle, \langle t_2, 0.8, 0.65\rangle, \langle t_3, 0.4, 0.7\rangle, \langle t_4, 0.42, 0.78\rangle\},$$

$$V = \{\langle k_1, 0.74, 0.32\rangle, \langle k_2, 0.7, 0.4\rangle, \langle k_3, 0.5, 0.6\rangle, \langle k_4, 0.2, 0.7\rangle, \langle k_5, 0.31, 0.45\rangle, \langle k_6, 0.4, 0.3\rangle\}.$$

Table 2.5: Approximations of P_yF_zS U with respect to A_fSs

	$\overline{\mathscr{S}}^{\mathbf{U}}(e_1)(k_i)$	$\underline{\mathscr{L}}^{\mathbf{U}}(e_1)(k_i)$	$\overline{\mathscr{S}}^{\mathbf{U}}(e_2)(k_i)$	$\underline{\mathscr{L}}^{\mathbf{U}}(e_2)(k_i)$	$\overline{\mathscr{S}}^{\mathbf{U}}(e_3)(k_i)$	$\underline{\mathscr{L}}^{\mathbf{U}}(e_3)(k_i)$
$k_{\mathbf{1}}$	$(0.9, 0.2)$	$(0.4, 0.7)$	$(0.4, 0.7)$	$(0.4, 0.7)$	$(0.8, 0.65)$	$(0.8, 0.65)$
$k_{\mathbf{2}}$	$(0.8, 0.65)$	$(0.42, 0.78)$	$(0.4, 0.7)$	$(0.4, 0.7)$	$(0.42, 0.78)$	$(0.42, 0.78)$
$k_{\mathbf{3}}$	$(0.9, 0.2)$	$(0.9, 0.2)$	$(0.42, 0.78)$	$(0.42, 0.78)$	$(0.9, 0.2)$	$(0.4, 0.7)$
$k_{\mathbf{4}}$	$(0.8, 0.65)$	$(0.4, 0.78)$	$(0.9, 0.2)$	$(0.9, 0.2)$	$(0.8, 0.65)$	$(0.8, 0.65)$
$k_{\mathbf{5}}$	$(0.42, 0.7)$	$(0.4, 0.78)$	$(0.9, 0.2)$	$(0.9, 0.2)$	$(0.42, 0.7)$	$(0.4, 0.78)$
$k_{\mathbf{6}}$	$(0.8, 0.65)$	$(0.8, 0.65)$	$(0.8, 0.65)$	$(0.4, 0.7)$	$(0.8, 0.65)$	$(0.8, 0.65)$

Table 2.6: Approximations of P_yF_zS V with respect to F_rS

	${}^{\mathbf{V}}\overline{\mathscr{S}}(e_1)(t_i)$	${}^{\mathbf{V}}\underline{\mathscr{L}}(e_1)(t_i)$	${}^{\mathbf{V}}\overline{\mathscr{S}}(e_2)(t_i)$	${}^{\mathbf{V}}\underline{\mathscr{L}}(e_2)(t_i)$	${}^{\mathbf{V}}\overline{\mathscr{S}}(e_3)(t_i)$	${}^{\mathbf{V}}\underline{\mathscr{L}}(e_3)(t_i)$
$t_{\mathbf{1}}$	$(0.74, 0.32)$	$(0.5, 0.6)$	$(0.31, 0.45)$	$(0.2, 0.7)$	$(0.5, 0.6)$	$(0.5, 0.6)$
$t_{\mathbf{2}}$	$(0.74, 0.3)$	$(0.2, 0.7)$	$(0.4, 0.3)$	$(0.4, 0.3)$	$(0.74, 0.3)$	$(0.2, 0.7)$
$t_{\mathbf{3}}$	$(0.74, 0.32)$	$(0.2, 0.7)$	$(0.74, 0.3)$	$(0.4, 0.4)$	$(0.5, 0.45)$	$(0.31, 0.6)$
$t_{\mathbf{4}}$	$(0.7, 0.4)$	$(0.2, 0.7)$	$(0.5, 0.6)$	$(0.5, 0.6)$	$(0.7, 0.4)$	$(0.31, 0.45)$

Table 2.7: Choice-values with respect to A_fSs

	$\hbar_1$	$\hbar_2$	$\hbar_3$	$\hbar_4$	$\hbar_5$	$\hbar_6$
$\overline{\mathbf{f}}(\overline{\mathbf{r}}^{\mathbf{U}}(\mathbf{e_1})(\hbar_i), \overline{\theta}^{\mathbf{U}}(\mathbf{e_1})(\hbar_i))$	0.83	0.57	0.83	0.57	0.37	0.57
$\underline{\mathbf{f}}(\underline{\mathbf{r}}^{\mathbf{U}}(\mathbf{e_1})(\hbar_i), \underline{\theta}^{\mathbf{U}}(\mathbf{e_1})(\hbar_i))$	0.36	0.34	0.83	0.33	0.33	0.57
$\overline{\mathbf{f}}(\overline{\mathbf{r}}^{\mathbf{U}}(\mathbf{e_2})(\hbar_i), \overline{\theta}^{\mathbf{U}}(\mathbf{e_2})(\hbar_i))$	0.36	0.36	0.34	0.83	0.83	0.57
$\underline{\mathbf{f}}(\underline{\mathbf{r}}^{\mathbf{U}}(\mathbf{e_2})(\hbar_i), \underline{\theta}^{\mathbf{U}}(\mathbf{e_2})(\hbar_i))$	0.36	0.36	0.34	0.83	0.83	0.36
$\overline{\mathbf{f}}(\overline{\mathbf{r}}^{\mathbf{U}}(\mathbf{e_3})(\hbar_i), \overline{\theta}^{\mathbf{U}}(\mathbf{e_3})(\hbar_i))$	0.57	0.34	0.83	0.57	0.37	0.57
$\underline{\mathbf{f}}(\underline{\mathbf{r}}^{\mathbf{U}}(\mathbf{e_3})(\hbar_i), \underline{\theta}^{\mathbf{U}}(\mathbf{e_3})(\hbar_i))$	0.57	0.34	0.36	0.57	0.33	0.57
$\delta_{\mathbf{i}}$	3.05	2.31	3.53	3.7	3.06	3.21

Table 2.8: Choice-values with respect to F_rS

	t_1	t_2	t_3	t_4
$\overline{\mathbf{f}}({}^{\mathbf{V}}\overline{\mathbf{r}}(\mathbf{e_1})(t_i), {}^{\mathbf{V}}\overline{\theta}(\mathbf{e_1})(t_i))$	0.69	0.7	0.69	0.64
$\underline{\mathbf{f}}({}^{\mathbf{V}}\underline{\mathbf{r}}(\mathbf{e_1})(t_i), {}^{\mathbf{V}}\underline{\theta}(\mathbf{e_1})(t_i))$	0.46	0.27	0.27	0.27
$\overline{\mathbf{f}}({}^{\mathbf{V}}\overline{\mathbf{r}}(\mathbf{e_2})(t_i), {}^{\mathbf{V}}\overline{\theta}(\mathbf{e_2})(t_i))$	0.43	0.55	0.7	0.46
$\underline{\mathbf{f}}({}^{\mathbf{V}}\underline{\mathbf{r}}(\mathbf{e_2})(t_i), {}^{\mathbf{V}}\underline{\theta}(\mathbf{e_2})(t_i))$	0.27	0.55	0.5	0.46
$\overline{\mathbf{f}}({}^{\mathbf{V}}\overline{\mathbf{r}}(\mathbf{e_3})(t_i), {}^{\mathbf{V}}\overline{\theta}(\mathbf{e_3})(t_i))$	0.46	0.7	0.52	0.64
$\underline{\mathbf{f}}({}^{\mathbf{V}}\underline{\mathbf{r}}(\mathbf{e_3})(t_i), {}^{\mathbf{V}}\underline{\theta}(\mathbf{e_3})(t_i))$	0.46	0.26	0.38	0.44
$\delta'_{\mathbf{i}}$	2.77	3.03	3.06	2.91

Table 2.5 and Table 2.6 present the approximations of P_yF_zSs U and V with respect to A_fSs and F_rS, respectively. Moving on to ***step*** 3 *of Algorithms 1 and 2, we compute choice-values δ and δ' as shown in Table 2.7 and Table 2.8.*

From Table 2.7, it is evident that the maximum value of $\delta_4 = 3.7$*, achieved by model* $\mathfrak{k}_4$*, indicating a decision in favor of model* $\mathfrak{k}_4$*. Similarly, in Table 2.8, the maximum value of* $\delta'_3 = 3.06$*, attained by color* $\mathfrak{t}_3$*, leading to a decision favoring color* $\mathfrak{t}_3$*.*
Therefore, the optimal choices are model $\mathfrak{k}_4$ *and color* $\mathfrak{t}_3$*, which the Foreign Minister will select. Consequently, the preferred showroom is* e_1*.*

In this chapter, we introduced the use of Soft Binary Relations to define the lower and upper approximations of Pythagorean Fuzzy Sets. The proposed methods were demonstrated through algorithms and examples, highlighting their utility in solving decision-making problems. The results showcased the flexibility and effectiveness of these approximations in computational intelligence. These methods extend the theoretical framework of Pythagorean Fuzzy Sets and pave the way for practical applications in decision support systems and future research in related fields.

Chapter 3

Rough q-Rung Orthopair Fuzzy Sets and their applications in decision-making

This chapter is structured as follows:

In Section 3.1, we explore the lower approximation (L_oA_p) and upper approximation (U_pA_p) of q-Rung Orthopair Fuzzy sets (${}^qROF_zS$s) using Crisp Binary Relations (C_rB_nRs) in relation to F_rS and A_fSs, along with a proof of their properties.

Section 3.2 introduces two types of q-Rung Orthopair Topological Spaces (${}^qROF_zT_pS$s) induced by C_rB_nRs.

In Section **??**, we delve into similarity relations between ${}^qROF_zS$s based on Binary Relations.

Next, Section 3.4 introduces the concepts of roughness and $A_cR_cD_g$s for q-Rung Orthopair M_mD_gs with respect to F_rS and A_fSs.

Section 3.5 presents an algorithm designed for solving decision-making problems using ${}^qROF_zS$s, followed by an illustrative example demonstrating the proposed method and its application in decision-making problems.

3.1 Rough q-Rung Orthopair Fuzzy Sets

In this section, we apply a Crisp Binary Relation from ξ_1 to ξ_2 to approximate a ${}^qROF_zS$ across ξ_1 using F_rS, resulting in two ${}^qROF_zS$s over ξ_2. Similarly, we approximate a ${}^qROF_zS$ over ξ_2 using A_fSs, leading to two ${}^qROF_zS$s over ξ_1. Additionally, we discuss several of their properties.

Definition 3.1.1. *Let J be a Crisp Binary Relation (C_rB_nR) from ξ_1 to ξ_2 and $U = \{\langle ƭ, U_Y(ƭ), U_N(ƭ)\rangle : ƭ \in \xi_2\}$ be a ${}^qROF_zS$ in ξ_2. Then we define L_oA_p $\underline{J}^U = (\underline{J}^{U_Y}, \underline{J}^{U_N})$ and U_pA_p $\overline{J}^U = (\overline{J}^{U_Y}, \overline{J}^{U_N})$ of $U = \{\langle ƭ, U_Y(ƭ), U_N(ƭ)\rangle : ƭ \in \xi_2\}$ with respect to A_fSs as follows:*

$$\underline{J}^U(ƙ) = \begin{cases} \left(\bigwedge_{ƭ\in ƙJ} U_Y(ƭ), \bigvee_{ƭ\in ƙJ} U_N(ƭ)\right) & if \quad ƙJ \neq \emptyset; \\ (1,0) & if \quad ƙJ = \emptyset. \end{cases}$$

and

$$\overline{J}^{U_Y}(ƙ) = \begin{cases} \left(\bigvee_{ƭ\in ƙJ} U_Y(ƭ), \bigwedge_{ƭ\in ƙJ} U_N(ƭ)\right) & if \quad ƙJ \neq \emptyset; \\ (0,1) & if \quad ƙJ = \emptyset. \end{cases}$$

where $ƙJ = \{ƭ \in \xi_2 : (ƙ, ƭ) \in J\}$, and is called the A_fS of $ƙ$ for all $ƙ \in \xi_1$.

It can be easily verified that $\underline{J}^U$, $\overline{J}^U$ are ${}^qROF_z$ subsets of ξ_1. And $\underline{J}^U, \overline{J}^U : {}^qROF_zS(\xi_2) \to {}^qROF_zS(\xi_1)$ are L_o and U_p rough ${}^qROF_z$ approximation operators, respectively, and the pair $(\underline{J}^U, \overline{J}^U)$ is called the rough ${}^qROF_zS$ with respect to A_fSs.

Definition 3.1.2. *Let J be a C_rB_nR from ξ_1 to ξ_2 and $U = \{\langle ƙ, U_Y(ƙ), U_N(ƙ)\rangle : ƙ \in \xi_1\}$ be a ${}^qROF_zS$ in ξ_1. Then we define L_oA_p ${}^U\underline{J} = ({}^{U_Y}\underline{J}, {}^{U_N}\underline{J})$ and U_pA_p ${}^U\overline{J} = ({}^{U_Y}\overline{J}, {}^{U_N}\overline{J})$ of $U = \{\langle ƙ, U_Y(ƙ), U_N(ƙ)\rangle : ƙ \in \xi_1\}$ with respect to F_rS as follows:*

$${}^U\underline{J}(ƭ) = \begin{cases} \left(\bigwedge_{ƙ\in Jƭ} U_Y(ƙ), \bigvee_{ƙ\in Jƭ} U_N(ƙ)\right) & if \quad Jƭ \neq \emptyset; \\ (1,0) & if \quad Jƭ = \emptyset. \end{cases}$$

and

$$^{U}\overline{J}(f) = \begin{cases} \left(\bigvee_{k \in Jf} U_Y(k), \bigwedge_{k \in Jf} U_N(k)\right) & if \quad Jf \neq \emptyset; \\ (0,1) & if \quad Jf = \emptyset. \end{cases}$$

where $Jf = \{k \in \xi_1 : (k, f) \in J\}$, *and is called the* F_rS *of* f *for all* $f \in \xi_2$.

It can be easily verified that $^{U}\underline{J}$, $^{U}\overline{J}$ are $^{q}ROF_z$ subsets of ξ_2. And $^{U}\underline{J}$, $^{U}\overline{J} : {}^{q}ROF_zS(\xi_1) \rightarrow {}^{q}ROF_zS(\xi_2)$ are L_o and U_p rough $^{q}ROF_z$ approximation operators, respectively, and the pair $(^{U}\underline{J}, {}^{U}\overline{J})$ is called the rough $^{q}ROF_zS$ with respect to F_rS. The example below demonstrates these concepts.

Example 3.1.3. *Suppose a student is considering buying a new laptop.*
Consider $\xi_1 = \{$*the set of available models*$\} = \{k_1, k_2, k_3, k_4\}$ *and* $\xi_2 = \{$*the colors of laptops*$\} = \{f_1, f_2, f_3\}$. *Let* $J \subseteq \xi_1 \times \xi_2$ *be defined as:*

$$J = \begin{array}{c} \\ f_1 \\ f_2 \\ f_3 \end{array} \begin{array}{c} \begin{array}{cccc} k_1 & k_2 & k_3 & k_4 \end{array} \\ \begin{pmatrix} 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 \end{pmatrix} \end{array}$$

representing the relation between laptop models and available colors at the shop.
Now consider for $q \geq 5$, $U \in {}^{q}ROF_zS(\xi_2)$ *and* $V \in {}^{q}ROF_zS(\xi_1)$, *where* U *represents the preference for colors and* V *represents the preference for models as specified by the student. They are defined as:*

$$U = \{\langle f_1, 0.9, 0.8\rangle, \langle f_2, 0.8, 0.6\rangle, \langle f_3, 0.6, 0.7\rangle\},$$
$$V = \{\langle k_1, 0.9, 0.5\rangle, \langle k_2, 0.7, 0.5\rangle, \langle k_3, 0.7, 0.6\rangle, \langle k_4, 0.4, 0.8\rangle\}.$$

Then L_oA_p *and* U_pA_ps *of* $^{q}ROF_zS$ U *with respect to* A_fSs k_iJ *are two* $^{q}ROF_zSs$ *on* ξ_1, *given by;*
$\underline{J}^U = \{\langle k_1, 0.8, 0.8\rangle, \langle k_2, 0.6, 0.8\rangle, \langle k_3, 0.6, 0.8\rangle, \langle k_4, 0.6, 0.7\rangle\}$,
$\overline{J}^U = \{\langle k_1, 0.9, 0.6\rangle, \langle k_2, 0.9, 0.7\rangle, \langle k_3, 0.9, 0.6\rangle, \langle k_4, 0.8, 0.6\rangle\}$.
Thus, $(\underline{J}^U, \overline{J}^U)$ *is a rough* $^{q}ROF_zS$ *with respect to* A_fSs.
Similarly, the L_oA_p *and* U_pA_ps *of* $^{q}ROF_zS$ V *with respect to* F_rS Jf_i *are two* $^{q}ROF_zSs$ *on* ξ_2,

given by;

${}^{V}\underline{J} = \{\langle \mathfrak{t}_1, 0.7, 0.6\rangle, \langle \mathfrak{t}_2, 0.4, 0.8\rangle, \langle \mathfrak{t}_3, 0.4, 0.8\rangle\}, \quad {}^{V}\overline{J} = \{\langle \mathfrak{t}_1, 0.9, 0.5\rangle, \langle \mathfrak{t}_2, 0.9, 0.5\rangle, \langle \mathfrak{t}_3, 0.7, 0.5\rangle\}.$

Thus, $({}^{V}\underline{J}, {}^{V}\overline{J})$ *is a rough* ${}^{q}ROF_zS$ *with respect to* F_rS.

Theorem 3.1.4. *Let* J *be a* C_rB_nR *from* ξ_1 *to* ξ_2*, that is,* $J \in P(\xi_1 \times \xi_2)$*. For any three* ${}^{q}ROF_zSs$ $U = \{\langle \mathfrak{t}, U_Y(\mathfrak{t}), U_N(\mathfrak{t})\rangle : \mathfrak{t} \in \xi_2\}$, $U_1 = \{\langle \mathfrak{t}, U_{1_Y}(\mathfrak{t}), U_{1_N}(\mathfrak{t})\rangle : \mathfrak{t} \in \xi_2\}$, *and* $U_2 = \{\langle \mathfrak{t}, U_{2_Y}(\mathfrak{t}), U_{2_N}(\mathfrak{t})\rangle : \mathfrak{t} \in \xi_2\}$ *of* ξ_2*, we have the following:*

i) $U_1 \subseteq U_2$ *implies* $\underline{J}^{U_1} \subseteq \underline{J}^{U_2}$

ii) $U_1 \subseteq U_2$ *implies* $\overline{J}^{U_1} \subseteq \overline{J}^{U_2}$

iii) $\underline{J}^{U_1 \cap U_2} = \underline{J}^{U_1} \cap \underline{J}^{U_2}$

iv) $\overline{J}^{U_1 \cap U_2} \subseteq \overline{J}^{U_1} \cap \overline{J}^{U_2}$

v) $\underline{J}^{U_1} \cup \underline{J}^{U_2} \subseteq \underline{J}^{U_1 \cup U_2}$

vi) $\overline{J}^{U_1} \cup \overline{J}^{U_2} = \overline{J}^{U_1 \cup U_2}$

vii) $\overline{J}^{1_{\xi_2}} = \underline{J}^{1_{\xi_2}} = 1_{\xi_1}$*, if* $\mathfrak{k}J \neq \emptyset$

viii) $\underline{J}^{U} = (\overline{J}^{U^c})^c$ *and* $\overline{J}^{U} = (\underline{J}^{U^c})^c$*, if* $\mathfrak{k}J \neq \emptyset$

ix) $\underline{J}^{0_{\xi_2}} = 0_{\xi_1} = \overline{J}^{0_{\xi_2}}$.

Proof.

i) Let $U_1 \subseteq U_2$, that is, for all $\mathfrak{t} \in \xi_2$, $U_{1_Y}(\mathfrak{t}) \leq U_{2_Y}(\mathfrak{t})$, and $U_{1_N}(\mathfrak{t}) \geq U_{2_N}(\mathfrak{t})$.
If $\mathfrak{k}J = \emptyset$, then $\underline{J}^{U_1} = (1, 0) = \underline{J}^{U_2}$.
If $\mathfrak{k}J \neq \emptyset$, then $\underline{J}^{U_{1_Y}}(\mathfrak{k}) = \bigwedge_{\mathfrak{t} \in \mathfrak{k}J} U_{1_Y}(\mathfrak{t}) \leq \bigwedge_{\mathfrak{t} \in \mathfrak{k}J} U_{2_Y}(\mathfrak{t}) = \underline{J}^{U_{2_Y}}(\mathfrak{k})$ and
$\underline{J}^{U_{1_N}}(\mathfrak{k}) = \bigvee_{\mathfrak{t} \in \mathfrak{k}J} U_{1_N}(\mathfrak{t}) \geq \bigvee_{\mathfrak{t} \in \mathfrak{k}J} U_{2_N}(\mathfrak{t}) = \underline{J}^{U_{2_N}}(\mathfrak{k})$.
Thus, $\underline{J}^{U_{1_Y}}(\mathfrak{k}) \leq \underline{J}^{U_{2_Y}}(\mathfrak{k})$ and $\underline{J}^{U_{1_N}}(\mathfrak{k}) \geq \underline{J}^{U_{2_N}}(\mathfrak{k})$. Hence, $\underline{J}^{U_1} \subseteq \underline{J}^{U_2}$.

ii) Let $U_1 \subseteq U_2$, that is, for all $\mathfrak{k} \in \xi_2$, $U_{1_Y}(\mathfrak{k}) \leq U_{2_Y}(\mathfrak{k})$, and $U_{1_N}(\mathfrak{k}) \geq U_{2_N}(\mathfrak{k})$.
If $\mathfrak{k}J = \emptyset$, then $\overline{J}^{U_1} = (0,1) = \overline{J}^{U_2}$.
If $\mathfrak{k}J \neq \emptyset$, then $\overline{J}^{U_{1_Y}}(\mathfrak{k}) = \bigvee_{\mathfrak{k} \in \mathfrak{k}J} U_{1_Y}(\mathfrak{k}) \leq \bigvee_{\mathfrak{k} \in \mathfrak{k}J} U_{2_Y}(\mathfrak{k}) = \overline{J}^{U_{2_Y}}(\mathfrak{k})$ and
$\overline{J}^{U_{1_N}}(\mathfrak{k}) = \bigwedge_{\mathfrak{k} \in \mathfrak{k}J} U_{1_N}(\mathfrak{k}) \geq \bigwedge_{\mathfrak{k} \in \mathfrak{k}J} U_{2_N}(\mathfrak{k}) = \overline{J}^{U_{2_N}}(\mathfrak{k})$.
Thus, $\overline{J}^{U_{1_Y}}(\mathfrak{k}) \leq \overline{J}^{U_{2_Y}}(\mathfrak{k})$ and $\overline{J}^{U_{1_N}}(\mathfrak{k}) \geq \overline{J}^{U_{2_N}}(\mathfrak{k})$. Hence, $\overline{J}^{U_1} \subseteq \overline{J}^{U_2}$.

iii) Consider $(\underline{J}^{U_{1_Y}} \cap \underline{J}^{U_{2_Y}})(\mathfrak{k}) = \underline{J}^{U_{1_Y}}(\mathfrak{k}) \wedge \underline{J}^{U_{2_Y}}(\mathfrak{k}) = (\bigwedge_{\mathfrak{k} \in \mathfrak{k}J} U_{1_Y}(\mathfrak{k})) \wedge (\bigwedge_{\mathfrak{k} \in \mathfrak{k}J} U_{2_Y}(\mathfrak{k}))$
$= \bigwedge_{\mathfrak{k} \in \mathfrak{k}J}(U_{1_Y}(\mathfrak{k}) \wedge U_{2_Y}(\mathfrak{k})) = \underline{J}^{U_1 \cap U_2}(\mathfrak{k})$, and
$(\underline{J}^{U_{1_N}} \cup \underline{J}^{U_{2_N}})(\mathfrak{k}) = \underline{J}^{U_{1_N}}(\mathfrak{k}) \vee \underline{J}^{U_{2_N}}(\mathfrak{k}) = (\bigvee_{\mathfrak{k} \in \mathfrak{k}J} U_{1_N}(\mathfrak{k})) \vee (\bigvee_{\mathfrak{k} \in \mathfrak{k}J} U_{2_N}(\mathfrak{k})) = \bigvee_{\mathfrak{k} \in \mathfrak{k}J}(U_{1_N}(\mathfrak{k}) \vee U_{2_N}(\mathfrak{k})) = \underline{J}^{U_1 \cup U_2}(\mathfrak{k})$.
Thus, $\underline{J}^{U_1 \cap U_2} = \underline{J}^{U_1} \cap \underline{J}^{U_2}$.

iv) Given that $U_1 \cap U_2 \subseteq U_1$ and $U_1 \cap U_2 \subseteq U_2$, it follows from part (ii) that $\overline{J}^{U_1 \cap U_2} \subseteq \overline{J}^{U_1}$ and $\overline{J}^{U_1 \cap U_2} \subseteq \overline{J}^{U_2}$. Therefore, we conclude that $\overline{J}^{U_1 \cap U_2} \subseteq \overline{J}^{U_1} \cap \overline{J}^{U_2}$.

v) Since $U_1 \subseteq U_1 \cup U_2$ and $U_2 \subseteq U_1 \cup U_2$, it follows from part (i) that $\underline{J}^{U_1} \subseteq \underline{J}^{U_1 \cup U_2}$ and $\underline{J}^{U_2} \subseteq \underline{J}^{U_1 \cup U_2}$. Therefore, we conclude that $\underline{J}^{U_1} \cup \underline{J}^{U_2} \subseteq \underline{J}^{U_1 \cup U_2}$.

vi) Consider
$(\overline{J}^{U_{1_Y}} \cup \overline{J}^{U_{2_Y}})(\mathfrak{k}) = \overline{J}^{U_{1_Y}}(\mathfrak{k}) \vee \overline{J}^{U_{2_Y}}(\mathfrak{k}) = (\bigvee_{\mathfrak{k} \in \mathfrak{k}J} U_{1_Y}(\mathfrak{k})) \vee (\bigvee_{\mathfrak{k} \in \mathfrak{k}J} U_{2_Y}(\mathfrak{k}))$
$= \bigvee_{\mathfrak{k} \in \mathfrak{k}J}(U_{1_Y}(\mathfrak{k}) \vee U_{2_Y}(\mathfrak{k})) = \overline{J}^{U_1 \cup U_2}(\mathfrak{k})$ and
$(\overline{J}^{U_{1_N}} \cap \overline{R}^{U_{2_N}})(\mathfrak{k}) = \overline{J}^{U_{1_N}}(\mathfrak{k}) \wedge \overline{J}^{U_{2_N}}(\mathfrak{k}) = (\bigwedge_{\mathfrak{k} \in \mathfrak{k}J} U_{1_N}(\mathfrak{k})) \wedge (\bigwedge_{\mathfrak{k} \in \mathfrak{k}J} U_{2_N}(\mathfrak{k})) = \bigwedge_{\mathfrak{k} \in \mathfrak{k}J}(U_{1_N}(\mathfrak{k}) \wedge U_{2_N}(\mathfrak{k})) = \overline{J}^{U_1 \cap U_2}(\mathfrak{k})$.
Thus, $\overline{J}^{U_1 \cup U_2} = \overline{J}^{U_1} \cup \overline{J}^{U_2}$.

vii) Since $\underline{J}^{1_{\xi_2}}(\mathfrak{k}) = \bigwedge_{\mathfrak{k} \in \mathfrak{k}J} 1(w) = \bigwedge_{\mathfrak{k} \in \mathfrak{k}J} 1 = 1$ and $\underline{J}^{0}(\mathfrak{k}) = \bigwedge_{\mathfrak{k} \in \mathfrak{k}J} 0(w) = \bigwedge_{\mathfrak{k} \in \mathfrak{k}J} 0 = 0$.
Thus, $\underline{J}^{1_{\xi_2}} = 1_{\xi_1}$. Similarly, we can prove that $\overline{J}^{1_{\xi_2}} = 1_{\xi_1}$.

viii) Consider
$\overline{J}^{U^c_Y}(\mathfrak{k}) = \bigvee_{\mathfrak{k} \in \mathfrak{k}J} U^c_Y(\mathfrak{k}) = \bigvee_{\mathfrak{k} \in \mathfrak{k}J} U_N(\mathfrak{k}) = \underline{J}^{U_N}(\mathfrak{k}) = (\underline{J}^{U_Y}(\mathfrak{k}))^c$ and
$\overline{J}^{U^c_N}(\mathfrak{k}) = \bigwedge_{\mathfrak{k} \in \mathfrak{k}J} U^c_N(\mathfrak{k}) = \bigwedge_{\mathfrak{k} \in \mathfrak{k}J} U_Y(\mathfrak{k}) = \underline{J}^{U_Y}(\mathfrak{k}) = (\underline{J}^{U_N}(\mathfrak{k}))^c$.

Thus, $\overline{J}^{U^c} = (\overline{J}^{U^c_Y}, \overline{J}^{U^c_N}) = ((\underline{J}^{U_Y})^c, (\underline{J}^{U_N})^c) = (\underline{J}^{U_Y}, \underline{J}^{U_N})^c = (\underline{J}^{U})^c$. Which gives that $(\overline{J}^{U^c})^c = \underline{J}^U$. Similarly, $\overline{J}^U = (\underline{J}^{U^c})^c$.

ix) The proof is straightforward.

$\square$

Theorem 3.1.5. *Let J be a C_rB_nR from ξ_1 to ξ_2, that is, $J \in P(\xi_1 \times \xi_2)$. For any three ${}^qROF_zSs$ $U = \{\langle \Bbbk, U_Y(\Bbbk), U_N(\Bbbk)\rangle : \Bbbk \in \xi_1\}$, $U_1 = \{\langle \Bbbk, U_{1_Y}(\Bbbk), U_{1_N}(\Bbbk)\rangle : \Bbbk \in \xi_1\}$, and $U_2 = \{\langle \Bbbk, U_{2_Y}(\Bbbk), U_{2_N}(\Bbbk)\rangle : \Bbbk \in \xi_1\}$ of ξ_1, we have the following:*

i) $U_1 \subseteq U_2$ *implies* ${}^{U_1}\underline{J} \subseteq {}^{U_2}\underline{J}$

ii) $U_1 \subseteq U_2$ *implies* ${}^{U_1}\overline{J} \subseteq {}^{U_2}\overline{J}$

iii) ${}^{U_1}\underline{J} \cap {}^{U_2}\underline{J} = {}^{U_1 \cap U_2}\underline{J}$

iv) ${}^{U_1 \cap U_2}\overline{J} \subseteq {}^{U_1}\overline{J} \cap {}^{U_2}\overline{J}$

v) ${}^{U_1 \cup U_2}\underline{J} \supseteq {}^{U_1}\underline{J} \cup {}^{U_2}\underline{J}$

vi) ${}^{U_1}\overline{J} \cup {}^{U_2}\overline{J} = {}^{U_1 \cup U_2}\overline{J}$

vii) ${}^{1_{\xi_1}}\underline{J} = 1_{\xi_2} = {}^{1_{\xi_1}}\overline{J}$, *if* $Jf \neq \emptyset$

viii) ${}^{U}\underline{J} = ({}^{U^c}\overline{J})^c$, *and* ${}^{U}\overline{J} = ({}^{U^c}\underline{J})^c$ *if* $Jf \neq \emptyset$

ix) ${}^{0_{\xi_1}}\underline{J} = 0_{\xi_2} = {}^{0_{\xi_1}}\overline{J}$.

Proof.
The proof can be derived using the same approach as in Theorem. 3.1.4. $\square$

The subsequent example shows that the equality fails in parts (iv) and (v) of Theorem 3.1.4.

Example 3.1.6. *Utilizing the information given in Example 3.1.3, define two* $^{q}ROF_zSs$ U_1, U_2 *on* ξ_2 *by:*

$U_1 = \{\langle t_1, 0.8, 0.6\rangle, \langle t_2, 0.1, 0.9\rangle, \langle t_3, 0.4, 0.7\rangle\}$, $U_2 = \{\langle t_1, 0.35, 0.65\rangle, \langle t_2, 0.65, 0.95\rangle, \langle t_3, 0.6, 0.87\rangle\}$ *for* $q = 5$. *Then*

$U_1 \cap U_2 = \{\langle t_1, 0.35, 0.65\rangle, \langle t_2, 0.1, 0.95\rangle, \langle t_3, 0.4, 0.87\rangle\}$, *and*

$U_1 \cup U_2 = \{\langle t_1, 0.8, 0.6\rangle, \langle t_2, 0.65, 0.9\rangle, \langle t_3, 0.6, 0.7\rangle\}$.

Table 3.1: Union of L_oA_ps and L_oA_ps of union of two $^{q}ROF_zS$s

	$\underline{\mathbf{J}}^{U_1}(\Bbbk_i)$	$\underline{\mathbf{J}}^{U_2}(\Bbbk_i)$	$(\underline{\mathbf{J}}^{U_1} \cup \underline{\mathbf{J}}^{U_2})(\Bbbk_i)$	$\underline{\mathbf{J}}^{\mathbf{U_1 \cup U_2}}(\Bbbk_i)$
$\Bbbk_{\mathbf{1}}$	$(0.1, 0.9)$	$(0.35, 0.95)$	$(0.35, 0.9)$	$(0.65, 0.9)$
$\Bbbk_{\mathbf{2}}$	$(0.4, 0.7)$	$(0.35, 0.87)$	$(0.4, 0.7)$	$(0.6, 0.7)$
$\Bbbk_{\mathbf{3}}$	$(0.1, 0.9)$	$(0.35, 0.95)$	$(0.35, 0.9)$	$(0.6, 0.9)$
$\Bbbk_{\mathbf{4}}$	$(0.1, 0.9)$	$(0.6, 0.95)$	$(0.6, 0.9)$	$(0.6, 0.9)$

Table 3.2: Intersection of U_pA_ps and U_pA_ps of intersection of two $^{q}ROF_zS$s

	$\overline{\mathbf{J}}^{U_1}(\Bbbk_i)$	$\overline{\mathbf{J}}^{U_2}(\Bbbk_i)$	$(\overline{\mathbf{J}}^{U_1} \cap \overline{\mathbf{J}}^{U_2})(\Bbbk_i)$	$\overline{\mathbf{J}}^{\mathbf{U_1 \cap U_2}}(\Bbbk_i)$
$\Bbbk_{\mathbf{1}}$	$(0.8, 0.6)$	$(0.65, 0.65)$	$(0.65, 0.65)$	$(0.35, 0.65)$
$\Bbbk_{\mathbf{2}}$	$(0.8, 0.6)$	$(0.6, 0.65)$	$(0.6, 0.65)$	$(0.4, 0.65)$
$\Bbbk_{\mathbf{3}}$	$(0.8, 0.6)$	$(0.65, 0.65)$	$(0.65, 0.65)$	$(0.4, 0.65)$
$\Bbbk_{\mathbf{4}}$	$(0.4, 0.7)$	$(0.65, 0.87)$	$(0.4, 0.87)$	$(0.4, 0.87)$

Now, observing Table 3.1, we can easily see that $(\underline{J}^{U_1} \cup \underline{J}^{U_2})(\Bbbk_1) \neq \underline{J}^{U_1 \cup U_2}(\Bbbk_1)$, $(\underline{J}^{U_1} \cup \underline{J}^{U_2})(\Bbbk_2) \neq \underline{J}^{U_1 \cup U_2}(\Bbbk_2)$, *and* $(\underline{J}^{U_1} \cup \underline{J}^{U_2})(\Bbbk_3) \neq \underline{J}^{U_1 \cup U_2}(\Bbbk_3)$. *Thus, the union of* L_oA_p*s of two* $^{q}ROF_zSs$ *is not equal to the* L_oA_p *of the union of two* $^{q}ROF_zSs$*, that is,* $\underline{J}^{U_1} \cup \underline{J}^{U_2} \neq \underline{J}^{U_1 \cup U_2}$.

Similarly, from Table 3.2, we can see that the intersection of U_pA_p*s of two* $^{q}ROF_zSs$ *is not* *equal to the U_pA_p of the intersection of two ${}^qROF_zSs$, that is, $\overline{J}^{U_1} \cap \overline{J}^{U_2} \neq \overline{J}^{U_1 \cap U_2}$. Hence, equality does not hold in parts (iv) and (v) of Theorem 3.1.4.*

Theorem 3.1.7. *Let J_1 and J_2 be two C_rB_nRs from ξ_1 to ξ_2 such that $J_1 \subseteq J_2$. Then, for any $U \in {}^qROF_zS(\xi_2)$, $\underline{J_2}^U \subseteq \underline{J_1}^U$ and $\overline{J_1}^U \subseteq \overline{J_2}^U$.*

Proof.

Since $J_1 \subseteq J_2$, we have $\mathfrak{k}J_1 \subseteq \mathfrak{k}J_2$.
Now if $\mathfrak{k}J_1 = \emptyset$, then $\underline{J_2}^{U_Y}(\mathfrak{k}) \leq 1 = \underline{J_1}^{U_Y}(\mathfrak{k})$, and $\underline{J_1}^{U_N}(\mathfrak{k}) = 0 \leq \underline{J_2}^{U_N}(\mathfrak{k})$. This implies that $\underline{J_2}^U \subseteq \underline{J_1}^U$.
If $\mathfrak{k}J_1 \neq \emptyset$, then $\underline{J_1}^{U_Y}(\mathfrak{k}) = \bigwedge_{\mathfrak{t} \in \mathfrak{k}J_1} U_Y(\mathfrak{t}) \geq \bigwedge_{\mathfrak{t} \in \mathfrak{k}J_2} U_Y(\mathfrak{t}) = \underline{J_2}^{U_Y}(\mathfrak{k})$, since $\mathfrak{k}J_1 \subseteq \mathfrak{k}J_2$ and $\underline{J_1}^{U_N}(\mathfrak{k}) = \bigvee_{\mathfrak{t} \in \mathfrak{k}J_1} U_N(\mathfrak{t}) \leq \bigvee_{\mathfrak{t} \in \mathfrak{k}J_2} U_N(\mathfrak{t}) = \underline{J_2}^{U_N}(\mathfrak{k})$, since $\mathfrak{k}J_1 \subseteq \mathfrak{k}J_2$. Thus, $\underline{J_2}^U \subseteq \underline{J_1}^U$.
Similarly, $\overline{J_1}^U \subseteq \overline{J_2}^U$. □

Theorem 3.1.8. *Let J_1 and J_2 be two C_rB_nRs from ξ_1 to ξ_2 such that $J_1 \subseteq J_2$. Then, for any $U \in {}^qROF_zS(\xi_1)$, ${}^U\underline{J_2} \subseteq {}^U\underline{J_1}$ and ${}^U\overline{J_1} \subseteq {}^U\overline{J_2}$.*

Proof.

The proof can be derived using the same approach as in Theorem. 3.1.7. □

Theorem 3.1.9. *Let J_1 and J_2 be two C_rB_nRs from ξ_1 to ξ_2. Then, for any $U \in {}^qROF_zS(\xi_2)$, the following are true:*

i) $\underline{J_1}^U \subseteq \underline{(J_1 \cap J_2)}^U$ *and* $\underline{J_2}^U \subseteq \underline{(J_1 \cap J_2)}^U$.

ii) $\overline{(J_1 \cap J_2)}^U \subseteq \overline{J_1}^U$ *and* $\overline{(J_1 \cap J_2)}^U \subseteq \overline{J_2}^U$.

Proof.

Theorem 3.1.7 directly lead to this conclusion. □

Similarly, we have the following.

Theorem 3.1.10. *Let J_1 and J_2 be two C_rB_nRs from ξ_1 to ξ_2. Then, for any $U \in {}^qROF_zS(\xi_1)$:*

i) ${}^U\underline{J_1} \subseteq {}^U\underline{(J_1 \cap J_2)}$, *and* ${}^U\underline{J_2} \subseteq {}^U\underline{(J_1 \cap J_2)}$.

ii) ${}^U\overline{(J_1 \cap J_2)} \subseteq {}^U\overline{J_1}$, *and* ${}^U\overline{(J_1 \cap J_2)} \subseteq {}^U\overline{J_2}$.

Theorem 3.1.11. *Let J be a C_rB_nR from ξ_1 to ξ_2 and let $\{U_i : i \in I\}$ be a family of ${}^qROF_zSs$ defined on ξ_2. Then the following properties hold:*

i) $\underline{J}^{(\bigcap_{i\in I} U_i)} = \bigcap_{i\in I} \underline{J}^{U_i}$

ii) $\bigcup_{i\in I} \underline{J}^{U_i} \subseteq \underline{J}^{(\bigcup_{i\in I} U_i)}$

iii) $\overline{J}^{(\bigcup_{i\in I} U_i)} = \bigcup_{i\in I} \overline{J}^{U_i}$

iv) $\overline{J}^{(\bigcap_{i\in I} U_i)} \subseteq \bigcap_{i\in I} \overline{J}^{U_i}$

Proof.

i) Let $U_i \in {}^qROF_zS(\xi_2)$, for $i \in I$. Then
$\underline{J}^{(\bigcap_{i\in I} U_{i_Y})}(\mathfrak{k}) = \bigwedge_{\mathfrak{t}\in\mathfrak{k}J}(\wedge_{i\in I}U_{i_Y}(\mathfrak{t})) = \bigwedge_{i\in I}(\wedge_{\mathfrak{t}\in\mathfrak{k}J}U_{i_Y}(\mathfrak{t})) = \bigcap_{i\in I} \underline{J}^{U_{i_Y}}(\mathfrak{k})$ and
$\underline{J}^{(\bigcup_{i\in I} U_{i_N})}(\mathfrak{k}) = \bigvee_{\mathfrak{t}\in\mathfrak{k}J}(\vee_{i\in I}U_{i_N}(\mathfrak{t})) = \bigvee_{i\in I}(\vee_{\mathfrak{t}\in\mathfrak{k}J}U_{i_N}(\mathfrak{t})) = \bigcup_{i\in I} \underline{J}^{U_{i_N}}(\mathfrak{k})$.
Thus, $\underline{J}^{(\bigcap_{i\in I} U_i)} = \bigcap_{i\in I} \underline{J}^{U_i}$.

ii) Since $U_i \subseteq \bigcup_{i\in I} U_i$ for each $i \in I$. Then $\underline{J}^{U_i} \subseteq \underline{J}^{(\bigcup_{i\in I} U_i)}$. Which implies that $\bigcup_{i\in I} \underline{J}^{U_i} \subseteq \underline{J}^{(\bigcup_{i\in I} U_i)}$.

iii) The proof can be derived using the same approach as in part (i).

iv) The proof can be derived using the same approach as in part (ii).

□

Theorem 3.1.12. *Let J be a C_rB_nR from ξ_1 to ξ_2 and $\{U_i : i \in I\}$ be a family of ${}^qROF_zSs$ defined on ξ_1. Then:*

i) ${}^{(\bigcap_{i\in I} U_i)}\underline{J} = \bigcap_{i\in I} {}^{U_i}\underline{J}$

ii) $\bigcup_{i\in I} {}^{U_i}\underline{J} \subseteq {}^{(\bigcup_{i\in I} U_i)}\underline{J}$

iii) ${}^{(\bigcup_{i\in I} U_i)}\overline{J} = \bigcup_{i\in I} {}^{U_i}\overline{J}$

iv) ${}^{(\bigcap_{i\in I} U_i)}\overline{J} \subseteq \bigcap_{i\in I} {}^{U_i}\overline{J}$.

Proof.

The proof can be derived using the same approach as in Theorem. 3.1.11. □

Theorem 3.1.13. *Let J be a Reflexive Binary Relation (R_fB_nR) over ξ. For any $U \in {}^qROF_zS(\xi)$, we have:*

i) $\underline{J}^U \leq U \leq \overline{J}^U$

ii) $\underline{J}^U \leq \overline{J}^U$.

Proof.

For $\mathfrak{k} \in \xi$,

i) Consider $\underline{J}^{U_Y}(\mathfrak{k}) = \bigwedge_{\mathfrak{t}\in\mathfrak{k}J} U_Y(\mathfrak{t}) \leq U_Y(\mathfrak{k})$, since $\mathfrak{k} \in \mathfrak{k}J$, and $\underline{J}^{U_N}(\mathfrak{k}) = \bigvee_{\mathfrak{t}\in\mathfrak{k}J} U_N(\mathfrak{t}) \geq U_N(\mathfrak{k})$, since $\mathfrak{k} \in \mathfrak{k}J$. Thus, $\underline{J}^U \leq U$.
Also, $\overline{J}^{U_Y}(\mathfrak{k}) = \bigvee_{\mathfrak{t}\in\mathfrak{k}J} U_Y(\mathfrak{t}) \geq U_Y(\mathfrak{k})$, since $\mathfrak{k} \in \mathfrak{k}J$, and $\overline{J}^{U_N}(\mathfrak{k}) = \bigwedge_{\mathfrak{t}\in\mathfrak{k}J} U_N(\mathfrak{t}) \leq U_N(\mathfrak{k})$, since $\mathfrak{k} \in \mathfrak{k}J$. Thus, $\overline{J}^U \geq U$.

ii) From part (i) we get that $\underline{J}^U \leq U \leq \overline{J}^U$ which implies that $\underline{J}^U \leq \overline{J}^U$.

□

Theorem 3.1.14. *Let J be a R_fB_nR over ξ. For any $U \in {}^qROF_zS(\xi)$, the following properties for L_oA_p and U_pA_ps with respect to F_rS hold:*

i) ${}^{U}\underline{J} \leq U \leq {}^{U}\overline{J}$

ii) ${}^{U}\underline{J} \leq {}^{U}\overline{J}$.

Proof.
The proof can be derived using the same approach as in Theorem. 3.1.13. □

3.2 q-Rung Orthopair Fuzzy Topologies induced by Reflexive Binary relations

The transition from rough approximations to q-Rung Orthopair Fuzzy Topological Spaces allows for a deeper analysis of uncertainty and relationships within data. Rough approximations provide a framework to define the lower and upper bounds of a q-Rung Orthopair Fuzzy Set, capturing the boundary regions of uncertainty. By introducing Fuzzy Topological Spaces, these boundaries are further analyzed through Topological properties such as interior, closure, and neighborhood. This transition enriches the theoretical framework, enabling the study of spatial relationships and structural patterns within Fuzzy data, which are critical for applications in decision-making and pattern recognition.
Cheng [17] introduced the concept of a Fuzzy Topological Space and generalized some fundamental notions of Topology. T$\ddot{u}$rkarslan et al. [56] presented the idea of q-Rung Orthopair Fuzzy Topological Spaces (${}^{q}ROF_zT_pS$s) and explored continuity between two ${}^{q}ROF_zT_pS$s. In this section, we introduce two types of ${}^{q}ROF_z$ Topologies induced by an R_lB_nR.

Definition 3.2.1. *[56] A family* $\mathfrak{A} \subseteq {}^{q}ROF_zS(\xi)$ *of* ${}^{q}ROF_zSs$ *on* ξ *is called a* ${}^{q}ROF_z$ *Topology on* ξ *if:*

1) $0, 1 \in \mathfrak{A}$

2) $U_1 \cap U_2 \in \mathfrak{A}$*, for all* $U_1, U_2 \in \mathfrak{A}$

3) $\bigcup_{i\in I} U_i \in \mathfrak{A}$*, for all* $U_i \in \mathfrak{A}$*,* $i \in I$.

If $\mathfrak{A}$ is a ${}^{q}ROF_z$ Topology on ξ, then $(\xi, \mathfrak{A})$ is called a ${}^{q}ROF_zT_pS_p$. The elements of $\mathfrak{A}$ are referred to as ${}^{q}ROF_z$ open sets.

Theorem 3.2.2. *If J is a R_fB_nR on ξ, then*

$$\mathfrak{T} = \{U \in {}^{q}ROF_zS(\xi) : \underline{J}^{U} = U\}$$

is a ${}^{q}ROF_z$ Topology on ξ.

Proof.
1) From Theorem 3.1.4, we have $\underline{J}^{0} = 0$ and $\underline{J}^{1} = 1$, which gives that $0, 1 \in \mathfrak{T}$.
2) If $U_1, U_2 \in \mathfrak{T}$, then $\underline{J}^{U_1} = U_1$ and $\underline{J}^{U_2} = U_2$. From Theorem 3.1.4, $\underline{J}^{U_1 \cap U_2} = (\underline{J}^{U_1} \cap \underline{J}^{U_2}) = (U_1 \cap U_2)$. This implies that $U_1 \cap U_2 \in \mathfrak{T}$.
(3) If $U_i \in \mathfrak{T}$, then $\underline{J}^{U_i} = U_i$, for each $i \in I$. Since the relation is Reflexive, so according to the Theorem 3.1.13, we have

$$\underline{J}^{(\bigcup_{i\in I} U_i)} \leq \bigcup_{i\in I} U_i. \tag{3.2.1}$$

Also, because $U_i \leq \bigcup_{i\in I} U_i$, so $\underline{J}^{U_i} \leq \underline{J}^{(\bigcup_{i\in} U_i)}$. This implies that $\bigcup_{i\in I} \underline{J}^{U_i} \leq \underline{J}^{(\bigcup_{i\in I} U_i)}$. Thus,

$$\bigcup_{i\in I} U_i \leq \underline{J}^{(\bigcup_{i\in I} U_i)}. \tag{3.2.2}$$

Thus from Equations (3.2.1) and (3.2.2), we get $\bigcup_{i\in I} U_i = \underline{J}^{(\bigcup_{i\in I} U_i)}$.
Hence, $\mathfrak{T}$ is a ${}^{q}ROF_z$ Topology on ξ. □

Theorem 3.2.3. *If J is a R_fB_nR on ξ, then*

$$\mathfrak{T}' = \{U \in {}^{q}ROF_zS(\xi) : {}^{U}\underline{J} = U\}$$

is a ${}^{q}ROF_z$ Topology on ξ.

