THREE DIMENSIONAL COMPLEX ROTATION GROUP

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by

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We accept this dissertation as conforming to the required standard.

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A WORD OF THANK

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CHAPTER ONE

INTRODUCTION

Three dimensional rotation group has been studied in detail in [1]. In this paper rotations have been discussed in terms of Euler angles. Another paper [2] developes isomorphism between complex orthogonal group O_3 and orthochronous Lorentz group L_+ without giving any explicit relationship in terms of angles.

In this thesis the representations of 4×4 Lorentz group have been discussed in terms of six real parameters ψ_1, ψ_2, θ_1, θ_2 and ϕ_1, ϕ_2 where ψ_1, θ_1, ϕ_1 represent Eulerian angles of rotations about the axes OZ, OX, OZ′. ψ_2, θ_2, ϕ_2 represent the angles which are related to pure Lorentz transformations along the same axes. 4×4 matrices of the Lorentz group in terms of the parameters given above are related to 3×3 matrices of O_3, thereby introducing the rotations of O_3 in terms of complex angles.

For example following the isomorphism developed in [2] a complex rotation about Z-axis has been shown isomorphic to the combination of a rotation about Z-axis and a Lorentz transformation about the same axis.

$$\begin{pmatrix} \cos(\psi_1 + i\psi_2) & -\sin(\psi_1 + i\psi_2) & 0 \\ \sin(\psi_1 + i\psi_2) & \cos(\psi_1 + i\psi_2) & 0 \\ 0 & 0 & 1 \end{pmatrix} \text{ is isomorphic to}$$

$$\begin{pmatrix} \cosh\psi_2 & 0 & 0 & \sinh\psi_2 \\ 0 & \cos\psi_1 & -\sin\psi_1 & 0 \\ 0 & \sin\psi_1 & \cos\psi_1 & 0 \\ \sin\psi_2 & 0 & 0 & \cosh\psi_2 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\psi_1 & -\sin\psi_1 & 0 \\ 0 & \sin\psi_1 & \cos\psi_1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cosh\psi_2 & 0 & 0 & \sinh\psi_2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \sinh\psi_2 & 0 & 0 & \cosh\psi_2 \end{pmatrix}$$

This gives arise to complex rotations. From this result the complex rotation matrix $g = g_\psi g_\theta g_\phi$ has been calculated.

Chapter Two consists of concepts about Lorentz group. Rotations and pure Lorentz transformations have been discussed. Preliminary concepts about complex orthogonal group has been given. In the end the isomorphism between O_3 and L_+ has been developed.

Chapter Three follows the work done by Gelfond and Ya'Shapiso in [1]. The operators A_1, A_2, A_3 have been calculated in terms of Euler angles $(\phi_1 \theta, \phi_2)$. Following [1] the operators A_1, A_2, A_3 and B_1, B_2, B_3 corresponding to rotations and Lorentz transformations respectively have been calculated in terms of parameters already mentioned.

In Chapter Four invariants of Lorentz group namely $\underline{J}^2 - \underline{K}^2$ and $\underline{J} \cdot \underline{K}$ have been derived in terms of the six parameters.

CHAPTER TWO

INTRODUCTION TO COMPLEX ROTATIONS

§2.1 LORENTZ TRANSFORMATIONS

A Lorentz Transformation is defined to be transformation of the type:

$$X'_\alpha = L_{\alpha\beta} X_\beta \qquad (2.1.1)$$

α, β takes on the values 0,1,2,3. Summation is implied over repeated indices. $L_{\alpha\beta}$ represents the elements of real matrix L called the matrix of Lorentz transformation. $x_o = ct$ is time co-ordinate and x_1, x_2, x_3 denote the space co-ordinate in Minkowski space.

The transformations (2.1.1) leave invariant $x_\alpha\, g_{\alpha\beta}\, x_\beta$. $g_{\alpha\beta}$ are the elements of the matrix g given by

$$g = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \qquad (2.1.2)$$

$$x_\alpha\, g_{\alpha\beta}\, g_\beta = x_o^2 - x_1^2 - x_2^2 - x_3^2$$

In matrix notation, Lorentz transformation (2.1.1) is written as

$$X' = LX \qquad (2.1.3)$$

with vector $x = (x_0, x_1, x_2, x_3)$ so that

$$x^{T'} g X' = x^T gX \qquad (2.1.4)$$

x^T denotes the transpose of x. Using (2.1.3) in (2.1.4)

$$x^T L^T g L\acute{x} = X^T gx \qquad (2.1.5)$$

(2.1.5) holds for all vectors X in the Minkowski space. Thus

$$L^T gL = g \qquad (2.1.6)$$

Therefore a Lorentz Transformation is defined by a four by four matrix satisfying the equation (2.1.6).

The Lorentz transformations may be divided into four subsets by studying the possible values for the determinant $|L|$ and the element L_{oo} of the matrix L. From (2.1.6)

$$\det(L^T)\ \det(g)\ \det(L) = \det(g)$$

or

$$\det(L^T)\ \det(L) = 1$$

$$[\det(L)]^2 = 1$$

$$\det L = \pm 1$$

when the matrix product in (2.1.6) is expanded and the element L_{oo} is written out, we get

$$L^2_{oo} = 1 + a^2_{10} + a^2_{20} + a^2_{30}$$

or $\quad L_{oo} \geq 1$

Thus $\quad L_{oo} \geq 1$ or $L_{oo} \leq 1$

Lorentz transformation with det $L = +1$ are named Proper Lorentz transformation and those with det $L = -1$ are named Improper Lorentz transformation; those with $L_{oo} \geq 1$ are called orthochronous and with $L_{oo} \leq 1$ are nonorthochronous. The proper orthochronous transformations are denoted by L_+.

§2.2 RELATIONSHIP OF SL(2C) WITH L_+

There is a simple relation between the elements of L_+ and another group called special linear group of two by two matrices with complex elements and determinant equal to one. This group has been named as SL(2C). With each point $x \equiv (x_o, x_1, x_2, x_3)$ of Minkowski space one may associate a 2×2 matrix M by setting

$$M = \sigma_\mu X_\mu = \begin{pmatrix} x_o + x_3 & x_1 - ix_2 \\ x_1 + ix_2 & x_o - x_3 \end{pmatrix} \qquad (2.2.1)$$

where $\quad \sigma_o = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma_2 = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}$

and $$\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad (2.2.2)$$

M is a Hermitian matrix if x is real and

$$\det M = x_o^2 - x_1^2 - x_2^2 - x_3^2$$

If M is transformed by a matrix A of SL(2C) to give M by the rule

$$M' = AMA^{+}$$

where det A = 1 and $A^{+} = (A^{T})^{*}$ the Hermitian conjugate of A, then M′ is also Hermitian and det M = det (M′). From (2.2.1) we get the inverse relationship

$$x_{\mu} = \frac{1}{2}\, t_{r}\, (\sigma_{\mu}\, M) \qquad (2.2.3)$$

If the matrix $M' = AMA^{+}$ is written as

$$M' = \sigma_{\mu}\, x'_{\mu}$$

we can calculate the component of the vector x′

$$\begin{aligned} x'_{\mu} &= \frac{1}{2}\, t_{r}(\sigma_{\mu} M') \\ &= \frac{1}{2}\, t_{r}(\sigma_{\mu} AMA^{+}) \\ &= \frac{1}{2}\, t_{r}(\sigma_{\mu} A \sigma_{\nu} x_{\nu} A^{+}) \\ &= \frac{1}{2}\, t_{r}(\sigma_{\mu} A \sigma_{\nu} A^{+}) x_{\nu} \end{aligned}$$

Comparing with (2.1.1)

$$L_{\mu\nu}(A) = \frac{1}{2}\, t_{r}(\sigma_{\mu} A \sigma_{\nu} A^{+}) \qquad (2.2.4)$$

Now from (2.2.3) $L_{\mu\nu} = \frac{1}{2}\, t_{r}(\sigma_{\mu} A \sigma_{\nu} A^{+})$

Putting

$$A = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}, \quad A^{+} = \begin{pmatrix} \alpha & \gamma \\ \beta & \delta \end{pmatrix}$$

and using (2.2.2) we get

$$L_{00} = \frac{1}{2}\, t_r(\sigma^0 A \sigma^0 A^+)$$

$$= \frac{1}{2}\, (\alpha\bar{\alpha} + \beta\bar{\beta} + \gamma\bar{\gamma} + \delta\bar{\delta})$$

$$L_{01} = \frac{1}{2}\, t_r(\sigma^0 A \sigma^1 A^+)$$

$$= \mathrm{Re}(\alpha\bar{\beta} + \gamma\bar{\delta})$$

$$L_{02} = \frac{1}{2}\, t_r(\sigma^0 A \sigma^2 A^+)$$

$$= \mathrm{Im}(\bar{\alpha}\beta + \bar{\gamma}\delta)$$

$$L_{03} = \frac{1}{2}\, t_r(\sigma^0 A \sigma^3 A^+)$$

$$= \frac{1}{2}\, (\alpha\bar{\alpha} - \beta\bar{\beta} + \gamma\bar{\gamma} - \delta\bar{\delta})$$

$$L_{10} = \frac{1}{2}\, t_r(\sigma^1 A \sigma^0 A^+)$$

$$= \mathrm{Re}(\alpha\bar{\gamma} + \beta\bar{\delta})$$

$$L_{11} = \frac{1}{2}\, t_r(\sigma^1 A \sigma^1 A^+)$$

$$= \mathrm{Re}(\beta\bar{\gamma} + \alpha\bar{\delta})$$

$$L_{12} = \frac{1}{2}\, t_r(\sigma^1 A \sigma^2 A^+)$$

$$= \mathrm{Im}(\beta\bar{\gamma} + \bar{\alpha}\delta)$$

$$L_{13} = \frac{1}{2}\, t_r(\sigma^1 A \sigma^3 A^+)$$

$$= \mathrm{Re}(\alpha\bar{\gamma} - \beta\bar{\delta})$$

$$L_{20} = \frac{1}{2}\, t_r(\sigma^2 A \sigma^0 A^+)$$

$$= \mathrm{Im}(\beta\bar{\delta} + \alpha\bar{\gamma})$$

$$L_{21} = \frac{1}{2}\, t_r(\sigma^2 A \sigma^1 A^+)$$

$$= \mathrm{Im}(\beta\bar{\gamma} + \alpha\bar{\delta})$$

$$L_{22} = \frac{1}{2}\, t_r(\sigma^2 A \sigma^2 A^+)$$

$$= \mathrm{Re}(\bar{\alpha}\beta - \beta\bar{\gamma})$$

$$L_{23} = \frac{1}{2}\, t_r(\sigma^2 A \sigma^3 A^+)$$

$$= I_m(\alpha\bar{\gamma} + \bar{\beta}\delta)$$

$$L_{30} = \frac{1}{2}\, t_r(\sigma^3 A \sigma^0 A^+)$$

$$= \frac{1}{2}\,(\alpha\bar{\alpha} + \beta\bar{\beta} - \gamma\bar{\gamma} - \delta\bar{\delta})$$

$$L_{31} = \frac{1}{2}\, t_r(\sigma^3 A \sigma^1 A^+)$$

$$= \mathrm{Re}(\alpha\bar{\beta} - \gamma\bar{\delta})$$

$$L_{32} = \frac{1}{2}\, t_r(\sigma^3 A \sigma^2 A^+)$$

$$= I_m(\bar{\alpha}\beta + \gamma\bar{\delta})$$

$$L_{33} = \frac{1}{2}\, t_r(\sigma^3 A \sigma^3 A^+)$$

$$= \frac{1}{2}(\alpha\bar{\alpha} - \beta\bar{\beta} - \gamma\bar{\gamma} + \delta\bar{\delta})$$

