FIXED POINTS OF MULTIVALUED MAPS

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TO

MY PARENTS

PREFACE

The theory of fixed points is concerned with the conditions which guarantee that a mapping T of a set X into itself admits one or more fixed points, that is, points x of X for which $x = Tx$. A large variety of the most important problems of applied mathematics reduce to finding solutions of nonlinear functional equation which can be formulated in terms of finding the fixed points of a nonlinear mapping. For example, the problem of solving the equation $p(z) = 0$, where $p(z)$ is complex polynomial, is equivalent to finding a fixed point of self mapping $z \longmapsto z-p(z)$ of C (Set of complex numbers). More generally, if $D: M \longmapsto R$ is any operator acting on a subset M of set of real numbers, to show that the equation $Du = 0$ $(u-\lambda\ Du = 0)$ has a solution, is equivalent to showing that mapping $y \longmapsto y-Dy$ $(y \longmapsto \lambda\ Dy)$ has a fixed point. For single valued self mappings, a general existence theory of fixed points has been constructed over a number of decades (associated with the names of Brouwer, Lofschefz, Schauder, Tychonoff, *and others). The fixed point theorem most frequently cited in the literature is Banach's* *contraction principle, which asserts that if X is a complete metric space and T a single valued contractive self mapping on X, then T has a unique fixed point in X. This theorem is simplest and one of the most versatile results in fixed point theory. Being based on an iteration process, it can be implemented on a computer to find the fixed point of a contrative mapping. Afterwards*, Browder, Fan, Jungck, Kirk, Kannan, Rhoades, Wong *and many others proved remarkable fixed point theorems. In 1969, the systematic study of Banach typefixed point Theorems of multivalued mappings had been started with the work of* Nadler, *who proved that any multivalued contractive mapping of a complete metric space* X *into the family of closed bounded subsets of X has a fixed point. He also established that every* (ε,λ)*-uniformly locally contractive mapping of an* ε*-chainable metric space X into the family of compact subsets of* X *has a fixed point. His work is followed by* Aubin *and* Siegel, Hu, Itoch and Takahashi, Kaneko, Massa *and many others. This dissertation is a continuation of these investigations and consists of three chapters.*

Each chapter beings with a brief introduction which summarises the material contained in that chapter.

Chapter 1, *is a survey aimed at clarifying the terminology to be used and recalling basic definitions and facts.*

Chapter 2, *is devoted to the study of coincidence points and common fixed points of hybrid contractions. A theorem on common fixed points of multivalued locally contractive mappings in* ε*-chainable metric space is also established. The results proved in* Section 2.2 *have appeared in Boll. U.M.I., (7)4-A (1990).*

Chapter 3, *deals with the study of fixed points of asymptotically regular multivalued mappings. It has been shown that for many of contractive type mappings there exists an asymptotically regular sequence. We exploite this property of contractive mappings to obtain their fixed points. Fixed points of Kannan type and Meir-Keeler type mappings are also obtained in this Chapter. The results proved in* Section 3.1, 3.2 *and* 3.3 *will appear in J. Austral. Math. Soc. (Series A).*

I am deeply grateful to my Supervisor Dr. Ismat Beg, *for suggesting the research topic and his valuable guidance throughout. It gives me great pleasure to acknowledge the excellent facilities provided by the Department of Mathematics, Quaid-i-Azam University, Islamabad, Pakistan. Finally, I offer my thanks to* Mr. Muhammad Sharaz *for skillful typing.*

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CONTENTS

CHAPTER ONE

PRELIMINARIES

The aim of this *chapter* is to present some basic concepts and to explain the terminology used throughout this dissertation. Some previously known results are given without proof. In section 1.1, we discuss the concept of Hausdorff metric on the family of closed bounded subsets of a metric space. In section 1.2 the notions of commuting and compatible single valued mappings are discussed and we extend the idea of single valued compatible mappings to multivalued mappings. Section 1.3 deals with multivalued contractions.

1.1. HAUSDORFF METRIC

Let (X,d) be a metric space and let $CB(X)$ denote the family of all nonempty bounded closed subsets of X. In order to make the family $CB(X)$ into a metric space, we need to have a measure of the "distance" between two sets A and B of $CB(X)$. One such notion of distance is

$$d(A,B) = \inf\{d(x,y): x \in A,\ y \in B\} .$$

This definition fails to discriminate sufficiently between sets. We would like the distance between two sets to be zero only if the two sets are same, both in shape and position. For this purpose the following concept is useful (cf., [36]).

DEFINITION 1.1.1

Let $A, B \in CB(X)$. For $\varepsilon > 0$ the sets $N(\varepsilon, A)$ and $E_{A,B}$ are defined as follow:

$$N(\varepsilon, A) = \{x \in X:\ d(x, A) < \varepsilon\},$$

$$E_{A,B} = \{\varepsilon:\ A \subseteq N(\varepsilon, B),\ B \subseteq N(\varepsilon, A)\},$$

where $d(x, A) = \inf\{d(x, y):\ y \in A\}$. The distance function H on CB(X) which is defined to be

$$H(A, B) = \begin{cases} \inf E_{A,B} & \text{if } E_{A,B} \neq \phi \\ +\infty & \text{if } E_{A,B} = \phi \end{cases}$$

is known as *Hausdorff metric* on X.

This concept enables us to fully exploit the distance properties of H. Specifically, it enables us to consider the convergence of sequences of elements in CB(X).

DEFINITION 1.1.2

A sequence $\{A_i\}$ of nonempty closed bounded subsets of X is said to converge to a set A if

$$\lim_{i\to\infty} H(A_i, A) = 0 .$$

DEFINITION 1.1.3

A sequence $\langle A_i \rangle$ in $CB(X)$ is said to be a Cauchy sequence if

$$H(A_i, A_j) \longrightarrow 0 \text{ as } i, j \longrightarrow \infty .$$

We note that the metric H actually depends on the metric for X. Two equivalent metrics for X may not generate equivalent Hausdorff metric for CB(X) (see [11, p. 131]). We shall not notate this dependency except where confusion may arise. It will be understood, unless otherwise stated, that the symbol H stands for the Hausdorff metric obtained from a fixed preassigned metric. We shall require the following well-known facts.

LEMMA 1.1.4 (Nadler [43])

Let $A, B \in CB(X)$ and $a \in A$. If $\eta > 0$, then there exists a $b \in B$ such that $d(a,b) \le H(A,B) + \eta$.

LEMMA 1.1.5 (Hu [23])

If $A, B \in CB(X)$ with $H(A,B) < \varepsilon$, then for each $a \in A$, there exists an element $b \in B$ such that $d(a,b) < \varepsilon$.

LEMMA 1.1.6 (Hu [23])

Let $\{A_n\}$ be a sequence of sets in CB(X) and $H(A_n, A) \longrightarrow 0$ for $A \in CB(X)$. If $x_n \in A_n$ $(n = 1,2,\ldots)$ and $d(x_n, x) \longrightarrow 0$, then $x \in A$.

LEMMA 1.1.7 (Dube [12])

Let $A, B \in CB(X)$, then for $a \in A$

$$d(a,B) \le H(A,B) .$$

REMARK 1.1.8 (Aubin [2])

The completeness of (X,d) implies that (CB(X),H) is complete.

1.2 COMPATIBLE MULTIVALUED MAPPINGS

The concept of compatible mappings has proven useful for generalizing in the context of metric space fixed point theory (see [1],[28],[29] and [30]).

DEFINITION 1.2.1

Two mappings $f,g: X \longrightarrow X$ are said to be *commuting* if $fgx = f(g(x)) = g(f(x)) = gfx$ for $x \in X$.

Sessa [58] generalized the concept of commuting mappings by calling self mappings f, g of a metric space X weakly commuting if and only if $d(fgx, gfx) \le d(fx, gx)$ for $x \in X$. Of course, commuting mappings are weakly commuting, but the converse is not true (see Sessa [58]). Many authors obtained nice fixed point theorems using this concept. However, since elementary functions as similar as $fx = x^3$, $gx = 2x^3$ are not weakly commutative, Jungck [28] introduced a less restrictive concept of compatible mappings. He also pointed out in [29] and [30] the potential of compatible mappings for generalized fixed point theorems.

DEFINITION 1.2.2

Mappings f, g $X \longrightarrow X$ are compatible if, whenever there is a sequence $\{x_n\} \subset X$ satisfying $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = u$, then $\lim_{n\to\infty} d(fgx_n, gfx_n) = 0$.

EXAMPLE 1.2.3 (Jungck [28])

Let $fx = x^3$, $gx = 2x$ with $X = R$. Then f and g are compatible but not weakly commuting.

EXAMPLE 1.2.4 (Jungck [28])

Let $fx = \cosh x$, $gx = \sinh x$ with $X = R$. Then $|fx_n - gx_n| = e^{-x_n} \longrightarrow 0$ if and only if $x_n \longrightarrow +\infty$. But fx_n, gx_n

$\longrightarrow +\infty$ as $x_n \longrightarrow +\infty$. Thus fx_n and gx_n do not converge to an element of X. Thus the condition of compatibility is satisfied vacuously, but f and g do not commute.

The concept of a pair of compatible single valued mappings is extended to one single valued and one multivalued mapping as follow.

DEFINITION 1.2.5

Mapping $T: X \longrightarrow CB(X)$, $f: X \longrightarrow X$ are said to be *compatible* if whenever there is a sequence $\{x_n\} \subset X$ satisfying $\lim_{n\to\infty} fx_n \in \lim_{n\to\infty} Tx_n$ (provided $\lim_{n\to\infty} fx_n$ and $\lim_{n\to\infty} Tx_n$ exist in X and $CB(X)$ respectively), then

$$\lim_{n\to\infty} H(fTx_n, Tfx_n) = 0 .$$

EXAMPLE 1.2.6

Let $X = [0,1]$ with the Euclidean metric, $Tx = [0, \frac{x}{x+10}]$, $fx = \frac{x}{5}$. Then f and T are compatible but not commuting.

1.3. MULTIVALUED CONTRACTIONS

DEFINITION 1.3.1

Let X be a metric space. A mapping $T: X \longrightarrow CB(X)$ is said to be *multivalued contraction* if there exists a constant α, $0 \le \alpha < 1$, such that for all $x, y \in X$,

$$H(Tx, Ty) \le \alpha\, d(x, y) \;.$$

DEFINITION 1.3.2

A point x is said to be a *fixed point* of a single valued mapping f(multivalued mapping T) provided $x = fx$($x \in Tx$). The point x is called *coincidence point* of f and T if $fx \in Tx$.

Nadler [43] generalized Banach contraction principle and proved the following fixed point result for multivalued contractions.

THEOREM 1.3.3 (Nadler [43])

Let X be a complete metric space. If $T: X \longrightarrow CB(X)$ is a multivalued contraction, then T has a fixed point.

Jungck [28] introduced the concept of commuting, mappings and improved the Banach contraction principle as follow.

THEOREM 1.3.4 (Jungck [27])

Let (X,d) be a complete metric space and $f,g\ X: \longrightarrow X$ two commuting mappings. If there exists a constant α: $0 \leq \alpha < 1$ such that $gX \subseteq fX$, $d(gx,gy) \leq \alpha\, d(fx,fy)$, then f and g have a unique common fixed point.

EXAMPLE 1.3.5

Let $X = R$ and $gx = 2x$, $fx = 3x$. Then $d(gx,gy) = 2|x-y| = \frac{2}{3}\ |3x-3y|$. In order to satisfy all hypotheses of Theorem 1.3.4 we can assume $\alpha = \frac{3}{4}$.

CHAPTER TWO

FIXED POINTS OF MULTIVALUED CONTRACTIONS

Since the appearance of celebrated Banach contraction principle in 1932, several generalization of this theorem in the setting of point to point mappings have been obtained. See Rhoades [50][56] and Kirk [37] for a complete survey of this subject. Jungck [27] generalized the Banach contraction principal by introducing a contraction condition for a pair of commuting mappings. He also pointed out the importance of commuting mappings for a generalized fixed point theorem. Subsequently a variety of extensions, generalizations and applications of this followed; e.g., see [19][33][46] and [52].

