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CONTINUOUS GROUPS

BY

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CHAPTER-1

INTRODUCTION

Continuous groups form an important branch of Mathematics with wide range of applications in Physics. Recently the theory of continuous groups has been applied to problems in various branches of Chemistry and Biology [1]. Special functions are also related to the theory of group representation and their important properties can be derived from this theory [3]. The representation of three dimensional rotation group , which is a continuous group, is extensively used in Quantum Mechanics and has been studied by I.M.Gelfand and Z.Ya. Shapiro [2]. They parameterize the group elements in terms of Euler angles ϕ,θ,ψ, where ϕ and ψ vary from 0 to 2π and θ varies from 0 to π. The rotation with Euler angles ϕ,θ,ψ is the product of three rotations: first rotation about axis OZ through an angle ϕ followed by a rotation about axis OX through an angle θ and finally a rotation about axis OZ through an angle ψ. The rotation matrix is given by (2.25) in terms of Euler angles ϕ,θ,ψ.

However the parameterization of the rotation group by Euler angles ϕ,θ,ψ, suffers from a number of significant shortcommings. If $\theta = 0$ only $\phi+\psi$ is determined while if $\theta=\pi$ $\phi-\psi$ is determined and thus at these singular points in the parametric space, ϕ and ψ no longer define a rotation matrix uniquely.

Singular points arise in any parametrization scheme for the rotation matrices. The Euler angle parametrization is made particularly inappropriate by the occurence of the singularity about the identity element of the group.

An alternate and more appropriate parameterization is obtained by making the first rotation through an angle α about the axis OX followed by the rotation through an angle β about the axis OY and finally a rotation through an angle γ about the axis OZ. The matrix of rotation is given by (4.14) where $-\pi \leq \alpha \leq \pi$, $-\pi \leq \beta \leq \pi$, $-\pi/2 \leq \gamma \leq \pi/2$.

We determined the differential operators corresponding to infinitesimal rotations about the coordinate axis in view of rotation matrix (4.14) and in this dissertation using these differential operators generalized spherical functions have been derived.

Second chapter is concerned with basic concepts about the continuous groups. Rotation group in terms of Euler angles has been described here. Infinitesimal transformations of the continuous group and group representations are also discussed in this chapter.

In the third chapter spherical functions are defined. Differential operator corresponding to infinitesimal rotations and differential equation of spherical functions have been derived. An explicit expression for spherical functions is also given.

In the last chapter we have determined differential operators corresponding to infinitesimal rotations. Generalized spherical functions and special values of these functions have been derived in this chapter.

CHAPTER-2

2.1 CONTINUOUS GROUP

By a group G we mean a set of elements $g_1, g_2, \ldots$ such that a form of group multiplication may be defined which associates a third element with any ordered pair. This multiplication must satisfy the four axioms:

1. The product of any two elements of the group is a unique element which also belongs to the group.
2. The associative law holds, that is,

   $$g_1(g_2g_3) = (g_1g_2)g_3$$

   for any three elements $g_1, g_2 g_3 \varepsilon G$.
3. There is an identity element e such that

   $$eg = ge = g \text{ for all } g \varepsilon G.$$
4. There is in the group an inverse, g^{-1} to each element g such that

   $$g\, g^{-1} = g^{-1} g = e$$

A group having a finite number of distinct elements - g - is said to be a (finite) group of order g. An infinite group is one which contains an infinite number of elements, and our group axioms can also be applied to infinite groups. The three dimensional real orthogonal matrices, the rotations in space, constitute a system of objects which satisfy the group axioms. These groups are called infinite groups and we divide all infinite groups into two categories: discrete groups and continuous groups.

A continuous group is a system of objects called group elements which can be characterized by parameters varying continuously in a certain region. Every set of values of the parameters within the region defines a group element; conversely, to every group element corresponds a set of values of the parameters within the specified region. These regions are called group space. There is one-to-one correspondence between the group elements and points in the group space.

We consider a group whose elements $g(\alpha)$ can be labelled by a single real parameter α . For these it must be possible to find a unique value γ of the parameter, so that

$$g(\gamma) = g(\beta)\, g(\alpha) \tag{2.1}$$

for all possible values of α and β. The group multiplication table, therefore, becomes a functional relation to determine γ from α and β.

$$\gamma = f(\beta,\alpha) \tag{2.2}$$

The associative law imposes a restriction on this function since

$$f(\gamma, f(\beta,\alpha)) = f(f(\gamma,\beta), \alpha) \tag{2.3}$$

for all α,β,γ.

There must also be a value of the parameter denoted by e, corresponding to the identity element and hence satisfying

$$f(\alpha,e) = f(e,\alpha) = \alpha \tag{2.4}$$

Since each group element must have an inverse, each parameter α has an associated parameter α^{-1} such that

$$f\ (\alpha,\alpha^{-1}) = f\ (\alpha^{-1},\alpha) = e \qquad (2.5)$$

The continuity requirement means that $f\ (\beta,\alpha)$ is a continuous function of both variables and that α^{-1} is a continuous function of α.

Neighbouring or adjacent group elements are those which differ by only small amounts in the values of all the parameters. If the parameter changes continuously, we say that the group element changes continuously. Groups whose elements can be denoted by n parameters are known as n-parametric groups.

If we confine our group elements to those which neighbour the identity we have an infinitesimal group. Such groups were studied extensively by Lie and are often called Lie groups.

Let G be some continuous group. Consider any neighbourhood V of the unit element of this group. We assume that by means of m real parameters $\alpha_1,\alpha_2,...,\alpha_m$, we can define every element of the neighbourhood V in such a way that

i. different elements g of V correspond to different set of values of the parameters $\alpha_1,\alpha_2,...,\alpha_m$

ii. as the parameters change continuously, the element g changes continuously and conversely as the element g changes continuously so do its parameters.

iii. if the elements g_1, g_2 and g_1g_2 lie in the neighbourhood V, then the parameters of the product g_1g_2 are continuous and differentiable functions of parameters of the factors.

If a group satisfies these conditions it is called a Lie group. An example of Lie group is the group consisting of all matrices of the form

$$A\ (\alpha) = \begin{pmatrix} 1 & \alpha \\ 0 & 1 \end{pmatrix}$$

For $\alpha = 0$, A (α) becomes identity. The inverse of A (α) is A ($-\alpha$). The relation between the parameters of the product takes the simple form

$$\gamma = \alpha + \beta$$

and this can be differentiated any number of times.

Let us consider a r-parameter Lie group of transformations

$$x_i' = f_i\ (x_1,\dots,x_n;a_1,\dots,a_r),\ i=1,2,\dots,n. \quad (2.6)$$

or symbolically

$$x_i' = f\ (x;a) \quad (2,7)$$

for which the functions f_i are analytic functions of the parameter a. If the parameters are not essential (by essential we mean that for an r-parameter group we cannot find a set of continuous parameters $a_1, a_2, \ldots, a_m$ with $m < r$) there exist parameter values

$$a_1+\varepsilon_1, a_2+\varepsilon_2, \ldots, a_r+\varepsilon_r,$$

where ε's are arbitrary small quantities which are functions of $a_1, a_2, \ldots, a_r$ such that

$$f_i\ (x,a) = f_i\ (x;\ a+\varepsilon) \qquad (2.8)$$

for all values of x. Expanding in terms of the small function ε_k, we have

$$0 = \sum_{k=1}^{r} \varepsilon_k\ (a)\ \frac{\partial f_i\ (x,a)}{\partial a_k} + \text{higher terms in the}$$

ε_k. $\qquad i = 1,2,\ldots,n. \qquad (2.9)$

If we let ε approaches zero we may write (since the higher terms in the expression (2.9) go to zero).

$$\sum_r \chi_k\ (a)\ \frac{\partial f_i (x,a)}{\partial a_k} = 0. (i = 1,2,\ldots,n) \qquad (2.10)$$

for all x and a.

where χ_k (a) are a set of r functions of the a's.

The transformation satisfy all the group requirements. Thus given a transformation given by parameters set a (equation (2.7)) we can find a parameter set $\bar{a}$ such that

$$\begin{aligned} x' &= f\ (x';\bar{a}) \\ &= f\ (f\ (x,a);\ \bar{a}\ ) = x \end{aligned} \qquad (2.11)$$

This means that equations (2.6) must be soluable for the x_i in terms of x_i', the condition being that the Jacobian is different from zero:

$$\begin{vmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & & \vdots \\ \frac{\partial f_n}{\partial x_1} & \cdots & \frac{\partial f_n}{\partial x_n} \end{vmatrix} \neq 0 \qquad (2.12)$$

If we perform in succession two transformations of the set

$$x_i' = f_i\ (x_1,..,x_n;\ a_1,...,a_r) \qquad (2.13)$$

$$x_i'' = f_i\ (x_1',...,\ x_n';\ b_1,..,b_r)$$

there exist a set of parameter values $c_1,...,\ c_r$ such that

$$x_i'' = f_i\ (x_1,...,x_n;\ c_1,...,c_r) \qquad (2.14)$$

The parameters c be functions of the parameters a and b.

$$c_k = \phi_k\ (a_1,a_2,...,a_k;b_1,b_2,...,b_r) \qquad (2.15)$$

We assume that the functions ϕ_k are analytic and that the $\bar{a}$ in equation (2.11) are analytic functions of a.

There also exist a set of parameter values a^0 which corresponds to the identity transformation

$$x' = f\ (x;\ a^0) = x \qquad (2.16)$$

In general argument we shall take a^0 equal to zero.

Now writing (2.13) through (2.15) we obtain

$$x_i' = f_i\ (x';b)$$

$$= f_i\ (f_1(x;a), \ldots, f_n(x;a);\ b)$$

$$= f_i\ (x;\ c)$$

$$= f_i\ (x, \phi\ (a;b))$$

hence symbolically

$$f(f(x;a);b) = f\ (x;\phi\ (a;b)) \qquad (2.17)$$

is an identity in x,a,b.

Let us look at some examples of continuous groups. Consider

$$x' = ax,\ a \neq 0.$$

The identity element: $a = 1$ and inverse element: $a^{-1} = \frac{1}{a}$. This is one parameter abelian group and the product element $c=ba$. c is an analytic function of a and b.

Let us consider linear group in two dimensions:

$$x' = a_1 x + a_2 y \qquad \begin{vmatrix} a_1 & a_2 \\ a_3 & a_4 \end{vmatrix} \neq 0$$

$$y' = a_3 x + a_4 y$$

If we consider x,y as components of a vector r, the transformations can be written in the form

$$r' = Ar$$

or $$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

Here the identity element: $A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = 1$. The inverse element: $A = A^{-1}$ and the product element $C = BA$. It is a four parameter group.

Consider another group (orthogonal group in two dimensions):

$$x' = a_1 x + a_2 y$$

$$y' = a_3 x + a_4 y$$

and consider only those transformations which leave x^2+y^2 invariant:

Now $$x'^2 + y'^2 = (a_1x + a_2y)^2 + (a_3x + a_4y)^2$$

$$= (a_1^2+a_3^2)x^2 + (a_2^2+a_4^2)y+2\ (a_1a_2 + a_3a_4)\ xy$$

Since we are considering only those transformations which leave $x^2 + y^2$ invariant. Therefore we have

$$a_1^2+a_3^2 = 1,\ a_2^2 + a_4^2 = 1,\ a_1a_2 + a_3a_4 = 0$$

The four parameters are subjected to three functional relations, so that we have a one parameter group. This is a group of rotation about OZ axis and can be written as

$$x' = x \cos\phi - y \sin\phi$$

$$y' = x \sin\phi + y \cos\phi$$

where ϕ is the angle of rotation about the OZ axis and $0 \leq \phi \leq 2\pi$.

In the next article we shall discuss three dimensional rotation group, which is a continuous group.

2.2 THREE DIMENSIONAL ROTATION GROUP

All rotations of space that leave a certain point O fixed constitute the elements of the rotation group. Let g,h,... be all rotations of the three dimensional space about the fixed point. Let G be the collection of all such rotations. The product gh of two rotations g,h is the rotation obtained by successive applications first of the rotation h and then of the rotation g. With this definition of the product of rotations G becomes a group; the identity element of the group G will be the rotation through zero angle, while the inverse of a given rotation g is the rotation that returns the space into initial position. The group G is called three dimensional rotation group.

Let us take a fixed orthogonal system of coordinates. Let e_1, e_2, e_3 be the unit vectors along the coordinate axes. A rotation g takes these vectors into three mutually orthogonal vectors which we denote by g_1, g_2, g_3 respectively. These vectors are completely determined by their projections on the axes of coordinates. Therefore denoting the projection of the vector g_k on the ith-axis by g_{ik}, the vectors g_1, g_2, g_3 are completely determined by the matrix

$$\begin{pmatrix} g_{11} & g_{12} & g_{13} \\ g_{21} & g_{22} & g_{23} \\ g_{31} & g_{32} & g_{33} \end{pmatrix} \qquad (2.18)$$

We denote the matrix (2.18) by g and call it the matrix of the rotation g.

The rotation g is completely determined by the vectors g_1, g_2, g_3 and therefore by its matrix g. In fact every vectors x in three dimensional space may be represented in the form

$$x = x_1 e_1 + x_2 e_2 + x_3 e_3$$

where x_1, x_2, x_3 are the projections of x on respective coordinate axes. The rotation g taken the vectors e_1, e_2, e_3 into vectors g_1, g_2, g_3 and consequently the vector x into the vector $x' = x_1 g_1 + x_2 g_2 + x_3 g_3$, completely determined by the numbers x_1, x_2, x_3 and vectors g_1, g_2, g_3.

We can express the projection x_1', x_2', x_3' of the vector x' on the coordinate axes in terms of the projections of the vector x. From the equations

$$\begin{aligned} x' &= x_1' e_1 + x_2' e_2 + x_3' e_3 = x_1 g_1 + x_2 g_2 + x_3 g_3 \\ &= x_1 (g_{11} e_1 + g_{21} e_2 + g_{31} e_3) + x_2 (g_{12} e_1 + g_{22} e_2 + g_{32} e_3) \\ &\quad + x_3 (g_{13} e_1 + g_{23} e_2 + g_{33} e_3) \end{aligned}$$

it follows that

$$
\begin{aligned}
x_1' &= g_{11}x_1 + g_{12}x_2 + g_{13}x_3 \\
x_2' &= g_{21}x_1 + g_{22}x_2 + g_{23}x_3 \qquad (2.19) \\
x_3' &= g_{31}x_1 + g_{32}x_2 + g_{33}x_3
\end{aligned}
$$

Consequently, the rotation g is determined by the linear transformation of the projections x_1, x_2, x_3 with matrix g. The successive application, first of a rotation h and then by a rotation g corresponds to the successive application of the linear transformation with matrix g. The result is the linear transformation whose matrix is the product gh of the matrices g and h. Thus the product gh of two rotations g,h corresponds to the product of their matrices. Let us now determine which matrices g are the matrices of rotations. The vectors e_1, e_2, e_3 can be transformed into the vectors g_1, g_2, g_3 by a rotation if and only if the latter three vectors are mutually orthogonal, normalized and have the same orientation as the vectors e_1, e_2, e_3, that is, if

$$\sum_{k=1}^{3} g_{ki}\, g_{kj} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases} \qquad (2.20)$$

and

$$\begin{vmatrix} g_{11} & g_{12} & g_{13} \\ g_{21} & g_{22} & g_{23} \\ g_{31} & g_{32} & g_{33} \end{vmatrix} = 1 \qquad (2.21)$$

A matrix g satisfying condition (2.20) is called orthogonal. So we conclude that the group G can be realized as the group of all orthogonal matrices of order three with determinant unity. Condition (2.20) means that

$$g'g = 1 \tag{2.22}$$

where
$$1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

is the unit matrix, while

$$g' = \begin{bmatrix} g_{11} & g_{21} & g_{31} \\ g_{12} & g_{22} & g_{32} \\ g_{13} & g_{23} & g_{33} \end{bmatrix}$$

is transposed matrix of g. But then g' is the inverse matrix of g, that is,

$$g' = g^{-1}$$

and so
$$g\,g' = g\,g^{-1} = 1$$

consequently, also

$$\sum_{k=1}^{3} g_{ik}\, g_{jk} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases} \tag{2.23}$$
$$(i,j = 1,2,3)$$

The Eulerian angle . ϕ, θ and ψ are also very convenient as parameters for describing rotations. To define these angles we introduce, in addition to the fixed system of coordinates xyz, the moving system of coordinates ξ ,η,ζ which follows the rotation. The straight line, along which the planes xoy and $\xi o \eta$ intersect is called nodal axis (Fig.1). We shall take the positive direction of nodal axis to be along the vector n . Let us denote by θ the angle between the axes OZ and O ζ ($0 \leq \theta \leq \pi$), by ϕ the angle between the nodal axis and the oξ axis and by ψ the angle between the ox and the nodal axis. The positive sense for measuring ϕ and ψ is indicated by arrows on (Fig.1). We shall denote by g (ϕ,θ,ψ), the rotation characterized by the Eulerian angles ϕ,θ,ψ. This rotation can be represented as the product of three rotations: the rotation g_ϕ about OZ axis, the rotation g_θ about OX axis and the rotation g_ψ about OZ axis, that is,

$$g\ (\phi,\theta,\psi) = g_\psi\ g_\theta\ g_\phi \tag{2.24}$$

Figure-2 illustrate this relation. On it we have represented the position of the moving coordinate system after the rotation g_ϕ (Fig.2,a) and $g_\theta g_\phi$ (Fig.2,b); the final position $g_\psi g_\theta g_\phi$ is represented in Figure-1.

