STUDY OF ENTANGLEMENT IN A MOVING FIVE-LEVEL ATOM

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Study of entanglement in a moving five level atom

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*Beginning with the name of **ALLAH ALMIGHTY**, the most beneficent and merciful, and the most sovereign among all of us.*

Read in the name of your Lord, who has created (all that exists)

Certificate

This is to certify that Mr. Muhammad Azeem has carried out research work for this dissertation under my supervision in Department of Physics, Quaid-i-Azam University, Islamabad.

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DEDICATED TO

DEAR

AMI, ABU

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Abstract

In this dissertation we study the dynamics of entanglement for a cascade type five level atom interacting with cavity field. We investigate time evolution of the von Neumann entropy and the atomic populations numerically on the degree of entanglement for the pure resonant case whether the atom is at rest or in a motion. We show that the influence of atomic motion plays an important role in the creation and destruction of entanglement and it is correlated with the atomic populations. We also determine the effect of decoherence on the system. Moreover we discuss these phenomena in terms of entanglement and disentanglement.

Contents

Chapter 1

Introduction

The branch of optics, in which quantum aspects of light and matter are studied is known as quantum optics. This field is used to investigate the phenomenon in which light interacts with matter by using semiclassical and quantum mechanical approach.

Light plays an important role to understand nature both classically and quantum mechanically. Since radiations are emitted and absorbed by atoms, the interaction of quantized electromagnetic radiation field with atoms acts as the most basic and central problem in quantum optics. In experiments the interaction between light and atoms provide an interesting window into the quantum world. One of the techniques, which is used in experiments is called laser spectroscopy.

Whereas, the systems are very complicated due to real atoms. Thus it is often mandatory to approximate the behavior of real atoms with simple quantum systems. For many purposes, the interaction of electromagnetic field plays a significant role with two atomic energy levels, so that in many theoretical analysis only two eigenstates of energy are required to represent an atom by quantum system.

Most of the existing atom-field interaction theories based on the models in which only a single two-level atom interacts with field. The Jaynes-Cummings model [1] is a particular model which shows how a single two-level atom interacts with a single-mode of quantized field.

In the present work we study the decoherence and entanglement via von Neumann entropy in a moving five-level atom interacting with cavity fields. We consider a closed system which consists of two sub-systems, i.e. five-level atom and single mode field. In this work we are

interested in the interaction of these two sub-systems, atom and field. Due to the interaction of these two sub-systems, the order of the whole system changes. Since entropy [2] is the measure of the disorder of a system, therefore we are more interested in time evolution of entropy. Thus we discuss the effect of interaction on entropy and have shown that how this entropy can tell us interesting things about the sub-system's dynamical behavior. Moreover, we investigate the populations of the atomic system which is under consideration. We compute both the von Neumann entropy and populations when initially the atom is in its excited state and field is in the coherent state. We also observe that the properties of the von Neumann entropy and the populations had an important effect due to atomic motion. Finally, we introduce the decoherence effect in all above cases and get some fascinating results.

In our work we used the density matrix approach [3] which provides computational convenience while stressing the statistical aspects of the problem. Since our treatment involves entire system, and we are interested only atomic part of the entire system, so field variables are eliminated by using reduced density matrix method.

In the present chapter we briefly describe the state of atom and field. We use the density matrix approach. The basic definition of entropy in quantum systems has also been introduced in this chapter.

In chapter two we introduce the Hamiltonian of the two-level atomic system. The Hamiltonian for interaction of the two-level atom with single-mode field is given in the dipole and rotating wave approximations in the interaction picture. Using this Hamiltonian, we solve the Jaynes Cummings model and find the unitary time evolution operator as well as the reduced density matrix operator for atom and field.

In chapter three we introduce the mathematical model of the system. As we are solving numerically the system so all the expressions given their are in general form.

In chapter four we show the time evolution of von Neumann entropy and the populations both for atom at rest and in motion. We also determine the effect of decoherence on the similar system. Results are plotted for a given system and at the end we discuss the results and give a conclusion.

1.1 Two-level atomic system

A quantum mechanical system is described by its wave function. Let $|a\rangle$ and $|b\rangle$ represents the upper- and lower-level of an atom respectively. The linear superposition of the upper- and lower-states for a two-level atom can be written in the form of state vector as

$$|\Psi\rangle = C_a\,|a\rangle + C_b\,|b\rangle\,, \tag{1.1}$$

where C_a and C_b are the coefficients and find the probability amplitudes of an atom in states $|a\rangle$ and $|b\rangle$, respectively.

1.2 Three-level atomic system

For a three-level atomic system $|a\rangle$, $|b\rangle$ and $|c\rangle$ represents the upper-level, medium-level and lower-level respectively. The linear superposition of the three-level atom may be written as

$$|\Psi\rangle = C_a\,|a\rangle + C_b\,|b\rangle + C_c\,|c\rangle\,, \tag{1.2}$$

where the coefficients C_a, C_b and C_c are helpful to find out the probability of an atom in the states $|a\rangle$, $|b\rangle$ and $|c\rangle$, respectively.

1.3 Types of atomic system

There are various types of atomic system. For the sake of understanding here we are discussing some of them.

1.3.1 Cascade-type configuration

Fig.(1-1) shows the three-level atom configuration which is known as cascade-type configuration where levels $|a\rangle$, $|b\rangle$ and $|c\rangle$ represent the upper, medium and lower levels and each lower-level is coupled to its next upper-level with one field of resonance frequency, i.e. ω_{bc} related to levels $|c\rangle$ and $|b\rangle$ and ω_{ab} related to levels $|b\rangle$ and $|a\rangle$.

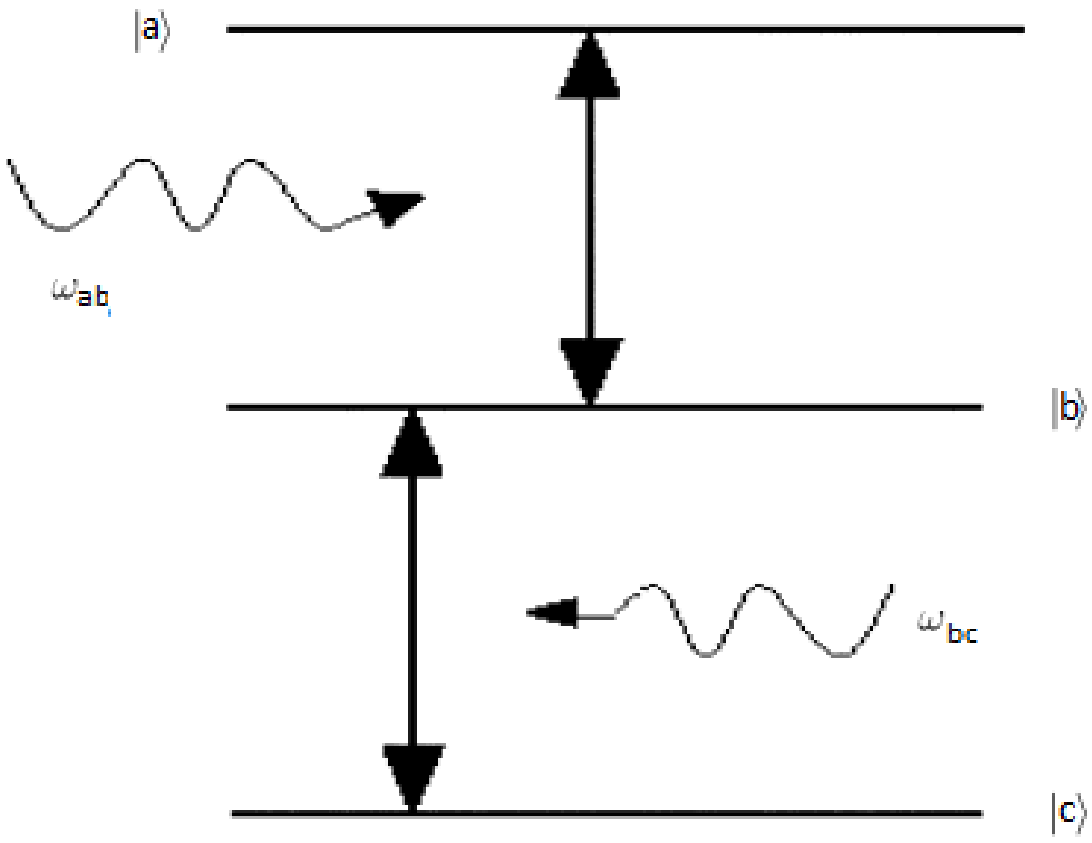

Figure 1-1: Diagram of a three-level atom in the Cascade-type configuration with fields of frequencies ω_{ab} and ω_{bc}.

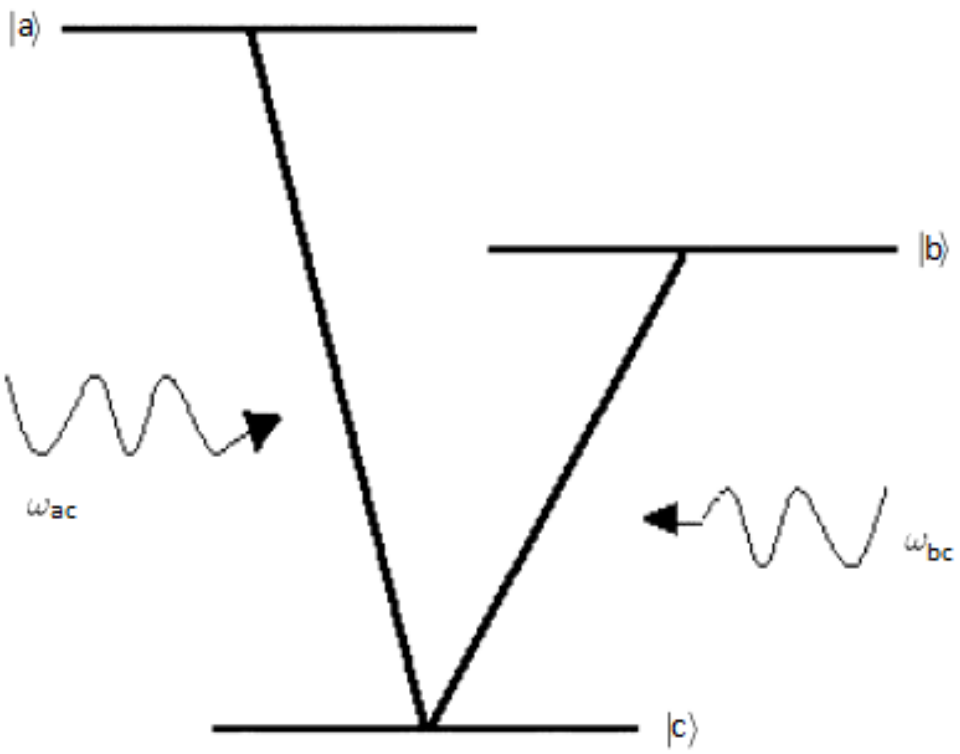

Figure 1-2: Diagram of a three-level atom in V-type configuration with fields of frequencies ω_{bc} and ω_{ac}.

1.3.2 V-type configuration

V-type configuration for a three-level atom is shown in Fig.(1-2), where a ground-level $|c\rangle$ is coupled to the medium-level $|b\rangle$ and the upper-level $|a\rangle$ with fields of resonance frequencies ω_{bc} and ω_{ac}.

1.4 Radiation field

Radiation field states can be explained in different ways.

1.4.1 Number state

A number state is also known as a Fock state. In quantum mechanics this is a quantum state which corresponds to a Fock space. The states of total radiation field in the cavity can be represented by the number of photons n_{k_1}, n_{k_2}, n_{k_3}, n_{k_4}, ... in the complete set of cavity modes k_1, k_2, k_3, k_4, ... A state of the total field can be denoted as

$$|n_{k_1}, n_{k_2}, n_{k_3}, n_{k_4}, ...\rangle . \tag{1.3}$$

In cavity the modes do not depend on each other so that the state of the total field is product of the individual modes

$$|n_{k_1}, n_{k_2}, n_{k_3}, n_{k_4}, ...\rangle = |n_{k_1}\rangle |n_{k_2}\rangle |n_{k_3}\rangle |n_{k_4}\rangle \ .... \tag{1.4}$$

A single mode number state or a Fock state [4] is represented by $|n\rangle$ where exactly n photons are excited. The vacuum state of the field is actually the state of the electromagnetic field in which no photons are excited. Number states are infact the eigenstates of number operator $\hat{N} = \hat{a}^\dagger \hat{a}$, i.e.

$$\hat{N} |n\rangle = \hat{a}^\dagger \hat{a} |n\rangle = n |n\rangle . \tag{1.5}$$

Physical properties of the single mode number state

- The uncertainty in the photon number is zero for the state $|n\rangle$

$$\triangle n = 0. \tag{1.6}$$

- Number states are complete

$$\sum_{n=0}^{\infty} |n\rangle \langle n| = 1, \tag{1.7}$$

and orthogonal

$$\langle m \,|n\rangle = \delta_{mn}. \tag{1.8}$$

1.4.2 Coherent state

The coherent state [5] is denoted by $|\alpha\rangle$ and is defined as

$$|\alpha\rangle = e^{-\frac{1}{2}|\alpha|^2} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n\rangle \,. \tag{1.9}$$

The coherent state is the eigenstate of the destruction operator $\hat{a}$ with an eigenvalue α, i.e.

$$\hat{a}\,|\alpha\rangle = \alpha\,|\alpha\rangle \,. \tag{1.10}$$

In general, α is a complex number because $\hat{a}$ is not a Hermitian operator.

Physical properties of the single mode coherent state

- The coherent state has minimum uncertainty, so that

$$\triangle p \triangle q = \frac{\hbar}{2}. \tag{1.11}$$

- Two coherent state corresponding to different eigenstates $|\alpha\rangle$ and $|\alpha'\rangle$ are not orthogonal to each other, i.e.

$$\langle\alpha \left|\alpha'\right\rangle = \exp\left(-\frac{1}{2}\left|\alpha\right|^2 + \alpha'\alpha^* - \frac{1}{2}\left|\alpha'\right|^2\right). \tag{1.12}$$

- In the coherent state $|\alpha\rangle$ mean number of photons can be calculated by the following expression

$$\langle\alpha|\,\hat{a}^\dagger\hat{a}\left|\alpha'\right\rangle = \left|\alpha\right|^2 . \tag{1.13}$$

- The set of all coherent state $|\alpha\rangle$ is a complete set

$$\pi \int |\alpha\rangle \langle\alpha| = 1. \tag{1.14}$$

1.5 Density matrix operator

The states of physical systems can be represented by parameters which describe a phenomenon correctly and conveniently. These parameters are logically consistent and also have a familiar, operational significance. Quantum mechanics deals usually with such a phenomena in which maximum information is available about the system under consideration. The state of the physical system with maximum information is referred as "pure state". Note that these systems are totally uncoupled from any external source. In quantum mechanics, the wave function of a pure state is represented by the coefficients c_i of the expansions of its state $|\Psi\rangle$ into eigenvectors $|\varphi_i\rangle$ of a complete set of operators,

$$|\Psi\rangle = c_i \, |\varphi_i\rangle \, . \tag{1.15}$$

Consider an observable O that is represented by a quantum-mechanical operator $\hat{O}$. For a pure state $|\Psi\rangle$, one can obtain the expectation value of an observable by the following expression i.e.

$$\left\langle \hat{O} \right\rangle = \left\langle \Psi \left| \hat{O} \right| \Psi \right\rangle . \tag{1.16}$$

While in many situations the wave function for the system is not known, but probabilities for having various different wave functions are known. Such a state is a statistical mixture or mixed state. So in that case where one have incomplete information of a state, it is then advantageous to introduce the density matrix operator $\hat{\rho}$. If $\hat{B}$ is an observable of the system, the average value of $\hat{B}$ is driven by the density matrix operator by the following relation

$$\left\langle \hat{B} \right\rangle = \mathrm{Tr} \left(\hat{\rho} \hat{B} \right) . \tag{1.17}$$

Where the trace of an operator is the sum of its diagonal matrix elements. If Ψ is the wave function of the system and let $\{|n\rangle\}$ span the Hilbert space containing the wave function Ψ, then,

$$\left\langle \hat{B} \right\rangle = \langle \Psi | \, \hat{B} \, | \Psi \rangle \, . \tag{1.18}$$

