Fixed Point Theorems for Contractive Type Mappings in Distance Spaces

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Contents

Abstract

We often come across nonlinear systems and their associated nonlinear equations and both pure and applied mathematics play a vital role in dealing with such kind of equations.

When we talk about non linear analysis, its fundamental importance in physical, biological, engineering and technological sciences can be seen in the formulation and analysis of various classes of equation.

As far as fixed point theory is concerned, its extensiveness can be seen by its applications in various fields. Theorems that are concerned with the existence and properties of fixed points are known as fixed point theorems. These theorems play a very important role for proving the existence and uniqueness of solutions for various mathematical models. Metric fixed point theory has taken on new dimensions since the inception of the well known Banach contraction principle, and this contraction principle gives us a really suitable base to find fixed point for self mappings. The main aim of this thesis is to add some more widely applicable results to the literature.

Preface

This thesis is organized in six chapters and each of them has a brief introduction.

Chapter 1 is a survey aimed to fix notions and notations. Therefore, it collects basic concepts and significant facts in the existing literature related to fixed points, some classical fixed point theorems, weaker form of contraction and some generalized versions of metric space.

Chapter 2 deals with the advancement of fixed point theory in some generalized versions of metric space such as gauge and b-gauge spaces and results are proved for both single-valued as well as multivalued mappings using different contraction condition and an application for integral equation of Volterra type is discuss to validate the given results.

Chapter 3 is devoted to fixed point theorems in the settings of generalized version of metric spaces precisely the vector-valued or generalized metric space, using different kind of contraction conditions for single-valued as well as multivalued mappings.

Chapter 4 is a dedicated to the enhancement of fixed point theory to obtain results for the existence and unique-ness of fixed point. The results are obtained for single-valued mapping using the uniform metric spaces enriched with the concept of graph. Different kind of contraction conditions are used to obtain some new and interesting results.

In Chapter 5 we have obtained fixed points and common fixed points for the family of bounded multivalued mappings as well as for the family of closed multivalued mappings. We have concluded our findings with an existence theorem for a system of integral equations.

Chapter 6 deals with the existence of proximity point for different kind of contraction-type conditions in metric spaces. New proximity theorems are proved for multivalued mappings.

Dedicated

To my family.

Chapter 1

Introduction and Preliminaries

In this chapter we will present some of those results which will be essential for the later chapters. This chapter has been solely dedicated to those definitions and theorems which will govern this dissertation, and a detailed survey in this chapter is aimed on fix notions and notations, and collections of basic concepts and significant facts in the existing literature.

Throughout this chapter $F-P$ represents a fixed point, $M-S$ represents a metric space, $C-S$ represents a Cauchy sequence.

1.1 Fixed points

Given a non-empty set X_d and a map T_f from X_d into itself, the problem of finding a point $e \in X_d$, such that $T_f(e) = e$ is considered as a fixed point problem, and the point $e \in X_d$ is called a fixed point of T_f.

A natural question is, under what conditions on X_d and T_f does a fixed point exist? Theorems which establish the existence (and uniqueness) of such points are called F-P theorems. Fixed point theorems enable us to find the existence of solutions for operator equations satisfying certain conditions.

1.1.1 Fixed point for single valued mappings

Let $X_d \neq \phi$ and T_f be a map from X_d to itself. Then we can say that a point $e \in X_d$ is called a $F-P$ of T_f if $T_f e = e$.

Example 1.1.1. Let $X_d = \mathbb{R}$ and $T_{f_1}, T_{f_2}, T_{f_3}$ be self mappings of X_d. Then,

(i) $T_{f_1}e = exp(e)$, T_{f_1} has no $F-P$.

(ii) $T_{f_2}e = \frac{e+1}{2}$, T_{f_2} has unique $F-P$.

(iii) $T_{f_3}e = [e]$, T_{f_3} has infinite many $F-P$'s.

1.1.2 Fixed points for multi valued mappings

Let X_d and Y_d be two $M-S$'s and $T_f : X_d \to P(Y_d)$, where $P(Y_d)$ is the power set of Y_d. Then for each element $e \in X_d$, $T_f e$ is a non-empty subset of Y_d, and we call T_f from X_d to $P(Y_d)$ a multivalued mapping.

Example 1.1.2. Let $X_d = [0,1]$ and $T_{f_1} : X_d \to Y = P(X_d)$ where $P(X_d)$ is the power set of X_d and T_{f_1} be defined as,

$$T_{f_1}e = \begin{cases} [0,1] & \text{if e=0,1;} \\ \{e\} & \text{otherwise.} \end{cases}$$

Then each $e \in X_d$ is a $F-P$ of T_f.

Example 1.1.3. Let $X_d = [0,1]$ and $T_{f_1} : X_d \to Y = P(X_d)$ where $P(X_d)$ is the power set of X_d) and T_{f_1} be defined as,

$$T_{f_1}e = \begin{cases} [0,1] & \text{if e=0,1;} \\ \{e^2, \sqrt{e}\} & \text{otherwise.} \end{cases}$$

Then 0 and 1 are $F-P$'s of T_{f_1}.

If T_f is a multivalued mapping and $e \in X_d$ is a point such that $e \in T_f e$ and $T_f e = \{e\}$ then e is the ***end point*** of T_f.

1.2 Banach and Nadler fixed point theorems

Banach and Nadler are considered as pioneers of metric $F-P$ theory for contractive type mappings. They both, with their exceptional contributions, have given new dimensions to $F-P$ theory for single and multivalued contractive type mappings. Their work has laid new foundations in the field of $F-P$ theory, and research has moved into a whole new era.

1.2.1 Contraction mappings

Let X_d be a complete $M-S$ and T_f a self mapping of X_d into itself, then the operator T_f is called contraction mapping if

$$d(T_f e, T_f \tilde{e}) < \alpha_f d(e, \tilde{e}) \tag{1.1}$$

for any element $e, \tilde{e} \in X_d$ and $\alpha_f \in (0,1)$.

1.2.2 Banach contraction principle

By a metric $F-P$ theorem we mean an existence result for a $F-P$ of mapping T_f, under conditions on a metric d, and which are not metric invariant. The most important and fundamental metric $F-P$ theorem is the Banach $F-P$ theorem, also known as the contraction mapping principle. It will not be false to state that the inception of the Banach contraction principle has opened many doors in metric fixed point theory. Experts in various fields have been continuously generalizing this famous contraction condition, which is quite promising. This theorem assures that every contraction from a complete $M-S$ into itself has a unique $F-P$.

Stefan Banach (1892-1945) was a Polish mathematician and founder of the great Polish School of Functional Analysis. This theorem first appeared in explicit form in Banach's PhD thesis (1922), which states that:

Theorem 1.2.1. Let $X_d \neq \phi$ be a complete $M-S$ with the metric d and let T_f from X_d to itself be a contraction on X_d, that is,

$$d(T_f(e), T_f(\tilde{e})) \leq \alpha_w d(e, \tilde{e}) \tag{1.2}$$

where $0 \leq \alpha_w < 1$ and $\forall\ e, \tilde{e} \in X_d$. Then T_f has a unique $F-P$.

In 1969 Kannan obtained unique $F-P$ for mappings T_f from X_d into itself satisfying:

$$d(T_f e, T_f \tilde{e}) = k[d(e, T_f e) + d(\tilde{e}, T_f \tilde{e})] \tag{1.3}$$

$\forall\ e, \tilde{e} \in X_d$, where $0 \leq k < 1/2$ and X_d is a complete $M-S$.

In 1972, Chatterjea quoted unique $F-P$ of a mappings T_f from X_d to itself satisfying:

$$d(T_f e, T_f \tilde{e}) = k[d(e, T_f \tilde{e}) + d(\tilde{e}, T_f e)] \tag{1.4}$$

$\forall\ e, \tilde{e} \in X_d$, where $0 \leq k < 1/2$ and X_d is a complete $M-S$.

Mappings satisfying contractive conditions (1.3) and (1.4) are called Kannan and Chatterjea mappings, respectively. It is observed that (1.3) and (1.4) do not imply continuity of a mapping T on its domain, which differentiates the nature of these mappings from the Banach contraction mapping. It is important to note that both of these $F-P$ result have great importance in modern fixed point theory for contractive type mappings.

1.2.3 Hausdorff distance

Let (X_d, d) be a $M-S$. Let $N_f(X_d)$ be the set of all nonempty subsets of X_d, $Cl_f(X_d)$ be the set of all the nonempty and closed subsets of X_d, $B_f(X_d)$ be the set of all nonempty and bounded subsets of X_d and $CB_f(X_d)$ as the set of all the nonempty, bounded and closed subsets of X_d. For an $a \in X_d, A_{2f} \in N_f(X_d)$,

$$d(a, A_{2f}) = \inf\{d(a, b) : b \in N_f(X_d)\}.$$

For $A_{1f}, A_{2f} \in B_f(X_d)$,

$$\delta(A_{1f}, A_{2f}) = \sup\{d(a, b) : a \in A_{1f}, b \in A_{2f}\}.$$

Note that δ satisfies all of the conditions of a metric, except $A_{1f} = A_{2f} \Rightarrow \delta(A_{1f}, A_{2f}) = 0$. For $A_{1f}, A_{2f} \in CB_f(X_d)$, the Hausdorff metric on $CB_f(X_d)$ is given as,

$$H_m(A_{1f}, A_{2f}) = \max\{\sup_{e \in A_{1f}} d(e, A_{2f}), \sup_{\tilde{e} \in A_{2f}} d(\tilde{e}, A_{1f})\}.$$

Thus the Hausdorff distance is a metric between two point sets.

Example 1.2.2. Let $X_d = \mathbb{R}$ and $A_{1f} = [0,1]$, $A_{2f} = [2,3]$ then $H_m(A_{1f}, A_{2f}) = 2$

For $A_{1f}, A_{2f} \in Cl_f(X_d)$, the generalized Hausdorff metric on $Cl_f(X_d)$ is given as,

$$H_m(A_{1f}, A_{2f}) = \begin{cases} \max\{\sup_{e \in A_{1f}} d(e, A_{2f}), \sup_{\tilde{e} \in A_{2f}} d(\tilde{e}, A_{1f})\} & \text{if the maximum exists} \\ \infty & \text{otherwise} \end{cases}$$

Many researchers working on this metric have made some exceptional contributions. Recently Czerwik [43] defined Hausdorff b-metric for the space of all nonempty, bounded and closed subsets of the b-$M-S$ (X_d, d, s), which is an extension of the notion of a b-$M-S$ (X_d, d, s). Consider that (X_d, d, s) as a b-$M-S$. For $e \in X_d$ and $A_{1f} \subset X_d$, $d(e, A_{1f}) = \inf\{d(e,a) : a \in A_{1f}\}$. For $A_{1f}, A_{2f} \in CB_f(X_d)$, the function $H_m : CB_f(X_d) \times CB_f(X_d) \to [0, \infty)$ can be defined by $H_m(A_{1f}, A_{2f}) = \max\left\{ \sup_{a_1 \in A_{1f}} d(a, A_{2f}), \sup_{a_2 \in A_{2f}} d(a_2, A_{1f})\right\}$ is called a Hausdorff b-metric influenced by the well known b-$M-S$ (X_d, d, s). All the properties of Hausdorff b-$M-S$ are the same as that of a Hausdorff metric, except the for triangle inequality, which in Hausdorff b-$M-S$ takes the form $H_m(A_{1f}, A_{2f}) \leq s[H_m(A_{1f}, \tilde{A}_3) + H_m(\tilde{A}_3, A_{2f})]$. Czerwik [43] also succeeded in extending the famous Nadler's $F-P$ theorem to the setting of Hausdorff b-$M-S$.

1.2.4 Nadler's fixed point theorem

In 1967 Nadler initiated the idea for multi-valued contraction mapping, and made use of the Hausdorff distance to give a multivalued version of the Banach contraction principle, which states that every multivalued contraction mapping defined on a complete $M-S$ has a $F-P$.

Theorem 1.2.3. Let (X_d, d) be a complete $M-S$ and $T_f : X_d \to CB_f(X_d)$ be a mapping such that

$$H_m(T_f e, T_f \tilde{e}) \leq \alpha_w d(e, \tilde{e}),$$

$\forall\ e, \tilde{e} \in X_d$ where $\alpha_w \in [0,1)$. Then T_f has a $F-P$.

Note that H_m is not a metric on the set of bounded subsets of X_d, which is illustrated below through an example.

Let $X_d = \mathbb{R}$, endowed with usual metric then $H_m(A_{1f}, A_{2f}) = 0$ but $A_{1f} \neq A_{2f}$ for $A_{1f} = [0, 1)$ and $A_{2f} = [0, 1]$. This implies that H_m is not a metric on Bounded subsets of $\mathbb{R}$.

Lemma 1.2.4. Let (X_d, d) be a $M - S$ and $A_{2f} \in CL_f(X_d)$. Then, for each $e \in X_d$ and $p > 1$, there exists a $b \in A_{2f}$ such that $d(e, b) \leq qd(e, B_f)$.

1.3 Some generalizations of a metric space

Since the axiomatic interpretation of a $M - S$ by French mathematician M. Frechet, in the year 1906, advancements in mathematics in general, and functional analysis in particular have taken new directions. Inspired by this natural idea, several researchers have tried different generalizations of this notion. In this section we shall limit our discussion to some of the generalizations which are essential for this dissertation.

1.3.1 b-metric space

Another important generalization of $M - S$, which will be the center of our discussion in one part of this dissertation is the b-$M - S$, which was introduced by Czerwik [42]. For more details regarding the definition, convergence and cauchyness in b-$M - S$ please read [42] and it would be important to note here that every $M - S$ is a b-$M - S$ but the converse is not true.

1.3.2 b-gauge spaces

In the recent past the Banach contraction principle has played a vital role in the advancement of metric fixed point theory. Several generalizations in different directions have proved the fruitfulness of this fundamental principle. One important concept in this regard is that of gauge spaces.
A Gauge space can be characterized by the fact that, if we consider two distinct points, then the distance between them may not be zero. This simple concept has been the center of interest for many researchers world wide. For more details on gauge spaces see [46]. The Banach contraction principle was

generalized to gauge spaces by Frigon [51] and Chis and Precup [38]. And for more details and recent developments in b-gauge spaces please check Ali *et al.*[21].

1.3.3 Vector valued metric space

Let X_d be a non-empty set and $\mathbb{R}^m$ is the set of all m-tuples of real numbers. If $\hat{\zeta}, \tilde{\eta} \in \mathbb{R}^m$, $\hat{\zeta} = (\hat{\zeta}_1, \hat{\zeta}_2, \ldots, \hat{\zeta}_m)$, $\tilde{\eta} = (\tilde{\eta}_1, \tilde{\eta}_2, \ldots, \tilde{\eta}_m)$ and $c \in \mathbb{R}$, then by $\hat{\zeta} \leq \tilde{\eta}$ (resp., $\hat{\zeta} < \tilde{\eta}$) we mean $\hat{\zeta}_i \leq \tilde{\eta}_i$ (resp., $\hat{\zeta}_i < \tilde{\eta}_i$) for $i \in \{1, 2, \ldots, m\}$ and by $\hat{\zeta} \leq c$ we mean that $\hat{\zeta}_i \leq c$ for $i \in \{1, 2, \ldots, m\}$. A mapping $d_v \colon X_d \times X_d \to \mathbb{R}^m$ is called a vector-valued metric on X_d if the following properties are satisfied:

(d_{v_1}) $d_v(e, \tilde{e}) \geq 0 \ \forall \ e, \tilde{e} \in X_d$; if $d_v(e, \tilde{e}) = 0$, then $e = \tilde{e}$;

(d_{v_2}) $d_v(e, \tilde{e}) = d_v(\tilde{e}, e) \ \forall \ e, \tilde{e} \in X_d$;

(d_{v_3}) $d_v(e, \tilde{e}) \leq d_v(e, \tilde{\tilde{e}}) + d_v(\tilde{\tilde{e}}, \tilde{e}) \ \forall \ e, \tilde{e}, \tilde{\tilde{e}} \in X_d$.

A set X_d equipped with a vector-valued metric d_v is called a generalized-$M-S$ and it is denoted by (X_d, d_v). The notions that are defined in the *C-M-S* are similar to those defined in usual $M-S$.

1.3.4 Uniform space

Consider X_d as a nonempty set. The nonempty family ϑ_f, which is actually the family of subsets of $X_d \times X_d$ is said to be the uniform structure of X_d, if the following properties are satisfied:

(i) if $G_f \in \vartheta_f$, then the diagonal $\{(e, e) | e \in X_d\}$ is contained in G_f;

(ii) if $G_f \in \vartheta_f$ and H_f is contained in $X_d \times X_d$ which contains G_f, then $H_f \in \vartheta_f$;

(iii) if G_f and $H_f \in \vartheta_f$, then $G_f \cap H_f \in \vartheta_f$;

(iv) if $G_f \in \vartheta_f$, then $\exists \ H_f \in \vartheta_f$, such that, whenever $(e, \tilde{e})$ and $(\tilde{e}, \tilde{\tilde{e}}) \in H_f$, then $(e, \tilde{\tilde{e}}) \in G_f$;

(v) if $G_f \in \vartheta_f$, then $\{(\tilde{e}, e)|(e, \tilde{e}) \in G_f\}$ also $\in \vartheta_f$.

Then we call the pair (X_d, ϑ_f) a uniform space and the element of ϑ_f is said to be the neighborhood. The pair (X_d, ϑ_f) is said to be quasi-uniform space (see e.g. [31, 100]) if we omit property (v). Let $\Delta_v = \{(e, e)|e \in X_d\}$ be the diagonal of a non-empty set X_d. For $\tilde{V_1}, \tilde{V_2} \in X_d \times X_d$, we shall use the following setting in the sequel

$$\tilde{V_1} \circ \tilde{V_2} = \{(e, \tilde{e})| \; there \; exists \tilde{\tilde{e}} \in X_d : (e, \tilde{\tilde{e}}) \in \tilde{V_2} \; and (\tilde{\tilde{e}}, \tilde{e}) \in \tilde{V_1}\}$$

and

$$\tilde{V_1}^{-1} = \{(e, \tilde{e})|(\tilde{e}, e) \in \tilde{V_1}\}.$$

For a subset $\tilde{V_1} \in \vartheta_f$, a pair of points e and $\tilde{e}$ are considered to be $\tilde{V_1}$-close if $(e, \tilde{e}) \in \tilde{V_1}$ and $(\tilde{e}, e) \in \tilde{V_1}$. Furthermore, a sequence $\{e_n\}$ in X_d is said to be a $C - S$ for ϑ_f, if for any $\tilde{V_1} \in \vartheta_f$, there exists $N_f \geq 1$ such that e_n and e_m are $\tilde{V_1}$-close for $n, m \geq N_f$. For (X_d, ϑ_f), there is a unique topology $\tau_f(\vartheta_f)$ on X_d generated by $\tilde{V_1}(e) = \{\tilde{e} \in X_d|(e, \tilde{e}) \in \tilde{V_1}\}$ where $\tilde{V_1} \in \vartheta_f$. A sequence $\{e_n\}$ in X_d is convergent to e for ϑ_f, denoted by $\lim_{n\to\infty} e_n = e$, if for any $\tilde{V_1} \in \vartheta_f$, $\exists \; n_0 \in \mathbb{N}$ such that $e_n \in \tilde{V_1}(e)$ for every $n \geq n_0$. A uniform space (X_d, ϑ_f) is called Hausdorff if the intersection of all the $\tilde{V_1} \in \vartheta_f$ is equal to Δ_v of X_d, that is, if $(e, \tilde{e}) \in \tilde{V_1} \; \forall \; \tilde{V_1} \in \vartheta_f$ implies $e = \tilde{e}$. If $\tilde{V_1} = \tilde{V_1}^{-1}$ then we shall say that a subset $\tilde{V_1} \in \vartheta_f$ is symmetrical. We shall assume that each $\tilde{V_1} \in \vartheta_f$ is symmetrical. For more details, see e.g. [4, 3, 2, 10].

1.4 Weaker forms of contractions

As we know, without any doubt, the Banach contraction principle is considered the most fundamental entity in metric $F - P$ theory and, from the point of view of research, this contraction principle is considered as an important tool both in pure and applied mathematics. Researchers worldwide have been constantly using different methods and techniques to generalize this famous contractive condition.

In this section we will discuss some of the weaker forms of contractions which will be useful in the later chapters.

1.4.1 F_w-contraction

Recently Wardowski in his work [see [99]] introduced a new family of mappings called F_w or $\mathfrak{F}_{\mathfrak{w}}$ family. Making use of these mappings from $\mathfrak{F}_{\mathfrak{w}}$ family, Wardowski succeeded in introducing a new contraction condition, which he named as the F_w-contraction. And which generalizes the Banach contraction in a different and nice way. Later on, many researchers worldwide generalized this result, see for example, [83, 41, 65, 85, 72, 76, 23, 67, 25, 66, 56]. Wardowski [99] introduced the $\mathfrak{F}_{\mathfrak{w}}$ family as: $\mathfrak{F}_{\mathfrak{w}}$ is the class of all functions $F_w : (0, \infty) \to \mathbb{R}$ satisfying the following three assumptions

(F_{w_1}) F_w is strictly increasing, that is, $\forall\ b_1, b_2 \in (0, \infty)$ with $b_1 < b_2$, we have $F_w(b_1) < F_w(b_2)$.

(F_{w_2}) For each sequence $\{\mathfrak{d}_n\}$ of positive real numbers we have $\lim_{n\to\infty} \mathfrak{d}_n = 0$ if and only if $\lim_{n\to\infty} F_w(\mathfrak{d}_n) = -\infty$.

(F_{w_3}) There exists $k \in (0, 1)$ such that $\lim_{\mathfrak{d}\to 0^+} \mathfrak{d}^k F(\mathfrak{d}) = 0$.

Here we have highlighted some of the examples of such functions.

- $F_{w_a} = \ln e\ \forall\ e \in (0, \infty)$.
- $F_{w_b} = e + \ln e\ \forall\ e \in (0, \infty)$.
- $F_{w_c} = -\frac{1}{\sqrt{e}}\ \forall\ e \in (0, \infty)$.

Further, Wardowski [99] introduced F_w-contraction and related $F - P$ theorem as given below:

Theorem 1.4.1. [99] Let (X_d, d) be a complete $M - S$ and let $T_f : X_d \to X_d$ is F_w-contraction, that is, $\exists\ F_w \in \mathfrak{F}_{\mathfrak{w}}$ and a $\tau_f > 0$ such that $\forall\ e, \tilde{e} \in X_d$ with $d(T_f e, T_f \tilde{e}) > 0$, we have

$$\tau_f + F_w(d(T_f e, T_f \tilde{e})) \leq F_w(d(e, \tilde{e})).$$

Then T_f has a unique $F - P$.

This theorem reduces to Banach contraction principle if T_f is F_{w_a}-contraction.

Many researchers have made some exceptional contributions and have further generalized Wardowski's F_w-contraction. For further details please consider [65], [85], [40]

1.4.2 $\alpha_w - \tilde{\psi}$-contraction

The notion of the $\alpha_w - \tilde{\psi}$-contraction principle is another contribution in the developments of metric $F - P$ theory. This technique was introduced by Samet *et. al.* [84], which generalizes the well known Banach contraction and is considered as unique in its own way.

$$\alpha_w d(T_f e, T_f \tilde{e}) \leq \tilde{\psi}(d(e, \tilde{e}))$$

Where Ψ denote the family of all non-decreasing functions, $\tilde{\psi} : [0, \infty) \rightarrow [0, \infty)$, such that, $\sum_{n=1}^{\infty} \tilde{\psi}^n(t) < \infty$, $\forall\ t > 0$, where $\tilde{\psi}^n$ is the n^{th} iterate of $\tilde{\psi}$. If $\tilde{\psi} \in \Psi$ then $\tilde{\psi}(t) < t$, $\forall\ t > 0$. And also if $T_f : X_d \rightarrow X_d$ and $\alpha_w : X_d \times X_d \rightarrow [0, \infty)$, we say that T_f is α_w-admissible, if for $e, \tilde{e} \in X_d$ and $\alpha_w(e, \tilde{e}) \geq 1$, we have $\alpha_w(T_f e, T_f \tilde{e}) \geq 1$. With this break through research in metric $F - P$ theory has widened to many different directions. The α_w-admissibility condition used by Samet [84], has been used quit frequently to discuss existence and uniqueness of $F - P$'s by different researchers in different ways. These two conditions introduced by Samet have been uploaded in a number of occasions by different experts, please see [11, 21, 23, 67, 25, 66, 56, 20, 24, 22, 57, 58].

Note that by taking $\alpha_w = 1\ \forall\ e, \tilde{e} \in X_d$ and $\tilde{\psi}(t) = at$ where $a \in [0, 1]$, the $\alpha_w - \tilde{\psi}$-contraction reduces to

$$d(T_f e, T_f \tilde{e}) \leq d(e, \tilde{e}).$$

1.4.3 Perov-contraction

Another generalization of our interest in this dissertation was initiated by Perov [74] in 1964, who generalized Banach contraction principle for contraction mappings on spaces enriched with vector-valued matrices. We denote $\overline{0}$ by the $m \times m$ matrix with all zero entries. A matrix A_{1f} is said to be convergent to zero if and only if $A_{1f}^n \rightarrow \overline{0}$ as $n \rightarrow \infty$ (see [96]). Also for some basic definitions and notions regarding Perov-contraction please se[[50], [78], [95]]

Theorem 1.4.2 ([74]). . Let (X_d, d) be a complete generalized-$M-S$ and the mapping $T_f \colon X_d \to X_d$ with the property that $\exists$ a matrix $A_{1f} \in M_{m,m}(R_+)$ such that $d(T_f(e), T_f(\tilde{e})) \leq A_{1f} d(e, \tilde{e})$ $\forall$ $e, \tilde{e} \in X_d$. If A_{1f} is a matrix convergent towards zero, then

(1) $Fix(T_f) = \{e^*\}$;

(2) the sequence of successive approximations $\{e_n\}$ such that, $e_n = f^n(e_0)$ is convergent, and it has the limit e^*, $\forall$ $e_0 \in X_d$.

1.4.4 Prešić contraction

In 1965 Prešić [75] also made a successful attempt to generalize the Banach contraction. His work is stated in the following theorem.

Theorem 1.4.3. [75] Let (X_d, d) be a complete $M-S$, k be a positive integer and $T_f : X_d^k \to X_d$ be a mapping such that

$$d(T_f(e_1, e_2, \cdots, e_k), T_f(e_2, e_3, \cdots, e_{k+1})) \leq \sum_{i=1}^{k} a_i d(e_i, e_{i+1}) \tag{1.5}$$

for every $e_1, e_2, \cdots, e_k, e_{k+1} \in X_d$, where $b_1, b_2, \cdots, a_k$ are nonnegative constants such that $\sum_{i=1}^{k} a_i < 1$. Then $\exists$ a unique point $e \in X_d$ such that $T_f(e, e, \cdots, e) = e$. Moreover, if $e_1, e_2, \cdots, e_k$ are arbitrary points in X_d and $\forall$ $n \in \mathbb{N}$ we have

$$e_{n+k} = T_f(e_n, e_{n+1}, \cdots, e_{n+k-1}) \tag{1.6}$$

then the sequence $\{e_n\}$ is convergent and $\lim e_n = T_f(\lim e_n, \lim e_n, \cdots, \lim e_n)$.

Note that a point $e \in X_d$ is known as $F-P$ of $T_f : X_d^k \to X_d$, if $T_f(e, e, \cdots, e) = e$.
Ćirić and Prešić [37] further generalized the above result, please see [37].

1.4.5 Proximal contraction

Fixed point theory focuses on the techniques to solve non-linear equations of the kind $T_f e = e$, where T_f is self mapping. But if T_f is not a self mapping, i.e $T_f : A_{1f} \to A_{2f}$ where A_{1f} and A_{2f} are non-empty subsets of a

$M-S$ X_d, then the equation $T_f e = e$ does not necessarily have solution. Consequently, it becomes the target to find out such an element e which is in any sense nearest to $T_f e$. In fact best approximation theory establishes an approximate solution to the equation $T_f e = e$. Best proximity theorems provide sufficient conditions for the existence of an element e such that the error $d(e, T_f e)$ is minimum. As it can be seen easily that best proximity point theorems are simple generalization of the $F-P$ theorems. Some best proximity theorems may also melt down to $F-P$ theorems, if the mapping is considered as a self mapping. In this regard Fan[48] proved a best proximity theorem considering a continuous self mapping. Many authors have extended this theorem in various directions and in this context Jleli *et al.* [54] introduced the notion of α_w-$\tilde{\psi}$-proximal contractive type mappings and proved some best proximity point theorems. Later on, Ali *et al.* [22] extended these notions to multivalued mappings. Many authors obtained best proximity point theorems in different settings, see for example [5, 6, 7, 13, 14, 15, 16, 44, 47, 45, 62, 77, 79, 97, 101]. Abkar and Gbeleh [7] and Al-Thagafi and Shahzad [14, 16] investigated best proximity points for multi-valued mappings.

Let (X_d, d) be a $M-S$. For $A_{1f}, A_{2f} \subseteq X_d$, we use the notions: $d(A_{1f}, A_{2f}) = \inf\{d(a,b) : a \in A_{1f},\ b \in A_{2f}\}$, $A_0 = \{a \in A_{1f} : d(a,b) = d(A_{1f}, A_{2f})\ for\ some\ b \in A_{2f}\}$, $B_0 = \{b \in A_{2f} : d(a,b) = d(A_{1f}, A_{2f})\ for\ some\ a \in A_{1f}\}$. A point $e^* \in X_d$ is considered to be a best proximity point of mapping $T_f : A_{1f} \to CL_f(A_{2f})$ if $d(e^*, T_f e^*) = d(A_{1f}, A_{2f})$. When $A_{1f} = A_{2f}$, the best proximity point reduces to $F-P$ of the mapping T_f.

Definition 1.4.4. [101] Let (A_{1f}, A_{2f}) be a pair of nonempty subsets of a $M-S$ (X_d, d) with $A_0 \neq \emptyset$. Then the pair (A_{1f}, A_{2f}) is considered to have the weak P-property if and only if for any $e_1, e_2 \in A_{1f}$ and $\tilde{e}_1, \tilde{e}_2 \in A_{2f}$,

$$\begin{cases} d(e_1, \tilde{e}_1) = d(A_{1f}, A_{2f}) \\ d(e_2, \tilde{e}_2) = d(A_{1f}, A_{2f}) \end{cases} \Rightarrow d(e_1, e_2) \leq d(\tilde{e}_1, \tilde{e}_2).$$

Lemma 1.4.5. Let (X_d, d) is a $M-S$ and $A_{2f} \in CL_f(X_d)$. Then $\forall\ e \in X_d$ and $q > 1$, there exists $b \in A_{2f}$ such that $d(e, b) \leq q d(e, A_{2f})$.

1.4.6 Jachymaski contraction

Jachymaski [52], generalized the Banach contraction principle on a complete $M-S$ enriched with a graph. He introduced the notion of Banach G-contraction as follows:

Definition 1.4.6 ([52])**.** Let (M,d) be a $M-S$, let $\triangle$ be the diagonal of the Cartesian product $X_d \times X_d$, and let G be a directed graph such that the set $V(G)$ of its vertices coincides with X_d and the set $E_g(G)$ of its edges contains loops; that is, $E_g(G) \supseteq \triangle$. Assume that G has no parallel edges. A mapping $T_f \colon X_d \to X_d$ is called a Banach G-contraction if (i) $e, \tilde{e} \in X_d$ $((e,\tilde{e}) \in E_g(G) \Rightarrow (T_f e, T_f \tilde{e}) \in E_g(G))$, (ii) $\exists\ \alpha_w$, $0 < \alpha_w < 1$ such that, $e, \tilde{e} \in X_d$, $(e,\tilde{e}) \in E_g(G) \Rightarrow d(T_f e, T_f \tilde{e}) \leq \alpha_w d(e,\tilde{e})$.