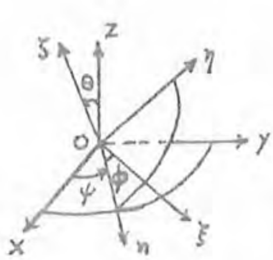


Fig.1

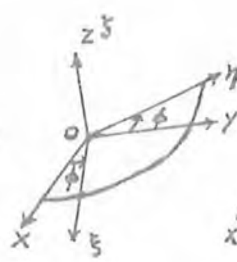


Fig. 2a

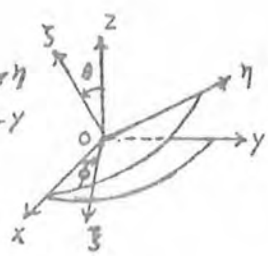


Fig.2b

We now determine the elements of the matrix of rotation $g(\phi,\theta,\psi)$ whose Euler angles are ϕ,θ,ψ. The matrices of rotations g_ϕ, g_θ, g_ψ have the form

$$g_\phi = \begin{bmatrix} \cos\phi & -\sin\phi & 0 \\ \mathrm{Sin}\,\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad g_\psi \begin{bmatrix} \cos\psi & -\sin\psi & 0 \\ \sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$g_\theta = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}$$

The matrix of an arbitrary rotation $g(\phi,\theta,\psi)$ is obtained in accordance with (2.24) by multiplying these three matrices. Performing the calculation, we obtain

$$g(\phi,\theta,\psi)=g_\psi g_\theta g_\phi =$$

$$\begin{pmatrix} \cos\phi\cos\psi - \cos\theta\sin\phi\sin\psi & -\sin\phi\cos\psi - \cos\theta\cos\phi\sin\psi & \sin\psi\sin\theta \\ \sin\psi\cos\phi + \cos\theta\cos\psi\sin\phi & -\sin\phi\sin\psi + \cos\theta\cos\phi\cos\psi & -\cos\psi\sin\theta \\ \sin\phi\sin\theta & \cos\phi\sin\theta & \cos\theta \end{pmatrix} \quad (2.25)$$

The angles ϕ and ψ vary from 0 to 2π while θ varies from 0 to π. Distinct triples of numbers, varying within these limits, correspond to distinct rotations, except in the cases $\theta=0$ and $\theta=\pi$. For $\theta=0$, our rotation is a rotation about the axis OZ through an angle $\phi+\psi$, and for $\theta=\pi$ it is rotation about the axis OZ through the angle $\phi-\psi$. Thus in these cases distinct pairs of numbers (ϕ,ψ) may correspond to the same rotation.

From (2.25) it follows that, if the rotation g is given by the angles ϕ,θ,ψ, then the rotation g^{-1} is given by the angles $\pi-\psi,\theta$ and $\pi-\phi$. In fact if ϕ,ψ,θ are replaced by $\pi-\psi$, $\pi-\phi,\theta$, the matrix g changes into the transposed matrix $g' = g^{-1}$.

2.3 INVARIANT INTEGRAL OVER THE ROTATION GROUP

Invariant integral is important for the theory of representation. We shall say that the function $\omega = f(g)$ is defined over the rotation group G, if for each rotation $g \in G$ there corresponds some number ω. If the rotation g is given by Euler angles ϕ, θ, ψ, then f(g) becomes

$$f(g) = f(\phi, \theta, \psi).$$

where
$$f(\phi+2\pi, \theta, \psi) = f(\phi, \theta, \psi),$$
$$f(\phi, \theta, \psi+2\pi) = f(\phi, \theta, \psi).$$

The integral

$$\int_0^{2\pi}\int_0^{\pi}\int_0^{2\pi} f(g)\ \omega(g)\ d\phi d\theta d\psi$$

$$= \int_0^{2\pi}\int_0^{\pi}\int_0^{2\pi} f(\phi,\theta,\psi)\ \omega\ (\phi,\theta,\psi)\ d\phi d\psi d\theta$$

is called the invariant integral of the function f(g) over the rotation group G, if the factor $\omega(g)$ is so chosen that for any function $f(g) = f(\phi,\theta,\psi)$, continues with respect to ϕ,θ,ψ, the following condition is satisfied

$$\int_0^{2\pi}\int_0^{\pi}\int_0^{2\pi} f\,(gg_0)\;\omega(g)\,d\phi d\theta d\psi = \int_0^{2\pi}\int_0^{\pi}\int_0^{2\pi} f(g)\;\omega(g)\;d\phi d\theta d\psi \qquad (2.26)$$

We prove that the function

$$\omega(g) = \sin\theta \text{ for } g = g_\psi g_\theta g_\phi \qquad (2.27)$$

satisfies (2.26), that is, the expression

$$\int_0^{2\pi}\int_0^{\pi}\int_0^{2\pi} f\,(g)\;\sin\theta d\phi d\theta d\psi \qquad (2.28)$$

is an invariant integral over the group G. For this purpose put $\bar{g} = gg_0$ and let $\bar{\phi},\bar{\theta},\bar{\psi}$ be Euler angles of the rotation $\bar{g}$. $\bar{\phi},\bar{\theta},\bar{\psi}$ are functions of the Euler angles ϕ,θ,ψ of the rotation g and condition (2.26) for integral (2.28) means that we must have

$$\int_0^{2\pi}\int_0^{\pi}\int_0^{2\pi} f(\bar{\phi},\bar{\theta},\bar{\psi})\quad\sin\theta\;d\phi d\theta d\psi$$

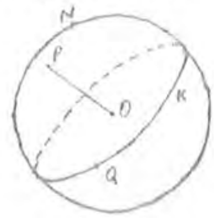
$$= \int_0^{2\pi}\int_0^{\pi}\int_0^{2\pi} f(\phi,\theta,\psi)\;\sin\theta\;d\phi d\theta d\psi \qquad (2.29)$$

If in the integral on the left, we change the variables of integration ϕ,θ,ψ to the variables $\bar{\phi},\bar{\theta},\bar{\psi}$, we see that it will be equal to the integral on the right hand side provided that this change of variable takes $\sin\theta\;d\phi d\theta d\psi$ into $\sin\theta\;d\bar{\phi}d\bar{\theta}d\bar{\psi}$, that is provided that

$$\sin\bar{\theta}\;d\bar{\phi}d\bar{\theta}d\bar{\psi} \;=\; \sin\theta\;d\phi d\theta d\psi \qquad (2.30)$$

To prove (2.30) let P be the point on the unit sphere to which the point N(0,0,1) is carried as a result of the rotation $g^{-1} = g'$ and denote by Q the point on the same sphere to which the point (1,0,0) is carried as a result of the rotation g . The points

P can be chosen arbitrarily on the sphere, and then the point Q



arbitrarily on the great circle K, whose plane is perpendicular to the radius OP. The spherical polar coordinates of the point P are $\frac{\pi}{2} - \phi, \theta$; consequently $\sin\theta \, d\psi d\theta$ is a spherical surface element at the point P. On the other hand $d\phi$ is an element of arc of the circle K. In fact, an increment $d\phi$ in ϕ, with fixed ψ and θ, corresponds to a rotation through $d\phi$ about OP, that is a displacement of $d\phi$ of the point Q.

But the points $\bar{P}$ and $\bar{Q}$ corresponding to a rotation $\bar{g} = gg_0$, are obtained from the points P and R by the rotation g_0. This rotation leaves both the surface element $\sin\theta \, d\psi d\theta$ and the element of arc $d\phi$ of the circle K invariant; it follows that their product $\sin\theta \, d\phi d\psi d\theta$ also remains invariant which proves formula (2.30).

It is evident that for any positive constant c, the multiple $\omega(g) = c \sin\theta$ also satisfies condition (2.26). Let us choose c so that the following condition is satisfied:

$$\int_0^{2\pi}\int_0^{\pi}\int_0^{2\pi} c \sin\theta \, d\phi d\theta d\psi = 1 \qquad (2.31)$$

that is $8\pi^2 c = 1$

Hence $c = \frac{1}{8\pi^2}$ and $\omega(g) = \frac{1}{8\pi^2} \sin\theta$.

We shall call the expression $\frac{1}{8\pi^2} \sin\theta \, d\phi \, d\theta \, d\psi$ the invariant volume element of the group G and denote it by dg, so that

$$dg = \frac{1}{8\pi^2} \sin\theta \, d\phi \, d\theta \, d\psi$$

using this notation we can write the invariant integral in the form $\int_G f(g)\, dg$, where the symbol $\int_G$ denotes integration over the range $0 \leq \phi \leq 2\pi$, $0 \leq \theta \leq \pi$, $0 \leq \psi \leq 2\pi$. Further we shall drop G. The condition (2.26) of invariance will then become

$$\int f(gg_0)\, dg = \int f(g)\, dg \qquad (2.32)$$

We further note that

$$\int f(g^{-1})\, dg = \int f(g)\, dg \qquad (2.33)$$

and

$$\int f(g_0 g)\, dg = \int f(g)\, dg \qquad (2.34)$$

Further putting $f_1(g) = f(g^{-1})$ and using formulae (2.32) and (2.33) we have

$$\int f(g_0 g)\, dg = \int f(g_0 g^{-1})\, dg = \int f_1(gg_0^{-1})\, dg$$

$$= \int f_1(g)\, dg = \int f(g^{-1})\, dg = \int f(g)\, dg$$

Finally we note that by virtue of condition (2.31)

$$\int dg = 1$$

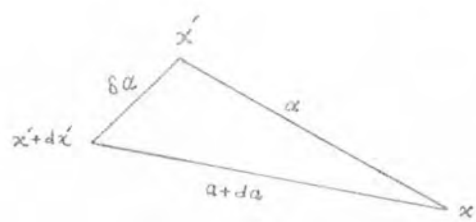
2.4 INFINITESIMAL TRANSFORMATION OF A CONTINUOUS GROUP

In equation (2.15) viz.

$$c_k = \phi_k\ (a_1, \ldots, a_k;\ b_1, \ldots,\ b_r) \tag{2.35}$$

We have expressed the parameters c of a product of transformations in terms of parameters a and b of the factors.

The transformation $x' = f(x;a)$ takes all points of the space from their initial positions x to final positions x'. Let us consider the gradual shift of the points of the space as we vary the parameters continuously from their initial values a=0. This leads us to the concept of infinitesimal transformations. We illustrate the method first for a one parameter group in one variable x.



Suppose that the transformation with parameter a takes x to x'. The neighbouring parameter value a+da will take the points x to points $x' + dx'$ (since f is analytic function of a). But we can also find a parameter value a very close to zero (that is to the identity) which takes x' to $x' + dx'$. Thus we have two alternative paths from x to $x' + dx'$.

$$x' + dx' = f(x;\ a + da) \tag{2.36}$$

Now $x' = f(x,a)$

$$x' + dx' = f(x'; \delta a) \tag{2.37}$$

Expanding the last equation we have

$$dx' = \left(\frac{\partial f(x';a)}{\partial a}\right)_{a=0} . \delta a \tag{2.38}$$

or $$dx' = u(x')\,\delta a \tag{2.38a}$$

From (2.35) we have

$$a + da = \phi(a; \delta a) \tag{2.39}$$

So that $da = \left(\frac{\partial \phi(a,b)}{\partial b}\right)_{b=0} . \delta a$

or $\delta a = \psi(a)\, da$

substituting in (2.38a) we get

$$dx' = u(x')\ \psi(a)\ da$$

Therefore $\dfrac{dx'}{u(x')} = \psi(a)\ da.$

Now integrating this equation from a=0 to a. The initial value of x' is x. Calling the integral of $\dfrac{1}{u(x')}$ the function $U(x')$, we have

$$U(x') - U(x) = \int_0^a \psi(a)\ da$$

If we introduce new variables $y=U(x)$ and let $\int_0^a \psi(a)\ da = t$, we get $y' - y = t$.

We now carry out the analogoes expansion in general case. Consider an r-parameter Lie group of transformation

$$x_i' = f_i\ (x_1,\ldots,x_n;\ a_1,\ldots,a_r) \qquad i = 1,\ldots,n.$$

We can reach a neighbouring point $x_i' + dx_i'$ near original point x_i' through small variation da_k of the parameter a_k as

$$x_i'+dx_i' = f_i(x_1,\ldots,x_n\ ;\ a_1+da_1,\ldots,a_r+da_r) \qquad (2.40)$$

$$i = 1,\ldots,n$$

or we may take

$$x_i' \quad f_i\ (x_1',\ldots,x_n';\ 0,\ldots,0)$$

which is the identity transformation and take a set of parameter variations δa_ℓ such that

$$x_i' + dx_i' = f_i\ (x_1',\ldots,x_n';\ \delta a_1,\ldots,\ \delta a_r)$$

so that

$$dx_i' = \sum_{k=1}^{r} \left[\frac{\partial f_i(x_1',\ldots,x_n';\ a_1,\ldots,a_r)}{\partial a_k} \right]_{a=0} \delta a_k$$

$$= \sum_{k=1}^{r} u_{ik}\ (x')\ \delta a_k \qquad (2.41)$$

From the equations (2.17), (2.40) and (2.41) we see that

$$a_\ell + da_\ell = \phi(a_1,\ldots,a_r;\ \delta a_1,\ldots,\ \delta a_r)$$

so that

$$da_\ell = \sum_{m=1}^{r} \left[\frac{\partial \phi_\ell}{\partial b_m} (a_1,\dots,a_r;\ b_1,\dots,b_r) \right]_{b=0} \delta a_m$$

$$da_\ell = \sum_{m=1}^{r} \Gamma_{\ell m} (a) \delta a_m$$

where $\Gamma_{\ell m} (a) = \delta_{\ell m}$

solving (2.41) for the δa's in terms of da's, we have

$$\delta a_k = \sum_{\ell=1}^{r} \psi_{k\ell} (a) \ da_\ell$$

where the matrices $\Psi\Gamma = 1$ and $\psi_{k\ell}(\theta) = \delta_{k\ell}$. Putting in (2.41) we get

$$dx_i = \sum_{k,\ell=1}^{r} u_{ik} (x') \ \psi_{k\ell} (a) \ da_\ell$$

or

$$\frac{\partial x_i}{\partial a_\ell} = \sum_{k=1}^{r} u_{ik} (x') \ \psi_{k\ell} (a) \qquad (2.42)$$

In equation (2.42) we may consider the x's as functions of the parameters a. The coordinates x are the initial values of the x's for a=0.

2.5 INFINITESIMAL OPERATOR OF THE CONTINUOUS GROUP

Any infinitesimal transformation is a linear combination of the r independent infinitesmal transformations. If we examine the change of a function F(x) under the infinitesimal transformation

$$x_i' = f_i \ (x_1,\ldots,x_n;\ a_1,\ldots,a_r) \qquad i=1,\ldots,n.$$

We find $dF = \sum_{i=1}^{n} \frac{\partial F}{\partial x_i} \ dx_i$

$$= \sum_{i=1}^{n} \frac{\partial F}{\partial x_i} \sum_{\ell=1}^{r} u_{i\ell}\ (x) \quad \delta a_\ell$$

$$= \sum_{\ell=1}^{r} \delta\alpha_\ell \sum_{i=1}^{r} \left(u_{i\ell}\ (x) \frac{\partial}{\partial x_i}\right) F$$

$$= \sum_{\ell=1}^{r} \delta a_\ell \quad X_\ell \ F \tag{2.43}$$

The operators $X_\rho = \sum_{i=1}^{n} u_{i\ell}(x) \frac{\partial}{\partial x_i}$

are called infinitesimal operator of the group. The operator $1 + \sum_\rho X_\rho \ \delta a_\ell$ is close to the identity operator. When we choose the function F to be one of the variables X_i, we find

$$x_i' = \left[1 + \sum_\rho X_\rho \ \delta a_\ell\right] x_i = x_i + \sum_\rho u_{i\ell}\ (x) \ \delta a_\ell$$

So we get back (2.41). We note that if we neglect higher order terms in the infinitesimals δa, the infinitesimal transformations commute with one another. In fact the result of two successive infinitesimal transformations is just the sum of the two.

Consider as an example, the group

$$x' = ax + b$$

The identity element has parameters a=1, b=0. The infinitesimal transformations are

$$x' = (1 + \delta a)\ x + \delta b$$

$$= x + x \cdot \delta a + \delta b$$

Thus the infinitesimal operators of the group are

$$X_1 = x \frac{\partial}{\partial x}, \quad X_2 = \frac{\partial}{\partial x}$$

We note that the commutator

$$[X_1, X_2] = X_1 X_2 - X_2 X_1$$

$$= x \frac{\partial}{\partial x} \left(\frac{\partial}{\partial x}\right) - \frac{\partial}{\partial x} \left(x \frac{\partial}{\partial x}\right)$$

$$= -\frac{\partial}{\partial x} = -X_2$$

gives no new operator.

As another example, take the group

$$x' = ax$$

$$y' = by$$

The identity element has parameters a = b = 1. The infinitesimal transformations are

$$x' = (1 + \delta a)\ x = x + x\,\delta a$$

$$y' = (1 + \delta b)\ y = y + y\,\delta b$$

So the infinitesimal operators of the group are

$$X_1 = x \frac{\partial}{\partial x}, \quad X_2 = y \frac{\partial}{\partial x}$$

$$[X_1, X_2] = 0$$ gives nothing new

For rotation group in two dimensions

$$x' = x \cos\phi - y \sin\phi$$

$$y' = x \sin\phi + y \cos\phi$$

We obtain the infinitesimal transformations by expanding in ϕ around $\phi = 0$:

$$x' = x - y\,\delta\phi$$

$$y' = x\,\delta\phi + y$$

The infinitesimal operator is

$$X = \frac{\partial}{\partial y} - y \frac{\partial}{\partial x}$$

2.6 REPRESENTATION OF GROUPS

By Representation of an abstract group we mean in general any group composed of concrete mathematical entities which is homomorphic to the original group. However we shall restrict our attention to the representation by square matrices.

The concept of group representation is a far-reaching generalization of the concept of exponential function. The exponential function e^{ax} can be defined as the continuous solution of the functional equation.

$$f\,(x + y) = f\,(x).f(y) \tag{2.44}$$

satisfying the initial condition $f'(0) = a$.

On generalizing this equation to any group G, we are led to consider scalar functions on G satisfying the relation.

$$f\,(g_1 g_2) = f(g_1)\,f\,(g_2) \tag{2.45}$$

However for commutative groups there are not many such functions, since equality (2.45) implies that they must satisfy the relation

$$f\,(g_1 g_2) = f(g_1)\,f(g_2) = f(g_2)\,f\,(g_1)$$

$$= f(g_2 g_1)$$

Therefore scalar functions satisfying equation (2.45) are inadequate for the purpose of expanding an arbitrary function F(g) on the group G. In order to obtain a sufficiently good supply of solutions of equation (2.45) we consider functions whose values are the matrices or linear transformations. Therefore we consider solutions of the functional equation

$$T_{g_1 g_2} = T_{g_1} T_{g_2}$$

where g_1 and g_2 are elements of the given group G and T_g is a function on this group. These solutions are called the representation of the group. Here we shall require that the operator T_g depend continuously on g and that all operators T_g be continuous in linear space S, that is, $\lim_{n\to\infty} x_n = x$ should imply that $\lim_{n\to\infty} T_g x_n = T_g x$. Further we shall require that the operators T_g have continuous inverses.

