MATH 219

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SOME STUDIES IN LEFT ALMOST SEMIGROUPS

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TO

MY HUSBAND

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PREFACE

In ternary operations the commutative law is given by abc = cba. M.A. Kazim and M. Naseerudin (1972) introduced braces on the left of this equation to get a new pseudo associative law, that is, (ab)c = (cb)a. It is called the left invertive law. A groupoid is called a left almost semigroup, abbreviated as LA-semigroup, if its elements satisfy the left invertive law. Similarly, a groupoid is called a right almost semigroup, abbreviated as RA-semigroup, if its elements satisfy the right invertive law, that is a(bc) = c(ba). A group is called an almost semigroup if it is both an LA-semigroup and an RA-semigroup.

An LA-semigroup is an algebraic structure midway between a groupoid and a commutative semigroup. Despite the fact that the structure is non-associative and non-commutative, it nevertheless possesses many interesting properties which we usually find in commutative and associative algebraic structures.

This thesis comprises four chapters. The first chapter contains only those definitions and results which are directly related to our study of the LA-semigroups. We have mentioned in this chapter the results without proofs in order to avoid making the dissertation unnecessarily bulky. We have avoided giving the text-book definition also by presuming that the reader is familiar with these definitions. However, one can refer for reference to several text-books, and one of them is: A.H. Clifford and G.B. Preston, The algebraic theory of semigroups, Amer. Math. Soc., Vols.1, 1961 and II, 1967.

In Chapter 2, we have described the structure of LA-semigroups by means of LA-semigroups and certain homomorphisms between them. Specifically, we have shown that an LA-semigroup G is a semilattice of LA-semigrups. Conversely we have shown that given a semilattice of LA-semigroups and a family of homomorphisms, with certain properties, an LA-semigroup can be defined which is a union of the given LA-semigroups.

In chapter 3, we have extended the results by Tamura and Kimura [33] that any commutative semigroup G is uniquely expressible as a semilattice of archimedean semigroups. We have generalized also the results of Hewitt and Zuckerman [11] that the following are mutually equivalent: (i) G is separative (ii) the archimedean components of G are cancellative (iii) G can be embedded in a union of groups. We have shown also in chapter 3, that any locally associative LA-semigroup G with left identity is uniquely expressible as a semilattice of archimedean components. Also it has been shown that G is separative if and only if the archimedean components of G are cancellative and G can be embedded in a union of LA-groups if and only if it is separative.

In chapter 4, an LA-semigroup G, which has a left regular band of LA-groups as an LA-semigroup of left quotients, is shown to be the LA-semigroup which is a left regular band of right reversible cancellative LA-semigroups. An alternative characterization is provided by unique spined products. These results are applied to the case where S is super abundant and where the set of idempotents form a left normal band.

The results contained in chapter 2, are published in Proceedings of Academy of Sciences 2, 28 (1991), 197-200. The results contained in chapter 3, are published in Semigroup Forum, 41 (1991) 155-164.

One separate paper, containing results from chapter 4, has already been submitted to journal for consideration of publication.

CHAPTER ONE

DEFINITIONS, EXAMPLES AND SURVEY

In ternary operations the commutative law is given by abc = cba. In 1972, Kazim and Naseerudin [15] have introduced braces on the left of this equation to get a new pseudo associative law, that is, (ab)c =(cb)a and proved several interesting results.

A left almost semigroup, abbreviated as LA-semigroup, is an algebraic structure midway between a groupoid and a commutative semigroup. An LA-semigroup is a non-commutative and non-associative algebraic structure. It has been defined in [15] and [28] as a groupoid G in which the left invertive law:

(1.1) $(ab)c = (cb)a$ for all a,b,c in G holds.

Naseerudin has investigated some basic characteristics of this structure in his doctoral thesis [28]. He has generalized some rudimentary but useful and important results of semigroup theory. Relationships between LA-semigroups and quasi-groups, semigroups, loops, monoids and groups have been established.

Kazim and Naseerudin, in their paper on almost semigroups [15] have shown that G is medial. That is,

(1.2) $(ab)(cd) = (ac)(bd)$ for all a,b,c,d in G.

Right almost semigroups can be defined dually. That is, a groupoid $(G,.)$ is called a right almost semigroup, abbreviated as an RA-semigroup, if it satisfies the right invertive law:

$$a(bc) = c(ba) \quad \text{for all } a,b,c \text{ in } G.$$

EXAMPLES 1.1

(i) Let $(Z,+)$ denote the group of integers under '+'. Define a binary operation $*$ in Z as follows:

$$x*y = y-x \text{ for every } x,y \text{ in } Z,$$

where '-' denotes the ordinary subtraction defined in Z. Then it is a routine matter to check that (Z,*) is an LA-semigroup.

(ii) Let (Q,+) denote the group of rational numbers under '+'. Let * be defined in Q as follows:

$$x*y = y-x \quad \text{for every } x,y \text{ in } Q.$$

Then it is easy to check that (Q,*) is an LA-semigroup.

(iii) Similarly (R,*), where (R,+) is a group of all real numbers under ordinary addition (+) and * is the binary operation defined by $x*y = y-x$, for every x,y in R, is an LA-semigroup.

(iv) Let $(\hat{Q},.)$ denote the group of all non-zero rational numbers under ordinary multiplication (.). Define a binary operation * in $\hat{Q}$ as follows:

$x*y = y \div x$ for every $x,y \in \hat{Q}$. Then it can be checked easily that $(\hat{Q},*)$ is an LA-semigroup.

REMARK 1.2

(i) Note that the binary operation '*' is not necessarily associative. For if we consider the additive group of integers, $(Z,+)$, and define

$a*b = b-a$ for all a,b in Z,

Then $(3*4)*5 = (4-3)*5 = 1*5 = 5-1 = 4$

and $3*(4*5) = 3*(5-4) = 3*1 = 1-3 = -2$.

Thus $3*(4*5) \neq (3*4)*5$ and so $(Z,*)$ is not a semigroup.

(ii) The binary operation '*' is not necessarily commutative. For

$3*4 = 4-3 = 1$

and $4*3 = 3-4 = -1$

implies that $3*4 \neq 4*3$.

The structural properties of LA-semigroups are studied in a number of important papers that have appeared since the introduction of this structure. In one of these papers Kazim and Naseeruddin [15] have tried to find out a condition under which an LA-semigroup can be converted into a group. They assert that an LA-semigroup G with left identity e will become a group if for each a in G there exist b and c in G such that $a(bc) = e = (ac)b$ holds in G. In [23] Mushtaq has shown that their assertion was not true. He provided a counter example to support his assertion. Kazim and

Naseerudin [15] have extensively used the identity $a(a(bc)) = e$ and $(a(bc))a = e$ which is not necessarily true as Mushtaq [23] has shown that $a(bc) = e$ does not necessarily imply that

$$a(a(bc)) = e \text{ and } (a(bc))\,a = e.$$

Consider, for instance, the following example of an LA-semigroup which satisfies the hypothesis of the theorem by Kazim and Naseerudin but which is not a group.

EXAMPLE 1.3

Let G = {a,b,c,d} and a binary operation (.) be defined in G as follows.

.	a	b	c	d
a	a	b	c	d
b	d	a	b	c
c	c	d	a	b
d	b	c	d	a

Then (G,.) is an LA-semigroup with left identity a because all the elements of G satisfy the left invertive law and ax = x for all x in G. Moreover, all the elements of G satisfy the identity

$$a(a(bc)) = e \quad \text{and} \quad (a(bc))\, a = e\ .$$

Thus, for each x in G, there exist y and z in G such that x(yz) = a = (xz)y. But (G,.) is not a group. It is not even a semigroup because we find at least two elements b and c in G such that (bb)c ≠ b(bc).

Mushtaq and Yusuf in [20] have defined an LA-semigroup defined by a commutative inverse semigroup. Let (G,.) be a commutative inverse semigroup. Define a binary operation * in G as follows:

$$a*b = b.a^{-1} \text{ for every } a,b \text{ in } G.$$

They have proved that (G,*) is an LA-semigroup and referred to this as an 'LA-semigroup defined by a commutative inverse semigroup'. In [20], the authors have described the structure of LA-semigroups defined by commutative inverse semigroups, by means of LA-semigroups defined by commutative groups and certain homomorphisms between them. Specifically, they have shown that if a commutative inverse semigroup G is a semilattice of the inverse semigroup G then the LA-semigroup defined by G is also a semilattice of LA-semigroups. Conversely they have shown that given a semilattice of LA-semigroups and a family of homomorphisms with certain properties, an LA-semigroup can be defined which is a union of the given LA-semigroups.

Mushtaq [22], has shown that conversely, provided that a necessary and sufficient condition is satisfied by an LA-semigroup, it can induce an Abelian group satisfying the condition $a.b = b*a^{-1}$ for all a,b in G. He also observed some additional characterstic of such LA-semigroups. Specifically, the author proved that in (G,.), the following conditions are equivalent:

(i) $a = (cc.ab)b$ for all a,b,c in G,

(ii) there exists an Abelian group (G,*) such that $a.b = b*a^{-1}$ for all a,b in G,

(iii) (G,.) is cancellation with left identity e and $a^2 = e$ for all a in G,

(iv) (G,.) has a left identity e and $a^2 = e$ for all a in G.

The notion of a left(right) translative

mapping (which is called a left(right) translation in semigroup theory) is natural and very useful. It is well-known [5] that each element of a semigroup induces a left and right translation. These translations play an important role, for example, in the theory of ideal extensions. A system of mappings T_u: $x \longrightarrow T_u(x)$ of a non-empty set G into itself, where u ranges over elements of a set U, is called commutable if $T_uT_v(x) = T_vT_u(x)$ holds for all u,v in U and x in G. A system of mappings $T_u:x \longrightarrow T_u(x)$ is transitive if $T_u(x) = G$ for all x in G, where the set of elements $T_u(x)$ for all u in U is denoted by $T_u(x)$. A system of mappings T: $x \longrightarrow T_u(x)$ of G into itself is called right translative, left translative or translative according as $T_u(xy) = xT_u(y)$, $T_u(xy) = T_u(x)y$ or $T_u(xy) = xT_u(y) = T_u(x)y$ holds for every x,y in G and u in U.

In [26], Mushtaq has defined translative mappings on LA-semigroups, and besides other things, he has shown that if there is a transitive system of translative mappings on an LA-semigroup with left identity then the structure is necessarily a commutative semigroup with identity. It has been shown also that a mapping T_u of a translative system of mappings over an LA-semigroup G is injective if the right cancellative law holds with respect to every element of $T_u(G)$. Also, every transitive system of translative mappings over a multiplicative LA-semigroup G with left identity has the form $x \longrightarrow T_u(x) = x + \theta(x)$, where + is an Abelian group operation on G and $\theta: U \longrightarrow G$ is a mapping of U onto G.

Mushtaq and Kamran [25] have shown that a cancellative LA-semigroup is a commutative semigroup if $a(bc) = (cb)a$ for all a,b,c in G. Further, it has been shown that G, with left identity, is a commutative monoid if and only if $(ab)c = b(ca)$ for all a,b,c in G.

Hewitt and Zukerman [11], surveyed the field of ternary operations and semigroups giving rise to them. In [13], Iqbal has generalized their results to invertive operations and studied the LA-semigroups connected with them. Apart from several interesting results, the main result he has proved is that an LA-semigroup is isomorphic to the direct product of a group all of whose elements are of order two and a semigroup under a special binary operation.

Analogous to Vagner-Preston Representation Theorem [5], Iqbal in [13] has proved that every inverse LA-semigroup has a faithfull representation as an inverse LA-semigroup of partial one-one mappings. Iqbal has also shown that the given partial ordering relation is the maximum idempotent-separating congruence on an inverse LA-semigroup.

In [13], a ternary operation on an LA-semigroup was introduced and the author generalized the results of Hewitt and Zukerman [11]. Some useful properties of this structure were studied and a relationship was established between LA-semigroups (S,.) and (S,o), defined on the same set S, such that x.(y.z) = xo(yoz) for all x,y,z in S. If in (S,.) and (S,o), x.(y.z) = xo(yoz) then we say that (S,.) and (S,o) are in relation R with each other. Iqbal [13] has shown that if (S,.) and (S,o) are related by R then (S,.) and (S,o) are isomorphic under certain conditions.

Translations and transformations play a vital role in the theory of semigroups. In [14] Kamran has shown that under certain conditions the set of left translations on a left almost semigroup forms a left almost semigroup. A parallel result to Cayley's theorem for the set of left translations defined on a left almost semigroup has been proved in [14]. In [14], the concepts of zeroids and idempoids in left almost semigroups are discussed in detail, and some interesting results have been proved.