Further note that a mapping $T_f \colon X_d \to X_d$ is called G-continuous, if $\forall$ sequence $\{e_n\}$ in X_d with $e_n \to e$ and $(e_n, e_{n+1}) \in E_g(G)$ $\forall\ n \in \mathbb{N}$, we have $fe_n \to fe$.

For some other interesting extensions of Banach G-contraction we refer to [98, 81, 55, 81, 28].

Chapter 2

Fixed Point Theorems in Gauge Spaces

Throughout this chapter $F-P$ represents a fixed point, $M-S$ represents a metric space, $C-S$ represents a Cauchy sequence. This chapter is aimed on the development of $F-P$ theory for single-valued and multivalued mappings in gauge and b-gauge spaces, where we will use different contraction type-conditions. This chapter has two sections. In the first section, we will introduce some F_w-type-contractions in the setting of gauge spaces and obtain some $F-P$ theorems for such mappings. We will utilize ideas like α_w-admissibility and Hardy-Rogers contraction mapping to prove our results.

We will derive some $F-P$ theorems for mapping in gauges spaces with a graph as the consequences of our results. To support and claim the validity of our results, we will discuss an application of our results by discussing a Volterra integral equation at the end of this section.

In the second section we will introduce some new $F-P$ theorems for multivalued operators satisfying an α_w-contraction in $b-$gauge spaces. We will make use of the Hardy-Rogers contraction frequently to prove our results.

At the end of this section we will discuss a possible application by discussing a Volterra integral equation, and we will also provide an example to support our results.

2.1 Fixed points of F_w-contractions in Gauge Spaces

In this section we generalize the results of Wardoski [99] on gauges space by involving the ideas of Samet's [84] on α-admissibility and the Hardy-Rogers contraction mapping.

Through out this section $\mathfrak{S}$ is directed set and X_d is a nonempty set enriched with a separating complete gauge structure $\{d_\mu : \mu \in \mathfrak{S}\}$.
We now present our first result of this section which is a generalization and modification of the well known F_w-contraction presented by Wardowski[99]. We generalize the F_w-contraction by omitting the property (F_{w_2}) of Definition 1.4.1, in Chapter 1 of this dissertation.

Theorem 2.1.1. Let $T_f : X_d \to X_d$ be a mapping for which we have F_w in $\mathfrak{S}$ and a $\tau_f > 0$ such that

$$\alpha_w(e,\tilde{e}) \geq 1 \Rightarrow \tau_f + F_w(d_\mu(T_f e, T_f \tilde{e})) \leq F_w(M(e,\tilde{e}))\ \forall\ \mu \in \mathfrak{S}, \tag{2.1}$$

where $M(e,\tilde{e}) := \max\{d_\mu(e,\tilde{e}), d_\mu(e,T_f\tilde{e}), d_\mu(e,T_f\tilde{e}), [d_\mu(e,T_f\tilde{e})+d_\mu(\tilde{e},T_f\tilde{e})]/2\}$, $\forall\ \mu \in \mathfrak{S}$, $\forall\ e,\tilde{e} \in X_d$, whenever $d_\mu(T_f e, T_f\tilde{e}) \neq 0$. Further, assume that the following conditions hold:

(i) $\exists$ an $e_0 \in X_d$ such that $\alpha_w(e_0, T_f e_0) \geq 1$;

(ii) $\forall\ e,\tilde{e} \in X_d$ with $\alpha_w(e,\tilde{e}) \geq 1$, we have $\alpha(T_f e, T_f\tilde{e}) \geq 1$;

(iii) for any sequence $\{e_n\}$ in X_d such that $\alpha_w(e_n, e_{n+1}) \geq 1\ \forall\ n \in \mathbb{N}$, and $e_n \to e$ as $n \to \infty$, then $\alpha(\hat{e_n}, e) \geq 1\ \forall\ n \in \mathbb{N}$.

Then T_f has a $F - P$.

Proof. By hypothesis (i), $\exists$ an $e_0 \in X$ with $\alpha_w(e_0, e_1) \geq 1$, where $e_1 = T_f e_0$.
From (2.1), we have

$$\begin{aligned}
\tau_f + F_w(d_\mu(e_1,e_2)) &= \tau_f + F_w(d_\mu(T_f e_0, T_f e_1)) \\
&\leq F_w(M(e,\tilde{e})) \\
&= F_w(\max\{d_\mu(e_0,e_1), d_\mu(e_0,T_f e_0), d_\mu(e_1,T_f e_1), [d_\mu(e_0,T_f e_1) + d_\mu(e_1,T_f e_0)]/2 \\
&= F_w(\max\{d_\mu(e_0,e_1) + d_\mu(e_1,e_2), [d_\mu(e_0,e_2) + 0]/2\}) \\
&= F_w(\max\{d_\mu(e_0,e_1), d_\mu(e_1,e_2)\})\ \forall \mu \in \mathfrak{S}.
\end{aligned}$$

If the maximum is $d_\mu(e_1, e_2)$ for all $\mu \in \mathfrak{S}$, then we have a contradiction. Therefore we have

$$\tau_f + F_w(d_\mu(e_1, e_2)) \quad \leq \quad F_w(d_\mu(e_0, e_1)), \ \forall \mu \in \mathfrak{S}$$

which, since F_w is increasing, implies that $d_\mu(e_1, e_2) \leq d_\mu(e_0, e_1)$ for all $\mu \in \mathfrak{S}$. Continuing the argument we obtain the fact that $d_\mu(e_n, e_{n+1})$ for all $\mu \in \mathfrak{S}$ is a monotone decreasing nonnegative sequence with limit $\gamma \geq 0$. Suppose that $\gamma > 0$. Then from Theorem 2.1.1, for all $n \in \mathbb{N}$,

$$\begin{aligned} F_w(\gamma) \leq F_w(d_\mu(e_n, T_f e_n)) &\leq F_w(d_\mu(e_{n-1}, T_f e_{n-1})) - \tau_f \ \forall \mu \in \mathfrak{S} \\ &\leq F_w(d_\mu(e_{n-2}, T_f e_{n-2})) - 2\tau_f \ \forall \mu \in \mathfrak{S} \\ &\leq \cdots \leq F_w(d_\mu(e_0, T_f e_0)) - n\tau_f, \ \forall \mu \in \mathfrak{S}. \end{aligned}$$

Furthermore,

$$F_w(d_\mu(e_n, e_{n+1})) \leq F_w(d_\mu(e_0, e_1)) - n\tau_f \ \forall \ n \in \mathbb{N} \text{ and } \mu \in \mathfrak{S}. \tag{2.2}$$

Since $\lim_{n\to\infty}[F_w(d_\mu(e_n, T_f e_n) - n\tau_f] = -\infty$ for all $\mu \in \mathfrak{S}$, $\exists$ an integer $n_1 \in \mathbb{N}$ such that

$$F_w(d_\mu(e_n, T_f e_n)) - n\tau_f < F_w(\gamma), \forall n > n_1 \text{ and } \mu \in \mathfrak{S}.$$

Then, for all $n > n_1$, we have

$$F_w(\gamma) \leq F_w(d_\mu(e_0, T_f e_0)) < F_w(\gamma) \ \forall \mu \in \mathfrak{S},$$

a contradiction. Therefore

$$\gamma = \lim_{n\to\infty} d_\mu(e_n, e_{n+1}) = 0 \ \forall \mu \in \mathfrak{S}.$$

Let $(d_\mu)_n = d_\mu(e_n, e_{n+1}) \ \forall \mu \in \mathfrak{S} \ \forall \ n \in \mathbb{N}$. From (F_{w_3}) there exists $k \in (0, 1)$ such that

$$\lim_{n\to\infty} (d_\mu)_n^k F_w((d_\mu)_n) = 0 \ \forall \mu \in \mathfrak{S}.$$

From (2.2) we have

$$(d_\mu)_n^k F((d_\mu)_n) - (d_\mu)_n^k F_w((d_\mu)_0) \leq -(d_\mu)_n^k n\tau_f \leq 0 \ \forall \ n \in \mathbb{N} \text{ and } \mu \in \mathfrak{S}. \tag{2.3}$$

Now if we let $n \to \infty$ in (2.3), we get

$$\lim_{n\to\infty} n(d_\mu)_n^k = 0 \ \forall \mu \in \mathfrak{S}. \tag{2.4}$$

This implies that $\exists\ n_1 \in \mathbb{N}$ such that $n(d_\mu)_n^k \leq 1 \ \forall\ n \geq n_1$ and $\mu \in \mathfrak{S}$. Thus, we have

$$(d_\mu)_n \leq \frac{1}{n^{1/k}}, \quad \forall\ n \geq n_1 \text{ and } \mu \in \mathfrak{S}. \tag{2.5}$$

To prove that $\{e_n\}$ is a *C-S*. Consider $m, n \in \mathbb{N}$ with $m > n > n_1$. By making use of triangle inequality and (2.5), we have

$$\begin{aligned} d_\mu(e_n, e_m) &\leq d_\mu(e_n, e_{n+1}) + d_\mu(e_{n+1}, e_{n+2}) + \cdots + d_\mu(e_{m-1}, e_m) \\ &= \sum_{i=n}^{m-1}(d_\mu)_i \leq \sum_{i=n}^{\infty}(d_\mu)_i \leq \sum_{i=n}^{\infty}\frac{1}{i^{1/k}} \ \forall \mu \in \mathfrak{S}. \end{aligned}$$

As we know that $\sum_{i=1}^{\infty}\frac{1}{i^{1/k}}$ is convergent series. Thus, $\lim_{n\to\infty} d_\mu(e_n, e_m) = 0 \ \forall \mu \in \mathfrak{S}$. Which implies that $\{e_n\}$ is a *C-S*. By completeness of X_d, $\exists\ e^* \in X_d$ such that $e_n \to e^*$ as $n \to \infty$. By condition (iii), we have $\alpha_w(e_n, e^*) \geq 1$ $\forall\ n \in \mathbb{N}$. We claim that $d_\mu(e^*, T_f e^*) = 0 \ \forall \mu \in \mathfrak{S}$. contradictory suppose that $d_\mu(e^*, T_f e^*) > 0$ for some μ, $\exists\ n_0 \in \mathbb{N}$ such that $d_\mu(e_n, T_f e^*) > 0 \ \forall$ $n \geq n_0$. Thus $\forall\ n \geq n_0$ by making use of triangular property and (2.1), we have

$$\begin{aligned} d_\mu(e^*, T_f e^*) &\leq d_\mu(e^*, e_{n+1}) + d_\mu(e_{n+1}, T_f e^*) \\ &= d_\mu(e^*, e_{n+1}) + d_\mu(T_f e_n, T_f e^*) \\ &< d_\mu(e^*, e_{n+1}) + a_\mu d_\mu(e_n, e^*) + b_\mu d_\mu(e_n, e_{n+1}) + c_\mu d_\mu(e^*, T_f e^*), \\ &\quad + e_\mu d_\mu(e_n, T_f e^*) + L_\mu d_\mu(e^*, e_{n+1}). \end{aligned} \tag{2.6}$$

Now if we let $n \to \infty$ in (2.6), we have

$$d_\mu(e^*, T_f e^*) \leq (c_\mu + e_\mu) d_\mu(e^*, T_f e^*) < d_\mu(e^*, T_f e^*).$$

Which is a contradiction. Thus $d_\mu(e^*, T_f e^*) = 0 \ \forall\ \mu \in \mathfrak{S}$. As X_d is separating we have $e^* = T_f e^*$. □

Consequences

Now, we derive some $F - P$ theorems for mappings on gauge space with a graph defined on X_d. Throughout this subsection we assume that G is a directed graph, such that the set of its vertices V_g coincides with X_d (i.e., $V_g = X_d$) and the set of its edges E_g is such that $E_g \supseteq \triangle$, where $\triangle = \{(e,e) : e \in X_d\}$. Let us also assume that G has no parallel edges. We can identify G with the pair (V_g, E_g).

The following corollaries can be obtained from our results by defining an $\alpha_w : X_d \times X_d \to [0,\infty)$ as:

$$\alpha_w(e,\tilde{e}) = \begin{cases} 1 & \text{if } (e,\tilde{e}) \in E_g \\ 0 & \text{otherwise.} \end{cases}$$

Corollary 2.1.2. Let $T_f : X_d \to X_d$ be a mapping for which we have an F_w in $\mathfrak{F_w}$ is continuous and a $\tau_f > 0$ such that

$$\begin{aligned}(e,\tilde{e}) \in E_g \Rightarrow \tau_f \quad &+ \quad F_w(d_\mu(T_f e, T_f \tilde{e})) \leq F_w\Big(\max\Big\{ d_\mu(e,\tilde{e}), d_\mu(e, T_f e), d_\mu(\tilde{e}, T_f \tilde{e}), \\ &\frac{d_\mu(e, T_f\tilde{e}) + d_\mu(\tilde{e}, T_f e)}{2} \Big\} + L d_\mu(\tilde{e}, T_f e) \Big) \ \forall \ \mu \in \mathfrak{S} \qquad (2.7)\end{aligned}$$

$\forall \ e, \tilde{e} \in X_d$ whenever $d_\mu(T_f e, T_f \tilde{e}) \neq 0$, where $L \geq 0$. Further suppose that the following conditions hold:

(i) $\exists$ an $e_0 \in X_d$ with $(e_0, T_f e_0) \in E_g$;

(ii) $\forall \ e, \tilde{e} \in X_d$ with $(e,\tilde{e}) \in E_g$ we have $(T_f e, T_f \tilde{e}) \in E_g$;

(iii) for any sequence $\{e_n\} \subseteq X_d$ such that $e_n \to e$ as $n \to \infty$ and $(e_n, e_{n+1}) \in E_g \ \forall \ n \in \mathbb{N}$, we have $(e_n, e) \in E_g \ \forall \ n \in \mathbb{N}$.

Then T_f has a $F - P$.

2.1.1 Application

Consider the Volterra integral equation of the form:

$$u(t) = \int_0^{f(t)} R(t, s, u(s))ds, \ t \in I = [0, \infty) \qquad (2.8)$$

where $f : I \to \mathbb{R}$ is a continuous function with $0 \leq f(t) \ \forall \ t \in I$ and $R : I \times I \times \mathbb{R} \to \mathbb{R}$ is a continuous and nondecreasing function(nondecreasing in the third coordinate). Let $X_d = (C[0,\infty), \mathbb{R})$ be the space of all real valued continuous functions. Define the family of pseudonorms by $\|u\|_n =$

$\max_{t\in[0,n]}\{|u(t)|\exp(-|\tau_f t|)\}$, $\forall\ n \in \mathbb{N}$. By making use of this family of pseudonorms we get a family of pseudo metrics as $d_n(u,v) = |u-v|_n$. Clearly, $\mathfrak{F}_{\mathfrak{w}} = \{d_n : n \in \mathbb{N}\}$ defines a gauge structure on X_d, which is complete and separating. Define the graph $G = (V_g, X_d)$, as $V_g = X_d$ and $E_g = \{(u,v) : u(t) \leq v(t)\ \forall t\}$.

Now we will introduce an existence theorem for the solution of **??**.

Theorem 2.1.3. Let $X_d = (C[0,\infty), \mathbb{R})$ and suppose the operator $T_f : X_d \to X_d$ is define by

$$T_f u(t) = \int_0^{f(t)} R(t,s,u(s))ds, t \in I = [0,\infty),$$

where $f : I \to \mathbb{R}$ is a continuous function with $0 \leq f(t)\ \forall\ t \in I$ and $R : I \times I \times \mathbb{R} \to \mathbb{R}$ is a continuous and nondecreasing function. Assume that the following conditions hold:

(i) $\exists$ a $\tau_f > 0$ and a $k : X_d \to [0,\infty)$ such that, $\forall\ (u,v) \in E_g$ and $t,s \in [0,n]$, we have

$$|R(t,s,u) - R(t,s,v)| \leq \frac{\exp(-\tau_f)}{k(u+v)} d_n(u,v)\ \forall\ n \in \mathbb{N};$$

moreover,

$$\Big| \int_0^{f(t)} \frac{1}{k(u(s)+v(s))} ds \Big| \leq exp|\tau_f t|$$

$\forall\ t \in I$;

(ii) $\exists$ a $u_0 \in X_d$ such that $(u_0, T_f u_0) \in E_g$;

(iii) $\forall\ u, v \in X_d$ with $(u,v) \in E_g$ we have $(T_f u, T_f v) \in E_g$;

(iv) for any sequence $\{u_n\} \subseteq X_d$ such that $u_n \to u$ as $n \to \infty$ and $(u_n, u_{n+1}) \in E_g\ \forall\ n \in \mathbb{N}$, we have $(u_n, u) \in E_g\ \forall\ n \in \mathbb{N}$.

Then the integral equation (2.8) has at least one solution.

Proof. For any $(u, v) \in E_g$ and $t \in [0, n]$ $\forall$ $n \geq 1$, we have

$$\begin{aligned}
|T_f u(t) - T_f v(t)| &\leq \int_0^{f(t)} |R(t, s, u(s)) - R(t, s, v(s))| ds \\
&\leq \int_0^{f(t)} \frac{\exp(-\tau_f)}{k(u(s) + v(s))} d_n(u, v) ds \\
&= \exp(-\tau_f) d_n(u, v) \int_0^{f(t)} \frac{1}{k(u(s) + v(s))} ds \\
&\leq \exp(|\tau_f t|) \exp(-\tau_f) d_n(u, v).
\end{aligned}$$

Thus, we have

$$|T_f u(t) - T_f v(t)| \exp(-|\tau_f t|) \leq \exp(-\tau_f) d_n(u, v).$$

Equivalently,

$$d_n(T_f u, T_f v) \leq \exp(-\tau_f) d_n(u, v).$$

Since the natural logarithm belongs to $\mathfrak{F}_{\mathfrak{w}}$, applying it to the above inequality, we get

$$\ln(d_n(T_f u, T_f v)) \leq \ln(\exp(-\tau_f) d_n(u, v)).$$

After some simplification, we get

$$\tau_f + \ln(d_n(T_f u, T_f v)) \leq \ln(d_n(u, v)) \;\; \forall \; n \in \mathbb{N}.$$

Thus T_f satisfies (2.1.2) with $a_n = 1$, and $b_n = c_n = e_n = L_n = 0$ $\forall$ $n \in \mathbb{N}$ and $F_w(u) = \ln u$. As R is nondecreasing, for $(u, v) \in E_g$, we have $(T_f u, T_f v) \in E_g$. Further, all of the other conditions of the above Corollary 2.1.2, follows immediately from the hypothesis of the theorem. Thus, $\exists$ a F-P of the operator T_f; that is, integral equation (2.8) has at least one solution. □

2.2 Fixed points of α_w-contractions in b-gauge spaces

In this section we introduce some multivalued α_w-contractions on b-gauge spaces. We also prove the existence of $F - P$'s of multivalued mappings satisfying one of these conditions. The applicability of our results is shown at the end of this section.

Through out this section, $\mathfrak{S}$ is directed set and X_d is a nonempty set enriched with a separating complete b_s-gauge structure $\mathfrak{F}_{\mathfrak{w}} = \{d_\mu : \mu \in \mathfrak{S}\}$. Further, $\alpha_w : X_d \times X_d \to [0, \infty)$ be a mapping. For each $d_\mu \in \mathfrak{F}_{\mathfrak{w}}$, $CL_\mu(X_d)$ denote the set of all nonempty closed subsets of X_d with respect to d_v. For each $\mu \in \mathfrak{S}$ and $A_{1f}, A_{2f} \in CL_\mu(X_d)$, the function $H_\mu : CL_\mu(X_d) \times CL_\mu(X_d) \to [0, \infty)$ defined by

$$H_\mu(A_{1f}, A_{2f}) = \begin{cases} \max\left\{ \sup_{e \in A_{1f}} d_\mu(e, A_{2f}), \sup_{\tilde{e} \in A_{2f}} d_\mu(\tilde{e}, A_{1f}) \right\}, & \text{if the maximum exists;} \\ \infty, & \text{otherwise} \end{cases}$$

is a generalized Hausdorff b_s-pseudo metric on $CL_\mu(X_d)$. We denote by $CL_f(X_d)$ the set of all nonempty closed subsets in the b_s-gauge space $(X_d, \mathfrak{T}(\mathfrak{F}_{\mathfrak{w}}))$.

Theorem 2.2.1. Let $T_f : X_d \to CL_f(X_d)$ be a mapping such that, $\forall\ \mu \in \mathfrak{S}$, we have

$$H_\mu(T_f e, T_f \tilde{e}) \leq a_\mu d_\mu(e, \tilde{e}) + b_\mu d_\mu(e, T_f e) + c_\mu d_\mu(\tilde{e}, T_f \tilde{e}) + e_\mu d_\mu(e, T_f \tilde{e}) + L_\mu d_\mu(\tilde{e}, T_f e)\ \ \forall\ \alpha_w(e, \tilde{e}) \geq 1 \tag{2.9}$$

where, $a_\mu, b_\mu, c_\mu, e_\mu, L_\mu \geq 0$, and $s^2 a_\mu + s^2 b_\mu + s^2 c_\mu + 2s^3 e_\mu < 1$.

Further, assume that the following conditions hold:

(i) $\exists$ an $e_0 \in X_d$ and an $e_1 \in T_f e_0$ such that $\alpha_w(e_0, e_1) \geq 1$;

(ii) if $\alpha_w(e, \tilde{e}) \geq 1$ then, for a $u \in T_f e$ and a $v \in T_f \tilde{e}$, we have $\alpha_w(u, v) \geq 1$;

(iii) if $\{e_n\}$ is a sequence in X_d such that $\alpha_w(e_n, e_{n+1}) \geq 1\ \forall\ n \in \mathbb{N}$ and $e_n \to e$ as $n \to \infty$, then $\alpha_w(e_n, e) \geq 1\ \forall\ n \in \mathbb{N}$;

(iv) $\forall\ \{q_\mu : q_\mu > 1\}_{\mu \in \mathfrak{S}}$ and $e \in X_d\ \exists\ \tilde{e} \in T_f e$ such that

$$d_\mu(e, \tilde{e}) \leq q_\mu d_\mu(e, T_f e)\ \ \forall\ \mu \in \mathfrak{S}.$$

Then T_f has a $F - P$.

Proof. By hypothesis (i), $\exists$ an $e_0, e_1 \in X_d$ such that $e_1 \in T_f e_0$ and $\alpha_w(e_0, e_1) \geq 1$. Now, it follows from (2.9) that

$$H_\mu(T_f e_0, T_f e_1) \leq a_\mu d_\mu(e_0, e_1) + b_\mu d_\mu(e_0, T_f e_0) + c_\mu d_\mu(e_1, T_f e_1) + e_\mu d_\mu(e_0, T_f e_1) + L_\mu d_\mu(e_1, T_f e_0)\ \forall\ \mu \in \mathfrak{S}. \tag{2.10}$$

Since $d_\mu(e_1, T_f e_1) \leq H_\mu(T_f e_0, T_f e_1)$ and $d_\mu(e_0, T_f e_1) \leq s[d_\mu(e_0, e_1) + d_\mu(e_1, T_f e_1)]$, from (2.10), we get

$$d_\mu(e_1, T_f e_1) \leq \frac{1}{\xi_\mu} d_\mu(e_0, e_1) \tag{2.11}$$

where, $\xi_\mu = \frac{1-c_\mu-se_\mu}{a_\mu+b_\mu+se_\mu} > 1$. Using hypothesis (iv) $\exists$ an $e_2 \in T_f e_1$ such that

$$d_\mu(e_1, e_2) \leq \sqrt{\xi_\mu} d_\mu(e_1, T_f e_1). \tag{2.12}$$

Combining (2.11) and (2.12) we get

$$d_\mu(e_1, e_2) \leq \frac{1}{\sqrt{\xi_\mu}} d_\mu(e_0, e_1) \ \ \forall \ \mu \in \mathfrak{S}. \tag{2.13}$$

Hypothesis (ii) implies that $\alpha_w(e_1, e_2) \geq 1$. Continuing in the same way, we get a sequence $\{e_m\}$ in X_d such that $\alpha_w(e_m, e_{m+1}) \geq 1$ and

$$d_\mu(e_m, e_{m+1}) \leq \Big(\frac{1}{\sqrt{\xi_\mu}}\Big)^m d_\mu(e_0, e_1) \ \ \forall \ \mu \in \mathfrak{S}.$$

For convenience we assume that $\eta_\mu = \frac{1}{\sqrt{\xi_\mu}}$ $\forall$ $\mu \in \mathfrak{S}$. We shall now show that $\{e_m\}$ is a *C-S*. For each $m, p \in \mathbb{N}$ and $\mu \in \mathfrak{S}$, we have

$$\begin{aligned} d_\mu(e_m, e_{m+p}) &\leq \sum_{i=m}^{m+p-1} s^i d_\mu(e_i, e_{i+1}) \\ &\leq \sum_{i=m}^{m+p-1} s^i (\eta_\mu)^i d_\mu(e_0, e_1) \\ &\leq \sum_{i=m}^{\infty} (s\eta_\mu)^i d_\mu(e_0, e_1) < \infty \ \ (As we know that \ s\eta_\mu < 1). \end{aligned}$$

This implies that $\{e_m\}$ is a *C-S* in X_d. By completeness of X_d, we have an $e^* \in X_d$ such that $e_m \to e^*$ as $m \to \infty$. By making use of hypothesis (iii), triangle inequality and (2.9), we have

$$\begin{aligned}
d_\mu(e^*, T_f e^*) &\leq sd_\mu(e^*, e_{m-1}) + sd_\mu(e_{m-1}, T_f e^*) \\
&\leq sd_\mu(e^*, e_{m-1}) + sH_\mu(T_f e_m, T_f e^*) \\
&\leq sd_\mu(e^*, e_{m-1}) + sa_\mu d_\mu(e_m, e^*) + sb_\mu d_\mu(e_m, T_f e_m) + \\
&\quad sc_\mu d_\mu(e^*, T_f e^*) + se_\mu d_\mu(e_m, T_f e^*) + sL_\mu d_n(e^*, T_f e_m) \\
&\leq sd_\mu(e^*, e_{m-1}) + sa_\mu d_\mu(e_m, e^*) + sb_\mu d_\mu(e_m, e_{m+1}) + \\
&\quad sc_\mu d_\mu(e^*, T_f e^*) + se_\mu d_\mu(e_m, T_f e^*) + sL_\mu d_\mu(e^*, e_{m+1}) \\
&\leq sd_\mu(e^*, e_{m-1}) + sa_\mu d_\mu(e_m, e^*) + sb_\mu d_\mu(e_m, e_{m+1}) + \\
&\quad sc_\mu d_\mu(e^*, T_f e^*) + se_\mu [sd_\mu(e_m, e^*) + sd_\mu(e^*, T_f e^*)] \\
&\quad + sL_\mu d_\mu(e^*, e_{m+1}) \ \ \forall\ \mu \in \mathfrak{S}.
\end{aligned}$$

Letting $m \to \infty$, we get

$$d_\mu(e^*, T_f e^*) \leq (sc_\mu + s^2 e_\mu) d_\mu(e^*, T_f e^*) \ \ \forall\ \mu \in \mathfrak{S},$$

which is only possible if $d_\mu(e^*, T_f e^*) = 0$. Since the structure $\{d_\mu : \mu \in \mathfrak{S}\}$ on X_d is separating, we have $e^* \in T_f e^*$. □

In case of single valued mapping $T_f : X_d \to X_d$, we have the following result:

Theorem 2.2.2. Let $T_f : X_d \to X_d$ be a mapping such that $\forall\ \mu \in \mathfrak{S}$, we have

$$d_\mu(T_f e, T_f \tilde{e}) \leq a_\mu d_\mu(e, \tilde{e}) + b_\mu d_\mu(e, T_f e) + c_\mu d_\mu(\tilde{e}, T_f \tilde{e}) + e_\mu d_\mu(e, T_f \tilde{e}) + L_\mu d_\mu(\tilde{e}, T_f e) \ \ \forall\ \alpha_w(e, \tilde{e}) \geq 1 \tag{2.14}$$

where, $a_\mu, b_\mu, c_\mu, e_\mu, L_\mu \geq 0$, and $sa_\mu + sb_\mu + sc_\mu + 2s^2 e_\mu < 1$.

Further, assume that the following conditions hold:

(i) $\exists$ an $e_0 \in X_d$ such that $\alpha_w(e_0, T_f e_0) \geq 1$;

(ii) if $\alpha_w(e, \tilde{e}) \geq 1$, then $\alpha_w(T_f e, T_f \tilde{e}) \geq 1$;

(iii) if $\{e_n\}$ is a sequence in X_d such that $\alpha_w(e_n, e_{n+1}) \geq 1\ \forall\ n \in \mathbb{N}$ and $e_n \to e$ as $n \to \infty$, then $\alpha_w(e_n, e) \geq 1\ \forall\ n \in \mathbb{N}$;

Then T_f has a $F-P$.

$\tilde{\Psi}_{s^2}$ denotes the family of nondecreasing functions $\tilde{\psi} : [0,\infty) \to [0,\infty)$ such that:

($\tilde{\psi}_1$) $\tilde{\psi}(0) = 0$;

($\tilde{\psi}_2$) $\tilde{\psi}(\rho t) = \rho\tilde{\psi}(t) < \rho t \ \forall \ \rho, t > 0$;

($\tilde{\psi}_3$) $\sum_{i=1}^{\infty} s^{2i}\tilde{\psi}^i(t) < \infty$, where $s \geq 1$.

Note that in the following theorems we have used the gauge structure with $s > 1$.

Theorem 2.2.3. Let $T_f : X_d \to CL_f(X_d)$ be a mapping such that $\forall \ \mu \in \mathfrak{S}$ we have

$$\begin{aligned} H_\mu(T_f e, T_f \tilde{e}) \quad &\leq \quad \tilde{\psi}_\mu(\max\{d_\mu(e,\tilde{e}), d_\mu(e,T_f e), d_\mu(\tilde{e},T_f\tilde{e}), \frac{1}{2s}[d_\mu(e,T_f\tilde{e}) + d_\mu(\tilde{e},T_f e)]\}) \\ &\quad +L_\mu d_\mu(\tilde{e},T_f e) \ \ \forall \ \alpha_w(e,\tilde{e}) \geq 1 \end{aligned} \tag{2.15}$$

where, $\tilde{\psi}_\mu \in \tilde{\Psi}_{s^2}$ and $L_\mu \geq 0$. Further, assume that the following conditions hold:

(i) $\exists$ an $e_0 \in X_d$ and $e_1 \in T_f e_0$ such that $\alpha_w(e_0, e_1) \geq 1$;

(ii) if $\alpha_w(e,\tilde{e}) \geq 1$, $\forall \ u \in T_f e$ and $v \in T_f \tilde{e}$, we have $\alpha_w(u,v) \geq 1$;

(iii) if $\{e_n\}$ is a sequence in X_d such that $\alpha_w(e_n, e_{n+1}) \geq 1 \ \forall \ n \in \mathbb{N}$ and $e_n \to e$ as $n \to \infty$, then $\alpha_w(e_n, e) \geq 1 \ \forall \ n \in \mathbb{N}$;

(iv) $\forall \ e \in X_d$, we have $\tilde{e} \in T_f e$ such that

$$d_\mu(e,\tilde{e}) \leq s d_\mu(e, T_f e) \ \ \forall \ \mu \in \mathfrak{S}.$$

Then T_f has a $F-P$.