Proof.

The proof can be derived using the same approach as in Theorem. 3.2.2. □

3.3 Similarity relations associated with Binary Relations

labelsec10 Here, we explore rough approximation based similarity relations among ${}^{q}ROF_zS$s and investigate their properties.

Definition 3.3.1. *Let J be a C_rB_nR over ξ. For $U_1, U_2 \in {}^{q}ROF_zS(\xi)$, the similarity relations $\underline{R}$, $\tilde{R}$ and R on ξ are:*

$U_1\tilde{R}U_2$ *if and only if* $\overline{J}^{U_1} = \overline{J}^{U_2}$

$U_1\underline{R}U_2$ *if and only if* $\underline{J}^{U_1} = \underline{J}^{U_2}$

U_1RU_2 *if and only if* $\underline{J}^{U_1} = \underline{J}^{U_2}$ *and* $\overline{J}^{U_1} = \overline{J}^{U_2}$.

Definition 3.3.2. *Let J be a C_rB_nR over ξ. For $U_1, U_2 \in {}^{q}ROF_zS(\xi)$, the similarity relations $\underline{r}$, $\tilde{r}$ and r on ξ are:*

$U_1\tilde{r}U_2$ *if and only if* ${}^{U_1}\overline{J} ={}^{U_2} \overline{J}$

$U_1\underline{r}U_2$ *if and only if* ${}^{U_1}\underline{J} ={}^{U_2} \underline{J}$

U_1rU_2 *if and only if* ${}^{U_1}\underline{J} ={}^{U_2} \underline{J}$ *and* ${}^{U_1}\overline{J} ={}^{U_2} \overline{J}$.

The aforementioned Binary Relations can be referred to as the lower ${}^{q}ROF_z$ similarity relation, upper ${}^{q}ROF_z$ similarity relation, and ${}^{q}ROF_z$ similarity relation, respectively.

Proposition 3.3.3. *The Binary Relations $\underline{R}$, $\tilde{R}$, R are E_qRs on ${}^{q}ROF_zS(\xi)$.*

Proof.

The proof is straightforward. □

Proposition 3.3.4. *The Binary Relations $\underline{r}$, $\tilde{r}$, r are E_qRs on ${}^qROF_zS(\xi)$.*

Proof.

The proof is straightforward. □

Theorem 3.3.5. *Let J be a C_rB_nR over ξ and $U_1, U_2, U_3, U_4 \in {}^qROF_zS(\xi)$. Then:*

i) $U_1\tilde{R}U_2$ if and only if $U_1\tilde{R}(U_1 \cup U_2)\tilde{R}U_2$

ii) If $U_1\tilde{R}U_2$ and $U_3\tilde{R}U_4$, then $(U_1 \cup U_3)\tilde{R}(U_2 \cup U_4)$

iii) If $U_1 \subseteq U_2$ and $U_2\tilde{R}0$, then $U_1\tilde{R}0$

iv) $(U_1 \cup U_2)\tilde{R}0$ if and only if $U_1\tilde{R}0$ and $U_2\tilde{R}0$

v) If $U_1 \subseteq U_2$ and $U_1\tilde{R}1$, then $U_2\tilde{R}1$

vi) If $(U_1 \cap U_2)\tilde{R}1$, then $U_1\tilde{R}1$ and $U_2\tilde{R}1$.

Proof.

i) If $U_1\tilde{R}U_2$, then $\overline{J}^{U_1} = \overline{J}^{U_2}$. According to the Theorem 3.1.4, $\overline{J}^{U_1 \cup U_2} = \overline{J}^{U_1} \cup \overline{J}^{U_2} = \overline{J}^{U_1} = \overline{J}^{U_2}$, so we have, $U_1\tilde{R}(U_1 \cup U_2)\tilde{R}U_2$.

Conversely, if $U_1\tilde{R}(U_1 \cup U_2)\tilde{R}U_2$, then $U_1\tilde{R}(U_1 \cup U_2)$ and $(U_1 \cup U_2)\tilde{R}U_2$. Which implies that $\overline{J}^{U_1} = \overline{J}^{U_1 \cup U_2}$ and $\overline{J}^{U_1 \cup U_2} = \overline{J}^{U_2}$. Thus, $\overline{J}^{U_1} = \overline{J}^{U_2}$. Hence, $U_1\tilde{R}U_2$.

ii) If $U_1\tilde{R}U_2$ and $U_3\tilde{R}U_4$, then $\overline{J}^{U_1} = \overline{J}^{U_2}$ and $\overline{J}^{U_3} = \overline{J}^{U_4}$. According to the Theorem 3.1.4, $\overline{J}^{U_1 \cup U_3} = \overline{J}^{U_1} \cup \overline{J}^{U_3} = \overline{J}^{U_2} \cup \overline{J}^{U_4} = \overline{J}^{U_2 \cup U_4}$. Thus, $(U_1 \cup U_3)\tilde{R}(U_2 \cup U_4)$.

iii) Let $U_1 \subseteq U_2$ and $U_2\tilde{R}0$. Then $\overline{J}^{U_2} = \overline{R}^0$. Also, since $U_1 \subseteq U_2$, so we have $\overline{J}^{U_1} \subseteq \overline{J}^{U_2} = \overline{J}^0$. But $\overline{J}^0 \subseteq \overline{J}^{U_1}$, so $\overline{J}^{U_1} = \overline{J}^0$. Hence, $U_1\tilde{R}0$.

iv) If $(U_1 \cup U_2)\tilde{R}0$, then $\overline{J}^{U_1} \cup \overline{J}^{U_2} = \overline{J}^{U_1 \cup U_2} = \overline{J}^0$. Since $\overline{J}^{U_1} \subseteq \overline{J}^{U_1} \cup \overline{J}^{U_2} = \overline{J}^0$, so we have $\overline{J}^{U_1} = \overline{J}^0$. Similarly, $\overline{J}^{U_2} = \overline{J}^0$. Hence, $U_1\tilde{R}0$ and $U_2\tilde{R}0$.

Conversely, if $U_1\tilde{R}0$ and $U_2\tilde{R}0$, then $\overline{J}^{U_1} = \overline{J}^0$ and $\overline{J}^{U_2} = \overline{J}^0$. According to the Theorem 3.1.4, $\overline{J}^{(U_1 \cup U_2)} = \overline{J}^{U_1} \cup \overline{J}^{U_2} = \overline{J}^0 \cup \overline{J}^0 = \overline{J}^0$. Hence, $U_1 \cup (U_2\tilde{R})0$.

v) If $U_1\tilde{R}1$, then $\overline{J}^{U_1} = \overline{J}^1$. Since $U_1 \subseteq U_2$, so $\overline{J}^1 = \overline{J}^{U_1} \subseteq \overline{J}^{U_2}$. But $\overline{J}^{U_2} \subseteq \overline{J}^1$ so, $\overline{J}^1 = \overline{J}^{U_1}$. Hence, $U_2\tilde{R}1$.

vi) If $U_1 \cap U_2\tilde{R}1$, then $\overline{J}^{U_1 \cap U_2} = \overline{J}^1$. According to the Theorem 1.2.3, we have $\overline{J}^{U_1} \cap \overline{J}^{U_2} \supseteq \overline{J}^{U_1 \cap U_2} = \overline{J}^1$. Thus, $\overline{J}^1 = \overline{J}^{U_1}$ and $\overline{J}^1 = \overline{J}^{U_2}$. Hence, $U_1\tilde{R}1$ and $U_2\tilde{R}1$.

□

Theorem 3.3.6. *Let J be a C_rB_nR over ξ and $U_1, U_2, U_3, U_4 \in {}^qROF_zS(\xi)$. Then:*

i) $U_1\tilde{r}U_2$ if and only if $U_1\tilde{r}(U_1 \cup U_2)\tilde{r}U_2$

ii) If $U_1\tilde{r}U_2$ and $U_3\tilde{r}U_4$, then $(U_1 \cup U_3)\tilde{r}(U_2 \cup U_4)$

iii) If $U_1 \subseteq U_2$ and $U_2\tilde{r}0$, then $U_1\tilde{r}0$

iv) $(U_1 \cup U_2)\tilde{r}0$ if and only if $U_1\tilde{r}0$ and $U_2\tilde{r}0$

v) If $U_1 \subseteq U_2$ and $U_1\tilde{r}1$, then $U_2\tilde{r}1$

vi) If $(U_1 \cap U_2)\tilde{r}1$, then $U_1\tilde{r}1$ and $U_2\tilde{r}1$.

Proof.

The proof can be derived using the same approach as in Theorem. 3.3.5 □

Theorem 3.3.7. *Let J be a C_rB_nR over ξ and $U_1, U_2, U_3, U_4 \in {}^qROF_zS(\xi)$. Then:*

i) $U_1\underline{R}U_2$ if and only if $U_1\underline{R}(U_1 \cap U_2)\underline{R}U_2$

ii) If $U_1\underline{R}U_2$ and $U_3\underline{R}U_4$, then $(U_1 \cap U_3)\underline{R}(U_2 \cap U_4)$

iii) If $U_1 \subseteq U_2$ and $U_2\underline{R}0$, then $U_1\underline{R}0$

iv) $(U_1 \cup U_2)\underline{R}0$ if and only if $U_1\underline{R}0$ and $U_2\underline{R}0$

v) If $U_1 \subseteq U_2$ and $U_1\underline{R}1$, then $U_2\underline{R}1$

vi) If $(U_1 \cap U_2)\underline{R}1$, then $U_1\underline{R}1$ and $U_2\underline{R}1$.

Proof.

The proof is straightforward. $\square$

Theorem 3.3.8. *Let J be a C_rB_nR over ξ and $U_1, U_2, U_3, U_4 \in {}^qROF_zS(\xi)$. Then:*

i) $U_1\underline{r}U_2$ *if and only if* $U_1\underline{r}(U_1 \cap U_2)\underline{r}U_2$

ii) *If* $U_1\underline{r}U_2$ *and* $U_3\underline{r}U_4$, *then* $(U_1 \cap U_3)\underline{r}(U_2 \cap U_4)$

iii) *If* $U_1 \subseteq U_2$ *and* $U_2\underline{r}0$, *then* $U_1\underline{r}0$

iv) $(U_1 \cup U_2)\underline{r}0$ *if and only if* $U_1\underline{r}0$ *and* $U_2\underline{r}0$

v) *If* $U_1 \subseteq U_2$ *and* $U_1\underline{r}1$, *then* $U_2\underline{r}1$

vi) *If* $(U_1 \cap U_2)\underline{r}1$, *then* $U_1\underline{r}1$ *and* $U_2\underline{r}1$.

Proof.

The proof is straightforward. $\square$

Theorem 3.3.9. *Let J be a C_rB_nR over ξ and $U_1, U_2 \in {}^qROF_zS(\xi)$. Then:*

i) U_1RU_2 *if and only if* $U_1\tilde{R}(U_1 \cup U_2)\tilde{R}U_2$ *and* $U_1\underline{R}(U_1 \cap U_2)\underline{R}U_2$

ii) *If* $U_1 \subseteq U_2$ *and* U_2R0, *then* U_1R0

iii) $(U_1 \cup U_2)R0$ *if and only if* U_1R0 *and* U_2R0

iv) *If* $(U_1 \cap U_2)R1$, *then* U_1R1 *and* U_2R1.

v) *If* $U_1 \subseteq U_2$ *and* U_1R1, *then* U_2R1

Proof.

Theorems 3.3.5 and 3.3.7 directly lead to this conclusion. $\square$

Theorem 3.3.10. *Let J be a C_rB_nR over ξ and $U_1, U_2 \in {}^qROF_zS(\xi)$. Then:*

i) $U_1 r U_2$ *if and only if* $U_1 \tilde{r}(U_1 \cup U_2)\tilde{r} U_2$ *and* $U_1 \underline{r}(U_1 \cap U_2)\underline{r} U_2$

ii) *If* $U_1 \subseteq U_2$ *and* $U_2 r 0$, *then* $U_1 r 0$

iii) $(U_1 \cup U_2) r 0$ *if and only if* $U_1 r 0$ *and* $U_2 r 0$

iv) *If* $(U_1 \cap U_2) r 1$, *then* $U_1 r 1$ *and* $U_2 r 1$.

v) *If* $U_1 \subseteq U_2$ *and* $U_1 r 1$, *then* $U_2 r 1$

Proof.
Theorems 3.3.6 and 3.3.8 directly lead to this conclusion. □

3.4 Accuracy Measure

The approximation of ${}^qROF_zSs$ introduces a novel approach for assessing the precision of M_mD_gs associated with ${}^qROF_zSs$ that characterize objects. This method allows us to determine how closely these sets describe the underlying entities. To begin, we define the $(\mathscr{A}, \mathscr{B})$-level cut set of a ${}^qROF_zS$ U.

Definition 3.4.1. *Let $U \in {}^qROF_zS(\xi)$ and consider $\mathscr{A}, \mathscr{B} \in [0,1]$ be such that $\mathscr{A}^q + \mathscr{B}^q \leq 1$ for $q \geq 1$. The $(\mathscr{A}, \mathscr{B})$-level cut set of ${}^qROF_zS$ U is defined as*

$$U_{\mathscr{A}}^{\mathscr{B}} = \{\Bbbk \in \xi : U_Y(\Bbbk) \geq \mathscr{A} \quad \text{and} \quad U_N(\Bbbk) \leq \mathscr{B}\}.$$

For example, if we consider
$U = \{\langle \Bbbk_1, 0.9, 0.4\rangle, \langle \Bbbk_2, 0.95, 0.3\rangle, \langle \Bbbk_3, 0.8, 0.5\rangle, \langle \Bbbk_4, 0.9, 0.2\rangle, \langle \Bbbk_5, 0.7, 0.65\rangle, \langle \Bbbk_6, 0.94, 0.42\rangle\}$ *and* $(\mathscr{A}, \mathscr{B}) = (0.7, 0.5) \in [0,1] \times [0,1]$ *such that* $\mathscr{A}^q + \mathscr{B}^q \leq 1$ *with* $q = 3$, *then*

$$U_{\mathscr{A}}^{\mathscr{B}} = \{\Bbbk_1, \Bbbk_2, \Bbbk_3, \Bbbk_4, \Bbbk_6\}.$$

The set $U_{\mathscr{A}} = \{\acute{k} \in \xi : U_Y(\acute{k}) \geq \mathscr{A}\}$ *represents the membership set of the* $\mathscr{A}$*-level cut of* U*, while* $U_{\dot{\mathscr{A}}} = \{\acute{k} \in \xi : U_Y(\acute{k}) > \mathscr{A}\}$ *denotes the membership set of the strong* $\mathscr{A}$*-level cut of* U*. Similarly,* $U^{\mathscr{B}} = \{\acute{k} \in \xi : U_N(\acute{k}) \leq \mathscr{B}\}$ *and* $U^{\dot{\mathscr{B}}} = \{\acute{k} \in \xi : U_N(\acute{k}) < \mathscr{B}\}$ *are the membership sets of the* $\mathscr{B}$*-level and strong* $\mathscr{B}$*-level cuts of* U*. Thus, the other cut sets of a* ${}^qROF_zS$ U *can be defined as follows:*

$$U^{\mathscr{B}}_{\dot{\mathscr{A}}} = \{\acute{k} \in \xi : U_Y(\acute{k}) > \mathscr{A} \quad and \quad U_N(\acute{k}) \leq \mathscr{B}\},$$

which is referred to as the $(\dot{\mathscr{A}}, \mathscr{B})$*-level cut set of* U*;*

$$U^{\dot{\mathscr{B}}}_{\mathscr{A}} = \{\acute{k} \in \xi : U_Y(\acute{k}) \geq \mathscr{A} \quad and \quad U_N(\acute{k}) < \mathscr{B}\},$$

which is referred to as the $(\mathscr{A}, \dot{\mathscr{B}})$*-level cut set of* U*;*

$$U^{\dot{\mathscr{B}}}_{\dot{\mathscr{A}}} = \{\acute{k} \in \xi : U_Y(\acute{k}) > \mathscr{A} \quad and \quad U_N(\acute{k}) < \mathscr{B}\},$$

which is referred to as the $(\dot{\mathscr{A}}, \dot{\mathscr{B}})$*-level cut set of* U*.*

Theorem 3.4.2. *Let* $U, V \in {}^qROF_zS(\xi)$ *and* $(\mathscr{A}, \mathscr{B}) \in [0,1] \times [0,1]$ *be such that* $\mathscr{A}^q + \mathscr{B}^q \leq 1$*, for* $q \geq 1$*. Then:*

i) $U^{\mathscr{B}}_{\mathscr{A}} = U_{\mathscr{A}} \cap U^{\mathscr{B}}$

ii) $(U^c)^{\mathscr{B}} = (U_{\dot{\mathscr{B}}})^c$, $(U^c)_{\mathscr{A}} = (U^{\dot{\mathscr{A}}})^c$

iii) $U \subseteq V$ *implies* $U^{\mathscr{B}}_{\mathscr{A}} \subseteq V^{\mathscr{B}}_{\mathscr{A}}$

iv) $(U \cap V)_{\mathscr{A}} = U_{\mathscr{A}} \cap V_{\mathscr{A}}$, $(U \cap V)^{\mathscr{B}} = U^{\mathscr{B}} \cap V^{\mathscr{B}}$, $(U \cap V)^{\mathscr{B}}_{\mathscr{A}} = U_{\mathscr{A}} \cap V^{\mathscr{B}}_{\mathscr{A}}$

v) $(U \cup V)_{\mathscr{A}} = U_{\mathscr{A}} \cup V_{\mathscr{A}}$, $(U \cup V)^{\mathscr{B}} = U^{\mathscr{B}} \cup V^{\mathscr{B}}$, $U_{\mathscr{A}} \cap V^{\mathscr{B}}_{\mathscr{A}} \subseteq (U \cup V)^{\mathscr{B}}_{\mathscr{A}}$

vi) $\mathscr{A}_1 \geq \mathscr{A}_2$ *and* $\mathscr{B}_1 \leq \mathscr{B}_2$ *implies* $U_{\mathscr{A}_1} \subseteq U_{\mathscr{A}_2}$, $U^{\mathscr{B}_1} \subseteq U^{\mathscr{B}_2}$, $U^{\mathscr{B}_1}_{\mathscr{A}_1} \subseteq U^{\mathscr{B}}_{\mathscr{A}_2}$.

Proof.

i) This follows directly from Definition 3.4.1.

ii) Let $U, V \in {}^qROF_zS(\xi)$ where $U = \{\langle \mathfrak{k}, U_Y(\mathfrak{k}), U_N(\mathfrak{k})\rangle : \mathfrak{k} \in \xi\}$. The complement U^c is defined as $U^c = \{\langle \mathfrak{k}, U_N(\mathfrak{k}), U_Y(\mathfrak{k})\rangle : \mathfrak{k} \in \xi\}$.
Hence, $\mathfrak{k} \in ((U^c)_{\mathscr{A}})^c$ if and only if $\mathfrak{k} \notin (U^c)_{\mathscr{A}} = \{\mathfrak{k} \in \xi : U_N(\mathfrak{k}) \geq \mathscr{A}\}$, which implies $U_N(\mathfrak{k}) < \mathscr{A}$, so $\mathfrak{k} \in U^{\dot{\mathscr{A}}}$. Therefore, $(U^c)_{\mathscr{A}} = (U^{\dot{\mathscr{A}}})^c$.
Similarly, $(U^c)^{\mathscr{B}} = (U_{\dot{\mathscr{B}}})^c$.

iii) This property directly follows from Definition 3.4.1.

iv) Let $k \in (U \cap V)_{\mathscr{A}}$. Then $U_Y(\mathfrak{k}) \wedge V_Y(\mathfrak{k}) \geq \mathscr{A}$ implies $U_Y(\mathfrak{k}) \geq \mathscr{A}$ and $V_Y(\mathfrak{k}) \geq \mathscr{A}$, thus $\mathfrak{k} \in U_{\mathscr{A}} \cap V_{\mathscr{A}}$.
For $\mathfrak{k} \in (U \cap V)^{\mathscr{B}}$, we have $U_N(\mathfrak{k}) \vee V_N(\mathfrak{k}) \leq \mathscr{B}$, which implies $U_N(\mathfrak{k}) \leq \mathscr{B}$ and $V_N(\mathfrak{k}) \leq \mathscr{B}$, hence $\mathfrak{k} \in U^{\mathscr{B}} \cap V^{\mathscr{B}}$.
Using property (i), we obtain:

$$(U \cap V)^{\mathscr{B}}_{\mathscr{A}} = (U \cap V)_{\mathscr{A}} \cap (U \cap V)^{\mathscr{B}} = (U_{\mathscr{A}} \cap V_{\mathscr{A}}) \cap (U^{\mathscr{B}} \cap V^{\mathscr{B}}) = U^{\mathscr{B}}_{\mathscr{A}} \cap V^{\mathscr{B}}_{\mathscr{A}}.$$

v) Let $\mathfrak{k} \in (U \cup V)_{\mathscr{A}}$. Then $U_Y(\mathfrak{k}) \vee V_Y(\mathfrak{k}) \geq \mathscr{A}$ implies $U_Y(\mathfrak{k}) \geq \mathscr{A}$ or $V_Y(\mathfrak{k}) \geq \mathscr{A}$, thus $\mathfrak{k} \in U_{\mathscr{A}} \cup V_{\mathscr{A}}$.
For $\mathfrak{k} \in (U \cup V)^{\mathscr{B}}$, we have $U_N(\mathfrak{k}) \wedge V_N(\mathfrak{k}) \leq \mathscr{B}$, which implies $U_N(\mathfrak{k}) \leq \mathscr{B}$ or $V_N(\mathfrak{k}) \leq \mathscr{B}$, hence $\mathfrak{k} \in U^{\mathscr{B}} \cup V^{\mathscr{B}}$.
Since $U \subseteq U \cup V$ and $V \subseteq U \cup V$, it follows that $U^{\mathscr{B}}_{\mathscr{A}} \subseteq (U \cup V)^{\mathscr{B}}_{\mathscr{A}}$ and $V^{\mathscr{B}}_{\mathscr{A}} \subseteq (U \cup V)^{\mathscr{B}}_{\mathscr{A}}$, which implies $U^{\mathscr{B}}_{\mathscr{A}} \cup V^{\mathscr{B}}_{\mathscr{A}} \subseteq (U \cup V)^{\mathscr{B}}_{\mathscr{A}}$.

vi) Suppose $\mathscr{A}_1 \geq \mathscr{A}_2$ and $\mathscr{B}_1 \leq \mathscr{B}_2$:
For $\mathfrak{k} \in U_{\mathscr{A}_1}$, we have $U_Y(\mathfrak{k}) \geq \mathscr{A}_1$. Since $\mathscr{A}_1 \geq \mathscr{A}_2$, it follows that $U_Y(\mathfrak{k}) \geq \mathscr{A}_2$, hence $\mathfrak{k} \in U_{\mathscr{A}_2}$.

For $\mathfrak{k} \in U^{\mathscr{B}_1}$, we have $U_N(\mathfrak{k}) \leq \mathscr{B}_1$. Since $\mathscr{B}_1 \leq \mathscr{B}_2$, it follows that $U_N(\mathfrak{k}) \leq \mathscr{B}_2$, hence $\mathfrak{k} \in U^{\mathscr{B}_2}$.

Therefore, $U_{\mathscr{A}_1} \subseteq U_{\mathscr{A}_2}$, $U^{\mathscr{B}_1} \subseteq U^{\mathscr{B}_2}$, and using property (i), we get $U^{\mathscr{B}_1}_{\mathscr{A}_1} \subseteq U^{\mathscr{B}_2}_{\mathscr{A}_2}$.

$\square$

If J is a C_rB_nR over ξ, then $\underline{J}^{U^{\mathscr{B}}_{\mathscr{A}}}$ represents the lower approximation (denoted L_oA_p) of the Crisp Set $U^{\mathscr{B}}_{\mathscr{A}}$. The expression $(\underline{J}^U(e))^{\mathscr{B}}_{\mathscr{A}}$ denotes the $(\mathscr{A}, \mathscr{B})$-level cut of $\underline{J}^U(e)$ with respect to the A_fSs. Hence,

$$\begin{aligned}(\underline{J}^U)^{\mathscr{B}}_{\mathscr{A}} &= \{\mathfrak{k} \in \xi : \underline{J}^{U_Y}(\mathfrak{k}) \geq \mathscr{A} \quad \textit{and} \quad \underline{J}^{U_N}(\mathfrak{k}) \leq \mathscr{B}\} \\ &= \{\mathfrak{k} \in \xi : \wedge_{\mathfrak{t} \in \mathfrak{k}J} U_Y(\mathfrak{t}) \geq \mathscr{A} \quad \textit{and} \quad \vee_{\mathfrak{t} \in \mathfrak{k}J}\, U_N(\mathfrak{t}) \leq \mathscr{B}\}\end{aligned}$$

and

$$\begin{aligned}(\overline{J}^U)^{\mathscr{B}}_{\mathscr{A}} &= \{\mathfrak{k} \in \xi : \overline{J}^{U_Y}(\mathfrak{k}) \geq \mathscr{A} \quad \textit{and} \quad \overline{J}^{U_N}(\mathfrak{k}) \leq \mathscr{B}\} \\ &= \{\mathfrak{k} \in \xi : \vee_{\mathfrak{t} \in \mathfrak{k}J} U_Y(\mathfrak{t}) \geq \mathscr{A} \quad \textit{and} \quad \wedge_{\mathfrak{t} \in \mathfrak{k}J}\, U_N(\mathfrak{t}) \leq \mathscr{B}\}.\end{aligned}$$

Similarly,

$$\begin{aligned}({}^U\underline{J})^{\mathscr{B}}_{\mathscr{A}} &= \{\mathfrak{t} \in \xi : {}^{U_Y}\underline{J}(\mathfrak{t}) \geq \mathscr{A} \quad \textit{and} \quad {}^{U_N}\underline{J}(\mathfrak{t}) \leq \mathscr{B}\} \\ &= \{\mathfrak{t} \in \xi : \wedge_{\mathfrak{k} \in J\mathfrak{t}} U_Y(\mathfrak{k}) \geq \mathscr{A} \quad \textit{and} \quad \vee_{\mathfrak{k} \in J\mathfrak{t}}\, U_N(\mathfrak{k}) \leq \mathscr{B}\}\end{aligned}$$

and

$$\begin{aligned}({}^U\overline{J})^{\mathscr{B}}_{\mathscr{A}} &= \{\mathfrak{t} \in \xi : {}^{U_Y}\overline{J}(\mathfrak{t}) \geq \mathscr{A} \quad \textit{and} \quad {}^{U_N}\overline{J}(e)(\mathfrak{t}) \leq \mathscr{B}\} \\ &= \{\mathfrak{t} \in \xi : \vee_{\mathfrak{k} \in J\mathfrak{t}} U_Y(\mathfrak{k}) \geq \mathscr{A} \quad \textit{and} \quad \wedge_{\mathfrak{k} \in J\mathfrak{t}}\, U_N(\mathfrak{k}) \leq \mathscr{B}\}\end{aligned}$$

with respect to F_rS.

Lemma 3.4.3. *Let J be a R_lB_nR on a non-empty universe ξ, and let $U \in {}^qROF_zS(\xi)$. Consider $\mathscr{A}, \mathscr{B} \in [0,1]$ be such that $\mathscr{A}^q + \mathscr{B}^q \leq 1$ for $q \geq 1$. Then,*

$$\underline{J}^{U_{\mathscr{A}}^{\mathscr{B}}} = (\underline{J}^U)_{\mathscr{A}}^{\mathscr{B}} \quad \textit{and} \quad \overline{J}^{U_{\mathscr{A}}^{\mathscr{B}}} = (\overline{J}^U)_{\mathscr{A}}^{\mathscr{B}}.$$

Proof. Let $\mathscr{A}, \mathscr{B} \in [0,1]$ be such that $\mathscr{A}^q + \mathscr{B}^q \leq 1$, for $q \geq 1$. Given $u \in \mathfrak{k}J$,

$$\begin{aligned}
(\underline{J}^U)_{\mathscr{A}}^{\mathscr{B}} &= \{\mathfrak{k} \in \xi : \underline{J}^{U_Y}(\mathfrak{k}) \geq \mathscr{A} \quad \text{and} \quad \underline{J}^{U_N}(\mathfrak{k}) \leq \mathscr{B}\} \\
&= \{\mathfrak{k} \in \xi : \wedge_{\mathfrak{t} \in \mathfrak{k}J} U_Y(\mathfrak{t}) \geq \mathscr{A} \quad \text{and} \quad \vee_{\mathfrak{t} \in \mathfrak{k}J}\, U_N(\mathfrak{t}) \leq \mathscr{B}\} \\
&= \{\mathfrak{k} \in \xi : U_Y(\mathfrak{t}) \geq \mathscr{A} \quad \text{and} \quad U_N(\mathfrak{t}) \leq \mathscr{B}, \text{ for all } \mathfrak{t} \in \mathfrak{k}J\} \\
&= \{\mathfrak{k} \in \xi : \mathfrak{k}J \subseteq U_{\mathscr{A}}^{\mathscr{B}}\} \\
&= \underline{J}^{U_{\mathscr{A}}^{\mathscr{B}}}(\mathfrak{k}).
\end{aligned}$$

Similarly, we can prove that $\overline{J}^{U_{\mathscr{A}}^{\mathscr{B}}} = (\overline{J}^U)_{\mathscr{A}}^{\mathscr{B}}$. □

Lemma 3.4.4. *Consider J, a R_lB_nR defined over a non-empty set ξ, and let $U \in {}^qROF_zS(\xi)$. Suppose $\mathscr{A}, \mathscr{B} \in [0,1]$ be such that $\mathscr{A}^q + \mathscr{B}^q \leq 1$ for $q \geq 1$. Then,*

$${}^{U_{\mathscr{A}}^{\mathscr{B}}}\underline{J} = ({}^U\underline{J})_{\mathscr{A}}^{\mathscr{B}} \qquad \textit{and} \qquad {}^{U_{\mathscr{A}}^{\mathscr{B}}}\overline{J} = ({}^U\overline{J})_{\mathscr{A}}^{\mathscr{B}}.$$

Proof.
The proof can be derived using the same approach as in Lemma 3.4.3. □

The $A_cR_cD_g$ and $R_fN_sD_g$ of a ${}^qROF_zS$ is defined below.

Definition 3.4.5. *Let J be a R_lB_nR defined on a non-empty set ξ. then the $A_cR_cD_g$ for the membership of $U \in {}^qROF_zS(\xi)$, considering the parameters $\mathscr{A}, \mathscr{B}, \mathscr{G}, \theta \in [0,1]$, is defined under the conditions $\mathscr{A} \leq \mathscr{G}$, $\mathscr{B} \geq \theta$, $\mathscr{A}^q + \mathscr{B}^q \leq 1$, $\mathscr{G}^q + \theta^q \leq 1$, for $q \geq 1$, and with respect* *to A_fSs:*

$$\delta^{(\mathscr{G},\theta)}_{(\mathscr{A},\mathscr{B})}(J^U) = \frac{\mid \underline{J}^{U^{\theta}_{\mathscr{G}}} \mid}{\mid \overline{J}^{U^{\mathscr{B}}_{\mathscr{A}}} \mid}$$

The $R_fN_sD_g$ for the membership of $U \in {}^qROF_zS(\xi)$ is defined as:

$$\eta^{(\mathscr{G},\theta)}_{(\mathscr{A},\mathscr{B})}(J^U) = 1 - \delta^{(\mathscr{G},\theta)}_{(\mathscr{A},\mathscr{B})}(J^U)$$

Similarly, the $A_cR_cD_g$ for the membership of $U \in {}^qROF_zS(\xi)$ with respect to F_rS can be defined as:

$$\delta^{(\mathscr{G},\theta)}_{(\mathscr{A},\mathscr{B})}({}^UJ) = \frac{|{}^{U^{\theta}_{\mathscr{G}}} \underline{J} \mid}{|{}^{U^{\mathscr{B}}_{\mathscr{A}}} \overline{J} \mid}$$

The $R_fN_sD_g$ for the membership of $U \in {}^qROF_zS(\xi)$ with respect to F_rS is defined as:

$$\eta^{(\mathscr{G},\theta)}_{(\mathscr{A},\mathscr{B})}({}^UJ) = 1 - \delta^{(\mathscr{G},\theta)}_{(\mathscr{A},\mathscr{B})}({}^UJ).$$

The concepts of F_rS and A_fS coincide when dealing with an E_qR. Moreover, $\underline{J}^{U^{\theta}_{\mathscr{G}}}$ includes elements from ξ that exhibit $\mathscr{G}$ as the as the minimal definite M_mD_g and θ as the maximal definite $N_nM_mD_g$ in U. Conversely, $\overline{J}^{U^{\mathscr{B}}_{\mathscr{A}}}$ comprises elements from ξ where $\mathscr{A}$ is the the minimal possible M_mD_g and $\mathscr{B}$ is the maximal possible $N_nM_mD_g$ in U.
In simpler terms, $\underline{J}^{U^{\theta}_{\mathscr{G}}}$ represents the union of E_q classes in ξ characterized by $\mathscr{G}$ as lowest definite M_mD_g and θ as the highest definite $N_nM_mD_g$ in the L_oA_p of U. Similarly, $\overline{J}^{U^{\mathscr{B}}_{\mathscr{A}}}$ denotes the union of E_q classes in ξ characterized by $\mathscr{A}$ as lowest possible M_mD_g and $\mathscr{B}$ as the highest possible $N_nM_mD_g$ in the U_pA_p of U.
Therefore, $(\mathscr{G},\theta)$ and $(\mathscr{A},\mathscr{B})$ act as thresholds that determine the levels of definite and potential fulfillment of the object u within U. Consequently, $\delta^{(\mathscr{G},\theta)}_{(\mathscr{A},\mathscr{B})}(J^U)$ can be interpreted as M_mD_g to which U is accurate, considering the threshold parameters $(\mathscr{G},\theta)$ and $(\mathscr{A},\mathscr{B})$.
These degrees are exemplified in the following example.

Example 3.4.6. *Let $\xi = \{k_1, k_2, k_3, k_4, k_5, k_6, k_7, k_8, k_9, k_{10}, k_{11}\}$ and $J \in P(\xi \times \xi)$ be such that*

the equivalence classes are given by: $E_1 = \{\Bbbk_1, \Bbbk_9\}$, $E_2 = \{\Bbbk_2, \Bbbk_4, \Bbbk_6, \Bbbk_7\}$, $E_3 = \{\Bbbk_3, \Bbbk_5, \Bbbk_8, \Bbbk_{10}\}$, $E_4 = \{\Bbbk_{11}\}$. *Define a* ${}^{q}ROF_zS$ $U : \xi \to [0,1]$, *for* $q = 4$, *by;*
$U = \{\langle \Bbbk_1, 0.8, 0.65\rangle, \langle \Bbbk_2, 0.6, 0.9\rangle, \langle \Bbbk_3, 0.64, 0.8\rangle, \langle \Bbbk_4, 0.44, 0.94\rangle, \langle \Bbbk_5, 0.65, 0.95\rangle, \langle \Bbbk_6, 0.54, 0.85\rangle, \langle \Bbbk_7, 0.64, 0.65\rangle, \langle \Bbbk_8, 0.7, 0.86\rangle, \langle \Bbbk_9, 1, 0\rangle, \langle \Bbbk_{10}, 0.3, 0.89\rangle, \langle \Bbbk_{11}, 0.4, 0.9\rangle\}$. *Take* $(\mathscr{G}, \theta) = (0.75, 0.76)$ *and* $(\mathscr{A}, \mathscr{B}) = (0.45, 0.85)$ *then* $(\mathscr{G}, \theta)-$*level and* $(\mathscr{A}, \mathscr{B})-$*level cuts* $U_{0.75}^{0.76}$ *and* $U_{0.45}^{0.85}$ *are, respectively,*
$U_{\mathscr{A}}^{\mathscr{B}} = U_{0.45}^{0.85} = \{\Bbbk : U_Y(\Bbbk) \geq 0.45, \quad U_N(\Bbbk) \leq 0.85\} = \{\Bbbk_1, \Bbbk_3, \Bbbk_6, \Bbbk_7, \Bbbk_9\}$,
$U_{\mathscr{G}}^{\theta} = U_{0.75}^{0.76} = \{\Bbbk_1, \Bbbk_9, \Bbbk_{11}\}$.
Then $\underline{J}^{U_{\mathscr{G}}^{\theta}} = \{\Bbbk \in \xi : \Bbbk J \subseteq U_{\mathscr{G}}^{\theta}\} = \{\Bbbk_1, \Bbbk_9, \Bbbk_{11}\}$ *and* $\overline{J}^{U_{\mathscr{A}}^{\mathscr{B}}} = \{\Bbbk \in \xi : \Bbbk J \cap U_{\mathscr{A}}^{\mathscr{B}} \neq \emptyset\}$ $= \{\Bbbk_1, \Bbbk_2, \Bbbk_3, \Bbbk_4, \Bbbk_5, \Bbbk_6, \Bbbk_7, \Bbbk_8, \Bbbk_9, \Bbbk_{10}\}$.
Thus $\delta_{(\mathscr{A},\mathscr{B})}^{(\mathscr{G},\theta)}(J^U) = \frac{|\underline{J}^{U_{\mathscr{G}}^{\theta}}|}{|\overline{J}^{U_{\mathscr{A}}^{\mathscr{B}}}|} = \frac{3}{10}$.

Theorem 3.4.7. *Let* J *be a* R_fB_nR *on* ξ, $U \in {}^{q}ROF_zS(\xi)$, *and* $\mathscr{A}, \mathscr{B}, \mathscr{G}, \theta \in [0,1]$ *be such that* $\mathscr{A} \leq \mathscr{G}$, $\mathscr{B} \geq \theta$, *and* $\mathscr{A}^q + \mathscr{B}^q \leq 1$, $\mathscr{G}^q + \theta^q \leq 1$, *for* $q \geq 1$. *Then*

$$0 \leq \delta_{(\mathscr{A},\mathscr{B})}^{(\mathscr{G},\theta)}(J^U) \leq 1$$

Proof.
Let $U \in {}^{q}ROF_zS(\xi)$ and $\mathscr{A}, \mathscr{B}, \mathscr{G}, \theta \in [0,1]$ be such that $\mathscr{A} \leq \mathscr{G}$, $\mathscr{B} \geq \theta$ and $\mathscr{A}^q + \mathscr{B}^q \leq 1$, $\mathscr{G}^q + \theta^q \leq 1$, for $q \geq 1$. Then $U_{\mathscr{G}}^{\theta} \subseteq U_{\mathscr{A}}^{\mathscr{B}}$ According to the Theorem 3.4.2. Now According to the Theorem 3.1.4, $\underline{J}^{U_{\mathscr{G}}^{\theta}} \subseteq \overline{J}^{U_{\mathscr{G}}^{\theta}} \subseteq \overline{J}^{U_{\mathscr{A}}^{\mathscr{B}}}$, so we have $|\ \underline{J}^{U_{\mathscr{G}}^{\theta}}\ | \leq |\ \overline{J}^{U_{\mathscr{A}}^{\mathscr{B}}}\ |$. Thus, $\frac{|\underline{J}^{U_{\mathscr{G}}^{\theta}}|}{|\overline{J}^{U_{\mathscr{A}}^{\mathscr{B}}}|} \leq 1$.
Hence, $0 \leq \delta_{(\mathscr{A},\mathscr{B})}^{(\mathscr{G},\theta)}(J^U) \leq 1$. □

Corollary 3.4.8. *Let* J *be a* R_fB_nR *on* ξ, $U \in {}^{q}ROF_zS(\xi)$, *and* $\mathscr{A}, \mathscr{B}, \mathscr{G}, \theta \in [0,1]$ *be such that* $\mathscr{A} \leq \mathscr{G}$, $\mathscr{B} \geq \theta$, *and* $\mathscr{A}^q + \mathscr{B}^q \leq 1$, $\mathscr{G}^q + \theta^q \leq 1$, *for* $q \geq 1$. *Then*

$$0 \leq \eta_{(\mathscr{A},\mathscr{B})}^{(\mathscr{G},\theta)}(J^U) \leq 1$$

Proof.

Theorem 3.4.7 and Definition 3.2.1 directly lead to this conclusion. . □

Theorem 3.4.9. *Let J be a R_lB_nR on ξ, and let $U, V \in {}^qROF_zS(\xi)$. Suppose $\mathscr{A}, \mathscr{B}, \mathscr{G}, \theta \in [0,1]$ such that $\mathscr{A} \leq \mathscr{G}$, $\mathscr{B} \geq \theta$, and $\mathscr{A}^q + \mathscr{B}^q \leq 1$, $\mathscr{G}^q + \theta^q \leq 1$ for $q \geq 1$. Then, if $U \leq V$, the following assertions hold with respect to the A_fSs:*

i) $\delta^{(\mathscr{G},\theta)}_{(\mathscr{A},\mathscr{B})}(J^U) \leq \delta^{(\mathscr{G},\theta)}_{(\mathscr{A},\mathscr{B})}(J^V)$, *whenever* $\overline{J}^{U^{\mathscr{B}}_{\mathscr{A}}} = \overline{J}^{V^{\mathscr{B}}_{\mathscr{A}}}$

ii) $\delta^{(\mathscr{G},\theta)}_{(\mathscr{A},\mathscr{B})}(J^U) \geq \delta^{(\mathscr{G},\theta)}_{(\mathscr{A},\mathscr{B})}(J^V)$ *whenever* $\underline{J}^{U^{\mathscr{B}}_{\mathscr{A}}} = \underline{J}^{V^{\mathscr{B}}_{\mathscr{A}}}$.

Proof.

i) Let $\mathscr{A}, \mathscr{B}, \mathscr{G}, \theta \in [0,1]$ be such that $\mathscr{A} \leq \mathscr{G}$, $\mathscr{B} \geq \theta$, and $\mathscr{A}^q + \mathscr{B}^q \leq 1$, $\mathscr{G}^q + \theta^q \leq 1$, for $q \geq 1$. Let $U, V \in {}^qROF_zS(\xi)$ be such that $U \leq V$ which implies $U^{\theta}_{\mathscr{G}} \subseteq V^{\theta}_{\mathscr{G}}$. Then According to the Theorem 1.2.3, $\underline{J}^{U^{\theta}_{\mathscr{G}}} \leq \underline{J}^{V^{\theta}_{\mathscr{G}}}$, this implies that $\frac{|\underline{J}^{U^{\theta}_{\mathscr{G}}}|}{|\overline{J}^{U^{\mathscr{B}}_{\mathscr{A}}}|} \leq \frac{|\underline{J}^{V^{\theta}_{\mathscr{G}}}|}{|\overline{J}^{V^{\theta}_{\mathscr{G}}}|}$, whenever $\overline{J}^{U^{\mathscr{B}}_{\mathscr{A}}} = \overline{J}^{V^{\mathscr{B}}_{\mathscr{A}}}$. Thus, $\delta^{(\mathscr{G},\theta)}_{(\mathscr{A},\mathscr{B})}(J^U) \leq \delta^{(\mathscr{G},\theta)}_{(\mathscr{A},\mathscr{B})}(J^V)$.

ii) The proof can be derived using the same approach as in part (i).

□

Corollary 3.4.10. *Let J be a R_lB_nR on ξ, and let $U, V \in {}^qROF_zS(\xi)$. Assume $\mathscr{A}, \mathscr{B}, \mathscr{G}, \theta \in [0,1]$ such that $\mathscr{A} \leq \mathscr{G}$, $\mathscr{B} \geq \theta$, and $\mathscr{A}^q + \mathscr{B}^q \leq 1$, $\mathscr{G}^q + \theta^q \leq 1$ for $q \geq 1$. If $U \leq V$, the following assertions hold with respect to the A_fSs:*

i) $\eta^{(\mathscr{G},\theta)}_{(\mathscr{A},\mathscr{B})}(J^U) \leq \eta^{(\mathscr{G},\theta)}_{(\mathscr{A},\mathscr{B})}(J^V)$, *whenever* $\overline{J}^{U^{\mathscr{B}}_{\mathscr{A}}} = \overline{J}^{V^{\mathscr{B}}_{\mathscr{A}}}$

ii) $\eta^{(\mathscr{G},\theta)}_{(\mathscr{A},\mathscr{B})}(J^U) \geq \eta^{(\mathscr{G},\theta)}_{(\mathscr{A},\mathscr{B})}(J^V)$, *whenever* $\underline{J}^{U^{\mathscr{B}}_{\mathscr{A}}} = \underline{J}^{V^{\mathscr{B}}_{\mathscr{A}}}$.

Proof.

Theorem 3.4.2 directly leads to this conclusion.. □

Theorem 3.4.11. *Let J_1 be a R_lB_nR on ξ, $U \in {}^qROF_zS(\xi)$, and let $\mathscr{A}, \mathscr{B}, \mathscr{G}, \theta \in [0,1]$ be such that $\mathscr{A} \leq \mathscr{G}$, $\mathscr{B} \geq \theta$, and $\mathscr{A}^q + \mathscr{B}^q \leq 1$, $\mathscr{G}^q + \theta^q \leq 1$ for $q \geq 1$. Suppose J_2 is another R_lB_nR on ξ such that $J_1 \subseteq J_2$. Then, with respect to the A_fSs, we have $\delta^{(\mathscr{G},\theta)}_{(\mathscr{A},\mathscr{B})}(J_1^U) \geq \delta^{(\mathscr{G},\theta)}_{(\mathscr{A},\mathscr{B})}(J_2^U)$.*

Proof.
Let $U \in {}^qROF_zS(\xi)$ and J_1, J_2 be two R_lB_nRs on ξ such that $J_1 \subseteq J_2$. According to Theorem 3.1.4, $\underline{J_1}^U \geq \underline{J_2}^U$ and $\overline{J_1}^U \leq \overline{J_2}^U$.
Applying Theorem 3.4.2, we find $\underline{J_2}^{U^\theta_\mathscr{G}} \supseteq \underline{J_2}^{U^\theta_\mathscr{G}}$ and $\overline{J_1}^{U^\mathscr{B}_\mathscr{A}} \subseteq \overline{J_2}^{U^\mathscr{B}_\mathscr{A}}$, which implies $|\underline{J_1}^{U^\theta_\mathscr{G}}| \geq |\underline{J_2}^{U^\theta_\mathscr{G}}|$ and $|\overline{J_1}^{U^\mathscr{B}_\mathscr{A}}| \leq |\overline{J_2}^{U^\mathscr{B}_\mathscr{A}}|$.
Dividing these inequalities gives us $\frac{|\underline{J_1}^{U^\theta_\mathscr{G}}|}{|\overline{J_1}^{U^\mathscr{B}_\mathscr{A}}|} \geq \frac{|\underline{J_2}^{U^\theta_\mathscr{G}}|}{|\overline{J_2}^{U^\mathscr{B}_\mathscr{A}}|}$. Hence, $\delta^{(\mathscr{G},\theta)}_{(\mathscr{A},\mathscr{B})}(J_1^U) \geq \delta^{(\mathscr{G},\theta)}_{(\mathscr{A},\mathscr{B})}(J_2^U)$. □

Corollary 3.4.12. *Let J be a R_lB_nR on a non-empty set ξ, $U \in {}^qROF_zS(\xi)$, and $\mathscr{A}, \mathscr{B}, \mathscr{G}, \theta \in [0,1]$ such that $\mathscr{A} \leq \mathscr{G}$, $\mathscr{B} \geq \theta$, and $\mathscr{A}^q + \mathscr{B}^q \leq 1$, $\mathscr{G}^q + \theta^q \leq 1$. If J_2 is another R_lB_nR on ξ such that $J_1 \subseteq J_2$, then $\eta^{(\mathscr{G},\theta)}_{(\mathscr{A},\mathscr{B})}(J_1^U) \geq \eta^{(\mathscr{G},\theta)}_{(\mathscr{A},\mathscr{B})}(J_2^U)$.*

Proof.
Theorem 3.4.11 directly leads to this conclusion. □

3.5 Application of proposed approach in Decision Making

The R_fS model by Pawlak is a qualitative framework that partitions a universe of objects into three distinct regions based on a C_rB_nR over the universe. A notable concern regarding Pawlak R_fS approximations is their perceived rigidity, where classifications are either entirely correct or definitive. F_zS theory offers a promising avenue to mitigate this issue. It represents a significant and successful extension of quantitative R_fS theory, particularly through the definitions of L_oA_p and U_pA_p sets in probabilistic R_fS. These sets are characterized by threshold parameters $(\mathscr{A}, \mathscr{B})$, where $\mathscr{A} > \mathscr{B}$, delineating three distinct regions for approximating subsets of the universe of objects [63].

Among probabilistic R_fS models, the Decision-Theoretic R_fS model (DTRS) was introduced in the early 1990s, drawing from established Bayesian decision procedures pioneered by Yao et al. [64] and Yao and Wong [65]. The DTRS model employs Bayesian decision theory concepts to compute probabilistic parameters that define rough regions. By integrating the notion of expected loss (conditional risk), the model allows users to base classification decisions on their specified cost considerations. It provides a systematic framework for determining these parameters within a probabilistic R_fS model.

In their work [54], Sun et al. proposed an approach for R_fS over dual universes using Bayesian decision-making techniques, further enriching the methodologies available for handling uncertainty and imprecision in decision-making scenarios.