Mushtaq [24] has proved that if an LA-semigroup contains the left cancellative LA-subsemigroup such that the LA-subsemigroup is contained in the centre of the LA-semigroup then it can be embedded in a commutative monoid whose cancellative elements form an Abelian group and the identity element of this group coincides with the identity element of the commutative monoid.

In [15], it has also been proved by Kazim and Naseerudin that in an LA-semigroup G the conditions:

(1.3) $b(ac) = (ab)c$

(1.4) $b(ca) = (ab)c$

are equivalent for all a,b,c in G.

In order to define associative powers in an LA-semigroup G we impose the condition (i) on G and call (1.3) or (1.4) a weak associative law. Notice that if $a = b = c$ in (1.3) then an LA-semigroup with the weak associative law becomes a locally associative LA-semigroup, that is, an LA-semigroup with the condition $(aa)a = a(aa)$ for all a in G. In [19], Mushtaq and Yusuf have defined a locally associative LA-semigroup G and have defined on it a relation ρ on G as follows:

$a \rho b$ if and only if $ab^n = b^{n+1}$ and $ba^n = a^{n+1}$ for some positive integer n.

They have shown that if G is a locally associative LA-semigroup with left identity, then ρ is a congruence on G and G/ρ is the maximal separative homomorphic image of G. (Refer to [19] for details) and hence all the results contained in [32] are true for this structure.

In [34], Tamura and Nordhal have called the semigroup satisfying the identity $(xy)^m = x^m y^m$ $(m \geq 2)$ as exponential m-subsemigroup.

It is important to note that an LA-semigroup G with weak associative property (1.3) or (1.4) is exponential. One can refer to [19] and [25] for more details about this property.

In [19], it has been shown that locally associative LA-semigroups are exponential. Several structural theorems are proved in this paper.

The following results are essential for our subsequent work and are referred to frequently. These results are proved in [18] and [21], and here we state these results without proofs.

THEOREM 1.4

In an LA-semigroup the left identiy is unique.

THEOREM 1.5

In an LA-semigroup the right identity becomes a two sided identity.

We may mention here that the converse of the above theorem is not necessarily true. That is, the left identity does not become the right identity.

As a consequence of the above theorem we have the following important result.

THEOREM 1.6

An LA-semigroup with right identity is a commutative monoid.

THEOREM 1.7

In an LA-semigroup G with left identity, a(bc) = b(ac) for all a,b,c in G.

THEOREM 1.8

An LA-semigroup with left identity and right inverses has two sided inverses.

A groupoid (G,.) is called a left almost group, abbreviated as LA-group, if:

(i) (G,.) is a left almost semigroup,

(ii) e.a = a for all a $\in$ G, and

(iii) a.a = e for all a $\in$ G.

EXAMPLE 1.9

Let G = {a,b,c,d} and (.) be the binary operation in G defined as follows.

.	a	b	c	d
a	a	b	c	d
b	d	a	b	c
c	c	d	a	b
d	b	c	d	a

Then G is an LA-group with left identity a, and every element of G has a left inverse and the elements satisfy the left invertive law.

THEOREM 1.10

An LA-group with right identity is an Abelian group.

THEOREM 1.11

A left cancellative LA-semigroup is a cancellative LA-semigroup.

THEOREM 1.12

In an LA-semigroup G with left identity, ab = cd implies that ba = dc for all a,b,c,d in G.

THEOREM 1.13

A finite LA-semigroup is a group provided a(bc) = (cb)a for all a,b,c in G.

THEOREM 1.14

If (G,.) is a commutative group then (G,*) is an LA-semigroup under *, where * is defined by: $a*b = a^{-1}.b = b^{-1}.a$ for every a,b in G, and by a^{-1} we mean the inverse of a.

THEOREM 1.15

A subset containing all the idempotent elements of an LA-semigroup with left identity e is a commutative subsemigroup with e as its identity.

Due to theorem 2.6, corollary 2.2 [21], we have the following useful results.

THEOREM 1.16

In a right cancellative LA-semigroup G every right identity of an idempotent element is its identity.

In theorem 3.10, 3.11, 3.12, [21] the following results have been proved.

THEOREM 1.17

If in an LA-semigroup G, ax = b has a unique solution for every a,b in G, then yc = d has also a unique solution for every c,d in G.

THEOREM 1.18

If in an LA-semigroup G with left identity e yc = d has a unique solution for every c,d in G, then ax = b has also a unique solution for every a,b in G.

THEOREM 1.19

If in an LA-semigroup G, ax = b has a unique solution for every a,b in G, then G is a commutative group.

The following example shows the existence of an LA-semigroup with more than one idempotent.

EXAMPLE 1.20

Let G = {a,b,c} and the binary operation (.) be defined in G as follows.

.	a	b	c
a	a	a	a
b	a	a	a
c	a	a	c

Then G is an LA-semigroup with more than one idempotent. An LA-semigroup with left identity can have idempotents other than the identity.

EXAMPLE 1.21

Let G = {e,f,a,b,c} and the binary operation (.) be defined as follows.

.	e	f	a	b	c
e	e	f	a	b	c
f	f	f	f	b	c
a	a	f	e	b	c
b	c	c	c	f	b
c	b	b	b	c	f

Then G is an LA-semigroup which has e as the left identity and f as an idempotent.

Note that ef = fe = f implies that $f \leq e$.
In [18], the following results have been proved.

THEOREM 1.22

An LA-semigroup with left identity e contains no idempotent such that $e \leq f$.

THEOREM 1.23

A subset containing all the idempotent elements of an LA-semigroup with left identity e, is a commutative subsemigroup with e as its identity.

EXAMPLE 1.24

Let G = {a,b,c} and a binary operation (.) be defined in G as follows.

.	a	b	c
a	c	c	b
b	b	b	b
c	b	b	b

Then (G,.) is a locally associative LA-semigroup. The above example shows that we can not define associative powers in G, as we do in semigroups. So in order to define associative powers, in a locally associative LA-semigroup we introduce the left identity.

Mushtaq and Yusuf [19] have proved the following results in this connection.

THEOREM 1.25

Every locally associative LA-semigroup with left identity has associative powers.

In [19], Mushtaq and Yusuf have defined a relation ρ (refer to page 12) on a locally associative LA-semigroup G with left identity.

Later in [19] it has been proved that the relation ρ is a congruence relation on a locally associative LA-semigroup with left identity.

A relation σ on a locally associative LA-semigroup G with left identity e is separative if and only if

$$ab \, \sigma \, a^2 \text{ and } ab \, \sigma \, b^2 \text{ implies } a \, \sigma \, b.$$

It was also proved in [19] that the relation ρ is separative.

In [20], Mushtaq and Yusuf have shown that if an LA-semigroup is defined by a commutative inverse semigroup [commutative group], then by defining a binary relation in the LA-semigroup, we can recover the commutative inverse semigroup [commutative group].

In chapter 2, we have described the structure of LA-semigroups by means of LA-semigroups and certain homomorphisms between them. Specifically, we have shown that an LA-semigroup G is a semilattice of LA-semigroups. Conversely we have shown that given a semilattice of LA-semigroups and a family of homomorphisms, with certain properties, an LA-semigroup can be defined which is a union of the given LA-semigroups.

In chapter 3 we have extended the results by Tamura and Kimura [33] that any commutative semigroup G is uniquely expressible as a semilattice of archimedean semigroups. We have generalized also the results of Hewitt and Zuckerman [11] that the following are mutually equivalent: (i) G is separative (ii) the archimedean components of G are cancellative (iii) G can be embedded in a union of groups.

We have shown in chapter 3, that any locally associative LA-semigroup G with left identity is uniquely expressible as a semilattice of archimedean components. Also it has been shown that G is separative if and only if the archimedean components of G are cancellative and G can be embedded in a union of LA-groups if and only if it is separative.

In chapter 4, an LA-semigroup G, which has a left regular band of LA-groups as an LA-semigroup of left quotients, has been shown to be the LA-semigroup which is a left regular band of right reversible cancellative LA-semigroups. An alternative characterization has been provided by unique spined products. These results have been applied to the case where S is super abundant and where the set of idempotents forms a left normal band.

CHAPTER TWO

SEMILATTICE STRUCTURE OF LA-SEMIGROUPS

To consider the decomposition of semigroups into groups, we need to recall from [5], the following theorem. It gives a number of conditions on G, each of which is equivalent to the assertion that G is a union of groups.

The following conditions are equivalent:

(i) G is a union of disjoint groups,

(ii) G is both left and right regular,

(iii) every left and every right ideal of G is semi-prime,

(iv) every H-class of G is a group.

These conditions, however, shed no light on the actual structure of G, and in article 4.2 [5], provide small illumination in this direction.

It is well known that a commutative inverse semigroup G is a union of groups. Due to [5], if E denotes the set of all idempotents of a commutative inverse semigroup G, then $G = \bigcup_{e \in E} G_e$ where each G_e is the group with identity element and $G_e G_f \subseteq G_{ef}$. Moreover, $e \neq f$ implies that $G_e \neq G_f$. Being a commutative band, E is a semilattice. Let Y be a semilattice isomorphic to E. Then $e_\alpha \leq e_\beta$ in Y if and only if $\alpha \leq \beta$ in Y. We write G_α for G_{e_α}; thus $G_\alpha G_\beta \subseteq G_{\alpha\beta}$. The elements of G_α will be denoted by $a_\alpha, b_\alpha, \ldots$.

Since by the Rees theorem [5], the structure of a completely simple semigroup is known, a semigroup which is a union of groups is a semilattice Y of semigroups $G_\alpha (\alpha \in Y)$ of a known structure. Even if we regard the structure of a semilattice as known, we still do not know the structure of G. For although we know that $G_\alpha G_\beta \subseteq G_{\alpha\beta}$, we are not in a position to say just how the product $a_\alpha b_\beta$ ($a_\alpha \in G_\alpha$, $b_\beta \in G_\beta$) lies in $G_{\alpha\beta}$, where $\alpha \neq \beta$. This is in general a complicated problem. But if we make the further assumption that the idempotent elements of G commute with each other then we can determine the structure. We observe by theorem 1.17 [5], that G is an inverse semigroup. We are thus dealing with inverse semigroups which are the union of groups.

Before we prove the results concerning LA-semigroups, we define the following terms.

An element a of an LA-semigroup G is called regular if (ax)a = a for some x in G. An LA-semigroup G is called left regular if, for any element a in G, there exists x in G such that x(aa) = a. Similarly, an LA-semigroup G is called right regular if for any element a in G, there exists x in G such that (aa)x = a. An LA-semigroup G is called regular if every element of G is regular.

In [20], Mushtaq and Yusuf have described the structure of LA-semigroups defined by commutative inverse semigroups, by means of LA-semigroups defined by commutative groups and certain homomorphims between them. Specifically, it has been shown that if a commutative inverse semigroup G is a semilattice of the inverse semigroups G_α then the LA-semigroup defined by G is also a semilattice of LA-semigroups. Conversely, it has also been shown that given a semilattice of LA-semigroups and a family of homomorphisms, with certain properties, an LA-semigroup can be defined which is the union of the given LA-semigroups. If G is an LA-semigroup and E denotes a set of all idempotent contained in G, then we call E to be a band. (It is important to point out here that E, being a subset of G is an LA-subsemigroup of G and is not associative as in the case of a band in semigroups) The main objective of this chapter is to refine these results and describe the structure of LA-semigroups by means of LA-semigroups and certain homomorphisms between them. Specifically, we shall show that an LA-semigroup G is a semilattice of LA-semigroups. Conversely, we shall show that given a semilattice of LA-semigroups and a family of homomorphisms, with certain properties, an LA-semigroup can be defined which is a union of the given LA-semigroups.

It is important to note that an LA-semigroup cannot contain a right identity because an LA-semigroup with a right identity becomes a commutative semigroup with two sided identity. A homomorphism between two LA-semigroups is defined in the same way as a homomorphism between two semigroups. That is a mapping f from an LA-semigroup (G,.) to an LA-semigroup (G,*) is called a homomorphism if (a.b)f = (a)f*(b)f, for all a,b in G.

With the necessary information and terminology in hand, we can now prove the following results.

THEOREM 2.1

Let an LA-semigroup G be a semilattice Y of LA-semigroups G_α, $\alpha \in Y$ whence each G_α has a unique idempotent e_α for α in Y. If $\alpha \geq \beta$, the mapping $\phi_{\alpha,\beta}$ defined by $a_\alpha \phi_{\alpha,\beta} = e_\beta a_\alpha$, $a_\alpha \in G_\alpha$ is a homomorphism of G_α into G_α.