Therefore $L_{\mu\nu}$ comes out to be

$$L_{\mu\nu} =$$

$$\begin{pmatrix} \frac{1}{2}\, \alpha\bar{\alpha} + \beta\bar{\beta} + \gamma\bar{\gamma} + \delta\bar{\delta} & Re(\alpha\bar{\beta} + \bar{\gamma}\delta) & I_m(\bar{\alpha}\beta - \gamma\bar{\delta}) & \frac{1}{2}(\alpha\bar{\alpha} - \beta\bar{\beta} + \gamma\bar{\gamma} - \delta\bar{\delta}) \\ Re(\alpha\bar{\gamma} + \beta\bar{\delta}) & Re(\alpha\bar{\delta} + \beta\bar{\gamma}) & I_m(\bar{\alpha}\delta + \beta\bar{\gamma}) & Re(\alpha\bar{\gamma} - \beta\bar{\delta}) \\ I_m(\alpha\bar{\gamma} + \beta\bar{\delta}) & I_m(\alpha\bar{\delta} + \beta\bar{\gamma}) & Re(\bar{\alpha}\delta - \beta\bar{\gamma}) & I_m(\alpha\bar{\gamma} - \beta\bar{\delta}) \\ \frac{1}{2}(\alpha\bar{\alpha} + \beta\bar{\beta} - \gamma\bar{\gamma} - \delta\bar{\delta}) & Re(\alpha\bar{\beta} - \gamma\bar{\delta}) & I_m(\bar{\alpha}\beta + \gamma\bar{\delta}) & \frac{1}{2}(\alpha\bar{\alpha} - \beta\bar{\beta} - \gamma\bar{\gamma} + \delta\bar{\delta}) \end{pmatrix} \quad (2.2.5)$$

§2.3 ROTATIONS AND PURE LORENTZ TRANSFORMATIONS

Two important subsets of L_+ are the spatial rotations and pure Lorentz transformations. A spatial rotation is a real linear transformation in three dimensional Eucledean space of type:

$$X_i' = R_{ij}\, x_j \qquad i,j = 1,2,3 \quad (2.3.1)$$

They leave invariant $x_1^2 + x_2^2 + x_3^2$. R_{ij} represents the elements of a 3×3 real matrix. If the spatial vector $\underline{x} = (x_1, x_2, x_3)$, then (2.3.1) may be written as

$$\underline{X}' = R\underline{x} \quad (2.3.2)$$

so that

$$X'^T X' = X^T X \quad (2.3.3)$$

Using (2.3.2) in (2.3.3)

$$X^T R^T R X = X^T X$$

which holds for all vectors $\underline{X}$ thus

$$R^T R = 1 \tag{2.3.4}$$

or $\quad R^T = R^{-1}$

From (2.3.4) one gets det R = ±1.

Rotations with det R = +1 are called Proper Rotations and those with det R = -1 are called Improper rotations. Spatial rotations form a three parameter lie group. These parameters can be expressed in terms of Euler angles or in terms of the angles of rotation about the three axes of co-ordinates. In the following a spatial rotation is given in terms of θ, the angle of rotation about a line whose direction is given by n (n_1, n_2, n_3) where $n_1^2 + n_2^2 + n_3^2 = 1$. This rotation also involves three parameters. On being the angle θ, the other two specifying the direction n.

A spatial rotation through an angle θ in a positive sense about the unit spatial vector n is given by

$$X_o' = X_o$$

$$X' = x \cos\theta + X\cdot n(1 - \cos\theta) + nXX \sin\theta \tag{2.3.5}$$

The rotation matrix R in (2.3.5) is given as

R =

$$\begin{pmatrix} \cos\theta + n_1^2(1-\cos\theta) & n_1n_2(1-\cos\theta) - n_3\sin\theta & n_1n_3(1-\cos\theta) + n_2\sin\theta \\ n_3\sin\theta + n_1n_2(1-\cos\theta) & \cos\ + n_2^2(1-\cos\theta) & n_2n_3(1-\cos\theta) - n_1\sin\theta \\ -n_2\sin\theta + n_2n_3(1-\cos\theta) & n_2n_3(1-\cos\theta) + n_1\sin\theta & \cos\theta + n_3^2(1-\cos\theta) \end{pmatrix}$$

The relations about the three axes of co-ordinates can be deduced as

$$R_{X_1}(\alpha) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & -\sin\alpha \\ 0 & \sin\alpha & \cos\alpha \end{pmatrix}$$

$$R_{X_2}(\beta) = \begin{pmatrix} \cos\beta & 0 & \sin\beta \\ 0 & 1 & 0 \\ -\sin\beta & 0 & \cos\beta \end{pmatrix}$$

and $$R_{X_3}(\gamma) = \begin{pmatrix} \cos\gamma & -\sin\gamma & 0 \\ \sin\gamma & \cos\gamma & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$R_{X_1}(\alpha)$ form a single parameter lie group; similarly $R_{X_2}(\beta)$ and $R_{X_3}(\gamma)$.

The elements of SL(2C) which correspond to spatial rotations are unitarly and conversely. The element A of SL(2C), which corresponds to spatial rotation (2.3.5) is given by

$$A = \cos\frac{1}{2}\theta - i\sin\frac{1}{2}\theta(\sigma\cdot n) \qquad (2.3.6)$$

These are matrices that are unitary. The unitary matrices form a subgroup of SL(2C) called SU(2), the spacial unitary group of dimension 2. It can be easily seen that the product of two unitary matrices is unitary.

The subject of pure Lorentz transformations is taken next. A pure Lorentz transformation of velocity V is positive direction of unit. Spatial vector n can be written in the form

$$x^{0\prime} = x^{0} \cosh \chi + \chi \cdot n \sinh \chi$$
$$x^{\prime} = x - X \cdot nn(1 - \cosh \chi) + x^{0} \cdot n \sinh \chi \qquad (2.3.7)$$

with $\tanh \chi = V$.

Transformations (2.3.7) also form a three parameter lie group. The matrix A which corresponds to pure Lorentz transformation is given by

$$A = \cosh \frac{1}{2} \chi - \sinh \frac{1}{2} \chi \ (\sigma \cdot n) \qquad (2.3.8)$$

Matrices given by (2.3.8) are Hermitian.

The general element of L can be expressed uniquely in the form

$$L = L_R \, L_P$$

with L_R, L_P respectively describing a spatial rotation and a pure Lorentz transformation.

The corresponding statement for the general element of A of SL(2C) is that it can be uniquely expressed as the product

$$A = UH$$

of a unitary matrix U and a Hermitian matrix H.

If $L_R = L(U)$ and $L_P = L(H)$, it follows that

$$L = L_R \, L_P = L(U) \; L(H)$$

$$= L(UH) = L(A).$$

§2.4 THE GROUP O_3

Orthogonal transformation in a complex space of three dimension are often referred to as complex rotations. Their relationship to L_+ is briefly treated in the following way:

Let p_μ and q_μ be the pair of orthogonal four vectors define

$$P = p_\mu \sigma_\mu = \begin{pmatrix} p_o + p_3 & p_1 - ip_2 \\ p_1 + ip_2 & p_o - p_3 \end{pmatrix}$$

and

$$q_\mu \sigma_\mu \quad \begin{pmatrix} q_o + q_3 & q_1 - iq_2 \\ q_1 + iq_2 & q_o - q_3 \end{pmatrix}$$

Under Lorentz transformation $L = L(A)$, let

$$P \to P' = APA^+$$

$$Q \to Q' = AQA^+$$

$$Q^{-1} \to Q'^{-1} = A^{+\prime} Q^{-1} A^{-1}$$

so that

$$PQ^{-1} \to P'Q'^{-1} = A(PQ^{-1})A^{-1} \qquad (2.4.1)$$

Writing

$$PQ^{-1} = \sigma_\mu Z_\mu = \sigma \cdot Z$$

where Z is complex 3 vector, such that

$$Z_1 = p_o q_1 - p_1 q_o - i(p_2 q_3 - q_2 p_3)$$

$$Z_2 = p_o q_2 - p_2 q_o - i(p_3 q_1 - p_1 q_3)$$

$$Z_3 = p_o q_3 - p_3 q_o - i(p_1 q_2 - p_2 q_1)$$

getting $$Z_j = R_{ij} Z_k \quad (2.4.2)$$

(2.4.1) and (2.4.2) together yield

$$R_{jk} = \frac{1}{2} t_r(\sigma_j A \sigma_k A^{-1}) \quad (2.4.3)$$

This relationship shows that there is 2-1 homomorphism $\pm A \leftrightarrow R(A)$ of SL(2C) on to O_3. Therefore it follows that O_3 is simply isomorphic to L.

Taking $A = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$ and $A^{-1} = \begin{pmatrix} \delta & \beta \\ \gamma & \alpha \end{pmatrix}$ we can calculate the elements of $R \in O_3$, where σ_i are elements of the Pauli spin Matrices.

$$R_{11} = \frac{1}{2} t_r(\sigma^1 A \sigma^1 A^{-1}) = \frac{1}{2}(\alpha^2 - \beta^2 - \gamma^2 + \delta^2)$$

$$R_{12} = \frac{1}{2} t_r(\sigma^1 A \sigma^2 A^{-1}) = \frac{1}{2} i(\alpha^2 + \beta^2 - \gamma^2 - \delta^2)$$

$$R_{13} = \frac{1}{2} t_r(\sigma^1 A \sigma^3 A^{-1}) = \gamma\delta - \alpha\beta$$

$$R_{21} = \frac{1}{2} t_r(\sigma^2 A \sigma^3 A^{-1}) = \frac{1}{2} (\alpha^2 + \beta^2 + \gamma^2 + \delta^2)$$

$$R_{22} = \frac{1}{2} t_r(\sigma^2 A \sigma^2 A^{-1}) = \frac{1}{2} (\alpha^2 + \beta^2 + \gamma^2 + \delta^2)$$

$$R_{23} = \frac{1}{2} t_r(\sigma^2 A \sigma^3 A^{-1}) = i(\alpha\beta + \gamma\delta)$$

$$R_{31} = \frac{1}{2}\, t_r(\sigma^3 A \sigma^1 A^{-1}) = (-\alpha\gamma + \beta\delta)$$

$$R_{32} = \frac{1}{2}\, t_r(\sigma^3 A \sigma^2 A^{-1}) = i(-\alpha\gamma - \beta\delta)$$

$$R_{33} = \frac{1}{2}\, t_r(\sigma^3 A \sigma^3 A^{-1}) = (\alpha\delta + \beta\gamma)$$

Therefore

$$R_{\gamma s} =$$

$$\begin{pmatrix} \frac{1}{2}(\alpha^2-\beta^2-\gamma^2+\delta^2) & \frac{1}{2}(\alpha^2+\beta^2-\gamma^2-\delta^2) & (\gamma\delta-\alpha\beta) \\ i(-\alpha^2+\beta^2-\gamma^2+\delta^2) & \frac{1}{2}(\alpha^2+\beta^2+\gamma^2+\delta^2) & i(\alpha\beta+\gamma\delta) \\ -(\alpha\gamma+\beta\gamma) & i(-\alpha\gamma-\beta\delta) & (\alpha\delta+\beta\gamma) \end{pmatrix} \qquad (2.4.4)$$

§2.5 COMPLEX ROTATIONS

Rotations can also be specified by unitary matrices of order two and determinant unity. The aggregate of all such matrices provides a group which is denoted by SU_2. Clearly

$$\begin{pmatrix} e^{i\theta-\psi/2} & 0 \\ 0 & e^{\frac{-i\theta+\psi}{2}} \end{pmatrix} \in SU_2$$

In this case $\alpha = e^{i\theta-\psi/2}$, $\delta = e^{-i\theta+\psi/2}$ and $\beta = \gamma = 0$.

