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REPRESENTATIONS OF LORENTZ GROUP AND THREE BRACKETS

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(continued)

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Ghazala Shaheen

ABSTRACT

In this dissertation, we have discussed "The representations of Three and four dimensional Euclidean groups". An attempt has been made to introduce three bracket, namely [X,Y,Z], analogous to the cummutator [A,B]. Some simple algebric properties of this bracket are given in Chapter Two. It has been shown that the invariants of the above groups can be represented by three brackets.

In Chapter One we have included all the definitions and basic results that are needed for the subsquent development of the subject.

The Second Chapter begins with the definition of a "Three dimensional rotation group". In this chapter we have introduced three brackets and have shown that

$$[J_-, J_3, J_+] = \underline{J}^2$$

which is the invariant of the group.

The Third Chapter contains a discussion on four-dimensional Euclidean group and Lorentz Group. It has been

shown that the invariants of these groups are related to three brackets. Also we have calculated the matrix representations of K_-, K_+ and K_3.

CHAPTER ONE

PRELIMINARIES

We use Dirac's ket and bra notations namely $|\rangle$ and $\langle|$ for vectors in Hilbert space throughout this dissertation.

§1.1 LINEAR VECTOR SPACES

<u>Definition:</u> If a set s of all elements $|a\rangle$, $|b\rangle$ $\cdots$ satisfies the following properties:

(A) (i) If $|a\rangle$ and $|b\rangle$ ε S. Then

$$(|a\rangle + |b\rangle) \; \varepsilon \; S.$$

(ii) If $|a\rangle$ and $|b\rangle$ ε S. Then

$$(|a\rangle + |b\rangle) = (|b\rangle + |a\rangle)$$

(cummutative law of addition).

(iii) $(|a\rangle + |b\rangle) + |c\rangle = |a\rangle + (|b\rangle + |c\rangle)$

— $|a\rangle$, $|b\rangle$, $|c\rangle$ ε S.

(associative law of addition).

(iv) There exists à null element $|o\rangle$ ε S

s + — $|a\rangle$ ε S, we have

$$|a\rangle + |o\rangle = |o\rangle + |a\rangle = |a\rangle.$$

(v) For every $|a\rangle \in S$, there exists an element $|a'\rangle$ such that $|a\rangle + |a'\rangle = |a'\rangle + |a\rangle$

$$= |o\rangle$$

i.e. $|a'\rangle = -|a\rangle$

(B) (i) $|a\rangle = |a\rangle$

(ii) For any $\alpha, \beta \in C$ (set of complex numbers)

$$(\alpha \cdot \beta)|a\rangle = \alpha(\beta|a\rangle)$$

(iii) If $|a\rangle \in S$ and α is a complex number then $\alpha|a\rangle \in S$.

(iv) $(\alpha + \beta)|a\rangle = \alpha|a\rangle + \beta|a\rangle$.

(Distributive law with respect to addition of complex numbers).

(v) $\alpha(|a\rangle + |b\rangle) = \alpha|a\rangle + \alpha|b\rangle$.

(Distributive law with respect to the addition of $|\rangle$).

A set of $|\rangle$ elements that has the properties (A) and (B) is called a linear vector space. The elements of the set S is called vectors and the complex numbers $\alpha, \beta, \gamma \cdots$ are called the scalars.
If the scalar of a vector space are complex numbers then the vector space is called complex vector space and if the scalars are real then the vector space is real. The scalar product of two vectors $|a\rangle$ and $|b\rangle$ is given by $\langle a|b\rangle$

and $\langle a|b\rangle = \overline{\langle b|a\rangle}$.

Inner Product Space: Let S be a (complex) linear space and let $\langle \,|\, \rangle : S \times S \to C$ be a map from the cartesian product set $S \times S$ to the set of complex numbers which has the following properties:

(i) $\langle x|x\rangle$ is real and non-negative where $\langle x|x\rangle$ is the square of the length of $|x\rangle \in S$.

(ii) $\langle x|x\rangle = 0$ iff $|x\rangle = 0$.

(iii) $\langle x|y\rangle = \langle y|x\rangle$ for all $x, y \in S$.

(iv) $\{\langle x| \cdot (|y\rangle + |z\rangle)\rangle\} = \langle x|y\rangle + \langle x|z\rangle$.

(v) $\langle \alpha x|y\rangle = \alpha\langle x|y\rangle$
and $\langle x|\alpha y\rangle = \bar{\alpha}\langle x|y\rangle$.

Then $(S, \langle \,|\, \rangle)$ is a complex inner product space.

Completness: A metric space (X, ϕ) is said to be complete if and only if every chanchy sequence converges to a point of the space.

Hilbert Space: An inner product space which is complete when considered as metric space is called a Hilbert space.

Note: We shall call the vectors $|\rangle$ of the Hilbert space S by Ket vectors.

§1.2 BRA VECTORS OR DUAL OF KET VECTORS

Whenever we have a set of vectors in any mathematical theory, we can set up a second set of vectors called the dual vectors.

The procedure of obtaining bra vectors is as follows:

Let S(c) be the vector space. Then the linear map $f:S(c) \to C$ is called a linear functional (operations).

Let $S^*(c) = \{f:S(c) \to C | f \text{ is linear}\}$.

We define addition and scalar multiplication in $S^*(c)$ by the following:

(1) $(f_1 + f_2)(s) = f_1(s) + f_2(s)$ — $s \in S(c)$.

and $f_1, f_2 \in S^*(c)$.

(2) $(\alpha f)(s) = \alpha\big(f(s)\big)$ — $\alpha \in C$.

Then under (1) and (2), $S^*(c)$ becomes a vector space over C. The vector-space $S^*(c)$ is called a dual space of S(c) and the vectors of $S^*(c)$ are the dual vectors. So the dual of ket vectors are known as bra-vectors, and they are denoted by $<|$.

§1.3 SCALAR PRODUCT

The scalar product of a bra-vector $<b|$ and a ket vector $|a>$ is written as $<b|a>$. A scalar product $<b|a>$ appears a complex number and an incomplete bracket expression denote a vector of the bra or ket according to whether it contains a first or second part of bracket.

The properties of the scalar product of ket and bra vectors will be, by definition the following:

(a) $\langle b|a\rangle = \overline{\langle a|b\rangle}$.

(b) If $|d\rangle = \alpha|a\rangle + \beta|b\rangle$

then $\langle c|d\rangle = \alpha\langle c|a\rangle + \beta\langle c|b\rangle$

and $\langle d|c\rangle = \bar{\alpha}\langle a|c\rangle + \bar{\beta}\langle b|c\rangle$.

(c) $\langle a|a\rangle \geq 0$, the equality sign appears only when $|a\rangle = 0$.

Definition: Two vectors $\langle a|$ and $|b\rangle$ are said to be orghoganol if their scalar product vanishes

$$\langle a|b\rangle = 0.$$

§1.4 LINEAR OPERATORS ON HILBERT SPACE

Let $\theta:H \rightarrow H$ be a map and H is a Hilbert space where vectors are denoted by Ket vectors or $|\rangle$. Then θ is said to be a linear operator iff

$$\theta(\alpha|a\rangle + \beta|b\rangle) = \alpha(\theta|a\rangle) + \beta(\theta|b\rangle)$$

for each α, β and $|a\rangle, |b\rangle \in H$.

The Adjoint of an Operator

Let T be a bounded linear operator on a Hilbert space H. Then we define

$\bar{T}:H \rightarrow H$ such that

$$\langle x|T|y\rangle = \overline{\langle y|\bar{T}|x\rangle} \quad (1)$$

where $x, y \in H$.

We call this $\bar{T}$ the adjoint of T and take the equation (1) as its definition. We can easily show that $\bar{T}$ is not only a map on H to H but actually it is a linear map.

Definition: If an operator is equal to its adjoint, it is called self adjoint operator, we also call it real linear operator.

Adjoint of the Product of Operators α, β.

Now we prove that the adjoining of the product the operators α, β is the product of the adjoints

If $\langle a| = \langle p|$ and $|b\rangle = \beta|Q\rangle$

then $\langle \bar{a}| = \bar{\alpha}|p\rangle$ and $|\bar{b}\rangle = \langle Q|\bar{\beta}$

i.e. $\overline{\langle p|\alpha\beta|Q\rangle} = \overline{\langle a|b\rangle} = \langle b|a\rangle$

$= \langle Q|\bar{\beta}\bar{\alpha}|p\rangle = \langle Q|\overline{\alpha\beta}|p\rangle$

Similarly $\therefore$ $\overline{\alpha\beta} = \bar{\beta}\ \bar{\alpha}$

$\overline{|b\rangle\langle a|} = |a\rangle\langle b|$.

This is an operator which while acting on a ket gives another ket.

Theorem 1: If A and B are operators on H, then

(a) $(\overline{\alpha A}) = \bar{\alpha}\bar{A}$.

(b) $(\overline{A + B}) = \bar{A} + \bar{B}$.

(c) $(\overline{AB}) = \bar{B}\bar{A}$.

<u>Theorem 2:</u> If T in a linear operator on H, then

$$(\bar{\bar{T}}) = T.$$

<u>Theorem 3:</u> Let T be an invertible bounded linear operator on H. Then $\bar{T}$ is also invertible and $(\bar{T})^{-1} = (\overline{T^{-1}})$.

The proof of the theorems are trivial.

<u>Unitary Operators</u>

<u>Definition:</u> An operator A in a finite dimensional Euclidean space R is said to be unitary if it preserves the scalar product i.e.

$$\langle Ax|Ay\rangle = \langle x|y\rangle \quad \text{for all } x,y \in R.$$

$$A\bar{A} = 1 \text{ or } \bar{A} = A^{-1}.$$

§1.5 <u>REPRESENTATIONS</u>

<u>Representations of Vectors</u>

We may decompose a vector with respect to some basis vectors $|a_i\rangle$ i.e.

$$|a\rangle = \sum_{i=1}^{n} \alpha_i |a_i\rangle \qquad (1)$$

Then we may regard the set of n numbers α_i's as representing the vector $|a\rangle$ with respect to the basis $|a_i\rangle$. The decomposition (1) is unique with respect to the given basis.