Proof. By hypothesis we have an $e_0 \in X_d$ and $e_1 \in T_f e_0$ such that $\alpha_w(e_0, e_1) \geq 1$. From (2.15), we get

$$\begin{aligned}
d_\mu(e_1, T_f e_1) &\leq H_\mu(T_f e_0, T_f e_1) \\
&\leq \tilde{\psi}_\mu(\max\{d_\mu(e_0, e_1), d_\mu(e_0, T_f e_0), d_\mu(e_1, T_f e_1), \\
&\quad \frac{1}{2s}[d_\mu(e_0, T_f e_1) + d_\mu(e_1, T_f e_0)]\}) + L_\mu d_\mu(e_1, T_f e_0) \\
&\leq \tilde{\psi}_\mu(\max\{d_\mu(e_0, e_1), d_\mu(e_0, e_1), d_\mu(e_1, T_f e_1), \\
&\quad \frac{1}{2s}[s(d_\mu(e_0, e_1) + d_\mu(e_1, T_f e_1))]\}) + L_\mu.0 \\
&= \tilde{\psi}_\mu(d_\mu(e_0, e_1)) \;\; \forall \mu \in \mathfrak{S}. \qquad (2.16)
\end{aligned}$$

Otherwise we have a contradiction. By hypothesis (iv), for $e_1 \in X_d$, we have an $e_2 \in T_f e_1$ such that

$$d_\mu(e_1, e_2) \leq s d_\mu(e_1, T_f e_1) \leq s\tilde{\psi}_\mu(d_\mu(e_0, e_1)) \;\; \forall \mu \in \mathfrak{S}. \qquad (2.17)$$

Applying $\tilde{\psi}_\mu$, we have

$$\tilde{\psi}_\mu(d_\mu(e_1, e_2)) \leq \tilde{\psi}_\mu(s\tilde{\psi}_\mu(d_\mu(e_0, e_1))) = s\tilde{\psi}^2_\mu(d_\mu(e_0, e_1)) \;\; \forall \mu \in \mathfrak{S}.$$

By hypothesis (ii), it is clear that $\alpha_w(e_1, e_2) \geq 1$. Again, from (2.15), we obtain the following inequality after some simplification.

$$d_\mu(e_2, T_f e_2) \leq H_\mu(T_f e_1, T_f e_2) \leq \tilde{\psi}_\mu(d_\mu(e_1, e_2)) \;\; \forall \mu \in \mathfrak{S}. \qquad (2.18)$$

By hypothesis (iv), for $e_2 \in X_d$, we have an $e_3 \in T_f e_2$ such that

$$d_\mu(e_2, e_3) \leq s d_\mu(e_2, T_f e_2) \leq s\tilde{\psi}_\mu(d_\mu(e_1, e_2)) \leq s^2\tilde{\psi}^2_\mu(d_\mu(e_0, e_1)) \;\; \forall \mu \in \mathfrak{S}. \qquad (2.19)$$

Clearly, $\alpha_w(e_2, e_3) \geq 1$. Proceeding in the same way, we get a sequence $\{e_m\}$ in X_d such that $\alpha_w(e_m, e_{m+1}) \geq 1$ and

$$d_\mu(e_m, e_{m+1}) \leq s^m\tilde{\psi}^m_\mu(d_\mu(e_0, e_1)) \;\; \forall \mu \in \mathfrak{S}.$$

We now show that $\{e_m\}$ is a Cauchy sequence. For $m, p \in \mathbb{N}$, we have

$$\begin{aligned}
d_\mu(e_m, e_{m+p}) &\leq \sum_{i=m}^{m+p-1} s^i d_\mu(e_i, e_{i+1}) \\
&\leq \sum_{i=m}^{m+p-1} s^{2i}\tilde{\psi}^i_\mu(d_\mu(e_0, e_1)) < \infty.
\end{aligned}$$

This implies that $\{e_m\}$ is a C-S in X_d. By completeness of X_d, we have an $e^* \in X_d$ such that $e_m \to e^*$ as $m \to \infty$. Using hypothesis (iv), triangle inequality, and (2.15), we have

$$\begin{aligned} d_\mu(e^*, T_f e^*) &\leq s d_\mu(e^*, e_{m-1}) + s d_\mu(e_{m-1}, T_f e^*) \\ &\leq s d_\mu(e^*, e_{m-1}) + s H_\mu(T_f e_m, T_f e^*) \\ &\leq s d_\mu(e^*, e_{m-1}) + s\tilde{\psi}_\mu(\max\{d_\mu(e_m, e^*), d_\mu(e_m, T_f e_m), d_\mu(e^*, T_f e^*), \\ &\quad \frac{1}{2s}[d_\mu(e_m, T_f e^*) + d_\mu(e^*, T_f e_m)]\}) + L_\mu d_\mu(e^*, T_f e_m) \\ &< s d_\mu(e^*, e_{m-1}) + s\max\{d_\mu(e_m, e^*), d_\mu(e_m, e_{m+1}), d_\mu(e^*, T_f e^*), \\ &\quad \frac{1}{2s}[d_\mu(e_m, T_f e^*) + d_\mu(e^*, e_{m+1})]\} + L_\mu d_\mu(e^*, e_{m+1}) \\ &\leq s d_\mu(e^*, e_{m-1}) + s\max\{d_\mu(e_m, e^*), d_\mu(e_m, e_{m+1}), d_\mu(e^*, T_f e^*), \\ &\quad \frac{1}{2s}[s d_\mu(e_m, e^*) + s d_\mu(e^*, T_f e^*) + d_\mu(e^*, e_{m+1})]\} \\ &\quad + L_\mu d_\mu(e^*, e_{m+1}) \ \ \forall\ \mu \in \mathfrak{S}. \end{aligned}$$

Now if we let $m \to \infty$, we get

$$d_\mu(e^*, T_f e^*) \leq s d_\mu(e^*, T_f e^*).$$

This is not possible, if $d_\mu(e^*, T_f e^*) > 0$. Thus, $d_\mu(e^*, T_f e^*) = 0\ \forall\ \mu \in \mathfrak{S}$. As we know that the structure $\{d_\mu : \mu \in \mathfrak{S}\}$ on X_d is separating, we have an $e^* \in T_f e^*$. □

By considering $T_f : X_d \to X_d$ in above theorem we get the following one.

Theorem 2.2.4. Let $T_f : X_d \to X_d$ be a mapping such that $\forall\ \mu \in \mathfrak{S}$, we have

$$\begin{aligned} d_\mu(T_f e, T_f \tilde{e}) &\leq \tilde{\psi}_\mu(\max\{d_\mu(e, \tilde{e}), d_\mu(e, T_f e), d_\mu(\tilde{e}, T_f \tilde{e}), \frac{1}{2s}[d_\mu(e, T_f \tilde{e}) + d_\mu(\tilde{e}, T_f e)]\}) \\ &\quad + L_\mu d_\mu(\tilde{e}, T_f e) \ \ \forall\ \alpha_w(e, \tilde{e}) \geq 1 \end{aligned} \tag{2.20}$$

where $\tilde{\psi}_\mu \in \tilde{\Psi}_{s^2}$. Assume that the following conditions hold:

(i) $\exists$ an $e_0 \in X_d$ such that $\alpha_w(e_0, T_f e_0) \geq 1$;

(ii) if $\alpha_w(e, \tilde{e}) \geq 1$, then $\alpha_w(T_f e, T_f \tilde{e}) \geq 1$;

(iii) if $\{e_n\}$ is a sequence in X_d such that $\alpha_w(e_n, e_{n+1}) \geq 1\ \forall\ n \in \mathbb{N}$ and $e_n \to e$ as $n \to \infty$, then $\alpha_w(e_n, e) \geq 1\ \forall\ n \in \mathbb{N}$.

Then T_f has a $F-P$.

We shall now introduce a $F-P$ theorem containing Feng-liu type α_w-contraction:

Theorem 2.2.5. Let $T_f : X_d \to CL_f(X_d)$ be a mapping such that $\forall\ \mu \in \mathfrak{S}$, we have

$$d_\mu(\tilde{e}, T_f\tilde{e}) \leq \tilde{\psi}_\mu(d_\mu(e,\tilde{e}))\ \ \forall\ \alpha_w(e,\tilde{e}) \geq 1 \text{ with } e \in X_d \text{ and } \tilde{e} \in T_f e \quad (2.21)$$

where, $\tilde{\psi}_\mu \in \tilde{\Psi}_{s^2}$. Further, assume that the following conditions hold:

(i) $\exists$ an $e_0 \in X_d$ and $e_1 \in T_f e_0$ such that $\alpha_w(e_0, e_1) \geq 1$;

(ii) $\forall\ e \in X_d$ and $\tilde{e} \in T_f e$ with $\alpha_w(e,\tilde{e}) \geq 1$, we have $\alpha_w(\tilde{e}, v) \geq 1\ \forall\ v \in T_f\tilde{e}$;

(iii) $\forall\ e \in X_d$, we have $\tilde{e} \in T_f e$ such that

$$d_\mu(e,\tilde{e}) \leq s d_\mu(e, T_f e)\ \ \forall\ \mu \in \mathfrak{S}.$$

Then T_f has a $F-P$, provided that $d_\mu(e, T_f e)$ is lower semi continuous, $\forall\ \mu \in \mathfrak{S}$.

Proof. By hypothesis (i) we have an $e_0 \in X_d$ and $e_1 \in T_f e_0$ such that $\alpha_w(e_0, e_1) \geq 1$. From (2.21), we get

$$d_\mu(e_1, T_f e_1) \leq \tilde{\psi}_\mu(d_\mu(e_0, e_1))\ \ \forall\ \mu \in \mathfrak{S}. \quad (2.22)$$

By hypothesis (iii), for $e_1 \in X_d$, we have an $e_2 \in T_f e_1$ such that

$$d_\mu(e_1, e_2) \leq s d_\mu(e_1, T_f e_1) \leq s\tilde{\psi}_\mu(d_\mu(e_0, e_1))\ \ \forall\ \mu \in \mathfrak{S}. \quad (2.23)$$

Applying $\tilde{\psi}_\mu$, we have

$$\tilde{\psi}_\mu(d_\mu(e_1, e_2)) \leq \tilde{\psi}_\mu(s\tilde{\psi}_\mu(d_\mu(e_0, e_1))) = s\tilde{\psi}^2_\mu(d_\mu(e_0, e_1))\ \ \forall\ \mu \in \mathfrak{S}.$$

By hypothesis (ii), it is clear that $\alpha_w(e_1, e_2) \geq 1$. Again from (2.21), we have

$$d_\mu(e_2, T_f e_2) \leq \tilde{\psi}_\mu(d_\mu(e_1, e_2))\ \ \forall\ \mu \in \mathfrak{S}. \quad (2.24)$$

By hypothesis (iii), for $e_2 \in X_d$, we have an $e_3 \in T_f e_2$ such that

$$d_\mu(e_2, e_3) \leq s d_\mu(e_2, T_f e_2) \leq s\tilde{\psi}_\mu(d_\mu(e_1, e_2)) \leq s^2 \tilde{\psi}_\mu^2(d_\mu(e_0, e_1)) \ \ \forall\ \mu \in \mathfrak{S}. \tag{2.25}$$

Clearly, $\alpha_w(e_2, e_3) \geq 1$. Proceeding in the same way, we get a sequence $\{e_m\}$ in X_d such that $\alpha_w(e_m, e_{m+1}) \geq 1$ and

$$d_\mu(e_m, e_{m+1}) \leq s^m \tilde{\psi}_\mu^m(d_\mu(e_0, e_1)) \ \ \forall\ \mu \in \mathfrak{S}.$$

We shall now show that $\{e_m\}$ is a *C-S*. For $m, p \in \mathbb{N}$, we have

$$\begin{aligned} d_\mu(e_m, e_{m+p}) &\leq \sum_{i=m}^{m+p-1} s^i d_\mu(e_i, e_{i+1}) \\ &\leq \sum_{i=m}^{m+p-1} s^{2i} \tilde{\psi}_\mu^i(d_\mu(e_0, e_1)) < \infty. \end{aligned}$$

This implies that $\{e_m\}$ is a *C-S* in X_d. By completeness of X_d, we have an $e^* \in X_d$ such that $e_m \to e^*$ as $m \to \infty$. Thus, we have $\lim_{m\to\infty} d_\mu(e_m, T_f e_m) = 0$. By lower semi continuity of $d_\mu(e, T_f e)$ and last fact, we conclude that $d_\mu(e^*, T_f e^*) = 0\ \forall\ \mu \in \mathfrak{S}$. As we know that the structure $\{d_\mu : \mu \in \mathfrak{S}\}$ on X_d is separating, thus $e^* \in T_f e^*$. □

For singlevalued mapping the above theorem reduces to following:

Theorem 2.2.6. Let $T_f : X_d \to X_d$ be a mapping such that $\forall\ \mu \in \mathfrak{S}$ we have

$$d_\mu(T_f e, T_f^2 e) \leq \tilde{\psi}_\mu(d_\mu(e, T_f e)) \ \ \forall\ \alpha_w(e, T_f e) \geq 1,\ e \in X_d$$

where, $\tilde{\psi}_\mu \in \tilde{\Psi}_{s^2}$. Further, assume that the following conditions hold:

(i) $\exists$ an $e_0 \in X_d$ such that $\alpha_w(e_0, T_f e_0) \geq 1$;

(ii) $\forall\ e \in X_d$ with $\alpha_w(e, T_f e) \geq 1$, we have $\alpha_w(T_f e, T_f^2 e) \geq 1$;

Then T_f has a $F-P$, provided $d_\mu(e, T_f e)$ is lower semi continuous, $\forall$ $\mu \in \mathfrak{S}$.

2.2.1 Application and Example

Consider the Volterra integral equation of the form:

$$u(t) = \int_a^t R(t, s, u(s))ds, \ t \in I = [0, \infty) \tag{2.26}$$

where $R : I \times I \times \mathbb{R} \to \mathbb{R}$ is a continuous and nondecreasing function(i.e nondecreasing in the third coordinate).

Let $X_d = (C[0, \infty), \mathbb{R})$. Define the family of b_2-pseudo norms by $\|u\|_n = \max_{t\in[0,n]}(u(t))^2$, $n \in \mathbb{N}$. By making use of this family of b_2-pseudonorms we get a family of b_2-pseudo metrics as $d_n(u, v) = |u - v|_n$. Clearly, $\mathfrak{F}_{\mathfrak{w}} = \{d_n : n \in \mathbb{N}\}$ defines a b_2-gauge structure on X_d, which is complete and separating. Define $\alpha_w : X_d \times X_d \to [0, \infty)$ by

$$\alpha_w(u, v) = \begin{cases} 1 & \text{if } u(t) \le v(t) \ \forall t \in I \\ 0 & \text{otherwise.} \end{cases}$$

Theorem 2.2.7. Let $X_d = (C[0, \infty), \mathbb{R})$ and let the operator $T_f : X_d \to X_d$ is define by

$$T_f u(t) = \int_0^t R(t, s, u(s))ds, t \in I = [0, \infty)$$

where $R : I \times I \times \mathbb{R} \to \mathbb{R}$ is a continuous and nondecreasing function. Assume that the following conditions hold:

(i) $\forall \ t, s \in [0, n]$ and $u, v \in X_d$ with $u(s) \le v(s)$, $\exists$ a continuous mapping $p : I \times I \to I$ such that

$$|R(t, s, u(s)) - R(t, s, v(s))| \le \sqrt{k(t, s)d_n(e, \tilde{e})} \ \forall \ n \in \mathbb{N};$$

(ii) $\sup_{t\ge0} \int_0^t \sqrt{k(t, s)}ds = a < \frac{1}{\sqrt{2}}$;

(iii) $\exists$ a $u_0 \in X_d$ such that

$$u_0(t) \le \int_0^t R(t, s, u_0(s))ds.$$

Then the integral equation (2.26) has at least one solution.

Proof. We shall first show that $\forall\ \alpha_w(u,v) \geq 1$, the inequalities (2.9) holds. For any $\alpha_w(u,v) \geq 1$ and $t \in [0,n]\ \forall\ n \geq 1$, we have

$$\begin{aligned}(T_f u(t) - T_f v(t))^2 &\leq \Big(\int_0^t |R(t,s,u(s)) - R(t,s,v(s))| ds\Big)^2 \\ &\leq \Big(\int_0^t \sqrt{k(t,s) d_n(u,v)} ds\Big)^2 \\ &= \Big(\int_0^t \sqrt{k(t,s)} ds\Big)^2 d_n(u,v) \\ &= a^2 d_n(u,v).\end{aligned}$$

Thus we get $d_n(T_f u, T_f v) \leq a^2 d_n(u,v)\ \forall\ \alpha_w(u,v) \geq 1$ and $n \in \mathbb{N}$ with $a^2 < 1/2$. This implies that (2.9) holds with $a_n = a^2$, and $b_n = c_n = e_n = L_n = 0$ $\forall\ n \in \mathbb{N}$. As R is nondecreasing, $\forall\ u \leq v$, we have $T_f u \leq T_f v$. Hence for $\alpha_w(u,v) \geq 1$, implies $\alpha_w(T_f u, T_f v) \geq 1$. Therefore, by Theorem 2.2.2, $\exists$ a F-P of the operator T_f; so that, the integral equation (2.26) has at least one solution. □

We now give an example to support of our result:

Example 2.2.8. Let $X_d = C([0,10],\mathbb{R})$ is the space of twice differentiable functions, enriched with the $d_n(e(t), \tilde{e}(t)) = \max_{t\in[0,n]}(e(t) - \tilde{e}(t))^2\ \forall\ n \in \{1,2,3,\cdots,10\}$. Consider the operator $T_f : X_d \to X_d$ is defined by $T_f e(t) = \frac{d^2 e(t)}{dt^2}$ and $\alpha_w : X_d \times X_d \to [0,\infty)$ is defined by

$$\alpha_w(e,\tilde{e}) = \begin{cases} 1 & \text{if } e,\tilde{e} \text{ are linear or constant functions} \\ 0 & \text{otherwise} \end{cases}$$

It is easy to see that 2.20 holds with $a_n = 1/2$ and $b_n = c_n = e_n = L_n = 0\ \forall$ $n \in \{1,2,3,\cdots,10\}$. For $e_0 = t$ and $e_1 = T_f e_0 = 0$, we have $\alpha_w(e_0, T_f e_0) = 1$. Further, $\forall\ \alpha_w(e,\tilde{e}) = 1$, we have $\alpha_w(T_f e, T_f \tilde{e}) = 1$. Moreover, $\forall$ sequence $\{e_m\}$ in X_d such that $\alpha_w(e_m, e_{m+1}) = 1\ \forall\ m \in \mathbb{N}$ and $e_m \to e$ as $m \to \infty$, then $\alpha_w(e_m, e) = 1\ \forall\ m \in \mathbb{N}$. Therefore, all of Theorem 2.2.2 conditions hold. Thus T_f has a $F - P$.

Chapter 3

Fixed Point Theorems in Vector-Valued Metric Spaces

Throughout this chapter $F-P$ represents a fixed point, $M-S$ represents a metric space, $C-S$ represents a Cauchy sequence. This chapter consists of two sections. In the first section we have proved some $F-P$ theorems for mappings in generalized $M-S$ enriched with graph. In this section we have proved theorems for a single metric and for two metrics as well. Along with results for singled valued mapping, We have also obtained results for multi-valued mappings for both, single metric and two metrics. To show the validity of our results, we have also constructed an example in this section. In the second section we have used a different approach to extend Perov's [74] $F-P$ theorem. We have used the Presic [75] type contraction condition in this section to prove our results. We have proved Perov's $F-P$ theorems for single and two metrics. An example is also constructed to show the importance of our results

3.1 Generalized metric space enriched with graph

This section deals with the generalized $M-S$. We will prove some $F-P$ theorems in this space with the help of graph.

Throughout this section, (X_d, d_g) is a generalized $M-S$ and we will denote $G=(V_g, E_g)$ as a directed graph such that the set V_g of its vertices coincides with X_d and the set E_g of its edges contains loops; so that, $E_g \supseteq \triangle$, where $\triangle$ is the diagonal of the Cartesian product $X_d \times X_d$.

Theorem 3.1.1. Let (X_d, d_g) be a complete generalized $M-S$ enriched with the graph G and let $T_f \colon X_d \to X_d$ be an edge preserving mapping with $A_{1f}, A_{2f} \in M_{m,m}(\mathbb{R}_+)$ such that

$$d_g(T_f e, T_f \tilde{e}) \leq A_{1f} d_g(e, \tilde{e}) + A_{2f} d_g(\tilde{e}, T_f e) \tag{3.1}$$

$\forall\ (e, \tilde{e}) \in E_g$. Assume that the following conditions hold:

(i) the matrix A_{1f} converges toward zero;

(ii) $\exists$ an $e_0 \in X_d$ such that $(e_0, T_f e_0) \in E_g$;

(iii) a. T_f is a G-continuous;
or
b. $\forall$ sequence $\{e_n\} \in X_d$ such that $e_n \to e$ and $(e_n, e_{n+1}) \in E_g$ $\forall$ $n \in \mathbb{N}$, we have $(e_n, e) \in E_g$ $\forall\ n \in \mathbb{N}$.

Then T_f has a $F-P$. Moreover, if $\forall\ e, \tilde{e} \in Fix(T_f)$, we have $(e, \tilde{e}) \in E_g$ and $A_{1f} + A_{2f}$ converges to zero, then we have a unique $F-P$.

Proof. By hypothesis (ii), we have $(e_0, T_f e_0) \in E_g$. Take $e_1 = T_f e_0$. From (3.1), we have

$$\begin{aligned} d_g(e_1, e_2) = d_g(T_f e_0, T_f e_1) &\leq A_{1f} d_g(e_0, e_1) + A_{2f} d_g(e_1, T_f e_0) \\ &= A_{1f} d_g(e_0, e_1). \end{aligned} \tag{3.2}$$

As T_f is edge preserving mapping, then $(e_1, e_2) \in E_g$, again from (3.1), we have

$$\begin{aligned} d_g(e_2, e_3) = d_g(T_f e_1, T_f e_2) &\leq A_{1f} d_g(e_1, e_2) + A_{2f} d_g(e_2, T_f e_1) \\ &\leq A_{1f}^2 d_g(e_0, e_1) \text{ (by using (3.2))} \end{aligned}$$

Proceeding in the same way, we get a sequence $\{e_n\} \subseteq X_d$, such that $e_n = T_f e_{n-1}$, $(e_{n-1}, e_n) \in E_g$ and

$$d_g(e_n, e_{n+1}) \leq A_{1f}^n d_g(e_0, e_1),\ \forall\ n \in \mathbb{N}.$$

Now $\forall\ n, m \in \mathbb{N}$. By making use of triangle inequality we get

$$\begin{aligned} d_g(e_n, e_{n+m}) &\leq \sum_{i=n}^{n+m-1} d_g(e_i, e_{i+1}) \\ &\leq \sum_{i=n}^{n+m-1} A_{1f}^{i} d_g(e_0, e_1) \\ &\leq A_{1f}^{n} \left(\sum_{i=0}^{\infty} A_{1f}^{i} \right) d_g(e_0, e_1) \\ &= A_{1f}^{n} (I - A_{1f})^{-1} d_g(e_0, e_1). \end{aligned}$$

Now if we let $n \to \infty$ in the above inequality we get, $d_g(e_n, e_{n+m}) \to 0$. As we know that A_{1f} is converging towards zero. Thus, the sequence $\{e_n\}$ is a *C-S*. As X_d is complete. Then $\exists$ an $e^* \in X_d$, such that $e_n \to e^*$. If hypothesis (iii.a) holds, then we have $T_f e_n \to T_f e^*$, so that $e_{n+1} \to T_f e^*$. Thus, $T_f e^* = e^*$. If (iii.b) holds, then we have $(e_n, e^*) \in E_g \ \ \forall n \in \mathbb{N}$. From (3.1), we have

$$d_g(e_{n+1}, T_f e^*) = d_g(T_f e_n, T_f e^*) \leq A_{1f} d_g(e_n, e^*) + A_{2f} d_g(e^*, T_f e_n) = A_{1f} d_g(e_n, e^*) + A_{2f} d_g(e^*, e_{n+1})$$

Now if we let $n \to \infty$, in the above inequality, we obtain $d_g(e^*, T_f e^*) = 0$. This shows that $e^* = T_f e^*$. Further, suppose that $e, \tilde{e} \in Fix(T_f)$ and $(e, \tilde{e}) \in E_g$, then by (3.1), we have

$$d_g(e, \tilde{e}) \leq A_{1f} d_g(e, \tilde{e}) + A_{2f} d_g(e, \tilde{e}).$$

So that,

$$(I - (A_{1f} + A_{2f})) d_g(e, \tilde{e}) \leq 0.$$

As we know that the matrix $I - (A_{1f} + A_{2f})$ is nonsingular, then $d_g(e, \tilde{e}) = 0$. Thus, we have $Fix(T_f) = \{e\}$. □

Remark 3.1.2. If we assume that $E_g = X_d \times X_d$ and $A_{2f} = \overline{0}$ then the above discussed theorems reduces to Theorem 1.4.2.

Example 3.1.3. Let $X_d = \mathbb{R}^2$ enriched with a generalized-metric defined by $d_g(e, \tilde{e}) = \begin{pmatrix} |e_1 - \tilde{e}_1| \\ |e_2 - \tilde{e}_2| \end{pmatrix}$ $\forall\ e = (e_1, e_2), \tilde{e} = (\tilde{e}_1, \tilde{e}_2) \in \mathbb{R}^2$. And an operator defined by

$$T_f : \mathbb{R}^2 \to \mathbb{R}^2, \quad T_f(e,\tilde{e}) = \begin{cases} \left(\frac{2e}{3} - \frac{\tilde{e}}{3} + 1, \frac{\tilde{e}}{3} + 1\right) & \text{for } (e,\tilde{e}) \in X_d \text{ with } e \leq 3 \\ \left(\frac{2e}{3} - \frac{\tilde{e}}{3} + 1, -\frac{5e}{3} + \frac{\tilde{e}}{3} + 1\right) & \text{for } (e,\tilde{e}) \in X_d \text{ with } e > 3. \end{cases}$$

If we consider $T_f(e,\tilde{e}) = (T_{f_1}(e,\tilde{e}), T_{f_2}(e,\tilde{e}))$, where

$$T_{f_1}(e,\tilde{e}) = \frac{2e}{3} - \frac{\tilde{e}}{3} + 1$$

and

$$T_{f_2}(e,\tilde{e}) = \begin{cases} \frac{\tilde{e}}{3} + 1 & \text{if } e \leq 3 \\ -\frac{5e}{3} + \frac{\tilde{e}}{3} + 1 & \text{if } e > 3, \end{cases}$$

then it can be easily seen that

$$|T_{f_1}(e_1,e_2) - T_{f_1}(\tilde{e}_1,\tilde{e}_2)| \leq \frac{2}{3}|e_1 - \tilde{e}_1| + \frac{1}{3}|e_2 - \tilde{e}_2|$$

and

$$|T_{f_2}(e_1,e_2) - T_{f_2}(\tilde{e}_1,\tilde{e}_2)| \leq \begin{cases} \frac{1}{3}|e_2 - \tilde{e}_2| & \text{if } e_1, \tilde{e}_1 \leq 3 \\ \frac{5}{3}|e_1 - \tilde{e}_1| + \frac{1}{3}|e_2 - \tilde{e}_2| & \text{otherwise.} \end{cases}$$

$\forall\ (e_1,e_2), (\tilde{e}_1,\tilde{e}_2) \in X_d$. A graph defined by $G = (V_g, E_g)$ such that $V_g = \mathbb{R}^2$ and $E_g = \{((e_1,e_2),(\tilde{e}_1,\tilde{e}_2)) : e_1, e_2, \tilde{e}_1, \tilde{e}_2 \in [0,3]\} \cup \{(\tilde{\tilde{e}}, \tilde{\tilde{e}}) : \tilde{\tilde{e}} \in \mathbb{R}^2\}$. Now $\forall\ (e,\tilde{e}) \in E_g$, we have

$$d_g(T_f e, T_f \tilde{e}) = \begin{pmatrix} |T_{f_1}(e_1,e_2) - T_{f_1}(\tilde{e}_1,\tilde{e}_2)| \\ |T_{f_2}(e_1,e_2) - T_{f_2}(\tilde{e}_1,\tilde{e}_2)| \end{pmatrix} \leq \begin{pmatrix} \frac{2}{3} & \frac{1}{3} \\ 0 & \frac{1}{3} \end{pmatrix} \begin{pmatrix} |e_1 - \tilde{e}_1| \\ |e_2 - \tilde{e}_2| \end{pmatrix} = A_{1f} d_g(e,\tilde{e}).$$

Moreover, it can be easily seen that all the other conditions of Theorem 3.1.1 hold, thus, T_f has a $F-P$, So that $e = T_f e = (T_{f_1} e, T_{f_2} e)$, where $e = (1.5, 1.5)$.

We shall now prove some results which deal with a mapping on two *C-M-S*.

Theorem 3.1.4. Let X_d be a non-empty set enriched with the graph G and two metrics d_g, σ. Let $T_f : (X_d, \sigma) \to (X_d, \sigma)$ be an edge preserving mapping with $A_{1f}, A_{2f} \in M_{m,m}(\mathbb{R}_+)$ such that

$$\sigma(T_f e, T_f \tilde{e}) \leq A_{1f}\sigma(e,\tilde{e}) + A_{2f}\sigma(\tilde{e}, T_f e)\ \forall\ (e,\tilde{e}) \in E_g. \tag{3.3}$$

Assume that the following conditions hold:

(i) the matrix A_{1f} converges towards zero;

(ii) $\exists$ an $e_0 \in X_d$ such that $(e_0, T_f e_0) \in E_g$;

(iii) $T_f : (X_d, d_g) \to (X_d, d_g)$ is a G-contraction;

(iv) $\exists$ an $\tilde{A}_3 \in M_{m,m}(\mathbb{R}_+)$ such that $d_g(T_f e, T_f \tilde{e}) \leq \sigma(e, \tilde{e}).\tilde{A}_3$, whenever, $\exists$ a path between e and $\tilde{e}$;

(v) (X_d, d_g) is complete generalize-$M-S$.

Then T_f has a $F-P$. Moreover, if, $\forall\, e, \tilde{e} \in Fix(T_f)$, we have $(e, \tilde{e}) \in E_g$ and $A_{1f} + A_{2f}$ converges to zero then we have a unique $F-P$.

Proof. By hypothesis (ii), we have $(e_0, T_f e_0) \in E_g$. Take an $e_1 = T_f e_0$. From (3.3), we have,

$$\begin{aligned} \sigma(e_1, e_2) = \sigma(T_f e_0, T_f e_1) &\leq A_{1f}\sigma(e_0, e_1) + A_{2f}\sigma(e_1, T_f e_0) \\ &= A\sigma(e_0, e_1). \end{aligned}$$

As T_f is edge preserving, then $(e_1, e_2) \in E_g$. Again from (3.3), we have

$$\begin{aligned} \sigma(e_2, e_3) = \sigma(T_f e_1, T_f e_2) &\leq A_{1f}\sigma(e_1, e_2) + A_{2f}\sigma(e_2, T_f e_1) \\ &= A_{1f}^2\sigma(e_0, e_1). \end{aligned}$$

Proceeding in the same way we get a sequence $\{e_n\}$ in X_d such that $e_n = T_f e_{n-1}, \quad (e_{n-1}, e_n) \in E_g$, and

$$\sigma(e_n, e_{n+1}) \leq A_{1f}^n\sigma(e_0, e_1) \quad \forall n \in \mathbb{N}.$$

We will now show that $\{e_n\}$ is a C-S in (X_d, σ). By making use of triangle inequality, we get

$$\begin{aligned} \sigma(e_n, e_{n+m}) &\leq \sum_{i=n}^{n+m-1} \sigma(e_i, e_{i+1}) \\ &\leq \sum_{i=n}^{n+m-1} A^i\sigma(e_0, e_1) \\ &\leq A_{1f}^n \left(\sum_{i=0}^{\infty} A_{1f}^i\right) \sigma(e_0, e_1) \\ &= A_{1f}^n(I - A_{1f})^{-1}\sigma(e_0, e_1), \end{aligned}$$

as we know that A_{1f} converges towards zero. Thus $\{e_n\}$ is a C-S in (X_d, σ). By the construction of this sequence, $\forall\ n, m \in \mathbb{N}$, we have a path between e_n and e_{n+m}. Now, if we use hypothesis (iv), we will obtain

$$\begin{aligned} d_g(e_{n+1}, e_{n+m+1}) &= d_g(T_f e_n, T_f e_{n+m}) \\ &\leq \sigma(e_n, e_{n+m}).C \\ &\leq A_{1f}^n (I - A_{1f})^{-1} \sigma(e_0, e_1).\tilde{A}_3 \end{aligned}$$

This shows that $\{e_n\}$ is also Cauchy in (X_d, d_g). As (X_d, d_g) is complete, so $\exists$ an $e^* \in X_d$, such that $e_n \to e^*$. By hypothesis (iii) we will have $\lim_{n\to\infty} d_g(T_f e_n, T_f e^*) = 0$. As $e_{n+1} = T_f e_n\ \forall\ n \in \mathbb{N}$. Thus, e^* is a F-P of T_f. Further suppose that $e, \tilde{e} \in Fix(T_f)$ and $(e, \tilde{e}) \in E_g$, then by (3.3), we have

$$\sigma(e, \tilde{e}) \leq A_{1f}\sigma(e, \tilde{e}) + A_{2f}\sigma(\tilde{e}, e).$$

So that,

$$(I - (A_{1f} + A_{2f}))\sigma(e, \tilde{e}) \leq 0.$$

As we know that, the matrix $I - (A_{1f} + A_{2f})$ is nonsingular, then $\sigma(e, \tilde{e}) = 0$. Thus, we obtain $Fix(T_f) = \{e\}$. □

We can also extend theorem 3.1.1 for multivalued mappings in the following way.