In the realm of decision-making problems, the ${}^qROF_zS$, introduced by Yager [62], has garnered attention for its operational capabilities and applications. Researchers have made substantial contributions to ${}^qROF_zS$ theory, with its applications spanning various domains and yielding numerous practical implementations.

In this chapter, we introduce an alternative approach to address decision-making problems using $R_f{}^qROF_zS$ through C_rB_nRs, extending upon methodologies of Chapter 2, Kanwal and Shabir [31], Sun et al. [54], and Hussain et al. [28]. This approach leverages only the data provided by the decision-making problem itself. Consequently, it mitigates the impact of subjective information on decision outcomes, fostering more objective results and averting paradoxical outcomes that may arise due to variations in subjective factors among experts.

The rough L_oA_p and U_pA_p are closely associated with the approximation of the universe's subset. Accordingly, we derive the closest values $\underline{J}^U(\Bbbk_i)$ and $\overline{J}^U(\Bbbk_i)$ relative to the A_fSs for each decision alternative $\Bbbk_i \in \xi$ within the universe ξ, using the ${}^qROF_zL_oA_p$ and ${}^qROF_zU_pA_p$ of the ${}^qROF_zS$ U. Hence, we define the choice-value λ_i for each decision alternative $\Bbbk_i$ in ξ with respect to the A_fSs as follows:

$$\lambda_i = S(\underline{J}^U(\Bbbk_i) \oplus \overline{J}^U(\Bbbk_i))$$

where S denotes the score function as defined in Definition 1.1.10. The decision alternative $\mathfrak{k}_i \in \xi$ with the maximum value of λ_i is selected as the optimal decision for the given decision-making problem, and the one with the minimum value of λ_i is considered the worst decision. If multiple objects $\mathfrak{k}_i \in \xi$ exhibit the same maximum (minimum) value of λ_i, one of them is taken as the optimal decision.

Here, we present two algorithms for the proposed model, which consist of the following steps:

One can use ring product operation $\otimes$ to perform Algorithm 3 and 4.

Algorithm 3

1: Using Definition 3.1.1 compute the $L_o{}^qROF_zS$ approximation $\underline{J}^U$ and $U_p{}^qROF_zS$ approximation $\overline{J}^U$ of a ${}^qROF_zS$ U with respect to the A_fSs.

2: Using the sum operation $\oplus$, compute the choice set as $T = \underline{J}^U \oplus \overline{J}^U$

3: Calculate the choice value using the score function defined in Definition 1.1.10:

$$\lambda_i = S(T(\mathfrak{k}_i))$$

4: The best decision is $\mathfrak{k}_m \in \xi$ if $\lambda_m = max_i \lambda_i$, $i = 1, 2, 3, ... \mid \xi \mid$.

5: The bad decision is $\mathfrak{k}_m \in \xi$ if $\lambda_m = min_i \lambda_i$, $i = 1, 2, 3, ... \mid \xi \mid$.

6: If m has multiple values, select any $\mathfrak{k}_m$ as the preferred or least preferred alternative.

Algorithm 4

1: Using Definition 3.1.2 compute the $L_o{}^qROF_zS$ approximation ${}^U\underline{J}$ and $U_p{}^qROF_zS$ approximation ${}^U\overline{J}$ of a ${}^qROF_zS$ U with respect to the F_rS.

2: Using the sum operation $\oplus$, compute the choice set as $T^{'} = {}^U\underline{J} \oplus {}^U\overline{J}$.

3: Calculate the choice value using the score function defined in Definition 1.1.10:

$$\lambda_i = S(T^{'}(\mathfrak{t}_i))$$

4: The best decision is $\mathfrak{t}_m \in \xi$ if $\lambda_m = max_i \lambda_i$, $i = 1, 2, 3, ... \mid \xi \mid$.

5: The bad decision is $\mathfrak{t}_m \in \xi$ if $\lambda_m = min_i \lambda_i$, $i = 1, 2, 3, ... \mid \xi \mid$.

6: If m has multiple values, select any $\mathfrak{t}_m$ as the preferred or least preferred alternative.

3.5.1 An application of the decision-making approach

In this section, we explore emergency decision-making within the framework of rough $^{q}ROF_{z}S$ *over dual universes. Effective emergency preparedness plans are crucial for ensuring rapid and efficient emergency responses while minimizing losses. Current research emphasizes qualitative evaluation criteria such as effectiveness, cost-efficiency, and adequacy of protection, among others, and offers enhancements to these metrics.*

This body of literature presents methodologies for assessing the relative importance of each criterion and indicator, determining the weighting of expert opinions, aggregating group judgments and opinions, and addressing related challenges. It also provides quantitative evaluations using established methods for emergency preparedness planning. Consequently, this research serves as a practical guide for decision-makers in selecting optimal emergency plans.

3.5.2 Problem Statement

The essential characteristics of an emergency preparedness plan are essentially described by the evaluation criteria and indicators for emergency decision-making. Consequently, expert scoring or pairwise comparisons are not relied upon for evaluating these indicators. The emergency preparedness plan encompasses attributes like specificity, comprehensiveness, and promptness in emergency response, among other relevant aspects, which are collectively represented as a set or universe referred to as ξ_2.

Here, ξ_2 *represents the entirety of characteristics defining the emergency preparedness plan, formally expressed as*

$\xi_2 = \{strong\ pertinence(f_1), soundness\ of\ personnel\ and\ resource\ allocation(f_2),$
$effective\ inter\text{-}sectoral\ collaboration(f_3), \ldots, reasonable\ cost(f_n)\}$. *Generally,* ξ_2 *is finite because the indicators that describe the fundamental features of the plan are limited in number. All emergency preparedness plans are organized into a particular grouping or category at the same time, denoted by* ξ_1, *where* $\xi_1 = \{k_1, k_2, \ldots, k_m\}$, *and each* k_i *represents an individual*

emergency plan. The relationship between the set of emergency preparedness plans ξ_1 *and the set of characteristics* ξ_2 *is encapsulated in a subset* $J \subseteq \xi_1 \times \xi_2$*. For any emergency plan* $ƙ \in \xi_1$*, its basic characteristics are represented by the* A_fS $ƙJ$*.*

The emergency decision-making process is structured as follows:

1. Initially, assume all emergency preparedness plans are characterized by a finite set of essential traits.

2. The principal characteristics are presented to decision-makers U*, denoted as a* ${}^qROF_z$ *subset of the universe* $\xi_2 = \{ƭ_j : j = 1, 2, \ldots, n\}$*, associated with an effective emergency plan are real-time information and online scenarios.*

3. Subsequently, decision-makers select one of the plans $ƙ_i \in \xi_1$ *(where* $i = 1, 2, \ldots, m$*), the optimal decision is determined by minimizing the risk of loss, and then carrying out that plan..*

Example 3.5.1. *Let* $\xi_1 = \{ƙ_1, ƙ_2, \ldots, ƙ_8\}$ *are eight plans for emergency preparedness that are tailored for particular types of non-traditional emergency situations. These plans are characterized by various essential attributes or evaluation indicators, represented by* ξ_2*. The set* ξ_2 *includes the following basic characteristics:*

$$\xi_2 = \{ƭ_1, ƭ_2, \ldots, ƭ_{15}\}$$

where $ƭ_1$*: Risk identification comprehensiveness,* $ƭ_2$*: Prevention and warning completeness,* $ƭ_3$*: Formation specifics,* $ƭ_4$*: Post-event disposal program completeness,* $ƭ_5$*: Scientific rescue program,* $ƭ_6$*: Good traceability of emergency resources,* $ƭ_7$*: Strong pertinence,* $ƭ_8$*: Comprehensiveness of elements of plan,* $ƭ_9$*: Competent rescue team members,* $ƭ_{10}$*: Clear response level,* $ƭ_{11}$*: Quick emergency handling,* $ƭ_{12}$*: Effective guarantee measures,* $ƭ_{13}$*: Reasonable rescue steps,* $ƭ_{14}$*: Responsibility clear among agencies,* $ƭ_{15}$*: Median cost of emergency resources.*

The relation $J \subseteq \xi_1 \times \xi_2$ *defines the main characteristics of each emergency preparedness plan* $ƙ \in \xi_1$*. Specifically, for each* $ƙ_i \in \xi_1$*, the characteristics* $ƙ_iJ$ *are specified based on the values assigned to each indicator* $ƭ_j \in \xi_2$*.*

This setup allows for a comprehensive evaluation of each emergency preparedness plan based on its specific attributes, ensuring that decision-makers can select the most suitable plan tailored to the requirements of the emergency event in question.

$$
J = \begin{array}{c} \\ t_{\mathbf{1}} \\ t_{\mathbf{2}} \\ t_{\mathbf{3}} \\ t_{\mathbf{4}} \\ t_{\mathbf{5}} \\ t_{\mathbf{6}} \\ t_{\mathbf{7}} \\ t_{\mathbf{8}} \\ t_{\mathbf{9}} \\ t_{\mathbf{10}} \\ t_{\mathbf{11}} \\ t_{\mathbf{12}} \\ t_{\mathbf{13}} \\ t_{\mathbf{14}} \\ t_{\mathbf{15}} \end{array}
\begin{array}{c} \begin{array}{cccccccc} \Bbbk_{\mathbf{1}} & \Bbbk_{\mathbf{2}} & \Bbbk_{\mathbf{3}} & \Bbbk_{\mathbf{4}} & \Bbbk_{\mathbf{5}} & \Bbbk_{\mathbf{6}} & \Bbbk_{\mathbf{7}} & \Bbbk_{\mathbf{8}} \end{array} \\
\begin{pmatrix}
0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 \\
0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 1 & 1 & 1 & 1 & 1 & 0 & 1 \\
0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\
1 & 0 & 1 & 1 & 0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 & 1 & 1 & 1 & 0 \\
1 & 1 & 0 & 1 & 0 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\
1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 \\
0 & 0 & 1 & 1 & 1 & 0 & 0 & 1 \\
1 & 1 & 0 & 0 & 0 & 1 & 1 & 0
\end{pmatrix} \end{array}
$$

This matrix provides an objective description of the characteristics associated with eight emergency preparedness plans designed for a specific type of emergency event. Similar to the previous analysis, no plan is strictly superior or inferior; the sole criterion for decision-making is the suitability of the plan.

When choosing an emergency preparedness plan, various types of loss functions or risks may be encountered during an unusual emergency event. Therefore, a team of experts expresses their preference for specific plan characteristics using a $^{q}ROF_{z}S$ (q-Rung Orthopair Fuzzy set), for $q = 3$;

$A = \{\langle \text{ŧ}_1, 0.9, 0.45\rangle, \langle \text{ŧ}_2, 0.8, 0.45\rangle, \langle \text{ŧ}_3, 0.4, 0.7\rangle, \langle \text{ŧ}_4, 0.62, 0.78\rangle, \langle \text{ŧ}_5, 0.74, 0.32\rangle, \langle \text{ŧ}_6, 0.7, 0.4\rangle,$
$\langle \text{ŧ}_7, 0.65, 0.46\rangle, \langle \text{ŧ}_8, 0.9, 0.1\rangle, \langle \text{ŧ}_9, 0.81, 0.65\rangle, \langle \text{ŧ}_{10}, 0.85, 0.31\rangle, \langle \text{ŧ}_{11}, 0.84, 0.3\rangle, \langle \text{ŧ}_{12}, 0.94, 0.23\rangle,$
$\langle \text{ŧ}_{13}, 0.74, 0.31\rangle, \langle \text{ŧ}_{14}, 0.74, 0.13\rangle, \langle \text{ŧ}_{15}, 0.12, 0.83\rangle\}.$

Given that ŧ_{15} *represents the cost characteristic, we will consider its complement to define a new* ${}^qROF_zS$. *Therefore, the new* ${}^qROF_zS$ *will focus on capturing the degree of cost-effectiveness or economy associated with each emergency preparedness plan.*

$A = \{\langle \text{ŧ}_1, 0.9, 0.45\rangle, \langle \text{ŧ}_2, 0.8, 0.45\rangle, \langle \text{ŧ}_3, 0.4, 0.7\rangle, \langle \text{ŧ}_4, 0.62, 0.78\rangle, \langle \text{ŧ}_5, 0.74, 0.32\rangle, \langle \text{ŧ}_6, 0.7, 0.4\rangle,$
$\langle \text{ŧ}_7, 0.65, 0.46\rangle, \langle \text{ŧ}_8, 0.9, 0.1\rangle, \langle \text{ŧ}_9, 0.81, 0.65\rangle, \langle \text{ŧ}_{10}, 0.85, 0.31\rangle, \langle \text{ŧ}_{11}, 0.84, 0.3\rangle, \langle \text{ŧ}_{12}, 0.94, 0.23\rangle,$
$\langle \text{ŧ}_{13}, 0.74, 0.31\rangle, \langle \text{ŧ}_{14}, 0.74, 0.13\rangle, \langle \text{ŧ}_{15}, 0.83, 0.12\rangle\}$

Table 3.3: Approximations of ${}^qROF_zS$ U with respect to A_fSs

	$\overline{\mathbf{J}}^{\mathbf{U}}(\text{ķ}_i)$	$\underline{\mathbf{J}}^{\mathbf{U}}(\text{ķ}_i)$	$\mathbf{T}(\text{ķ}_\mathbf{i})$	$\lambda_\mathbf{i}$
$\text{ķ}_\mathbf{1}$	$(0.94, 0.1)$	$(0.4, 0.78)$	$(0.8, 0.08)$	0.7587
$\text{ķ}_\mathbf{2}$	$(0.85, 0.12)$	$(0.65, 0.46)$	$(0.7, 0.06)$	0.6677
$\text{ķ}_\mathbf{3}$	$(0.9, 0.1)$	$(0.74, 0.65)$	$(0.8, 0.07)$	0.7344
$\text{ķ}_\mathbf{4}$	$(0.94, 0.1)$	$(0.74, 0.45)$	$(0.81, 0.05)$	0.7696
$\text{ķ}_\mathbf{5}$	$(0.9, 0.13)$	$(0.7, 0.65)$	$(0.76, 0.09)$	0.7209
$\text{ķ}_\mathbf{6}$	$(0.84, 0.12)$	$(0.7, 0.65)$	$(0.7, 0.08)$	0.6735
$\text{ķ}_\mathbf{7}$	$(0.94, 0.12)$	$(0.62, 0.78)$	$(0.8, 0.09)$	0.7424
$\text{ķ}_\mathbf{8}$	$(0.9, 0.1)$	$(0.74, 0.45)$	$(0.8, 0.05)$	0.7345

In this context, we utilized the sum operation $\oplus$ *to compute* T, *and* λ_i *was determined based on Definition 1.1.10. All computations were performed using Python.*

According to Table 3.3, it is evident that plan ķ_4 *emerges as the optimal decision, achieving the highest score of* 0.7696.

This chapter expanded the theoretical understanding of q-Rung Orthopair Fuzzy Sets by *introducing rough approximations and exploring their applications in decision-making. The proposed methods and algorithms were validated through illustrative examples and computational experiments. These contributions enrich the theory of Fuzzy Sets and provide practical tools for addressing real-world problems in various domains.*

Chapter 4

Graphical ranking technique for Generalized Rough q-Rung Orthopair Fuzzy Sets based on Soft Binary Relations and corresponding decision-making

We've structured this chapter as follows:

In Section 4.1, we provide an exposition on L_oA_p and U_pA_ps of ${}^qROF_zSs$, employing S_fB_nRs concerning F_rS and A_fSs, along with a demonstration of their properties. Section 4.2 is dedicated to the introduction of two varieties of q-Rung Orthopair Fuzzy Topologies, established through S_fB_nRs. Furthermore, Section 4.3 delves into the introduction of similarity relations among ${}^qROF_zSs$, grounded on S_fB_nRs. Then, we introduce an algorithm specifically crafted to address decision-making challenges using ${}^qROF_zSs$. Additionally, we present a practical example to exemplify the utilization of this approach and to showcase its effectiveness in real-world decision-making scenarios.

4.1 Approximating a q-Rung Orthopair Fuzzy set by Soft Binary Relation

In Chapter 3, we have introduced the concept of approximating a ${}^{q}ROF_zS$ *using* C_rB_nRs*. Here, we extend this idea by utilizing a* S_fB_nR *from a finite universe* ξ_1 *to another finite universe* ξ_2 *to approximate a* ${}^{q}ROF_zS$ *in* ξ_2 *through* A_fSs*. Consequently, we derive two* ${}^{q}ROF_zSs$ *on* ξ_1*. Similarly, by approximating a* ${}^{q}ROF_zS$ *in* ξ_1 *through* F_rS*, we obtain two* ${}^{q}ROF_zSs$ *on* ξ_2*.*

Definition 4.1.1. *Let* $(\mathscr{S}, D)$ *represent a* S_fB_nR *from* ξ_1 *to* ξ_2 *and* $U = \{\langle t, U_Y(t), U_N(t)\rangle : t \in \xi_2\}$ *be a* ${}^{q}ROF_zS$ *in* ξ_2*. Then, the* L_oA_p $\underline{\mathscr{S}}^U$ *and the* U_pA_p $\overline{\mathscr{S}}^U$ *of* $U = \{\langle t, U_Y(t), U_N(t)\rangle : t \in \xi_2\}$ *with respect to* A_fSs *are defined as follows, for all* $k \in \xi_1$*:*

$$\underline{\mathscr{S}}^U(e)(k) = \begin{cases} (\bigwedge_{t\in k\mathscr{S}(e)} U_Y(t), \bigvee_{t\in k\mathscr{S}(e)} U_N(t)) & if\ k\mathscr{S}(e) \neq \emptyset; \\ (1,0) & if\ k\mathscr{S}(e) = \emptyset. \end{cases}$$

and

$$\overline{\mathscr{S}}^U(e)(k) = \begin{cases} (\bigvee_{t\in k\mathscr{S}(e)} U_Y(t), \bigwedge_{t\in k\mathscr{S}(e)} U_N(t)) & if\ k\mathscr{S}(e) \neq \emptyset; \\ (0,1) & if\ k\mathscr{S}(e) = \emptyset. \end{cases}$$

Here, $k\mathscr{S}(e) = \{t \in \xi_2 \mid (k, t) \in \mathscr{S}(e)\}$ *is termed as the* A_fS *of* k *for all* $k \in \xi_1$ *and* $e \in D$*.*

Definition 4.1.2. *Let* $(\mathscr{S}, D)$ *be a* S_fB_nR *from* ξ_1 *to* ξ_2 *and* $U = \{\langle k, U_Y(k), U_N(k)\rangle : k \in \xi_1\}$ *be a* ${}^{q}ROF_zS$ *in* ξ_1*. Then, the* L_oA_p ${}^{U}\underline{\mathscr{S}}$ *and the* U_pA_p ${}^{U}\overline{\mathscr{S}}$ *of* $U = \{\langle k, U_Y(k), U_N(k)\rangle : k \in \xi_1\}$ *with respect to* F_rS *(*F_rSs*) are defined as follows, for all* $t \in \xi_2$*:*

$${}^{U}\overline{\mathscr{S}}(e)(t) = \begin{cases} (\bigvee_{k\in \mathscr{S}(e)t} U_N(k), \bigwedge_{k\in \mathscr{S}(e)t} U_N(k)) & if\ \mathscr{S}(e)t \neq \emptyset; \\ (0,1) & if\ \mathscr{S}(e)t = \emptyset. \end{cases}$$

and

$$^{U}\underline{\mathscr{S}}(e)(\mathfrak{t}) = \begin{cases} (\bigwedge_{\mathfrak{k}\in\mathscr{S}(e)\mathfrak{t}} U_Y(\mathfrak{k}), \bigvee_{\mathfrak{k}\in\mathscr{S}(e)\mathfrak{t}} U_N(\mathfrak{k})) & \text{if } \mathscr{S}(e)\mathfrak{t} \neq \emptyset; \\ (1,0) & \text{if } \mathscr{S}(e)\mathfrak{t} = \emptyset. \end{cases}$$

Here, $\mathscr{S}(e)\mathfrak{t} = \{\mathfrak{k} \in \xi_1 \mid (\mathfrak{k}, \mathfrak{t}) \in \mathscr{S}(e)\}$, *and is called the* F_rS *of* $\mathfrak{t}$ *for all* $\mathfrak{t} \in \xi_2$ *and* $e \in D$.

Here, we have $\underline{\mathscr{S}}^U : D \to {}^qROF_zS(\xi_1)$, $\overline{\mathscr{S}}^U : D \to {}^qROF_zS(\xi_1)$, $^U\underline{\mathscr{S}} : D \to {}^qROF_zS(\xi_2)$ *and* $^U\overline{\mathscr{S}} : D \to {}^qROF_zS(\xi_2)$. *The triplet* $(\xi_1, \xi_2, \mathscr{S})$ *will be termed as a Generalized Soft Approximation Space (*$GS_fA_pS_p$*) based on* $^qROF_zSs$. *In the following example, we explain the above concepts.*

Example 4.1.3. *Let* $\xi_1 = \{\mathfrak{k}_1, \mathfrak{k}_2, \mathfrak{k}_3, \mathfrak{k}_4\}$, $\xi_2 = \{\mathfrak{t}_1, \mathfrak{t}_2, \mathfrak{t}_3, \mathfrak{t}_4\}$, *and the set of attributes* $D = \{e_1, e_2\}$. *Define a mapping* $\mathscr{S} : D \to P(\xi_1 \times \xi_2)$ *by,*

$$\mathscr{S}(e_1) = \begin{array}{c} \\ \mathfrak{t}_1 \\ \mathfrak{t}_2 \\ \mathfrak{t}_3 \\ \mathfrak{t}_4 \end{array} \begin{array}{c} \begin{array}{cccc} \mathfrak{k}_1 & \mathfrak{k}_2 & \mathfrak{k}_3 & \mathfrak{k}_4 \end{array} \\ \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix} \end{array} \quad \text{and} \quad \mathscr{S}(e_2) = \begin{array}{c} \\ \mathfrak{t}_1 \\ \mathfrak{t}_2 \\ \mathfrak{t}_3 \\ \mathfrak{t}_4 \end{array} \begin{array}{c} \begin{array}{cccc} \mathfrak{k}_1 & \mathfrak{k}_2 & \mathfrak{k}_3 & \mathfrak{k}_4 \end{array} \\ \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 \end{pmatrix} \end{array}$$

represent the S_fB_nRs *from* ξ_1 *to* ξ_2 *for the attributes* e_1, e_2, *respectively. Now let* $U \in {}^qROF_zS(\xi_2)$ *and* $V \in {}^qROF_zS(\xi_1)$, *for* $q = 3$, *defined by:*

$$U = \{\langle \mathfrak{t}_1, 0.685, 0.233\rangle, \langle \mathfrak{t}_2, 0.785, 0.221\rangle, \langle \mathfrak{t}_3, 0.765, 0.315\rangle, \langle \mathfrak{t}_4, 0.686, 0.370\rangle\},$$
$$V = \{\langle \mathfrak{k}_1, 0.52, 0.53\rangle, \langle \mathfrak{k}_2, 0.40, 0.45\rangle, \langle \mathfrak{k}_3, 0.51, 0.51\rangle, \langle \mathfrak{k}_4, 0.49, 0.52\rangle\}.$$

Table 4.1 shows that the L_oA_p $\underline{\mathscr{S}}^U$ *and the* U_pA_p $\overline{\mathscr{S}}^U$ *of* $^qROF_zS$ U *with respect to* A_fSs $\mathfrak{k}_i\mathscr{S}(e_j)$ *are two* $^qROF_zSs$ *on* ξ_1. *Similarly Table 4.2 shows that* L_oA_p $^V\underline{\mathscr{S}}$ *and the* U_pA_p $^V\overline{\mathscr{S}}$ *of* $^qROF_zS$ V *with respect to* F_rSs $\mathscr{S}(e_j)\mathfrak{t}_i$ *are two* $^qROF_zSs$ *on* ξ_2, *where* $1 \leq i \leq 4$ *and* $1 \leq j \leq 2$.

Table 4.1: Approximating a ${}^{q}ROF_{z}S$ with respect to $A_{f}S$s

	$\overline{\mathscr{S}}^{\mathbf{U}}(e_1)(k_i)$	$\underline{\mathscr{S}}^{\mathbf{U}}(e_1)(k_i)$	$\overline{\mathscr{S}}^{\mathbf{U}}(e_2)(k_i)$	$\underline{\mathscr{S}}^{\mathbf{U}}(e_2)(k_i)$
$k_{\mathbf{1}}$	$(0.785, 0.221)$	$(0.685, 0.233)$	$(0.765, 0.315)$	$(0.686, 0.370)$
$k_{\mathbf{2}}$	$(0.785, 0.221)$	$(0.765, 0.315)$	$(0.785, 0.221)$	$(0.685, 0.370)$
$k_{\mathbf{3}}$	$(0.785, 0.221)$	$(0.685, 0.370)$	$(0.765, 0.315)$	$(0.765, 0.315)$
$k_{\mathbf{4}}$	$(0.765, 0.315)$	$(0.686, 0.370)$	$(0, 1)$	$(1, 0)$

Table 4.2: Approximating a ${}^{q}ROF_{z}S$ with respect to $F_{r}S$s

	${}^{\mathbf{V}}\overline{\mathscr{S}}(e_1)(t_i)$	${}^{\mathbf{V}}\underline{\mathscr{S}}(e_1)(t_i)$	${}^{\mathbf{V}}\overline{\mathscr{S}}(e_2)(t_i)$	${}^{\mathbf{V}}\underline{\mathscr{S}}(e_2)(t_i)$
$t_{\mathbf{1}}$	$(0.52, 0.51)$	$(0.51, 0.53)$	$(0.40, 0.45)$	$(0.40, 0.45)$
$t_{\mathbf{2}}$	$(0.52, 0.45)$	$(0.40, 0.53)$	$(0.40, 0.45)$	$(0.40, 0.45)$
$t_{\mathbf{3}}$	$(0.49, 0.45)$	$(0.40, 0.52)$	$(0.52, 0.45)$	$(0.40, 0.53)$
$t_{\mathbf{4}}$	$(0.51, 0.51)$	$(0.49, 0.52)$	$(0.52, 0.45)$	$(0.40, 0.53)$

In the following theorem, we have presented some basic operations on this newly obtained collection of ${}^{q}ROF_{z}Ss$ on ξ_2.

Theorem 4.1.4. *Let $(\xi_1, \xi_2, \mathscr{S})$ be a $GS_{f}A_{p}S_{p}$. For any three ${}^{q}ROF_{z}Ss$, $U = \{\langle t, U_Y(t), U_N(t)\rangle : t \in \xi_2\}$, $U_1 = \{\langle t, U_{1_Y}(t), U_{1_N}(t)\rangle : t \in \xi_2\}$, and $U_2 = \{\langle t, U_{2_Y}(t), U_{2_N}(t)\rangle : t \in \xi_2\}$ of ξ_2, we have the following:*

i) $U_1 \subseteq U_2$ *implies* $\underline{\mathscr{S}}^{U_1} \subseteq \underline{\mathscr{S}}^{U_2}$

ii) $U_1 \subseteq U_2$ *implies* $\overline{\mathscr{S}}^{U_1} \subseteq \overline{\mathscr{S}}^{U_2}$

iii) $\underline{\mathscr{S}}^{U_1 \cap U_2} = \underline{\mathscr{S}}^{U_1} \cap \underline{\mathscr{S}}^{U_2}$

iv) $\underline{\mathscr{L}}^{U_1\cup U_2} \supseteq \underline{\mathscr{L}}^{U_1} \cup \underline{\mathscr{L}}^{U_2}$

v) $\overline{\mathscr{S}}^{U_1\cap U_2} \subseteq \overline{\mathscr{S}}^{U_1} \cap \overline{\mathscr{S}}^{U_2}$

vi) $\overline{\mathscr{S}}^{U_1\cup U_2} = \overline{\mathscr{S}}^{U_1} \cup \overline{\mathscr{S}}^{U_2}$

vii) $\underline{\mathscr{L}}^{1_{\xi_2}} = 1_{\xi_1} = \overline{\mathscr{S}}^{1_{\xi_2}}$, *if* $\mathfrak{k}\mathscr{S}(e) \neq \emptyset$

viii) $\underline{\mathscr{L}}^{U} = (\overline{\mathscr{S}}^{U^c})^c$ *and* $\overline{\mathscr{S}}^{U} = (\underline{\mathscr{L}}^{U^c})^c$, *if* $\mathfrak{k}\mathscr{S}(e) \neq \emptyset$

ix) $\underline{\mathscr{L}}^{0_{\xi_2}} = 0_{\xi_1} = \overline{\mathscr{S}}^{0_{\xi_2}}$.

Proof.

i) Let $U_1 \subseteq U_2$, that is, for all $\mathfrak{t} \in \xi_2$, $U_{1_Y}(\mathfrak{t}) \leq U_{2_Y}(\mathfrak{t})$, and $U_{1_N}(\mathfrak{t}) \geq U_{2_N}(\mathfrak{t})$.
If $\mathfrak{k}\mathscr{S}(e) = \emptyset$, then $\underline{\mathscr{L}}^{U_1} = (1,0) = \underline{\mathscr{L}}^{U_2}$.
If $\mathfrak{k}\mathscr{S}(e) \neq \emptyset$, then $\underline{\mathscr{L}}^{U_{1_Y}}(e)(\mathfrak{k}) = \bigwedge_{\mathfrak{t}\in\mathfrak{k}\mathscr{S}(e)} U_{1_Y}(\mathfrak{t}) \leq \bigwedge_{\mathfrak{t}\in\mathfrak{k}\mathscr{S}(e)} U_{2_Y}(\mathfrak{t}) = \underline{\mathscr{L}}^{U_{2_Y}}(e)(\mathfrak{k})$ and $\underline{\mathscr{L}}^{U_{1_N}}(e)(\mathfrak{k}) = \bigvee_{\mathfrak{t}\in\mathfrak{k}\mathscr{S}(e)} U_{1_N}(\mathfrak{t}) \geq \bigvee_{\mathfrak{t}\in\mathfrak{k}\mathscr{S}(e)} U_{2_N}(\mathfrak{t}) = \underline{\mathscr{L}}^{U_{2_N}}(e)(\mathfrak{k})$.
Thus, $\underline{\mathscr{L}}^{U_{1_Y}}(e)(\mathfrak{k}) \leq \underline{\mathscr{L}}^{U_{2_Y}}(e)(\mathfrak{k})$ and $\underline{\mathscr{L}}^{U_{1_N}}(e)(\mathfrak{k}) \geq \underline{\mathscr{L}}^{U_{2_N}}(e)(\mathfrak{k})$. Hence, $\underline{\mathscr{L}}^{U_1} \subseteq \underline{\mathscr{L}}^{U_2}$.

ii) Let $U_1 \subseteq U_2$, that is, for all $\mathfrak{t} \in \xi_2$, $U_{1_Y}(\mathfrak{t}) \leq U_{2_Y}(\mathfrak{t})$, and $U_{1_N}(\mathfrak{t}) \geq U_{2_N}(\mathfrak{t})$.
If $\mathfrak{k}\mathscr{S}(e) = \emptyset$, then $\overline{\mathscr{S}}^{U_1} = (0,1) = \overline{\mathscr{S}}^{U_2}$.
If $\mathfrak{k}\mathscr{S}(e) \neq \emptyset$, then $\overline{\mathscr{S}}^{U_{1_Y}}(e)(\mathfrak{k}) = \bigvee_{\mathfrak{t}\in\mathfrak{k}\mathscr{S}(e)} U_{1_Y}(\mathfrak{t}) \leq \bigvee_{\mathfrak{t}\in\mathfrak{k}\mathscr{S}(e)} U_{2_Y}(\mathfrak{t}) = \overline{\mathscr{S}}^{U_{2_Y}}(e)(\mathfrak{k})$ and $\overline{\mathscr{S}}^{U_{1_N}}(e)(\mathfrak{k}) = \bigwedge_{\mathfrak{t}\in\mathfrak{k}\mathscr{S}(e)} U_{1_N}(\mathfrak{t}) \geq \bigwedge_{\mathfrak{t}\in\mathfrak{k}\mathscr{S}(e)} U_{2_N}(\mathfrak{t}) = \overline{\mathscr{S}}^{U_{2_N}}(e)(\mathfrak{k})$.
Thus, $\overline{\mathscr{S}}^{U_{1_Y}}(e)(\mathfrak{k}) \leq \overline{\mathscr{S}}^{U_{2_Y}}(e)(\mathfrak{k})$ and $\overline{\mathscr{S}}^{U_{1_N}}(e)(\mathfrak{k}) \geq \overline{\mathscr{S}}^{U_{2_N}}(e)(\mathfrak{k})$. Hence, $\overline{\mathscr{S}}^{U_1} \subseteq \overline{\mathscr{S}}^{U_2}$.

iii) Consider $(\underline{\mathscr{L}}^{U_{1_Y}} \cap \underline{\mathscr{L}}^{U_{2_Y}})(e)(\mathfrak{k}) = \underline{\mathscr{L}}^{U_{1_Y}}(e)(\mathfrak{k}) \wedge \underline{\mathscr{L}}^{U_{2_Y}}(e)(\mathfrak{k}) = (\bigwedge_{\mathfrak{t}\in\mathfrak{k}\mathscr{S}(e)} U_{1_Y}(\mathfrak{t})) \wedge (\bigwedge_{\mathfrak{t}\in\mathfrak{k}\mathscr{S}(e)} U_{2_Y}(\mathfrak{t})) = \bigwedge_{\mathfrak{t}\in\mathfrak{k}\mathscr{S}(e)}(U_{1_Y}(\mathfrak{t}) \wedge U_{2_Y}(\mathfrak{t})) = \underline{\mathscr{L}}^{U_1\cap U_2}(e)(\mathfrak{k})$, and
$(\underline{\mathscr{L}}^{U_{1_N}} \cup \underline{\mathscr{L}}^{U_{2_N}})(e)(\mathfrak{k}) = \underline{\mathscr{L}}^{U_{1_N}}(e)(\mathfrak{k}) \vee \underline{\mathscr{L}}^{U_{2_N}}(e)(\mathfrak{k}) = (\bigvee_{\mathfrak{t}\in\mathfrak{k}\mathscr{S}(e)} U_{1_N}(\mathfrak{t})) \vee (\bigvee_{\mathfrak{t}\in\mathfrak{k}\mathscr{S}(e)} U_{2_N}(\mathfrak{t})) = \bigvee_{\mathfrak{t}\in\mathfrak{k}\mathscr{S}(e)}(U_{1_N}(\mathfrak{t}) \vee U_{2_N}(\mathfrak{t})) = \underline{\mathscr{L}}^{U_1\cup U_2}(e)(\mathfrak{k})$.
Thus, $\underline{\mathscr{L}}^{U_1\cap U_2} = \underline{\mathscr{L}}^{U_1} \cap \underline{\mathscr{L}}^{U_2}$.

iv) Given that $U_1 \subseteq U_1 \cup U_2$ and $U_2 \subseteq U_1 \cup U_2$, it follows from part (i) that $\underline{\mathscr{S}}^{U_1} \subseteq \underline{\mathscr{S}}^{U_1 \cup U_2}$ and $\underline{\mathscr{S}}^{U_2} \subseteq \underline{\mathscr{S}}^{U_1 \cup U_2}$. Therefore, we conclude that $\underline{\mathscr{S}}^{U_1} \cup \underline{\mathscr{S}}^{U_2} \subseteq \underline{\mathscr{S}}^{U_1 \cup U_2}$.

v) Given that $U_1 \cap U_2 \subseteq U_1$ and $U_1 \cap U_2 \subseteq U_2$, it follows from part (ii) that $\overline{\mathscr{S}}^{U_1 \cap U_2} \subseteq \overline{\mathscr{S}}^{U_1}$ and $\overline{\mathscr{S}}^{U_1 \cap U_2} \subseteq \overline{\mathscr{S}}^{U_2}$. Therefore, we conclude that $\overline{\mathscr{S}}^{U_1 \cap U_2} \subseteq \overline{\mathscr{S}}^{U_1} \cap \overline{\mathscr{S}}^{U_2}$.

vi) Consider $(\overline{\mathscr{S}}^{U_{1_Y}} \cup \overline{\mathscr{S}}^{U_{2_Y}})(e)(\mathfrak{k}) = \overline{\mathscr{S}}^{U_{1_Y}}(e)(\mathfrak{k}) \vee \overline{\mathscr{S}}^{U_{2_Y}}(e)(\mathfrak{k}) = (\bigvee_{\mathfrak{k} \in \mathfrak{k}\mathscr{S}(e)} U_{1_Y}(\mathfrak{k})) \vee (\bigvee_{\mathfrak{k} \in \mathfrak{k}\mathscr{S}(e)} U_{2_Y}(\mathfrak{k})) = \bigvee_{\mathfrak{k} \in \mathfrak{k}\mathscr{S}(e)}(U_{1_Y}(\mathfrak{k}) \vee U_{2_Y}(\mathfrak{k})) = \overline{\mathscr{S}}^{U_1 \cup U_2}(e)(\mathfrak{k})$ and $(\overline{\mathscr{S}}^{U_{1_N}} \cap \overline{\mathscr{S}}^{U_{2_N}})(e)(\mathfrak{k}) = \overline{\mathscr{S}}^{U_{1_N}}(e)(\mathfrak{k}) \wedge \overline{\mathscr{S}}^{U_{2_N}}(e)(\mathfrak{k}) = (\bigwedge_{\mathfrak{k} \in \mathfrak{k}\mathscr{S}(e)} U_{1_N}(\mathfrak{k})) \wedge (\bigwedge_{\mathfrak{k} \in \mathfrak{k}\mathscr{S}(e)} U_{2_N}(\mathfrak{k})) = \bigwedge_{\mathfrak{k} \in \mathfrak{k}\mathscr{S}(e)}(U_{1_N}(\mathfrak{k}) \wedge U_{2_N}(\mathfrak{k})) = \overline{\mathscr{S}}^{U_1 \cap U_2}(e)(\mathfrak{k})$. Thus, $\overline{\mathscr{S}}^{U_1 \cup U_2} = \overline{\mathscr{S}}^{U_1} \cup \overline{\mathscr{S}}^{U_2}$.

vii) Since $\underline{\mathscr{S}}^{1_W}(e)(\mathfrak{k}) = \bigwedge_{\mathfrak{k} \in \mathfrak{k}\mathscr{S}(e)} 1(w) = \bigwedge_{\mathfrak{k} \in \mathfrak{k}\mathscr{S}(e)} 1 = 1$ and $\underline{\mathscr{S}}^{0}(e)(\mathfrak{k}) = \bigwedge_{\mathfrak{k} \in \mathfrak{k}\mathscr{S}(e)} 0(w) = \bigwedge_{\mathfrak{k} \in \mathfrak{k}\mathscr{S}(e)} 0 = 0$. Thus, $\underline{\mathscr{S}}^{1_{\xi_2}} = 1_{\xi_1}$.
Similarly, we can prove that $\overline{\mathscr{S}}^{1_{\xi_2}} = 1_{\xi_1}$.

viii) Consider $\overline{\mathscr{S}}^{U_Y^c}(e)(\mathfrak{k}) = \bigvee_{\mathfrak{k} \in \mathfrak{k}\mathscr{S}(e)} U_Y^c(\mathfrak{k}) = \bigvee_{\mathfrak{k} \in \mathfrak{k}\mathscr{S}(e)} U_N(\mathfrak{k}) = \underline{\mathscr{S}}^{U_N}(e)(\mathfrak{k}) = (\underline{\mathscr{S}}^{U_Y}(e)(\mathfrak{k}))^c$ and $\overline{\mathscr{S}}^{U_N^c}(e)(\mathfrak{k}) = \bigwedge_{\mathfrak{k} \in \mathfrak{k}\mathscr{S}(e)} U_N^c(\mathfrak{k}) = \bigwedge_{\mathfrak{k} \in \mathfrak{k}\mathscr{S}(e)} U_Y(\mathfrak{k}) = \underline{\mathscr{S}}^{U_Y}(e)(\mathfrak{k}) = (\underline{\mathscr{S}}^{U_N}(e)(\mathfrak{k}))^c$.
Thus, $\overline{\mathscr{S}}^{U^c} = (\overline{\mathscr{S}}^{U_Y^c}, \overline{\mathscr{S}}^{U_N^c}) = ((\underline{\mathscr{S}}^{U_Y})^c, (\underline{\mathscr{S}}^{U_N})^c) = (\underline{\mathscr{S}}^{U_Y}, \underline{\mathscr{S}}^{U_N})^c = (\underline{\mathscr{S}}^{U})^c$. Which gives that $(\overline{\mathscr{S}}^{U^c})^c = \underline{\mathscr{S}}^{U}$.
Similarly, $\overline{\mathscr{S}}^{U} = (\underline{\mathscr{S}}^{U^c})^c$.

ix) The proof is straightforward.

□

Theorem 4.1.5. *Let $(\xi_1, \xi_2, \mathscr{S})$ be a $GS_fA_pS_p$. For any three ${}^qROF_zSs$, U, U_1, and U_2 of ξ_1, we have the following:*

i) $U_1 \subseteq U_2$ *implies* ${}^{U_1}\underline{\mathscr{S}} \subseteq {}^{U_2}\underline{\mathscr{S}}$

ii) $U_1 \subseteq U_2$ *implies* ${}^{U_1}\overline{\mathscr{S}} \subseteq {}^{U_2}\overline{\mathscr{S}}$

iii) ${}^{U_1}\underline{\mathscr{S}} \cap {}^{U_2}\underline{\mathscr{S}} = {}^{U_1 \cap U_2}\underline{\mathscr{S}}$

iv) ${}^{U_1\cap U_2}\overline{\mathscr{S}} \subseteq {}^{U_1}\overline{\mathscr{S}} \cap {}^{U_2}\overline{\mathscr{S}}$

v) ${}^{U_1\cup U_2}\underline{\mathscr{S}} \supseteq {}^{U_1}\underline{\mathscr{S}} \cup {}^{U_2}\underline{\mathscr{S}}$

vi) ${}^{U_1}\overline{\mathscr{S}} \cup {}^{U_2}\overline{\mathscr{S}} = {}^{U_1\cup U_2}\overline{\mathscr{S}}$

vii) ${}^{1_{\xi_1}}\underline{\mathscr{S}} = 1_{\xi_2} = {}^{1_{\xi_1}}\overline{\mathscr{S}}$, *if* $\mathscr{S}(e)\mathscr{t} \neq \emptyset$

viii) ${}^{U}\underline{\mathscr{S}} = ({}^{U^c}\overline{\mathscr{S}})^c$, *and* ${}^{U}\overline{\mathscr{S}} = ({}^{U^c}\underline{\mathscr{S}})^c$ *if* $\mathscr{S}(e)\mathscr{t} \neq \emptyset$

ix) ${}^{0_{\xi_1}}\underline{\mathscr{S}} = 0_{\xi_2} = {}^{0_{\xi_1}}\overline{\mathscr{S}}$.

Proof.
The proof can be derived using the same approach as in Theorem. 4.1.4. □

In the following example, we demonstrate that ${}^{U_1\cap U_2}\overline{\mathscr{S}}$ is not equal to ${}^{U_1}\overline{\mathscr{S}}\cap{}^{U_2}\overline{\mathscr{S}}$, and ${}^{U_1\cup U_2}\underline{\mathscr{S}}$ is not equal to ${}^{U_1}\underline{\mathscr{S}} \cup {}^{U_2}\underline{\mathscr{S}}$ as stated in Theorem 4.1.4.

Example 4.1.6. *Utilizing the information given in Example 4.1.3, define two* ${}^qROF_zS$*s* U_1, U_2 *on* ξ_2 *by:*
$U_1 = \{\langle \mathscr{t}_1, 0.85, 0.65\rangle, \langle \mathscr{t}_2, 0.65, 0.95\rangle, \langle \mathscr{t}_3, 0.5, 0.9\rangle, \langle \mathscr{t}_4, 0.6, 0.9\rangle\}$,
$U_2 = \{\langle \mathscr{t}_1, 0.7, 0.8\rangle, \langle \mathscr{t}_2, 0.79, 0.7\rangle, \langle \mathscr{t}_3, 0.6, 0.87\rangle, \langle \mathscr{t}_4, 0.5, 0.65\rangle\}$.
Then, $U_1 \cup U_2 = \{\langle \mathscr{t}_1, 0.85, 0.65\rangle, \langle \mathscr{t}_2, 0.79, 0.7\rangle, \langle \mathscr{t}_3, 0.6, 0.87\rangle, \langle \mathscr{t}_4, 0.6, 0.65\rangle\}$,
$U_1 \cap U_2 = \{\langle \mathscr{t}_1, 0.7, 0.8\rangle, \langle \mathscr{t}_2, 0.65, 0.95\rangle, \langle \mathscr{t}_3, 0.5, 0.9\rangle, \langle \mathscr{t}_4, 0.5, 0.9\rangle\}$.

Table 4.3: Union of L_oA_ps and L_oA_p of union of two ${}^qROF_zS$s

	$(\underline{\mathscr{S}}^{U_1} \cup \underline{\mathscr{S}}^{U_2})(e_1)(\mathscr{k}_i)$	$(\underline{\mathscr{S}}^{U_1} \cup \underline{\mathscr{S}}^{U_2})(e_2)(\mathscr{k}_i)$	$\underline{\mathscr{S}}^{\mathbf{U_1\cup U_2}}(e_1)(\mathscr{k}_i)$	$\underline{\mathscr{S}}^{\mathbf{U_1\cup U_2}}(e_2)(\mathscr{k}_i)$
$\mathscr{k}_\mathbf{1}$	$(0.7, 0.8)$	$(0.5, 0.87)$	$(0.79, 0.7)$	$(0.6, 0.87)$
$\mathscr{k}_\mathbf{2}$	$(0.6, 0.87)$	$(0.5, 0.87)$	$(0.6, 0.87)$	$(0.6, 0.87)$
$\mathscr{k}_\mathbf{3}$	$(0.6, 0.8)$	$(0.6, 0.87)$	$(0.6, 0.7)$	$(0.6, 0.87)$
$\mathscr{k}_\mathbf{4}$	$(0.5, 0.87)$	$(1, 0)$	$(0.6, 0.87)$	$(1, 0)$

Table 4.4: Intersection of U_pA_ps and U_pA_p of intersection of two ${}^qROF_zS$s

	$(\overline{\mathscr{S}}^{U_1} \cap \overline{\mathscr{S}}^{U_2})(e_1)(\Bbbk_i)$	$(\overline{\mathscr{S}}^{U_1} \cap \overline{\mathscr{S}}^{U_2})(e_2)(\Bbbk_i)$	$\overline{\mathscr{S}}^{\mathbf{U_1 \cap U_2}}(e_1)(\Bbbk_i)$	$\overline{\mathscr{S}}^{\mathbf{U_1 \cap U_2}}(e_2)(\Bbbk_i)$
$\Bbbk_\mathbf{1}$	$(0.79, 0.7)$	$(0.6, 0.9)$	$(0.7, 0.8)$	$(0.5, 0.9)$
$\Bbbk_\mathbf{2}$	$(0.65, 0.9)$	$(0.79, 0.65)$	$(0.65, 0.9)$	$(0.7, 0.8)$
$\Bbbk_\mathbf{3}$	$(0.79, 0.65)$	$(0.5, 0.9)$	$(0.7, 0.8)$	$(0.5, 0.9)$
$\Bbbk_\mathbf{4}$	$(0.6, 0.9)$	$(0, 1)$	$(0.5, 0.9)$	$(0, 1)$

From Table 4.3, for $e = e_1$, $\Bbbk = \Bbbk_1$, *we have* $(\underline{\mathscr{S}}^{U_1} \cup \underline{\mathscr{S}}^{U_2})(e)(\Bbbk) \neq \underline{\mathscr{S}}^{U_1 \cup U_2}(e)(\Bbbk)$ *and for* $e = e_2$, $\Bbbk = \Bbbk_2$, $(\underline{\mathscr{S}}^{U_1} \cup \underline{\mathscr{S}}^{U_2})(e)(\Bbbk) \neq \underline{\mathscr{S}}^{U_1 \cup U_2}(e)(\Bbbk)$. *That is,* $\underline{\mathscr{S}}^{U_1} \cup \underline{\mathscr{S}}^{U_2} \neq \underline{\mathscr{S}}^{U_1 \cup U_2}$, *in general.*

Similarly, from Table 4.4, we see that $\overline{\mathscr{S}}^{U_1} \cap \overline{\mathscr{S}}^{U_2} \neq \overline{\mathscr{S}}^{U_1 \cap U_2}$.

Hence, equality does not hold in part (iv) and (v) of the Theorem 4.1.4.

Theorem 4.1.7. *Let* $(\xi_1, \xi_2, \mathscr{S}_1)$ *and* $(\xi_1, \xi_2, \mathscr{S}_2)$ *be two* $GS_fA_pS_p$*s such that* $(\xi_1, \xi_2, \mathscr{S}_1) \subseteq (\xi_1, \xi_2, \mathscr{S}_2)$, *that is, for all* $e \in D$, $\mathscr{S}_1(e) \subseteq \mathscr{S}_2(e)$. *Then, for any* $U \in {}^qROF_zS(\xi_2)$, $\underline{\mathscr{S}_2}^U \subseteq \underline{\mathscr{S}_1}^U$ *and* $\overline{\mathscr{S}_1}^U \subseteq \overline{\mathscr{S}_2}^U$.

Proof.

If $\Bbbk\mathscr{S}_1(e) = \emptyset$, then $\underline{\mathscr{S}_2}^{U_Y}(e)(\mathfrak{k}) \leq 1 = \underline{\mathscr{S}_1}^{U_Y}(e)(\mathfrak{k})$, and $\underline{\mathscr{S}_1}^{U_N}(e)(\mathfrak{k}) = 0 \leq \underline{\mathscr{S}_2}^{U_N}(e)(\mathfrak{k})$. This implies that $\underline{\mathscr{S}_2}^U \subseteq \underline{\mathscr{S}_1}^U$.