Addition of two vectors is represented by the addition of their components, e.g.

$$|a\rangle + |b\rangle = \sum_{i=1}^{n} \alpha_i |a_i\rangle + \sum_{i=1}^{n} \beta_i |b_i\rangle$$

$$= \sum_{i=1}^{n} (\alpha_i + \beta_i) |a_i\rangle$$

and similarly the multiplication of a vector by a number is represented by the multiplication of its components by this number, e.g.

$$x|a\rangle = x \sum_{i=1}^{n} \alpha_i |a_i\rangle = \sum_{i=1}^{n} (x\alpha_i) |a_i\rangle$$

Here we have first fixed the basis. The question arises what happens to the set of numbers that represents the vectors when one changes the basis? This we will discuss later.

The Representation of a Linear Operator in an n-dimensional space

Let $|a_i\rangle$ $(i = 1,2,\cdots,n)$ denote the basis vectors of S_n. Let us consider a linear operator F. Then $F|a_i\rangle$ is also a vector of S_n and therefore it may be written as

$$F|a_i\rangle = \sum_{j=1}^{n} F_i^j |a_j\rangle.$$

The components of $F|a_i\rangle$ have two indicies one, the superscript, identifies the components of the vector that is being decomposed. The other is subscript, identifies the vector that is decomposed. Thus F_i^j is the jth component of the ith vector $F|a_i\rangle$.

Now we consider the case of multiplication of F of an arbitrary vector $|a\rangle$ i.e. not necessarily a basis vector. Let

$$|b\rangle = F|a\rangle$$

where $$|a\rangle = \sum_{i=1}^{n} \alpha^i |a_i\rangle$$

and $$|b\rangle = \sum_{i=1}^{n} \beta^i |a_i\rangle$$

Then $$\sum_{i=1}^{n} \beta^i |a_i\rangle = \sum_{i=1}^{n} F(\alpha^i |a_i\rangle)$$

$$= \sum_{i=1}^{n} \sum_{j=1}^{n} F_i^j \alpha^i |a_j\rangle.$$

or $$\beta^j = \sum_{i=1}^{n} F_i^j \alpha^i$$

by using Einstein convention, we may write

$$\beta^j = F_i^j \alpha^i.$$

The set of numbers F_i^j represents the operator F.

The numbers F_i^j can be arranged in a table

$$\begin{pmatrix} F_1^1 & F_2^1 & F_3^1 & \cdots\cdots & F_n^1 \\ F_1^2 & F_2^2 & F_3^2 & \cdots\cdots & F_n^2 \\ \vdots & \vdots & \vdots & & \vdots \\ F_1^n & F_2^n & F_3^n & \cdots\cdots & F_n^n \end{pmatrix}$$

§1.6 CHANGE OF BASIS IN AN n-DIMENSIONAL SPACE

In the previous section, we have examined the representations of vectors and linear operators with respect to a fixed basis. Now we consider the case when the basis are changed. Let A be a linear operator represented in the basis $|a_i\rangle$ by the matrix A with det (A) $\neq 0$. By considering a set of vector $|a_i'\rangle$

$$|a_i'\rangle = A|a_i\rangle = \sum_{j=1}^{n} A_i^j |a_j\rangle \qquad (1.6.1)$$

$$(i = 1,2,\cdots,n)$$

As det (A) $\neq 0$ $\therefore$ A^{-1} exists

$$\therefore \quad (A_k^i)(A^{-1})_j^k = \delta_j^i$$

Multiply both sides of equation (1.6.1) by $(A^{-1})_k^i$

$$\sum_{i=1}^{n} (A^{-1})_k^i |a_i'\rangle = |a_k\rangle$$

or

$$|a_k\rangle = \sum_{i=1}^{n} (A^{-1})_k^i \; |a_i'\rangle . \qquad (1.6.2)$$

Now we show that $|a_i'\rangle$ are linearly independent. If not, then

$$\sum_{i=1}^{n} \alpha_i |a_i'\rangle = 0 \text{ where all } \alpha_i \neq 0$$

or

$$\sum_{j=1}^{n} \alpha_i \; A_i^j |a_j\rangle = 0$$

linear independence of $|a_j\rangle$ implies that

$$A_i^j \; \alpha_i = 0 \quad (j = 1,2,\cdots,n)$$

As det (A) $\neq 0$ $\therefore$ $\alpha_i = 0$.

Therefore $|a_i'>$ are linearly independent.

Now we have two basis. Question arises, What is the relation between the representations of a vector or an operator in new and old basis?. We consider a vector $|b>$. Then :

$$|b> = \sum_{i=1}^{n} B_i |a_i> \qquad (1.6.3)$$

and

$$|b> = \sum_{i=1}^{n} B_i' |a_i'> \qquad (1.6.4)$$

Using equation (1.6.2), equation (1.6.3) may be written as

$$|b> = \sum B_i \, (A^{-1})_i^j |a_j'> \qquad (1.6.5)$$

by comparing equations (1.6.4) and (1.6.5)

$$B_i' = B_i \, (A^{-1})_i^j$$

or in matrix form $B' = BA^{-1}$.

Now we consider the case of change of basis in linear operator. Let F be any linear operator which is represented by the matrix F in the old basis. Then from equation (1.6.1)

$$F|a_i> = F\left(\sum_{j=1}^{n} A_i^j \; |a_j> \right)$$

$$= \sum_{k=1}^{n} A_i^j \; F_j^k \; |a_k>$$

$$= \sum_{m=1}^{n} (A^{-1})_k^m \; A_i^j \; F_j^k \; |a_m'> \qquad (1.6.6)$$

In the new basis F is represented by the matrix F′, defined by

$$F|a'_i> = \sum F'^m_i \; |a'_m> \qquad (1.6.7)$$

by comparing (1.6.6) and (1.6.7)

$$F'^m_i = (A^{-1})^m_k \; A^j_i \; F^k_j$$

or in the matrix form

$$F' = A^{-1} \; F \; A.$$

CHAPTER TWO

REPRESENTATIONS OF THREE-DIMENSIONAL ROTATION GROUP

§2.1 ROTATION IN THREE-DIMENSIONAL SPACE

A roation in three-dimensional Euclidean space is given by

$$x'_i = \sum_{j=1}^{3} g_{ij}\, x_j \qquad (2.1.1)$$

where (x_1, x_2, x_3) and (x'_1, x'_2, x'_3) are the co-ordinates of the same point with respect to two orthogonal co-ordinate system (Ox_1, Ox_2, Ox_3) and (Ox'_1, Ox'_2, Ox'_3) having the same vertex O. So that

$$x_1^2 + x_2^2 + x_3^2 = x_1'^2 + x_2'^2 + x_3'^2 \qquad (2.1.2)$$

Equation (2.1.1) may be written as

$$X' = gX \qquad (2.1.3)$$

where g is a 3×3 rotation matrix and it is assumed that $\det(g) \neq 0$. If g_{ij} be the elements of the infintesimal rotation matrix, g may be written as

$$g_{ij} = \delta_{ij} + \varepsilon_{ij} \qquad (2.1.4)$$

$$(i, j = 1, 2, 3)$$

where $$\delta_{ij} = \begin{matrix} 1 & i = j \\ 0 & i \neq j \end{matrix}$$

So δ_{ij} are the elements of the unit matrix. The rotation matrix g form a group. Their composition is given by the law of the product of two matrices.

Substituting the values of X'^T and X' from (2.1.3) in $X'^T X' = X^T X$, we get

$$X^T g^T gX = X^T X$$

which is true for all X. Therefore we have

$$g^T g = I \qquad (2.1.5)$$

where I is 3 × 3 unit matrix, which becomes the identity element of the group g^T defines the inverse of g, viz

$$g^T = g^{-1} \qquad (2.1.6)$$

From (2.1.5), we have

$$g_{ij} \cdot g_{ji} = \delta_{ij} \qquad (2.1.7)$$

Using (2.1.4) we obtain

$$(\delta_{ij} + \varepsilon_{ij})(\delta_{ji} + \varepsilon_{ji}) = \delta_{ij}$$

By neglecting the squared term on left hand side, we have

$$\varepsilon_{ji} + \varepsilon_{ij} = 0$$

so that $$\varepsilon_{ij} = -\varepsilon_{ji} \qquad (2.1.8)$$

From equation (2.1.6), (2.1.7) and (2.1.8), we get

$$(g^{-1})_{ij} = \delta_{ij} - \varepsilon_{ij} \qquad (2.1.9)$$

Definition: Let g,h,k are all possible rotations of three dimensional space about a fixed point. Let G denote the aggregate of all such rotations. We shall define the product hk of two rotations h and k to be the rotation obtained by successive applications, first of k and then of rotation h. It can be easily proved that the set of rotations G is a group under the definition of the product of rotations. The unit element of this group will be the rotation through zero angle, and the inverse of a given rotation g is the rotation that returns the space into the initial position. Then G is called the three dimensional rotation group

§2.2 BASIC INFINTESIMAL ROTATION OPERATORS REPRESENTING THE GROUP OF ELEMENTS g

We consider the function $f(x) = f(x_1, x_2, x_3)$. If we substitute for the x_k in $f(x)$ their values in terms of x_i' as obtainable from (2.1.3), we obtain a new function $f_1(x')$ under $x \to x' = gx$. The transformation operator which carries $f(x)$ to another function $f(x')$ denoted by T_g. Thus we have associated a transformation operator T_g with each rotation g. In view of the definition of T_g, we write

$$T_g\, f(x) = f(x')$$

$$= f(g^{-1} x) \quad \text{(under the rotation } x = gx'\text{)}$$

Dropping dashes, we have in the transformed co-ordinate system

$$T_g \, f(x) = f(g^{-1} x) \qquad (2.2.2)$$

T_g form a group of linear operator representing the group of rotation matrices g. To prove this we will show that the product of the rotation g_1 and g_2 correspond to the product of the transformation operator T_{g_1} and T_{g_2}.