Thus by a representation of the group G, we shall mean a continuous function T_g on this group, taking values in the group of non-singular continuous linear transformation of the linear space S and satisfying the functional equation

$$T_{g_1 g_2} = T_{g_1} T_{g_2} \tag{2.46}$$

It follows from the above equation that

$$T_{g^{-1}} = T_g^{-1} \tag{2.47}$$

and $$T_e = E$$

where e is the identity in the group and E is the identity operator in linear space S. The equations (2.46) and (2.47) shows that T_g is a homomorphic mapping of G of the non-singular continuous transformation of linear space S.

A representation T_g is called faithful if only for the identity element e of G we have $T_e = E$ and trivial or identity transformation if $T_g = E$ for all elements $g \in G$.

The linear speace S, in which the operator T_g act, is called the space of representation T_g. If this space is finite dimensional, then the representation T_g is also called finite dimentional. The number of rows and columns in the matrix is called the dimensionality of the representation.

A subspace S_1 of the space of representation T_g is called invariant if $x \in S_1$ implies that for all $g \in G$ we have $T_g\, x \in S_1$. In other words, all operators of the representation T_g transform vectors of the subspace S_1 into vectors of the same subspace.

For every representation T_g there are at least two invariants subspaces, namely the null subspace and the whole space S of this representation. These invariant subspaces are called trivial. If a representation T_g possesses only trivial invariant subspace it is called irreducible. A representation with non-trivial invariant subspaces is reducible,

Suppose that in a space S we are given a scalar product (x,y). The operator A^* is called Hermition-adjoint relative to this scalar product if for any two vectors x and y of S we have

$$(Ax,y) = (x,A^*y)$$

If T_g is a representation of the group G in the space S, then $T^*_{g^{-1}}$ is also a representation of this group. Since $(AB)^*=B^*A^*$, wer have

$$T^*_{g_1^{-1}} \; T^*_{g_2^{-1}} = \left| T_{g_2^{-1}} \; T_{g_1^{-1}} \right|^* = T^*_{g_2^{-1}g_1^{-1}}$$

$$= T^*_{(g_1g_2)^{-1}}$$

The representation $T^*_{g^{-1}}$ is called Hermition-adjoint to T_g.

A representation T_g of a group G in a space S is called unitary relative to the scalar product (x,y) if the operators T_g leave the scalar product invariant, that is, if all vectors x and y of S and all elements g of G one has the equality

$$(T_g \, x, \; T_g \, y) = (x,y)$$

In this case we have $T^*_g \; T_g = E$ and therefore $T^*_g = T^{-1}_g = T_{g^{-1}}$

2.7 THE MATRIX NOTATION FOR REPRESENTATION

A representation T_g of a group G has been defined as an operator function on this group satisfying the functional equation

$$T_{g_1 g_2} = T_{g_1} T_{g_2} \qquad (2.48)$$

In order to pass from abstract functions taking numerical values, we make use of matrix notation for operators.

We first consider the case when the space representation T_g is finite dimensional. In this space we choose a basis $e_1, \ldots, e_n$. The operator T_g transform the basis element e_j into $T_g\, e_j$. Decomposing $T_g\, e_j$ into basis elements, we get

$$T_g e_j = \sum_{i=1}^{n} T_{ij}\,(g)\, e_j \qquad (2.49)$$

Thus with each operator of representation T_g we associate the matrix

$$T_g = T_{ij}\,(g) \quad 1 \leq i,\ j \leq n. \qquad (2.50)$$

or what is the same thing, a collection of n^2 numerical functions T_{ij} (g) on the group .From the continuity of T_g it follows that the functions T_{ij} (g) are continuous.

Since the multiplication of operators means the multiplication of corresponding matrices from the functional equation we get the system of n^2 equalities

$$T_{ij}(g_1 g_2) = \sum_{k=1}^{n} T_{ik}(g_1)\, T_{kj}(g_2) \qquad (2.51)$$

$$1 \leq i,\ j \leq n$$

In this way we can define an n-dimensional representation of the group G as the collection of n^2 continuous numerical functions $T_{ij}(g)$, $g \in G$, satisfying the system of functional equations (2.51) and such that $\det (T_{ij}(g)) \neq 0$.

The matrix notation depends on the choice of the basis $\{e_i\}$ in the representation space. If A is a non-singular operator, mapping the space S onto itself and $f_i = Ae_i$, then in the basis $\{f_i\}$ the matrix of representation T_g has the form

$$(A^{-1})\ T_g\ (A) \qquad (2.52)$$

Here A denotes the matrix of operator A in the basis $\{e_k\}$ that is the matrix such that

$$f_j = \sum_{i=1}^{n} a_{ij}\, e_i \qquad (2.53)$$

Thus on going over to another basis in the space S the matrix T_g is replaced by another matrix.

If $T_{ij}(g)$ is the matrix of the representation T_g then $\overline{T_{ij}(g)}$ is the matrix of certain representation $\overline{T}_g$. $\overline{T}_g$ depends not only on T_g but also on the choice of the basis in space S.

If the space of representation T_g is Euclidean, then we choose an orthogonal normalized (that is orthonormal) basis in it:

$$(e_i\ ,\ e_j) = \delta_{ij} \tag{2.54}$$

where δ_{ij} is the kronecker symbol defined as

as $$\delta_{ij} = \begin{cases} 0 & \text{for } i \neq j \\ 1 & \text{for } i = j \end{cases}$$

In this case the matrix elements are calculated according to the formula

$$T_{ij}\ (g) = (T_g\ e_j,\ e_i) \tag{2.55}$$

The above formula is obtained by taking scalar product of both sides of equation (2.49) with e_i.

We now consider an infinite-dimensional representation T_g. In this case we take the representation space to be Hilbert space. In this Hilbert space we choose an orthonormal basis $\{e_i\}$, $i = 1,\ldots,n,\ldots$ In this basis we have

$$T_g\ e_j = \sum_{i=1}^{\infty} T_{ij}\ (g)\ e_i \tag{2.56}$$

On taking the scalar product of both sides of (2.56) with e_i, we get

$$T_{ij}(g) = (T_g e_j, e_i), \quad 1 \leq i, j < \infty \tag{2.57}$$

Thus with each operator T_g we associate the infinite matrix T_g with element $T_{ij}(g)$.

Now we shall show that in multiplying operators, the matrices are multiplied according to the usual rule, that is,

$$T_{ij}(g_1 g_2) = \sum_{k=1}^{\infty} T_{ik}(g_1)\, T_{kj}(g_2) \tag{2.58}$$

From (2.57) we have

$$\begin{aligned} T_{ij}(g_1 g_2) &= (T_{g_1 g_2} e_j, e_i) \\ &= (T_{g_1} T_{g_2} e_j, e_i) \\ &= (T_{g_2} e_j, T^*_{g_1} e_i) \end{aligned} \tag{2.59}$$

The scalar product of vectors $T_{g_2} e_j$ and $T^*_{g_1} e_i$ expressed as follows:

$$\begin{aligned} (T_{g_2} e_j, T_{g_1} e_i) &= \sum_{k=1}^{\infty} (T_{g_2} e_j, e_k)\overline{T^*_{g_1} e_i, e_k)} \\ &= \sum_{k=1}^{\infty} (T_{g_2} e_j, e_k)\, (T_{g_1} e_k, e_i) \\ &= \sum_{k=1}^{\infty} T_{ik}(g_1)\, T_{kj}(g_2) \end{aligned} \tag{2.60}$$

From (2.59) and (2.60) we get

$$T_{ij}(g_1 g_2) = \sum_{k=1}^{\infty} T_{ik}(g_1)\, T_{kj}(g_2)$$

Consider an irreducible representation $g \to T_g$ of weight ℓ. T_{mn}, the elements of the matrix T_g where $-\ell \leq m, n \leq \ell$ are functions of g. If we multiply g by an arbitrary rotation g_1, then T_{mn} goes into a different function of g, equal to $T_{mn}(gg_1)$. This transformation of functions T_{mn} depends upon g_1. We shall denote it by U_g. We can write

$$U_{g_1} T_{mn}(g) = T_{mn}(gg_1) \tag{2.61}$$

We have $U_{g_2} U_{g_1} T_{mn}(g) = U_{g_2} T_{mn}(gg_1) = T_{mn}(gg_2g_1)$

$$= U_{g_2 g_1} T_{mn}(g)$$

The relation $U_{g_2} U_{g_1} = U_{g_2 g_1}$ (2.62)

holds for the transformation U_{g_1}. The function $T_{mn}(gg_1)$ is an element of the matrix T_{gg_1}. Because T_g form a representation of the rotation group; therefore,

$$T_{gg_1} = T_g T_{g_1}$$

Equating the elements of the matrices on the left and right hand sides of this equality, we find that

$$T_{mn}(gg_1) = \sum_{s=-\ell}^{\ell} T_{ms}(g)\, T_{sn}(g_1)$$

This implies that

$$U_{g_1} T_{mn}(g) = \sum_{s=-\ell}^{\ell} T_{ms}(g)\, T_{sn}(g) \tag{2.63}$$

Therefore the transformation U_{g_1} carries an element of the m-th row of the matrix T_g into the linear combination of the element of the same row of T_g with coefficients which depend upon g_1. It follows from (2. 62) and (2.63) that for every m the transformation U_{g_1} comprise a (2 ℓ+ 1) -dmensional representation of the rotation group.

CHAPTER-3

3.1 SPHERICAL FUNCTIONS

Consider a function

$$f(x) = f(x_1, x_2, x_3).$$

We write the rotation g in the form

$$x' = gx$$

$$x'_i = \sum_{k=1}^{3} g_{ik} x_k \qquad (3.1)$$

If we substitute for the x_k in $f(x_1, x_2, x_3)$ their values in terms of x_i as obtainable from (3.1) we obtain a new function $f_1(x'_1, x'_2, x'_3)$. We shall say that the function f goes into function f_1 under the rotation g. The transformation which carries f into f_1 will be denoted by T_g. Therefore for every rotation g there exist a transformation T_g on the function f(x) and this transformation carries the function f into f_1, where f_1 is obtained from f by replacing x by its expression in terms of x'. We have, therefore,

$$T_g f(x) = f_1(x) \text{ where } f_1(x) = f(g^{-1}x)$$

It is clear that the transformation T_g is linear

$$T_g [f(x) + g(x)] = f_1(x) + g_1(x)$$

$$= T_g f(x) + T_g g(x)$$

And $T_g \left[m\, f(x)\right] = m\, f_1\, (x)$

$$= m\, T_g\, f\, (x)$$

To show that the product of two transformations T_{g_1} and T_{g_2} correspond to the product of rotations g_1 and g_2, consider the two rotations g_1 and g_2 taken consecutively, we have

$$x' = g_1\, (x)$$

$$x'' = g_2\, (x)$$

By first rotation f(x) goes into

$$T_{g_1} f(x) = f\, (g^{-1}\, x), \tag{3.2}$$

and as a result of second rotation, f (x) goes into

$$\begin{aligned} T_{g_2} T_{g_1} f(x) &= T_{g_2}\, f(g_1^{-1}\, x) \\ &= f\left[(g_1^{-1}\, x\, g_2^{-1}\, x)\right] \\ &= f\left[(g_2 g_1)^{-1}\, x\right] \\ &= T_{g_2 g_1} f(x). \end{aligned}$$

Thus we have

$$T_{g_2 g_1} = T_{g_2}\, T_{g_1}$$

Now if we have a sphere with center at the origin, it will goes into itself under all rotations. It will be convenient to limit ourselves to functions defined on the surface of such a sphere. Because of this reason, we shall suppose that $x_1^2 + x_2^2 + x_3^2 = 1$, that is, x lies on the surface of the unit sphere. It will be convenient to suppose that the vector $\vec{x}$ is given by its spherical coordinates θ and φ and to set

$$x_1 = \sin\theta\cos\phi$$

$$x_2 = \sin\theta\sin\phi$$

$$x_3 = \cos\theta$$

We limit ourselves also to functions f (θ,φ), which have the property that the square of their absolute values is integrable over the surface of the sphere and we define scalar product for such functions by the formula

$$(f,g) = \int_0^{2\pi}\int_0^{\pi} f(\theta,\phi)\,\overline{g(\theta,\phi)}\,\sin\theta\,d\theta\,d\phi$$

If we denote the coordinates of the vector x by θ,φ and the coordinates of the vector x′ by θ′,φ′, then we have

$$\begin{aligned}(T_g f, T_g g) &= \int_0^{2\pi}\int_0^{\pi} f(\theta,\phi)\,\overline{g(\theta,\phi)}\sin\theta\,d\theta\,d\phi \\ &= \int_0^{2\pi}\int_0^{\pi} f(\theta',\phi')\,\overline{g(\theta',\phi')}\sin\theta'\,d\theta'\,d\phi' \\ &= (f,g)\end{aligned}$$

This implies that the transformations T_g are unitary. By the transformations just introduced we can construct irreducible representations with arbitrary integer weights ℓ. In order to do this we construct finite dimensional spaces consisting of functions, in which the transformations T_g are irreducible representations of the rotation group with a given weight ℓ, consists of linear combinations of $2\ell+1$ functions $f_m(x)$ $(-\ell \leq m \leq \ell)$. We shall choose the functions in such a way that they are a canonical basis for this representation. Functions on the sphere which belong to a space in which an irreducible representation of weight ℓ is realized are called spherical functions of the ℓ-th order. The functions $f_m(x)$ which form a canonical basis in space are called principal spherical functions of the ℓ-th order.

3.2 DIFFERENTIAL OPERATORS CORRESPONDING TO INFINITESIMAL ROTATIONS

In article 3.1 we defined the linear transformations T_g in a certain space of functions on the surface of the unit sphere in three dimensional space. We now construct the transformations A_1, A_2 and A_3 corresponding to infinitesimal rotations about the coordinate axes in this space.

First we find the operator A_3 which correspond to an infinitesimal rotation aboutOZ-axis. We examine a rotation g through the angle α.

We have $T_g\ f(x) = f\ (g^{-1}x)$. Therefore, for the rotation g around the OZ-axis we have

$$T_g f(\theta,\phi) = f(\theta, \phi - \alpha).$$

Expanding $f(\theta,\phi-\alpha)$ in powers of α, we have

$$f(\theta,\phi-\alpha) = f(\theta,\phi) - \alpha \frac{\partial f(\theta,\phi)}{\partial \phi} + \ldots \qquad (3.3)$$

Therefore $A_3 f = -\frac{\partial f(\theta,\phi)}{\partial \phi}$

since $T_g = E + \alpha A_3 + \ldots$

So the operator A_3 has the form

$$A_3 = -\frac{\partial}{\partial \phi} \qquad (3.4)$$

For small rotation g through an angle α around any fixed axis,

$$T_g f(\theta,\phi) = f(\theta',\phi'),$$

where θ' and ϕ' depend upon the angle α and are equal to θ and ϕ respectively for $\alpha = 0$.

Expanding $T_g f = f(\theta',\phi')$ in a power series with respect to α, we have

$$f(\theta',\phi') = f(\theta,\phi) + \left(\frac{\partial f}{\partial \theta}\frac{d\theta'}{d\alpha} + \frac{\partial f}{\partial \phi}\cdot\frac{d\phi'}{d\alpha}\right)\Big|_{\alpha=0} \alpha + \ldots \qquad (3.5)$$

Therefore the operator A corresponding to a given infinitesimal rotation has the form

$$A = a(\theta,\phi)\frac{\partial}{\partial \theta} + b(\theta,\phi)\frac{\partial}{\partial \phi} \qquad (3.6)$$

where $a(\theta,\phi) = \frac{d\theta'}{d\alpha}\Big|_{\alpha=0}$, $b(\theta,\phi) = \frac{d\phi'}{d\alpha}\Big|_{\alpha=0}$ (3.7)

We now find the differential operator A_1 which corresponds to an infinitesimal rotation around the OX-axis. If g is a rotation through the angle α around the OX axis, then g^{-1} is a rotation through the angle $-\alpha$ around the same axis. Hence the vector $x' = g^{-1}x$ has the coordinates

$$x_1' = x$$

$$x_2' = x_2 \cos\alpha + x_3 \sin\alpha \qquad (3.8)$$

$$x_3' = -x_2 \sin\alpha + x_3 \cos\alpha$$

The functions $\frac{dx_k}{d\alpha}\Big|_{\alpha=0}$, therefore, have the form

$$\frac{dx_1}{d\alpha}\Big|_{\alpha=0} = 0$$

$$\frac{dx_2}{d\alpha}\Big|_{\alpha=0} = x_3 \qquad (3.9)$$

$$\frac{dx_3}{d\alpha}\Big|_{\alpha=0} = -x_2$$

Now we have

$$x_1 = \sin\theta \cos\phi$$

$$x_2 = \sin\theta \sin\phi$$

$$x_3 = \cos\theta \qquad (3.10)$$

Differentiating equations (3.10) which connects the cartesian coordinates with spherical coordinates and using (3.9), we find for $\alpha = 0$, the following equations

$$\left.\frac{dx_1}{d\alpha}\right|_{\alpha=0} = \cos\theta \cos\phi \frac{d\theta}{d\alpha} - \sin\theta \sin\phi \frac{d\phi}{d\alpha}$$

or $$\cos\theta \cos\phi \frac{d\theta}{d\alpha} - \sin\theta \sin\phi \frac{d\phi}{d\alpha} = 0 \qquad (3.11)$$

$$\left.\frac{dx_2}{d\alpha}\right|_{\alpha=0} = \cos\theta \sin\phi \frac{d\theta}{d\alpha} + \sin\theta \cos\phi \frac{d\phi}{d\alpha}$$

or $$\cos\theta \sin\phi \frac{d\theta}{d\alpha} + \sin\theta \cos\phi \frac{d\phi}{d\alpha} = \cos\theta \qquad (3.12)$$

$$\left.\frac{dx_3}{d\alpha}\right|_{\alpha=0} = -\sin\theta \frac{d\theta}{d\alpha}$$

or $$-\sin\theta \frac{d\theta}{d\alpha} = -\sin\theta \sin\phi$$

or $$\frac{d\theta}{d\alpha} = \sin\phi \qquad (3.13)$$

From (3.11) and (3.13) we have

$$\frac{d\phi}{d\alpha} = \cot\theta \cos\phi$$

Substituting in equations (3.6) and (3.7) the values found in this way for $\left.\frac{d\theta}{d\alpha}\right|_{\alpha=0}$ and $\left.\frac{d\phi}{d\alpha}\right|_{\alpha=0}$. We find that A_1 is the differential operator defined by the formula

$$A_1 = \sin\phi \frac{\partial}{\partial\theta} + \cot\theta \cos\phi \frac{\partial}{\partial\phi} \qquad (3.14)$$

