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PROPAGATION OF LOVE-TYPE WAVES
IN
INHOMOGENEOUS MEDIA

BY

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DEDICATED TO

My Auntie

whose prayers have always been a source of great inspiration for me and whose sustained hope in me led me to where I stand today.

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ABSTRACT

In this thesis we first investigate the effect of inhomogeneity on the propagation of Love-type waves in a layer trapped between two half-spaces. The inhomogeneity is due to linear variation of density (or rigidity) in the lower half-space (or layer). The Love-type waves are excited in the layer due to a line source situated at the interface between the layer and the lower half-space. Using the Fourier transform and Green's function method the dispersion relations are derived and it is proved that if we take the inhomogeneity factor equal to zero, these dispersion relations reduce to standard dispersion relations. The analytic solution of the transmitted field in the layer is also presented.

The discussion is further extended to a point source excitation. The point source is taken at an interface between a homogeneous layer and inhomogeneous half-space. The upper surface of the layer is taken stress-free. The transmitted wave and the dispersion relation for the Love waves are calculated analytically.

At the end the wave-operator corresponding to a horizontally polarized shear wave travelling in an elastic layer trapped between two half-spaces is studied. The Green's function for the problem is derived and is then used to obtain a spectral representation of the operator.

LIST OF PUBLICATIONS

(M. AHMAD)

RESEARCH PAPERS:

(This thesis is based on papers 1 to 4)

1. Field due to a point source in a layer over an inhomogeneous medium (co-authors S. Asghar and F. D. Zaman), IL NUOVO CIMENTO C, 14(6), 569-573, 1991.
2. Dispersion of Love-type waves in a vertically inhomogeneous intermediate layer (co-authors S. Asghar and F. D. Zaman), J. Phys. Earth, 38(3), 213-221, 1990.
3. Love-type waves due to a line source in an inhomogeneous layer trapped between two half-spaces (co-authors F. D. Zaman and S. Asghar), BULLETTIN DI GEOFISICA TEORICA ED APPLICATA, 32, 167-174, 1990.
4. Spectral representation of a Love-type operator (co-authors F. D. Zaman and S. Asghar) submitted.
5. Spectral representation of Love wave operator for a layered infinite strip (co-authors F. D. Zaman and S. Asghar), Technical Report No. 113, KFUPM, June, 1989.
6. Diffraction of SH-waves by a plane crack in a thick plate (co-authors F. D. Zaman and S. Asghar), Revista Matematica Argentina, 32, 1988.
7. Diffraction of SH-waves in a layered plate (co-authors F. D. Zaman and S. Asghar), J. Tech. Phys., 282, 2, 143-152, 1987.

CONTENTS

CHAPTER ZERO

GENERAL INTRODUCTION

Seismic surface waves play an important role in the field of seismology. One of the principal aims of theoretical seismology is to investigate the effects of velocity changes and of discontinuities or boundaries on the propagation of elastic waves through the earth and to identify its subsurface structure in terms of elastic parameters by comparing the theoretical results with seismological observations. The role of Love waves is important in this context.

Since the first long-period seismograph measured horizontal motion only, the presence of large transverse components in the 'main tremor' was one of the first established facts of seismology. However, the homogeneous, semi-infinite elastic solid model did not admit disturbances of this nature, whose effect is confined to the surface only. It was not until 1911 that an explanation of these waves was provided by Love [29], who showed that these waves consisted of horizontally polarized shear waves in a surface layer overlying a half-space whose rigidity is less than that of the lower half-space. These waves can be regarded as SH-waves continually reflected between the outer surface and the interface. Love waves are totally reflected at the interface and therefore they can travel horizontally without continual loss of energy downwards.

Stoneley [37] proposed the propagation of generalized type of Love waves in a homogeneous medium of finite depth sandwiched between two semi-infinite isotropic media. He showed that the existence of Love-type waves is possible, if the wave length is not very large or the thickness of the middle layer is not too thin. He also showed that Love-type waves can exist when the distortional wave velocity in the upper semi-infinite medium is more or is less than that in the lower semi-infinite medium.

Hudson [22] investigated the existence of Love waves in a layer of finite depth, with upper surface free from stress and the lower boundary rigidly fixed. He found that Love waves exist for any variation in density, such that it is integrable as a function of depth and any variation in rigidity such that it is a piecewise continuous function of depth. He also proved that the periods of waves of given wavelength are reduced if the rigidity is increased or the density decreased in some range of depth. Finally he showed that the last result holds when the depth of the layer becomes infinite.

Dutta [14,15,16], solved the problem of propagation of Love waves in a non-homogeneous thin layer lying over a semi-infinite medium. In his first paper he showed that the existence of Love-type wave is possible in a layer of variable rigidity and density. In the second paper he discussed the problem in which the rigidity and the density of the trapped layer vary exponentially, and in the third, the possibility of propagation of Love waves in a non-homogeneous internal stratum of finite depth lying between two semi-infinite isotropic media in two different cases. In the first case the rigidity and density vary as $\cosh^2(z/\lambda)$ and in the second case as $(1+z/\lambda)^2$, where λ is a constant and z is the depth measured from the common interface of the upper medium and the internal stratum. He showed that the Love-type wave is possible if the product of the wave number and the depth of the internal stratum is sufficiently large. Later on Sinha [34,35], Bhattacharya [8], Chattopadhyay [10] and Kar [26], among others, studied problems of the propagation of Love-type waves propagating in a non-homogeneous layer lying deep in the earth, in which either the rigidity or density or both vary in different forms. In all these papers, they found that the conditions under which Love-type waves can propagate are the same in each case, viz. the velocity of propagation must be greater than the shear velocity in the internal stratum and is less than the shear velocity in each of the isotropic semi-infinite media. The dispersion equations in all these cases reduce to the standard dispersion equation for a homogeneous intermediate layer when the non-homogeneous factors are vanishingly small.

The propagation of Love waves in a fluid saturated porous layer lying between two elastic half-spaces has been considered by Paul [33]. He found that in the case of small porosity factor, the wavelength of the Love wave propagated in a fluid saturated porous layer increases (or decreases) in comparison with the case of elastic intermediate layer according as the density of the solid is greater (or less) than that of the fluid filling the pores.

Vlaar [38] derived the expressions for the field from an SH-point source in a stratified heterogeneous layer of finite depth. He found that for a periodic disturbance, the contribution to the far field is mainly due to at most a finite number of unattenuated normal Love modes. In a second paper [39], he studied the propagation of waves due to the presence of an SH-point source in the interior of a piecewise continuously stratified half-space. He assumed that the physical parameters governing the wave propagation i.e. the rigidity and the density are arbitrary piecewise continuous functions of depth with constant finite limiting values as the depth goes to infinity. It is found that for the time harmonic case the final expression is given in the form of a finite residue series plus a branch line integral the first representing the normal mode contribution to the field and the second giving the body waves. The field expression appears to have a symmetrical form with respect to field point depth and source depth.