If $\alpha \geq \beta \geq \gamma$ then $\phi_{\alpha,\beta}\, \phi_{\beta,\gamma} = \phi_{\alpha,\gamma}$. Moreover, $\phi_{\alpha,\alpha}$ is the identity mapping of G_α.

If $a_\alpha \in G_\alpha$ and $b_\beta \in G_\beta$, then $a_\alpha b_\beta = (a_\alpha \phi_{\alpha,\gamma})(b_\beta \phi_{\beta,\gamma})$ whence $\gamma = \alpha\beta$.

PROOF

First note that $\phi_{\alpha,\beta}$ maps G_α into G_β because α,β being idempotents commute and $a_\alpha \phi_{\alpha,\beta} = e_\beta a_\alpha \in G_\beta G_\alpha \subseteq G_{\beta\alpha} = G_{\alpha\beta} \subseteq G_\beta$.

Let $a_\alpha, b_\alpha \in G_\alpha$, then $(a_\alpha b_\alpha) \phi_{\alpha,\beta} = e_\beta(a_\alpha b_\alpha) = (e_\beta e_\beta)(a_\alpha b_\alpha) = (e_\beta a_\alpha)(e_\beta b_\alpha) = (a_\alpha \phi_{\alpha,\beta})(b_\alpha \phi_{\alpha,\beta})$. Thus $\phi_{\alpha,\beta}$ is a homomorphism from G_α to G_β. If $\alpha \geq \beta \geq \gamma$, then for any a_α in G_α, $(a_\alpha \phi_{\alpha,\beta})\phi_{\beta,\gamma} = (e_\beta a_\alpha)\phi_{\beta,\gamma} = e_\gamma(e_\beta a_\alpha) = (e_\gamma e_\gamma)(e_\beta a_\alpha) = (e_\gamma e_\beta)(e_\gamma a_\alpha) = e_\gamma(e_\gamma a_\alpha) = e_\gamma a_\alpha$, as e_γ is the left identity of G_γ and $e_\gamma a_\alpha \in G_\gamma$. Thus $(a_\alpha \phi_{\alpha,\beta})\phi_{\beta,\gamma} = a_\alpha \phi_{\alpha,\gamma}$ and $\phi_{\alpha,\beta}\phi_{\beta,\gamma} = \phi_{\alpha,\gamma}$. As $a_\alpha \phi_{\alpha,\alpha} = e_\alpha a_\alpha = a_\alpha$, therefore $\phi_{\alpha,\alpha}$ is the identity map of G_α.

In an LA-semigroup the product of idempotents is an idempotent, so $a_\alpha b_\beta = (e_\alpha a_\alpha)(e_\beta b_\beta) = (e_\alpha e_\beta)(a_\alpha b_\beta) = e_\gamma(a_\alpha b_\beta) = (e_\gamma e_\gamma)(a_\alpha b_\beta) = (e_\gamma a_\alpha)(e_\gamma b_\beta) = (a_\alpha \phi_{\alpha,\gamma})(b_\beta \phi_{\beta,\gamma})$.

THEOREM 2.2

Let Y be a semilattice, and to each element α

of Y assign an LA-semigroup G_α with left identity e_α and no other idempotent such that G_α and G_β are disjoint if $\alpha \neq \beta$ in Y. To each pair of elements α,β of Y such that $\alpha > \beta$, assign a homomorphism $\phi_{\alpha,\beta}$ of G_α into G_β such that if $\alpha > \beta > \gamma$ then $\phi_{\alpha,\beta}\phi_{\beta,\gamma} = \phi_{\alpha,\gamma}$. Let $\phi_{\alpha,\alpha}$ be the identity epi-morphism of G_α. Let G be the union of all LA-semigroups G_α, $\alpha \in Y$ and define the product of any two elements a_α, b_β of G ($a_\alpha \in G_\alpha$ and $b_\beta \in G_\beta$) by $a_\alpha . b_\beta = (a_\alpha \ \phi_{\alpha,\gamma})(b_\beta \ \phi_{\beta,\alpha})$ where $\gamma = \alpha\beta$ in Y. Then G is an LA-semigroup which is a semilattice Y of LA-semigroups G_α, $\alpha \in Y$.

PROOF

The converse statement has already been established in theorem 2.1. Since $a_\alpha \ \phi_{\alpha,\alpha\beta} \in G_{\alpha\beta}$ and $b_\beta\phi_{\beta,\alpha\beta} \in G_{\alpha\beta}$, therefore $(a_\alpha \ \phi_{\alpha,\alpha\beta})(b_\beta \ \phi_{\beta.\alpha\beta})$ belongs to $G_{\alpha\beta}$.

Now $(a_\alpha b_\beta) \ c_\gamma = \{(a_\alpha \ \phi_{\alpha,\alpha\beta})(b_\beta \ \phi_{\beta,\alpha\beta})\}c_\gamma$

$= \{(a_\alpha \ \phi_{\alpha,\alpha\beta}) \ \phi_{\alpha\beta,\alpha\beta\gamma} \ (b_\beta \ \phi_{\beta,\alpha\beta})\phi_{\alpha\beta,\alpha\beta\gamma}\} \ c_\gamma \ \phi_{\gamma,\alpha\beta\gamma}$

$= \{(a_\alpha \ \phi_{\alpha,\alpha\beta\gamma})(b_\beta \ \phi_{\beta,\alpha\beta\gamma})\} \ (c_\gamma \ \phi_{\gamma,\alpha\beta\gamma})$

$= \{(c_\gamma \ \phi_{\gamma,\alpha\beta\gamma})(b_\beta \ \phi_{\beta,\alpha\beta\gamma})\}(a_\alpha \ \phi_{\alpha,\alpha\beta\gamma})$ and $(c_\gamma b_\beta)a_\alpha$

$= \{(c_\gamma \ \phi_{\gamma,\gamma\beta})(b_\beta \ \phi_{\beta,\gamma\beta})\} \ a_\alpha$

$= \{(c_\gamma \ \phi_{\gamma,\gamma\beta}) \ \phi_{\gamma,\alpha\beta\gamma}(b_\beta\phi_{\beta,\gamma\beta}) \ \phi_{\alpha\beta,\alpha\beta\gamma}\} \ (a_\alpha \ \phi_{\alpha,\alpha\beta\gamma})$

$= \{(c_\gamma \ \phi_{\gamma,\alpha\beta\gamma})(b_\beta \ \phi_{\beta,\alpha\beta\gamma})\}(a_\alpha\phi_{\alpha,\alpha\beta\gamma})$ implies that $(a_\alpha b_\beta)c_\gamma = (c_\gamma b_\beta)a_\alpha$.

Moreover, $e_\alpha e_\beta = (e_\alpha \ \phi_{\alpha,\alpha\beta})(e_\beta \ \phi_{\beta,\alpha\beta}) = e_{\alpha\beta}e_{\alpha\beta} = e_{\alpha\beta}$ and $e_\alpha e_\beta = e_{\alpha\beta} = e_{\beta\alpha} = e_\beta e_\alpha$. Hence G is an LA-semigroup with commuting idempotents and is a union of LA-semigroups, each having a left identity.

Now we shall prove the last theorem which describes the structure of an LA-semigroup defined by a commutative inverse semigroup.

THEOREM 2.3

An LA-semigroup G is a union, $\bigcup_{e\in E} G_e$, of LA-semigroups G_e, where G_e is the LA-semigroup with left identity e. Moreover, E is a commutative sub-semigroup of the LA-semigroup.

PROOF

Since the idempotents of an LA-semigroup with left identity commute, it implies that $G_eG_f \subseteq G_{ef}$ where e and f, being left identities in G_e and G_f respectively, are the idempotents. This implies that G is an LA-semigroup which is a union of LA-semigroups G_e. Moreover, E is a commutative subsemigroup of the LA-semigroup by the result mentioned in the beginning of this chapter.

CHAPTER THREE

DECOMPOSITION OF A LOCALLY ASSOCIATIVE LA-SEMIGROUP

In [33], Tamura and Kimura proved that any commutative semigroup G is uniquely expressible as a semilattice of archimedean semigroups. Later in [11], Hewitt and Zuckerman proved that the following conditions are mutually equivalent:
(i) G is separative, (ii) the archimedean components of G are cancellative, (iii) G can be embedded in a union of groups. In this chapter, we have extended their results to a locally associative LA-semigroup G which, as we know, is not an associative structure.

Note also that a locally associative

LA-semigroup does not necessarily have associative powers.

EXAMPLE 3.1

For example, in a locally associative LA-semigroup G = {a, b, c}, defined by the table:

.	a	b	c
a	c	c	b
b	b	b	b
c	b	b	b

$a(a(aa)) = c \neq b = (a(aa))a.$

Next, we prove the following theorems.

THEOREM 3.2

A locally associative LA-semigroup G with left identity e has associative powers.

PROOF

For any element a in G we let $a^1 = a$ and a^{n+1}

$= a^na$, where n is a positive integer. The identity $aa^n = a^{n+1}$ is true for n = 1 and 2 by [2]. So assume that the identity holds for some n > 2. Then by theorem 1.7 we have $aa^{n+1} = a(a^na) = a^n(aa) = (aa^{n-1})(aa)$. But because S is medial, $(aa^{n-1})(aa) = (aa)(a^{n-1}a)$. Thus $aa^{n+1} = (aa)(a^{n-1}a) = (aa)a^n = (a^na)a$ by the left invertive law. Hence by induction it follows that $aa^n = a^{n+1}$.

Now we shall show that for all a in G and for all positive integers m, n

(3.1) $\quad a^ma^n = a^{m+n}$.

According to $aa^n = a^{n+1}$, the result is true for m = 1. Suppose that (3.1) holds for some m>1 also. Then by the left invertive law and (3.1), we have $a^{m+1}a^n = (a^ma)a^n = (a^na)a^m = a^{n+1}\ a^m = (aa^n)a^m = (a^ma^n)a = a^{m+n}a = a^{m+n+1}$. Hence, the result (3.1) follows by induction. Thus, the sub-structure generated by a is associative.

Due to [19], if G is a locally associative LA-semigroup with left identity e, then for all a in G and for all positive integers m,n

(3.2) $\quad (a^m)^n = a^{mn}$

It is important to mention that in [19] it has been shown also that

(3.3) $(ab)^m = a^m b^m$ for all a,b in G and all positive integers m.

The result is true for n=1, let n = 2. Then $(ab)^2 = (ab)(ab) = (a^2b^2)$, by (1.2). Thus the result is true for n = 2, suppose the result is true for n = k, that is $(ab)^k = a^kb^k$. Then $(ab)^{k+1} = (ab)^k(ab) = (a^ka)(b^kb)$ by (1.2). Thus $(ab)^{k+1} = a^{k+1}b^{k+1}$. Hence by induction, the result is true for all positive integers.

Before we prove the next result, we consider an example which shows that there exists a locally associative LA-semigroup with left identity that is not associative.

EXAMPLE 3.3

For instance, if G = {a,b,c,d} and the binary operation (.) is defined as follows

.	a	b	c	d
a	d	d	b	d
b	d	d	a	d
c	a	b	c	d
d	d	d	d	d

G is a locally associative LA-semigroup with left identity c and $(ac)c = a \neq b = a(cc)$.

THEOREM 3.4

If G is a locally associative LA-semigroup with left identity e then $H = \{a \in G: ae = a\}$ is a commutative subsemigroup of G with identity e. Moreover, for any a in G and positive integer $n \geq 2$, a^n is in H.

PROOF

Let x,y belong to H. Then xe = x, ye = y and since G is medial, (xy)e = (xy)(ee) = (xe)(ye) = xy. Also xy = (xe)y = (ye)x = yx by virtue of (1.1). Now, let x,y,z be in H. Then xe = x and so because of (1.2), we have x(yz) = (xe)(yz) = (xy)(ez) = (xy)z. Thus H is a commutative semigroup with identity e.

Let a belong to G and $n \geq 2$. It follows from

(1.1) and the fact $aa^n = a^{n+1}$ that $a^n e = (a^{n-1}a)\ e = (ea)a^{n-1} = aa^{n-1} = a^n$. This shows that a^n is in H.

In theorem 5 [19] Mushtaq and Yusuf have proved the following result.

LEMMA 3.5

Let G be a locally associative LA-semigroup with left identity. If there exists positive integers m and n such that $ab^m = b^{m+1}$ and $ba^n = a^{n+1}$, then $a\rho b$.