Nadler [43] was the first to combine the ideas of multivalued mappings and contractions. He proved some remarkable fixed point results for multivalued contractions. He also introduced the idea of multivalued locally contractions and generalized a fixed point theorem of Edelstein [15]. Afterwards, Dube and Singh [13], Iseki [24], Ray [47] Itoh and Takahashi [26], Aubin and Siegel [3], Hu [23] and Massa [41], Kaneko [33][35] and many others have studied fixed point theorems for multivalued contractive type mappings. In this chapter we extend the idea of Jungck [27] to multivalued contractions and obtain some new fixed point theorems. In section 2.1, we study coincidence points of three mappings. The structure of common fixed points is also discussed. Section 2.2 deals with the existence of common fixed points of a pair of locally contractive multivalued mappings (not necessarily commuting). The results proved in this section appeared in Boll. U.M.I., (7) 4-A, (1990). In section 2.3 we continue the study of coincidence points, common fixed points of two multivalued mappings that are compatible with two single valued mappings and satisfying a contractive type condition.

2.1. COINCIDENCE POINTS OF HYBRID CONTRACTIONS

In this section, Theorem 2.1.2 considers a multivalued mapping that commutes with two single valued mappings and satisfies a general multivalued contraction type condition. Theorems 2.1.4, 2.1.6 involve an analogous contractive definition for two multivalued mappings which commute with a single valued mapping. If for a multivalued mapping T, $C_T = \{f: X \longrightarrow X \mid TX \subseteq fX \text{ and } (\forall x \in X)\ fTx = Tfx\}$, we have the following Lemma.

LEMMA 2.2.1

Let X be a metric space and $T: X \longrightarrow CB(X)$ a continuous mapping. Let $f \in C_T$ and continuous such that f and T have a coincidence point z in X. If $\lim_{n\to\infty} f^n z = t < \infty$, then t is a common fixed point of f and T.

PROOF

Obviously, $fz \in Tz$ implies that $f^2z \in fTz = Tfz$. Therefore $f^{n+1}z \in Tf^nz$. It follows that $t \in Tt$. Moreover $ft = f \lim_{n\to\infty} f^n z = \lim_{n\to\infty} f^{n+1}z = t$. Hence t is a common fixed point of f and T.

THEOREM 2.1.2

Let X be a metric space and $T: X \longrightarrow CB(X)$ a continuous mapping. Let $f, g \in C_T$ and continuous such that the following condition is satisfied:

$$H(Tx, Ty) \le Ad(fx, gy) + B(d(fx, Tx) + d(gy, Ty)) + C(d(fx, Ty) + d(gy, Tx)) + D(1 + d(fx, gy))^{-1} d(fx, Tx)\, d(gy, Ty), \quad (2.1.1)$$

for all $x, y \in X$, $A, B, C, D \ge 0$ and $0 < \dfrac{A+B+C}{1-B-C-D} < 1$. Then there is a common coincidence point of f and T, and g and T.

PROOF

Assume that $M = \frac{(A+B+C)}{(1-B-C-D)}$. Let x_o be an arbitrary but a fixed element of X. We shall construct two sequences $\{x_n\}$ and $\{y_n\}$ as follow. Let $x_1 \in X$ be such that $y_1 = fx_1 \in Tx_o$. Using Lemma 1.1.4 and the fact that $TX \subseteq gX$, we may choose $y_2 = gx_2 \in Tx_1$, such that

$$d(y_1,y_2) = d(fx_1,gx_2) \le H(Tx_o,Tx_1) + A+B+C .$$

Since $Tx \subseteq fX$, we may choose $x_3 \in X$, such that $y_3 = fx_3 \in Tx_2$ and $d(y_2,y_3) = d(gx_2,fx_3) \le H(Tx_1,Tx_2) + \frac{(A+B+C)^2}{(1-B-C-D)}$.

By induction we produce two sequences of points of X, such that

$$\begin{aligned} y_{2k+1} &= fx_{2k+1} \in Tx_{2k} \\ y_{2k+2} &= gx_{2k+2} \in Tx_{2k+1} , \end{aligned} \qquad (2.1.2)$$

where k is any natural number. Furthermore,

$$\begin{aligned} d(y_{2k+1}, y_{2k+2}) &= d(fx_{2k+1}, gx_{2k+2}) \\ &\le H(Tx_{2k}, Tx_{2k+1}) + \frac{(A+B+C)^{2k+1}}{(1-B-C-D)^{2k}} , \end{aligned}$$

and

$$\begin{aligned} d(y_{2k+2}, y_{2k+3}) &= d(gx_{2k+2}, fx_{2k+3}) \\ &\le H(Tx_{2k+1}, Tx_{2k+2}) + \frac{(A+B+C)^{2k+2}}{(1-B-C-D)^{2k+1}} . \end{aligned}$$

Hence

$$d(fx_{2k+1}, fx_{2k+2}) \le Ad(fx_{2k+1}, gx_{2k}) + B\{d(fx_{2k+1}, Tx_{2k+1})$$
$$+ d(gx_{2k}, Tx_{2k})\} + C\{d(fx_{2k+1}, Tx_{2k})$$
$$+ d(gx_{2k}, Tx_{2k+1})\} + D(1+d(fx_{2k+1}, gx_{2k}))^{-1}$$
$$d(fx_{2k+1}, Tx_{2k+1})d(gx_{2k}, Tx_{2k}) + \frac{(A+B+C)^{2k+1}}{(1-B-C-D)^{2k}}$$
$$\le (A+B+C)\ d(fx_{2k+1}, gx_{2k})$$
$$+ (B+C+D)\ d(fx_{2k+1}, gx_{2k+2}) + \frac{(A+B+C)^{2k+1}}{(1-B-C-D)^{2k}}$$
$$\le M\ d(fx_{2k+1}, gx_{2k}) + M^{2k+1}.$$

Similarly,

$$d(gx_{2k}, fx_{2k+1}) \le M\ d(fx_{2k-1}, gx_{2k}) + M^{2k}.$$

It further implies that

$$d(y_n, y_{n+1}) \le M^{n-1}\ d(fx_1, gx_2) + (n-1)M^n .$$

For $p \ge 1$ and $m = n + p$, we have

$$d(y_{n+1}, y_{m+1}) \le d(y_{n+1}, y_{n+2}) + d(y_{n+2}, y_{n+3}) + \ldots + d(y_{n+p}, y_{n+p+1})$$
$$\le \{M^n d(fx_1, gx_2) + nM^{n+1}\} + \{M^{n+1} d(fx_1, gx_2) + (n+1)M^{n+2}\}$$
$$+ \ldots + \{M^{n+p-1} d(fx_1, gx_2) + (n+p-1)M^{n+p}\}$$
$$\le \sum_{i=n}^{n+p-1} M^i d(fx_1, gx_2) + \sum_{i=n}^{n+p-1} iM^{i+1} .$$

It follows that the sequence $\{y_n\}$ is a Cauchy sequence. Hence there exists z in X such that $y_n \longrightarrow z$. Therefore $fx_{2k+1} \rightarrow z$ and $gx_{2k+2} \longrightarrow z$. The continuity of T implies that

$Tfx_{2k+1} \longrightarrow Tz$ and $Tgx_{2k+2} \longrightarrow Tz$.

From (2.1.2), we have

$$gfx_{2k+1} \in gTx_{2k} \subseteq Tgx_{2k}$$

$$fgx_{2k+2} \in fTx_{2k+1} \subseteq Tfx_{2k+1} .$$

Since f and g are continuous, by letting $k \longrightarrow \infty$, we obtain

$$gz \in Tz \quad \text{and} \quad fz \in Tz .$$

This completes the proof of the Theorem.

COROLLARY 2.1.3

Let X be a complete metric space and $T: X \longrightarrow CB(X)$ a continuous mapping. Let $f,g \in C_T$ and continuous such that (2.1.1) is satisfied. Moreover, assume that

$$\{fz,gz\} \subset Tz \text{ implies } \lim_{n\to\infty} f^n z = \lim_{n\to\infty} g^n z = t < \infty.$$

Then t is a common fixed point of f,g and T.

THEOREM 2.1.4

Let S,T be two mappings from a complete metric space X into CB(X) and let $f \in C_S \cap C_T$ be a continuous mapping. Suppose that for all $x,y \in X$,

$$\begin{aligned} H(Sx,Ty) \le Ad(fx,fy) &+ B\,\{d(fx,Sx) + d(fy,Ty)\} \\ &+ C\,\{d(fx,Ty) + d(fy,Sx)\} \\ &+ D\,\{1+d(fx,fy)\}^{-1}\, d(fx,Sx)\, d(fy,Ty) . \end{aligned} \tag{2.1.3}$$

where $A, B, C, D \geq 0$ and $0 < \frac{A+B+C}{1-B-C-D} < 1$. Then there exists a common coincidence point of f and T and f and S.

PROOF

Define $M = \frac{(A+B+C)}{(1-B-C-D)}$. Let x_o be an arbitrary but fixed element of X. We shall construct two sequences $\{x_n\}$ and $\{y_n\}$ as follows.

Let $x_1 \in X$ be such that $y_1 = fx_1 \in Sx_0$. Using the Lemma 1.1.4 and the fact that $TX \subseteq fX$, we may choose $x_2 \in X$ such that $y_2 = fx_2 \in Tx_1$ and $d(y_1, y_2) = d(fx_1, fx_2) \leq H(Sx_0, Tx_1) + (A+B+C)$. Since $SX \subseteq fX$, we may choose $x_3 \in X$ such that $y_3 = fx_3 \in Sx_2$ and $d(y_2, y_3) = d(fx_2, fx_3) \leq H(TX_1, Sx_2) + \frac{(A+B+C)^2}{1-B-C-D}$. By induction we produce two sequences of points of X such that

$$y_{2k+1} = fx_{2k+1} \in Sx_{2k},$$

and

$$y_{2k+2} = fx_{2k+2} \in Tx_{2k+1}, \qquad (2.1.4)$$

where k is any positive integer. Furthermore,

$$d(y_{2k+1}, y_{2k+2}) = d(fx_{2k+1}, fx_{2k+2})$$
$$\leq H(Sx_{2k}, Tx_{2k+1}) + \frac{(A+B+C)^{2k+1}}{(1-B-C-D)^{2k}}$$

and

$$d(y_{2k+2}, y_{2k+3}) = d(fx_{2k+2}, fx_{2k+3})$$
$$\leq H(Tx_{2k+1}, Sx_{2k+2}) + \frac{(A+B+C)^{2k+2}}{(1-B-C-D)^{2k+1}} .$$

Hence,

$$d(fx_{2k+1}, fx_{2k+2}) \leq A\, d(fx_{2k}, fx_{2k+1})$$
$$+ B\, \{d(fx_{2k}, Sx_{2k}) + d(fx_{2k+1}, Tx_{2k+1})\}$$
$$+ C\{d(fx_{2k}, Tx_{2k+1}) + d(fx_{2k+1}, Sx_{2k})\}$$
$$+ D\, \{1+d(fx_{2k}, fx_{2k+1})\}^{-1}\, d(fx_{2k}, Sx_{2k})\, d(fx_{2k+1}, Tx_{2k+1})$$
$$+ \frac{(A+B+C)^{2k+1}}{(1-B-C-D)^{2k}}$$
$$\leq (A+B+C)\, d(fx_{2k}, fx_{2k+1})$$
$$+ (B+C+D)\, d(fx_{2k+1}, fx_{2k+2}) + \frac{(A+B+C)^{2k+1}}{(1-B-C-D)^{2k}}$$

Therefore $d(fx_{2k+1}, fx_{2k+2}) \leq M d(fx_{2k}, fx_{2k+1}) + M^{2k+1}$.

Similarly,

$$d(fx_{2k}, fx_{2k+1}) \leq H(Tx_{2k-1}, Sx_{2k}) + \frac{(A+B+C)^{2k}}{(1-B-C-D)^{2k-1}}$$
$$\leq H(Sx_{2k}, Tx_{2k-1}) + \frac{(A+B+C)^{2k}}{(1-B-C-D)^{2k-1}} .$$

Therefore $d(fx_{2k}, fx_{2k+1}) \le Md(fx_{2k-1}, fx_{2k}) + M^{2k}$.