Now let us calculate the expressions for $R_{\gamma s}$ and $L_{\mu\nu}$, for this special unitary matrix. $R_{\gamma s}$ and $L_{\mu\nu}$ will come out to be

$$R_{\gamma s} = \begin{pmatrix} \cos(\theta + i\psi) & -\sin(\theta + i\psi) & 0 \\ \sin(\theta + i\psi) & \cos(\theta + i\psi) & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad (2.5.1)$$

$$L_{\mu\nu} = \begin{pmatrix} \cosh\psi & 0 & 0 & \sinh\psi \\ 0 & \cos\theta & -\sin\theta & 0 \\ 0 & \sin\theta & \cos\theta & 0 \\ \sinh\psi & 0 & 0 & \cosh\psi \end{pmatrix} \qquad (2.5.2)$$

Equation (2.5.2) can be written as follows

$$L_{\mu\nu} = \begin{pmatrix} \cosh\psi & 0 & 0 & \sinh\psi \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \sinh\psi & 0 & 0 & \cosh\psi \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta & 0 \\ 0 & \sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad (2.5.3)$$

As in section 2.4, we have shown that complex orthogonal group is isomorphic to Lorentz group, therefore from (2.5.1) and (2.5.3) we can write

$$\begin{pmatrix} \cos(\theta + i\psi) & -\sin(\theta + i\psi) & 0 \\ \sin(\theta + i\psi) & \cos(\theta + i\psi) & 0 \\ 0 & 0 & 1 \end{pmatrix} \stackrel{\text{Iso}}{=}$$

$$\begin{pmatrix} \cosh\psi & 0 & 0 & \sinh\psi \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \sinh\psi & 0 & 0 & \sinh\psi \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta & 0 \\ 0 & \sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad (2.5.4)$$

The matrices of rotation g_ϕ, g_θ, $g_\psi (\phi, \theta, \psi$ are the Euler Angles) around the axes OZ, OX, OZ´ have the form

$$g_\phi = \begin{pmatrix} \cos\theta & -\sin\phi & 0 \\ \sin\phi_1 & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad g_\theta = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix}$$

and $$g_\psi = \begin{pmatrix} \cos\psi & -\sin\psi & 0 \\ \sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Taking ϕ, θ, ψ be complex angles

$$\phi = \phi_1 + i\phi_2$$
$$\theta = \theta_1 + i\theta_2$$
$$\psi = \psi_1 + i\psi_2$$

Using (2.5.4) the equivalent values of rotations g_ϕ, g_θ and g_ψ are

$$g_\theta = \begin{pmatrix} \cos(\phi_1 + i\phi_2) & -\sin(\phi_1 + i\phi_2) & 0 \\ \sin(\phi_1 + i\phi_2) & \cos(\phi_1 + i\phi_2) & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \underline{\underline{Iso}}$$

$$\begin{pmatrix} \cosh\phi_2 & 0 & 0 & \sinh\phi_2 \\ 0 & \cos\phi_1 & -\sin\phi_1 & 0 \\ 0 & \sin\phi_1 & \cos\phi_1 & 0 \\ \sinh\phi_2 & 0 & 0 & \cosh\phi_2 \end{pmatrix}$$

$$g_\theta = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos(\theta_1 + i\theta_2) & -\sin(\theta_1 + i\theta_2) \\ 0 & \sin(\theta_1 + i\theta_2) & \cos(\theta_1 + i\theta_2) \end{pmatrix} \quad \underline{\underline{\text{Iso}}}$$

$$\begin{pmatrix} \cosh\theta_2 & \sinh\theta_2 & 0 & 0 \\ \sinh\theta_2 & \cosh\theta_2 & 0 & 0 \\ 0 & 0 & \cos m_1 & -\sin m_1 \\ 0 & 0 & \sin m_1 & \cos m_1 \end{pmatrix}$$

$$g_\psi = \begin{pmatrix} \cos(\psi_1 + i\psi_2) & -\sin(\psi_1 + i\psi_2) & 0 \\ \sin(\psi_1 + i\psi_2) & \cos(\psi_1 + i\psi_2) & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \underline{\underline{\text{Iso}}}$$

$$\begin{pmatrix} \cosh\psi_2 & 0 & 0 & \sinh\psi_2 \\ 0 & \cos\psi_1 & -\sin\psi_1 & 0 \\ 0 & \sin\psi_1 & \cos\psi_1 & 0 \\ \sinh\psi_2 & 0 & 0 & \cosh\psi_2 \end{pmatrix}$$

Upon carrying out these rotations one after another their matrices are multiplied and hence we have the complex rotation matrix

$$g = g_\psi g_\theta g_\phi \quad \underline{\underline{\text{Iso}}}$$

$$\begin{pmatrix}
\cosh\psi_2\cosh\theta_2\cosh\phi_2 & \cosh\psi_2\sinh\theta_2\cos\phi_1 & -\cosh\psi_2\sinh\theta_2\sin\phi_1 & \cosh\psi_2\cosh\theta_2\sinh\phi_2 \\
\cosh\psi_2\cosh\theta_1\sinh\phi_2 & \sinh\psi_2\sin\theta_1\sin\phi_1 & \sinh\psi_2\sin\theta_1\cos\phi_1 & \sinh\psi_2\cos\theta_1\cosh\phi_2 \\
\cos\psi_1\sinh\theta_2\cosh\phi_2 & \cos\psi_1\cosh\theta_2\cos\phi_1 & -\cos\psi_1\cosh\theta_2\sin\phi_1 & \cos\psi_1\sinh\theta_2\sinh\phi_2 \\
\sin\psi_1\sin\theta_1\sinh\phi_2 & -\sin\psi_1\cos\theta_1\sin\phi_1 & -\sin\psi_1\cos\theta_1\cos\phi_1 & \sin\psi_1\sin\theta_1\cosh\phi_2 \\
\sin\psi_1\sin\theta_2\cosh\phi_2 & \sin\psi_1\cosh\theta_2\cos\phi_1 & -\sin\psi_1\cosh\theta_2\sin\phi_1 & \sin\psi_1\sinh\theta_2\sinh\phi_2 \\
-\cos\psi_1\sin\theta_1\sinh\phi_2 & \cos\psi_1\cos\theta_1\sin\phi_1 & \cos\psi_1\cos\theta_1\cos\phi_1 & -\sin\psi_1\sin\theta_1\cosh\phi_2 \\
\sinh\psi_2\cosh\theta_2\cosh\phi_2 & -\sinh\psi_2\sinh\theta_2\cos\phi_1 & -\sinh\psi_2\sinh\theta_2\sin\phi_1 & \sinh\psi_2\cosh\theta_2\sinh\phi_2 \\
+\cosh\psi_2\cos\theta_1\sinh\phi_2 & \cosh\psi_2\sin\theta_1\sin\phi_1 & +\cosh\psi_2\sin\theta_1\cos\psi_1 & +\cosh\psi_2\cos\theta_1\cosh\phi_2
\end{pmatrix} \qquad (\cdot 5.5)$$

CHAPTER THREE

§3.1 BASIC INFINITESIMAL ROTATION AND LORENTZ TRANSFORMATION OPERATORS

Rotations $a_1(\psi)$, $a_2(\psi)$ and $a_3(\psi)$ and Lorentz transformations $b_1(\psi)$, $b_2(\psi)$ and $b_3(\psi)$, around and along the axes OX^1, OX^2, OX^3 can be written as follows.

$$a_1(\psi) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos(\psi) & -\sin\psi \\ 0 & 0 & \sin\psi & \cos\psi \end{pmatrix},$$

$$a_2(\psi) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\psi & 0 & \sin\psi \\ 0 & 0 & 1 & 0 \\ 0 & -\sin\psi & 0 & \cos\psi \end{pmatrix},$$

$$a_3(\psi) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\psi & -\sin\psi & 0 \\ 0 & \sin\psi & \cos\psi & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

$$b_1(\psi) = \begin{pmatrix} \cosh\psi & \sinh\psi & 0 & 0 \\ \sinh\psi & \cosh\psi & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

$$b_2(\psi) = \begin{pmatrix} \cosh\psi & 0 & \sinh\psi & 0 \\ 0 & 1 & 0 & 0 \\ \sinh\psi & 0 & \cosh\psi & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

and $$b_3(\psi) = \begin{pmatrix} \cosh\psi & 0 & 0 & \sinh\psi \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \sinh\psi & 0 & 0 & \cosh\psi \end{pmatrix}.$$

They satisfy the relations

$$a_k(\psi_1)\ a_k(\psi_2) = a_k(\psi_1 + \psi_2)$$

$$b_k(\psi_1)\ b_k(\psi_2) = b_k(\psi_1 + \psi_2)$$

where $k = 1,2,3$.

The infinitesimal matrices a_1, a_2, a_3 and b_1, b_2, b_3 of the group L are defined by

$$a_k = \left.\frac{d\ a_k(\psi)}{d}\right|_{\psi=o}$$

$$b_k = \left.\frac{d\ b_k(\psi)}{d\psi}\right|_{\psi=o}$$

Explicitly they are given by

$$a_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad a_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix},$$

$$a_3 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

and $$b_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad b_2 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

and $$b_3 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

They are related to the rotations and Lorentz transformation $a_k(\psi)$ and $b_k(\psi)$ by

$$a_k(\psi) = \exp(\psi a_k),\ b_k = \exp(\psi b_k)$$

And they satisfy the commotation relations

$$[a_i\ a_j] = \varepsilon_{ijk}\ a_k$$

$$[b_i\ b_j] = \varepsilon_{ijk}\ a_k$$

$$[a_i, b_j] = \varepsilon_{ijk}\ b_k$$

Here $[a\ b] = ab - ba$. ε_{ijk} is antisymmetric tensor.

§3.2 DIFFERENTIAL OPERATORS CORRESPONDING TO INFINITESIMAL ROTATIONS

Let $g \to T_g$ by an irreducible representation of weight ℓ. The element T_{mn} of the matrix T_g $(-\ell \leq m, n \leq \ell)$ are the function of rotation g i.e. $[T_{mn}] = T_{mn}(g)$.

Multiply g by an arbitrary rotation g_1. Then $T_{mn}(g)$ goes into different function of g, which is equal to $T_{mn}(gg_1)$. This transformation of the function T_{mn} depend upon g_1, denoting it by Ug_1, we can write

$$Ug_1 \, T_{mn}(g) = T_{mn}(gg_1) \tag{3.2.1}$$

It can be verified that $Ug_2Ug_1 = Ug_2Ug_1$ holds. In fact $Ug_2Ug_1(T_{mn}(g)) = Ug_2 \, T_{mn}(gg_1) = T_{mn}(gg_1g_2)$

$$= Ug_2g_1(T_{mn}(g)) \tag{3.2.2}$$

Now, we consider the function $T_{mn}(gg_1)$. This is an element of the matrix Tgg_1. Since the matrix form a representation of the rotation group. Therefore it follows

$$Tgg_1 = TgTg_1$$

Equating the elements of the matrices on the left and right side of the equality

$$T_{mn}(gg_1) = \sum_{s=-\ell}^{s=\ell} T_{ms}(g) \, T_{sn}(g_1) \tag{3.2.3}$$

or equally

$$Ug_1 \, T_{mn}(g) = \sum_{s=-\ell}^{s=\ell} T_{ms}(g) \, T_{sn}(g_1) \tag{3.2.4}$$

It follows from (3.2.2) and (3.2.4) that for energy m the transformation Ug_1 comprise a $(2\ell + 1)$ dimensional representation of the rotation group.

Now we try to find the transformation A_k $(k = 1,2,3)$ which correspond to the infinitesimal rotation about the coordinate axes in the representation. For this purpose we take g_1 as the rotation through an angle α about the fixed axes. Then expand $Ug_1\ T_{mn}(g) = T_{mn}(gg_1)$ in the power of α.