Theorem 3.1.5. Let (X_d, d_g) be a complete generalized $M-S$ enriched with the graph G and let $\mathbf{T}_f \colon X_d \to Cl_f(X_d)$ be a multi-valued mapping with $A_{1f}, A_{2f} \in M_{m,m}(\mathbb{R}_+)$, such that $\forall\ (e, \tilde{e}) \in E_g$ and a $u \in \mathbf{T}_f e$, there exists a $v \in \mathbf{T}_f \tilde{e}$ satisfying

$$d_g(u, v) \leq A_{1f} d_g(e, \tilde{e}) + A_{2f} d_g(\tilde{e}, u). \tag{3.4}$$

Assume that the following conditions hold:

(i) the matrix A_{1f} converges towards zero;

(ii) $\exists$ an $e_0 \in X_d$ and $e_1 \in \mathbf{T}_f e_0$ such that $(e_0, e_1) \in E_g$;

(iii) $\forall\ u \in \mathbf{T}_f e$ and $v \in \mathbf{T}_f \tilde{e}$ with $d_g(u, v) \leq A_{1f} d_g(e, \tilde{e})$ we have $(u, v) \in E_g$ whenever $(e, \tilde{e}) \in E_g$;

(iv) $\forall$ sequence $\{e_n\}$ in X_d such that $e_n \to e$ and $(e_n, e_{n+1}) \in E_g$ $\forall\ n \in \mathbb{N}$, we have $(e_n, e) \in E_g$ $\forall\ n \in \mathbb{N}$.

Proof. By the above hypothesis *(ii)*, we have an $e_0 \in X_d$ and $e_1 \in \boldsymbol{T}_f e_0$ with $(e_0, e_1) \in E_g$. From (3.4), for $(e_0, e_1) \in E_g$, we have an $e_2 \in \boldsymbol{T}_f e_1$ such that

$$\begin{aligned} d_g(e_1, e_2) &\leq A_{1f} d_g(e_0, e_1) + A_{2f} d_g(e_1, e_1) \\ &= A_{1f} d_g(e_0, e_1). \end{aligned} \tag{3.5}$$

By hypothesis *(iii)* and (3.5), we have $(e_1, e_2) \in E_g$. Again from (3.4), for $(e_1, e_2) \in E_g$ and an $e_2 \in \boldsymbol{T}_f e_1$, we have an $e_3 \in \boldsymbol{T}_f e_2$ such that

$$\begin{aligned} d_g(e_2, e_3) &\leq A_{1f} d_g(e_1, e_2) + A_{2f} d_g(e_2, e_2) \\ &\leq A_{1f}^2 d_g(e_0, e_1) \quad (\text{ by using } (3.5)). \end{aligned}$$

Proceeding in the same way, we will get a sequence $\{e_n\}$ in X_d such that $e_n \in \boldsymbol{T}_f e_{n-1}$, $(e_{n-1}, e_n) \in E_g$ and

$$d_g(e_n, e_{n+1}) \leq A_{1f}^n d_g(e_0, e_1), \ \forall \ \in \mathbb{N}.$$

For each $n, m \in \mathbb{N}$. By making use of triangle inequality we will have,

$$\begin{aligned} d_g(e_n, e_{n+m}) &\leq \sum_{i=n}^{n+m-1} d_g(e_i, e_{i+1}) \\ &\leq \sum_{i=n}^{n+m-1} A_{1f}^i d_g(e_0, e_1) \\ &\leq A_{1f}^n \left(\sum_{i=0}^{\infty} A_{1f}^i \right) d_g(e_0, e_1) \\ &= A_{1f}^n (I - A_{1f})^{-1} d_g(e_0, e_1). \end{aligned}$$

As we know that the matrix A_{1f} converges towards 0. Thus the sequence $\{e_n\}$ is a Cauchy sequence in X_d. As X_d is complete. Then $\exists$ an $e^* \in X_d$, such that $e_n \to e^*$. By hypothesis (iv) we have $(e_n, e^*) \in E_g$, $\forall\ n \in \mathbb{N}$. From (3.4), for $(e_n, e^*) \in E_g$ and $e_{n+1} \in \boldsymbol{T}_f e_n$ we have $w^* \in \boldsymbol{T}_f e^*$ such that

$$d_g(e_{n+1}, w^*) \leq A_{1f} d_g(e_n, e^*) + A_{2f} d_g(e^*, e_{n+1}).$$

Now, if, we let $n \to \infty$ in the above inequality, we get $d_g(e^*, w^*) = 0$, so that, $e^* = w^*$. Thus $e^* \in \boldsymbol{T}_f e^*$. $\square$

Theorem 3.1.1 is extended for multivalued mappings for two C-M-S in the following theorem.

Theorem 3.1.6. Let X_d be a non-empty set enriched with the graph G and two metrics d_g, σ. Let $\mathbf{T}_f \colon X_d \to Cl_f(X_d)$ be a multi-valued mapping with $A_{1f}, A_{2f} \in M_{n,n}(\mathbb{R}_+)$, such that $\forall\ (e, \tilde{e}) \in E_g$ and $u \in \mathbf{T}_f e$ there exists $v \in \mathbf{T}_f \tilde{e}$ satisfying

$$\sigma(u, v) \leq A_{1f}\sigma(e, \tilde{e}) + A_{2f}\sigma(\tilde{e}, u). \tag{3.6}$$

Assume that the following conditions hold:

(i) the matrix A_{1f} converges towards zero;

(ii) $\exists$ an $e_0 \in X_d$ and $e_1 \in \mathbf{T}_f e_0$ such that $(e_0, e_1) \in E_g$;

(iii) $\forall\ u \in \mathbf{T}_f e$ and $v \in \mathbf{T}_f \tilde{e}$ with $\sigma(u, v) \leq A_{1f}\sigma(e, \tilde{e})$ we have $(u, v) \in E_g$ whenever $(e, \tilde{e}) \in E_g$;

(iv) (X_d, d_g) is complete generalize $M - S$;

(v) $\exists$ a $\tilde{A}_3 \in M_{m,m}(\mathbb{R}_+)$ such that $d_g(e, \tilde{e}) \leq \sigma(e, \tilde{e}).\tilde{A}_3$, whenever, $\exists$ a path between e and $\tilde{e}$;

(vi) $\forall$ sequence $\{e_n\}$ in X_d such that $e_n \to e$ and $(e_n, e_{n+1}) \in E_g\ \forall\ n \in \mathbb{N}$, we have $(e_n, e) \in E_g\ \forall\ n \in \mathbb{N}$.
Then $\mathbf{T}_f$ has a $F - P$.

Proof. By hypothesis (ii), we have an $e_0 \in X_d$ and $e_1 \in \boldsymbol{T}_f e_0$ such that $(e_0, e_1) \in E_g$. From (3.6), for $(e_0, e_1) \in E_g$ and $e_1 \in \boldsymbol{T}_f e_0$, we have $e_2 \in \boldsymbol{T}_f e_1$ such that

$$\begin{aligned} \sigma(e_1, e_2) &\leq A_{1f}\sigma(e_0, e_1) + A_{2f}\sigma(e_1, e_1) \\ &= A_{1f}\sigma(e_0, e_1). \end{aligned}$$

By hypothesis (iii) and above inequality, we will obtaian $(e_1, e_2) \in E_g$. Again from (3.6) for $(e_1, e_2) \in E_g$, and an $e_2 \in \boldsymbol{T}_f e_1$, we have an $e_3 \in \boldsymbol{T}_f e_2$ such that

$$\begin{aligned} \sigma(e_2, e_3) &\leq A_{1f}\sigma(e_1, e_2) + A_{2f}\sigma(e_2, e_2) \\ &\leq A_{1f}^2\sigma(e_0, e_1). \end{aligned}$$

Proceeding in the same way, we will obtain a sequence $\{e_n\} \in X_d$ such that $e_n \in \boldsymbol{T}_f e_{n-1}, \quad (e_{n-1}, e_n) \in E_g$ and

$$\sigma(e_n, e_{n+1}) \leq A_{1f}^{n} \sigma(e_0, e_1) \ \forall \ n \in \mathbb{N}.$$

We will now show that $\{e_n\}$ is a *C-S* in (X_d, σ). Let $n, m \in \mathbb{N}$, then by making use of triangle inequality we get

$$\begin{aligned}
\sigma(e_n, e_{n+m}) &\leq \sum_{i=n}^{n+m-1} \sigma(e_i, e_{i+1}) \\
&\leq \sum_{i=n}^{n+m-1} A_{1f}^{i} \sigma(e_0, e_1) \\
&\leq A_{1f}^{n} \left(\sum_{i=0}^{\infty} A_{1f}^{i} \right) \sigma(e_0, e_1) \\
&= A_{1f}^{n} (I - A_{1f})^{-1} \sigma(e_0, e_1). \qquad (3.7)
\end{aligned}$$

As we know that the matrix A_{1f} converges towards zero. Thus $\{e_n\}$ is a *C-S* in (X_d, σ). Clearly, $\forall \ m, n \in \mathbb{N} \ \exists$ a path between e_n and e_{n+m}. By making use of hypothesis (v) we get,

$$\begin{aligned}
d_g(e_n, e_{n+m}) &\leq \sigma(e_n, e_{n+m}).\tilde{A}_3 \\
&\leq A_{1f}^{n-1} (I - A_{1f})^{-1} \sigma(e_0, e_1).\tilde{A}_3 \text{ (by using (3.7))}.
\end{aligned}$$

Thus $\{e_n\}$ is also a *C-S* in (X_d, d_g). As (X_d, d_g) is complete, $\exists$ an $e^* \in X_d$, such that $e_n \to e^*$. By hypothesis (vi) we have $(e_n, e^*) \in E_g \ \forall \ n \in \mathbb{N}$. From (3.4), for $(e_n, e^*) \in E_g$ and $e_{n+1} \in \boldsymbol{T}_f e_n$ we have a $w^* \in \boldsymbol{T}_f e^*$ such that

$$\sigma(e_{n+1}, w^*) \leq A_{1f} \sigma(e_n, e^*) + A_{2f} \sigma(e^*, e_{n+1}).$$

Now, if, we let $n \to \infty$ in the above inequality we get $\sigma(e^*, w^*) = 0$. This implies that $e^* \in \boldsymbol{T}_f e^*$.

□

3.2 Presic-Perov type fixed point theorems

In this section we introduce a new generalization of a generalized $M - S$. Then we use our generalized $M - S$ to prove $F - P$ theorems for mappings satisfying a contractive condition, which is a mixture of Perov and Presic contractions.

Throughout this section, we use the following vector-valued/generalized $M-S$. Let X_d be a non-empty set and $\Re_+{}^m$ is the set of all m-tuples with non-negative real numbers. If $\zeta, \eta \in [0,\infty)^m$, such that $\zeta = (\zeta_1, \zeta_2, ...\zeta_m)$, and $\eta = (\eta_1, \eta_2, ...\eta_m)$, then $\zeta \leq \eta (\zeta < \eta)$ means $\sum_{i=1}^m \zeta_i \leq \sum_{i=1}^m \eta_i (\sum_{i=1}^m \zeta_i < \sum_{i=1}^m \eta_i)$, respectively. Further, for $c \in [0,\infty)$, $\zeta \leq c$ means $\sum_{i=1}^m \zeta_i \leq mc$. A mapping $d : X_d \times X_d \to \Re_+{}^m$ is called a vector-valued/generalized-metric on X_d in the sense of summation, if the following properties are satisfied:

(d_1) $d_v(e,\tilde{e}) \geq 0 \ \forall \ e, \tilde{e} \in X_d$; if $d_v(e,\tilde{e}) = 0$, then $e = \tilde{e}$;

(d_2) $d_v(e,\tilde{e}) = d_v(\tilde{e}, e) \ \forall \ e, \tilde{e}, \tilde{\tilde{e}} \in X_d$;

(d_3) $d_v(e,\tilde{e}) \leq d_v(e,\tilde{\tilde{e}}) + d_v(\tilde{\tilde{e}},\tilde{e}) \ \forall \ e, \tilde{e}, \tilde{\tilde{e}} \in X_d$.

Then the pair (X_d, d_v) is called vector-valued/generalized $M-S$ in the sense of summation.

Remark 3.2.1. Note that every vector-valued/generalized $M-S$ in the sense of Perov is also a vector-valued/generalized $M-S$ in the sense of summation but converse is not true in general. To see, we consider the following most simple example.

Example 3.2.2. Let $X_d = \{(1,0),(0,1),(0,2)\}$. Define the vector-valued/generalized-metric $d_v : X_d \times X_d \to \Re_+{}^2$ as $d_v((1,0),(0,1)) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} = d_v((0,1),(1,0))$, $d_v((0,1),(0,2)) = \begin{pmatrix} 0 \\ 1 \end{pmatrix} = d_v((0,2),(0,1))$, $d_v((1,0),(0,2)) = \begin{pmatrix} 0 \\ 2 \end{pmatrix} = d_v((1,0),(0,2))$ and $d_v((1,0),(1,0)) = d_v((0,1),(0,1)) = d_v((0,2),(0,2)) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$. It is easy to see that d_v is generalized-metric on X_d in the sense of summation, but not in the sense of Perov.

We now prove some results regarding the extension of Perov's $F-P$ theorem in the light of Ćirić and Prešić contractive type condition.

Theorem 3.2.3. Let (X_d, d_v) be a complete generalized $M-S$, k be a positive integer and let $T_f : X_d^k \to X_d$ be a mapping with $A_{1f} \in M_{m,m}(\Re_+)$ such that

$$d_v(T_f(e_1, e_2, \cdots, e_k), T_f(e_2, e_3, \cdots, e_{k+1})) \leq A_{1f} \max\{d_v(e_i, e_{i+1}) : i = 1, 2, \cdots k\} \quad (3.8)$$

$\forall$ $((e_1, e_2, \cdots, e_k), (e_2, e_3, \cdots, e_{k+1})) \in X_d^k \times X_d^k$. Assume that the following conditions hold:

(i) $\exists$ a nonsingular matric D_f such that $A_{1f} = D_f^k$;

(ii) the matrix A_{1f} converges toward zero.

Then $\exists$ a point $e \in X_d$ such that $e = T_f(e, e, \cdots, e)$. Moreover, for any arbitrary $e_1, e_2, \cdots, e_k \in X_d$, $\exists$ a sequence $\{e_n\}$ in X_d such that $e_{k+n} = T_f(e_n, e_{n+1}, \cdots, e_{k+n-1})$ $\forall$ $n \in \mathbb{N}$ and $\{e_n\}$ converges to a fixed point of T_f.

Proof. Let $(e_1, e_2, \cdots, e_k) \in X_d^k$, we construct a sequence $\{e_{k+n}\}$ such that $e_{k+n} = T_f(e_n, e_{n+1}, \cdots, e_{k+n-1})$ $\forall$ $n \in \mathbb{N}$. From (3.8), we have

$$\begin{aligned} d_v(e_{k+n}, e_{k+n+1}) &= d_v(T_f(e_n, e_{n+1}, \cdots, e_{k+n-1}), T_f(e_{n+1}, e_{n+2}, \cdots, e_{k+n})) \\ &\leq A_{1f} \max\{d(e_i, e_{i+1}) : i = n, n+1, \cdots n+k\} \ \forall \ n \in \mathbb{N} \quad (3.9) \end{aligned}$$

Thus, from (3.9), we obtain

$$d(e_{k+n}, e_{k+n+1}) \leq A_{1f}^n \max\{d(e_i, e_{i+1}) : i = 1, 2, \cdots k\} \ \forall \ n \in \mathbb{N}. \quad (3.10)$$

We denote $d_{vn} = d(e_n, e_{n+1})$ $\forall$ $n \in \mathbb{N}$. We will show by induction that

$$d_n \leq D_f^n \theta \ \forall \ n \in \mathbb{N}$$

where, $A_{1f} = D_f^k$ and $\theta = \max\{D_f^{-1} d_{v1}, D_f^{-2} d_{v2}, \cdots, D_f^{-k} d_{vk}\}$. Consider $d_{v1} \leq D_f \theta$, $d_{v2} \leq D_f^2 \theta, \cdots, d_{v(m+k-1)} \leq D_f^{m+k-1} \theta$. We will show that the above inequality also holds for $m+k \in \mathbb{N}$. From (3.9), we have

$$\begin{aligned} d_{v(m+k)} &\leq A_{1f} \max\{d_{vm}, d_{v(m+1)}, \cdots, d_{v(m+k-1)}\} \\ &\leq D_f^k \max\{D_f^m \theta, D_f^{m+1} \theta, \cdots, D_f^{v(m+k-1)} \theta\} \\ &= D_f^{k+m} \theta. \end{aligned}$$

Thus, we have $d_{vn} \leq D_f^n \theta \; \forall \, n \in \mathbb{N}$. For $n, p \in \mathbb{N}$, by making use of triangular inequality, we have

$$\begin{aligned} d_v(e_n, e_{n+p}) &\leq d_v(e_n, e_{n+1}) + d_v(e_{n+1}, e_{n+2}) + \cdots + d_v(e_{n+p-1}, e_{n+p}) \\ &\leq [D_f^n + D_f^{n+1} + \cdots + D_f^{n+p-1}]\theta \\ &\leq D_f^n (I - D_f)^{-1} \theta. \end{aligned}$$

Now, if, we let $n \to \infty$, in above inequality, we obtain $d_v(e_n, e_{n+p}) \to 0$. Thus, $\{e_n\}$ is a *C-S* in X_d. As we know that X_d is complete, we have $e^* \in X_d$ such that $e_n \to e^*$. By making use of triangle inequality, we have

$$\begin{aligned} d_v(e^*, T_f(e^*, e^*, \cdots, e^*)) &\leq d_v(e^*, e_{k+n}) + d_v(e_{k+n}, T_f(e^*, e^*, \cdots, e^*)) \\ &= d_v(e^*, e_{k+n}) + d_v(T_f(e_n, e_{n+1}, \cdots, e_{k+n-1}), T_f(e^*, e^*, \cdots, e^*)) \\ &\leq d_v(e^*, e_{k+n}) + d_v(T_f(e_n, e_{n+1}, \cdots, e_{k+n-1}), T_f(e_{n+1}, \cdots, e_{k+n-1}, e^*)) \\ &\quad + d_v(T_f(e_{n+1}, \cdots, e_{k+n-1}, e^*), T_f(e_{n+2}, \cdots, e_{k+n-1}, e^*, e^*)) \\ &\quad + \cdots + d_v(T_f(e_{k+n-1}, e^* \cdots, e^*), T_f(e^*, e^*, \cdots, e^*)) \\ &\leq d_v(e^*, e_{k+n}) + A_{1f} \max\{d_v(e_n, e_{n+1}), d_v(e_{n+1}, e_{n+2}), \cdots, \\ &\quad d_v(e_{k+n-1}, e_{k+n-2}), d_v(e_{k+n-1}, e^*)\} \\ &\quad + A_{1f} \max\{d_v(e_{n+1}, e_{n+2}), d_v(e_{n+2}, e_{n+3}), \cdots, d_v(e_{k+n-1}, e^*)\} \\ &\quad + \cdots + A_{1f} d_v(e_{k+n-1}, e^*). \end{aligned}$$

Now, if, we let $n \to \infty$ in above inequality, we get $d_v(e^*, T_f(e^*, e^*, \cdots, e^*)) = 0$. Thus, $e^* = T_f(e^*, e^*, \cdots, e^*)$. □

Example 3.2.4. Let $X_d = \Re$ be enriched with a generalized-metric define by $d_v(e, \tilde{e}) = \begin{pmatrix} |e - \tilde{e}| \\ |e - \tilde{e}| \end{pmatrix} \; \forall \, e, \tilde{e} \in X_d$. Consider the operator $T_f : X_d \times X_d \to X_d$ such that

$$T_f(e, \tilde{e}) = \frac{e}{4} - \frac{\tilde{e}}{4} + 1$$

It can be easily seen that

$$|T_f(e_1, e_2) - T_f(e_2, e_3)| \leq \frac{1}{4}|e_1 - e_2| + \frac{1}{4}|e_2 - e_3|$$

Thus, $\forall\ (e_1, e_2), (e_2, e_3) \in X_d \times X_d$, we have

$$
\begin{aligned}
d_v(T_f(e_1,e_2), T_f(e_2,e_3)) = \begin{pmatrix} |T_f(e_1,e_2) - T_f(e_2,e_3)| \\ |T_f(e_1,e_2) - T_f(e_2,e_3)| \end{pmatrix} &\leq \begin{pmatrix} \frac{1}{4}|e_1 - e_2| + \frac{1}{4}|e_2 - e_3| \\ \frac{1}{4}|e_1 - e_2| + \frac{1}{4}|e_2 - e_3| \end{pmatrix} \\
&= \begin{pmatrix} \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} \end{pmatrix} \begin{pmatrix} |e_1 - e_2| \\ |e_2 - e_3| \end{pmatrix} \\
&\leq \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix} \max\left\{ \begin{pmatrix} |e_1 - e_2| \\ |e_1 - e_2| \end{pmatrix}, \begin{pmatrix} |e_2 \\ |e_2 \end{pmatrix} \right. \\
&= A_{1f} \max\{d_v(e_1,e_2), d_v(e_2,e_3)\}
\end{aligned}
$$

where, we have $D_f = \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 \\ 0 & \frac{1}{\sqrt{2}} \end{pmatrix}$ such that $A_{1f} = D_f^2$. Thus, by Theorem 3.2.3, we have an $e \in X_d$ such that $e = T_f(e, e)$.

We now extend our above results for two *C-M-S*.

Theorem 3.2.5. Let X_d be a nonempty set enriched with two *C-M-S* d_v, σ_v and let k be a positive integer, $T_f : X_d^k \to X_d$ be a mapping with $A_{1f} \in M_{m,m}(\Re)$ such that

$$\sigma_v(T_f(e_1, e_2, \cdots, e_k), T_f(e_2, e_3, \cdots, e_{k+1})) \leq A_{1f} \max\{\sigma_v(e_i, e_{i+1}) : i = 1, 2, \cdots, k\} \quad (3.14)$$

$\forall\ ((e_1, e_2, \cdots, e_k), (e_2, e_3, \cdots, e_{k+1})) \in X_d^k \times X_d^k$. Assume that the following conditions hold:

(i) $\exists$ a nonsingular matric D_f such that $A_{1f} = D_f^k$;

(ii) the matrix A_{1f} converges toward zero;

(iii) (X_d, d_v) is a complete generalized $M - S$;

(iv) $\exists$ a matrix $C \in M_{m,m}(\Re)$ such that $d_v(e, \tilde{e}) < C\sigma_v(e, \tilde{e})\ \forall\ e, \tilde{e} \in X_d$.

Then $\exists$ a point $e \in X_d$ such that $e = T_f(e, e, \cdots, e)$. Moreover, for any arbitrary $e_1, e_2, \cdots, e_k \in X_d$, $\exists$ a sequence $\{e_n\}$ in X_d such that $e_{k+n} = T_f(e_n, e_{n+1}, \cdots, e_{k+n-1})\ \forall\ n \in \mathbb{N}$ and $\{e_n\}$ converges to fixed point of T_f.

Proof. Let $(e_1, e_2, \cdots, e_k) \in X_d^k$, we construct a sequence $\{e_{k+n}\}$ such that $e_{k+n} = T_f(e_n, e_{n+1}, \cdots, e_{k+n-1}) \; \forall \; n \in \mathbb{N}$. From (3.11), we have

$$\begin{aligned} \sigma_v(e_{k+n}, e_{k+n+1}) &= \sigma_v(T_f(e_n, e_{n+1}, \cdots, e_{k+n-1}), T_f(e_{n+1}, e_{n+2}, \cdots, e_{k+n})) \\ &\leq A_{1f} \max\{\sigma_v(e_i, e_{i+1}) : i = n, n+1, \cdots n+k\} \; \forall \; n \in \mathbb{N} \end{aligned} \quad (3.12)$$

Thus, from (3.12), we get

$$\sigma_v(e_{k+n}, e_{k+n+1}) \leq A_{1f}^{\,n} \max\{\sigma_v(e_i, e_{i+1}) : i = 1, 2, \cdots k\} \; \forall \; n \in \mathbb{N}. \quad (3.13)$$

We denote $\sigma_{vn} = \sigma_v(e_n, e_{n+1}) \; \forall \; n \in \mathbb{N}$. We will show by induction that

$$\sigma_{vn} \leq D_f^n \theta \; \forall \; n \in \mathbb{N}$$

where, $A_{1f} = D_f^k$ and $\theta = \max\{D_f^{-1}\sigma_{v1}, D_f^{-2}\sigma_{v2}, \cdots, D_f^{-k}\sigma_{vk}\}$. Consider $\sigma_{v1} \leq D_f\theta$, $\sigma_{v2} \leq D_f^2\theta, \cdots, \sigma_{v(m+k-1)} \leq D_f^{m+k-1}\theta$. We will show that above inequality also holds for $m + k \in \mathbb{N}$. from (3.9), we have

$$\begin{aligned} \sigma_{v(m+k)} &\leq A_{1f} \max\{\sigma_{vm}, \sigma_{v(m+1)}, \cdots, \sigma_{v(m+k-1)}\} \\ &\leq D_f^k \max\{D_f^m\theta, D_f^{m+1}\theta, \cdots, D_f^{m+k-1}\theta\} \\ &= D_f^{k+m}\theta. \end{aligned}$$

Thus, we have $\sigma_{vn} \leq D_f^n\theta \; \forall \; n \in \mathbb{N}$. For $n, p \in \mathbb{N}$, by making use of triangular inequality, we have

$$\begin{aligned} \sigma_v(e_n, e_{n+p}) &\leq \sigma_v(e_n, e_{n+1}) + \sigma_v(e_{n+1}, e_{n+2}) + \cdots + \sigma_v(e_{n+p-1}, e_{n+p}) \\ &\leq [D_f^n + D_f^{n+1} + \cdots + D_f^{n+p-1}]\theta \\ &\leq D_f^n(I - D_f)^{-1}\theta. \end{aligned}$$

Now, if, we let $n \to \infty$, in above inequality, we get $\sigma_v(e_n, e_{n+p}) \to 0$. By making use of hypothesis (iv), $\forall \; n, p \in \mathbb{N}$, we have $d_v(e_n, e_{n+p}) \leq C\sigma_v(e_n, e_{n+p})$. Thus, $\{e_n\}$ is a *C-S* in (X_d, d_v). As we know that (X_d, d_v) is complete, we have an $e^* \in X_d$ such that $e_n \to e^*$. By making use of triangle inequality, we have

$$\begin{aligned}
\sigma_v(e^*, T_f(e^*, e^*, \cdots, e^*)) &\leq \sigma_v(e^*, e_{k+n}) + \sigma_v(e_{k+n}, T_f(e^*, e^*, \cdots, e^*)) \\
&= \sigma_v(e^*, e_{k+n}) + \sigma_v(T_f(e_n, e_{n+1}, \cdots, e_{k+n-1}), T_f(e^*, e^*, \cdots, e^*)) \\
&\leq \sigma_v(e^*, e_{k+n}) + \sigma_v(T_f(e_n, e_{n+1}, \cdots, e_{k+n-1}), T_f(e_{n+1}, \cdots, e_{k+n-1}, e^*)) \\
&\quad +\sigma_v(T_f(e_{n+1}, \cdots, e_{k+n-1}, e^*), T_f(e_{n+2}, \cdots, e_{k+n-1}, e^*, e^*)) \\
&\quad + \cdots + \sigma_v(T_f(e_{k+n-1}, e^* \cdots, e^*), T_f(e^*, e^*, \cdots, e^*)) \\
&\leq \sigma_v(e^*, e_{k+n}) + A_{1f} \max\{\sigma_v(e_n, e_{n+1}), \sigma_v(e_{n+1}, e_{n+2}), \cdots, \\
&\quad \sigma_v(e_{k+n-1}, e_{k+n-2}), \sigma_v(e_{k+n-1}, e^*)\} \\
&\quad +A_{1f} \max\{\sigma_v(e_{n+1}, e_{n+2}), \sigma_v(e_{n+2}, e_{n+3}), \cdots, \sigma_v(e_{k+n-1}, e^*)\} \\
&\quad + \cdots + A_{1f}\sigma_v(e_{k+n-1}, e^*).
\end{aligned}$$

Now, if, we let $n \to \infty$ in above inequality, we will obtain $\sigma_v(e^*, T_f(e^*, e^*, \cdots, e^*)) = 0$. So that, $e^* = T_f(e^*, e^*, \cdots, e^*)$. □

Chapter 4

Fixed point theorems in uniform spaces enriched with E_S-distance

Throughout this chapter $F-P$ represents a fixed point, $M-S$ represents a metric space, $C-S$ represents a Cauchy sequence. In this chapter we introduce the concepts of an F_{w_G}-contraction and a $\tilde{\psi}_G$-contraction in uniform space enriched with a graph, to discuss the existence and uniqueness of fixed points for mappings satisfying these conditions. We shall also introduce a common $F-P$ theorem for pair of mappings satisfying the notion of a $\tilde{\psi}_G$-contraction in uniform space enriched with a graph.

This chapter has two sections. In the first section we have investigated the existence and uniqueness of fixed points for F_w-contractions in uniform spaces. Using the concept of an F_{w_G}-contraction mappings results are proved in the setting of S-complete Hausdorff uniform spaces enriched with a graph and an E_s-distance.

The second section deals with the investigation of $F-P$ theorems and common $F-P$ theorems satisfying the $\tilde{\psi}_G$-contraction condition in uniform spaces. Results in this section are also proved in the setting of an S-complete Hausdorff uniform space enriched with graph and E_s-distance. An example related to our results is also constructed towards the end of this chapter.

The following play a very important role in this chapter:

Definition 4.0.6. Let us consider a uniform space (X_d, υ). A function $p_f : X_d \times X_d \longrightarrow [0, \infty)$ is considered to be an E_s-distance if

(i) p_f is an A-distance,

(ii) $p_f(e, \tilde{e}) \leq s[p_f(e, \tilde{\tilde{e}}) + p_f(\tilde{\tilde{e}}, \tilde{e})], \forall e, \tilde{e}, \tilde{\tilde{e}} \in X_d$ for some $s \geq 1$.

Example 4.0.7. Let us consider a uniform space (X_d, υ) and let a b-metric on X_d be d . Then it can be clearly seen that, (X_d, υ_d) is a uniform space where υ_d is a set of all subsets of $X_d \times X_d$ containing a "band" $U_\epsilon = \{(e, \tilde{e}) \in X_d^2 | d(e, \tilde{e}) < \epsilon\}$ for some $\epsilon > 0$. Moreover, if $\upsilon \subseteq \upsilon_d$, then d is an E_s-distance on (X_d, υ).

Throughout this chapter $G = (V_g, E_g)$ is considered as a directed graph such that the set of its vertices V_g coincides with X_d (i.e., $V_g = X_d$) and the set of its edges E_g is such that $E_g \supseteq \triangle$, where $\triangle = \{(e, e) : e \in X_d\}$. Further, assume that G has no parallel edges.