Ghosh [19] developed a Green's function technique to find the displacement field in the case of Love waves due to a line source placed at the interface between a substratum, consisting of a homogeneous half-space and a homogeneous surface layer with a gradually sloping top. He also discussed the propagation of Love waves excited by a line source lying at the interface of the two by allowing the layer or the substratum to be inhomogeneous with linear variation of rigidity [20]. Chattopadhyay et al. [13] used Green's function method to find the dispersion relations for Love waves in a homogeneous layer over a half-space in which the density is varying slowly with depth.

We observe that the investigations of calculating dispersion relations and the transmitted wave in a trapped layer model due to a line source has not been made so far. This in mind, the model consisting of a trapped layer between two half-spaces is considered. We study the effect of density variation in the lower half-space for a trapped layer model, when a Love-type wave is generated due to a line source at the interface between the lower half-space and the layer. Green's function method and Fourier transform is used to calculate the dispersion relations and the transmitted field in the layer. We note that the dispersion relations obtained in this problem reduce to the dispersion relations for the layer over an inhomogeneous half-space (Chattopadhyay [13]) and for the trapped layer (Ewing et al [17]), when μ_1 (rigidity of the upper half-space) and ε (inhomogeneity) are taken to be zero respectively. As a second problem, we take layer to be of variable density and calculate the transmitted field and the dispersion relations for the same model. The dispersion relations so obtained reduce to the dispersion relations for a trapped layer if we put ε equal to zero [17]. The field in the layer for both the cases is calculated analytically. These observations have been published [41].

The effect of inhomogeneity due to rigidity variation in the lower half-space/layer, for a trapped layer model is also discussed. For the inhomogeneous half-space the dispersion relations reduce to the dispersion relations obtained by Ghosh [20], if we put μ_1 equal to zero and for the trapped layer (Ewing et al. [17]), when inhomogeneity becomes zero. For the inhomogeneous layer the dispersion relations also reduce to the dispersion relations for trapped layer by taking ε equal to zero. The transmitted field is also calculated analytically. The inhomogeneous trapped layer problem (due to variable rigidity) has been published [5].

Shear and explosive point sources have been used extensively in geophysics (Ewing et al. [17]; Burridge [9]; Aki and Richards [2]; Ben-Menahem and Singh [7]). The behaviour of the medium in the neighbourhood of an underground explosion can be regarded as perfectly elastic outside a certain sphere surrounding the source. Thus, a thorough understanding of motion in an elastic medium subject to an applied force, is an essential part of wave propagation theory needed to interpret seismic waves. This will help understand the propagation of Love waves in the layer from a point source, which is very important in the description of seismic sources and their effects. This in view, we consider the propagation of Love-type wave due to line source and a point source placed at an interface between a layer and a lower inhomogeneous half-space. The upper surface of the layer is stress free. We note that Chattopadhyay [13] has attempted this problem but only calculated the dispersion relations for the case of a line source without presenting the transmitted field in the layer. We complete this problem in the sense that the transmitted field is also calculated in the layer. Using the results obtained for the line source, the problem is further extended to a point source excitation. Mathematically the problem involves three variables. The method adopted to solve it is to take an additional Fourier transform and reduce the problem to that of a line source situation in the transformed plane. The extra integral appearing in this process is solved analytically using asymptotic methods. The dispersion relations and the transmitted field are then calculated. It is interesting to note that this procedure can be fairly easily extendable to the problems of other configurations, once the solution for the line source [5,11,12] incidence is known. In a nut shell, it has been observed that wave problems for line source incidence differ from those of plane wave incidence [3,4,40] by a multiplicative factor [25] in case the line source is taken to a far off distance and the solution for a point source can be obtained once the results for a line source are known. Therefore, a whole range of problems starting from a plane wave incidence to a spherical wave (point source) can be tackled in a systematic way by applying the technique of solving this specific problem. For example the method can be used to find the results for some realistic models such as [3,4,40]. The point source solution has been published [6].

The spectral representation of an operator arising from certain wave propagation problems is very useful in dealing with transmission, reflection and diffraction of waves across a horizontal discontinuity. This approach is based upon two steps. Firstly, we find the Green's function associated with the problem. Secondly, we integrate it around a large circle enclosing all the singularities. In case of finite depth problems, the sum of residues at the poles gives the representation of the delta function as a series in eigenfunctions. However, if we deal with the wave propagation in media with infinite depth the Green's function, in addition to poles, possesses branch point singularities. The resulting representation then involves the sum of residues at the poles and a branch cut integral over a portion of the real axis (Friedman [18] and Stakgold [36]).

Kazi [27] has used this approach to obtain spectral representations of the operator arising from horizontally polarized shear waves propagating in a soft layer of uniform thickness overlying a harder half-space. Kazi's results were subsequently applied to various problems involving Love waves (Kazi [28], Niazi and Kazi [32] and Madja, Chin and Followill [30]).

However, the spectral representation of horizontally polarized shear waves in a trapped layer between two half-spaces had not been attempted so far. Such a model could adequately describe a soft layer lying deep under the earth, which is most general from a physical and mathematical point of view. In this thesis we also present the spectral representation of such model. we have obtained a complete set of proper and improper eigenfunctions belonging to the operator, in terms of which the displacement field may be represented generally. we see that the spectrum of eigenvalues for trapped layer model is the disjoint union of the discrete and continuous spectra. Due to the similarity between the wave and the dispersion relation, we call it a Love-type wave and the operator, arising from the equation of motion, a Love-type operator.

Before going on to that discussion, in chapter one of this thesis we present the basic concepts and definitions of Love and Love-type waves. A brief description of the Fourier transform and Green's function technique is also given.

In chapter two, we calculate the field due to a line source in a layer trapped between two half-spaces when the density of the lower half-space (or layer) varies linearly with depth.

In chapter three, the problems have been tackled, when the rigidity of lower half-space (or layer) changes with depth.

Chapter four, is devoted to the problems of line and point source in a layer over an inhomogeneous medium.

Chapter five is concerned with the spectral representation of the two dimensional Love-type wave operator, associated with the propagation of monochromatic SH-waves in a layer trapped between two half-spaces.

CHAPTER ONE

PRELIMINARIES

1.1 LOVE WAVES

The criterion for surface waves is that the propagating disturbance decays exponentially with distance from the surface. If the surface waves consist of pure Rayleigh waves they should have both vertical and horizontal components, the former predominating. This is not found to be the case in practice, the vertical component sometimes being completely absent. For Rayleigh waves the direction of vibration of the horizontal components should be parallel to the direction of propagation, whereas horizontal components parallel to the wave front are often found. However, experimental data [24,17], particularly collected from seismological observations, have shown that SH-surface waves may occur along free surfaces. An analytical solution of this question was provided by Love [29], who showed that SH-waves are possible if the half-space is covered by a layer of a different material as shown in figure 1.1. He showed that these waves can propagate through such an outer layer without penetrating in to the interior. Waves of this type have become known as Love waves.