As a result of first rotation, we get

$$T_{g_1} \, f(x) = f(g_1^{-1} x)$$

and as a result of the second rotation g_2 we get

$$T_{g_2} \, f(x) = f(g_2^{-1} x)$$

$$\therefore \quad T_{g_1} T_{g_2} \, f(x) = f(g_2^{-1} g_1^{-1} x)$$

$$= f\left((g_1 g_2)^{-1} x\right)$$

$$= T_{g_1 g_2} \, f(x) .$$

Also if I is the identity element of the rotation group, then

$$T_I \, f(x) = f(x) .$$

Now we make use of (2.2.2) in order to obtain T explicitly with the help of (2.1.9). We write (2.2.2) in the form

$$T_g \, f(x_i) = f\left((\delta_{ij} - \varepsilon_{ij}) x_j\right)$$

Expanding the right hand side of the equation by Taylor's theorem. We have

$$T_g\, f(x_i) = f(x_i)\ \delta_{ij} - \varepsilon_{ij}\ x_j \frac{\partial}{\partial x_j} f(x_j)\ \delta_{ij} + \frac{(\varepsilon_{ij}\ x_j)^2}{2} \cdot \frac{\partial^2}{\partial x_i^2} \left(f(x_j)\delta_{ij}\right) + \cdots$$

Neglecting the terms involving ε_{ij}^2, we have

$$T_g\ f(x_i) = \delta_{ij}\ f(x_i) - \varepsilon_{ij}\ x_j \frac{\partial}{\partial x_i} f(x_i)$$

$$= \left(\delta_{ij} - \varepsilon_{ij}\ x_j \frac{\partial}{\partial x_i}\right) f(x_i)$$

By putting $\quad A_{ij} = -\left(x_j \cdot \frac{\partial}{\partial x_i} - x_i \frac{\partial}{\partial x_j}\right)$

we get

$$T_g = 1 + \frac{1}{2}\ \varepsilon_{ij}\ A_{ij} \tag{2.2.3}$$

Then A_{ij} are called the generators of the group representations. To convert the operator A_{ij} into hermitian operator we define

$$J_i = i\ \varepsilon_{ijk}\ A_{jk} \ . \qquad (i,j,k = 1,2,3) \tag{2.2.4}$$

where ε_{ijk} is antisymmetric tensor in i,j,k so that

$$\varepsilon_{ijk} = \begin{cases} 0 \text{ if any two indices are equal} \\ \pm 1 \text{ when all different depending on even or odd permutation of } 1,2,3. \end{cases}$$

Thus from (2.2.4)

$$J_1 = -i\left(x_2 \cdot \frac{\partial}{\partial x_3} - x_3 \cdot \frac{\partial}{\partial x_2}\right) \tag{2.2.5}$$

$$J_2 = -i\left(x_3 \cdot \frac{\partial}{\partial x_1} - x_1 \cdot \frac{\partial}{\partial x_3}\right) \tag{2.2.6}$$

$$J_3 = -i\left(x_1\cdot\frac{\partial}{\partial x_2} - x_2\cdot\frac{\partial}{\partial x_1}\right) \quad (2.2.7)$$

J_1, J_2 and J_3 can be identified as the components of the angular momentum operator of a particle.

§2.3 COMMUTATION RELATIONS

Using the above expression for J_1, J_2 and J_3, we can prove the following commutation relations:

$$[J_1,J_2] = i\ J_3 \quad (2.3.1)$$

$$[J_2,J_3] = i\ J_1 \quad (2.3.2)$$

$$[J_3,J_1] = i\ J_2 \quad (2.3.3)$$

where $[J_1,J_2] = J_1J_2 - J_2J_1$ etc. Or in the compact form

$$[J_i,J_j] = i\ \varepsilon_{ijk}\ J_k \quad (2.3.4)$$

Again we define

$$J_\pm = J_1 \pm i\ J_2 \quad (2.3.5)$$

RESULTS

(I) $[J_+,J_-] = 2J_3$ (2.3.6)

PROOF $[J_+,J_-] = [J_1 + iJ_2,\ J_1 - iJ_2]$

$= [J_1,J_1] - i[J_1,J_2] + i[J_2,J_1] + [J_2,J_2]$

$= i[J_2,J_1] - i[J_1,J_2]$

(as $[J_i,J_i] = 0$)

Using (2.3.1)

$$= i(-iJ_3) - i(iJ_3)$$

$$= J_3 + J_3 = 2J_3.$$

(II) $$[J_-, J_3] = J_- \qquad (2.3.7)$$

$$= [J_1 - iJ_2, J_3] = [J_1, J_3] - i[J_2, J_3]$$

$$= -iJ_2 - i(iJ_1) = -iJ_2 + J_1 = J_-.$$

(III) $$[J_3, J_+] = J_+ \qquad (2.3.8)$$

$$[J_3, J_+] = [J_3, J_1 + iJ_2]$$

$$= [J_3, J_1] + i[J_3, J_2]$$

$$= iJ_2 + i(-J_1) = J_1 + iJ_2 = J_+.$$

§2.4

In this section we determine the matrices J_3, J_+, J_- and J^2 which satisfy the above cummutation relations. First we prove the following lemma.

<u>Lemma</u>: Let $|m>$ be an eigen-vector of J_3 with corresponding eigen-value m, i.e.

$$J_3|m> = m|m>.$$

Then

(i) The vector $J_+|m>$ is either the null vector or an eigen-vector of J_3 with corresponding eigen-value $(m+1)$.

(ii) The vector $J_-|m\rangle$ is either the null vector or an eigen-vector of J_3 with corresponding eigen-value $(m-1)$.

PROOF: $$J_3(J_+|m\rangle) = J_3J_+|m\rangle$$

$$= (J_+J_3 + J_+)|m\rangle$$

We have used

$$[J_3, J_+] = J_+$$

Thus $$J_3(J_+|m\rangle) = J_+J_3|m\rangle + J_+|m\rangle$$

$$= (m+1)\ J_+|m\rangle$$

Similarly from $[J_-, J_3] = J_-$

We have

$$J_3(J_-|m\rangle) = (m-1)\ J_-|m\rangle$$

So we conclude the following two results

$$J_+|m\rangle = |m+1\rangle \qquad (2.4.1)$$

$$J_-|m\rangle = |m-1\rangle \qquad (2.4.2)$$

Because J_3 is hermitian, all its eigen-values are real. Since we are dealing with a space of finite dimension, eigen values of J_3 are finite in number. Hence by operating on an eigen-vector of J_3 with J_+ or J_- a sufficient number of times we can arrive at maximum or minimum value.

Note: $J^2 = J_1^2 + J_2^2 + J_3^2$ commutes with every J and therefore the invariant of the group.

§2.5 MATRIX ELEMENTS OF J_3

We assume a representation in which J_3 is diagonal i.e. if the base vectors are normalized the matrix of J_3 are

$$\langle m' | J_3 | m \rangle = \delta_{mm'}$$

as

$$J_3 = \frac{1}{2} (J_+ J_- - J_- J_+)$$

Taking trace of both sides, we get

$$\text{Trace } J_3 = 0$$

Therefore the sum of the eigen-values of J_3 is zero, and we can write

$$\sum_m \langle m | J_3 | m \rangle = 0 \sum_m m$$

We have assumed that finite set of vectors $|m\rangle$ have corresponding finite set of eigen-values running from minimum upto maximum m_{max}. Also $J_- |m_{max}\rangle$ generates as eigen-vector of J_3 with eigen-values $m_{max} - 1$; Thus the successive eigen-values of J_3 differ by unity. The set of eigen-values is therefore

$$m_{min}, m_{min} + 1, \cdots, m_{max} - 1, m_{max}$$

If there are S members of the set then

$$m_{max} = m_{min} + S - 1$$

If we rewrite the series of eigen-values as

$$m_{max}, m_{max} - 1, \cdots, m_{max} - (S - 1)$$

Since the sum is zero, we have

$$S \cdot m_{max} - \frac{1}{2} S(S - 1) = 0.$$

Discounting the trivial solution S = 0, we get

$$m_{max} = \frac{1}{2} (S - 1)$$

As S is an integer, m_{max} and therefore all the eigen-values of J_3 are either integers or half integers. Writing

$m_{max} = j$, we have $s = 2j + 1$ and $m = j, j-1, \cdots,$

This gives the number of independent eigen-vectors of J_3. When the maximum values is j; the corresponding set of eigen values is

$$j, j-1, j-2, \cdots, -j+2, -j+1, -j$$

Note: This is convenient place to make a small but necessary change in the notation. Here after we will write the eigen-vectors of J_3 as $|j,m\rangle$ instead of $|m\rangle$ to indicate it corresponds to the eigen-value m of the set whose maximum is j.