The operator A_2 which correspond to an infinitesimal rotation about Oy axis can be obtained by replacing by $\phi - \frac{\pi}{2}$ for ϕ in (3.14). Therefore the differential operator A_2 is defined by the formula

$$A_2 = -\cos\phi \ \frac{\partial}{\partial\theta} + \cot\theta \ \sin\phi \ \frac{\partial}{\partial\phi} \qquad (3.15)$$

We now determine the transformations H_+, H_- and H_3, using the expressions for A_1, A_2 and A_3. We have

$$H_+ = H_1 + i\,H_2 = i\,A_1 - A_2$$

$$= i\sin\phi \frac{\partial}{\partial\theta} + i\cot\theta \ \cos\phi \ \frac{\partial}{\partial\phi} + \cos\phi \ \frac{\partial}{\partial\theta}$$

$$-\cot\theta \ \sin\phi \ \frac{\partial}{\partial\phi}$$

$$= (\cos\phi + i\sin\phi)\frac{\partial}{\partial\theta} + i\cot\theta\ (\cos\phi + i\sin\phi)\frac{\partial}{\partial\phi}$$

$$= e^{i\phi}\left(\frac{\partial}{\partial\theta} + i\cot\theta \ \frac{\partial}{\partial\phi}\right) \qquad (3.16)$$

$$H_- = H_1 - i\,H_2 = i\,A_1 + A_2$$

$$= i\sin\phi \ \frac{\partial}{\partial\theta} + i\cot\theta \ \cos\phi \ \frac{\partial}{\partial\phi} - \cos\phi\ \frac{\partial}{\partial\theta}$$

$$+ \cot\theta \ \sin\phi \ \frac{\partial}{\partial\phi}$$

$$= -(\cos\phi - i\sin\phi)\frac{\partial}{\partial\theta} + i\cot\theta\ (\cos\phi - i\sin\phi)\frac{\partial}{\partial\phi}$$

$$= e^{-i\phi}\left(-\frac{\partial}{\partial\theta} + i\cot\theta \ \frac{\partial}{\partial\phi}\right) \qquad (3.17)$$

And $H_3 = i\,A_3 = -i\,\frac{\partial}{\partial\phi}$ (3.18)

3.3 DIFFERENTIAL EQUATION OF THE SPHERICAL FUNCTIONS

Functions on the sphere which lie in an invariant subspace in which an irreducible representation of weight ℓ operates are called spherical functions of the ℓ-th order. Functions forming the canonical basis in this subspace (that is, the eigenvectors of the transformation H_3) are called principal spherical functions of the ℓ-th order. Principal spherical functions are denoted by $Y_\ell^m(\theta,\phi)$ where m is the corresponding eigenfunction of the operator H_3 with corresponding eigenvalue m. Using expression (3.18) for H_3 we obtain

$$H_3\, Y_\ell^m(\theta,\phi) = -i\frac{\partial}{\partial\phi} Y_\ell^m(\theta,\phi) = m\, Y_\ell^m(\theta,\phi)$$

or
$$\frac{\partial}{\partial\phi} Y_\ell^m(\theta,\phi) = im\, Y_\ell^m(\theta,\phi)$$

From this we have

$$Y_\ell^m(\theta,\phi) = e^{im\phi} F_\ell^m(\theta) \tag{3.19}$$

Thus $Y_\ell^m(\theta,\phi)$ depends on ϕ.

Since $Y_\ell^m(\theta,\phi)$ are normalized eigenfunctions of the transformation H_3 it follows that

$$\int_0^{2\pi}\int_0^{\pi} |\, Y_\ell^m(\theta,\phi)\,|^2 \sin\theta \, d\theta \, d\phi \tag{3.20}$$

As $\int_0^{2\pi} | e^{im\phi} | d\phi = 2\pi$, we can set

$$Y_\ell^m (\theta,\phi) = \frac{1}{\sqrt{2\pi}} F_\ell^m (\theta) e^{im\phi} \tag{3.21}$$

So we can write the normalization (3.20) as

$$\int_0^\pi | F_\ell^m (\theta) |^2 \sin\theta \, d\theta = 1 \tag{3.22}$$

We now derive the differential equations for spherical functions and the functions $F_\ell^m (\theta)$. The vectors f in a space in which an irreducible representation with weight ℓ acts satisfy the equation

$$H^2 f = \ell(\ell+1) f$$

where $H^2 = H_1^2 + H_2^2 + H_3^2$

We note that $H_1^2 + H_2^2 = \frac{1}{2} (H_+ H_- + H_- H_+)$ substituting H_+, H_- from (3.16) and (3.17) we get

$$H_1^2 + H_2^2 = \frac{1}{2} \Big[e^{i\phi} (\frac{\partial}{\partial\theta} + i \cot\theta \frac{\partial}{\partial\phi}) e^{-i\phi} (-\frac{\partial}{\partial\theta} + i \cot\theta \frac{\partial}{\partial\phi})$$

$$+ e^{-i\phi} (-\frac{\partial}{\partial\theta} + i \cot\theta \frac{\partial}{\partial\phi}) e^{i\phi} (\frac{\partial}{\partial\theta} + i \cot\theta \frac{\partial}{\partial\theta}) \Big]$$

$$= \frac{1}{2} \Big[(e^{i\phi} \frac{\partial}{\partial\theta} + i e^{i\phi} \cot\theta \frac{\partial}{\partial\phi}) (-e^{-i\phi} \frac{\partial}{\partial\theta} + i e^{-i\phi} \cot\theta \frac{\partial}{\partial\phi})$$

$$+ (-e^{-i\phi} \frac{\partial}{\partial\theta} + e^{-i\phi} \cot\theta \frac{\partial}{\partial\theta}) (e^{i\phi} \frac{\partial}{\partial\theta} + e^{i\phi} \cot\theta \frac{\partial}{\partial\phi})$$

$$= - \frac{\partial}{\partial\theta^2} - \cot\theta \frac{\partial}{\partial\theta} - \cot^2\theta \frac{\partial^2}{\partial\phi^2}$$

Thus $H_1^2 + H_2^2 = - \frac{\partial}{\partial\theta^2} - \cot\theta \frac{\partial}{\partial\theta} - \cot^2\theta \frac{\partial^2}{\partial\phi^2}$

Adding to this expression $H_3^2 = - \frac{\partial^2}{\partial\phi^2}$, we get $H_1^2 + H_2^2 + H_3^2 =$

$- \frac{\partial^2}{\partial\theta^2} - \cot\theta \frac{\partial}{\partial\theta} - \cot^2\theta \frac{\partial^2}{\partial\phi^2} - \frac{\partial^2}{\partial\phi^2}$. Therefore we have

$$- H^2 = \frac{1}{\sin\theta} \frac{\partial}{\partial\theta} (\sin\theta \frac{\partial}{\partial\theta}) + \frac{1}{\sin^2\theta} \frac{\partial^2}{\partial\phi^2}$$

The equation $\{-H^2 + \ell(\ell+ 1) f \}$ takes the form

$$\frac{1}{\sin\theta} \frac{\partial}{\partial\theta} (\sin\theta \frac{\partial f}{\partial\theta}) + \frac{1}{\sin^2\theta} \frac{\partial^2 f}{\partial\phi^2} + \ell(\ell+1) f = 0 \quad (3.23)$$

This equation is called the equation of spherical functions of the ℓ-th order.

We now turn to principal spherical functions. Putting the expression for $Y_\ell^m (\theta,\phi)$ from (3.19) into (3.23) we obtain

$$\frac{1}{\sin} \frac{d}{d\theta} (\sin\theta \frac{dF_\ell^m (\theta)}{d\theta}) + \left\{ \ell(\ell+1) - \frac{m^2}{\sin^2\theta} \right\} F_\ell^m (\theta)=0$$

(3.24)

Setting a new variable $\mu= \cos\theta$ and replacing $F_\ell^m (\theta)$ by $P_\ell^m (\mu)$, we get

$$\left\{ (1-\mu^2)\ P_\ell^{m'}(\mu) \right\}' + \left\{ \ell(\ell+1) - \frac{m^2}{1-\mu^2} \right\} P_\ell^m(\mu) = 0 \qquad (3.25)$$

And finally the principal spherical functions have the form

$$Y_\ell^m(\theta,\phi) = \frac{1}{\sqrt{2\pi}}\ e^{im\phi}\ P_\ell^m\ (\cos\theta) \qquad (3.26)$$

where $P_\ell^m(\mu)$ satisfies (3.25)

3.4 AN EXPLICIT EXPRESSION FOR SPHERICAL FUNCTIONS

We shall obtain here an explicit expression for the principal spherical functions. In order to find the canonical basis for the irreducible representation of weight ℓ, we begin with simultaneous solution of the equations

$$H_3\ f = \ell\ f \qquad (3.27)$$

$$H_+\ f = 0 \qquad (3.28)$$

We determine the function $Y_\ell^\ell(\theta,\phi)$ (the eigenfunctions of the transformation H_3 which has the largest eigenvalue), From first of these equations, we have

$$H_3\ Y_\ell^\ell\ (\theta,\phi) = -\ i\ \frac{\partial Y_\ell^\ell\ (\theta,\phi)}{\partial\phi} = \ell Y_\ell^\ell\ (\theta,\phi)$$

From this we obtain as in article 3.3,

$$Y_\ell^\ell\ (\theta,\phi) = \frac{1}{\sqrt{2\pi}}\ e^{i\ell\phi}\ F_\ell^\ell\ (\theta) \qquad (3.29)$$

From (3.29), (3.28) and (3.16) we have

$$e^{i\phi}\left[\frac{\partial}{\partial\theta}+i\cot\theta\,\frac{\partial}{\partial\phi}\right]\frac{1}{\sqrt{2\pi}}\,e^{i\ell\phi}F_\ell^\ell(\theta)=0$$

$$e^{i\phi}\,e^{i\ell\phi}\left[\frac{dF_\ell^\ell(\theta)}{d}+i\cot\theta\,(i\ell)\,F_\ell^\ell(\theta)\right]=0$$

$$\frac{dF_\ell^\ell(\theta)}{d\theta}-\ell\cot\theta\;F_\ell^\ell(\theta)=0$$

$$\frac{dF_\ell^\ell(\theta)}{F_\ell^\ell(\theta)}-\ell\cot\theta\,d\theta=0$$

Integration yields

$$\log F_\ell^\ell(\theta)=\log c\sin^\ell\theta$$

Therefore the general solution has the form

$$F_\ell^\ell(\theta)=c\sin^\ell(\theta) \qquad (3.30)$$

Therefore for every ℓ there exists only one irreducible representation of weight ℓ, since in the opposite case the equation $H_+\,Y_\ell^\ell(\theta,\phi)=0$ would possess for some ℓ at least two linearly independent solutions of the form

$$e^{i\ell\phi}\quad\Phi(\theta)$$

Before determining $Y_{\ell}^{m}(\theta,\phi)$ for $m < \ell$, let us normalize the function $F_{\ell}^{\ell}(\theta) = c \sin^{\ell}\theta$. Let us choose the constant c so that the normalization condition (3.22) is satisfied:

$$\int_0^{\pi} \left| F_{\ell}^{\ell}(\theta) \right|^2 \sin\theta \; d\theta = 1$$

computing this integral,

$$c^2 \int_0^{\pi} \sin^{2\ell+1}\theta \; d\theta = 1$$

We shall use the formula

$$\int \sin^n \theta \; d\theta = -\frac{1}{n} \sin^{n-1}\theta \cos\theta + \frac{n-1}{n} \int \sin^{n-2}\theta \; d\theta$$

we have

$$c^2 \left[-\frac{1}{2\ell+1} \sin^{2\ell}\theta \cos\theta \Big|_0^{\pi} + \frac{2}{2\ell+1} \int_0^{\pi} \sin^{2\ell-1}\theta \; d\theta \right]$$

$$c^2 \left[\frac{2\ell}{2\ell+1} \left| -\frac{1}{2\ell-1} \sin^{2\ell-2}\theta \cos\theta \right|_0^{\pi} + \frac{2\ell-2}{2\ell-1} \cdot \int_0^{\pi} \sin^{2\ell-3}\theta \; d\theta \right] = 1$$

$$c^2 \left[\frac{2\ell}{2\ell+1} \cdot \frac{2\ell-2}{2\ell-1} \left| -\frac{1}{2\ell-3} \sin^{2\ell-4}\theta \cos\theta \right|_0^{\pi} + \frac{2\ell-4}{2\ell-1} \int_0^{\pi} \sin^{2\ell-5}\theta \; d\theta \right] = 1$$

$$c^2 \left[\frac{2\ell}{2\ell+1} \; \frac{2\ell-2}{2\ell-1} \Big| \frac{2\ell-4}{2\ell-3} \int_0^{\pi} \sin^{2\ell-5}\theta \; d\theta \right] = 1$$

and so on, therefore we have

$$c^2 \left[\frac{2\ell.2(\ell-1)\ .2(\ell-2)\ldots 2\ell.2(\ell-1).2(\ell-2)\ldots}{(2\ell+1)\ (2\ell)\ (2\ell-1)\ (2\ell-2)\ (2\ell-3)\ldots}\right] = 1$$

or $$c^2 \left\{\frac{2^{2\ell+1}(\ell!)^2}{(2\ell+1)\,!}\right\} = 1$$

or $$c^2 \left[\frac{2^{2\ell}\ .2\ (\ell!)^2}{(2\ell+1)\ (2\ell)!}\right] = 1$$

or $$c^2 = \frac{(2\ell+1)\ (2\ell)!}{2^{2\ell}\ .2\ (\ell!)^2}$$

or $$c = \pm\ \frac{1}{2^{\ell}\ell!} \sqrt{\frac{2\ell+1}{2}}\ .\ \sqrt{2\ell!} \qquad (3.31)$$

One ordinarily choose the sign of c so that

$$c = (-1)^{\ell}\ \frac{1}{2^{\ell}\ell!} \sqrt{\frac{2\ell+1}{2}}\ \ \sqrt{2\ell!}$$

It follows that

$$Y_{\ell}^{\ell}\ (\theta,\phi) = \frac{1}{\sqrt{2\pi}}\ e^{i\ell\phi}\ \frac{(-1)^{\ell}}{2^{\ell}\ \ell!} \sqrt{\frac{2\ell+\ 1}{2}}\ \ \sqrt{2\ell!}\ \ \sin^{\ell}\theta$$

$$= \frac{c}{\sqrt{2\pi}}\ e^{i\ell\phi}\ \ \sin^{\ell}\theta \qquad (3.33)$$

We now find the remaining functions $f_m = Y_{\ell}^{m}\ (\phi,\theta)$ of the canonical basis. For this purpose we employ the formula

$$H_-\ f_m\ = a_m\ f_{m-1} \qquad (3.34)$$

where $$a_m\ = \sqrt{(\ell+m)\ (\ell-m+1)} \qquad (3.35)$$

We have $H_{-} = e^{-i\phi} \left(- \frac{\partial}{\partial\theta} + i \cot\theta \frac{\partial}{\partial\phi}\right)$

It follows that

$$e^{-i\phi}\left(- \frac{\partial}{\partial\theta} + i \cot\theta \frac{\partial}{\partial\phi}\right) Y_{\ell}^{m} (\theta,\phi) = a_m Y_{\ell}^{m-1} (\theta,\phi)$$

we replace $Y_{\ell}^{m} (\theta,\phi)$ by $\frac{1}{\sqrt{2\pi}} e^{im\phi} F_{\ell}^{m} (\theta)$ and obtain

$$e^{-i\phi} \left(- \frac{\partial}{\partial\theta} \frac{1}{\sqrt{2\pi}} e^{im\phi} F_{\ell}^{m} (\theta) + i \cot\theta\frac{\partial}{\partial\phi} \frac{1}{\sqrt{2\pi}} e^{im\phi} F_{\ell}^{m} (\theta)\right.$$

$$= a_m \frac{1}{\sqrt{2\pi}} e^{i(m-1)\phi} F_{\ell}^{m-1} (\theta)$$

$$e^{-i\phi} \left(- \frac{1}{\sqrt{2\pi}} \frac{\partial}{\partial\theta} e^{im\phi} F_{\ell}^{m} (\theta) + i \cot\theta ((im) \frac{1}{\sqrt{2\pi}}e^{im\phi} F_{\ell}^{m} (\theta)\right.$$

$$= a_m \frac{1}{\sqrt{2\pi}} e^{i(m-1)\phi} F_{\ell}^{m-1} (\theta)$$

$$e^{-i\phi} \frac{1}{\sqrt{2\pi}} e^{im\phi} \left[- \frac{\partial}{\partial\theta} F_{\ell}^{m} (\theta) -m \cot\theta F_{\ell}^{m} (\theta) \right].$$

$$= a_m \frac{1}{\sqrt{2\pi}} e^{i(m-1)\phi} F_{\ell}^{m-1} (\theta)$$

Dividing by $\frac{1}{\sqrt{2\pi}} e^{i\phi (m-1)}$ we get

$$- \frac{dF_{\ell}^{m} (\theta)}{d\theta} - m \cot\theta F_{\ell}^{m} (\theta) = a_m F_{\ell}^{m-1} (\theta) \tag{3.36}$$

setting cosθ θ=μ and writing $F_{\ell}^{m} (\theta) = P_{\ell}^{m} (\mu)$ we get

$$-\sin\theta \, d\theta = d\mu$$

$$\frac{d\theta}{d\mu} = - \frac{1}{\sin\theta} = - \frac{1}{\sqrt{1-\mu^2}}$$

Therefore $-\frac{dP_\ell^m(\mu)}{d\mu} \cdot \frac{d\mu}{d\theta} - m\cot\theta\, P_\ell^m(\mu) = a_m P_\ell^{m-1}(\mu)$

$$\sqrt{1-\mu^2}\;\frac{dP_\ell^m(\mu)}{d\mu} - m\frac{\mu}{\sqrt{1-\mu^2}}\,P_\ell^m(\mu) = a_m P_\ell^{m-1}(\mu)$$

$$\sqrt{1-\mu^2}\left[\frac{d\,P_\ell^m(\mu)}{d\mu} - m\frac{\mu}{1-\mu^2}\,P_\ell^m(\mu)\right] = a_m P_\ell^{m-1}(\mu) \qquad (3.37)$$