A mapping $T_f : X_d \to X_d$ is p_{f_G}-continuous if $\forall$ sequence $\{e_n\} \subseteq X_d$ with $(e_n, e_{n+1}) \in E_g$ $\forall$ $n \in \mathbb{N}$ and $\lim_{n\to\infty} p_f(e_n, e) = 0$, then we have $\lim_{n\to\infty} p_f(T_f e_n, T_f e) = 0$.

4.1 Fixed point theorem for F_w-contractions on uniform space

This section deals with $F - P$ of F_{w_G}-contraction mappings in the settings of S-complete Housdorff uniform space enriched with graph and E_s-distance.

Definition 4.1.1. Let (X_d, υ) be a uniform space enriched with the graph G and p_{f_E} is an E_s-distance on X_d. A mapping $T_f : X_d \to X_d$ is a F_{w_G}-contraction, if $\exists$ $F_w \in \mathfrak{F}_{\mathfrak{w}_s}$ and a $\tau_f > 0$, such that, $\forall$ $(e, \tilde{e}) \in E_g$, we have

$$\tau_f + F_w(sp_f(T_f e, T_f y)) \leq F_w(p_{f_E}(e, \tilde{e})), \tag{4.1}$$

whenever $\min\{p_{f_E}(T_f e, T_f y), p_{f_E}(e, \tilde{e})\} > 0$.

Theorem 4.1.2. Let (X_d, υ) be an S-complete Hausdorff uniform space enriched with the graph G and p_{f_E} is an E_s-distance on X_d. Let $T_f : X_d \to X_d$ be an F_{w_G}-contraction satisfying the following conditions:

(i) T_f is edge preserving, So that, for $(e, \tilde{e}) \in E_g$, we have $(T_f e, T_f y) \in E_g$;

(ii) $\exists$ an $e_0 \in X_d$ such that $(e_0, T_f e_0) \in E_g$ and $(T_f e_0, e_0) \in E_g$;

(iii) T_f is ${p_f}_{E_G}$-continuous, or, for any sequence $\{e_n\} \subseteq X_d$ such that $e_n \to e$ as $n \to \infty$ and $(e_n, e_{n+1}) \in E_g$ $\forall$ $n \in \mathbb{N}$, we have $(e_n, e) \in E_g$ $\forall$ $n \in \mathbb{N}$.

Then T_f has a $F - P$.

Proof. By hypothesis (ii), $\exists$ an $e_0 \in X_d$ such that $(e_0, e_1) = (e_0, T_f e_0) \in E_g$. From (4.1), we have

$$\tau_f + F_w(s{p_f}_E(e_1, e_2)) = \tau_f + F_w(s{p_f}_E(T_f e_0, T_f e_1)) \leq F_w({p_f}_E(e_0, e_1)) \quad (4.2)$$

As T_f is edge preserving, for $(e_0, e_1) \in E_g$, we have $(e_1, e_2) \in E_g$, From (4.1), we will get

$$\tau_f + F_w(sp(e_2, e_3)) = \tau_f + F_w(sp(T_f e_1, T_f e_2)) \leq F_w({p_f}_E(e_1, e_2)) \quad (4.3)$$

Moving forward with same procedure, we would get a sequence $\{e_n\} \subset X_d$ such that

$$e_n = T_f e_{n-1},\ e_{n-1} \neq e_n \text{ and } (e_{n-1}, e_n) \in E_g\ \forall\ n \in \mathbb{N}.$$

Furthermore,

$$\tau_f + F_w(sp(e_n, e_{n+1})) \leq F_w({p_f}_E(e_{n-1}, e_n))\ \forall\ n \in \mathbb{N}. \quad (4.4)$$

Here we can use the property F_{w_4} and get

$$\tau_f + F_w(s^n {p_f}_E(e_n, e_{n+1})) \leq F_w(s^{n-1} {p_f}_E(e_{n-1}, e_n)).$$

Now, if, we let ${p_f}_{E_n} = {p_f}_E(e_n, e_{n+1})$, $\forall$ $n \in \mathbb{N}$ and after small simplification, we would get

$$F_w(s^n p_{E_n}) \leq F_w({p_f}_{E_0}) - n\tau_f\ \forall\ n \in \mathbb{N}. \quad (4.5)$$

Now if we let $n \to \infty$ in (4.5), we get $\lim_{n\to\infty} F_w(s^n p_{E_n}) = -\infty$. Thus, by property (F_{w_2}), we have $\lim_{n\to\infty} s^n p_{E_n} = 0$. From (F_{w_3}) $\exists$ a $k \in (0, 1)$ such that

$$\lim_{n\to\infty} (s^n p_{E_n})^k F_w(s^n p_{E_n}) = 0.$$

From (4.5) we have

$$(s^n p_{E_n})^k F(s^n p_{E_n}) - (s^n p_{E_n})^k F_w(p_{f_{E_0}}) \leq -(s^n p_{E_n})^k n\tau_f \leq 0 \text{ for each } n \in \mathbb{N}. \tag{4.6}$$

Now, if, we let $n \to \infty$ in (4.6), we get

$$\lim_{n\to\infty} n(s^n p_{E_n})^k = 0. \tag{4.7}$$

This implies that $\exists$ an $n_1 \in \mathbb{N}$ such that $n(s^n p_{E_n})^k \leq 1 \ \forall\ n \geq n_1$. Thus, we have

$$s^n p_{E_n} \leq \frac{1}{n^{1/k}}, \quad \forall\ n \geq n_1. \tag{4.8}$$

To show that $\{e_n\}$ is a p_{f_E}-*C-S*, consider

$$S_n = \sum_{i=n_1}^{n} \frac{1}{i^{1/k}}.$$

As we know that $\sum_{i=1}^{\infty} \frac{1}{i^{1/k}}$ is convergent series. Thus, $\exists\ S \in [0, \infty)$ such that $\lim_{n\to\infty} S_n = S$. Consider $m, n \in \mathbb{N}$ with $m > n > n_1$. By making use of triangle inequality and (4.8), we have

$$\begin{aligned} p_{f_E}(e_n, e_m) &\leq s^n p_E(e_n, e_{n+1}) + s^{n+1} p_{f_E}(e_{n+1}, e_{n+2}) + \cdots + s^{m-1} p_{f_E}(e_{m-1}, e_m) \\ &= \sum_{i=n_1}^{m-1} s^i p_{E_i} - \sum_{i=n_1}^{n-1} s^i p_{E_i} \\ &\leq S_{m-1} - S_{n-1}. \end{aligned}$$

Thus, $\lim_{n,m\to\infty} p_{f_E}(e_n, e_m) = 0$. In a similar way, we show that $\lim_{n,m\to\infty} p_{f_E}(e_m, e_n) = 0$. Thus, $\{e_n\}$ is a p_{f_E}-*C-S*. As (X_d, υ) is S-complete, $\exists$ an $e^* \in X_d$ such that $\lim_{n\to\infty} p_{f_E}(e_n, e^*) = 0$. By condition (iii), when T is $p_{f_{E_G}}$-continuous, we have $\lim_{n\to\infty} p_{f_E}(e_{n+1}, T_f e^*) = 0$. As $\lim_{n\to\infty} p_{f_E}(e_n, e^*) = 0$ and $\lim_{n\to\infty} p_{f_E}(e_n, T_f e^*) = 0$. Thus by Lemma-(a)[4, 3] we have an $e^* = T_f e^*$. By condition (iii), when we have $(e_n, e^*) \in E_g\ \forall\ n \in \mathbb{N}$. From (4.1), we have

$$\tau_f + F_w(s p_E(T_f e_n, T_f e^*)) \leq F_w(p_{f_E}(e_n, e^*)).$$

This implies that $s p_E(e_{n+1}, T_f e^*) < p_{f_E}(e_n, e^*)$. Now, if, we let $n \to \infty$, we have $\lim_{n\to\infty} p_{f_E}(e_{n+1}, T_f e^*) = 0$. Again, by Lemma-(a)[4, 3] we obtain $e^* = T_f e^*$. □

4.2 Fixed and common fixed point theorems for $\tilde{\psi}_G$-contraction in uniform space

In this section we will investigate the existence and uniqueness of $F-P$'s as well as common fixed points for $\tilde{\psi}_G$-contraction mapping in the setting of S-complete Housdorff uniform space enriched with a graph and E_s-distance.

Definition 4.2.1. Let (X_d, υ) be a uniform space enriched with the graph G and p_{f_E} be an E_s-distance on X_d. A mapping $T_f : X_d \to X_d$ is a $\tilde{\psi}_G$-contraction mapping if $\forall$ $(e, \tilde{e}) \in E_g$, we have

$$p_{f_E}(T_f e, T_f y) \leq \tilde{\psi}(p_{f_E}(e, \tilde{e})) \tag{4.9}$$

where $\tilde{\psi} \in \tilde{\Psi}$.

Note that throughout this section $\tilde{\Psi}$ be the family of functions $\tilde{\psi}$: $[0, \infty) \to [0, \infty)$ satisfying the following conditions:

($\tilde{\Psi}_1$) $\tilde{\psi}$ is nondecreasing;

($\tilde{\Psi}_2$) $\sum_{n=1}^{+\infty} s^n \tilde{\psi}^n(t) < \infty$ $\forall$ $t > 0$, where $\tilde{\psi}^n$ is the n^{th} iterate of $\tilde{\psi}$.

Theorem 4.2.2. Let (X_d, υ) be a S-complete Hausdorff uniform enriched with the graph G and p_{f_E} is an E_s-distance on X_d. Let $T : X_d \to X_d$ be a $\tilde{\psi}_G$-contraction mapping satisfying the following conditions:

(i) T_f is edge preserving, So that, for $(e, \tilde{e}) \in E_g$, we have $(T_f e, T_f y) \in E_g$;

(ii) $\exists$ an $e_0 \in X_d$ such that $(e_0, T_f e_0) \in E_g$ and $(T_f e_0, e_0) \in E_g$;

(iii) T_f is $p_{f_{E_G}}$-continuous, or, for any sequence $\{e_n\} \subseteq X_d$ such that $e_n \to e$ as $n \to \infty$ and $(e_n, e_{n+1}) \in E_g$ $\forall$ $n \in \mathbb{N}$, we have $(e_n, e) \in E_g$ $\forall$ $n \in \mathbb{N}$.

Then T_f has a $F-P$.

Proof. By hypothesis (ii) of the above theorem we have an $e_0 \in X_d$ such that $(e_0, T_f e_0) \in E_g$. Define the sequence $\{e_n\}$ in X_d by $e_{n+1} = T_f e_n$ $\forall$ $n \in \mathbb{N} \cup \{0\}$. If $e_{n_0} = e_{n_0+1}$ for some n_0, then e_{n_0} is a F-P of T_f. So, we can assume that $e_n \neq e_{n+1}\ \forall\ n$. As we know that T_f is edge preserving, we have

$$(e_0, e_1) = (e_0, T_f e_0) \in E_g \Rightarrow (T_f e_0, T_f e_1) = (e_1, e_2) \in E_g.$$

Inductively, we would have

$$(e_n, e_{n+1}) \in E_g,\ \forall\ n \in \mathbb{N} \cup \{0\}. \tag{4.10}$$

From (4.9) and (4.10), it follows that for all $n \in \mathbb{N} \cup \{0\}$, we have

$$p_{f_E}(e_{n+1}, e_{n+2}) = p_{f_E}(T_f e_n, T_f e_{n+1}) \leq \tilde{\psi}(p_{f_E}(e_n, e_{n+1})). \tag{4.11}$$

Proceeding with the same steps, we obtain

$$p_{f_E}(e_n, e_{n+1}) \leq \tilde{\psi}^n(p_{f_E}(e_0, e_1)), \text{ for all } n \in \mathbb{N}.$$

As we know that p_{f_E} is an E_s-distance then for an $m > n$, we have

$$\begin{aligned} p_{f_E}(e_n, e_m) &\leq s^n p(e_n, e_{n+1}) + s^{n+1} p_{f_E}(e_{n+1}, e_{n+2}) + \cdots + s^{m-1} p_{f_E}(e_{m-1}, e_m) \\ &\leq s^n \tilde{\psi}^n(p_{f_E}(e_0, e_1)) + s^{n+1} \tilde{\psi}^{n+1}(p_{f_E}(e_0, e_1)) + \cdots + s^{m-1} \tilde{\psi}^{m-1}(p_{f_E}(e_0, e_1)). \end{aligned} \tag{4.12}$$

To show that $\{e_n\}$ is a p_{f_E}-C-S, consider

$$S_n = \sum_{k=0}^{n} s^k \tilde{\psi}^k(p_{f_E}(e_0, e_1)).$$

Thus from (4.12) we have

$$p_{f_E}(e_n, e_m) \leq S_{m-1} - S_{n-1}. \tag{4.13}$$

As we know that $\tilde{\psi} \in \tilde{\Psi}$, $\exists\ S \in [0, \infty)$ such that $\lim_{n\to\infty} S_n = S$. Thus by (4.13) we have

$$\lim_{n,m\to\infty} p_{f_E}(e_n, e_m) = 0. \tag{4.14}$$

As we know that p_{f_E} is not symmetric then by repeating the same argument we have $\lim_{n,m\to\infty} p_{f_E}(e_m, e_n) = 0$. Hence the sequence $\{e_n\}$ is a p_{f_E}-Cauchy in the S-complete space X_d. Thus, $\exists$ an $e^* \in X_d$ such that $\lim_{n\to\infty} p_{f_E}(e_n, e^*) = 0$. By condition (iii), when we have T a $p_{f_{E_G}}$-continuous, we get $\lim_{n\to\infty} p_{f_E}(T_f e_n, T_f e^*) = 0$, which implies that $\lim_{n\to\infty} p_{f_E}(e_{n+1}, T_f e^*) =$

0. Hence we have $\lim_{n\to\infty} p_{f_E}(e_n, e^*) = 0$ and $\lim_{n\to\infty}(e_n, T_f e^*) = 0$. Thus by Lemma-(a)[4, 3] we have $e^* = Te^*$. By condition (iii), when we have $(e_n, e^*) \in E_g \ \forall\ n \in \mathbb{N}$, then from (4.9)

$$p_{f_E}(e_{n+1}, T_f e^*) = p_{f_E}(T_f e_n, Te^*) \le \tilde{\psi}(p_{f_E}(e_n, e^*)) < p_{f_E}(e_n, e^*). \quad (4.15)$$

Now, if, we let $n \to \infty$ in the inequality above, we have $\lim_{n\to\infty} p_{f_E}(e_{n+1}, T_f e^*) = 0$. Thus by repeating the same arguments as above we have $e^* = T_f e^*$. □

Example 4.2.3. Let $X_d = [0, 1]$ be enriched with a graph $G = (V_g, E_g)$ with $V_g = X_d$ and $E_g = \Big\{(e, \tilde{e}) : e, \tilde{e} \in \{\frac{1}{n+1} : n \in \mathbb{N}\} \cup \{0\}\Big\} \cup \{(e, e) : e \in X_d\}$, and b-metric $d(e, \tilde{e}) = (e - \tilde{e})^2$ with $s = 2$. Define $\upsilon = \{U_\epsilon | \epsilon > 0\}$. It is easy to see that (X_d, υ) is a uniform space. Define $T_f : X_d \to X_d$ by

$$T_f e = \begin{cases} 0 \ \text{ if } e = 0 \\ \frac{1}{3n+1} \ \text{ if } e = \frac{1}{n} : n > 1 \\ \sqrt{e} \ \text{ otherwise.} \end{cases} \quad (4.16)$$

Take $\tilde{\psi}(t) = \frac{t}{3} \ \forall\ t \ge 0$. It can easily be seen that T_f is edge preserving and a $\tilde{\psi}_G$-contraction. Also for an $e_0 = \frac{1}{2}$ we have $(e_0, T_f e_0) \in E_g$ and $(T_f e_0, e_0) \in E_g$. Moreover for any sequence $\{e_n\}$ in X_d with $e_n \to e$ as $n \to \infty$ and $(e_{n-1}, e_n) \in E_g \ \forall\ n \in \mathbb{N}$ we have $(e_n, e) \in E_g \ \forall\ n \in \mathbb{N}$. Therefore by Theorem 4.2.2, T_f has a $F - P$.

We consider the following condition to discuss the uniqueness of a $F - P$:

(H) For all $e, \tilde{e} \in \text{Fix}(T_f)$, $\exists\ \tilde{\tilde{e}} \in X_d$ such that $(\tilde{\tilde{e}}, e) \in E_g$ and $(\tilde{\tilde{e}}, \tilde{e}) \in E_g$.

Here, $\text{Fix}(T_f)$ denotes the set of all $F - P$'s of T_f.

The following theorem guarantees the uniqueness of a $F - P$.

Theorem 4.2.4. Adding the condition (H) in the hypothesis of Theorem 4.2.2, we obtain the uniqueness of $F - P$ of T_f.

Proof. Suppose, on the contrary, that $c_1, c_2 \in X_d$ are two distinct F-P's of T_f. From (H), $\exists\ \tilde{\tilde{e}} \in X_d$ such that

$$(\tilde{\tilde{e}}, c_1) \in E_g \text{ and } (\tilde{\tilde{e}}, c_2) \in E_g. \quad (4.17)$$

By making use of fact that T_f is edge preserving, from (4.17), we have

$$(T_f^n \tilde{\tilde{e}}, c_1) \in E_g \text{ and } (T_f^n \tilde{\tilde{e}}, c_2) \in E_g, \ \forall \ n \in \mathbb{N} \cup \{0\}. \tag{4.18}$$

We define the sequence $\{\tilde{\tilde{e}}_n\}$ in X_d by $\tilde{\tilde{e}}_{n+1} = T_f \tilde{\tilde{e}}_n = T_f^n \tilde{\tilde{e}}_0 \ \forall \ n \in \mathbb{N} \cup \{0\}$ and $\tilde{\tilde{e}}_0 = \tilde{\tilde{e}}$. From (4.18) and (4.9), we have

$$p_{f_E}(\tilde{\tilde{e}}_{n+1}, c_1) = p_{f_E}(T_f \tilde{\tilde{e}}_n, T_f c_1) \leq \tilde{\psi}(p_{f_E}(\tilde{\tilde{e}}_n, c_1)), \tag{4.19}$$

$\forall \ n \in \mathbb{N} \cup \{0\}$. This implies that

$$p_{f_E}(\tilde{\tilde{e}}_n, c_1) \leq \tilde{\psi}^n(p_{f_E}(\tilde{\tilde{e}}_0, c_1)), \ \forall \ n \in \mathbb{N}.$$

Now if we let $n \to \infty$ in the above inequality, we obtain

$$\lim_{n \to \infty} p_{f_E}(\tilde{\tilde{e}}_n, c_1) = 0. \tag{4.20}$$

Similarly, we have

$$\lim_{n \to \infty} p_{f_E}(\tilde{\tilde{e}}_n, c_2) = 0. \tag{4.21}$$

From (4.20) and (4.21) together with Lemma-(a)[4, 3], it follows that $c_1 = c_2$. Thus, F-P of T_f is unique.

In the following definition we will define $\tilde{\Psi}_G$-contraction for pair of mappings □

Definition 4.2.5. Let (X_d, υ) be a uniform space enriched with the graph G. A pair of two self mappings $T_f, S_f : X_d \to X_d$ are considered to be a $\tilde{\psi}_G$-contraction pair if $\forall \ (e, \tilde{e}) \in E_g$, we have

$$\max\{p_{f_E}(T_f e, S_f y), p_{f_E}(S_f e, T_f y)\} \leq \tilde{\psi}(p_{f_E}(e, \tilde{e})), \tag{4.22}$$

where $\tilde{\psi} \in \tilde{\Psi}$.

Theorem 4.2.6. Let (X_d, υ) be a S-complete Hausdorff uniform space enriched with the graph G and p_{f_E} is an E_s-distance on X_d. Suppose that the pair of $T_f, S_f : X_d \to X_d$ is $\tilde{\psi}_G$-contraction pair satisfying the following conditions.

(i) (T_f, S_f) is edge preserving pair, So that, $\forall \ (e, \tilde{e}) \in E_g$, we have $(T_f e, S_f y) \in E_g$ and $(S_f e, T_f y) \in E_g$;

(ii) $\exists$ an $e_0 \in X_d$ such that $(e_0, T_f e_0) \in E_g$ and $(T_f e_0, e_0) \in E_g$;

(iii) for any sequence $\{e_n\}$ in X_d with $e_n \to e$ as $n \to \infty$ and $(e_n, e_{n+1}) \in E_g$ $\forall\ n \in \mathbb{N} \cup \{0\}$, then $(e_n, e) \in E_g$ $\forall\ n \in \mathbb{N} \cup \{0\}$.

Then T_f and S_f have a common $F-P$.

Proof. By hypothesis (ii) of the above theorem, we have an $e_0 \in X_d$ such that $(e_0, T_f e_0) \in E_g$ and $(T_f e_0, e_0) \in E_g$. As we know that (T_f, S_f) is an edge preserving pair, then we can construct a sequence such that

$T_f e_{2n} = e_{2n+1},\ S_f e_{2n+1} = e_{2n+2}$ and $(e_n, e_{n+1}) \in E_g,\ (e_{n+1}, e_n) \in E_g,\ \forall\, n \in \mathbb{N}\cup\{0\}$.

From (4.22) $\forall\ n \in \mathbb{N} \cup \{0\}$, we have

$$\begin{aligned} p_{f_E}(e_{2n+1}, e_{2n+2}) &= p_{f_E}(T_f e_{2n}, S_f e_{2n+1}) \\ &\leq \max\{p_{f_E}(T_f e_{2n}, S_f e_{2n+1}), p_{f_E}(S_f e_{2n}, T_f e_{2n+1})\} \\ &\leq \tilde{\psi}(p_{f_E}(e_{2n}, e_{2n+1})). \end{aligned}$$

Hence, we conclude that

$$p_{f_E}(e_{2n+1}, e_{2n+2}) \leq \tilde{\psi}(p_{f_E}(e_{2n}, e_{2n+1})). \tag{4.23}$$

Similarly, we obtain

$$\begin{aligned} p_{f_E}(e_{2n+2}, e_{2n+3}) &= p_{f_E}(S_f e_{2n+1}, T_f e_{2n+2}) \\ &\leq \max\{p_{f_E}(T_f e_{2n+1}, S_f e_{2n+2}), p_{f_E}(S e_{2n+1}, T_f e_{2n+2})\} \\ &\leq \tilde{\psi}(p_{f_E}(e_{2n+1}, e_{2n+2})). \end{aligned}$$

Hence, we have

$$p_{f_E}(e_{2n+2}, e_{2n+3}) \leq \tilde{\psi}(p_{f_E}(e_{2n+1}, e_{2n+2})). \tag{4.24}$$

Thus, from (4.23) and (4.24), and by induction, we get

$$p_{f_E}(e_n, e_{n+1}) \leq \tilde{\psi}^n(p_{f_E}(e_0, e_1)),\ \forall\ n \in \mathbb{N}. \tag{4.25}$$

We now show that $\{e_n\}$ is a p_{f_E}-*C-S*. As we know that p_{f_E} is an E_s-distance then for $m > n$, we have

$$\begin{aligned} p_{f_E}(e_n, e_m) &\leq s^n p_{f_E}(e_n, e_{n+1}) + s^{n+1} p_{f_E}(e_{n+1}, e_{n+2}) + \cdots + s^{m-1} p_{f_E}(e_{m-1}, e_m) \\ &\leq s^n \tilde{\psi}^n(p_{f_E}(e_0, e_1)) + s^{n+1}\tilde{\psi}^{n+1}(p_{f_E}(e_0, e_1)) + \cdots + s^{m-1}\tilde{\psi}^{m-1}(p_{f_E}(e_0, e_1)). \end{aligned} \tag{4.26}$$

We shall now consider

$$S_n = \sum_{k=0}^{n} s^k \tilde{\psi}^k (p_{f_E}(e_0, e_1)).$$

Thus, from (4.26) we have

$$p_{f_E}(e_n, e_m) \le S_{m-1} - S_{n-1}. \tag{4.27}$$

As we know that $\tilde{\psi} \in \tilde{\Psi}$, $\exists\ S \in [0, \infty)$ such that $\lim_{n\to\infty} S_n = S$. Thus, by (4.27) we have

$$\lim_{n,m\to\infty} p_{f_E}(e_n, e_m) = 0. \tag{4.28}$$

As we know that p_{f_E} is not symmetric then by repeating the same argument we have

$$\lim_{n,m\to\infty} p_{f_E}(e_m, e_n) = 0. \tag{4.29}$$

Hence the sequence $\{e_n\}$ is a p_{f_E}-Cauchy in the S-complete space X_d. Thus, $\exists$ an $e^* \in X_d$ such that $\lim_{n\to\infty} p_{f_E}(e_n, e^*) = 0$ which implies $\lim_{n\to\infty} T_f e_{2n} = \lim_{n\to\infty} S_f e_{2n+1} = e^*$. By assumption (iii), we have $(e_n, e^*) \in E_g$. Thus, by making use of triangle inequality and (4.22), we have

$$\begin{aligned} p_{f_E}(e_n, T_f e^*) &\le sp_E(e_n, e_{2n+2}) + sp_E(e_{2n+2}, T_f e^*) \\ &= sp_E(e_n, e_{2n+2}) + sp_E(S_f e_{2n+1}, T_f e^*) \\ &\le sp_E(e_n, e_{2n+2}) + s \max\{p_{f_E}(T_f e_{2n+1}, S_f e^*), p_{f_E}(S e_{2n+1}, T_f e^*)\} \\ &\le sp_E(e_n, e_{2n+2}) + s\tilde{\psi}(p_{f_E}(e_{2n+1}, e^*)) \end{aligned} \tag{4.30}$$

Now, if, we let $n \to \infty$ in (4.30), we have $p_{f_E}(e_n, T_f e^*) = 0$. Hence we have $\lim_{n\to\infty} p_{f_E}(e_n, e^*) = 0$ and $\lim_{n\to\infty} p_{f_E}(e_n, T_f e^*) = 0$. Thus by Lemma-(a)[4, 3] we have an $e^* = Te^*$. Analogously, we can derive $e^* = S_f e^*$. Therefore $e^* = T_f e^* = S_f e^*$. □

Remark 4.2.7. Note that Theorem 4.2.6 is valid if one replace condition (ii) with

(ii)' $\exists\ e_0 \in X_d$ such that $(e_0, S_f e_0) \in E_g$ and $(S_f e_0, e_0) \in E_g$.

Example 4.2.8. Let $X_d = [0, 1]$ be enriched with a graph $G = (V_g, E_g)$ with $V_g = X_d$ and $E_g = \Big\{(e, \tilde{e}) : e, \tilde{e} \in \{\frac{1}{n+1} : n \in \mathbb{N}\} \cup \{0\}\Big\} \cup \{(e, e) : e \in X_d\}$, and dislocated $M-S$ $d(e,\tilde{e}) = \max\{e,\tilde{e}\}$. Define $\upsilon = \{U_\epsilon | \epsilon > 0\}$, where $U_\epsilon = \{(e,\tilde{e}) \in X_d^2 : d(e,\tilde{e}) < d(e,e) + \epsilon\}$. It is easy to see that (X_d, υ) is a uniform space. Define $T_f : X_d \to X_d$ by

$$T_f e = \begin{cases} 0 \ \text{ if } e = 0 \\ \frac{1}{2n+1} \ \text{ if } e = \frac{1}{n} : n > 1 \\ e^2 \ \text{ otherwise} \end{cases} \tag{4.31}$$

and $S : X_d \to X_d$ by

$$Se = \begin{cases} 0 \ \text{ if } e = 0 \\ \frac{1}{2n} \ \text{ if } e = \frac{1}{n} : n > 1 \\ \sqrt{e} \ \text{ otherwise} \end{cases} \tag{4.32}$$

Take $\tilde{\psi}(t) = \frac{t}{2}\ \forall\ t \geq 0$. Further, it can be easily seen that (T_f, S_f) is edge preserving and $\tilde{\psi}_G$-contraction pair. Also for $e_0 = \frac{1}{2}$ we have $(e_0, T_f e_0) \in E_g$ and $(T_f e_0, e_0) \in E_g$. Moreover for any sequence $\{e_n\}$ in X_d with $e_n \to e$ as $n \to \infty$ and $(e_n, e_{n+1}) \in E_g\ \forall\ n \in \mathbb{N} \cup \{0\}$ we have $(e_n, e) \in E_g\ \forall\ n \in \mathbb{N}$. Therefore by Theorem 4.2.6, T_f and S_f have a common $F-P$.

We use the following condition, to discuss the uniqueness of a common $F-P$.

(I) For each $e, \tilde{e} \in CFix(T_f, S_f)$, we have $(e,\tilde{e}) \in E_g$, where $CFix(T_f, S_f)$ is the set of all common $F-P$'s of T_f and S_f.

Theorem 4.2.9. Adding the condition (I) in the hypothesis of Theorem 4.2.6, we obtain the uniqueness of common $F-P$ of T_f and S_f.

Proof. On the contrary suppose that $c_1, c_2 \in X_d$ are two distinct common *F-P*'s of T_f and S_f. From (I) and (4.22) we have

$$p_{f_E}(c_1,c_2) = \max\{p_{f_E}(T_{fc_1}, S_{fc_2}), p_{f_E}(S_{fc_1}, T_{fc_2})\} \leq \tilde{\psi}(p_{f_E}(c_1,c_2)) < p_{f_E}(c_1,c_1),$$

which is impossible for $p_{f_E}(c_1,c_2) > 0$. Consequently, we have $p_{f_E}(c_1,c_2) = 0$. Analogously, one can show that $p_{f_E}(c_1,c_1) = 0$. Thus we have $c_1 = c_2$, which is a contradiction to our assumption. Hence T_f and S_f have a unique common *F-P*. □

Chapter 5

Common Fixed Point Theorems for Family of Mappings

Throughout this chapter $F-P$ represents a fixed point, $M-S$ represents a metric space, $C-S$ represents a Cauchy sequence. The purpose of this chapter is to introduce a new contraction conditions for a sequence of multifunction and prove corresponding $F-P$ theorems. We will also give a common $F-P$ theorem for sequence of bounded multifunctions by making use of δ-distance. To conclude our findings we establish an existence theorem for a system of integral equations.

This chapter consists of three section. In the first section we have discussed common $F-P$ theorems for family of closed multivalued mappings satisfying F_w-type contraction. Our results are proved in a complete $M-S$ and the sequence of mappings under consideration satisfy the $F_{w_{\alpha w}}$- contraction of Hardy-roger type. An example towards the end of this section shows the validity of our results.

In the second section, we have proved results for the family of bounded multivalued mappings satisfying the F_w-type contraction. Here also, we have considered a complete $M-S$ and the sequence of mapping considered in this section are also Hardy-Rogers type. We have also constructed an example in this section to validate our results.

As a consequence of our results we have established an existence theorem for volterra integral equation in the third section.

5.1 Family of closed multivalued mappings satisfying F_w type contractions and related common fixed point theorems

In this section we will prove some common $F-P$ theorems for $F_{w_{\alpha_w}}$-contraction and $F^*_{w_{\alpha_w}}$-contraction mappings for Hardy-roger type in a complete $M-S$.

We begin this section by introducing definitions for sequence of mapping, which would be useful regarding the proofs of our results in the sections.

Definition 5.1.1. Let $\alpha_w : X_d \times X_d \to [0, \infty)$. A sequence of mappings $\{T_{f_i} : X_d \to N_f(X_d)\}_{i=1}^{\infty}$ is α_w-admissible sequence if $\forall$ $e \in X_d$ and $\tilde{e} \in T_{f_i}e$ for some $i \in \mathbb{N}$ such that $\alpha_w(e, \tilde{e}) \geq 1$, then we have $\alpha_w(\tilde{e}, \tilde{\tilde{e}}) \geq 1$ $\forall$ $\tilde{\tilde{e}} \in T_{f_{i+1}}\tilde{e}$. A sequence of mappings $\{T_{f_i} : X_d \to N_f(X_d)\}_{i=1}^{\infty}$ is α_{w_*}-admissible sequence if $\forall$ $e, \tilde{e} \in X_d$ with $\alpha_w(e, \tilde{e}) \geq 1$, we have $\alpha_{w_*(T_{f_i}e, T_{f_j}y)} \geq 1$ $\forall$ $i, j \in \mathbb{N}$, where $\alpha_{w_*(T_{f_i}e, T_{f_j}y)} = \inf\{\alpha_w(u, v) : u \in T_{f_i}e \text{ and } v \in T_{f_j}y\}$.