§2.6 MATRIX ELEMENTS OF J^2

As
$$J^2 = J_1^2 + J_2^2 + J_3^2$$
$$= \frac{1}{2} (J_+J_- + J_-J_+) + J_3^2$$

Matrix element will be given by

$$<j,m'|J^2|j,m> = <j,m'|\frac{1}{2}(J_+J_- + J_-J_+) + J_3^2|j,m>$$

$$= \frac{1}{2} <j,m'|J_+J_-|j,m> + \frac{1}{2} <j,m'|J_-J_+|j,m>$$

$$+ <j,m'|J_3^2|j,m>$$

$$= \frac{1}{2} <j,m'|J_+|j,m''><j,m''|J_-|j,m>$$

$$+ \frac{1}{2}<j,m'|J_-|j,m'''><j,m'''|J_+|j,m> + m^2$$

$$= \frac{1}{2} <j,m|J_+|j,m-1><j,m-1|J_-|j,m>$$

$$+ \frac{1}{2} <j,m|J_-|j,m+1><j,m+1|J_+|j,m> + m^2$$

We write

$$<jm|J_+|j,m-1><j,m-1|J_-|j,m> = \phi(m) \qquad (2.6.1)$$

so that

$$<j,m|J_-|j,m+1><j,m+1|J_+|j,m> = \phi(m+1) \qquad (2.6.2)$$

Therefore we have

$$<j,m|J^2|j,m> = \frac{1}{2}\phi(m) + \frac{1}{2}\phi(m+1) + m^2$$

Also using $J_+J_- - J_-J_+ = 2J_3$

$$\therefore \quad <j,m|J_+J_-|j,m> - <j,m|J_-J_+|j,m> = 2m$$

or $$\phi(m) - \phi(m+1) = 2m \qquad (2.6.3)$$

or $$\phi(m+1) - \phi(m) = -2m \qquad (2.6.4)$$

$$\therefore \quad \Delta\phi(m) = -2m$$

$$\phi(m) = \Delta^{-1}(-2m) + A = A - m(m-1) \qquad (2.6.5)$$

It follows from equation (2.6.1) that

$$\phi(j+1) = 0$$

Therefore by putting m = (j + 1) in Equation (2.6.4)

$$A = j(j+1)$$

$$\therefore \qquad \phi(m) = j(j+1) - m(m+1)$$

substituting the values of $\phi(m)$ and $\phi(m+1)$, we get

$$\langle j,m|J^2|j,m\rangle = j(j+1)$$

§2.7 MATRIX ELEMENTS OF J_+, J_-

Let $|j,m\pm 1\rangle = \alpha_\pm (J_\pm |j,m\rangle)$

where $\alpha_\pm$ are to be determined. Then

$$\overline{|j,m\mp 1\rangle}\,|j,m\pm 1\rangle = \overline{\alpha_\pm J_\pm |j,m\rangle}\;\alpha_\pm J_\pm |j,m\rangle \qquad (2.7.1)$$

$$= \bar{\alpha}_\pm \alpha_\pm \overline{|j,m\rangle}\, \bar{J}_\pm J_\pm |j,m\rangle$$

$$= \bar{\alpha}_\pm \alpha_\pm \langle j,m|(J_1 \mp iJ_2)(J_1 \pm iJ_2)|j,m\rangle$$

$$= \bar{\alpha}_\pm \alpha_\pm \langle j,m|J^2 - J_3^2 \mp J_3|j,m\rangle$$

$$= \bar{\alpha}_\pm \alpha_\pm \left(j(j+1) - m^2 \mp m\right)$$

$$= \bar{\alpha}_\pm \alpha_\pm \left(\sqrt{(j \mp m)(j \pm m+1)}\right)^2$$

The right hand side of the equation (2.7.1) should be equal to 1. Therefore

$$\alpha_{\pm} = \frac{1}{\sqrt{(j \mp m)(j \pm m+1)}}$$

Hence

$$J_{\pm}|j,m\rangle = \sqrt{(j \mp m)(j \pm m+1)}\ |j,m \pm 1\rangle \qquad (2.7.2)$$

We may now immediately obtain the matrix components of J_1 and J_2. The general element of the matrix of J_1 is

$$\langle j',m'|J_1|j,m\rangle$$

$$\frac{1}{2}\langle j',m'|J_+|j,m\rangle + \frac{1}{2}\langle j',m'|J_-|j,m\rangle \qquad (2.7.3)$$

is seen from (2.7.2) to be zero unless

$$j' = j \text{ and } m' = m \pm 1$$

Equation (2.7.3) reduces to

$$\langle j,m+1|J_1|j,m\rangle = \frac{1}{2}\langle j,m+1|J_+|j,m\rangle$$

$$= \frac{1}{2}\langle j,m+1|\sqrt{(j-m)(j+m+1)}\,|j,m+1\rangle$$

$$= \frac{1}{2}\sqrt{(j-m)(j+m+1)} \qquad (2.7.4)$$

or

$$\langle j,m-1|J_1|j,m\rangle = \frac{1}{2}\langle j,m-1|J_-|j,m\rangle$$

$$= \frac{1}{2}\sqrt{(j+m)(j-m+1)} \qquad (2.7.5)$$

Similarly the non-vanishing components of J_2 are found to have the values

$$\langle j,m+1|J_2|j,m\rangle = -\frac{1}{2}i\sqrt{(j-m)(j+m+1)} \qquad (2.7.6)$$

and

$$\langle j,m-1|J_2|j,m\rangle = \frac{1}{2}i\sqrt{(j+m)(j-m+1)} \qquad (2.7.7)$$

§2.8 THE THREE BRACKETS

In this section we introduce the idea of 3-bracket cummutator analagous to 2-bracket cummutator which is known to be $[X_\alpha, X_\beta] = \varepsilon_{\alpha\beta} X_\alpha X_\beta \quad (\alpha, \beta = 1,2)$

$$= X_\alpha X_\beta - X_\beta X_\alpha$$

equivalently

$$[X_\alpha, X_\beta] = \begin{vmatrix} X_\alpha & X_\beta \\ X_\alpha & X_\beta \end{vmatrix}$$

where $\begin{vmatrix} X_\alpha & X_\beta \\ X_\alpha & X_\beta \end{vmatrix}$ is an ordinary determinant of order 2, but it does not obey the property that if two rows or two columns of a determinant are identical the value of the determinant is zero. This property holds only when X_α commutes with X_β, for $\alpha, \beta = 1,2$.

Now we define

$$[X_\alpha, X_\beta, X_\gamma] = \varepsilon_{\alpha\beta\gamma} X_\alpha X_\beta X_\gamma \qquad (\alpha, \beta, \gamma = 1,2,3)$$

or $$[X,Y,Z] = X[Y,Z] + Y[Z,X] + Z[X,Y]$$

or $$[X,Y,Z] = [X,Y]Z + [Z,X]Y + [Y,Z]X$$

or $$[X,Y,Z] = \begin{vmatrix} X & Y & Z \\ X & Y & Z \\ X & Y & Z \end{vmatrix}$$

Properties of $[X,Y,Z]$

(1) When one of X,Y,Z, say Z, is constant, then

$$[X,Y,Z] = [X,Y]Z.$$

(2) Interchange of any two X,Y,Z changes the sign of the three bracket because the interchange of two rows or of two columns changes the sign of the determinant, i.e.

$$[X,Y,Z] = -[Y,X,Z].$$

(3) $[\alpha X,Y,Z] = \alpha[X,Y,Z]$, i.e.

$$\begin{vmatrix} \alpha X & Y & Z \\ \alpha X & Y & Z \\ \alpha X & Y & Z \end{vmatrix} = \alpha \begin{vmatrix} X & Y & Z \\ X & Y & Z \\ X & Y & Z \end{vmatrix}.$$

(4) $[X_1 + X_2, Y, Z] = [X_1, Y, Z] + [X_2, Y, Z]$

by using the property of the determinant that

$$\begin{vmatrix} a_1+\alpha_1 & a_2 & a_3 \\ b_1+\beta_1 & b_2 & b_3 \\ c_1+\gamma_1 & c_2 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} + \begin{vmatrix} \alpha_1 & a_2 & a_3 \\ \beta_1 & b_2 & c_3 \\ \gamma_1 & c_2 & c_3 \end{vmatrix}$$

(5) $[X, \alpha X + Y, Z] = [X,Y,Z]$

because the value of a determinant is unaltered if to each element of a row (or column) is added a constant multiple of the corresponding element of another row (or column).

Now we apply this three bracket to angular momentum operator J_-, J_3 and J_+ and show that $[J_-, J_3, J_+]$ is an invariant of the group.

By definition

$$[J_-, J_3, J_+] = [J_-, J_3]J_+ + [J_3, J_+]J_- + [J_+, J_-]J_3$$

Using (2.3.6), (2.3.7) and (2.3.8)

$$= J_-J_+ + J_+J_- + 2J_3J_3$$

$$= (J_-J_+ + J_+J_-) + 2J_3^2$$

$$= 2(J_1^2 + J_2^2) + 2J_3^2$$

$$= 2(J_1^2 + J_2^2 + J_3^2) = 2J^2.$$

which proves that $[J_-, J_3, J_+]$ is an invariant of three dimensional rotation group and is equal to $2J^2$.

CHAPTER THREE

REPRESENTATION OF FOUR DIMENSIONAL LORENTZ GROUP

§3.1 FOUR DIMENSIONAL ENCLIDEAN GROUP

We consider a four dimensional Euclidean space in which a point is given by $x \equiv (x_1, x_2, x_3, x_4)$. The pure rotation group whose invariant in the metric

$$x_1^2 + x_2^2 + x_3^2 + x_4^2 \qquad (3.1.1)$$

may be studied in terms of three dimensional rotation group, discussed in Chapter Two. Analogous to (2.2.3), we obtain a rotation operator

$$T_g = 1 + \frac{1}{2} \varepsilon_{ij} A_{ij} \qquad (3.1.2)$$

where i,j now take on the values 1 to 4 instead of 1 to 3. The six infintesimal generators corresponding to rotations are given by

$$J_1 = -i\left(x_2 \cdot \frac{\partial}{\partial x_3} - x_3 \cdot \frac{\partial}{\partial x_2}\right)$$

$$J_2 = -i\left(x_3 \cdot \frac{\partial}{\partial x_1} - x_1 \cdot \frac{\partial}{\partial x_3}\right)$$

$$J_3 = -i\left(x_1 \cdot \frac{\partial}{\partial x_2} - x_2 \cdot \frac{\partial}{\partial x_1}\right)$$

and $$N_1 = -i\left(x_4 \cdot \frac{\partial}{\partial x_1} - x_1 \cdot \frac{\partial}{\partial x_4}\right)$$

$$N_2 = -i\left(x_4 \cdot \frac{\partial}{\partial x_2} - x_2 \cdot \frac{\partial}{\partial x_4}\right)$$

$$N_3 = -i\left(x_4 \cdot \frac{\partial}{\partial x_3} - x_3 \cdot \frac{\partial}{\partial x_4}\right)$$

The operator $J \equiv (J_p)$ and $N = (N_p)$ satisfy the following commutation relations

$$[J_i, J_j] = i\,\varepsilon_{ijk}\,J_k$$

$$[N_i, N_j] = i\,\varepsilon_{ijk}\,J_k$$

and $$[J_i, N_i] = 0 \quad i = 1,2,3.$$

Here J, N and $J^2 + N^2$ commutes with every J and N and are the invariant of the group.