Since we know $P_\ell^\ell(\mu)$, this formula gives us possibility of computing each $P_\ell^m(\mu)$ in turn. For this purpose we make the substitution

$$P_\ell^m(\mu) = (1-\mu^2)^{-\frac{m}{2}}\,U_m(\mu) \qquad (3.38)$$

Putting in (3.37), we get

$$\sqrt{1-\mu^2}\left[\frac{d}{d\mu}(1-\mu^2)^{-\frac{m}{2}}U_m(\mu) - m\frac{\mu}{1-\mu^2}(1-\mu^2)^{-\frac{m}{2}}U_m(\mu)\right] = a_m(1-\mu^2)^{-\frac{(m-1)}{2}}U_{m-1}(\mu)$$

$$\sqrt{1-\mu^2}\left[-\frac{m}{2}(1-\mu^2)^{-\frac{m}{2}-1}(-2\mu)\,U_m(\mu) + (1-\mu^2)^{-\frac{m}{2}}\frac{d\,U_m(\mu)}{d\mu} - m\frac{\mu}{(1-\mu^2)}(1-\mu^2)^{-\frac{m}{2}}U_m(\mu)\right] = a_m(1-\mu^2)^{-\frac{(m-1)}{2}}U_{m-1}(\mu)$$

$$\sqrt{1-\mu^2}\quad m(1-\mu^2)^{-\frac{m-2}{2}}U_m(\mu)\,(1-1) + (1-\mu^2)^{-\frac{m}{2}+\frac{1}{2}}\frac{d\,U_m(\mu)}{d\mu} = a_m(1-\mu^2)^{-\frac{(m-1)}{2}}U_{m-1}(\mu)$$

$$(1-\mu^2)^{-\frac{m}{2}+\frac{1}{2}} \frac{d\, U_m(\mu)}{d\mu} = a_m\, (1-\mu^2)^{-\frac{m}{2}-1}\, U_{m-1}(\mu)$$

$$U_{m-1}(\mu) = \frac{1}{a_m\, (1-\mu^2)^{-\frac{(m-1)}{2}}}\, (1-\mu^2)^{-\frac{m}{2}+\frac{1}{2}} \frac{d\, U_m\, (\mu)}{d\mu}$$

$$U_{m-1}\, (\mu) = \frac{1}{am} \quad \frac{d\, U_m\, (\mu)}{d\mu} \tag{3.39}$$

from formula (3.30) , we have

$$P_\ell^\ell\ (\mu) = c\, (1-\mu^2)\ \frac{\ell}{2}$$

But we have

$$P_\ell^\ell\ (\mu) = (1-\mu^2)^{-\frac{\ell}{2}}\ U_\ell\ (\mu)$$

Therefore $U_\ell = \dfrac{P_\ell^\ell\ (\mu)}{(1-\mu^2)^{-\frac{\ell}{2}}}$

$$U_\ell(\mu) = \frac{c\, (1-\mu^2)^{\frac{\ell}{2}}}{(1-\mu^2)^{-\frac{\ell}{2}}}$$

$$U_\ell\ (\mu) = c\ (1-\mu^2)^\ell \tag{3.40}$$

It follows from (3.39) that

$$U_{\ell-1}(\mu) = \frac{c}{a_\ell}\ \frac{d}{d\mu}\ (1-\mu^2)^\ell$$

$$U_{\ell-2}(\mu) = \frac{c}{a_\ell a_{\ell-1}}\ \frac{d^2}{d\mu^2}\ (1-\mu^2)^\ell$$

and so on. We have

$$U_m(\mu) = \frac{c}{a_\ell a_{\ell-1}\cdots a_m}(1-\mu^2)^{-\frac{m}{2}} \frac{d^{\ell-m}}{d\mu^{\ell-m}}(1-\mu^2)^\ell \qquad (3.41)$$

Here $m = \ell,\ \ell-1, \ell-2, \ldots$ We note that for $m \leq -\ell-1$, we obtain $P_\ell^m(\mu) = 0$. We now replace c and a_m by their values as computed and put $(-1)^\ell$ under the differentiation sign, we get

$$P_\ell^m(\mu) = \sqrt{\frac{(\ell+m)!}{(\ell-m)!}\ \frac{2\ell+1}{2}}\ \frac{1}{2^\ell \ell!}(1-\mu^2)^{-\frac{m}{2}}$$

$$\cdot \frac{d^{\ell-m}}{d\mu^{\ell-m}}(\mu^2-1) \qquad (3.41)$$

In particular, the function $P_\ell^0(\mu)$ which is denoted by $P_\ell(\mu)$ has the form

$$P_\ell(\mu) = \sqrt{\frac{2\ell+1}{2}}\ \frac{1}{2^\ell \ell!}\ \frac{d^\ell}{d\mu^\ell}(\mu^2-1)^\ell \qquad (3.42)$$

The polynomial $P_\ell(\mu)$ is called the normalized legendre ploynomial of ℓ-th order, and the functions $P_\ell^m(\mu)$ are called the normalized associated Legendre functions.

We have therefore proved that the principal spherical functions of the ℓ-th order have the form

$$Y_\ell^m(\theta,\phi) = \frac{1}{\sqrt{2\pi}}\ e^{im\phi}\ P_\ell^m(\cos\theta).$$

where the functions $P_\ell^m(\mu)$ are defined by the formula (3.41). Linear combination of the functions $Y_\ell^m(\theta,\phi)$ with a given fixed ℓ form a $(2\ell+1)$ dimensional space of functions which is invariant with respect to rotations of the sphere and in which is realized the irreducible representation of the rotation group with weight ℓ.

3.5 RECURRENCE RELATIONS FOR POLYNOMIALS AND FUNCTIONS OF LEGENDRE WITH ONE AND THE SAME VALUE ℓ.

Two recurrence relations, in which the functions $P_\ell^m(\mu)$ and their first derivatives enter, are contained in the formulas of transformation for the principal spherical functions:

$$H_- Y_\ell^m(\theta,\phi) = a_m Y_\ell^{m-1}(\theta,\phi)$$

$$H_+ Y_\ell^m(\theta,\phi) = a_{m+1} Y_\ell^{m+1}(\theta,\phi)$$

The first of these formulas has been employed already at the beginning of articel 3.4, for obtaining a recurrence relation. To do this we wrote

$$Y_\ell^m(\theta,\phi) = \frac{e^{im\phi}}{\sqrt{2\pi}} P_\ell^m(\cos\theta).$$

and obtained the formula, viz (3.37)

$$\sqrt{1-\mu^2}\;\frac{dP_\ell^m(\mu)}{d\mu} = m\frac{\mu}{\sqrt{1-\mu^2}}P_\ell^m(\mu) + \sqrt{(\ell+m)(\ell-m+1)}\;P_\ell^{m-1}(\mu) \tag{3.43}$$

The analogous relation $H_+ Y_\ell^m(\theta,\phi) = a_{m+1} Y_\ell^{m+1}(\theta,\phi)$ gives a different recurrence formula. Using (3.16) and (3.21), we have

$$e^{i\phi}\left(\frac{\partial}{\partial\theta} + i\cot\theta\frac{\partial}{\partial\phi}\right)\frac{1}{\sqrt{2\pi}}e^{im\phi}F_\ell^m(\theta) = a_{m+1}\frac{1}{\sqrt{2\pi}}e^{i(m+1)}F_\ell^{m+1}(\theta)$$

$$e^{i\phi}e^{im\phi}\left|\frac{d}{d\theta}F_\ell^m(\theta) - m\cot\theta F_\ell^m(\theta)\right| = a_{m+1}\frac{1}{\sqrt{2\pi}}e^{i(m+1)\phi}F_\ell^{m+1}(\theta)$$

$$\frac{d}{d\theta}F_\ell^m(\theta) - m\cot\theta\, F_\ell^m(\theta) = a_{m+1}F_\ell^{m+1}(\theta)$$

Setting $\cos\theta = \mu$ and writing $F_\ell^m(\theta) = P_\ell^m(\mu)$, we have

$$-\sin\theta\, d\theta = d\mu$$

$$-\frac{d\mu}{d\theta} = -\sin\theta = \sqrt{1-\mu^2}$$

Therefore $$\frac{d\,P_\ell^m(\mu)}{d\mu}\frac{d\mu}{d\theta} - m\cot\theta\, P_\ell^m(\mu) = a_{m+1}P_\ell^{m+1}(\theta)$$

$$-\sqrt{1-\mu^2}\,\frac{dP_\ell^m(\mu)}{d\mu} = m\frac{\mu}{\sqrt{1-\mu^2}}P_\ell^m(\mu) + a_{m+1}P_\ell^{m+1}(\mu)$$

$$-\sqrt{1-\mu^2}\,\frac{dP_\ell^m(\mu)}{d\mu} = m\frac{\mu}{\sqrt{1-\mu^2}}P_\ell^m(\mu) + \sqrt{(\ell+m+1)(\ell-m)}\,P_\ell^{m+1}(\mu) \quad (3.44)$$

Combining formulas (3.43) and (3.44), we obtain a connection between three consecutive normalized Legendre functions

$$\sqrt{(\ell+m+1)(\ell-m)}\; P_\ell^{m+1}(\mu) + 2m \frac{\mu}{\sqrt{1-\mu^2}} P_\ell^m(\mu)$$

$$+\sqrt{(\ell+m)(\ell-m+1)}\; P_\ell^{m-1}(\mu) = 0 \tag{3.45}$$

Formula (3.44) can be used to obtain a different and more expanded expression for the associated Legendre functions. Setting m=0 in this formula we obtain

$$\sqrt{\ell(\ell+m)}\; P_\ell^1(\mu) = -\sqrt{1-\mu^2}\; \frac{dP_\ell(\mu)}{d\mu}$$

$$P_\ell^1(\mu) = -\frac{1}{a_1}\sqrt{(1-\mu^2)}\; \frac{dP_\ell(\mu)}{d\mu}$$

In general we set

$$P_\ell^m(\mu) = (1-\mu^2)^{\frac{m}{2}}\; \nu_m(\mu) \tag{3.46}$$

Putting in (3.43), we have

$$-\sqrt{1-\mu^2}\; \frac{d}{d\mu}(1-\mu^2)^{\frac{m}{2}}\; \nu_m(\mu) = \frac{m\mu}{\sqrt{1-\mu^2}}(1-\mu^2)\; \nu_m(\mu)$$

$$+\sqrt{(\ell+m+1)(\ell-m)}\;(1-\mu^2)^{\frac{m+1}{2}}\; \nu_{m+1}(\mu)$$

$$-\sqrt{1-\mu^2}\left[(1-\mu^2)^{\frac{m}{2}}\frac{d\nu_m(\mu)}{d\mu} + \nu_m(\mu)\left(\frac{m}{2}\right)\right.$$

$$\left.(1-\mu^2)^{\frac{m}{2}-1}(-2\mu)\right] = m\mu\,(1-\mu^2)^{\frac{m}{2}-\frac{1}{2}}\;\nu_m(\mu)$$

$$+\sqrt{(\ell+m+1)(\ell-m)}\;(1-\mu^2)^{\frac{m+1}{2}}\;\nu_{m+1}(\mu)$$

or $$-\left[(1-\mu^2)^{\frac{m}{2}+\frac{1}{2}} \frac{d\nu_m(\mu)}{d\mu} - \nu_m(\mu)\ m\ \mu\ (1-\mu)^{\frac{m-1}{2}}\right]$$

$$= m\mu\ (1-\mu^2)^{\frac{m-1}{2}}\ \nu_m\ (\mu) + \sqrt{(\ell+m+1)(\ell-m)}\ (1-\mu)^{\frac{m+1}{2}} \nu_{m+1}(\mu)$$

Or $$-\ (1-\mu^2)^{\frac{m+1}{2}} \frac{d\nu_m(\mu)}{d\mu} + m\mu\ \ (1-\mu^2)^{\frac{m-1}{2}}\ \nu_m\ (\mu)$$

$$-\ m\mu\ (1-\mu^2)^{\frac{m-1}{2}}\ \nu_m\ (\mu) = \sqrt{(\ell+m+1)(\ell-m)}\ (1-\mu^2)^{\frac{m+1}{2}} \nu_{m+1}(\mu)$$

$$-\ (1-\mu^2)^{\frac{m+1}{2}} \frac{d\nu_m\ (\mu)}{d\mu} = \sqrt{(\ell+m+1)(\ell-m)}\ \ (1-\mu^2)^{\frac{m+1}{2}} \nu_{m+1}(\mu)$$

$$\nu_{m+1}(\mu) = \frac{-(1-\mu^2)^{\frac{m+1}{2}}}{a_{m+1}\ (1-\mu^2)^{\frac{m+1}{2}}} \quad \frac{d\nu_m\ (\mu)}{d\mu}$$

$$\nu_{m+1}(\mu) = -\ \frac{1}{a_{m+1}} \quad \frac{d\nu_m\ (\mu)}{d\mu} \tag{3.47}$$

Now $\nu_0\ (\mu) = P_\ell^0\ (\mu)$

$$= \sqrt{\frac{2\ell+1}{2}}\ \frac{1}{2^\ell \ell\ !} \quad \frac{d^\ell (\mu^2-1)^\ell}{d\mu^\ell}$$

We have

$$\nu_1\ (\mu) \quad = -\ \frac{1}{a_1} \quad \frac{d\nu_0\ (\mu)}{d\mu}$$

$$= \frac{1}{a_1}\ \frac{d}{d\mu}\sqrt{\frac{2\mu+1}{2}}\ \frac{1}{2^\ell \ell!} \quad \frac{d^\ell}{d\mu^\ell}\ (\mu^2-1)^\ell$$

$$= -\ \frac{1}{a_1}\ \sqrt{\frac{2\ell+1}{2}}\ \frac{1}{2^\ell \ell!} \quad \frac{d^{\ell+1}}{d\mu^{\ell+1}}\ (\mu^2-1)^\ell$$

$$\nu_2(\mu) = -\frac{1}{a_2}\frac{d}{d\mu} \quad -\frac{1}{a_1}\sqrt{\frac{2\ell+1}{2}}\;\frac{1}{2^{\ell}\ell!}\;\frac{d^{\ell+1}}{d\mu^{\ell+1}}(\mu^2-1)^{\ell}$$

$$= (-1)^2\;\frac{1}{a_1a_2}\sqrt{\frac{2\ell+1}{2}}\;\frac{1}{2^{\ell}\ell!}\;\frac{d^{\ell+2}}{d\mu^{\ell+2}}(\mu^2-1)^{\ell}$$

and so on.

Therefore we have

$$\nu_m(\mu) = (-1)^m\;\frac{1}{a_1a_2\cdots a_m}\sqrt{\frac{2\ell+1}{2}}\;\frac{1}{2^{\ell}\ell!}\;\frac{d^{\ell+m}}{d\mu^{\ell+m}}(\mu^2-1)^{\ell}.$$

Substituting $\nu_m(\mu)$ is (3.46) and replacing a_m by their values we obtain the following expression for associated Legendre functions:

$$P_\ell^m(\mu) = (-1)^m\sqrt{\frac{(\ell-m)!}{(\ell+m)!}}\sqrt{\frac{2\ell+1}{2}}\;\frac{1}{2^{\ell}\ell\,!}\;(1-\mu^2)^{\frac{m}{2}}$$

$$\frac{d^{m+\ell}}{d\mu^{m+\ell}}(\mu^2-1)^{\ell} \qquad (3.49)$$

Comparing (3.41) and (3.49), we see that, except for a constant factor, one of them goes into the other when m is replaced by -m. It follows that

$$P_\ell^m(\mu) = (-1)^m P_\ell^{-m}(\mu)$$

That is, the normalized associated Legender functions with value m differing in sign are proportional to each other. Finally we note that if we replace

$$\sqrt{\frac{2\ell+1}{2}}\ \frac{1}{2^{\ell}\ell\,!}\ \frac{d^{\ell}}{d\mu^{\ell}}\ (\mu^2-1)^{\ell}$$

in (3.49) by $P_\ell(\mu)$, then we obtain

$$P_\ell^m(\mu) = (-1)^m\sqrt{\frac{(\ell-m)!}{(\ell+m)!}}\ (1-\mu^2)^{\frac{m}{2}}\ \frac{d^m P_\ell(\mu)}{d\mu^m} \qquad (3.50)$$

This formula is well-known expression of the normalized associated Legendre functions by means of the normalized Legendre polynomial of the same order.

CHAPTER-4

4.1 DIFFERENTIAL OPERATORS CORRESPONDING TO INFINITESIMAL ROTATIONS

First we find the transformations I_1, I_2 and I_3 which correspond to infinitesimal rotations about the coordinate axes. We take g_1 as the rotation through an angle α_1 around a fixed axis and then expand $U_{g_1} T_{mn}(g) = T_{mn}(gg_1)$ in powers of α_1..