To derive the period equation (see Achenbach [1], Hudson [23] and Miklowitz [31]), we consider time harmonic plane waves in a single homogeneous layer overlying a homogeneous half-space. Writing the rigidities of the layer and half-space as μ_1, μ_2, shear velocities as β_1, β_2 and their densities as ρ_1, ρ_2 respectively, the y-component of the displacement (v) satisfies the wave equation

$$\frac{\partial^2 v_1}{\partial x^2} + \frac{\partial^2 v_1}{\partial z^2} + k_1^2 v_1 = 0 \tag{1.1}$$

in the layer $0 \leq z \leq h$ and

$$\frac{\partial^2 v_2}{\partial x^2} + \frac{\partial^2 v_2}{\partial z^2} + k_2^2 v_2 = 0 \tag{1.2}$$

in the half-space $z > h$, where $k_i = \frac{\omega}{\beta_i}$, $i = 1, 2$, ω being the angular frequency.

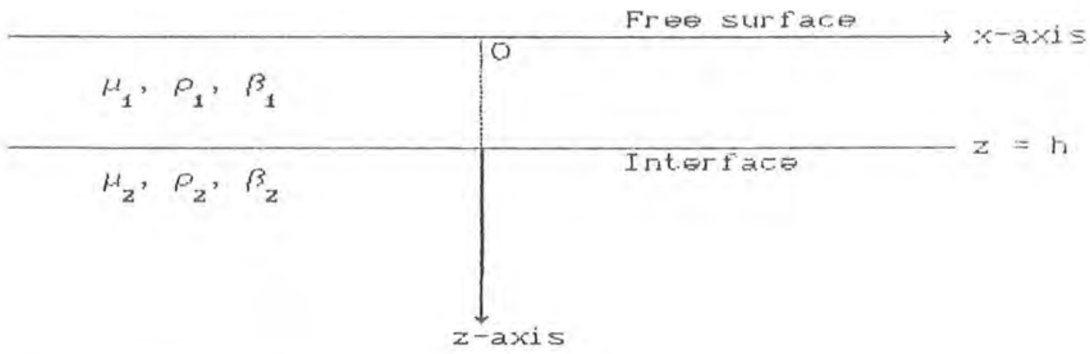

Figure 1.1. Love Waves Model.

The wave motion in the layer can be represented by

$$v_1 = [A \cos\gamma_1 z + B \sin\gamma_1 z] \exp\{i(\omega t - kx)\}, \qquad (1.3)$$

and the solution, that decays with depth, of the equation (1.2) for the half-space is given by

$$v_2 = C e^{-\gamma_2 z} \exp\{i(\omega t - kx)\}, \qquad (1.4)$$

where

$$\gamma_1 = k\left(\frac{c^2}{\beta_1^2} - 1\right)^{1/2}, \qquad \gamma_2 = k\left(1 - \frac{c^2}{\beta_2^2}\right)^{1/2}.$$

The condition of vanishing stress at the free surface z = 0, yields

$$v_1 = A \cos\gamma_1 z \exp\{i(\omega t - kx)\}. \qquad (1.5)$$

As the shear stresses and displacements are continuous at z = h,

$$\mu_1\gamma_1 A \sin\gamma_1 h = \mu_2\gamma_2 C\, e^{-\gamma_2 h}, \qquad (1.6)$$

and

$$A \cos\gamma_1 h = C\, e^{-\gamma_2 h}. \qquad (1.7)$$

From equations (1.6) and (1.7), we have

$$\tan\gamma_1 h = \frac{\mu_2\gamma_2}{\mu_1\gamma_1},$$

or

$$\tan\left[\left(\frac{c^2}{\beta_1^2} - 1\right)^{1/2} kh\right] = \frac{\mu_2\left[1 - c^2/\beta_2^2\right]^{1/2}}{\mu_1\left[c^2/\beta_1^2 - 1\right]^{1/2}}, \qquad (1.8)$$

which is the dispersion relation for Love waves. The left hand side of equation (1.8) is positive for $c = \beta_2$, while it is negative for $c = \beta_1$. Apparently, we can thus find a real root in the interval $\beta_1 < c \le \beta_2$. No real root exists if $\beta_2 < \beta_1$. Equation (1.8) shows that Love waves are dispersive and it is not possible to propagate Love waves in a high velocity layer.

1.2 LOVE-TYPE WAVES

Ewing, Jardetsky and Press [17] have derived a dispersion relation for the waves travelling in a layer trapped between two half-spaces. Here, we completely follow Ewing et al. We consider a homogeneous elastic layer $0 \leq z \leq h$ of uniform thickness h sandwiched between two homogeneous half-spaces $z < 0$ and $z > h$. We choose the coordinate system in such a way that the interface between the upper half-space and the layer coincides with the x-axis. The z-axis lies vertically downward. The subscripts, 1, 2 and 3 refer respectively to the upper medium, the intermediate layer and the lower medium. The rigidity,

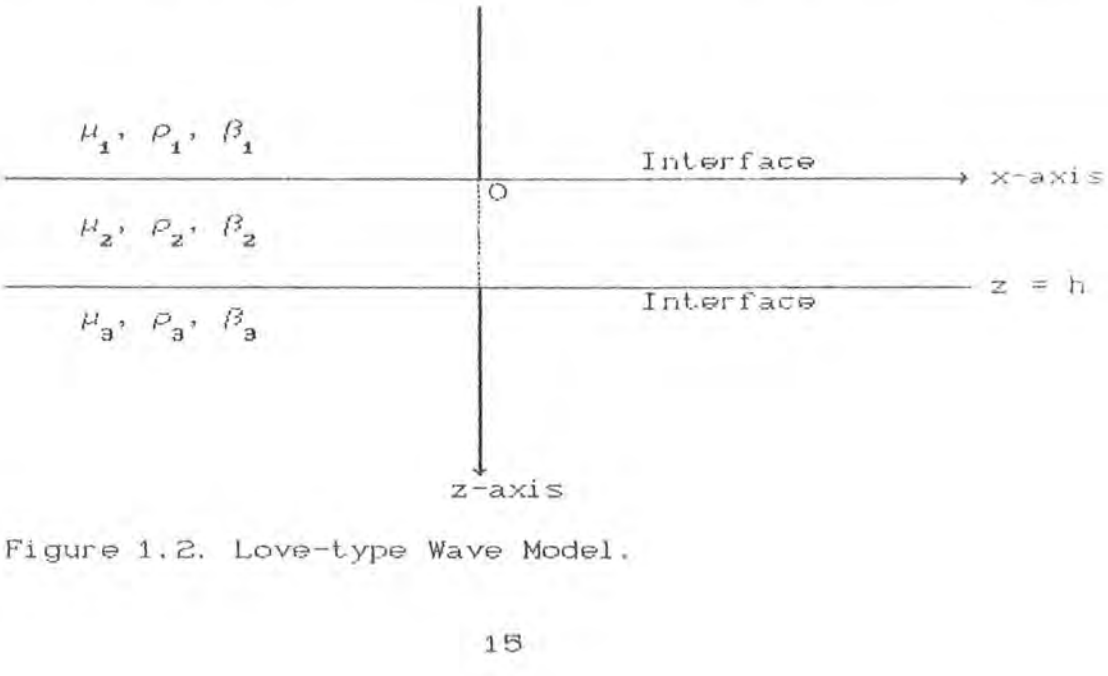

Figure 1.2. Love-type Wave Model.

shear velocity and density of the respective medium are denoted by μ_i, β_i and ρ_i for i = 1, 2, 3 respectively.