$|\Psi\rangle$ may expand into its basis to obtain,

$$|\Psi\rangle = \sum_n |n\rangle \, \langle n \, | \Psi \rangle \, , \tag{1.19}$$

substitute Eq.(1.19) into Eq.(1.18),

$$\left\langle \hat{B} \right\rangle = \sum_p \sum_n \rho_{np} B_{pn} = \mathrm{Tr} \left(\hat{\rho} \hat{B} \right) . \tag{1.20}$$

Here, $\rho_{np} = \langle p \, | \Psi \rangle^* \, \langle n \, | \Psi \rangle = b_p^* b_n$, projection of wave function on the basis vector $|n\rangle$ is denoted by the coefficient b_n. $\rho_{nn} = |\langle \Psi \, | n \rangle|^2 = b_n^* b_n = P_n$, shows the diagonal elements of $\hat{\rho}$. Also P_n is the probability of finding the system in the state $|n\rangle$. From Eq.(1.18) and Eq.(1.20) the density operator of state $|\Psi\rangle$ is given by [6, 7]

$$\hat{\rho} = |\Psi\rangle \, \langle \Psi| \, . \tag{1.21}$$

1.5.1 Properties of the density matrix operator

- $\hat{\rho}$ is Hermitian,

$$\rho_{nm} = \rho_{mn}^* . \tag{1.22}$$

- $\hat{\rho}$ is positive definite,

$$\langle a \, |\hat{\rho}| \, a \rangle \geq 0. \tag{1.23}$$

- Trace of the density matrix operator is always 1, i.e.

$$\mathrm{Tr} \, (\hat{\rho}) = 1. \tag{1.24}$$

1.5.2 Temporal evolution of density matrix operator

Evolution of density matrix operator within time is governed by the Shrodinger equation. The time dependent Shrodinger equation is given by

$$\frac{\partial |\Psi\rangle}{\partial t} = -\frac{i}{\hbar}\hat{H}|\Psi\rangle . \tag{1.25}$$

The solution of the above equation gives the wave function at time t in terms of the wave function at time $t = 0$ describe as

$$|\Psi(t)\rangle = \hat{U}(t)|\Psi(0)\rangle , \tag{1.26}$$

where $\hat{U}(t)$ is the Unitary time-evolution operator. The corresponding formula for the density matrix is

$$\frac{\partial \hat{\rho}}{\partial t} = -\frac{i}{\hbar}\left[\hat{H}\hat{\rho} - \hat{\rho}\hat{H}\right], \tag{1.27}$$

and its solution is

$$\hat{\rho}(t) = \hat{U}(t)\,\hat{\rho}(0)\,\hat{U}^{\dagger}(t) . \tag{1.28}$$

Eq.(1.27) is the Schrödinger equation for the density operator $\hat{\rho}$ [8].

1.5.3 Density matrix operator for two-level atom

For the two-level atomic system where the state of the system is given by $|\Psi\rangle = C_a|a\rangle + C_b|b\rangle$, the density matrix operator can be written as

$$\begin{aligned}
\hat{\rho}_{Atom} &= |\Psi\rangle\langle\Psi| \\
&= [C_a|a\rangle + C_b|b\rangle]\,[C_a^*\langle a| + C_b^*\langle b|] \\
&= |C_a|^2|a\rangle\langle a| + C_aC_b^*|a\rangle\langle b| + C_bC_a^*|b\rangle\langle a| + |C_b|^2|b\rangle\langle b| .
\end{aligned} \tag{1.29}$$

The matrix elements are,

$$\rho_{aa} = \langle a|\hat{\rho}|a\rangle = |C_a|^2 \text{, probability of being in upper level} \tag{1.30a}$$

$$\rho_{ab} = \langle a|\hat{\rho}|b\rangle = C_a C_b^* \text{, proportional to the complex dipole moment} \tag{1.30b}$$

$$\rho_{ab} = \rho_{ba}^* \tag{1.30c}$$

$$\rho_{bb} = \langle b|\hat{\rho}|b\rangle = |C_b|^2 \text{, probability of being in lower level} \tag{1.30d}$$

Therefore in matrix notation the density matrix operator for two-level system can be written as

$$\hat{\rho} = \begin{bmatrix} \rho_{aa} & \rho_{ab} \\ \rho_{ba} & \rho_{bb} \end{bmatrix}. \tag{1.31}$$

1.5.4 Density matrix operator for radiation field

Number state

For a field in one of the number state $|n\rangle$, where m photons are present, the density matrix operator is simply given by

$$\hat{\rho} = |n\rangle\langle n|. \tag{1.32}$$

Coherent state

The density matrix operator for one of the coherent state $|\alpha\rangle$ can be similarly constructed as

$$\hat{\rho} = |\alpha\rangle\langle\alpha|. \tag{1.33}$$

1.5.5 Reduced density matrix operator

The density matrix operator of the whole system (atom-field) obviously contains too much information and has little to tell us directly about particular sub-systems behavior. Therefore to characterize a given sub-system we need a statistical operator. Such an operator is called the reduced density matrix operator.

Let us take a density matrix operator $\hat{\rho}$ which describes the interaction between atom and single mode electromagnetic field. By tracing the total density matrix operator $\hat{\rho}$ over the field we get the reduced density matrix operator which describes the properties of the atomic system.

$$\hat{\rho}_{Atom} = Tr_{Field}(\hat{\rho}). \tag{1.34}$$

Similarly we can get the reduced density matrix operator which describes the behavior of the field system by tracing the total density matrix operator $\hat{\rho}$ over the atom.

$$\hat{\rho}_{Field} = Tr_{Atom}(\hat{\rho}). \tag{1.35}$$

1.6 Entropy

In general entropy is measure of the disorder of the system. According to the 2^{nd} law of thermodynamics if the energy in the form of heat dQ is added to a system held at a constant temperature T, the change in entropy dS is given by

$$dS = dQ/T. \tag{1.36}$$

In statistical mechanics entropy is defined as

$$S = k_B \ln W, \tag{1.37}$$

where W is the number of accessible states of the system and k_B is the Boltzmann constant.

1.6.1 Von Neumann entropy

The von Neumann entropy is infact the quantum mechanical entropy [9]. To describe entropy quantum mechanically, one has to distinguish between observables and states. Observables, like position, momentum, energy, etc. are mathematically described by self-adjoint operators in Hilbert space. In the same way states of the system are characterized by a density matrix $\hat{\rho}$. The expectation value of an observable $\hat{O}$ in the state $\hat{\rho}$ is $\left\langle \hat{O} \right\rangle = \mathrm{Tr}\left(\hat{\rho}\hat{O}\right)$.

Entropy is different from most physical quantities. It is not an observable; it means that

there does not exist an operator with the property that its expectation value in some state would be its entropy. It is rather than a function of state. If a quantum state is described by the density matrix operator $\hat{\rho}$, its entropy is defined by

$$S(\rho) = -\operatorname{Tr}(\rho \ln \rho). \tag{1.38}$$

If $\hat{\rho}$ describes a pure state then $S = 0$ and if $\hat{\rho}$ describes a mixed state then $S \neq 0$. In this sense entropy measures deviations from pure state behavior. Since we are dealing with the interaction of atomic system, the von Neumann entropy of atom is defined as [10, 11]

$$S(\hat{\rho}_{Atom}) = -\operatorname{Tr}_{Atom}(\hat{\rho}_{Atom} \ln \hat{\rho}_{Atom}), \tag{1.39}$$

where the reduced density matrix operator is given by

$$\hat{\rho}_{Atom} = Tr_{Field}(\hat{\rho}). \tag{1.40}$$

1.6.2 Shannon entropy

As the entropy of field can be given by the following relation

$$S(\hat{\rho}_{Field}) = -\operatorname{Tr}_{Field}(\hat{\rho}_{Field} \ln \hat{\rho}_{Field}). \tag{1.41}$$

But it may be difficult to evaluate field system's entropy by Eq.(1.41) since this requires diagonalization of the reduced density matrix operator for the field. Therefore we use entropy called Shannon entropy [12] which measures the fluctuations in the observable $\hat{O}$ through the relation

$$\begin{aligned} S(\hat{\rho}_{Field}) &= \tilde{S}\left(\hat{\rho}_{Field}; \hat{O}_{Field}\right) \\ &= -\sum_{\alpha} (\rho_{Field})_{\alpha\alpha} \ln (\rho_{Field})_{\alpha\alpha}, \end{aligned} \tag{1.42}$$

where we have

$$\hat{O}_{Field}\left|\alpha\right\rangle = \alpha\left|\alpha\right\rangle, \tag{1.43}$$

$$\left(\rho_{Field}\right)_{\alpha\alpha} = \left\langle\alpha\right|\hat{\rho}_{Field}\left|\alpha\right\rangle. \tag{1.44}$$

If our observable is $\hat{O} = \hat{a}^{\dagger}\hat{a}$ then

$$\begin{aligned} S\left(\hat{\rho}_{Field}\right) &= \tilde{S}\left(\hat{\rho}_{Field}; \hat{a}^{\dagger}\hat{a}\right) \\ &= -\sum_{n}\left(\rho_{Field}\right)_{nn}\ln\left(\rho_{Field}\right)_{nn}. \end{aligned} \tag{1.45}$$

1.7 Population of the state

Population of the state represents the physical meaning of the matrix elements of the density matrix operator.

$$\rho_{nn}\left(t\right) = \sum_{j} p_j \rho_{nn}^{j}\left(t\right) = \sum_{j} p_j \left|c_n^j\left(t\right)\right|^2. \tag{1.46}$$

Where $\left|c_n^j\left(t\right)\right|^2$ is the probability that for a system in the state $\left|\Psi_j\right\rangle$ a measurement of the observable whose eigenbasis is $\{\left|u_n\right\rangle\}$ will leave the system in the state $\left|u_n\right\rangle$. Therefore, $\rho_{nn}\left(t\right)$ represents the average probability of finding the system in the state $\left|u_n\right\rangle$. $\rho_{nn}\left(t\right)$ is known as the population of the state $\left|u_n\right\rangle$. The population depends upon the chosen basis.

1.8 Entanglement

In many applications of quantum information processing the most famous and fundamental quantity is quantum-mechanical entanglement [13, 14, 15] between the subsystems which are spatially separated. Entanglement is a purely quantum-mechanical effect which has no classical counterpart. In quantum mechanics we can define the entanglement as a correlation of the properties of two particles, no matter how they are far apart. For the sake of understanding consider two quantum mechanical systems B and C respectively. If the systems are entangled

with each other, we can say that the values of certain properties of the system B are correlated with system C. The properties are interlinked even that the two systems are spatially separated leading to the phrase "spooky action at a distance".

1.9 Decoherence in quantum mechanics

One of the most fundamental property of quantum mechanics is the quantum superposition. The principle of quantum mechanics which provides more fascinating window of quantum world is given by Pauli Dirac in his book [16]

"....any two or more states may be superposed to give a new state". But when Dirac gave the basic quantum mechanics principles, some famous scientists cause odd situation which did create challenge for quantum mechanics. In 1935 Shrodinger and Einstein put their theoretical experiments [17] which was a real problem for quantum-mechanical basic postulates. Quantum-mechanical basic principles are going to be used to build a quantum computer, for this it is an important thing to get a close box system where one can easily apply the basic quantum-mechanical laws. But real systems can't be detached from extraneous interactions and such type of interactions with systems are called decoherence [18, 19, 20]. The decoherence in quantum-mechanical systems ruin the superposition of quantum states and change the quantum superposition into statistical mixture of states. Hence one can conclude that decoherence vanishes the quantum-mechanical phenomenon in systems.

Chapter 2

Atom-field interaction

In this chapter the reduced density matrix operators for atom and field from the density matrix operator for the entire interacting system of atom and field at time t are calculated. To obtain this we first calculate the density matrix operator for the whole interacting system at time t using the Eq.(1.28). We use the simplest and important quantum optical model called Jaynes Cummings model [1].

2.1 Jaynes Cummings model

The Jaynes Cummings model consists of two energy levels atom with upper-level $|a\rangle$ and lower-level $|b\rangle$ separated in energy by $\hbar\omega_o$ interacting with quantized electromagnetic field with frequency ω as shown in Figure(2-1). The matrix representation of upper- and lower-level is

$$|a\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \text{and} \quad |b\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix}. \tag{2.1}$$

The quantum states $|a\rangle$ and $|b\rangle$ are the eigenstates of non-interacting atomic Hamiltonian $\hat{H}_A$ with energy eigenvalues E_a and E_b i.e.

$$\begin{aligned} \hat{H}_A |a\rangle &= E_a |a\rangle , \\ \hat{H}_A |b\rangle &= E_b |b\rangle , \end{aligned} \tag{2.2}$$

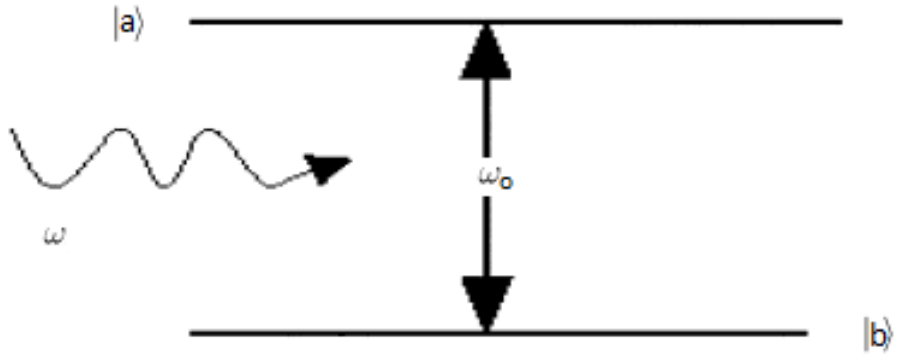

Figure 2-1: Interaction of a two-level atom with the single mode electromagnetic wave.

with

$$E_a - E_b = \hbar\omega_o. \tag{2.3}$$

Since the total atomic energy eigenstate is complete, i.e. $|a\rangle\langle a| + |b\rangle\langle b| = 1$. Thus multiplying the atomic Hamiltonian from right with unity and using the Eq.(2.2)

$$\begin{aligned} \hat{H}_A &= \hat{H}_A\left(|a\rangle\langle a| + |b\rangle\langle b|\right) \\ &= E_a|a\rangle\langle a| + E_b|b\rangle\langle b| \\ &= E_a\hat{\sigma}_{aa} + E_b\hat{\sigma}_{bb,} \end{aligned} \tag{2.4}$$

where $\hat{\sigma}_{aa} = |a\rangle\langle a|$ and $\hat{\sigma}_{bb} = |b\rangle\langle b|$. In matrix form

$$\hat{\sigma}_{aa} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad \hat{\sigma}_{bb} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}. \tag{2.5}$$

The right side of Eq.(2.4) can also be written as

$$
\begin{aligned}
E_a\hat{\sigma}_{aa} + E_b\hat{\sigma}_{bb} &= \frac{1}{2}\left(E_a - E_b\right)\left(\hat{\sigma}_{aa} - \hat{\sigma}_{bb}\right) + \frac{1}{2}\left(E_a + E_b\right)\left(\hat{\sigma}_{aa} + \hat{\sigma}_{bb}\right) \\
&= \frac{1}{2}\hbar\omega_o\left(\hat{\sigma}_{aa} - \hat{\sigma}_{bb}\right) = \frac{1}{2}\hbar\omega_o\hat{\sigma}_z,
\end{aligned} \tag{2.6}
$$

where $\hat{\sigma}_z = \hat{\sigma}_{aa} - \hat{\sigma}_{bb}$ is the Pauli spin operator and $\hat{\sigma}_{aa} + \hat{\sigma}_{bb} = I$. Also $\frac{1}{2}\left(E_a + E_b\right)$ is a constant energy term, we therefore can neglect it. In matrix form

$$
\hat{\sigma}_z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}. \tag{2.7}
$$

