SOLUTION OF LAPLACE EQUATION IN FOUR DIMENSIONS

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ISLAMABAD

1982

SOLUTION OF LAPLACE EQUATION IN FOUR DIMENSIONS

A DISSERTATION SUBMITTED IN PARTIAL FULFILMENT OF THE DEGREE OF

Master of Philosophy in Mathematics

WE ACCEPT THIS DISSERTATION AS CONFORMING TO THE REQUIRED STANDARD

(1)

(2)

(3)

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ACKNOWLEDGMENTS

I owe my profoundest thanks to God Almighty, who gave me the courage to complete this dissertation.

There is no word to pay humble thanks to my eminent and affectionate supervisor Dr. C. M. Hussain, Associate Professor, Department of Mathematics, Quaid-i-Azam University, Islamabad, for his continuous guidance and valuable support at all stages in the completion of this work.

I am also thankful to Dr. S. M. Yousaf, Dean Faculty of Natúral Sciences and Chairman Department of Mathematics, Quaid-i-Azam University, Islamabad, for providing me necessary facilities in preparation of this dissertation.

WAHEED-UR-REHMAN

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CHAPTER I

INTRODUCTION

A second order partial differential equation in n-independent variables $(x_1, x_2, \ldots, x_n)$ namely

$$\nabla^2 U = 0 \qquad (1.1)$$

where ∇^2 stands for $\frac{\partial^2}{\partial x_1^2} + \ldots + \frac{\partial^2}{\partial x_n^2}$ is known as an n-dimensional Laplace's Equation and $\frac{\partial^2}{\partial x_1^2} + \ldots + \frac{\partial^2}{\partial x_n^2}$ is named as n-dimensional Laplacian operator. The co-ordinates $(x_1, x_2, \ldots x_n)$ span n-dimensional Euclidean space. In this dissertation we are dealing with case when $n = 4$. Treatment of the Laplace's Equation for dimensions one, two and three is well known. However these cases will be considered briefly so as to provide understanding to the case of dimensions four. The two-dimensional Laplace equation is commonly written as

$$\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} = 0 \qquad (1.2)$$

where x,y are independent variables. Equation (1.2) is also known as the Laplace's Equation in the plane. Similarly,

$$\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} + \frac{\partial U}{\partial z^2} = 0 \qquad (1.3)$$

is called Laplace's Equation in space. The corresponding equation for $n = 4$ is

$$\frac{\partial^2 U}{\partial x_1^2} + \frac{\partial^2 U}{\partial x_2^2} + \frac{\partial^2 U}{\partial x_3^2} + \frac{\partial^2 U}{\partial x_4^2} = 0 \qquad (1.4)$$

where (x_1, x_2, x_3) are the space coordinates given in equation (1.3). The given four dimensional Laplace's Equation can be easily transformed to well known three dimensional wave equation if we change $x_1 \to x$, $x_2 \to y$, $x_3 \to z$ and $x_4 \to$ ict Here x, y, z are the usual space coordinates, c represents the wave velocity and t the time. Then the equation (1.4) becomes the familiar wave equation or in other words we can say that Four dimensional Laplace's Equation is closely related to the familiar wave equation.

The non-homogeneous equation corresponding to Laplace's Equation is

$$\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} = \nabla^2 U = f(x,y) \qquad (1.5)$$

called Poisson's Equation in the plane. Similarly Poisson Equation in the space and in the four dimensions are respectively

$$\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} + \frac{\partial^2 U}{\partial z^2} = f(x,y,z) \qquad (1.6)$$

and
$$\frac{\partial^2 U}{\partial x_1^2} + \frac{\partial^2 U}{\partial x_2^2} + \frac{\partial^2 U}{\partial x_3^2} + \frac{\partial^2 U}{\partial x_4^2} = f(x_1, x_2, x_3, x_4)$$

The function U is called the Potential function. Also these equations are named as Potential Equation. The Laplace's Equation occurs in mathematical Physics particularly this Partial differential equation is of fundamental importance in electrostatics, hydrodynamics and the theory of attraction. It evidently arises when we attempt to find a solution of the equation of wave propagation. Moreover, in some branches of Physics the Field Equation can be reduced to Laplace's Equation. Thus Laplace's equation is very useful to solve various problems of mathematical Physics.

In chapter II we shall discuss the preliminaries for example Gamma Function, hyper-geomatric function, Legendre functions, Associated legendre functions and Bessel's Functions. Because we obtain the polynomial of these functions when we use different parameters to solve the Laplace's Equation. The detailed discussion of different co-ordinates is given in Chapter III. And also in Chapter III we are concerned with the solution of Laplace's Equation when n = 2,3. When we use different

Co-ordinates. General solution of Laplace's Equation for three dimensions due to Whittaker is also given. In Chapter IV we are dealing with the solution of Laplace's Equation in four dimension. We solve this equation by using different parameters.

CHAPTER II

"PRELIMINARIES"

This chapter contain some elementary result which will be used later on.

§ 2.1: The Gamma Function (Γ): In the definition of Hyper geomatric function, Bessel function and Legender function, Associated Legender function we make use of Gamma function. Which is considered a generalization of a factorial function. We know that if n is a Positive Integer then $n! = n(n-1)(n-2)\ldots\ldots$ 2.1

If n is not a Positive integer then the above equation is not satisfied. This is a function which is defined for all values of the variable α and equals $(\alpha-1)!$ Where α is positive integer is known as "Gamma Function" $\Gamma(\alpha)$ which is defined as

$$\Gamma(\alpha) = \int_0^\infty x^{\alpha-1} e^{-x} dx \qquad (2.1)$$

The following formulae are given here for convenience in reference

$$\Gamma(x) = \lim_{n\to\infty} \frac{n!\, n^x}{x(x+1)\ldots x(x+n)} \qquad (2.2)$$

$$\frac{1}{\Gamma(x)} = x\, e^{\gamma x} \prod_{n=1}^{\infty} \left(1+\frac{x}{n}\right) e^{-x/n} \qquad (2.3)$$

where $\gamma = \lim_{m\to\infty} \left(\sum_{l=1}^{m} \frac{1}{l} - \log m \right) = 0.577$

$$\Gamma(x+1) = x\Gamma(x) \qquad (2.4)$$

If n is positive integer, $\Gamma(n+1) = n!$ (2.5)

$$\Gamma(1) = 1, \quad \Gamma(\tfrac{1}{2}) = 1/\sqrt{\pi} \qquad (2.6)$$

and $F(\alpha,\beta,\gamma,1) = \frac{\Gamma(\gamma)\Gamma(\gamma-\alpha-\beta)}{\Gamma(\gamma-\alpha)\Gamma(\gamma-\beta)}$ (2.7)

where $F(\alpha,\beta,\gamma,1)$ is a Hyper Geometric Function and will be defined in the next section. Now we also give the definition of Beta function which also make use in the hypergeomatric function.

Beta function

$$\beta(p,q) = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)} \qquad (2.8)$$

If $p,q > 0$, $\beta(p,q) = \int_0^1 x^p (1-x)^{q-1}\, dx$

§ 2.2: SYSTEM OF ORTHOGONAL FUNCTIONS: A set of functions $\{\phi_i(x)\}$ $i = 1,2,\ldots,n$ is said to be orthogonal in the interval [a,b] if the scaler product

$$(\phi_i, \phi_j) = \int_a^b \phi_i(x)\, \phi_j(x)\, dx = 0 \qquad \text{if } i \neq j \tag{2.9}$$

$$\int_a^b [\phi_i(x)]^2\, dx \neq 0 \qquad \text{if } i = j \tag{2.10}$$

If $$\int_a^b [\phi_i(x)]^2\, dx = a_i \neq 0$$

We can normalize the $\phi_i(x)$ that is find a new set of functions

$$\phi_i(x) = \frac{\phi_i(x)}{\sqrt{a_i}}$$ which are said to be orthonormal

in the interval [a,b]. If

$$\int_a^b \phi_i(x)\, \phi_j(x)\, dx = \delta_{ij} \tag{2.11}$$

where $\delta_{ij} = 0$ when $i \neq j$

$= 1$ when $i = j$

Let $\{\phi_n(x)\}$ be the sequence of orthogonal functions then the members of the sequence are mutually orthogonal on [a,b]. If ϕ is an integrable function then the norm of ϕ is the real number

$$||\phi|| = \left[\int_a^b |\phi(x)|^2 \, dx\right]^{\frac{1}{2}} \tag{2.12}$$

$$||\phi|| \geq 0 \qquad \text{for any } \phi$$

If $||\phi_n|| = 1$ $n = 1,2,3,\ldots$ then $\{\phi_n(x)\}$ is said to be an orthonormal on $[a,b]$. Let us attempt to expand the arbitrary function $f(x)$ in term of the arthonormal function $\phi_n(x)$ that is, let us write

$$f(x) = \sum_{n=1}^{\infty} a_n \phi_n(x) \tag{2.13}$$

where a_n's are known constants. Since $\phi_n(x)$ are arthonormal functions. By inner product property

$<\phi_i , \phi_j> = \delta_{ij}$ we have

$$< f , \phi_k > = < \sum_{n=1}^{\infty} a_n \phi_n, \phi_k >$$

$$= \sum_{n=1}^{\infty} a_n < \phi_n, \phi_k > = \sum_{n=1}^{\infty} a_n \delta_{nk}$$

where $\delta_{nk} = 0$ if $n \neq k$

$= 1$ if $n = k$

Taking $n = k$

$$< f , \phi_k > = \sum_{n=1}^{\infty} a_k \tag{2.14}$$

Thus (2.13) can be written as

$$f(x) = \sum_{n=1}^{\infty} (f, \phi_k) \, (\phi_k(x))$$

So we have expressed function f(x) in terms of arthonormal function, but we do not always get such expression. Therefore when any function can be expressed in term of arthonormal functions for all n then arthonormal set of function $\{\phi_n(x)\}$ is said to be complete over the interval [a,b] or we can say that the sequence $\{\phi_n(x)\}$ form a complete arthonormal set of over [a,b].

The trigonometric function sin nx and cos nx, the complete exponential functions e^{inx} k = 0,1,2,... the hyper-geometric functions, the Bessel Functions, the Legendre functions and Associated Legendre Functions (discussed in the next section) are examples of complete arthonormal set.

It is very useful that we can expand an arbitrary function in term of hypergeometric, Bessel, Legender and Associated Legenders, functions.

Let $\{\phi_n(x)\}$ be an arthonormal set on [a,b]. If f(x) is an integrable function on [a,b] then the numbers

$$a_n = \int_a^b f(x) \; \phi_n(x) \, dx, \quad n=1,2,3,\ldots.$$

are called the Fourier co-efficients of f relative to $\{\phi_n(x)\}$.

The series $\sum_{n=1}^{\infty} a_n\,_n(x)$ is called the Fourier series of f(x) relative to $\{\phi_n(x)\}$. The series need not be convergent, however to each integrable function f(x) there correspond its Formal Fourier Series.

§ 2.3 HYPERGEOMATRIC FUNCTIONS

As the hyper geomatric function is frequently employed in connection with the solution of Laplace's Equation in four dimensions on account of some of its properties will be given here. The function is define by mean of the hyper-geomatric series

$$F(\alpha,\beta,\gamma,x) = 1 + \frac{\alpha\cdot\beta}{1.\gamma} x + \frac{\alpha(\alpha+1)\ (\beta+1)}{1.2.\gamma(\gamma+1)} x^2 +$$

$$\frac{\alpha(\alpha+1)\ (\alpha+2)\beta(\beta+1)\ (\beta+2)}{1.2.3.\gamma(\gamma+1)(\gamma+2)} x^3 + \ldots.$$

which is absolutely convergent of $|x| < 1$, it converges obsolutely if $\gamma - \alpha - \beta > 0$.

Function $F(\alpha,\beta,\gamma,x)$ is a solution of the differential equation

$$x(1-x)y'' + \{\gamma-(\alpha+\beta+1)x\}\ y' - \alpha\beta\ y = 0 \qquad (2.15)$$

is known as Gauss's Equation or the Hypergeomatric Equation. The second solution is given by $y = C_0 x^{1-\gamma} F[\alpha-\gamma+1, \beta-\gamma+1, 2-\gamma, x]$ where γ is an integer if $\gamma = 1$ the solutions are identical and if γ tends to any other integral value one of the integrals usually ceases to exist.

The above solution can easily be derived by the usual method of Frobenas for solving the differential equation in series.