PROOF

For the sake of definitions assume that m<n; then we can multiply $ab^m = b^{m+1}$ by b^{n-m} to obtain

$$b^{n-m}(ab^m) = b^{n-m}b^{m+1}$$

$$= b^{n+1}, \text{ by } (3.1)$$

imply $\quad b^{n-m}(ab^m) = b^{n+1}$

imply $\quad a(b^{n-m}b^m) = b^{n+1}$, by theorem 1.7

Hence by (1) $ab^n = b^{n+1}$. Thus $ab^m = b^{m+1}$ imply $ab^n = b^{n+1}$. Since $ba^n = a^{n+1}$, have $a\rho b$.

LEMMA 3.6

The relation ρ on any locally associative LA-semigroup G with left identity is a congruence relation.

PROOF

Evidently ρ is reflexive and symmetric. For transitivity we may proceed as follows. Let $a\rho b$ and $b\rho c$ so that there exist positive integers n and m such that $ab^n = b^{n+1}$, $ba^n = a^{n+1}$ and $bc^m = c^{m+1}$, $cb^m = b^{m+1}$.

Let $k = (n+1)(m+1)-1$, that is, $k = n(m+1) + m$. Then by (3.1), (3.2) and (3.3), $ac^k = ac^{n(m+1)+m} = a(c^{n(m+1)}c^m) = a\{(c^{m+1})^n c^m\} = a\{(bc^m)^n c^m\} = a\{(b^n c^{mn})c^m)\}$.

Hence, $ac^k = a\{(c^m c^{mn})b^n\}$, by definition of an LA-semigroup. Then by (3.1) and theorem 1.7,

$$
\begin{aligned}
ac^k &= a(c^{m(n+1)}b^n) \\
&= c^{m(n+1)}(ab^n) \\
&= c^{m(n+1)}b^{n+1} = (ec^{m(n+1)})b^{n+1} \\
&= (b^{n+1}c^{m(n+1)})e
\end{aligned}
$$

$$= (bc^m)^{n+1}e$$

$$= (c^{m+1})^{n+1}e$$

$$= c^{(m+1)(n+1)}e.$$

Thus, $ac^k = (cc^{mn+n+m})e$

$= c^{mn+n+m+1} = c^{k+1}$, by (3.1) and remark 2 in [19]. Therefore, $ac^k = c^{k+1}$.

Similarly, it can be proved that $ca^k = a^{k+1}$, thus showing that ρ is an equivalence relation.

To show that ρ is compatible, assume that for some positive integer $n, ab^n = b^{n+1}$ and $ba^n = a^{n+1}$. Let c belong to G. Then, by (3.3) and (1.2)

$$\begin{aligned}(ac)(bc)^n &= (ac)(b^nc^n),\\ &= (ab^n)cc^n),\\ &= b^{n+1}c^{n+1}\\ &= (bc)^{n+1}.\end{aligned}$$

Thus

$$(ac)(bc)^n = (bc)^{n+1}. \qquad \text{(i)}$$

Similarly, $(bc)(ac)^n = (bc)(a^nc^n) = (ba^n)(cc^n)$

$$\begin{aligned}&= (ba^n)(c^{n+1})\\ &= a^{n+1}c^{n+1}\\ &= (ac)^{n+1}.\end{aligned}$$

This implies that

$$(bc)(ac)^n = (ac)^{n+1}. \qquad \text{(ii)}$$

From (i) and (ii) we conclude that ρ is compatible. Thus ρ is a congruence relation on G.

A congruence ρ on a groupoid is called separative if $a^2 \ \rho \ ab$ and $ab \ \rho \ b^2$ implies that $a \ \rho \ b$.

THEOREM 3.7

Let ρ and σ be separative congruences on a locally associative LA-semigroup G with left identity. If $\rho \cap (H\times H) \subseteq \sigma \cap (H\times H)$, then $\rho \subseteq \sigma$.

PROOF

Let $a \ \rho \ b$. Then $(a^2(ab))^2 \ \rho \ (a^2(ab))(a^2b^2) \rho \ (a^2b^2)^2$.

It follows from theorem 3.4, (1.1) and (1.2), $(a^2(ab))^2, (a^2b^2)^2$ belong to H and $(a^2(ab))(a^2b^2) = a^4((ab)b^2) = a^4(b^3a) = b^3a^5$ belong to H. Then $(a^2(ab))^2 \ \sigma \ (a^2(ab))(a^2b^2)\sigma(a^2b^2)^2$ and so $a^2(ab)\sigma$

a^2b^2. Since $a^2b^2 \ \rho \ a^4$ and by theorem 3.4, a^2b^2, a^4 is in H, we have $a^2b^2 \ \sigma \ a^4$. Since, G is medial, $a^2b^2 = (ab)^2$ and so $(a^2)^2 \ \sigma \ a^2(ab)\sigma(ab)^2$. Thus, we have $a^2 \ \sigma$ ab. Finally $a^2 \ \rho \ b^2$ and again by theorem 3.4, we have a^2, b^2 is in H. Then we obtain $a^2 \ \sigma \ ab \ \sigma \ b^2$ and so $a \ \sigma \ b$.

A groupoid is said to be separative if the identity map defined on it is a separative congruence.

THEOREM 3.8

A locally associative LA-semigroup G with left identity is a commutative semigroup with identity if it is separative.

PROOF

By virtue of theorem 3.4, we need only to show that G = H. Let a belong to S. Then since G is medial it follows from theorem 3.4, that $(ae)^2 = (ae)(ae) = a^2e = a^2$. Now by the fact that G is medial and by theorem 3.2, we have $((ae)a)^2 =$

$(ae)^2a^2 = a^2a^2 = (a^2)^2$ and $(a^2)^2 = (aa^2)(ea) = (ae)(a^2a) = a^2((ae)a)$, by theorem 1.7 and since G is separative $(ae)a = a^2$. Moreover, we have $(ae)^2 = (ae)a = a^2$ which implies that a = ae. Thus G = H.

We define a relation η on G as follows. Let a,b be in G. Then we say that a η b if and only if each of the elements a and b divides some power of the other.

That is, a η b if and only if $b^m = ax$ for some x and $a^n = by$ for some y and positive integers m, n.

THEOREM 3.9

Let G be a locally associative LA-semigroup with left identity. Then the relation η on G is the least semilattice congruence on G.

PROOF

The relation η is obviously reflexive and symmetric. To show transitivity, let a η b and b η c, where a, b, c are in G. Then b^m/a and a^n/b

for some positive integers m and n. This implies that $ax = b^m$ and $by = c^n$ for some x and y. Then $c^{nm} = (c^n)^m = (by)^m = b^m y^m$ by (3.2) and (3.3). So $c^{nm} = b^m y^m = (ax)y^m = (y^m x)a = (e(y^m x))a = (a(y^m x))e$. Now $c^{nm} = c^{nm-1}c = (cc^{nm-1})e = c^{nm}e$ implies that $ec^{nm} = e(a(y^m x))$ by (3.3). That is $c^{nm} = a(y^m x)$. If $nm = k$ and $y^m x = z$ then $c^k = az$. Similarly, $bx' = a^{m'}$ and $cy' = b^{n'}$ implies that $a^{k'} = cz'$.

Next, let a, b,c belong to G and a η b. Then by (3.3) and (1.2), $(bc)^m = b^m c^m = (ax)c^m = (ax)(cc^{m-1}) = (ac)(xc^{m-1})$ and so $(bc)^m = (ac)y$, where $y = xc^{m-1}$. Thus ac η bc. Similarly it can be shown that ca η cb. This proves that η is a congruence relation on G.

Now to show that η is a semilattice congruence on G, first we show that

(3.4) a η b implies ab η a.

Let a η b. Then $ax = b^m$ and $by = a^n$ for some x and y. So by (3.4) and (1.2), $(ab)^m = a^m b^m = a^m(ax) = a(a^m x)$. Also, by (3.3) and (1.2), $a^n = by$ implies that $a^{n+2} = a^2 a^n = (aa)(by) = (ab)(ay)$. Hence ab η a.

Next we show that,

(3.5) $ab\ \eta\ ba$ for all $a, b \in G$.

By (3.3), theorem 1.7 and by (1.1), $(ab)^2 = a^2b^2 = a^2(bb) = b(a^2b) = b((ba)a) = (ba)(ba) = (ba)^2$. Hence $ab\ \eta\ ba$. Also

(3.6) $(ab)c\ \eta\ a(bc)$ for all $a, b, c \in G$.

By the medial law $(ab)c = (cb)\,a\ \eta\ (bc)a = (ac)b\ \eta\ b(ac) = a(bc)$ by theorem 1.7. Therefore η is a semilattice congruence on G.

Now to show that η is the least semilattice congruence on G we need to show that η is contained in any other idempotent ρ on G. Let $a\ \eta\ b$, then there exist positive integers m and n and elements x and y in G such that $ax = b^m$ and $by = a^n$. Since $a\rho a^2$ and $b\rho b^2$, we infer that $ax\rho b$ and $by\rho a$. Also, since $b\rho b^2$ and ρ is compatible, we get $by\rho b^2y$. Now $by\rho a$ implies that $(by)x_1 = a^m$ and $ay_1 = (by)^n$. Thus $b^ma^m = (ba)^m = b^m((by)x_1) = (b^{m-1}b)((by)x_1) = (b^{m-1}b)((x_1y)b) = (b^{m-1}(x_1y))(bb) = (b^{m-1}(x_1y))b^2 = (b^2(x_1y)b^{m-1}) = x_1(b^2y))b^{m-1}$ by medial law and theorem 1.7. So $(ba)^m = (x_1(b^2y))b^{m-1} = ((ex_1)(b^2y))b^{m-1} = (((b^2y)x_1)e)b^{m-1} = (b^{m-1}e)((b^2y)x_1) = (b^2y)((b^{m-1}e)x_1)$ by (1.1) and theorem 1.7. If we let $z = (b^{m-1}e)x_1$, then $(ba)^m =$

$(b^2y)z$ implies that $(b^2y)\rho(ba)$. Similarly it can be shown that $(a^2x)\rho(ax)$. Thus

$a\rho(by)\rho(b^2y)\rho(ba)\rho(a^2x)\rho(ax)\rho b$ implies that $a\rho b$.

That is $\eta \subseteq \rho$. Thus η is the least semilattice congruence on G.

THEOREM 3.10

Let G be a locally associative LA-semigroup with left identity. Then G/η is a maximal semilattice homomorphic image of G.

PROOF

Evidently $a \eta a^2$ for any a in G implies that G/η is idempotent. Now by theorem 2.10 in [21], G/η is commutative and it follows that G/η is a semilattice. Since by theorem 3.9, G/η is the least semilattice congruence on G, it follows from proposition 1.7 in [5] that G/η is the maximal semilattice homomorphic image of G.

We say that G is archimedean if for any two elements of G, each divides some power of the other. This leads us to the following theorem.

THEOREM 3.11

If G is a locally associative LA-semigroup with left identity then G is uniquely expressible as a semilattice Y of archimedean locally associative LA-semigroups G_α ($\alpha \varepsilon$ y) with the left identity. The semilattice y is isomorphic to the maximal semilattice homomorphic image G/η of G and G_α (α belongs to y) are the equivalence classes of G mod η.

PROOF

Let η be the equivalence relation defined on G as in theorem 3.9. Then by theorem 3.4, G/η is a semilattice and G is homomorphic to G/η. G is a semilattice of archimedean locally associative

LA-semigroups with left identity will follow when we show that each equivalence class A on G mod η is an archimedean locally associative LA-subsemigroup (with left identity) of G. A is a locally associative LA-semigroup (with left) identity of S is clear. Let a,b ε A, then a η b implies that $ax = b^m$ and $by = a^n$ for some x,y ε S and some positive integers m,n. Then $a(bx) = b(ax) = bb^m = b^{m+1}$ and $b(ay) = a(by) = aa^n = a^{n+1}$. This implies that b^{m+1}/bx, bx/b. That is, (bx) η b and so bx ε A. Similarly, ay ε A. Thus b^{m+1}/a and a^{n+1}/b are relative to A, whence A is archimedean.

For uniqueness, let G be a semilattice Y of archimedean locally associative LA-semigroups (with left identity) G_α (α belongs to Y). Once we show that G_α are the equivalence classes of S mod η our job is done because then Y is isomorphic to G/η follows immediately. Let a,b be in G. We have to show that aηb if and only if a,b ε G_α. Now each divides a power of the other. Since G_α is archimedean, a η b by definition. Conversely, let a η b and a belong to G_α, b belong to G_β.