Hence Eq.(2.4) becomes

$$
\hat{H}_A = \frac{1}{2}\hbar\omega_o\hat{\sigma}_z. \tag{2.8}
$$

In treating the quantized electromagnetic field [21] it is suitable to introduce non-Hermitian operators $\hat{a}^\dagger$ and $\hat{a}$ known as creation and annihilation operators. These operators play the role of lowering and raising the excitation of the field by $\hbar\omega$. The Hamiltonian of the quantized radiation field is

$$
\hat{H}_F = \hbar\omega\left(\hat{a}^\dagger\hat{a} + \frac{1}{2}\right), \tag{2.9}
$$

where ω is the frequency of the radiation field and $\frac{1}{2}\hbar\omega$ is the zero point energy. Since it is the constant term, thus for convenience we neglect it. Therefore, we can write

$$
\hat{H}_F = \hbar\omega\hat{a}^\dagger\hat{a}. \tag{2.10}
$$

In the atom-field interaction we suppose that the atom is located at a fixed point in an electromagnetic field. If we remark the atom as an electric dipole with moment $\hat{\boldsymbol{\mu}}$, then the interaction energy $\hat{H}_I$ may be taken as

$$
\hat{H}_I = -\hat{\boldsymbol{\mu}} \,.\, \hat{\mathbf{E}}, \tag{2.11}
$$

where $\hat{\mathbf{E}}$ is the electric field operator given by

$$\hat{\mathbf{E}} = \hat{\epsilon}\varepsilon \left(\hat{a} + \hat{a}^{\dagger}\right). \tag{2.12}$$

We make a representation of $\hat{\boldsymbol{\mu}}$ in terms of operators by multiplying it with unity on the left and on the right using the relation $|a\rangle\langle a| + |b\rangle\langle b| = 1$, we get

$$\hat{\boldsymbol{\mu}} = \boldsymbol{\mu}_{aa}|a\rangle\langle a| + \boldsymbol{\mu}_{ab}|a\rangle\langle b| + \boldsymbol{\mu}_{ba}|b\rangle\langle a| + \boldsymbol{\mu}_{bb}|b\rangle\langle b|, \tag{2.13}$$

where $\boldsymbol{\mu}_{ab} = \langle a|\hat{\boldsymbol{\mu}}|b\rangle$, the electric dipole transition matrix element. Here $\boldsymbol{\mu}_{aa}$ and $\boldsymbol{\mu}_{bb}$, which shows the expectation values of the dipole moment $\hat{\boldsymbol{\mu}}$ in the upper and lower states, must vanish from considerations of symmetry for states of definite parity, because the dipole moment has odd parity. Hence the Eq.(2.13) becomes

$$\begin{aligned} \hat{\boldsymbol{\mu}} &= \boldsymbol{\mu}_{ab}|a\rangle\langle b| + \boldsymbol{\mu}_{ba}|b\rangle\langle a| \\ &= \boldsymbol{\mu}_{ab}\hat{\sigma}_{ab} + \boldsymbol{\mu}_{ba}\hat{\sigma}_{ba}, \end{aligned} \tag{2.14}$$

substitute the values of $\hat{\boldsymbol{\mu}}$ and $\hat{\mathbf{E}}$ in Eq.(2.11) from Eq.(2.12) and Eq.(2.14), we get

$$\begin{aligned} \hat{H}_I &= -\boldsymbol{\mu}_{ab}\hat{\sigma}_{ab}.\hat{\epsilon}\varepsilon\left(\hat{a} + \hat{a}^{\dagger}\right) - \boldsymbol{\mu}_{ba}\hat{\sigma}_{ba}.\hat{\epsilon}\varepsilon\left(\hat{a} + \hat{a}^{\dagger}\right) \\ &= \hbar p^{ab}\hat{\sigma}_{ab}\left(\hat{a} + \hat{a}^{\dagger}\right) + \hbar p^{ba}\hat{\sigma}_{ba}\left(\hat{a} + \hat{a}^{\dagger}\right). \end{aligned} \tag{2.15}$$

where we have

$$p^{ab} = \frac{-\boldsymbol{\mu}_{ab}.\hat{\epsilon}\varepsilon}{\hbar}. \tag{2.16}$$

Since the transition take place between two levels and it corresponds a $\triangle n = 0$ transition of a real atom, we may take $\boldsymbol{\mu}_{ab}$ to be real vector. For $\boldsymbol{\mu}_{ab} = \boldsymbol{\mu}_{ba}$,

$$p = p^{ab} = p^{ba}. \tag{2.17}$$

Now the Eq.(2.15) becomes

$$\hat{H}_I = \hbar p \left(\hat{\sigma}_{ab} + \hat{\sigma}_{ba}\right) \left(\hat{a} + \hat{a}^\dagger\right). \tag{2.18}$$

We establish two atomic operators $\hat{\sigma}_+$ and $\hat{\sigma}_-$ given by

$$\hat{\sigma}_+ = \hat{\sigma}_{ab} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \tag{2.19}$$

and

$$\hat{\sigma}_- = \hat{\sigma}_{ba} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}. \tag{2.20}$$

The operators $\hat{\sigma}_+$ and $\hat{\sigma}_-$ are the Pauli spin-flip operators which raise and lower the excitation of the atom by $\hbar\omega_o$, i.e. $\hat{\sigma}_+$ operator takes an atom in the lower state into the upper state, whereas $\hat{\sigma}_-$ takes an atom in the upper state into the lower state. Since we consider two states,

$$\begin{aligned} \hat{\sigma}_+ |b\rangle &= |a\rangle, \qquad \hat{\sigma}_- |b\rangle = 0. \\ \hat{\sigma}_+ |a\rangle &= 0, \qquad \hat{\sigma}_- |a\rangle = |b\rangle. \end{aligned} \tag{2.21}$$

Hence Eq.(2.18) becomes

$$\begin{aligned} \hat{H}_I &= \hbar p \left(\hat{\sigma}_+ + \hat{\sigma}_-\right) \left(\hat{a} + \hat{a}^\dagger\right) \\ &= \hbar p \left(\hat{\sigma}_+\hat{a} + \hat{\sigma}_+\hat{a}^\dagger + \hat{\sigma}_-\hat{a} + \hat{\sigma}_-\hat{a}^\dagger\right). \end{aligned} \tag{2.22}$$

The interaction energy in above equation consists of four terms. The term $\hat{\sigma}_+\hat{a}$ describes the process in which the atom is making transition from the lower state into the upper state and a photon is absorbed. Similarly the term $\hat{\sigma}_-\hat{a}^\dagger$ describes the process in which the atom is taken from upper state to the lower state and a photon is created. The energy is conserved in both process. The term $\hat{\sigma}_+\hat{a}^\dagger$ describes the process in which the atom makes the transition

from the lower state into the upper state and a photon is created resulting in the gain of $2\hbar\omega$. The term $\hat{\sigma}_{-}\hat{a}$ describes the process in which the atom makes transition from upper-level into the lower-level and a photon is absorbed resulting in the loss of approximately $2\hbar\omega$ in energy. Dropping the energy non-conserving terms corresponds to the rotating wave approximation, the Hamiltonian in Eq.(2.22) can be written as

$$\hat{H}_I = \hbar p \left(\hat{\sigma}_{+}\hat{a} + \hat{\sigma}_{-}\hat{a}^{\dagger} \right). \tag{2.23}$$

The total energy of the interacted system in the two-level atom and the single-mode field is

$$\hat{H} = \hat{H}_A + \hat{H}_F + \hat{H}_I. \tag{2.24}$$

By putting the Eq.(2.8) for $\hat{H}_A$, Eq.(2.10) for $\hat{H}_F$, Eq.(2.23) for $\hat{H}_I$ in Eq.(2.24), we reach at

$$\begin{aligned} \hat{H} &= \frac{1}{2}\hbar\omega_o\hat{\sigma}_z + \hbar\omega\hat{a}^{\dagger}\hat{a} + \hbar p \left(\hat{\sigma}_{+}\hat{a} + \hat{\sigma}_{-}\hat{a}^{\dagger} \right) \\ &= \hat{H}_o + \hat{H}_I. \end{aligned} \tag{2.25}$$

Where we have,

$$\hat{H}_o = \frac{1}{2}\hbar\omega_o\hat{\sigma}_z + \hbar\omega\hat{a}^{\dagger}\hat{a}, \tag{2.26}$$

and

$$\hat{H}_I = \hbar p \left(\hat{\sigma}_{+}\hat{a} + \hat{\sigma}_{-}\hat{a}^{\dagger} \right). \tag{2.27}$$

As we work in Interaction Picture. The Hamiltonian in the interaction picture is given by

$$\hat{H}^I = \exp\left(\frac{i\hat{H}_o t}{\hbar} \right) \hat{H}_I \exp\left(-\frac{i\hat{H}_o t}{\hbar} \right). \tag{2.28}$$

Substituting the values of $\hat{H}_o$ from Eq.(2.26) and $\hat{H}_I$ from Eq.(2.27) in above Equation, we have

$$\begin{aligned}\hat{H}^I &= \hbar p \exp\left(i\frac{1}{2}\omega_o t\hat{\sigma}_z + i\omega\hat{a}^\dagger\hat{a}t\right)\left[\hat{\sigma}_+\hat{a} + \hat{\sigma}_-\hat{a}^\dagger\right]\exp\left(-i\frac{1}{2}\omega_o t\hat{\sigma}_z - i\omega\hat{a}^\dagger\hat{a}t\right) \\ &= \hbar p \exp\left(i\frac{1}{2}\omega_o t\hat{\sigma}_z\right)\hat{\sigma}_+\exp\left(-i\frac{1}{2}\omega_o t\hat{\sigma}_z\right)\exp\left(i\omega\hat{a}^\dagger\hat{a}t\right)\hat{a}\exp\left(-i\omega\hat{a}^\dagger\hat{a}t\right) + \\ &\quad \hbar p \exp\left(i\frac{1}{2}\omega_o t\hat{\sigma}_z\right)\hat{\sigma}_-\exp\left(-i\frac{1}{2}\omega_o t\hat{\sigma}_z\right)\exp\left(i\omega\hat{a}^\dagger\hat{a}t\right)\hat{a}^\dagger\exp\left(-i\omega\hat{a}^\dagger\hat{a}t\right). \quad (2.29)\end{aligned}$$

Using the Baker-Hausdorf identity

$$\exp\left(\eta\hat{A}\right)\hat{B}\exp\left(-\eta\hat{A}\right) = \hat{B} + \eta\left[\hat{A},\hat{B}\right] + \frac{\eta^2}{2!}\left[\hat{A},\left[\hat{A},\hat{B}\right]\right] + \ ..... \quad (2.30)$$

We calculate

$$\exp\left(i\frac{1}{2}\omega_o t\hat{\sigma}_z\right)\hat{\sigma}_+\exp\left(-i\frac{1}{2}\omega_o t\hat{\sigma}_z\right) = \hat{\sigma}_+\exp\left(i\omega_o t\right) \quad (2.31a)$$

$$\exp\left(i\frac{1}{2}\omega_o t\hat{\sigma}_z\right)\hat{\sigma}_-\exp\left(-i\frac{1}{2}\omega_o t\hat{\sigma}_z\right) = \hat{\sigma}_-\exp\left(-i\omega_o t\right) \quad (2.31b)$$

$$\exp\left(i\omega\hat{a}^\dagger\hat{a}t\right)\hat{a}\exp\left(-i\omega\hat{a}^\dagger\hat{a}t\right) = \hat{a}\exp\left(-i\omega t\right) \quad (2.31c)$$

$$\exp\left(i\omega\hat{a}^\dagger\hat{a}t\right)\hat{a}^\dagger\exp\left(-i\omega\hat{a}^\dagger\hat{a}t\right) = \hat{a}^\dagger\exp\left(i\omega t\right). \quad (2.31d)$$

Substituting the above Equations in Eq.(2.29), we get

$$\begin{aligned}\hat{H}^I &= \hbar p\left(\hat{\sigma}_+\hat{a}e^{i(\omega_o-\omega)t} + \hat{\sigma}_-\hat{a}^\dagger e^{-i(\omega_o-\omega)t}\right) \\ &= \hbar p\left(\hat{\sigma}_+\hat{a}e^{i(\Delta)t} + \hat{\sigma}_-\hat{a}^\dagger e^{-i(\Delta)t}\right), \quad (2.32)\end{aligned}$$

where $\Delta = \omega_o - \omega$. At exact resonance, i.e. $\omega = \omega_o$

$$\begin{aligned}\hat{H}^I &= \hbar p\left(\hat{\sigma}_+\hat{a} + \hat{\sigma}_-\hat{a}^\dagger\right) \\ &= \hbar p\begin{bmatrix} 0 & \hat{a} \\ \hat{a}^\dagger & 0 \end{bmatrix}. \quad (2.33)\end{aligned}$$

The unitary time evolution operator is defined as

$$\begin{aligned}\hat{U}(t) &= \exp\left(-i\hat{H}^I t/\hbar\right) \\ &= 1-\frac{1}{1!}\left(\frac{it}{\hbar}\hat{H}^I\right)+\frac{1}{2!}\left(\frac{it}{\hbar}\hat{H}^I\right)^2-\frac{1}{3!}\left(\frac{it}{\hbar}\hat{H}^I\right)^3+.....,\end{aligned} \tag{2.34}$$

where $\hat{H}^I$ is the complete Hamiltonian in the interaction picture. Substitute the value of $\hat{H}^I$ from Eq.(2.33) into Eq.(2.34) and after some calculation we get

$$\hat{U}(t)=\begin{bmatrix} \cos\left(pt\sqrt{\hat{a}^\dagger\hat{a}+1}\right) & -i\frac{\sin\left(pt\sqrt{\hat{a}^\dagger\hat{a}+1}\right)}{\sqrt{\hat{a}^\dagger\hat{a}+1}}\hat{a} \\ -i\hat{a}^\dagger\frac{\sin\left(pt\sqrt{\hat{a}^\dagger\hat{a}+1}\right)}{\sqrt{\hat{a}^\dagger\hat{a}+1}} & \cos\left(pt\sqrt{\hat{a}^\dagger\hat{a}}\right) \end{bmatrix}. \tag{2.35}$$

2.1.1 Reduced density matrix operator of field

For the interacting system, the density matrix operator of atom and field at time t is given by

$$\hat{\rho}_{Atom\otimes Field}(t)=\hat{U}(t)\,\hat{\rho}_{Atom\otimes Field}(t)\,\hat{U}^\dagger(t). \tag{2.36}$$

If initially the atom and field are decoupled, we can write

$$\hat{\rho}_{Atom\otimes Field}(0)=\hat{\rho}_{Atom}(0)\otimes\hat{\rho}_{Field}(0). \tag{2.37}$$

And if the initial atomic state is represented by Eq.(1.31), then

$$\hat{\rho}_{Atom\otimes Field}(0)=\begin{bmatrix} \rho_{aa}\rho(0) & \rho_{ab}\rho(0) \\ \rho_{ba}\rho(0) & \rho_{bb}\rho(0) \end{bmatrix}. \tag{2.38}$$