We take the axis OZ as the axis of rotation and suppose that g is an arbitrary rotation, with Euler angles ϕ, θ, ψ. Also we suppose that g_1 is the rotation through an angle α_1 around the axis OZ. Then the rotation gg_1 has Euler angles $\phi + \alpha_1, \theta, \psi$. Consequently we have

$$T_{mn}(gg_1) = T_{mn}(\phi + \alpha_1, \theta, \psi)$$

$$= T_{mn}(\phi, \theta, \psi) + \alpha_1 \frac{\partial T_{mn}}{\partial \phi} + \ldots$$

and the transformation I_3 is the differential operator

$$I_3 = \frac{\partial}{\partial \phi} \qquad (4.1)$$

The expansion $T_{mn}(gg_1) = T_{mn}(\phi', \theta', \psi')$, in general, has the form,

$$T_{mn}(\phi',\theta',\psi') = T_{mn}(\phi,\theta,\psi) + \alpha_1 \left\{ \frac{\partial T_{mn}}{\partial \phi} \frac{d\phi'}{d\alpha_1} + \frac{\partial T_{mn}}{\partial \theta} \frac{d\theta'}{d\alpha_1} + \frac{\partial T_{mn}}{\partial \psi} \frac{d\psi'}{d\alpha_1} \right\}_{\alpha_1=0} + \dots \qquad (4.2)$$

We shall now determine $\left.\frac{d\phi'}{d\alpha_1}\right|_{\alpha_1=0}$, $\left.\frac{d\theta'}{d\alpha_1}\right|_{\alpha_1=0}$

and $\left.\frac{d\psi'}{d\alpha_1}\right|_{\alpha_1=0}$, when g is the rotation through an angle α_1 around the axis OX. As was in article 2.2, we have

$$g(\phi,\theta,\psi) = g_{ik}(\phi,\theta,\psi) =$$

$$\begin{bmatrix} \cos\phi \cos\psi - \cos\theta \sin\phi \sin\psi & -\sin\phi \sin\psi - \cos\theta \cos\phi \sin\psi & \sin\psi \sin\theta \\ \sin\psi \cos\phi + \cos\theta \cos\psi \sin\phi & -\sin\phi \sin\psi + \cos\theta \cos\phi \cos\psi & -\cos\psi \sin\theta \\ \sin\phi \sin\theta & \cos\phi \sin\theta & \cos\theta \end{bmatrix} \qquad (4.3)$$

The angles ϕ and ψ vary from o to 2π and the angle θ varies from o to π. The matrix of rotation gg_1 is given by certain values of Euler angles ϕ',θ',ψ'. These values depend upon the rotation angle α_1 and for $\alpha_1=0$, they are equal to ϕ,θ,ψ. Expanding the matrix gg_1 in powers of α_1, we have

$$gg_1 = \left[g_{ik}(\phi,\theta,\psi)\right] + \alpha_1 \left[\frac{\partial g_{ik}}{\partial \phi} \frac{d\phi'}{d\alpha_1} + \frac{\partial g_{ik}}{\partial \theta} \cdot \frac{d\theta'}{d\alpha_1} + \frac{\partial g_{ik}}{\partial \psi} \frac{d\psi'}{d\alpha_1} \right]_{\alpha_1=0} + \dots \qquad (4.4)$$

Since g_1 is the rotation through an angle α_1 around the axis OX, its matrix is equal to

$$g_1 = \begin{vmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha_1 & -\sin\alpha_1 \\ 0 & \sin\alpha_1 & \cos\alpha_1 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} + \alpha_1 \begin{vmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{vmatrix} + \ldots$$

$$\text{Therefore } gg_1 = g_{ik}(\phi,\theta,\psi) + \alpha_1 \begin{vmatrix} 0 & g_{13} & -g_{12} \\ 0 & g_{23} & -g_{22} \\ 0 & g_{33} & -g_{32} \end{vmatrix} + \ldots \tag{4.5}$$

Now equating the expressions (4.4) and (4.5) for the matrix gg_1 and equating the coefficients of α_1 in these two expressions, we obtain equations from which

$$\frac{d\phi'}{d\alpha_1}\Big|_{\alpha_1=0}, \quad \frac{d\theta'}{d\alpha_1}\Big|_{\alpha_1=0} \quad \text{and} \quad \frac{d\psi'}{d\alpha_1}\Big|_{\alpha_1=0}$$

are defined.

We take g_{33} from (4.4), (4.5) and corresponding expressions from (4.3) and differentiating these we have

$$\left\{\frac{d\phi'}{d\alpha_1}\frac{\partial}{\partial\phi} + \frac{d\theta'}{d\alpha_1}\frac{\partial}{\partial\theta} + \frac{d\psi'}{d\alpha_1}\frac{\partial}{\partial\psi}\right\}_{\alpha_1=0} \cos\theta = -\cos\phi\,\sin\theta$$

$$-\sin\theta\,\frac{d\theta'}{d\alpha_1}\Big|_{\alpha_1=0} = -\cos\phi\,\sin\theta$$

$$\text{or} \quad \frac{d\theta'}{d\alpha_1}\Big|_{\alpha_1=0} = \cos\phi \tag{4.6}$$

Now take, g_{31}, to determine $\frac{d\phi'}{d\alpha_1}\Big|_{\alpha_1=0}$, we have

$$\left\{\frac{d\phi'}{d\alpha_1}\frac{\partial}{\partial\phi} + \frac{d\theta'}{d\alpha_1}\frac{\partial}{\partial\theta} + \frac{d\psi'}{d\alpha_1}\frac{\partial}{\partial\psi}\right\}_{\alpha_1=0} \sin\phi\ \sin\theta = 0$$

$$\sin\theta\ \cos\phi\ \frac{d\phi'}{d\alpha_1}\Big|_{\alpha_1=0} + \sin\phi\ \cos\theta\ \frac{d\theta'}{d\alpha_1}\Big|_{\alpha_1=0} = 0$$

or $$\sin\theta\ \cos\phi\ \frac{d\phi'}{d\alpha_1}\Big|_{\alpha_1=0} = -\sin\phi\cos\theta\ \cos\phi$$

$$\frac{d\phi'}{d\alpha_1}\Big|_{\alpha_1=0} = -\frac{\sin\phi\ \cos\theta\ \cos\phi}{\sin\theta\ \ \cos\phi} = -\sin\phi\ \cot\theta \qquad (4.7)$$

To determine $\frac{d\psi'}{d\alpha_1}\Big|_{\alpha_1=0}$ we consider g_{13}, we have

$$\left\{\frac{d\phi'}{d\alpha_1}\frac{\partial}{\partial\phi} + \frac{d\theta'}{d\alpha_1}\frac{\partial}{\partial\theta} + \frac{d\psi'}{d\alpha_1}\frac{\partial}{\partial\psi}\right\}_{\alpha_1=0} \sin\psi\ \sin\theta = \cos\psi\ \sin\phi + \cos\theta\ \sin\psi\ \cos\phi$$

$$\cos\theta\ \sin\psi\ \frac{d\theta'}{d\alpha_1}\Big|_{\alpha_1=0} + \cos\psi\ \sin\theta\ \frac{d\psi'}{d\alpha_1}\Big|_{\alpha_1=0} = \cos\psi\ \sin\phi + \cos\theta\ \sin\psi\ \cos\phi$$

or $$\cos\theta\ \sin\psi\ \cos\phi + \cos\psi\ \sin\theta\ \frac{d\psi'}{d\alpha_1}\Big|_{\alpha_1=0} = \cos\psi\ \sin\phi + \cos\theta\ \sin\psi\ \cos\phi$$

or $$\cos\psi\ \sin\theta\ \frac{d\psi'}{d\alpha_1}\Big|_{\alpha_1=0} = \cos\psi\ \sin\phi + \cos\theta\ \sin\psi\ \cos\phi - \cos\theta\ \sin\psi\ \cos\phi$$

$$= \cos\psi\ \sin\phi$$

or
$$\left.\frac{d\psi'}{d\alpha_1}\right|_{\alpha_1=0} = \frac{\cos\psi\ \sin\phi}{\cos\psi\ \sin\theta} = \frac{\sin\phi}{\sin\theta} \tag{4.8}$$

Putting expressions (4.6), (4.7) and (4.8) in (4.2), we have the differential operator corresponding to an infinitesional rotation about the axis OX:

$$I_1 = -\cot\theta\ \sin\phi\ \frac{\partial}{\partial\phi} + \cos\phi\ \frac{\partial}{\partial\theta} + \frac{\sin\phi}{\sin\theta}\ \frac{\partial}{\partial\psi} \tag{4.9}$$

Similarly we can compute the operator A . It has the form

$$I_2 = -\cot\theta\ \cos\phi\ \frac{\partial}{\partial\phi} - \sin\theta\ \frac{\partial}{\partial\theta} + \frac{\cos\phi}{\sin\theta}\ \frac{\partial}{\partial\psi} \tag{4.10}$$

The operators A_1, A_2, A_3 ca now be determined:

$$A_+ = A_1 + iA_2 = iI_1 - I_2 = e^{-i\phi}\ (\cot\theta\ \frac{\partial}{\partial\phi} + i\ \frac{\partial}{\partial\theta} - \frac{}{\sin\theta}\ \frac{\partial}{\partial\psi}) \tag{4.11}$$

$$A_- = A_1 - iA_2 = iI_1 + I_2 = e^{i\phi}\ (-\cot\theta\frac{\partial}{\partial\phi} + i\frac{\partial}{\partial\theta} + \frac{}{\sin\theta}\ \frac{\partial}{\partial\psi}) \tag{4.12}$$

$$A_3 = i\ \frac{\partial}{\partial\phi} \tag{4.13}$$

We produce the operator

$$A^2 = A_1^2 + A_2^2 + A_3^2$$

from the above relations

we have $A_1^2 + A_2^2 = \frac{1}{2}\ (A_+\ A_- + A_-\ A_+)$

Therefore we have

$$-A^2 = \frac{\partial^2}{\partial\theta^2} + \cot\theta \frac{\partial}{\partial\theta} + \frac{1}{\sin^2\theta} \quad \frac{\partial^2}{\partial\phi^2} - 2\cos\theta \frac{\partial^2}{\partial\phi\partial\psi} + \frac{\partial^2}{\partial\psi} \}$$

Since $A^2U + \ell\ (\ell+1)U$.

Therefore the above equation has the form

$$\frac{\partial^2 U}{\partial\theta^2} + \cot\theta \frac{\partial U}{\partial\theta} + \frac{1}{\sin^2\theta} \left\{\frac{\partial^2 U}{\partial\phi^2} - 2\cos\theta \frac{\partial^2 U}{\partial\phi\partial\psi} + \frac{\partial^2 U}{\partial\phi^2}\right\}$$

$$+ \ \ell(\ell+1)\ U = 0.$$

We have found the differential operators corresponding to infinitesimal rotations, in terms of Euler angles ϕ,θ,ψ. Now for $\theta = 0$, our rotation is a rotation about the axis OZ through an angle $\phi+\psi$ and for $\theta = \pi$, it is a rotation about the axis OZ through an angle $\phi-\psi$. Thus in these cases, the distinct pair of numbers (ϕ,ψ) correspond to same rotation.

Now an alternate parameterization is obtained by making the first rotation through an angle α about the axis OX followed by a rotation through an angle β about the axis OY and finally a rotation through an angle γ about the axis OZ, where $-\pi \le \alpha \le \pi$, $-\pi < \beta \le \pi$, $-\frac{\pi}{2} \le \gamma \le \frac{\pi}{2}$

The matrices of rotations g_α, g_β, g_γ have the form

$$g_\alpha = \begin{vmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & -\sin\alpha \\ 0 & \sin\alpha & \cos\alpha \end{vmatrix}, \quad g_\beta = \begin{vmatrix} \cos\beta & 0 & \sin\beta \\ 0 & 1 & 0 \\ -\sin\beta & 0 & \cos\beta \end{vmatrix}$$

$$g_\gamma = \begin{vmatrix} \cos\gamma & -\sin\gamma & 0 \\ \sin\gamma & \cos\gamma & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

Therefore the matrix of rotation $g(\alpha,\beta,\gamma) = g_\alpha g_\beta g_\gamma$ is given by

$$g(\alpha,\beta,\gamma) = \begin{vmatrix} \cos\beta\ \cos\gamma & -\cos\beta\ \sin\gamma & \sin\beta \\ \sin\alpha\ \sin\beta\ \cos\gamma + \cos\alpha\ \cos\gamma & -\sin\alpha\ \sin\beta\ \sin\gamma + \cos\alpha\ \cos\gamma & -\sin\alpha\ \cos\beta \\ -\cos\alpha\ \sin\beta\ \cos\gamma + \sin\alpha\ \sin\gamma & \cos\alpha\ \sin\beta\ \sin\gamma + \sin\alpha\ \cos\gamma & \cos\alpha\ \cos\beta \end{vmatrix} \tag{4.14}$$

where α and β vary from $-\pi$ to π and β varies from $\frac{-\pi}{2}$ to $\frac{\pi}{2}$.

Now we shall find the differential operator corresponding to infinitesimal rotation about the coordinate axes, using the matrix of rotation (4.14). First we shall find the operator A_1, A_2 and A_3. We take g_1 as the rotation through an angle α_2 and expand $Ug_1\ T_{mn}(g) = T_{mn}(gg_1)$ in powers of α_2, we have

$$T_{mn}(\alpha',\beta',\gamma') = T_{mn}(\alpha,\beta,\gamma) + \alpha_2 \left\{ \frac{\partial T_{mn}}{\partial\alpha} \frac{d\alpha'}{d\alpha_2} + \frac{\partial T_{mn}}{\partial\beta} \frac{d\beta'}{d\alpha_2} + \frac{\partial T_{mn}}{\partial\gamma} \frac{d\gamma'}{d\alpha_2} \right\}_{\alpha_2=0} + \dots \tag{4.15}$$

Now we find $\left.\frac{d\alpha'}{d\alpha_2}\right|_{\alpha_2=0}$, $\left.\frac{d\beta'}{d\alpha_2}\right|_{\alpha_2=0}$ and $\left.\frac{d\gamma'}{d\alpha_2}\right|_{\alpha_2=0}$

and consider g as the function of α,β,γ, which is given by (4.14).

Expanding the matrix gg_1 in powers of α_2, we have

$$gg_1 = \left|g_{ik}\ (\alpha,\beta,\gamma)\right| + \left|\frac{\partial g_{ik}}{\partial\alpha}\frac{d\alpha'}{d\alpha_2} + \frac{\partial g_{ik}}{\partial\beta}\frac{d\beta'}{d\alpha_2} + \frac{\partial g_{ik}}{\partial\gamma}\frac{d\gamma'}{d\alpha_2}\right|_{\alpha_2=0} + \ldots \quad (4.16)$$

consider the matrix g_1, which correspond to the infinitesimal rotation through an angle α_2 about the axis OX. Its matrix is equal to

$$g_1 = \begin{vmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha_2 & -\sin\alpha_2 \\ 0 & \sin\alpha_2 & \cos\alpha_2 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} + \alpha_2 \begin{vmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{vmatrix} + \ldots$$

Therefore

$$gg_1 = g_{ik}\ (\alpha,\beta,\gamma) + \alpha_2 \begin{vmatrix} 0 & g_{13} & -g_{12} \\ 0 & g_{23} & -g_{22} \\ 0 & g_{33} & -g_{32} \end{vmatrix} + \ldots \quad (4.17)$$

Let us take g_{13} from (4.16), (4.17) and taking the corresponding expressions from (4.14) and differentiating these we have

$$\left\{\frac{d\alpha'}{d\alpha_2}\frac{\partial}{\partial\alpha} + \frac{d\beta'}{d\alpha_2}\frac{\partial}{\partial\beta} + \frac{d\gamma'}{d\alpha_2}\frac{\partial}{\partial\gamma}\right\}_{\alpha_2=0} \sin\beta = \cos\beta \quad \sin\gamma$$

$$\cos \left.\frac{d\beta}{d\alpha_2}\right|_{\alpha_2=0} = \cos\beta \sin\gamma$$

$$\text{or} \quad \left.\frac{d\beta}{d\alpha_2}\right|_{\alpha_2=0} = \sin\gamma \qquad (4.18)$$

Now take g_{23} from (4.16),(4.17) and taking the corresponding expressions from (4.14) and differentiating these we have

$$\left\{\frac{d\alpha'}{d\alpha_2}\frac{\partial}{\partial\alpha} + \frac{d\beta'}{d\alpha_2}\frac{\partial}{\partial\beta} + \frac{d\gamma'}{d\alpha_2}\frac{\partial}{\partial\gamma}\right\}_{\alpha_2=0} -\sin\alpha\cos\beta = \sin\alpha\sin\beta\sin\gamma + \cos\alpha\cos\gamma$$

$$-\cos\alpha\cos\beta \left.\frac{d\alpha'}{d\alpha_2}\right|_{\alpha_2=0} +\sin\beta\sin\alpha \left.\frac{d\beta'}{d\alpha_2}\right|_{\alpha_2=0} = \sin\alpha\sin\beta\sin\gamma - \cos\alpha\cos\gamma$$

$$-\cos\alpha\cos\beta \left.\frac{d\alpha'}{d\alpha_2}\right|_{\alpha_2=0} = \sin\alpha\sin\beta\sin\gamma - \sin\beta\sin\alpha\sin\gamma - \cos\alpha\cos\gamma$$

$$\left.\frac{d\alpha'}{d\alpha_2}\right|_{\alpha_2=0} = \frac{\cos\alpha\cos\gamma}{\cos\alpha\cos\beta} = \frac{\cos\gamma}{\cos\beta} \qquad (4.19)$$

Now take g_{32}, to determine $\left.\frac{d\gamma'}{d\alpha_2}\right|_{\alpha_2=0}$, we have

$$\left\{\frac{d\alpha'}{d\alpha_2}\frac{\partial}{\partial\alpha} + \frac{d\beta'}{d\alpha_2}\frac{\partial}{\partial\beta} + \frac{d\gamma'}{d\alpha_2}\frac{\partial}{\partial\gamma}\right|_{\alpha_2=0} \cos\alpha\sin\beta\sin\gamma + \sin\alpha\cos\gamma = \cos\alpha\cos\beta$$

$$-\sin\alpha\ \sin\beta\ \sin\gamma\ \frac{d\alpha'}{d\alpha_2}\bigg|_{\alpha_2=0} + \cos\alpha\ \cos\gamma\ \frac{d\alpha'}{d\alpha_2}\bigg|_{\alpha_2=0} + \cos\alpha\ \cos\beta\ \sin\gamma\ \frac{d\beta'}{d\alpha_2}\bigg|_{\alpha_2=0}$$

$$+ \cos\alpha\ \sin\beta\ \cos\ \gamma\frac{d\gamma'}{d\alpha_2}\bigg|_{\alpha_2=0} - \sin\alpha\ \sin\gamma\ \frac{d\gamma'}{d\alpha_2}\bigg|_{\alpha_2=0} = \cos\alpha\ \ \cos\beta$$

$$\text{or } -\sin\alpha\ \sin\beta\ \sin\gamma\ \frac{\cos\gamma}{\cos\beta} + \cos\alpha\ \cos\gamma\ \frac{\cos\gamma}{\cos\beta} + \cos\alpha\ \cos\beta\ \sin^2\gamma$$

$$+ (\cos\alpha\ \sin\beta\ \cos\gamma - \sin\alpha\ \sin\ \gamma\ \frac{d\gamma'}{d\alpha_2}\bigg|_{\alpha_2=0} = \cos\alpha\ \cos\beta$$

$$\text{or } \cos\ \beta\ \ (\cos\alpha\ \cos\beta\ \cos\gamma - \sin\ \alpha\sin\gamma)\frac{d\gamma'}{d\alpha_2}\bigg|_{\alpha_2=0} = \cos\alpha\ \ \cos^2\beta$$

$$+ \sin\alpha\ \sin\beta\ \sin\gamma\ \cos\ \gamma - \cos\alpha\ \cos^2\gamma - \cos\alpha\ \cos^2\beta\ \sin^2\gamma$$

$$= \cos\alpha\ \cos^2\beta\ (1-\sin^2\ \gamma) + \sin\alpha\ \sin\beta\ \sin\gamma\ \cos\gamma - \ \cos\alpha\ \cos^2\gamma$$

$$= \cos\alpha\ \cos^2\beta\ \cos^2\gamma + \sin\alpha\ \sin\beta\ \sin\gamma\ \cos\ \ \gamma - \cos\alpha\ \cos^2\gamma$$

$$= -\ \cos\alpha\ \cos^2\gamma\ (1-\cos^2\beta\ ) + \sin\alpha\ \sin\beta\ \sin\gamma\ \cos\gamma$$

$$= -\ \cos\alpha\ \cos^2\gamma\ \sin^2\beta + \sin\alpha\ \sin\beta\ \sin\gamma\ \cos\gamma$$

$$= -\ \cos\gamma\ \sin\beta\ (\cos\alpha\ \sin\beta\ \cos\gamma\ \ - \sin\alpha\ \sin\gamma\ )$$

$$\text{or } \frac{d\gamma}{d\alpha_2}\bigg|_{\alpha_2=0} = -\ \tan\ \beta\ \ \cos\ \ \gamma \qquad (4.20)$$