It further implies that,

$$\begin{aligned} d(y_n, y_{n+1}) &\le M\, d(y_{n-1}, y_n) + M^n \\ &\le M^{n-1}\, d(y_1, y_2) + (n-1)M^n \\ &\le M^{n-1}\, d(fx_1, fx_2) + (n-1)M^n. \end{aligned}$$

For $p \ge 1$, we have

$$\begin{aligned} d(y_{n+1}, y_{n+p+1}) &\le d(y_{n+1}, y_{n+2}) + d(y_{n+2}, y_{n+3}) + \ldots + d(y_{n+p}, y_{n+p+1}) \\ &\le \{M^n d(fx_1, fx_2) + n\, M^{n+1}\} \\ &\quad + \{M^{n+1} d(fx_1, fx_2) + (n+1)\, M^{n+2}\} \\ &\quad + \ldots + \{M^{n+p-1} d(fx_1, fx_2) + (n+p-1)M^{n+p}\} \\ &\le \sum_{i=n}^{n+p-1} M^i d(fx_1, fx_2) + \sum_{i=n}^{n+p-1} i\, M^{i+1}. \end{aligned}$$

It follows that the sequence $\{y_n\}$ is a Cauchy sequence. Hence there exists z in X such that $y_n \to z$. Therefore $fx_{2k+1} \to z$ and $fx_{2k+2} \longrightarrow z$. From (2.1.4), we have

$$f^2x_{2k+1} = ffx_{2k+1} \in fSx_{2k} \subseteq Sfx_{2k},$$

and

$$f^2x_{2k+2} = ffx_{2k+2} \in fTx_{2k+1} \subseteq Tfx_{2k+1}.$$

Now using Lemma 1.1.7,

$$\begin{aligned} d(fz, Sz) &\le d(fz, f^2x_{2k+2}) + d(f^2x_{2k+2}, Sz) \\ &\le d(fz, f^2x_{2k+2}) + H(Tfx_{k+1}, Sz) \\ &\le d(fz, f^2x_{2k+2}) + Ad(fz, f^2x_{2k+1}) \\ &\quad + B\{d(fz, Sz) + d(f^2x_{2k+1}, Tfx_{2k+1})\} \end{aligned}$$

$$+ C(d(fz,\ Tfx_{2k+1}) + d(f^2x_{2k+1},\ Sz))$$

$$+ D(1 + d(fz,\ f^2x_{2k+1}))^{-1} d(fz,\ Sz)\ d(f^2x_{2k+1},\ Tfx_{2k+1})$$

$$\leq d(fz,\ f^2x_{2k+2}) + Ad(fz,\ f^2x_{2k+1})$$

$$+ B(d(fz,\ Sz) + d(f^2x_{2k+1},\ f^2x_{2k+2}))$$

$$+ C(d(fz,\ f^2x_{2k+2}) + d(f^2x_{2k+1},\ Sz))$$

$$+ D(1 + d(fz,\ f^2x_{2k+1}))^{-1} d(fz,\ Sz)\ d(f^2x_{2k+1},\ f^2x_{2k+2})$$

Since f is continuous, by letting $k \longrightarrow \infty$, we obtain $d(fz, Sz) \leq (B + C)\ d\ (fz,\ Sz)$. Thus $fz \in Sz$. Similarly,

$$d(fz,\ Tz) \leq d(fz,\ f^2x_{2k+1}) + d(f^2x_{2k+1},\ Tz)$$

$$\leq d(fz,\ f^2x_{2k+1}) + H(Sfx_{2k}\ \ ,\ Tz)$$

$$\leq (B + C)\ d(fz,\ Tz).$$

Therefore $fz \in Tz$. Hence z is a coincidence point of f and S and f and T.

COROLLARY 2.1.5

Let S, T be continuous mappings from a complete metric space X into CB(X) and $f \in C_S \cap C_T$ be a continuous mapping. Assume that (2.1.3) is satisfied. If $f(z) \in Sz \cap Tz$ implies $\lim_{n \to \infty} f^n z = t$ then t is a common fixed point of S, T and f.

PROOF

Clearly, $fz \in Sz$ implies that $f^2z \in fSz \subseteq Sfz$.

Therefore $f^{n+1}z \in Sf^{n}z$. It follows that $t \in St$. Similarly $t \in Tt$. Moreover $ft = f \lim_{n\to\infty} f^{n}z = \lim_{n\to\infty} f^{n+1}z = t$. Hence t is a common fixed point of f, S and T.

In the following theorem the continuity of f, is not required.

THEOREM 2.1.6

Let S,T be two mappings from a metric space X into $CB(X)$ and let $f : X \longrightarrow X$ be a mapping such that fX is complete, $TX \subseteq fX$ and $SX \subseteq fX$. Suppose that (2.1.3) is satisfied, then there exists a common coincidence point of f and T, and f and S.

PROOF

As in the proof of Theorem 2.1.4, construct the Cauchy sequence $y_n = fx_n$ in X. By our hypothesis it follows that there exists a point u in X such that $y_n \longrightarrow z = fu$. Now using Lemma 1.1.7,

$$\begin{aligned} d(fu,Tu) &\le d(fu,fx_{2k+1}) + d(fx_{2k+1},Tu) \\ &\le d(fu,fx_{2k+1}) + H(Sx_{2k},Tu) \\ &\le d(fu,fx_{2k+1}) + A\, d(fx_{2k},fu) \\ &\quad + B\{d(fx_{2k},Sx_{2k}) + d(fu,Tu)\} \\ &\quad + C\{d(fx_{2k},Tu) + d(fu,Sx_{2k})\} \end{aligned}$$

$$+ D(1+d(fx_{2k},fu))^{-1}d(fx_{2k},Sx_{2k})d(fu,Tu)$$
$$\le d(fu,fx_{2k+1}) + A\, d(fx_{2k},fu)$$
$$+ B(d(fx_{2k},fx_{2k+1}) + d(fu,Tu))$$
$$+ C(d(fx_{2k},Tu) + d(fu,fx_{2k+1})) .$$
$$+ D(1+d(fx_{2k},fu))^{-1}d(fx_{2k},fx_{2k+1})d(fu,Tu)$$

Letting $k \longrightarrow \infty$, we obtain

$$d(fu,Tu) \le (B+C)\, d(fu,Tu) .$$

Hence $fu \in Tu$. Similarly,

$$d(fu,Su) \le d(fu,fx_{2k+2}) + d(fx_{2K+2},Su)$$
$$\le d(fu,fx_{2k+2}) + H(Tx_{2k+1},Su)$$
$$\le (B+C)\, d(fu,Su) .$$

Hence $fu \in Su$.

A particular case (when $f = I$ and $S = T$) of the above theorem is also a generalization of the theorem of Ray [47], since T is not assumed to have closed graph. It further illustrates that the compactness of Tx is not necessary for a theorem of Aubin and Siegel [3].

2.2. FIXED POINTS OF MULTIVALUED LOCALLY CONTRACTIONS

In this section a fixed point theorem for a pair of multivalued locally contractive mappings in ε-chainable metric space is proved. The theorem thus established, extend results of Edelstein [15], Hu [23], Ko and Tsai [38], Kuhfittig [39], Nadler [43] and Reich [48].

DEFINITION 2.2.1

A metric space (X,d) is said to be *ε-chainable* if and only if given x,y in X, there is an ε-chain from x to y (i.e., a finite set of points $x = z_o, z_1, z_2, z_3, \ldots, z_n = y$ such that $d(z_{j-1}, z_j) < \varepsilon$ for all $j = 1,2,3,\ldots,n$.

DEFINITION 2.2.2

A mapping $T: X \longrightarrow CB(X)$ is called an *(ε,λ)-uniformly locally contractive* mapping (where $\varepsilon > 0$ and $0 < \lambda < 1$) if $x,y \in X$ and $d(x,y) < \varepsilon$, then $H(Tx,Ty) \le \lambda\, d(x,y)$.

DEFINITION 2.2.3

A mapping $k : (0,\varepsilon) \longrightarrow [0,1)$ is said to have *property (P)* if for each t in the domain of k there exists $\delta(t)>0$, $s(t)<1$, such that

$0 \le r-t < \delta(t)$ implies $k(r) \le s(t)<1$ (cf.,[23] and [48]).

Nadler [43] proved the following important theorem.

THEOREM 2.2.4

Let X be a complete ε-chainable metric space. If $T: X \longrightarrow C(X)$ (family of compact subsets of X) is an (ε,λ)-uniformly locally contractive mapping, then T has a fixed point.

In the present section we generalize Theorem 2.2.4, to a pair of multivalued mappings.

THEOREM 2.2.5

Let X be a complete ε-chainable metric space and $T_1: X \longrightarrow CB(X)$, $T_2: X \longrightarrow CB(X)$ be two mappings that satisfy the following condition:

$$0<d(x,y)<\varepsilon \text{ implies } H(T_i x, T_j y) < k(d(x,y))d(x,y) \qquad (2.2.1)$$

for i,j = 1,2. Where $k:(o,\varepsilon) \longrightarrow [0,1)$ is a function having property (P). Then there exists a common fixed point of T_1 and T_2.

PROOF

Let x_o be an arbitrary, but a fixed element of X. We shall construct a sequence $\{x_n\}$ of points of X as follow. Let $x_1 \in X$ be such that $x_1 \in T_1 x_o$ and let

$$x_o = z_{(1,0)}, z_{(1,1)}, z_{(1,2)}, \ldots, z_{(1,m)} = x_1 \in T_1 x_o$$

be an arbitrary ε-chain from x_o to x_1. Rename x_1 as $z_{(2,0)}$.

Since $z_{(2,0)} \in T_1 z_{(1,0)}$ and

$$H(T_1 z_{(1,0)}, T_2 z_{(1,1)}) < k[d(z_{(1,0)}, z_{(1,1)})] d(z_{(1,0)}, z_{(1,1)})$$

$$< d(z_{(1,0)}, z_{(1,1)}) < \varepsilon.$$

Lemma 1.1.5, implies that there exists

$z_{(2,1)} \in T_2\, z_{(1,1)}$ such that

$$d(z_{(2,0)}, z_{(2,1)}) < k\{d(z_{(1,0)}, z_{(1,1)})] \; d(z_{(1,0)}, z_{(1,1)})$$

$$< d(z_{(1,0)}, z_{(1,1)}) < \varepsilon.$$

Since $z_{(2,1)} \in T_2 z_{(1,1)}$ and

$$H(T_2 z_{(1,1)}, T_2 z_{(1,2)}) < k[d(z_{(1,1)}, z_{(1,2)})] \; d(z_{(1,1)}, z_{(1,2)})$$

$$< d(z_{(1,1)}, z_{(1,2)}) < \varepsilon.$$

We may choose an element $z_{(2,2)} \in T_2 z_{(1,2)}$ such that

$$d(z_{(2,1)}, z_{(2,2)}) < k[d(z_{(1,1)}, z_{(1,2)})] \; d(z_{(1,1)}, z_{(1,2)})$$

$$< d(z_{(1,1)}, z_{(1,2)}) < \varepsilon$$

Thus we obtain a finite set of points

$$x_1 = z_{(2,0)}, z_{(2,1)}, z_{(2,2)}, \ldots, z_{(2,m)} = x_2 \in T_2 x_1,$$

such that

$z_{(2,0)} \in T_1 z_{(1,0)}$ and $z_{(2,j)} \in T_2 z_{(1,j)}$ for $j = 1,2,3 \ldots m-1$,

with $d(z_{(2,j)} z_{(2,j+1)}) < k[d(z_{(1,j)}, z_{(1,j+1)})] d(z_{(1,j)}, z_{(1,j+1)})$

$$< d(z_{(1,j)}, z_{(1,j+1)}) < \varepsilon.$$

In particular, $z_{(2,m)} \in T_2 z_{(1,m)} = T_2 x_1$ and we let $x_2 = z_{(2,m)}$. Thus the set of points

$$x_1 = z_{(2,0)}, \quad z_{(2,1)}, \quad z_{(2,2)}, \ldots, z_{(2,m)} = x_2 \in T_2 x_1$$

is an ε-chain from x_1 to x_2.

Rename x_2 as $z_{(3,0)}$, then by the same procedure we obtain an ε-chain

$$x_2 = z_{(3,0)}, \quad z_{(3,1)}, \quad z_{(3,2)}, \ldots, \quad z_{(3,m)} = x_3 \in T_1 x_2$$

from x_2 to x_3.

Inductively, we obtain

$$x_{2n} = z_{(2n+1,0)}, z_{(2n+1,1)}, z_{(2n+1,2)}, \ldots, z_{(2n+1m)} = x_{2n+1} \in T_1 x_{2n}$$

and

$$x_{2n+1} = z_{(2n+2,0)}, z_{(2n+2,1)}, z_{(2n+2,2)}, \ldots, z_{(2n+2,m)} = x_{2n+2} \in T_2 x_{2n+1}$$

with

$$d(z_{(n+1,j)}, z_{(n+1,j+1)}) < k[d(z_{(n,j)}, z_{(n,j+1)})] d(z_{(n,j)}, z_{(n,j+1)})$$
$$< d(z_{(n,j)}, z_{(n,j+1)}) < \varepsilon, \qquad (2.2.2)$$

for $j = 0\ 1,2,3,\ldots,m-1$ and $n = 0\ 1,2,\ldots$.