If $\mathfrak{k}\mathscr{S}_1(e) \neq \emptyset$, then $\underline{\mathscr{S}_1}^{U_Y}(e)(\mathfrak{k}) = \bigwedge_{\mathfrak{t} \in \mathfrak{k}\mathscr{S}_1(e)} U_Y(\mathfrak{t}) \geq \bigwedge_{\mathfrak{t} \in \mathfrak{k}\mathscr{S}_2(e)} U_Y(\mathfrak{t}) = \underline{\mathscr{S}_2}^{U_Y}(e)(\mathfrak{k})$, and $\underline{\mathscr{S}_1}^{U_N}(e)(\mathfrak{k}) = \bigvee_{\mathfrak{t} \in \mathfrak{k}\mathscr{S}_1(e)} U_N(\mathfrak{t}) \leq \bigvee_{\mathfrak{t} \in \mathfrak{k}\mathscr{S}_2(e)} U_N(\mathfrak{t}) = \underline{\mathscr{S}_2}^{U_N}(e)(\mathfrak{k})$. Thus, $\underline{\mathscr{S}_2}^U \subseteq \underline{\mathscr{S}_1}^U$. Similarly, $\overline{\mathscr{S}_1}^U \subseteq \overline{\mathscr{S}_2}^U$. □

Similarly, the dual of the Theorem 4.1.7 can be given, that is, the same result holds with respect to F_rSs.

Theorem 4.1.8. *Let $(\xi_1, \xi_2, \mathscr{S}_1)$ and $(\xi_1, \xi_2, \mathscr{S}_2)$ be two $GS_fA_pS_ps$ such that $(\xi_1, \xi_2, \mathscr{S}_1) \subseteq (\xi_1, \xi_2, \mathscr{S}_2)$, that is, $\mathscr{S}_1(e) \subseteq \mathscr{S}_2(e)$, for all $e \in D$. Then, for any $U \in {}^qROF_zS(\xi_1)$, ${}^U\underline{\mathscr{L}}_2 \subseteq {}^U\underline{\mathscr{L}}_1$ and ${}^U\overline{\mathscr{S}_1} \subseteq {}^U\overline{\mathscr{S}_2}$.*

Proof.
The proof can be derived using the same approach as in Theorem. 4.1.7. □

Theorem 4.1.9. *Let $(\xi_1, \xi_2, \mathscr{S}_1)$ and $(\xi_1, \xi_2, \mathscr{S}_2)$ be two $GS_fA_pS_ps$. Then, for any $U \in {}^qROF_zS(\xi_2)$, the following are true:*

i) $\underline{\mathscr{L}}_1^U \subseteq \underline{(\mathscr{S}_1 \cap \mathscr{S}_2)}^U$ *and* $\underline{\mathscr{L}}_2^U \subseteq \underline{(\mathscr{S}_1 \cap \mathscr{S}_2)}^U$.

ii) $\overline{(\mathscr{S}_1 \cap \mathscr{S}_2)}^U \subseteq \overline{\mathscr{S}_1}^U$ *and* $\overline{(\mathscr{S}_1 \cap \mathscr{S}_2)}^U \subseteq \overline{\mathscr{S}_2}^U$.

Proof.
The proof is a direct consequence of Theorem 4.1.7. □

Similarly, we have the dual of the Theorem 4.1.9.

Theorem 4.1.10. *Let $(\xi_1, \xi_2, \mathscr{S}_1)$ and $(\xi_1, \xi_2, \mathscr{S}_2)$ be two $GS_fA_pS_ps$. Then, for any $U \in {}^qROF_zS(\xi_1)$, the following are true:*

i) ${}^U\underline{\mathscr{L}}_1 \subseteq {}^U\underline{(\mathscr{S}_1 \cap \mathscr{S}_2)}$, *and* ${}^U\underline{\mathscr{S}_2} \subseteq {}^U\underline{(\mathscr{S}_1 \cap \mathscr{S}_2)}$.

ii) ${}^U\overline{(\mathscr{S}_1 \cap \mathscr{S}_2)} \subseteq {}^U\overline{\mathscr{S}_1}$ *and* ${}^U\overline{(\mathscr{S}_1 \cap \mathscr{S}_2)} \subseteq {}^U\overline{\mathscr{S}_2}$.

Theorem 4.1.11. *Let $(\xi_1, \xi_2, \mathscr{S})$ be a $GS_fA_pS_p$ and $\{U_i \mid i \in I\}$ be a family of ${}^qROF_zSs$ defined on ξ_2. Then the following hold with respect to A_fSs:*

i) $\underline{\mathscr{S}}^{(\bigcap_{i \in I} U_i)} = \bigcap_{i \in I} \underline{\mathscr{S}}^{U_i}$

ii) $\bigcup_{i \in I} \underline{\mathscr{S}}^{U_i} \subseteq \underline{\mathscr{S}}^{(\bigcup_{i \in I} U_i)}$

iii) $\overline{\mathscr{S}}^{(\bigcup_{i \in I} U_i)} = \bigcup_{i \in I} \overline{\mathscr{S}}^{U_i}$

iv) $\overline{\mathscr{S}}^{(\bigcap_{i\in I} U_i)} \subseteq \bigcap_{i\in I} \overline{\mathscr{S}}^{U_i}$.

Proof.

i) Let $U_i \in {}^qROF_zS(\xi_2)$, for $i \in I$. Then $\underline{\mathscr{S}}^{(\bigcap_{i\in I} U_{i_Y})}(e)(\mathfrak{k}) = \bigwedge_{\mathfrak{t}\in\mathfrak{k}\mathscr{S}(e)}(\wedge_{i\in I} U_{i_Y}(\mathfrak{t})) = \bigwedge_{i\in I}(\wedge_{\mathfrak{t}\in\mathfrak{k}\mathscr{S}(e)} U_{i_Y}(\mathfrak{t})) = \bigcap_{i\in I} \underline{\mathscr{S}}^{U_{i_Y}}(e)(\mathfrak{k})$ and $\underline{\mathscr{S}}^{(\bigcup_{i\in I} U_{i_N})}(e)(\mathfrak{k}) = \bigvee_{\mathfrak{t}\in\mathfrak{k}\mathscr{S}(e)}(\vee_{i\in I} U_{i_N}(\mathfrak{t})) = \bigvee_{i\in I}(\vee_{\mathfrak{t}\in\mathfrak{k}\mathscr{S}(e)} U_{i_N}(\mathfrak{t})) = \bigcup_{i\in I} \underline{\mathscr{S}}^{U_{i_N}}(e)(\mathfrak{k})$. Thus, $\underline{\mathscr{S}}^{(\bigcap_{i\in I} U_i)} = \bigcap_{i\in I} \underline{\mathscr{S}}^{U_i}$.

ii) Given that $U_i \subseteq \bigcup_{i\in I} U_i$ for each $i \in I$, it follows that $\underline{\mathscr{S}}^{U_i} \subseteq \underline{\mathscr{S}}^{(\bigcup_{i\in I} U_i)}$. Consequently, $\bigcup_{i\in I} \underline{\mathscr{S}}^{U_i} \subseteq \underline{\mathscr{S}}^{(\bigcup_{i\in I} U_i)}$.

iii) The proof can be derived using the same approach as in part (i).

iv) The proof can be derived using the same approach as in part (ii).

□

We can easily verify that the dual of the Theorem 4.1.11 also holds, that is, the axioms of the Theorem 4.1.11 hold with respect to F_rSs.

Theorem 4.1.12. *Let $(\xi_1, \xi_2, \mathscr{S})$ be a $GS_fA_pS_p$ and $\{U_i \mid i \in I\}$ be a family of ${}^qROF_zSs$ defined on ξ_1. Then the following hold with respect to F_rSs:*

i) ${}^{(\bigcap_{i\in I} U_i)}\underline{\mathscr{S}} = \bigcap_{i\in I} {}^{U_i}\underline{\mathscr{S}}$

ii) $\bigcup_{i\in I} {}^{U_i}\underline{\mathscr{S}} \subseteq {}^{(\bigcup_{i\in I} U_i)}\underline{\mathscr{S}}$

iii) ${}^{(\bigcup_{i\in I} U_i)}\overline{\mathscr{S}} = \bigcup_{i\in I} {}^{U_i}\overline{\mathscr{S}}$

iv) ${}^{(\bigcap_{i\in I} U_i)}\overline{\mathscr{S}} \subseteq \bigcap_{i\in I} {}^{U_i}\overline{\mathscr{S}}$.

Proof.

The proof can be derived using the same approach as in Theorem. 4.1.11. □

Here, we give the relationship between a ${}^{q}ROF_zS$ and its approximations based on S_fR_fR. In fact, a ${}^{q}ROF_zS$ is sandwiched between its L_oA_p and U_pA_p.

Theorem 4.1.13. *Let $(\mathscr{S}, D)$ be a S_fR_fR over ξ. For any $U \in {}^{q}ROF_zS(\xi)$, the following properties of the L_oA_p and the U_pA_p with respect to A_fSs hold, for $e \in D$:*

i) $\underline{\mathscr{S}}^{U}(e) \leq U \leq \overline{\mathscr{S}}^{U}(e)$.

ii) $\underline{\mathscr{S}}^{U}(e) \leq \overline{\mathscr{S}}^{U}(e)$.

Proof.

For $\mathfrak{k} \in \xi$,

i) Consider
$\underline{\mathscr{S}}^{U_Y}(e)(\mathfrak{k}) = \bigwedge_{\mathfrak{t} \in \mathfrak{k}\mathscr{S}(e)} U_Y(\mathfrak{t}) \leq U_Y(\mathfrak{k})$, since $\mathfrak{k} \in \mathfrak{k}\mathscr{S}(e)$, and $\underline{\mathscr{S}}^{U_N}(e)(\mathfrak{k}) = \bigvee_{\mathfrak{t} \in \mathfrak{k}\mathscr{S}(e)} U_N(\mathfrak{t}) \geq U_N(\mathfrak{k})$, since $\mathfrak{k} \in \mathfrak{k}\mathscr{S}(e)$. Thus, $\underline{\mathscr{S}}^{U}(e) \leq U$.
Also, $\overline{\mathscr{S}}^{U_Y}(e)(\mathfrak{k}) = \bigvee_{\mathfrak{t} \in \mathfrak{k}\mathscr{S}(e)} U_Y(\mathfrak{t}) \geq U_Y(\mathfrak{k})$, since $\mathfrak{k} \in \mathfrak{k}\mathscr{S}(e)$, and $\overline{\mathscr{S}}^{U_N}(e)(\mathfrak{k}) = \bigwedge_{\mathfrak{t} \in \mathfrak{k}\mathscr{S}(e)} U_N(\mathfrak{t}) \leq U_N(\mathfrak{k})$, since $\mathfrak{k} \in \mathfrak{k}\mathscr{S}(e)$. Thus, $\overline{\mathscr{S}}^{U}(e) \geq U$.

ii) From part (i), we get that $\underline{\mathscr{S}}^{U}(e) \leq U \leq \overline{\mathscr{S}}^{U}(e)$ which implies that $\underline{\mathscr{S}}^{U}(e) \leq \overline{\mathscr{S}}^{U}(e)$.

□

Theorem 4.1.14. *Let $(\mathscr{S}, D)$ be an S_fR_fR (S_fR_fR) over ξ. For any $U \in {}^{q}ROF_zS(\xi)$, the following properties of the L_oA_p and the U_pA_ps with respect to F_rSs hold, for $e \in D$:*

i) ${}^{U}\underline{\mathscr{S}}(e) \leq U \leq {}^{U}\overline{\mathscr{S}}(e)$

ii) ${}^{U}\underline{\mathscr{S}}(e) \leq {}^{U}\overline{\mathscr{S}}(e)$.

Proof.

The proof can be derived using the same approach as in Theorem. 4.1.13. □

4.2 q-Rung Orthopair Fuzzy Topologies induced by Soft Reflexive Relations

Cheng [17] introduced the idea of Fuzzy Topological spaces, which generalize fundamental notions from Topology. Tǜrkarslan [56] et al. extended this idea to ${}^qROF_zT_pS$s and explored the continuity between two ${}^qROF_zT_pS$s. In Chapter 3 of this work, we developed ${}^qROF_zT_pS$s that are induced by R_fB_nRs.

In this chapter, we introduce and analyze two types of ${}^qROF_zT_pS$s induced by a S_fR_fR.

Definition 4.2.1. *[56] A collection τ of ${}^qROF_zSs$ on ξ is called a q-Rung Orthopair Fuzzy Topology (${}^qROF_zT_p$, for short) on ξ if it satisfies:*

1) $0, 1 \in \tau$

2) $U_1 \cap U_2 \in \tau$, *for all* $U_1, U_2 \in \tau$

3) $\bigcup_{i\in I} U_i \in \tau$, *for all* $U_i \in \tau$, $i \in I$.

If τ is a ${}^qROF_zT_p$ on ξ, then the pair (ξ, τ) is called a ${}^qROF_zT_pS$. The elements of τ are called ${}^qROF_z$ open sets.

Theorem 4.2.2. *If $(\mathscr{S}, D)$ is a S_fR_fR on ξ, then*

$$G_e = \{U \in {}^qROF_zS(\xi) \mid \underline{\mathscr{S}}^U(e) = U\}$$

is a ${}^qROF_zT_p$ on ξ for each $e \in D$.

Proof.

i) From Theorem 4.1.4, for each $e \in D$, we have $\underline{\mathscr{S}}^0(e) = 0$ and $\underline{\mathscr{S}}^1(e) = 1$, which gives that $0, 1 \in G_e$.

ii) If $U_1, U_2 \in G_e$, then $\underline{\mathscr{S}}^{U_1}(e) = U_1$ and $\underline{\mathscr{S}}^{U_2} = U_2$. According to the Theorem 4.1.4 $(U_1 \cap U_2) = (\underline{\mathscr{S}}^{U_1} \cap \underline{\mathscr{S}}^{U_2})(e) = \underline{\mathscr{S}}^{U_1 \cap U_2}(e)$. This implies that $U_1 \cap U_2 \in G_e$.

iii) If $U_i \in G_e$, then $\underline{\mathscr{L}}^{U_i} = U_i$, for each $i \in I$. Since S is an S_fR_fR, so according to the Theorem 4.1.13, we have

$$\underline{\mathscr{L}}^{(\bigcup_{i\in I} U_i)}(e) \subseteq \bigcup_{i\in I} U_i. \tag{4.2.1}$$

Also, because $U_i \subseteq \bigcup_{i\in I} U_i$, so $\underline{\mathscr{L}}^{U_i}(e) \subseteq \underline{\mathscr{L}}^{(\bigcup_{i\in I} U_i)}(e)$. Which implies that $\bigcup_{i\in I} \underline{\mathscr{L}}^{U_i}(e) \subseteq \underline{\mathscr{L}}^{(\bigcup_{i\in I} U_i)}(e)$. Thus,

$$\bigcup_{i\in I} U_i \subseteq \underline{\mathscr{L}}^{(\bigcup_{i\in I} U_i)}(e). \tag{4.2.2}$$

Thus, from Equations (4.2.1) and (4.2.2), we get $\bigcup_{i\in I} U_i = \underline{\mathscr{L}}^{(\bigcup_{i\in I} U_i)}(e)$.

Hence, G_e is a ${}^qROF_zT_p$ on ξ.

□

Theorem 4.2.3. *If $(\mathscr{S}, D)$ is an S_fR_fR on ξ, then*

$$G_e^{'} = \{U \in {}^qROF_zS(\xi) \mid {}^U\underline{\mathscr{L}}(e) = U\}$$

is a ${}^qROF_zT_p$ on ξ for each $e \in D$.

Proof.

The proof can be derived using the same approach as in Theorem. 4.2.2. □

4.3 Similarity Relations Associated with Soft Binary Relations

Similarity relations (S_mRs) play an important role for checking symmetry between objects. In this section, we define S_mR between two ${}^qROF_zS$s based on corresponding L_oA_p and U_pA_ps.

Definition 4.3.1. *Let $(\xi, \mathscr{S})$ be a $S_fA_pS_p$. For $U_1, U_2 \in {}^qROF_zS(\xi)$, we define*

$U_1\underline{R}U_2$ *if and only if* $\underline{\mathscr{L}}^{U_1} = \underline{\mathscr{L}}^{U_2}$

$U_1\tilde{R}U_2$ *if and only if* $\overline{\mathscr{S}}^{U_1} = \overline{\mathscr{S}}^{U_2}$

U_1RU_2 *if and only if* $\underline{\mathscr{S}}^{U_1} = \underline{\mathscr{S}}^{U_2}$ *and* $\overline{\mathscr{S}}^{U_1} = \overline{\mathscr{S}}^{U_2}$.

Definition 4.3.2. *Let* $(\xi, \mathscr{S})$ *be a* $S_fA_pS_p$. *For* $U_1, U_2 \in {}^qROF_zS(\xi)$, *we define*

$U_1\underline{r}U_2$ *if and only if* ${}^{U_1}\underline{\mathscr{S}} = {}^{U_2}\underline{\mathscr{S}}$

$U_1\tilde{r}U_2$ *if and only if* ${}^{U_1}\overline{\mathscr{S}} = {}^{U_2}\overline{\mathscr{S}}$

U_1rU_2 *if and only if* ${}^{U_1}\underline{\mathscr{S}} = {}^{U_2}\underline{\mathscr{S}}$ *and* ${}^{U_1}\overline{\mathscr{S}} = {}^{U_2}\overline{\mathscr{S}}$.

We call these Binary Relations as the lower ${}^qROF_zS_mR$, upper ${}^qROF_zS_mR$, and ${}^qROF_zS_mR$, respectively.

Proposition 4.3.3. *The relations* $\underline{R}$, $\tilde{R}$, R *are* E_qRs *on* ${}^qROF_zS(\xi)$.

Proof.
The proof is straightforward. □

Proposition 4.3.4. *The relations* $\underline{r}$, $\tilde{r}$, r *are* E_qRs *on* ${}^qROF_zS(\xi)$.

Proof.
The proof is straightforward. □

Theorem 4.3.5. *Let* $(\xi, \mathscr{S})$ *be a* $S_fA_pS_p$ *and* $U_1, U_2, U_3, U_4 \in {}^qROF_zS(\xi)$. *Then:*

i) $U_1\tilde{R}U_2$ *if and only if* $U_1\tilde{R}(U_1 \cup U_2)\tilde{R}U_2$

ii) If $U_1\tilde{R}U_2$ *and* $U_3\tilde{R}U_4$, *then* $(U_1 \cup U_3)\tilde{R}(U_2 \cup U_4)$

iii) If $U_1 \subseteq U_2$ *and* $U_2\tilde{R}0$, *then* $U_1\tilde{R}0$

iv) $(U_1 \cup U_2)\tilde{R}0$ *if and only if* $U_1\tilde{R}0$ *and* $U_2\tilde{R}0$

v) If $U_1 \subseteq U_2$ *and* $U_1\tilde{R}1$, *then* $U_2\tilde{R}1$

vi) If $(U_1 \cap U_2)\tilde{R}1$, then $U_1\tilde{R}1$ and $U_2\tilde{R}1$.

Proof.

i) If $U_1\tilde{R}U_2$, then $\overline{\mathscr{S}}^{U_1} = \overline{\mathscr{S}}^{U_2}$. According to the Theorem 4.1.4, $= \overline{\mathscr{S}}^{U_1} = \overline{\mathscr{S}}^{U_2} = \overline{\mathscr{S}}^{U_1} \cup \overline{\mathscr{S}}^{U_2} = \overline{\mathscr{S}}^{U_1 \cup U_2}$, so we have $U_1\tilde{R}(U_1 \cup U_2)\tilde{R}U_2$.

Conversely, if $U_1\tilde{R}(U_1 \cup U_2)\tilde{R}U_2$, then $U_1\tilde{R}(U_1 \cup U_2)$ and $(U_1 \cup U_2)\tilde{R}U_2$. This implies that $\overline{\mathscr{S}}^{U_1} = \overline{\mathscr{S}}^{U_1 \cup U_2}$ and $\overline{\mathscr{S}}^{U_2} = \overline{\mathscr{S}}^{U_1 \cup U_2}$. Thus, $\overline{\mathscr{S}}^{U_1} = \overline{\mathscr{S}}^{U_2}$. Hence, $U_1\tilde{R}U_2$.

ii) If $U_1\tilde{R}U_2$ and $U_3\tilde{R}U_4$, then $\overline{\mathscr{S}}^{U_1} = \overline{\mathscr{S}}^{U_2}$ and $\overline{\mathscr{S}}^{U_3} = \overline{\mathscr{S}}^{U_4}$. According to the Theorem 4.1.4, $\overline{\mathscr{S}}^{U_1 \cup U_3} = \overline{\mathscr{S}}^{U_1} \cup \overline{\mathscr{S}}^{U_3} = \overline{\mathscr{S}}^{U_2} \cup \overline{\mathscr{S}}^{U_4} = \overline{\mathscr{S}}^{U_2 \cup U_4}$. Thus, $(U_1 \cup U_3)\tilde{R}(U_2 \cup U_4)$.

iii) Let $U_1 \subseteq U_2$ and $U_2\tilde{R}0$. Then $\overline{\mathscr{S}}^{U_2} = \overline{\mathscr{S}}^{0}$. Also, since $U_1 \subseteq U_2$, so we have $\overline{\mathscr{S}}^{U_1} \subseteq \overline{\mathscr{S}}^{U_2} = \overline{\mathscr{S}}^{0}$. But $\overline{\mathscr{S}}^{0} \subseteq \overline{\mathscr{S}}^{U_1}$, so $\overline{\mathscr{S}}^{U_1} = \overline{\mathscr{S}}^{0}$. Hence, $U_1\tilde{R}0$.

iv) If $(U_1 \cup U_2)\tilde{R}0$, then $\overline{\mathscr{S}}^{0} = \overline{\mathscr{S}}^{U_1 \cup U_2} = \overline{\mathscr{S}}^{U_1} \cup \overline{\mathscr{S}}^{U_2}$. Since $\overline{\mathscr{S}}^{U_1} \subseteq \overline{\mathscr{S}}^{U_1} \cup \overline{\mathscr{S}}^{U_2} = \overline{\mathscr{S}}^{0}$, so we have $\overline{\mathscr{S}}^{U_1} = \overline{\mathscr{S}}^{0}$. Similarly, $\overline{\mathscr{S}}^{U_2} = \overline{\mathscr{S}}^{0}$. Hence, $U_1\tilde{R}0$ and $U_2\tilde{R}0$.

Conversely, if $U_1\tilde{R}0$ and $U_2\tilde{R}0$, then $\overline{\mathscr{S}}^{U_1} = \overline{\mathscr{S}}^{0}$ and $\overline{\mathscr{S}}^{U_2} = \overline{\mathscr{S}}^{0}$. According to the Theorem 1.2.3, $\overline{\mathscr{S}}^{0} = \overline{\mathscr{S}}^{0} \cup \overline{\mathscr{S}}^{0} = \overline{\mathscr{S}}^{U_1} \cup \overline{\mathscr{S}}^{U_2} = \overline{\mathscr{S}}^{(U_1 \cup U_2)}$. Hence, $(U_1 \cup U_2)\tilde{R}0$.

v) If $U_1\tilde{R}1$, then $\overline{\mathscr{S}}^{U_1} = \overline{\mathscr{S}}^{1}$. Since $U_1 \subseteq U_2$, so $\overline{\mathscr{S}}^{1} = \overline{\mathscr{S}}^{U_1} \subseteq \overline{\mathscr{S}}^{U_2}$. But $\overline{\mathscr{S}}^{U_2} \subseteq \overline{\mathscr{S}}^{1}$ so, $\overline{\mathscr{S}}^{1} = \overline{\mathscr{S}}^{U_1}$. Hence, $U_2\tilde{R}1$.

vi) If $(U_1 \cap U_2)\tilde{R}1$, then $\overline{\mathscr{S}}^{U_1 \cap U_2} = \overline{\mathscr{S}}^{1}$. According to the Theorem 4.1.4, we have $\overline{\mathscr{S}}^{U_1} \cap \overline{\mathscr{S}}^{U_2} \supseteq \overline{\mathscr{S}}^{U_1 \cap U_2} = \overline{\mathscr{S}}^{1}$. Thus, $\overline{\mathscr{S}}^{1} = \overline{\mathscr{S}}^{U_1}$ and $\overline{\mathscr{S}}^{1} = \overline{\mathscr{S}}^{U_2}$. Hence, $U_1\tilde{R}1$ and $U_2\tilde{R}1$.

□

Theorem 4.3.6. *Let $(\xi, \mathscr{S})$ be a $S_fA_pS_p$ and $U_1, U_2, U_3, U_4 \in {}^qROF_zS(\xi)$. Then:*

i) $U_1\tilde{r}U_2$ if and only if $U_1\tilde{r}(U_1 \cup U_2)\tilde{r}U_2$

ii) If $U_1\tilde{r}U_2$ and $U_3\tilde{r}U_4$, then $(U_1 \cup U_3)\tilde{r}(U_2 \cup U_4)$

iii) If $U_1 \subseteq U_2$ and $U_2\tilde{r}0$, then $U_1\tilde{r}0$

iv) $(U_1 \cup U_2)\tilde{r}0$ if and only if $U_1\tilde{r}0$ and $U_2\tilde{r}0$

v) If $U_1 \subseteq U_2$ and $U_1\tilde{r}1$, then $U_2\tilde{r}1$

vi) If $(U_1 \cap U_2)\tilde{r}1$, then $U_1\tilde{r}1$ and $U_2\tilde{r}1$.

Proof.

The proof can be derived using the same approach as in Theorem. 4.3.5 □

Theorem 4.3.7. *Let* $(\xi, \mathscr{S})$ *be a* $S_fA_pS_p$ *and* $U_1, U_2, U_3, U_4 \in {}^qROF_zS(\xi)$*. Then:*

i) $U_1\underline{R}U_2$ if and only if $U_1\underline{R}(U_1 \cap U_2)\underline{R}U_2$

ii) If $U_1\underline{R}U_2$ and $U_3\underline{R}U_4$, then $(U_1 \cap U_3)\underline{R}(U_2 \cap U_4)$

iii) If $U_1 \subseteq U_2$ and $U_2\underline{R}0$, then $U_1\underline{R}0$

iv) $(U_1 \cup U_2)\underline{R}0$ if and only if $U_1\underline{R}0$ and $U_2\underline{R}0$

v) If $U_1 \subseteq U_2$ and $U_1\underline{R}1$, then $U_2\underline{R}1$

vi) If $(U_1 \cap U_2)\underline{R}1$, then $U_1\underline{R}1$ and $U_2\underline{R}1$.

Proof.

The proof is straightforward. □

Theorem 4.3.8. *Let* $(\xi, \mathscr{S})$ *be a* $S_fA_pS_p$ *and* $U_1, U_2, U_3, U_4 \in {}^qROF_zS(\xi)$*. Then:*

i) $U_1\underline{r}U_2$ if and only if $U_1\underline{r}(U_1 \cap U_2)\underline{r}U_2$

ii) If $U_1\underline{r}U_2$ and $U_3\underline{r}U_4$, then $(U_1 \cap U_3)\underline{r}(U_2 \cap U_4)$

iii) If $U_1 \subseteq U_2$ and $U_2\underline{r}0$, then $U_1\underline{r}0$

iv) $(U_1 \cup U_2)\underline{r}0$ if and only if $U_1\underline{r}0$ and $U_2\underline{r}0$

v) If $U_1 \subseteq U_2$ *and* $U_1\underline{r}1$*, then* $U_2\underline{r}1$

vi) If $(U_1 \cap U_2)\underline{r}1$*, then* $U_1\underline{r}1$ *and* $U_2\underline{r}1$*.*

Proof.

The proof is straightforward. □

Theorem 4.3.9. *Let* $(\xi, \mathscr{S})$ *be a* $S_f A_p S_p$ *and* $U_1, U_2 \in {}^qROF_zS(\xi)$*. Then:*

i) $U_1 R U_2$ *if and only if* $U_1\underline{R}(U_1 \cap U_2)\underline{R}U_2$ *and* $U_1\tilde{R}(U_1 \cup U_2)\tilde{R}U_2$

ii) If $U_1 \subseteq U_2$ *and* U_2R0*, then* U_1R0

iii) $(U_1 \cup U_2)R0$ *if and only if* U_1R0 *and* U_2R0

iv) If $(U_1 \cap U_2)R1$*, then* U_1R1 *and* U_2R1*.*

v) If $U_1 \subseteq U_2$ *and* U_1R1*, then* U_2R1

Proof.

Theorems 4.3.5 and 4.3.7 directly lead to this conclusion. □

Theorem 4.3.10. *Let* $(\xi, \mathscr{S})$ *be a* $S_f A_p S_p$ *and* $U_1, U_2 \in {}^qROF_zS(\xi)$*. Then:*

i) $U_1 r U_2$ *if and only if* $U_1\underline{r}(U_1 \cap U_2)\underline{r}U_2$ *and* $U_1\tilde{r}(U_1 \cup U_2)\tilde{r}U_2$

ii) If $U_1 \subseteq U_2$ *and* U_2r0*, then* U_1r0

iii) $(U_1 \cup U_2)r0$ *if and only if* U_1r0 *and* U_2r0

iv) If $(U_1 \cap U_2)r1$*, then* U_1r1 *and* U_2r1*.*

v) If $U_1 \subseteq U_2$ *and* U_1r1*, then* U_2r1

Proof.

Theorems 4.3.6 and 4.3.8 directly lead to this conclusion. □

4.4 Application in Decision-Making

In decision-making problems (DMPs), different experts have produced different evaluation results. Yager [62] introduced the ${}^{q}ROF_zS$ and described some of its operations. So far, many researchers have accomplished numerous works in ${}^{q}ROF_zS$ theory and many applications have appeared in different aspects.

An S_fB_nR is a parameterized family of Binary Relations on a universe, offering significant utility in decision-making problems (DMPs). This concept generalizes ordinary Binary Relations on a set. In rough set theory, traditional rough approximations address single Binary Relations. However, rough approximations in the sense of S_fB_nRs can handle multiple Binary Relations. Pawlak's rough set theory can be considered a special case of soft rough sets due to S_fB_nRs.

Peng et al. [47] introduced the concept of $P_yF_zS_fS$, outlined basic operations, and demonstrated its application in DMPs. Kanwal and Shabir [31] defined L_oA_p and U_pA_p of a F_zS in a S_mG using S_fB_nR and applied it to a real-life problem. In Chapter 2, we defined approximations of P_yF_zS over dual universes and presented an application of this new structure. In Chapter 3, we introduced generalized rough ${}^{q}ROF_zS$s and explored their applications in DMPs.

The transition from rough approximations to graphical representation provides an intuitive and visual approach to ranking alternatives. Graphical methods simplify the comparison process by visually representing the relationships between alternatives, making them particularly useful in handling complex uncertainties. By assigning weights to edges based on fuzzy measures, this approach ensures a systematic and efficient analysis of alternatives, enhancing decision-making processes.

In [33], Khan et al. developed a technique for ranking and selecting ${}^{q}ROF_zD_g$s, proposing a graphical method to solve DMPs. Their method is based on the hesitancy index and entropy of ${}^{q}ROF_zD_g$s and was validated through numerical examples. Khan et al. [33] also defined a modified Hamming distance and entropy for ${}^{q}ROF_zD_g$s, further contributing to the field.

Definition 4.4.1. *[33] For two $^qROF_zD_g$s $U = (U_Y(\Bbbk), U_N(\Bbbk))$, $V = (V_Y(\Bbbk), V_N(\Bbbk))$, the modified Hamming distance defined by Khan et al. [33] is:*

$$D_H(U,V) = \frac{1}{2}(\mid U_Y^q(\Bbbk) - V_Y^q(\Bbbk) \mid + \mid U_N^q(\Bbbk) - V_N^q(\Bbbk) \mid) \times (1 - \frac{1}{2} \mid \pi_U^q - \pi_V^q \mid)$$

Definition 4.4.2. *[33] For a $^qROF_zD_g$ $U = (U_Y(\Bbbk), U_N(\Bbbk))$, the entropy measure ($E_U$) defined by Khan et al. [33] is:*

$$E_U(U) = \frac{l_U}{\mathfrak{t}_U}$$

where l_U is the minimum of the Hamming distances of U from extreme $^qROF_zD_g$s $(0,1)$, $(1,0)$ *and $\mathfrak{t}_U$ is the maximum of the Hamming distances of U from extreme $^qROF_zD_g$s* $(0,1)$, $(1,0)$.

We now propose a novel method to address decision-making problems (DMPs) based on $^qROF_zS$ theory using S_fB_nRs. This approach extends the existing methodologies presented by Kanwal and Shabir [31], Hussain et al. [25], and in Chapters 2 and 3 of this work. The proposed method relies solely on the data and information provided by the decision-maker, without requiring any additional information from decision-makers or other sources. This eliminates the influence of subjective information on the decision results, leading to more objective outcomes and avoiding paradoxical results for the same DMPs.

The corresponding lower and upper approximations (L_oA_p and U_pA_p) are crucial in the decision-making process as they are closest to the main set to be approximated. Therefore, we derive two $^qROF_zD_g$s, $\underline{\mathscr{L}}^U(e)(\mathfrak{k})$ and $\overline{\mathscr{F}}^U(e)(\mathfrak{k})$, with respect to the A_fSs for the decision alternative $\mathfrak{k}$ in the universe ξ. This refined approach enhances the decision-making process by ensuring that the results are consistent, reliable, and devoid of subjective biases. Using ring sum operation $\oplus$ given in Definition 1.1.9, we calculate $\mathscr{S}^U(e_i)(\mathfrak{k})$ by $\mathscr{S}^U(e_i)(\mathfrak{k}) = \underline{\mathscr{L}}^U(e_i)(\mathfrak{k}) \oplus \overline{\mathscr{F}}^U(e_i)(\mathfrak{k})$ for each alternative u in ξ, for $e_i \in D$. Then we find $\mathscr{S}^U(\mathfrak{k}) = \mathscr{S}^U(e_1)(\mathfrak{k}) \otimes \mathscr{S}^U(e_2)(\mathfrak{k}) \otimes \mathscr{S}^U(e_3)(\mathfrak{k}) ... \otimes \mathscr{S}^U(e_n)(\mathfrak{k})$ for $\mathfrak{k}$ in ξ and for $e_1, e_2, e_3, ..., e_n \in D$.

Then following the technique given by Khan et al. [33] we represent the ${}^qROF_zD_g$s $\mathscr{S}^U(\mathfrak{k})$ on graph. Find the ${}^qROF_zD_g$s which lie above, below or on the straight line $\mathscr{S}^{U_Y} = \mathscr{S}^{U_N}$, (in general the line $y = x$). To decide the best alternative, we calculate the H_sD of the ${}^qROF_zD_g$ $\mathscr{S}^U(\mathfrak{k})$. So the ${}^qROF_zD_g$ lying below the straight line and having a lesser H_sD is ranked as the best alternative and the ${}^qROF_zD_g$ lying above the straight line and having higher H_sD is ranked as the best alternative. In case the H_sD is same for two or more best alternatives then the ${}^qROF_zD_g$ with a lesser E_U value is ranked best. Similarly, for the ${}^qROF_zD_g$s lying on or above the straight line we decide accordingly. The detailed steps for the proposed approach are given below.

Algorithm 5

1: Compute the L_oA_p $\underline{\mathscr{S}}^U$ and U_pA_p $\overline{\mathscr{S}}^U$ of a ${}^qROF_zS$ U with respect to the A_fSs and find $\mathscr{S}^U(e_i)(\mathfrak{k})$ by $\mathscr{S}^U(e_i)(\mathfrak{k}) = \underline{\mathscr{S}}^U(e_i)(\mathfrak{k}) \oplus \overline{\mathscr{S}}^U(e_i)(\mathfrak{k})$.

2: Represent the ${}^qROF_zD_g$s $\mathscr{S}^U(\mathfrak{k})$ on the graph calculated using the ring product operation $\mathscr{S}^U(\mathfrak{k}) = \mathscr{S}^U(e_1)(\mathfrak{k}) \otimes \mathscr{S}^U(e_2)(\mathfrak{k}) \otimes \mathscr{S}^U(e_3)(\mathfrak{k})... \otimes \mathscr{S}^U(e_n)(\mathfrak{k})$ for $\mathfrak{k}$ in ξ and for $e_1, e_2, e_3, ..., e_n \in E$.

3: Rank $\mathfrak{k}$ as the best alternative if the $\mathscr{S}^U(\mathfrak{k})$ lies below the straight line $\mathscr{S}^{U_Y} = \mathscr{S}^{U_N}$ and has least H_sD. Select $\mathfrak{k}$ with lesser E_U value if H_sD is same for two or more best alternatives.

4: Rank $\mathfrak{k}$ as the best alternative if the $\mathscr{S}^U(\mathfrak{k})$ lies on the straight line $\mathscr{S}^{U_Y} = \mathscr{S}^{U_N}$ and has least H_sD. Select $\mathfrak{k}$ with lesser E_U value if H_sD is same for two or more best alternatives.

5: Rank $\mathfrak{k}$ as the best alternative if the $\mathscr{S}^U(\mathfrak{k})$ lies above the straight line $\mathscr{S}^{U_Y} = \mathscr{S}^{U_N}$ and has highest H_sD. Select $\mathfrak{k}$ with highest E_U value if H_sD is same for two or more best alternatives.

6: Getting best alternatives from Steps (3) to (5), the alternative lying below the straight line $\mathscr{S}^{U_Y} = \mathscr{S}^{U_N}$ is ranked higher than the alternative lying on the line and the alternative lying on the straight line $\mathscr{S}^{U_Y} = \mathscr{S}^{U_N}$ is ranked higher than the alternative lying above the line.

Algorithm 6

1: Compute the L_oA_p $\underline{\mathscr{S}}^U$ and U_pA_p $\overline{\mathscr{S}}^U$ of a ${}^qROF_zS$ U with respect to the F_rSs and find ${}^U\mathscr{S}(e_i)(\mathfrak{k})$ by ${}^UF(e_i)(\mathfrak{k}) = {}^U\underline{\mathscr{S}}(e_i)(\mathfrak{k}) \oplus {}^U\overline{\mathscr{S}}(e_i)(\mathfrak{k})$.

2: Represent the ${}^qROF_zD_g$s ${}^UF(\mathfrak{k})$ on the graph calculated using the ring product operation ${}^UF(\mathfrak{k}) = {}^UF(e_1)(\mathfrak{k}) \otimes {}^UF(e_2)(\mathfrak{k}) \otimes {}^UF(e_3)(\mathfrak{k}) ... \otimes {}^UF(e_n)(\mathfrak{k})$ for $\mathfrak{k}$ in ξ and for $e_1, e_2, e_3, ..., e_n \in E$.

3: Rank $\mathfrak{k}$ as the best alternative if the ${}^U\mathscr{S}(\mathfrak{k})$ lies below the straight line ${}^{U_Y}F = {}^{U_N}F$ and has least H_sD. Select $\mathfrak{k}$ with lesser E_U value if H_sD is same for two or more best alternatives.

4: Rank $\mathfrak{k}$ as the best alternative if the ${}^U\mathscr{S}(\mathfrak{k})$ lies on the straight line ${}^{U_Y}F = {}^{U_N}F$ and has least H_sD. Select $\mathfrak{k}$ with lesser E_U value if H_sD is same for two or more best alternatives.

5: Rank $\mathfrak{k}$ as the best alternative if the ${}^U\mathscr{S}(\mathfrak{k})$ lies above the straight line ${}^{U_Y}F = {}^{U_N}F$ and has highest H_sD. Select $\mathfrak{k}$ with highest E_U value if H_sD is same for two or more best alternatives.

6: Getting best alternatives from Steps (3) to (4), the alternative lying below the straight line ${}^{U_Y}F = {}^{U_N}F$ is ranked higher than the alternative lying on the line and the alternative lying on the straight line ${}^{U_Y}F = {}^{U_N}F$ is ranked higher than the alternative lying above the line.

4.4.1 An Application of the Decision-Making Approach:

Recent research in decision-making focuses on selecting the best alternative from a set of comparable options. In this context, we present an example illustrating our proposed approach for choosing a suitable candidate to appoint in a multinational company.

Our method leverages the advanced framework of ${}^qROF_zS$ theory using S_fB_nRs to ensure an objective and reliable selection process. Unlike traditional methods that might rely heavily on subjective opinions or additional external information, our approach strictly utilizes the data provided by the decision-maker. This not only eliminates bias but also enhances the consistency and accuracy of the decision-making process.

For instance, when selecting a candidate for a position in a multinational company, various criteria must be evaluated, such as educational qualifications, relevant experience, technical skills, and cultural fit. By applying our proposed method, we can systematically evaluate each candidate against these criteria using the ${}^qROF_zS$ framework. This method allows us to identify the candidate who most closely aligns with the desired qualifications and organizational needs, ensuring a fair and thorough selection process.

The effectiveness of this approach lies in its ability to objectively rank candidates by calculating lower and upper approximations (L_oA_p and U_pA_p) with respect to the decision alternatives. These approximations provide a robust basis for making informed decisions, thus helping the multinational company appoint the best possible candidate for the role.

Example 4.4.3. *A multi-national company in Pakistan wants to hire an officer for a vacant position of officer in the company. A list of* 10 *short listed candidates is given which are classified in two finite sets, named as platinum and diamond based on their qualification and experience. The sets* $\xi_1 = \{\text{the platinum group}\} = \{k_1, k_2, k_3, k_4, k_5, k_6\}$, $\xi_2 = \{\text{the diamond group}\} = \{t_1, t_2, t_3, t_4\}$, *and the set of parameters* $D = \{e_1, e_2, e_3\}$, *where* e_1 =*education,* e_2 =*experience and* e_3 =*computer knowledge.*

Panel of selection analyzes and compares the qualifications of the candidates and gives a relationship between the two groups. Define a S_fB_nR $\mathscr{S} : D \to P(\xi_1 \times \xi_2)$ *by*

$$\mathscr{S}(e_1) = \begin{array}{c} \\ t_1 \\ t_2 \\ t_3 \\ t_4 \end{array} \begin{array}{c} \begin{array}{cccccc} k_1 & k_2 & k_3 & k_4 & k_5 & k_6 \end{array} \\ \begin{pmatrix} 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \end{pmatrix} \end{array}, \quad \mathscr{S}(e_2) = \begin{array}{c} \\ t_1 \\ t_2 \\ t_3 \\ t_4 \end{array} \begin{array}{c} \begin{array}{cccccc} k_1 & k_2 & k_3 & k_4 & k_5 & k_6 \end{array} \\ \begin{pmatrix} 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \end{pmatrix} \end{array}, \text{ and}$$

$$\mathscr{S}(e_3) = \begin{array}{c} \\ t_1 \\ t_2 \\ t_3 \\ t_4 \end{array} \begin{array}{c} \begin{array}{cccccc} k_1 & k_2 & k_3 & k_4 & k_5 & k_6 \end{array} \\ \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \end{pmatrix} \end{array}$$

represent the relation between candidates of platinum group and diamond group with respect to the parameters e_1, e_2, e_3, *respectively.*

The company owner gives preferences for the candidates of the two groups in the form of two ${}^qROF_zSs$, V *and* U, *where*

$V = \{\langle t_1, 0.7, 0.4\rangle, \langle t_2, 0.8, 0.65\rangle, \langle t_3, 0.95, 0.4\rangle, \langle t_4, 0.4, 0.3\rangle\}$,

$U = \{\langle k_1, 0.5, 0.95\rangle, \langle k_2, 0.55, 0.87\rangle, \langle k_3, 0.98, 0.35\rangle, \langle k_4, 0.5, 0.95\rangle, \langle k_5, 0.93, 0.3\rangle, \langle k_6, 0.8, 0.65\rangle\}$.

Table 4.5: Approximation sets of ${}^qROF_zS$ V with respect to A_fSs

	$\overline{\mathscr{S}}^{\mathbf{V}}(e_1)(k_i)$	$\underline{\mathscr{L}}^{\mathbf{V}}(e_1)(k_i)$	$\overline{\mathscr{S}}^{\mathbf{V}}(e_2)(k_i)$	$\underline{\mathscr{L}}^{\mathbf{V}}(e_2)(k_i)$	$\overline{\mathscr{S}}^{\mathbf{V}}(e_3)(k_i)$	$\underline{\mathscr{L}}^{\mathbf{V}}(e_3)(k_i)$
$k_{\mathbf{1}}$	$(0.95, 0.4)$	$(0.7, 0.65)$	$(0.95, 0.4)$	$(0.95, 0.4)$	$(0.8, 0.65)$	$(0.8, 0.65)$
$k_{\mathbf{2}}$	$(0.8, 0.3)$	$(0.4, 0.65)$	$(0.95, 0.4)$	$(0.95, 0.4)$	$(0.4, 0.3)$	$(0.4, 0.3)$
$k_{\mathbf{3}}$	$(0.7, 0.4)$	$(0.7, 0.4)$	$(0.4, 0.3)$	$(0.4, 0.3)$	$(0.95, 0.4)$	$(0.7, 0.4)$
$k_{\mathbf{4}}$	$(0.95, 0.3)$	$(0.4, 0.65)$	$(0.7, 0.4)$	$(0.7, 0.4)$	$(0.8, 0.65)$	$(0.8, 0.65)$
$k_{\mathbf{5}}$	$(0.95, 0.3)$	$(0.4, 0.4)$	$(0.7, 0.4)$	$(0.7, 0.4)$	$(0.95, 0.3)$	$(0.4, 0.4)$
$k_{\mathbf{6}}$	$(0.8, 0.65)$	$(0.8, 0.65)$	$(0.95, 0.4)$	$(0.8, 0.65)$	$(0.8, 0.65)$	$(0.8, 0.65)$

Table 4.6: Approximation sets of ${}^qROF_zS$ U with respect to F_rSs

	${}^{\mathbf{U}}\overline{\mathscr{S}}(e_1)(t_i)$	${}^{\mathbf{U}}\underline{\mathscr{L}}(e_1)(t_i)$	${}^{\mathbf{U}}\overline{\mathscr{S}}(e_2)(t_i)$	${}^{\mathbf{U}}\underline{\mathscr{L}}(e_2)(t_i)$	${}^{\mathbf{U}}\overline{\mathscr{S}}(e_3)(t_i)$	${}^{\mathbf{U}}\underline{\mathscr{L}}(e_3)(t_i)$
$t_{\mathbf{1}}$	$(0.98, 0.35)$	$(0.5, 0.95)$	$(0.93, 0.3)$	$(0.5, 0.95)$	$(0.98, 0.35)$	$(0.98, 0.35)$
$t_{\mathbf{2}}$	$(0.8, 0.65)$	$(0.5, 0.95)$	$(0.8, 0.65)$	$(0.8, 0.65)$	$(0.8, 0.65)$	$(0.5, 0.95)$
$t_{\mathbf{3}}$	$(0.93, 0.3)$	$(0.5, 0.95)$	$(0.8, 0.65)$	$(0.5, 0.95)$	$(0.98, 0.3)$	$(0.93, 0.35)$
$t_{\mathbf{4}}$	$(0.93, 0.3)$	$(0.5, 0.95)$	$(0.98, 0.35)$	$(0.98, 0.35)$	$(0.93, 0.3)$	$(0.55, 0.87)$

Table 4.7: Calculating the ${}^{q}ROF_zS$ $\mathscr{S}^V$ with respect to A_fSs

	$\mathscr{S}^{\mathbf{V}}(\mathbf{e_1})(\mathbf{k_i})$	$\mathscr{S}^{\mathbf{V}}(\mathbf{e_2})(\mathbf{k_i})$	$\mathscr{S}^{\mathbf{V}}(\mathbf{e_3})(\mathbf{k_i})$	$\mathscr{S}^{\mathbf{V}}(\mathbf{k_i})$	$\mathbf{H_s}(\mathscr{S}^{\mathbf{V}}(\mathbf{k_i}))$
$\mathbf{k_1}$	$(0.968, 0.26)$	$(0.993, 0.16)$	$(0.913, 0.423)$	$(0.836, 0.457)$	0.684
$\mathbf{k_2}$	$(0.816, 0.195)$	$(0.993, 0.16)$	$(0.499, 0.09)$	$(0.404, 0.231)$	0.973
$\mathbf{k_3}$	$(0.828, 0.16)$	$(0.499, 0.09)$	$(0.968, 0.16)$	$(0.4, 0.207)$	0.975
$\mathbf{k_4}$	$(0.953, 0.195)$	$(0.828, 0.16)$	$(0.913, 0.423)$	$(0.720, 0.422)$	0.820
$\mathbf{k_5}$	$(0.953, 0.12)$	$(0.828, 0.16)$	$(0.953, 0.12)$	$(0.752, 0.196)$	0.828
$\mathbf{k_6}$	$(0.913, 0.423)$	$(0.976, 0.26)$	$(0.913, 0.423)$	$(0.813, 0.543)$	0.671

Table 4.8: Calculating the ${}^{q}ROF_zS$ ${}^{U}F$ with respect to F_rSs

	${}^{\mathbf{U}}\mathscr{S}(\mathbf{e_1})(\mathbf{t_i})$	${}^{\mathbf{U}}\mathscr{S}(\mathbf{e_2})(\mathbf{t_i})$	${}^{\mathbf{U}}\mathscr{S}(\mathbf{e_3})(\mathbf{t_i})$	${}^{\mathbf{U}}\mathscr{S}(\mathbf{t_i})$	$\mathbf{H_s}({}^{\mathbf{U}}\mathbf{F}(\mathbf{t_i}))$
$\mathbf{t_1}$	$(0.928, 0.332)$	$(0.939, 0.285)$	$(0.999, 0.122)$	$(0.922, 0.386)$	0.541
$\mathbf{t_2}$	$(0.831, 0.617)$	$(0.913, 0.423)$	$(0.831, 0.617)$	$(0.631, 0.771)$	0.662
$\mathbf{t_3}$	$(0.939, 0.285)$	$(0.831, 0.617)$	$(0.996, 0.105)$	$(0.777, 0.633)$	0.652
$\mathbf{t_4}$	$(0.939, 0.285)$	$(0.999, 0.122)$	$(0.942, 0.261)$	$(0.884, 0.348)$	0.644

Approximation sets of ${}^{q}ROF_zS$ V *are given in the Table 4.5 with respect to* A_fSs *based on* S_fB_nRs $\mathscr{S}(e_i)$ *and approximation sets of* ${}^{q}ROF_zS$ U *are given in the Table 4.6 with respect to* F_rSs *based on* S_fB_nRs $\mathscr{S}(e_i)$*. Using Algorithm we can see the* ${}^{q}ROF_zS$ $\mathscr{S}^V$ *and the* ${}^{q}ROF_zS$ ${}^{U}F$ *calculated in the Tables 4.7 and 4.8, respectively.*

From Figure 4.1, we can see that the ${}^{q}ROF_zD_gs$ *for* t_1, t_3, t_4 *lie below the straight line* $U_Y = U_N$ *and the relationship between their corresponding hesitancy degrees, from Table 4.8, is;*

$$H_sD(t_1) < H_sD(t_4) < H_sD(t_3)$$

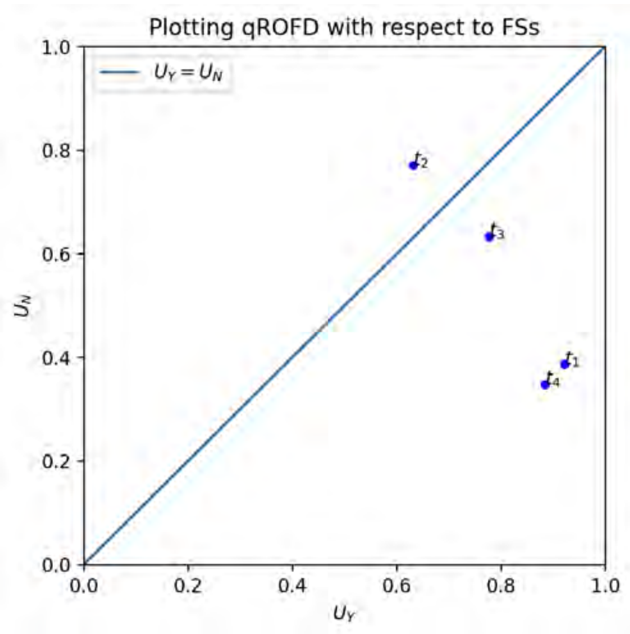

Figure 4.1

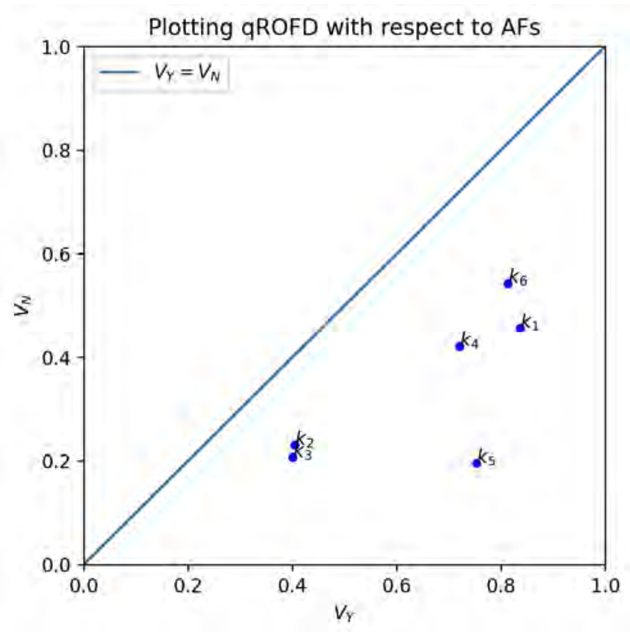

Figure 4.2

Thus, t_1 is the best choice. Hence, the company should go for the candidate t_1.