§ 2.4 LEGENDRE AND ASSOCIATED LEGENDRE FUNCTION: The differential equation

$$(1-x^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} - \lambda y = 0 \qquad (2.16)$$

is called the Legender equation. The end of the interval $[-1,1]$ are singular points of the differential equation if we write $\lambda = -n(n+1)$, n being a positive integer the legender equation takes the form

$$(1-x^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + n(n+1)y = 0 \qquad (2.17)$$

On solving legendre equation in series we get two solutions. First $P_n(x)$ is an algebraic function of x of degree n and is given by

$$P_n(x) = \frac{1.3.5....(2n-1)}{1.2.3...n} [x^n - \frac{n(n-1)}{2(2n-1)} x^{n-2} +$$

$$\frac{n(n-1)\ (n-2)\ (n-3)}{(2n-1)\ (2n-3)} x^{n-4} - ....]$$

$$= \frac{1.3.5...\ (2n-1)}{1.2.3....n} x^n \quad 1 - \frac{n(n-1)}{2(2n-1)} x^{-2}$$

$$+ \frac{n(n-1)(n-2)(n-3)}{2.4(2n-1)(2n-3)} x^{-4} - ...$$

By definition of Hypergeometric function $F(\alpha,\beta,\gamma,x)=1+ \frac{\alpha.\beta}{\gamma} x$

$$+ \frac{\alpha(\alpha+1)\beta(\beta+1)}{2!\,\gamma(\gamma+1)} x^2 + ... \qquad (2.18)$$

Using equation (2.18) $P_n(x)$ can be written as

$$P_n(x) = \frac{(2n)!}{2^n n!n!} x^n F(-\frac{n}{2}, \frac{1-n}{2}, \frac{1-n}{2}, \frac{1}{x^2}) \qquad (2.19)$$

here $P_n(x)$ is a Polynomial of degree n so bounded at the end points that is at x = +1 and x = -1 therefore $P_0(x)$, $P_1(x)$, $P_2(x)$,... are called Legendre function of first kind.

There is a second solution $Q_n(x)$ of the form

$$Q_n(x) = B\{\frac{1}{x^{n+1}} + \frac{(n+1)(n+2)}{2(2n+3)x^{n+3}} +$$

$$\frac{(n+1)(n+2)(n+3)(n+4)}{2.4(2n+3)(2n+5)\ x^{n+5}} + ...\}$$

Using equation (2.18) we have

$$Q_n(x) = B\frac{1}{x^{n+1}} F(\frac{n+1}{2}, \frac{n}{2}+1, \frac{2n+3}{2}, \frac{1}{x^2}) \qquad (2.20)$$

where $Q_0(x)$, $Q_1(x)$,... are called Legender function of the second kind. The following results are important involving Legendre Polynomials

$$P_n(1) = 1 \qquad (2.21)$$

$$P_n(-1) = (-1)^n \qquad (2.22)$$

$$P_n(-x) = (-1)^n P_n(x) \qquad (2.23)$$

$$P_n(0) = \left\{ \begin{array}{ll} 0 & \text{If n is odd} \\ (-1)^{n/2} \dfrac{1.3.5...(n-1)}{2.4.6...2n} & \text{if n is even} \end{array} \right\}$$

(2.24)

$$\int P_n(x)dx = \frac{P_{n+1}(x) - P_{n-1}(x)}{2n+1} \qquad (2.25)$$

The following recurrance formulae for Legendres function are given which will be used in this dissertation.

$$(n+1)P_{n+1}(x) - (2n+1)xP_n(x)+nP_{n-1}(x) = 0 \qquad (2.26)$$

$$P'_{n+1}(x) - xP'_n(x) = (n+1)P_n(x) \qquad (2.27)$$

$$xP_n'(x) - P_{n-1}'(x) = nP_n(x) \tag{2.28}$$

$$(x^2-1)P_n'(x) = nxP_n(x) - nP_{n-1}(x) \tag{2.29}$$

$$P_{n+1}'(x) - P_{n-1}'(x) = (2n+1)\ P_n(x) \tag{2.30}$$

ORTHOGONALITY OF LEGENDRE POLYNOMIALS: It is very useful that Legendre Polynomials form a complete arthogonal set, so that any arbitrary function can be expanded in terms of the Legendre Polynomials. We shall show that the function $\{P_n(x)\}$ are arthogonal in the interval $[-1,1]$ and that

$$\int_{-1}^{+1} P_n(x)\ P_m(x)\ dx = 0 \quad \text{if } m \neq n$$

where $m,n = 0,1,2,\ldots$

and $$\int_{-1}^{+1} P_n(x)\ P_m(x)\ dx = \frac{2}{2n+1} \quad \text{If } m = n$$

i.e. $$\int_{-1}^{+1} [P_n(x)]^2\ dx = \frac{2}{2n+1}, \quad n = 0,1,2,\ldots$$

Let $P_n(x)$ and $P_m(x)$ be two Legendre Polynomials then they satisfy the Legendre equation

(a) $$\frac{d}{dx}\left[(1-x^2)\frac{d}{dx}P_n(x)\right] + n(n+1)P_n(x) = 0$$

(b) $$\frac{d}{dx}[(1-x^2)\frac{d}{dx}P_m(x)] + m(n+1)\ P_m(x) = 0$$

Now multiplying (a) by $P_m(x)$ and (b) by $P_n(x)$ and substracting we have

$$[\ m(m+1) - n(n+1)]\ P_n(x)P_m(x) =$$

$$P_m\ \frac{d}{dx}[\ (1-x^2)\ \frac{d}{dx}P_n(x)] - P_n(x)\ \frac{d}{dx}[(1-x^2)\frac{d}{dx}P_m(x)]$$

Integrating the whole equation over the interval $[-1,1]$ we have

$$m(m+1)-n(n+1)\int_{-1}^{+1} P_n(x)\ P_m(x)\ dx =$$

$$\int_{-1}^{+1} P_m(x)\ \frac{d}{dx}[(1-x^2)\ \frac{d}{dx}P_n(x)]\ dx$$

$$-\int_{-1}^{+1} P_n(x)\ \frac{d}{dx}[(1-x^2)\ \frac{d}{dx}P_m(x)]\ dx$$

Integrating by parts we have

$$(m^2+m-n^2-n)\int_{-1}^{+1} P_m(x)P_n(x)\ dx =$$

$$P_m(x)\ [\ (1-x^2)\frac{d}{dx}P_n(x)]\ \Big|_{-1}^{+1} - \int_{-1}^{+1} [(1-x^2)\frac{d}{dx}P_n(x)]\ [\frac{d}{dx}P_m(x)]dx$$

$$- P_n(x) \left[(1-x^2) \frac{d}{dx} P_m(x) \right]_{-1}^{+1} +$$

$$\int_{-1}^{+1} \left[(1-x^2) \frac{d}{dx} P_m(x)\right] \left[\frac{d}{dx} P_n(x)\right] dx$$

Solving R.H.S. we get

$$(m^2 + m - n^2 - n) \int_{-1}^{+1} P_n(x) P_m(x) \, dx = 0$$

If we assume that $m \neq n$ then $n^2 + m - n^2 - n \neq 0$

Then clearly $\int_{-1}^{+1} P_n(x) P_m(x) \, dx = 0$ for $m \neq n$

Now consider

$\int_{-1}^{+1} [P_n(x)]^2 \, dx$, From the definition of $P_n(x)$

$$\int_{-1}^{+1} [P_n(x)]^2 \, dx = \frac{1}{2^{2n}(n!)^2} \int_{-1}^{+1} \frac{d^n}{dx^n}(x^2-1)^n \frac{d^n}{dx^n}(x^2-1)^n \, dx$$

Integrating by parts R.H.S. we get

$$\int_{-1}^{+1} [P_n(x)]^2 \, dx = \frac{1}{2^{2n}(n!)^2} \left[\frac{d^n}{dx^n} (x^2-1)^n \frac{d^{n-1}}{dx^{n-1}} (x^2-1)^n \right|_{-1}^{+1}$$

$$- \frac{1}{2^{2n}(n!)^2} \int_{-1}^{+1} \frac{d^{n-1}}{dx^{n-1}} (x^2-1)^n \frac{d^{n+1}}{dx^{n+1}} (x^2-1)^n \, dx \;]$$

Repeating this for n times we get

$$\int_{-1}^{+1} [P_n(x)]^2 = \frac{(-1)^n}{2^{2n}(n!)^2} \int_{-1}^{+1} (x^2-1)^n \frac{d^{2n}}{dx^{2n}} (x^2-1)^n \, dx$$

$$= \frac{(-1)^n}{2^{2n}(n!)^2} \int_{-1}^{+1} (x^2 - 1)^n \, 2n! \, dx$$

$$= \frac{2.2n!}{2^{2n}(n!)^2} \int_{0}^{1} (1-x^2)^n \, dx$$

Let $x = \mathrm{Sin}\theta$ $\quad dx = \mathrm{Cos}\theta \, d\theta$

$$(1 - x^2)^n = (1-\mathrm{Sin}^2\theta)^2 = (\mathrm{Cos}^2\theta)^n$$

Therefore $(1-x^2) \, dx = \mathrm{Cos}^{2n+1}\theta \;\; d\theta$

Therefore $\displaystyle\int_{-1}^{+1} [P_n(x)]^2 \, dx = \frac{2.2n!}{2^{2n}(n!)^2} \int_{-0}^{\pi/2} \mathrm{Cos}^{2n+1}\theta \, d\theta$

$$= \frac{2.2n! \quad 2n(2n-2) \quad \ldots. \quad 4.2}{2^{2n}(n!)^2 \; (2n+1) \; (2n-1) \ldots \; 5.3}$$

$$= \frac{2.2 \; n! \; 2^n . \; n!}{2^{2n}(n!)^2(2n+1)(2n-1) \ldots \; 5.3}$$

$$= \frac{2}{(2n+1)}$$

Thus we get the following result

$$\int_{-1}^{+1} P_n(x) \; P_m(x) \; dx = \frac{2}{2n+1} \quad \delta \; nm$$

where $\delta \, mn = \{ \begin{matrix} 0 & \text{If } m \neq n \\ 1 & \text{If } m = n \end{matrix} \}$

So $P_0(x)$, $P_1(x)$ .... form an orthogonal set and $\sqrt{\frac{2n+1}{2}}$ form an orthonarmal set. Since $P_n(x)$ and $Q_n(x)$ are linearity independent. Thus a general solution of the Legendre equation is given by

$$y = A.P_n(x) + B \; Q_n(x)$$

where A and B are arbitrary constants.

The differential equation

$$(1-x^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + [\,n(n+1) - \frac{m^2}{1-x^2}\,]\,y = 0 \qquad (2.31)$$

is called Associated Legendre equation, its solution are given by

$$y_1 = P_n^m(x) = (1-x^2)^{m/2}\frac{d^n}{dx^n}P_n(x) \qquad (2.32)$$

$$y_2 = Q_n^m(x) = (1-x^2)^{m/2}\frac{d^n}{dx^n}Q_n(x) \qquad (2.33)$$

the function $P_n^m(x)$ are called as Associated Legendre function of the first kind, and $Q_n^m(x)$ are known as Associated Legendre function of the second kind.

The Legendre Polynomial $P_n(x)$, $n = 0,1,2,\ldots.$ are particular solution of the differential equation

$$(1-x^2)\frac{dy^2}{dx^2} - 2x\frac{dy}{dx} + n(n+1)\,y = 0 \qquad (2.34)$$

Here $y = P_n(x)$ satisfies the above equation differentiating the equation (2.34) m times by Leibnitz theorem we have

$$(1-x^2)\frac{d^{m+2}y}{dx^{m+2}} - 2(m+1)\,xy^{m+1} + [\,n(n+1) - m(m+1)\,]\,y^m = 0$$

$$(2.35)$$

Putting $y = (1-x^2)^{m/2}\,U$

Where $U = \frac{d^m}{dx^m} P_n(x)$ in equation (2.31) which gives us the equation.

$$(1-x^2)U'' - 2(m+1)U' + [n(n+1) - m(m+1)]U = 0 \quad (2.36)$$

Comparing equation (2.35) and (2.36) we see that $y = (1-x^2)^{m/2}U$ is the solution of Legendre Associated equation.

Thus $P_n^m(x) = y = (1-x^2)^{m/2} \frac{d^m}{dx^m} P_n(x)$

$$0 \leqslant m \leqslant n$$

and $Q_n^m(x) = y = (1-x^2)^{m/2} \frac{d^m}{dx^m} Q_n(x)$

are solution of equation (2.31)

Rodrigues formula for $P_n^m(x)$ is

$$P_n^m(x) = (1-x^2)^{m/2} \frac{d^{n+m}}{dx^{n+m}} (x^2-1)^n \quad (2.37)$$

We also have the following results

$$P_n^0(x) = P_n(x) \quad (2.38)$$

$$P_n^m(x) = 0 \qquad \text{If } m > n \quad (2.39)$$

$$P_n^{-m}(x) = \frac{(-1)^m (n-m)!}{(n+m)!} P_n^m(x) \qquad (2.40)$$

Recurrence formulae for $P_n^m(x)$ are

$$(n+1-m)P_{n+1}^m(x) - (m+1)x\, P_n^m(x) + (n+m)P_{n-1}^m(x) = 0 \qquad (2.41)$$

$$P_n^{m+2}(x) - \frac{2(m+1)}{(1-x^2)^{\frac{1}{2}}} P_n^{m+1}(x) + (n-m)(n+m+1)P_n^m(x)=0 \qquad (2.42)$$

ORTHOGONALITY OF ASSOCIATED LEGENDRE FUNCTIONS: The Associated Legendre functions also form an orthogonal set.