Consequently, we obtain a sequence $\{x_n\}$ of points of X with

$$x_1 = z_{(1,m)} = z_{(2,0)} \in T_1 x_0 \ ,$$

$$x_2 = z_{(2,m)} = z_{(3,0)} \in T_2 x_1 \ ,$$

$$x_3 = z_{(3,m)} = z_{(4,0)} \in T_1 x_2 \ ,$$

. . . .

. . . .

. . . .

$$x_{2n+1} = z_{(2n+1,m)} = z_{(2n+2,0)} \in T_1 x_{2n},$$

$$x_{2n+2} = z_{(2n+2,m)} = z_{(2n+3,0)} \in T_2 x_{2n+1}.$$

Then we have

(i) $\lim_{n\to\infty} d(z_{(n,j)}, z_{(n,j+1)}) = 0$, for $j = 0, 1, 2, \ldots m-1$,

(ii) $\lim_{n\to\infty} d(x_{n-1}, x_n) = 0$

and

(iii) $\{x_n\}$ is a Cauchy sequence.

<u>Proof of (i)</u>

From (2.2.2), we see that $\lim_{n\to\infty} d(z_{(n,j)}, z_{(n,j+1)})$ exists and must be a number in $[0,\varepsilon)$. Let $\lim_{n\to\infty} d(z_{(n,j)}, z_{(n,j+1)}) = t$. If $t>0$, then by definition of k there exists $\delta(t) > 0$, $s(t) < 1$ such that $0 \le r - t < \delta(t)$ implies $k(r) \le s(t) < 1$. For this $\delta(t) > 0$, there exists an integer N such that $0 \le d(z_{(n,j)}, z_{(n,j+1)}) - t < \delta(t)$ for $n > N$. Hence

$$k[d(z_{(n,j)}, z_{(n,j+1)})] \le s(t) < 1 \text{ whenever } n \ge N .$$

Let $M = \max \{k_0, k_1, \ldots k_N, s(t)\} < 1$ where $k_i = k[d(z_{(i,j)}, z_{(i,j+1)})]$ for $i = 0\ 1, 2, \ldots N$. Then

$$d(z_{(n,j)}, z_{(n,j+1)}) < k[d(z_{(n-1,j)}, z_{(n-1,j+1)})]$$
$$d(z_{(n-1j)}, z_{(n-1,j+1)})$$

$$\leq M\, d(z_{(n-1,j)},\ z_{(n-1,j+1)}) \text{ for } n = 1,2,3\ldots$$

Thus $d(z_{(n,j)},\ z_{(n,j+1)}) \leq M^n d(z_{(0,j)},\ z_{(0,j+1)}) \longrightarrow 0,$

that is a contradiction to the fact that $t > 0$. Consequently,

$$t = \lim_{n\to\infty} d(z_{(n,j)},\ z_{(n,j+1)}) = 0.$$

Proof of (ii)

$$d(x_{n-1}, x_n) = d(z_{(n,0)},\ z_{(n,m)})$$

$$\leq \sum_{j=0}^{m-1} d(z_{(n,j)} z_{(n,j+1)}) \longrightarrow 0.$$

Proof of (iii)

Assume that $\{x_n\}$ is not a Cauchy sequence. Then there exists a number $t > 0$ (We may assume $t < \varepsilon$ without loss of generality) and two sequences $\{n_j\},\{m_j\}$ of natural number with $n_j < m_j$ and such that

$$d(x_{n_j}, x_{m_j}) \geq t,\ d(x_{n_j}, x_{m_j-1}) < t \quad \text{for } j = 1,2,3,\ldots,$$

Then $t \leq d(x_{n_j}\ x_{m_j}) \leq d(x_{n_j}, x_{m_j-1}) + d(x_{m_j-1}, x_{m_j})$

$$< t + d(x_{m_j-1}, x_{m_j}).$$

Since letting $j \longrightarrow \infty$, $d(x_{m_j-1}, x_{m_j}) \longrightarrow 0$, we get

$$\lim_{j\to\infty} d(x_{n_j}, x_{m_j}) = t \in (0,\varepsilon).$$

For this $t > 0$, by definition of k there exists $\delta(t) > 0$, $s(t) < 1$ such that $0 \le r-t < \delta(t)$ implies $k(r) \le s(t) < 1$. For this $\delta(t)>0$, there exists an integer N such that $j \ge N$ implies $0 \le d(x_{n_j}, x_{m_j}) - t < \delta(t)$ and hence $k[d(x_{n_i}, x_{m_j})] < s(t)$ if $i \ge N$. Thus

$$
\begin{aligned}
d(x_{n_j}, x_{m_j}) &\le d(x_{n_j}, x_{n_j+1}) + d(x_{n_j+1}, x_{m_j+1}) + d(x_{m_j+1}, x_{m_j}) \\
&\le d(x_{n_j}, x_{n_j+1}) + k[d(x_{n_j}, x_{m_j})]\, d(x_{n_j}, x_{m_j}) \\
&\quad + d(x_{m_j+1}, x_{m_j})
\end{aligned}
$$

Since $k[d(x_{n_j}, x_{m_j})] < s(t)$, by letting $j \longrightarrow \infty$, we get

$t \le s(t) < t$. Hence a contradiction. Thus $\{x_n\}$ is a Cauchy sequence. Since X is a complete metric space, therefore $x_n \longrightarrow p \in X$. It implies that $x_{2n} \longrightarrow p$ and $x_{2n+1} \longrightarrow p$ Hence there exists an integer $N_1 > 0$ such that $n > N_1$ implies $d(x_{2n}, p) < \varepsilon$ and $d(x_{2n+1}, p) < \varepsilon$. Thus for $n \ge N_1$, we have

$$H(T_1 x_{2n}, T_2 p) < \varepsilon \text{ and } H(T_2 x_{2n+1}, T_1 p) < \varepsilon .$$

Consequently $H(T_1 x_{2n}, T_2 p) \longrightarrow 0$ and $H(T_2 x_{2n+1}, T_1 p) \longrightarrow 0$.

Since $x_{2n+1} \in T_1 x_{2n}$ and $x_{2n+2} \in T_2 x_{2n+1}$ with $d(x_{2n+1}, p) \longrightarrow 0$ and $d(x_{2n+2}, p) \longrightarrow 0$. Lemma 1.1.5 implies that $p \in T_1 p$ and $p \in T_2 p$.

COROLLARY 2.2.6

Let (X,d) be a complete metric space. Suppose $T_1: X \longrightarrow CB(X)$, $T_2: X \longrightarrow CB(X)$ are two mappings such that for all x,y in X,

$$H(T_i x, T_j y) < k[d(x,y)]d(x,y) \text{ for } i,j = 1,2,$$

where $k : (0,\infty) \longrightarrow [0,1)$ is a function having property (P). Then T_1 and T_2 have a common fixed point.

The condition that $H(T_i x, T_j y) < k(d(x,y))d(x,y)$ as stated in Theorem 2.2.5 and Corollary 2.2.6 can be replaced by $H(Tx,Ty) \leq k[d(x,y)]d(x,y)$ without affecting the validity of the Theorem; Because if the function $k: (0,b) \longrightarrow [0,1)$ has property(P) then the function $g: (0,b) \longrightarrow [0,1)$ defined by $g(t) = (k(t))^{1/2}$ also has property (P).

Several other results may also be seen to follow as immediate corollaries to Theorem 2.2.5. Included among these are following:

Edelstein, Theorem 5.2 [15], Hu, theorem 2 and 3 [23],
Ko and Tsai [38], Kuhfittig, Theorem 1 [39],
Nadler, Theorem 5 and 6 [43] and Reich Theorem 1 [48].

2.3. COINCIDENCE POINTS OF COMPATIBLE MULTIVALUED MAPPINGS

In Chapter 1, the concept of compatible single valued mappings has been extended to multivalued mappings. In this section we use it to obtain coincidence points for four mappings, satisfying a contractive type condition. Our result generalizes the theorems of Jungck [27], Keneko [33] and Nadler [43].

THEOREM 2.3.1

Let X be a complete metric space. Let $f,g: X \longrightarrow X$ and $S,T: X \longrightarrow CB(X)$ be continuous mappings such that f is compatible with S, and g is compatible with T. If $SX \subseteq gX$, $TX \subseteq fX$ and for all $x,y \in X$,

$$H(Sx,Ty) \leq \lambda\, d(fx,gy), \quad \text{where} \quad 0 < \lambda < 1.$$

Then there is a common coincidence point of f and S, and g and T.

PROOF

Let x_o be an arbitrary but fixed element of X. We shall construct two sequences $\{x_n\}$, $\{y_n\}$ of elements in X and a sequence $\{A_n\}$ of elements in $CB(X)$ as follow.

Let $x_1 \in X$ be such that $y_1 = gx_1 \in Sx_o$. Using the Lemma 1.1.4 and the fact that $Tx \subseteq fX$, we may choose $x_2 \in X$ such that $y_2 = fx_2 \in Tx_1 = A_1$ and $d(y_1,y_2) = d(gx_1, fx_2) \leq H(Sx_o, Tx_1) + \lambda$.

Since $SX \subseteq gX$, we may choose $x_3 \in X$ such that

$y_3 = gx_3 \in Sx_2 = A_2$ and $d(y_2, y_3) = d(fx_2, gx_3) \leq H(Tx_1, Sx_2) + \lambda^2$. By induction we produce the sequences $\{x_n\}, \{y_n\}$, and $\{A_n\}$, such that

$$y_{2k+1} = gx_{2k+1} \in Sx_{2k} = A_{2k} ,$$

and

$$y_{2k+2} = fx_{2k+2} \in Tx_{2k+1} = A_{2k+1} ,$$

where k is any positive integer. Furthermore,

$$\begin{aligned} d(y_{2k+1} , y_{2k+2}) &= d(gx_{2k+1} , gx_{2k+2}) \\ &\leq H(Sx_{2k}, Tx_{2k+1}) + \lambda^{2k+1} , \end{aligned}$$

and

$$\begin{aligned} d(y_{2k+2} , y_{2k+3}) &= d(fx_{2k+2} , gx_{2k+3}) \\ &\leq H(Tx_{2k+1} , Tx_{2k+2}) + \lambda^{2k+2} . \end{aligned}$$

Hence,

$$\begin{aligned} d(gx_{2k+1} , fx_{2k+2}) &\leq \lambda\, d(fx_{2k}, gx_{2k+1}) + \lambda^{2k+1} \\ &\leq \lambda\{H(Tx_{2k-1}, Sx_{2k}) + \lambda^{2k}\} + \lambda^{2k+1} \\ &\leq \lambda^2 d(gx_{2k-1}, fx_{2k}) + 2\, \lambda^{2k+1} \\ &\leq \lambda^{2k}\, d(gx_1, fx_2) + 2k\, \lambda^{2k+1} . \end{aligned} \quad (2.3.1)$$

Similarly,

$$d(fx_{2k+2}, gx_{2k+3}) \leq \lambda^{2k+1}\, d(gx_1, fx_2) + (2k+1)\, \lambda^{2k+2}$$

(2.3.2)

Then $\{y_n\}$ is a Cauchy sequence. For this, let $m > n$ and m and n are of opposite parity, that is $n = 2p + 1$ and $m = 2q$ for $p > 0$, $q > 0$.

$$
\begin{aligned}
d(y_m, y_n) &= d(y_{2p+1}, y_{2p}) \\
&= d(gx_{2p+1}, fx_{2q}) \\
&\le d(gx_{2p+1}, fx_{2p+2}) + d(fx_{2p+2}, gx_{2p+3}) \\
&\quad +\ldots+ d(gx_{2q-1}, fx_{2q}) .
\end{aligned}
$$

Now (2.3.1) and (2.3.2) implies that

$$
\begin{aligned}
d(y_m, y_n) \le\ & (\lambda^{2p} d(gx_1, fx_2) + 2p\ \lambda^{2+1}) + (\lambda^{2p+1} d(gx_1, fx_2) + \\
& + (2p+1)\ \lambda^{2p+2}) +\ldots+ (\lambda^{2q-2} d(gx_1, fx_2) + \\
& + (2q-2)\ \lambda^{2q-1}) .
\end{aligned}
$$

It further implies that

$$
d(y_{2p+1}, y_{2q}) \le \sum_{i=2p}^{2q-2} [\lambda^i d(gx_1, fx_2) + i\ \lambda^{i+1}] . \qquad (2.3.3)
$$