Similarly, From Figure 4.2, we can see that the ${}^qROF_zD_g$s for $k_1, k_2, k_3, k_4, k_5, k_6$ lie below the *straight line* $V_Y = V_N$ *and the relationship between their corresponding hesitancy degrees, from Table 4.7, is;*

$$H_sD(k_6) < H_sD(k_1) < H_sD(k_4) < H_sD(k_5) < H_sD(k_2) < H_sD(k_4) < H_sD(k_3)$$

Thus, k_6 *is the best choice. Hence, the company should go for the candidate* k_6*.*

This chapter presented a graphical ranking technique for generalized rough q-Rung Orthopair Fuzzy Sets, offering a visual and systematic approach to decision-making. The method was demonstrated through examples, highlighting its ability to handle complex uncertainties. This contribution not only enhances the theoretical understanding of Fuzzy Sets but also provides a practical tool for ranking alternatives in real-world applications.

Chapter 5

Approximation of Pythagorean Fuzzy Ideals over dual universes based on Soft Binary Relation

The concept of dual universes provides a comprehensive framework for representing Pythagorean Fuzzy Ideals by considering two separate but related universes: one for the membership values and another for the non-membership values. This dual representation enables a deeper analysis of the relationships between elements, capturing their uncertainty more effectively. To facilitate the approximation process within this framework, soft binary relations are employed to define the interactions between elements in the dual universes. These relations allow for the systematic computation of lower and upper approximations, ensuring a structured and precise analysis of Pythagorean Fuzzy Ideals. The chapter delves into the rough approximations of various types of S_bS_mGs within a S_mG, employing soft compatible relations. It explores upper and L_oA_ps of Pythagorean Fuzzy S_bS_mGs ($P_yF_zS_bS_mG$), Pythagorean Fuzzy Left Ideals ($P_yF_zL_fI_d$), Pythagorean Fuzzy Right Ideals ($P_yF_zR_iI_d$), Pythagorean Fuzzy Interior Ideals ($P_yF_zI_tI$), and Pythagorean Fuzzy B_iI_ds ($P_yF_zB_iI_d$) in a S_mG, considering both A_fSs and F_rS.

In particular, it demonstrates that the upper approximation (U_pA_p) of a Pythagorean Fuzzy Soft SubS_mG ($P_yF_zS_bS_mG$), Pythagorean Fuzzy Soft Left Ideal ($P_yF_zL_fI_d$), Pythagorean Fuzzy Soft Right Ideal ($P_yF_zR_iI_d$), Pythagorean Fuzzy Soft Interior Ideal ($P_yF_zI_tI$), and Pythagorean Fuzzy Soft Bi-Ideal ($P_yF_zB_iI_d$) in a S_mG indeed results in a Pythagorean Fuzzy Soft S_bS_mG (Pythagorean Fuzzy Soft Left Ideal, Pythagorean Fuzzy Soft Right Ideal, Pythagorean Fuzzy Soft Interior Ideal, or Pythagorean Fuzzy Soft Bi-Ideal, respectively).
Furthermore, the chapter provides concrete examples that illustrate scenarios where this assertion does not hold true, offering a comprehensive view of the conditions under which the upper approximation may fail to yield the expected Pythagorean Fuzzy Soft structures. This critical analysis extends to the discussion of lower approximations (L_oA_p) as well, thereby presenting a complete picture of the behavior and properties of these approximations within the framework of Pythagorean Fuzzy Soft Sets in S_mGs.
This chapter contributes to the comprehension of R_fS-based methodologies in examining S_bS_mGs, utilizing soft compatible relations ($S_fC_mR_l$) as a framework.

5.1 Approximations of Pythagorean Fuzzy Ideals in S_mGs by Soft Binary Relation

Definition 5.1.1. *A $P_yF_zS_fS$ (F, D) over a S_mG M is referred to as a Pythagorean Fuzzy Soft S_bS_mG ($P_yF_zS_fS_bS_mG$) of M if it satisfies the conditions: $\mathscr{S}_Y(e)(mn) \geq \mathscr{S}_Y(e)(m) \wedge \mathscr{S}_Y(e)(n)$ and $\mathscr{S}_N(e)(mn) \leq \mathscr{S}_N(e)(m) \vee \mathscr{S}_N(e)(n)$, for all $m, n \in M$ and $e \in D$.*

Example 5.1.2. *Let $M = \{\mathfrak{k}_1, \mathfrak{k}_2\}$ be a S_mG with the multiplication table given by:*

Table 5.1: Multiplication table

$\cdot$	$\mathfrak{k}_1$	$\mathfrak{k}_2$
$\mathfrak{k}_1$	$\mathfrak{k}_1$	$\mathfrak{k}_2$
$\mathfrak{k}_2$	$\mathfrak{k}_2$	$\mathfrak{k}_2$

Let $D = \{e_1, e_2\}$ be a set of parameters. Define a $P_yF_zS_fS$ (F, D) over M as follows:

$$\mathscr{S}(e_1)(t_i) = \{\langle t_1, 0.8, 0.4\rangle, \langle t_2, 0.5, 0.6\rangle, \langle t_1t_1, 0.8, 0.4\rangle, \langle t_1t_2, 0.5, 0.6\rangle, \langle t_2t_2, 0.5, 0.6\rangle\},$$

$$\mathscr{S}(e_2)(t_i) = \{\langle t_1, 0.7, 0.3\rangle, \langle t_2, 0.6, 0.5\rangle, \langle t_1t_1, 0.7, 0.3\rangle, \langle t_1t_2, 0.6, 0.5\rangle, \langle t_2t_2, 0.6, 0.5\rangle\}$$

where $t_i \in M$. We need to verify that $(\mathscr{S}, D)$ satisfies the conditions of a $P_yF_zS_fS_bS_mG$ over M:

For $e_1 \in D$: $\mathscr{S}_Y(e_1)(t_1t_2) = 0.5 \geq \mathscr{S}_Y(e_1)(t_1) \wedge \mathscr{S}_Y(e_1)(t_2) = 0.8 \wedge 0.5 = 0.5$, $\mathscr{S}_N(e_1)(t_1t_2) = 0.6 \leq \mathscr{S}_N(e_1)(t_1) \vee \mathscr{S}_N(e_1)(t_2) = 0.4 \vee 0.6 = 0.6$, $\mathscr{S}_Y(e_1)(t_2t_2) = 0.5 \geq \mathscr{S}_Y(e_1)(t_2) \wedge \mathscr{S}_Y(e_1)(t_2) = 0.5 \wedge 0.5 = 0.5$, $\mathscr{S}_N(e_1)(t_2t_2) = 0.6 \leq \mathscr{S}_N(e_1)(t_2) \vee \mathscr{S}_N(e_1)(t_2) = 0.6 \vee 0.6 = 0.6$
For $e_2 \in D$: $\mathscr{S}_Y(e_2)(t_1t_2) = 0.6 \geq \mathscr{S}_Y(e_2)(t_1) \wedge \mathscr{S}_Y(e_2)(t_2) = 0.7 \wedge 0.6 = 0.6$, $\mathscr{S}_N(e_2)(t_1t_2) = 0.5 \leq \mathscr{S}_N(e_2)(t_1) \vee \mathscr{S}_N(e_2)(t_2) = 0.3 \vee 0.5 = 0.5$, $\mathscr{S}_Y(e_2)(t_2t_2) = 0.6 \geq \mathscr{S}_Y(e_2)(t_2) \wedge \mathscr{S}_Y(e_2)(t_2) = 0.6 \wedge 0.6 = 0.6$, $\mathscr{S}_N(e_2)(t_2t_2) = 0.5 \leq \mathscr{S}_N(e_2)(t_2) \vee \mathscr{S}_N(e_2)(t_2) = 0.5 \vee 0.5 = 0.5$.

This example demonstrates that the given $P_yF_zS_fS$ satisfies the conditions for being a $P_yF_zS_fS_bS_mG$ of the S_mG M.

Definition 5.1.3. *A $P_yF_zS_fS$ $(\mathscr{S}, D)$ over M is called*

1) *a Pythagorean Fuzzy Soft Left Ideal ($P_yF_zS_fL_fI_d$) if, for each $e \in D$, $\mathscr{S}(e)$ is a $P_yF_zL_fI_d$ of M.*

2) *a Pythagorean Fuzzy Soft Right Ideal ($P_yF_zS_fR_fI_d$) if, for each $e \in D$, $\mathscr{S}(e)$ is a $P_yF_zR_iI_d$ of M.*

3) *a Pythagorean Fuzzy Soft Interior Ideal ($P_yF_zS_fI_tI_d$) if, for each $e \in D$, $\mathscr{S}(e)$ is a $P_yF_zI_tI_d$ of M.*

4) *a Pythagorean Fuzzy Soft Bi-Ideal ($P_yF_zS_fB_iI_d$) if, for each $e \in D$, $\mathscr{S}(e)$ is a $P_yF_zB_iI_d$ of M.*

Theorem 5.1.4. *Let $(\mathscr{S}, D)$ be a $S_fC_mR_l$ from a S_mG M_1 to a S_mG M_2.*

1) If U_2 is a $P_yF_zS_bS_mG$ of M_2, then $(\overline{\mathscr{S}}^{U_2}, D)$ is a $P_yF_zS_fS_bS_mG$ of M_1.

2) If U_2 is a $P_yF_zL_fI_d$ of M_2, then $(\overline{\mathscr{S}}^{U_2}, D)$ is a $P_yF_zS_fL_fI_d$ of M_1.

3) If U_2 is a $P_yF_zR_fI_d$ of M_2, then $(\overline{\mathscr{S}}^{U_2}, D)$ is a $P_yF_zS_fR_iI_d$ of M_1.

Proof.

1) We assume that U_2 is a $P_yF_zS_bS_mG$ of M_2. Now for $a, b \in M_1$,

$$\begin{aligned}
\overline{\mathscr{S}}^{U_{2_Y}}(e)(a) \wedge \overline{\mathscr{S}}^{U_{2_Y}}(e)(b) &= \Big(\vee_{m \in a\mathscr{S}(e)} U_{2_Y}(m) \Big) \wedge \Big(\vee_{n \in b\mathscr{S}(e)} U_{2_Y}(n) \Big) \\
&= \vee_{m \in a\mathscr{S}(e)} \vee_{n \in b\mathscr{S}(e)} \Big(U_{2_Y}(m) \wedge U_{2_Y}(n) \Big) \\
&\leq \vee_{m \in a\mathscr{S}(e)} \vee_{n \in b\mathscr{S}(e)} \Big(U_{2_Y}(mn) \Big) \\
&\leq \vee_{mn \in (ab)\mathscr{S}(e)} \Big(U_{2_Y}(mn) \Big) = \vee_{m' \in (ab)\mathscr{S}(e)} \Big(U_{2_Y}(m') \Big) \\
&= \overline{\mathscr{S}}^{U_{2_Y}}(e)(ab).
\end{aligned}$$

Similarly for $a, b \in M_1$,

$$\begin{aligned}
\overline{\mathscr{S}}^{U_{2_N}}(e)(a) \vee \overline{\mathscr{S}}^{U_{2_N}}(e)(b) &= \Big(\wedge_{m \in a\mathscr{S}(e)} U_{2_N}(m) \Big) \vee \Big(\wedge_{n \in b\mathscr{S}(e)} U_{2_N}(n) \Big) \\
&= \wedge_{m \in a\mathscr{S}(e)} \wedge_{n \in b\mathscr{S}(e)} \Big(U_{2_N}(m) \vee U_{2_N}(n) \Big) \\
&\geq \wedge_{m \in a\mathscr{S}(e)} \wedge_{n \in b\mathscr{S}(e)} \Big(U_{2_N}(mn) \Big) \\
&\geq \wedge_{mn \in (ab)\mathscr{S}(e)} \Big(U_{2_N}(mn) \Big) = \wedge_{m' \in (ab)\mathscr{S}(e)} \Big(U_{2_N}(m') \Big) \\
&= \overline{\mathscr{S}}^{U_{2_N}}(e)(ab).
\end{aligned}$$

Hence, $\overline{\mathscr{S}}^{U_2}(e)$ is a $P_yF_zS_bS_mG$ of M_1 for all $e \in D$, so $(\overline{\mathscr{S}}^{U_2}, D)$ is a $P_yF_zS_fS_bS_mG$ of M_1.

2) Assume that U_2 is a $P_yF_zL_fI_d$ of M_2. Now for $a, b \in M_1$,

$$\overline{\mathscr{S}}^{U_{2_Y}}(e)(b) = \Big(\vee_{n \in b\mathscr{S}(e)} U_{2_Y}(n) \Big) \leq \vee_{m \in a\mathscr{S}(e)} \vee_{n \in b\mathscr{S}(e)} U_{2_Y}(mn)$$

$$\leq \vee_{mn \in (mn)\mathscr{S}(e)} \Big(U_{2_Y}(mn) \Big) = \vee_{m' \in (ab)\mathscr{S}(e)} \Big(U_{2_Y}(m') \Big) = \overline{\mathscr{S}}^{U_{2_Y}}(e)(ab).$$

Similarly for $a, b \in M_1$,

$$\overline{\mathscr{S}}^{U_{2_N}}(e)(b) = \Big(\wedge_{n \in b\mathscr{S}(e)} U_{2_N}(n) \Big) \geq \wedge_{m \in a\mathscr{S}(e)} \wedge_{n \in b\mathscr{S}(e)} U_{2_N}(mn)$$

$$\geq \wedge_{mn \in (ab)\mathscr{S}(e)} \Big(U_{2_N}(mn) \Big) = \wedge_{n' \in (ab)\mathscr{S}(e)} \Big(U_{2_N}(n') \Big) = \overline{\mathscr{S}}^{U_{2_N}}(e)(ab).$$

Hence, $\overline{\mathscr{S}}^{U_2}(e)$ is a $P_yF_zL_fI_d$ of M_1 for all $e \in D$, so $(\overline{\mathscr{S}}^{U_2}, D)$ is a $P_yF_zS_fL_fI_d$ of M_1.

3) The proof can be derived using the same approach as in part (2).

□

In Theorem 5.1.4 from part 1, $S_fC_mR_l$s from M_1 to M_2 are given, and U_2 is a $P_yF_zS_bS_mG$ in M_2. After combining them, we get generalized $P_yF_zS_fS_bS_mG$s in M_1. Similarly, if we take a $P_yF_zL_fI_d$ or $P_yF_zR_iI_d$ U_2 of M_2, then we get generalized $P_yF_zS_fL_fI_d$ or $P_yF_zS_fR_iI_d$ of M_1.

Theorem 5.1.5. *Let $(\mathscr{S}, D)$ be a $S_fC_mR_l$ from a S_mG M_1 to a S_mG M_2:*

1) If U_1 is a $P_yF_zS_bS_mG$ of M_1, then $({}^{U_1}\overline{\mathscr{S}}, D)$ is a $P_yF_zS_fS_bS_mG$ of M_2

2) If U_1 is a $P_yF_zL_fI_d$ ($P_yF_zR_iI_d$) of M_1, then $({}^{U_1}\overline{\mathscr{S}}, D)$ is a $P_yF_zS_fL_fI_d$ ($P_yF_zS_fR_iI_d$) of M_2, respectively.

Proof.

The proof can be derived using the same approach as in the Theorem 5.1.4. □

In Theorem 5.1.5 from part 1, $S_fC_mR_l$s from M_1 to M_2 are given, and U_1 is a $P_yF_zS_bS_mG$ in M_1. After combining them, we get generalized $P_yF_zS_fS_bS_mG$s in M_2. Similarly, if we take a

$P_yF_zL_fI_d$ or $P_yF_zL_fI_d\ U_1$ of M_1, then we get generalized $P_yF_zS_fL_fI_d$ or $P_yF_zS_fL_fI_d$ of M_2. Now, we show that the converses of parts of the above Theorems 5.1.4 and 5.1.5 generally do not hold.

Example 5.1.6. *Let $M_1 = \{k_1, k_2, k_3, k_4, k_5\}$ and $M_2 = \{t_1, t_2, t_3, t_4, t_5\}$ be two S_mGs, where the multiplication tables of M_1 and M_2 are presented in Tables 5.2 and 5.3, respectively. Let $D = \{e_1, e_2\}$. Define a S_fB_nR $\mathscr{S} : D \to P(M_1 \times M_2)$ by:*

$$\mathscr{S}(e_1) = \begin{array}{c} \\ t_1 \\ t_2 \\ t_3 \\ t_4 \\ t_5 \end{array} \begin{array}{c} \begin{array}{ccccc} k_1 & k_2 & k_3 & k_4 & k_5 \end{array} \\ \begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 \end{pmatrix} \end{array}, \quad \mathscr{S}(e_2) = \begin{array}{c} \\ t_1 \\ t_2 \\ t_3 \\ t_4 \\ t_5 \end{array} \begin{array}{c} \begin{array}{ccccc} k_1 & k_2 & k_3 & k_4 & k_5 \end{array} \\ \begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 1 \end{pmatrix} \end{array}$$

Then, $(\mathscr{S}, D)$ is a $S_fC_mR_l$ from M_1 to M_2.

$$\begin{aligned}
k_1\mathscr{S}(e_1) &= \{t_1\}, & \mathscr{S}(e_1)t_1 &= \{k_1, k_2, k_4\}, \\
k_2\mathscr{S}(e_1) &= \{t_1, t_2, t_5\}, & \mathscr{S}(e_1)t_2 &= \{k_2\}, \\
k_3\mathscr{S}(e_1) &= \{t_3, t_5\}, & \mathscr{S}(e_1)t_3 &= \{k_3, k_4\}, \\
k_4\mathscr{S}(e_1) &= \{t_1, t_3, t_4, t_5\}, & \mathscr{S}(e_1)t_4 &= \{k_4\}, \\
k_5\mathscr{S}(e_1) &= \{t_5\}, & \mathscr{S}(e_1)t_5 &= \{k_2, k_3, k_4, k_5\}, \\
k_1\mathscr{S}(e_2) &= \{t_1\}, & \mathscr{S}(e_2)t_1 &= \{k_1, k_2, k_4\}, \\
k_2\mathscr{S}(e_2) &= \{t_1, t_2, t_3, t_5\}, & \mathscr{S}(e_2)t_2 &= \{k_2\}, \\
k_3\mathscr{S}(e_2) &= \{t_3, t_5\}, & \mathscr{S}(e_2)t_3 &= \{k_2, k_3, k_4\}, \\
k_4\mathscr{S}(e_2) &= \{t_1, t_3, t_4, t_5\}, & \mathscr{S}(e_2)t_4 &= \{k_4\}, \\
k_5\mathscr{S}(e_2) &= \{t_5\}, & \mathscr{S}(e_2)t_5 &= \{k_2, k_3, k_4, k_5\}.
\end{aligned}$$

Table 5.2: Multiplication table for M_1

.	$\mathfrak{k}_1$	$\mathfrak{k}_2$	$\mathfrak{k}_3$	$\mathfrak{k}_4$	$\mathfrak{k}_5$
$\mathfrak{k}_1$	$\mathfrak{k}_2$	$\mathfrak{k}_2$	$\mathfrak{k}_4$	$\mathfrak{k}_4$	$\mathfrak{k}_4$
$\mathfrak{k}_2$	$\mathfrak{k}_2$	$\mathfrak{k}_2$	$\mathfrak{k}_4$	$\mathfrak{k}_4$	$\mathfrak{k}_4$
$\mathfrak{k}_3$	$\mathfrak{k}_4$	$\mathfrak{k}_4$	$\mathfrak{k}_3$	$\mathfrak{k}_4$	$\mathfrak{k}_3$
$\mathfrak{k}_4$	$\mathfrak{k}_4$	$\mathfrak{k}_4$	$\mathfrak{k}_4$	$\mathfrak{k}_4$	$\mathfrak{k}_4$
$\mathfrak{k}_5$	$\mathfrak{k}_4$	$\mathfrak{k}_4$	$\mathfrak{k}_3$	$\mathfrak{k}_4$	$\mathfrak{k}_3$

Table 5.3: Multiplication table for M_2

.	$\mathfrak{t}_1$	$\mathfrak{t}_2$	$\mathfrak{t}_3$	$\mathfrak{t}_4$	$\mathfrak{t}_5$
$\mathfrak{t}_1$	$\mathfrak{t}_1$	$\mathfrak{t}_5$	$\mathfrak{t}_3$	$\mathfrak{t}_4$	$\mathfrak{t}_5$
$\mathfrak{t}_2$	$\mathfrak{t}_1$	$\mathfrak{t}_2$	$\mathfrak{t}_3$	$\mathfrak{t}_4$	$\mathfrak{t}_5$
$\mathfrak{t}_3$	$\mathfrak{t}_1$	$\mathfrak{t}_5$	$\mathfrak{t}_3$	$\mathfrak{t}_4$	$\mathfrak{t}_5$
$\mathfrak{t}_4$	$\mathfrak{t}_1$	$\mathfrak{t}_5$	$\mathfrak{t}_3$	$\mathfrak{t}_4$	$\mathfrak{t}_5$
$\mathfrak{t}_5$	$\mathfrak{t}_1$	$\mathfrak{t}_5$	$\mathfrak{t}_3$	$\mathfrak{t}_4$	$\mathfrak{t}_5$

1) Define a P_yF_zS $U_1 : M_2 \to [0,1]$ by

$U_1 = \{\langle \mathfrak{t}_1, 0.5, 0.4\rangle, \langle \mathfrak{t}_2, 0.4, 0.5\rangle, \langle \mathfrak{t}_3, 0.3, 0.7\rangle, \langle \mathfrak{t}_4, 1, 0\rangle, \langle \mathfrak{t}_5, 0.1, 0.8\rangle\}$.

Then, U_1 is not a $P_yF_zS_bS_mG$ of M_2 because if we take $a = \mathfrak{t}_1, b = \mathfrak{t}_2$, then $U_{1_Y}(\mathfrak{t}_1\mathfrak{t}_2) = 0.1 \ngeq 0.4 = U_{1_Y}(\mathfrak{t}_1) \wedge U_{1_Y}(\mathfrak{t}_2)$ and $U_{1_N}(\mathfrak{t}_1\mathfrak{t}_2) \nleq U_{1_N}(\mathfrak{t}_1) \vee U_{1_N}(\mathfrak{t}_2)$. U_pA_p of U_1 is given in Table 5.4.

Table 5.4: U_pA_p of U_1

	$\overline{\mathscr{S}}^{U_{1_Y}}(e_1)$	$\overline{\mathscr{S}}^{U_{1_N}}(e_1)$	$\overline{\mathscr{S}}^{U_{1_Y}}(e_2)$	$\overline{\mathscr{S}}^{U_{1_N}}(e_2)$
$\mathfrak{k}_1$	0.5	0.4	0.5	0.4
$\mathfrak{k}_2$	0.5	0.4	0.5	0.4
$\mathfrak{k}_3$	0.3	0.7	0.3	0.7
$\mathfrak{k}_4$	1	0	1	0
$\mathfrak{k}_5$	0.1	0.8	0.1	0.8

Clearly, $\overline{\mathscr{S}}^{U_1}(e_1)$ *and* $\overline{\mathscr{S}}^{U_1}(e_2)$ *are* $P_yF_zS_bS_mGs$ *of* M_1*, so* $(\overline{\mathscr{S}}^{U_1}, D)$ *is a* $P_yF_zS_fS_bS_mG$ *of* M_1*.*

2) *Define a* P_yF_zS $U_2 : M_1 \to [0,1]$ *by*
$U_2 = \{\langle \mathfrak{k}_1, 0.2, 0.7\rangle, \langle \mathfrak{k}_2, 0.7, 0.3\rangle, \langle \mathfrak{k}_3, 0.8, 0.2\rangle, \langle \mathfrak{k}_4, 0, 1\rangle, \langle \mathfrak{k}_5, 0.9, 0.1\rangle\}$*. Then,* U_2 *is not a* $P_yF_zS_bS_mG$ *of* M_1 *because if we take* $a = \mathfrak{k}_2, b = \mathfrak{k}_3$*, then* $U_{2_Y}(\mathfrak{k}_2\mathfrak{k}_3) = 0 \ngeq 0.7 = U_{2_Y}(\mathfrak{k}_2) \wedge U_{2_Y}(\mathfrak{k}_3)$ *and* $U_{2_N}(\mathfrak{k}_2\mathfrak{k}_3) = 1 \nleq 0.8 = U_{2_N}(\mathfrak{k}_2) \vee U_{2_N}(\mathfrak{k}_3)$*.* U_pA_p *of* U_2 *is given in Table 5.5.*

Table 5.5: U_pA_p of U_2

	${}^{U_{2Y}}\overline{\mathscr{S}}(e_1)$	${}^{U_{2N}}\overline{\mathscr{S}}(e_1)$	${}^{U_{2Y}}\overline{\mathscr{S}}(e_2)$	${}^{U_{2N}}\overline{\mathscr{S}}(e_2)$
$\mathfrak{t}_1$	0.7	0.3	0.7	0.3
$\mathfrak{t}_2$	0.7	0.3	0.7	0.3
$\mathfrak{t}_3$	0.8	0.2	0.8	0.2
$\mathfrak{t}_4$	0	1	0	1
$\mathfrak{t}_5$	0.9	0.1	0.9	0.1

Clearly, $\overline{\mathscr{S}}^{U_2}(e_1)$ *and* $\overline{\mathscr{S}}^{U_2}(e_2)$ *are* $P_yF_zS_bS_mGs$ *of* M_2*, so* $(\overline{\mathscr{S}}^{U_2}, D)$ *is a* $P_yF_zS_fS_bS_mG$ *of* M_2*.*

3) *Define a* $P_yF_zSU_3 : M_2 \to [0,1]$ *by*
$U_3 = \{\langle \mathfrak{t}_1, 0.5, 0.4\rangle, \langle \mathfrak{t}_2, 0.4, 0.6\rangle, \langle \mathfrak{t}_3, 0.3, 0.6\rangle, \langle \mathfrak{t}_4, 1, 0\rangle, \langle \mathfrak{t}_5, 0.1, 0.8\rangle\}$*. Then,* U_3 *is not a* $P_yF_zL_fI_d$ *of* M_2 *because if we take* $a = \mathfrak{t}_1, b = \mathfrak{t}_2$*, then* $U_{3_Y}(\mathfrak{t}_1\mathfrak{t}_2) = 0.1 \ngeq= 0.4 = U_{3_Y}(\mathfrak{t}_2)$ *and* $U_{3_N}(\mathfrak{t}_1\mathfrak{t}_2) = 0.8 \nleq 0.6 = U_{3_N}(\mathfrak{t}_2)$*.* U_pA_p *of* U_3 *is given in Table 5.6.*

Table 5.6: U_pA_p of U_3

	$\overline{\mathscr{S}}^{U_{3_Y}}(e_1)$	$\overline{\mathscr{S}}^{U_{3_N}}(e_1)$	$\overline{\mathscr{S}}^{U_{3_Y}}(e_2)$	$\overline{\mathscr{S}}^{U_{3_N}}(e_2)$
$\mathfrak{k}_1$	0.5	0.4	0.5	0.4
$\mathfrak{k}_2$	0.5	0.4	0.5	0.4
$\mathfrak{k}_3$	0.3	0.6	0.3	0.6
$\mathfrak{k}_4$	1	0	1	0
$\mathfrak{k}_5$	0.1	0.8	0.1	0.8

Clearly, $\overline{\mathscr{S}}^{U_3}(e_1)$ *and* $\overline{\mathscr{S}}^{U_3}(e_2)$ *are* $P_yF_zL_fI_ds$ *of* M_1*, so* $(\overline{\mathscr{S}}^{U_3}, D)$ *is a* $P_yF_zS_fL_fI_d$ *of* M_1.

4) Define a P_yF_zS $U_4 : M_1 \to [0,1]$ *by*
$U_4 = \{\langle \mathfrak{k}_1, 0.2, 0.7\rangle, \langle \mathfrak{k}_2, 0.7, 0.2\rangle, \langle \mathfrak{k}_3, 0.8, 0.1\rangle, \langle \mathfrak{k}_4, 0, 1\rangle, \langle \mathfrak{k}_5, 0.9, 0.1\rangle\}$. *Then,* U_4 *is not a* $P_yF_zL_fI_d$ *of* M_1 *because if we take* $a = \mathfrak{k}_1, b = \mathfrak{k}_3$ *then* $U_{4_Y}(\mathfrak{k}_1\mathfrak{k}_3) = 0 \ngeq 0.8 = U_{4_Y}(\mathfrak{k}_3)$ *and* $U_{4_N}(\mathfrak{k}_1\mathfrak{k}_3) = 1 \nleq 0.7 = U_{4_N}(\mathfrak{k}_3)$*.* U_pA_p *of* U_4 *is given in Table 5.7.*

Table 5.7: U_pA_p of U_4

	${}^{U_{4_Y}}\overline{\mathscr{S}}(e_1)$	${}^{U_{4_N}}\overline{\mathscr{S}}(e_1)$	${}^{U_{4_Y}}\overline{\mathscr{S}}(e_2)$	${}^{U_{4_N}}\overline{\mathscr{S}}(e_2)$
$\mathfrak{t}_1$	0.7	0.2	0.7	0.2
$\mathfrak{t}_2$	0.7	0.2	0.7	0.2
$\mathfrak{t}_3$	0.8	0.1	0.8	0.1
$\mathfrak{t}_4$	0	1	0	1
$\mathfrak{t}_5$	0.9	0.1	0.9	0.1

Clearly, ${}^{U_4}\overline{\mathscr{S}}(e_1)$ *and* ${}^{U_4}\overline{\mathscr{S}}(e_2)$ *are* $P_yF_zL_fI_ds$ *of* M_2*, so* $({}^{U_4}\overline{\mathscr{S}}, D)$ *is a* $P_yF_zS_fL_fI_d$ *of* M_2.

Example 5.1.7. *Consider the* S_mGs *and* S_fB_nR *from Example 5.1.6.*
Define a P_yF_zS $U : M_2 \to [0,1]$ *by*
$U = \{\langle \mathfrak{t}_1, 0.7, 0.2\rangle, \langle \mathfrak{t}_2, 0.7, 0.3\rangle, \langle \mathfrak{t}_3, 0.8, 0.1\rangle, \langle \mathfrak{t}_4, 0, 1\rangle, \langle \mathfrak{t}_5, 0.9, 0.1\rangle\}$. *Then,* U *is a* $P_yF_zL_fI_d$ *of*

M_2 L_oA_p of U is given in Table 5.8. But $\underline{\mathscr{L}}^U(e_1)$ is not a $P_yF_zL_fI_d$ of M_1 because if we take $a = \Bbbk_1, b = \Bbbk_3$ then $U_Y(\Bbbk_1\Bbbk_3) = 0 \ngeq 0.8 = U_Y(\Bbbk_3)$ and $U_N(\Bbbk_1\Bbbk_3) = 1 \nleq 0.1 = U_N(\Bbbk_3)$.

Table 5.8: L_oA_p of U

	$\underline{\mathscr{L}}^{U_Y}(e_1)$	$\underline{\mathscr{L}}^{U_N}(e_1)$
$\Bbbk_1$	0.7	0.2
$\Bbbk_2$	0.7	0.3
$\Bbbk_3$	0.8	0.1
$\Bbbk_4$	0	1
$\Bbbk_5$	0.9	0.1

The aforementioned example demonstrates that if a soft relation is a C_mR_l, the L_oA_p of a $P_yF_zL_fI_d$ is not necessarily a $P_yF_zS_fL_fI_d$. However, we have the following result.

Theorem 5.1.8. *Consider $(\mathscr{S}, D)$ constitutes a $S_fC_{mp}R_l$ from a S_mG M_1 to a S_mG M_2.*

1) *If U_2 is a $P_yF_zS_bS_mG$ of M_2, then $(\underline{\mathscr{L}}^{U_2}, D)$ is a $P_yF_zS_fS_bS_mG$ of M_1*

2) *If U_2 is a $P_yF_zL_fI_d$ ($P_yF_zR_iI_d$) of M_2, then $(\underline{\mathscr{L}}^{U_2}, D)$ is a $P_yF_zS_fL_fI_d$ ($P_yF_zS_fR_iI_d$) of M_1.*

Proof.

1) We assume that U_2 is a $P_yF_zS_bS_mG$ of M_2. Now for $a, b \in M_1$,

$$\begin{aligned}
\underline{\mathscr{L}}^{U_{2_Y}}(ab) &= \wedge_{m' \in (ab)\mathscr{S}(e)} U_{2_Y}\left(m'\right) = \wedge_{m' \in (a)\mathscr{S}(e).(b)\mathscr{S}(e)} U_{2_Y}\left(m'\right) \\
&= \wedge_{m \in (a)\mathscr{S}(e).n \in (b)\mathscr{S}(e)} U_{2_Y}(mn) \geq \wedge_{m \in (a)\mathscr{S}(e)} \wedge_{n \in (b)\mathscr{S}(e)} \left(U_{2_Y}(m) \wedge U_{2_Y}(n)\right) \\
&\geq \left(\wedge_{m \in a\mathscr{S}(e)} U_{2_Y}(m)\right) \wedge \left(\wedge_{n \in b\mathscr{S}(e)} U_{2_Y}(n)\right) = \underline{\mathscr{L}}^{U_{2_Y}}(e)(a) \wedge \underline{\mathscr{L}}^{U_{2_Y}}(e)(b).
\end{aligned}$$

Similarly for $a, b \in M_1$,

$$\underline{\mathscr{L}}^{U_{2_N}}(e)(ab) = \vee_{m' \in (ab)\mathscr{S}(e)} U_{2_N}\left(m'\right) = \vee_{m' \in (a)\mathscr{S}(e).(b)\mathscr{S}(e)} U_{2_N}\left(m'\right)$$
$$= \vee_{m \in (a)\mathscr{S}(e).n \in (b)\mathscr{S}(e)} U_{2_N}(mn) \leq \vee_{m \in (a)\mathscr{S}(e)} \vee_{n \in (b)\mathscr{S}(e)} \left(U_{2_N}(m) \vee U_{2_N}(n)\right)$$
$$\leq \left(\vee_{m \in a\mathscr{S}(e)} U_{2_N}(m)\right) \vee \left(\vee_{n \in b\mathscr{S}(e)} U_{2_N}(n)\right) = \underline{\mathscr{L}}^{U_{2_N}}(e)(a) \vee \underline{\mathscr{L}}^{U_{2_N}}(e)(b).$$

Hence, $\underline{\mathscr{L}}^{U_2}(e)$ is a $P_yF_zS_bS_mG$ of M_1 for all $e \in D$, so $(\underline{\mathscr{L}}^{U_2}, D)$ is a $P_yF_zS_fS_bS_mG$ of M_1.

2) Assume that U_2 is a $P_yF_zL_fI_d$ of M_2. Now for $a, b \in M_1$,

$$\underline{\mathscr{L}}^{U_{2_Y}}(e)(ab) = \wedge_{m' \in (ab)\mathscr{S}(e)} U_{2_Y}\left(m'\right) = \wedge_{m' \in (a)\mathscr{S}(e).(b)\mathscr{S}(e)} U_{2_Y}\left(m'\right)$$
$$= \wedge_{m \in (a)\mathscr{S}(e).n \in (b)\mathscr{S}(e)} U_{2_Y}(mn) \geq \wedge_{n \in (b)\mathscr{S}(e)} U_{2_Y}(n)$$
$$= \underline{\mathscr{L}}^{U_{2_Y}}(e)(b).$$

Similarly for $a, b \in M_1$,

$$\underline{\mathscr{L}}^{U_{2_N}}(e)(ab) = \vee_{m' \in (ab)\mathscr{S}(e)} U_{2_N}\left(m'\right) = \vee_{m' \in (a)\mathscr{S}(e).(b)\mathscr{S}(e)} U_{2_N}\left(m'\right)$$
$$= \vee_{m \in (a)\mathscr{S}(e).n \in (b)\mathscr{S}(e)} U_{2_N}(mn) \leq \vee_{n \in (b)\mathscr{S}(e)} U_{2_N}(n)$$
$$= \underline{\mathscr{L}}^{U_{2_N}}(e)(b).$$

Hence, $\underline{\mathscr{L}}^{U_2}(e)$ is a $P_yF_zL_fI_d$ of M_1 for all $e \in D$, so $(\underline{\mathscr{L}}^{U_2}, D)$ is a $P_yF_zS_fL_fI_d$ of M_1.

□

Theorem 5.1.9. *Suppose $(\mathscr{S}, D)$ constitutes a $S_fC_{mp}R_l$ from a S_mG M_1 to a S_mG M_2.*

1) If U_1 is a $P_yF_zS_bS_mG$ of M_1, then $({}^{U_1}\overline{\mathscr{S}}, D)$ is a $P_yF_zS_fS_bS_mG$ of M_2

2) If U_1 is a $P_yF_zL_fI_d$ ($P_yF_zR_iI_d$) of M_1, then $({}^{U_1}\overline{\mathscr{S}}, D)$ is a $P_yF_zS_fL_fI_d$ ($P_yF_zS_fR_iI_d$) of M_2.

Proof.

The proof can be derived using the same approach as in the Theorem 5.1.8. □

Example 5.1.10. *Consider two S_mGs, $M_1 = \{\mathfrak{k}_1, \mathfrak{k}_2, \mathfrak{k}_3, \mathfrak{k}_4\}$ and $M_2 = \{\mathfrak{t}_1, \mathfrak{t}_2, \mathfrak{t}_3, \mathfrak{t}_4\}$, with their respective multiplication tables presented in Tables 5.9 and 5.10. Let $D = \{e_1, e_2\}$.*

Table 5.9: Multiplication table for M_1

.	$\mathfrak{k}_1$	$\mathfrak{k}_2$	$\mathfrak{k}_3$	$\mathfrak{k}_4$
$\mathfrak{k}_1$	$\mathfrak{k}_1$	$\mathfrak{k}_1$	$\mathfrak{k}_1$	$\mathfrak{k}_4$
$\mathfrak{k}_2$	$\mathfrak{k}_1$	$\mathfrak{k}_2$	$\mathfrak{k}_1$	$\mathfrak{k}_4$
$\mathfrak{k}_3$	$\mathfrak{k}_1$	$\mathfrak{k}_1$	$\mathfrak{k}_3$	$\mathfrak{k}_4$
$\mathfrak{k}_4$	$\mathfrak{k}_4$	$\mathfrak{k}_4$	$\mathfrak{k}_4$	$\mathfrak{k}_4$

Table 5.10: Multiplication table for M_2

.	$\mathfrak{t}_1$	$\mathfrak{t}_2$	$\mathfrak{t}_3$	$\mathfrak{t}_4$
$\mathfrak{t}_1$	$\mathfrak{t}_1$	$\mathfrak{t}_2$	$\mathfrak{t}_3$	$\mathfrak{t}_4$
$\mathfrak{t}_2$	$\mathfrak{t}_2$	$\mathfrak{t}_2$	$\mathfrak{t}_2$	$\mathfrak{t}_2$
$\mathfrak{t}_3$	$\mathfrak{t}_3$	$\mathfrak{t}_3$	$\mathfrak{t}_3$	$\mathfrak{t}_3$
$\mathfrak{t}_4$	$\mathfrak{t}_4$	$\mathfrak{t}_3$	$\mathfrak{t}_2$	$\mathfrak{t}_1$

Define a S_fB_nR $\mathscr{S} : D \to P(M_1 \times M_2)$ by:

$$\mathscr{S}(e_1) = \begin{array}{c} \\ \mathfrak{t}_1 \\ \mathfrak{t}_2 \\ \mathfrak{t}_3 \\ \mathfrak{t}_4 \end{array} \begin{array}{c} \begin{array}{cccc} \mathfrak{k}_1 & \mathfrak{k}_2 & \mathfrak{k}_3 & \mathfrak{k}_4 \end{array} \\ \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \end{array}, \quad \mathscr{S}(e_2) = \begin{array}{c} \\ \mathfrak{t}_1 \\ \mathfrak{t}_2 \\ \mathfrak{t}_3 \\ \mathfrak{t}_4 \end{array} \begin{array}{c} \begin{array}{cccc} \mathfrak{k}_1 & \mathfrak{k}_2 & \mathfrak{k}_3 & \mathfrak{k}_4 \end{array} \\ \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \end{array}$$

Then, $(\mathscr{S}, D)$ *forms a* $S_fC_{mp}R_l$ *from* M_1 *to* M_2*.*

$$\Bbbk_1\mathscr{S}(e_1) = \{t_2, t_3\},$$
$$\Bbbk_2\mathscr{S}(e_1) = \{t_2, t_3\},$$
$$\Bbbk_3\mathscr{S}(e_1) = \{t_2, t_3\},$$
$$\Bbbk_4\mathscr{S}(e_1) = \{t_2, t_3\},$$
$$\Bbbk_1\mathscr{S}(e_2) = \{t_2\},$$
$$\Bbbk_2\mathscr{S}(e_2) = \{t_2\},$$
$$\Bbbk_3\mathscr{S}(e_2) = \{t_2\},$$
$$\Bbbk_4\mathscr{S}(e_2) = \{t_2\}.$$

1) *Define a* P_yF_zS $U_2 : M_2 \to [0,1]$ *by* $U_2 = \{\langle t_1, 0.2, 0.7\rangle, \langle t_2, 0.4, 0.5\rangle, \langle t_3, 0.6, 0.2\rangle, \langle t_4, 0.8, 0.1\rangle\}$*. Then,* U_2 *is not a* $P_yF_zS_bS_mG$ *of* M_2 *because if we take* $a = t_4$ *and* $b = t_4$*, we find* $U_{2_Y}(t_4t_4) = 0.2 \ngeq 0.8 = U_{2_Y}(t_4) \wedge U_{2_Y}(t_4)$ *and* $U_{2_N}(t_4t_4) = 0.7 \nleq 0.1 = U_{2_N}(t_4) \vee U_{2_N}(t_4)$*. The* L_oA_p *of* U_2 *is presented in Table 5.11.*

Table 5.11: L_oA_p of U_2

	$\underline{\mathscr{S}}^{U_{2_Y}}(e_1)$	$\underline{\mathscr{S}}^{U_{2_N}}(e_1)$	$\underline{\mathscr{S}}^{U_{2_Y}}(e_2)$	$\underline{\mathscr{S}}^{U_{2_N}}(e_2)$
$\Bbbk_1$	0.2	0.5	0.4	0.5
$\Bbbk_2$	0.2	0.5	0.4	0.5
$\Bbbk_3$	0.2	0.5	0.4	0.5
$\Bbbk_4$	0.2	0.5	0.4	0.5

Clearly, $\underline{\mathscr{S}}^{U_2}(e_1)$ *and* $\underline{\mathscr{S}}^{U_2}(e_2)$ *are* $P_yF_zS_bS_mG$*s of* M_1*, so* $(\underline{\mathscr{S}}^{U_2}, D)$ *is a* $P_yF_zS_fS_bS_mG$ *of* M_1*.*

2) *Define a* P_yF_zS $U_2 : M_2 \to [0,1]$ *by*

$U_2 = \{\langle ŧ_1, 0.2, 0.7\rangle, \langle ŧ_2, 0.4, 0.5\rangle, \langle ŧ_3, 0.6, 0.2\rangle, \langle ŧ_4, 0.8, 0.1\rangle\}$. *Then,* U_2 *is not a* $P_yF_zL_fI_d$ *of* M_2 *because if we take* $a = ŧ_2, b = ŧ_3$*, then* $U_{2_Y}(ŧ_2ŧ_3) = 0.4 \ngeq 0.6 = U_{2_Y}(ŧ_3)$ *and* $U_{2_N}(ŧ_2ŧ_3) = 0.5 \nleq 0.2 = U_{2_N}(ŧ_3)$. L_oA_p *of* U_2 *is presented in the Table 5.12.*

Table 5.12: L_oA_p of U_2

	$\underline{\mathscr{L}}^{U_{2_Y}}(e_1)$	$\underline{\mathscr{L}}^{U_{2_N}}(e_1)$	$\underline{\mathscr{L}}^{U_{2_Y}}(e_2)$	$\underline{\mathscr{L}}^{U_{2_N}}(e_2)$
k_1	0.4	0.5	0.4	0.5
k_2	0.4	0.5	0.4	0.5
k_3	0.4	0.5	0.4	0.5
k_4	0.4	0.5	0.4	0.5

Clearly, $\underline{\mathscr{L}}^{U_2}(e_1)$ *and* $\underline{\mathscr{L}}^{U_2}(e_2)$ *are* $P_yF_zL_fI_d$*s of* M_1*, so* $(\underline{\mathscr{L}}^{U_2}, D)$ *is a* $P_yF_zS_fL_fI_d$ *of* M_1.