Let $P_n^m(x)$ and $P_l^m(x)$ be two distinct associated Legendre Polynomials satisfying the Associated Legendre equations

$$\left[\frac{d}{dx}\left\{ (1-x^2)\frac{d}{dx} - \frac{m^2}{1-x^2} \right] P_n^m(x) = -\,n(n+1)\; P_n^m(x) \qquad \ldots \qquad (I)$$

$$\left[\frac{d}{dx}\left\{ (1-x^2)\frac{d}{dx} - \frac{m^2}{1-x^2} \right] P_l^m(x) = -\,l(l+1)\; P_l^m(x) \qquad \ldots \qquad (II)$$

Multiplying equation I by $P_l^m(x)$ and II by $P_n^m(x)$ and substracting we get

$$[\, l(l+1) - n(n+1)\,]\; P_n^m(x)\; P_l^m(x) =$$

$$P_l^m(x)\; \frac{d}{dx}\, [(1-x^2)\; \frac{d}{dx}\;]\; P_n^m(x) \; -$$

$$P_n^m(x)\; \frac{d}{dx}\, [\,(1-x^2)\; \frac{d}{dx}\;]\; P_l^m(x)$$

Integrating the whole equation over the interval [-1,1] we have

$$[l(l+1) - n(n+1)] \int_{-1}^{+1} P_n^m(x)\ P_l^m(x)\ dx =$$

$$\int_{-1}^{+1} [P_l^m(x) \frac{d}{dx} \{(1-x^2) \frac{d}{dx}\} P_n^m(x) -$$

$$P_n^m(x) \frac{d}{dx} \{(1-x^2) \frac{d}{dx}\} P_l^m(x)]\ dx$$

Integrating by parts R.H.S. which gives us

$$P_l^m(x) \{ (1-x^2) \frac{d}{dx} P_n^m(x) \} \Big|_{-1}^{+1}$$

$$- \int_{-1}^{+1} \{(1-x^2) \frac{d}{dx} P_n^m(x) \} \frac{d}{dx} P_l^m(x)\ dx$$

$$- P_n^m(x) \{ (1-x^2) \frac{d}{dx} P_l^m(x) \} \Big|_{-1}^{+1} +$$

$$\int_{-1}^{+1} [\{(1-x^2)\frac{d}{dx} P_l^m(x) \} \frac{d}{dx} P_n^m(x)]\ dx$$

Since $\int_{-1}^{+1} P_n^m(x)\ P_l^m(x)\ dx = 0$ when $n \neq l$

Take $n = l$

$$\int_{-1}^{+1} [\, P_n^m(x)]^2 \, dx = \int_{-1}^{+1} P_n^m(x) \, P_n^m(x) \, dx$$

$$= \frac{(n+m)!}{(-1)^m (n-m)!} \int_{-1}^{+1} P_n^m(x) \, P_n^{-m}(x) \, dx$$

using the Rodrignes formula for $P_n^m(x)$ we have

$$= \frac{(n+m)!}{(-1)^m (n-m)!} \int_{-1}^{+1} (1-x^2)^{m/2} \frac{d^m}{dx^m} P_n(x)$$

$$\times \, [\, (1-x^2)^{-m/2} \frac{d^{-m}}{dx^{-m}} P_n(x) \,] \, dx$$

$$= \frac{(n+m)!}{(-1)^m (n-m)!} \int_{-1}^{+1} \frac{d^m}{dx^m} P_n(x) \, \frac{d^{-m}}{dx^{-m}} P_n(x) \, dx$$

$$= \frac{(n+m)!}{(-1)^m (n-m)!} \left[\frac{d^{-m}}{dx^{-m}} P_n(x) \, \frac{d^{m-1}}{dx^{m-1}} P_n(x) \right] \Bigg|_{-1}^{+1}$$

$$- \int_{-1}^{+1} \frac{d^{m-1}}{dx^{m-1}} P_n(x) \, \frac{d^{-m+1}}{dx^{-m+1}} P_n(x) \, dx$$

$$= \frac{(n+m)!}{(-1)^m (n-m)!} \; - \left[\int_{-1}^{+1} \frac{d^{-m+1}}{dx^{-m+1}} P_n(x) \, \frac{d^{m-1}}{dx^{m-1}} P_n(x) \, dx \right]$$

Repeating this process m-time we get

$$= \frac{(n+m)!(-1)^m}{(-1)^m(n-m)!} \int_{-1}^{+1} P_n(x)\ P_n(x)\ dx$$

$$= \frac{2(n+m)}{(n-m)!(2n+1)}$$

Thus $$\int_{-1}^{+1} P_n^m(x)\ P_1^m(x)\ dx = \frac{2(n+m)!\ \delta_{n1}}{(n-m)!(2n+1)} \qquad (2.43)$$

Where $$\delta n1 = \left\{ \begin{matrix} 0 & n \neq 1 \\ 1 & n = 1 \end{matrix} \right\}$$

§2.5 BESSEL FUNCTIONS: The differential equation

$$x^2 \frac{d^2y}{dx^2} + x\frac{dy}{dx} + (x^2 - n^2)\ y = 0 \qquad (2.44)$$

is called the Bessel Equation of order n. The solution of this equation are the Bessel Function. Some special solution of this equation are the Bessel, Newmann and Hankle Functions

$J_n(x)$, $y_n(x)$, $H_n^{(1)}(x)$, $H_n^{(2)}(x)$ respectively. The later three are the linear combination of the first $J_n(x)$ is define as

$$J_n(x) = \sum_{k=0}^{\infty} \frac{(-1)^k (\frac{x}{2})^{2k+n}}{K!\ \Gamma(n+k+1)} \qquad (2.45)$$

and is known as Bessel Function of the first kind of order n.

$$J_{-n}(x) = \sum_{k=0}^{\infty} \frac{(-1)^k (\frac{x}{2})^{2k-n}}{K!\,\Gamma(-n+k+1)} \tag{2.46}$$

are also solution of Bessel equation and are called Bessel Function of first kinds of order -n. The function $J_n(x)$ and $J_{-n}(x)$ are linearly independent provided n is not an integer or zero. If n is a Positive integer then we have the following results:

$$J_{-n}(x) = (-1)^n J_n(x) \tag{2.47}$$

i.e. $J_n(x)$ and $J_{-n}(x)$ are linear dependent for n as an integer. Thus general solution of the Bessel's differential equation is

$$y = A\,J_n(x) + B\,J_{-n}(x) \tag{2.48}$$

where A and B are arbitrary constants. The following results for Bessel function can be verified.

$$\frac{d}{dx}[x^n J_n(x)] = x^n J_{n-1}(x) \tag{2.49}$$

$$\frac{d}{dx}[x^{-n} J_n(x)] = -x^{-n} J_{n+1}(x) \tag{2.50}$$

$$x\, J_n'(x) = J_n(x) - x\, J_{n+1}(x) \tag{2.51}$$

Adding and substracting (2.50) and (2.51) gives us

$$J_{n-1}(x) + J_{n+1}(x) = \frac{2n}{x} J_n(x) \tag{2.52}$$

$$J_{n-1}(x) - J_{n+1}(x) = 2J_n'(x) \tag{2.53}$$

To find the general solution of Bessel equation when n is an integer. We have to find a second solution of the equation which will be linearly independent to $J_n(x)$. For this second solution $y_n(x)$ valid for all values of n is defined by

$$y_n(x) = \frac{J_n(x)\ \mathrm{Cos}\ n\pi - J_{-n}(x)}{\mathrm{Sin}\ n\pi} \tag{2.54}$$

It is linear combination of $J_n(x)$ and $J_{-n}(x)$. If n is an integer or equal to zero, then $J_n(x)$ will take an indeterminate form thus we can define $y_n(x)$ as

$$y_n(x) = \lim_{\rho \to n} \frac{J_\rho(x)\ \mathrm{Cos}\ 1\pi - J_{-\rho}(x)}{\mathrm{Sin}\rho\ \pi} \tag{2.55}$$

called Bessel function of the second kind of order n as weber function or Neumann function Function. It is very useful to have a third type of Bessel functions which are solution of the Bessel differential equation known as the Hankle Function of the first and second kind define as

$$H_n^{(1)}(x) = J_n(x) + i\, y_n(x) \quad - \qquad (2.56)$$

$$H_n^{(2)}(x) = J_n(x) - i\, y_n(x) \qquad (2.57)$$

These are called Bessel function of the third kind.

$$x\, y'' + 2xy' + (x^2 - l(l+1)\; y = 0$$

$$l = 0,1,2,3,\ldots.$$

Putting $y(x) = \dfrac{u(x)}{\sqrt{x}}$

$$x^2 \frac{d^2u}{dx^2} + x \frac{du}{dx} + [\, x^2 - (l + \tfrac{1}{2})^2\,]\, u = 0$$

$$\rightarrow \quad y(x) = C_1\, J_{l+\frac{1}{2}}(x) + C_2\, J_{-l-\frac{1}{2}}(x)$$

Called Spherical Bessels Functions.

§ 2.6 ROOTS OF BESSEL'S FUNCTIONS AND THEIR ORTHOGONAL PROPERTIES:

Consider $x^2 \dfrac{d^2y}{dx^2} + x \dfrac{dy}{dx} + (k^2x^2 - n^2)\, y = 0$ (2.58)

Set $\rho = kx$

$$\rho^2 \frac{d^2y}{d\rho^2} + \rho \frac{dy}{d\rho} + (\rho^2 - n^2)\; y = 0$$

Therefore $y = J_n(\rho) = J_n(kx)$ is a solution of (2.58).

From (2.58)

$$\frac{d}{dx}\left[x \frac{d}{dx} J_n(kx) \right] + (k^2 x - \frac{n^2}{x}) J_n(kx) = 0$$

Writting $k = \lambda , \mu$

$$\frac{d}{dx}\left[x \frac{d}{dx} J_n(\lambda x) \right] + \left[\lambda^2 x - \frac{n^2}{x} \right] J_n(\lambda x) = 0$$

$$\frac{d}{dx}\left[x \frac{d}{dx} J_n(\mu x) \right] + \left[\mu^2 x - \frac{n^2}{x} \right] J_n(\mu x) = 0$$

Multiplying first by with $J_n(\mu x)$ and second with $J_n(\lambda x)$ and substract

$$(\lambda^2 - \mu^2)\, x\, J_n(\lambda x)\, J_n(\mu x) =$$

$$\frac{d}{dx}\left[x\, J_n(\lambda x)\, J_n'(\mu x) - x\, J_n(\mu x)\, J_n'(\lambda x) \right]$$

$$(\lambda^2 - \mu^2) \int_0^a x\, J_n(\lambda x)\, J_n(\mu x)\, dx$$

$$= \Big| x\, (J_n(\lambda x)\, J_n'(\mu x) - J_n(\mu x) J_n'(\lambda x)) \Big|_{n=0}^{n=a}$$

Now the lowest power of n in the series for

$$x \left[J_n(\lambda x)\, J_n'(\mu x) - J_n(\mu x)\, J_n'(\lambda x) \right]$$

is x^{2n+2}, that the right hand side vanishes for $n = 0$

Provided $n > -1$ and we have

$$\int_0^a x\, J_n(\lambda x)\, J_n(\mu x)\, dx$$

$$= \frac{a}{\lambda - \mu^2} \left[\mu J_n(\lambda a)\, J_n'(\mu a) - \lambda J_n(\mu a)\, J_n'(\lambda a)\right]$$

For each fixed n, $J_n(\xi)$ has infinite numbers of +ive roots and these can be ordered $0 < \xi_{n1} < \xi_{n2} < \ldots < \xi_{nm} < \xi_{nm+1} < \ldots$

If $x > -1$ settin g

$$\lambda = \lambda_{nm} = \frac{\xi_{nm}}{a} \qquad \text{and}$$

$$\mu = \mu_{nm'} = \frac{\xi_{nm'}}{a}$$

where ξ_{nm} and $\xi_{nm'}$ are two distinct positive roots of $J_n(\xi)$

So $J_n(\xi)=0$, $\displaystyle\int_0^a x\, J_n\left(\xi_{nm} \frac{x}{a}\right) J_n\left(\xi_{nm'} \frac{x}{a}\right) dx = 0$

$$m \neq m'$$

Now we evaluate

$$\int_0^a x \left[J_n\left(\xi_{nm} \frac{x}{a}\right)\right]^2 dx$$

$$x^2 y'' + xy' + (k^2x^2 - n^2) y = 0$$

Multiplying by 2y'

$$2x^2y'y'' + 2xy'^2 + 2(k^2x^2 - n^2) yy' = 0$$

or
$$\frac{d}{dx} [x^2(\frac{dy}{dx})^2 - n^2y^2 + k^2x^2y^2] = 2k^2x y^2$$

or
$$2k^2 \int_0^a xy^2dx = x^2y'^2 - n^2y^2 + k^2x^2y^2$$

$$\int_0^a x J_n(kx)dx = \frac{x^2}{2} [(1- \frac{n^2}{k^2x^2}) J_n^2(kx) + \{ J_n'(kx)\}^2]$$

Using $n J_n(x) + x J_n'(x) = x J_{n-1}(x)$ and $n J_n(x) - x J_n'(x) = x J_{n+1}(x)$

$$\int_0^a x J_n^2(kx) dx = \frac{x^2}{2} [J_n^2(kx) - J_{n-1}(k) J_{n+1}(k)]$$

or
$$\int_0^a x J_n^2(kx) dx =$$

$$= \frac{a^2}{2} [1- \frac{n^2}{k^2 a^2}) J_n^{\,2}(ka) + \{ J_n'(ka)\}^2]$$

$$= \frac{a^2}{2} [J_n^2(ka) - J_{n-1}(ka) J_{n+1}(ka)]$$

If $k = \frac{\xi_{nm}}{a}$ is a root of $J_n(\xi)$

Therefore $J_n(\xi) = 0$

Then $\int_b^a x J_n^2(\xi_{nm} \frac{x}{a}) dx =$

$$\frac{a^2}{2} [J_n'(\xi_{nm})]^2$$

$$= \frac{a^2}{2} [J_{n+1}(\xi_{nm})]^2$$

CHAPTER III

SOLUTION OF LAPLACE'S EQUATION IN TWO AND THREE DIMENSIONS:

In different Co-ordinates system Laplace's Equation takes different forms. So we have solution of this equation in each Co-ordinate system.