Similarly, if m and n are of like parity, then

$$
\begin{aligned}
d(y_{2p}, y_{2q}) &\le d(fx_{2p}, fx_{2p}) \\
&\le d(fx_{2p}, gx_{2p+1}) + d(gx_{2p+1}, fx_{2p+2}) \\
&\quad +\ldots+ d(gx_{2q-1}, fx_{2q}) . \\
&\le \sum_{i=2p-1}^{2q-2} [\lambda^i d(gx_1, fx_2) + i\ \lambda^{i+1}] \qquad (2.3.4)
\end{aligned}
$$

and

$$
d(y_{2p+1}, y_{2q+1}) \le \sum_{i=2p}^{2q-1} [\lambda^i d(gx_1, fx_2) + i\ \lambda^{i+1}] . \qquad (2.3.5)
$$

It follows that the sequence $\{y_n\}$ is a Cauchy sequence. Hence there exists z in x such that $y_n \longrightarrow z$. Therefore,

$$gx_{2k+1} \longrightarrow z \quad \text{and} \quad fx_{2k+2} \longrightarrow z. \text{ Moreover,}$$

$$gfx_{2k+2} \longrightarrow gz \ , \quad Sfx_{2k+2} \longrightarrow Sz \ ,$$

$$fgx_{2k+1} \longrightarrow fz \ , \quad \text{and} \quad Tgx_{2k+1} \longrightarrow Tz \ .$$

Now using (2.3.3), we have

$$\begin{aligned} H(A_{2p+1}, A_{2q}) &= H(Tx_{2p+1}, Sx_{2q}) \\ &\leq \lambda\, d(gx_{2p+1}, fx_{2q}) \\ \leq \sum_{i=2p}^{2q-2} &[\lambda^{i+1}\, d(gx_1, fx_2) + i\, \lambda^{i+2}] \ . \end{aligned}$$

Similarly, using (2.3.4) and (2.3.5), we have

$$H(A_{2p}, A_{2q}) \leq \sum_{i=2p-1}^{2q-2} [\lambda^{i+1}\, d(gx_1, fx_2) + i\, \lambda^{i+2}],$$

and

$$H(A_{2p+1}, A_{2q+1}) \leq \sum_{i=2p}^{2q-1} [\lambda^{i+1}\, d(gx_1, fx_2) + i\, \lambda^{i+2}].$$

Hence $\{A_n\}$ is a Cauchy sequence. Since $(CB(X),H)$ is a complete metric space, therefore there exists $A \in CB(X)$ such that $A_n \longrightarrow A$.

It implies that,

$$Tx_{2k+1} \longrightarrow A, \ Sx_{2k+2} \longrightarrow A \ ,$$

and

$$\begin{aligned} d(z,A) &= \lim_{n\to\infty} d(y_n, A) \\ &\leq \lim_{n\to\infty} H(A_{n-1}, A) = 0 \ . \end{aligned}$$

It further implies that

$$\lim_{k\to\infty} fx_{2k} = z \in A = \lim_{k\to\infty} Sx_{2k} .$$

Hence by compatibility of f and S, we have

$$\lim_{k\to\infty} H(fSx_{2k}, Sfx_{2k}) = 0. \text{ Therefore}$$

$$\lim_{k\to\infty} d(fy_{2k+1}, Sy_{2k}) = 0. \text{ Hence}$$

$$d(fz, Sz) = 0. \text{ That is } fz \in Sz.$$

Similarly, $gz \in Tz$.

COROLLARY 2.3.2

If in addition to the hypothesis of the theorem 1, the following condition is satisfied:

$$fz \in Sz,\ gz \in Tz \text{ implies } \lim_{n\to\infty} f^n z = \lim_{n\to\infty} g^n z = t .$$

Then t is a common fixed point of f, g, S and T.

PROOF

Obviously $f \lim_{n\to\infty} f^n z = ft$ and

$$g \lim_{n\to\infty} g^n z = gt. \text{ Therefore } t = ft = gt.$$

Consider a constant sequence $u_n = z$, then $u_n \to z$, $fu_n \longrightarrow fz$ and $Su_n \longrightarrow Sz$. By theorem $\lim_{n\to\infty} fu_n \in \lim_{n\to\infty} Su_n$, therefore by compatibility of f and S, we have

$$H(f\ Sz,\ S\ fz) = \lim_{n\to\infty} H(f\ Su_n,\ S\ fu_n) = 0 .$$

It implies that

$$f\ fz \in f\ Sz = S\ fz\ . \qquad (2.3.6)$$

Now consider an other constant sequence $v_n = fz$ then $v_n \longrightarrow fz$, $fv_n \longrightarrow f^2z$ and $Sv_n \longrightarrow Sfz$. By (2.3.6) $\lim_{n\to\infty} fv_n \in \lim_{n\to\infty} Sv_n$, therefore compatibility of f and S implies that

$$H(f\ S\ fz,\ S\ f^2z) = \lim_{n\to\infty} H(f\ Sv_n,\ Sfv_n) = 0\ .$$

Hence $f^3z \in f\ S\ fz = Sf^2z$. Consequently, we obtain, $f^{n+1} \in S\ f^nz$. It further implies that $t \in St$. Similarly, $t \in Tt$.

EXAMPLE 2.3.3

Let $X = [0,\ \infty]$ be with Euclidean metric, $Sx = [0,x]$, $Tx = [0,\ Sx]$, $fx = 4x$ and $gx = 20x$. We have for all $x,y \in X$:

$$\begin{aligned} H(Sx,\ Ty) &= |x - 5y| \\ &= \frac{1}{4}\ |4x - 20y| \\ &= \frac{1}{4}\ d(fx,\ gy). \end{aligned}$$

It is easily seen that for any $\lambda \in [\frac{1}{4},1]$, all the hypotheses of Theorem 2.3.1 (and Corollary 2.3.2) are satisfied. Since $f \neq g$ and $S \neq T$, we cannot apply the theorems of Kaneko [33]. Moreover, S and T are not contractions, therefore the result of Nadler [43] is also not applicable (even in the case $Sx = Tx = [0,x]$ and $f = g = 4x$).

EXAMPLE 2.3.4

Let $X = [0,1]$ be with Euclidean metric, $Sx = [\frac{x}{x+10}]$, $Tx = [0, \frac{x}{x+16}]$ $fx = \frac{x}{5}$ and $gx = \frac{x}{8}$. For $x, y \in X$, we have

$$
\begin{aligned}
H(Sx, Ty) &= \left|\frac{x}{x+10} - \frac{y}{y+16}\right| \\
&= \left|\frac{x(y+16) - y(x+10)}{(x+10)(y+16)}\right| \\
&\leq \left|\frac{16x - 10y}{160}\right| \\
&= \left|\frac{x}{10} - \frac{y}{16}\right| \\
&= \frac{1}{2}\left|\frac{x}{5} - \frac{y}{8}\right| \\
&= \frac{1}{2}\, d(fx, gy) .
\end{aligned}
$$

It suffices to assume $\lambda = \frac{1}{2}$, in order to satisfy all assumptions of Theorem 2.3.1 (and Corollary 2.3.2).

Note that $f \neq g$, $S \neq T$, $fS \neq Sf$ and $gT \neq Tg$. Previously known results are not applicable to this example.

CHAPTER THREE

FIXED POINTS OF ASYMPTOTICALLY REGULAR MULTIVALUED MAPPINGS

Let T be a single valued self mapping on a metric space X. A sequence $\{x_n\}$ in X is said to be asymptotically T-regular if $d(x_n, Tx_n) \longrightarrow 0$. The presence of a sequence $\{x_n\}$ for which $d(x_n, Tx_n) \longrightarrow 0$ is related to some property of T (see [3],[16],[17],[50],[51], and [54]) and hence exploited to obtain fixed point of T. The aim of the present chapter is to bring out the thrust of a similar assumption for multivalued mappings. The weakly dissipative multivalued mappings recently introduced by Aubin and Siegel in [3] do satisfy such an assumption. As stated in Aubin and Siegel [3], such fixed point theorems have application to control theory, system theory and optimization problems. Moreover, such a sequence (for multivalued mappings) has been used by Itoh and Takahashi [26] and Rhoades, Singh and Kulshrestha [52], Kaneko [35] and Rhoades [56], have compared these contractive conditions. Most of the contractive conditions used imply the asymptotic regularity of mappings under consideration. So the study of such mappings play an important role in fixed point theory. In section 3.1 we prove the existence of a common fixed point of two multivalued mappings satisfying a contractive type condition in a metric space. In section 3.2 a class of multivalued mappings is introduced which is larger (even in the case of single valued mappings) than those that Wong [63] refers to as Kannan mappings. The fixed point theorems therein, are proved under less restrictive hypothesis and for wider classes than the results of Shiau, Tan and Wong [60]. In section 3.3 we obtain coincidence theorem for a pair of compatible multivalued mappings. The structure of common fixed points of these mappings is also studied. Section 3.4 deals with the study of fixed points of Meir-Keeler type multivalued mappings. The results proved in sections 3.1, 3.2 and 3.3 will appear in J. Austral. Math. Soc., (Series A).

3.1 COMMON FIXED POINT OF MULTIVALUED GENERALIZED CONTRACTIONS

Wong [62] extended the result of Hardy and Rogers [18] by showing that two self mappings S and T on a complete metric space, satisfying a contractive type condition have a common fixed point. In this section we extend this result of Wong to the case when S and T are multivalued and satisfy a more general contractive type condition.

THEOREM 3.1.1

Let X be a complete metric space, $S : X \longrightarrow CB(X)$ and $T: X \longrightarrow CB(X)$. If there exists a constant α, $0 \le \alpha < 1$, such that for each $x, y \in X$,

$$H(Tx, Sy) \le \alpha \max \{d(x,y),\ d(x,Sx),\ d(y,Ty),\ \frac{d(x,Ty) + d(y,Sx)}{2}\}. \qquad (3.1.1)$$

Then there exists a common fixed point of S and T.

PROOF

Assume that $\beta = \sqrt{\alpha}$.

Let x_o be an arbitrary but fixed element of X and choose $x_1 \in Sx_o$, then

$$H(Sx_o, Tx_1) < \beta \max \{d(x_o,x_1),\ d(x_o,Sx_o),\ d(x_1,Tx_1),\ \frac{d(x_o,Tx_1) + d(x_1,Sx_o)}{2}\}$$

Lemma 1.1.5 implies that there exists a point $x_2 \in Tx_1$ such that,

$$d(x_1,x_2) < \beta \max \{d(x_o,x_1),\ d(x_o,Sx_o),\ d(x_1,Tx_1),\ \frac{d(x_o,Tx_1) + d(x_1,Sx_o)}{2}\}$$

$$< \beta \max \{d(x_o,x_1),\ d(x_1,x_2),\ \frac{d(x_o,x_1) + d(x_1,x_2)}{2}\}$$

If $d(x_1,x_2) > d(x_o,x_1)$. Then

$d(x_1,x_2) \le \beta\, d(x_1,x_2)$, a contradiction. Thus

$d(x_1,x_2) < \beta\, d(x_0,x_1)$.

Now,

$$H(Tx_1,Sx_2) < \beta \max\{d(x_1,x_2),\ d(x_1,Tx_1),\ d(x_2,Sx_2),\ \frac{d(x_1,Sx_2) + d(x_2,Tx_1)}{2}\}.$$

Again using Lemma 1.1.5, we obtain a point $x_3 \in Sx_2$ such that,

$$d(x_2,x_3) < \beta \max\{d(x_1,x_2),\ d(x_1,Tx_1),\ d(x_2,Sx_2),\ \frac{d(x_1,Sx_2) + d(x_2,Tx_1)}{2}\}$$

$$< \beta\, d(x_1,x_2) .$$

By induction we produce a sequence $\{x_n\}$ of points of X, such that, for $k \ge 0$

$$x_{2k+1} \in Sx_{2k} \ ,\ x_{2k+2} \in Tx_{2k+1}$$

and

$$d(x_n,x_{n+1}) \le \beta\, d(x_{n-1},x_n)$$

$$\le \beta^n\, d(x_0,x_1).$$

Furthermore, for $m > n$

$$d(x_n,x_m) \le d(x_n,x_{n+1}) + d(x_{n+1},x_{n+2}) + \dots$$

$$+ d(x_{m-1},x_m)$$

$$\le \{\beta^n+\beta^{n+1} + \dots + \beta^{m-1}\}\, d(x_0,x_1) .$$

It follows that $\{x_n\}$ is a Cauchy sequence and there exists a point $t \in X$ such that $x_n \longrightarrow t$. It further implies that $x_{2k+1} \longrightarrow t$, and $x_{2k+2} \longrightarrow t$. Thus we have,

$$d(t,St) \le d(t,x_{2k+2}) + d(x_{2k+2},St),$$

$$\le d(t,x_{2k+2}) + H(Tx_{2k+1},St),$$

$$\le d(t,x_{2k+2}) + \beta \max \{d(x_{2k+1},t), d(t,St),$$

$$\le d(x_{2k+1},x_{2k+2}), \quad \frac{d(t,x_{2k+2}) + (x_{2k+1},St)}{2} \}.$$

Letting $k \longrightarrow \infty$, we have $d(t,St) \le \beta\, d(t,St)$. Hence $t \in St$. Similarly,

$$d(t,Tt) \le d(t,x_{2k+1}) + H(Sx_{2k},Tt) \le \beta\, d(t,Tt).$$

Therefore $t \in Tt$.