The sequence of mappings is considered to be strictly α_w-admissible and strictly α_{w_*}-admissible if we have strict inequality in the above definition.

Remark 5.1.2.

(i) Note that if a sequence of mappings $\{T_{f_i} : X_d \to N_f(X_d)\}_{i=1}^{\infty}$ is strictly α_{w_*}-admissible sequence, then it is strictly α_w-admissible sequence.

(ii) When $\{T_{f_i}\}_{i=1}^{\infty}$ is a constant sequence Definition 5.1.1 coincide with definition of α_w-admissible and α_{w_*}-admissible given in [67, Page 4] and [25, Page 1] respectively. Furthermore, if T is a singlevalued mapping then these definition 5.1.1 coincide with [84, Definition 2.2].

Definition 5.1.3. Let (X_d, d_r) be a $M-S$ and $\alpha_w : X_d \times X_d \to [0, \infty)$ be a function. A sequence of mappings $\{T_{f_i} : X_d \to Cl_f(X_d)\}_{i=1}^{\infty}$ is an $F_{w_{\alpha_w}}$-contraction of Hardy-Rogers-type, if $\exists$ $F_w \in \mathfrak{F}_{\mathfrak{w}}$ and a $\tau_f > 0$ such that $\forall\ i, j \in \mathbb{N}$, we have

$$\tau_f + F_w(\alpha_w(e, \tilde{e})H_m(T_{f_i}e, T_{f_j}y)) \leq F_w(N_f(e, \tilde{e})), \tag{5.1}$$

$\forall\ e, \tilde{e} \in X_d$, whenever $\min\{\alpha_w(e, \tilde{e})H_m(T_{f_i}e, T_{f_j}y), N_f(e, \tilde{e})\} > 0$, where

$N_f(e, \tilde{e}) = b_1 d_r(e, \tilde{e}) + b_2 d_r(e, T_{f_i}e) + b_3 d_r(\tilde{e}, T_{f_j}y) + b_4 d_r(e, T_{f_j}y) + L d_r(\tilde{e}, T_{f_i}e)$,

with $b_1, b_2, b_3, b_4, L \geq 0$ satisfying $b_1 + b_2 + b_3 + 2b_4 = 1$ and $b_3 \neq 1$.

Now we will prove a result for $F_{w_{\alpha_w}}$-contraction of Hardy-Rogers-type.

Theorem 5.1.4. Let (X_d, d_r) be a complete $M - S$ and let $\{T_{f_i} : X_d \to Cl_f(X_d)\}_{i=1}^{\infty}$ be an $F_{w_{\alpha_w}}$-contraction of Hardy-Rogers-type satisfying the following conditions:

(i) $\{T_{f_i}\}_{i=1}^{\infty}$ is strictly α_w-admissible sequence;

(ii) $\exists\ e_0 \in X_d$ and $e_1 \in T_{f_i}e_0$ for some $i \in \mathbb{N}$ with $\alpha_w(e_0, e_1) > 1$;

(iii) for any sequence $\{e_n\} \subseteq X_d$ such that $e_n \to e$ as $n \to \infty$ and $\alpha_w(e_n, e_{n+1}) > 1\ \forall\ n \in \mathbb{N}$, we have $\alpha_w(e_n, e) > 1\ \forall\ n \in \mathbb{N}$.

Then the mappings in the sequence $\{T_{f_i}\}_{i=1}^{\infty}$ have a common $F - P$.

Proof. By hypothesis (ii), we assume without loss of generality that $\exists\ e_0 \in X_d$ and $e_1 \in T_{f_1}e_0$ with $\alpha_w(e_0, e_1) > 1$. If $e_1 \in T_{f_i}e_1\ \forall i \in \mathbb{N}$, then e_1 is a common fixed point. Let $e_1 \notin T_{f_2}e_1$, as $\alpha_w(e_0, e_1) > 1\ \exists\ e_2 \in T_{f_2}e_1$ such that

$$d_r(e_1, e_2) \leq \alpha_w(e_0, e_1)H_m(T_{f_1}e_0, T_{f_2}e_1). \tag{5.2}$$

As we know that F_w is increasing, we have

$$F_w(d_r(e_1, e_2)) \leq F_w(\alpha_w(e_0, e_1)H_m(T_{f_1}e_0, T_{f_2}e_1)). \tag{5.3}$$

From (5.1) we have

$$\begin{aligned}\tau_f + F_w(d_r(e_1,e_2)) &\leq \tau_f + F_w(\alpha_w(e_0,e_1)H_m(T_{f_1}e_0, T_{f_2}e_1))\\ &\leq F_w\Big(b_1 d_r(e_0,e_1) + b_2 d_r(e_0, T_{f_1}e_0) + b_3 d_r(e_1, T_{f_2}e_1) +\\ &\quad b_4 d_r(e_0, T_{f_2}e_1) + L d_r(e_1, T_{f_1}e_0)\Big)\\ &\leq F_w\Big(b_1 d_r(e_0,e_1) + b_2 d_r(e_0,e_1) + b_3 d_r(e_1,e_2) +\\ &\quad b_4 d_r(e_0,e_2) + L.0\Big)\\ &\leq F_w\Big(b_1 d_r(e_0,e_1) + b_2 d_r(e_0,e_1) + b_3 d_r(e_1,e_2) +\\ &\quad b_4(d_r(e_0,e_1) + d_r(e_1,e_2))\Big)\\ &= F_w\Big((b_1+b_2+b_4)d_r(e_0,e_1) + (b_3+b_4)d_r(e_1,e_2)\Big). \quad (5.4)\end{aligned}$$

As we know that F_w is increasing, we get from above that

$$d_r(e_1,e_2) < (b_1+b_2+b_4)d_r(e_0,e_1) + (b_3+b_4)d_r(e_1,e_2).$$

So that,

$$(1-b_3-b_4)d_r(e_1,e_2) < (b_1+b_2+b_4)d_r(e_0,e_1).$$

As $b_1+b_2+b_3+2b_4 = 1$, thus we have

$$d_r(e_1,e_2) < d_r(e_0,e_1).$$

From (5.4), we have

$$\tau_f + F_w(d_r(e_1,e_2)) \leq F_w(d_r(e_0,e_1)).$$

If $e_2 \in T_{f_i}e_2$ $\forall i \in \mathbb{N}$ then e_2 is a common *F-P*. Let $e_2 \notin T_{f_3}e_2$. As we know that $\{T_{f_i}\}_{i=1}^{\infty}$ is strictly α_w-admissible, we have $\alpha_w(e_1,e_2) > 1$. There exists $e_3 \in T_{f_3}e_2$ such that

$$d_r(e_2,e_3) \leq \alpha_w(e_1,e_2)H_m(T_{f_2}e_1, T_{f_3}e_2). \quad (5.5)$$

As we know that F_w is increasing, we have

$$F_w(d_r(e_2,e_3)) \leq F_w(\alpha_w(e_1,e_2)H_m(T_{f_2}e_1, T_{f_3}e_2)). \quad (5.6)$$

From (5.1) we have

$$\begin{aligned}
\tau_f + F_w(d_r(e_2,e_3)) &\leq \tau_f + F_w(\alpha_w(e_1,e_2)H_m(T_{f_2}e_1, T_3e_2)) \\
&\leq F_w\Big(b_1 d_r(e_1,e_2) + b_2 d_r(e_1, T_{f_2}e_1) + b_3 d_r(e_2, T_{f_3}e_2) + \\
&\quad b_4 d_r(e_1, T_{f_3}e_2) + L d_r(e_2, T_{f_2}e_1)\Big) \\
&\leq F_w\Big(b_1 d_r(e_1,e_2) + b_2 d_r(e_1,e_2) + b_3 d_r(e_2,e_3) + \\
&\quad b_4 d_r(e_1,e_3) + L.0\Big) \\
&\leq F_w\Big(b_1 d_r(e_1,e_2) + b_2 d_r(e_1,e_2) + b_3 d_r(e_2,e_3) + \\
&\quad b_4(d_r(e_1,e_2) + d_r(e_2,e_3))\Big) \\
&= F_w\Big((b_1+b_2+b_4)d_r(e_1,e_2) + (b_3+b_4)d_r(e_2,e_3)\Big). \qquad (5.7)
\end{aligned}$$

As we know that F_w is increasing, we get from above that

$$d_r(e_2,e_3) < (b_1+b_2+b_4)d_r(e_1,e_2) + (b_3+b_4)d_r(e_2,e_3).$$

So that,

$$(1-b_3-b_4)d_r(e_2,e_3) < (b_1+b_2+b_4)d_r(e_1,e_2).$$

As $b_1+b_2+b_3+2b_4 = 1$, thus we have

$$d_r(e_2,e_3) < d_r(e_1,e_2).$$

Now from (5.7) we have

$$\tau_f + F_w(d_r(e_2,e_3)) \leq F_w(d_r(e_1,e_2)).$$

So we have

$$F_w(d_r(e_2,e_3)) \leq F_w(d_r(e_1,e_2)) - \tau_f \leq F_w(d_r(e_0,e_1)) - 2\tau_f.$$

Proceeding in the same way we get a sequence $\{e_n\} \subset X_d$ such that

$$e_n \in T_{f_n}e_{n-1},\ e_{n-1} \neq e_n \text{ and } \alpha_w(e_{n-1},e_n) > 1\ \forall\ n \in \mathbb{N}.$$

Furthermore,

$$F_w(d_r(e_n,e_{n+1})) \leq F_w(d_r(e_0,e_1)) - n\tau_f\ \forall\ n \in \mathbb{N}. \qquad (5.8)$$

Now if we let $n \to \infty$ in (5.8) we get $\lim_{n\to\infty} F_w(d_r(e_n, e_{n+1})) = -\infty$. Thus by property (F_{w_2}), we have $\lim_{n\to\infty} d_r(e_n, e_{n+1}) = 0$. Let $d_{r_n} = d_r(e_n, e_{n+1})$ $\forall\ n \in \mathbb{N}$. From (F_{w_3}) $\exists\ k \in (0,1)$ such that

$$\lim_{n\to\infty} d_{r_n}^k F_w(d_{r_n}) = 0.$$

From (5.8) we have

$$d_{r_n}^k F(d_{r_n}) - d_{r_n}^k F_w(d_{n_0}) \leq -d_{r_n}^k n\tau_f \leq 0\ \forall\ n \in \mathbb{N}. \tag{5.9}$$

Now if we let $n \to \infty$ in (5.9) we get,

$$\lim_{n\to\infty} n d_{r_n}^k = 0. \tag{5.10}$$

This implies that $\exists\ n_1 \in \mathbb{N}$ such that $n d_{r_n}^k \leq 1\ \forall\ n \geq n_1$. Thus we have

$$d_{r_n} \leq \frac{1}{n^{1/k}}, \quad \forall\ n \geq n_1. \tag{5.11}$$

To prove that $\{e_n\}$ is a *C-S*. Consider $m, n \in \mathbb{N}$ with $m > n > n_1$. By making use of triangle inequality and (5.11), we have

$$\begin{aligned} d_r(e_n, e_m) &\leq d_r(e_n, e_{n+1}) + d_r(e_{n+1}, e_{n+2}) + \cdots + d_r(e_{m-1}, e_m) \\ &= \sum_{i=n}^{m-1} d_{r_i} \leq \sum_{i=n}^{\infty} d_{r_i} \leq \sum_{i=n}^{\infty} \frac{1}{i^{1/k}}. \end{aligned}$$

As we know that $\sum_{i=1}^{\infty} \frac{1}{i^{1/k}}$ is convergent series. Thus, $\lim_{n\to\infty} d_r(e_n, e_m) = 0$. Which implies that $\{e_n\}$ is a *C-S*. As (X_d, d_r) is complete, there exists $e^* \in X_d$ such that $e_n \to e^*$ as $n \to \infty$. By condition (iii) we have $\alpha_w(e_n, e^*) > 1\ \forall\ n \in \mathbb{N}$. We claim that $d_r(e^*, T_{f_i}e^*) = 0\ \forall i \in \mathbb{N}$. contradictory suppose that $d_r(e^*, T_{f_{i_0}}e^*) > 0$ for some $i_0 \in \mathbb{N}$, $\exists\ n_0 \in \mathbb{N}$ such that $d_r(e_n, T_{f_{i_0}}e^*) > 0\ \forall\ n \geq n_0$. For each $n \geq n_0$ and for above i_0 we have

$$\begin{aligned} d_r(e^*, T_{f_{i_0}}e^*) &\leq d_r(e^*, e_{n+1}) + d_r(e_{n+1}, T_{f_{i_0}}e^*) \\ &< d_r(e^*, e_{n+1}) + \alpha_w(e_n, e^*) H_m(T_{f_{n+1}}e_n, T_{f_{i_0}}e^*) \\ &< d_r(e^*, e_{n+1}) + b_1 d_r(e_n, e^*) + b_2 d_r(e_n, e_{n+1}) + b_3 d_r(e^*, T_{f_{i_0}}e^*) \\ &\quad + b_4 d_r(e_n, T_{f_i}e^*) + L d_r(e^*, e_{n+1}). \end{aligned} \tag{5.12}$$

Now if we let $n \to \infty$ in (5.12) we have

$$d_r(e^*, T_{f_{i_0}}e^*) \leq (b_3 + b_4) d_r(e^*, T_{f_{i_0}}e^*) < d_r(e^*, T_{f_{i_0}}e^*).$$

Which is a contradiction. Thus $d_r(e^*, T_{f_i}e^*) = 0\ \forall i \in \mathbb{N}$. □

Example 5.1.5. Let $X_d = \mathbb{N}$ be enriched with the usual metric $d_r(e, \tilde{e}) = |e - \tilde{e}|$ $\forall$ $e, \tilde{e} \in X_d$. Define $\{T_{f_i} : X_d \to C(X_d)\}_{i=1}^{\infty}$ by

$$T_{f_i}e = \begin{cases} \{0,1\} & \text{if } e = 0,1 \\ \{2e-2, 2e\} & \text{if } e > 1 \end{cases}$$

and $\alpha_w : X_d \times X_d \to [0, \infty)$ by

$$\alpha_w(e, \tilde{e}) = \begin{cases} 2 & \text{if } e, \tilde{e} \in \{0,1\} \\ \frac{1}{4} & \text{if } e, \tilde{e} > 1 \\ 0 & \text{otherwise.} \end{cases}$$

Take $F_w(e) = e + \ln e$ $\forall$ $e \in (0, \infty)$. Under this F_w condition (5.1.1) reduces to

$$\frac{\alpha_w(e,\tilde{e})H_m(T_{f_i}e, T_{f_j}y)}{N_f(e,\tilde{e})} \exp(\alpha_w(e,\tilde{e})H_m(T_{f_i}e, T_{f_j}y) - N_f(e,\tilde{e})) \leq \exp(-\tau_f) \tag{5.13}$$

$\forall$ $e, \tilde{e} \in X_d$ with $\min\{\alpha_w(e,\tilde{e})H_m(T_{f_i}e, T_{f_j}y), N_f(e,\tilde{e})\} > 0$. Assume that $b_1 = 1, b_2 = b_3 = b_4 = L = 0$ and $\tau_f = \frac{1}{2}$. Clearly, $\min\{\alpha_w(e,\tilde{e})H_m(T_{f_i}e, T_{f_j}y), d_r(e,\tilde{e})\} > 0$ $\forall$ $e, \tilde{e} > 1$ with $e \neq \tilde{e}$. From (5.28) $\forall$ $e, \tilde{e} > 1$ with $e \neq \tilde{e}$ we have

$$\frac{1}{4}\exp(-\frac{1}{2}|e-\tilde{e}|) < \exp(-\frac{1}{2}).$$

Thus $\{T_f\}_{i=1}^{\infty}$ is an α_w-F_w-contraction of Hardy-Rogers-type with $F_w(e) = e + \ln e$. For $e_0 = 1$ we have $e_1 = 0 \in T_{f_1}e_0$ such that $\alpha_w(e_0, e_1) > 1$. Moreover, it is easy to see that T_{f_i} is strictly α_w-admissible sequence and for any sequence $\{e_n\} \subseteq X_d$ such that $e_n \to e$ as $n \to \infty$ and $\alpha_w(e_n, e_{n+1}) > 1$ $\forall$ $n \in \mathbb{N}$, we have $\alpha_w(e_n, e) > 1$ $\forall$ $n \in \mathbb{N}$. Therefore, by Theorem 5.1.4 T_f has a common $F - P$ in X_d.

Definition 5.1.6. Let (X_d, d_r) be a $M - S$ and $\alpha_w : X_d \times X_d \to [0, \infty)$ be a function. A sequence of mappings $\{T_{f_i} : X_d \to Cl_f(X_d)\}_{i=1}^{\infty}$ is an $F_{w_{\alpha_w}}*$-contraction of Hardy-Rogers-type, if $\exists$ $F_w \in \mathfrak{F}_{\mathfrak{w}}$ and a $\tau_f > 0$ such that $\forall$ $i, j \in \mathbb{N}$, we have

$$\tau_f + F_w(\alpha_{w*(T_{f_i}e, T_{f_j}y)H_m(T_{f_i}e, T_{f_j}y))}) \leq F_w(N_f(e,\tilde{e})), \tag{5.14}$$

$\forall\ e, \tilde{e} \in X_d$, whenever $\min\{\alpha_{w_*(T_{f_i}e, T_{f_j}y) H_m(T_{f_i}e, T_{f_j}y)}, N_f(e, \tilde{e})\} > 0$, where

$N_f(e, \tilde{e}) = b_1 d_r(e, \tilde{e}) + b_2 d_r(e, T_{f_i}e) + b_3 d_r(\tilde{e}, T_{f_j}y) + b_4 d_r(e, T_{f_j}y) + L d_r(\tilde{e}, T_{f_i}e)$,

with $b_1, b_2, b_3, b_4, L \geq 0$ satisfying $b_1 + b_2 + b_3 + 2b_4 = 1$ and $b_3 \neq 1$.

Now prove a theorem for $F^*_{w_{\alpha_w}}$-contraction of Hardy-Rogers-type satisfying.

Theorem 5.1.7. Let (X_d, d_r) be a complete $M - S$ and let $\{T_{f_i} : X_d \to N_f(X_d)\}_{i=1}^{\infty}$ be an $F^*_{w_{\alpha_w}}$-contraction of Hardy-Rogers-type satisfying the following conditions:

(i) $\{T_{f_i}\}_{i=1}^{\infty}$ is strictly α_{w_*}-admissible sequence;

(ii) $\exists$ an $e_0 \in X_d$ and $e_1 \in T_{f_i}e_0$ for some $i \in \mathbb{N}$ with $\alpha_w(e_0, e_1) > 1$;

(iii) for any sequence $\{e_n\} \subseteq X_d$ such that $e_n \to e$ as $n \to \infty$ and $\alpha_w(e_n, e_{n+1}) > 1\ \forall\ n \in \mathbb{N}$, we have $\alpha_w(e_n, e) > 1\ \forall\ n \in \mathbb{N}$.

Then the mappings in a sequence $\{T_{f_i}\}_{i=1}^{n}$ have a common $F - P$.

Proof. The proof of this theorem runs along the same lines as the proof of Theorem 5.2.2. □

Remark 5.1.8. Theorems 5.1.4 and 5.2.5 can be further generalized, if we use the following contractive condition instead of the one used in these theorems. Let (X_d, d_r) be a $M - S$ and $\alpha_w : X_d \times X_d \to [0, \infty)$ be a function. A sequence of mappings $\{T_{f_i} : X_d \to Cl_f(X_d)\}_{i=1}^{\infty}$ is an $F_{w_{\alpha_w}}$-contraction of Roades-type, if $\exists\ F_w \in \mathfrak{F}_{\mathfrak{w}}$ and a $\tau_f > 0$ such that $\forall\ i, j \in \mathbb{N}$, we have

$$\alpha(e, \tilde{e}) \geq 1 \Rightarrow \tau_f + F_w(H_m(T_{f_i}e, T_{f_j}\tilde{e})) \leq F_w(M(e, \tilde{e})), \tag{5.15}$$

$\forall\ e, \tilde{e} \in X_d$, where $M(e, \tilde{e}) = \max\{d_r(e, \tilde{e}), d_r(e, T_{f_i}e), d_r(\tilde{e}, T_{f_j}\tilde{e}), [d_r(e, T_{f_j}\tilde{e}) + d_r(\tilde{e}, T_{f_i}e)]/2\}$.

The above mentioned theorems can be proved in a similar fashion as that of Theorem 2.1.1.

5.2 A Family of bounded multivalued mappings satisfying F_w type contractions and related end point theorems

In this section we will discuss common $F-P$ theorems for sequence of a bounded multivalued mappings satisfying $F_{w_{\alpha_w}}$-contraction and $F^*_{w_{\alpha_w}}$-contraction in a complete metric space.

Definition 5.2.1. Let (X_d, d_r) be a $M-S$ and $\alpha_w : X_d \times X_d \to [0, \infty)$ be a function. A sequence of mappings $\{T_{f_i} : X_d \to B_f(X_d)\}_{i=1}^{\infty}$ is an $F_{w_{\alpha_w}}$-contraction of Hardy-Rogers-type, if $\exists$ $F_w \in \mathfrak{F}_{\mathfrak{w}}$ and a $\tau_f > 0$ such that $\forall$ $i, j \in \mathbb{N}$, we have

$$\tau_f + F_w(\alpha_w(e, \tilde{e})\delta(T_{f_i}e, T_{f_j}y)) \leq F_w(N_f(e, \tilde{e})), \tag{5.16}$$

$\forall$ $e, \tilde{e} \in X_d$, whenever $\min\{\alpha_w(e, \tilde{e})\delta(T_{f_i}e, T_{f_j}y), N_f(e, \tilde{e})\} > 0$, where

$$N_f(e, \tilde{e}) = b_1 d_r(e, \tilde{e}) + b_2 d_r(e, T_{f_i}e) + b_3 d_r(\tilde{e}, T_{f_j}y) + b_4 d_r(e, T_{f_j}y) + L d_r(\tilde{e}, T_{f_i}e),$$

with $b_1, b_2, b_3, b_4, L \geq 0$ satisfying $b_1 + b_2 + b_3 + 2b_4 = 1$ and $b_3 \neq 1$.

It would be interesting to see whether the conclusions of Theorem 5.1.4 hold for bounded subsets of X_d. We will show that the conclusions of Theorem 5.1.4 still hold for bounded subsets of X_d provided that the Housdorff distance $H_m(A_{1f}, A_{2f})$ in definition 5.1.3 is replaced with $\delta(A_{1f}, A_{2f})$ and the strict inequality in (ii) of Theorem 5.1.4 is replaced by the soft inequality. More precisely we have the following result. Now we will prove a result $F_{w_{\alpha_w}}$-contraction of Hardy-Rogers-type satisfying.

Theorem 5.2.2. Let (X_d, d_r) be a complete $M-S$ and let $\{T_{f_i} : X_d \to B_f(X_d)\}_{i=1}^{\infty}$ be an $F_{w_{\alpha_w}}$-contraction of Hardy-Rogers-type satisfying the following conditions:

(i) $\{T_{f_i}\}_{i=1}^{\infty}$ is α_w-admissible sequence;

(ii) $\exists$ $e_0 \in X_d$ and $e_1 \in T_{f_i}e_0$ for some $i \in \mathbb{N}$ with $\alpha_w(e_0, e_1) \geq 1$;

(iii) for any sequence $\{e_n\} \subseteq X_d$ such that $e_n \to e$ as $n \to \infty$ and $\alpha_w(e_n, e_{n+1}) \geq 1$ $\forall$ $n \in \mathbb{N}$, we have $\alpha_w(e_n, e) \geq 1$ $\forall$ $n \in \mathbb{N}$.

Then the mappings in the sequence $\{T_{f_i}\}_{i=1}^{\infty}$ have a common $F-P$.

Proof. As we know by hypothesis (ii), we assume without loss of generality that $\exists\ e_0 \in X_d$ and $e_1 \in T_{f_1}e_0$ with $\alpha_w(e_0, e_1) \geq 1$. If $e_1 \in T_{f_i}e_1\ \forall i \in \mathbb{N}$, then e_1 is a common *F-P*. Let $e_1 \notin T_{f_2}e_1$. As $\alpha_w(e_0, e_1) \geq 1$, $\exists\ e_2 \in T_{f_2}e_1$ such that

$$d_r(e_1, e_2) \leq \alpha_w(e_0, e_1)\delta(T_{f_1}e_0, T_{f_2}e_1). \tag{5.17}$$

As we know that F_w is increasing, we have

$$F_w(d_r(e_1, e_2)) \leq F_w(\alpha_w(e_0, e_1)\delta(T_{f_1}e_0, T_{f_2}e_1)). \tag{5.18}$$

From (5.16) we have

$$\begin{aligned}
\tau_f + F_w(d_r(e_1, e_2)) &\leq \tau_f + F_w(\alpha_w(e_0, e_1)\delta(T_{f_1}e_0, T_{f_2}e_1)) \\
&\leq F_w\Big(b_1 d_r(e_0, e_1) + b_2 d_r(e_0, T_{f_1}e_0) + b_3 d_r(e_1, T_{f_2}e_1) + \\
&\quad b_4 d_r(e_0, T_{f_2}e_1) + L d_r(e_1, T_{f_1}e_0)\Big) \\
&\leq F_w\Big(b_1 d_r(e_0, e_1) + b_2 d_r(e_0, e_1) + b_3 d_r(e_1, e_2) + \\
&\quad b_4 d_r(e_0, e_2) + L.0\Big) \\
&\leq F_w\Big(b_1 d_r(e_0, e_1) + b_2 d_r(e_0, e_1) + b_3 d_r(e_1, e_2) + \\
&\quad b_4(d_r(e_0, e_1) + d_r(e_1, e_2))\Big) \\
&= F_w\Big((b_1 + b_2 + b_4)d_r(e_0, e_1) + (b_3 + b_4)d_r(e_1, e_2)\Big). \qquad (5.19)
\end{aligned}$$

As we know that F_w is increasing, we get from above that

$$d_r(e_1, e_2) < (b_1 + b_2 + b_4)d_r(e_0, e_1) + (b_3 + b_4)d_r(e_1, e_2).$$

So that,

$$(1 - b_3 - b_4)d_r(e_1, e_2) < (b_1 + b_2 + b_4)d_r(e_0, e_1).$$

As $b_1 + b_2 + b_3 + 2b_4 = 1$, thus we have

$$d_r(e_1, e_2) < d_r(e_0, e_1).$$

Now from (5.19), we have

$$\tau_f + F_w(d_r(e_1, e_2)) \leq F_w(d_r(e_0, e_1)).$$

If $e_2 \in T_{f_i}e_2$ $\forall i \in \mathbb{N}$ then e_2 is a common F-P. Let $e_2 \notin T_{f_3}e_2$, As we know that $\{T_{f_i}\}_{i=1}^{\infty}$ is α_w-admissible, we have $\alpha_w(e_1, e_2) \geq 1$. There exists $e_3 \in T_{f_3}e_2$ such that

$$d_r(e_2, e_3) \leq \alpha_w(e_1, e_2)\delta(T_{f_2}e_1, T_{f_3}e_2). \tag{5.20}$$

As we know that F_w is increasing, we have

$$F_w(d_r(e_2, e_3)) \leq F_w(\alpha_w(e_1, e_2)\delta(T_{f_2}e_1, T_{f_3}e_2)). \tag{5.21}$$

From (5.16) we have

$$\begin{aligned}
\tau_f + F_w(d_r(e_2, e_3)) &\leq \tau_f + F_w(\alpha_w(e_1, e_2)\delta(T_{f_2}e_1, T_{f_3}e_2)) \\
&\leq F_w\Big(b_1 d_r(e_1, e_2) + b_2 d_r(e_1, T_{f_2}e_1) + b_3 d_r(e_2, T_{f_3}e_2) + \\
&\quad b_4 d_r(e_1, T_{f_3}e_2) + L d_r(e_2, T_{f_2}e_1)\Big) \\
&\leq F_w\Big(b_1 d_r(e_1, e_2) + b_2 d_r(e_1, e_2) + b_3 d_r(e_2, e_3) + \\
&\quad b_4 d_r(e_1, e_3) + L.0\Big) \\
&\leq F_w\Big(b_1 d_r(e_1, e_2) + b_2 d_r(e_1, e_2) + b_3 d_r(e_2, e_3) + \\
&\quad b_4(d_r(e_1, e_2) + d_r(e_2, e_3))\Big) \\
&= F_w\Big((b_1 + b_2 + b_4)d_r(e_1, e_2) + (b_3 + b_4)d_r(e_2, e_3)\Big) \qquad (5.22)
\end{aligned}$$

As we know that F_w is increasing, we get from above that

$$d_r(e_2, e_3) < (b_1 + b_2 + b_4)d_r(e_1, e_2) + (b_3 + b_4)d_r(e_2, e_3).$$

So that,

$$(1 - b_3 - b_4)d_r(e_2, e_3) < (b_1 + b_2 + b_4)d_r(e_1, e_2).$$

As $b_1 + b_2 + b_3 + 2b_4 = 1$, thus we have

$$d_r(e_2, e_3) < d_r(e_1, e_2).$$

Now from (5.22) we have

$$\tau_f + F_w(d_r(e_2, e_3)) \leq F_w(d_r(e_1, e_2)).$$

So we have

$$F_w(d_r(e_2, e_3)) \leq F_w(d_r(e_1, e_2)) - \tau_f \leq F_w(d_r(e_0, e_1)) - 2\tau_f.$$

Proceeding in the same way we get a sequence $\{e_n\} \subset X_d$ such that

$$e_n \in T_{f_n} e_{n-1},\ e_{n-1} \neq e_n \text{ and } \alpha_w(e_{n-1}, e_n) \geq 1\ \forall\ n \in \mathbb{N}.$$

Furthermore,

$$F_w(d_r(e_n, e_{n+1})) \leq F_w(d_r(e_0, e_1)) - n\tau_f\ \forall\ n \in \mathbb{N}. \tag{5.23}$$

Now if we let $n \to \infty$ in (5.23) we get $\lim_{n\to\infty} F_w(d_r(e_n, e_{n+1})) = -\infty$. Thus, by property (F_{w_2}), we have $\lim_{n\to\infty} d_r(e_n, e_{n+1}) = 0$. Let $d_{r_n} = d(e_n, e_{n+1})$ $\forall\ n \in \mathbb{N}$. From (F_{w_3}) $\exists\ k \in (0, 1)$ such that

$$\lim_{n\to\infty} d_{r_n}^k F_w(d_{r_n}) = 0.$$

From (5.23) we have

$$d_{r_n}^k F(d_{r_n}) - d_{r_n}^k F_w(d_{r_0}) \leq -d_{r_n}^k n\tau_f \leq 0\ \forall\ n \in \mathbb{N}. \tag{5.24}$$

Now if we let $n \to \infty$ in (5.24) we get

$$\lim_{n\to\infty} n d_{r_n}^k = 0. \tag{5.25}$$

This implies that $\exists\ n_1 \in \mathbb{N}$ such that $n d_{r_n}^k \leq 1\ \forall\ n \geq n_1$. Thus we have

$$d_{r_n} \leq \frac{1}{n^{1/k}}, \quad \forall\ n \geq n_1. \tag{5.26}$$

To prove that $\{e_n\}$ is a *C-S*. Consider $m, n \in \mathbb{N}$ with $m > n > n_1$. By making use of triangle inequality and (5.26) we have

$$\begin{aligned} d_r(e_n, e_m) &\leq d_r(e_n, e_{n+1}) + d_r(e_{n+1}, e_{n+2}) + \cdots + d_r(e_{m-1}, e_m) \\ &= \sum_{i=n}^{m-1} d_{r_i} \leq \sum_{i=n}^{\infty} d_{r_i} \leq \sum_{i=n}^{\infty} \frac{1}{i^{1/k}}. \end{aligned}$$