§ 3.1 SOLUTION OF LAPLACE'S EQUATION IN TWO DIMENSIONS:

I. CARTISION CO-ORDINATE: In Cartesion Coordinate (x,y) the two dimensional Laplaces' Equation is given by

$$\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} = 0 \qquad (3.1)$$

Using method of separations of variables, by assuming

$$U = X(x)\, Y(y)\ , \qquad \frac{\partial U}{\partial y} = X\frac{\partial Y}{\partial y}$$

$$\frac{\partial U}{\partial x} = Y\frac{\partial X}{\partial x}\ , \qquad \frac{\partial^2 U}{\partial y^2} = X\frac{\partial^2 Y}{\partial y^2}$$

$$\frac{\partial^2 U}{\partial x^2} = Y\frac{\partial^2 X}{\partial x^2}\ ,$$

Hence $\dfrac{\partial^2 U}{\partial x^2} + \dfrac{\partial^2 U}{\partial y^2} = 0$

will take the form

$$Y \frac{\partial^2 X}{\partial x^2} + X \frac{\partial^2 Y}{\partial y^2} = 0$$

Deviding both sides by XY we have

$$\frac{1}{X} \frac{\partial^2 X}{\partial x^2} + \frac{1}{Y} \frac{\partial^2 Y}{\partial y^2} = 0$$

or

$$\frac{1}{X} \frac{\partial^2 X}{\partial x^2} = - \frac{1}{Y} \frac{\partial^2 Y}{\partial y^2} = - P^2 \text{ (say)}$$

Then

$$\frac{\partial^2 X}{\partial x^2} = - P^2 X \qquad (3.2)$$

and

$$\frac{\partial^2 Y}{\partial y^2} = P^2 Y \qquad (3.3)$$

From (3.2)

$$X = A \operatorname{Cos} Px + B \operatorname{Sin} Px$$

and From (3.3) $Y = Ce^{Py} + De^{-Py}$

Hence $U = X(x)\, Y(y) = (A \operatorname{Cos} Px + B \operatorname{Sin} Px)(Ce^{Py}+De^{Py})$

(3.4)

Where A, B, C and D are arbitrary constants. On the other hand differential equation

$$\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial x^2} = 0$$

which we reduce to the canonical form

$$\frac{\partial^2 U}{\partial \alpha \partial \beta} = 0$$

By writing $x + iy = \alpha$, $x - iy = \beta$

The complete solution of

$$\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} = 0 \text{ is } U = f(x + iy) + g(x-iy)$$

where f and g are arbitrary functions. It is evident that if $u + iv = f(x + iy)$ and $u - iv = g(x - iy)$
Then U and v are solutions of Laplace's Equation and they also satify the equation

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} , \quad \frac{\partial u}{\partial y} = - \frac{\partial v}{\partial x}$$ and are called Congugate Functions.

II. PLANE POLAR CO-ORDINATES: Plane Polar Co-ordinates (r, θ) are related to Cartesion coordinates by equations

$$x = r \operatorname{Cos}\theta \quad , y = r \operatorname{Sin}\theta$$

So that $\tan\theta = y/x$ and $x^2 + y^2 = r^2$ and using chain rule we have

$$\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} = \frac{\partial^2 U}{\partial r^2} + \frac{1}{r}\frac{\partial U}{\partial r} + \frac{1}{r^2}\frac{\partial^2 U}{\partial \theta^2} = 0 \qquad (3.4)$$

Now we solve this equation by putting $U = r^n F(\theta)$

The above equation gives $F''(\theta) + n^2 F(\theta) = 0$

So that $F(\theta) = A \text{ Cos } n\theta + B \text{ Sin } n\theta$

Thus $U = r^n (A \text{ Cos } n\theta + B \text{ Sin } n\theta)$ (3.5)

Where A, B are arbitrary constants.

§ 3.2 SOLUTION OF LAPLACE'S EQUATION IN THREE DIMENSIONS USING CYLINDRICAL POLAR AND CARTISION COORDINATES.

I. Now we are considering Solution of Laplace's Equation in three dimensions, using different coordinates. First we use cylindrical coordinates (r, θ, z) by using the equations $x = r \text{ Cos}\theta$, $y = r \text{ Sin}\theta$ and $z = z$

The Laplace's Equation has the form

$$\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} + \frac{\partial^2 U}{\partial z^2} = \frac{\partial^2 U}{\partial r^2} + \frac{1}{r}\frac{\partial U}{\partial r} + \frac{1}{r^2}\frac{\partial^2 U}{\partial \theta^2} + \frac{\partial^2 U}{\partial z^2} = 0 \qquad (3.6)$$

By separation of variables its solution is given by

$$U(r, \theta, z) = R_{(r)}\, \Theta_{(\theta)}\, Z_{(z)}$$

$$\frac{R''}{R} + \frac{1}{r}\,\frac{R'}{R} + \frac{1}{r^2}\,\frac{\Theta''}{\Theta} = -\frac{Z''}{Z}$$

R.H.S. contain only Z. Therefore

$$\frac{Z''}{Z} = P^2 \quad \text{(say)}$$

where P is constant

$$Z = Ae^{Pz} + Be^{-Pz} \quad \text{then}$$

$$\frac{R''}{R} + \frac{1}{r} + \frac{R'}{R} + \frac{1}{r^2}\,\frac{\Theta''}{\Theta} = -P^2$$

$$r^2\,\frac{R''}{R} + r\,\frac{R'}{R} + P^2 r^2 = \frac{\Theta''}{\Theta}$$

Since the R.H.S. of the equation given above contain only Θ

Therefore

$$-\frac{\Theta''}{\Theta} = m^2$$

Thus $\Theta = A_1 \cos m\,\theta + B_1 \sin m\,\theta$

Now consider the equation

$$\frac{r^2R''}{R} + \frac{rR'}{R} + P^2r^2 - m^2 = 0$$

or $$r^2R'' + rR' + (P^2r^2 - m^2)\, R = 0$$

This is the well known equation of Bessel function and its solution is

$$A_2\, J_m(Pr) + B_2\, Y_m(Pr)$$

Where $J_m(Pr)$ and $Y_m(Pr)$ are Bessel's Functions. Which we have the already discussed in Chapter 2, and thus the solution of (3.6) is $U = R_r\, \Theta_\theta\, Z_z = (Ae^{Pz} + Be^{-Pz})(A_1 Cosm\theta + B_1 Sinm\theta)$

$$x\ (\ A_2\, J_m(Pr) + B_2\, Y_m(Pr)) \qquad (3.7)$$

II. <u>CARTISION COORDINATES:</u>

In Cartision Coordinates Laplace's Equation

$$\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} + \frac{\partial^2 U}{\partial z^2} = 0$$

evidently possesses particular solution of the form

$$U = u(x)v(y)w(z) \qquad (3.8)$$

Then Laplace's Equation takes the form

$$\frac{1}{u}\frac{d^2U}{dx^2} + \frac{1}{v}\frac{d^2v}{dy^2} + \frac{1}{w}\frac{d^2w}{dz^2} = 0$$

Hence, if we write

$$\frac{1}{u}\frac{d^2u}{dx^2} = l^2, \quad \frac{1}{v}\frac{d^2v}{dy^2} = m^2 \quad \text{and} \quad \frac{1}{w}\frac{d^2w}{dz^2} = n^2$$

Where $\quad l^2 + m^2 + n^2 = 0$

Equation (3.6) will be satisfied. The preceding equations may be satisfied by taking $u = e^{lx}$, $v=e^{my}$ and $w = e^{nz}$. l, m, n being constants. Hence

$$U = e^{lx + my + nz} \tag{3.9}$$

is a particular solution of the Laplaces' Equation. The usefulness of Particular Solution of Equation (3.6) is increased by the fact that since the equation is linear, the sum of any number of solutions is also a solution. We have in fact

$$\nabla^2(U + V + W + \dots) = \nabla^2 U + \nabla^2 V + \nabla^2 W + \dots.$$

which is zero if U,V,W ... are solution of Laplace's Eqn.

§ 3.3 SOLUTION OF LAPLACE EQUATION IN THREE DIMENSION USING SPHERICAL POLAR COORDINATES.

We now transform Laplace's Equation into Polar Coordinates (r, θ, ϕ), where $x = r \sin\theta \cos\phi$,

$y = r \sin\theta \ \sin\phi$

$z = r \cos\theta$ clearly $x^2 + y^2 + z^2 = r^2$

Let $r \sin\theta = \rho$ Then

$x = \rho \cos\phi$ $y = \rho \sin\phi$ and $z = r \cos\theta$

First of all

$$\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} = \frac{\partial^2 U}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial U}{\partial \rho} + \frac{1}{\rho^2}\frac{\partial^2 U}{\partial \phi^2}$$

Then $$\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} + \frac{\partial^2 U}{\partial z^2} = \frac{\partial^2 U}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial U}{\partial \rho} + \frac{1}{\rho^2}\frac{\partial^2 U}{\partial \phi^2} + \frac{\partial^2 U}{\partial z^2} \qquad (3.10)$$

Now $$\frac{\partial^2 U}{\partial \rho^2} + \frac{\partial^2 U}{\partial z^2} = \frac{\partial^2 U}{\partial r^2} + \frac{1}{r}\frac{\partial U}{\partial r} + \frac{1}{r^2}\frac{\partial^2 U}{\partial \theta^2} \qquad (a)$$

Because $z = r \cos\theta$ and $\rho = r \sin\theta$ and $\tan\theta = \rho/z$

$z^2 + \rho^2 = r^2$

Then

$$\frac{\partial U}{\partial \rho} = \sin\theta \frac{\partial U}{\partial r} + \frac{\cos\theta}{r}\frac{\partial U}{\partial \theta} \qquad (b)$$

Putting (a) and (b) in equation (3.10) we have

$$\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} + \frac{\partial^2 U}{\partial z^2} = \frac{\partial^2 U}{\partial r^2} + \frac{2}{r}\frac{\partial U}{\partial r} + \frac{1}{r^2}\frac{\partial^2 U}{\partial \theta^2} + \frac{\cot\theta}{r^2}\frac{\partial U}{\partial \theta} + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2 U}{\partial \theta^2} = 0$$

Multiplying above equation by $r^2 \sin\theta$ we have

$$r^2 \sin\theta \frac{\partial^2 U}{\partial r^2} + 2r \sin\theta \frac{\partial U}{\partial r} + \sin\theta \frac{\partial^2 U}{\partial \theta^2}$$

$$+ \cos\theta \frac{\partial U}{\partial \theta} + \operatorname{cosec}\theta \frac{\partial^2 U}{\partial \phi^2} = 0$$

or $$\sin\theta \frac{\partial}{\partial r} \left(r^2 \frac{\partial U}{\partial r} \right) + \frac{\partial}{\partial \theta} \left(\sin\theta \frac{\partial U}{\partial \theta} \right)$$

$$+ \operatorname{cosec}\theta \frac{\partial^2 U}{\partial \phi^2} = 0 \qquad (3.11)$$

Now we solve this equation by method of separation of variables. By putting

$$U = u(r)v(\theta)w(\phi)$$

We write (3.11) as $$uvw \left\{ \frac{1}{u} \sin\theta \frac{\partial}{\partial r} \left(r^2 \frac{\partial u}{\partial \theta}\right) + \right.$$

$$\left. \frac{1}{v} \frac{\partial}{\partial \theta} \left(\sin\theta \frac{\partial v}{\partial \theta}\right) + \frac{1}{w} \operatorname{cosec}\theta \frac{\partial^2 w}{\partial \phi^2} \right) \} = 0$$

Now separating the variables we have

$$\frac{d^2 w}{d\phi^2} + m^2 \phi = 0$$

$$\frac{d}{dr}\left(r^2 \frac{du}{dr}\right) = n(n+1)u$$

$$\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{dv}{d\theta}\right)+\left[n(n+1)-\frac{m^2}{\sin^2\theta}\right]v=0$$

Where m, and n are constants.

$$\frac{d^2v}{d\theta^2}+\cot\theta\frac{dv}{d\theta}+\left[n(n+1)-\frac{m^2}{\sin^2\theta}\right]v=0 \qquad (3.12)$$

If we omit v the above equation is called Legendre Operator

In this equation we put $\cos\theta = \sigma$

$$\frac{dv}{d\theta}=\frac{dv}{d\sigma}\frac{d\sigma}{d\theta}=-\sin\theta\frac{dv}{d\sigma}$$

and $$\frac{d^2v}{d\theta^2}=\frac{d}{d\theta}\left(-\sin\theta\frac{dv}{d\sigma}\right)=\sin^2\theta\frac{d^2v}{d\sigma^2}-\cos\theta\frac{dv}{d\sigma}$$

i.e.

$$\frac{d^2v}{d\theta^2}=\sin^2\theta\frac{d^2v}{d\sigma^2}-\cos\theta\frac{dv}{d\sigma}$$

Thus equation (3.12) will take the form

$$(1-\sigma^2)\frac{d^2v}{d\sigma^2}-2\sigma\frac{dv}{d\sigma}+\left[n(n+1)-\frac{m^2}{1-\sigma^2}\right]v=0 \qquad (3.13)$$

When m = 0 this reduces to Legendre's Equation. The equation

$\frac{d}{dr}\left(r^2\frac{du}{dr}\right)=n(n+1)u$ is satisfied by

$u=r^n$ and $u=r^{-(n+1)}$

Hence we have particular solution of Laplace's Equation of this types

$$U(r,\theta,\phi) = r^n \operatorname{Cos}(m\phi+\varepsilon)\, P_n^m(\sigma)$$

$$U(r,\theta,\phi) = r^{-(n+1)} \operatorname{Cos}(m\phi+\varepsilon)\, P_n^m(\sigma)$$

Where $\operatorname{Cos}(m\phi+\varepsilon)$ is a solution of the equation containing w and $P_n^m(\sigma)$ is a solution of equation (3.13). A comparison of these two solutions suggests the following theorem.