Now we define

$$J_{\pm} = J_1 \pm iJ_2$$

and $$N_{\pm} = N_1 \pm iN_2$$

By applying three brackets on N_-, N_3 and N_+ we have

$$[N_-, N_3, N_+] = 2 \cdot J \cdot N \quad \text{(Invariant of the Group).}$$

Also we have $$[N_- + J_-,\ N_3 + J_3,\ N_+ + J_+]$$

$$= 8\left(\frac{J^2 + N^2}{2} + J \cdot N\right)$$

$$= 8 \text{ (sum of the invariants of the group).}$$

This result can easily be proved by using the definition of three bracket and cummutation relations.

3.2 LORENTZ GROUP

The Lorentz transformation differ from the transformation of four-dimensional Euclidean group by reality condition, which can be expressed by putting $x_4 = ix_o$. So that now x_o is real instead of x_4. From (3.1.1) the Lorentz transformation leaves $x_1^2 + x_2^2 + x_3^2 - x_o^2$ = invariant The six infenterimal basic operators are given by

$$J = (J_1, J_2, J_3), \quad \underline{k} \equiv (k_1, k_2, k_3)$$

J are given in (2.2.5), (2.2.6) and (2.2.7). They generate pure rotation and are the angular momentum operator. $\underline{k}$ generates pure Lorentz transformation. They are given by

$$k_1 = -i\left(x_o \cdot \frac{\partial}{\partial x_1} + x_1 \cdot \frac{\partial}{\partial x_o}\right) \qquad (3.2.1)$$

$$k_2 = -i\left(x_o \cdot \frac{\partial}{\partial x_2} + x_2 \cdot \frac{\partial}{\partial x_o}\right) \qquad (3.2.2)$$

$$k_3 = -i\left(x_o \cdot \frac{\partial}{\partial x_3} + x_3 \cdot \frac{\partial}{\partial x_o}\right) \qquad (3.2.3)$$

The operators $J \equiv (J_p)$ and $k \equiv (k_p)$ $(p = 1,2,3)$ satisfies the following commutation relations

$$[J_p, J_q] = i \cdot \varepsilon_{pqr} J_r \qquad (3.2.4)$$

$$[k_p, k_q] = -i \varepsilon_{pqr} J_r \qquad (3.2.5)$$

and

$$[J_p, k_q] = i \varepsilon_{pqr} k_r \qquad (3.2.6)$$

$\underline{J}\cdot\underline{k}$ and $J^2 - K^2$ commutes with every $\underline{J}$ and $\underline{K}$ and are therefore invariants of the group. We define $K_{\pm} = K_1 \pm iK_2$ and obtain following results.

<u>RESULT:</u>

1. $$[J_3, K_+] = [J_3, K_1 + iK_2]$$
$$= [J_3, K_1] + i[J_3, K_2]$$
$$= K_1 + iK_2 = K_+$$

2. $$[K_-, J_3] = [K_1 - iK_2, J_3] = [K_1, J_3] - i[K_2, J_3]$$
$$= K_1 - iK_2 = K_-$$

3. $$[K_-, K_+] = [K_1 - iK_2, K_1 + iK_2]$$
$$= [K_1, K_1] + i[K_1, K_2] - i[K_2, K_1] - i^2[K_3, K_2]$$
$$= 2i[K_1, K_2] = 2J_3.$$

Similarly by using (3.2.4), (3.2.5) and (3.2.6) we have

4. $[K_3, K_+] = -J_+$

5. $[K_3, K_-] = J_-$

6. $[J_-, K_+] = -2K_3$

7. $[J_-, J_3] = J_-$

8. $[J_-, K_3] = K_-$

9. $[J_-, K_-] = 0$.

10. $[J^2, K_\alpha] = [J_\lambda \cdot J_\lambda, K_\alpha]$

$$= J_\lambda [J_\lambda, K_\alpha] + [J_\lambda, K_\alpha] J_\lambda$$

$$= i\ \varepsilon_{\lambda\alpha\mu}\ J_\lambda K_\mu + i\ \varepsilon_{\lambda\alpha\mu}\ K_\mu J_\lambda$$

By using $(J_\lambda K_\mu - K_\mu J_\lambda) = i\ \varepsilon_{\lambda\mu\theta}\ K_\theta$ we have

$$[J^2, K_\alpha] = -\varepsilon_{\lambda\alpha\mu}\ \varepsilon_{\lambda\mu\theta}\ K_\theta + 2i\ \varepsilon_{\lambda\alpha\mu}\ K_\mu J_\lambda$$

$$= 2K_\alpha + 2i\ \varepsilon_{\alpha\lambda\mu}\ K_\lambda J_\mu .$$

In a similar way

$$[K^2, K_\alpha] = 2K_\alpha + 2i\ \varepsilon_{\alpha\lambda\mu}\ K_\lambda J_\mu$$

so $\quad [J^2, K_\alpha] = [K^2, K_\alpha]$

or $\quad [J^2 - K^2, K_\alpha] = 0 \qquad$ (A)

Also $\quad [J^2, J_\alpha] = 0$

and $\quad [K^2, J_\alpha] = 0$

$$[J^2 - K^2, J_\alpha] = 0 \qquad \text{(B)}$$

From equations (A) and (B) $J^2 - K^2$ commutes with every J_α and K_α and hence the invariant of the group.

11. $[J_\alpha \cdot K_\alpha, J_\beta] = J_\alpha [K_\alpha, J_\beta] + [J_\alpha, J_\beta] K_\alpha$

$$= -i\ J_\alpha K_\gamma + iJ_\gamma K_\alpha = 0 \quad ??$$

$$= -J_\alpha [J_\beta, K_\alpha] + [J_\alpha, J_\beta] K_\alpha$$

Similarly we can prove that $= -i\epsilon_{\beta\alpha\gamma} J_\alpha K_\gamma + i\epsilon_{\alpha\beta\gamma} J_\gamma K_\alpha$

$$= i\epsilon_{\alpha\beta\gamma} J_\alpha K_\gamma + i\epsilon_{\alpha\beta\gamma} J_\gamma K_\alpha$$

$$= i\epsilon_{\alpha\beta\gamma} J_\alpha K_\gamma + i\epsilon_{\gamma\beta\alpha} J_\alpha K_\gamma$$

$$= i\epsilon_{\alpha\beta\gamma} J_\alpha K_\gamma - i\epsilon_{\alpha\beta\gamma} J_\alpha K_\gamma = 0$$

$$[J_\alpha K_\alpha, K_\beta] = 0$$

so $J_\alpha K_\alpha$ commutes with every J_β and K_β and hence the invariant of the group.

§3.3

In this section we operate three brackets on $J = (J_p)$ and $K = (K_p)$ and obtain invariants of Lorentz group.

1. $[K_-, K_3, K_+] = -2(K \cdot J)$

PROOF:

$$= K_-[K_3, K_+] + K_3[K_+, K_-] + K_+[K_-, K_3]$$

$$= K_-(-J_+) + K_3(-2J_3) + K_+(-J_-)$$

$$= -K_-J_+ - 2K_3J_3 - K_+J_-$$

$$= -(K_-J_+ + 2K_3J_3 + K_+J_-)$$

$$= \{(K_-J_+ + K_+J_-) + 2K_3J_3\} - \{2K_1J_1 + 2K_2J_2 + 2K_3J_3\}$$

$$= -2\{K_1J_1 + K_2J_2 + K_3J_3\}$$

$$= -2(K \cdot J).$$

(Invariant of the Lorentz Group)

2. Also $[J_-, K_3, J_+] = 2(J \cdot K)$

$$= 2 \text{ (Invariant)}.$$

By using the following property of three bracket

$$[A + X, Y, Z] = [A, Y, Z] + [X, Y, Z]$$

we have 3. $[J_- + iK_-, J_3 + iK_3, J_+ + iK_+] = 4\{(J^2 - K^2) + i(J\cdot K)\}$

So $[J_- + iK_-, J_3 + iK_3, J_+ + iK_+]$

$$= [J_-, J_3 + iK_3, J_+ + iK_+] + i[K_-, J_3 + iK_3, J_+ + iK_+]$$

$$= [J_-, J_3, J_+ + iK_+] + i[J_-, K_3, J_+ + iK_+]$$
$$+ i[K_-, J_3, J_+ + iK_+] + i^2[K_-, K_3, J_+ + iK_+]$$

$$= [J_-, J_3, J_+] + i[J_-, J_3, K_+] + i[J_-, K_3, J_+]$$
$$+ i^2[J_-, K_3, K_+] + i[K_-, J_3, J_+] + i^2[K_-, J_3, K_+]$$
$$+ i^2[K_-, K_3, J_+] + i^3[K_-, K_3, K_+]$$

$$= J_-[J_3, J_+] + J_3[J_+, J_-] + J_+[J_-, J_3] + iJ_-[J_3, K_+]$$
$$+ iJ_3[K_+, J_-] + iK_+[J_-, J_3] + iJ_-[K_3, J_+] + iK_3[J_+, J_-]$$
$$+ iJ_+[K_3, J_-] - J_-[K_3, K_+] - K_3[K_+, J_-] - K_+[K_3, J_-]$$
$$+ iK_-[J_3, J_+] + iJ_3[J_+, K_-] + iJ_+[K_-, J_3]$$
$$- K_-[J_3, K_+] + J_3[K_+, K_-] + K_+[K_-, J_3]$$
$$- K_-[K_3, J_+] - K_3[J_+, K_-] - J_+[K_-, K_3]$$
$$- iK_-[K_3, K_+] - iK_3[K_+, K_-] - iK_+[K_-, K_3]$$