Let OZ be the axes of rotation and g be an arbitrary rotation with Euler angles ϕ_1, θ, ϕ_2. Let g_1 be the rotation through an angle α about Z-axis. Then rotation gg_1 has Eulerian angles $(\phi_1 + \alpha, \theta, \phi_2)$.

$$T_{mn}(gg_1) = T_{mn}(\phi_1 + \alpha, \theta, \phi_2) + \alpha \frac{\partial T_{mn}}{\partial \phi_1} + \cdots$$

In general $T_{mn}(gg_1) = T_{mn}(\phi_1', \theta', \phi_2')$

has the form

$$T_{mn}(\phi_1', \theta', \phi_2') = T_{mn}(\phi_1, \theta, \phi_2)$$

$$+ \alpha \left\{ \frac{\partial T}{\partial \phi_1} \frac{d\phi_1'}{d\alpha} + \frac{\partial T_{mn}}{\partial \theta} \frac{d\theta'}{d\alpha} + \frac{\partial T_{mn}}{\partial \phi_2} \frac{d\phi_2'}{d\alpha} \right\}_{\alpha=0} \tag{3.2.5}$$

We next determine

$$\left.\frac{d\phi_1'}{d\alpha}\right|_{\alpha=0}, \quad \left.\frac{d\theta'}{d\alpha}\right|_{\alpha=0}, \quad \left.\frac{d\phi_2'}{d\alpha}\right|_{\alpha=0}$$

for the case in which g_1 is the rotation through an angle α about OX. For this purpose we consider the matrix of rotation g itself as the function of Euler angles.

$$g(\phi_1,\theta,\phi_2) = [g_{ik}(\phi_1,\theta,\phi_2)] = \begin{pmatrix} g_{11} & g_{12} & g_{13} \\ g_{21} & g_{22} & g_{23} \\ g_{31} & g_{32} & g_{33} \end{pmatrix}$$

or $\qquad g =$

$$\begin{pmatrix} \cos\phi_2\cos\phi_1 - \cos\theta\sin\phi_2\sin\phi_1 & -\cos\phi_2\sin\phi_1 - \cos\theta\sin\phi_2\cos\phi_1 & \sin\phi_2 \sin\theta \\ \sin\phi_2\cos\phi_1 + \cos\theta\cos\phi_2\sin\phi_1 & -\sin\phi_2\sin\phi_1 + \cos\theta\cos\phi_2\cos\phi_1 & -\cos\phi_2\sin\theta \\ \sin\phi_1 \sin\theta & \cos\phi_1 \sin\theta & \cos\theta \end{pmatrix} \tag{3.2.6}$$

The matrix of rotation gg_1 is given by certain values of Euler angles ϕ_1',θ',ϕ_2' and these values depend upon the rotation angle α. For $\alpha = o$, they are equal to ϕ_1,θ,ϕ_2. Expanding the matrix gg_1 in powers of α, we obtain

$$gg_1 = g_{ik}(\phi_1,\theta,\phi_2) + \alpha \left| \frac{\partial g_{ik}}{\partial \phi_1}\frac{d\phi_1}{d\alpha} + \frac{\partial g_{ik}}{\partial \theta}\frac{d\theta'}{d\alpha} + \frac{\partial g_{ik}}{\partial \phi_2}\frac{d\phi_2'}{d\alpha} \right|_{\alpha=o} + \cdots \tag{3.2.7}$$

Since g_1 is the rotation through an angle α about OX. Therefore its matrix values will be

$$g_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & -\sin\alpha \\ 0 & \sin\alpha & \cos\alpha \end{pmatrix}$$

And for the small values of α, we expand g_1 by Taylor series upto first order.

$$g_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & -\sin\alpha \\ 0 & \sin\alpha & \cos\alpha \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \alpha \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} + \cdots$$

And consequently

$$gg_1 = g_{ik}(\phi_1, \theta, \phi_2) + \alpha \begin{pmatrix} 0 & g_{13} & -g_{12} \\ 0 & g_{23} & -g_{22} \\ 0 & g_{33} & -g_{32} \end{pmatrix} \qquad (3.2.8)$$

Equating the expression (3.2.7) and (3.2.8) for the matrix gg_1 and equating the coefficients of α in these two expressions we obtain equations for which

$$\left.\frac{d\phi_1'}{d\alpha}\right|_{\alpha=0}, \quad \left.\frac{d\theta'}{d\alpha}\right|_{\alpha=0} \quad \text{and} \quad \left.\frac{d\phi_2'}{d\alpha}\right|_{\alpha=0}$$

are defined.

It is sufficient for our purpose to take the three simplest elements of the matrices by which α is multiplied in formula (3.2.7) and (3.2.8). We use the lowest right, the lower left, and the upper right and upper left elements. Taking the corresponding expressions for g_{ik} from formula (3.2.6) and differentiating these, we obtain the equations

$$-\sin\theta \left.\frac{d\theta'}{d\alpha}\right|_{\alpha=o} = -\cos\phi_1 \sin\theta$$

$$\cos\phi_1 \sin\theta \left.\frac{d\phi_1'}{d\alpha}\right|_{\alpha=o} + \sin\phi_1 \cos\theta \left.\frac{d\theta'}{d\alpha}\right|_{\alpha=o} = 0$$

$$\cos\phi_2 \sin\theta \left.\frac{d\phi'}{d\alpha}\right|_{\alpha=o} + \sin\phi_2 \cos\theta \left.\frac{d\theta'}{d\alpha}\right|_{\alpha=o}$$

$$= \cos\phi_2 \sin\phi_1 + \cos\theta \sin\phi_2 \cos\phi_1$$

From these equations, we infer that

$$\left.\frac{d\theta'}{d\alpha}\right|_{\alpha=o} = \cos\phi_1 \;,\; \left.\frac{d\phi_1'}{d\alpha}\right|_{\alpha=o} = -\sin\phi_1 \cot\theta$$

$$\left.\frac{d\phi_2'}{d\alpha}\right|_{\alpha=o} = \frac{\sin\phi_1}{\sin\theta}$$

Substituting these values in formula (   ) we find the differential operator which corresponds to an infinitesimal rotation about the axis OX

$$A_1 = -\cot\theta \sin\phi_1 \frac{\partial}{\partial\phi_1} + \frac{\sin\phi_1}{\sin\theta} \frac{\partial}{\partial\phi_2} + \cos\phi_1 \frac{\partial}{\partial\theta} \qquad (3.2.9)$$

The operators A_2 and A_3 are computed in much the same way, they have the form

$$A_2 = -\cot\theta \cos\phi_1 \frac{\partial}{\partial\phi_1} + \frac{\cos\phi_1}{\sin\theta} \frac{\partial}{\partial\phi_2} - \sin\phi_1 \frac{\partial}{\partial\theta} \qquad (3.2.10)$$

and $A_3 = \frac{\partial}{\partial\phi_1}$ (3.2.11)

§3.3 DIFFERENTIAL OPERATORS CORRESPONDING TO ROTATIONS

In Chapter Two an isomorphism have been developed between complex orthogonal group and Lorentz group. This isomorphism can be used to calculate the operators which corresponds to complex rotations by calculating operators corresponding to rotations and Lorentz transformations.

For this purpose the method mentioned in the last section can be used. In this case arbitrary rotation is g given by the equation (2.5.5).

Let OZ be the axis of rotation and g_1 be an arbitrary rotation through an angle α, then it can be written as in the last section,

$$gg_1 = \left| g_{1k}(\phi,\theta,\psi) \right| + \alpha \left| \frac{\partial g_{1k}}{\partial \psi_1}\frac{d\psi_1}{d\alpha} + \frac{\partial g_{1k}}{\partial \psi_2}\frac{d\psi}{d\alpha} + \frac{\partial g_{1k}}{\partial \theta_1}\frac{d\theta_1}{d\alpha} + \frac{\partial g_{1k}}{\partial \theta_2}\frac{d\theta_2}{d\alpha} + \frac{\partial g_{1k}}{\partial \phi_1}\frac{d\phi_1}{d\alpha} + \frac{\partial g_{1k}}{\partial \phi_2}\frac{d\phi_2}{d\alpha} \right| \tag{3.3.1}$$

On the other hand since g_1 is the rotation through an angle α around X-axis it axis is equal to

$$g_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos\alpha & -\sin\alpha \\ 0 & 0 & \sin\alpha & \cos\alpha \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} + \alpha \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \tag{3.3.2}$$

Consequently

$$gg_1 = \left|g_{1k}(\psi,\theta,\phi)\right| + \alpha \begin{vmatrix} 0 & 0 & g_{14} & -g_{13} \\ 0 & 0 & g_{24} & -g_{23} \\ 0 & 0 & g_{34} & -g_{33} \\ 0 & 0 & g_{44} & -g_{44} \end{vmatrix}$$

Equating the epxressions (3.3.1) and (3.3.3) for the matrix gg_1 and equating the coefficients of α in these two expressions, we obtain equations from which

$$\left.\frac{d\psi_1}{d\alpha}\right|_{\alpha=o}, \left.\frac{d\psi_2}{d\alpha}\right|_{\alpha=o}, \left.\frac{d\theta_1}{d\alpha}\right|_{\alpha=o}, \frac{d\theta_2}{d\alpha}, \frac{d\phi_1}{d\alpha} \text{ and } \left.\frac{d\phi_2}{d\alpha}\right|_{\alpha=o}$$

are defined.

For this purpose we equate all the elements of the matrices by which α is multiplied in formula (3.3.1) and (3.3.3).

When we will do this, we get sixteen equations, eluminating from all these equations $d\theta_1/d\alpha$ and $d\theta_2/d\alpha$ we obtain a set of sixteen equation which are given below:

$$\sin\theta_1 \sin\phi_2 \frac{d\psi_1}{d\alpha} + \cosh\theta_2 \cosh\phi_2 \frac{d\theta_2}{d\alpha} + \sinh\theta_2 \sinh\phi_2 \frac{d\phi_2}{d\alpha} = 0$$

$$-\cos\theta_1 \sin\phi_1 \frac{d\psi_1}{d\alpha} + \sinh\theta_2 \cos\phi_1 \frac{d\theta_2}{d\alpha} - \cosh\theta_2 \sin\phi_1 \frac{d\phi_1}{d\alpha} = 0$$

$$\cos\theta_1 \sinh\phi_2 \frac{d\psi_2}{d\alpha} + \sinh\theta_2 \cosh\phi_2 \frac{d\theta_2}{d\alpha} + \cosh\theta_2 \sinh\phi_2 \frac{d\phi_2}{d\alpha} = 0$$

$$-\sin\theta_1 \sin\phi_1 \frac{d\psi_2}{d\alpha} + \cosh\theta_2 \cos\phi_1 \frac{d\theta_2}{d\alpha} - \sinh\theta_2 \sin\phi_1 \frac{d\phi_1}{d\alpha} = 0$$

$$-\sinh\theta_2\cosh\phi_2\frac{d\psi_1}{d\alpha}+\cos\theta_1\sinh\phi_2\frac{d\theta_1}{d\alpha}+\sin\theta_1\cosh\phi_2\frac{d\phi_2}{d\alpha}=0$$

$$-\cosh\theta_2\cosh\phi_2\frac{d\psi_2}{d\alpha}+\sin\theta_1\sinh\phi_2\frac{d\theta_1}{d\alpha}-\cos\theta_1\cosh\phi_2\frac{d\phi_2}{d\alpha}=0$$

$$-\cosh\theta_2\cos\phi_1\frac{d\psi_1}{d\alpha}+\sin\theta_1\sin\phi_1\frac{d\theta_1}{d\alpha}-\cos\theta_1\cos\phi_1\frac{d\phi_1}{d\alpha}=0$$