As we know that $\sum_{i=1}^{\infty} \frac{1}{i^{1/k}}$ is convergent series. Thus $\lim_{n\to\infty} d_r(e_n, e_m) = 0$. Which implies that $\{e_n\}$ is a *C-S*. As (X_d, d_r) is complete so $\exists\ e^* \in X_d$ such that $e_n \to e^*$ as $n \to \infty$. By condition (iii) we have $\alpha_w(e_n, e^*) \geq 1\ \forall$ $n \in \mathbb{N}$. We claim that $d_r(e^*, T_{f_i} e^*) = 0\ \forall i \in \mathbb{N}$. contradictory suppose that $d_r(e^*, T_{f_{i_0}} e^*) > 0$ for some $i_0 \in \mathbb{N}$, $\exists\ n_0 \in \mathbb{N}$ such that $d_r(e_n, T_{f_{i_0}} e^*) > 0\ \forall$

$n \geq n_0$. For each $n \geq n_0$ and for above i_0, we have

$$\begin{aligned} d_r(e^*, T_{f_{i_0}} e^*) &\leq d_r(e^*, e_{n+1}) + d_r(e_{n+1}, T_{f_{i_0}} e^*) \\ &< d_r(e^*, e_{n+1}) + \alpha_w(e_n, e^*)\delta(T_{f_{n+1}} e_n, T_{f_{i_0}} e^*) \\ &< d_r(e^*, e_{n+1}) + b_1 d_r(e_n, e^*) + b_2 d_r(e_n, e_{n+1}) + b_3 d_r(e^*, T_{f_{i_0}} e^*), \\ &\quad + b_4 d_r(e_n, T_{f_i} e^*) + L d_r(e^*, e_{n+1}). \end{aligned} \tag{5.27}$$

Now if we let $n \to \infty$ in (5.27) we have

$$d_r(e^*, T_{f_{i_0}} e^*) \leq (b_3 + b_4) d_r(e^*, T_{f_{i_0}} e^*) < d_r(e^*, T_{f_{i_0}} e^*).$$

Which is a contradiction. Thus $d_r(e^*, T_{f_i} e^*) = 0 \ \forall i \in \mathbb{N}$. □

Example 5.2.3. Let $X_d = \{0, 1, 2, 3, ...\}$ and

$$d_r(e, \tilde{e}) = \begin{cases} 0 & \text{if } e = \tilde{e} \\ e + \tilde{e} & \text{if } e \neq \tilde{e} \end{cases}$$

. Define $\{T_{f_i} : X_d \to B_f(X_d)\}_{i=1}^{\infty}$ by

$$T_i e = \begin{cases} \{0\} & \text{if } e = 0 \\ \{0, 1, 2, 3, ..., e\} & \text{if } e \neq 0 \end{cases}$$

and $\alpha_w : X_d \times X_d \to [0, \infty)$ by

$$\alpha_w(e, \tilde{e}) = \begin{cases} 1 & \text{if } e = \tilde{e} = 0 \\ \frac{1}{2} & \text{if } e, \tilde{e} > 1 \\ 0 & \text{otherwise.} \end{cases}$$

Take $F_w(e) = e + \ln(e) \ \forall \ e \in (0, \infty)$. Under this F_w condition (5.16) reduces to

$$\frac{\alpha_w(e, \tilde{e})\delta(T_{f_i} e, T_{f_j} y)}{N_f(e, \tilde{e})} \exp(\alpha_w(e, \tilde{e})\delta(T_{f_i} e, T_{f_j} y) - N_f(e, \tilde{e})) \leq \exp(-\tau_f) \tag{5.28}$$

$\forall \ e, \tilde{e} \in X_d$ with $\min\{\alpha_w(e, \tilde{e})\delta(T_{f_i} e, T_{f_j} y), N_f(e, \tilde{e})\} > 0$. Assume that $b_1 = 1, b_2 = b_3 = b_4 = L = 0$ and $\tau_f = \frac{1}{2}$. Clearly $\min\{\alpha_w(e, \tilde{e})\delta(T_{f_i} e, T_{f_j} y), d_r(e, \tilde{e})\} > 0 \ \forall \ e, \tilde{e} > 1$ with $e \neq \tilde{e}$. From (5.16) $\forall \ e, \tilde{e} > 1$ with $e \neq \tilde{e}$, we have

$$\frac{1}{2} \exp(-\frac{1}{2}(e + \tilde{e})) < \exp(-\frac{1}{2}).$$

Thus T is an $F_{w_{\alpha_w}}$-contraction of Hardy-Roger-type with $F_w(e) = e + \ln e$. For $e_0 = 1$, we have $e_1 = 0 \in T_{f_1}e_0$ such that $\alpha_w(e_0, e_1) \geq 1$. Moreover, it can be easily seen that T_f is α_w-admissible mapping and for any sequence $\{e_n\} \subseteq X_d$ such that $e_n \to e$ as $n \to \infty$ and $\alpha_w(e_n, e_{n+1}) \geq 1 \; \forall \; n \in \mathbb{N}$, we have $\alpha_w(e_n, e) \geq 1 \; \forall \; n \in \mathbb{N}$. Therefore by Theorem 5.2.2 T_f has a $F - P$ in X_d.

Definition 5.2.4. Let (X_d, d_r) be a $M - S$ and $\alpha_w : X_d \times X_d \to [0, \infty)$ be a function. A sequence of mappings $\{T_{f_i} : X_d \to B_f(X_d)\}_{i=1}^{\infty}$ is an $F_{w_{\alpha_w}}*$-contraction of Hardy-Rogers-type, if $\exists \; F_w \in \mathfrak{F_w}$ and a $\tau_f > 0$ such that $\forall \; i, j \in \mathbb{N}$, we have

$$\tau_f + F_w(\alpha_{w*(T_{f_i}e, T_{f_j}y)} \delta(T_{f_i}e, T_{f_j}y)) \leq F_w(N_f(e, \tilde{e})), \tag{5.29}$$

$\forall \; e, \tilde{e} \in X_d$, whenever $\min\{\alpha_{w*(T_{f_i}e, T_{f_j}y)} \delta(T_{f_i}e, T_{f_j}y), N_f(e, \tilde{e})\} > 0$, where

$$N_f(e, \tilde{e}) = b_1 d_r(e, \tilde{e}) + b_2 d_r(e, T_{f_i}e) + b_3 d_r(\tilde{e}, T_{f_j}y) + b_4 d_r(e, T_{f_j}y) + L d_r(\tilde{e}, T_{f_i}e),$$

with $b_1, b_2, b_3, b_4, L \geq 0$ satisfying $b_1 + b_2 + b_3 + 2b_4 = 1$ and $b_3 \neq 1$.

Now we will prove a result $F_{w_{\alpha_w}}*$-contraction of Hardy-Rogers-type satisfying.

Theorem 5.2.5. Let (X_d, d_r) be a complete $M - S$ and let $\{T_{f_i} : X_d \to B_f(X_d)\}_{i=1}^{\infty}$ be and $F_{w_{\alpha_w}}*$-contraction of Hardy-Rogers-type satisfying the following conditions:

(i) $\{T_i\}_{i=1}^{\infty}$ is α_{w_*}-admissible sequence;

(ii) $\exists \; e_0 \in X_d$ and $e_1 \in T_i e_0$ for some $i \in \mathbb{N}$ with $\alpha_w(e_0, e_1) \geq 1$;

(iii) for any sequence $\{e_n\} \subseteq X_d$ such that $e_n \to e$ as $n \to \infty$ and $\alpha_w(e_n, e_{n+1}) \geq 1 \; \forall \; n \in \mathbb{N}$, we have $\alpha_w(e_n, e) \geq 1 \; \forall \; n \in \mathbb{N}$.

Then the mappings in a sequence $\{T_{f_i}\}_{i=1}^{n}$ have a common $F - P$.

Proof. The proof of this theorem runs along the same lines as the proof of Theorem 5.2.2. □

Remark 5.2.6. Theorems 5.2.1 and 5.2.5 can be further generalized if we use the following contractive condition instead of the one used in these theorems.

Let (X_d, d_r) be a $M - S$ and $\alpha_w : X_d \times X_d \to [0, \infty)$ be a function. A sequence of mappings $\{T_{f_i} : X_d \to B_f(X_d)\}_{i=1}^{\infty}$ is an $F_{w_{\alpha_w}}$-contraction of Roades-type, if $\exists\, F_w \in \mathfrak{F}_{\mathfrak{w}}$ and a $\tau_f > 0$ such that $\forall\, i, j \in \mathbb{N}$, we have

$$\alpha_w(e, \tilde{e}) \geq 1 \Rightarrow \tau_w + F_w(\delta(T_{f_i}e, T_{f_j}\tilde{e})) \leq F_w(M(e, \tilde{e})), \tag{5.30}$$

$\forall\, e, \tilde{e} \in X_d$, where $M(e, \tilde{e}) = \max\{d_r(e, \tilde{e}), d(e, T_{f_i}e), d(\tilde{e}, T_{f_j}\tilde{e}), [d(e, T_{f_j}\tilde{e}) + d(\tilde{e}, T_{f_i}e)]/2\}$.

The above mentioned theorems can be proved in a similar fashion as that of Theorem 2.1.1.

5.3 Application

In this section, as a consequence of our result we establish an existence theorem for a system of integral equations. Let $X_d = (C[a, b], \mathbb{R})$ be the space of all real valued continuous functions defined on $[a, b]$. Note that X_d is complete [69] with respect to the metric $d_{\tau_f}(e, \tilde{e}) = \sup_{t \in [a,b]}\{|e(t) - \tilde{e}(t)| \exp(-|\tau_f t|)\}$.

Consider the system of integral equations of the form

$$e(t) = f(t) + \int_a^b K_i(t, s, e(s))ds, \tag{5.31}$$

for $t, s \in [a, b]$ and $i \in \{1, 2, 3, \cdots, N_f\}$ with $N_f \in \mathbb{N}$. Where $K_i : [a, b] \times [a, b] \times \mathbb{R} \to \mathbb{R}$ and $f : [a, b] \to \mathbb{R}$ are continuous functions.

Theorem 5.3.1. Let $X_d = (C[a, b], \mathbb{R})$ and let $\{T_{f_i} : X_d \to X_d\}_{i=1}^{N_f}$ be the operators defined as

$$T_{f_i}e(t) = f(t) + \int_a^b K_i(t, s, e(s))ds, \tag{5.32}$$

for $t, s \in [a, b]$. Where $K_i : [a, b] \times [a, b] \times \mathbb{R} \to \mathbb{R}$ and $f : [a, b] \to \mathbb{R}$ are continuous functions. Assume that $\exists\, \gamma : X_d \to (0, \infty)$ and $\alpha_w : X_d \times X_d \to (0, \infty)$ and following conditions hold:

(i) $\forall\ i,j \in \{1,2,3,\cdots,N_f\}\ \exists\ \tau_f > 0$ such that

$$|K_i(t,s,e) - K_j(t,s,\tilde{e})| \leq \frac{exp{-}\tau_f}{\gamma(e+\tilde{e})}|e-\tilde{e}|$$

$\forall\ t,s \in [a,b]$ and $e,\tilde{e} \in X_d$. Moreover,

$$\Big|\int_a^b \frac{\exp(|\tau_f s|)}{\gamma(e+\tilde{e})}ds\Big| \leq \frac{\exp(|\tau_f t|)}{\alpha_w(e,\tilde{e})}$$

$\forall\ t \in [a,b]$;

(ii) for $e,\tilde{e} \in X_d$, $\alpha_w(e,\tilde{e}) \geq 1$ implies $\alpha_w(T_{f_i}e, T_{f_j}y) \geq 1\ \forall\ i,j \in \{1,2,3,\cdots,N_f\}$;

(iii) $\exists\ e_0 \in X_d$ such that $\alpha_w(e_0, T_{f_i}e_0) \geq 1$ for some $i \in \{1,2,3,\cdots,N_f\}$;

(iv) for any sequence $\{e_n\} \subseteq X_d$ such that $e_n \to e$ as $n \to \infty$ and $\alpha_w(e_n,e_{n+1}) \geq 1\ \forall\ n \in \mathbb{N}$, we have $\alpha_w(e_n,e) \geq 1\ \forall\ n \in \mathbb{N}$.

Then the system of integral equations (5.31) has a solution in X_d.

Proof. First we show that $\{T_{f_i}\}$ is an $F_{w_{\alpha_w}}$-contraction of Hardy-Rogers-type. For each $i,j \in \{1,2,3,\cdots,N_f\}$, we have

$$\begin{aligned}
|T_{f_i}e(t) - T_{f_j}y(t)| &\leq \int_a^b |K_i(t,s,e(s)) - K_j(t,s,\tilde{e}(s))|ds \\
&\leq \int_a^b \frac{\exp(-\tau_f)}{\gamma(e(s)+\tilde{e}(s))}|e(s)-\tilde{e}(s)|ds \\
&= \int_a^b \frac{\exp(-\tau_f)\exp(|\tau_f s|)}{\gamma(e(s)+\tilde{e}(s))}|e(s)-\tilde{e}(s)|\exp(-|\tau_f s|)ds \\
&\leq \exp(-\tau_f)d_{\tau_f}(e,\tilde{e})\int_a^b \frac{\exp(|\tau_f s|)}{\gamma(e(s)+\tilde{e}(s))}ds \\
&\leq \frac{\exp(|\tau_f)t|}{\alpha_w(e,\tilde{e})}\exp(-\tau_f)d_{\tau_f}(e,\tilde{e}).
\end{aligned}$$

Thus we have

$$\alpha_w(e,\tilde{e})|T_{f_i}e(t) - T_{f_j}y(t)|\exp(-|\tau_f t|) \leq \exp(-\tau_f)d_{\tau_f}(e,\tilde{e}).$$

Equivalently,

$$\alpha_w(e,\tilde{e})d_{\tau_f}(T_{f_i}e, T_{f_j}y) \leq \exp(-\tau_f)d_{\tau_f}(e,\tilde{e}).$$

Clearly natural logarithm belongs to $\mathfrak{F}_{\mathfrak{w}}$. Applying it on above inequality we get

$$\ln(\alpha_w(e,\tilde{e})d_{\tau_f}(T_{f_i}e, T_{f_j}y)) \leq \ln(\exp(-\tau_f)d_{\tau_f}(e,\tilde{e})),$$

after some simplification we get

$$\tau_f + \ln(\alpha_w(e,\tilde{e})d_{\tau_f}(T_{f_i}e, T_{f_j}y)) \leq \ln(d_{\tau_f}(e,\tilde{e})).$$

Thus $\{T_{f_i}\}_{i=1}^{N_f}$ is an $F_{w_{\alpha_w}}$-contraction of Hardy-Rogers-type with $b_1 = 1$, $b_2 = b_3 = b_4 = L = 0$ and $F_w(e) = \ln e$. Therefore by 5.2.2 it follows that the system of operators (5.32) have a common F-P, So that, the system of integral equations (5.31) has a solution in X_d. □

Chapter 6

Existence of Best proximity points for F_w-proximal contractions

Throughout this chapter $F-P$ represents a fixed point, $M-S$ represents a metric space, $C-S$ represents a Cauchy sequence. In this chapter we introduce the notions of an $F_{w_{\alpha_w}}$-proximal contractions for Hardy-Rogers type mappings as well as for Ciric-type mappings. We also discuss the existence of best proximity for nonself multivalued mappings satisfying at least one of these conditions, along with few other conditions.
This chapter is divided into two section. In the first section, we will discuss best proximity point theorems for Hardy-Rogers type an $F_{w_{\alpha_w}}$-proximal contraction. We will discuss our results on the closed subsets of a complete $M-S$.
In the second section we will prove some best proximity point theorems for Ciric type an $F_{w_{\alpha_w}}$-proximal contraction. Towards the end of this chapter, we will prove theorems as a consequence of our results in this chapter.

We begin this chapter with the following definitions.

Definition 6.0.2. Let A_{1f} and A_{2f} be two nonempty subsets of a $M-S$ (X_d, d_r). A mapping $T_f : A_{1f} \to CL_f(X_d)$ is called strictly α_w-proximal admissible if $\exists$ a mapping $\alpha_w : A_{1f} \times A_{1f} \to [0, \infty)$ such that

$$\begin{cases} \alpha_w(e_1, e_2) > 1 \\ d_r(u_1, \tilde{e}_1) = d_r(A_{1f}, A_{2f}) \\ d_r(u_2, \tilde{e}_2) = d_r(A_{1f}, A_{2f}) \end{cases} \Rightarrow \alpha_w(u_1, u_2) > 1,$$

where $e_1, e_2, u_1, u_2 \in A_{1f}$ and $\tilde{e}_1 \in T_f e_1$, $\tilde{e}_2 \in T_f e_2$.

6.1 Best proximity point theorems for Hardy Rogers type F_w-proximal contraction

In this section we will investigate a best proximity point for an $F_{w_{\alpha_w}}$-proximal contraction of Hardy-Rogers type among the non-empty closed subsets of a complete $M - S$.

Definition 6.1.1. Let (X_d, d_r) be a $M - S$, $A_{1f}, A_{2f} \subseteq X_d$ and $\alpha_w : A_{1f} \times A_{1f} \to [0, \infty)$ be a function. A mapping $T_f : A_{1f} \to CL_f(A_{2f})$ is an $F_{w_{\alpha_w}}$-proximal contraction of Hardy-Rogers-type, if $\exists$ an $F_w \in \mathfrak{F}_{\mathfrak{w}}$ and a $\tau_f > 0$ such that

$$\tau_f + F_w(\alpha_w(e, \tilde{e}) H_m(T_f e, T_f y)) \leq F_w(N_f(e, \tilde{e})), \tag{6.1}$$

$\forall$ $e, \tilde{e} \in A_{1f}$, whenever $\min\{\alpha_w(e, \tilde{e}) H_m(T_f e, T_f y), N_f(e, \tilde{e})\} > 0$, where

$$\begin{aligned} N_f(e, \tilde{e}) \;=\; & b_1 d_r(e, \tilde{e}) + b_2[d_r(e, T_f e) - d_r(A_{1f}, A_{2f})] + b_3[d_r(\tilde{e}, T_f y) - d_r(A_{1f}, A_{2f})] \\ & + b_4[d_r(e, T_f y) - d_r(A_{1f}, A_{2f})] + L[d_r(\tilde{e}, T_f e) - d_r(A_{1f}, A_{2f})], \end{aligned}$$

with $b_1, b_2, b_3, b_4, L \geq 0$ satisfying $b_1 + b_2 + b_3 + 2b_4 = 1$ and $b_3 \neq 1$.

Now we will prove a theorem for a $F_{w_{\alpha_w}}$-proximal contraction of Hardy-Rogers-type.

Theorem 6.1.2. Let A_{1f} and A_{2f} be nonempty closed subsets of a complete $M - S$ (X_d, d_r). Further suppose that A_0 is nonempty and $T : A_{1f} \to CL_f(A_{2f})$ is an $F_{w_{\alpha_w}}$-proximal contraction of Hardy-Rogers-type satisfying the following conditions:

(i) $T_f e \subseteq B_0$ $\forall$ $e \in A_0$ and (A_{1f}, A_{2f}) satisfies the weak P-property;

(ii) T_f is strictly α_w-proximal admissible;

(iii) $\exists\ e_0, e_1 \in A_0$ and $\tilde{e}_1 \in T_f e_0$ such that

$$\alpha_w(e_0, e_1) > 1 \ \text{ and } \ d_r(e_1, \tilde{e}_1) = d_r(A_{1f}, A_{2f}).$$

(iv) T_f is continuous, or, for any sequence $\{e_n\} \subseteq A_{1f}$ such that $e_n \to e$ as $n \to \infty$ and $\alpha_w(e_n, e_{n+1}) > 1\ \forall\ n \in \mathbb{N}$, we have $\alpha_w(e_n, e) > 1\ \forall\ n \in \mathbb{N}$.

Then T_f has a best proximity point.

Proof. As we know by hypothesis (iii), $\exists\ e_0, e_1 \in A_0$ and $\tilde{e}_1 \in T_f e_0$ such that

$$d_r(e_1, \tilde{e}_1) = d_r(A_{1f}, A_{2f}) \ \ \textit{and}\ \alpha_w(e_0, e_1) > 1, \tag{6.2}$$

If $\tilde{e}_1 \in T_f e_1$, then e_1 is a best proximity point of T_f. Let $\tilde{e}_1 \notin T_f e_1$. As $\alpha_w(e_0, e_1) > 1$, by Lemma 1.4.5 $\exists\ \tilde{e}_2 \in T_f e_1$ such that

$$d_r(\tilde{e}_1, \tilde{e}_2) \leq \alpha_w(e_0, e_1) H_m(T_f e_0, T_f e_1). \tag{6.3}$$

As we know that F_w is strictly increasing, we have

$$F_w(d_r(\tilde{e}_1, \tilde{e}_2)) \leq F_w(\alpha_w(e_0, e_1) H_m(T_f e_0, T_f e_1)). \tag{6.4}$$

From (6.1), we have

$$\begin{aligned} \tau_f + F_w(d_r(\tilde{e}_1, \tilde{e}_2)) &\leq \tau_f + F_w(\alpha_w(e_0, e_1) H_m(T_f e_0, T_f e_1)) \\ &\leq F_w\Big(b_1 d_r(e_0, e_1) + b_2[d_r(e_0, T_f e_0) - d_r(A_{1f}, A_{2f})] + b_3[d_r(e_1, T_f e_1) - d_r(A_{1f}, \\ &\quad b_4[d_r(e_0, T_f e_1) - d_r(A_{1f}, A_{2f})] + L[d_r(e_1, T_f e_0) - d_r(A_{1f}, A_{2f})]\Big) \\ &\leq F_w\Big(b_1 d_r(e_0, e_1) + b_2 d_r(e_0, e_1) + b_3 d_r(\tilde{e}_1, \tilde{e}_2), \\ &\quad b_4[d_r(e_0, e_1) + d_r(\tilde{e}_1, \tilde{e}_2)] + L.0\Big) \\ &= F_w\Big((b_1 + b_2 + b_4) d_r(e_0, e_1) + (b_3 + b_4) d_r(\tilde{e}_1, \tilde{e}_2)\Big). \end{aligned}$$

As we know that

$$d_r(e_0, T_f e_0) \leq d_r(e_0, e_1) + d_r(e_1, \tilde{e}_1) + d_r(\tilde{e}_1, T_f e_0) = d_r(e_0, e_1) + d_r(A_{1f}, A_{2f}) + 0,$$

$$d_r(e_1, T_f e_1) \leq d_r(e_1, \tilde{e}_1) + d_r(\tilde{e}_1, \tilde{e}_2) + d_r(\tilde{e}_2, T_f e_1) = d_r(A_{1f}, A_{2f}) + d_r(\tilde{e}_1, \tilde{e}_2) + 0,$$

$$d_r(e_0, T_f e_1) \leq d_r(e_0, e_1)+d_r(e_1, \tilde{e}_1)+d_r(\tilde{e}_1, \tilde{e}_2)+d_r(\tilde{e}_2, T_f e_1) = d_r(e_0, e_1)+d_r(A_{1f}, A_{2f})+d_r(\tilde{e}_1, \tilde{e}_2)+0$$

$$d_r(e_1, T_f e_0) \leq d_r(e_1, \tilde{e}_1) + d_r(\tilde{e}_1, T_f e_0) = d_r(A_{1f}, A_{2f}) + 0.$$

As we know that F_w is strictly increasing, we get from (6.5) that

$$d_r(\tilde{e}_1, \tilde{e}_2) < (b_1 + b_2 + b_4)d_r(e_0, e_1) + (b_3 + b_4)d_r(\tilde{e}_1, \tilde{e}_2).$$

So that,

$$(1 - b_3 - b_4)d_r(\tilde{e}_1, \tilde{e}_2) < (b_1 + b_2 + b_4)d_r(e_0, e_1).$$

As $b_1 + b_2 + b_3 + 2b_4 = 1$, thus we have

$$d_r(\tilde{e}_1, \tilde{e}_2) < d_r(e_0, e_1).$$

Now, from (6.5), we have

$$\tau_f + F_w(d_r(\tilde{e}_1, \tilde{e}_2)) \leq F_w(d_r(e_0, e_1)). \tag{6.6}$$

As $\tilde{e}_2 \in T_f e_1 \subseteq B_0$, $\exists$ an $e_2 \neq e_1 \in A_0$ such that

$$d_r(e_2, \tilde{e}_2) = d_r(A_{1f}, A_{2f}), \tag{6.7}$$

for otherwise e_1 is a best proximity point. As (A_{1f}, A_{2f}) satisfies the weak P-property. From (6.2) and (6.7), we have

$$0 < d_r(e_1, e_2) \leq d_r(\tilde{e}_1, \tilde{e}_2).$$

By applying F_w, we get

$$F_w(d_r(e_1, e_2)) \leq F_w(d_r(\tilde{e}_1, \tilde{e}_2)). \tag{6.8}$$

Thus from (6.6) and (6.8), we have

$$\tau_f + F_w(d_r(e_1, e_2)) \leq \tau_f + F_w(d_r(\tilde{e}_1, \tilde{e}_2)) \leq F_w(d_r(e_0, e_1)). \tag{6.9}$$

As T_f is strictly α_w-proximal admissible, and we know that $\alpha_w(e_0, e_1) > 1$, $d_r(e_1, \tilde{e}_1) = d_r(A_{1f}, A_{2f})$ and $d_r(e_2, \tilde{e}_2) = d_r(A_{1f}, A_{2f})$, then $\alpha_w(e_1, e_2) > 1$. Thus we have

$$d_r(e_2, \tilde{e}_2) = d_r(A_{1f}, A_{2f}) \quad \textit{and} \ \alpha_w(e_1, e_2) > 1. \tag{6.10}$$

If $\tilde{e}_2 \in T_f e_2$, then e_2 is a best proximal point of T. Let $\tilde{e}_2 \notin T_f e_2$. As $\alpha_w(e_1, e_2) > 1$. There exists an $\tilde{e}_3 \in T_f e_2$ such that

$$d_r(\tilde{e}_2, \tilde{e}_3) \leq \alpha_w(e_1, e_2) H_m(T_f e_1, T_f e_2). \tag{6.11}$$

As we know that, F_w is strictly increasing, we have

$$F_w(d_r(\tilde{e}_2, \tilde{e}_3)) \leq F_w(\alpha_w(e_1, e_2) H_m(T_f e_1, T_f e_2)). \tag{6.12}$$

From (6.1), we have

$$\begin{aligned}
\tau_f + F_w(d_r(\tilde{e}_2, \tilde{e}_3)) &\leq \tau_f + F_w(\alpha_w(e_1, e_2) H_m(T_f e_1, T_f e_2)) \\
&\leq F_w\Big(b_1 d_r(e_1, e_2) + b_2[d_r(e_1, T_f e_1) - d_r(A_{1f}, A_{2f})] + b_3[d_r(e_2, T_f e_2) - d_r(A_{1f} \\
&\quad b_4[d_r(e_1, T_f e_2) - d_r(A_{1f}, A_{2f})] + L[d_r(e_2, T_f e_1) - d_r(A_{1f}, A_{2f})]\Big) \\
&\leq F_w\Big(b_1 d_r(e_1, e_2) + b_2 d_r(e_1, e_2) + b_3 d_r(\tilde{e}_2, \tilde{e}_3), \\
&\quad b_4[d_r(e_1, e_2) + d_r(\tilde{e}_2, \tilde{e}_3)] + L.0\Big) \\
&= F_w\Big((b_1 + b_2 + b_4) d_r(e_1, e_2) + (b_3 + b_4) d_r(\tilde{e}_2, \tilde{e}_3)\Big).
\end{aligned}$$

As we know that F_w is strictly increasing, we get from above that

$$d_r(\tilde{e}_2, \tilde{e}_3) < (b_1 + b_2 + b_4) d_r(e_1, e_2) + (b_3 + b_4) d_r(\tilde{e}_2, \tilde{e}_3).$$

So that,

$$(1 - b_3 - b_4) d_r(\tilde{e}_2, \tilde{e}_3) < (b_1 + b_2 + b_4) d_r(e_1, e_2).$$

As $b_1 + b_2 + b_3 + 2b_4 = 1$, thus we have

$$d_r(\tilde{e}_2, \tilde{e}_3) < d_r(e_1, e_2).$$

Now from (6.13), we have

$$\tau_f + F_w(d_r(\tilde{e}_2, \tilde{e}_3)) \leq F_w(d_r(e_1, e_2)).$$

As $\tilde{e}_3 \in T_f e_2 \subseteq B_0$, $\exists\ e_3 \neq e_2 \in A_0$ such that

$$d_r(e_3, \tilde{e}_3) = d_r(A_{1f}, A_{2f}), \tag{6.14}$$

for otherwise e_2 is a best proximity point. As (A_{1f}, A_{2f}) satisfies the weak P-property. From (6.7) and (6.14), we have

$$0 < d_r(e_2, e_3) \leq d_r(\tilde{e}_2, \tilde{e}_3).$$

By applying F_w, we get

$$F_w(d_r(e_2, e_3)) \le F_w(d_r(\tilde{e}_2, \tilde{e}_3)).$$

Thus, we have

$$\tau_f + F_w(d_r(e_2, e_3)) \le \tau_f + F_w(d_r(\tilde{e}_2, \tilde{e}_3)) \le F_w(d_r(e_1, e_2)). \qquad (6.15)$$

So we get

$$F_w(d_r(e_2, e_3)) \le F_w(d_r(\tilde{e}_2, \tilde{e}_3)) \le F_w(d_r(e_1, e_2)) - \tau_f \le F_w(d_r(e_0, e_1)) - 2\tau_f.$$

As T_f is strictly α_w-proximal admissible, As we know that $\alpha_w(e_1, e_2) > 1$, $d_r(e_2, \tilde{e}_2) = d_r(A_{1f}, A_{2f})$ and $d_r(e_3, \tilde{e}_3) = d_r(A_{1f}, A_{2f})$, then $\alpha_w(e_2, e_3) > 1$. Proceeding in the same way, we get sequences $\{e_n\}$ in A_0 and $\{\tilde{e}_n\}$ in B_0, where $\tilde{e}_n \in T_f e_{n-1}$ $\forall\, n \in \mathbb{N}$ such that

$$d_r(e_n, \tilde{e}_n) = d_r(A_{1f}, A_{2f}) \ \ \textit{and} \ \alpha_w(e_{n-1}, e_n) > 1. \qquad (6.16)$$

Furthermore,

$$F_w(d_r(e_n, e_{n+1})) \le F_w(d_r(\tilde{e}_n, \tilde{e}_{n+1})) \le F_w(d_r(e_0, e_1)) - n\tau_f \ \forall\, n \in \mathbb{N}. \qquad (6.17)$$

Now if we let $n \to \infty$ in (6.17), we get $\lim_{n\to\infty} F_w(d_r(e_n, e_{n+1})) = \lim_{n\to\infty} F_w(d_r(\tilde{e}_n, \tilde{e}_{n+1})) - \infty$. Thus, by property (F_{w_2}), we have $\lim_{n\to\infty} d_r(e_n, e_{n+1}) = 0$. Let $d_{r_n} = d_r(e_n, e_{n+1})$ $\forall\, n \in \mathbb{N}$. From (F_{w_3}) $\exists\, k \in (0, 1)$ such that

$$\lim_{n\to\infty} d_{r_n}^k F_w(d_{r_n}) = 0.$$

From (6.17) we have

$$d_{r_n}^k F(d_{r_n}) - d_{r_n}^k F_w(d_0) \le -d_{r_n}^k n\tau_f \le 0 \ \forall\, n \in \mathbb{N}. \qquad (6.18)$$

Now if we let $n \to \infty$ in (6.18), we get

$$\lim_{n\to\infty} n d_{r_n}^k = 0.$$

This implies that $\exists$ an $n_1 \in \mathbb{N}$ such that $n d_{r_n}^k \le 1$ $\forall\, n \ge n_1$. Thus, we have

$$d_{r_n} \le \frac{1}{n^{1/k}}, \quad \forall\, n \ge n_1. \qquad (6.19)$$

To prove that $\{e_n\}$ is a C-S in A_{1f}. Consider $m, n \in \mathbb{N}$ with $m > n > n_1$. By making use of triangle inequality and (6.19), we have

$$\begin{aligned} d_r(e_n, e_m) &\leq d_r(e_n, e_{n+1}) + d_r(e_{n+1}, e_{n+2}) + \cdots + d_r(e_{m-1}, e_m) \\ &= \sum_{i=n}^{m-1} d_{r_i} \leq \sum_{i=n}^{\infty} d_{r_i} \leq \sum_{i=n}^{\infty} \frac{1}{i^{1/k}}. \end{aligned}$$

As we know that $\sum_{i=1}^{\infty} \frac{1}{i^{1/k}}$ is convergent series. Thus, $\lim_{n\to\infty} d_r(e_n, e_m) = 0$. Which implies that $\{e_n\}$ is a C-S in A_{1f}. Similarly, we see that $\{\tilde{e}_n\}$ is a C-S in A_{2f}. As we know that A_{1f} and A_{2f} are closed subsets of a complete M-S, $\exists\ e^*$ in A and $\tilde{e}^*$ in A_{2f} such that $e_n \to e^*$ and $\tilde{e}_n \to \tilde{e}^*$ as $n \to \infty$. By the (6.16), we conclude that $d_r(e^*, \tilde{e}^*) = d_r(A_{1f}, A_{2f})$ as $n \to \infty$. By hypothesis (iv), when T_f is continuous, we have $\tilde{e}^* \in T_f e^*$, As we know that $\tilde{e}_n \in Te_{n-1}$. Hence $d_r(A_{1f}, A_{2f}) \leq d_r(e^*, T_f e^*) \leq d_r(e^*, \tilde{e}^*) = d_r(A_{1f}, A_{2f})$. Therefore e^* is a best proximity point of the mapping T_f. By hypothesis (iv), when $\alpha_w(e_n, e^*) > 1\ \forall\ n \in \mathbb{N}$, then by using triangular property, we have

$$\begin{aligned} d_r(e^*, Te^*) &\leq d_r(e^*, \tilde{e}_{n+1}) + d_r(\tilde{e}_{n+1}, T_f e^*) \\ &< d_r(e^*, \tilde{e}_{n+1}) + \alpha_w(e_n, e^*) H_m(Te_n, T_f e^*) \\ &< d_r(e^*, \tilde{e}_{n+1}) + b_1 d_r(e_n, e^*) + b_2[d_r(e_n, T_f e_n) - d_r(A_{1f}, A_{2f})] + \\ &\quad b_3[d_r(e^*, T_f e^*) - d_r(A_{1f}, A_{2f})] + b_4[d_r(e_n, T_f e^*) - d_r(A_{1f}, A_{2f})] \\ &\quad + L[d_r(e^*, T_f e_n) - d_r(A_{1f}, A_{2f})] \\ &\leq d_r(e^*, \tilde{e}_{n+1}) + b_1 d_r(e_n, e^*) + b_2[d_r(e_n, \tilde{e}_{n+1}) - d_r(A_{1f}, A_{2f})] + \\ &\quad b_3[d_r(e^*, T_f e^*) - d_r(A_{1f}, A_{2f})] + b_4[d_r(e_n, T_f e^*) - d_r(A_{1f}, A_{2f})] \\ &\quad + L[d_r(e^*, \tilde{e}_{n+1}) - d_r(A_{1f}, A_{2f})]. \end{aligned} \tag{6.20}$$

Now if we let $n \to \infty$ in (6.20), we have

$$d_r(e^*, T_f e^*) \leq d_r(A_{1f}, A_{2f}) + (b_3 + b_4)[d_r(e^*, T_f e^*) - d_r(A_{1f}, A_{2f})].$$

This implies that

$$d_r(e^*, T_f e^*) \leq d_r(A_{1f}, A_{2f}).$$

Thus, we conclude that $d_r(e^*, T_f e^*) = d_r(A_{1f}, A_{2f})$. □

6.2 Best proximity point theorem for Ciric type F_w-proximal contraction

In this section, we will investigate a best proximity point for a $F_{w_{\alpha_w}}$-proximal contraction of Ciric type on the nonempty closed subsets of a complete $M-S$.