COROLLARY 3.1.2

Let X be a complete metric space and $T: X \longrightarrow CB(X)$. If there exists a constant α, $0 \le \alpha < 1$, such that for each $x,y \in X$ $H(Tx,Ty) \le \alpha \max \{d(x,y), d(x,Tx), d(y,Ty), \frac{d(x,Ty) + d(y,Tx)}{2}\}$. Then there exists a sequence $\{x_n\}$ which is asymptotically T-regular and converges to a fixed point of T.

REMARK 3.1.3

Theorem 3.1 improved the results of Kaneko [35], which considered the mapping T of a reflexive Banach space X into the family of weakly compact subsets of X. The proximinality of the set Tx is a consequence of his assumption and it is used in his proof. No such assumption is required in Theorem 3.1.1.

REMARK 3.1.4

In [47], Ray proved a fixed point theorem for a multivalued mapping T: $X \longrightarrow CB(X)$ satisfying:

$$H(Tx,Ty) \le a\, d(x,y) + b\{d(x,Tx) + d(y,Ty)\}$$
$$+ c\{d(x,Ty) + d(y,Tx)\},$$

where a,b and c are non-negative real numbers and $0<a+2b+c<1$. Theorem 3.1.1 (even in the particular case for $S = T$) is a generalization of the theorem of Ray [47] since T is not assumed to have closed graph. It also illustrates that the compactness of Tx is not necessary for the theorem of Aubin and Siegel [3].

Several other results may also be seen to follow as immediate corollaries to Theorem 3.1.1. Included among these are Dube theorem 1 [12], Dube and Singh theorem 1 [13], Hardy and Rogers [18], Iseki [24], Nadler theorem 5 [43], and Wong [62].

3.2. FIXED POINT OF KANNAN TYPE MULTIVALUED MAPPINGS

In this section we consider the mapping $T: X \longrightarrow CB(X)$ satisfying the condition:

$$H(Tx, TY) \leq \alpha_1 (d(x, Tx))\, d(x, Tx) + \alpha_2 (d(y, Ty))\, d(y, Ty),$$

where $\alpha_i : R \longrightarrow [0,1)$, $(i = 1,2)$. Mapping T is not a special case of the mapping considered in section 3.1. In 1968 Kannan [31] had established a fixed point theorem for a single valued mapping T defined on a complete metric space X satisfying

$$d(Tx, Ty) \leq \alpha \{d(x, Tx) + d(y, Ty)\},$$

where $0 < \alpha < \frac{1}{2}$ and $x, y \in X$. Within the context of a complete metric space the assumption $0 < \alpha < \frac{1}{2}$ is crucial even to the existence part of this result, but within a more restrictive yet quite natural setting, an elaborate fixed point theory exists for the case $\alpha = \frac{1}{2}$. Mappings of this wider class were studied by Kannan in [32]. In recent years, Beg and Azam [4], Shiau Tan and Wong [60] and Wong [62] have also studied such mapping.

THEOREM 3.2.1

Let X be a complete metric space and $T: X \longrightarrow CB(X)$ a mapping satisfying:

$$H(Tx,Ty) \le \alpha_1(d(x,Tx))\, d(x,Tx) + \alpha_2(d(y,Ty))\, d(y,Ty), \quad (3.2.1)$$

for all $x, y \in X$, where $\alpha_i : R \longrightarrow [0,1)$ $(i = 1,2)$.
If there exists an asymptotically T-regular sequence $\{x_n\}$ in X, then T has a fixed point x^* in X. Moreover $Tx_n \longrightarrow Tx^*$.

PROOF

By hypothesis, we have

$$H(Tx_n, Tx_m) \le \alpha_1(d(x_n, Tx_n))\, d(x_n, Tx_n)$$
$$+\alpha_2(d(x_m, Tx_m))\, d(x_m, Tx_m).$$

Thus $\{Tx_n\}$ is a Cauchy sequence. Since $(CB(X),H)$ is complete, there exists a $K^* \in CB(X)$, such that

$H(Tx_n, K^*) \longrightarrow 0$. Let $x^* \in K^*$, then

$$\begin{aligned} d(x^*, Tx^*) &\le H(K^*, Tx^*) \\ &= \lim_{n\to\infty} H(Tx_n, Tx^*) \\ &\le \lim_{n\to\infty} \{\alpha_1(d(x_n, Tx_n))\, d(x_n, Tx_n) \\ &\quad + \alpha_2(d(x^*, Tx^*))\, d(x^*, Tx^*)\} \\ &\le \alpha_2(d(x^*, Tx^*))\, d(x^*, Tx^*). \end{aligned}$$

It further implies that,

$$\{1-\alpha_2(d(x^*, Tx^*))\}\, d(x^*, Tx^*) \le 0.$$

Therefore $d(x^*, Tx^*) = 0$. Thus $x^* \in Tx^*$. Now,

$$H(K, Tx^*) = \lim_{n\to\infty} H(Tx_n, Tx^*)$$
$$\le \alpha_2(d(x^*, Tx^*))\, d(x^*, Tx^*)$$
$$\le d(x^*, Tx^*) = 0 .$$

It follows that,

$$Tx^* = K^* = \lim_{n\to\infty} Tx_n .$$

<u>THEOREM 3.2.2</u>

Let X be a complete metric space and $T: X \longrightarrow CB(X)$ a mapping satisfying (3.2.1). If there exists an asymptotically T-regular sequence $\{x_n\}$ in X and Tx_n is compact for each n, then each cluster point of $\{x_n\}$ is a fixed point of T.

<u>PROOF</u>

Let $y_n \in Tx_n$ be such that $d(x_n, y_n) = d(x_n, Tx_n)$. Obviously, a cluster point of $\{x_n\}$ is a cluster point of $\{y_n\}$. If y^* is such a cluster point of $\{x_n\}$ and $\{y_n\}$, then with x^* (as in Theorem 3.2.1),

$$d(y_n, Tx^*) \le H(Tx_n, Tx^*)$$
$$\le \alpha_1(d(x_n, Tx_n))\, d(x_n, Tx_n)$$
$$+ \alpha_2(d(x^*, Tx^*))\, d(x^*, Tx^*)$$
$$\le \alpha_1(d(x_n, Tx_n))\, d(x_n, Tx_n).$$

Therefore $y^* \in Tx^*$. Now,

$$d(y^*, Ty^*) \le H(Tx^*, Ty^*)$$
$$\le \alpha_1(d(x^*, Tx^*))\, d(x^*, Tx^*)$$
$$+ \alpha_2(d(y^*, Ty^*))\, d(y^*, Ty^*).$$

It follows that, $(1-\alpha_2(d(y^*, Ty^*)))\, d(y^*, Ty^*) \le 0$.
Hence $y^* \in Ty^*$.

Theorems 3.2.1, and 3.2.2 generalize results of Shiau, Tan and Wong [60]. Here we desire to emphasize that not only that our T belongs to a wider class of mappings but also that the hypothesis of compactness of Tx (in theorem 1 of [60]) is dropped.

3.3. FIXED POINTS OF GENERALIZED MULTIVALUED f-CONTRACTIONS

Jungck [28] introduced a contraction condition for single valued compatible mappings on a metric space. He also pointed out in [29] and [30] the potential of compatible mappings for generalized fixed point theorems. Subsequently a variety of extensions, generalizations and applications of this followed; e.g. see [1],[57] and [59]. This section is a continuation of these investigations for multivalued compatible mappings.

DEFINITION 3.3.1

Let X be a metric space. Mappings $T: X \longrightarrow CB(X)$,

$f: X \longrightarrow X$ are compatible if, whenever there is a sequence $\{x_n\} \subset X$ satisfying $\lim fx_n \in \lim Tx_n$ (provided $\lim fx_n$ exists in X and $\lim Tx_n$ exists in $CB(X)$), then $\lim H(fTx_n, Tfx_n) = 0$.

If T is a single valued self mapping on X, this definition of compatibility becomes that of Jungck [28]. Let $X = R$, with Euclidean metric, $Tx = [\frac{x^2}{4}, \frac{x^2}{2}]$, $fx = \frac{x^2}{8}$. Then f and T are compatible but they do not commute.

If $\phi: (0,\infty) \longrightarrow [0,1)$ is a function having property (P) (see definition 2.2.3).

The following Theorem is a generalization of Hu theorem 2[23], Jungck [27], Kaneko [33] and Nadler theorem 5[43].

THEOREM 3.3.1

Let T be a mapping from a complete metric space X into $CB(X)$. Let $f: X \longrightarrow X$ be a continuous mapping such that $TX \subseteq fX$. If f and T are compatible and for all $x, y \in X$,

$$H(Tx, Ty) < \phi(d(fx, fy))\, d(fx, fy), \qquad (3.3.1)$$

then there exists a sequence $\{x_n\}$ which is asymptotically T-regular with respect to f, and fx_n converges to a coincidence point of f and T.

PROOF

Let x_o be an arbitrary, but fixed element of X. We shall construct two sequences $\{x_n\}$ and $\{y_n\}$ of points of X as follow. Let $y_o = fx_o$ and $x_1 \in X$ be such that $y_1 = fx_1 \in Tx_o$, then inequality (3.3.1) implies that

$$H(Tx_o, Tx_1) < \phi(d(fx_o, fx_1))\, d(fx_o, fx_1).$$

Using the Lemma 1.1.5 and the fact that $TX \subseteq fX$, we may choose $x_2 \in X$ such that $y_2 = fx_2 \in Tx_1$ and

$$\begin{aligned} d(y_1, y_2) &= d(fx_1, fx_2) \\ &< \phi(d(fx_o, fx_1))\, d(fx_o, fx_1) \\ &< d(fx_o, fx_1). \end{aligned}$$

By induction we produce two sequences of points of X such that $y_n = fx_n \in Tx_{n-1}$, $n \geq 0$. Furthermore,

$$\begin{aligned} d(y_{n+1}, y_{n+2}) &= d(fx_{n+1}, fx_{n+2}) \\ &< \phi(d(fx_n, fx_{n+1}))\, d(fx_n, fx_{n+1}) \\ &< d(fx_n, fx_{n+1}) = d(y_n, y_{n+1}) . \end{aligned}$$

It follows that the sequence $\{d(y_n, y_{n+1})\}$ is decreasing and converges to its greatest lower bound which we denote by t. Now $t \geq 0$; in fact $t = 0$. Otherwise by property (P) of ϕ, there exists $\delta(t) > 0$, $s(t) < 1$, such that,

$$0 \leq r-t < \delta(t) \text{ implies } \phi(r) \leq s(t) .$$

For this $\delta(t) > 0$, there exists a natural number N such that,

$$0 \le d(y_n, y_{n+1})) - t < \delta(t), \text{ whenever } n \ge N.$$

Hence,

$$\phi(d(y_n, y_{n+1})) \le s(t), \text{ whenever } n \ge N.$$

Let $$K = \max \{\phi(d(y_o, y_1)), \phi(d(y_1, y_2)), \ldots, \phi(y_{N-1}, y_N), s(t)\}.$$

Then, for $n = 1,2,3,\ldots$

$$\begin{aligned} d(y_n, y_{n+1}) &< \phi(d(y_{n-1}, y_n)) \, d(y_{n-1}, y_n) \\ &\le K \, d(y_{n-1}, y_n) \\ &\le K^n \, d(y_o, y_1) \longrightarrow 0 \text{ as } n \longrightarrow \infty, \end{aligned}$$

which contradicts the assumption that $t > 0$. Consequently,

$$\lim_{n \to \infty} d(y_n, y_{n+1}) = 0 ,$$

which implies that, $d(fx_n, Tx_n) \longrightarrow 0$.

Hence the sequence $\{x_n\}$ is asymptotically T-regular with respect to f.