Now define $\mathscr{S}_1 : D \to P(M_1 \times M_2)$ *by:*

$$\mathscr{S}_1(e_1) = \begin{array}{c} \\ ŧ_1 \\ ŧ_2 \\ ŧ_3 \\ ŧ_4 \end{array}\begin{array}{c} \begin{array}{cccc} k_1 & k_2 & k_3 & k_4 \end{array} \\ \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \end{array}, \quad \mathscr{S}_1(e_2) = \begin{array}{c} \\ ŧ_1 \\ ŧ_2 \\ ŧ_3 \\ ŧ_4 \end{array}\begin{array}{c} \begin{array}{cccc} k_1 & k_2 & k_3 & k_4 \end{array} \\ \begin{pmatrix} 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \end{pmatrix} \end{array}$$

Then, $(\mathscr{S}_1, D)$ *is* $S_fC_{mp}R_l$ *from* M_1 *to* M_2.

$$\mathscr{S}_1(e_1)\mathfrak{k}_1 = \{\Bbbk_4\},$$
$$\mathscr{S}_1(e_1)\mathfrak{k}_2 = \{\Bbbk_4\},$$
$$\mathscr{S}_1(e_1)\mathfrak{k}_3 = \{\Bbbk_4\},$$
$$\mathscr{S}_1(e_1)\mathfrak{k}_4 = \{\Bbbk_4\}$$
$$\mathscr{S}_1(e_2)\mathfrak{k}_1 = \{\Bbbk_1, \Bbbk_4\}$$
$$\mathscr{S}_1(e_2)\mathfrak{k}_2 = \{\Bbbk_1, \Bbbk_4\}$$
$$\mathscr{S}_1(e_2)\mathfrak{k}_3 = \{\Bbbk_1, \Bbbk_4\}$$
$$\mathscr{S}_1(e_2)\mathfrak{k}_4 = \{\Bbbk_1, \Bbbk_4\}.$$

1) *Define a* P_yF_zS $U_1 : M_1 \to [0,1]$ *by* $U_1 = \{\langle \Bbbk_1, 0.1, 0.9\rangle, \langle \Bbbk_2, 0.3, 0.6\rangle, \langle \Bbbk_3, 0.5, 0.4\rangle, \langle \Bbbk_4, 0.7, 0.2\rangle\}$. *Then,* U_1 *is not a* $P_yF_zS_bS_mG$ *of* M_1 *because if we take* $a = \Bbbk_2, b = \Bbbk_3$ *then* $U_{1_Y}(\Bbbk_2\Bbbk_3) = 0.1 \ngeq 0.3 = U_Y(\Bbbk_2) \wedge U_{1_Y}(\Bbbk_3)$ *and* $U_{1_N}(\Bbbk_2\Bbbk_3) = 0.9 \nleq 0.6 = U_{1_N}(\Bbbk_2) \vee U_{1_N}(\Bbbk_3)$. L_oA_p *of* U_1 *is given in Table 5.13.*

Table 5.13: L_oA_p of U_1

	${}^{U_{1_Y}}\underline{\mathscr{S}_1}(e_1)$	${}^{U_{1_N}}\underline{\mathscr{S}_1}(e_1)$	${}^{U_{1_Y}}\underline{\mathscr{S}_1}(e_2)$	${}^{U_{1_N}}\underline{\mathscr{S}_1}(e_2)$
$\mathfrak{k}_1$	0.7	0.2	0.1	0.9
$\mathfrak{k}_2$	0.7	0.2	0.1	0.9
$\mathfrak{k}_3$	0.7	0.2	0.1	0.9
$\mathfrak{k}_4$	0.7	0.2	0.1	0.9

Clearly, ${}^{U_1}\underline{\mathscr{S}_1}(e_1)$ *and* ${}^{U_1}\underline{\mathscr{S}_1}(e_2)$ *are* $P_yF_zS_bS_mGs$ *of* M_2, *so* $({}^{U_1}\underline{\mathscr{S}_1}, D)$ *is a* $P_yF_zS_fS_bS_mG$ *of* M_2.

2) *Define a* P_yF_zS $U_1 : M_1 \to [0,1]$ *by* $U_1 = \{\langle \Bbbk_1, 0.1, 0.8\rangle, \langle \Bbbk_2, 0.3, 0.6\rangle, \langle \Bbbk_3, 0.5, 0.5\rangle, \langle \Bbbk_4, 0.7, 0.2\rangle\}$. *Then,* U_1 *is not a* $P_yF_zL_fI_d$

of M_1 because if we take $a = \Bbbk_2, b = \Bbbk_3$ then $U_{1_Y}(\Bbbk_2\Bbbk_3) = 0.1 \ngeq 0.5 = U_{1_Y}(\Bbbk_3)$ and $U_{1_N}(\Bbbk_2\Bbbk_3) = 0.8 \nleq 0.5 = U_{1_N}(\Bbbk_3)$. L_oA_p of U_1 is given in Table 5.14.

Table 5.14: L_oA_p of U_1

	${}^{U_{1_Y}}\underline{\mathscr{S}_1}(e_1)$	${}^{U_{1_N}}\underline{\mathscr{S}_1}(e_1)$	${}^{U_{1_Y}}\underline{\mathscr{S}_1}(e_2)$	${}^{U_{1_N}}\underline{\mathscr{S}_1}(e_2)$
$\mathfrak{k}_1$	0.7	0.2	0.1	0.8
$\mathfrak{k}_2$	0.7	0.2	0.1	0.8
$\mathfrak{k}_3$	0.7	0.2	0.1	0.8
$\mathfrak{k}_4$	0.7	0.2	0.1	0.8

Clearly, ${}^{U_1}\underline{\mathscr{S}_1}(e_1)$ and ${}^{U_1}\underline{\mathscr{L}}(e_2)$ are $P_yF_zL_fI_d$ of M_2, so $({}^{U}\underline{\mathscr{L}}, D)$ is a $P_yF_zS_fL_fI_d$ of M_2.

Theorem 5.1.11. *Suppose $(\mathscr{S}, D)$ is a S_fB_nR from a S_mG M_1 to a S_mG M_2; that is, $\mathscr{S} : D \to P(M_1 \times M_2)$. Then, for a $P_yF_zR_iI_d$ $U_1 = \langle U_{1_Y}, U_{1_N}\rangle$ and for a $P_yF_zL_fI_d$ $U_2 = \langle U_{2_Y}, U_{2_N}\rangle$ of M_2, $\overline{\mathscr{S}}^{U_1U_2} \subseteq \overline{\mathscr{S}}^{U_1} \cap \overline{\mathscr{S}}^{U_2}$.*

Proof.

Since U_1 is a $P_yF_zR_iI_d$, so $U_1U_2 \subseteq U_1$ and U_2 is $P_yF_zL_iI_d$ of M_2, so $U_1U_2 \subseteq U_2$.
Thus $U_1U_2 \subseteq U_1 \cap U_2$. It follows from Theorem 2.1.4, $\overline{\mathscr{S}}^{U_{1_Y}U_{2_Y}}(e) \subseteq \overline{\mathscr{S}}^{U_{1_Y}\cap U_{2_Y}}(e) \subseteq \overline{\mathscr{S}}^{U_{1_Y}}(e) \cap \overline{\mathscr{S}}^{U_{2_Y}}(e)$.
Hence, $\overline{\mathscr{S}}^{U_{1_Y}U_{2_Y}}(e) \subseteq \overline{\mathscr{S}}^{U_{1_Y}}(e) \cap \overline{\mathscr{S}}^{U_{2_Y}}(e)$.
Also, $\overline{\mathscr{S}}^{U_{1_N}}(e) \cap \overline{\mathscr{S}}^{U_{2_N}}(e) \subseteq \overline{\mathscr{S}}^{U_{1_N}\cap U_{2_N}}(e) \subseteq \overline{\mathscr{S}}^{U_{1_N}U_{2_N}}(e)$.
Hence, $\overline{\mathscr{S}}^{U_{1_N}U_{2_N}}(e) \supseteq \overline{\mathscr{S}}^{U_{1_N}}(e) \cap \overline{\mathscr{S}}^{U_{2_N}}(e)$ □

Theorem 5.1.12. *Suppose $(\mathscr{S}, D)$ is a S_fB_nR from a S_mG M_1 to a S_mG M_2; that is, $\mathscr{S} : D \to P(M_1 \times M_2)$. Then, for a $P_yF_zR_iI_d$ $U_1 = \langle U_{1_Y}, U_{1_N}\rangle$ and for a $P_yF_zL_fI_d$ $U_2 = \langle U_{2_Y}, U_{2_N}\rangle$ of M_1, ${}^{U_1U_2}\overline{\mathscr{S}} \subseteq {}^{U_1}\overline{\mathscr{S}} \cap {}^{U_2}\overline{\mathscr{S}}$.*

Proof.

It follows from Theorem 5.1.11. □

Theorem 5.1.13. *If* $(\mathscr{S}, D)$ *constitutes a* $S_fC_mR_l$ *from* S_mG M_1 *to* S_mG M_2, *and if* U_2 *represents a* $P_yF_zI_tI$ *in* M_2, *then* $(\overline{\mathscr{S}}^{U_2}, D)$ *forms a* $P_yF_zS_fI_tI$ *in* M_1.

Proof.

Suppose that U_2 is a $P_yF_zI_tI$ of M_2. Thus, U_2 is a $P_yF_zS_bS_mG$ of M_2, so according to the Theorem 5.1.4, $(\overline{\mathscr{S}}^{U_2}, D)$ is a $P_yF_zS_fS_bS_mG$ of M_1. Now for $a, b, c \in M_1$,

$$\begin{aligned}\overline{\mathscr{S}}^{U_{2_Y}}(e)(c) &= \vee_{n\in c\mathscr{S}(e)}U_{2_Y}(n) \leq \vee_{m\in a\mathscr{S}(e)} \vee_{n\in c\mathscr{S}(e)} \vee_{o\in b\mathscr{S}(e)}U_{2_Y}(mno)\\ &\leq \vee_{(mno)\in(acb)\mathscr{S}(e)}U_{2_Y}(mno) = m' \in acb\mathscr{S}(e)U_{2_Y}(m')\\ &= \overline{\mathscr{S}}^{U_{2_Y}}(e)(acb).\end{aligned}$$

Similarly, for $a, b, c \in M_1$,

$$\begin{aligned}\overline{\mathscr{S}}^{U_{2_N}}(e)(c) &= \wedge_{n\in c\mathscr{S}(e)}U_{2_N}(n) \geq \wedge_{m\in a\mathscr{S}(e)} \wedge_{n\in c\mathscr{S}(e)} \wedge_{o\in b\mathscr{S}(e)}U_{2_N}(mno)\\ &\geq \wedge_{(mno)\in(acb)\mathscr{S}(e)}U_{2_Y}(mno) = m' \in acb\mathscr{S}(e)U_{2_N}(m')\\ &= \overline{\mathscr{S}}^{U_{2_N}}(e)(acb).\end{aligned}$$

Hence, $\overline{\mathscr{S}}^{U_2}(e)$ is a $P_yF_zI_tI$ of M_1 for all $e \in D$, so $(\overline{\mathscr{S}}^{U_2}, D)$ is a $P_yF_zS_fI_tI$ of M_1. $\square$

Consider the following example to illustrate that the converse of the preceding result is not true, generally.

Example 5.1.14. *Let* $M_1 = \{\mathfrak{t}_1, \mathfrak{t}_2, \mathfrak{t}_3\}$ *and* $M_2 = \{\mathfrak{k}_1, \mathfrak{k}_2, \mathfrak{k}_3\}$ *denote two* S_mGs, *with their respective multiplication tables depicted in Tables 5.15 and 5.16.*

Table 5.15: Multiplication table for M_1

.	$\mathfrak{t}_1$	$\mathfrak{t}_2$	$\mathfrak{t}_3$
$\mathfrak{t}_1$	$\mathfrak{t}_1$	$\mathfrak{t}_2$	$\mathfrak{t}_3$
$\mathfrak{t}_2$	$\mathfrak{t}_1$	$\mathfrak{t}_2$	$\mathfrak{t}_3$
$\mathfrak{t}_3$	$\mathfrak{t}_1$	$\mathfrak{t}_2$	$\mathfrak{t}_3$

Table 5.16: Multiplication table for M_2

.	$\Bbbk_1$	$\Bbbk_2$	$\Bbbk_3$
$\Bbbk_1$	$\Bbbk_1$	$\Bbbk_1$	$\Bbbk_3$
$\Bbbk_2$	$\Bbbk_1$	$\Bbbk_2$	$\Bbbk_3$
$\Bbbk_3$	$\Bbbk_1$	$\Bbbk_3$	$\Bbbk_3$

Let $D = \{e_1, e_2\}$. Define a $S_f B_n R$ $\mathscr{S} : D \to P(M_1 \times M_2)$ by:

$$\mathscr{S}(e_1) = \begin{array}{c} \\ t_1 \\ t_2 \\ t_3 \end{array} \begin{array}{c} \begin{array}{ccc} \Bbbk_1 & \Bbbk_2 & \Bbbk_3 \end{array} \\ \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \end{array}, \quad \mathscr{S}(e_2) = \begin{array}{c} \\ t_1 \\ t_2 \\ t_3 \end{array} \begin{array}{c} \begin{array}{ccc} \Bbbk_1 & \Bbbk_2 & \Bbbk_3 \end{array} \\ \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \end{array}$$

In conclusion, $(\mathscr{S}, D)$ constitutes a $S_f C_m R_l$ mapping from M_1 to M_2.

$$t_1\mathscr{S}(e_1) = \{\Bbbk_1, \Bbbk_2, \Bbbk_3\},$$
$$t_2\mathscr{S}(e_1) = \{\Bbbk_1, \Bbbk_2\},$$
$$t_3\mathscr{S}(e_1) = \{\Bbbk_1, \Bbbk_3\},$$
$$t_1\mathscr{S}(e_2) = \{\Bbbk_1, \Bbbk_2, \Bbbk_3\},$$
$$t_2\mathscr{S}(e_2) = \{\Bbbk_1, \Bbbk_2\},$$
$$t_3\mathscr{S}(e_2) = \{\Bbbk_3\}.$$

Define a $P_y F_z S$ $U_2 : M_2 \to [0, 1]$ by
$U_2 = \{\langle \Bbbk_1, 0, 0.9\rangle, \langle \Bbbk_2, 0.1, 0.8\rangle, \langle \Bbbk_3, 0.1, 0.9\rangle\}$. Then, U_2 is not a $P_y F_z I_t I$ of M_2 because if we take $a = \Bbbk_2, c = \Bbbk_3, b = \Bbbk_1$, then $U_{2_Y}(\Bbbk_2\Bbbk_3\Bbbk_1) = 0 \ngeq 0.1 = U_{2_Y}(\Bbbk_3)$ and $U_{2_N}(\Bbbk_2\Bbbk_3\Bbbk_1) = 0.9 \nleq 0.8 = U_{2_N}(\Bbbk_3)$. $U_p A_p$ of U_2 is given in Table 5.17.

Table 5.17: U_pA_p of U_2

	$\overline{\mathscr{S}}^{U_{2Y}}(e_1)$	$\overline{\mathscr{S}}^{U_{2N}}(e_1)$	$\overline{\mathscr{S}}^{U_{2Y}}(e_2)$
$\mathfrak{t}_1$	0.1	0.8	0.1
$\mathfrak{t}_2$	0.1	0.8	0.1
$\mathfrak{t}_3$	0.1	0.9	0.1

Clearly, $\overline{\mathscr{S}}^{U_2}(e_1)$ *and* $\overline{\mathscr{S}}^{U_2}(e_2)$ *are* $P_yF_zI_tIs$ *of* M_1*, so* $(\overline{\mathscr{S}}^{U_2}, D)$ *is a* $P_yF_zS_fI_tI$ *of* M_1*.*

Theorem 5.1.15. *Let* $(\mathscr{S}, D)$ *represent a* $S_fC_mR_l$ *from a* S_mG M_1 *to a* S_mG M_2*. If* U_1 *is a* $P_yF_zI_tI$ *of* M_1*, then* $({}^{U_1}\overline{\mathscr{S}}, D)$ *forms a* $P_yF_zS_fI_tI$ *over* M_2*.*

Proof.
It follows from Theorem 5.1.13. □

Consider the following example to illustrate that the converse of the preceding result is not true, generally.

Example 5.1.16. *Consider the* S_mGs *of Example 5.1.14 and let* $D = \{e_1, e_2\}$*. Define a* S_fB_nR $\mathscr{S}_1 : D \to P(M_1 \times M_2)$ *by:*

$$\mathscr{S}_1(e_1) = \begin{array}{c} \\ t_1 \\ t_2 \\ t_3 \end{array}\begin{array}{c} \begin{array}{ccc} k_1 & k_2 & k_3 \end{array} \\ \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \end{array}, \quad \mathscr{S}_1(e_2) = \begin{array}{c} \\ t_1 \\ t_2 \\ t_3 \end{array}\begin{array}{c} \begin{array}{ccc} k_1 & k_2 & k_3 \end{array} \\ \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 0 \end{pmatrix} \end{array}$$

Then, $(\mathscr{S}_1, D)$ is a $S_f C_{mp} R_l$ from M_1 to M_2.

$$\mathscr{S}_1(e_1)\mathfrak{k}_1 = \{\mathfrak{t}_1, \mathfrak{t}_2, \mathfrak{t}_3\},$$
$$\mathscr{S}_1(e_1)\mathfrak{k}_2 = \{\mathfrak{t}_1, \mathfrak{t}_2\},$$
$$\mathscr{S}_1(e_1)\mathfrak{k}_3 = \{\mathfrak{t}_1, \mathfrak{t}_3\},$$
$$\mathscr{S}_1(e_2)\mathfrak{k}_1 = \{\mathfrak{t}_1, \mathfrak{t}_2\},$$
$$\mathscr{S}_1(e_2)\mathfrak{k}_2 = \{\mathfrak{t}_2\},$$
$$\mathscr{S}_1(e_2)\mathfrak{k}_3 = \{\mathfrak{t}_2, \mathfrak{t}_3\}.$$

Define a $P_y F_z S$ $U_1 : M_1 \to [0,1]$ by
$U_1 = \{\langle \mathfrak{t}_1, 0, 0.9\rangle, \langle \mathfrak{t}_2, 0.1, 0.8\rangle, \langle \mathfrak{t}_3, 0.1, 0.9\rangle\}$. *Then, U_1 is not a $P_y F_z I_t I$ of M_2 because if we take $a = \mathfrak{t}_2, c = \mathfrak{t}_3, b = \mathfrak{t}_1$, then $U_{1_Y}(\mathfrak{t}_2\mathfrak{t}_3\mathfrak{t}_1) = 0 \not\geq 0.1 = U_{1_Y}(\mathfrak{t}_3)$ and $U_{1_N}(\mathfrak{t}_2\mathfrak{t}_3\mathfrak{t}_1) = 0.9 \not\leq 0.8 = U_{1_N}(\mathfrak{t}_3)$. $U_p A_p$ of U_1 is presented in Table 5.18.*

Table 5.18: $U_p A_p$ of U_1

	${}^{U_{1_Y}}\overline{\mathscr{S}_1}(e_1)$	${}^{U_{1_N}}\overline{\mathscr{S}_1}(e_1)$	${}^{U_{1_Y}}\overline{\mathscr{S}_1}(e_2)$
$\mathfrak{k}_1$	0.1	0.8	0.1
$\mathfrak{k}_2$	0.1	0.8	0.1
$\mathfrak{k}_3$	0.1	0.9	0.1

Clearly, ${}^{U_1}\overline{\mathscr{S}_1}(e_1)$ and ${}^{U_1}\overline{\mathscr{S}_1}(e_2)$ are $P_y F_z I_t I s$ of M_2, so $({}^{U_1}\overline{\mathscr{S}_1}, D)$ is a $P_y F_z S_f I_t I$ of M_2.

Theorem 5.1.17. *Suppose $(\mathscr{S}, D)$ constitutes a $S_f C_{mp} R_l$ from a $S_m G$ M_1 to a $S_m G$ M_2. If U_2 is a $P_y F_z I_t I$ of M_2, then $(\underline{\mathscr{S}}^{U_2}, D)$ is a $P_y F_z S_f I_t I$ of M_1.*

Proof.
Suppose that U_2 is a $P_y F_z I_t I$ of M_2, Thus, U_2 is a $P_y F_z S_b S_m G$ of M_2, so according to the

Theorem 5.1.5, $(\underline{\mathscr{L}}^{U_2}, D)$ is a $P_yF_zS_fS_bS_mG$ of M_1. Now for $a, c, b \in M_1$,

$$\begin{aligned}\underline{\mathscr{L}}^{U_{2_Y}}(acb) &= \wedge_{m'\in(acb)\mathscr{S}(e)}U_{2_Y}\big(m'\big) = \wedge_{m'\in a\mathscr{S}(e).c\mathscr{S}(e).b\mathscr{S}(e)}U_{2_Y}\big(m'\big)\\ &= \wedge_{m\in a\mathscr{S}(e).n\in c\mathscr{S}(e).c\in b\mathscr{S}(e)}U_{2_Y}\big(mno\big) \geq \wedge_{n\in c\mathscr{S}(e)}U_{2_Y}\big(n\big)\\ &= \underline{\mathscr{L}}^{U_{2_Y}}(e)(m).\end{aligned}$$

Similarly, for $a, b, c \in M_1$,

$$\begin{aligned}\underline{\mathscr{L}}^{U_{2_N}}(acb) &= \vee_{m'\in(acb)\mathscr{S}(e)}U_{2_N}\big(m'\big) = \vee_{m'\in a\mathscr{S}(e).b\mathscr{S}(e).c\mathscr{S}(e)}U_{2_N}\big(m'\big)\\ &= \vee_{m\in a\mathscr{S}(e).n\in c\mathscr{S}(e).c\in b\mathscr{S}(e)}U_{2_N}\big(mno\big) \leq \vee_{n\in c\mathscr{S}(e)}U_{2_N}\big(n\big)\\ &= \underline{\mathscr{L}}^{U_{2_N}}(e)(c).\end{aligned}$$

Therefore, $\underline{\mathscr{L}}^{U_2}(e)$ is a $P_yF_zI_tI$ of M_1 for all $e \in D$, so $(\underline{\mathscr{L}}^{U_2}, D)$ is a $P_yF_zS_fI_tI$ of M_1. □

In the following example, we demonstrate that the converse of the above theorem does not hold true.

Example 5.1.18. *Consider the S_mGs of Example 5.1.10 and let $D = \{e_1, e_2\}$. Define a S_fB_nR $\mathscr{S} : D \to P(M_1 \times M_2)$ by:*

$$\mathscr{S}(e_1) = \begin{array}{c} \\ \Bbbk_1 \\ \Bbbk_2 \\ \Bbbk_3 \\ \Bbbk_4 \end{array}\!\!\begin{array}{c} \begin{array}{cccc} \mathfrak{f}_1 & \mathfrak{f}_2 & \mathfrak{f}_3 & \mathfrak{f}_4 \end{array} \\ \begin{pmatrix} 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \end{pmatrix} \end{array}, \quad \mathscr{S}(e_2) = \begin{array}{c} \\ \Bbbk_1 \\ \Bbbk_2 \\ \Bbbk_3 \\ \Bbbk_4 \end{array}\!\!\begin{array}{c} \begin{array}{cccc} \mathfrak{f}_1 & \mathfrak{f}_2 & \mathfrak{f}_3 & \mathfrak{f}_4 \end{array} \\ \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \end{array}$$

Then, $(\mathscr{S}, D)$ *constitutes a* $S_fC_{mp}R_l$ *from* M_1 *to* M_2*.*

$$\begin{aligned}
\mathfrak{k}_1\mathscr{S}(e_1) &= \{\mathfrak{t}_2, \mathfrak{t}_3\},\\
\mathfrak{k}_2\mathscr{S}(e_1) &= \{\mathfrak{t}_2, \mathfrak{t}_3\},\\
\mathfrak{k}_3\mathscr{S}(e_1) &= \{\mathfrak{t}_2, \mathfrak{t}_3\},\\
\mathfrak{k}_4\mathscr{S}(e_1) &= \{\mathfrak{t}_2, \mathfrak{t}_3\},\\
\mathfrak{k}_1\mathscr{S}(e_2) &= \{\mathfrak{t}_2\},\\
\mathfrak{k}_2\mathscr{S}(e_2) &= \{\mathfrak{t}_2\},\\
\mathfrak{k}_3\mathscr{S}(e_2) &= \{\mathfrak{t}_2\},\\
\mathfrak{k}_4\mathscr{S}(e_2) &= \{\mathfrak{t}_2\}.
\end{aligned}$$

Define a P_yF_zS $U_2 : M_2 \to [0,1]$ *by*
$U_2 = \{\langle \mathfrak{t}_1, 0.4, 0.5\rangle, \langle \mathfrak{t}_2, 0.6, 0.3\rangle, \langle \mathfrak{t}_3, 0.8, 0.1\rangle, \langle \mathfrak{t}_4, 0.1, 0\rangle\}$*. Then,* U_2 *is not a* $P_yF_zI_tI$ *of* M_2 *because if we take* $a = \mathfrak{t}_2, c = \mathfrak{t}_3, b = \mathfrak{t}_4$*, then* $U_{2_Y}(\mathfrak{t}_2\mathfrak{t}_3\mathfrak{t}_4) = 0.6 \ngeq 0.8 = U_{2_Y}(\mathfrak{t}_3)$ *and* $U_{2_N}(\mathfrak{t}_2\mathfrak{t}_3\mathfrak{t}_4) = 0.3 \nleq 0.1 = U_{2_N}(\mathfrak{t}_3)$*.* L_oA_p *of* U_2 *is presented in Table 5.19.*

Table 5.19: L_oA_p of U_2

	$\underline{\mathscr{S}}^{U_{2_Y}}(e_1)$	$\underline{\mathscr{S}}^{U_{2_N}}(e_1)$	$\underline{\mathscr{S}}^{U_{2_Y}}(e_2)$
$\mathfrak{k}_1$	0.6	0.3	0.6
$\mathfrak{k}_2$	0.6	0.3	0.6
$\mathfrak{k}_3$	0.6	0.3	0.6
$\mathfrak{k}_4$	0.6	0.3	0.6

It is clear from Table 5.19 that $\underline{\mathscr{S}}^{U_2}(e_1)$ *and* $\underline{\mathscr{S}}^{U_2}(e_2)$ *are* $P_yF_zI_tIs$ *of* M_1*, so* $(\underline{\mathscr{S}}^{U_2}, D)$ *is a* $P_yF_zS_fI_tI$ *of* M_1*.*

Theorem 5.1.19. *Suppose* $(\mathscr{S}, D)$ *constitutes a* $S_fC_{mp}R_l$ *from a* S_mG M_1 *to a* S_mG M_2*. If* U_1 *is a* $P_yF_zI_tI$ *of* M_1*, then* $({}^{U_1}\underline{\mathscr{S}}, D)$ *is a* $P_yF_zS_fI_tI$ *of* M_2*.*

Proof.

The proof can be derived using the same approach as in the Theorem 5.1.17. □

Example 5.1.20. *Consider the S_mGs of Example 5.1.10. Then define $\mathscr{S} : D \to P(M_1 \times M_2)$ by:*

$$\mathscr{S}(e_1) = \begin{array}{c} \\ k_1 \\ k_2 \\ k_3 \\ k_4 \end{array}\!\!\begin{array}{c} \begin{array}{cccc} t_1 & t_2 & t_3 & t_4 \end{array} \\ \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \end{pmatrix} \end{array}, \quad \mathscr{S}(e_2) = \begin{array}{c} \\ k_1 \\ k_2 \\ k_3 \\ k_4 \end{array}\!\!\begin{array}{c} \begin{array}{cccc} t_1 & t_2 & t_3 & t_4 \end{array} \\ \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \end{pmatrix} \end{array}$$

Then, $(\mathscr{S}, D)$ constitutes a $S_fC_{mp}R_l$ from M_1 to M_2.

$$\begin{aligned}
\mathscr{S}(e_1)t_1 &= \{k_4\}, \\
\mathscr{S}(e_1)t_2 &= \{k_4\}, \\
\mathscr{S}(e_1)t_3 &= \{k_4\}, \\
\mathscr{S}(e_1)t_4 &= \{k_4\}, \\
\mathscr{S}(e_2)t_1 &= \{k_1, k_4\}, \\
\mathscr{S}(e_2)t_2 &= \{k_1, k_4\}, \\
\mathscr{S}(e_2)t_3 &= \{k_1, k_4\}, \\
\mathscr{S}(e_2)t_4 &= \{k_1, k_4\}.
\end{aligned}$$

Define a P_yF_zS $U_1 : M_1 \to [0,1]$ by
$U_1 = \{\langle k_1, 0, 1\rangle, \langle k_2, 0.3, 0.7\rangle, \langle k_3, 0.5, 0.4\rangle, \langle k_4, 0.7, 0.2\rangle\}$. Then, U_1 is not a $P_yF_zI_tI$ of M_1 because if we take $a = k_3, c = k_2, b = k_1$, then $U_{1_Y}(k_3k_2k_1) = 0 \ngeq 0.3 = U_{1_Y}(k_2)$ and $U_{1_N}(k_3k_2k_1) = 1 \nleq 0.7 = U_{1_N}(k_2)$. L_oA_p of U_1 is given in Table 5.20.

Table 5.20: L_oA_p of U_1

	${}^{U_{1_Y}}\underline{\mathscr{L}}(e_1)$	${}^{U_{1_N}}\underline{\mathscr{L}}(e_1)$	${}^{U_{1_Y}}\underline{\mathscr{L}}(e_2)$	${}^{U_{1_N}}\underline{\mathscr{L}}(e_2)$
$\mathfrak{t}_1$	0.7	0.2	0	1
$\mathfrak{t}_2$	0.7	0.2	0	1
$\mathfrak{t}_3$	0.7	0.2	0	1
$\mathfrak{t}_4$	0.7	0.2	0	1

Clearly, ${}^{U_1}\underline{\mathscr{L}}(e_1)$ *and* ${}^{U_1}\underline{\mathscr{L}}(e_2)$ *are* $P_yF_zI_tIs$ *of* M_2*, so* $({}^{U_1}\underline{\mathscr{L}}, D)$ *is a* $P_yF_zS_fI_tI$ *of* M_2*.*

Now, we present properties for $P_yF_zB_iI_d$s of a S_mG.

Theorem 5.1.21. *Suppose* $(\mathscr{S}, D)$ *constitutes a* $S_fC_mR_l$ *from a* S_mG M_1 *to a* S_mG M_2*. If* U_2 *is a* $P_yF_zB_iI_d$ *of* M_2 *then* $(\overline{\mathscr{S}}^{U_2}, D)$ *is a* $P_yF_zS_fB_iI_d$ *of* M_1*.*

Proof.

Suppose that U_2 is a $P_yF_zB_iI_d$ of M_2. Thus, U_2 is an $P_yF_zS_bS_mG$ of M_2, so according to the Theorem 5.1.4, $(\overline{\mathscr{S}}^{U_2}, D)$ is a $P_yF_zS_fS_bS_mG$ of M_1. Now for $a, b, c \in M_1$,

$$\begin{aligned}
\overline{\mathscr{S}}^{U_{2_Y}}(e)(a) \wedge \overline{\mathscr{S}}^{U_{2_Y}}(e)(b) &= \Big(\vee_{m\in a\mathscr{S}(e)} U_{2_Y}(m)\Big) \wedge \Big(\vee_{o\in b\mathscr{S}(e)} U_{2_Y}(o)\Big) \\
&= \vee_{m\in a\mathscr{S}(e)} \vee_{o\in b\mathscr{S}(e)} \Big(U_{2_Y}(m) \wedge U_{2_Y}(o)\Big) \\
&\leq \vee_{m\in a\mathscr{S}(e)} \vee_{n\in c\mathscr{S}(e)} \vee_{o\in b\mathscr{S}(e)} \Big(U_{2_Y}(mno)\Big) \\
&\leq \vee_{mno\in(acb)\mathscr{S}(e)} \Big(U_{1_Y}(mno)\Big) \\
&= \vee_{m'\in(acb)\mathscr{S}(e)} \Big(U_{2_Y}(m')\Big) \\
&= \overline{\mathscr{S}}^{U_{2_Y}}(e)(acb).
\end{aligned}$$

Similarly for $a, b, c \in M_1$,

$$\begin{aligned}
\overline{\mathscr{S}}^{U_{2_N}}(e)(a) \vee \overline{\mathscr{S}}^{U_{2_N}}(e)(b) &= \Big(\wedge_{m \in a\mathscr{S}(e)} U_{2_N}(m) \Big) \vee \Big(\wedge_{o \in b\mathscr{S}(e)} U_{2_N}(o) \Big) \\
&= \wedge_{m \in a\mathscr{S}(e)} \wedge_{o \in b\mathscr{S}(e)} \Big(U_{2_N}(m) \vee U_{2_N}(o) \Big) \\
&\geq \wedge_{m \in \mathscr{S}(e)} \wedge_{n \in c\mathscr{S}(e)} \wedge_{o \in a\mathscr{S}(e)} \Big(U_{2_N}(mno) \Big) \\
&\geq \wedge_{mno \in (acb)\mathscr{S}(e)} \Big(U_{2_N}(mno) \Big) \\
&= \wedge_{m' \in (acb)\mathscr{S}(e)} \Big(U_{2_N}(m') \Big) \\
&= \overline{\mathscr{S}}^{U_{2_N}}(e)(acb).
\end{aligned}$$

Hence, $\overline{\mathscr{S}}^{U_2}(e)$ is a $P_yF_zB_iI_d$ of M_1 for all $e \in D$, $(\overline{\mathscr{S}}^{U_2}, D)$ is a $P_yF_zS_fB_iI_d$ of M_1. □

Theorem 5.1.22. *Suppose $(\mathscr{S}, D)$ is a $S_fC_mR_l$ from a S_mG M_1 to a S_mG M_2. If U_1 is a $P_yF_zB_iI_d$ of M_1 then $({}^{U_1}\overline{\mathscr{S}}, D)$ is a $P_yF_zS_fB_iI_d$ of M_2.*

Proof.
The proof can be derived using the same approach as in the Theorem 5.1.21. □

Example 5.1.23. *Consider the S_mGs and soft relations from the Example 5.1.6.*
Define a P_yF_zS $U_2 : M_2 \to [0,1]$ by
$U_2 = \{\langle t_1, 1, 0\rangle, \langle t_2, 0.5, 0.4\rangle, \langle t_3, 0.7, 0.3\rangle, \langle t_4, 0.9, 0.1\rangle, \langle t_5, 0.2, 0.7\rangle\}$. Then, U_2 is not a $P_yF_zB_iI_d$ of M_2 because if we take $a = t_1, c = t_3, b = t_2$, then $U_{2_Y}(t_1t_3t_2) = 0.2 \ngeq 0.5 = U_{2_Y}(t_1) \wedge U_{2_Y}(t_2)$ and $U_{2_N}(t_1t_3t_2) = 0.7 \nleq 0.4 = U_{2_N}(t_1) \vee U_{2_N}(t_2)$. U_pA_p of U_2 is presented in the Table 5.21.

Table 5.21: U_pA_p of U_2

	$\overline{\mathscr{S}}^{U_{2_Y}}(e_1)$	$\overline{\mathscr{S}}^{U_{2_N}}(e_1)$	$\overline{\mathscr{S}}^{U_{2_Y}}(e_2)$	$\overline{\mathscr{S}}^{U_{2_N}}(e_2)$
$\Bbbk_1$	1	0	1	0
$\Bbbk_2$	1	0	1	0
$\Bbbk_3$	0.7	0.3	0.7	0.3
$\Bbbk_4$	1	0	1	0
$\Bbbk_5$	0.2	0.7	0.2	0.7

Clearly, $\overline{\mathscr{S}}^{U_2}(e_1)$ *and* $\overline{\mathscr{S}}^{U_2}(e_2)$ *are* $P_yF_zB_iI_d$*s of* M_1*, so* $(\overline{\mathscr{S}}^{U_2}, D)$ *is a* $P_yF_zS_fB_iI_d$ *of* M_1*. Define a* P_yF_zS $U_1 : M_1 \rightarrow [0,1]$ *by*
$U_1 = \{\langle \Bbbk_1, 1, 0\rangle, \langle \Bbbk_2, 0.5, 0.4\rangle, \langle \Bbbk_3, 0.9, 0.1\rangle, \langle \Bbbk_4, 0.7, 0.3\rangle, \langle \Bbbk_5, 0.2, 0.7\rangle\}$*. Then,* U_1 *is not a* $P_yF_zB_iI_d$ *of* M_1 *because if we take* $a = \Bbbk_1, c = \Bbbk_2, b = \Bbbk_3$ *then* $U_{1_Y}(\Bbbk_1\Bbbk_2\Bbbk_3) = 0.7 \ngeq 0.9 = U_{1_Y}(\Bbbk_1) \wedge U_{1_Y}(\Bbbk_3)$ *and* $U_{1_N}(\Bbbk_1\Bbbk_2\Bbbk_3) = 0.3 \nleq 0.1 = U_{1_N}(\Bbbk_1) \vee U_{1_N}(\Bbbk_3)$*.* U_pA_p *of* U_1 *is given in Table 5.22.*

Table 5.22: U_pA_p of U_1

	${}^{U_{1_Y}}\overline{\mathscr{S}}(e_1)$	${}^{U_{1_N}}\overline{\mathscr{S}}(e_1)$	${}^{U_{1_Y}}\overline{\mathscr{S}}(e_2)$
$\mathfrak{t}_1$	1	0	1
$\mathfrak{t}_2$	0.5	0.4	0.5
$\mathfrak{t}_3$	0.9	0.1	0.9
$\mathfrak{t}_4$	0.7	0.7	0.7
$\mathfrak{t}_5$	0.9	0.1	0.9

Clearly, ${}^{U}\overline{\mathscr{S}}(e_1)$ *and* ${}^{U}\overline{\mathscr{S}}(e_2)$ *are* $P_yF_zB_iI_d$*s of* M_2*, so* $({}^{U}\overline{\mathscr{S}}, D)$ *is a* $P_yF_zS_fB_iI_d$ *of* M_2*.*

Theorem 5.1.24. *Let* $(\mathscr{S}, D)$ *constitutes a* $S_fC_{mp}R_l$ *from a* S_mG M_1 *to a* S_mG M_2*. If* U_2 *is a* $P_yF_zB_iI_d$ *of* M_2*, then* $(\underline{\mathscr{S}}^{U_2}, D)$ *is a* $P_yF_zS_fB_iI_d$ *of* M_1*.*

Proof.
Suppose that U_2 is a $P_yF_zB_iI_d$ of M_2. Thus, U_2 is a $P_yF_zS_bS_mG$ of M_2, so according to the

Theorem 5.1.5, $(\underline{\mathscr{L}}^{U_2}, D)$ is a $P_yF_zS_fS_bS_mG$ of M_1. Now for $a, b, c \in M_1$,

$$\begin{aligned}
\underline{\mathscr{L}}^{U_{2_Y}}(acb) &= \wedge_{m' \in (acb)\mathscr{S}(e)} U_{2_Y}\big(m'\big) = \wedge_{m' \in a\mathscr{S}(e).c\mathscr{S}(e).b\mathscr{S}(e)} U_{2_Y}\big(m'\big) \\
&= \wedge_{m \in a\mathscr{S}(e).o \in c\mathscr{S}(e).n \in b\mathscr{S}(e)} U_{2_Y}\big(mon\big) \\
&\geq \Big(\wedge_{m \in a\mathscr{S}(e)} U_{2_Y}(m)\Big) \wedge \Big(\wedge_{n \in b\mathscr{S}(e)} U_{1_Y}(n)\Big) \\
&= \Big(\underline{\mathscr{L}}^{U_{2_Y}}(e)(a)\Big) \wedge \Big(\underline{\mathscr{L}}^{U_{2_Y}}(e)(b)\Big).
\end{aligned}$$

Similarly, for $a, b, c \in M_1$,

$$\begin{aligned}
\underline{\mathscr{L}}^{U_{2_N}}(acb) &= \vee_{m' \in (acb)\mathscr{S}(e)} U_{2_N}\big(m'\big) = \vee_{m' \in a\mathscr{S}(e).c\mathscr{S}(e).b\mathscr{S}(e)} U_{2_N}\big(m'\big) \\
&= \vee_{m \in a\mathscr{S}(e).o \in c\mathscr{S}(e).n \in a\mathscr{S}(e)} U_{2_N}\big(mon\big) \\
&\leq \Big(\vee_{m \in a\mathscr{S}(e)} U_{2_N}(m)\Big) \vee \Big(\vee_{n \in b\mathscr{S}(e)} U_{2_N}(n)\Big) \\
&= \Big(\underline{\mathscr{L}}^{U_{2_N}}(e)(a)\Big) \vee \Big(\underline{\mathscr{L}}^{U_{2_N}}(e)(b)\Big).
\end{aligned}$$

Hence, $\underline{\mathscr{L}}^{U_2}(e)$ is a $P_yF_zB_iI_d$ of M_1 for all $e \in D$, so $(\underline{\mathscr{L}}^{U_2}, D)$ is a $P_yF_zS_fB_iI_d$ of M_1. □

Example 5.1.25. *Consider the S_mGs of Example 5.1.10. Define a S_fB_nR $\mathscr{S} : D \to P(M_1 \times M_2)$ by:*

$$\mathscr{S}(e_1) = \begin{array}{c} \\ k_1 \\ k_2 \\ k_3 \\ k_4 \end{array}\!\!\begin{array}{c} \begin{array}{cccc} t_1 & t_2 & t_3 & t_4 \end{array} \\ \begin{pmatrix} 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \end{pmatrix} \end{array}, \quad \mathscr{S}(e_2) = \begin{array}{c} \\ k_1 \\ k_2 \\ k_3 \\ k_4 \end{array}\!\!\begin{array}{c} \begin{array}{cccc} t_1 & t_2 & t_3 & t_4 \end{array} \\ \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \end{array}$$

Then, $(\mathscr{S}, D)$ *forms a* $S_f C_{mp} R_l$ *from* M_1 *to* M_2.

$$\Bbbk_1\mathscr{S}(e_1) = \{t_2, t_3\},$$

$$\Bbbk_2\mathscr{S}(e_1) = \{t_2, t_3\},$$

$$\Bbbk_3\mathscr{S}(e_1) = \{t_2, t_3\},$$

$$\Bbbk_4\mathscr{S}(e_1) = \{t_2, t_3\},$$

$$\Bbbk_1\mathscr{S}(e_2) = \{t_2\},$$

$$\Bbbk_2\mathscr{S}(e_2) = \{t_2\},$$

$$\Bbbk_3\mathscr{S}(e_2) = \{t_2\},$$

$$\Bbbk_4\mathscr{S}(e_2) = \{t_2\}.$$

Define a $P_y F_z S$ $U_2 : M_2 \to [0,1]$ *by* $U_2 = \{\langle t_1, 0.7, 0.2\rangle, \langle t_2, 0.2, 0.7\rangle, \langle t_3, 0.1, 0.9\rangle, \langle t_4, 0.4, 0.5\rangle\}$. *Then,* U_2 *is not a* $P_y F_z B_i I_d$ *of* M_2 *because if we take* $a = t_1, c = t_4, b = t_1$, *then* $U_{2_Y}(t_1 t_4 t_1) = 0.4 \ngeq 0.7 = U_{2_Y}(t_1) \wedge U_{2_Y}(t_1)$ *and* $U_{2_N}(t_1 t_4 t_1) = 0.4 \nleq 0.2 = U_{2_N}(t_1) \vee U_{2_N}(t_1)$. $L_o A_p$ *of* U_2 *is presented in the Table 5.23.*

Table 5.23: $L_o A_p$ of U_2

	$\underline{\mathscr{S}}^{U_{2_Y}}(e_1)$	$\underline{\mathscr{S}}^{U_{2_N}}(e_1)$	$\underline{\mathscr{S}}^{U_{2_Y}}(e_2)$	$\underline{\mathscr{S}}^{U_{2_N}}(e_2)$
$\Bbbk_1$	0.1	0.9	0.2	0.7
$\Bbbk_2$	0.1	0.9	0.2	0.7
$\Bbbk_3$	0.1	0.9	0.2	0.7
$\Bbbk_4$	0.1	0.9	0.2	0.7

Clearly, $\underline{\mathscr{S}}^{U_2}(e_1)$ *and* $\underline{\mathscr{S}}^{U_2}(e_2)$ *are* $P_y F_z B_i I_d s$ *of* M_1, *so* $(\underline{\mathscr{S}}^{U_2}, D)$ *is a* $P_y F_z S_f B_i I_d$ *of* M_1.

Theorem 5.1.26. *Let* $(\mathscr{S}, D)$ *constitutes a* $S_f C_{mp} R_l$ *from a* $S_m G$ M_1 *to a* $S_m G$ M_2. *If* U_1 *is a* $P_y F_z B_i I_d$ *of* M_1 *then* $({}^{U_2}\overline{\mathscr{S}}, D)$ *is a* $P_y F_z S_f B_i I_d$ *of* M_2.

Proof.

It follows from Theorem 5.1.24. □

Example 5.1.27. *Consider the S_mGs from the Example 5.1.10. Define a S_fB_nR $\mathscr{S}: D \to P(M_1 \times M_2)$ by:*

$$\mathscr{S}(e_1) = \begin{array}{c} \\ \&_1 \\ \&_2 \\ \&_3 \\ \&_4 \end{array}\!\!\begin{array}{c} \begin{array}{cccc} f_1 & f_2 & f_3 & f_4 \end{array} \\ \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \end{pmatrix} \end{array}, \quad \mathscr{S}(e_2) = \begin{array}{c} \\ \&_1 \\ \&_2 \\ \&_3 \\ \&_4 \end{array}\!\!\begin{array}{c} \begin{array}{cccc} f_1 & f_2 & f_3 & f_4 \end{array} \\ \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \end{pmatrix} \end{array}$$

Then, $(\mathscr{S}, D)$ forms a $S_fC_{mp}R_l$ from M_1 to M_2.

$$\mathscr{S}(e_1)f_1 = \{\&_4\},$$
$$\mathscr{S}(e_1)f_2 = \{\&_4\},$$
$$\mathscr{S}(e_1)f_3 = \{\&_4\},$$
$$\mathscr{S}(e_1)f_4 = \{\&_4\},$$
$$\mathscr{S}(e_2)f_1 = \{\&_1, \&_4\},$$
$$\mathscr{S}(e_2)f_2 = \{\&_1, \&_4\},$$
$$\mathscr{S}(e_2)f_3 = \{\&_1, \&_4\},$$
$$\mathscr{S}(e_2)f_4 = \{\&_1, \&_4\}.$$

Define a P_yF_zS $U_1 : M_1 \to [0,1]$ by
$U_1 = \{\langle \&_1, 0.1, 0.9\rangle, \langle \&_2, 0.8, 0.1\rangle, \langle \&_3, 0.6, 0.3\rangle, \langle \&_4, 0.7, 0.3\rangle\}$. Then, U_1 is not a $P_yF_zB_iI_d$ of M_1 because if we take $a = \&_2, c = \&_1, b = \&_3$, then $U_{1_Y}(\&_2\&_1\&_3) = 0.1 \ngeq 0.6 = U_{1_Y}(\&_2) \wedge U_{1_Y}(\&_3)$ and $U_{1_N}(\&_2\&_1\&_3) = 0.9 \nleq 0.3 = U_{1_N}(\&_2) \wedge U_{1_N}(\&_3)$. L_oA_p of U_1 is given in Table 5.24.

Table 5.24: L_oA_p of U_1

	${}^{U_{1_Y}}\underline{\mathscr{L}}(e_1)$	${}^{U_{1_N}}\underline{\mathscr{L}}(e_1)$	${}^{U_{1_Y}}\underline{\mathscr{L}}(e_2)$	${}^{U_{1_N}}\underline{\mathscr{L}}(e_2)$
$\mathfrak{k}_1$	0.7	0.3	0.1	0.9
$\mathfrak{k}_2$	0.7	0.3	0.1	0.9
$\mathfrak{k}_3$	0.7	0.3	0.1	0.9
$\mathfrak{k}_4$	0.7	0.3	0.1	0.9

Clearly, ${}^{U_1}\underline{\mathscr{L}}(e_1)$ *and* ${}^{U_1}\underline{\mathscr{L}}(e_2)$ *are* $P_yF_zB_iI_ds$ *of* M_2*, so* $({}^{U_1}\underline{\mathscr{L}}, D)$ *is a* $P_yF_zS_fB_iI_d$ *of* M_2*.*

This chapter developed a framework for approximating Pythagorean Fuzzy Ideals over dual universes using Soft Binary Relations. The proposed methods and algorithms provide valuable tools for analyzing algebraic structures and solving complex decision-making problems. These contributions extend the theoretical understanding of Fuzzy Ideals and demonstrate their practical relevance in diverse applications.

Chapter 6

Approximation of q-Rung Orthopair Fuzzy Ideals over dual Universes based on Soft Binary Relation

The transition from Pythagorean Fuzzy Ideals to q-Rung Orthopair Fuzzy Ideals represents a significant generalization, enabling the handling of higher levels of uncertainty. This additional flexibility allows q-Rung Orthopair Fuzzy Ideals to model situations with greater uncertainty and hesitation more effectively. By accommodating a broader range of values for membership and non-membership degrees, this framework extends the applicability of Fuzzy Ideals to more complex and diverse decision-making scenarios.

In this chapter, we discuss the rough approximations of ${}^qROF_zS_bS_mG$s, ${}^qROF_z$ left (right) Ideals, ${}^qROF_zI_tI_d$s, and ${}^qROF_zB_iI_d$s within a S_mG concerning both A_fSs and F_rS, utilizing a $S_fC_mR_l$. We demonstrate that the U_pA_p of a ${}^qROF_zS_bS_mG$, ${}^qROF_z$ left (right) Ideal, ${}^qROF_zI_tI_d$, and ${}^qROF_zB_iI_d$ within a S_mG constitutes a ${}^qROF_zS_fS_bS_mG$, ${}^qROF_z$ soft left (right) Ideal, ${}^qROF_zS_fI_tI_d$, and ${}^qROF_zS_fB_iI_d$, and provide examples illustrating that the converse is not necessarily true. Furthermore, we establish similar results for L_oA_p.

6.1 Approximations of q-Rung Orthopair Fuzzy Ideals in S_mGs by Soft Binary Relation

Definition 6.1.1. *A $^qROF_zS$ $U = \{\langle m, U_Y(m), U_N(m)\rangle : m \in M\}$ in M is called a $^qROF_z$ S_bS_mG of M if it satisfies the following:*

1) $U_Y(mn) \geq U_Y(m) \wedge U_Y(n)$

2) $U_N(mn) \leq U_N(m) \vee U_N(n)$

for all $m, n \in M$.