Since a η b by definition we have $ax = b^m$ and $by = a^n$ for some x,y in G and positive integers m and n. Let x belong to S_τ. Then ax belongs to $G_{\alpha\tau}$ and b^m belongs to G_β, so that $\alpha\tau = \beta$. Hence $a \geq \beta$ in the semilattice Y. By symmetry $\beta \geq \alpha$, and hence $\alpha = \beta$.

THEOREM 3.12

If G is a locally associative LA-semigroup with left identity, then G is separative if and only if its archimedean components are cancellative.

PROOF

Let G be separative. Then by theorem 3.8, G is a commutative semigroup with identity and so by theorem 4.16 [5] the archimedean components G_α of G are cancellative.

Conversely, let every archimedean component G_α of G be cancellative. Let a,b belong to G such that $a^2 = b^2 = ab$. If a belongs to G_α and b belongs to G_β, where α,β are in Y, then a^2 belongs to G_α and b^2 belongs to G_β such that $\alpha = \beta$. Using the cancellation in G, we conclude that a = b. Thus, G is separative.

THEOREM 3.13

If G is a locally associative LA-semigroup with left identity, then G can be embedded in a semigroup which is a union of groups if and only if G is separative.

PROOF

Suppose that G can be embedded in a semigroup Q which is a union of groups. Let a,b belong to G such that $a^2 = b^2 = ab$. If H_x denotes the maximal subgroup of Q containing x, then a^2 belongs to H_a, b^2 belongs to H_b, so that $H_a = H_b$. But $a^2 = ab$ implies that a = b. Hence S is separative.

Conversely, assume that G is separative. Then by theorem 3.8, G is a commutative semigroup with identity and so by the well-known result in [5], G can be embedded in a semigroup which is a union of groups.

CHAPTER FOUR

CHARACERTIZATION OF LA-SEMIGROUP BY A SPINED PRODUCT

In this chapter we characterize LA-semigroups S which have an LA-semigroup Q of left quotients, where Q is an $\mathbb{R}$-unipotent LA-semigroup which is a band of LA-semigroups.

$\mathbb{R}$-unipotent semigroups were studied by several authors (see for example [8] and [9]). Bailes [2] characterized $\mathbb{R}$-unipotent semigroups which are bands of groups. This characterization extended the structure of inverse semigroups which are semilattices of groups. Recently, Gould [9], studied the semigroup S which has a semigroup Q of left quotients where Q is an inverse semigroup which is a semilattice of groups. However, many definitions of semigroups of quotients have been proposed and studied. For a survey, the reader may consult Weinert [36]. These definitions have been motivated by corresponding definitions in ring theory. In this chapter we are concerned with a concept of semigroups of left quotients adopted by Fountain and Petrich [7]. The definition proposed there, is restricted to completely 0-simple semigroups of left quotients. The idea is that a completely 0-simple semigroup Q, containing a subsemigroup S, is a semigroup of left quotients of S if every element q in Q can be written as $q = a^{-1}b$ for some elements a,b in S with $a^2 \neq 0$ and a^{-1} is the inverse of a in the group $\mathcal{H}$-class H_a of Q. In this case S is called a left order in Q. This definition and its dual were used by Fountain and Petrich [7], to characterize a semigroup S which has a completely 0-simple semigroup of quotients. An extension of this definition and its dual was used by Gould [8] to obtain a necessary and sufficient condition for a semigroup S to have a bisimple inverse ω-semigroup of left quotients. This extended definition was used by Gould in [9] also to characterize semigroups S which have a semigroup Q of left quotients, where Q is an inverse semigroup which is a semilattice of groups. In this chapter we have considered the corresponding problem for $\mathbb{R}$-unipotent LA-semigroups which are band of LA-groups.

After preliminary results, we have obtained a necessary and sufficient condition for an LA-semigroups S to have an LA-semigroup Q of left quotients where Q is an $\mathbb{R}$-unipotent LA-semigroup which is a band of LA-groups. An $\mathbb{R}$-unipotent LA-semigroup is an LA-semigroup whose set of idempotents is a left regular band in which (ef)e = ef, for any idempotents e and f in S.

For an LA-semigroup S, any two elements a,b in S are $\mathbb{R}^*$-related if they are related by Green's relation $\mathbb{R}$ in some over LA-semigroup of S. The dual relation of $\mathbb{R}^*$ is $\mathcal{L}^*$. It is easy to see that $\mathbb{R}^*$ is a left and $\mathcal{L}^*$is a right congruence. Thus the intersection of $\mathbb{R}^*$ and $\mathcal{L}^*$ is an equivalence relation denoted by $\mathcal{H}^*$.

We say that an over-LA-semigroup Q of an LA-semigroup S is an LA-semigroup of left quotients of S if for any element q of Q, there exist a,b in S such that $q = a^{-1}b$ where a^{-1} is the left inverse of a in an LA-subgroup of Q. If Q is an LA-semigroup of left quotients of an LA-semigroup S, then S is said to be a left order in Q.

An LA-semigroup S is right reversible if for any a,b in S, there exists x,y in S such that xa = yb.

It is known now [20] that if Q is an $\mathbb{R}$-unipotent LA-semigroup which is a band of LA-groups, then Q can be written as a disjoint union of LA-groups G_α, $\alpha \in Y$, that is, $Q = \bigcup_{\alpha \in Y} G_\alpha$, where Y is a band isomorphic to the band of idempotents of Q. In particular Y is left regular; so we may call Q in this case a left regular band of LA-groups.

This result has been used together with the characterization of $\mathbb{R}$-unipotent LA-semigroups which are bands of LA-groups in terms of spined product to obtain an alternative structure for an LA-semigroup S to have a left regular band of LA-groups as an LA-semigroup of left quotients. At the end the case where the left orders are in a class of $\mathbb{R}^*$-unipotent LA-semigroups has been discussed.

PROPOSITION 4.1

S is a left regular band Y of right reversible left cancellative LA-semigroups S_α: $\alpha \in Y$ with left identity.

PROOF

Let Q be an $\mathbb{R}$-unipotent LA-semigroup with set of idempotents E. The set E is a left regular band. So every $\mathbb{R}$-class in Q contains a unique idempotent. Consider Q to be the semilattice Y of LA-semigroups G_α : $(\alpha \in Y)$ where for any $\alpha, \beta \in Y$, $G_\alpha \cap G_\beta = \phi$ if $\alpha \neq \beta$ and $Q = \bigcup_{\alpha \in Y} G_\alpha$, $G_\alpha G_\beta \subseteq G_{\alpha\beta}$ such that $E = Y$.

Now let S be an LA-semigroup which is a left order in Q. Put $S_\alpha = S \cap G_\alpha$ for any α in Y. It follows that for any α in Y, a in G_α, there exist x,y in S, with $a = x^{-1}y$ where x in S_β, y in S_γ, for some β, γ in Y. Since x^{-1} in G_β, y in G_γ, then $\alpha = \beta\gamma$ and xy in $S_\beta S_\gamma \subseteq S_{\beta\gamma} = S_\alpha$ so that S_α is non-empty for any α in Y. Clearly for any α in Y; S_α is an LA-subsemigroup of S. Now to show that S_α is cancellative. Let a,b,c belong to S_α and let ac = bc. Since a,b,c are in S_α, therefore a,b,c belong to G_α also. This implies that c' is in G_α. That is

$(ac)c' = (bc)c'$, this implies that $(c'c)a = (c'c)b$ thus $a = b$.

Now to show that S_α is right reversible let α be in Y and a,b belong to S_α. Choose s in S_α. Since b^{-1} in G_α, this implies that $(sa)b^{-1}$ is in G_α. By the ordering of S in Q, there exists x in S_β and y in S_γ for some β,γ in Y such that $(sa)b^{-1} = x^{-1}y$. This implies that $\alpha = \beta\gamma$ and $(x^{-1}y)b = \{(sa)b^{-1}\}b = (bb^{-1})(sa) = e_\alpha(sa)$. Thus

$$\begin{aligned}\{e_\alpha(sa)\}x &= \{(x^{-1}y)b\}x \\ &= (xb)(x^{-1}y) \\ &= (xx^{-1})(by)\end{aligned}$$

That is $\quad \{e_\alpha(sa)\}x = e_\beta(by)$.

This implies that $(sa)x = e_\beta(by) = b(e_\beta y)$ by theorem 1.7; and so $\alpha\beta = \alpha$ and $(sa)x = e_\beta(by)$.

Let z be in S_β. Then

$$\begin{aligned}\{(sa)x\}z &= \{(sa)(e_\beta x)\}z \\ &= \{(se_\beta)(ax)\}z = \{z(ax)\}(se_\beta)\end{aligned}$$

$= \{a(zx)\}(se_\beta) = (as)\{(zx)e_\beta\}$

$= (as)\{(e_\beta x)z\}$

$= (as)(xz) = \{(xz)s\}a.$

Since $(sa)x = e_\beta(by)$, therefore $\{(sa)x\}z = \{e_\beta(by)\}z$ implies that $\{(xz)s\}a = \{b(e_\beta y)\}z = \{z(e_\beta y)\}b$. It is clear that $(xz)s$ is in $S_\beta S_\alpha \subseteq S_{\beta\alpha} = S_\alpha$. Similarly $z(e_\beta y)$ is in $S_\beta S_\alpha \subseteq S_{\beta\alpha} = S_\alpha$. This shows that S is right reversible.

COROLLARY 4.2

For any α in Y; G_α is an LA-group of left quotients of S_α.

PROOF

For any α in Y, let g be in G_α and choose a in S_α. Since ag is in G_α, there exists x in S_β and y in S_γ for some β, γ in Y such that $ag = x^{-1}y$. Then by theorem 1.12, $ga = yx^{-1}$. Notice that x^{-1} is in G_β, $\beta\gamma = \alpha$, we have $(ga)x = (yx^{-1})x$. This implies that $(xa)g = (xx^{-1})y = e_\beta y$. Let b belong to S_β. Then, $\{(xa)g\}b = (e_\beta y)b = (by)e_\beta$ and $(bg)(xa) =$

$(by)(b^{-1}b)$ imply that $(bx)(ga) = (bb^{-1})(yb) = e_\beta(yb) = y(e_\beta b) = yb$ because of (1.2) and theorem 1.7. It follows that $\beta\alpha = \alpha$. Now since yb is in $S_\gamma S_\beta$ and $S_\gamma S_\beta \subseteq S_{\gamma\beta} = S_\alpha$ therefore $g = e_\alpha g = \{(bx)a\}^{-1}(yb)$.

COROLLARY 4.3

If q belongs to Q, then there exist a,b in S with $a\mathbb{R}b$ in Q and $q = a^{-1}b$.

PROOF

This follows from corollary 4.2 and from the fact that every two elements in G_α are $\mathcal{H}$-related.

LEMMA 4.4

If α belongs to Y and a,b are elements of S_α, then $a\mathbb{R}^*b$ in S.

PROOF

If α belongs to Y and a,b are in S_α and s is in S_λ, t is in S_μ for some λ, μ in Y, with sa = ta, then $S_{\lambda\alpha} = S_{\mu\alpha}$. Put $\beta = \lambda\alpha = \mu\alpha$. Since sa,ta are in S_β and S_β is a right reversible cancellative LA-semigroup with left identity, as = at implies that there exist m,n in S_β such that m(as) = n(at). Then by (1.3), a(ms) = a(nt). Now sm is in $S_\lambda S_\beta$ and $S_\lambda S_\beta = S_\lambda S_{\lambda\alpha} = S_{\lambda\alpha} = S_\beta$, therefore tn is in $S_\mu S_\beta \subseteq S_\mu S_{\mu\alpha} = S_{\mu\alpha} = S_\beta$.

And again by the right reversability of S_β, there exist μ, ν in S_β with $\mu(sm) = \nu(tn)$ such that $s(\mu m) = t(\nu n)$ or $(\mu m)s = (\nu n)t$. This means that $(as)(\mu m) = (at)(\nu n)$ where μm, νt, as, νn, at are in S_β and as = at implies that $\mu m = \nu t$ as S_β is cancellative. This implies that $\mu m = \nu t = k$ (say). Hence ks = kt or (ks)b = (kt)b. That is (bs)k = (bt)k. Since k, bs, bt are in S_β, therefore by right cancellation in S_β we have bs = bt or sb = st. Thus a $\mathbb{R}^*$ b in S.

COROLLARY 4.5

$a \mathcal{H}^* a^2$ for any element a in S.