Theorem (3.1): If $U = F(r, \theta, \phi)$ is a solution of Laplace's equation then the function $U = \frac{1}{r} \cdot F(\frac{1}{r}, \theta, \phi)$ is also a solution of Laplace's Equation.

Proof: Consider the equation

$$\operatorname{Sin}\theta \frac{\partial}{\partial r}\left(r^2 \frac{\partial U}{\partial r}\right) + \frac{\partial}{\partial \theta}\left(\operatorname{Sin}\theta \frac{\partial U}{\partial \theta}\right) + \operatorname{Cosec}\theta \frac{\partial^2 U}{\partial \phi^2} = 0$$

Putting $r = 1/s$ it becomes

$$\operatorname{Sin}\theta \left(s^2 \frac{\partial^2 U}{\partial s^2}\right) + \frac{\partial}{\partial \theta}\left(\operatorname{Sin}\theta \frac{\partial U}{\partial \theta}\right) + \operatorname{Cosec}\theta \frac{\partial^2 U}{\partial \phi^2} = 0$$

A further substitution $U = sV$ reduces this to the form

$$\operatorname{Sin}\theta \left(s^2 \frac{\partial^2 V}{\partial s^2} + 2s \frac{\partial V}{\partial s}\right) + \frac{\partial}{\partial \theta}\left(\operatorname{Sin}\theta \frac{\partial V}{\partial \theta}\right) + \operatorname{Cosec}\theta \frac{\partial^2 V}{\partial \phi^2} = 0$$

Which is the same as (3.11) but with s, V written in place of r and U respectively. Hence the theorem is proved. The preceding theorem enables us to derive, one solution of Laplace's equation from another by means of the transformation of Coordinates

$$x' = \frac{x}{r^2}\ ,\ y' = \frac{y}{r^2}\ ,\ z' = \frac{z}{r^2}$$

This transformation is known as inversion. It was applied with great success to electrostatic problems by Lord Keluin. The preceding theorem may also enunciated as follows:

Theorem:- If F(x, y, z) is a solution of Laplace's Equation then $\frac{1}{r} F\left(\frac{x}{r^2}, \frac{y}{r^2}, \frac{z}{r^2}\right)$ is also a solution of Laplace's equation.

On the other hand we can also prove another theorem which is as follows:

Theorem:- If U is any solution of Laplace's Equation of degree n then

$$\frac{\partial^q U}{\partial x^q} + \frac{\partial^r U}{\partial y^r} + \frac{\partial^t U}{\partial z^t}$$ is also a solution of Laplace's Equation of degree n - (q + r + t).

Proof:- Since $\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} + \frac{\partial^2 U}{\partial z^2} = 0$

Then differentiating the above equation q time, with respect to x r times with respect to y and t times with respect to z we have

$$\frac{\partial^{q+r+t+2} U}{\partial x^{q+2} \partial y^{r} \partial z^{t}} + \frac{\partial^{q+r+t+2} U}{\partial x^{q} \partial y^{r+2} \partial z^{t}} + \frac{\partial^{q+r+t+2} U}{\partial x^{q} \partial y^{r} \partial z^{t+2}} = 0$$

i.e. $$\nabla^2 \left(\frac{\partial^{q+r+t} U}{\partial x^{q} \partial y^{r} \partial z^{t}} \right) = 0$$

which proves the theorem.

§ 3.4: ORDINARY SOLUTION OF LAPLACE'S EQUATION IN THREE DIMENSIONS

The most important solution of Laplace's Equation of degree n, where n is a positive integer are those which are Polynomials of degree n in (x,y,z). This kind of solution which together with the corresponding solution of negative degree $-(n+1)$ obtained on multiplication by $r^{-(2n+1)}$ may be spoken of as Ordinary or Complete Solution of Laplace's Equation will now be considered.

The most general homogeneous Polynomials of degree n contains $\frac{1}{2}(n+1)(n+2)$ arbitrary coefficients and if the expression be put in Laplace's equation there arises an expression of degree n-2 equated to zero. Since the coefficient of each term, involving

$x^{\alpha}\ y^{\beta}\ z^{\gamma}$, where $\alpha+\beta+\gamma = n-2$ must be zero, $\frac{1}{2}n(n-1)$ relation must be satisfied between the coefficients of the original polynomials $\frac{1}{2}n(n+1)(n+2)$ in number, in order that it may be a solution of Laplace's equation. If all these relations are independent of one another, $\frac{1}{2}n(n-1)$ of the Coefficients can be determined in terms of the remainder, and thus the most general solution of Laplace's Equation of the prescribed type contains $\frac{1}{2}n(n+1)(n+2) - \frac{1}{2}n(n-1)$ or $2n+1$, independent solution of Laplace's Equation of the prescribed type, any other solution of Laplace's Equation of the type would be a linear function of these. For example, three independent solution of Laplace's Equation of degree 1 are x, y, z and of degree 2, the expressions y^2-z^2, z^2-x^2, yz, zx, xy are five independent solution of Laplace's Equation.

By substitution of $r\sin\theta\cos\phi$, $r\sin\theta\sin\phi$, $r\cos\theta$, for x, y, z respectively, in the most general homogeneous polynomials of degree n in (x, y, z), and by expressing the terms in $\cos^p\phi\,\sin^q\phi$ in cosines and since of multiples of ϕ, and rearranging the result in terms each involving only one such multiple. It is seen that if $P_n(x,y,z)$ is the most general homogeneous polynomial of degree n, for $\nabla^2 P_n(x, y, z)$, an expression is obtained which employing the transformation of ∇^2.

$$\frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2} + \frac{\cot\theta}{r^2}\frac{\partial}{\partial\theta} + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\phi^2}$$

$$= [r^n\{A_o\mu_o + \sum_{m=\phi}^{n} A_{n\,m}\cos m\phi + \beta_m V_m \sin n\phi\}] \qquad (3.14)$$

Where $\mu_o, \mu_1, \mu_2, \ldots, v_1, v_2, \ldots$ are functions of θ only and $A_o, A_1, \ldots, \beta_o, \beta_1, \beta_2, \ldots$ are 2n+1 arbitrary constants. This reduce to

$$r^{n-2}\left[\frac{d}{d\mu}\left\{(1-\mu^2)\frac{d}{d\mu}\right\}+n(n+1)\right]A_o\mu_o +$$

$$\sum_{m=1}^{n} r^{n-2}\left\{\frac{d}{d\mu}\left[(1-\mu^2)\frac{d}{d\mu}\right\}+n(n+1)-\frac{m^2}{1-\mu^2}\right]$$

$$x\ (A_m V_m \cos m\,\phi + \beta_m V_m \sin m\,\phi).$$

This will have the value zero, if all the constants vanish except one, say A_m, and if

$$\frac{d}{d\mu}\left[\left\{(1-\mu^2)\frac{d}{d\mu}\right\}+n(n+1)-\frac{m^2}{1-\mu^2}\right]\mu_m = 0$$

This equation has been shown to have only one solution $\alpha_m P_n^m(\mu)$ which does not involve logrithmic infinities. It thus appear that there exist the 2n+1 solutions of Laplace's equation $r^n P_n(\mu)$, $r^n P_n^m(\mu) \begin{matrix}\cos\\ \sin\end{matrix} m\,\phi$, where $m = 1,2,3,\ldots n$ and these are independent of one another, as no linear relation

$$\alpha_o P_n(\mu) + \sum_{m=1}^{\infty} (\alpha_m \cos m\,\phi + \beta_m \sin m\,\phi)\, P_n^m(\mu) = 0 \quad (3.15)$$

can exist between them. This is seen by multiplying by $\cos m\,\phi$ or by sinmϕ and integerating for ϕ over the interval $(0,\pi)$, which would prove that $\alpha_m = 0$, $\beta_m = 0$ and this for all value of m.

To show that there can not be more than 2n+1 solution of Laplace's Equation of the type. If we assume that $P_n(x, y, z)$ is a solution of Laplace's Equation we have

$$A_o U_o + \sum_{m=1}^{n} (U_m A_m \cos m\phi + V_m \beta_m \sin m\phi) = 0$$

where U_m denotes

$$\left[\frac{d}{d\mu} \{(1-\mu^2)\frac{d}{d\mu}\} + (n(n+1) - \frac{m^2}{1-\mu^2}) \right] \mu_m$$

and for V_m there is similar expression with V_m instead of μ_m. From this equation, as before, we see that $A_m U_m = 0$, $\beta_m V_m = 0$. Since the equation hold for all values of ϕ. Hence, if A_m or β_m is not zero. We must have $V_m = 0$ or $U_m = 0$ and therefore μ_m, V_m have the values $\alpha'_m P_n^m(\mu)$, $\beta'_m P_n^m(\mu)$ and so the solution of Laplace's Equation is a linear function of the 2n+1 independent solutions already found. It has now been proved that the number of independent ardinary solution of degree is 2n+1.

§ 3.5 ANOTHER METHOD OF FINDING 2n+1 INDEPENDENT ORDINARY SOLUTIONS OF LAPLACES EQUATION OF DEGREE n.

Let f(ax + by + cz) be a differentiable function such that

$a^2 + b^2 + c^2 = 0$, where a, b, c are constants. Then we prove that the given function f(ax + by + cz) satisfies Laplace's equation. Therefore

$$\frac{\partial f}{\partial x} = af'(ax + by + cz),$$

$$\frac{\partial^2 f}{\partial x^2} = a^2 f''(ax + by + cz)$$

Similarly $$\frac{\partial^2 f}{\partial y^2} = b^2 f''(ax + by + cz),$$

$$\frac{\partial^2 f}{\partial z^2} = c^2 f''(ax + by + cz)$$

Therefore

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = (a^2 + b^2 + c^2) f''(ax + by + cz) = 0$$

because $a^2 + b^2 + c^2 = 0$

Thus the function f(ax + by + cz) satisfies the Laplace's equation. In particular let $U = (z + ix\cos\theta + iy\sin\theta)^n$ (3.16)

Where θ is an arbitrary constant

$$\frac{\partial U}{\partial x} = n(z + ix\cos\theta + iy\sin\theta)^{n-1}\, i\cos\theta$$

$$\frac{\partial^2 U}{\partial x^2} = -n(n-1)(z + ix\cos\theta + iy\sin\theta)^{n-2}\cos\theta$$

Similarly $$\frac{\partial^2 U}{\partial y^2} = -n(n-1)\sin^2\theta(z+ix\cos\theta+iy\sin\theta)^{n-2}$$

$$\frac{\partial^2 U}{\partial z^2} = n(n-1)(z + ix\cos\theta + iy\sin\theta)^{n-2}$$

Therefore

$$\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} + \frac{\partial^2 U}{\partial z^2} =$$

$$-n(n-1)[z + ix\cos\theta + iy\sin\theta]^{n-2}$$

$$(\cos^2\theta + \sin^2\theta - 1) = 0$$

Hence the result. If for any value of θ we expand the expression (3.16) in power of x, y, z the real and imaginary parts will each be a solution of Laplace's Equation of degree n, we have

$$(z + ix\cos\theta + iy\sin\theta)^n = r^n[\cos\psi + i\sin\psi\cos(\phi-\theta)]^n$$

the expansion of the expression $[\cos\psi + i\sin\psi\cos(\phi+\theta)]^n$ in cosine of multiples of $\phi - \theta$. Then by writing $\cos\psi = \mu$ we have

$$(z + ix\cos\theta + iy\sin\theta)^n = r^n\{P_n(\mu) + 2\sum_{m=1}^{n} e^{\frac{1}{2}m\pi i} \cdot \frac{n!(1-\mu^2)^{\frac{1}{2}m}}{(n-m)!}\frac{d^m P_n(\mu)}{d\mu^m}\cos m(\phi-\theta)\}$$

Since the R.H.S. of this equation is an expression which for every value of θ , satisfies Laplace's Equation. The coefficients of $\cos m\theta$, $\sin m\theta$ are each separately solutions of Laplace's equation. We thus obtain 2n+1 solutions of Laplace's Equation $r^n P_n(\mu)$, $r^n P_n^m(\mu) \cos m\phi$, $r^n P_n^m(\mu) \sin m\phi$, where $m = 1,2,\ldots,n$.