$$-\sinh\theta_2\cos\phi_1\frac{d\psi_2}{d\alpha}-\cos\theta_1\sin\phi_1\frac{d\theta_1}{d\alpha}-\sin\theta_1\cos\phi_1\frac{d\phi_1}{d\alpha}=0$$

$$\sin\theta_1\cos\phi_1\frac{d\psi_2}{d\alpha}-\cosh\theta_2\sin\phi_1\frac{d\theta_2}{d\alpha}-\sinh\theta_2\cos\phi_1\frac{d\phi_1}{d\alpha}=\cosh\theta_2\sinh\phi_2$$

$$-\cos\theta_1\cos\phi_1\frac{d\psi_1}{d\alpha}-\sinh\theta_2\sin\phi_1\frac{d\theta_2}{d\alpha}-\cosh\theta_2\cos\phi_1\frac{d\phi_1}{d\alpha}=\sinh\theta_2\sinh\phi_2$$

$$\cos\theta_1\cosh\phi_2\frac{d\psi_2}{d\alpha}+\sinh\theta_2\sinh\phi_2\frac{d\theta_2}{d\alpha}+\cosh\theta_2\cosh\phi_2\frac{d\phi_2}{d\alpha}=\sinh\theta_2\sin\phi_1$$

$$\sin\theta_1\cosh\phi_2\frac{d\psi_1}{d\alpha}+\cosh\theta_2\sinh\phi_2\frac{d\theta_2}{d\alpha}+\sinh\theta_2\cosh\phi_2\frac{d\phi_2}{d\alpha}=\cosh\theta_2\sin\phi_1$$

$$\sinh\theta_2\sin\phi_1\frac{d\psi_2}{d\alpha}-\cos\theta_1\cos\phi_1\frac{d\theta_1}{d\alpha}+\sin\theta_1\sin\phi_1\frac{d\phi_1}{d\alpha}=-\cos m_1\cosh\phi_2$$

$$\cosh\theta_2\sin\phi_1\frac{d\psi_1}{d\alpha}+\sin\theta_1\cos\phi_1\frac{d\theta_1}{d\alpha}+\cos\theta_1\sin\phi_1\frac{d\phi_1}{d\alpha}=\sin\theta_1\cosh\phi_2$$

$$-\cosh\theta_2\sinh\phi_2\frac{d\psi_2}{d}+\sin\theta_1\cosh\phi_2\frac{d\theta_1}{d}-\cos\theta_1\sinh\phi_2\frac{d\phi_2}{d}=\sin\theta_1\cos\phi_1$$

$$-\sinh \theta_2 \sinh \phi_2 \frac{d\psi_1}{d\alpha} + \cos \theta_1 \cosh \phi_2 \frac{d\theta_1}{d\alpha} + \sin \theta_1 \sinh \phi_2 \frac{d\phi_2}{d\alpha}$$

$$= \cos \theta_1 \cos \phi_1 \qquad (3.3.4)$$

Selecting one equation from the first equations of (3.3.4) involving the variables $d\psi_1/d\alpha$, $d\theta_1/d\alpha$ and $d\phi_1/d\alpha$ and one equation from the second eight equations involving the same variables

$$\cosh \theta_2 \sin \phi_1 \frac{d\psi_1}{d\alpha} + \sin \theta_1 \cos \phi_1 \frac{d\theta_1}{d\alpha} + \cos \theta_1 \sin \phi_1 \frac{d\phi_1}{d\alpha}$$

$$= \sin \theta_1 \cosh \phi_2$$

$$-\cosh \theta_2 \cos \phi_1 \frac{d\psi_1}{d\alpha} + \sin \theta_1 \sin \phi_1 \frac{d\theta_1}{d\alpha} - \cos \theta_1 \cos \phi_1 \frac{d\phi_1}{d\alpha} = 0$$

Multiplying the first equation by $\cos \phi_1$, second by $\sin \phi_1$ and adding

$$\sin \theta_1 \frac{d\theta_1}{d\alpha} = \cos \phi_1 \sin \theta_1 \cosh \phi_2$$

$$\frac{d\theta_1}{d\alpha} = \cos \phi_1 \cosh \phi_2$$

As is done in the above selecting two equations for variables $\frac{d\psi_2}{d\alpha}$, $\frac{d\theta_2}{d\alpha}$ and $\frac{d\phi_2}{d\alpha}$

$$\cos \theta_1 \sinh \phi_2 \frac{d\psi_2}{d\alpha} + \sinh \theta_2 \cosh \phi_2 \frac{d\theta_2}{d\alpha} + \cosh \theta_2 \sinh \phi_2 \frac{d\phi_2}{d\alpha} = 0$$

$$\cos \theta_1 \cosh \phi_2 \frac{d\psi_2}{d\alpha} + \sinh \theta_2 \sinh \phi_2 \frac{d\theta_2}{d\alpha} + \cosh \theta_2 \cosh \phi_2 \frac{d\phi_2}{d\alpha}$$

$$= \sinh \theta_2 \sin \phi_1$$

Multiplying first by $\cosh \phi_2$, second by $\sinh \phi_2$ and substracting;

$$\frac{d\theta_2}{d\alpha} = -\sin\phi_1 \sinh\phi_2$$

Selecting one equation from the first eight equations of (3.3.4) involving the variables $d\psi_2/d\alpha$, $d\theta_2/d\alpha$ and $d\phi_1/d\alpha$ and one equation from the second eight equations involving the same variables

$$\sin\theta_1 \sin\phi_1 \frac{d\psi_2}{d\alpha} + \cosh\theta_2 \cos\phi_1 \frac{d\theta_2}{d\alpha} - \sinh\theta_2 \sin\phi_1 \frac{d\phi_1}{d\alpha} = 0$$

$$\sin\theta_1 \cos\phi_1 \frac{d\psi_2}{d\alpha} - \cosh\theta_2 \sin\phi_1 \frac{d\theta_2}{d\alpha} - \sinh\theta_2 \cos\phi_1 \frac{d\phi_1}{d\alpha}$$

$$= \cosh\theta_2 \sinh\phi_2$$

Multiplying first equation by $\sin\phi_1$, second by $\cos\phi_1$ and adding

$$\sin\phi_1 \frac{d\psi_2}{d\alpha} - \sinh\theta_2 \frac{d\phi_1}{d\alpha} = \cos\phi_1 \sinh\theta_2 \sinh\phi_2 \tag{3.3.5}$$

As is done in the above selecting two equations from (3.3.4) for variables $d\psi_2/d\alpha$, $d\theta_1/d\alpha$ and $d\phi_1/d\alpha$

$$\sinh\theta_2 \sin\phi_1 \frac{d\psi_2}{d\alpha} - \cos\theta_1 \cos\phi_1 \frac{d\theta_1}{d\alpha} + \sin\theta_1 \sin\phi_1 \frac{d\phi_1}{d\alpha}$$

$$= -\cos\theta_1 \cosh\phi_2$$

$$-\sinh\theta_2 \cos\phi_1 \frac{d\psi_2}{d\alpha} - \cos\theta_1 \sin\phi_1 \frac{d\theta_1}{d\alpha} - \sin\theta_1 \cos\phi_1 \frac{d\phi_1}{d\alpha} = 0$$

Multiplying first equation by $\sin\phi_1$ and second by $\cos\phi_1$ and substracting

$$-\sinh\theta_2 \frac{d\psi_2}{d\alpha} + \sin\theta_1 \frac{d\phi_1}{d\alpha} = -\cos\theta_1 \sin\phi_1 \cosh\phi_2 \tag{3.3.6}$$

Multiplying (3.3.5) by sin θ_1, (3.3.6) by sinh θ_2 and substracting

$$\frac{d\phi_1}{d\alpha} = - \frac{1}{(\sin^2\theta_1 + \sinh^2\theta_2)} [\sin\theta_1 \cos\theta_1 \sin\phi_1 \cosh\phi_2 + \sinh\theta_2 \cosh\theta_2 \cos\phi_1 \sinh\phi_2]$$

Multiplying (3.3.5) by sinh θ_2, (3.3.6) by sin θ_1 and adding

$$\frac{d\psi_2}{d\alpha} = - \frac{1}{(\sin^2\theta_1 + \sinh^2\theta_2)} [\cos\theta_1 \sinh\theta_2 \sin\phi_1 \cosh\phi_2 - \sin\theta_1 \cosh\theta_2 \cos\phi_1 \sinh\phi_2]$$

Selecting two equations from (3.3.4) for the variables $d\psi_1/d\alpha$, $d\theta_2/d\alpha$ and $d\phi_2/d\alpha$

$$\sin\theta_1 \sinh\phi_2 \frac{d\psi_1}{d\alpha} + \cosh\theta_2 \cosh\phi_2 \frac{d\theta_2}{d\alpha} + \sinh\theta_2 \sinh\phi_2 \frac{d\phi_2}{d\alpha} = 0$$

$$\sin\theta_1 \cosh\phi_2 \frac{d\psi_1}{d\alpha} + \cosh\theta_2 \sinh\phi_2 \frac{d\theta_2}{d\alpha} + \sinh\theta_2 \cosh\phi_2 \frac{d\phi_2}{d\alpha} = \cosh\theta_2 \sin\phi_1$$

Multiplying first equation by sinh ϕ_2, second by cosh ϕ_2 and substracting

$$-\sin\theta_1 \frac{d\psi_1}{d\alpha} - \sinh\theta_2 \frac{d\phi_2}{d\alpha} = -\cosh\theta_2 \sin\phi_1 \cosh\phi_2 \qquad (3.3.7)$$

Selecting two equations from (3.3.4) for the variables $d\psi_1/d\alpha$, $d\theta_1/d\alpha$ and $d\phi_2/d\alpha$

$$-\sinh\theta_2 \cosh\phi_2 \frac{d\psi_1}{d\alpha} + \cos\theta_1 \sinh\phi_2 \frac{d\theta_1}{d\alpha} + \sin\theta_1 \cosh\phi_2 \frac{d\phi_2}{d\alpha} = 0$$

$$-\sinh\theta_2 \sinh\phi_2 \frac{d\psi_1}{d\alpha} + \cos\theta_1 \cosh\phi_2 \frac{d\theta_1}{d\alpha} + \sin\theta_1 \sinh\phi_2 \frac{d\phi_2}{d\alpha} = \cos\theta_1 \cos\phi_1$$

Multiplying first equation by cosh ϕ_2, second by sinh ϕ_2 and substracting

$$-\sinh\theta_2 \frac{d\psi_1}{d\alpha} + \sin\theta_1 \frac{d\phi_2}{d\alpha} = -\cos\theta_1 \cos\phi_1 \sinh\phi_2 \qquad (3.3.8)$$

Multiplying (3.3.7) by sin θ_1, (3.3.8) by sinh θ_2 and adding

$$\frac{d\psi_1}{d\alpha} = \frac{1}{(\sin^2\theta_1 + \sinh^2\theta_2)} [\sin\theta_1 \cosh\theta_2 \sin\phi_1 \cosh\phi_2 + \sinh\theta_2 \cos\theta_1 \cos\phi_1 \sinh\phi_2]$$

Multiplying (3.3.7) by sinh θ_2, (3.3.8) by sin θ_1 and substracting

$$\frac{d\phi_2}{d\alpha} = -\frac{1}{(\sin^2\theta_1 + \sin^2\theta_2)} [\sin\theta_1 \cos\theta_1 \cos\phi_1 \sinh\phi_2 - \sinh\theta_2 \cosh\theta_2 \sin\phi_1 \cosh\phi_2]$$

Hence the operator corresponding to rotation about X-axis is

$$A_1 = \frac{1}{(\sin^2\theta_1 + \sinh^2\theta_2)} [\sin\theta_1\cosh\theta_2\sin\phi_1\cosh\phi_2 + \sinh\theta_2\cos\theta_1\cos\phi_1\sinh\phi_2] \frac{\partial}{\partial\psi_1} - \frac{1}{(\sin^2\theta_1 + \sin^2\theta_2)}$$
$$[\cos\theta_1\sinh\theta_2\sin\phi_1\cosh\phi_2 - \sin\theta_1\cosh\theta_2\cos\phi_1\sinh\phi_2] \frac{\partial}{\partial\psi_2}$$
$$+ \cos\phi_1 \cosh\phi_2 \frac{\partial}{\partial\phi_1} - \sin\phi_1 \sinh\phi_2 \frac{\partial}{\partial\theta_2}$$
$$- \frac{1}{(\sin^2\theta_1 + \sinh^2\theta_2)}[\sin\theta_1\cos\theta_1\sin\phi_1\cosh\phi_2 + \sinh\theta_2\cosh\phi_2$$
$$\cos\phi_1\sinh\phi_2] \frac{\partial}{\partial\phi_1} - \frac{1}{(\sin^2\theta_1 + \sinh^2\theta_2)} [\sin\theta_1\cos\theta_1\cos\phi_1\sinh\phi_2$$
$$- \sinh\theta_2\cosh\theta_2\sin\phi_1\cosh\phi_2] \frac{\partial}{\partial\phi_2}$$

Similarly the operators A_2 and A_3, corresponding to rotations about X and Y axes respectively can be computed.