Substituting the expression (4.18),4.19) and (4.20) in (4.15) we obtain

$$A_1 = \frac{\cos\gamma}{\cos\beta} \frac{\partial}{\partial\alpha} + \sin\gamma \frac{\partial}{\partial\beta} - \tan\beta \cos\gamma \frac{\partial}{\partial\gamma} \tag{4.21}$$

Now the operator A_2 is the differential operator which correspond to infinitesimal rotation about the axis oy, and is computed in much the same way. It has the form

$$A_2 = -\frac{\sin\gamma}{\cos\beta} \frac{\partial}{\partial\alpha} + \cos\gamma \frac{\partial}{\partial\beta} + \tan\beta \sin\gamma \frac{\partial}{\partial\gamma} \tag{4.22}$$

Similarly, we find

$$A_3 = \frac{\partial}{\partial\gamma} \tag{4.23}$$

We now determine the operators H_+, H_- and H_3. We have

$$H_+ = H_1 + iH_2 = iA_1 - A_2$$

$$H_- = H_1 - iH_2 = iA_1 + A_2$$

$$H_3 = iA_3 = i \frac{\partial}{\partial\gamma}$$

Therefore

$$H_+ = e^{-i\gamma} \left(\frac{1}{\cos\beta} i \frac{\partial}{\partial\alpha} - \frac{\partial}{\partial\beta} - i \tan\beta \frac{\partial}{\partial\gamma}\right) \tag{4.24}$$

$$H_- = e^{i\gamma} \left(\frac{1}{\cos\beta} i \frac{\partial}{\partial\alpha} + \frac{\partial}{\partial\beta} - i \tan\beta \frac{\partial}{\partial\gamma}\right) \tag{4.25}$$

$$H_3 = i \frac{\partial}{\partial\gamma} \tag{4.26}$$

In the next article we shall use these operator to determine generalized spherical functions.

4.2 GENERALIZED SPHERICAL FUNCTIONS

For every value of m, the functions T_{mn} (α,β,γ) are eigen-functions of the operator H_3 with corresponding eigenvalue n. In particular the functions $T_{m\ell}$ (α,β,γ) correspond to maximum eigenvalue ℓ. These functions must satisfy the equation

$$H_+ \; T_{m\ell} \; (\alpha,\beta,\gamma) \;=\; 0 \tag{4.27}$$

From equation (4.24), we have

$$H_+ = e^{-i\gamma}\left(\frac{1}{\cos\beta}\; i\;\frac{\partial}{\partial\alpha} - \frac{\partial}{\partial\beta} - i\;\tan\beta\frac{\partial}{\partial\gamma}\right) \tag{4.28}$$

From (4.27) and (4.28), we get

$$e^{-i\gamma}\;\left(\frac{1}{\cos\beta}\; i\;\frac{\partial}{\partial\alpha} - \frac{\partial}{\partial\beta} - i\tan\beta\;\frac{\partial}{\partial\gamma}\right)\; T_{m\ell}\;(\alpha,\beta,\gamma) = 0 \tag{4.29}$$

Substituting $T_{m\ell}\;(\alpha,\beta,\gamma) = e^{-im\alpha}\; U_{m\ell}(\beta)\; e^{-i\ell\gamma}$ in equation (4.29)

we obtain

$$e^{-i\gamma}\left(\frac{1}{\cos\beta}\; i\;\frac{\partial}{\partial\alpha} - \frac{\partial}{\partial\beta} - i\;\tan\beta\;\frac{\partial}{\partial\gamma}\right)\; e^{-im\alpha}.U_{m\ell}\;(\beta)\; e^{-i\ell\gamma} = 0$$

or $e^{-i\gamma}\left(\frac{1}{\cos\beta}\; i\;\frac{\partial}{\partial\alpha}\; e^{-im\alpha}U_{m\ell}(\beta)e^{-i\ell\gamma} - \frac{\partial}{\partial\beta}\; e^{-im\alpha}\; U_{m\ell}(\beta)\; e^{-i\ell\gamma}\right.$

$$\left. - i\tan\beta\;\frac{\partial}{\partial\gamma}\; e^{-im\alpha}\; U_{m\ell}(\beta)\; e^{-i\ell\gamma}\;\right) \;=\; 0$$

$$\text{or } e^{-i\gamma}\left(\frac{m}{\cos\beta}\, e^{-im\alpha}\, U_{m\ell}(\beta)\, e^{-i\ell\gamma} - \frac{\partial}{\partial\theta}\, e^{-im\alpha}\, U_{m\ell}(\beta)\right.$$

$$\left. e^{-i\ell\gamma} - \ell\tan\beta\; e^{-im\alpha}\, U_{m\ell}(\beta)\, e^{-i\ell\gamma}\right) = 0$$

Cancelling out the common factor $e^{-im\alpha}\, U_{m\ell}(\beta)\, e^{-i\gamma}$,we obtain

$$-\frac{d\,U_{m\ell}(\beta)}{d\beta} + \left(\frac{m}{\cos\beta} - \ell\tan\beta\right) U_{m\ell}(\beta) = 0$$

$$\text{or} \quad \frac{d\,U_{m\ell}(\beta)}{d\beta} - \left(\frac{m}{\cos\beta} - \ell\tan\beta\right) U_{m\ell}(\beta) = 0 \qquad (4.30)$$

$$\text{or} \quad \frac{d\,U_{m\ell}(\beta)}{U_{m\ell}(\beta)} - \left(\frac{m - \ell\sin\beta}{\cos\beta}\right) d\beta = 0$$

Integrating, we get,

$$\log U_{m\ell}(\beta) - m\log\tan\left(\frac{\beta}{2} + \frac{\pi}{4}\right) + \ell\log\sec\beta = \log c_m$$

$$\text{or } \log U_{m\ell}(\beta) = m\log\tan\left(\frac{B}{2} + \frac{\pi}{4}\right) - \ell\log\sec_\beta + \log c_m$$

$$= \log c_m + \log\tan^m\left(\frac{\beta}{2} + \frac{\pi}{4}\right) - \log\sec^{\ell}\beta$$

$$= \log c_m \frac{\tan^m\left(\frac{\beta}{2} + \frac{\pi}{4}\right)}{\sec^{\ell}\beta}$$

$$\text{Therefore } U_{m\ell}(\beta) = c_m \frac{\cos^{\ell}\beta}{\cot^m\left(\frac{\beta}{2} + \frac{\pi}{4}\right)} \qquad (4.31)$$

Let us consider again equation (4.30), we have,

$$\frac{dU_{m\ell}(\beta)}{U_{m\ell}(\beta)} - \left(\frac{m}{\cos\beta} - \ell\tan\beta\right) U_{m\ell}(\beta) = 0$$

Integrating, we also have,

$$U_{m\,\ell}(\beta) = c_m \frac{(\sec\beta + \tan\beta)^m}{\sec^{\ell}\beta}$$

$$= c_m \frac{\left(\frac{1+\sin\beta}{\cos\beta}\right)^m}{\sec^{\ell}\beta}$$

$$= c_m\,(1+\sin\beta)^m \cos^{\ell-m}\beta \tag{4.32}$$

For later computation it is convenient to introduce the variable $\mu = \sin\beta$. Denote the function $U_{m\ell}$ (arc $\sin\mu$) by the symbol $P_{m\ell}(\mu)$.

Since $\mu = \sin\beta$

Therefore $\cos\beta = \sqrt{1-\mu^2}$

and $\cos^{\ell-m}\beta = (1-\mu^2)^{\frac{\ell-m}{2}}$

Hence we have

$$P_{m\ell}(\mu) = c_m\,(1-\mu)^m\,(1-\mu^2)^{\frac{\ell-m}{2}}$$

$$\text{or } P_{m\,\ell}(\mu) = c_m\,(1-\mu)^{\frac{\ell-m}{2}}\,(1+\mu)^{\frac{\ell+m}{2}} \tag{4.33}$$

In this way we have defined, except for the constant factor c_m, all of the elements of the right-hand column of the matrix T_g, that is, $T_{m\,\ell}(\alpha,\beta,\gamma)$. Leaving c_m for the moment undetermined, we find all of the other elements $T_{mn}(\alpha,\beta,\gamma)$. To do this, we apply the pperator H_- to the functions T_m (α,β,γ) and make use of the fact that

$$H_- T_{mn} (\alpha,\beta,\gamma) = a_n T_{m,n-1} (\alpha,\beta,\gamma) \tag{4.34}$$

where $a_n = \sqrt{(\ell+n)(\ell-n+1)}$

We substitute the operator

$$H_- = e^{-i\gamma}\left(\frac{1}{\cos\beta} i \frac{\partial}{\partial\alpha} + \frac{\partial}{\partial\beta} - i\tan\beta \frac{\partial}{\partial\gamma}\right).$$

and $T_{mn} (\alpha,\beta,\gamma) = e^{-im\alpha} U_{mn}(\beta) e^{-in\gamma}$

in the equation (4.34), we have

$$e^{i\gamma} \left(\frac{1}{\cos\beta} i \frac{\partial}{\partial\alpha} + \frac{\partial}{\partial\beta} - i \tan\beta \frac{\partial}{\partial\gamma}\right) e^{-im\alpha} U_{mn} (\beta) e^{-in\gamma}$$

$$= a_n e^{-im\alpha} U_{m,n-1}(\beta) e^{-i(n-1)\gamma}$$

$$e^{i} \left(\frac{1}{\cos\beta} i \frac{\partial}{\partial\alpha} e^{-im\alpha} U_{mn} (\beta) e^{-in\gamma} + \frac{\partial}{\partial\beta} e^{-im\alpha} U_{mn}(\beta) e^{-in\gamma}\right.$$

$$\left.- i \tan\beta \frac{\partial}{\partial\gamma} e^{-im\alpha} U_{mn} (\beta) e^{-in\gamma}\right)$$

$$= a_n e^{-im\alpha} U_{mn} (\beta) e^{-in\gamma}$$

$$e^{im\alpha} e^{-i\gamma(n-1)} \left(\frac{m}{\cos\beta} U_{mn} (\beta)+\frac{\partial}{\partial\beta} U_{mn} (\beta) - n\tan\beta U_{mn} (\beta) \right)$$

$$= a_n e^{-im\alpha} e^{-i\gamma(n-1)} U_{m,n-1} (\beta)$$

or $$\frac{d\,U_{mn}(\beta)}{d\beta} + \left(\frac{m}{\cos\beta} - n\tan\beta\right) U_{mn}(\beta) = a_n\, U_{m,n-1}(\beta)$$

or $$\frac{d\,U_{mn}(\beta)}{d\beta} + \frac{m-n\sin\beta}{\cos}\, U_{mn}(\beta) = a_n\, U_{m,n-1}(\beta) \qquad (4.35)$$

Upon introducing the variable $\mu = \sin\beta$ and the notation $U_{mn}(\beta) = P_{mn}(\mu)$, we have

$$\sin\beta = \mu$$

$$\cos\beta\, d\beta = d\mu$$

or $$\frac{d\mu}{d\beta} = \cos\beta = \sqrt{1-\mu^2}$$

Equation (4.35) becomes, using,

$$\frac{d\,U_{mn}(\beta)}{d\beta} = \frac{d\,P_{mn}(\mu)}{d\mu}\, \frac{d\mu}{d\theta}$$

$$\sqrt{1-\mu^2}\, \frac{d\,P_{mn}(\mu)}{d\mu} + \frac{m-n\mu}{\sqrt{1-\mu^2}}\, P_{mn}(\mu) = a_n\, P_{m,n-1}(\mu) \qquad (4.36)$$

We set

$$P_{mn}(\mu) = (1-\mu)^{-\frac{n-m}{2}}\, (1+\mu)^{-\frac{n+m}{2}}\, \nu_{mn}(\mu) \qquad (4.37)$$

Substituting this expression for $P_{mn}(\mu)$ in (4.36), we have

$$\sqrt{1-\mu^2}\, \frac{d}{d\mu}\, (1-\mu)^{-\frac{n-m}{2}}\, (1+\mu)^{-\frac{n+m}{2}}\, \nu_{mn}(\mu) + \frac{m - n\mu}{\sqrt{1-\mu^2}}$$

$$(1-\mu)^{-\frac{n-m}{2}}(1+\mu)^{-\frac{n+m}{2}}\nu_{mn}(\mu) = a_n\,(1-\mu)^{-\frac{n-m-1}{2}}(1+\mu)^{-\frac{n+m-1}{2}}\nu_{m,n-1}(\mu)$$

$$\sqrt{1-\mu^2}\left[\frac{n-m}{2}(1-\mu)^{-\frac{n-m}{2}-1}(1+\mu)^{-\frac{n+m}{2}}\nu_{mn}(\mu) - \frac{n+m}{2}(1+\mu)^{-\frac{n+m}{2}-1}\right.$$

$$\left.(1-\mu)^{-\frac{n-m}{2}}\nu_{mn}(\mu) + (1-\mu)^{-\frac{n-m}{2}}(1+\mu)^{-\frac{n+m}{2}}\frac{d}{d\mu}\nu_{mn}(\mu)\right]$$

$$+\frac{m-n}{\sqrt{1-\mu^2}}(1-\mu)^{-\frac{n-m}{2}}(1+\mu)^{-\frac{n+m}{2}}\cdot\nu_{mn}(\mu)$$

$$= a_n\,(1-\mu)^{-\frac{n-m-1}{2}}(1+\mu)^{-\frac{n+m-1}{2}}$$

or

$$\sqrt{1-\mu^2}\left[\frac{n-m}{2}(1-\mu)^{-\frac{n-m+2}{2}}(1+\mu)^{-\frac{n+m}{2}}\nu_{mn}(\mu)\right.$$

$$-\frac{n+m}{2}(1+\mu)^{-\frac{n+m+2}{2}}(1-\mu)^{-\frac{n-m}{2}}\nu_{mn}(\mu)$$

$$\left.+(1-\mu)^{-\frac{n-m}{2}}(1+\mu)^{-\frac{n+m}{2}}\frac{d\,\nu_{mn}}{d\mu}\right| + (m-n\mu)$$

$$(1-\mu)^{-\frac{n-m+1}{2}}(1+\mu)^{-\frac{n+m+1}{2}}\nu_{mn}(\mu) = a_n\,(1-\mu)^{-\frac{n-m-1}{2}}$$

$$(1+\mu)^{-\frac{n+m-1}{2}}\nu_{m,n-1}(\mu)$$

or

$$\frac{n-m}{2}(1-\mu)^{-\frac{n-m+1}{2}}(1+\mu)^{-\frac{n+m-1}{2}}\nu_{mn}(\mu) - \frac{n+m}{2}(1+\mu)^{-\frac{n+m+1}{2}}$$

$$(1-\mu)^{-\frac{n-m+1}{2}}\nu_{mn}(\mu) + (1-\mu)^{-\frac{n-m-1}{2}}(1+\mu)^{-\frac{n+m-1}{2}}\frac{d\,\nu_{mn}(\mu)}{d\mu}$$

$$+ (m-n\mu)\ (1-\mu)^{-\frac{n-m+1}{2}} (1+\mu)^{-\frac{n+m+1}{2}}\ \nu_{mn}(\mu)$$

$$= a_n\ (1-\mu)^{-\frac{n-m-1}{2}}\ (1+\mu)^{-\frac{n+m+1}{2}}\ \nu_{mn}(\mu)$$

or $$\frac{n-m}{2}\ (1-\mu)^{-1}\ \nu_{mn}(\mu) - \frac{n+m}{2}\ (1+\mu)^{-1}\ \nu_{mn}(\mu) + \frac{d\ \nu_{mn}(\mu)}{d\mu}$$

$$+ (m-n\mu)\ (1-\mu)^{-1}\ (1+\mu)^{-1}\ \nu_{mn}(\mu) = a_n\ \nu_{m,n-1}(\mu)$$

or $$\frac{n-m}{2}\ (1+\mu)\ \nu_{mn}(\mu) - \frac{n+m}{2}\ (1-\mu)\nu_{mn}(\mu) + (1-\mu^2)\ \frac{d\ \nu_{mn}(\mu)}{d\mu}$$

$$+ (1-\mu^2)\ \frac{d\ \nu_{mn}(\mu)}{d\mu} + m\nu_{mn}\ (\mu) - n\ \ \nu_{mn}(\mu)$$

$$= (1-\mu^2)\ a_n\ \nu_{m,n-1}(\mu)$$

or $$\frac{n-m}{2}\ \nu_{mn}(\mu) + \frac{n-m}{2}\ \nu_{mn}(\mu) - \frac{n+m}{2}\ \nu_{mn}(\mu)$$

$$+ \frac{n+m}{2}\ \mu\nu_{mn}(\mu) + (1-\mu^2)\ \frac{d\ \nu_{mn}(\mu)}{d\mu} + m\ \nu_{mn}\ (\mu)$$

$$- n\mu\ \nu_{mn}(\mu) = (1-\mu^2)\ a_n\ \nu_{m,n-1}(\mu)$$

or $$-m\ \nu_{mn}(\mu) + n\ \mu\nu_{mn}(\mu) + m\ \nu_{mn}(\mu) - n\mu\ \nu_{mn}(\mu)$$

$$+ (1-\mu^2)\ \frac{d\ \nu_{mn}(\mu)}{d\mu} = (1-\mu^2)\ a_n\ \nu_{m,n-1}(\mu)$$

or $$\frac{d\ \nu_{mn}(\mu)}{d\mu} = a_n\ \nu_{m,n-1}(\mu) \qquad (4.38)$$

The functions $P_{m\ell}(\mu)$ have already been determined in equation (4.33). Rewriting these functions in form (4.37), we obtain

$$P_{m\ell}(\mu) = c_m (1-\mu)^{-\frac{\ell-m}{2}} (1+\mu)^{-\frac{\ell+m}{2}} \nu_{m\ell}(\mu)$$