These solutions of Laplace's Equation are obviously independents and therefore form a system of the required kind. The general solution of Laplace's Equation of degree n is thus

$$r^n [a_o P_n(\mu) + \sum_{m=1}^{n} a_m \cos m\phi + b_m \sin m\phi) P_n^m(\mu)]$$

Where a_o, a_m , b_m are 2n+1 arbitrary constants. This expression when the values of r, μ , ϕ in terms of x, y, z are substituted, is the most general solutions of Laplace's Equation of the prescribed type. If $Y_n(x,y,z)$ be a solution of Laplace's Equation of degree n, $\frac{\partial Y_n}{\partial \phi}$ is also a solution of Laplace's Equation. This follows at once from the last expression.

Since $\frac{\partial}{\partial \phi} = x \frac{\partial}{\partial y} - y \frac{\partial}{\partial x}$, it follows that

$x \frac{\partial y_n}{\partial y} - y \frac{\partial y_n}{\partial x}$ is a solution of Laplace's Equation. Clearly

$y \frac{\partial y_n}{\partial z} - z \frac{\partial y_n}{\partial y}$ and $z \frac{\partial y_n}{\partial x} - x \frac{\partial y_n}{\partial z}$ are also solutions of

Laplace's Equation. Further negative solution $-y_n-1$ of Laplace

Equation has the corresponding property. In the next solution we discuss the general solution of Laplace's Equation due to Whittaker.

§ 3.6: A GENERAL SOLUTION OF LAPLACE'S EQUATION: IN THREE DIMENSIONS.

Let $U(x,y,z)$ be a solution of Laplace's Equation. Which can be expanded into power series in three variables valid for points of (x,y,z) sufficiently near a given point (x_o, y_o, z_o), accordingly we write

$$x = x_o + X, \quad y = y_o + Y \text{ and } z = z_o + Z$$

and we assume the expansion

$$U = a_o + a_1X + b_1Y + c_1Z + a_2X^2 + b_2Y^2 + c_2Z^2 +$$

$$2d_2XZ + 2e_2ZX + 2f_2XY + \dots$$

is being supposed that this series is absolutely convergent when ever

$$|X| + |Y| + |Z| \leqslant a$$

where a is some positive constant. If this expansion exists, U is said to be analytic at (x_o, y_o, z_o) and the above series converges uniformely throughout the domain indicated and differentiated term by term with respect to X, Y, Z any number of time at points inside the domain. Now we substitute the above expansion in Laplace's Equation i.e.

$$\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} + \frac{\partial^2 U}{\partial z^2} = 0$$

and equate to zero, the coefficient of the various power of X, Y and Z we get an infinit site of linear relation between the coefficient of which $a_2 + b_2 + c_2 = 0$. Which is taken as typical.

There are $\frac{1}{2}n(n-1)$ of these relation between the $\frac{1}{2}(n+2)(n+1)$. Coefficient of the terms of degree n in the expansion of U. So that there are only $\frac{1}{2}(n+2)(n+1) - \frac{1}{2}n(n-1) = 2n + 1$ independent coefficients in the terms of degree n in U. Hence the terms of degree n in U must be a linear combination of 2n+1 linearly independent particular solutions of Laplace's Equation. These solutions being each of degree n in x, y and z.

To find a set of such solution, consider

$(z + ix \cos\mu + iy \sin\mu)^n$, it is a solution of Laplace's Equation. Which we expand in a series of sine, and cosins of multiples μ thus,

$$\sum_{m=0}^{n} g_m(X,Y,Z) \cos m\mu + \sum_{m=1}^{n} h_m(X,Y,Z) \sin m\mu$$

the function $g_m(X,Y,Z)$ and $h_m(X,Y,Z)$ are independent of μ. Thus the highest power of in

$$\sum_{m=0}^{n} g_m(X,Y,Z) \quad \text{and} \quad h_m(X,Y,Z)$$

is z^{n-m} and the f rmer function is an even function of y and the later an odd function. Hence the functions are linearly independent. They therefore form a set of 2n+1 function of the type sought.

Now we have

$$(z + ix\cos\mu + iy\sin\mu)^n$$

$$= \sum_{m=0}^{n} g_m(x,y,z)\cos m\mu + \sum_{m=0}^{n} h_m(x,y,z)\sin m\mu$$

We apply the Fourier's rule on the above equation we have

$$\pi g_m(x,y,z) = \int_{-\pi}^{+\pi} (z + ix\cos\mu + iy\sin\mu)^n \cos m\mu \, d\mu$$

$$\pi h_m(x,y,z) = \int_{-\pi}^{+\pi} (z + ix\cos\mu + iy\sin\mu)^n \sin m\mu \, d\mu$$

and so any linear combination of the 2n+1 solutions can be written in the form

$$\int_{-\pi}^{+\pi} (z + ix\cos\mu + iy\sin\mu)^n f_n(\mu)\, d\mu$$

where $f_n(\mu)$ is a rational function of $e^{i\mu}$. Now it is readily verified that, if the terms of degree n in the expression assumed for U be written in this form, the series of terms under the integral sign converges uniformly if $|X| + |Y| + |Z|$ be sufficiently small. So therefore we write

$$U = \int_{-\pi}^{+\pi} \sum_{n=0}^{\infty} (z + ix\cos\mu + iy\sin\mu)^n f_n(\mu)\, d\mu$$

But any expression of this form may be written

$$U = \int_{-\pi}^{+\pi} F(z + ix\cos\mu + iy\sin\mu, \mu)\, d\mu$$

Where F is a function such that differentiations with regard to x,y or z under the sign of integration are permissible. And conversely, if F be any function of this type. U is a solution of Laplace's Equation. We write the above result also in the form

$$U = \int_{-\pi}^{+\pi} f(z + ix\cos\mu + iy\sin\mu, \mu)\, d\mu$$

On absorbing the terms $-z_o - ix_o \cos \mu - iy_o \sin \mu$ into the second variable and, if differentiation under the sign of integration are permissibles this gives a general solution of Laplace's Equation. That is to say, "every solution of Laplace's Equation which is analytic throughout the interior of some sphere is expressible by an integral of the form given above.

CHAPTER IV

SOLUTION OF LAPLACE EQUATION IN FOUR DIMENSIONS

In this Chapter we are concerned with the solution of Laplace equation in four dimensions. We solve this equation by two different methods, using different parameters.

§ 4.1 SOLUTION OF LAPLACE EQUATION IN FOUR DIMENSIONAL USING SPHERICAL POLAR COORDINATES:

We solve the equation

$$\frac{\partial^2 U}{\partial x_1^2} + \frac{\partial^2 U}{\partial x_2^2} + \frac{\partial^2 U}{\partial x_3^2} + \frac{\partial^2 U}{\partial x_4^2} = 0 \qquad (4.1)$$

in four independent variables x_1, x_2, x_3 and x_4 where x_1, x_2 and x_3 are the usual cartesian coordinates x,y and z and x_4 may be defined as $x_4 = ict$, t being the time coordinate and c the wave velocity.

We transform the equation by using new variables which are named as four dimensional polar coordinates (r, Ψ, θ, ϕ) defined on the sphere $x_1^2 + x_2^2 + x_3^2 + x_4^2 = r^2$,

$$x_1 = r \sin\psi \sin\theta \cos\phi$$

$$x_2 = r \sin\psi \sin\theta \sin\phi$$

$$x_3 = r \sin\psi \cos\theta$$

$$x_4 = r \cos\psi$$

Consider $\frac{\partial^2 U}{\partial x_1^2} + \frac{\partial^2 U}{\partial x_2^2} = 0$

$$x_1 = r \sin\psi \; \sin\theta \; \cos\phi$$

$$x_2 = r \sin\psi \; \sin\theta \; \sin\phi$$

Here we put $r \sin \psi \sin \theta = P$

Then $x_1 = P \cos\phi$

$$x_2 = P \sin\phi$$

implies that $x_1^2 + x_2^2 = P^2$ and $\tan\phi = \frac{x_2}{x_1}$

Then by using plane polar coordinates we have

$$\frac{\partial^2 U}{\partial x_1^2} + \frac{\partial^2 U}{\partial x_2^2} = \frac{\partial^2 U}{\partial P^2} + \frac{1}{P} + \frac{\partial U}{\partial P} + \frac{1}{P^2}\frac{\partial^2 U}{\partial \phi^2} \qquad (4.2)$$

Again we consider

$$\frac{\partial^2 U}{\partial x_1^2} + \frac{\partial^2 U}{\partial x_2^2} + \frac{\partial^2 U}{\partial x_3^2} = \frac{\partial^2 U}{\partial P^2} + \frac{1}{P}\frac{\partial U}{\partial P} + \frac{1}{P^2}\frac{\partial^2 U}{\partial \phi^2} + \frac{\partial^2 U}{\partial x_3^2}$$

$$= \frac{\partial^2 U}{\partial P^2} + \frac{\partial^2 U}{\partial x_3^2} + \frac{1}{P}\frac{\partial U}{\partial P} + \frac{1}{P^2}\frac{\partial^2 U}{\partial \phi^2} \qquad (4.3)$$

Now consider $\frac{\partial^2 U}{\partial P^2} + \frac{\partial^2 U}{\partial x_3^2}$

$$P = r \sin\psi \sin\theta$$

$$x_3 = r \sin\psi \cos\theta$$

Let $r \sin\psi = q$. Then we have

$$P = q \sin\theta$$

$$x_3 = q \cos\theta$$

$$P^2 + x_3^2 = q^2 \text{ and } \tan\theta = \frac{P}{x_3}$$

These are plane polar coordinates. Therefore

$$\frac{\partial^2 U}{\partial P^2} + \frac{\partial^2 U}{\partial x_3^2} = \frac{\partial^2 U}{\partial q^2} + \frac{1}{q}\frac{\partial U}{\partial q} + \frac{1}{q^2}\frac{\partial^2 U}{\partial \theta^2}$$

and
$$\frac{\partial U}{\partial P} = \frac{\partial U}{\partial q} \cdot \frac{\partial q}{\partial P} + \frac{\partial U}{\partial \theta} \cdot \frac{\partial \theta}{\partial P} = \sin \frac{\partial U}{\partial q} + \frac{\cos\theta}{r \sin\psi}\frac{\partial U}{\partial \theta}$$

Putting these values in equation (4.3), we have

$$\frac{\partial^2 U}{\partial x_1^2} + \frac{\partial^2 U}{\partial x_2^2} + \frac{\partial^2 U}{\partial x_3^2} = \frac{\partial^2 U}{\partial q^2} + \frac{1}{q}\frac{\partial U}{\partial q} + \frac{1}{q^2}\frac{\partial^2 U}{\partial \theta^2}$$

$$+ \frac{1}{r \sin\Psi \sin\theta} [\sin\theta \frac{\partial U}{\partial q} + \frac{\cos\theta}{r \sin\Psi} \frac{\partial U}{\partial \theta}]$$

$$+ \frac{1}{r^2 \sin^2\theta \sin^2\Psi} \frac{\partial^2 U}{\partial \phi^2}$$

$$= \frac{\partial^2 U}{\partial q^2} + \frac{1}{q}\frac{\partial U}{\partial q} + \frac{1}{q^2}\frac{\partial^2 U}{\partial \theta^2}$$

$$+ \frac{1}{r \sin\psi}\frac{\partial U}{\partial q} + \frac{\cot\theta \operatorname{cosec}^2\psi}{r^2}\frac{\partial U}{\partial \theta}$$

$$+ \frac{1}{r^2 \sin^2\theta \sin^2\phi}\frac{\partial^2 U}{\partial \phi^2}$$

Similarly we have

$$\frac{\partial^2 U}{\partial x_1^2} + \frac{\partial^2 U}{\partial x_2^2} + \frac{\partial^2 U}{\partial x_3^2} + \frac{\partial^2 U}{\partial x_4^2} = \frac{\partial^2 U}{\partial q^2} + \frac{1}{q}\frac{\partial U}{\partial q}$$

$$+ \frac{1}{q^2}\frac{\partial^2 U}{\partial \theta^2} + \frac{1}{r \sin\psi}\frac{\partial U}{\partial q} + \frac{\cot\theta \operatorname{cosec}^2\theta}{r^2}\frac{\partial U}{\partial \theta}$$

$$+ \frac{1}{r^2 \sin^2\theta \sin^2\psi}\frac{\partial^2 U}{\partial \phi^2} + \frac{\partial^2 U}{\partial x_4^2} = 0 \qquad (4.4)$$

Considering $\frac{\partial^2 U}{\partial q^2} + \frac{\partial^2 U}{\partial x_4^2}$

and using $q = r \sin\psi$

$x_4 = r \cos\psi$

$q^2 + x_4^2 = r^2$

$\tan\psi = \frac{q}{x_4}$

which are again the plane polar coordinates, we have,

$$\frac{\partial^2 U}{\partial q^2} + \frac{\partial^2 U}{\partial x_4^2} = \frac{\partial U}{\partial r^2} + \frac{1}{r} + \frac{\partial U}{\partial r} + \frac{1}{r^2} \quad \frac{\partial^2 U}{\partial \psi^2}$$

and
$$\frac{\partial U}{\partial q} = \frac{\partial U}{\partial r} \ \frac{\partial r}{\partial q} + \frac{\partial U}{\partial \psi} \cdot \frac{\partial \psi}{\partial q}$$

$$= \sin\psi \frac{\partial U}{\partial r} + \frac{\cos\psi}{r} \ \frac{\partial U}{\partial \psi}$$