- *A $^qROF_zS$ U in a S_mG M is called a q-Rung Orthopair Fuzzy Left Ideal ($^qROF_zL_fI_d$) of M if it satisfies $U_Y(mn) \geq U_Y(n)$ and $U_N(mn) \leq U_Y(n)$ for all $m, n \in M$.*
- *A $^qROF_zS$ U in a S_mG M is called a q-Rung Orthopair Fuzzy Right Ideal ($^qROF_zL_fI_d$) of M if it satisfies $U_Y(mn) \geq U_Y(m)$ and $U_N(mn) \leq U_Y(m)$ for all $m, n \in M$.*
- *If U is both a $^qROF_zL_fI_d$ and a $^qROF_zR_iI_d$ then U is called a q-Rung Orthopair Fuzzy Ideal ($^qROF_zI_ds$).*
- *A $^qROF_zS_bS_mG$ U in M is called a q-Rung Orthopair Fuzzy Interior Ideal ($^qROF_zI_tI_d$ of M if it satisfies $U_Y(nam) \geq U_Y(a)$ and $U_N(nam) \leq U_N(a)$ all $m, n, a \in M$.*
- *A $^qROF_zS_bS_mG$ U in M is called a q-Rung Orthopair Fuzzy B_iI_d ($^qROF_zB_iI_d$) of M if it satisfies the following:*

1) $U_Y(nam) \geq U_Y(n) \wedge U_Y(m)$

2) $U_N(nam) \leq U_N(n) \vee U_N(m)$

for all $a, m, n \in M$.

Definition 6.1.2. *A $^qROF_zS_fS$ $(\mathscr{S}, D)$ over a S_mG M is called a q-Rung Orthopair Fuzzy Soft S_bS_mG ($^qROF_zS_fS_bS_mG$) of M if it satisfies: $\mathscr{S}_Y(e)(mn) \geq \mathscr{S}_Y(e)(m) \wedge \mathscr{S}_Y(e)(n)$ and $\mathscr{S}_N(e)(mn) \leq \mathscr{S}_N(e)(m) \vee \mathscr{S}_N(e)(n)$, for all $m, n \in M$ and $e \in D$.*

Example 6.1.3. *Consider a S_mG $M = \{k_1, k_2, k_3\}$ with the binary operation defined in the Table 6.1:*

$\cdot$	$ƙ_1$	$ƙ_2$	$ƙ_3$
$ƙ_1$	$ƙ_1$	$ƙ_2$	$ƙ_3$
$ƙ_2$	$ƙ_2$	$ƙ_2$	$ƙ_3$
$ƙ_3$	$ƙ_3$	$ƙ_3$	$ƙ_3$

Table 6.1: Multiplication table

Let $D = \{e_1\}$ be a set with one parameter. Define a ${}^{q}ROF_zS_fS$ as follows, for $q = 3$:

$$\mathscr{S}(e_1)(ƙ_i) = \{\langle ƙ_1, 0.3, 0.7\rangle, \langle ƙ_2, 0.5, 0.5\rangle, \langle ƙ_3, 0.7, 0.3\rangle\}$$

for $ƙ_i \in M$. Now, let's check if $(\mathscr{S}, D)$ is a ${}^{q}ROF_zS_fS_bS_mG$ of M:

$$\begin{aligned}
F_Y(e_1)(ƙ_1ƙ_1) = F_Y(e_1)(ƙ_1) = 0.7 \quad &\geq F_Y(e_1)(ƙ_1) \wedge F_Y(e_1)(ƙ_1) = 0.7 \wedge 0.7 = 0.7,\\
F_Y(e_1)(ƙ_1ƙ_2) = F_Y(e_1)(ƙ_2) = 0.5 \quad &\geq F_Y(e_1)(ƙ_1) \wedge F_Y(e_1)(ƙ_2) = 0.7 \wedge 0.5 = 0.5,\\
F_Y(e_1)(ƙ_1ƙ_3) = F_Y(e_1)(ƙ_3) = 0.3 \quad &\geq F_Y(e_1)(ƙ_1) \wedge F_Y(e_1)(ƙ_3) = 0.7 \wedge 0.3 = 0.3,\\
F_Y(e_1)(ƙ_2ƙ_2) = F_Y(e_1)(ƙ_2) = 0.5 \quad &\geq F_Y(e_1)(ƙ_2) \wedge F_Y(e_1)(ƙ_2) = 0.5 \wedge 0.5 = 0.5,\\
F_Y(e_1)(ƙ_2ƙ_3) = F_Y(e_1)(ƙ_3) = 0.3 \quad &\geq F_Y(e_1)(ƙ_2) \wedge F_Y(e_1)(ƙ_3) = 0.5 \wedge 0.3 = 0.3,\\
F_Y(e_1)(ƙ_3ƙ_3) = F_Y(e_1)(ƙ_3) = 0.3 \quad &\geq F_Y(e_1)(ƙ_3) \wedge F_Y(e_1)(ƙ_3) = 0.3 \wedge 0.3) = 0.3.
\end{aligned}$$

and

$$\begin{aligned}
F_N(e_1)(ƙ_1ƙ_1) = F_N(e_1)(ƙ_1) = 0.5 \quad &\leq F_N(e_1)(ƙ_1) \vee F_N(e_1)(ƙ_1) = 0.5 \vee 0.5 = 0.5,\\
F_N(e_1)(ƙ_1ƙ_2) = F_N(e_1)(ƙ_2) = 0.5 \quad &\leq F_N(e_1)(ƙ_1) \vee F_N(e_1)(ƙ_2) = 0.5 \vee 0.5 = 0.5,\\
F_N(e_1)(ƙ_1ƙ_3) = F_N(e_1)(ƙ_3) = 0.7 \quad &\leq F_N(e_1)(ƙ_1) \vee F_N(e_1)(ƙ_3) = 0.5 \vee 0.7 = 0.7,\\
F_N(e_1)(ƙ_2ƙ_2) = F_N(e_1)(ƙ_2) = 0.5 \quad &\leq F_N(e_1)(ƙ_2) \vee F_N(e_1)(ƙ_2) = 0.5 \vee 0.5 = 0.5,\\
F_N(e_1)(ƙ_2ƙ_3) = F_N(e_1)(ƙ_3) = 0.7 \quad &\leq F_N(e_1)(ƙ_2) \vee F_N(e_1)(ƙ_3) = 0.5 \vee 0.7 = 0.7,\\
F_N(e_1)(ƙ_3ƙ_3) = F_N(e_1)(ƙ_3) = 0.7 \quad &\leq F_N(e_1)(ƙ_3) \vee F_N(e_1)(ƙ_3) = 0.7 \vee 0.7 = 0.7.
\end{aligned}$$

Since both conditions are satisfied, $(\mathscr{S}, D)$ *is a* ${}^qROF_zS_fS_bS_mG$ *of* M.

Definition 6.1.4. *A* ${}^qROF_zS_fS$ $(\mathscr{S}, D)$ *over* M *is called*

i) a q-Rung Orthopair Fuzzy Soft Left Ideal (${}^qROF_zS_fL_fI_d$*) if, for each* $e \in D$, $\mathscr{S}(e)$ *is a* ${}^qROF_zL_fI_d$ *of* M.

ii) a q-Rung Orthopair Fuzzy Soft Right Ideal (${}^qROF_zS_fR_iI_d$*) if, for each* $e \in D$, $\mathscr{S}(e)$ *is a* ${}^qROF_zR_iI_d$ *of* M.

iii) a q-Rung Orthopair Fuzzy Soft Interior Ideal (${}^qROF_zS_fI_tI_d$*) if, for each* $e \in D$, $\mathscr{S}(e)$ *is a* ${}^qROF_zI_tI_d$ *of* M.

iv) a q-Rung Orthopair Fuzzy Soft Bi-Ideal (${}^qROF_zS_fB_iI_d$*) if, for each* $e \in D$, $\mathscr{S}(e)$ *is a* ${}^qROF_zB_iI_d$ *of* M.

Theorem 6.1.5. *Let* $(\mathscr{S}, D)$ *be a* $S_fC_mR_l$ *from a* S_mG M_1 *to a* S_mG M_2.

1) If U_2 *is a* ${}^qROF_zS_bS_mG$ *of* M_2*, then* $(\overline{\mathscr{S}}^{U_2}, D)$ *is a* ${}^qROF_zS_fS_bS_mG$ *of* M_1.

2) If U_2 *is a* ${}^qROF_zL_fI_d$ *of* M_2*, then* $(\overline{\mathscr{S}}^{U_2}, D)$ *is a* ${}^qROF_zS_fL_fI_d$ *of* M_1.

3) If U_2 *is a* ${}^qROF_zR_fI_d$ *of* M_2*, then* $(\overline{\mathscr{S}}^{U_2}, D)$ *is a* ${}^qROF_zS_fR_iI_d$ *of* M_1.

Proof.

1) We assume that U_2 is a ${}^qROF_zS_bS_mG$ of M_2. Now for $a, b \in M_1$,

$$\begin{aligned}
\overline{\mathscr{S}}^{U_{2_Y}}(e)(a) \wedge \overline{\mathscr{S}}^{U_{2_Y}}(e)(b) &= \left(\vee_{m \in a\mathscr{S}(e)} U_{2_Y}(m) \right) \wedge \left(\vee_{n \in b\mathscr{S}(e)} U_{2_Y}(n) \right) \\
&= \vee_{m \in a\mathscr{S}(e)} \vee_{n \in b\mathscr{S}(e)} \left(U_{2_Y}(m) \wedge U_{2_Y}(n) \right) \\
&\leq \vee_{m \in a\mathscr{S}(e)} \vee_{n \in b\mathscr{S}(e)} \left(U_{2_Y}(mn) \right) \\
&\leq \vee_{mn \in (ab)\mathscr{S}(e)} \left(U_{2_Y}(mn) \right) \\
&= \vee_{m' \in (ab)\mathscr{S}(e)} \left(U_{2_Y}(m') \right) = \overline{\mathscr{S}}^{U_{2_Y}}(e)(ab).
\end{aligned}$$

Similarly for $a, b \in M_1$,

$$\begin{aligned}
\overline{\mathscr{S}}^{U_{2_N}}(e)(a) \vee \overline{\mathscr{S}}^{U_{2_N}}(e)(b) &= \Big(\wedge_{m \in a\mathscr{S}(e)} U_{2_N}(m) \Big) \vee \Big(\wedge_{n \in b\mathscr{S}(e)} U_{2_N}(n) \Big) \\
&= \wedge_{m \in a\mathscr{S}(e)} \wedge_{n \in b\mathscr{S}(e)} \Big(U_{2_N}(m) \vee U_{2_N}(n) \Big) \\
&\geq \wedge_{m \in a\mathscr{S}(e)} \wedge_{n \in b\mathscr{S}(e)} \Big(U_{2_N}(mn) \Big) \\
&\geq \wedge_{mn \in (ab)\mathscr{S}(e)} \Big(U_{2_N}(mn) \Big) \\
&= \wedge_{m' \in (ab)\mathscr{S}(e)} \Big(U_{2_N}(m') \Big) = \overline{\mathscr{S}}^{U_{2_N}}(e)(ab).
\end{aligned}$$

Hence, $\overline{\mathscr{S}}^{U_2}(e)$ is a ${}^qROF_zS_bS_mG$ of M_1 for all $e \in D$, so $(\overline{\mathscr{S}}^{U_2}, D)$ is a ${}^qROF_zS_fS_bS_mG$ of M_1.

2) Assume that U_2 is a ${}^qROF_zL_fI_d$ of M_2. Now for $a, b \in M_1$,

$$\begin{aligned}
\overline{\mathscr{S}}^{U_{2_Y}}(e)(b) &= \Big(\vee_{n \in b\mathscr{S}(e)} U_{2_Y}(n) \Big) \leq \vee_{m \in a\mathscr{S}(e)} \vee_{n \in b\mathscr{S}(e)} U_{2_Y}(mn) \\
&\leq \vee_{mn \in (mn)\mathscr{S}(e)} \Big(U_{2_Y}(mn) \Big) = \vee_{m' \in (ab)\mathscr{S}(e)} \Big(U_{2_Y}(m') \Big) = \overline{\mathscr{S}}^{U_{2_Y}}(e)(ab).
\end{aligned}$$

Similarly for $a, b \in M_1$,

$$\begin{aligned}
\overline{\mathscr{S}}^{U_{2_N}}(e)(b) &= \Big(\wedge_{n \in b\mathscr{S}(e)} U_{2_N}(n) \Big) \geq \wedge_{m \in a\mathscr{S}(e)} \wedge_{n \in b\mathscr{S}(e)} U_{2_N}(mn) \\
&\geq \wedge_{mn \in (ab)\mathscr{S}(e)} \Big(U_{2_N}(mn) \Big) = \wedge_{n' \in (ab)\mathscr{S}(e)} \Big(U_{2_N}(n') \Big) = \overline{\mathscr{S}}^{U_{2_N}}(e)(ba).
\end{aligned}$$

Hence, $\overline{\mathscr{S}}^{U_2}(e)$ is a ${}^qROF_zS_bS_mG$ of M_1 for all $e \in D$, so $(\overline{\mathscr{S}}^{U_2}, D)$ is a ${}^qROF_zS_fS_bS_mG$ of M_1.

3) The proof can be derived using the same approach as in part (2).

□

In Theorem 6.1.5 from part 1, $S_fC_mR_l$s from M_1 to M_2 are given, and U_2 is a ${}^qROF_zS_bS_mG$ in M_2. After combining them, we get generalized ${}^qROF_zS_fS_bS_mG$s in M_1. Similarly, if we take a ${}^qROF_zL_fI_d$ or ${}^qROF_zL_fI_d$ U_2 of M_2, then we get generalized ${}^qROF_zS_fL_fI_d$ or ${}^qROF_zS_fL_fI_d$ of M_1.

Theorem 6.1.6. *Let $(\mathscr{S}, D)$ be a $S_fC_mR_l$ from a S_mG M_1 to a S_mG M_2:*

1) If U_1 is a ${}^qROF_zS_bS_mG$ of M_1, then $({}^{U_1}\overline{\mathscr{S}}, D)$ is a ${}^qROF_zS_fS_bS_mG$ of M_2

2) If U_1 is a ${}^qROF_zL_fI_d$ (${}^qROF_zR_iI_d$) of M_1, then $({}^{U_1}\overline{\mathscr{S}}, D)$ is a ${}^qROF_zS_fL_fI_d$ (${}^qROF_zS_fR_iI_d$) of M_2.

Proof.
The proof is similar to the proof of Theorem 6.1.5. □

In Theorem 6.1.6 from part 1, $S_fC_mR_l$s from M_1 to M_2 are given, and U_1 is a ${}^qROF_zS_bS_mG$ in M_1. After combining them, we get generalized ${}^qROF_zS_fS_bS_mG$s in M_2. Similarly, if we take a ${}^qROF_zL_fI_d$ or ${}^qROF_zR_iI_d$ U_1 of M_1, then we get generalized ${}^qROF_zS_fL_fI_d$ or ${}^qROF_zS_fR_iI_d$ of M_2.
Now, we show that the converses of parts of Theorems 6.1.5 and 6.1.6 do not hold in general.

Example 6.1.7. *Let $M_1 = \{\Bbbk_1, \Bbbk_2, \Bbbk_3, \Bbbk_4, \Bbbk_5\}$ and $M_2 = \{\mathfrak{t}_1, \mathfrak{t}_2, \mathfrak{t}_3, \mathfrak{t}_4, \mathfrak{t}_5\}$ represent two S_mGs, with their respective multiplication tables shown in Tables 6.2 and 6.3.*

Table 6.2: Multiplication table for M_1

.	$\Bbbk_1$	$\Bbbk_2$	$\Bbbk_3$	$\Bbbk_4$	$\Bbbk_5$
$\Bbbk_1$	$\Bbbk_2$	$\Bbbk_2$	$\Bbbk_4$	$\Bbbk_4$	$\Bbbk_4$
$\Bbbk_2$	$\Bbbk_2$	$\Bbbk_2$	$\Bbbk_4$	$\Bbbk_4$	$\Bbbk_4$
$\Bbbk_3$	$\Bbbk_4$	$\Bbbk_4$	$\Bbbk_3$	$\Bbbk_4$	$\Bbbk_3$
$\Bbbk_4$	$\Bbbk_4$	$\Bbbk_4$	$\Bbbk_4$	$\Bbbk_4$	$\Bbbk_4$
$\Bbbk_5$	$\Bbbk_4$	$\Bbbk_4$	$\Bbbk_3$	$\Bbbk_4$	$\Bbbk_3$

Table 6.3: Multiplication table for M_2

.	$\mathfrak{t}_1$	$\mathfrak{t}_2$	$\mathfrak{t}_3$	$\mathfrak{t}_4$	$\mathfrak{t}_5$
$\mathfrak{t}_1$	$\mathfrak{t}_1$	$\mathfrak{t}_5$	$\mathfrak{t}_3$	$\mathfrak{t}_4$	$\mathfrak{t}_5$
$\mathfrak{t}_2$	$\mathfrak{t}_1$	$\mathfrak{t}_2$	$\mathfrak{t}_3$	$\mathfrak{t}_4$	$\mathfrak{t}_5$
$\mathfrak{t}_3$	$\mathfrak{t}_1$	$\mathfrak{t}_5$	$\mathfrak{t}_3$	$\mathfrak{t}_4$	$\mathfrak{t}_5$
$\mathfrak{t}_4$	$\mathfrak{t}_1$	$\mathfrak{t}_5$	$\mathfrak{t}_3$	$\mathfrak{t}_4$	$\mathfrak{t}_5$
$\mathfrak{t}_5$	$\mathfrak{t}_1$	$\mathfrak{t}_5$	$\mathfrak{t}_3$	$\mathfrak{t}_4$	$\mathfrak{t}_5$

Let $D = \{e_1, e_2\}$. Define a S_fB_nR $\mathscr{S} : D \to P(M_1 \times M_2)$ by:

$$\mathscr{S}(e_1) = \begin{array}{c} \\ \mathfrak{t}_1 \\ \mathfrak{t}_2 \\ \mathfrak{t}_3 \\ \mathfrak{t}_4 \\ \mathfrak{t}_5 \end{array} \begin{array}{c} \begin{array}{ccccc} \mathfrak{k}_1 & \mathfrak{k}_2 & \mathfrak{k}_3 & \mathfrak{k}_4 & \mathfrak{k}_5 \end{array} \\ \begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 \end{pmatrix} \end{array}, \quad \mathscr{S}(e_2) = \begin{array}{c} \\ \mathfrak{t}_1 \\ \mathfrak{t}_2 \\ \mathfrak{t}_3 \\ \mathfrak{t}_4 \\ \mathfrak{t}_5 \end{array} \begin{array}{c} \begin{array}{ccccc} \mathfrak{k}_1 & \mathfrak{k}_2 & \mathfrak{k}_3 & \mathfrak{k}_4 & \mathfrak{k}_5 \end{array} \\ \begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 1 \end{pmatrix} \end{array}$$

Then, $(\mathscr{S}, D)$ is a $S_fC_mR_l$ from M_1 to M_2.

$$\begin{aligned}
\mathfrak{k}_1\mathscr{S}(e_1) &= \{\mathfrak{t}_1\}, & \mathscr{S}(e_1)\mathfrak{t}_1 &= \{\mathfrak{k}_1, \mathfrak{k}_2, \mathfrak{k}_4\}, \\
\mathfrak{k}_2\mathscr{S}(e_1) &= \{\mathfrak{t}_1, \mathfrak{t}_2, \mathfrak{t}_5\}, & \mathscr{S}(e_1)\mathfrak{t}_2 &= \{\mathfrak{k}_2\}, \\
\mathfrak{k}_3\mathscr{S}(e_1) &= \{\mathfrak{t}_3, \mathfrak{t}_5\}, & \mathscr{S}(e_1)\mathfrak{t}_3 &= \{\mathfrak{k}_3, \mathfrak{k}_4\}, \\
\mathfrak{k}_4\mathscr{S}(e_1) &= \{\mathfrak{t}_1, \mathfrak{t}_3, \mathfrak{t}_4, \mathfrak{t}_5\}, & \mathscr{S}(e_1)\mathfrak{t}_4 &= \{\mathfrak{k}_4\}, \\
\mathfrak{k}_5\mathscr{S}(e_1) &= \{\mathfrak{t}_5\}, & \mathscr{S}(e_1)\mathfrak{t}_5 &= \{\mathfrak{k}_2, \mathfrak{k}_3, \mathfrak{k}_4, \mathfrak{k}_5\}, \\
\mathfrak{k}_1\mathscr{S}(e_2) &= \{\mathfrak{t}_1\}, & \mathscr{S}(e_2)\mathfrak{t}_1 &= \{\mathfrak{k}_1, \mathfrak{k}_2, \mathfrak{k}_4\}, \\
\mathfrak{k}_2\mathscr{S}(e_2) &= \{\mathfrak{t}_1, \mathfrak{t}_2, \mathfrak{t}_3, 5\}, & \mathscr{S}(e_2)\mathfrak{t}_2 &= \{\mathfrak{k}_2\}, \\
\mathfrak{k}_3\mathscr{S}(e_2) &= \{\mathfrak{t}_3, \mathfrak{t}_5\}, & \mathscr{S}(e_2)\mathfrak{t}_3 &= \{\mathfrak{k}_2, \mathfrak{k}_3, \mathfrak{k}_4\}, \\
\mathfrak{k}_4\mathscr{S}(e_2) &= \{\mathfrak{t}_1, \mathfrak{t}_3, \mathfrak{t}_4, \mathfrak{t}_5\}, & \mathscr{S}(e_2)\mathfrak{t}_4 &= \{\mathfrak{k}_4\}, \\
\mathfrak{k}_5\mathscr{S}(e_2) &= \{\mathfrak{t}_5\}, & \mathscr{S}(e_2)\mathfrak{t}_5 &= \{\mathfrak{k}_2, \mathfrak{k}_3, \mathfrak{k}_4, \mathfrak{k}_5\}.
\end{aligned}$$

1) Define a ${}^{q}ROF_zS$ $U_1 : M_2 \to [0,1]$ *by*

$U_1 = \{\langle \mathfrak{t}_1, 0.5, 0.4\rangle, \langle \mathfrak{t}_2, 0.6, 0.5\rangle, \langle \mathfrak{t}_3, 0.4, 0.7\rangle, \langle \mathfrak{t}_4, 0.8, 0.7\rangle, \langle \mathfrak{t}_5, 0.1, 0.8\rangle\}$. *Then,* U_1 *is not a* ${}^{q}ROF_zS_bS_mG$ *of* M_2 *because if we take* $a = \mathfrak{t}_1, b = \mathfrak{t}_2$, *then* $U_{1_Y}(\mathfrak{t}_1\mathfrak{t}_2) = 0.1 \not\geq 0.5 = U_{1_Y}(\mathfrak{t}_1) \wedge U_{1_Y}(\mathfrak{t}_2)$ *and* $U_{1_N}(\mathfrak{t}_1\mathfrak{t}_2) = 0.8 \not\leq 0.6 = U_{1_N}(\mathfrak{t}_1) \vee U_{1_N}(\mathfrak{t}_2)$. U_pA_p *of* U_1 *is given in Table 6.4.*

Table 6.4: U_pA_p of U_1

	$\overline{\mathscr{S}}^{U_{1_Y}}(e_1)$	$\overline{\mathscr{S}}^{U_{1_N}}(e_1)$	$\overline{\mathscr{S}}^{U_{1_Y}}(e_2)$	$\overline{\mathscr{S}}^{U_{1_N}}(e_2)$
$\mathfrak{k}_1$	0.5	0.4	0.5	0.4
$\mathfrak{k}_2$	0.6	0.4	0.6	0.4
$\mathfrak{k}_3$	0.4	0.7	0.4	0.7
$\mathfrak{k}_4$	0.8	0.4	0.8	0.4
$\mathfrak{k}_5$	0.1	0.7	0.1	0.8

Clearly, $\overline{\mathscr{S}}^{U_1}(e_1)$ *and* $\overline{\mathscr{S}}^{U_1}(e_2)$ *are* ${}^{q}ROF_zS_bS_mGs$ *of* M_1, *so* $(\overline{\mathscr{S}}^{U_1}, D)$ *is a* ${}^{q}ROF_zS_fS_bS_mG$ *of* M_1.

2) Define a ${}^{q}ROF_zS$ $U_2 : M_1 \to [0,1]$ *by*

$U_2 = \{\langle \mathfrak{k}_1, 0.5, 0.9\rangle, \langle \mathfrak{k}_2, 0.7, 0.3\rangle, \langle \mathfrak{k}_3, 0.8, 0.2\rangle, \langle \mathfrak{k}_4, 0.5, 0.8\rangle, \langle \mathfrak{k}_5, 0.9, 0.1\rangle\}$. *Then,* U_2 *is not a* ${}^{q}ROF_zS_bS_mG$ *of* M_1 *because if we take* $a = \mathfrak{k}_2, b = \mathfrak{k}_3$, *then* $U_{2_Y}(\mathfrak{k}_2\mathfrak{k}_3) = 0.5 \not\geq 0.7 = U_{2_Y}(\mathfrak{k}_2) \wedge U_{2_Y}(\mathfrak{k}_3)$ *and* $U_{2_N}(\mathfrak{k}_2\mathfrak{k}_3) = 0.3 \not\leq 0.8 = U_{2_N}(\mathfrak{k}_2) \vee U_{2_N}(\mathfrak{k}_3)$. U_pA_p *of* U_2 *is given in Table 6.5.*

Table 6.5: U_pA_p of U_2

	${}^{U_{2_Y}}\overline{\mathscr{S}}(e_1)$	${}^{U_{2_N}}\overline{\mathscr{S}}(e_1)$	${}^{U_{2_Y}}\overline{\mathscr{S}}(e_2)$	${}^{U_{2_N}}\overline{\mathscr{S}}(e_2)$
$\mathfrak{t}_1$	0.8	0.2	0.8	0.2
$\mathfrak{t}_2$	0.7	0.3	0.7	0.3
$\mathfrak{t}_3$	0.8	0.2	0.8	0.2
$\mathfrak{t}_4$	0.5	0.8	0.5	0.8
$\mathfrak{t}_5$	0.9	0.1	0.9	0.1

Clearly, $\overline{\mathscr{S}}^{U_2}(e_1)$ *and* $\overline{\mathscr{S}}^{U_2}(e_2)$ *are* ${}^{q}ROF_zS_bS_mGs$ *of* M_2*, so* $(\overline{\mathscr{S}}^{U_2}, D)$ *is a* ${}^{q}ROF_zS_fS_bS_mG$ *of* M_2*.*

3) Define a ${}^{q}ROF_zS$ $U_3 : M_2 \to [0,1]$ *by*
$U_3 = \{\langle \mathfrak{t}_1, 0.5, 0.4\rangle, \langle \mathfrak{t}_2, 0.6, 0.5\rangle, \langle \mathfrak{t}_3, 0.4, 0.7\rangle, \langle \mathfrak{t}_4, 0.8, 0.7\rangle, \langle \mathfrak{t}_5, 0.1, 0.8\rangle\}$*. Then,* U_3 *is not a* ${}^{q}ROF_zL_fI_d$ *of* M_2 *because if we take* $a = \mathfrak{t}_1, b = \mathfrak{t}_2$*, then* $U_{3_Y}(\mathfrak{t}_1\mathfrak{t}_2) = 0.1 \ngeq 0.6 = U_{3_Y}(\mathfrak{t}_2)$ *and* $U_{3_N}(\mathfrak{t}_1\mathfrak{t}_2) = 0.8 \nleq 0.5 = U_{3_N}(\mathfrak{t}_2)$*.* U_pA_p *of* U_3 *is given in Table 6.6.*

Table 6.6: U_pA_p of U_3

	$\mathfrak{k}_1$	$\mathfrak{k}_2$	$\mathfrak{k}_3$	$\mathfrak{k}_4$	$\mathfrak{k}_5$
$\overline{\mathscr{S}}^{U_{3Y}}(e_1)$	0.5	0.6	0.4	0.8	0.1
$\overline{\mathscr{S}}^{U_{3N}}(e_1)$	0.4	0.4	0.7	0.4	0.7
$\overline{\mathscr{S}}^{U_{3Y}}(e_2)$	0.5	0.6	0.4	0.8	0.1
$\overline{\mathscr{S}}^{U_{3N}}(e_2)$	0.4	0.4	0.7	0.4	0.8

Clearly, $\overline{\mathscr{S}}^{U_3}(e_1)$ *and* $\overline{\mathscr{S}}^{U_3}(e_2)$ *are* ${}^{q}ROF_zL_fI_ds$ *of* M_1*, so* $(\overline{\mathscr{S}}^{U_3}, D)$ *is a* ${}^{q}ROF_zS_fL_fI_d$ *of* M_1*.*

4) Define a ${}^{q}ROF_zS$ $U_4 : M_1 \to [0,1]$ *by*
$U_4 = \{\langle \mathfrak{k}_1, 0.5, 0.9\rangle, \langle \mathfrak{k}_2, 0.7, 0.3\rangle, \langle \mathfrak{k}_3, 0.8, 0.2\rangle, \langle \mathfrak{k}_4, 0.5, 0.8\rangle, \langle \mathfrak{k}_5, 0.9, 0.1\rangle\}$*. Then,* U_4 *is not a* ${}^{q}ROF_zL_fI_d$ *of* M_1 *because if we take* $a = \mathfrak{k}_1, b = \mathfrak{k}_3$ *then* $U_{4_Y}(\mathfrak{k}_1\mathfrak{k}_3) = 0.5 \ngeq 0.8 = U_{4_Y}(\mathfrak{k}_3)$ *and* $U_{4_N}(\mathfrak{k}_1\mathfrak{k}_3) = 1 \nleq 0.7 = U_{4_N}(\mathfrak{k}_3)$*.* U_pA_p *of* U_4 *is given in Table 6.7.*

Table 6.7: U_pA_p of U_4

	${}^{U_{4Y}}\overline{\mathscr{S}}(e_1)$	${}^{U_{4N}}\overline{\mathscr{S}}(e_1)$	${}^{U_{4Y}}\overline{\mathscr{S}}(e_2)$	${}^{U_{4N}}\overline{\mathscr{S}}(e_2)$
$\mathfrak{t}_1$	0.8	0.2	0.8	0.2
$\mathfrak{t}_2$	0.7	0.3	0.7	0.3
$\mathfrak{t}_3$	0.8	0.2	0.8	0.2
$\mathfrak{t}_4$	0.5	0.8	0.5	0.8
$\mathfrak{t}_5$	0.9	0.1	0.9	0.1

Clearly, ${}^{U_4}\overline{\mathscr{S}}(e_1)$ *and* ${}^{U_4}\overline{\mathscr{S}}(e_2)$ *are* ${}^{q}ROF_zL_fI_ds$ *of* M_2*, so* $({}^{U_4}\overline{\mathscr{S}}, D)$ *is* ${}^{q}ROF_zS_fL_fI_d$ *of* M_2*.*

Example 6.1.8. *Consider the two* S_mGs *and* S_fB_nR *of Example 6.1.7. Define a* ${}^{q}ROF_zS$ $U : M_2 \to [0, 1]$ *by*
$U = \{\langle ŧ_1, 0.5, 0.4\rangle, \langle ŧ_2, 0.6, 0.5\rangle, \langle ŧ_3, 0.4, 0.7\rangle, \langle ŧ_4, 0.8, 0.7\rangle, \langle ŧ_5, 0.1, 0.8\rangle\}$*. Then,* U *is not a* ${}^{q}ROF_zL_fI_d$ *of* M_2*.* L_oA_p *of* U *is given in Table 6.8. But* $\underline{\mathscr{S}}^U(e_1)$ *is not a* ${}^{q}ROF_zL_fI_d$ *of* M_1 *because if we take* $a = ŧ_1, b = ŧ_2$*, then* $U_Y(ŧ_1ŧ_2) = 0.1 \ngeq 0.6 = U_Y(ŧ_2)$ *and* $U_N(ŧ_1ŧ_2) = 0.8 \nleq 0.5 = U_N(ŧ_2)$*.*

Table 6.8: L_oA_p of U

	$\underline{\mathscr{S}}^{U_Y}(e_1)$	$\underline{\mathscr{S}}^{U_N}(e_1)$
$ƙ_1$	0.5	0.4
$ƙ_2$	0.1	0.8
$ƙ_3$	0.1	0.8
$ƙ_4$	0.1	0.8
$ƙ_5$	0.1	0.8

This example illustrates that if the soft relation is compatible, then the lower approximation of a generalized rough Fuzzy lower Ideal (${}^{q}ROF_zL_fI_d$) is not necessarily a generalized rough Fuzzy soft lower Ideal (${}^{q}ROF_zS_fL_fI_d$). However, the following theorem confirms the validity of another result.

Theorem 6.1.9. *Suppose* $(\mathscr{S}, D)$ *is a* $S_fC_{mp}R_l$ *from a* S_mG M_1 *to a* S_mG M_2*.*

1) *If* U_2 *is a* ${}^{q}ROF_zS_bS_mG$ *of* M_2*, then* $(\underline{\mathscr{S}}^{U_2}, D)$ *is a* ${}^{q}ROF_zS_fS_bS_mG$ *of* M_1

2) *If* U_2 *is a* ${}^{q}ROF_zL_fI_d$ *(*${}^{q}ROF_zR_iI_d$*) of* M_2*, then* $(\underline{\mathscr{S}}^{U_2}, D)$ *is a* ${}^{q}ROF_zS_fL_fI_d$ *(*${}^{q}ROF_zS_fR_iI_d$*) of* M_1*, respectively.*

Proof.

1) We assume that U_2 is a ${}^qROF_zS_bS_mG$ of M_2. Now for $a, b \in M_1$,

$$\underline{\mathscr{L}}^{U_{2_Y}}(ab) = \wedge_{m' \in (ab)\mathscr{S}(e)} U_{2_Y}\left(m'\right) = \wedge_{m' \in (a)\mathscr{S}(e).(b)\mathscr{S}(e)} U_{2_Y}\left(m'\right)$$
$$= \wedge_{m \in (a)\mathscr{S}(e).n \in (b)\mathscr{S}(e)} U_{2_Y}(mn) \geq \wedge_{m \in (a)\mathscr{S}(e)} \wedge_{n \in (b)\mathscr{S}(e)} \left(U_{2_Y}(m) \wedge U_{2_Y}(n)\right)$$
$$\geq \left(\wedge_{m \in a\mathscr{S}(e)} U_{2_Y}(m)\right) \wedge \left(\wedge_{n \in b\mathscr{S}(e)} U_{2_Y}(n)\right) = \underline{\mathscr{L}}^{U_{2_Y}}(e)(a) \wedge \underline{\mathscr{L}}^{U_{2_Y}}(e)(b).$$

Similarly for $a, b \in M_1$,

$$\underline{\mathscr{L}}^{U_{2_N}}(e)(ab) = \vee_{m' \in (ab)\mathscr{S}(e)} U_{2_N}\left(m'\right) = \vee_{m' \in (a)\mathscr{S}(e).(b)\mathscr{S}(e)} U_{2_N}\left(m'\right)$$
$$= \vee_{m \in (a)\mathscr{S}(e).n \in (b)\mathscr{S}(e)} U_{2_N}(mn) \leq \vee_{m \in (a)\mathscr{S}(e)} \vee_{n \in (b)\mathscr{S}(e)} \left(U_{2_N}(m) \vee U_{2_N}(n)\right)$$
$$\leq \left(\vee_{m \in a\mathscr{S}(e)} U_{2_N}(m)\right) \vee \left(\vee_{n \in b\mathscr{S}(e)} U_{2_N}(n)\right) = \underline{\mathscr{L}}^{U_{2_N}}(e)(a) \vee \underline{\mathscr{L}}^{U_{2_N}}(e)(b).$$

Hence, $\underline{\mathscr{L}}^{U_2}(e)$ is a ${}^qROF_zS_bS_mG$ of M_1 for all $e \in D$, so $(\underline{\mathscr{L}}^{U_2}, D)$ is a ${}^qROF_zS_fS_bS_mG$ of M_1.

2) Assume that U_2 is a ${}^qROF_zL_fI_d$ of M_2. Now for $a, b \in M_1$,

$$\underline{\mathscr{L}}^{U_{2_Y}}(e)(ab) = \wedge_{m' \in (ab)\mathscr{S}(e)} U_{2_Y}\left(m'\right) = \wedge_{m' \in (a)\mathscr{S}(e).(b)\mathscr{S}(e)} U_{2_Y}\left(m'\right)$$
$$= \wedge_{m \in (a)\mathscr{S}(e).n \in (b)\mathscr{S}(e)} U_{2_Y}(mn) \geq \wedge_{n \in (b)\mathscr{S}(e)} U_{2_Y}(n)$$
$$= \underline{\mathscr{L}}^{U_{2_Y}}(e)(b).$$

Similarly for $a, b \in M_1$,

$$\underline{\mathscr{S}}^{U_{2_N}}(e)(ab) = \vee_{m' \in (ab)\mathscr{S}(e)} U_{2_N}\left(m'\right) = \vee_{m' \in (a)\mathscr{S}(e).(b)\mathscr{S}(e)} U_{2_N}\left(m'\right)$$

$$= \vee_{m \in (a)\mathscr{S}(e).n \in (b)\mathscr{S}(e)} U_{2_N}(mn) \leq \vee_{n \in (b)\mathscr{S}(e)} U_{2_N}(n)$$

$$= \underline{\mathscr{S}}^{U_{2_N}}(e)(b).$$

Hence, $\underline{\mathscr{S}}^{U_2}(e)$ is a ${}^qROF_zL_fI_d$ of M_1 for all $e \in D$, so $(\underline{\mathscr{S}}^{U_2}, D)$ is a ${}^qROF_zS_fL_fI_d$ of M_1.

□

Theorem 6.1.10. *Suppose $(\mathscr{S}, D)$ is a $S_fC_{mp}R_l$ from a S_mG M_1 to a S_mG M_2.*

1) *If U_1 is a ${}^qROF_zS_bS_mG$ of M_1, then $({}^{U_1}\underline{\mathscr{S}}, D)$ is a ${}^qROF_zS_fS_bS_mG$ of M_2*

2) *If U_1 is a ${}^qROF_zL_fI_d$ (${}^qROF_zR_iI_d$) of M_1, then $({}^{U_1}\underline{\mathscr{S}}, D)$ is a ${}^qROF_zS_fL_fI_d$ (${}^qROF_zS_fR_iI_d$) of M_2, respectively.*

Proof.
The proof can be derived using the same approach as in the Theorem 6.1.9. □

Example 6.1.11. *Let $M_1 = \{\Bbbk_1, \Bbbk_2, \Bbbk_3, \Bbbk_4\}$ and $M_2 = \{ŧ_1, ŧ_2, ŧ_3, ŧ_4\}$ represent two S_mGs, with their multiplication tables provided in Tables 6.9 and 6.10, respectively. Consider $D = \{e_1, e_2\}$.*

Table 6.9: Multiplication table for M_1

.	$\Bbbk_1$	$\Bbbk_2$	$\Bbbk_3$	$\Bbbk_4$
$\Bbbk_1$	$\Bbbk_1$	$\Bbbk_1$	$\Bbbk_1$	$\Bbbk_4$
$\Bbbk_2$	$\Bbbk_1$	$\Bbbk_2$	$\Bbbk_1$	$\Bbbk_4$
$\Bbbk_3$	$\Bbbk_1$	$\Bbbk_1$	$\Bbbk_3$	$\Bbbk_4$
$\Bbbk_4$	$\Bbbk_4$	$\Bbbk_4$	$\Bbbk_4$	$\Bbbk_4$

Table 6.10: Multiplication table for M_2

.	$\mathfrak{t}_1$	$\mathfrak{t}_2$	$\mathfrak{t}_3$	$\mathfrak{t}_4$
$\mathfrak{t}_1$	$\mathfrak{t}_1$	$\mathfrak{t}_2$	$\mathfrak{t}_3$	$\mathfrak{t}_4$
$\mathfrak{t}_2$	$\mathfrak{t}_2$	$\mathfrak{t}_2$	$\mathfrak{t}_2$	$\mathfrak{t}_2$
$\mathfrak{t}_3$	$\mathfrak{t}_3$	$\mathfrak{t}_3$	$\mathfrak{t}_3$	$\mathfrak{t}_3$
$\mathfrak{t}_4$	$\mathfrak{t}_4$	$\mathfrak{t}_3$	$\mathfrak{t}_2$	$\mathfrak{t}_1$

Define a S_fB_nR $\mathscr{S} : D \to P(M_1 \times M_2)$ by

$$\mathscr{S}(e_1) = \begin{array}{c} \\ t_1 \\ t_2 \\ t_3 \\ t_4 \end{array} \begin{array}{c} \begin{array}{cccc} k_1 & k_2 & k_3 & k_4 \end{array} \\ \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \end{array}, \quad \mathscr{S}(e_2) = \begin{array}{c} \\ t_1 \\ t_2 \\ t_3 \\ t_4 \end{array} \begin{array}{c} \begin{array}{cccc} k_1 & k_2 & k_3 & k_4 \end{array} \\ \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \end{array}$$

Then, $(\mathscr{S}, D)$ forms a $S_fC_{mp}R_l$ from M_1 to M_2.

$$k_1\mathscr{S}(e_1) = \{t_2, t_3\},$$
$$k_2\mathscr{S}(e_1) = \{t_2, t_3\},$$
$$k_3\mathscr{S}(e_1) = \{t_2, t_3\},$$
$$k_4\mathscr{S}(e_1) = \{t_2, t_3\},$$
$$k_1\mathscr{S}(e_2) = \{t_2\},$$
$$k_2\mathscr{S}(e_2) = \{t_2\},$$
$$k_3\mathscr{S}(e_2) = \{t_2\},$$
$$k_4\mathscr{S}(e_2) = \{t_2\}.$$

1) Define a ${}^qROF_zS$ $U_1 : M_2 \to [0, 1]$ by
$U_1 = \{\langle t_1, 0.9, 0.7\rangle, \langle t_2, 0.6, 0.5\rangle, \langle t_3, 0.7, 0.4\rangle, \langle t_4, 0.8, 0.3\rangle\}$. *Then U_1 is not a ${}^qROF_zS_bS_mG$ of M_2 because if we take $a = t_4, b = t_3$, then $U_{1_Y}(t_4t_3) = 0.6 \ngeq 0.7 = U_{1_Y}(t_4) \wedge U_{1_Y}(t_3)$*

and $U_{1_N}(\mathfrak{t}_4\mathfrak{t}_3) = 0.5 \nleq 0.4 = U_{1_N}(\mathfrak{t}_4) \vee U_{1_N}(\mathfrak{t}_3)$. L_oA_p of U_1 is given in Table 6.11.

Table 6.11: L_oA_p of U_1

	$\underline{\mathscr{L}}^{U_{1_Y}}(e_1)$	$\underline{\mathscr{L}}^{U_{1_N}}(e_1)$	$\underline{\mathscr{L}}^{U_{1_Y}}(e_2)$	$\underline{\mathscr{L}}^{U_{1_N}}(e_2)$
$\mathfrak{k}_1$	0.6	0.5	0.6	0.5
$\mathfrak{k}_2$	0.6	0.5	0.6	0.5
$\mathfrak{k}_3$	0.6	0.5	0.6	0.5
$\mathfrak{k}_4$	0.6	0.5	0.6	0.5

Clearly, $\underline{\mathscr{L}}^{U_1}(e_1)$ and $\underline{\mathscr{L}}^{U_1}(e_2)$ are ${}^qROF_zS_bS_mGs$ of M_1, so $(\underline{\mathscr{L}}^{U_1}, D)$ is a ${}^qROF_zS_fS_bS_mG$ of M_1.

2) Define a ${}^qROF_zS$ $U_2 : M_2 \to [0,1]$ by
$U_2 = \{\langle \mathfrak{t}_1, 0.5, 0.7\rangle, \langle \mathfrak{t}_2, 0.5, 0.8\rangle, \langle \mathfrak{t}_3, 0.6, 0.7\rangle, \langle \mathfrak{t}_4, 0.8, 0.4\rangle\}$. Then, U_2 is not a ${}^qROF_zL_fI_d$ of M_2 because if we take $a = \mathfrak{t}_2, b = \mathfrak{t}_3$, then $U_{2_Y}(\mathfrak{t}_2\mathfrak{t}_3) = 0.5 \ngeq 0.6 = U_{2_Y}(\mathfrak{t}_3)$ and $U_{2_N}(\mathfrak{t}_2\mathfrak{t}_3) = 0.8 \nleq 0.7 = U_{2_N}(\mathfrak{t}_3)$. U_pA_p of U_2 is given in Table 6.12.

Table 6.12: L_oA_p of U_2

	$\underline{\mathscr{L}}^{U_{2_Y}}(e_1)$	$\underline{\mathscr{L}}^{U_{2_N}}(e_1)$	$\underline{\mathscr{L}}^{U_{2_Y}}(e_2)$	$\underline{\mathscr{L}}^{U_{2_N}}(e_2)$
$\mathfrak{k}_1$	0.5	0.8	0.5	0.8
$\mathfrak{k}_2$	0.5	0.8	0.5	0.8
$\mathfrak{k}_3$	0.5	0.8	0.5	0.8
$\mathfrak{k}_4$	0.5	0.8	0.5	0.8

Clearly, $\underline{\mathscr{L}}^{U_2}(e_1)$ and $\underline{\mathscr{L}}^{U_2}(e_2)$ are ${}^qROF_zL_fI_ds$ of M_1, so $(\underline{\mathscr{L}}^{U_2}, D)$ is a ${}^qROF_zS_fL_fI_d$ of M_1.
Now define $\mathscr{S}_1 : D \to P(M_1 \times M_2)$ by:

$$\mathscr{S}_1(e_1) = \begin{array}{c} \\ ƭ_1 \\ ƭ_2 \\ ƭ_3 \\ ƭ_4 \end{array} \begin{array}{c} \begin{array}{cccc} ƙ_1 & ƙ_2 & ƙ_3 & ƙ_4 \end{array} \\ \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \end{array}, \quad \mathscr{S}_1(e_2) = \begin{array}{c} \\ ƭ_1 \\ ƭ_2 \\ ƭ_3 \\ ƭ_4 \end{array} \begin{array}{c} \begin{array}{cccc} ƙ_1 & ƙ_2 & ƙ_3 & ƙ_4 \end{array} \\ \begin{pmatrix} 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \end{pmatrix} \end{array}$$

Then, $(\mathscr{S}_1, D)$ forms a $S_fC_{mp}R_ls$ from M_1 to M_2.

$$\mathscr{S}_1(e_1)ƭ_1 = \{ƙ_4\},$$
$$\mathscr{S}_1(e_1)ƭ_2 = \{ƙ_4\},$$
$$\mathscr{S}_1(e_1)ƭ_3 = \{ƙ_4\},$$
$$\mathscr{S}_1(e_1)ƭ_4 = \{ƙ_4\}$$
$$\mathscr{S}_1(e_2)ƭ_1 = \{ƙ_1, ƙ_4\}$$
$$\mathscr{S}_1(e_2)ƭ_2 = \{ƙ_1, ƙ_4\}$$
$$\mathscr{S}_1(e_2)ƭ_3 = \{ƙ_1, ƙ_4\}$$
$$\mathscr{S}_1(e_2)ƭ_4 = \{ƙ_1, ƙ_4\}.$$

1) *Define a ${}^qROF_zS$ $U_3 : M_1 \to [0,1]$ by $U_3 = \{\langle ƙ_1, 0.5, 0.9\rangle, \langle ƙ_2, 0.7, 0.6\rangle, \langle ƙ_3, 0.6, 0.4\rangle, \langle ƙ_4, 0.7, 0.2\rangle\}$. Then, U_3 is not a ${}^qROF_zS_bS_mG$ of M_1 because if we take $a = ƙ_2, b = ƙ_3$ then $U_{3_Y}(ƙ_2ƙ_3) = 0.5 \ngeq 0.6 = U_{3_Y}(ƙ_2) \wedge U_{3_Y}(ƙ_3)$ and $U_{3_N}(ƙ_2ƙ_3) = 0.9 \nleq 0.6 = U_{3_N}(ƙ_2) \vee U_{3_N}(ƙ_3)$. L_oA_p of U_3 is given in Table 6.13.*

Table 6.13: L_oA_p of U_3

	${}^{U_{3_Y}}\underline{\mathscr{S}_1}(e_1)$	${}^{U_{3_N}}\underline{\mathscr{S}_1}(e_1)$	${}^{U_{3_Y}}\underline{\mathscr{S}_1}(e_2)$	${}^{U_{3_N}}\underline{\mathscr{S}_1}(e_2)$
$\mathfrak{t}_1$	0.7	0.2	0.5	0.9
$\mathfrak{t}_2$	0.7	0.2	0.5	0.9
$\mathfrak{t}_3$	0.7	0.2	0.5	0.9
$\mathfrak{t}_4$	0.7	0.2	0.5	0.9

Clearly, ${}^{U_3}\underline{\mathscr{S}_1}(e_1)$ *and* ${}^{U_3}\underline{\mathscr{S}_1}(e_2)$ *are* ${}^{q}ROF_zS_bS_mGs$ *of* M_2*, so* $({}^{U_3}\underline{\mathscr{S}_1}, D)$ *is a* ${}^{q}ROF_zS_fS_bS_mG$ *of* M_2*.*

2) *Define a* ${}^{q}ROF_zS$ $U_4 : M_1 \to [0,1]$ *by*
$U_4 = \{\langle \Bbbk_1, 0.1, 0.8\rangle, \langle \Bbbk_2, 0.3, 0.6\rangle, \langle \Bbbk_3, 0.5, 0.5\rangle, \langle \Bbbk_4, 0.7, 0.2\rangle\}$*. Then,* U_4 *is not a* ${}^{q}ROF_zL_fI_d$ *of* M_1 *because if we take* $a = \Bbbk_2, b = \Bbbk_3$ *then* $U_{4_Y}(\Bbbk_2\Bbbk_3) = 0.1 \not\geq 0.5 = U_{4_Y}(\Bbbk_3)$ *and* $U_{4_N}(\Bbbk_2\Bbbk_3) = 0.8 \not\leq 0.5 = U_{4_N}(\Bbbk_3)$*.* L_oA_p *of* U_4 *is given in Table 6.14.*

Table 6.14: L_oA_p of U_4

	${}^{U_{4_Y}}\underline{\mathscr{S}_1}(e_1)$	${}^{U_{4_N}}\underline{\mathscr{S}_1}(e_1)$	${}^{U_{4_Y}}\underline{\mathscr{S}_1}(e_2)$	${}^{U_{4_N}}\underline{\mathscr{S}_1}(e_2)$
$\mathfrak{k}_1$	0.7	0.2	0.1	0.8
$\mathfrak{k}_2$	0.7	0.2	0.1	0.8
$\mathfrak{k}_3$	0.7	0.2	0.1	0.8
$\mathfrak{k}_4$	0.7	0.2	0.1	0.8

Clearly, ${}^{U_4}\underline{\mathscr{S}_1}(e_1)$ *and* ${}^{U_4}\underline{\mathscr{S}_1}(e_2)$ *are* ${}^{q}ROF_zL_fI_d$ *of* M_2*, so* $({}^{U_4}\underline{\mathscr{S}_1}, D)$ *is a* ${}^{q}ROF_zS_fL_fI_d$ *of* M_2*.*

Theorem 6.1.12. *Suppose* $(\mathscr{S}, D)$ *is a* S_fB_nR *from a* S_mG M_1 *to a* S_mG M_2*; that is,* $\mathscr{S} : D \to P(M_1 \times M_2)$*. Then, for a* ${}^{q}ROF_zR_iI_d$ $U_1 = \langle U_{1_Y}, U_{1_N}\rangle$ *and for a* ${}^{q}ROF_zL_fI_d$ $U_2 = \langle U_{2_Y}, U_{2_N}\rangle$ *of* M_2*,* $\overline{\mathscr{S}}^{U_1U_2} \subseteq \overline{\mathscr{S}}^{U_1} \cap \overline{\mathscr{S}}^{U_2}$*.*

Proof.
Since U_1 is a ${}^{q}ROF_zS_fR_iI_d$, so $U_1U_2 \subseteq U_1$ and U_2 is ${}^{q}ROF_zS_fL_fI_d$ of M_2, so $U_1U_2 \subseteq U_2$. Thus $U_1U_2 \subseteq U_1 \cap U_2$. It follows from Theorem 4.1.4, $\overline{\mathscr{S}}^{U_{1_Y}U_{2_Y}}(e) \subseteq \overline{\mathscr{S}}^{U_{1_Y}\cap U_{2_Y}}(e) \subseteq \overline{\mathscr{S}}^{U_{1_Y}}(e) \cap \overline{\mathscr{S}}^{U_{2_Y}}(e)$.
Hence, $\overline{\mathscr{S}}^{U_{1_Y}U_{2_Y}}(e) \subseteq \overline{\mathscr{S}}^{U_{1_Y}}(e) \cap \overline{\mathscr{S}}^{U_{2_Y}}(e)$.
Also, $\overline{\mathscr{S}}^{U_{1_N}}(e) \cap \overline{\mathscr{S}}^{U_{2_N}}(e) \subseteq \overline{\mathscr{S}}^{U_{1_N}\cap U_{2_N}}(e) \subseteq \overline{\mathscr{S}}^{U_{1_N}U_{2_N}}(e)$.
Hence, $\overline{\mathscr{S}}^{U_{1_N}U_{2_N}}(e) \supseteq \overline{\mathscr{S}}^{U_{1_N}}(e) \cap \overline{\mathscr{S}}^{U_{2_N}}(e)$. □

Theorem 6.1.13. *Suppose* $(\mathscr{S}, D)$ *is a* S_fB_nR *from a* S_mG M_1 *to a* S_mG M_2*; that is,* $\mathscr{S} : D \to P(M_1 \times M_2)$*. Then, for a* ${}^qROF_zR_iI_d$ $U_1 = \langle U_{1_Y}, U_{1_N}\rangle$ *and for a* ${}^qROF_zL_fI_d$ $U_2 = \langle U_{2_Y}, U_{2_N}\rangle$ *of* M_1*,* ${}^{U_1U_2}\overline{\mathscr{S}} \subseteq {}^{U_1}\overline{\mathscr{S}} \cap {}^{U_2}\overline{\mathscr{S}}$*.*

Proof.