Assuming that all the displacements are independent of the y-coordinate and the time variations are given by the factor $\exp(i\omega t)$. The equations of motion for the upper and lower media reduce to

$$\frac{\partial^2 v_{1,3}}{\partial x^2} + \frac{\partial^2 v_{1,3}}{\partial z^2} + k_{1,3}^2 v_{1,3} = 0, \qquad (1.9)$$

and for the intermediate layer

$$\frac{\partial^2 v_2}{\partial x^2} + \frac{\partial^2 v_2}{\partial z^2} + k_2^2 v_2 = 0, \qquad (1.10)$$

where $k_i = \frac{\omega}{\beta_i}$, i = 1, 2, 3, ω being the angular frequency.

The boundary conditions are

(a) At $z = 0$, $-\infty < x < \infty$,

$$v_1 = v_2,$$

$$\mu_1 \frac{\partial v_1}{\partial z} = \mu_2 \frac{\partial v_2}{\partial z}. \qquad (1.11a)$$

(b) At $z = h$, $-\infty < x < \infty$,

$$v_2 = v_3,$$

$$\mu_2 \frac{\partial v_2}{\partial z} = \mu_3 \frac{\partial v_3}{\partial z}. \qquad (1.11b)$$

(c) $v_1 \longrightarrow 0$ as $z \longrightarrow -\infty$,

$$v_3 \longrightarrow 0 \quad \text{as} \quad z \longrightarrow \infty. \qquad (1.11c)$$

The solutions of the equation (1.9) subject to the boundary conditions (1.11c) for the upper and lower media are

$$v_1 = A\, e^{\sigma_1 z} \exp\{i(\omega t - kx)\}, \tag{1.12}$$

$$v_3 = B\, e^{-\sigma_3 z} \exp\{i(\omega t - kx)\}, \tag{1.13}$$

where

$$\sigma_{1,3} = k\left(1 - \frac{c^2}{\beta^2_{1,3}}\right)^{1/2},$$

and k is the wave number. For the intermediate layer, the solution of equation (1.10) is

$$v_2 = [C \cos\sigma_2 z + D \sin\sigma_2 z] \exp\{i(\omega t - kx)\}, \tag{1.14}$$

with

$$\sigma_2 = k\left(\frac{c^2}{\beta^2_2} - 1\right)^{1/2}.$$

As the displacements and shear stresses are continuous at $z = 0$ and $z = h$, therefore

$$A = C, \tag{1.15}$$

$$\mu_1\sigma_1 A = \mu_2\sigma_2 D, \tag{1.16}$$

$$C \cos\sigma_2 h + D \sin\sigma_2 h = B\, e^{-\sigma_3 h}, \tag{1.17}$$

$$-\mu_2\sigma_2 C \sin\sigma_2 h + \mu_2\sigma_2 D \cos\sigma_2 h = -\mu_3\sigma_3 B e^{-\sigma_3 h}. \tag{1.18}$$

Solving the above four equations, we obtain the dispersion equation for the trapped layer

$$\tan\sigma_2 h = \frac{\mu_2\sigma_2(\mu_1\sigma_1 + \mu_3\sigma_3)}{\mu_2^2\sigma_2^2 - \mu_1\mu_3\sigma_1\sigma_3}. \tag{1.19}$$

Real roots of equation (1.19) occur when $\beta_2 < c < \beta_1$ or β_3. Since the dispersion relation (1.19) for the trapped layer is similar to the dispersion relation for Love waves (equation 1.8) the waves in the trapped layer are termed as Love-type waves. Love-type waves exist when $\beta_3 > \beta_2 > \beta_1$, but may also exist when $\beta_3 > \beta_1 > \beta_2$.

1.3 EXPONENTIAL FOURIER TRANSFORM

In solving partial differential equations Fourier and Laplace transforms are very useful. We use the exponential Fourier transform in this thesis. Avoiding details we summarise some important notions in the theory of the exponential Fourier transform (see Graff [21]).

The exponential Fourier transform and its inverse is defined by

$$F(\xi) = \mathcal{F}_E[f(x)] = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x)e^{i\xi x}dx, \tag{1.20}$$

$$f(x) = \mathcal{F}_E^{-1}[F(\xi)] = \int_{-\infty}^{\infty} F(\xi)e^{-i\xi x}d\xi. \tag{1.21}$$

The exponential Fourier transform integral in equation (1.20) is frequently expressed as a sum of two integrals, each defined in semi-infinite range, as follows:

$$F(\xi) = F_+(\xi) + F_-(\xi), \tag{1.22}$$

where

$$F_+(\xi) = \frac{1}{2\pi}\int_0^{\infty} f(x)e^{i\xi x}dx,$$

and

$$F_-(\xi) = \frac{1}{2\pi}\int_{-\infty}^{0} f(x)e^{i\xi x}dx.$$

Depending upon the behaviour of f(x) as $x \longrightarrow \infty$ and $x \longrightarrow -\infty$, it is possible that only one of the two integrals defining $F_+(\xi)$ and $F_-(\xi)$ exists while the other does not.

The exponential Fourier transform of a derivative is related in a simple manner to the exponential Fourier transform of the function itself. Let us confine our attention to function f(x) which vanishes as $|x| \longrightarrow \infty$. By employing the definition (1.20), the exponential Fourier transform of $\frac{df}{dx}$ is then given by

$$\mathscr{F}_E[\frac{df}{dx}] = \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{df}{dx}\, e^{i\xi x}dx. \tag{1.23}$$

Integrations by parts,

$$\mathscr{F}_E[\frac{df}{dx}] = \frac{1}{2\pi}\{[e^{i\xi x}f(x)]_{x=-\infty}^{x=\infty} - i\xi\int_{-\infty}^{\infty}f(x)e^{i\xi x}dx\}.$$

Thus

$$\mathscr{F}_E[\frac{df}{dx}] = -i\xi F(\xi). \tag{1.24}$$

Similarly, assuming that all derivatives of f(x) up to the (n-1)st vanish at $\pm\infty$, we find

$$\mathscr{F}_E[\frac{d^n f}{dx^n}] = (-i\xi)^n F(\xi). \tag{1.25}$$

1.4 GREEN'S FUNCTION TECHNIQUE

The Green's function technique may be found in many standard books (see e.g. Stakgold [36]). To illustrate this technique, let us consider the one dimensional boundary value problem

$$-\frac{d^2u}{dx^2} = f(x), \quad 0 < x < 1; \quad u(0) = a, \quad u(1) = b, \tag{1.26}$$

where $f(x)$ is the source density. The three quantities $\{f(x);a,b\}$ are known collectively as the data for the problem. The data consists of the boundary data a, b and of the forcing function $f(x)$.