Definition 6.2.1. Let (X_d, d_r) be a $M-S$, A_{1f}, A_{2f} are nonempty subsets in X_d and $\alpha_w : A \times A_{1f} \to [0, \infty)$ be a function. A mapping $T : A_{1f} \to CL_f(A_{2f})$ is a $F_{w_{\alpha_w}}$-proximal contraction of Ciric-type, if $\exists$ continuous F_w in $\mathfrak{F}_\mathfrak{w}$ and a $\tau_f > 0$ such that

$$\tau_f + F_w(\alpha_w(e,\tilde{e})H_m(T_f e, T_f y)) \leq F_w(M(e,\tilde{e})), \tag{6.21}$$

$\forall\ e, \tilde{e} \in A_{1f}$, whenever $\min\{\alpha_w(e,\tilde{e})H_m(T_f e, T_f y), M(e,\tilde{e})\} > 0$, where

$$\begin{aligned} M(e,\tilde{e}) \quad = \quad & \max\Big\{d_r(e,\tilde{e}), d_r(e, T_f e) - d_r(A_{1f}, A_{2f}), d_r(\tilde{e}, T_f y) - d_r(A_{1f}, A_{2f}), \\ & \frac{d_r(e, T_f y) + d_r(\tilde{e}, T_f e) - 2d_r(A_{1f}, A_{2f})}{2}\Big\} + L[d_r(\tilde{e}, T_f e) - d_r(A_{1f}, A_{2f})] \end{aligned}$$

and $L \geq 0$.

Now we will prove a result for a $F_{w_{\alpha_w}}$-proximal contraction of Ciric-type.

Theorem 6.2.2. Let A_{1f} and A_{2f} be nonempty closed subsets of a complete $M-S$ (X_d, d_r). Further suppose that A_0 is nonempty and $T_f : A_{1f} \to CL_f(A_{2f})$ is an $F_{w_{\alpha_w}}$-proximal contraction of Ciric-type satisfying the following conditions:

(i) $T_f e \subseteq B_0\ \forall\ e \in A_0$ and (A_{1f}, A_{2f}) satisfies the weak P-property;

(ii) T_f is strictly α_w-proximal admissible;

(iii) $\exists\ e_0, e_1 \in A_0$ and $\tilde{e}_1 \in T_f e_0$ such that

$$\alpha_w(e_0, e_1) > 1 \quad \text{and} \quad d_r(e_1, \tilde{e}_1) = d_r(A_{1f}, A_{2f}).$$

(iv) T_f is continuous, or, for any sequence $\{e_n\} \subseteq A$ such that $e_n \to e$ as $n \to \infty$ and $\alpha_w(e_n, e_{n+1}) > 1\ \forall\ n \in \mathbb{N}$, we have $\alpha_w(e_n, e) > 1\ \forall\ n \in \mathbb{N}$.

Then T_f has a best proximity point.

Proof. As we know by hypothesis (iii), $\exists\ e_0, e_1 \in A_0$ and $\tilde{e}_1 \in T_f e_0$ such that

$$d_r(e_1, \tilde{e}_1) = d_r(A_{1f}, A_{2f}) \;\; and \; \alpha_w(e_0, e_1) > 1, \tag{6.22}$$

If $\tilde{e}_1 \in T_f e_1$, then e_1 is a best proximity point of T. Let $\tilde{e}_1 \notin T_f e_1$. As $\alpha_w(e_0, e_1) > 1$, by Lemma 1.4.5 $\exists\ \tilde{e}_2 \in T_f e_1$ such that

$$d_r(\tilde{e}_1, \tilde{e}_2) \leq \alpha_w(e_0, e_1) H_m(Te_0, Te_1). \tag{6.23}$$

As we know that F_w is strictly increasing, we have

$$F_w(d_r(\tilde{e}_1, \tilde{e}_2)) \leq F_w(\alpha_w(e_0, e_1) H_m(T_f e_0, T_f e_1)). \tag{6.24}$$

From (6.21), we have

$$\begin{aligned}
\tau_f + F_w(d_r(\tilde{e}_1, \tilde{e}_2)) &\leq \tau_f + F_w(\alpha_w(e_0, e_1) H_m(T_f e_0, T_f e_1)) \\
&\leq F_w\Big(\max\Big\{d_r(e_0, e_1), d_r(e_0, T_f e_0) - d_r(A_{1f}, A_{2f}), d_r(e_1, T_f e_1) - d_r(A_{1f}, A_{2f} \\
&\quad \frac{d_r(e_0, T_f e_1) + d_r(e_1, T_f e_0) - 2d_r(A_{1f}, A_{2f})}{2}\Big\} + L[d_r(e_1, T_f e_0) - d_r(A_{1f}, A_{2f} \\
&\leq F_w\Big(\max\Big\{d_r(e_0, e_1), d_r(e_0, e_1), d_r(\tilde{e}_1, \tilde{e}_2), \frac{d_r(e_0, e_1) + d_r(\tilde{e}_1, \tilde{e}_2)}{2}\Big\} + L.0\Big) \\
&= F_w\Big(\max\{d_r(e_0, e_1), d_r(\tilde{e}_1, \tilde{e}_2)\}\Big) \\
&= F_w(d_r(e_0, e_1)), \qquad (6.
\end{aligned}$$

for other choose of max, we have a contraction. Note that, we use the following facts in above inequalities:

$$d_r(e_0, T_f e_0) \leq d_r(e_0, e_1) + d_r(e_1, \tilde{e}_1) + d_r(\tilde{e}_1, T_f e_0) = d_r(e_0, e_1) + d_r(A_{1f}, A_{2f}) + 0,$$

$$d_r(e_1, T_f e_1) \leq d_r(e_1, \tilde{e}_1) + d_r(\tilde{e}_1, \tilde{e}_2) + d_r(\tilde{e}_2, T_f e_1) = d_r(A_{1f}, A_{2f}) + d_r(\tilde{e}_1, \tilde{e}_2) + 0,$$

$$d_r(e_0, T_f e_1) \leq d_r(e_0, e_1) + d_r(e_1, \tilde{e}_1) + d_r(\tilde{e}_1, \tilde{e}_2) + d_r(\tilde{e}_2, T_f e_1) = d_r(e_0, e_1) + d_r(A_{1f}, A_{2f}) + d_r(\tilde{e}_1, \tilde{e}_2) + 0$$

$$d_r(e_1, T_f e_0) \leq d_r(e_1, \tilde{e}_1) + d_r(\tilde{e}_1, T_f e_0) = d_r(A_{1f}, A_{2f}) + 0.$$

As $\tilde{e}_2 \in Te_1 \subseteq B_0$, $\exists\ e_2 \neq e_1 \in A_0$ such that

$$d_r(e_2, \tilde{e}_2) = d_r(A_{1f}, A_{2f}), \tag{6.26}$$

for otherwise e_1 is a best proximity point. As (A_{1f}, A_{2f}) satisfies the weak P-property. From (6.22) and (6.26), we have

$$0 < d_r(e_1, e_2) \leq d_r(\tilde{e}_1, \tilde{e}_2).$$

By applying F_w, we get

$$F_w(d_r(e_1, e_2)) \leq F_w(d_r(\tilde{e}_1, \tilde{e}_2)). \tag{6.27}$$

Thus from (6.25) and (6.27), we have

$$\tau_f + F_w(d_r(e_1, e_2)) \leq \tau_f + F_w(d_r(\tilde{e}_1, \tilde{e}_2)) \leq F_w(d_r(e_0, e_1)). \tag{6.28}$$

As T is strictly α_w-proximal admissible, As we know that $\alpha_w(e_0, e_1) > 1$, $d_r(e_1, \tilde{e}_1) = d_r(A_{1f}, A_{2f})$ and $d_r(e_2, \tilde{e}_2) = d_r(A_{1f}, A_{2f})$, then $\alpha_w(e_1, e_2) > 1$. Thus we have

$$d_r(e_2, \tilde{e}_2) = d_r(A_{1f}, A_{2f}) \quad \textit{and} \ \alpha_w(e_1, e_2) > 1. \tag{6.29}$$

If $\tilde{e}_2 \in T_f e_2$, then e_2 is a best proximity point of T_f. Let $\tilde{e}_2 \notin T_f e_2$. As $\alpha_w(e_1, e_2) > 1$. There exists $\tilde{e}_3 \in T_f e_2$ such that

$$d_r(\tilde{e}_2, \tilde{e}_3) \leq \alpha_w(e_1, e_2) H_m(T_f e_1, T_f e_2). \tag{6.30}$$

As we know that, F_w is strictly increasing, we have

$$F_w(d_r(\tilde{e}_2, \tilde{e}_3)) \leq F_w(\alpha_w(e_1, e_2) H_m(T_f e_1, T_f e_2)). \tag{6.31}$$

From (6.21), we have

$$\begin{aligned}
\tau_f + F_w(d_r(\tilde{e}_2, \tilde{e}_3)) &\leq \tau_f + F_w(\alpha_w(e_1, e_2) H_m(T_f e_1, T_f e_2)) \\
&\leq F_w\Big(\max\Big\{d_r(e_1, e_2), d_r(e_1, T_f e_1) - d_r(A_{1f}, A_{2f}), d_r(e_2, T_f e_2) - d_r(A_{1f}, A_{2f} \\
&\quad \frac{d_r(e_1, T_f e_2) + d_r(e_2, T_f e_1) - 2d_r(A_{1f}, A_{2f})}{2}\Big\} + L[d_r(e_2, T_f e_1) - d_r(A_{1f}, A_{2f} \\
&\leq F_w\Big(\max\Big\{d_r(e_1, e_2), d_r(e_1, e_2), d_r(\tilde{e}_2, \tilde{e}_3), \frac{d_r(e_1, e_2) + d_r(\tilde{e}_2, \tilde{e}_3)}{2}\Big\} + L.0\Big) \\
&= F_w\Big(\max\{d_r(e_1, e_2), d_r(\tilde{e}_2, \tilde{e}_3)\}\Big) \\
&= F_w(d_r(e_1, e_2)), \qquad (6.
\end{aligned}$$

otherwise we have a contradiction. As $\tilde{e}_3 \in T_f e_2 \subseteq B_0$, $\exists\ e_3 \neq e_2 \in A_0$ such that

$$d_r(e_3, \tilde{e}_3) = d_r(A_{1f}, A_{2f}), \tag{6.33}$$

otherwise e_2 is a best proximity point. As (A_{1f}, A_{2f}) satisfies the weak P-property. From (6.26) and (6.33), we have

$$0 < d_r(e_2, e_3) \leq d_r(\tilde{e}_2, \tilde{e}_3).$$

By applying F_w, we get

$$F_w(d_r(e_2, e_3)) \leq F_w(d_r(\tilde{e}_2, \tilde{e}_3)).$$

Thus, we have

$$\tau_f + F_w(d_r(e_2, e_3)) \leq \tau_f + F_w(d_r(\tilde{e}_2, \tilde{e}_3)) \leq F_w(d_r(e_1, e_2)). \tag{6.34}$$

So we get

$$F_w(d_r(e_2, e_3)) \leq F_w(d_r(\tilde{e}_2, \tilde{e}_3)) \leq F_w(d_r(e_1, e_2)) - \tau_f \leq F_w(d_r(e_0, e_1)) - 2\tau_f.$$

As T_f is strictly α_w-proximal admissible, As we know that $\alpha_w(e_1, e_2) > 1$, $d_r(e_2, \tilde{e}_2) = d_r(A_{1f}, A_{2f})$ and $d_r(e_3, \tilde{e}_3) = d_r(A_{1f}, A_{2f})$, then $\alpha_w(e_2, e_3) > 1$. Proceeding in the same way, we get sequences $\{e_n\}$ in A_0 and $\{\tilde{e}_n\}$ in B_0, where $\tilde{e}_n \in T_f e_{n-1}$ $\forall$ $n \in \mathbb{N}$ such that

$$d_r(e_n, \tilde{e}_n) = d_r(A_{1f}, A_{2f}) \;\; and \; \alpha_w(e_{n-1}, e_n) > 1. \tag{6.35}$$

Furthermore,

$$F_w(d_r(e_n, e_{n+1})) \leq F_w(d_r(\tilde{e}_n, \tilde{e}_{n+1})) \leq F_w(d_r(e_0, e_1)) - n\tau_f \; \forall \; n \in \mathbb{N}. \tag{6.36}$$

Now if we let $n \to \infty$ in (6.36), we get $\lim_{n\to\infty} F_w(d_r(e_n, e_{n+1})) = \lim_{n\to\infty} F_w(d_r(\tilde{e}_n, \tilde{e}_{n+1})) - \infty$. Thus, by property (F_{w_2}), we have $\lim_{n\to\infty} d_r(e_n, e_{n+1}) = 0$. Let $d_{r_n} = d(e_n, e_{n+1})$ $\forall$ $n \in \mathbb{N}$. From (F_{w_3}) $\exists$ $k \in (0, 1)$ such that

$$\lim_{n\to\infty} d_{r_n}^k F_w(d_{r_n}) = 0.$$

From (6.36) we have

$$d_{r_n}^k F(d_{r_n}) - d_{r_n}^k F_w(d_0) \leq -d_{r_n}^k n\tau_f \leq 0 \; \forall \; n \in \mathbb{N}. \tag{6.37}$$

Now if we let $n \to \infty$ in (6.37), we get

$$\lim_{n\to\infty} n d_{r_n}^k = 0.$$

This implies that $\exists$ $n_1 \in \mathbb{N}$ such that $n d_{r_n}^k \leq 1$ $\forall$ $n \geq n_1$. Thus, we have

$$d_{r_n} \leq \frac{1}{n^{1/k}}, \quad \forall \; n \geq n_1. \tag{6.38}$$

To prove that $\{e_n\}$ is a *C-S* in A_{1f}. Consider $m, n \in \mathbb{N}$ with $m > n > n_1$. By making use of triangle inequality and (6.38), we have

$$\begin{aligned} d_r(e_n, e_m) &\leq d_r(e_n, e_{n+1}) + d_r(e_{n+1}, e_{n+2}) + \cdots + d_r(e_{m-1}, e_m) \\ &= \sum_{i=n}^{m-1} d_{r_i} \leq \sum_{i=n}^{\infty} d_{r_i} \leq \sum_{i=n}^{\infty} \frac{1}{i^{1/k}}. \end{aligned}$$

As we know that $\sum_{i=1}^{\infty} \frac{1}{i^{1/k}}$ is convergent series. Thus, $\lim_{n\to\infty} d_r(e_n, e_m) = 0$. Which implies that $\{e_n\}$ is a *C-S* in A_{1f}. Similarly, we see that $\{\tilde{e}_n\}$ is a *C-S* in A_{2f}. As we know that A_{1f} and A_{2f} are closed subsets of a complete *M-S*, $\exists$ e^* in A_{1f} and $\tilde{e}^*$ in A_{2f} such that $e_n \to e^*$ and $\tilde{e}_n \to \tilde{e}^*$ as $n \to \infty$. By the (6.35), we conclude that $d_r(e^*, \tilde{e}^*) = d_r(A_{1f}, A_{2f})$ as $n \to \infty$. By hypothesis (iv), when T_f is continuous, we have $\tilde{e}^* \in T_f e^*$, As we know that $\tilde{e}_n \in T_f e_{n-1}$. Hence $d_r(A_{1f}, A_{2f}) \leq d_r(e^*, T_f e^*) \leq d_r(e^*, \tilde{e}^*) = d_r(A_{1f}, A_{2f})$. Therefore e^* is a best proximity point of the mapping T_f. Now if we consider hypothesis (iv), when $\alpha_w(e_n, e^*) > 1$ $\forall$ $n \in \mathbb{N}$. We claim that $d_r(A_{1f}, A_{2f}) = d_r(e^*, T_f e^*)$. contradictory assume that $d_r(A_{1f}, A_{2f}) \neq d_r(e^*, T_f e^*)$. By making use of triangle inequality, we have

$$\begin{aligned} \tau_f + F_w(d_r(\tilde{e}_{n+1}, T_f e^*)) &\leq \tau_f + F_w(\alpha_w(e_n, e^*) H_m(T_f e_n, T_f e^*)) \\ &\leq F_w\Big(\max \Big\{ d_r(e_n, e^*), d_r(e_n, T_f e_n) - d_r(A_{1f}, A_{2f}), d_r(e^*, T_f e^*) - d_r(A_{1f} \\ &\quad \frac{d_r(e^*, T_f e_n) + d_r(e_n, T_f e^*) - 2d_r(A_{1f}, A_{2f})}{2} \Big\} \\ &\quad + L[d_r(e^*, T_f e_n) - d_r(A_{1f}, A_{2f})] \Big) \\ &\leq F_w\Big(\max \Big\{ d_r(e_n, e^*), d_r(e_n, \tilde{e}_{n+1}) - d_r(A_{1f}, A_{2f}), d_r(e^*, T_f e^*) - d_r(A_{1f} \\ &\quad \frac{d_r(e^*, \tilde{e}_{n+1}) + d_r(e_n, T_f e^*) - 2d_r(A_{1f}, A_{2f})}{2} \Big\} \\ &\quad + L[d_r(e^*, \tilde{e}_{n+1}) - d_r(A_{1f}, A_{2f})] \Big). \end{aligned}$$

Now if we let $n \to \infty$ in the above inequality, we have

$$\tau_f + F_w(d_r(\tilde{e}^*, T_f e^*)) \leq F_w\Big(\max \Big\{ d_r(e^*, T_f e^*) - d_r(A_{1f}, A_{2f}), \frac{d_r(e^*, T_f e^*) - d_r(A_{1f}, A_{2f})}{2} \Big\} \Big) \tag{6.39}$$

As

$$d_r(e^*, T_f e^*) \leq d_r(e^*, \tilde{e}^*) + d_r(\tilde{e}^*, T_f e^*).$$

Thus, we have

$$d_r(e^*, T_f e^*) - d_r(A_{1f}, A_{2f}) \leq d_r(\tilde{e}^*, T_f e^*).$$

By making use of above inequality, (F_{w_1}) and (6.39), we get

$$\begin{aligned}\tau_f + F_w(d_r(e^*, T_f e^*) - d_r(A_{1f}, A_{2f})) &\leq \tau_f + F_w(d_r(\tilde{e}^*, T_f e^*)) \\ &\leq F_w\Big(\max\Big\{d_r(e^*, T_f e^*) - d_r(A_{1f}, A_{2f}), \\ &\quad \frac{d_r(e^*, T_f e^*) - d_r(A_{1f}, A_{2f})}{2}\Big\}\Big).\end{aligned}$$

This implies that

$$d_r(e^*, T_f e^*) - d_r(A_{1f}, A_{2f}) < d_r(e^*, T_f e^*) - d_r(A_{1f}, A_{2f}).$$

This is a contradiction to our assumption. Thus, we conclude that $d_r(e^*, T_f e^*) = d_r(A_{1f}, A_{2f})$. □

Example 6.2.3. Let $X_d = \mathbb{R}\times\mathbb{R}$ be enriched with a metric $d_r((e_1, e_2), (\tilde{e}_1, \tilde{e}_2)) = |e_1 - \tilde{e}_1| + |e_2 - \tilde{e}_2| \ \forall \ e, \tilde{e} \in X_d$. Take $A_{1f} = \{(0, e) : e \in \mathbb{R}\}$ and $A_{2f} = \{(1, e) : e \in \mathbb{R}\}$. Define $T_f : A_{1f} \to CL_f(A_{2f})$ by

$$T_f(0, e) = \begin{cases} \{(1, e^2)\} & \text{if } e < 0 \\ \{(1, 0), (1, 1)\} & \text{if } 0 \leq e \leq 1 \\ \{(1, e - 1), (1, e)\} & \text{if } e > 1 \end{cases}$$

and $\alpha_w : A_{1f} \times A_{1f} \to [0, \infty)$ by

$$\alpha_w((0, e), (0, \tilde{e})) = \begin{cases} 2 & \text{if } e, \tilde{e} \in [0, 1] \\ \frac{1}{2} & \text{if } e, \tilde{e} \in \mathbb{N} - \{1\} \\ 0 & \text{otherwise.} \end{cases}$$

Take $F_w(e) = e + \ln e \ \forall \ e \in (0, \infty)$. Under this F_w, condition (6.1) reduces to

$$\frac{\alpha_w(e, \tilde{e}) H_m(T_f e, T_f y)}{N_f(e, \tilde{e})} \exp(\alpha_w(e, \tilde{e}) H_m(T_f e, T_f y) - N_f(e, \tilde{e})) \leq \exp(-\tau_f) \tag{6.40}$$

$\forall \ e, \tilde{e} \in A_{1f}$ with $\min\{\alpha_w(e, \tilde{e}) H_m(T_f e, T_f y), N_f(e, \tilde{e})\} > 0$. Assume that $b_1 = 1, b_2 = b_3 = b_4 = L = 0$ and $\tau_f = \frac{1}{2}$. Clearly, $\min\{\alpha_w(e, \tilde{e}) H_m(T_f e, T_f y), d_r(e, \tilde{e})\} >$

$0\ \forall\ e, \tilde{e} \in \mathbb{N} - \{1\}$ with $e \neq \tilde{e}$. From (6.40) $\forall\ e, \tilde{e} \in \mathbb{N} - \{1\}$ with $e \neq \tilde{e}$, we have

$$\frac{1}{2}\exp(-\frac{1}{2}|e - \tilde{e}|) < \exp(-\frac{1}{2}).$$

Thus, T_f is a $F_{w_{\alpha_w}}$-proximal contraction of Hardy-Rogers-type with $F_w(e) = e + \ln e$. Note that $A_0 = A_{1f}$, $B_0 = A_{2f}$ and $Te \subseteq B_0\ \forall\ e \in A_0$. Also, the pair (A_{1f}, A_{2f}) satisfies the weak P-property. If $e_0, e_1 \in \{(0, e) : 0 \leq e \leq 1\}$, then $Te_0, Te_1 = \{(1, 0), (1, 1)\}$. Take $\tilde{e}_1 \in Te_0$, $\tilde{e}_2 \in Te_1$ and $u_1, u_2 \in A_{1f}$ such that $d_r(u_1, \tilde{e}_1) = d_r(A_{1f}, A_{2f})$ and $d_r(u_2, \tilde{e}_2) = d_r(A_{1f}, A_{2f})$. Then we have $u_1, u_2 \in \{(0, 0), (0, 1)\}$. Hence T_f is strictly α_w-proximal admissible mapping. For $e_0 = (0, 1) \in A_0$ and $\tilde{e}_1 = (1, 0) \in T_f e_0$ in B_0, we have $e_1 = (0, 0) \in A_0$ such that $d_r(e_1, \tilde{e}_1) = d_r(A_{1f}, A_{2f})$ and $\alpha_w(e_0, e_1) = 2 > 1$. Moreover, for any sequence $\{e_n\} \subseteq A_{1f}$ such that $e_n \to e$ as $n \to \infty$ and $\alpha_w(e_n, e_{n+1}) > 1\ \forall\ n \in \mathbb{N}$, we have $\alpha_w(e_n, e) > 1\ \forall\ n \in \mathbb{N}$. Therefore, by Theorem 6.1.2, T_f has a best proximity point.

Consequences

When we take $X_d = A_{1f} = A_{2f}$, we get the following $F - P$ theorems from our results:

Theorem 6.2.4. Let (X_d, d_r) be a complete $M - S$. Assume $T_f : X_d \to CL_f(X_d)$ is a mapping for which there exist $F_w \in \mathfrak{F}_{\mathfrak{w}}$ and a $\tau_f > 0$ such that

$$\tau_f + F_w(\alpha_w(e, \tilde{e})H_m(T_f e, T_f \tilde{e})) \leq F_w(N_f(e, \tilde{e})),$$

$\forall\ e, \tilde{e} \in X_d$, whenever $\min\{\alpha_w(e, \tilde{e})H_m(T_f e, T_f \tilde{e}), N_f(e, \tilde{e})\} > 0$, where

$$N_f(e, \tilde{e}) = b_1 d_r(e, \tilde{e}) + b_2 d_r(e, T_f e) + b_3 d_r(\tilde{e}, T_f \tilde{e}) + b_4 d_r(e, T_f \tilde{e}) + L d_r(\tilde{e}, T_f e),$$

with $b_1, b_2, b_3, b_4, L \geq 0$ satisfying $b_1 + b_2 + b_3 + 2b_4 = 1$ and $b_3 \neq 1$. Further suppose that the following conditions hold:

(i) T_f is strictly α_w-admissible, So that, if $\alpha_w(e, \tilde{e}) > 1$, then $\alpha_w(a, b) > 1$ $\forall\ a \in T_f e$ and $b \in T_f \tilde{e}$;

(ii) $\exists\ e_0 \in X_d$ and $e_1 \in T_f e_0$ such that $\alpha_w(e_0, e_1) > 1$;

(iii) T_f is continuous, or, for any sequence $\{e_n\} \subseteq X_d$ such that $e_n \to e$ as $n \to \infty$ and $\alpha_w(e_n, e_{n+1}) > 1 \ \forall \ n \in \mathbb{N}$, we have $\alpha_w(e_n, e) > 1 \ \forall \ n \in \mathbb{N}$.

Then T_f has a $F - P$.

Theorem 6.2.5. Let (X_d, d_r) be a complete $M - S$. Assume $T_f : X_d \to CL_f(X_d)$ is a mapping for which there exist a continuous F_w in $\mathfrak{F_w}$ and a $\tau_f > 0$ such that

$$\tau_f + F_w(\alpha_w(e, \tilde{e}) H_m(T_f e, T_f \tilde{e})) \leq F_w(M(e, \tilde{e})),$$

$\forall \ e, \tilde{e} \in X_d$, whenever $\min\{\alpha_w(e, \tilde{e}) H_m(T_f e, T_f \tilde{e}), M(e, \tilde{e})\} > 0$, where

$$M(e, \tilde{e}) \quad = \quad \max\left\{d_r(e, \tilde{e}), d_r(e, T_f e), d_r(\tilde{e}, T_f y), \frac{d_r(e, T_f y) + d_r(\tilde{e}, T_f e)}{2}\right\} + L d_r(\tilde{e}, T_f e)$$

and $L \geq 0$. Further suppose that the following conditions hold:

(i) T_f is strictly α_w-admissible, So that, if $\alpha_w(e, \tilde{e}) > 1$, then $\alpha_w(a, b) > 1$ $\forall \ a \in T_f e$ and $b \in T_f \tilde{e}$;

(ii) $\exists$ an $e_0 \in X_d$ and $e_1 \in T_f e_0$ such that $\alpha_w(e_0, e_1) > 1$;

(iii) T_f is continuous, or, for any sequence $\{e_n\} \subseteq X_d$ such that $e_n \to e$ as $n \to \infty$ and $\alpha_w(e_n, e_{n+1}) > 1 \ \forall \ n \in \mathbb{N}$, we have $\tilde{\alpha_w}(e_n, e) > 1 \ \forall \ n \in \mathbb{N}$.

Then T_f has a $F - P$.

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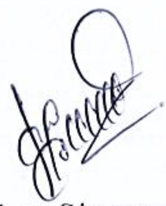

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By

Fahim Ud Din

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This is to certify that the research work presented in this thesis entitled **Fixed Point Theorems for Contractive Type Mappings in Distance Spaces** was conducted by Mr. **Fahim Ud DIn** under the kind supervision of **Dr. Tayyab Kamran**. No part of this thesis has been submitted anywhere else for any other degree. This thesis is submitted to the Department of Mathematics, Quaid-I-Azam University, Islamabad in partial fulfillment of the requirements for the degree of Doctor of Philosophy in field of Mathematics from Department of Mathematics, Quaid-i-Azam University Islamabad, Pakistan.

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