This shows that

$$\nu_{m\ell}(\mu) = c_m (1-\mu)^{\ell-m} (1+\mu)^{\ell+m} \qquad (4.39)$$

From this and formula (4.38), we have

$$\nu_{m,n-1}(\mu) = \frac{1}{a_n} \frac{d\,\nu_{mn}(\mu)}{d\mu}$$

$$\nu_{m,\ell-1}(\mu) = \frac{1}{a_\ell} \frac{d}{d\mu} c_m (1-\mu)^{\ell-m} (1+\mu)^{\ell+m}$$

$$\nu_{m,\ell-2}(\mu) = \frac{c_m}{a_\ell a_{\ell-1}} \frac{d^2\mu}{d\mu^2} (1-\mu)^{\ell-m} (1+\mu)^{\ell+m}$$

and so on, therefore,

$$\nu_{m,\ell-(\ell-n)}(\mu) = \frac{c_m}{a_\ell\, a_{\ell-1}\cdots a_{n+1}} \frac{d^{\ell-n}}{d\mu^{\ell-n}} (1-\mu)^{\ell-m} (1+\mu)^{\ell+m}$$

or $$\nu_{mn}(\mu) = \frac{c_m}{a_\ell\, a_{\ell-1}\cdots a_{n+1}} \frac{d^{\ell-n}}{d\mu^{\ell-n}} (1-\mu)^{\ell-m} (1+\mu)^{\ell+m}$$

Therefore the functions P_{mn} (μ) which we shall also designate as $P_{mn}(\mu)$ have the form

$$P^{\ell}_{mn}(\mu) = \frac{c_m}{a_\ell a_{\ell-1}\cdots a_{n+1}} (1-\mu)^{-\frac{n-m}{2}} (1+\mu)^{-\frac{n+m}{2}} \frac{d^{\ell-n}}{d\mu^{\ell-n}}$$

$$(1-\mu)^{\ell-m} (1+\mu)^{\ell+m} \qquad (4.40)$$

Substituting now in $T_{mn}(\alpha,\beta,\gamma) = e^{-im\alpha} U_{mn}(\mu) e^{-in\gamma}$, P^{ℓ}_{mn} ($\sin\beta$) for $U_{mn}(\beta)$, we obtain the function $T_{mn}(\alpha,\beta,\gamma)$ for all values of the indices m and n.

$$T_{mn}(\alpha,\beta,\gamma) = e^{-im\alpha} \frac{c_m}{a_\ell\, a_{\ell-1}\cdots a_{n+1}} e^{-in\gamma} (1-\mu)^{-\frac{n-m}{2}}$$

$$(1+\mu)^{-\frac{n+m}{2}} \frac{d^{\ell-n}}{d\mu^{\ell-n}} (1-\mu)^{\ell-m} (1+\mu)^{\ell+m}$$

Now we determine the constant c_m in the expression for $T_{mn}(\alpha,\beta,\gamma)$ from the condition that the rotation with angles (α,β,γ), zero; $g_0 = g(0,0,0)$, correspond under the representation to the matrix E. This implies that $T_{mn}(0,0,0) = 1$. Therefore

$$P^{\ell}_{mm}(0) = \frac{c_m}{a_\ell\, a_\ell \cdots a_{m+1}} \frac{d^{\ell-m}}{d\mu^{\ell-m}} \cdot (1-\mu)^{\ell-m} (1+\mu)^{\ell+m} = 1 \qquad (4.41)$$

Now we shall calculate $\frac{d^{\ell-m}}{d^{\ell-m}} (1-\mu)^{\ell-m} (1+\mu)^{\ell+m}$

We have

$$\frac{d^{\ell-m}}{d\mu^{\ell-m}} (1-\mu)^{\ell-m} (1+\mu)^{\ell+m} = \frac{d^{\ell-m}}{d\mu^{\ell-m}} (1-\mu)^{\ell-m} (1+\mu)^{\ell+m}$$

$$+ \binom{\ell-m}{1} \frac{d^{\ell-m-1}}{d\mu^{\ell-m-1}} (1-\mu)^{\ell-m} \frac{d}{d\mu} (1+\mu)^{\ell+m}$$

$$+ \binom{\ell-m}{2} \frac{d^{\ell-m-2}}{d\mu^{\ell-m-2}} (1-\mu)^{\ell-m} \frac{d^2}{d\mu^2} (1+\mu)^{\ell+m}$$

$$+ \binom{\ell-m}{3} \frac{d^{\ell-m-3}}{d\mu^{\ell-m-2}} (1-\mu)^{\ell-m} \frac{d^3}{d\mu^3} (1+\mu)^{\ell+m}$$

$$+ \cdots + (1-\mu)^{\ell-m} \frac{d^{\ell-m}}{d\mu^{\ell-m}} (1+\mu)^{\ell+m}$$

or

$$\frac{d^{\ell-m}}{d\mu^{\ell-m}} (1-\mu)^{\ell-m} (1+\mu)^{\ell+m} = (-1)^{\ell-m} (\ell-m)! \ (1+\mu)^{\ell+m}$$

$$+ (-1)^{\ell-m-1} (\ell-m)(\ell-m)(\ell-m-1)\cdots 2(1-\mu)(\ell+m)(1+\mu)^{\ell+m-1}$$

$$+ \frac{(\ell-m)(\ell-m-1)}{2!} (-1)^{\ell-m-2} (\ell-m)(\ell-m-1)\cdots 3(1-\mu)^2 (\ell+m)(\ell+m-1)$$

$$(1+\mu)^{\ell+m-2} + \frac{(\ell-m)(\ell-m-1)(\ell-m-2)}{3!} (-1)^{\ell-m-3} (\ell-m)(\ell-m-1)\cdots 4$$

$$\cdot(1-\mu)^3 (\ell+m)(\ell+m-1)(\ell+m-2)(1-\mu)^{\ell+m-3} + \cdots +$$

$$(1-\mu)^{\ell-m} (\ell+m)(\ell+m-1)(\ell+m-2)\cdots(2m+1)(1+\mu)^{2m}$$

Therefore for $\mu = 0$, we have

$$\frac{d^{\ell-m}}{d\mu^{\ell-m}} (1-\mu)^{\ell-m} (1+\mu)^{\ell+m} = (-1)^{\ell-m} (\ell-m)! + (-1)^{\ell-m-1} (\ell-m)!(\ell-m)$$

$$(\ell+m) + (-1)^{\ell-m-2} \frac{(\ell-m)(\ell-m-1)}{2!} \frac{(\ell-m)!}{2!} (\ell+m)(\ell+m-1)$$

$$+ (-1)^{\ell-m-3} \frac{(\ell-m)(\ell-m-1)(\ell-m-2)}{3!} \frac{(\ell-m)!}{3!} (\ell+m)(\ell+m-1)(\ell+m-2)$$

$$+ \cdots + (\ell+m)(\ell+m-1)(\ell+m-2) \cdots (2m+1)$$

or

$$\frac{d^{\ell-m}}{d\mu^{\ell-m}} (1-\mu)^{\ell-m} (1+\mu)^{\ell+m} = (-1)^{\ell-m} (\ell-m)! + (-1)^{\ell-m-1} (\ell-m)(\ell-m)!$$

$$\frac{(\ell+m)!}{(\ell+m-1)!} + \frac{(\ell-m)(\ell-m-1)}{2!} (-1)^{\ell-m-2} \frac{(\ell-m)!}{2!} \frac{(\ell+m)!}{(\ell+m-2)!}$$

$$+ \frac{(\ell-m)(\ell-m-1)(\ell-m-2)}{3!} (-1)^{\ell-m-3} \frac{(\ell-m)!}{3!} \frac{(\ell+m)!}{(\ell+m-3)!} + \cdots + \frac{(\ell+m)!}{2m!}$$

or

$$\frac{d^{\ell-m}}{d\mu^{\ell-m}} (1-\mu)^{\ell-m} (1+\mu)^{\ell+m} = \sum_{r=o}^{\ell-m} \binom{\ell-m}{r} (-1)^{\ell-m-r} \frac{(\ell-m)!(\ell+m)!}{r!(\ell+m-r)!}$$

Hence

$$P^{\ell}_{mm}(o) = \frac{C_m}{a_\ell a_{\ell-1} \cdots a_{m+1}} \left[\sum_{r=o}^{\ell-m} \binom{\ell-m}{r} (-1)^{\ell-m-r} (\ell-m)! \binom{\ell+m}{r} \right] \tag{4.42}$$

As we have

$$a_n = \sqrt{(\ell+n)(\ell-n+1)}$$

Therefore $a_\ell = \sqrt{2\ell}$, $a_{\ell-1} = \sqrt{(2\ell-1)\cdot 2}$ and so on.

$$a_{m+1} = \sqrt{(\ell+m+1)(\ell-m)}$$

Thus we get

$$a_\ell \, a_{\ell-1} \cdots a_{m+1} = \sqrt{\frac{2\ell!(\ell-m)!}{(\ell+m)!}} \tag{4.43}$$

Replacing $a_\ell a_{\ell-1} \cdots a_{m+1}$ by their known values, we have

$$C_m = \frac{a_\ell a_{\ell-1} \cdots a_{m+1}}{\sum_{r=o}^{\ell-m} (-1)^{\ell-m-r} \binom{\ell-m}{r} \binom{\ell+m}{\gamma} (\ell-m)!}$$

$$= \sqrt{\frac{2\ell!(\ell-m)!}{(\ell+m)!}} \; \frac{1}{\sum_{r=o}^{\ell-m} (-1)^{\ell-m-r} \binom{\ell-m}{r} \binom{\ell+m}{r} (\ell-m)!} \tag{4.44}$$

Finally substituting the known values for C_m and $a_\ell a_{\ell-1} \cdots a_{n+1}$ in the expression for $P^\ell_{mn}(\mu)$ we have

$$P^\ell_{mn}(\mu) = \sqrt{\frac{2\ell!(\ell-m)!}{(\ell+m)!}} \; \frac{1}{\sum_{r=o}^{\ell-m} (-1)^{\ell-m-r} \binom{\ell-m}{r} \binom{\ell+m}{r} (\ell-m)!} \sqrt{\frac{(\ell+n)!}{2\ell!(\ell-n)}}$$

$$(1-\mu)^{-(n-m)/2} \, (1+\mu)^{-(n+m)/2} \, \frac{d^{\ell-n}}{d\mu^{\ell-n}} (1-\mu)^{\ell-m} \, (1+\mu)^{\ell+m}$$

or

$$P^{\ell}_{mn}(\mu) = \frac{1}{\sum_{r=o}^{\ell-m} (-1)^{\ell-m-r} \binom{\ell-m}{r}\binom{\ell+m}{r}} \sqrt{\frac{(\ell-m)!(\ell+n)!}{(\ell+m)!(\ell-n)!}}$$

$$(1-\mu)^{-(n-m)/2} (1+\mu)^{-(n+m)/2} \frac{d^{\ell-n}}{d\mu^{\ell-n}} (1-\mu)^{\ell-m} (1+\mu)^{\ell+n} \tag{4.45}$$

Thus the matrix corresponding under the irreducible representation with weight ℓ to the rotation with angles α,β,γ is written in the canonical basis as follows:

$$T^{\ell}_{g} = \left[T^{\ell}_{mn}(\alpha,\beta,\gamma)\right] , \quad (n,m = -\ell,-\ell+1,\cdots,\ell)$$

where

$$T^{\ell}_{mn}(\alpha,\beta,\gamma) = e^{-im\alpha} P^{\ell}_{mn}(\sin \beta) e^{-in\gamma}$$

and

$$P^{\ell}_{mn}(\mu) = A(1-\mu)^{-(n-m)/2} (1+\mu)^{-(n+m)/2} \frac{d^{\ell-n}}{d\mu^{\ell-n}} (1-\mu)^{\ell-m}(1+\mu)^{\ell+m} \tag{4.46}$$

where

$$A = \frac{1}{\sum_{r=o}^{\ell-m} (-1)^{\ell-m-r}\binom{\ell-m}{r}\binom{\ell+m}{r}} \sqrt{\frac{(\ell+n)!}{(\ell+m)!(\ell-n)!(\ell-m)!}} \tag{4.47}$$

The functions $T^{\ell}_{mn}(\alpha,\beta,\gamma)$ will be referred to in the sequel as generalized spherical functions. For ℓ an integer and $m = 0$, the function $T^{\ell}_{mn}(\alpha,\beta,\gamma)$ have the form

$$T^{\ell}_{on}(\alpha,\beta,\gamma) = e^{-in\alpha} \frac{1}{\sum_{r=o}^{\ell} (-1)^{\ell-r} \binom{\ell}{r}^2 \ell!} \sqrt{\frac{\ell+n!}{(\ell-n)!}} (1-\mu^2)^{-(n/2)}$$

$$\frac{d^{\ell-n}}{d\mu^{\ell-n}} (1-\mu^2)^{\ell} \tag{4.48}$$

In particular, for $\mu = 0$,

$$P^{\ell}_{oo}(\mu) = \frac{1}{\left[\sum_{r=o}^{\ell} (-1)^{\ell-r} \binom{\ell}{r}^2\right]} \frac{1}{\ell!} \frac{d^{\ell}}{d\mu^{\ell}} (1-\mu^2)^{\ell}$$

$$= \frac{1}{\sum_{r=o}^{\ell} (-1)^{\ell-r} \binom{\ell}{r}^2 \ell!} \sum_{r=o}^{\ell} (-1)^{\ell-r} \binom{\ell}{r}^2 \ell! = 1$$

Therefore $P^{\ell}_{oo}(\mu)$, for $\mu = 0$, that is, $\sin\beta = 0$ or $\beta = 0$ is 1. We now look at the functions $P^{\ell}_{mn}(\mu)$ for $m = \ell$, we have

$$P^{\ell}_{\ell n} = \sqrt{\frac{(\ell+n)!}{2\ell!(\ell-n)!}} (1-\mu)^{-(n-\ell)/2} (1+\mu)^{-(n+\ell)/2} \frac{d^{\ell-n}}{d\mu^{\ell-n}} (1-\mu)^{2\ell}$$

$$= \sqrt{\frac{(\ell+n)!}{(\ell+n)!(\ell-n)!}} (1-\mu)^{(\ell-n)/2} (1+\mu)^{(-\ell-n)/2} \frac{2\ell!}{(\ell+n)!} (1+\mu)^{\ell+n}$$

$$= \sqrt{\frac{2\ell!}{(\ell+n)!(\ell-n)!}} (1-\mu)^{(\ell-n)/2} (1+\mu)^{(\ell+n)/2} \tag{4.49}$$

and $$P^{\ell}_{\ell\ell}(\mu) = (1+\mu)^{\ell} \tag{4.50}$$

$$P^{\ell}_{\ell,-\ell}(\mu) = (1-\mu)^{\ell} \tag{4.51}$$

$$P^{\ell}_{\ell o}(\mu) = \sqrt{2/(\ell!)} (1-\mu^2)^{\ell/2} \tag{4.52}$$

BIBLIOGRAPHY

[1] Brian G. Wybourne — Classical groups for Physicists John Wiley and Sons Inc. New York 1973.

[2] I.M. Gelfand and Z.Ya Shapiro — Representation of the Group of Rotation in Three Dimensional Space and their Applications. Uspekhi Matem. Nauk, 7,3-117(1952), English Translation Amer. Math. Soc. Translations,(2) 2, 207-316(1956).

[3] Willard Miller, Jr. — Lie Theory and Special Function. Academic Press, New York, 1968.

[4] G. Ya Lyubarskii — The Application of Group Theory in Physics. Pergamon Press, New. York, 1960.

[5] Morton Hamermesh — Group Theory and its Application to Physical Problems. Addison-Wesley Publishing Company, Inc. Reading Mass. 1962.

[6] N.J. Vilenkin — Special Functions and Theory of Group Representations. American Mathematical Society, 1968.

[7] M.A. Naimark — Linear Representation of the Lorentz Group. Pergamon Press Limited, London, 1964.

[8] James D. Talman — Special Functions. A Group Theoretical Approach. W.A. Bengmans Inc. New York, 1968.

[9] Wolker Heine — Group Theory in Quantum Mechanics Pergamon Press, New York, 1960.

[10] Michael Tinkham — Group Theory and Quantum Mechanics. McGraw-Hill Company, New York, 1964.

[11] E.P. Winger — Group Theory and its Application to the Quantum Mechanics of Atomic Spectra. Academic Press, N.Y. 1959.

[12] Hermannweyl — The Theory of Groups and Quantum Mechanics. Dover Publications, Inc. N.Y., 1950.

[13] Hermann Boerner — Representations of Groups. North-Holland Publishing Co. Amsterdam, 1970.

[14] Paul H.E. Meijer and Edmond Bauer — Group Theory. The Applications to Quantum Mechanics. North-Holland Publishing Company. Amsterdam, 1962.

[15] B.L. Van der Waerdon — Group Theory and Quantum Mechanics. Springer-Verlag Berlin, Heidelberg, 1974.

[16] G.G. Hall — Applied Group Theory. Longmans, Green and Co. Ltd. London, 1967.

[17] P.A. Rowlatt — Group Theory and Elementary Particles. Longmans, Green and Co. Ltd. London, 1966.