We are concerned not only with the solution of system (1.26) for the specific data but also with finding a suitable form for the solution that exhibits its dependence on the data. Thus as we change the data our expression for the solution remains useful. The feature of system (1.26) that enables us to achieve this goal is its linearity, as reflected in the superposition principle: If $u_1(x)$ is a solution for the data $\{f_1(x);a_1,b_1\}$ and $u_2(x)$ for the data $\{f_2(x);a_2,b_2\}$, then $Au_1(x) + Bu_2(x)$ is a solution for the data $\{Af_1(x)+Bf_2(x);\ Aa_1+Ba_2,\ Ab_1+Bb_2\}$. Thus the superposition principle permits us to decompose complicated data into possibly simpler parts, to solve each of the simpler boundary value problems and to reassemble these solutions to find the solution of the original problem. One decomposition of the data which is often used is

$$\{f(x);a,b\} = \{f(x);0,0\} + \{0;a,b\}. \tag{1.27}$$

The problem with data $\{f(x);0,0\}$ is an inhomogeneous equation with homogeneous boundary conditions; the problem with data $\{0;a,b\}$ is a homogeneous equation with inhomogeneous boundary conditions.

Since we want to solve the problem as compactly as possible for arbitrary data $\{f(x);a,b\}$, the differential operator and the boundary operators appearing on the left side of the equality sign in system (1.26) are kept fixed; no one is proposing to solve all differential equations with arbitrary boundary conditions at one stroke.

To solve the system (1.26) for arbitrary data, we introduce an accessory problem where, instead of a distributed density of sources, there is only a concentrated source of unit strength at $x = \zeta$ and where the boundary data vanishes. This solution of the accessory problem is known as the Green's function and is denoted by $G(x,\zeta)$. Here ζ is the position of the source and x is the observation point. We usually regard ζ as a parameter and x as the running variable.

Now we construct $G(x,\zeta)$ on the basis of the information available so far. Since there are no sources in $0 < x < \zeta$ and in $\zeta < x < 1$, we have

$$-\frac{d^2G(x,\zeta)}{dx^2} = 0 \tag{1.28}$$

in both intervals. Therefore, we can write

$$\left.\begin{aligned} G(x,\zeta) &= A_1x + B_1, \quad 0 \le x < \zeta, \\ &= A_2x + B_2, \quad \zeta < x \le 1. \end{aligned}\right\} \tag{1.29}$$

Here A_1, A_2, B_1, B_2 are constants, which are independent of x; they may however depend on the parameter ζ. Taking into account the fact that $G(x,\zeta)$ vanishes at $x = 0$ and $x = 1$, we find that

$$\left.\begin{aligned} G(x,\zeta) &= A_1 x, && 0 \le x < \zeta, \\ &= B_2(1-x), && \zeta < x \le 1. \end{aligned}\right\} \qquad (1.30)$$

The jump condition for $G'(x,\zeta)$ i.e.

$$G'(x,\zeta)\Big|_{x=\zeta^+} - G'(x,\zeta)\Big|_{x=\zeta^-} = -1 \qquad (1.31)$$

and continuity of $G(x,\zeta)$ at $x = \zeta$ enable us to calculate A_1 and B_2 in equation (1.30) from simultaneous equations $-B_2 - A_1 = -1$ and $A_1\zeta = B_2(1 - \zeta)$. Thus $B_2 = \zeta$ and $A_1 = 1 - \zeta$, so that

$$\left.\begin{aligned} G(x,\zeta) &= (1-\zeta)x, && 0 \le x < \zeta, \\ &= (1-x)\zeta, && \zeta < x \le 1. \end{aligned}\right\} \qquad (1.32)$$

1.4.1 Delta Function Formulation

The Green's function $G(x,\zeta)$ associated with the problem

$$-\frac{d^2u}{dx^2} = f(x), \quad 0 < x < 1; \quad u(0) = a, \quad u(1) = b \qquad (1.26)$$

satisfies the following equation

$$-\frac{d^2G(x,\zeta)}{dx^2} = \delta(x-\zeta), \quad 0 < x < 1, \quad 0 < \zeta < 1;$$

$$G(0,\zeta) = G(1,\zeta) = 0. \qquad (1.33)$$

Multiplying the equation (1.26) by $G(x,\zeta)$, equation (1.33) by u, subtracting and integrating from 0 to 1, we obtain

$$\int_0^1 (Gu'' - uG'')dx = -\int_0^1 f(x)G(x,\zeta)dx + \int_0^1 \delta(x-\zeta)u(x)dx,$$

which reduces to

$$u(\zeta) = \int_0^1 f(x)G(x,\zeta)dx + (Gu' - uG')\Big|_0^1.$$

Since $G(x,\zeta)$ vanishes at the end points, and

$$G'(x,\zeta)\Big|_{x=0} = 1 - \zeta, \qquad G'(x,\zeta)\Big|_{x=1} = -\zeta,$$

we have

$$u(\zeta) = \int_0^1 f(x)G(x,\zeta)dx + (1-\zeta)a + \zeta b.$$

Interchanging the labels x and ζ and using the symmetry of the Green's function

$$u(x) = \int_0^1 G(x,\zeta)f(\zeta)d\zeta + (1-x)a + xb. \qquad (1.34)$$

Now equation (1.34) satisfies the system (1.26) with data $\{f(x);a,b\}$.

CHAPTER TWO

EFFECT OF DENSITY VARIATION FOR A TRAPPED LAYER MODEL

2.1 INTRODUCTION

We construct and solve two problems here. In the first problem a homogeneous layer is sandwiched between homogeneous and inhomogeneous half-spaces (see figure 2.1). The inhomogeneity in the lower half-space is due to the linear variation of density. The propagation of Love-type waves due to a line source lying at the interface of the layer and lower inhomogeneous half-space is studied. The dispersion relation is obtained using Fourier transform and Green's function methods. It is found that the dispersion relation for the layer over the half-space (Chattopadhyay [13]) and the results for the trapped layer (Ewing et al. [17]) can be obtained, as special cases.

In the second problem, the layer is considered to be of variable density and is sandwiched between two homogeneous half-spaces. The Love-type waves are produced in the inhomogeneous layer due to a line source excitation. The dispersion equation of the Love-type waves so obtained is found to be in agreement with the homogeneous case. Finally, the transmitted wave in the layer is calculated analytically for both the problems.