$$= J_-J_+ + J_3(2J_3) + J_+J_- + iJ_-K_+ + 2iJ_3K_3 + iK_+J_-$$
$$+ iJ_-K_+ + 2iK_3J_3 + iJ_+K_- + J_-J_+ - 2K_3^2 - K_+K_- + iK_-J_+$$
$$- 2iJ_3K_3 + iJ_+K_- - K_-K_+ + 2J_3^2 - K_+K_- - K_-K_+ - 2K_3^2$$
$$+ J_+J_- + iK_-J_+ + 2K_3J_3 + iK_+J_-$$

$$= 2J_-J_+ + 4J_3^2 + 2J_-J_+ + 2iJ_-K_+ + 2iK_+J_- + 4iK_3J_3$$
$$2iJ_+K_- - 4K_3^2 - 2K_+K_- + 2iK_-J_+ - 2K_-K_+ + 4iJ_3K_3$$

$$= 2(J_-J_+ + J_+J_- + 2J_3^2) - 2(K_+K_- + K_-K_+ + 2K_3^2)$$

$$+ 2i(J_-K_+ + J_+K_- + 2J_3K_3) + 2i(K_+J_- + K_-J_+ + 2K_3J_3)$$

$$= 4J^2 - 4K^2 + 4iJ\cdot K + 4iK\cdot J$$

$$= 4\{(J^2 - K^2) + i(J\cdot K) + i(K\cdot J)\}$$

As $(J^2 - K^2)$ and $J\cdot K$ commutes with every J and K, therefore sum of the invariants of the group is the invaraint i.e.

$$[J_- + iK_-, J_3 + iK_3, J_+ + iK_+]$$

is the invariant.

4. By using the fact that $J^2 - K^2$ is the invariant of the group, we have

$$\left(J^2, [K^2, K_\alpha]\right) = \left(K^2, [J^2, K_\alpha]\right).$$

§3.4 MATRIX ELEMENTS OF K_+, K_- AND K_3

In this section we evaluate the matrix elements of K_+, K_- and K_3 but we first prove the following lemma.

Lemma: For non-zero matrix elements $\langle j,m|K|j',m'\rangle$, we have the selection rule $j' = j \pm 1, j$.

PROOF: For the sake of convenience we use tensor notation here and write

$$J^2 = J_\ell \cdot J_\ell$$

so that a summation with respect to the repeated index in a term over the range 1,2,3 is implied. Thus

$$[J^2, K_p] = [J_\ell J_\ell, K_p]$$

$$= J_\ell [J_\ell, K_p] + [J_\ell, K_p] J_\ell$$

Using (3.2.6) and $\varepsilon_{\ell pq}\, \varepsilon_{\ell qr} = 2\, \delta_{pr}$

we obtain

$$[J^2, K_p] = -2i\, \varepsilon_{p\ell q}\, J_\ell\, K_q - 2K_p \qquad (3.4.1)$$

We shall use the above equation to evaluate the double cummutator $[J^2, [J^2, K_p]]$, which we denote by C. Thus by (3.4.1

$$C = -2i\, \varepsilon_{p\ell q}\, [J^2, J_\ell K_q] - 2[J^2, K_p]$$

Because J^2 commutes with J, the first term on the right hand side simplifies to $-2i\, \varepsilon_{p\ell q}\, J_\ell [J^2, K_q]$.
If we apply (3.4.1) again to this term, we get

$$C = -4\varepsilon_{p\ell q}\, \varepsilon_{qrs}\, J_\ell J_r K_s + 4i\varepsilon_{p\ell q}\, J_\ell K_q - 2J^2\, K_p + 2K_p J^2$$

Replacing $\varepsilon_{p\ell q}$ and $\varepsilon_{qp\ell}$ and using

$$\varepsilon_{qp\ell}\, \varepsilon_{qrs} = \delta_{pr}\, \delta_{\ell s} - \delta_{ps}\, \delta_{\ell r}$$

we obtain

$$C = -4J_\ell (J_p K_\ell - i\varepsilon_{p\ell q}\, K_q) + 2J^2 K_p + 2K_p J^2$$

or $$C = 2J^2 K_p + 2K_p J^2 - 4J\, K\, J_p \qquad (3.4.2)$$

Also $$C = J^4 K_p - 2J^2 K_p J^2 + K_p J^4 \qquad (3.4.3)$$

Therefore equating the above two equations

$$J^4 K_p - 2J^2 K_p J^2 + K_p J^4 = 2J^2 K_p + 2K_p J^2 - 4J\ K\ J_p \qquad (3.4.4)$$

Taking the matrix elements of (3.4.4) referred to the states $\langle j,m|$ and $|j',m'\rangle$ we get

$$\{j^2(j+1)^2 - 2j(j+1)(j'+1)j' + j'^2(j'+1)^2\}$$

$$\langle j,m|K_p|j',m'\rangle$$

$$= \{2j(j+1) + 2j'(j'+1)\}\langle j,m|K_p|j',m'\rangle \qquad (3.4.5)$$

The last term on right side of (3.4.4) does not contribute anything for $j' = j$ for $J\ K$ is a scalar and if

$$\langle j,m|K_p|j',m'\rangle \neq 0$$

we obtain from (3.4.5)

$$\{(j + j' + 1)^2 - 1\}((j' - j)^2 - 1) = 0$$

We have supposed $j' \neq j$ and $j', j > 0$. Therefore the first bracket is non-zero. The second gives

$$j' = j \pm 1.$$

Also $$[J_p, K_p] = 0 \qquad p = 1,2,3$$

Therefore we obtain elements for $j' = j$. This proves the lemma.

Matrix Elements of K_-

As $[J_3, K_-] = -K_-$

Therefore $J_3K_- - K_-J_3 = -K_-$ (3.4.6)

Operating $\langle j,m|$ and $|j',m'\rangle$ on both sides of (3.4.6), we have

$$\langle j,m|J_3K_-|j',m'\rangle - \langle j,m|K_-J_3|j',m'\rangle$$
$$= -\langle j,m|K_-|j',m'\rangle$$

or $$\langle j,m|J_3|j,m\rangle\langle j,m|K_-|j',m'\rangle - \langle j,m|K_-|j',m'\rangle$$
$$\langle j',m'|J_3|j',m'\rangle = -\langle j,m|K_-|j',m'\rangle$$

or $$(m - m' + 1)\langle j,m|K_-|j',m'\rangle = 0$$

Then $$\langle j,m|K_-|j',m'\rangle \neq 0 \Rightarrow m' = m + 1$$

and hence

$$\langle j,m|K_+|j',m'\rangle \neq 0 \Rightarrow m' = m - 1$$

Hence the non-vanishing components of K_1 and K_2 are therefore $m' = m \pm 1$. Since K_3 commutes with J_3. The only non-vanishing of K_3 are those for which $m = m'$.

Now we shall obtain the matrix of K_- as $[J_-, K_-] = 0$. Therefore $J_-K_- = K_-J_-$ and $\langle j,m-1|J_-|j,m\rangle$

$$= \sqrt{(j+m)(j-m+1)}$$
$$(j \geq m \geq -j+1)$$

$$\therefore \quad \langle j,m-1|J_-K_-|j',m+1\rangle = \langle j,m-1|K_-J_-|j',m+1\rangle$$

or $$\langle j,m-1|J_-|j,m\rangle\langle j,m|K_-|j',m+1\rangle$$
$$= \langle j,m-1|K_-|j',m\rangle\langle j',m|J_-|j',m+1\rangle$$

or $$\sqrt{(j+m)(j-m+1)}\ \langle j,m|K_-|j',m+1\rangle$$

$$= \sqrt{(j'+m+1)(j'-m)}\ \langle j,m-1|K_-|j',m\rangle \qquad (3.4.7)$$

$$(j'-j = 0,\pm 1)$$

If $j' = j$

$$\frac{\langle j,m|K_-|j,m+1\rangle}{\sqrt{(j-m)(j+m+1)}} = \frac{\langle j,m-1|K_-|j,m\rangle}{\sqrt{(j+m)(j-m+1)}} \qquad (3.4.8)$$

As the ratio is independent of m, so we shall denote this by $\langle j|K_-|j\rangle$. Hence we find for the dependence on m of the elements of K_- diagonal in j

$$\langle j,m|K_-|j,m+1\rangle = \sqrt{(j-m)(j+m+1)}\ \langle j|K_-|j\rangle \qquad (3.4.9)$$

If $j' = j-1$, then equation (3.4.7) becomes

$$\sqrt{(j+m)(j-m+1)}\ \langle j,m|K_-|j-1,m+1\rangle$$

$$= \sqrt{(j-m-1)(j+m)}\ \langle j,m-1|K_-|j-1,m\rangle$$

Multiplying both sides by $\sqrt{(j-m)/(j+m)}$ and rewriting, we get

$$\frac{\langle j,m|K_-|j-1,m+1\rangle}{\sqrt{(j-m)(j-m-1)}} = \frac{\langle j,m-1|K_-|j,m\rangle}{\sqrt{(j-m)(j-m+1)}}$$

Again this ratio is independent of m. Hence

$$\langle j,m-1|K_-|j-1,m\rangle = \langle j|K_-|j-1\rangle$$

$$\langle j,m|K_-|j-1,m+1\rangle = \sqrt{(j-m)(j-m-1)}\ \langle j|K_-|j-1\rangle \qquad (3.4.10)$$

If $j' = j+1$ then equation (3.4.7) becomes

$$\sqrt{(j+m)(j-m+1)}\ \langle j,m|K_-|j+1,m+1\rangle = \langle j,m-1|K_-|j+1,m\rangle$$

$$\sqrt{(j+m+2)(j+m+1)}\ \times$$

Multiplying by $\sqrt{(j+m+1)(j-m+1)}$ and rewriting gives

$$\frac{\langle j,m|K_-|j+1,m+1\rangle}{\sqrt{(j+m+2)(j+m+1)}} = \frac{\langle j,m-1|K_-|j+1,m\rangle}{\sqrt{(j+m+1)(j+m)}}$$

$$= \langle j|K_-|j+1\rangle$$

$$\langle j,m|K_-|j+1,m+1\rangle = \sqrt{(j+m+1)(j+m+2)}\ \langle j|K_-|j+1\rangle \qquad (3.4.11)$$