The proof can be derived using the same approach as in the Theorem 6.1.12. □

Theorem 6.1.14. *Let* $(\mathscr{S}, D)$ *be a* $S_fC_mR_l$ *from a* S_mG M_1 *to a* S_mG M_2*. If* U_2 *is a* ${}^qROF_zI_tI$ *of* M_2*, then* $(\overline{\mathscr{S}}^{U_2}, D)$ *is a* ${}^qROF_zS_fI_tI$ *of* M_1*.*

Proof.

Suppose that U_2 is a ${}^qROF_zI_tI$ of M_2. Thus, U_2 is a ${}^qROF_zS_bS_mG$ of M_2, so according to the Theorem 6.1.5, $(\overline{\mathscr{S}}^{U_2}, D)$ is a ${}^qROF_zS_fS_bS_mG$ of M_1. Now for $a, b, c \in M_1$,

$$\begin{aligned}\overline{\mathscr{S}}^{U_{2_Y}}(e)(c) &= \vee_{n\in c\mathscr{S}(e)}U_{2_Y}(n) \leq \vee_{m\in a\mathscr{S}(e)} \vee_{n\in c\mathscr{S}(e)} \vee_{o\in b\mathscr{S}(e)}U_{2_Y}(mno)\\ &\leq \vee_{(mno)\in(acb)\mathscr{S}(e)}U_{2_Y}(mno) = m' \in acb\mathscr{S}(e)U_{2_Y}(m')\\ &= \overline{\mathscr{S}}^{U_{2_Y}}(e)(acb).\end{aligned}$$

Similarly, for $a, b, c \in M_1$,

$$\begin{aligned}\overline{\mathscr{S}}^{U_{2_N}}(e)(c) &= \wedge_{n\in c\mathscr{S}(e)}U_{2_N}(n) \geq \wedge_{m\in a\mathscr{S}(e)} \wedge_{n\in c\mathscr{S}(e)} \wedge_{o\in b\mathscr{S}(e)}U_{2_N}(mno)\\ &\geq \wedge_{(mno)\in(acb)\mathscr{S}(e)}U_{2_Y}(mno) = m' \in acb\mathscr{S}(e)U_{2_N}(m')\\ &= \overline{\mathscr{S}}^{U_{2_N}}(e)(acb).\end{aligned}$$

Hence, $\overline{\mathscr{S}}^{U_2}(e)$ is a ${}^qROF_zI_tI$ of M_1 for all $e \in D$, so $(\overline{\mathscr{S}}^{U_2}, D)$ is a ${}^qROF_zS_fI_tI$ of M_1. □

In the following example, we show that the converse of above theorem does not hold, in general.

Example 6.1.15. *Consider two* S_mG*s, denoted by* $M_1 = \{f_1, f_2, f_3\}$ *and* $M_2 = \{k_1, k_2, k_3\}$*, with their multiplication operations specified as follows.*

Table 6.15: Multiplication table for M_1

.	$\mathfrak{t}_1$	$\mathfrak{t}_2$	$\mathfrak{t}_3$
$\mathfrak{t}_1$	$\mathfrak{t}_1$	$\mathfrak{t}_2$	$\mathfrak{t}_3$
$\mathfrak{t}_2$	$\mathfrak{t}_1$	$\mathfrak{t}_2$	$\mathfrak{t}_3$
$\mathfrak{t}_3$	$\mathfrak{t}_1$	$\mathfrak{t}_2$	$\mathfrak{t}_3$

Table 6.16: Multiplication table for M_2

.	$\mathfrak{k}_1$	$\mathfrak{k}_2$	$\mathfrak{k}_3$
$\mathfrak{k}_1$	$\mathfrak{k}_1$	$\mathfrak{k}_1$	$\mathfrak{k}_3$
$\mathfrak{k}_2$	$\mathfrak{k}_1$	$\mathfrak{k}_2$	$\mathfrak{k}_3$
$\mathfrak{k}_3$	$\mathfrak{k}_1$	$\mathfrak{k}_3$	$\mathfrak{k}_3$

Let $D = \{e_1, e_2\}$: Define a $S_f B_n R$ $\mathscr{S} : D \to P(M_1 \times M_2)$ by:

$$\mathscr{S}(e_1) = \begin{array}{c} \\ \mathfrak{t}_1 \\ \mathfrak{t}_2 \\ \mathfrak{t}_3 \end{array}\!\!\begin{array}{c} \begin{array}{ccc} \mathfrak{k}_1 & \mathfrak{k}_2 & \mathfrak{k}_3 \end{array} \\ \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \end{array}, \quad \mathscr{S}(e_2) = \begin{array}{c} \\ \mathfrak{t}_1 \\ \mathfrak{t}_2 \\ \mathfrak{t}_3 \end{array}\!\!\begin{array}{c} \begin{array}{ccc} \mathfrak{k}_1 & \mathfrak{k}_2 & \mathfrak{k}_3 \end{array} \\ \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \end{array}$$

Then, $(\mathscr{S}, D)$ forms a $S_f C_{mp} R_l$ from M_1 to M_2.

$$\mathfrak{t}_1\mathscr{S}(e_1) = \{\mathfrak{k}_1, \mathfrak{k}_2, \mathfrak{k}_3\},$$
$$\mathfrak{t}_2\mathscr{S}(e_1) = \{\mathfrak{k}_1, \mathfrak{k}_2\},$$
$$\mathfrak{t}_3\mathscr{S}(e_1) = \{\mathfrak{k}_1, \mathfrak{k}_3\},$$
$$\mathfrak{t}_1\mathscr{S}(e_2) = \{\mathfrak{k}_1, \mathfrak{k}_2, \mathfrak{k}_3\},$$
$$\mathfrak{t}_2\mathscr{S}(e_2) = \{\mathfrak{k}_1, \mathfrak{k}_2\},$$
$$\mathfrak{t}_3\mathscr{S}(e_2) = \{\mathfrak{k}_3\}.$$

Define a $^qROF_zS$ $U_2 : M_2 \to [0, 1]$ by
$U_2 = \{\langle \mathfrak{k}_1, 0.3, 0.8\rangle, \langle \mathfrak{k}_2, 0.5, 0.6\rangle, \langle \mathfrak{k}_3, 0.6, 0.7\rangle\}$. *Then, U_2 is not a $^qROF_zI_tI_d$ of M_2 because if*

we take $a = \Bbbk_2, c = \Bbbk_3, b = \Bbbk_1$, *then* $U_{2_Y}(\Bbbk_2\Bbbk_3\Bbbk_1) = 0.3 \ngeq 0.6 = U_{2_Y}(\Bbbk_3)$ *and* $U_{2_N}(\Bbbk_2\Bbbk_3\Bbbk_1) = 0.8 \nleq 0.7 = U_{2_N}(\Bbbk_3)$. U_pA_p *of* U_2 *is given in Table 6.17.*

Table 6.17: U_pA_p of U_2

	$\overline{\mathscr{S}}^{U_{2_Y}}(e_1)$	$\overline{\mathscr{S}}^{U_{2_N}}(e_1)$	$\overline{\mathscr{S}}^{U_{2_Y}}(e_2)$	$\overline{\mathscr{S}}^{U_{2_N}}(e_2)$
$\mathfrak{t}_1$	0.6	0.6	0.6	0.6
$\mathfrak{t}_2$	0.5	0.6	0.5	0.6
$\mathfrak{t}_3$	0.6	0.7	0.6	0.7

Clearly, $\overline{\mathscr{S}}(e_1)^{U_2}$ *and* $\overline{\mathscr{S}}^{U_2}(e_2)$ *are* ${}^qROF_zI_tI_ds$ *of* M_1, *so* $(\overline{\mathscr{S}}^{U_2}, D)$ *is a* ${}^qROF_zS_fI_tI_d$ *of* M_1.

Theorem 6.1.16. *Let* $(\mathscr{S}, D)$ *be a* $S_fC_mR_l$ *from a* S_mG *of* M_1 *to a* S_mG M_2. *If* U_1 *is a* ${}^qRO_zI_tI$ *of* M_1, *then* $({}^{U_1}\overline{\mathscr{S}}, D)$ *is a* ${}^qRO_zS_fI_tI$ *of* M_2.

Proof.
The proof can be derived using the same approach as in the Theorem 6.1.14. □

In the following example, we show that the converse of above theorem is not true, in general.

Example 6.1.17. *Consider the* S_mGs *of Example 6.1.15 and let* $D = \{e_1, e_2\}$. *Define a* S_fB_nR $\mathscr{S}_1 : D \to P(M_1 \times M_2)$ *by:*

$$\mathscr{S}_1(e_1) = \begin{array}{c} \\ \mathfrak{t}_1 \\ \mathfrak{t}_2 \\ \mathfrak{t}_3 \end{array}\begin{array}{c} \begin{array}{ccc} \Bbbk_1 & \Bbbk_2 & \Bbbk_3 \end{array} \\ \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \end{array}, \quad \mathscr{S}_1(e_2) = \begin{array}{c} \\ \mathfrak{t}_1 \\ \mathfrak{t}_2 \\ \mathfrak{t}_3 \end{array}\begin{array}{c} \begin{array}{ccc} \Bbbk_1 & \Bbbk_2 & \Bbbk_3 \end{array} \\ \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 0 \end{pmatrix} \end{array}$$

Then, $(\mathscr{S}_1, D)$ forms a $S_fC_mR_l$ from M_1 to M_2.

$$\mathscr{S}_1(e_1)\Bbbk_1 = \{t_1, t_2, t_3\},$$
$$\mathscr{S}_1(e_1)\Bbbk_2 = \{t_1, t_2\},$$
$$\mathscr{S}_1(e_1)\Bbbk_3 = \{t_1, t_3\},$$
$$\mathscr{S}_1(e_2)\Bbbk_1 = \{t_1, t_2\},$$
$$\mathscr{S}_1(e_2)\Bbbk_2 = \{t_2\},$$
$$\mathscr{S}_1(e_2)\Bbbk_3 = \{t_2, t_3\}.$$

Define a ${}^qROF_zS$ $U_1 : M_1 \to [0,1]$ by
$U_1 = \{\langle t_1, 0.4, 0.9\rangle, \langle t_2, 0.5, 0.7\rangle, \langle t_3, 0.6, 0.8\rangle\}$. *Then, U_1 is not a ${}^qROF_zI_tI_d$ of M_2 because if we take $a = t_2, c = t_3, b = t_1$, then $U_{1_Y}(t_2t_3t_1) = 0.4 \not\geq 0.6 = U_{1_Y}(t_3)$ and $U_{1_N}(t_2t_3t_1) = 0.9 \not\leq 0.8 = U_{1_N}(t_3)$. U_pA_p of U_1 is given in Table 6.18.*

Table 6.18: U_pA_p of U_1

	${}^{U_{1_Y}}\overline{\mathscr{S}_1}(e_1)$	${}^{U_{1_N}}\overline{\mathscr{S}_1}(e_1)$	${}^{U_{1_Y}}\overline{\mathscr{S}_1}(e_2)$	${}^{U_{1_N}}\overline{\mathscr{S}_1}(e_2)$
$\Bbbk_1$	0.6	0.7	0.5	0.7
$\Bbbk_2$	0.5	0.7	0.5	0.7
$\Bbbk_3$	0.6	0.8	0.5	0.7

Clearly, ${}^{U_1}\overline{\mathscr{S}_1}(e_1)$ and ${}^{U_1}\overline{\mathscr{S}_1}(e_2)$ are ${}^qROF_zI_tI_d$s of M_2, so $({}^{U_1}\overline{\mathscr{S}_1}, D)$ is a ${}^qROF_zS_fI_tI_d$ of M_2.

Now, let's discuss some results regarding the L_oA_ps of a ${}^qROF_zI_tI_d$ of a S_mG.

Theorem 6.1.18. *Suppose $(\mathscr{S}, D)$ constitutes a $S_fC_{mp}R_l$ from a S_mG M_1 to a S_mG M_2. If U_2 is a ${}^qROF_zI_tI$ of M_2, then $(\underline{\mathscr{S}}^{U_2}, D)$ is a ${}^qROF_zS_fI_tI$ of M_1.*

Proof.
Suppose that U_2 is a ${}^qROF_zI_tI$ of M_2, Thus, U_2 is a ${}^qROF_zS_bS_mG$ of M_2, so according to the

Theorem 6.1.6, $(\underline{\mathscr{S}}^{U_2}, D)$ is a ${}^qROF_zS_fS_bS_mG$ of M_1. Now for $a, c, b \in M_1$,

$$\begin{aligned}
\underline{\mathscr{S}}^{U_{2_Y}}(acb) &= \wedge_{m' \in (acb)\mathscr{S}(e)} U_{2_Y}\left(m'\right) = \wedge_{m' \in a\mathscr{S}(e).c\mathscr{S}(e).b\mathscr{S}(e)} U_{2_Y}\left(m'\right) \\
&= \wedge_{m \in a\mathscr{S}(e).n \in c\mathscr{S}(e).o \in b\mathscr{S}(e)} U_{2_Y}\left(mno\right) \geq \wedge_{n \in c\mathscr{S}(e)} U_{2_Y}\left(n\right) \\
&= \underline{\mathscr{S}}^{U_{2_Y}}(e)(m).
\end{aligned}$$

Similarly, for $a, b, c \in M_1$,

$$\begin{aligned}
\underline{\mathscr{S}}^{U_{2_N}}(acb) &= \vee_{m' \in (acb)\mathscr{S}(e)} U_{2_N}\left(m'\right) = \vee_{m' \in a\mathscr{S}(e).b\mathscr{S}(e).c\mathscr{S}(e)} U_{2_N}\left(m'\right) \\
&= \vee_{m \in a\mathscr{S}(e).n \in c\mathscr{S}(e).c \in b\mathscr{S}(e)} U_{2_N}\left(mno\right) \leq \vee_{n \in c\mathscr{S}(e)} U_{2_N}\left(n\right) \\
&= \underline{\mathscr{S}}^{U_{2_N}}(e)(c).
\end{aligned}$$

Hence, $\underline{\mathscr{S}}^{U_2}(e)$ is a ${}^qROF_zI_tI$ of M_1 for all $e \in D$, so $(\underline{\mathscr{S}}^{U_2}, D)$ is a ${}^qROF_zS_fI_tI$ of M_1. □

In the Example 6.1.19, we show that the converse of above theorem which is not true.

Example 6.1.19. *Consider the S_mGs of Example 6.1.11.*
Let $D = \{e_1, e_2\}$: Define a S_fB_nR $\mathscr{S} : D \to P(M_1 \times M_2)$ by:

$$\mathscr{S}(e_1) = \begin{array}{c} \\ \mathscr{k}_1 \\ \mathscr{k}_2 \\ \mathscr{k}_3 \\ \mathscr{k}_4 \end{array} \begin{array}{c} \begin{array}{cccc} \mathscr{t}_1 & \mathscr{t}_2 & \mathscr{t}_3 & \mathscr{t}_4 \end{array} \\ \begin{pmatrix} 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \end{pmatrix} \end{array}, \quad \mathscr{S}(e_2) = \begin{array}{c} \\ \mathscr{k}_1 \\ \mathscr{k}_2 \\ \mathscr{k}_3 \\ \mathscr{k}_4 \end{array} \begin{array}{c} \begin{array}{cccc} \mathscr{t}_1 & \mathscr{t}_2 & \mathscr{t}_3 & \mathscr{t}_4 \end{array} \\ \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \end{array}$$

Then, $(\mathscr{S}, D)$ forms a $S_f C_{mp} R_l$ from M_1 to M_2.

$$\mathfrak{k}_1 \mathscr{S}(e_1) = \{\mathfrak{t}_2, \mathfrak{t}_3\},$$
$$\mathfrak{k}_2 \mathscr{S}(e_1) = \{\mathfrak{t}_2, \mathfrak{t}_3\},$$
$$\mathfrak{k}_3 \mathscr{S}(e_1) = \{\mathfrak{t}_2, \mathfrak{t}_3\},$$
$$\mathfrak{k}_4 \mathscr{S}(e_1) = \{\mathfrak{t}_2, \mathfrak{t}_3\},$$
$$\mathfrak{k}_1 \mathscr{S}(e_2) = \{\mathfrak{t}_2\},$$
$$\mathfrak{k}_2 \mathscr{S}(e_2) = \{\mathfrak{t}_2\},$$
$$\mathfrak{k}_3 \mathscr{S}(e_2) = \{\mathfrak{t}_2\},$$
$$\mathfrak{k}_4 \mathscr{S}(e_2) = \{\mathfrak{t}_2\}.$$

Define a ${}^qROF_zS$ $U_2 : M_2 \to [0,1]$ by
$U_2 = \{\langle \mathfrak{t}_1, 0.4, 0.8\rangle, \langle \mathfrak{t}_2, 0.6, 0.7\rangle, \langle \mathfrak{t}_3, 0.8, 0.3\rangle, \langle \mathfrak{t}_4, 0.5, 0.6\rangle\}$. *Then, U_2 is not a ${}^qROF_zI_tI_d$ of M_2 because if we take $a = \mathfrak{t}_2, c = \mathfrak{t}_3, b = \mathfrak{t}_4$, then $U_{2_Y}(\mathfrak{t}_2\mathfrak{t}_3\mathfrak{t}_4) = 0.6 \not\geq 0.8 = U_{2_Y}(\mathfrak{t}_3)$ and $U_{2_N}(\mathfrak{t}_2\mathfrak{t}_3\mathfrak{t}_4) = 0.7 \not\leq 0.3 = U_{2_N}(\mathfrak{t}_3)$. L_oA_p of U_2 is given in Table 6.19.*

Table 6.19: L_oA_p of U_2

	$\underline{\mathscr{S}}^{U_{2_Y}}(e_1)$	$\underline{\mathscr{S}}^{U_{2_N}}(e_1)$	$\underline{\mathscr{S}}^{U_{2_Y}}(e_2)$	$\underline{\mathscr{S}}^{U_{2_N}}(e_2)$
$\mathfrak{k}_1$	0.6	0.7	0.6	0.7
$\mathfrak{k}_2$	0.6	0.7	0.6	0.7
$\mathfrak{k}_3$	0.6	0.7	0.6	0.7
$\mathfrak{k}_4$	0.6	0.7	0.6	0.7

Clearly, $\underline{\mathscr{S}}^{U_2}(e_1)$ and $\underline{\mathscr{S}}^{U_2}(e_2)$ are ${}^qROF_zI_tI_d$s of M_1, so $(\underline{\mathscr{S}}^{U_2}, D)$ is a ${}^qROF_zS_fI_tI_d$ of M_1.

Theorem 6.1.20. *Suppose $(\mathscr{S}, D)$ is a $S_f C_{mp} R_l$ from a S_mG M_1 to a S_mG M_2. If U_1 is a ${}^qROF_zI_tI_d$ of M_1, then $({}^{U_1}\underline{\mathscr{S}}, D)$ is a ${}^qROF_zS_fI_tI_d$ of M_2.*

Proof.

The proof can be derived using the same approach as in the Theorem 6.1.18. □

Example 6.1.21. *Consider the S_mGs from the Example 6.1.11.*

Define a S_fB_nR $\mathscr{S} : D \rightarrow P(M_1 \times M_2)$ by:

$$\mathscr{S}(e_1) = \begin{array}{c} \\ k_1 \\ k_2 \\ k_3 \\ k_4 \end{array}\begin{array}{c} \begin{array}{cccc} t_1 & t_2 & t_3 & t_4 \end{array} \\ \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \end{pmatrix} \end{array}, \quad \mathscr{S}(e_2) = \begin{array}{c} \\ k_1 \\ k_2 \\ k_3 \\ k_4 \end{array}\begin{array}{c} \begin{array}{cccc} t_1 & t_2 & t_3 & t_4 \end{array} \\ \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \end{pmatrix} \end{array}$$

Then, $(\mathscr{S}, D)$ forms a $S_fC_{mp}R_l$ from M_1 to M_2.

$$\mathscr{S}(e_1)t_1 = \{k_4\},$$

$$\mathscr{S}(e_1)t_2 = \{k_4\},$$

$$\mathscr{S}(e_1)t_3 = \{k_4\},$$

$$\mathscr{S}(e_1)t_4 = \{k_4\},$$

$$\mathscr{S}(e_2)t_1 = \{k_1, k_4\},$$

$$\mathscr{S}(e_2)t_2 = \{k_1, k_4\},$$

$$\mathscr{S}(e_2)t_3 = \{k_1, k_4\},$$

$$\mathscr{S}(e_2)t_4 = \{k_1, k_4\}.$$

Define a $^qROF_zS$ $U_1 : M_1 \rightarrow [0, 1]$ by

$U_1 = \{\langle k_1, 0.4, 0.9\rangle, \langle k_2, 0.8, 0.7\rangle, \langle k_3, 0.5, 0.4\rangle, \langle k_4, 0.7, 0.6\rangle\}$. *Then, U_1 is not a $^qROF_zI_tI_d$ of M_1 because if we take $a = k_3, c = k_2, b = k_1$, then $U_{1_Y}(k_3k_2k_1) = 0.4 \ngeq 0.8 = U_{1_Y}(k_2)$ and $U_{1_N}(k_3k_2k_1) = 0.9 \nleq 0.7 = U_{1_N}(k_2)$. L_oA_p of U_1 is given in Table 6.20.*

Table 6.20: L_oA_p of U_1

	${}^{U_{1_Y}}\underline{\mathscr{L}}(e_1)$	${}^{U_{1_N}}\underline{\mathscr{L}}(e_1)$	${}^{U_{1_Y}}\underline{\mathscr{L}}(e_2)$	${}^{U_{1_N}}\underline{\mathscr{L}}(e_2)$
$\mathfrak{t}_1$	0.7	0.6	0.4	0.9
$\mathfrak{t}_2$	0.7	0.6	0.4	0.9
$\mathfrak{t}_3$	0.7	0.6	0.4	0.9
$\mathfrak{t}_4$	0.7	0.6	0.4	0.9

Clearly, ${}^{U_1}\underline{\mathscr{L}}(e_1)$ *and* ${}^{U_1}\underline{\mathscr{L}}(e_2)$ *are* ${}^{q}ROF_zI_tI_ds$ *of* M_2*, so* $({}^{U_1}\underline{\mathscr{L}}, D)$ *is a* ${}^{q}ROF_zS_fI_tI_d$ *of* M_2*.*

Theorem 6.1.22. *Suppose* $(\mathscr{S}, D)$ *constitutes a* $S_fC_mR_l$ *from a* S_mG M_1 *to a* S_mG M_2*. If* U_2 *is a* ${}^{q}ROF_zB_iI_d$ *of* M_2 *then* $(\overline{\mathscr{S}}^{U_2}, D)$ *is a* ${}^{q}ROF_zS_fB_iI_d$ *of* M_1*.*

Proof.

Suppose that U_2 is a ${}^{q}ROF_zB_iI_d$ of M_2. Thus, U_2 is a ${}^{q}ROF_zS_bS_mG$ of M_2, so according to the Theorem 6.1.5, $(\overline{\mathscr{S}}^{U_2}, D)$ is a ${}^{q}ROF_zS_fS_bS_mG$ of M_1. Now for $a, b, c \in M_1$,

$$\begin{aligned}
\overline{\mathscr{S}}^{U_{2_Y}}(e)(a) \wedge \overline{\mathscr{S}}^{U_{2_Y}}(e)(b) &= \Big(\vee_{m \in a\mathscr{S}(e)} U_{2_Y}(m) \Big) \wedge \Big(\vee_{c \in b\mathscr{S}(e)} U_{2_Y}(o) \Big) \\
&= \vee_{m \in a\mathscr{S}(e)} \vee_{o \in b\mathscr{S}(e)} \Big(U_{2_Y}(m) \wedge U_{2_Y}(o) \Big) \\
&\leq \vee_{m \in a\mathscr{S}(e)} \vee_{n \in c\mathscr{S}(e)} \vee_{o \in b\mathscr{S}(e)} \Big(U_{2_Y}(mno) \Big) \\
&\leq \vee_{mno \in (acb)\mathscr{S}(e)} \Big(U_{1_Y}(mno) \Big) \\
&= \vee_{m' \in (acb)\mathscr{S}(e)} \Big(U_{2_Y}(m') \Big) \\
&= \overline{\mathscr{S}}^{U_{2_Y}}(e)(acb).
\end{aligned}$$

Similarly for $a, b, c \in M_1$,

$$\begin{aligned}
\overline{\mathscr{S}}^{U_{2_N}}(e)(a) \vee \overline{\mathscr{S}}^{U_{2_N}}(e)(b) &= \Big(\wedge_{m \in a\mathscr{S}(e)} U_{2_N}(m) \Big) \vee \Big(\wedge_{o \in b\mathscr{S}(e)} U_{2_N}(o) \Big) \\
&= \wedge_{m \in a\mathscr{S}(e)} \wedge_{o \in b\mathscr{S}(e)} \Big(U_{2_N}(m) \vee U_{2_N}(o) \Big) \\
&\geq \wedge_{m \in \mathscr{S}(e)} \wedge_{n \in c\mathscr{S}(e)} \wedge_{o \in a\mathscr{S}(e)} \Big(U_{2_N}(mno) \Big) \\
&\geq \wedge_{mno \in (acb)\mathscr{S}(e)} \Big(U_{2_N}(mno) \Big) \\
&= \wedge_{m' \in (acb)\mathscr{S}(e)} \Big(U_{2_N}(m') \Big) \\
&= \overline{\mathscr{S}}^{U_{2_N}}(e)(acb).
\end{aligned}$$

Hence, $\overline{\mathscr{S}}^{U_2}(e)$ is a ${}^qROF_zB_iI_d$ of M_1 for all $e \in D$, $(\overline{\mathscr{S}}^{U_2}, D)$ is a ${}^qROF_zS_fB_iI_d$ of M_1. □

Theorem 6.1.23. *Suppose $(\mathscr{S}, D)$ is a $S_fC_mR_l$ from a S_mG M_1 to a S_mG M_2. If U_1 is a ${}^qROF_zB_iI_d$ of M_1 then $({}^{U_1}\overline{\mathscr{S}}, D)$ is a ${}^qROF_zS_fB_iI_d$ of M_2.*

Proof.
The proof can be derived using the same approach as in the Theorem 6.1.22. □

Example 6.1.24. *Consider the S_mGs and soft relations illustrated in Example 6.1.7. Define a ${}^qROF_zS$ $U_2 : M_2 \to [0,1]$ by $U_2 = \{\langle t_1, 0.8, 0.5\rangle, \langle t_2, 0.6, 0.7\rangle, \langle t_3, 0.7, 0.3\rangle, \langle t_4, 0.9, 0.1\rangle, \langle t_5, 0.4, 0.8\rangle\}$. Then, U_2 is not a ${}^qROF_zB_iI_d$ of M_2 because if we take $a = t_1, c = t_3, b = t_2$, then $U_{2_Y}(t_1t_3t_2) = 0.4 \ngeq 0.6 = U_{2_Y}(t_1) \wedge U_{2_Y}(t_2)$ and $U_{2_N}(t_1t_3t_2) = 0.8 \nleq 0.7 = U_{2_N}(t_1) \vee U_{2_N}(t_2)$. U_pA_p of U_2 is given in Table 6.21.*

Table 6.21: U_pA_p of U_2

	$\overline{\mathscr{S}}^{U_{2_Y}}(e_1)$	$\overline{\mathscr{S}}^{U_{2_N}}(e_1)$	$\overline{\mathscr{S}}^{U_{2_Y}}(e_2)$	$\overline{\mathscr{S}}^{U_{2_N}}(e_2)$
$\Bbbk_1$	0.8	0.5	0.8	0.9
$\Bbbk_2$	0.8	0.5	0.8	0.3
$\Bbbk_3$	0.7	0.3	0.7	0.3
$\Bbbk_4$	0.9	0.1	0.9	0.1
$\Bbbk_5$	0.4	0.8	0.4	0.8

Clearly, $\overline{\mathscr{S}}^{U_2}(e_1)$ *and* $\overline{\mathscr{S}}^{U_2}(e_2)$ *are* ${}^qROF_zB_iI_ds$ *of* M_1*, so* $(\overline{\mathscr{S}}^{U_2}, D)$ *is a* ${}^qROF_zS_fB_iI_d$ *of* M_1*.*
Define a ${}^qROF_zS$ $U_1 : M_1 \to [0,1]$ *by*
$U_1 = \{\langle \Bbbk_1, 0.8, 0.5\rangle, \langle \Bbbk_2, 0.5, 0.4\rangle, \langle \Bbbk_3, 0.9, 0.3\rangle, \langle \Bbbk_4, 0.7, 0.6\rangle, \langle \Bbbk_5, 0.4, 0.7\rangle\}$*. Then,* U_1 *is not a* ${}^qROF_zB_iI_d$ *of* M_1 *because if we take* $a = \Bbbk_1, c = \Bbbk_2, b = \Bbbk_3$ *then* $U_{1_Y}(\Bbbk_1\Bbbk_2\Bbbk_3) = 0.7 \ngeq 0.8 = U_{1_Y}(\Bbbk_1) \wedge U_{1_Y}(\Bbbk_3)$ *and* $U_{1_N}(\Bbbk_1\Bbbk_2\Bbbk_3) = 0.6 \nleq 0.5 = U_{1_N}(\Bbbk_1) \vee U_{1_N}(\Bbbk_3)$*.* U_pA_p *of* U_1 *is given in Table 6.22.*

Table 6.22: U_pA_p of U_1

	${}^{U_{1_Y}}\overline{\mathscr{S}}(e_1)$	${}^{U_{1_N}}\overline{\mathscr{S}}(e_1)$	${}^{U_{1_Y}}\overline{\mathscr{S}}(e_2)$	${}^{U_{1_N}}\overline{\mathscr{S}}(e_2)$
$\mathfrak{t}_1$	0.8	0.4	0.8	0.4
$\mathfrak{t}_2$	0.5	0.4	0.5	0.4
$\mathfrak{t}_3$	0.9	0.3	0.9	0.3
$\mathfrak{t}_4$	0.7	0.6	0.7	0.6
$\mathfrak{t}_5$	0.9	0.3	0.9	0.3

Clearly, ${}^{U_1}\overline{\mathscr{S}}(e_1)$ *and* ${}^{U_1}\overline{\mathscr{S}}(e_2)$ *are* ${}^qROF_zB_iI_ds$ *of* M_2*, so* $({}^{U_1}\overline{\mathscr{S}}, D)$ *is a* ${}^qROF_zS_fB_iI_d$ *of* M_2*.*

Theorem 6.1.25. *Let* $(\mathscr{S}, D)$ *be a* $S_fC_{mp}R_l$ *from a* S_mG M_1 *to a* S_mG M_2*. If* U_2 *is a* ${}^qROF_zB_iI_d$ *of* M_2*, then* $(\underline{\mathscr{S}}^{U_2}, D)$ *is a* ${}^qROF_zS_fB_iI_d$ *of* M_1*.*

Proof.

Suppose that U_2 is a ${}^{q}ROF_zB_iI_d$ of M_2. Thus, U_2 is a ${}^{q}ROF_zS_bS_mG$ of M_2, so according to the Theorem 6.1.6, $(\underline{\mathscr{L}}^{U_2}, D)$ is a ${}^{q}ROF_zS_fS_bS_mG$ of M_1. Now for $a, b, c \in M_1$,

$$\begin{aligned}
\underline{\mathscr{L}}^{U_{2_Y}}(acb) &= \wedge_{m' \in (acb)\mathscr{S}(e)} U_{2_Y}\big(m'\big) = \wedge_{m' \in a\mathscr{S}(e).c\mathscr{S}(e).b\mathscr{S}(e)} U_{2_Y}\big(m'\big) \\
&= \wedge_{m \in a\mathscr{S}(e).o \in c\mathscr{S}(e).n \in b\mathscr{S}(e)} U_{2_Y}\big(mon\big) \\
&\geq \Big(\wedge_{m \in a\mathscr{S}(e)} U_{2_Y}(m)\Big) \wedge \Big(\wedge_{n \in b\mathscr{S}(e)} U_{1_Y}(n)\Big) \\
&= \Big(\underline{\mathscr{L}}^{U_{2_Y}}(e)(a)\Big) \wedge \Big(\underline{\mathscr{L}}^{U_{2_Y}}(e)(b)\Big).
\end{aligned}$$

Similarly, for $a, b, c \in M_1$,

$$\begin{aligned}
\underline{\mathscr{L}}^{U_{2_N}}(acb) &= \vee_{m' \in (acb)\mathscr{S}(e)} U_{2_N}\big(m'\big) = \vee_{m' \in a\mathscr{S}(e).c\mathscr{S}(e).b\mathscr{S}(e)} U_{2_N}\big(m'\big) \\
&= \vee_{m \in a\mathscr{S}(e).o \in c\mathscr{S}(e).n \in a\mathscr{S}(e)} U_{2_N}\big(mon\big) \\
&\leq \Big(\vee_{m \in a\mathscr{S}(e)} U_{2_N}(m)\Big) \vee \Big(\vee_{n \in b\mathscr{S}(e)} U_{2_N}(n)\Big) \\
&= \Big(\underline{\mathscr{L}}^{U_{2_N}}(e)(a)\Big) \vee \Big(\underline{\mathscr{L}}^{U_{2_N}}(e)(b)\Big).
\end{aligned}$$

Hence, $\underline{\mathscr{L}}^{U_2}(e)$ is a ${}^{q}ROF_zB_iI_d$ of M_1 for all $e \in D$, so $(\underline{\mathscr{L}}^{U_2}, D)$ is a ${}^{q}ROF_zS_fB_iI_d$ of M_1. $\square$

Example 6.1.26. *Consider the S_mGs of Example 6.1.11. Define a S_fB_nR $\mathscr{S} : D \to P(M_1 \times M_2)$ by:*

$$\mathscr{S}(e_1) = \begin{array}{c} \\ \Bbbk_1 \\ \Bbbk_2 \\ \Bbbk_3 \\ \Bbbk_4 \end{array} \begin{array}{c} \begin{array}{cccc} t_1 & t_2 & t_3 & t_4 \end{array} \\ \begin{pmatrix} 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \end{pmatrix} \end{array}, \quad \mathscr{S}(e_2) = \begin{array}{c} \\ \Bbbk_1 \\ \Bbbk_2 \\ \Bbbk_3 \\ \Bbbk_4 \end{array} \begin{array}{c} \begin{array}{cccc} t_1 & t_2 & t_3 & t_4 \end{array} \\ \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \end{array}$$

Then, $(\mathscr{S}, D)$ *forms a* $S_fC_{mp}R_l$ *from* M_1 *to* M_2.

$$\Bbbk_1\mathscr{S}(e_1) = \{\text{ŧ}_2, \text{ŧ}_3\},$$
$$\Bbbk_2\mathscr{S}(e_1) = \{\text{ŧ}_2, \text{ŧ}_3\},$$
$$\Bbbk_3\mathscr{S}(e_1) = \{\text{ŧ}_2, \text{ŧ}_3\},$$
$$\Bbbk_4\mathscr{S}(e_1) = \{\text{ŧ}_2, \text{ŧ}_3\},$$
$$\Bbbk_1\mathscr{S}(e_2) = \{\text{ŧ}_2\},$$
$$\Bbbk_2\mathscr{S}(e_2) = \{\text{ŧ}_2\},$$
$$\Bbbk_3\mathscr{S}(e_2) = \{\text{ŧ}_2\},$$
$$\Bbbk_4\mathscr{S}(e_2) = \{\text{ŧ}_2\}.$$

Define a ${}^qROF_zS$ $U_2 : M_2 \to [0,1]$ *by*
$U_2 = \{\langle \text{ŧ}_1, 0.7, 0.5\rangle, \langle \text{ŧ}_2, 0.4, 0.8\rangle, \langle \text{ŧ}_3, 0.3, 0.9\rangle, \langle \text{ŧ}_4, 0.6, 0.7\rangle\}$. *Then,* U_2 *is not a* ${}^qROF_zB_iI_d$ *of* M_2 *because if we take* $a = \text{ŧ}_1, c = \text{ŧ}_4, b = \text{ŧ}_1$, *then* $U_{2_Y}(\text{ŧ}_1\text{ŧ}_4\text{ŧ}_1) = 0.6 \ngeq 0.7 = U_{2_Y}(\text{ŧ}_1) \wedge U_{2_Y}(\text{ŧ}_1)$ *and* $U_{2_N}(\text{ŧ}_1\text{ŧ}_4\text{ŧ}_1) = 0.7 \nleq 0.5 = U_{2_N}(\text{ŧ}_1) \vee U_{2_N}(\text{ŧ}_1)$. L_oA_p *of* U_2 *is given in Table 6.23.*

Table 6.23: L_oA_p of U_2

	$\underline{\mathscr{S}}^{U_{2_Y}}(e_1)$	$\underline{\mathscr{S}}^{U_{2_N}}(e_1)$	$\underline{\mathscr{S}}^{U_{2_Y}}(e_2)$	$\underline{\mathscr{S}}^{U_{2_N}}(e_2)$
$\Bbbk_1$	0.3	0.9	0.4	0.8
$\Bbbk_2$	0.3	0.9	0.4	0.8
$\Bbbk_3$	0.3	0.9	0.4	0.8
$\Bbbk_4$	0.3	0.9	0.4	0.8

Clearly, $\underline{\mathscr{S}}^{U_2}(e_1)$ *and* $\underline{\mathscr{S}}^{U_2}(e_2)$ *are* ${}^qROF_zB_iI_ds$ *of* M_1, *so* $(\underline{\mathscr{S}}^{U_2}, D)$ *is a* ${}^qROF_zS_fB_iI_d$ *of* M_1.

Theorem 6.1.27. *Let* $(\mathscr{S}, D)$ *be a* $S_fC_{mp}R_l$ *from a* S_mG M_1 *to a* S_mG M_2. *If* U_1 *is a* ${}^qROF_zB_iI_d$ *of* M_1 *then* $({}^{U_2}\overline{\mathscr{S}}, D)$ *is a* ${}^qROF_zS_fB_iI_d$ *of* M_2.

Proof.

The proof can be derived using the same approach as in the Theorem 6.1.25. □

Example 6.1.28. *Consider the S_mGs from the Example 6.1.11. Define a S_fB_nR $\mathscr{S} : D \rightarrow P(M_1 \times M_2)$ by:*

$$\mathscr{S}(e_1) = \begin{array}{c c} & \begin{array}{cccc} \mathfrak{t}_1 & \mathfrak{t}_2 & \mathfrak{t}_3 & \mathfrak{t}_4 \end{array} \\ \begin{array}{c} \Bbbk_1 \\ \Bbbk_2 \\ \Bbbk_3 \\ \Bbbk_4 \end{array} & \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \end{pmatrix} \end{array}, \quad \mathscr{S}(e_2) = \begin{array}{c c} & \begin{array}{cccc} \mathfrak{t}_1 & \mathfrak{t}_2 & \mathfrak{t}_3 & \mathfrak{t}_4 \end{array} \\ \begin{array}{c} \Bbbk_1 \\ \Bbbk_2 \\ \Bbbk_3 \\ \Bbbk_4 \end{array} & \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \end{pmatrix} \end{array}$$

Then, $(\mathscr{S}, D)$ forms a $S_fC_{mp}R_l$ from M_1 to M_2.

$$\begin{aligned} \mathscr{S}(e_1)\mathfrak{t}_1 &= \{\Bbbk_4\}, \\ \mathscr{S}(e_1)\mathfrak{t}_2 &= \{\Bbbk_4\}, \\ \mathscr{S}(e_1)\mathfrak{t}_3 &= \{\Bbbk_4\}, \\ \mathscr{S}(e_1)\mathfrak{t}_4 &= \{\Bbbk_4\}, \\ \mathscr{S}(e_2)\mathfrak{t}_1 &= \{\Bbbk_1, \Bbbk_4\}, \\ \mathscr{S}(e_2)\mathfrak{t}_2 &= \{\Bbbk_1, \Bbbk_4\}, \\ \mathscr{S}(e_2)\mathfrak{t}_3 &= \{\Bbbk_1, \Bbbk_4\}, \\ \mathscr{S}(e_2)\mathfrak{t}_4 &= \{\Bbbk_1, \Bbbk_4\}. \end{aligned}$$

Define a ${}^qROF_zS$ $U_1 : M_1 \rightarrow [0, 1]$ by
$U_1 = \{\langle \Bbbk_1, 0.5, 0.8\rangle, \langle \Bbbk_2, 0.8, 0.6\rangle, \langle \Bbbk_3, 0.6, 0.7\rangle, \langle \Bbbk_4, 0.7, 0.6\rangle\}$. *Then, U_1 is not a ${}^qROF_zB_iI_d$ of M_1 because if we take $a = \Bbbk_2, c = \Bbbk_1, b = \Bbbk_3$, then $U_{1_Y}(\Bbbk_2\Bbbk_1\Bbbk_3) = 0.5 \ngeq 0.6 = U_{1_Y}(\Bbbk_2) \wedge U_{1_Y}(\Bbbk_3)$ and $U_{1_N}(\Bbbk_2\Bbbk_1\Bbbk_3) = 0.8 \nleq 0.7 = U_{1_N}(\Bbbk_2\Bbbk_3) \wedge U_{1_N}(\Bbbk_3)$. L_oA_p of U_1 is given in Table 6.24.*

Table 6.24: L_oA_p of U_1

	${}^{U_{1_Y}}\underline{\mathscr{L}}(e_1)$	${}^{U_{1_N}}\underline{\mathscr{L}}(e_1)$	${}^{U_{1_Y}}\underline{\mathscr{L}}(e_2)$	${}^{U_{1_N}}\underline{\mathscr{L}}(e_2)$
$\mathfrak{t}_1$	0.7	0.6	0.5	0.8
$\mathfrak{t}_2$	0.7	0.6	0.5	0.8
$\mathfrak{t}_3$	0.7	0.6	0.5	0.8
$\mathfrak{t}_4$	0.7	0.6	0.5	0.8

Clearly, ${}^{U_1}\underline{\mathscr{L}}(e_1)$ and ${}^{U_1}\underline{\mathscr{L}}(e_2)$ are ${}^{q}ROF_zB_iI_d s$ of M_2, so $({}^{U_1}\underline{\mathscr{L}}, D)$ is a ${}^{q}ROF_zS_fB_iI_d$ of M_2.

This chapter developed a framework for approximating q-Rung Orthopair Fuzzy Ideals over dual universes using Soft Binary Relations. The methods and algorithms introduced here extend the theoretical understanding of Fuzzy Ideals, enabling a more flexible and nuanced analysis of uncertainty. Practical applications in decision-making scenarios validate the effectiveness of these approximations, paving the way for further exploration in both theory and practice.

Conclusion 6.2. *The culmination of this thesis marks a significant advancement in the understanding and application of information systems, S_mG theory, R_fSs, Fuzzy sets, and their extensions. Each chapter has contributed unique insights and theoretical frameworks that lay the groundwork for further exploration and practical implementation in various domains.*

Chapter 1 provided a comprehensive overview of foundational concepts, including Binary Relations, rough sets, S_mGs, Soft Sets, Fuzzy Soft Sets, Pythagorean Fuzzy sets, Pythagorean Fuzzy Soft Sets, Pythagorean Fuzzy Ideals, and q-Rung Orthopair Fuzzy sets. This literature review set the stage for subsequent chapters by establishing a solid theoretical foundation.

Building upon this foundation, Chapter 2 delved into the lower and upper approximations of Pythagorean Fuzzy sets, leveraging soft Binary Relations and exploring their properties. The proposed algorithm for decision-making problems using Pythagorean Fuzzy sets offers practical utility and opens avenues for further research in computational intelligence and decision *support systems.*

In Chapter 3, the exploration extended to q-Rung Orthopair Fuzzy Sets, where Crisp Binary Relations were employed to examine their lower and upper approximations. The introduction of q-Rung Orthopair Topological Spaces and similarity relations among q-Rung Orthopair Fuzzy Sets expands the theoretical framework and prompts further investigation into their applications in pattern recognition and classification.

Chapter 4 continued the investigation by exploring lower and upper approximations of qROF sets using soft Binary Relations. The elucidation of q-Rung Orthopair Fuzzy Topologies and similarity relations among q-Rung Orthopair Fuzzy sets contributes to the existing body of literature on Fuzzy Topology and enhances our understanding of Fuzzy set theory.

Navigating through rough approximations in Chapter 5 shed light on the analysis of S_bS_mGs *within a* S_mG*, employing soft compatible relations. The identification of upper and lower approximations for various types of* S_bS_mGs*, along with illustrative examples, offers valuable insights for researchers and practitioners in algebraic structures and mathematical modeling.*

Looking ahead, future research directions could focus on expanding the theoretical frameworks proposed in this thesis to address complex real-world problems in diverse domains. Incorporating machine learning techniques, such as deep learning and reinforcement learning, could enhance the performance of decision-making algorithms in uncertain environments. Additionally, exploring applications in fields such as data mining, image processing, and bioinformatics holds promise for advancing knowledge and technology.

In conclusion, the contributions of this thesis extend beyond the realms of theory, offering practical solutions and inspiring further inquiry. By bridging the gap between theory and practice, we pave the way for interdisciplinary collaboration and innovation, driving progress in the ever-evolving landscape of information systems and computational intelligence.

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