Putting these values in equation (4.4) we get,

$$\frac{\partial^2 U}{\partial x_1^2} + \frac{\partial^2 U}{\partial x_2^2} + \frac{\partial^2 U}{\partial x_3^2} + \frac{\partial^2 U}{\partial x_4^2} = \frac{\partial^2 U}{r^2} + \frac{1}{r} \ \frac{\partial U}{\partial r} + \frac{1}{r^2} \frac{\partial^2 U}{\partial \psi^2}$$

$$+ \frac{2}{r \sin\psi} \left(\sin\psi \frac{\partial U}{\partial r} + \frac{\cos\psi}{r} \frac{\partial U}{\partial \psi} \right)$$

$$+ \frac{1}{r^2 \sin^2\psi} \ \frac{\partial^2 U}{\partial \theta^2} + \frac{\cot\theta \ \operatorname{cosec}^2\psi}{r^2} \ \frac{\partial U}{\partial \theta}$$

$$+ \frac{1}{r \ \sin^2\theta \sin^2\psi} \ \frac{\partial U}{\partial \phi^2} = 0$$

or
$$\frac{\partial^2 U}{\partial x_1^2} + \frac{\partial^2 U}{\partial x_2^2} + \frac{\partial^2 U}{\partial x_3^2} + \frac{\partial^2 U}{\partial x_4^2} = \frac{\partial^2 U}{\partial r^2} + \frac{3}{r} \frac{\partial U}{\partial r}$$

$$+ \frac{1}{r^2} \ \frac{\partial U}{\partial \psi^2} + \frac{2\cot\psi}{r^2} \ \frac{\partial U}{\partial \psi} + \frac{1}{r^2 \sin^2\psi} \frac{\partial^2 U}{\partial \theta^2}$$

$$+ \frac{\cot\theta \operatorname{cosec}^2\psi}{r^2} \frac{\partial U}{\partial \theta} + \frac{1}{r^2 \sin^2\theta \sin^2\psi} \ \frac{\partial^2 U}{\partial \phi^2} = 0$$

$$= \frac{\partial^2 U}{\partial r^2} + \frac{3}{r} \ \frac{\partial U}{\partial r} + \frac{1}{r^2} + \frac{\partial^2 U}{\partial \psi^2} + 2 \ \cot\psi \ \frac{\partial U}{\partial \psi}$$

$$+ \frac{1}{r^2 \sin^2\psi} \frac{\partial^2 U}{\partial \theta^2} + \frac{\cot\theta}{r^2 \sin^2\psi} \frac{\partial U}{\partial \theta}$$

$$+ \frac{1}{r^2 \sin^2\psi \sin^2\theta} \frac{\partial^2 U}{\partial \phi^2} = 0 \qquad (4.5)$$

We solve the above equation by the method of separation of variables.

Putting $U(r, \psi, \theta, \phi) = R(r)\psi(\psi)\Theta(\theta)\Phi(\phi)$

Then (4.5) takes the form

$$\left[\frac{R''}{R} + \frac{3}{r}\frac{R'}{R} + \frac{1}{r^2}\frac{\psi''}{\psi} + \frac{2\cot\psi}{r^2}\frac{\psi'}{\psi}\right.$$

$$\left. + \frac{1}{r^2\sin^2\psi}\frac{\Theta''}{\Theta} + \frac{\cot\theta}{r^2\sin^2\psi}\frac{\Theta'}{\Theta}\right] r^2\sin^2\psi\sin^2\theta = -\frac{\Phi''}{\Phi} \qquad (4.6)$$

R.H.S. is independent of R, θ, ψ and is a function of ϕ only, therefore each side of (4.6) must be equal to a constant say m^2.

Therefore $-\frac{\Phi''}{\Phi} = \text{Constant} = m^2$

That is, $\frac{d^2\Phi}{d\phi^2} + m^2\Phi = 0$

Thus $\Phi = A\cos m\phi + B\sin\phi \qquad (4.7)$

In order that Φ be single valued, m must be an integer for $0 \leqslant \phi \leqslant 2\pi$. After normalizing the above equation we have

$$\Phi = \frac{1}{\sqrt{2\pi}} e^{im\phi} \tag{4.8}$$

Putting $\frac{\Phi''}{\Phi} = -m^2$ in (4.6), we have

$$\left[\frac{R''}{R} + \frac{3}{r}\frac{R'}{R} + \frac{1}{r^2}\frac{\Psi''}{\Psi} + \frac{2\cot\psi}{r^2}\frac{\Psi'}{\Psi}\right] r^2 \sin^2\psi$$

$$= -\left[\frac{\Theta''}{\Theta} + \cot\theta \frac{\Theta'}{\Theta} - \frac{m^2}{\sin^2\theta}\right]$$

R.H.S. of above equation is called Legendre Operator and it must be constant. Hence

$$-\left(\frac{\Theta''}{\Theta} + \cot\theta\frac{\Theta'}{\Theta} - \frac{m^2}{\sin^2\theta}\right) = n(n+1) \tag{4.9}$$

Let $\cos\theta = \mu$ and $\Theta = V$

$$\frac{d\Theta}{d\theta} = \frac{d\Theta}{d\mu} \cdot \frac{d\mu}{d\theta} = -\sin\theta\frac{d\Theta}{d\mu}$$

$$\frac{d^2\Theta}{d\theta^2} = \frac{d}{d\theta}\left(-\sin\theta\frac{d\Theta}{d\mu}\right)$$

$$= -\cos\theta\frac{d\Theta}{d\mu} + \sin^2\theta\frac{d^2\Theta}{d\mu^2}$$

Putting these values in the equation (4.9), we get

$$\sin^2\theta\frac{d^2\Theta}{d\mu^2} - 2\cos\theta\frac{d\Theta}{d\mu} + \left[(n(n+1) - \frac{m^2}{1-\mu^2}\right]\Theta = 0$$

As we have $\Theta = V$, therefore

$$(1-\mu^2)\frac{d^2V}{d\mu^2} - 2\mu\frac{dV}{d\mu} + [n(n+1) - \frac{m^2}{1-\mu^2}]\, V = 0$$

Equation (4.9) can be written as

$$\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}[\sin\theta\frac{\partial\Theta}{\partial\theta}] + [\, n(n+1) - \frac{m^2}{\sin^2\theta}]\Theta = 0$$

This equation is well known Associated Legendre equation where $0 \leq \theta \leq \pi$ and has a solution

$$\Theta(\theta) = Anm\, P_n^m(\cos\theta)$$

The Associated Legendre polynomials are defined by

$$P_n^m(\cos\theta) = (-1)^{|m|}(1-\cos^2\theta)^{|m|/2}\frac{d^{|m|}P_n(\cos\theta)}{d(\cos\theta)^{|m|}}$$

and $P_n(\cos\theta) = \dfrac{1}{2^n n!}\dfrac{d^n(\cos^2\theta-1)}{d(\cos\theta)^n}$

is Legendre's polynomial of the first kind and is a solution of Legendre equation

$$\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\frac{\partial\Theta}{\partial\theta}) + n(n+1)\,\Theta = 0$$

Now we denote $Y_{nm}(\Theta,\phi)$ by $Y_{nm}(\Theta,\phi) = A_{nm}\,\Theta(\theta)\,\Phi(\phi)$.

$$Y_{nm}(\theta, \phi) = A_{nm}\, e^{im\phi}\, P_n^m(\cos\theta)$$

where $0 \leqslant \phi \leqslant 2\pi, \quad 0 \leqslant \theta \leqslant \pi$

$Y_{nm}(\theta, \phi)$ are called spherical Harmonics and are normalized by the condition

$$A_{nm}^2 \int_0^{2\pi} d\phi \int_0^{\pi} [P_n^m(\cos\theta)]^2 \sin\theta\, d\theta = 1$$

where we calculate A_{nm}^2 as

$$A_{nm}^2 = \frac{(2n+1)\ (n-|m|)}{4\pi(n+|m|)!}$$

The normalized Harmic function is given by

$$Y_{nm}(\theta, \phi) = \sqrt{\frac{(2n+1)(n-|m|)!}{4\pi(n+|m|)!}}\ .\ P_n^m(\cos\theta)\, e^{im\phi} \quad (4,10)$$

Now consider

$$\frac{R''}{R} + \frac{3}{r}\frac{R'}{R} + \frac{1}{r^2}\frac{\Psi''}{\Psi} + \frac{2\cot\psi}{r^2}\frac{\Psi'}{\Psi} - \frac{n(n+1)}{r^2\sin^2\Psi} = 0$$

$$\Rightarrow \frac{R''}{R} + \frac{3}{R} = -\frac{1}{r^2}\left(\frac{\Psi''}{\Psi} + 2\cot\psi\frac{\Psi'}{\Psi} - \frac{n(n+1)}{r^2\sin^2\Psi}\right)$$

Multiplying by r^2

$$\frac{r^2R''}{R} + \frac{3rR'}{R} = -\left(\frac{\Psi''}{\Psi} + 2\cot\psi \frac{\Psi'}{\Psi} - \frac{n(n+1)}{\sin^2\psi} \right)$$

or
$$\frac{r^2R''}{R} + \frac{3rR'}{R} + \frac{\Psi''}{\Psi} + 2\cot\psi \frac{\Psi'}{\Psi} - \frac{n(n+1)}{\sin^2\psi} = 0$$

We put $R(r)\Psi(\psi) = r^{2n'} F(\psi)$ (4.11)

Then
$$\frac{d^2F}{d\psi^2} + 2\cot\psi \frac{dF}{d\psi} + \left[4n'(n'+1) - \frac{n(n+1)}{\sin^2\psi} \right] F = 0 \qquad (4.12)$$

By substituting $\cos^2\psi = \xi$, we have

$$\frac{dF}{d\psi} = \frac{dF}{d\xi} \cdot \frac{d\xi}{d\psi} = -2 \sin\psi \cos\psi \frac{dF}{d\xi}$$

$$\frac{d^2F}{d\psi^2} = \frac{d}{d\psi}\left(\frac{dF}{d\psi} \right) = \frac{d}{d\psi}\left(- \sin 2\psi \frac{dF}{d\xi}\right) = \sin^2 2\psi \frac{d^2F}{d\xi^2} - 2\cos 2\psi \frac{dF}{d\xi}$$

since
$$\sin 2\psi = 2 \sin\psi \cos\psi$$

$$\sin^2 2\psi = 4 \sin^2\psi \cos^2\psi$$

$$\cos 2\psi = 2 \cos^2\psi - 1$$

(4.12) takes the form

$$4 \sin^2\psi \cos^2 + \frac{d^2F}{d\xi^2} - \left[2(2\cos^2\psi - 1) + 4\cos^2\psi \right] \frac{dF}{d\xi}$$

$$+ \left[4n'(n'+1) - \frac{n(n+1)}{\sin^2\psi} \right] F = 0$$

Then $$\xi(1-\xi)\frac{d^2F}{d\xi^2} + (\tfrac{1}{2} - 2\xi)\frac{dF}{d\xi}$$

$$+ [\, n'(n'+1) - \frac{n(n+1)}{4(1-\xi)}]\, F = 0$$

Putting $F = (1-\xi)^{\frac{n}{2}} K$

$$\frac{dF}{d\xi} = (1-\xi)^{n/2}\frac{dK}{d\xi} - \frac{n}{2}(1-\xi)^{\frac{n}{2}-1} K$$

$$\frac{d^2F}{d\xi^2} = (1-\xi)^{\frac{n}{2}}\frac{d^2K}{d\xi^2} - \frac{n}{2}(1-\xi)^{\frac{n}{2}-1}\frac{dK}{d\xi}$$

$$- \frac{n}{2}(1-\xi)^{\frac{n}{2}-1}\frac{dK}{d\xi} + \frac{n(n-2)}{4}(1-\xi)^{\frac{n}{2}-2} K = (1-\xi)^{\frac{n}{2}}\frac{d^2K}{d\xi^2} - n(1-\xi)^{\frac{n}{2}-1}$$

$\frac{dK}{d\xi} + \frac{n(n-2)}{4}(1-\xi)^{\frac{n}{2}-2} K$, then we have

$$\xi(1-\xi)\frac{d^2K}{d\xi^2} + [-n\xi + \frac{1}{2} - 2\xi]\frac{dK}{d\xi}$$

$$+ [\, n'(n'+1) - \frac{n(n+1)}{4(1-\xi)} - \frac{n}{4(1-\xi)}$$

$$+ \frac{4n\xi}{4(1-\xi)} + \frac{n(n-2)\xi}{4(1-\xi)}]\, K = 0$$

$$\xi(1-\xi)\frac{d^2K}{d\xi^2} + [\tfrac{1}{2} - \xi(n+2)]\frac{dK}{d\xi}$$

$$+ [\, n'(n'+1) - \frac{n(n+2)}{4}]\, K = 0$$

It is a Hypergeometric equation which can be written in the form

$$\xi(1-\xi)\frac{d^2K}{d\xi^2} + [\gamma-(\alpha+\beta+1)\xi]\frac{dK}{d\xi} - \alpha\beta K = 0 \qquad (4.13)$$

where α, β and γ are constants. The solution of the above equation can be written as $F(\alpha, \beta, \gamma, \xi)$.