2.2 INHOMOGENEOUS HALF-SPACE

In this section, investigations are presented when the density of the lower half-space varies linearly with the depth. The effect of the density variation on the propagation of Love-type waves generated from a line source in the layer is studied.

2.2.1 Formulation of the Problem

Consider a homogeneous elastic layer $0 \le z \le h$ with uniform thickness h sandwiched between homogeneous and inhomogeneous half-spaces given by $z < 0$ and $z > h$ respectively. A line source of disturbance is assumed to be situated at (0,h). The geometry of the problem is shown in figure 2.1. The subscripts 1, 2 and 3 refer respectively

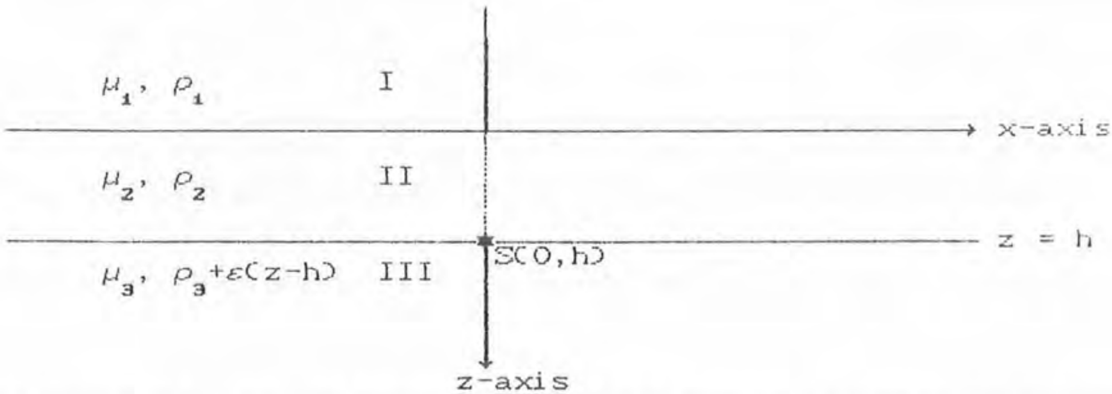

Figure 2.1. Geometry of the problem with an inhomogeneous lower half-space.

to the upper medium, intermediate layer and lower inhomogeneous substratum. The rigidity and density of media 1 to 3 are denoted by μ_i and ρ_i (i=1,2,3) respectively. The time harmonic variation is taken exp(iωt) and can be suppressed throughout. We can write the equations of motion for the upper medium and the intermediate layer as:

$$\frac{\partial^2 v_1}{\partial x^2} + \frac{\partial^2 v_1}{\partial z^2} + \frac{\rho_1 \omega^2}{\mu_1} v_1 = 0, \qquad (2.1)$$

$$\frac{\partial^2 v_2}{\partial x^2} + \frac{\partial^2 v_2}{\partial z^2} + \frac{\rho_2 \omega^2}{\mu_2} v_2 = \frac{4\pi}{\mu_2} \delta(x)\delta(z-h), \qquad (2.2)$$

where ω is the angular frequency and δ is the usual Dirac delta function. For the lower inhomogeneous medium, the equation of motion is

$$\frac{\partial^2 v_3}{\partial x^2} + \frac{\partial^2 v_3}{\partial z^2} + \frac{\rho_3 \omega^2}{\mu_3} v_3 = 0. \qquad (2.3)$$

To study the inhomogeneity effect, we replace ρ_3 by $\rho_3+\varepsilon(z-h)$, ε being a small parameter. Thus, we can rewrite equations (2.1) to (2.3) as:

$$\frac{\partial^2 v_1}{\partial x^2} + \frac{\partial^2 v_1}{\partial z^2} + k_1^2 v_1 = 0, \qquad (2.4)$$

$$\frac{\partial^2 v_2}{\partial x^2} + \frac{\partial^2 v_2}{\partial z^2} + k_2^2 v_2 = \frac{4\pi}{\mu_2} \delta(x)\delta(z-h), \qquad (2.5)$$

$$\frac{\partial^2 v_3}{\partial x^2} + \frac{\partial^2 v_3}{\partial z^2} + k_3^2 v_3 = -\frac{\varepsilon\omega^2}{\mu_3}(z-h) v_3, \qquad (2.6)$$

where

$$k_i^2 = \frac{\rho_i \omega^2}{\mu_i}, \quad i = 1,\ 2,\ 3. \tag{2.7}$$

The physical considerations of the problem lead to the following boundary conditions:

(a) At $z = 0, \quad -\infty < x < \infty,$

$$v_1 = v_2,$$

$$\mu_1 \frac{\partial v_1}{\partial z} = \mu_2 \frac{\partial v_2}{\partial z}. \tag{2.8a}$$

(b) At $z = h, \quad -\infty < x < \infty,$

$$v_2 = v_3,$$

$$\mu_2 \frac{\partial v_2}{\partial z} = \mu_3 \frac{\partial v_3}{\partial z}. \tag{2.8b}$$

(c) $v_1 \longrightarrow 0 \quad \text{as} \quad z \longrightarrow -\infty,$

$$v_3 \longrightarrow 0 \quad \text{as} \quad z \longrightarrow \infty. \tag{2.8c}$$

2.2.2 Solution of the Problem

Taking Fourier transforms, the equations of motion (2.4) to (2.6) become

$$\frac{d^2 V_1}{dz^2} - \alpha_1^2 V_1 = 0, \tag{2.9}$$

$$\frac{d^2 V_2}{dz^2} - \alpha_2^2 V_2 = \frac{2}{\mu_2}\, \delta(z-h), \tag{2.10}$$

$$\frac{d^2 V_3}{dz^2} - \alpha_3^2 V_3 = -\frac{\varepsilon\omega^2}{\mu_3}(z-h) V_3, \tag{2.11}$$

with

$$\alpha_i^2 = \xi^2 - k_i^2. \tag{2.12}$$

Let $G_2(z,z_o)$ be the Green's function for the inhomogeneous equation (2.10), where z_o is an arbitrary point in the layer. The equation satisfied by $G_2(z,z_o)$ is

$$G_2''(z,z_o) - \alpha_2^2 G_2(z,z_o) = \delta(z-z_o), \tag{2.13a}$$

together with homogeneous boundary conditions

$$G_2'(z,z_o) = 0, \quad \text{at} \quad z = 0,\ h, \tag{2.13b}$$

where prime denotes the differentiation with respect to z. As the boundary conditions in our problem are of an inhomogeneous nature, we follow Stakgold [36] and multiply equation (2.10) by $G_2(z,z_o)$, equation (2.13a) by $V_2(\xi,z)$, subtract and integrate over $0 \le z \le h$ to obtain