(ii) Assume that $\{fx_n\}$ is not a Cauchy sequence. Then there exists a positive number t^* and subsequences $\{n(i)\}$, $\{m(i)\}$ of the natural numbers with $n(i) < m(i)$ and such that,

$$d(y_{n(i)}, y_{m(i)}) \ge t^* , \; d(y_{n(i)}, y_{m(i)-1}) < t^* \text{ for } i = 1,2,3\ldots .$$

Then

$$\begin{aligned} t^* &\le d(y_{n(i)}, y_{m(i)}) \\ &\le d(y_{n(i)}, y_{m(i)-1}) + d(y_{m(i)-1}, y_{m(i)}). \end{aligned}$$

Letting $i \to \infty$ and using the fact that $d(y_{n(i)}, y_{m(i)-1}) < t^*$, we obtain $\lim_{i\to\infty} d(y_{n(i)}, y_{m(i)}) = t^*$. For this $t^* > 0$, there exists $\delta(t^*) > 0$, $s(t^*) < 1$, such that

$$0 \le r-t < \delta(t^*) \text{ implies } \phi(r) \le s(t^*).$$

For this $\delta(t^*) > 0$, there exists a natural number N_o such that,

$$i \ge N_o \text{ implies } 0 \le d(y_{n(i)}, y_{m(i)})-t^* < \delta(t^*) .$$

Hence $\qquad \phi(d(y_{n(i)}, y_{m(i)}) \le s(t^*) \text{for } i \ge N_o.$

Thus
$$\begin{aligned} d(y_{n(i)}, y_{m(i)}) &\le d(y_{n(i)}, y_{n(i)+1}) \\ &\quad + d(y_{n(i)+1}, y_{m(i)+1}) \\ &\quad + d(y_{m(i)+1}, y_{m(i)}) \\ &\le d(y_{n(i)}, y_{n(i)+1}) \\ &\quad + \phi(d(y_{n(i)}, y_{m(i)}))\, d(y_{n(i)}, y_{m(i)}) \\ &\quad + d(y_{m(i)+1}, y_{m(i)}) \\ &\le d(y_{n(i)}, y_{n(i)+1}) \\ &\quad + s(t^*)\, d(y_{n(i)}, y_{m(i)}) \\ &\quad + d(y_{m(i)+1}, y_{m(i)}) . \end{aligned}$$

Letting $i \longrightarrow \infty$, we get $t \le s(t^*)t^* < t^*$, a contradiction. Hence $\{fx_n\}$ is a Cauchy sequence. By completeness of the space, there exists an element $p \in X$ such that $d(y_n, p) \longrightarrow 0$. Continuity of f implies that $d(fy_n, fp) \longrightarrow 0$. Hence

$$H(Ty_n, Tp) < \phi(d(fy_n, fp))\, d(fy_n, fp)$$
$$< d(fy_n, fp) \longrightarrow 0 .$$

Inequality (3.3.1) and the fact that $\{fx_n\}$ is a Cauchy sequence imply that there exists $A \in CB(X)$ such that $Tx_n \longrightarrow A$. Furthermore,

$$d(p, A) \leq \lim_{n\to\infty} H(Tx_{n-1}, Tx_n) = 0 .$$

Now

$$d(fy_{n+1}, Ty_n) \leq H(fTx_n, Tfx_n) .$$

Letting $n \longrightarrow \infty$, we obtain

$$d(fp, Tp) = 0. \text{ Hence } fp \in Tp .$$

EXAMPLE 3.3.2

Let $X = [0,\infty)$ with the Euclidean metric $Tx = [0,x]$ and $fx = 10^4 x$. Then f and T do not satisfy the condition of the theorems in [23],[27] and [43]. Considering the function $\phi(x) = c$, where $10^{-4} < c < 1$, it is easily seen that all the hypotheses of Theorem 5.1 are valid. Thus f and T have a coincidence point.

EXAMPLE 3.3.3

Let $X = [0,1]$ with the Euclidean metric, $Tx = [0, \frac{x}{x+10}]$ and $fx = \frac{x}{5}$. Then all the hypotheses of Theorem

3.3.1 are valid and $f(0) \in T(0)$. Any previously known result is not applicable to this example since $fTx \neq Tfx$ at $x \neq 0$.

COROLLARY 3.3.4

If, in addition to the hypotheses of Theorem 3.3.1 the mapping f satisfies: for all $x, y \in X$,

$$d(fx,fy) \le \gamma \max \{d(x,y),\ d(x,fx),\ d(y,fy),\ \frac{d(x,fy) + d(y,fx)}{2}\}, \qquad (3.3.2)$$

where $0 \le \gamma < 1$, then there exists a common fixed point of f and T.

PROOF

Let $\beta = \sqrt{\gamma}$. As in the proof of Theorem 3.3.1 there is a coincidence point p of f and T. Define the iterative sequence $\{t_n\}$ as follow:

$t_o = p$ and $t_n = ft_{n-1} = f^n t_o$, $n = 1, 2, \ldots$ Now inequality (3.3.2) implies that

$$\begin{aligned} d(t_n, t_{n+1}) &= d(ft_{n-1}, ft_n) \\ &< \beta \max \{d(t_{n-1}, t_n),\ d(t_n, t_{n+1}),\ \frac{d(t_{n-1}, t_{n+1})}{2}\} \\ &< \beta\, d(t_{n-1}, t_n) \\ &< \beta^n\, d(t_o, t_1) \ . \end{aligned}$$

It further implies that $\{t_n\}$ is a Cauchy sequence. By the completeness of X, we have

$$f^n t_o \longrightarrow x^* \in X.$$

Now, consider a constant sequence $\{u_n\} \subset X$ as follows:

$$u_n = t_o, \text{ then}$$

$$\lim_{n\to\infty} fu_n = ft_o \in Tt_o = \lim_{n\to\infty} Tu_n .$$

Thus by the compatibility of f and T,

$$H(fTt_o, Tft_o) = \lim_{n\to\infty} H(fTu_n, Tfu_n) = 0 .$$

Hence, $f^2t_o = fft_o \in fTt_o = Tft_o$. Choose an other constant sequence,

$$v_n = ft_o. \text{ Then}$$

$$\lim_{n\to\infty} fv_n = f^2t_o \in Tft_o = \lim_{n\to\infty} Tv_n, \text{ and}$$

$$H(fTft_o, Tf^2t_o) = \lim_{n\to\infty} H(fTv_n, Tfv_n) = 0.$$

Thus $$f^3t_o = ff^2t_o \in fTft_o = Tf^2t_o .$$

Consequently, we have

$$f^{n+1}t_o \in Tf^nt_o .$$

Using (3.3.1), we get

$$\lim_{n\to\infty} Tf^nt_o = Tx^*.$$

Hence by Lemma 1.1.6, we obtain,

$$x^* \in Tx^* .$$

Moreover,

$$fx^* = f \lim_{n\to\infty} f^n t_o = \lim_{n\to\infty} f^{n+1} t_o = x^* .$$

Hence x^* is a common fixed point of f and T.

In Theorem 3.3.1 our hypothesis that f is continuous implies that T is continuous. And we use the continuity of f and T in our proof. In the next theorem we show that if fX is complete then the continuity and compatibility of f and T is not required.

THEOREM 3.3.5

Let T be a mapping of a metric space X into CB(X). Let $f: X \longrightarrow X$ be a mapping such that $TX \subseteq fX$, fX is complete and the condition (3.3.1) is satisfied. Then (i) there exists a sequence $\{x_n\}$ which is asymptotically T-regular with respect to f and (ii) f and T have a coincidence point.

PROOF

Examining the proof of Theorem 3.3.1, the only change is that the completeness of fX allows one to obtain $z \in X$ such that $fx_n \longrightarrow p = fz$. Then

$$
\begin{aligned}
d(fz,Tz) &\le d(fz,fx_{n+1}) + d(fx_{n+1},Tz) \\
&\le d(fz,fx_{n+1}) + H(Tx_n,Tz) \\
&\le d(fz,fx_{n+1}) + \phi(d(fx_n,fz))\, d(fx_n,fz) \\
&\le d(fz,fx_{n+1}) + d(fx_n,fz).
\end{aligned}
$$

Letting $n \longrightarrow \infty$, we obtain

$$d(fz,Tz) \le d(fz,p) + d(p,fz) = 0 \ .$$

Hence $fz \in Tz$.

COROLLARY 3.3.6

If, in addition to the hypotheses of Theorem 3.3.5 f satisfies (3.3.2) and, f and T are compatible. Then $\{fx_n\}$ converges to a coincidence point (say p) of f and T, and $\{f^n p\}$ converges to common fixed point of f and T.

PROOF

By Theorem 3.3.5, there exists $z \in X$ such that $fz \in Tz$. As in Corollary 3.3.4, compatibility of f and T implies that,

$$ffz = fTz = Tfz \ .$$

Since $fx_n \longrightarrow fz$ (see Theorem 3.3.5.), hence fx_n converges to a coincidence point of f and T.

Now, inequality (3.3.2) implies that $\{f^n z\}$ is a

Cauchy sequence. Let $f^n z \longrightarrow x^*$. Since, (as in Corollary 3.3.4) $f^{n+1}z \in Tf^n z$ Therefore,

$$d(x^*, Tx^*) \le d(x^*, f^{n+1}z) + H(Tf^n z, Tx^*)$$
$$\le d(x^*, f^{n+1}z) + \phi(d(f^n z, x^*))\, d(f^n z, x^*)$$
$$\le d(x^*, f^{n+1}z) + d(f^n z, x^*).$$

Letting $n \longrightarrow \infty$, we obtain

$$d(x^*, Tx^*) = 0 \text{ i.e. } x^* \in Tx^*.$$

Moreover,

$$d(x^*, fx^*) \le d(x^*, f^{n+1}z) + d(f^{n+1}z, fx^*)$$
$$\le d(x^*, f^{n+1}z) + \gamma \max \{d(f^n z, x^*),$$
$$d(f^n z, f^{n+1}z),\ d(x^*, fx^*), \frac{d(f^n z, fx^*) + d(fz^{n+1}, x^*)}{2} \}$$

Letting $n \longrightarrow \infty$, we have

$$d(x^*, fx^*) \le \gamma\, d(x^*, fx^*). \text{ Hence } x^* = fx^*.$$

We show that the assumption of $TX \subseteq fX$ (Theorem 3.3.5) and compatibility of f and T (Corollary 3.3.6) cannot be dropped.

EXAMPLE 3.3.7

Let $X = [0,\infty)$ with the Euclidean metric, $Tx = [0,e^x]$, $fx = 5e^x$ and $\phi(x) = \frac{1}{4}$. $TX \nsubseteq fX$, all other assumptions of Theorem 3.3.5 are satisfied but f and T have no coincidence point.

EXAMPLE 3.3.8

Let $X = R$ with the Euclidean metric, $Tx = [0,\frac{|x|}{3}]$, $fx = \frac{x+3}{2}$ and $\phi(x) = \frac{2}{3}$. Then all the hypothesis of Theorem 3.3.5 are satisfied and $f(-2) \in T(-2)$. Moreover f and T are not compatible, but other assumptions of Corollary 3.3.6 are satisfied. $f^n(-2) \longrightarrow 3$ and 3 is not common fixed point of f and T.

3.4. FIXED POINTS OF MEIR-KEELER TYPE MULTIVALUED MAPPINGS

In 1969 Meir and Keeler [42] established a remarkable fixed point theorem for a single valued mapping $T: X \longrightarrow X$ that satisfies the following condition;

Given $\varepsilon > 0$, there exists a $\delta > 0$ such that

$\varepsilon \le d(x,y) < \varepsilon + \delta$ implies $d(Tx,Ty) < \varepsilon$.

In 1981 Park and Bae [46] extended it to a pair of commuting single valued mappings $f,T: X \longrightarrow X$ satisfying the following condition.

Given $\varepsilon > 0$, there exists a $\delta > 0$ such that

$\varepsilon \le d(fx,fy) < \varepsilon + \delta$ implies $d(Tx,Ty) < \varepsilon$

and $Tx = Ty$ when $fx = fy$.

Afterwards some extensions, generalizations and applications of this followed; e.g., see [19] and [57]. In this section we continue these investigations for multivalued compatible mappings. The extension, however, is different and made in such a way as to generalize the theorem of Park and Bae [46] and includes as corollaries the results of Hadzic [19] and Jungck [27]. Furthermore, the technique of Meir-Keeler is applied to Kannan type multivalued contractive mappings.

THEOREM 3.4.1

Let X be a complete metric space and $T: X \longrightarrow CB(X)$, $f: X \longrightarrow X$ be a continuous and compatible mappings such that $TX \subseteq fX$ and the following condition is satisfied.