Now we shall determine the dependence of matrix K_3 on m. We have

$$[J_+,K_-] = 2K_3$$

$$J_+K_- - K_-J_+ = 2K_3$$

by applying $\langle j,m|$, $|j',m'\rangle$ on both sides

$$2\langle j,m|K_3|j',m'\rangle = \langle j,m|(J_+K_- - K_-J_+)|j',m'\rangle$$

but m' should be equal to m. As K_3 commutes with J_3, therefore the only non-vanishing components of K_3 are those for $m' = m$.

$$\therefore \quad 2\langle j,m|K_3|j',m\rangle = \sqrt{(j+m)(j-m+1)}\ \langle j,m-1|K_-|j',m\rangle$$

$$-\sqrt{(j'+m+1)(j'-m)}\ \langle j,m|K_-|j',m+1\rangle$$

$$= \sqrt{(j+m)(j-m+1)}\cdot\sqrt{(j+m)(j-m+1)}\langle j|K_-|j\rangle$$

$$-\sqrt{(j-m)(j+m+1)}\cdot\sqrt{(j+m)(j+m+1)}\langle j|K_-|j\rangle$$

$$= 2m\langle j|K_-|j\rangle$$

$$\therefore \quad \langle j,m|K_3|j,m\rangle = m\langle j|K_-|j\rangle \tag{3.4.12}$$

For $j' = j - 1$

$$\langle j,m|K_3|j-1,m\rangle = (j^2 - m^2)\langle j|K_-|j-1\rangle \tag{3.4.13}$$

If $j' = j + 1$

$$\langle j,m|K_3|j+1,m\rangle = \sqrt{(j+1)^2 - m^2}\,\langle j|K_-|j+1\rangle \tag{3.4.14}$$

Since K_3 is real

$$\langle j|K|j-1\rangle = \langle j-1|K|j\rangle$$

the matrix $\langle j|K_-|j-1\rangle$ as we have defined above is hermitian.

Similarly for K_+, i.e.

(i) $\langle j-1,m+1|K_+|j,m\rangle = \sqrt{(j-m)(j-m-1)}\,\langle j-1|K_+|j\rangle$

(ii) $\langle j,m+1|K_+|j,m\rangle = \sqrt{(j-m)(j+m+1)}\,\langle j|K_+|j\rangle$

(iii) $\langle j+1|K_+|j,m\rangle = \sqrt{(j+m+1)(j+m+2)}\,\langle j+1|K_+|j\rangle$

§3.5 MATRIX ELEMENTS OF $[J^2,K_+]$, $[J^2,K_-]$, $[J^2,K_3]$ $[J_-,K_3,J_+]$

As $[J^2,K_+] = J^2K_+ - K_+J^2$

therefore

$$\langle j,m|[J^2,K_+]|j',m-1\rangle$$

$$= \langle j,m|J^2K_+|j',m-1\rangle - \langle j,m|K_+J^2|j',m-1\rangle$$

$$= j(j+1)\langle j,m|K_+|j',m-1\rangle - \langle j'(j'+1)\langle j,m|K_+|j',m-1\rangle$$

$\langle j,m|K_3|j,m\rangle = m\langle j|K_-|j\rangle$ (3.4.12)

For $j' = j - 1$

$\langle j,m|K_3|j-1,m\rangle = (j^2 - m^2)\langle j|K_-|j-1\rangle$ (3.4.13)

Now if $j' = j$

then the matrix

$$\langle j,m|[J^2,K_+]|j,m-1\rangle = 0 \qquad (3.5.1)$$

If $j' = j-1$

then

$$\langle j,m|[J^2,K_+]|j-1,m-1\rangle = 2\sqrt{(j+m)(j-m+1)}\ \langle j|K_+|j\rangle \qquad (3.5.2)$$

Now if $j' = j+1$

$$\langle j,m|[J^2,K_+]|j+1,m-1\rangle$$

$$= -2(j+1)\sqrt{(j-m)(j+m+1)}\ \langle j|K_+|j+1\rangle \qquad (3.5.3)$$

Similarly we can calculate matrix element of $[J^2,K_-]$ and $[J^2,K_3]$, i.e.

$$\langle j,m|[J^2,K_-]|j,m+1\rangle = 0 \qquad (3.5.4)$$

$$\langle j,m|[J^2,K_-]|j-1,m+1\rangle = 2j\sqrt{(j-m)(j-m-1)}\ \langle j|K_-|j-1\rangle \qquad (3.5.5)$$

and $$\langle j,m|[J^2,K_-]|j+1,m+1\rangle$$

$$= 2j(j+1)\sqrt{(j+m+1)(j+m+2)}\ \langle j|K_-|j+1\rangle \qquad (3.5.6)$$

and similarly

$$\langle j,m|[J^2,K_3]|j,m\rangle = 0 \qquad (3.5.7)$$

$$\langle j,m|[J^2,K_3]|j-1,m\rangle = 2j\sqrt{j^2-m^2}\ \langle j|K_3|j-1\rangle \qquad (3.5.8)$$

and

$$\langle j,m|[J^2,K_3]|j+1,m\rangle = -2(j+1)\sqrt{(j+1)^2 - m^2}\ \langle j|K|j+1\rangle \tag{3.5.9}$$

As $\quad [J_+, K_3, J_+] = 2(J\cdot K)$

or $\quad J_-K_+ + J_+K_- + 2J_3K_3 = 2J\cdot K$

Applying $\langle j,m|$

$$\langle j,m|J_-K_+|j,m\rangle + \langle j,m|J_+K_-|j,m\rangle + 2\langle j,m|J_3K_3|j,m\rangle$$

$$= 2\langle j,m|J\cdot K|j,m\rangle$$

or $\quad (j+m+1)(j-m)\langle j|K|j\rangle + (j+m)(j-m+1)\langle j|K|j\rangle + 2m^2\langle j|K|j\rangle$

$$= 2C \qquad \text{(C is any constant)}$$

$$= (2j^2 + 2j)\ \langle j|K|j\rangle = 2C$$

$$C = j(j+1)\ \langle j|K|j\rangle \tag{3.5.10}$$

<u>Note 1</u>: This is a convenient place to make a necessary change in notation. Hence, after we shall write B_j instead of $\langle j|K_-|j\rangle$, C'_j instead of $\langle j|K_-|j-1\rangle$ and D'_{j+1} instead of $\langle j|K_-|j+1\rangle$. Similarly A_j instead of $\langle j|K_+|j\rangle$, C_j instead of $\langle j-1|K_+|j\rangle$ and D_{j+1} instead of $\langle j+1|K_+|j\rangle$.

<u>Note 2</u>: It is easy to verify that

$$A_j = B_j$$

and $\quad C'_j = -C_j$

and $\quad D'_{j+1} = -D_{j+1}$.

§3.6 DETERMINATION OF THE OPERATOR K_+, K_- AND K_3

In this section we will apply $|j,m>$ to the operator K_+, K_-, K_3 and obtain some vector $K_+|j,m>$, $K_-|j,m>$ and $K_3|j,m>$. First we calculate $K_+|j,m>$. As

$$K_+J_+ = J_+K_+$$

operating $|j,m-1>$ on both sides, we have

$$K_+|j,m><j,m|J_+|j',m-1>$$

$$= J_+|j',m><j',m|K_+|j,m-1> \qquad (j' = j+1, j, j-1)$$

$$= \sqrt{(j+m)(j-m-1)}\ |j-1,m+1><j-1,m|K_+|j,m-1>$$

$$+ \sqrt{(j-m)(j+m+1)}\ |j,m+1><j,m|K_+|j,m-1>$$

$$+ \sqrt{(j-m+1)(j+m+2)}\ |j+1,m+1><j+1,m|K_+|j,m-1>$$

By using the results of previous section and by dividing $\sqrt{(j+m)(j-m+1)}$ we have

$$K_+|j,m> = \sqrt{(j-m)(j-m-1)}\ C_j|j-1,m+1> - \sqrt{(j-m)(j+m+1)}\ A_j|j,m+1>$$

$$+ \sqrt{(j+m+1)(j+m+2)}\ \ D_{j+1}|j+1,m+1> \qquad (3.6.1)$$

Now we calculate $K_-|j,m>$

As $\qquad K_-J_- = J_-K_-$

Operating $|j,m+1>$ on both sides we have

$$\sqrt{(j-m)(j+m+1)}\ K_-|j,m> = \sqrt{(j-m)(j+m-1)}\,|j-1,m-1><j-1,m|K_-|j,m+1$$

$$+ \sqrt{(j+m)(j-m+1)}\,|j,m-1><j,m|K_-|j,m+1>$$

$$+ \sqrt{(j+m+1)(j+m+2)}\,|j+1,m-1><j+1,m|K_-|j,m$$

Using (3.4.9), (3.4.10) and (3.4.11), we have

$$K_-|j,m\rangle = -\sqrt{(j+m)(j+m-1)}\ C_j|j-1,m-1\rangle - \sqrt{(j+m)(j-m+1)}\ A_j|j,m-1\rangle$$
$$- \sqrt{(j-m+1)(j-m+2)}\ D_{j+1}|j+1,m-1\rangle \qquad (3.6.2)$$

Similarly for K_3 we have

$$2K_3 = [J_+,K_-] = J_+K_- - K_-J_+$$

Operating $|j,m\rangle$ on both sides and using previous results, we have

$$K_3|j,m\rangle = \sqrt{(j-m(j+m)}\ C_j|j-1,m\rangle - m\ A_j|j,m\rangle$$
$$- \sqrt{(j+m+1)(j-m+\ )}\ D_{j+1}|j+1,m\rangle \qquad (3.6.3)$$

§3.7 DETERMINATION OF A_j, C_j AND D_{j+1}

Before calculating A_j, C_j, D_{j+1}, we first consider the case when the basis are changed. Let us suppose that the basis vectors $|j,m\rangle$ are replaced by vectors $(j,m) = \omega(j)|j,m\rangle$ where $\omega(j)$ is arbitrary numerical factor depending on j only.