PROOF

Let a belong to S_α, s belong to S_λ and t belong to S_μ with $a^2s = a^2t$. Clearly a^2 is in S_α and $\alpha\lambda = \alpha\mu$ (= γ say). Choose k in S_γ and write $k(a^2s) = k(a^2t)$. Then $k\{(aa)s\} = k\{(aa)t\}$ implies that $(aa)(ks) = (aa)(kt)$ or $(ak)(as) = (ak)(at)$. That is ak is in $S_\alpha S_\gamma = S_\alpha S_{\alpha\lambda} = S_{\alpha\alpha\lambda} = S_\gamma$ where as, at belong to S_γ and S_γ is cancellative. Hence as = at implies that sa = ta and this implies that (sa)a = (ta)a. Thus $a^2s = a^2t$ and $a \mathcal{L}^* a^2$ in S. Therefore by the dual of the fact that for any two elements a,b in an LA-semigroup S, the following two conditions are equivalent:(1) $a\mathcal{R}^*b$ in S (ii) sa = ta if and only if sb = tb. But $a\mathcal{R}^*a^2$ by lemma 4.4 and hence $a\mathcal{H}^*a^2$ in S.

Returning now to the product in Q, it can be seen that the product in Q is an extension of that in S. It is immediate from the definition of the product in Q that $G_\alpha G_\beta \subseteq G_{\alpha\beta}$ for any α, β in Y. Therefore Q is a left regular band of LA-groups G_α, where α belongs to Y. From its construction, Q is an LA-semigroup of left quotients of S. In conclusion we have established the following result.

THEOREM 4.6

An LA-semigroup S has a left regular band of LA-groups as an LA-semigroup of left quotients if and only if S is a left regular band of right reversible cancellative LA-semigroups.

Theorem 4.6 shows that, if S is a left regular band of right reversible, left cancellative LA-semigroups, then for any decomposition of S as a left regular band of right reversible, cancellative LA-semigroups, we can construct Q, where Q is a left regular band of LA-groups.

Now we provide an alternative characterization of an LA-semigroup S which has an LA-semigroup Q of left quotients, where Q is a left regular band of LA-groups. This characterization will be in terms of spined products. Recall that, if E is a band and M is an LA-semigroup with a semilattice congruence τ and an LA-semigroup isomorphism ϕ: $E/\varepsilon \longrightarrow M/\tau$, where ε is the minimum semilattice congruence on E, then the sub-direct product

$$P = \{e,x) \in E \times M : e\,\varepsilon^{\#}\,\phi = x\tau^{\#}\}$$

is called a spined product of E and M. We call a sub-direct product S of ExM a punched spined product of E and M if S is subset of spined product of E and M such that for any e in E, there exists x in M with (e,x) in S and for any y in M, there exists f in E with (f,y) in S. The aim of this discussion is to show that the left orders, which have been characterized earlier, are in fact punched spined products.

Let Q be an $\mathbb{R}$-unipotent LA-semigroup and E be its band of idempotents. Let ε be the minimum semilattice congruence on E. For any e in E, denote the ε-class containing e, by $\bar{e}$ or E(e). Write Y = {E(e): e $\in$ E}. Since E is left regular, therefore E(e) is a left zero semigroup.

REMARK 4.7

Let $\gamma = \{(x,y) \in Q \times Q: \gamma(x) = \gamma(y)\}$. It is well known that γ is the minimum inverse semigroup congruence on Q, and $\gamma/E = \varepsilon$. Suppose that Q is a band of LA-groups then Q/γ is a semilattice of LA-groups, and we can write $Q/\gamma = \bigcup_{\bar{e} \in Y} H_{\bar{e}}$, where $H_{\bar{e}}$ is the group $\mathcal{H}$-class in Q/γ containing $\bar{e}$. Moreover, Q is a spined product P of E and Q/γ, that is

$$Q = P = \bigcup_{\bar{e} \in Y} (E(e) \times H_{\bar{e}}) = \{(x^{-1}x, x\gamma): x \in Q\}.$$

We emphasize that P is a semilattice of the direct products $E(e) \times H_{\bar{e}}$ where $\bar{e}$ belongs to Y and the product P is reduced from the Cartesian product $E \times Q/\gamma$. Moreover $(f, x^{-1}\gamma)$ is an inverse of $(e, x\gamma)$ for any f in E(e). In particular, for any $(f, y\gamma)$ in $E(e) \times H_{\bar{e}}$; $(f, y\gamma)\ \mathcal{H}\ (f, \bar{e})$ in P and the inverse of $(f, y\gamma)$ in $H_{(f,\bar{e})}$ is $(f, y^{-1}\gamma)$. We refer the reader to [2] and [31] for further details.

Let S be an LA-semigroup which has P as an LA-semigroup of left quotients. For any $\bar{e}$ in Y, define a subset $M_{\bar{e}}$ of Q/γ by the rule: m in $M_{\bar{e}}$ if and only if m belongs to Q/γ and (f,m) is in S for some f in E(e).

LEMMA 4.8

For any $\bar{e}$ in Y, $M_{\bar{e}}$ is a left cancellative LA-semigroup.

PROOF

Let e belong to E and $(e, a\gamma)$ be in $E(e) \times H_{\bar{e}}$. Since P is an LA-semigroup of left quotients of S, then there exist $(k, x\gamma)$, $(g, y\gamma)$ in S and $(f, x^{-1}\gamma)$, the inverse of $(k, x\gamma)$, in an LA-subgroup of P, that is, $f \in E(k)$ such that

$$(e, a\gamma) = (f, x^{-1}\gamma)(g, y\gamma).$$

It follows that $e = fg$, and $fe = e$, $ke = kg$, where ke is in $E(f)E(e) \subseteq E(fe) \subseteq E(e)$ and $(x\gamma)(y\gamma)$ belongs to $H_{\bar{f}\bar{g}} = H_{\bar{e}}$. Therefore $(k, x\gamma)(g, y\gamma) = (kg, (x\gamma)(y\gamma))$ belongs to $S \cap (E(e) \times H_{\bar{e}})$. Hence $(x\gamma)(y\gamma) = xy\gamma \in M_{\bar{e}}$ and so $M_{\bar{e}}$ is non-empty. Clearly, $M_{\bar{e}}$ is an LA-subsemigroup of $H_{\bar{e}}$, where $H_{\bar{e}}$ is an LA-group and $M_{\bar{e}}$ is a left cancellative LA-semigroup.

LEMMA 4.9

For any $\bar{e}$ in Y, $M_{\bar{e}}$ is reversible.

PROOF

Let $a\gamma, b\gamma$ be in $M_{\bar{e}}$ and g,h in E(e) so that $(g,a\gamma),(h,b\gamma)$ are in S. Choose $c\gamma$ in $M_{\bar{e}}$ and take $(k,c\gamma)$ from S for some k in E(e). Let $(n,b^{-1}\gamma)$ be the inverse of $(h,b\gamma)$ in an LA-subgroup of P. That is, n in E(h) and

$$\{(k,c\gamma)(g,a\gamma)\}(n,b^{-1}\gamma) \text{ belongs to } E(e)xH_{\bar{e}}.$$

By the left ordering of S in P, there exist $(f,q\gamma),(i,d\gamma)$ in S, and $(t,q^{-1}\gamma)$ the inverse of $(f,q\gamma)$ in an LA-subgroup of P, that is, t belongs to E(f) such that $\{(k,c\gamma)(g,a\gamma)\}(n,b^{-1}\gamma) = (t,q^{-1}\gamma)(i,d\gamma)$. This implies that $[\{(k,c\gamma)(g,a\gamma)\}(n,b^{-1}\gamma)](h,b\gamma) = \{(t,q^{-1}\gamma)(i,d\gamma)\}(h,b\gamma)$. That is

$$\{(h,b\gamma)(n,b^{-1}\gamma)\}\{(k,c\gamma)(g,a\gamma)\} = \{(h,b\gamma)(i,d\gamma)\}(t,q^{-1}\gamma)$$

implies that

$$(h,\bar{e})\{(k,c\gamma)(g,a\gamma)\} = \{(h,b\gamma)(i,d\gamma)\}(t,q^{-1}\gamma).$$

That is,

$$\{(k,c\gamma)(g,a\gamma)\} = \{(h,b\gamma)(i,d\gamma)\}(t,q^{-1}\gamma) \quad \text{or}$$

$$\{(k,\bar{e})(k,c\gamma)\}(g,a\gamma) = \{(h,b\gamma)(i,d\gamma)\}(t,q^{-1}\gamma)$$

or $$\{(g,a\gamma)(k,c\gamma)\}(k,\bar{e}) = \{(h,b\gamma)(i,d\gamma)\}(t,q^{-1}\gamma)$$

or $$(f,q\gamma)[\{(g,a\gamma)(k,c\gamma)\}(k,\bar{e})] = (f,q\gamma)[\{h,b\gamma)(i,d\gamma)\} \ (t,q^{-1}\gamma)]$$

implies that

$$\{(g,a\gamma)(k,c\gamma)\}\{(f,q\gamma)(k,\bar{e})\} = \{(h,b\gamma)(i,d\gamma)\}\{(f,q\gamma)(t,q^{-1}\gamma)\} \text{ and}$$

$[\{(f,q\gamma)(k,\bar{e})\}(k,c\gamma)](g,a\gamma)= (h,b\gamma)(i,d\gamma)\}(f,\bar{f})$ and $[\{(f,q\gamma)(k,\bar{e})\}(k,c\gamma)](g,a\gamma) = [(f,\bar{f})(i,d\gamma)](h,b\gamma)$.

That is $\{(fk,q\gamma.\bar{e})(k,c\gamma)\}(g,a\gamma) = (fi,\bar{f}.d\gamma)(h,b\gamma)$ implies that

$$\{(fk,(q\gamma.\bar{e})c\gamma)\}(g,a\gamma) = (fi,\bar{f}.d\gamma)(h,b\gamma).$$

By theorem 1.11, we have $(g,a\gamma)\{(fk),(q\gamma.\bar{e})c\gamma)\} = (h,b\gamma)(fi,\bar{f}d\gamma)$ $[(g,a\gamma)\{(fk,(q\gamma.\bar{e})c\gamma)\}(j,v\gamma) = [(h,b\gamma)(fi,\bar{f}d\gamma)](j,v\gamma)$. That is $\{(j,v\gamma)(fk,(q\gamma.\bar{e})c\gamma)\}(g,a\gamma)= \{(j,v\gamma)(fi,\bar{f}d\gamma)\}(h,b\gamma)$ and $(jfk, v\gamma\{(q\gamma.\bar{e})c\gamma\})(g,a\gamma)= (jfi, v\gamma(\bar{f}d\gamma))(h,b\gamma)$ and $(jfk,(q\gamma.\bar{e})(v\gamma.c\gamma))(g,a\gamma)= (jfi, v\gamma(\bar{f}d\gamma))(h,b\gamma)$ and $(jfk, (q\gamma v\gamma)c\gamma)(g,a\gamma) = (jfi, v\gamma(\bar{f}d\gamma))(h,b\gamma)$ and $(jfk, (qv\gamma.)c\gamma)(g,a\gamma) = (jfi, v\gamma(\bar{f}d\gamma))(h,b\gamma)$ and $(jfk, (qv)c.\gamma)(g,a\gamma) = (jfi, v\gamma(\bar{f}d\gamma))(h,b\gamma)$.

Recall that $k = ti$, and notice that $fk = fi, tk = k$. so that $E(f)E(e) \subseteq E(e)$ and $jfi = jfk$, $ef = efe$ are in $E(e)$. Moreover $(v\gamma)(\bar{f}d\gamma) = (v\gamma((\bar{e}\bar{f})d\gamma)) = v\gamma(\bar{e}(d\gamma)) = v\gamma d\gamma = vd.\gamma$ (as $\bar{e}$ is the left identity) therefore $(qv)c$, $vd\gamma$ are in $M_{\bar{e}}$.

Now we put $M = \bigcup_{\bar{e} \in Y} M_{\bar{e}}$, M is a semilattice Y of reversible left cancellative LA-semigroup $M_{\bar{e}}$ with left identity, where $\bar{e}$ belongs to Y. It is easy to note that $\bigcup_{\bar{e} \in Y} (E(e) \times M_{\bar{e}})$ is a spined product containing S. Moreover, we have

LEMMA 4.10

(i) For any $e \in E$, there exists $x\gamma$ in $H_{\bar{e}}$ with $(e, x\gamma)$ in S,

(ii) For any f in E, $y\gamma$ in $M_{\bar{f}}$, there exists g in $E(f)$ with $(g, y\gamma) \in S$.