For $\varepsilon > 0$ there exists a $\delta > 0$ such that

$\varepsilon \le d(fx,fy) < \varepsilon + \delta$ implies $d(u,v) < \varepsilon$,

$u \in Tx$, $v \in Ty$ and $fx = fy$ when $Tx = Ty$. (3.4.1)

Then T and f have a common fixed point.

Proof

Let $x_o \in X$, consider the following sequences x_n, y_n in

X and A_n in CB(X), $y_n = fx_n \in Tx_{n-1} = A_{n-1}$ $n \geq 0$ (which is possible due to the hypothesis $TX \subseteq fX$.) Then for each $\varepsilon > 0$ there exists a $\delta > 0$ such that $\varepsilon \leq d(fx_m, fx_n) < \varepsilon + \delta$ implies $d(fx_{m+1}, fx_{n+1}) < \varepsilon$. It follows that $d(y_n, y_{n+1}) < d(y_{n-1}, y_n)$. Thus the sequence $\{d(y_n, y_{n+1})\}$ is non-increasing and converges to greatest lower bound of its range, which we denote by r.

Now $r \geq 0$; infact $r = 0$. If otherwise $r > 0$, pick N so that $n \geq N$ implies $r \leq d(y_n, y_{n+1}) < r + \delta$. It implies that $d(y_{n+1}, y_{n+2}) < r$ which is a contradiction to the fact that $r = \inf_n d(y_n, y_{n+1})$. Hence $d(fx_n, Tx_n) \leq d(fx_n, fx_{n+1}) \longrightarrow 0$. Now we show that $\{y_n\}$ is a Cauchy sequences. Suppose that $d(y_n, y_{n+1}) = 0$ for some $n > 0$. Then $d(y_m, y_{m+1}) = 0$ for all $m > n$, otherwise $d(y_n, y_{n+1}) = 0 < d(y_{n+1}, y_{n+2})$, a contradiction. Hence $\{y_n\}$ is a Cauchy sequence.

Now assume that $d(y_n, y_{n+1}) \neq 0$ for each n. Define $\varepsilon' = 2\varepsilon$ and choose (without loss of generality) δ, $0 < \delta < \varepsilon$ such that (3.4.1) is satisfied. Since $d(y_n, y_{n+1}) \longrightarrow 0$, there exists an integer N such that $d(y_i, y_{i+1}) < \delta/6$ for $i \geq N$. We now let $q > p > N$ and show that $d(y_p, y_q) \leq \varepsilon'$, to prove that $\{y_n\}$ is indeed Cauchy. Suppose that

$$d(y_p, y_q) \geq 2\varepsilon = \varepsilon' . \tag{3.4.2}$$

We first show that there exists an integer $m > p$ such that

$$\varepsilon + \frac{\delta}{3} < d(y_p, y_m) < \varepsilon + \delta , \qquad (3.4.3)$$

with p and m are of opposite parity. Let k be the smallest integer greater than p such that

$$d(y_p, y_k) > \varepsilon + \frac{\delta}{2} , \qquad (3.4.4)$$

(which is possible due to (3.4.2) as $\delta < \varepsilon$). Moreover

$$d(y_p, y_k) < \varepsilon + \frac{2\delta}{3} . \qquad (3.4.5)$$

For otherwise,

$$\varepsilon + \frac{2\delta}{3} \le d(y_p, y_{k-1}) + d(y_{k-1}, y_k) .$$

Since $k-1 \ge p \ge N$, therefore $d(y_{k-1}, y_k) < \delta/6$. It implies that,

$$d(y_p, y_{k-1}) > \varepsilon + \frac{\delta}{2} , \qquad (3.4.6)$$

which is a contradiction to the fact that k be the smallest such that (3.4.4) is satisfied. Thus

$$\varepsilon + \frac{\delta}{2} < d(y_p, y_k) < \varepsilon + \frac{2\delta}{3} . \qquad (3.4.7)$$

If p and k are of opposite parity we can let k = m in (3.4.7) to obtain (3.4.3).

If p and k are of like parity, p and k+1 are of opposite parity. In this event,

$$d(y_p, y_{k+1}) \le d(y_p, y_k) + d(y_k, y_{k+1})$$
$$\le \varepsilon + \frac{2\delta}{3} + \frac{\delta}{6} = \varepsilon + \frac{5\delta}{6} .$$

Moreover,

$$d(y_p, y_k) \le d(y_p, y_{K+1}) + d(y_{K+1}, y_k)$$
$$d(y_p, y_k) - d(y_{K+1}, y_k) \le d(y_p, y_{K+1})$$
$$\varepsilon + \delta/2 - \delta/6 < d(y_p, y_{K+1})$$
$$\varepsilon + \frac{\delta}{3} < d(y_p, y_{K+1}) .$$

Thus

$$\varepsilon + \delta/3 < d(y_p, y_{K+1}) < \varepsilon + \frac{5\delta}{6} .$$

Putting $m = k+1$, we obtain (3.4.3).

Hence (3.4.3) holds.

Now,

$$\varepsilon + \frac{\delta}{3} < d(y_p, y_m) \le d(y_p, y_{p+1}) + d(y_{p+1}, y_{m+1}) + d(y_{m+1}, y_m)$$
$$< \frac{\delta}{6} + \varepsilon + \frac{\delta}{6} = \varepsilon + \frac{\delta}{3} .$$

Thus contradiction.

Hence $\{y_n\} = \{fx_n\}$ is a Cauchy sequence. By completness of the space, there exists an element $t \in X$ such that $d(y_n, t) \longrightarrow 0$, continuity of f implies that $d(fy_n, ft) \longrightarrow 0$. Hence $H(Ty_n, Tt) \le \sup\{d(u,v),\ u\in Ty_n,\ v\in Tt\} < d(fy_n, ft) \longrightarrow 0$.

Since $\{fx_n\}$ is a Cauchy sequence in X, and

$$H(A_m, A_n) = H(Tx_m, Tx_n)$$
$$\le \sup\{d(u,v),\ u \in Tx_m,\ v \in Tx_n\}$$
$$< d(fx_m, fx_n).$$

It follows that $\{A_n\}$ is a Cauchy sequence in $CB(X)$. By completeness of $CB(X)$, there exists an $A \in CB(X)$ such that $H(A_n, A) \longrightarrow 0$. Since $y_{n+1} \in A_n$ and $d(y_{n+1}, t) \rightarrow 0$, Lemma 1.1.6 implies that $t \in A$, that is $\lim_{n\to\infty} fx_n \in \lim_{n\to\infty} Tx_n$ compatibility of f and T further implies that

$$\lim_{n\to\infty} H(fTx_n, Tfx_n) = 0.$$

Since $d(fy_{n+1}, Ty_n) \le H(fTx_n, Tfx_n)$. Therefore $ft \in Tt$, that is $\lim_{n\to\infty} fy_n \in \lim_{n\to\infty} Ty_n$ and $\lim_{n\to\infty} H(fTy_n, Tfy_n) = H(fTt,\ Tft) = 0$. Let $c = ft$. Then by (3.4.1) we have

$$\begin{aligned} d(c,fc) &\le \sup\{d(u,v) \colon u \in Tt,\ v \in fTt\} \\ &\le \sup\{d(u,v) \colon u \in Tt,\ v \in Tft\} \\ &< d(ft,fft) = d(c,fc) . \end{aligned}$$

Thus $c = fc$. Now

$$\begin{aligned} d(c,Tc) &\le d(ft,Tft) \\ &\le \sup\{d(u,v) \colon u \in Tt,\ v \in Tft\} \\ &< d(ft,fft) = d(c,fc) = 0 . \end{aligned}$$

Hence $c = Tc$.

DEFINITION 3.4.2

A mapping $f \colon X \longrightarrow X$ is said to be a *selection* of a multivalued mapping $T \colon X \longrightarrow CB(X)$ if $fx \in Tx$ for all $x \in X$.

THEOREM 3.4.3

Let K be a compact subset of a complete metric space X and $T: K \longrightarrow CB(K)$ a mapping satisfying the following condition:

Given $\varepsilon > 0$, there exists a $\delta > 0$ such that for all $x, y \in K$.

$$\varepsilon \le \max\{d(x,Tx),\ d(y,Ty)\} < \varepsilon + \delta \text{ implies } H(Tx,Ty) < \varepsilon \tag{3.4.8}$$

Then there exists a subset K^* of K such that $Tx^* = K^*$ for each $x^* \in K^*$. Moreover corresponding to each $x^* \in K$, there exists a selection of T having a unique fixed point x^*.

PROOF

Let x_o be an arbitrary but a fixed element of X. We shall construct two sequences $\{x_n\}$ and $\{r_n\}$ of elements in X and R respectively. Tx_o is closed subset of K and therefore is compact. There exists a point $x_1 \in Tx_o$ such that $d(x_o, x_1) = d(x_o, Tx_o) = r_o$. Similarly there exists $x_2 \in Tx_1$ such that,

$d(x_1, x_2) = d(x_1, Tx_1) = r_1$. By induction we prove sequences $\{x_n\}$ and $\{r_n\}$ such that $x_n \in Tx_{n-1}$, $d(x_n, x_{n+1}) = d(x_n, Tx_n) = r_n$, $n \ge 0$. Using (3.4.8), we have

$$d(x_n, Tx_n) \le H(Tx_{n-1}, Tx_n)$$

$$< \max\{d(x_{n-1}, Tx_{n-1}),\ d(x_n, Tx_n)\}. \qquad (3.4.9)$$

If $d(x_n, Tx_n) > d(x_{n-1}, Tx_{n-1})$ then (3.4.9) implies that $d(x_n, Tx_n) < d(x_n, Tx_n)$, a contradiction. It follows that

$$d(x_n, Tx_n) < d(x_{n-1}, Tx_{n-1}) .$$

Thus $\{r_n\}$ a monotone non increasing sequence of non negative real numbers. Therefore r_n converges to $\inf\{r_n : n \ge 0\}$. We claim that $\inf\{r_n : n \ge 0\} = r > 0$. Since $r_n \longrightarrow r$. Pick N so that $n \ge N$ implies that

$$r \le r_n < r + \delta.$$

It follows that

$$r_{n+1} \le H(Tx_n, Tx_{n+1}) < r, \text{ which is a contradiction}$$

to the assumption that $\inf\{r_n : n \ge 0\} = r$. Hence $r_n \longrightarrow 0$. That is

$$d(x_n, Tx_n) \longrightarrow 0$$

It follows that $H(Tx_n, Tx_m) \to 0$. By completeness of $(CB(X), H)$ there exists a set $K^* \in CB(X)$ such that $H(Tx_n, K^*) \longrightarrow 0$. Let $x^* \in K^*$, then $x^* \in Tx^*$. If not let $d(x^*, Tx^*) = c > 0$, then

$$c = d(x^*, Tx^*) \le H(Tx^*, K^*)$$

$$\le H(Tx^*, Tx_n) + H(Tx_n, K^*)$$

$$< \max\{d(x^*, Tx^*),\ d(x_n, Tx_n)\} +$$

$$H(Tx_n, K^*) .$$

By letting $n \longrightarrow \infty$, we have $c < c$, a contradiction. Hence

$$x^* \in Tx^* .$$

Now,

$$H(Tx^*,K) = \lim_{n\to\infty} H(Tx^*,Tx_n)$$

$$< \lim_{n\to\infty} \max\{d(x^*,Tx^*),\ d(x_n,Tx_n)\} = 0 .$$

Hence $Tx^* = K$ for all $x^* \in K^*$.

Now we will prove that T has a selection having a unique fixed point.

For each $u \in K$, Tu is compact. Therefore for $x^* \in K$ there exists $u_x* \in Tu$ such that

$$d(x^*,\ u_x*) = d(x^*,Tu) . \qquad (3.4.10)$$

Define a single valued self mapping $f: K \longrightarrow K$ as $fu = u_x*$. Then for each $u \in K$ we have $fu = u_x* \in Tu$, that is f is a selection of T. Let $fx^* = v$ $(= x^*_x*)$ then $d(x^*,\ v) = d(x^*,Tx^*) = 0$. This implies that

$$v = x^* = fx^* .$$

Now,

$$\begin{aligned} d(fu,fv) &\le d(fu,x^*) + d(x^*,fv) \\ &\le d(u_x*,x^*) + d(x^*,v_x*) \\ &\le d(x^*,Tu) + d(x^*,Tv) \\ &\le H(Tx^*,Tu) + H(Tx^*,Tv) \\ &< d(u,Tu) + d(v,Tv) \\ &< d(u,fu) + d(v,fv) . \end{aligned}$$

It follows that the fixed point of f is unique.

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