To get reduced density matrix operator for field we trace $\hat{\rho}_{Atom\otimes Field}(t)$ over atom [11]

$$\hat{\rho}_{Field}(t)=\mathrm{Tr}_{Atom}\left(\hat{U}(t)\,\hat{\rho}_{Atom\otimes Field}(0)\,\hat{U}^\dagger(t)\right), \tag{2.39}$$

putting the values from Eq.(2.38) and Eq.(2.35) in Eq.(2.39), we have

$$\hat{\rho}_{Field}(t) = \mathrm{Tr}_{Atom}\left(\left[\begin{array}{cc} \cos\left(pt\sqrt{\hat{a}^{\dagger}\hat{a}+1}\right) & -i\frac{\sin\left(pt\sqrt{\hat{a}^{\dagger}\hat{a}+1}\right)}{\sqrt{\hat{a}^{\dagger}\hat{a}+1}}\hat{a} \\ -i\hat{a}^{\dagger}\frac{\sin\left(pt\sqrt{\hat{a}^{\dagger}\hat{a}+1}\right)}{\sqrt{\hat{a}^{\dagger}\hat{a}+1}} & \cos\left(pt\sqrt{\hat{a}^{\dagger}\hat{a}}\right) \end{array}\right]\times \left[\begin{array}{cc} \rho_{aa}\rho(0) & \rho_{ab}\rho(0) \\ \rho_{ba}\rho(0) & \rho_{bb}\rho(0) \end{array}\right]\left[\begin{array}{cc} \cos\left(pt\sqrt{\hat{a}^{\dagger}\hat{a}+1}\right) & i\frac{\sin\left(pt\sqrt{\hat{a}^{\dagger}\hat{a}+1}\right)}{\sqrt{\hat{a}^{\dagger}\hat{a}+1}}\hat{a} \\ i\hat{a}^{\dagger}\frac{\sin\left(pt\sqrt{\hat{a}^{\dagger}\hat{a}+1}\right)}{\sqrt{\hat{a}^{\dagger}\hat{a}+1}} & \cos\left(pt\sqrt{\hat{a}^{\dagger}\hat{a}}\right) \end{array}\right]\right) \quad (2.40)$$

$$\begin{aligned} \hat{\rho}_{Field}(t) = {} & \rho_{aa}\left[\cos\left(pt\sqrt{\hat{a}^{\dagger}\hat{a}+1}\right)\rho(0)\cos\left(pt\sqrt{\hat{a}^{\dagger}\hat{a}+1}\right)\right. \\ & \left.+\hat{a}^{\dagger}\frac{\sin\left(pt\sqrt{\hat{a}^{\dagger}\hat{a}+1}\right)}{\sqrt{\hat{a}^{\dagger}\hat{a}+1}}\rho(0)\frac{\sin\left(pt\sqrt{\hat{a}^{\dagger}\hat{a}+1}\right)}{\sqrt{\hat{a}^{\dagger}\hat{a}+1}}\hat{a}\right] \\ & +\rho_{bb}\left[\cos\left(pt\sqrt{\hat{a}^{\dagger}\hat{a}+1}\right)\rho(0)\cos\left(pt\sqrt{\hat{a}^{\dagger}\hat{a}}\right)\right. \\ & \left.+\frac{\sin\left(pt\sqrt{\hat{a}^{\dagger}\hat{a}+1}\right)}{\sqrt{\hat{a}^{\dagger}\hat{a}+1}}\hat{a}\rho(0)\hat{a}^{\dagger}\frac{\sin\left(pt\sqrt{\hat{a}^{\dagger}\hat{a}+1}\right)}{\sqrt{\hat{a}^{\dagger}\hat{a}+1}}\right] \\ & +i\rho_{ab}\left[\cos\left(pt\sqrt{\hat{a}^{\dagger}\hat{a}+1}\right)\rho(0)\hat{a}^{\dagger}\frac{\sin\left(pt\sqrt{\hat{a}^{\dagger}\hat{a}+1}\right)}{\sqrt{\hat{a}^{\dagger}\hat{a}+1}}\right. \\ & \left.-\hat{a}^{\dagger}\frac{\sin\left(pt\sqrt{\hat{a}^{\dagger}\hat{a}+1}\right)}{\sqrt{\hat{a}^{\dagger}\hat{a}+1}}\rho(0)\cos\left(pt\sqrt{\hat{a}^{\dagger}\hat{a}}\right)\right] \\ & +i\rho_{ba}\left[\cos\left(pt\sqrt{\hat{a}^{\dagger}\hat{a}+1}\right)\rho(0)\frac{\sin\left(pt\sqrt{\hat{a}^{\dagger}\hat{a}+1}\right)}{\sqrt{\hat{a}^{\dagger}\hat{a}+1}}\hat{a}\right. \\ & \left.-\frac{\sin\left(pt\sqrt{\hat{a}^{\dagger}\hat{a}+1}\right)}{\sqrt{\hat{a}^{\dagger}\hat{a}+1}}\hat{a}\rho(0)\cos\left(pt\sqrt{\hat{a}^{\dagger}\hat{a}+1}\right)\right] \end{aligned} \quad (2.41)$$

2.1.2 Reduced density matrix operator for atom

Similarly to get reduced density matrix operator for atom we trace $\hat{\rho}_{Atom\otimes Field}(t)$ over field [22]

$$\begin{aligned}\hat{\rho}_{Atom}(t) &= \mathrm{Tr}_{Field}\left(\hat{U}(t)\,\hat{\rho}_{Atom\otimes Field}(0)\,\hat{U}^{\dagger}(t)\right)\\ &= \sum_{n}\langle n|\,\hat{U}(t)\,\hat{\rho}_{Atom\otimes Field}(0)\,\hat{U}^{\dagger}(t)\,|n\rangle\,. \end{aligned} \tag{2.42}$$

Again substituting the values from Eq.(2.38) and Eq.(2.35), we get

$$\hat{\rho}_{Atom}(t) = \begin{bmatrix} \alpha(t) & \gamma(t) \\ \gamma^{*}(t) & \beta(t) \end{bmatrix}, \tag{2.43}$$

where

$$\begin{aligned}\alpha(t) &= \sum_{n}\left[\rho_{aa}\cos^{2}\left(pt\sqrt{n+1}\right)+\rho_{bb}\sin^{2}\left(pt\sqrt{n}\right)\right]\rho_{nn}(0)\\ &\quad +i\sum_{n}\left[\rho_{ab}\cos\left(pt\sqrt{n+1}\right)\sin\left(pt\sqrt{n+1}\right)\rho_{nn+1}(0)\right.\\ &\quad \left.-\rho_{ba}\sin\left(pt\sqrt{n+1}\right)\cos\left(pt\sqrt{n+1}\right)\rho_{n+1n}(0)\right]\end{aligned} \tag{2.44}$$

$$\begin{aligned}\beta(t) &= \sum_{n}\left[\rho_{aa}\sin^{2}\left(pt\sqrt{n+1}\right)+\rho_{bb}\cos^{2}\left(pt\sqrt{n}\right)\right]\rho_{nn}(0)\\ &\quad +i\sum_{n}\left[\rho_{ba}\cos\left(pt\sqrt{n+1}\right)\sin\left(pt\sqrt{n+1}\right)\rho_{n+1n}(0)\right.\\ &\quad \left.-\rho_{ab}\sin\left(pt\sqrt{n+1}\right)\cos\left(pt\sqrt{n+1}\right)\rho_{nn+1}(0)\right]\end{aligned} \tag{2.45}$$

$$\begin{aligned}\gamma(t) &= \sum_{n}\left[\rho_{ab}\cos\left(pt\sqrt{n+1}\right)\cos\left(pt\sqrt{n}\right)\rho_{nn}(0)\right.\\ &\quad \left.+\rho_{ba}\sin\left(pt\sqrt{n+1}\right)\sin\left(pt\sqrt{n}\right)\rho_{n+1n-1}(0)\right]\\ &\quad +i\sum_{n}\left[\rho_{aa}\cos\left(pt\sqrt{n+1}\right)\sin\left(pt\sqrt{n}\right)\rho_{nn-1}(0)\right.\\ &\quad \left.-\rho_{bb}\sin\left(pt\sqrt{n+1}\right)\cos\left(pt\sqrt{n}\right)\rho_{n+1n}(0)\right]\end{aligned} \tag{2.46}$$

$$
\begin{aligned}
\gamma^{*}(t) &= \sum_{n} \left[\rho_{ba} \cos\left(pt\sqrt{n+1}\right) \cos\left(pt\sqrt{n}\right) \rho_{nn}(0) \right. \\
&\quad \left. + \rho_{ab} \sin\left(pt\sqrt{n+1}\right) \sin\left(pt\sqrt{n}\right) \rho_{n+1n-1}(0) \right] \\
&\quad + i \sum_{n} \left[\rho_{bb} \cos\left(pt\sqrt{n}\right) \sin\left(pt\sqrt{n+1}\right) \rho_{nn+1}(0) \right. \\
&\quad \left. - \rho_{aa} \sin\left(pt\sqrt{n}\right) \cos\left(pt\sqrt{n+1}\right) \rho_{n-1n}(0) \right] \qquad (2.47)
\end{aligned}
$$

Chapter 3

Study of entanglement in a moving five-level atom

In the previous chapter we have calculated the reduced density operator for two-level atom and field. Now in this chapter we will consider the cascade-type single five-level atomic system and one mode cavity field. As we are solving the system numerically so all of the expressions given below are in general form.

3.1 Mathematical modeling

We are taking into consideration cascade-type single five-level atom having transition energies ω_p ($p = 1,....,5$), between the levels. Where $\omega_5 < \omega_4 < \omega_3 < \omega_2 < \omega_1$. The atom is interacting with a single-mode field. This field mode can be described by the non-Hermitian operators $\hat{a}^\dagger$ and $\hat{a}$ known as creation and annihilation operators and frequency Ω.

3.1.1 Hamiltonian of the system

The total Hamiltonian $\hat{H}_T$ in the rotating wave approximation for the given system is

$$\hat{H}_T = \hat{H}_{Atom-Field} + \hat{H}_{Int}. \tag{3.1}$$

Where $\hat{H}_{Atom-Field}$ is the Hamiltonian for the non-interacting atom and field, and the interaction part is $\hat{H}_{Int}$. We can define $\hat{H}_{Atom-Field}$ as

$$\hat{H}_{Atom-Field} = \sum_{p} \omega_p \hat{\sigma}_{p,p} + \Omega \hat{a}^{\dagger} \hat{a}, \tag{3.2}$$

where $\hbar = 1$ and $\hat{\sigma}_{p,p} = |p\rangle \langle p|$ are known as population operators. And the interaction Hamiltonian for the non-resonant case is given as

$$\hat{H}_{Int} = \sum_{p=1}^{4} \lambda_p h(z) \left[\hat{a} e^{-i\Delta_p t} \hat{\sigma}_{p,p+1} + \left(\hat{a} e^{-i\Delta_p t} \hat{\sigma}_{p,p+1} \right)^{\dagger} \right]. \tag{3.3}$$

We can define detuning parameter as

$$\Delta_p = \Omega + \omega_{p+1} - \omega_p, \tag{3.4}$$

and the coupling constant for atom and field is λ_p, where $h(z)$ represents the shape function of the field mode and atomic motion is along the z-axis. Now consider Eq.(3.3) and perform the operation we get

$$\begin{aligned} \hat{H}_{Int} = h(z) \Big[& \left(e^{-i\Delta_1 t} \hat{a} |0\rangle \langle 1| + e^{i\Delta_1 t} |1\rangle \langle 0| \hat{a}^{\dagger} \right) \lambda_1 + \left(e^{-i\Delta_2 t} \hat{a} |1\rangle \langle 2| + e^{i\Delta_2 t} |2\rangle \langle 1| \hat{a}^{\dagger} \right) \lambda_2 \\ & + \left(e^{-i\Delta_3 t} \hat{a} |2\rangle \langle 3| + e^{i\Delta_3 t} |3\rangle \langle 2| \hat{a}^{\dagger} \right) \lambda_3 + \left(e^{-i\Delta_4 t} \hat{a} |3\rangle \langle 4| + e^{i\Delta_4 t} |4\rangle \langle 3| \hat{a}^{\dagger} \right) \lambda_4 \Big]. \end{aligned} \tag{3.5}$$

Since detuning can not be negative it may be zero or positive so here we are taking $t > 0$, $\Delta_1 \geq 0$, $\Delta_2 \geq 0$, $\Delta_3 \geq 0$, $\Delta_4 \geq 0$.

3.1.2 Wave function for the system

For the construction of wave function we assume that initially at time $t = 0$ the atom was in its upper-state $|0\rangle$ and the field was in coherent state. Thus the wave function can be written into its field and atomic parts as follows

$$|\Psi(t=0)\rangle = \sum_{n} q_n |0, n\rangle. \tag{3.6}$$

Where coherent state is defined as

$$q_n = \exp\left(-\frac{\bar{n}}{2}\right)\frac{\alpha^n}{\sqrt{n!}}, \tag{3.7}$$

and $\bar{n} = |\alpha|^2$ known as initial mean photon number. To construct wave function at time t we may proceed as follows

$$|\Psi(t)\rangle = \hat{U}(t)\,|\Psi(t=0)\rangle\,. \tag{3.8}$$

Where $\hat{U}(t)$ is the unitary time-evolution operator given by

$$\hat{U}(t) = \exp\left(-i\hat{H}_{Int}t/\hbar\right). \tag{3.9}$$

It should be noted that in our calculations we are taking $\hbar = 1$ also find eigenvalues and eigenvectors of interaction Hamiltonian. By placing these values in Eq.(3.9) we get that form of unitary operator which can apply on wave function at time $t = 0$ and then we construct wave function at any time t.

3.1.3 Density matrix formation

Once a wave function has been establish at time t one can develop density matrix by the following expression

$$\hat{\rho}(t) = |\Psi(t)\rangle\langle\Psi(t)|\,. \tag{3.10}$$

Hence a given quantum state is described by the density operator $\hat{\rho}(t)$.

3.1.4 Von Neumann entropy

The quantum mechanical entropy is also known as von Neumann entropy i.e.

$$S_{v_N} = -\operatorname{Tr}(\rho\ln\rho)\,. \tag{3.11}$$

For all the pure states which fulfil the condition $\hat{\rho}^2 = \hat{\rho}$, it gives zero i.e. $S_{v_N} = 0$. As the $\hat{\rho}$ is the given quantum state that's why various pure states can not be distinguished by this

entropy. So that to find the von Neumann entropy we take trace of the total density matrix operator over the field and get a reduced density matrix operator given below

$$\hat{\rho}_{Atom}(t) = \mathrm{Tr}_{Field}\left(\hat{\rho}(t)\right). \tag{3.12}$$

Then the von Neumann entropy becomes

$$S_{Atom} = -\mathrm{Tr}_{Atom}\left\{\hat{\rho}_{Atom}(t)\ln\hat{\rho}_{Atom}(t)\right\}. \tag{3.13}$$

3.1.5 Decoherence effect

To study the decoherence effect we introduce a factor $\frac{\gamma}{2}$ in the unitary time-evolution operator as given below

$$\hat{U}(t) = \exp\left(-\frac{\gamma}{2}\left(\hat{H}_{Int}\right)^2 t/\hbar - i\hat{H}_{Int}t/\hbar\right). \tag{3.14}$$

Then we use this unitary operator and make calculations same as above pattern and find some useful results which shows that how system is going from quantum-mechanical to classical.

Chapter 4

Results and discussion

In this dissertation we study the dynamics of entanglement via von Neumann entropy for a five-level atom in a pure resonant case interacting with cavity field both for atom at rest and in motion. We also determine the effect of decoherence on the similar system. Further more, we explore the atomic population dynamics of the system and relate it with entanglement.

We consider cascade type five-level atom interacting with field which is in the coherent state with mean photon number $|\alpha|^2 = 25$. The plots of von Neumann entropy with and without atomic motion along with atomic population are given in Fig.(4-1) to Fig.(4-8). For the case of atom at rest von Neumann entropy shows non periodic oscillatory behavior with range of oscillations between 1 and $\ln 5$. The entanglement remained non-zero throughout the scaled time. Whereas, atomic population shows oscillatory behavior with non-zero positive values without reaching its maximum value of 1 indicating that system remains entangled.

We also consider atom in a motion. A periodic oscillatory behavior is observed with entanglement value between 1 and zero showing maximum entanglement and disentanglement of the system. Correspondingly, we can see that atomic population ρ_{11} approaches value 1 at disentanglement point of von Neumann entropy and also at $\rho_{11} = 1$. Other populations (ρ_{33} and ρ_{55}) are zero. So we can see that the atomic motion play a role of entanglement and disentanglement generator for such type of models.

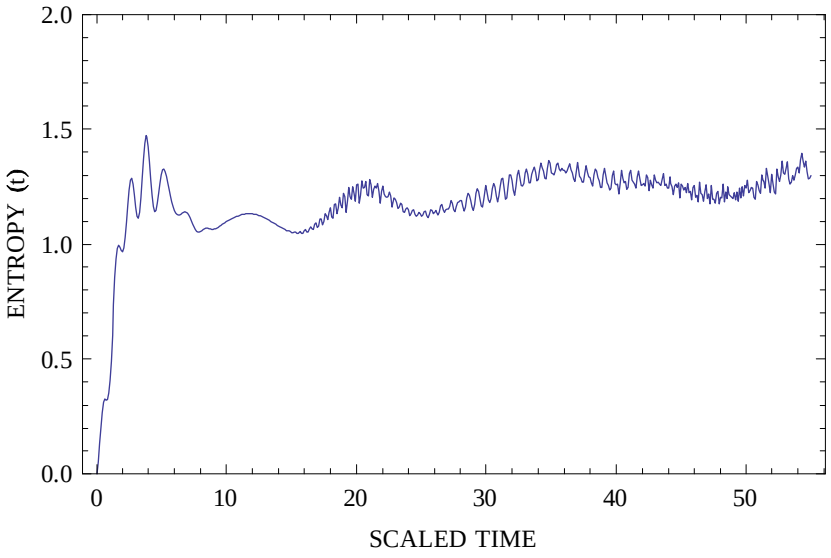

Figure 4-1: The time evolution of the von Neumann entropy for the pure resonant case. The atomic motion is neglected.