PROOF

(i) Let e belong to E and $(e, a\gamma)$ be in $E(e) \times M_{\bar{e}}$. Then $(e, a\gamma) = (f, x^{-1}\gamma)(g, y\gamma)$, where $(f, x\gamma), (g, y\gamma)$ are in S and $(f, x^{-1}\gamma)$ is the inverse of $(f, x\gamma)$ in

$H_{(f,\bar{f})}$ of P. Therefore $e = fg$ and $(f,x\gamma)(g,y\gamma) = (fg,(x\gamma)(y\gamma)) = (e,(xy)\gamma)$ in S.

(ii) The proof is straightforward.

Now it follows that S is a punched spined product and the following result is established.

PROPOSITION 4.11

Let P be a left regular band of LA-groups and S be an LA-semigroup. If P is an LA-semigroup of left quotients of S, then S is a punched spined product of a left regular band and a semilattice of reversible, cancellative LA-semigroups.

PROOF

For the converse of proposition 4.11, let S be a punched spined product of a left regular band E and a semilattice Y of reversible, cancellative LA-semigroups M_α where α belongs to Y. By corollary 4.2, there is an LA-group of left quotients G_α of M_α for any α in Y. We may assume that $G_\alpha \cap G_\beta = \phi$ for all α,β in Y, $\alpha \neq \beta$. Let $T = \bigcup_{\alpha\in Y} G_\alpha$. Define a product (.) in T by

$$a^{-1}b.c^{-1}d = (xa)^{-1}yd$$

where, if a,b in M_α; c,d in M_β, then x,y in $M_{\alpha\beta}$ are chosen such that xb = yc. Then T is an LA-semigroup of left quotients of M where $M = \bigcup_{\alpha\in Y} M_\alpha$. That is, T is a semilattice of LA-groups. Put $P = \bigcup_{\alpha\in Y} (E_\alpha x G_\alpha)$. Since $E_\alpha x G_\alpha$ is an LA-semigroup so is P, which is a band of LA-groups and whose set of idempotents is an LA-subsemigroup isomorphic to E. Therefore P is a left regular band of LA-groups. In fact we have:

LEMMA 4.12

P is an LA-semigroup of left quotients of S.

PROOF

Let α belong to Y and (e,m) be in $E_\alpha x G_\alpha$. Recall that S is a punched spined product of E and M. Since e is in E_α there exists an element z in M_α such that (e,z) in S. As m in G_α and M is a left order in G_α, there exists an element z in M_α such that (e,z) in S. As m is from G_α and M is a left order in G_α there exist x,y in M_α, such that m = $x^{-1}y$ and hence there exist f,g in E_α with (f,x) and (g,y) in S. Notice that x^{-1} belongs G_α and there exist u,ν in M_α with $x^{-1} = u^{-1}v$ and $(uz)x^{-1} = uz(u^{-1}\nu) = (uu^{-1})(zv) = zv$.

Let i,j be in E_α so that (i,u) and (j,v) are in S. Clearly (ei,uz) = (e,uz) in S (since E(e) are left zero semigroups). Now $(e,(uz)^{-1})$ is the inverse of (e,uz) in $H_{(e,\bar{e})}$ of P and (ejg, (zv)y) = (e,(zv)y) (since E(e) is a left zero semigroup).

Moreover, $(e,m) = (e,x^{-1}y) = (e,\ (u^{-1}v)u)$
$= (e,\ \{(uz)^{-1}(zv)y\}) = (e,\ (u^{-1}z^{-1})\{(zv)y\}$
$= (e,\ \{u^{-1}(zv)\}\{z^{-1}y\}) = (e,\ \{z(u^{-1}v)\}\{z^{-1}y\})$
$= (e,\ (zz^{-1})\{(u^{-1}v)y\}) = (e,\ (u^{-1}v)y)$. This implies that $(e,m) = (e,\ (uz)^{-1}\{(zv)y\})$

$$= (e,\ (uz^{-1})(e,\ (zv)y).$$

Now the converse of 4.11 is evident. In conclusion we have the following result.

THEOREM 4.13

An LA-semigroup S has a left regular band of LA-groups as an LA-semigroup of left quotients if and only if S is a punched spined product of a left regular band and a semilattice of reversible, left cancellative LA-semigroups.

The following corollary is an immediate consequence of theorem 4.13.

COROLLARY 4.14

If S is a spined product of a left regular band and a semilattice of right reversible left cancellative LA-semigroups, then S has a left regular band of LA-groups as an LA-semigroup of left quotients.

For the rest of this chapter, let S be a spined product of a left regular band E and a semilattice Y of cancellative LA-semigroups M_α: where α belongs to Y. Put $E = \bigcup_{\alpha \in Y} E_\alpha$, $M = \bigcup_{\alpha \in Y} M_\alpha$ and $S = \bigcup_{\alpha \in Y} (E_\alpha x M_\alpha)$.

LEMMA 4.15

The relation $\mathcal{H}^*$ is the greatest semilattice congruence on M all of whose classes are cancellative.

PROOF

By the fact that M is a semilattice of cancellative LA-semigroups, then $\mathcal{H}^*$ is the greatest band congruence on M all of whose classes are cancellative. The relation γ defined on M by the rule (a,b) is in γ if and only if a,b are in M_α for some α belonging to Y is a band congruence on M all of whose classes are cancellative. Therefore $\gamma \subseteq \mathcal{H}^*$. Now for any elements a,b in M, we have (ab,ba) in γ. Hence ab $\mathcal{H}^*$ba and $M/\mathcal{H}^*$ is a semilattice.

Identify the semilattice $M/\mathcal{H}^*$ by J, that is, M is a semilattice J of $\mathcal{H}^*_j$, where j belongs to J. For each j in J, let $Z_j = \{\alpha \in Y, M_\alpha \subseteq \mathcal{H}^*_j\}$. Readily, Z_j is a sub-semilattice of Y for any j in J. Put $F_j = \bigcup_{\alpha \in Z_j} E_\alpha$ and $S_j = \bigcup_{\alpha \in Z_j} (E_\alpha \times M_\alpha)$.

Now we come to the final result.

PROPOSITION 4.16

The following statements concerning the LA-semigroup S are equivalent.

(i) Each $\mathcal{H}^*$-class of M is reversible

(ii) For any a,b in M, there exist x,y in M with
$xa = yb$ and $x \, \mathcal{H}^* y \, \mathcal{H}^* ab$

(iii) S_j is right reversible for any j in J.

(iv) There is an over-LA-semigroup T of S which is a left regular band X of right reversible left cancellative LA-semigroups T_α, where α belongs to X and for any j in J, $\mathcal{H}^*_j$ is isomorphic to T_α for some α in X.

PROOF

Recall that $\mathcal{H}^*$ is a semilattice congruence on M. (i) $\Leftrightarrow$ (ii)

If (i) holds and a,b are in M, then ab, ba are in H^*_{ab} and there exist c,b in H^*_{ab} with

$$c(ab) = d(ba)$$

or $\quad a(cb) = b(da) \quad$ by theorem 1.7.

Also $\quad cb$ in H^*_{ab} , $H^*_a \subseteq H^*_{ab}$

$$da \in H^*_{ab}\ ,\ H^*_b \subseteq H^*_{ab}.$$

Put $x = cb$ and $y = da$ to get $ax = by$ or $xa = yb$ (by theorem 1.12) and $x\ \mathcal{H}^*y\ \mathcal{H}^*ab$. Hence (ii) holds.

If (ii) holds, z belongs to M and a,b are in H^*_z, then in particular there exist x,y in M with $xa = yb$ and $x\ \mathcal{H}^*y\ \mathcal{H}^*ab$. Since $a^2\ \mathcal{H}^*ab$, then $a\ \mathcal{H}^*ab$ and x,y are in H^*_z so (i) holds.

(i) ⇒ (iii)

If (i) holds and j is in J, (e,a), (f,b) belong to S_j, such that (e,a) belongs to $E_\alpha x M_\alpha$, (f,b) belongs to $E_\beta \times M_\beta$, say, where M_α and M_β are subsets of H^*_j. Then a,b are in H^*_j with $xa = yb$, where x is in M_λ, y is in M_μ for some λ,μ in Z_j. It follows that $\lambda\alpha = \mu\beta$. Let g belong to E_λ, h belong to E_μ and s belong to $M_{\lambda\alpha} = M_{\mu\beta}$. Then $gehf$ is in $E_\lambda E_\alpha E_\mu E_\beta \subseteq E_{\lambda\alpha}$, sx is in $M_{\lambda\alpha}M_\lambda \subseteq M_{\lambda\alpha}$, sy is in $M_{\mu\beta}M_\beta \subseteq M_{\mu\beta}$ whence $(sx)a = (sy)b$. The elements $(gehf,sx)$, $(gehf,sy)$ are in $E_{\lambda\alpha} \times M_{\lambda\alpha}$ so that they are in S_j. Moreover, $(gehf,(sx)a) = (gehf,(sy)b)$ (S_j is right reversible) that is $(gehfe,\ (sx)a) = (gehff,(sy)b)$ and $(gehf,sx)(e,a) = (gehf,sy)(f,b)$. Hence (iii) holds.

If (iii) holds and a,b are in H_j^*, then for some α,β in Z_j, a belongs to M_α, b belongs to M_β. Let e be in E_α, f be in E_β, so that (e,a),(f,b) are in S_j. Then there exist (g,x),(h,y) in S_j with (g,x)(e,a) = (h,y)(f,b). In particular, x,y belong to H_j^*, xa = yb and (i) holds.

(i) ⇒ (iv)

If (i) holds, then by lemma 4.15, H_j^* and hence $\{e\}xH_j^*$ is a reversible, left cancellative LA-semigroup for any e in E_α, α in Z_j, for any j in J, α in Z_j, put

$$N_\alpha = \bigcup_{\alpha \in E_\alpha} (\{e\} \times H_j^*) \text{ so that } F_j \times H_j^* = \bigcup_{\alpha \in Z_j} N_\alpha$$

and $$T = \bigcup_{j \in J} (F_j \times H_j^*) = \bigcup_{j \in J} (\bigcup_{\alpha \in Z_j} N_\alpha) = \bigcup_{j \in J} (\bigcup_{\alpha \in Z_j} (\bigcup_{e \in E_\alpha} (\{e\} \times H_j^*)))$$

is a left regular band of reversible, left cancellative LA-semigroups. Clearly, for any j in J, α in Z_j, e in E_α ; $\{e\}xH_j^* = H_j^*$ and S is an LA-subsemigroup of T. Hence (iv) holds.

If (iv) holds, then trivially (i) holds.

An LA-semigroup S is abundant if each

$\mathbb{R}^*$-class and each $\mathcal{L}^*$-class of S contains an idempotent. If a is an element of S, then a^+ and a^* denote typical idempotents in $\mathbb{R}^*_a$ and $\mathcal{L}^*_a$ respectively. An LA-semigroup S is super abundant if each $\mathcal{H}^*$-class contains an idempotent. Next we consider the class of abundant LA-semigroups in which the set of idempotents form a left regular band. In this case every $\mathbb{R}^*$-class of S contains a unique idempotent. Thus S is called $\mathbb{R}^*$-unipotent. The objective is to characterize a class of $\mathbb{R}^*$-unipotent LA-semigroups which have an LA-semigroup Q of left quotients where Q is a left regular band of LA-groups. This is the special case of the subject matter discussed previously.

LEMMA 4.17

Let S be an $\mathbb{R}^*$-unipotent LA-semigroup then: (i) S is super abundant if and only if $\mathbb{R}^* = \mathcal{H}^*$ on S. (ii) S is a band of a left cancellative LA-monoid if and only if S is super abundant and $\mathcal{H}^*$ is a congruence on S.

Henceforth by S we shall mean an $\mathbb{R}^*$-unipotent LA-semigroup.

PROPOSITION 4.18

If S is a left regular band Y of reversible, left cancellative LA-semigroups S_α, where α is in Y then the following statements are equivalent.

(i) S is super abundant

(ii) for every α in Y, a in S_α, there exists an idempotent e_γ in S_γ for some y in Y with $e_\gamma \mathcal{L}^* a$ and $S_\gamma S_\alpha \subseteq S_\alpha$.

PROOF (i) $\Rightarrow$ (ii)

Let α belong to Y, a belong to S_α and a $\mathbb{R}^* e_\gamma$, where e_γ is an idempotent in S_γ. Since $\mathbb{R}^* = \mathcal{H}^*$ by lemma 4.17 therefore a $\mathcal{L}^* e_\gamma$ and $e_\gamma a = a$. That is $S_\gamma S_\alpha \subseteq S_\alpha$.