§3.4 OPERATORS CORRESPONDING TO LORENTZ TRANSFORMATIONS

Let us calculate the operator corresponding to Lorentz transformation under which the variable x is transformed. In this case our g_1 will be

$$g_1 = \begin{pmatrix} \cosh\alpha & \sinh\alpha & 0 & 0 \\ \sinh\alpha & \cosh\alpha & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} + \alpha \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} + \cdots$$

Therefore

$$gg_1 = \left| g_{ik}(\phi,\theta,\psi) \right| + \begin{pmatrix} g_{12} & g_{11} & 0 & 0 \\ g_{22} & g_{21} & 0 & 0 \\ g_{32} & g_{31} & 0 & 0 \\ g_{42} & g_{41} & 0 & 0 \end{pmatrix} + \cdots \qquad (3.4.1)$$

Equating the expressions (3.3.1) and (3.4.1) for the matrix gg_1 and equating the coefficients of α in these two expressions we obtain equations from which

$\left.\frac{d\psi_1}{d\alpha}\right|_{\alpha=0}$, $\left.\frac{d\psi_2}{d\alpha}\right|_{\alpha=0}$, $\left.\frac{d\theta_1}{d\alpha}\right|_{\alpha=0}$, $\left.\frac{d\theta_2}{d\alpha}\right|_{\alpha=0}$, $\left.\frac{d\phi_1}{d\alpha}\right|_{\alpha=0}$ and $\left.\frac{d\phi_2}{d\alpha}\right|_{\alpha=0}$ are defined.

For this purpose we equate all the elements of the matrices by which α is multiplied in formula (3.3.1) and (3.4.1) When we do this, we get sixteen equations, eluminating from these equations $\frac{d\theta_1}{d\alpha}$ and $\frac{d\theta_2}{d\alpha}$, we obtain a set of sixteen equations.

$$\sin\theta_1 \sinh\phi_2 \frac{d\psi_1}{d\alpha} + \cosh\theta_2 \cosh\phi_2 \frac{d\theta_2}{d\alpha} + \sinh\theta_2 \sinh\phi_2 \frac{d\phi_2}{d\alpha} = \cosh\theta_2 \cos\phi_1$$

$$\cos\theta_1 \sinh\phi_2 \frac{d\psi_2}{d\alpha} + \sinh\theta_2 \cosh\phi_2 \frac{d\theta_2}{d\alpha} + \cosh\theta_2 \sinh\phi_2 \frac{d\phi_2}{d\alpha} = \sinh\theta_2 \cos\phi_1$$

$$-\cos\theta_1 \sin\phi_1 \frac{d\psi_1}{d\alpha} + \sinh\theta_2 \cos\phi_1 \frac{d\theta_2}{d\alpha} - \cosh\theta_2 \sin\phi_1 \frac{d\phi_1}{d\alpha} = \sinh\theta_2 \cosh\phi_2$$

$$\sin\theta_1 \sin\phi_1 \frac{d\psi_2}{d\alpha} + \cosh\theta_2 \cos\phi_1 \frac{d\theta_2}{d\alpha} - \sinh\theta_2 \sin\phi_1 \frac{d\phi_1}{d\alpha} = \cosh\theta_2 \cosh\phi_2$$

$$-\sinh\theta_2 \cosh\phi_2 \frac{d\psi_1}{d\alpha} + \cos\theta_1 \sinh\phi_2 \frac{d\theta_1}{d\alpha} + \sin\theta_1 \cosh\phi_2 \frac{d\phi_2}{d\alpha} = \cos\theta_1 \sin\phi_1$$

$$-\cosh\theta_2 \cosh\phi_2 \frac{d\psi_2}{d\alpha} + \sin\theta_1 \sinh\phi_2 \frac{d\theta_1}{d\alpha} - \cos\theta_1 \cosh\phi_2 \frac{d\phi_2}{d\alpha} = -\cos\theta_1 \sin\phi_1$$

$$-\cosh\theta_2 \cos\phi_1 \frac{d\psi_1}{d\alpha} + \sin\theta_1 \sin\phi_1 \frac{d\theta_1}{d\alpha} - \cos\theta_1 \cos\phi_1 \frac{d\phi_1}{d\alpha} = \sin\theta_1 \sinh\phi_2$$

$$-\sinh\theta_2 \cos\phi_1 \frac{d\psi_2}{d\alpha} - \cos\theta_1 \sin\phi_1 \frac{d\theta_1}{d\alpha} - \sin\theta_1 \cos\phi_1 \frac{d\phi_1}{d\alpha} = -\cos\theta_1 \sinh\phi_2$$

$$-\sin\theta_1 \cos\phi_1 \frac{d\psi_2}{d\alpha} - \cosh\theta_2 \sin\phi_1 \frac{d\theta_2}{d\alpha} - \sinh\theta_2 \cos\phi_1 \frac{d\phi_1}{d\alpha} = 0$$

$$\cos\theta_1 \cos\phi_1 \frac{d\psi_1}{d\alpha} - \sinh\theta_2 \sin\phi_1 \frac{d\theta_2}{d\alpha} - \cosh\theta_2 \cos\phi_1 \frac{d\phi_1}{d\alpha} = 0$$

$$\cos\theta_1 \cosh\phi_2 \frac{d\psi_2}{d\alpha} + \sinh\theta_2 \sinh\phi_2 \frac{d\theta_2}{d\alpha} + \cosh\theta_2 \cosh\phi_2 \frac{d\phi_2}{d\alpha} = 0$$

$$\sin\theta_1 \cosh\phi_2 \frac{d\psi_1}{d\alpha} + \cosh\theta_2 \sinh\phi_2 \frac{d\theta_2}{d\alpha} + \sinh\theta_2 \cosh\phi_2 \frac{d\phi_2}{d\alpha} = 0$$

$$\sinh\theta_2 \sin\phi_1 \frac{d\psi_2}{d\alpha} - \cos\theta_1 \cos\phi_1 \frac{d\theta_1}{d\alpha} + \sin\theta_1 \sin\phi_1 \frac{d\phi_1}{d\alpha} = 0$$

$$\cosh\theta_2 \sin\phi_1 \frac{d\psi_1}{d\alpha} + \sin\theta_1 \cos\phi_1 \frac{d\theta_1}{d\alpha} + \cos\theta_1 \sin\phi_1 \frac{d\phi_1}{d\alpha} = 0$$

$$-\cosh\theta_2 \sinh\phi_2 \frac{d\psi_2}{d\alpha} + \sin\theta_1 \cosh\phi_2 \frac{d\theta_1}{d\alpha} - \cos\theta_1 \sinh\phi_2 \frac{d\phi_2}{d\alpha} = 0$$

$$-\sinh m_2 \sinh\phi_2 \frac{d\psi_1}{d\alpha} + \cos\theta_1 \cosh\phi_2 \frac{d\theta_1}{d\alpha} + \sin\theta_1 \sinh\phi_2 \frac{d\phi_2}{d\alpha} = 0 \qquad (3.4.2)$$

Selecting one equation from the first eight equations of (3.4.2) involving the variables $\frac{d\psi_1}{d\alpha}$, $\frac{d\theta_1}{d\alpha}$, $\frac{d\phi_1}{d\alpha}$ and one equation from the second eight equations involving the same variables.

$$-\cosh\theta_2 \cos\phi_1 \frac{d\psi_1}{d\alpha} + \sin\theta_1 \sin\phi_1 \frac{d\theta_1}{d\alpha} - \cos\theta_1 \cos\phi_1 \frac{d\phi_1}{d\alpha} = \sin\theta_1 \sinh\phi_2$$

$$\cosh\theta_2 \sin\phi_1 \frac{d\psi_1}{d\alpha} + \sin\theta_1 \cos\phi_1 \frac{d\theta_1}{d\alpha} + \cos\theta_1 \sin\phi_1 \frac{d\phi_1}{d\alpha} = 0$$

Multiplying first by $\sin\phi_1$, second by $\cos\phi_1$ and adding

$$\frac{d\theta_1}{d\alpha} = \sin\phi_2 \sin\phi_1$$

As is done in the above, selecting the equations for the variables $\frac{d\psi_2}{d\alpha}$, $\frac{d\theta_2}{d\alpha}$ $\frac{d\phi_2}{d\alpha}$

$$\cos\theta_1 \cosh\phi_2 \frac{d\psi_2}{d\alpha} + \sinh\theta_2 \sinh\phi_2 \frac{d\theta_2}{d\alpha} + \cosh\theta_2 \cosh\phi_2 \frac{d\phi_2}{d\alpha} = 0$$

$$\cos\theta_1 \sinh\phi_2 \frac{d\psi_2}{d\alpha} + \sinh\theta_2 \cosh\phi_2 \frac{d\theta_2}{d\alpha} + \cosh\theta_2 \sinh\phi_2 \frac{d\phi_2}{d\alpha} = \sinh\theta_2 \sin\phi_1$$

Multiplying first equation by $\sinh\phi_2$, second by $\cosh\phi_2$ and substracting

$$\frac{d\theta_2}{d\alpha} = \cos\phi_1 \cosh\phi_2 \ .$$

Selecting two equations for the variables $\frac{d\psi_2}{d\alpha}$, $\frac{d\theta_2}{d\alpha}$ and $\frac{d\phi_1}{d\alpha}$ from (3.4.2)

$$\sin\theta_1 \cos\phi_1 \frac{d\psi_2}{d\alpha} - \cosh\theta_2 \sin\phi_1 \frac{d\theta_2}{d\alpha} - \sinh\theta_2 \cos\phi_1 \frac{d\phi_1}{d\alpha} = 0$$

$$\sin\theta_1 \sin\phi_1 \frac{d\psi_2}{d\alpha} - \cosh\theta_2 \cos\phi_1 \frac{d\theta_2}{d\alpha} - \sinh\theta_2 \sin\phi_1 \frac{d\phi_1}{d\alpha} = \cosh\theta_2 \cosh\phi_2$$

Multiplying first equation by $\cos\phi_1$, second by $\sin\phi_1$ and adding

$$\sin\theta_1 \frac{d\psi_2}{d\alpha} - \sinh\theta_2 \frac{d\phi_1}{d\alpha} = \cosh\theta_2 \sin\phi_1 \cosh\phi_2 \qquad (3.4.3)$$