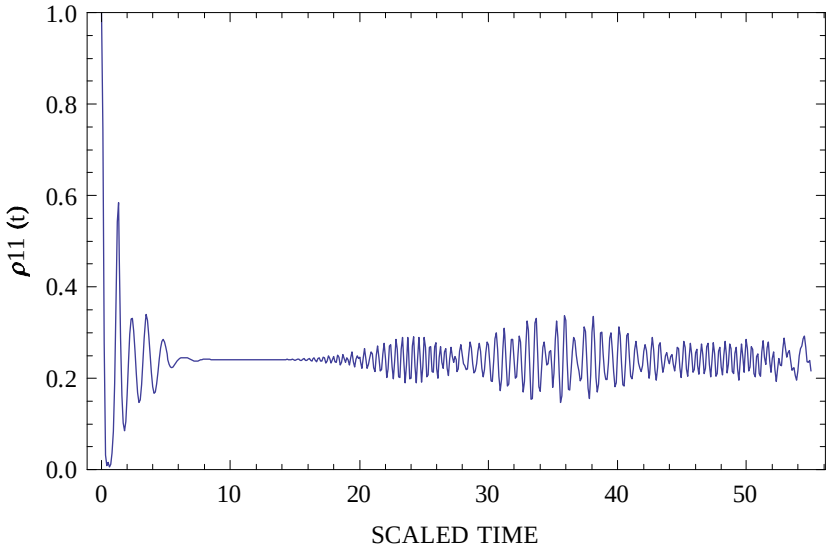

Figure 4-2: The time evolution of the atomic population ρ_{11} for the pure resonant case. The atomic motion is neglected.

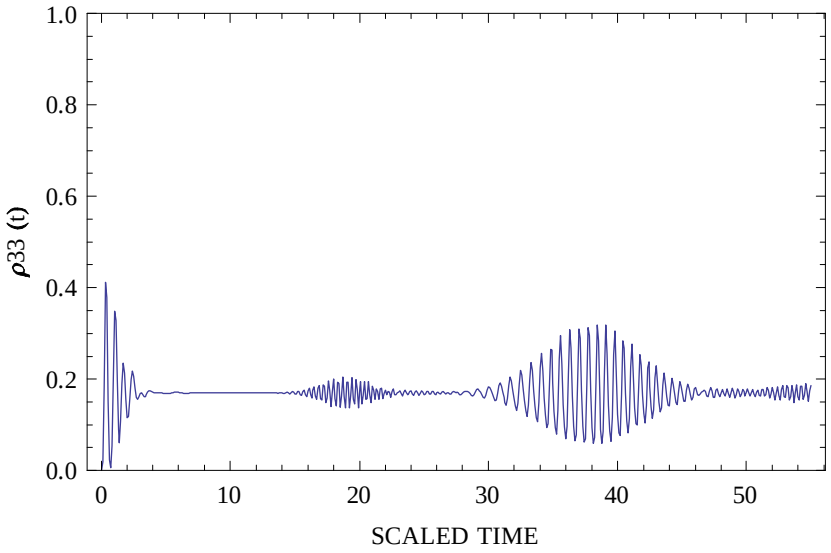

Figure 4-3: The time evolution of the atomic population ρ_{33} for the pure resonant case. The atomic motion is neglected.

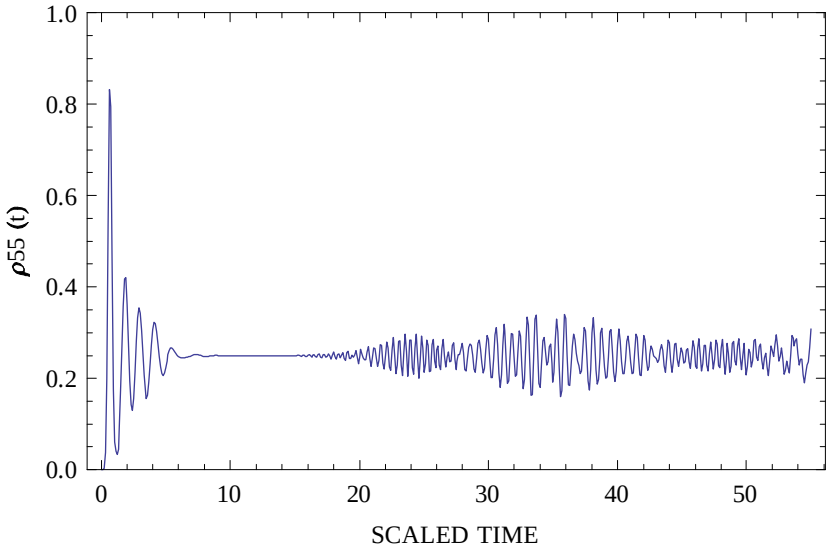

Figure 4-4: The time evolution of the atomic population ρ_{55} for the pure resonant case. The atomic motion is neglected.

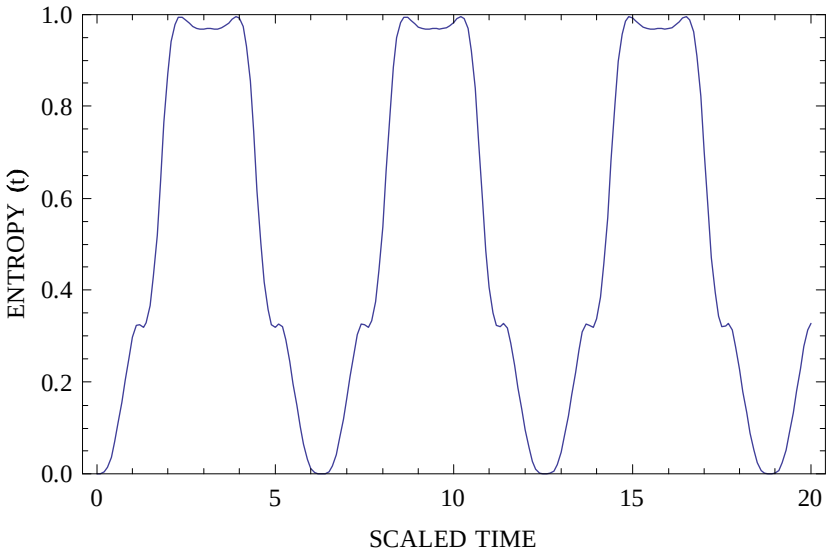

Figure 4-5: The time evolution of the von Neumann entropy for the pure resonant case. The atomic motion is considered.

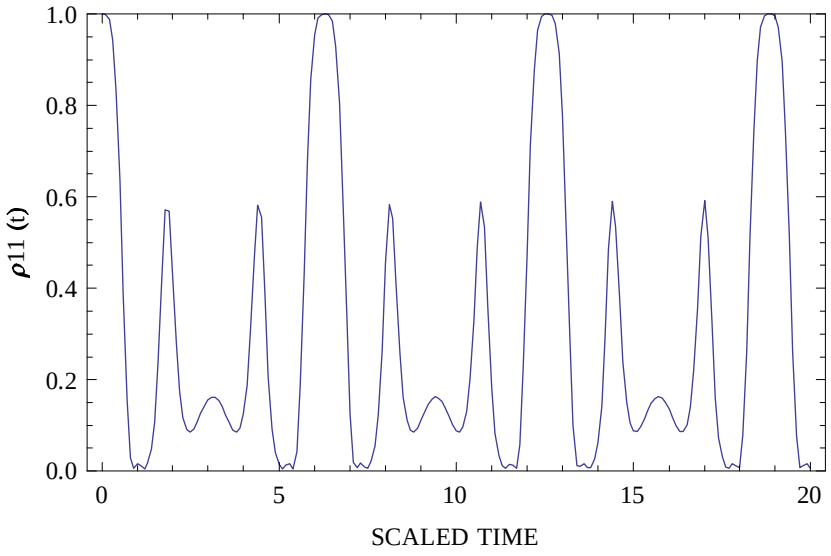

Figure 4-6: The time evolution of the atomic population ρ_{11} for the pure resonant case. The atomic motion is considered.

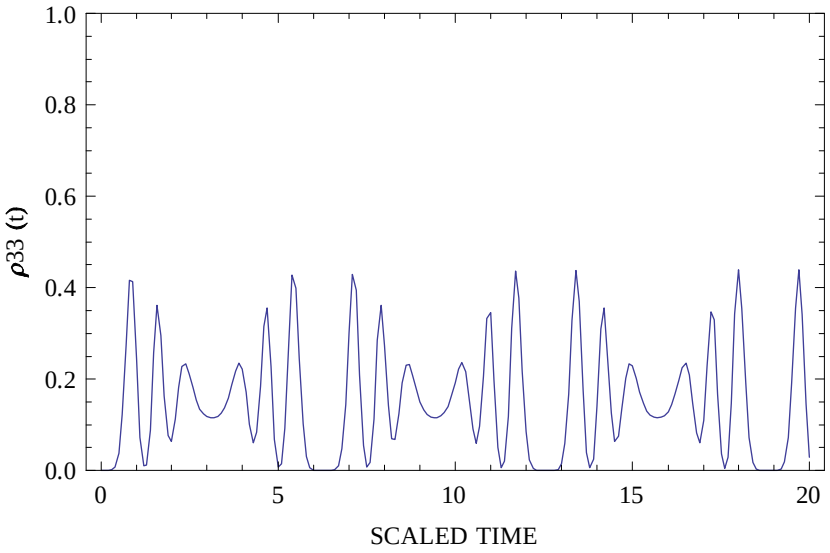

Figure 4-7: The time evolution of the atomic population ρ_{33} for the pure resonant case. The atomic motion is considered.

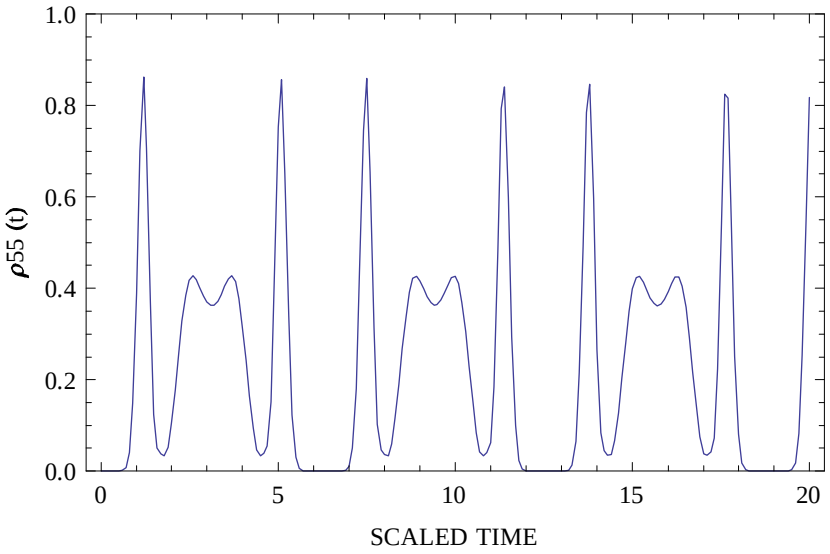

Figure 4-8: The time evolution of the atomic population ρ_{55} for the pure resonant case. The atomic motion is considered.

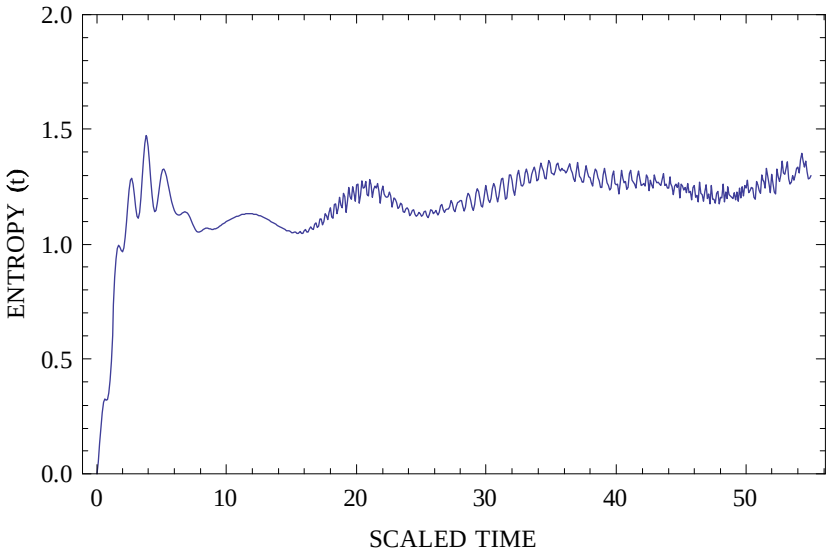

Figure 4-9: The effect of decoherence on the time evolution of the von Neumann entropy with decoherence parameter $\gamma = 0$. The atomic motion is neglected.

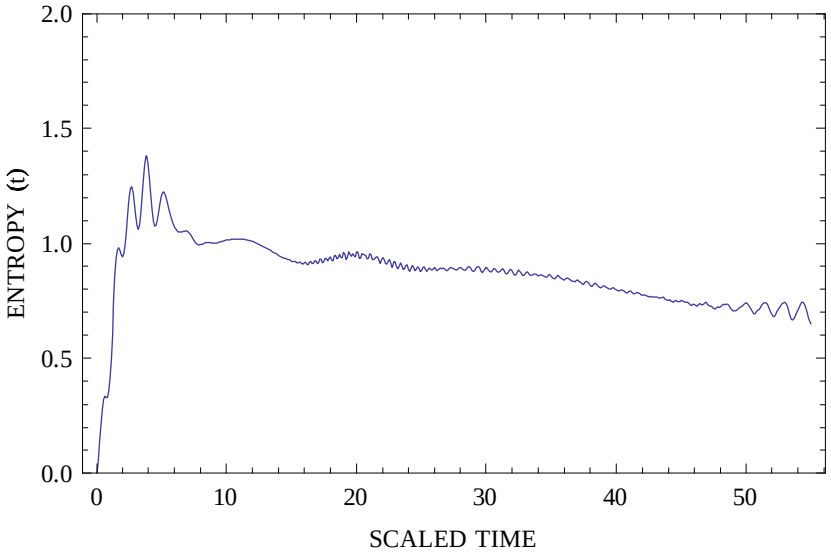

Figure 4-10: The effect of decoherence on the time evolution of the von Neumann entropy with decoherence parameter $\gamma = 0.001$. The atomic motion is neglected.

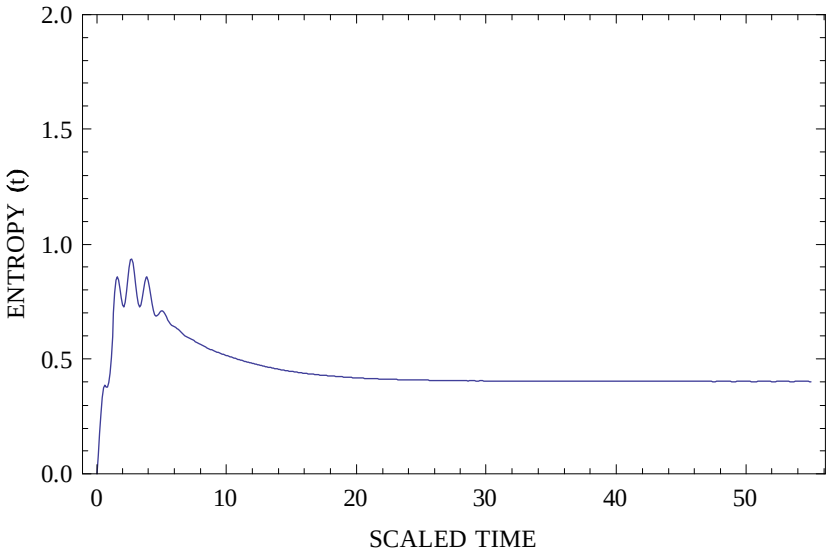

Figure 4-11: The effect of decoherence on the time evolution of the von Neumann entropy with decoherence parameter $\gamma = 0.01$. The atomic motion is neglected.