$$G_2(h,z_o)\left[V_2'\right]_{z=h} - G_2(0,z_o)\left[V_2'\right]_{z=0}$$

$$= \frac{2}{\mu_2} G_2(h,z_o) - V_2(z_o). \tag{2.14}$$

Again, let $G_1(z,z_o)$ and $G_3(z,z_o)$ be Green's functions corresponding to upper homogeneous and lower inhomogeneous half-spaces satisfying

$$G_1'(z,z_o) = 0, \quad \text{at} \quad z = 0;$$

$$G_1(z,z_o) \longrightarrow 0 \quad \text{as} \quad z \longrightarrow -\infty,$$

$$G_3'(z,z_o) = 0, \quad \text{at} \quad z = h;$$

$$G_3(z,z_o) \longrightarrow 0 \quad \text{as} \quad z \longrightarrow \infty.$$

By the same procedure as before, we obtain

$$G_1(0,z_o)\left[V_1'\right]_{z=0} = -\, V_1(z_o), \tag{2.15}$$

$$G_3(h,z_o)\left[V_3'\right]_{z=h} = V_3(z_o) + \frac{\varepsilon\omega^2}{\mu_3}\int_h^{\infty}(z-h)V_3(z)G_3(z,z_o)dz. \tag{2.16}$$

Interchanging z and z_o and using the symmetry of the Green's function in equations (2.14)—(2.16), we obtain

$$V_2(z) = G_2(z,0)\left[V_2'\right]_{z=0} - G_2(z,h)\left[V_2'\right]_{z=h} + \frac{2}{\mu_2}\,G_2(z,h), \tag{2.17}$$

$$V_1(z) = -\, G_1(z,0)\left[V_1'\right]_{z=0}, \tag{2.18}$$

$$V_3(z) = G_3(z,h)\left[V_3'\right]_{z=h}$$

$$-\,\frac{\varepsilon\omega^2}{\mu_3}\int_h^{\infty}(z_o-h)V_3(z_o)G_3(z,z_o)dz_o. \tag{2.19}$$

Now the boundary conditions (2.8a) give us

$$\left[V_2'\right]_{z=0} = \frac{1}{A}\left\{G_2(h,0)\left[V_2'\right]_{z=h} - \frac{2}{\mu_2}\,G_2(h,0)\right\}, \tag{2.20}$$

where

$$A = G_2(0,0) + \frac{\mu_2}{\mu_1}\,G_1(0,0), \tag{2.21}$$

Similarly, the boundary conditions (2.8b) yield

$$\left[V_2'\right]_{z=h} = \frac{2}{\mu_2}\left\{\frac{C}{AB-G_2^2(h,0)}\right\} + \frac{\varepsilon\omega^2 A}{\mu_3(AB-G_2^2(h,0))}$$

$$\times\int_h^{\infty}(z_o-h)V_3(z_o)G_3(h,z_o)dz_o, \tag{2.22}$$

where

$$B = G_2(h,h) + \frac{\mu_2}{\mu_3}\,G_3(h,h), \tag{2.23}$$

$$C = G_2(h,h)A - G_2^2(h,0). \tag{2.24}$$

Substituting the value of $[V_2']_{z=0}$ and $[V_2']_{z=h}$ in (2.17), the expression for $V_2(z)$ is

$$V_2(z) = \frac{2}{\mu_3}\left\{\frac{G_2(z,h)D - G_2(z,0)E}{AB-G_2^2(h,0)}\right\} + \frac{\varepsilon\omega^2}{\mu_3}\left\{\frac{G_2(z,0)G_2(h,0)}{AB-G_2^2(h,0)}\right.$$

$$\left.\frac{- G_2(z,h)A}{}\right\}\int_h^\infty (z_o-h)V_3(z_o)G_3(h,z_o)dz_o, \tag{2.25}$$

where

$$D = G_3(h,h)A, \tag{2.26}$$

$$E = G_3(h,h)G_2(h,0). \tag{2.27}$$

The continuity condition and the value of $[V_2']_{z=h}$, gives $V_3(z)$, which can be written as

$$V_3(z) = \frac{2}{\mu_3}\left\{\frac{G_3(z,h)C}{AB-G_2^2(h,0)}\right\} + \frac{\mu_2\varepsilon\omega^2}{\mu_3^2}\left\{\frac{G_3(z,h)A}{AB-G_2^2(h,0)}\right\}\int_h^\infty (z_o-h)$$

$$\times V_3(z_o)G_3(h,z_o)dz_o - \frac{\varepsilon\omega^2}{\mu_3}\int_h^\infty (z_o-h)V_3(z_o)$$

$$\times G_3(z,z_o)dz_o. \tag{2.28}$$

Now, $V_3(z)$ is to be determined from equation (2.28) by the method of successive approximations. The value of $V_3(z)$ derived from (2.28) when substituted in (2.25) gives the value of $V_2(z)$. Since we are interested only in the value of $V_2(z)$, which will give the displacement at any point in the layer and since square and higher powers of ε are to be neglected, as a first approximation we take

$$V_3(z) = \frac{2}{\mu_3}\left\{\frac{G_3(z,h)C}{AB-G_2^2(h,0)}\right\}. \qquad (2.29)$$

Putting this value back in the right hand side of equation (2.25), we obtain $V_2(z)$ in the following form

$$V_2(z) = \frac{2}{\mu_3}\left\{\frac{G_2(z,h)D-G_2(z,0)E}{AB-G_2^2(h,0)}\right\} + \frac{2\varepsilon\omega^2}{\mu_3^2}\left\{\frac{G_2(z,0)G_2(h,0)}{(AB-G_2^2(h,0))^2}\right.$$

$$\left.\frac{-G_2(z,h)A}{}\right\}\int_h^\infty \{C(z_o-h)G_3(z_o,h)\}G_3(h,z_o)dz_o. \qquad (2.30)$$

In order to determine the value of $V_2(z)$ from equation (2.30), the expressions for G_1, G_2 and G_3 are needed.