Selecting two equations for the variables $\frac{d\psi_2}{d\alpha}$, $\frac{d\theta_1}{d\alpha}$ and $\frac{d\phi_1}{d\alpha}$ from (3.4.2)

$$-\sinh\theta_2 \cos\phi_1 \frac{d\psi_2}{d\alpha} - \cos\theta_1 \sin\phi_1 \frac{d\theta_1}{d\alpha} - \sin\theta_1 \cos\phi_1 \frac{d\phi_1}{d\alpha} = -\cos\theta_1 \sinh\phi_2$$

$$\sinh\theta_2 \sin\phi_1 \frac{d\psi_2}{d\alpha} - \cos\theta_1 \cos\phi_1 \frac{d\theta_1}{d\alpha} + \sin\theta_1 \sin\phi_1 \frac{d\phi_1}{d\alpha} = 0$$

Multiplying first equation by $\cos\phi_1$, second by $\sin\phi_1$ and substracting

$$-\sinh\theta_2 \frac{d\psi_2}{d\alpha} - \sin\theta_1 \frac{d\phi_1}{d\alpha} = -\cos\theta_1 \cos\phi_1 \sinh\phi_2 \qquad (3.4.4)$$

Multiplying (3.4.3) by $\sin\theta_1$, (3.4.4) by $\sinh\theta_2$ and adding

$$\frac{d\phi_1}{d\alpha} = \frac{1}{(\sin^2\theta_1 + \sinh^2\theta_2)} [\sin\theta_1 \cos\theta_1 \cos\phi_1 \sinh\phi_2 - \sinh\theta_2 \cosh\theta_2 \sin\phi_1 \cosh\phi_2]$$

Multiplying (3.4.3) by $\sinh\theta_2$, (3.4.4) by $\sin\theta_1$ and substracting

$$\frac{d\psi_2}{d\alpha} = \frac{1}{(\sin^2\theta_1 + \sinh^2\theta_2)} [\cos\theta_1 \sinh\theta_2 \cos\phi_1 \sinh\phi_2 + \sin\theta_1 \cosh\theta_2 \sin\phi_1 \cosh\phi_2]$$

Selecting two equations for the variables $\frac{d\psi_1}{d\alpha}$ $\frac{d\theta_2}{d\alpha}$ and $\frac{d\phi_2}{d\alpha}$ from (3.4.2)

$$\sin\theta_1 \sinh\phi_2 \frac{d\psi_1}{d\alpha} + \cosh\theta_2 \cosh\phi_2 \frac{d\theta_2}{d\alpha} + \sinh\theta_2 \sinh\phi_2 \frac{d\phi_2}{d\alpha} = \cosh\theta_2 \cos\phi_1$$

$$\sin\theta_1 \cosh\phi_2 \frac{d\psi_1}{d\alpha} + \cosh\theta_2 \sinh\phi_2 \frac{d\theta_2}{d\alpha} + \sinh\theta_2 \cosh\phi_2 \frac{d\phi_2}{d\alpha} = 0$$

Multiplying first equation by $\sin\phi_2$, second equation by $\cosh\phi_2$ and substracting

$$-\sin\theta_1 \frac{d\psi_1}{d\alpha} - \sinh\phi_2 \frac{d\phi_2}{d\alpha} = \cosh\theta_2 \cos\phi_1 \sinh\phi_2 \qquad (3.4.5)$$

Selecting two equations for the variables $\frac{d\psi_1}{d\alpha}$, $\frac{d\theta_1}{d\alpha}$ and $\frac{d\phi_2}{d\alpha}$ from (3.4.2)

$$-\sinh\theta_2 \cosh\phi_2 \frac{d\psi_1}{d\alpha} + \cos\theta_1 \sinh\phi_2 \frac{d\theta_1}{d\alpha} + \sin\theta_1 \cosh\phi_2 \frac{d\phi_2}{d\alpha} = -\cos\theta_1 \sin\phi_1$$

$$-\sinh\theta_2 \sinh\phi_2 \frac{d\psi_1}{d\alpha} + \cos\theta_1 \cosh\phi_2 \frac{d\theta_1}{d\alpha} + \sin\theta_1 \sinh\phi_2 \frac{d\phi_2}{d\alpha} = 0$$

Multiplying first equation by $\cosh\phi_2$, second by $\sinh\phi_2$ and substracting

$$-\sinh\theta_2 \frac{d\psi_1}{d\alpha} + \sin\theta_1 \frac{d\phi_2}{d\alpha} = -\cos\theta_1 \sin\phi_1 \cosh\phi_2 \qquad (3.4.6)$$

Multiplying (3.4.5) by $\sin\theta_1$, (3.4.6) by $\sinh\theta_2$ and substracting

$$\frac{d\phi_2}{d\alpha} = \frac{1}{(\sin^2\theta_1 + \sinh^2\theta_2)} [\sin\theta_1 \cos\theta_1 \sin\phi_1 \cosh\phi_2 + \sinh\theta_2 \cosh\theta_2 \cos\phi_1 \sinh\phi_2]$$

Multiplying (3.4.5) by $\sinh\theta_2$, (3.4.6) by $\sin\theta_1$ and adding

$$\frac{d\psi_1}{d\alpha} = \frac{1}{(\sin^2\theta_1 + \sinh^2\theta_2)} [\cos\theta_1 \sinh\theta_2 \sin\phi_1 \cosh\phi_2 - \sin\theta_1 \cosh\theta_2 \cos\phi_1 \sinh\phi_2]$$

Hence the operator corresponding to Lorentz transformation under which the variable x is transformed is:

$$B_1 = \frac{1}{(\sin^2\theta_1 + \sinh^2\theta_2)} [\cos\theta_1 \sinh\theta_2 \sin\phi_1 \cosh\phi_2$$

$$- \sin\theta_1 \cosh\theta_2 \cos\phi_1 \sinh\phi_2] \frac{\partial}{\partial\psi_1} + \frac{1}{(\sin^2\theta_1 + \sinh^2\theta_2)}$$

$$[\cos\theta_1 \sinh\theta_2 \cos\phi_1 \sinh\phi_2 + \sin\theta_1 \cosh\theta_2 \sin\phi_1 \cosh\phi_2]$$

$$\frac{\partial}{\partial\psi_2} + \sinh\phi_2 \sin\phi_1 \frac{\partial}{\partial\theta_1} + \cos\phi_1 \cosh\phi_2 \frac{\partial}{\partial\theta_2}$$

$$+ \frac{1}{(\sin^2\theta_1 + \sinh^2\theta_2)} [\sin\theta_1 \cos\theta_1 \cos\phi_1 \sinh\phi_2$$

$$- \sinh\theta_2 \cosh\theta_2 \sin\phi_1 \cosh\phi_2] \frac{\partial}{\partial\phi_1}$$

$$- \frac{1}{(\sin^2\theta_1 + \sinh^2\theta_2)} [\sin\theta_1 \cos\theta_1 \sin\phi_1 \sinh\phi_2$$

$$- \sinh\theta_2 \cosh\theta_2 \cos\phi_1 \sinh\phi_2] \frac{\partial}{\partial\phi_2}$$

Similarly it can be calculated the other two operators B_2 and B_3 under which the variables y and z are transformed.

CHAPTER FOUR

§4.1 GENERATORS OF LORENTZ GROUP

The Lorentz group is generated by six operators $\underline{J} \equiv (J_1, J_2, J_3)$ and $\underline{K} \equiv (K_1, K_2, K_3)$. The operators $\underline{J}$, form the operators of three dimension rotation group and therefore generate pure rotations. The operators $\underline{K}$ generate pure Lorentz transformations.

The operators $\underline{J}$ can be obtained from the operators corresponding to rotations

$$A_1 = \frac{1}{(\sin^2\theta_1 + \sinh^2\theta_2)} [\sin\theta_1 \cosh\theta_2 \sin\phi_1 \cosh\phi_2 + \sinh\theta_2 \cos\theta_1 \cos\phi_1 \sinh\phi_2] \frac{\partial}{\partial\psi_1} - \frac{1}{(\sin^2\theta_1 + \sinh^2\theta_2)} [\cos\theta_1 \sinh\theta_2 \sin\phi_1 \cosh\phi_2 - \sin\theta_1 \cosh\theta_2 \cos\phi_1 \sinh\phi_2] \frac{\partial}{\partial\psi_2} + \cos\phi_1 \cosh\phi_2 \frac{\partial}{\partial\theta_1} - \sin\phi_1 \sinh\phi_2 \frac{\partial}{\partial\theta_2} - \frac{1}{(\sin^2\theta_1 + \sinh^2\theta_2)} [\sin\theta_1 \cos\theta_1 \sin\phi_1 \cosh\phi_2 + \sinh\theta_2 \cosh\theta_2 \cos\phi_1 \sinh\phi_2] \frac{\partial}{\partial\phi_1} - \frac{1}{(\sin^2\theta_1 + \sinh^2\theta_2)} [\sin\theta_1 \cos\theta_1 \cos\phi_1 \sinh\phi_2 - \sinh\theta_2 \cosh\theta_2 \sin\phi_1 \cosh\phi_2] \frac{\partial}{\partial\phi_2}$$

$$A_2 = -\frac{1}{(\sin^2\theta_1 + \sinh^2\theta_2)} [\cos\theta_1 \sinh\theta_2 \sin\phi_1 \sinh\phi_2$$

$$- \sin\theta_1 \cosh\theta_2 \cos\phi_1 \cosh\phi_2] \frac{\partial}{\partial\psi_1} - \frac{1}{(\sin^2\theta_1 + \sinh^2\theta_2)}$$

$$[\cos\theta_1 \sinh\theta_2 \cos\phi_1 \cosh\phi_2 + \sin\theta_1 \cosh\theta_2 \sin\phi_1 \sinh\phi_2]$$

$$\frac{\partial}{\partial\psi_2} - \cosh\phi_2 \sin\phi_1 \frac{\partial}{\partial\theta_1} - \cos\phi_1 \sinh\phi_2 \frac{\partial}{\partial\theta_2}$$

$$- \frac{1}{(\sin^2\theta_1 + \sinh^2\theta_2)} [\sin\theta_1 \cos\theta_1 \cos\phi_1 \cosh\phi_2$$

$$- \sinh\theta_2 \cosh\theta_2 \sin\phi_1 \sinh\phi_2] \frac{\partial}{\partial\phi_1}$$

$$+ \frac{1}{(\sin^2\theta_1 + \sinh^2\theta_2)} [\sin\theta_1 \cos\theta_1 \sin\phi_1 \sinh\phi_2$$

$$+ \sinh\theta_2 \cosh\theta_2 \cos\phi_1 \cosh\phi_2] \frac{\partial}{\partial\phi_2}$$

$$A_3 = \frac{\partial}{\partial\phi_1} \tag{4.1.1}$$

$$B_1 = \frac{1}{(\sin^2\theta_1 + \sinh^2\theta_2)} [\cos\theta_1 \sinh\theta_2 \sin\phi_1 \cosh\phi_2$$

$$- \sin\theta_1 \cosh\theta_2 \cos\phi_1 \sinh\phi_2] \frac{\partial}{\partial\psi_1} + \frac{1}{(\sin^2\theta_1 + \sinh^2\theta_2)}$$

$$[\cos\theta_1 \sinh\theta_2 \cos\phi_1 \sinh\phi_2 + \sin\theta_1 \cosh\theta_2 \sin\phi_1 \cosh\phi_2]$$

$$\frac{\partial}{\partial\psi_2} + \sinh\phi_2 \sin\phi_1 \frac{\partial}{\partial\theta_1} + \cos\phi_1 \cosh\phi_2 \frac{\partial}{\partial\theta_2}$$

$$+ \frac{1}{(\sin^2\theta_1 + \sinh^2\theta_2)} [\sin\theta_1 \cos\theta_1 \cos\phi_1 \sinh\phi_2$$

$$- \sinh\theta_2 \cosh\theta_2 \sin\phi_1 \cosh\phi_2] \frac{\partial}{\partial\phi_1} -$$