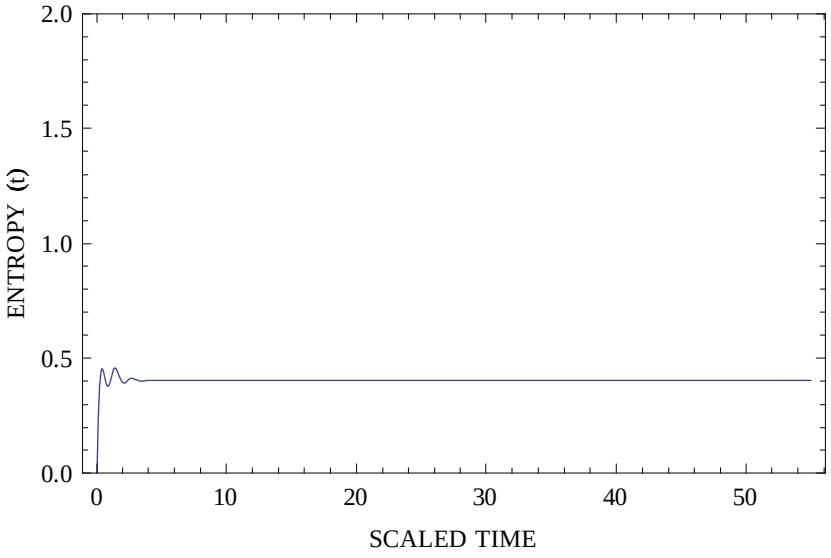

Figure 4-12: The effect of decoherence on the time evolution of the von Neumann entropy with decoherence parameter $\gamma = 0.1$. The atomic motion is neglected.

We also explore the effect of decoherence by introducing decoherence parameter γ in the dynamics of the atom-field interaction. It can be seen in Fig.(4-9) to Fig.(4-12) that by increasing decoherence in the system, the entanglement reduces with non-oscillatory constant behavior in

the static case and it reduces with periodic oscillatory behavior for the case of atom in motion as shown in Fig.(4-13) to Fig.(4-16).

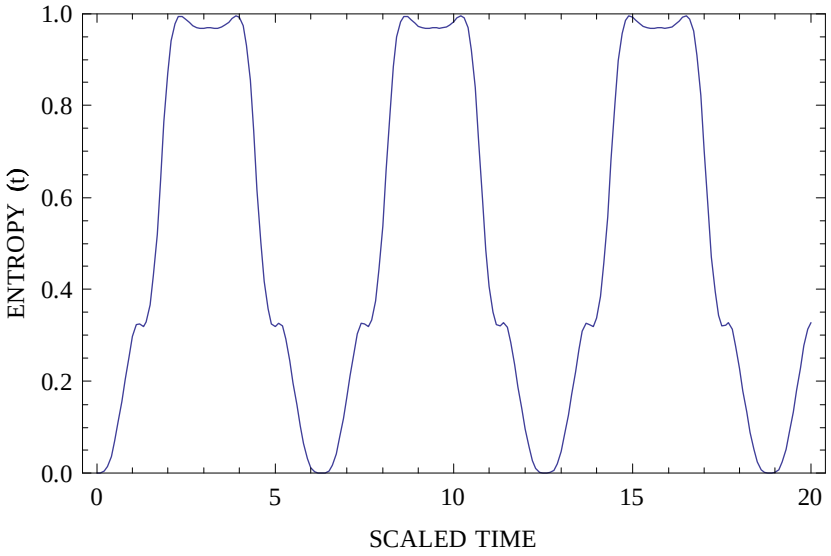

Figure 4-13: The effect of decoherence on the time evolution of the von Neumann entropy with decoherence parameter $\gamma = 0$. The atomic motion is considered.

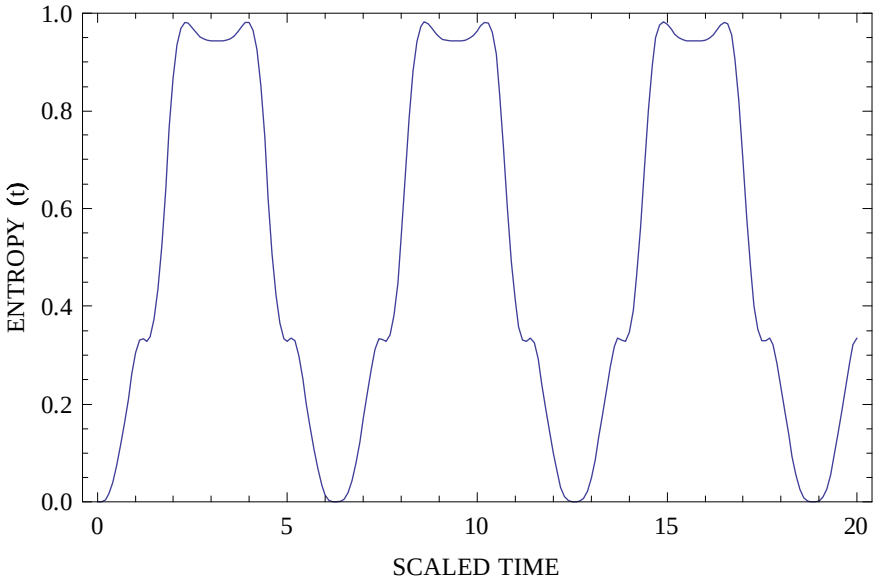

Figure 4-14: The effect of decoherence on the time evolution of the von Neumann entropy with decoherence parameter $\gamma = 0.001$. The atomic motion is considered.

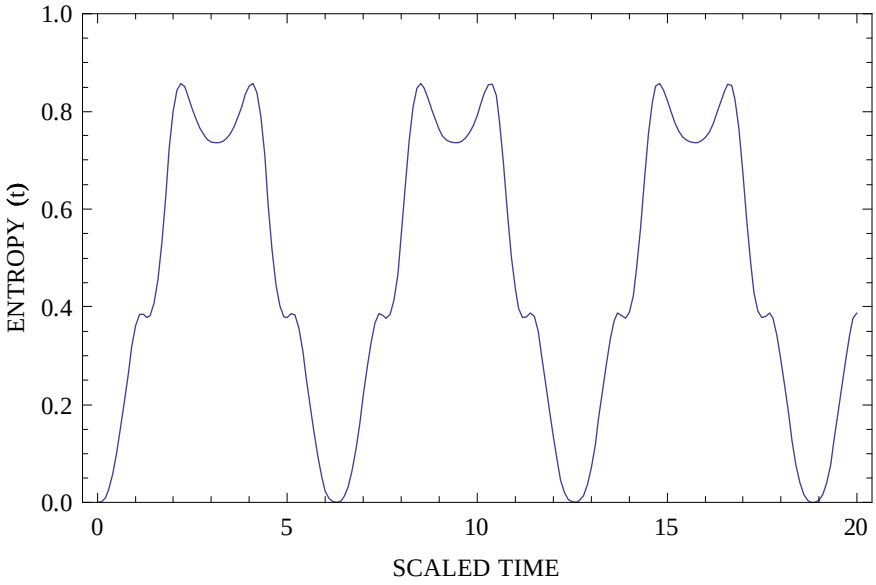

Figure 4-15: The effect of decoherence on the time evolution of the von Neumann entropy with decoherence parameter $\gamma = 0.01$. The atomic motion is considered.

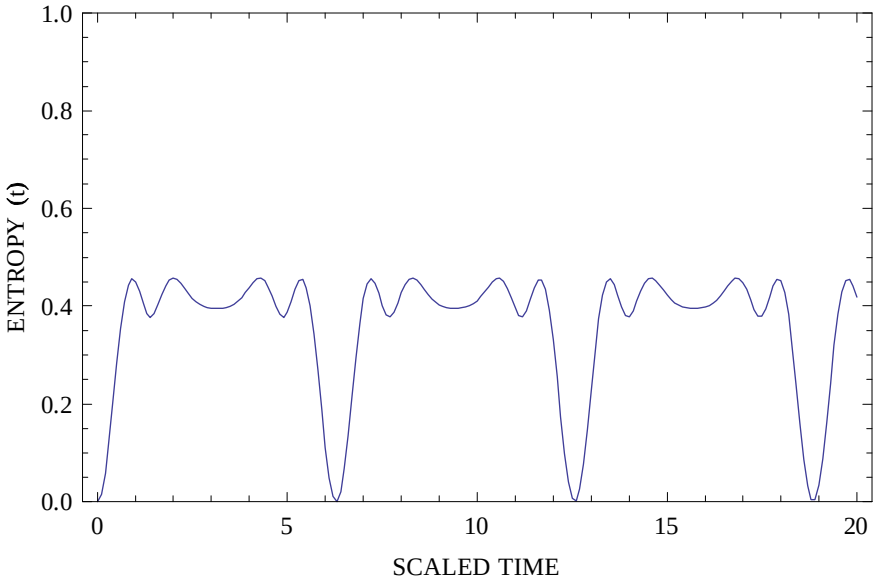

Figure 4-16: The effect of decoherence on the time evolution of the von Neumann entropy with decoherence parameter $\gamma = 0.1$. The atomic motion is considered.

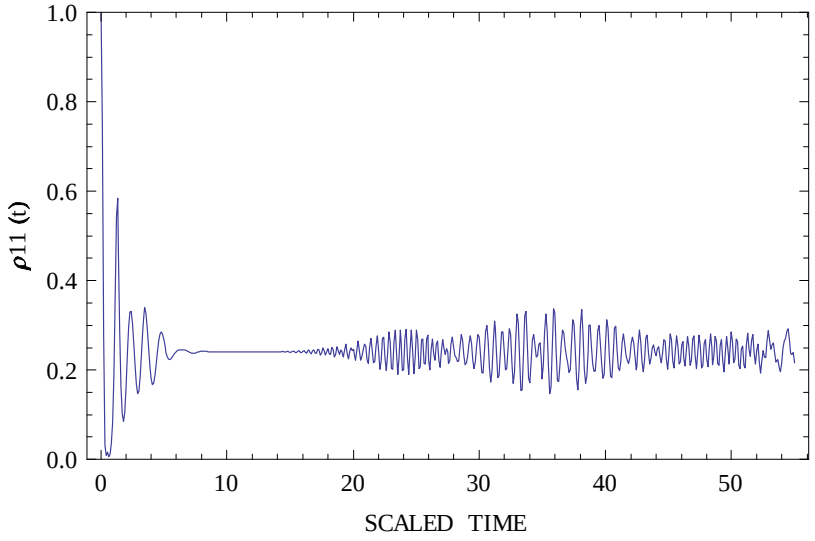

Figure 4-17: The effect of decoherence on the time evolution of the atomic population ρ_{11} with decoherence parameter $\gamma = 0$. The atomic motion is neglected.

In the static case the population continues to attain constant non-zero value due to decoherence effect while for the atomic motion case, its value also reduces except for the case of disentanglement where it attains maximum value of 1 even in the presence of decoherence. The results are shown in given Fig.(4-17) to Fig.(4-40).

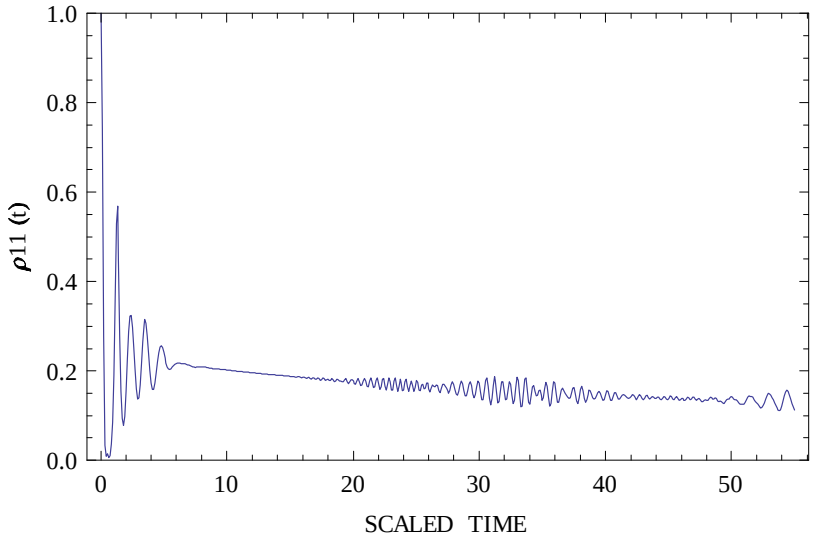

Figure 4-18: The effect of decoherence on the time evolution of the atomic population ρ_{11} with decoherence parameter $\gamma = 0.001$. The atomic motion is neglected.

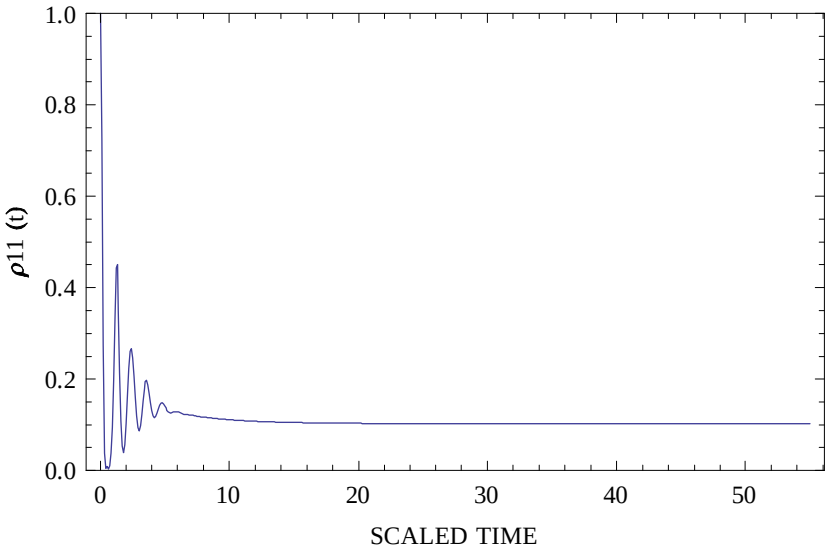

Figure 4-19: The effect of decoherence on the time evolution of the atomic population ρ_{11} with decoherence parameter $\gamma = 0.01$. The atomic motion is neglected.

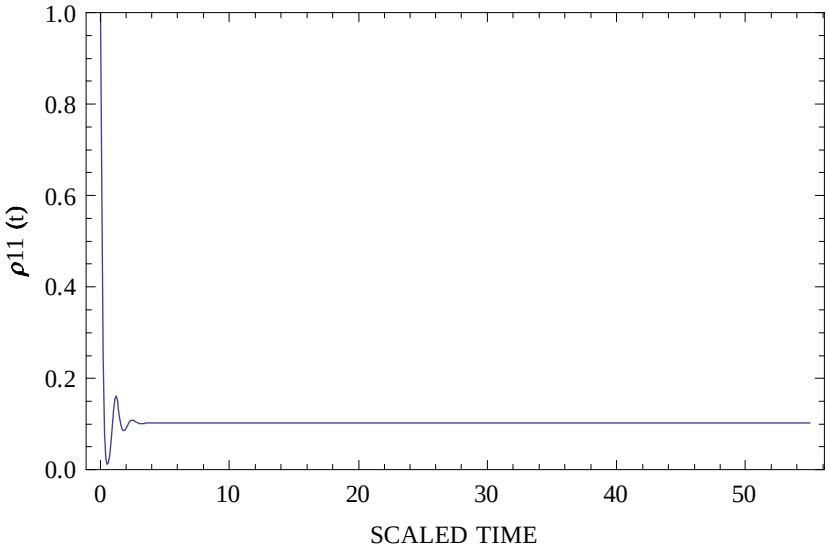

Figure 4-20: The effect of decoherence on the time evolution of the atomic population ρ_{11} with decoherence parameter $\gamma = 0.1$. The atomic motion is neglected.