(ii) $\Rightarrow$ (i)

Let a belong to S_α where $a\mathbb{R}^* e_\delta$, e_δ is an idempotent in S_δ. Then $e_\delta a = a$, that is, $\delta\alpha = \alpha$. It follows that $\alpha S\alpha = \alpha$ and $\alpha\delta = \alpha$. In particular, ae_δ

belongs to S_α. By reversability of S_α , $xa = y(ae_\delta)$, for some x,y in S_α, that is, $(xa)e_\delta = \{y(ae_\delta)\}e_\delta$ and by the left cancellation in S_α this implies that $x = y$ and $xa = xae_\delta$. Thus $a = ae_\delta$.

Now let e_γ be an idempotent in S_γ with $e_\gamma \mathcal{L}^* a$ and $S_\gamma S_\alpha \subseteq S_\alpha$. Since $ae_\gamma = a = ae_\delta$ and $e_\gamma \mathcal{L}^* a$, then $ae_\gamma = (ae_\delta)e_\gamma = (e_\gamma e_\delta)a$. This implies that $e_\gamma a = (e_\gamma e_\delta)a$ and $e_\gamma = e_\gamma e_\delta$. Recall that $e_\gamma a$ belongs to $S_\gamma S_\alpha \subseteq S_\alpha$, a is in S_α. We have

$$u(e_\gamma a) = va$$

and $$e_\gamma(ua) = va \qquad \text{by theorem 1.7.}$$

or $$(e_\gamma e_\gamma)(ua) = va$$

and $$(e_\gamma u)(e_\gamma a) = va \qquad \text{by (1.2).}$$

Similarly $$(e_\gamma u)(e_\gamma a) = va$$

$$e_\gamma\{(e_\gamma u)a\} = va$$

$$e_\gamma\{(au)e_\gamma\} = va$$

$$(au)(e_\gamma e_\gamma) = va$$

$$(au)e_\gamma = va$$

$$(e_\gamma u)a = va.$$

This implies that $e_\gamma a = a$ since $e_\gamma a = a = e_\delta a$ and $e_\gamma \mathbb{R}^* a$.

Since $$e_\gamma a = a$$

$$(e_\gamma a)e_\delta = ae_\delta$$

$$(e_\gamma a)e_\delta = a(e_\delta e_\delta)$$
$$(e_\gamma a)(e_s e_s) = e_\delta(ae_s)$$
$$(e_\gamma e_\delta)(ae_s) = e_\delta(ae_s).$$

This implies that $e_\gamma e_\delta = e_s$.

Hence $e_\gamma = e_\delta$ and $a\ \mathcal{L}^* e_\delta$, that is $a\ \mathcal{H}^* e_\delta$ and (i) holds.

LEMMA 4.19

If S is super abundant in which for any elements a,b in S, there exist x,y in S with xa = yb and $x\ \mathcal{H}^* y\ \mathcal{H}^* ab$ then each $\mathcal{H}^*$-class in S is right reversible.

PROOF

This is immediate from the fact that each $\mathcal{H}^*$-class of S is a left cancellative LA-monoid.

PROPOSITION 4.20

If S is a band of cancellative LA-monoids, then the following statements are equivalent.

(i) Each $\mathcal{H}^*$-class in S is right reversible.

(ii) For any a,b in S, there exist elements x,y in S with xa = yb and $x\mathcal{H}^* y\ \mathcal{H}^* ab$.

PROOF

(i) ⇒ (ii)

By Lemma 4.17, S is super abundant on which $\mathcal{H}^*$ is a congruence. Let a belong to H^*_e, b belong to H^*_f, for some idempotents e,f in S. Then ab belongs to H^*_{ef} and (ab)a belongs to $H^*_{efe} = H^*_{ef}$. But H^*_{ef} is right reversible, so there exist u,v in H^*_{ef} such that u(ab) = v{(ab)a}. Then, by theorem 1.7. a(ub) = (ab)(va) or $a\{(bu)e_{ef}\} = \{(va)b\}a$. This implies that $(bu)(ae_{ef}) = \{(va)b\}a$ and and $(ba)(ue_{ef}) = \{(va)b\}a$ which further implies that $\{(ue_{ef})a\}b = \{(va)b\}a$.

Let $y = (ue_{ef})a$ belong to $H^*_{efe} = H^*_{ef}$. Then x = $(va)b \in H^*_{efef} = H^*_{ef}$ xa = yb and $x\ \mathcal{H}^* y\ \mathcal{H}^* ab$

(i) ⇒ (ii)

This is Lemma 4.19.

In fact any of the statements of proposition 4.20 is a consequence of S to have LA-semigroup Q of left quotients where Q is a left regular band of LA-groups. The following Lemma demonstrates this result.

LEMMA 4.21

Let S be super abundant which is a left regular band of right reversible left cancellative LA-semigroups. Then for any elements a,b in S, there exist x,y in S with xa = yb and $x\ \mathcal{H}^{*} y\ \mathcal{H}^{*} ab$.

PROOF

Put $S = \bigcup_{\alpha \in Y} S_{\alpha}$, where Y is a left regular band and S_{α} is a right reversible left cancellative LA-semigroup for any α in Y. Let a,b belong to S; a belong to S_{α}, b belong to S_{β}, say. Then ab belongs to $S_{\alpha\beta}$, and (ab)a belongs to $S_{(\alpha\beta)\alpha} = S_{\alpha\beta}$, and there exist u,v in $S_{\alpha\beta}$ with u{(ab)a} = v(ab) where x = (ua)b is in $S_{\alpha\beta\alpha\beta} = S_{\alpha\beta}$ and $y = (ve_{\alpha\beta})a$ belongs to $S_{\alpha\beta\alpha} = S_{\alpha\beta}$. But every two elements in $S_{\alpha\beta}$ are $\mathbb{R}^{*}$-related (Lemma 4.20). Then the result follows from the fact that $\mathbb{R}^{*} = \mathcal{H}^{*}$ on S.

Now we consider the construction of S_α in S as given in the folloing proposition.

PROPOSITION 4.22

Let S be super abundant with band of idempotents E and $E = \bigcup_{\alpha \in y} E_\alpha$ be the maximal semilattice decomposition of E. For each α in Y, define

$$S_\alpha = \{x \in S : x^+, x^* \in E_\alpha\}.$$

Then:

(i) S_α is a maximal abundant LA-subsemigroup of S which contains E_α as its set of idempotents such that $\mathbb{R}^*(S_\alpha) \subseteq \mathbb{R}^*(S)$ and $\mathcal{L}^*(S_\alpha) \subseteq \mathcal{L}^*(S)$

(ii) $S_\alpha \cap S_\beta = \phi$ if $\alpha \neq \beta$

(iii) S is a semilattice of S_α; where α belongs to Y

(iv) $S_\alpha = E_\alpha x H_e^*$, where H_e^* is the $\mathcal{H}^*$-class in S containing e, and e belongs to E_α.

Now let S be super abundant with set of idempotents E. Retain the notations of proposition 4.22. Assign to each α in Y, a left cancellative LA-monoid $M_\alpha = H_e^*$ for some fixed e in E_α. By the fact that if e,f are $\mathcal{L}$-related idempotents in an LA-semigroup S, then $H_e^* = H_f^*$ implies $M_\alpha^* = H_f$ for any f in E_α. By proposition 4.22, $S_\alpha = E_\alpha x M_\alpha$. Denote the identity of M_α by e_α and put $M = \bigcup_{\alpha \in Y} M_\alpha$. Define a product (.)on M by $x.y = e_{\alpha\beta}xy$, for any x in M_α, y in M_β. Then

$(x.y).z = \{e_{\alpha\beta}(xy)\}.z$ where

$e_{\alpha\beta}(xy)$ belongs to $M_{\alpha\beta}$, z belong to M_γ. Also

$$
\begin{aligned}
(x.y).z &= e_{\alpha\beta\gamma}\{(xy)z\} \\
&= e_{\alpha\beta\gamma}\{(zy)x\} \\
&= \{e_{\beta\gamma}(zy)\}(e_\alpha x) \\
&= (z.y)x.
\end{aligned}
$$

Hence M is a semilattice y of the left cancellative LA-monoids. M_α where $\alpha \in Y$.

Moreover, we have the following lemma.

LEMMA 4.23

S is in one-to-one correspondence with

$$P = \bigcup_{\alpha \in Y} (E_\alpha x M_\alpha)$$

PROOF

Define ϕ: $P \longrightarrow S$ by $(e,a)\phi = ea$. It is obvious that ϕ is a well-defined map. Let (e,x) belong to $E_\alpha \times M_\alpha$ and (f,y) belong to $E_\beta \times M_\beta$ such that $ex = fy$. We can verify that $e \; \mathbb{R}^* ex$ and $f \; \mathbb{R}^* fy$. Consider $e(ex) = e\{(ee)x\} = e\{xe)e\} = (xe)(ee) = (xe)e = (ee)x = ex$. This implies $e(ex) = ex$. That is $e \; \mathbb{R}^* ex$. Similarly $f \; \mathbb{R}^* fy$. Therefore $e = f$ and $E_\alpha = E_\beta$, that is, $\alpha = \beta$. Thus $ex = fy$ implies that

$e_\alpha(ex) = e_\alpha(fy)$

or $\quad e(e_\alpha x) = f(e_\alpha y)$

or $\quad e_\alpha x = e_\alpha y$

or $\quad x = y$

Thus, ϕ is one-to-one.

For surjectivity, let x belong to S, where $x \; \mathbb{R}^* \; x^+$; x^+ belongs to E_α, say. Then $(x^+, e_\alpha x)$ is in $E_\alpha \times M_\alpha$, and $(x^+, e_\alpha x)\phi = x^+(e_\alpha x) = x^+ x = x$. Hence, ϕ is surjective.

Recall that a band E is a left normal band if $efg = egf$ for any idempotents e,f,g in E. Clearly left normal bands are left regular. To improve the result of Lemma 4.23, we impose the condition of left normality on E.

PROPOSITION 4.24

If E is left normal, then $P = \bigcup_{\alpha \in Y} (E_\alpha \times M_\alpha)$ is isomorphic to S.

PROOF

From the proof of Lemma 4.23, we have the bijection ϕ: $P \longrightarrow S$ defined by $(e,a)\phi = ea$ for any (e,a) in P. To show that ϕ is a homomorphism, let (e,x) belong to $E_\alpha \times M_\alpha$ and (f,y) be in $E_\beta \times M_\beta$. Then $\{(e,x)\cdot(f,y)\}\phi = (ef, e_{\alpha\beta}xy)\phi = (ef)(e_{\alpha\beta}xy) = (ef)(xy)$ where $ef, e_{\alpha\beta}$ belong to $E_{\alpha\beta}$ and $(e,x)\phi(f,y)\phi = (ex)(fy)$. Notice that $ex \; \mathbb{R}^* e$ implies that $efe \; x \; \mathbb{R}^* \; efe$. That is, $efex \; \mathbb{R}^* ef$ or $efex \; \mathcal{L}^* ef$ because $\mathbb{R}^* = H^*$ on S. That is, $efexef = efx$. This implies that $efexfy = efxy$ or $efxfy = efxy$.

Now let i be in E such that $xf \; \mathcal{H}^* i$. Then, in particular we have $xfi = xf$. That is $xfi = xff$ which implies that $i = if$. Therefore $efxfy = efixfy = eifxfy$, because E is left normal, and $efxfy = eixfy$, because $if = i$. Thus $efxfy = exfy$. Hence ϕ

is an isomorphism.

As an immediate consequence of proposition 4.24, we have the following corollary.

COROLLARY 4.25

If E is left normal, then S is a spined product of a left regular band and a semilattice of Y of left cancellative LA-monoids M_α; where α belongs to Y and M_α's are $\mathcal{H}^*$-classes of S.

Now directly from theorem 4.6 proposition 4.20, proposition 4.24, proposition 4.18, and lemma 4.12, we have

THEOREM 4.26

Let S be super abundant in which the set of idempotents is a left normal band. Then the following statements are equivalent.

(i) S is a left order in a left regular band of LA-groups,

(ii) S is a left regular band of right reversible and left cancellative LA-semigroups,

(iii) For any a,b in S, there exists x,y in S with $xa = yb$ and $x \mathcal{H}^* y \mathcal{H}^* ab$,

(iv) Each $\mathcal{H}^*$-class in S is right reversible,

(v) S is a spined product of a left regular band and a semilattice of right reversible and cancellative LA-semigroups.

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