$$- \frac{1}{(\sin^2\theta_1 + \sinh^2\theta_2)} [\sin\theta_1 \cos\theta_1 \sin\phi_1 \cosh\phi_2$$

$$+ \sinh\theta_2 \cosh\theta_2 \cos\phi_1 \sinh\phi_2] \frac{\partial}{\partial\phi_2}$$

$$B_2 = \frac{1}{(\sin^2\theta_1 + \sinh^2\theta_2)} [\sin\theta_1 \cosh\theta_2 \sin\phi_1 \sinh\phi_2$$

$$+ \cos\theta_1 \sinh\theta_2 \cos\phi_1 \cosh\phi_2] \frac{\partial}{\partial\psi_1} + \frac{1}{(\sin^2\theta_1 + \sinh^2\theta_2)}$$

$$[\sin\theta_1 \cosh\theta_2 \cos\phi_1 \cosh\phi_2 - \cos\theta_1 \sinh\theta_2 \sin\phi_1 \sinh\phi_2]$$

$$\frac{\partial}{\partial\psi_2} + \cos\phi_1 \sinh\phi_2 \frac{\partial}{\partial\theta_1} - \sin\phi_1 \cosh\phi_2 \frac{\partial}{\partial\theta_2}$$

$$- \frac{1}{(\sin^2\theta_1 + \sinh^2\theta_2)} [\sin\theta_1 \cos\theta_1 \sin\phi_1 \sinh\phi_2$$

$$+ \sinh\theta_2 \cosh\theta_2 \cos\phi_1 \cosh\phi_2] \frac{\partial}{\partial\phi_1} - \frac{1}{(\sin^2\theta_1 + \sinh^2\theta_2)}$$

$$[\sin\theta_1 \cos\theta_1 \cos\phi_1 \cosh\phi_2 - \sinh\theta_2 \cosh\theta_2 \sin\phi_1 \sinh\phi_2]$$

$$\frac{\partial}{\partial\phi_2}$$

$$B_3 = \frac{\partial}{\partial\phi_2} \tag{4.1.2}$$

From these operators it can be calculated

$$A\cdot B = \frac{\sin\theta_1 \cos\theta_1 \sinh\theta_2 \cosh\theta_2}{\sin^2\theta_1 + \sinh^2\theta_2} \left(\frac{\partial^2}{\partial\psi_1^2} - \frac{\partial^2}{\partial\psi_2^2} + \frac{\partial^2}{\partial\phi_1^2} - \frac{\partial^2}{\partial\phi_2^2}\right)$$

$$+ \frac{1}{(\sin^2\theta_1 + \sinh^2\theta_2)} \sinh\theta_2 \cosh\theta_2 \frac{\partial}{\partial\theta_1}$$

$$- \frac{1}{(\sin^2\theta_1 + \sinh^2\theta_2)} \sin\theta_1 \cos\theta_1 \frac{\partial}{\partial\theta_2}$$

$$A^2 = \frac{1}{(\sin^2\theta_1 + \sinh^2\theta_2)^2} [\sin^2\theta_1 \cosh^2\theta_2 \cosh^2\phi_2$$

$$+ \cos^2\theta_1 \sinh^2\theta_2 \sinh^2\phi_2] \frac{\partial^2}{\partial\psi_1^2} + \frac{1}{(\sin^2\theta_1 + \sinh^2\theta_2)^2}$$

$$[\cos^2\theta_1 \sinh^2\theta_2 \cosh^2\phi_2 + \sin^2\theta_1 \cosh^2\theta_2 \sinh^2\phi_2] \frac{\partial}{\partial\psi_2^2}$$

$$+ \cosh^2\phi_2 \frac{\partial^2}{\partial\theta_1^2} + \sinh^2\phi_2 \frac{\partial^2}{\partial\theta_2^2} + \frac{\partial^2}{\partial\phi_1^2}$$

$$+ \frac{1}{(\sin^2\theta_1 + \sinh^2\theta_2)} [\sin^2\theta_1 \cos^2\theta_1 \cosh^2\phi_2$$

$$+ \sinh^2\theta_2 \cosh^2\theta_2 \sinh^2\phi_2] \frac{\partial^2}{\partial\phi_1^2} + \frac{1}{(\sin^2\theta_1 + \sinh^2\theta_2)}$$

$$[\sin^2\theta_1 \cos^2\theta_1 \sinh^2\phi_2 + \sinh^2\theta_2 \cosh^2\theta_2 \cosh^2\phi_1] \frac{\partial^2}{\partial\phi_2^2}$$

$$+ \frac{\sin\theta_1 \cos\theta_1}{(\sin^2\theta_1 + \sinh^2\theta_2)} \frac{\partial}{\partial\theta_1} - \frac{\sinh\theta_2 \cosh\theta_2}{(\sin^2\theta_1 + \sinh^2\theta_2)} \frac{\partial}{\partial\theta_2^2}$$

$$B^2 = \frac{1}{(\sin^2\theta_1 + \sinh^2\theta_2)^2} [\sin^2\theta_1 \cosh^2\theta_2 \sinh^2\phi_2$$

$$+ \cos^2\theta_1 \sinh^2\theta_2 \cosh^2\phi_2] \frac{\partial^2}{\partial\psi_1^2} + \frac{1}{(\sin^2\theta_1 + \sinh^2\theta_2)^2}$$

$$[\cos^2\theta_1 \sinh^2\theta_2 \sinh^2\phi_2 + \sin^2\theta_1 \cosh^2\theta_2 \cosh^2\phi_2] \frac{\partial^2}{\partial\psi_2^2}$$

$$+ \sinh^2\phi_2 \frac{\partial^2}{\partial\theta_1^2} + \cosh^2\phi_2 \frac{\partial^2}{\partial\theta_2^2} + \frac{1}{(\sin^2\theta_1 + \sinh^2\theta_2)^2}$$

$$[\sin^2\theta_1 \cos^2\theta_1 \sinh^2\phi_2 + \sinh^2\theta_2 \cosh^2\theta_2 \cosh^2\phi_2] \frac{\partial^2}{\partial\phi_1^2}$$

$$+ \frac{1}{(\sin^2\theta_1 + \sinh^2\phi_2)} [\sin^2\theta_1 \cos^2\theta_1 \cosh^2\phi_2$$

$$+ \sinh^2\theta_2 \cosh^2\theta_2 \sinh^2\phi_2] \frac{\partial}{\partial\phi_2^2} + \frac{\partial^2}{\partial\phi_2^2}$$

$$- \frac{\sin\theta_1 \cos\theta_1}{\sin^2\theta_1 + \sinh^2\theta_2} \frac{\partial}{\partial\theta_1} + \frac{\sinh\theta_2 \cosh\theta_2}{\sin^2\theta_1 + \sinh^2\theta_2} \frac{\partial}{\partial\theta_2}$$

$$A^2 - B^2 = \frac{\sin^2\theta_1 \cosh^2\theta_2 - \cos^2\theta_1 \sinh^2\theta_2}{(\sin^2\theta_1 + \sinh^2\theta_2)^2} \left(\frac{\partial^2}{\partial\psi_1^2} - \frac{\partial^2}{\partial\psi_2^2}\right) + \frac{\partial^2}{\partial\theta_1^2}$$

$$- \frac{\partial^2}{\partial\theta_2^2} + \frac{\partial^2}{\partial\phi_1^2} - \frac{\partial^2}{\partial\phi_2^2} + \frac{\sin^2\theta_1 \cos^2\theta_1 - \sinh^2\theta_2 \cosh^2\theta_2}{(\sin \quad _1 + \sinh \quad _2)}$$

$$\left(\frac{\partial^2}{\partial\phi_1^2} - \frac{\partial^2}{\partial\phi_2^2}\right) + \frac{2\sin\theta_1 \cos\theta_1}{\sin^2\theta_1 + \sinh^2\theta_2} \frac{\partial}{\partial\theta_1}$$

$$- \frac{2\sinh\theta_2 \cosh\theta_2}{\sin^2\theta_1 + \sinh^2\theta_2} \frac{\partial}{\partial\theta_2}$$

The operator A_i and B_j $(i,j = 1,2,3)$ satisfy the commutation relations

$$[A_i\ A_j] = \varepsilon_{ijk}\ A_k$$

$$[B_i\ B_j] = -\varepsilon_{ijk}\ A_k$$

$$[A_i\ B_j] = \varepsilon_{ijk}\ B_k$$

§4.2 BOUNDARY CONDITION ON THE OPERATORS

Let us consider the form of operator under the boundary condition $\theta_2 = \phi_2 = \psi_2 = 0$.

$$A_1\Big|_{\theta_2=\phi_2=\psi_2=0} = \frac{\sin\phi_1}{\sin\theta_1}\frac{\partial}{\partial\psi_1} + \cos\phi_1\frac{\partial}{\partial\theta_1} - \cot\theta_1\sin\phi_1\frac{\partial}{\partial\phi_1}$$

$$A_2\Big|_{\theta_2=\phi_2=\psi_2=0} = \frac{\cos\phi_1}{\sin\theta_1}\frac{\partial}{\partial\psi_1} - \sin\phi_1\frac{\partial}{\partial\theta_1} - \cot\theta_1\cos\phi_1\frac{\partial}{\partial\phi_1}$$

$$A_3\Big|_{\theta_2=\phi_2=\psi_2=0} = \frac{\partial}{\partial\phi_1}$$

and

$$B_1\Big|_{\theta_2=\phi_2=\psi_2=0} = B_2\Big|_{\theta_2=\phi_2=\psi_2=0} = B_3\Big|_{\theta_2=\phi_2=\psi_2=0} = 0$$

$$\bar{A}\cdot\bar{B}\Big|_{\theta_2=\phi_2=\psi_2=0} = 0$$

$$A^2\Big|_{\theta_2=\phi_2=\psi_2=0} = \frac{1}{\sin^2\theta_1}\frac{\partial^2}{\partial\psi_1^2} + \frac{\partial^2}{\partial\theta_1^2} + \frac{\partial^2}{\partial\phi_1^2} + \cot^2\theta_1\frac{\partial^2}{\partial\phi_1^2} + \cot\theta_1\frac{\partial}{\partial\theta_1}$$

$$B^2\Big|_{\theta_2=\phi_2=\psi_2=0} = -\cot\theta_1\frac{\partial}{\partial\theta_1}$$

$$(A^2 - B^2)\Big|_{\theta_2=\phi_2=\psi_2=0} = \frac{1}{\sin^2\theta_1}\frac{\partial^2}{\partial\psi_1^2} + \frac{\partial^2}{\partial\theta_1^2} + \frac{\partial^2}{\partial\phi_1^2} + \cot^2\theta_1\frac{\partial^2}{\partial\phi_1^2} + 2\cot\theta_1\frac{\partial}{\partial\theta_1}$$

§4.3 INVARIANTS OF LORENTZ GROUP

Define:

$$J_1 = iA_1$$

$$J_2 = iA_2$$

$$J_3 = iA_3$$

$$K_1 = -iB_1$$

$$K_2 = -iB_2$$

$$K_3 = -iB_3$$

Then these operators J_i and J_j $(i,j = 1,2,3)$ satisfy the commutation relations

$$[J_i\ J_j] = i\varepsilon_{ijk}\ J_k$$

$$[J_i\ K_j] = i\varepsilon_{ijk}\ K_k$$

$$[K_i\ K_j] = -i\varepsilon_{ijk}\ J_k$$

$$\underline{J}^2 - \underline{K}^2 = -(A^2 - B^2)$$

$$\underline{J}\cdot\underline{K} = A\cdot B$$

$\underline{J}^2 - \underline{K}^2$ and $\underline{J}\cdot\underline{K}$ are the invariants of Lorentz group and they commute with J_1, J_2 and J_3.

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