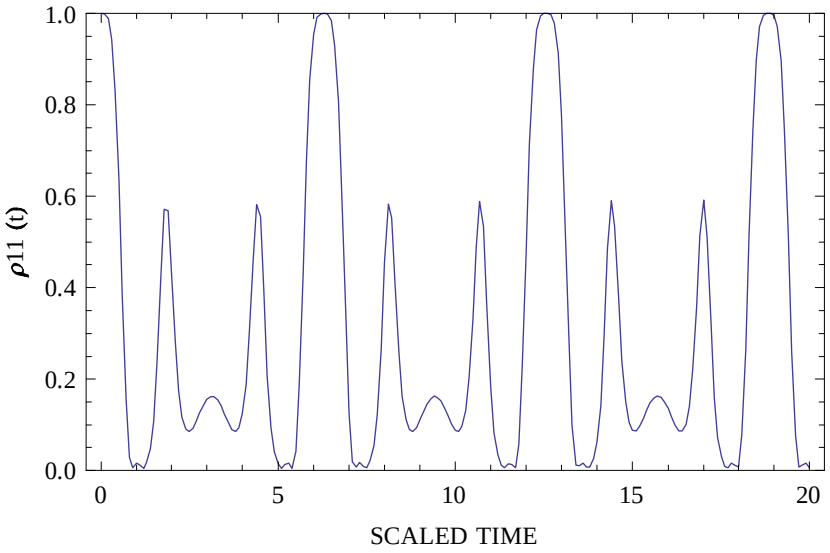

Figure 4-21: The effect of decoherence on the time evolution of the atomic population ρ_{11} with decoherence parameter $\gamma = 0$. The atomic motion is considered.

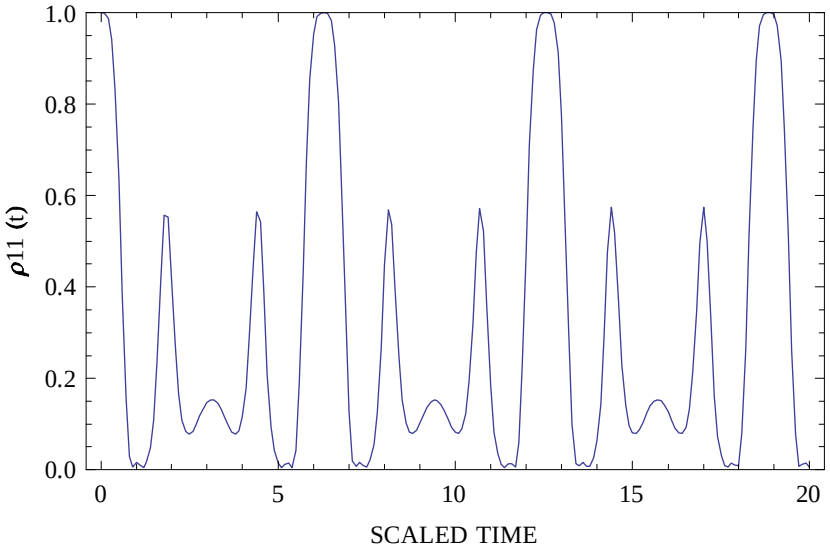

Figure 4-22: The effect of decoherence on the time evolution of the atomic population ρ_{11} with decoherence parameter $\gamma = 0.001$. The atomic motion is considered.

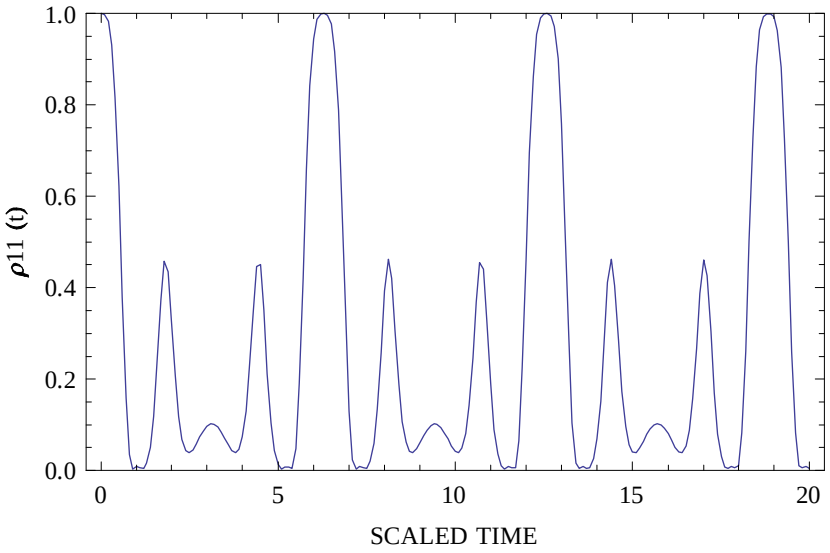

Figure 4-23: The effect of decoherence on the time evolution of the atomic population ρ_{11} with decoherence parameter $\gamma = 0.01$. The atomic motion is considered.

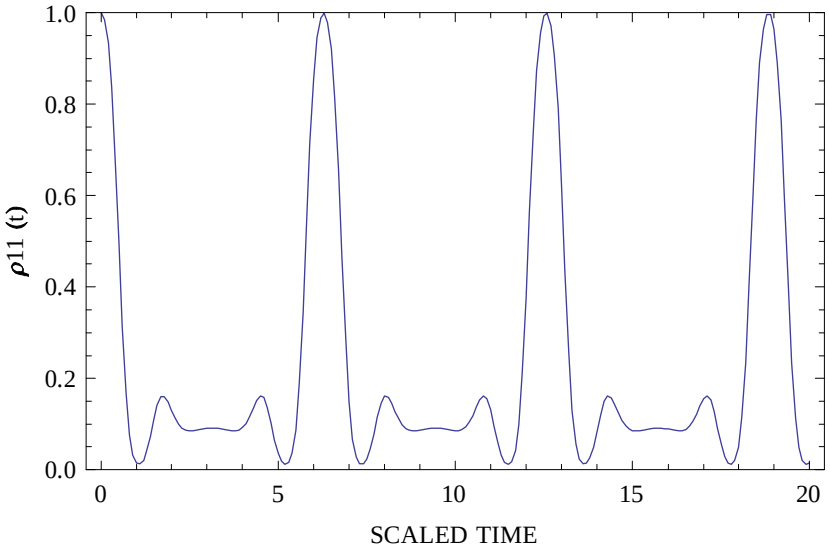

Figure 4-24: The effect of decoherence on the time evolution of the atomic population ρ_{11} with decoherence parameter $\gamma = 0.1$. The atomic motion is considered.

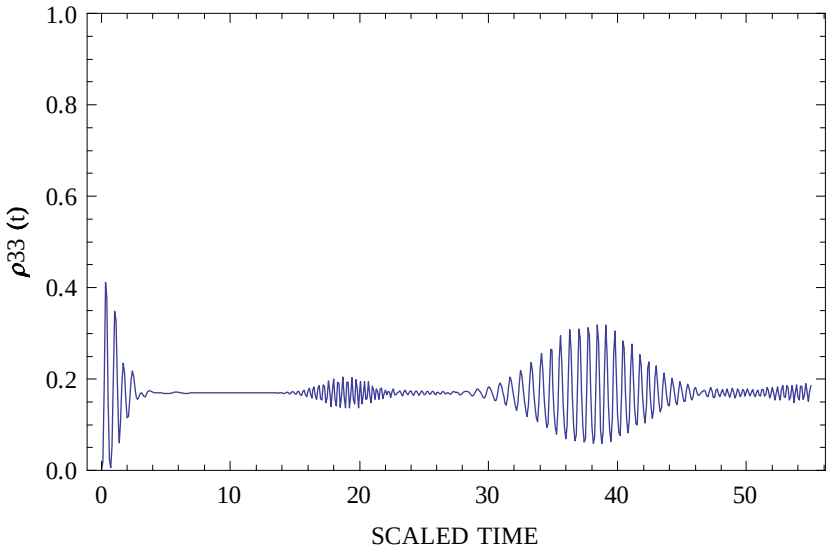

Figure 4-25: The effect of decoherence on the time evolution of the atomic population ρ_{33} with decoherence parameter $\gamma = 0$. The atomic motion is neglected.

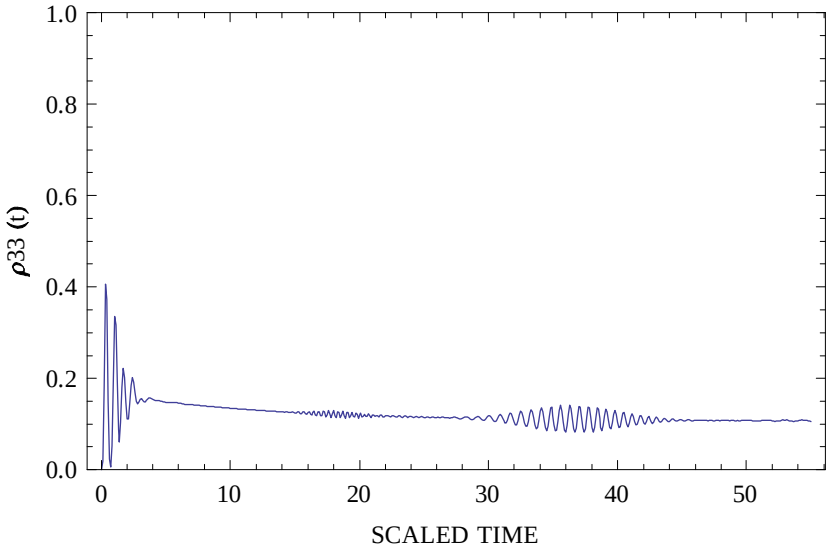

Figure 4-26: The effect of decoherence on the time evolution of the atomic population ρ_{33} with decoherence parameter $\gamma = 0.001$. The atomic motion is neglected.

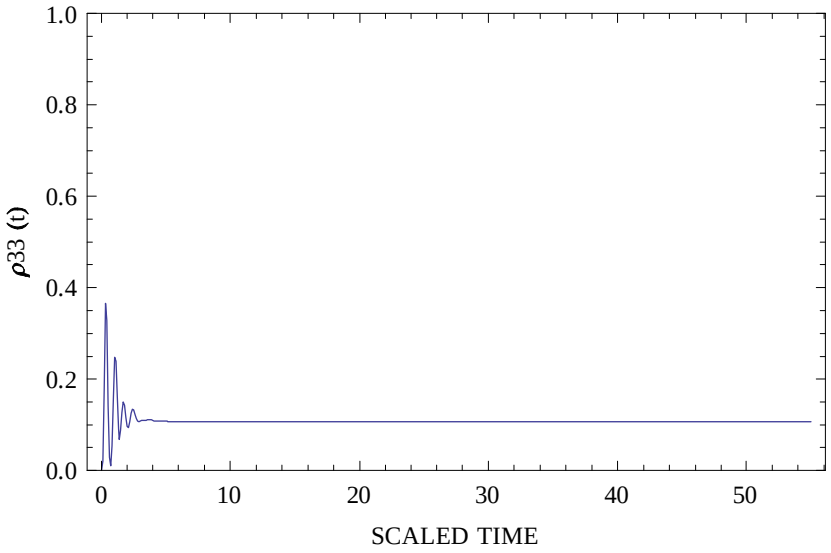

Figure 4-27: The effect of decoherence on the time evolution of the atomic population ρ_{33} with decoherence parameter $\gamma = 0.01$. The atomic motion is neglected.

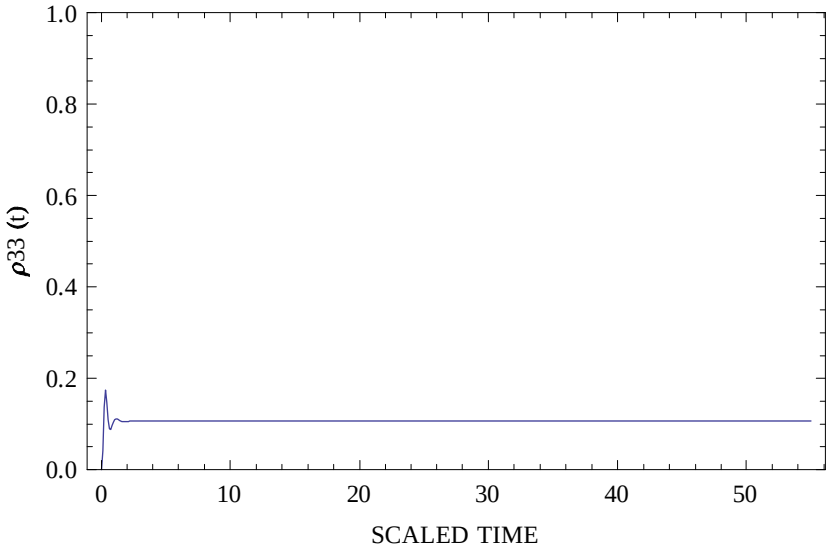

Figure 4-28: The effect of decoherence on the time evolution of the atomic population ρ_{33} with decoherence parameter $\gamma = 0.1$. The atomic motion is neglected.

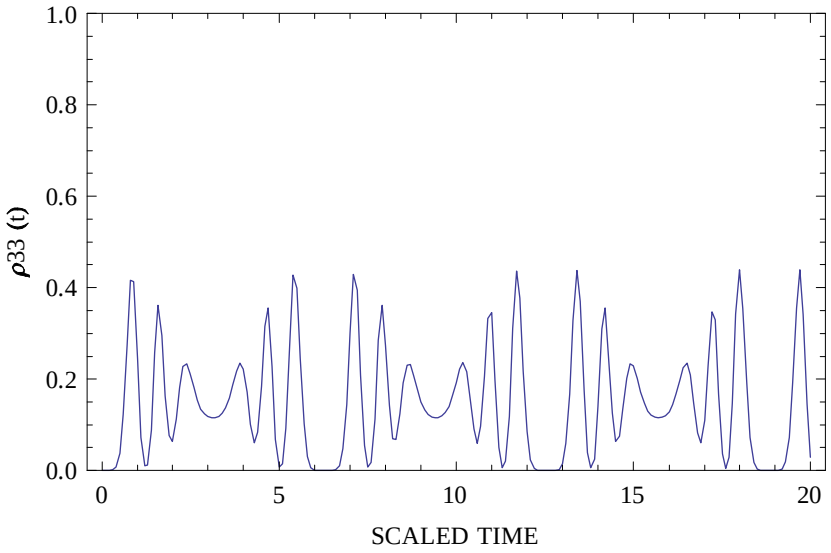

Figure 4-29: The effect of decoherence on the time evolution of the atomic population ρ_{33} with decoherence parameter $\gamma = 0$. The atomic motion is considered.

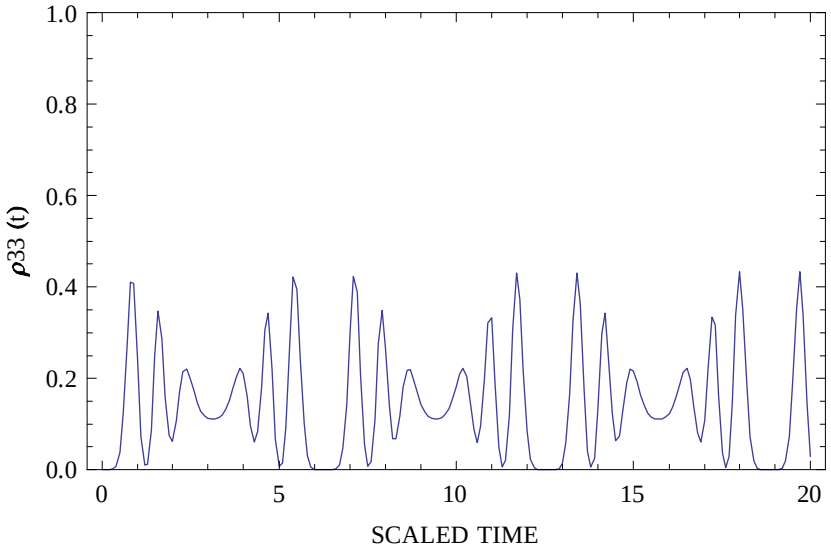

Figure 4-30: The effect of decoherence on the time evolution of the atomic population ρ_{33} with decoherence parameter $\gamma = 0.001$. The atomic motion is considered.