If one multiplies both sides of (3.6.1)-(3.6.3) by $\omega(j)$ and goes over the vector (j,m) then the co-efficient A_j remain unchanged, while C_j and D_{j+1} go into

$$C_j' = \frac{\omega(j)}{\omega(j-1)}\ C_j\ , \quad D_{j+1}' = \frac{\omega(j)}{\omega(j+1}\ D_{j+1} \qquad (3.7.1)$$

or $\qquad C_j'\ D_j' = C_j\ D_j$

i.e. the product remains unchanged. Let j_o be the least of the weight j. The factor $\omega(j)$ may obviously be so chosen that $C_j' = D_j'$ for $j \geq j_o+1$. In fact by (3.7.1) this equality is equivalent to

$$\frac{\omega(j)}{\omega(j-1)} C_j = \frac{\omega(j-1)}{\omega(j)} D_j$$

Hence
$$\left(\frac{\omega(j)}{\omega(j-1)}\right)^2 = \frac{D_j}{C_j}$$

so
$$\omega(j) = \sqrt{\prod_{k=j_o+1}^{j} \frac{D_k}{C_k}} \qquad j \geq j_o+1$$

We shall suppose that this replacement of the basis has already been carried out from very beginning so that $C_j = D_j$

$$\therefore \quad C_{j+1} = D_{j+1}$$

It remains to determine A_j and C_j.

Let us apply $|j,m\rangle$ to both sides of $[K_-,K_+] = 2J_3$. Then we have

$$\begin{aligned} 2m|j,m\rangle = {} & \sqrt{(j-m)(j-m-1)}\, C_j\, K_-|j-1,m+1\rangle \\ & - \sqrt{(j-m)(j+m+1)}\, A_j\, K_-|j+1,m+1\rangle \\ & + \sqrt{(j+m+1)(j-m+2)}\, D_{j+1}\, K_-|j+1,m+1\rangle \\ & + \sqrt{(j+m)(j+m-1)}\, C_j\, K_+|j-1,m-1\rangle \\ & + \sqrt{(j+m)(j-m+1)}\, A_j\, K_+|j,m-1\rangle \\ & + \sqrt{(j-m+1)(j-m+2)}\, D_{j+1}\, K_+|j+1,m-1\rangle \end{aligned}$$

By using (3.6.1) and (3.6.2), and then we compare the terms involving $|j-2,m\rangle$ we have

$$2(j+1)\ A_j\ C_j - 2(j-1)\ A_{j-1}\ C_j = 0 \qquad (3.7.2)$$

Also comparing the terms involving $|j,m\rangle$ we have

$$(2j-1)\ C_j\ D_j - C_{j+1}\ D_{j+1} - A_j^2 = 1 \qquad (3.7.3)$$

Now we take $j = j_o$. Then $C_{j_o} = 0$ as $j_o - 1$ cannot appear. Then there are following two cases:

(i) C_j does not vanish for $j = j_o + n$ $(n = 1, 2, \cdots, m)$

(ii) C_j vanish for some of the values $j = j_o + n$. Let us consider 1st case. Then (3.7.2) reduces to

$$A_j(j+1) - (j-1)\ A_{j-1} = 0 \quad \text{for } j = j_o + n \qquad (3.7.4)$$

Multiplying both sides by j and introducing $j(j+1)A_j = P_j$, we have

$$P_j - P_{j-1} = 0$$

P_j does not depend on j i.e. a constant $P_j = i\ j_o\alpha$. Hence $j_o \neq 0$ as of $j_o = 0$ then it follows from (3.7.4) that $A_j = 0$, $j \geq 1$, so

$$A_j = \frac{i j_o \alpha}{j(j+1)} \qquad (3.7.5)$$

Now let us consider (3.7.3) and multiplying both sides of it by $(2j+1)$ and introducing

$$\sigma_j = (2j-1)(2j+1)C_j^2 \qquad (3.7.6)$$

So we have

$$\sigma_j - \sigma_{j+1} - (2j+1)A_j^2 = (2j+1)$$

$$2(j+1)\ A_j\ C_j - 2(j-1)\ A_{j-1}\ C_j = 0 \qquad (3.7.2)$$

Also comparing the term involving $|j,m\rangle$ we have

$$(2j-1)\ C_j\ D_j - C_{j+1}\ D_{j+1} - A_j^2 = 1 \qquad (3.7.3)$$

Using (3.7.5), we have

$$\sigma_{j_o} - \sigma_j = \frac{(j^2 - j_o^2)(j^2 - \alpha^2)}{j^2}$$

as $C_j = 0$ and $\sigma_{j_o} = 0$ so we have

$$\sigma_j = - \frac{(j^2 - j_o^2)(j^2 - \alpha^2)}{j^2}$$

combining with (3.7.6), we have

$$C_j = \frac{i}{j} \sqrt{\frac{(j^2 - j_o^2)(j^2 - \alpha^2)}{(4j^2 - 1)}} \qquad (3.7.7)$$

Using (3.7.5), (3.7.7) and the fact that $C_j = D_j$, we have

$$K_+ |j,m\rangle = \sqrt{(j-m)(j-m-1)} \cdot \frac{i}{j} \sqrt{\frac{(j^2 - j_o^2)(j^2 - C^2)}{4j^2 - 1}} \, |j-1,m+1\rangle$$

$$= \sqrt{(j-m)(j+m+1)} \, \frac{i j_o C}{j(j+1)} \, |j,m+1\rangle$$

$$+ \sqrt{(j+m+1)(j+m+2)} \cdot \frac{i}{(j+1)} \sqrt{\frac{\{(j+1)^2 - j_o^2\}\{(j+1)^2 - C^2\}}{4(j+1)^2 - 1}}$$

$$\times \, |j+1,m+1\rangle \qquad (3.7.8)$$

and

$$K_- |j,m\rangle = - \sqrt{(j+m)(j+m-1)} \, \frac{i}{j} \sqrt{\frac{(j^2 - j_o^2)(j^2 - C^2)}{4j^2 - 1}} \, |j-1,m-1\rangle$$

$$- \sqrt{(j+m)(j-m+1)} \, \frac{i j_o C}{j(j+1)} \, |j,m-1\rangle$$

$$- \sqrt{(j-m+1)(j-m+2)} \, \frac{i}{(j+1)} \sqrt{\frac{\{(j+1)^2 - j_o^2\}\{(j+1)^2 - C^2\}}{4(j+1)^2 - 1}}$$

$$\times \, |j+1,m-1\rangle \qquad (3.7.9)$$

and

$$K_3|j,m> = \sqrt{(j-m)(j+m)}\ \frac{i}{j}\sqrt{\frac{(j^2 - j_o^2)(j^2 - C^2)}{4j^2 - 1}}\ |j,m-1>$$

$$- \frac{miC}{j(j+1)}|j,m> - \sqrt{(j+m+1)(j-m+1)}$$

$$\times \frac{i}{(j+1)}\sqrt{\frac{\{(j+1)^2 - j_o^2\}\{(j+1)^2 - C^2\}}{4(j+1)^2 - 1}}\ |j+1,m> \qquad (3.7.10)$$

§3.8 CONDITION OF BEING UNITARY

Theorem: If the representation $g \to T_g$ of the Lorentz Group is unitary then the pair (j_o,α) determining it satisfies one of the following conditions:

(1) α is purely imaginary and j_o is an arbitrary non-negative integer or semi-integer.

(2) α is a real number in the interval $0 \leq \alpha \leq 1$ and $j_o = 0$.

PROOF: Combining the relation

$$(K_3|j,m>, |j,m>) = (|j,m>, K_3|j,m>)$$

with (3.6.3) and taking account of the mutual orthogonality of $|j,m>$ we have

$$-m\,A_j = -m\,\bar{A}_j$$

$$\therefore \quad A_j = \bar{A}_j$$

$\therefore$ A is real.

From (3.7.5) it follows that this is only possible in the following cases:

(1) α is pure imaginary and j_o is arbitrary.

(2) α is arbitrary, $j_o = 0$

Similarly combining the relation

$$(K_3|j,m>, |j-1,m>) = (|j,m>, K_3|j-1,m>)$$

with (3.6.3), we obtain

$$\sqrt{(j-m)(j+m)}\; C_j = -\sqrt{(j-m)(j+m)}\; \bar{C}_j$$

$$\therefore \quad C_j = -\bar{C}_j$$

$\therefore$ C_j is purely imaginary. As

$$C_j = \frac{i}{j}\sqrt{\frac{(j^2 - j_o^2)(j^2 - \alpha^2)}{4j^2 - 1}}$$

$$\therefore \quad \frac{1}{j}\sqrt{\frac{(j^2 - j_o^2)(j^2 - \alpha^2)}{4j^2 - 1}} \text{ must be real.}$$

The expression under square root sign must be positive. It is possible only when α^2 is real i.e. when α is real or purely imaginary. In the second case $-\alpha^2 \geq 0$. In the first case of real α we must have $j_o = 0$. Therefore the expression

$$\frac{1}{j}\sqrt{\frac{(j^2 - j_o^2)(j^2 - \alpha^2)}{4j^2 - 1}} \text{ takes the form}$$

$$\sqrt{\frac{j^2 - \alpha^2}{4j^2 - 1}}$$

where $\alpha^2 \geq 0$. This latter expression must be real for all $j = 0,1,2,\cdots$. Obviously this is possible if $\alpha^2 \leq 1$ which proves the theorem.

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where $\alpha^2 \geq 0$. This latter expression must be real for all $j = 0,1,2,\cdots$. Obviously this is possible if $\alpha^2 \leq 1$ which proves the theorem.

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