Thus the solution of the Laplace Equation

$$\frac{\partial^2 U}{\partial x_1^2} + \frac{\partial^2 U}{\partial x_2^2} + \frac{\partial^2 U}{\partial x_3^2} + \frac{\partial^2 U}{\partial x_4^2} = 0$$

is

$$U(r, \psi, \theta, \phi) = R(r)\Psi(\psi)\Theta(\theta)\Phi(\phi)$$

$$= r^{2n'} F(\psi)\, Y_{nm}(\theta, \phi)$$

$$= \sqrt{\frac{(2n+1)(n-|m|)}{4\pi(n+|m|)!}}\, P_n^m(\cos\theta)\, e^{im\phi}$$

$$r^{2n'} F(\alpha, \beta, \gamma, \cos^2\psi) \qquad (4.14)$$

$\gamma = \frac{1}{2}$

$\alpha = -n' + \frac{n}{2}$

$\beta = n' + \frac{1}{2}n + 1$

§ 4.2 SOLUTION OF LAPLACE EQUATION IN FOUR DIMENSIONS USING CYLINDRICAL POLAR COORDINATES.

Now we solve the Laplace Equation by another method using new coordinates.

$$x_1 = \rho \cos \phi \,, \qquad x_2 = \rho \sin \phi$$

$$x_3 = \sigma \cos \psi \qquad x_4 = \sigma \sin \psi$$

From these coordinates we have

$$x_1^2 + x_2^2 = \rho^2 \,, \qquad \tan \phi = \frac{x_2}{x_1}$$

$$x_3^2 + x_4^2 = \sigma^2 \,, \qquad \tan \psi = \frac{x_4}{x_3}$$

then we have

$$\frac{\partial^2 U}{\partial x_1^2} + \frac{\partial^2 U}{\partial x_2^2} = \frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho \frac{\partial U}{\partial \rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 U}{\partial \phi^2}$$

and

$$\frac{\partial^2 U}{\partial x_3^2} + \frac{\partial^2 U}{\partial x_4^2} = \frac{1}{\sigma}\frac{\partial}{\partial \sigma}\left(\sigma \frac{\partial U}{\partial \sigma}\right) + \frac{1}{\sigma^2}\frac{\partial^2 U}{\partial \psi^2}$$

combining these two equations we have

$$\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial x_2^2} + \frac{\partial^2 U}{\partial x_3^2} + \frac{\partial^2 U}{\partial x_4^2} = \frac{\partial^2 U}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial U}{\partial \rho} + \frac{1}{\rho^2}\frac{\partial^2 U}{\partial \phi^2}$$

$$+ \frac{\partial^2 U}{\partial \sigma^2} + \frac{1}{\sigma}\frac{\partial U}{\partial \sigma} + \frac{1}{\sigma^2} + \frac{\partial^2 U}{\partial \psi^2} = 0 \tag{4.15}$$

This equation is satisfied by

$$U=\cos(m\phi+\varepsilon)\cos(P\psi+\eta)F(\rho)G(\sigma)$$

Putting the value of U in equation (4.15), we get

$$\frac{d^2F}{d\rho^2} + \frac{1}{\rho}\frac{dF}{d\rho} + (K^2 - \frac{m^2}{\rho^2})F = 0 \tag{4.16}$$

$$\frac{d^2G}{d\sigma^2} + \frac{1}{\sigma}\frac{dG}{d\sigma} - (K^2 + \frac{P^2}{\sigma^2})G = 0 \tag{4.17}$$

where K is a constant. If we put $K\rho = \xi$ and $iK\sigma = \zeta$ we have

$$\frac{dF}{d\rho} = \frac{dF}{d\xi} \cdot \frac{d\xi}{d\rho} = K\frac{dF}{d\xi}$$

$$\frac{d^2F}{d\rho^2} = K\frac{d^2F}{d\xi^2} \cdot \frac{d\xi}{d\rho} = K^2\frac{d^2F}{d\xi^2}$$

and
$$\frac{dG}{d\sigma} = \frac{dG}{d\zeta} \cdot \frac{d\zeta}{d\sigma} = iK\frac{dG}{d\zeta}$$

$$\frac{d^2G}{d\sigma^2} = -K^2\frac{d^2G}{d\zeta^2}$$

From (4.16) and (4.17) we get

$$K^2\frac{d^2F}{d\xi^2} + \frac{K^2}{\xi}\frac{dF}{d\xi} + (K^2 - \frac{m^2K^2}{\xi^2})F = 0$$

or $$\frac{d^2F}{d\xi^2} + \frac{1}{\xi}\frac{dF}{d\xi} + (1 - \frac{m^2}{\xi^2}) F = 0$$

or $$\xi^2 \frac{d^2F}{d\xi^2} + \frac{dF}{d\xi} + (\xi^2 - m^2) F = 0 \qquad (4.18)$$

$$- K^2 \frac{d^2G}{d\zeta^2} - \frac{K^2}{\zeta}\frac{dG}{d\zeta} - (K^2 - \frac{P^2K^2}{\zeta}) = 0$$

or $$\zeta^2 \frac{d^2G}{d\zeta^2} + \zeta\frac{dG}{d\zeta} + (\zeta^2 - P^2) G = 0 \qquad (4.19)$$

The resultant two equations are the well-known Bessel equations.

Now we make the further substitution

$$\rho = r \sin\theta \ , \quad \sigma = r \cos\theta$$

that is, $x_1 = r \sin\theta \cos\phi$

$x_2 = r \sin\theta \sin\phi$

$x_3 = r \cos\theta \cos\psi$

$x_4 = r \cos\theta \sin\psi$

clearly we have

$$x_1^2 + x_2^2 + x_3^2 + x_4^2 = r^2$$

Then by using cylindrical polar coordinates we have

$$\frac{\partial^2 U}{\partial \rho^2} + \frac{\partial^2 U}{\partial \sigma^2} = \frac{1}{r}\frac{\partial}{\partial r}\left(r \frac{\partial U}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 U}{\partial \theta^2}$$

$$\frac{1}{\rho}\frac{\partial U}{\partial \rho} = \frac{1}{r \sin\theta}\left[\frac{\partial U}{\partial \theta} \cdot \frac{\partial \theta}{\partial \rho} + \frac{\partial U}{\partial r} \cdot \frac{\partial r}{\partial \rho}\right]$$

$$\frac{1}{\sigma}\frac{\partial U}{\partial \sigma} = \frac{1}{r \cos\theta}\left[\frac{\partial U}{\partial \theta} \cdot \frac{\partial \theta}{\partial \sigma} + \frac{\partial U}{\partial r} \cdot \frac{\partial r}{\partial \sigma}\right]$$

As $\rho = r \sin\theta$, $\sigma = r \cos\theta$

Then $\rho^2 + \sigma^2 = r^2$, $\tan\theta = \frac{\rho}{\sigma}$

$$\frac{1}{\rho}\frac{U}{\partial \rho} + \frac{1}{\sigma}\frac{U}{\partial \sigma} = \frac{\partial}{r}\frac{\partial U}{\partial r} + \frac{1}{r^2}(\text{Cot}\theta - \tan\theta)\frac{\partial U}{\partial \theta}$$

Substituting the value of $\frac{\partial^2 U}{\partial \rho^2} + \frac{\partial^2 U}{\partial \sigma^2}$,

$\frac{1}{\rho}\frac{\partial U}{\partial \rho} + \frac{1}{\sigma}\frac{\partial U}{\partial \sigma}$ in equation (4.15) we have

$$\frac{\partial^2 U}{\partial r^2} + \frac{3}{r}\frac{\partial U}{\partial r} + \frac{1}{r^2}\frac{\partial^2 U}{\partial \theta^2} + \frac{2 \cot 2\theta}{r^2}\frac{\partial U}{\partial \theta}$$

$$+ \frac{1}{r^2 \sin^2\theta}\frac{\partial^2 U}{\partial \phi^2} + \frac{1}{r^2 \cos^2\theta}\frac{\partial^2 U}{\partial \psi^2} = 0 \qquad (4.20)$$

Putting $U = r^{2n} \cos(m\phi + \varepsilon) \cos(P\psi + \eta) F(\theta)$ in equation (4.20) we have

$$\frac{d^2F}{d\theta^2} + 2 \cot 2\theta \frac{dF}{d\theta} + [\, 4n(n+1) - \frac{m^2}{\sin^2\theta} - \frac{P^2}{\cos^2\theta} \,] F = 0 \qquad (4,21)$$

The equation (4.21) is unaltered if we replace n by -(n+1). Hence solution is given by

$$U = r^{-2n-2} \cos (m\phi + \varepsilon)\cos(P\psi + \eta)F(\theta).$$

This suggests the following theorem.

Theorem: If $F(r, \theta, \phi, \psi)$ is a solution of equation (4.20). Then $U = \frac{1}{r^2} F(\frac{1}{r}, \theta, \phi, \psi)$ is also a solution.

This is easily varified by a slight extension of the method used in the case of the corresponding theorem for Laplace's equation.

The theorem can also be stated as follows:

If $U = F(x_1, x_2, x_3, x_4)$ is a solution of equation (4.20), then $U = \frac{1}{r^2} F(\frac{x_1}{r^2}, \frac{x_2}{r^2}, \frac{x_3}{r^2}, \frac{x_4}{r^2})$ is also a solution, where $r^2 = x_1^2 + x_2^2 + x_3^2 + x_4^2$.

The transformation

$$x_1' = \frac{x_1}{r^2}, \quad x_2' = \frac{x_2}{r^2}, \quad x_3' = \frac{x_3}{r^2}, \quad x_4' = \frac{x_4}{r^2}$$

is called generalized inversion.

If in equation (4.21), we write $P^2 = m^2$ and $\cos 2\theta = \mu$. Then equation (4.21) can be written as (when we substitute the values of F, $\frac{dF}{d\theta}$ and $\frac{d^2F}{d\theta^2}$)

$$4(1-\mu^2)\frac{d^2F}{d\mu^2} - 8\frac{dF}{d\mu} + [(n(n+1) - \frac{m^2}{1-\mu^2})]F = 0$$

or

$$\frac{d}{d\mu}[(1-\mu^2)\frac{dF}{d\mu}] + [n(n+1) - \frac{m^2}{1-\mu^2}]F = 0$$

The above equation is well-known Polynomial of associated Legendre function. We thus have solution of equation (4.20) of the form

$$U = r^{2n} P_n^m(\mu) \cos(m\phi + \varepsilon)\cos(P\psi + \eta)$$

and since ε and η are arbitrary constants, we may deduce that

$$r^{2n} P_n^m(\mu) \cos(m\phi \pm \psi) \text{ and}$$

$$r^{2n} P_n^m(\mu) \sin(m\phi \pm \psi),$$

are solution of equation (4.20).

Now comparing these solutions of (4.20) with the corresponding solution (3. 11) we have the following theorem:

Theorem: If $U = f(r)\ \theta, \phi)$ is a solution of (3.11) then the solution $f(r^2, 2\theta, \phi \pm \psi)$ is a solution of equation (4.20).

This may be varified by direct comparison of the differential equation to be satisfied by the function f in two cases. In the general case when $P^2 \neq m^2$, then substituting $\sin^2\phi = \xi$ in the equation

$$\frac{d^2F}{d\theta^2} + 2 \cot 2\theta \frac{dF}{d\theta} + [4n(n+1) - \frac{m^2}{\sin^2\theta} - \frac{P^2}{\cos^2\theta}] F = 0$$

we have

$$\sin 2\theta \frac{d^2F}{d\xi^2} + 4 \cos 2\theta \frac{dF}{d\xi} + [4n(n+1) - \frac{P^2}{1-\xi} - \frac{m^2}{\xi}] F=0$$

or
$$4 \xi(1-\xi) \frac{d^2F}{d\xi^2} + 4 [2(1-\xi) - 1] \frac{dF}{d\xi}$$

$$+ [4n(n+1) - \frac{P^2}{1-\xi} - \frac{m^2}{\xi}] F = 0$$

or
$$\xi (1-\xi) \frac{d^2F}{d\xi^2} + (1-2\xi) \frac{dF}{d\xi} + [n(n+1) - \frac{P^2}{4(1-\xi)} - \frac{m^2}{4\xi}] F=0 \tag{4.22}$$

Putting $F = \xi^{\frac{m}{2}} (1-\xi)^{\frac{P}{2}} G$ in equation (4.22) we obtain the hypergeometric equation

$$\xi\ (1-\xi)\ \frac{d^2G}{d\xi^2} + (\ \gamma - (\ \alpha + \beta + 1)\ \xi\ )\ \frac{dG}{d\xi} = -\alpha\beta G = 0 \tag{4.23}$$

where $\gamma = m + 1$

$$\alpha = n + 1 + \tfrac{1}{2}\ (m + P)$$

$$\beta = -n + \tfrac{1}{2}\ (m + P)$$

Hence the solution of the equation (4.23) is $F(\alpha, \beta, \gamma, \xi)$. Therefore we have solution of (4.1) of the type

$$U = r^{2n} \cos(m\phi + \varepsilon) \cos(P\ \psi + \eta) \sin^{m}\theta\ \cos^{P}\theta\ F(\alpha, \beta, \gamma, \sin^{2}_{\theta}) \tag{4.24}$$

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