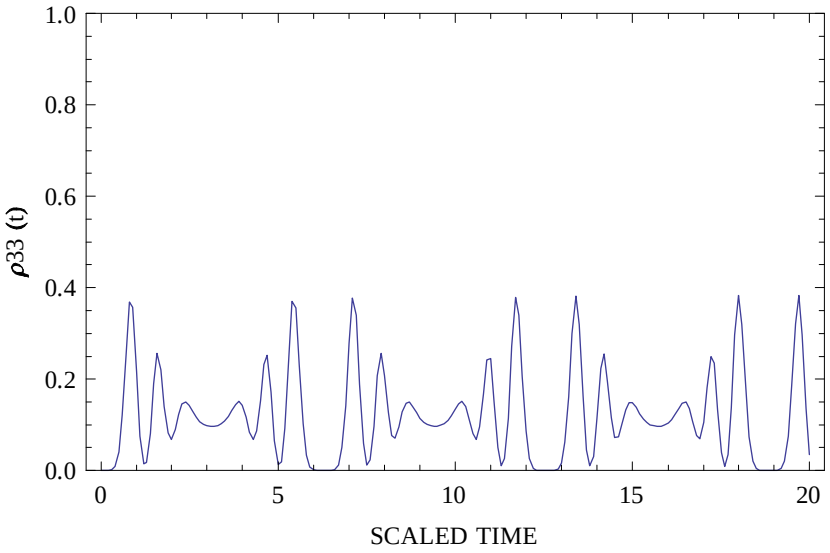

Figure 4-31: The effect of decoherence on the time evolution of the atomic population ρ_{33} with decoherence parameter $\gamma = 0.01$. The atomic motion is considered.

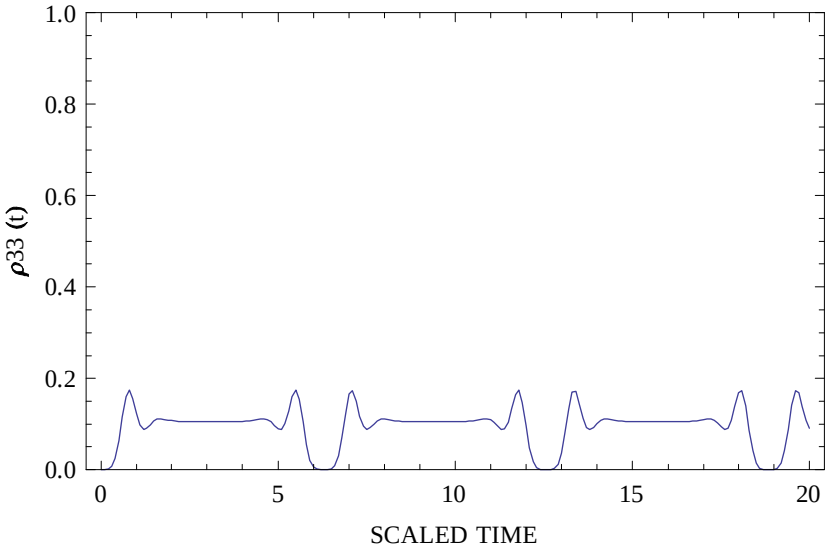

Figure 4-32: The effect of decoherence on the time evolution of the atomic population ρ_{33} with decoherence parameter $\gamma = 0.1$. The atomic motion is considered.

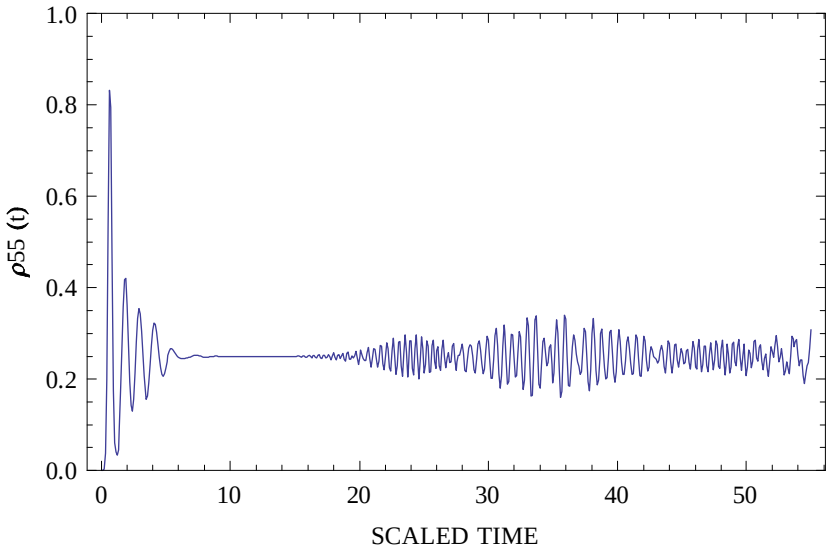

Figure 4-33: The effect of decoherence on the time evolution of the atomic population ρ_{55} with decoherence parameter $\gamma = 0$. The atomic motion is neglected.

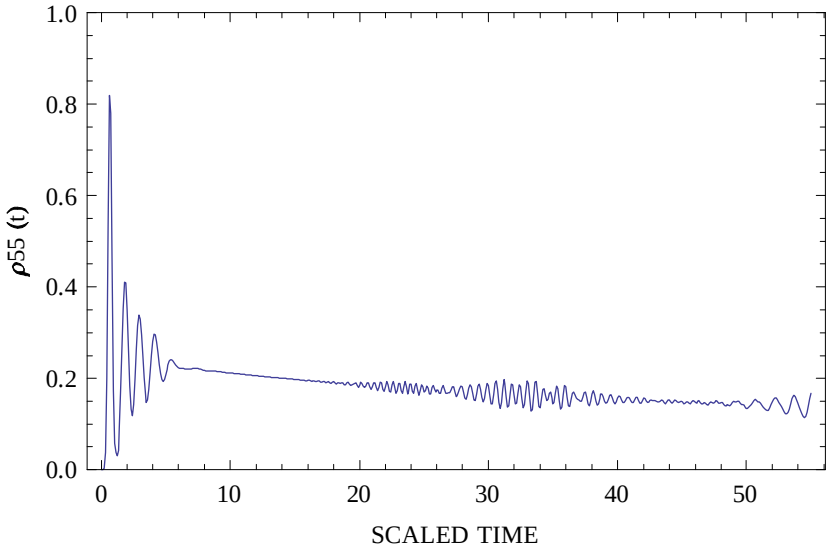

Figure 4-34: The effect of decoherence on the time evolution of the atomic population ρ_{55} with decoherence parameter $\gamma = 0.001$. The atomic motion is neglected.

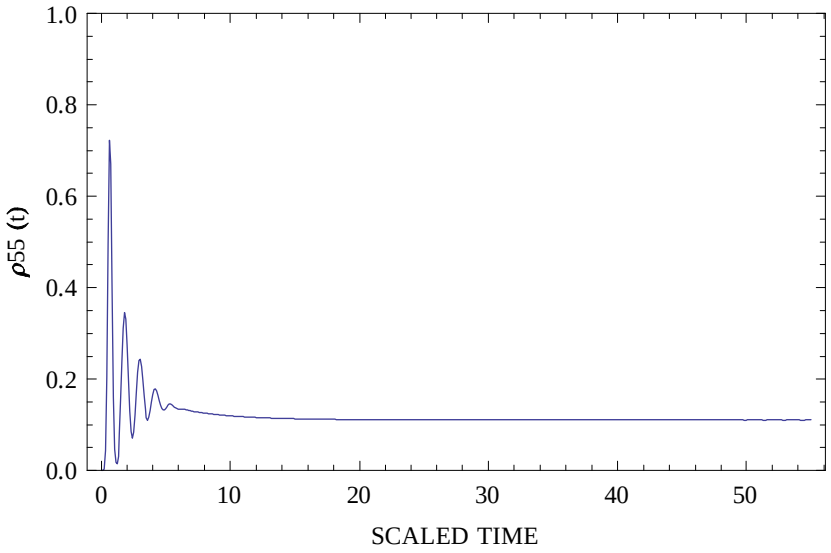

Figure 4-35: The effect of decoherence on the time evolution of the atomic population ρ_{55} with decoherence parameter $\gamma = 0.01$. The atomic motion is neglected.

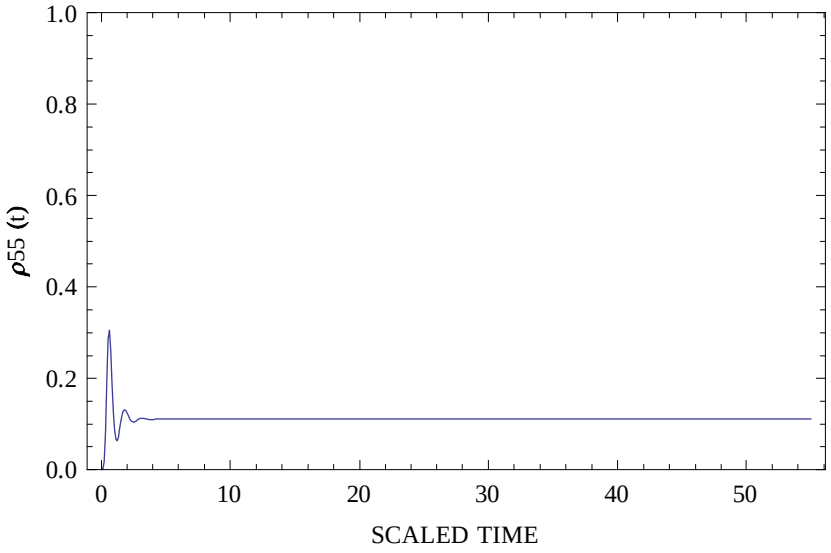

Figure 4-36: The effect of decoherence on the time evolution of the atomic population ρ_{55} with decoherence parameter $\gamma = 0.1$. The atomic motion is neglected.

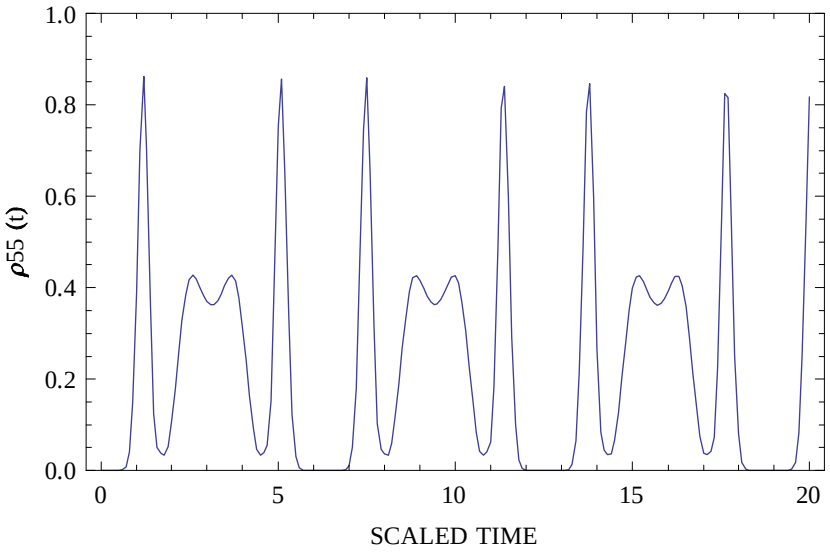

Figure 4-37: The effect of decoherence on the time evolution of the atomic population ρ_{55} with decoherence parameter $\gamma = 0$. The atomic motion is considered.

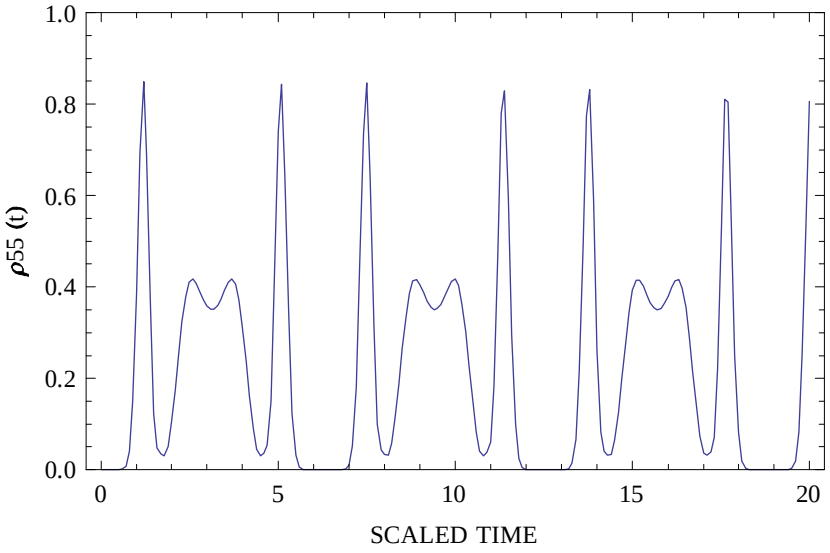

Figure 4-38: The effect of decoherence on the time evolution of the atomic population ρ_{55} with decoherence parameter $\gamma = 0.001$. The atomic motion is considered.

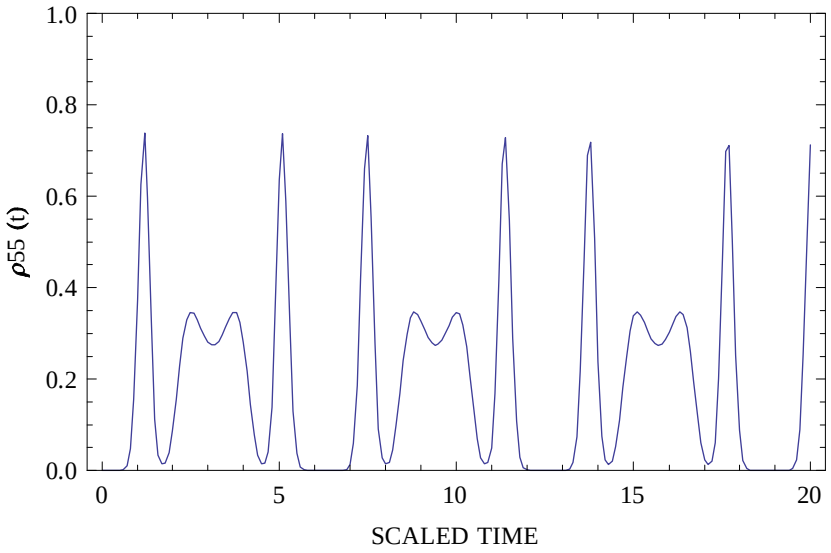

Figure 4-39: The effect of decoherence on the time evolution of the atomic population ρ_{55} with decoherence parameter $\gamma = 0.01$. The atomic motion is considered.

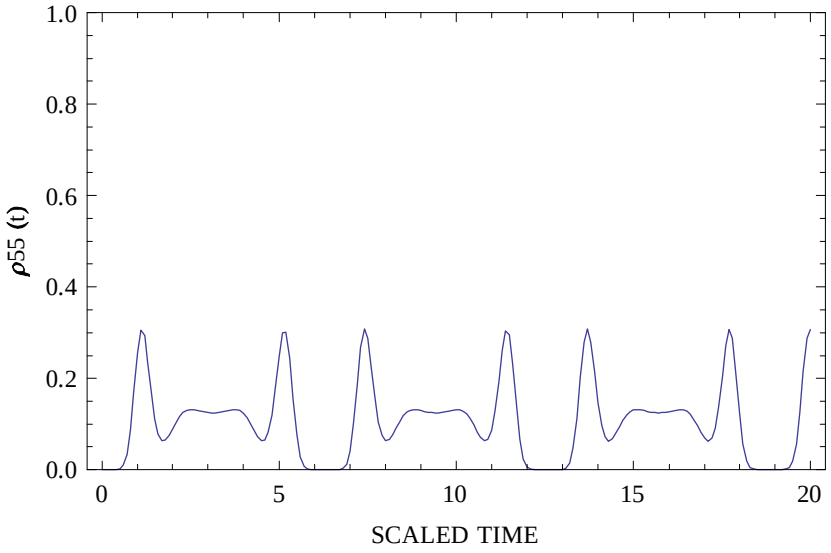

Figure 4-40: The effect of decoherence on the time evolution of the atomic population ρ_{55} with decoherence parameter $\gamma = 0.1$. The atomic motion is considered.

In the end, we can conclude that this research will lead us to explore the behavior of entanglement via different entanglement scenario for various atom-field interaction models which will be helpful to increase our understanding in the field of quantum information.

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