To find these Green's functions, we follow the procedure outlined by Stakgold [36]. It is known that $G_2(z,z_o)$ satisfies the equation (2.13a). To solve it, consider the solutions of the equation

$$\frac{d^2U}{dz^2} - \alpha_2^2 U = 0. \qquad (2.31)$$

Two independent solutions of equation (2.31) vanishing at $z = -\infty$ and $z = \infty$ respectively are

$$U_1 = e^{\alpha_2 z}, \qquad U_1 = e^{-\alpha_2 z}.$$

Therefore, the solution of equation (2.13a) for an infinite medium is

$$\frac{U_1(z)U_2(z_o)}{W} \quad \text{for } z < z_o,$$

$$\frac{U_1(z_o)U_2(z)}{W} \quad \text{for } z > z_o,$$

where the Wronskian W is given by

$$W = U_1(z)U_2'(z) - U_2(z)U_1'(z) = -2\alpha_2.$$

Thus the solution of equation (2.13a) for an infinite medium is

$$-\frac{e^{-\alpha_z|z-z_o|}}{2\alpha_z}.$$

Since the conditions $G_z'(z,z_o) = 0$ at $z = 0$ and $z = h$ are also to be satisfied, we take

$$G_z(z,z_o) = -\frac{1}{2\alpha_z}e^{-\alpha_z|z-z_o|} + \mathcal{A}e^{\alpha_z z} + \mathcal{B}e^{-\alpha_z z}.$$

Using the above conditions, we get

$$G_z(z,z_o) = -\frac{1}{2\alpha_z}\left[e^{-\alpha_z|z-z_o|} + e^{\alpha_z z}\left\{\frac{e^{-\alpha_z(h+z_o)} + e^{-\alpha_z(h-z_o)}}{e^{\alpha_z h} - e^{-\alpha_z h}}\right\} + e^{-\alpha_z z}\left\{\frac{e^{\alpha_z(h-z_o)} + e^{-\alpha_z(h-z_o)}}{e^{\alpha_z h} - e^{-\alpha_z h}}\right\}\right]. \quad (2.32)$$

Therefore

$$G_z(z,h) = -\frac{1}{\alpha_z}\frac{(e^{\alpha_z z} + e^{-\alpha_z z})}{(e^{\alpha_z h} - e^{-\alpha_z h})},$$

$$G_z(z,0) = -\frac{1}{\alpha_z}\frac{(e^{\alpha_z(z-h)} + e^{-\alpha_z(z-h)})}{(e^{\alpha_z h} - e^{-\alpha_z h})},$$

$$G_z(h,h) = -\frac{1}{\alpha_z}\frac{(e^{\alpha_z h} + e^{-\alpha_z h})}{(e^{\alpha_z h} - e^{-\alpha_z h})},$$

$$G_z(0,0) = -\frac{1}{\alpha_z}\frac{(e^{\alpha_z h} + e^{-\alpha_z h})}{(e^{\alpha_z h} - e^{-\alpha_z h})}.$$

Similarly, it can be shown that

$$G_1(z,z_o) = -\frac{1}{2\alpha_1}\left[e^{-\alpha_1|z-z_o|} + e^{\alpha_1(z+z_o)}\right], \quad (2.33)$$

$$G_3(z,z_o) = -\frac{1}{2\alpha_3}\left[e^{-\alpha_3|z-z_o|} + e^{-\alpha_3(z+z_o-2h)}\right], \qquad (2.34)$$

and

$$G_1(0,0) = -\frac{1}{\alpha_1}, \qquad G_3(h,z_o) = -\frac{1}{\alpha_3}e^{-\alpha_3(z_o-h)},$$

$$G_3(h,h) = -\frac{1}{\alpha_3}.$$

Using the values of G_1, G_2 and G_3 in equation (2.30), we arrive at

$$V_2(z) = \frac{-2(\mu_2\alpha_2\cosh\alpha_2 z + \mu_1\alpha_1\sinh\alpha_2 z)}{\mu_1\mu_3\alpha_1\alpha_2^2\alpha_3\{AB-G_2^2(h,0)\}(1-\varepsilon J_1)\sinh\alpha_2 h},$$

or

$$V_2(z) = \frac{-2(\mu_2\alpha_2\cosh\alpha_2 z + \mu_1\alpha_1\sinh\alpha_2 z)}{F_1(\xi,h)}, \qquad (2.35)$$

where

$$F_1(\xi,h) = \mu_1\mu_3\alpha_1\alpha_2^2\alpha_3\{AB-G_2^2(h,0)\}(1-\varepsilon J_1)\sinh\alpha_2 h,$$

$$J_1 = \frac{\omega^2(\mu_2\alpha_2\cosh\alpha_2 h + \mu_1\alpha_1\sinh\alpha_2 h)}{4\mu_1\mu_3\alpha_1\alpha_2^2\alpha_3^3(AB-G_2^2(h,0))\sinh\alpha_2 h}.$$

2.2.3 Transmitted Waves

The transmitted waves in the layer can be calculated by taking the inverse Fourier transform of equation (2.35). Thus, the displacement in the layer is given by

$$v_2(x,z) = \int_{-\infty}^{\infty} \frac{-2(\mu_2\alpha_2\cosh\alpha_2 z + \mu_1\alpha_1\sinh\alpha_2 z)}{F_1(\xi,h)} e^{-i\xi x}d\xi. \qquad (2.36)$$

The expression in the denominator equated to zero gives the dispersion relation of Love-type waves for a homogeneous trapped layer due to the presence of the inhomogeneity in the lower half-space. This is

$$\tan\hat{\alpha}_2 h = \frac{\mu_2\hat{\alpha}_2(\mu_1\alpha_1 + \mu_3\alpha_3)}{\mu_2^2\,\hat{\alpha}_2^2 - \mu_1\mu_3\alpha_1\alpha_3} - \frac{\varepsilon\omega^2(\mu_2\hat{\alpha}_2 + \mu_1\alpha_1\tan\hat{\alpha}_2 h)}{4\alpha_3^2(\mu_2^2\hat{\alpha}_2^2 - \mu_1\mu_3\alpha_1\alpha_3)}, \tag{2.37}$$

where

$$\hat{\alpha}_2 = \sqrt{k_2^2 - \xi^2}.$$

We observe that the dispersion relation (2.37) reduces to the dispersion relation by Chattopadhyay [13], and Ewing et al. [17], in the limits μ_1 and ε equal to zero respectively.

The transmitted waves can now be obtained by solving the integral in equation (2.36). We note that the poles of the integrand are the roots $p_{1,n}$ $(n=1,2,3,\ldots)$ of the equation $F_1(\xi,h) = 0$. Thus, calculating this integral at these poles, using the residue method, the transmitted Love-type wave is given by:

$$v_2(x,z) = 4\pi\sum_{n=1}^{\infty}\frac{(\mu_2\hat{\alpha}_{2,1,n}\cos\hat{\alpha}_{2,1,n}z + \mu_1\alpha_{1,1,n}}{\left.\dfrac{dF_1(\xi,h)}{d\xi}\right|_{\xi=p_{1,n}}}$$

$$\times\ \frac{\sin\hat{\alpha}_{2,1,n}z)}{}\exp(-ip_{1,n}x), \tag{2.38}$$

where

$$\hat{\alpha}_2\Big|_{\xi=p_{1,n}} = \hat{\alpha}_{2,1,n}, \qquad \alpha_1\Big|_{\xi=p_{1,n}} = \alpha_{1,1,n}.$$

Equation (2.38) gives the transmitted travelling waves in the layer in the x-axis direction.

Besides the poles, we have the branch points which give rise to body waves and are of no interest to us for the present study. However, these calculations can be made with the help of the saddle point method considering the asymptotic behaviour of the integrals.

2.3 INHOMOGENEOUS LAYER

For the inhomogeneous trapped layer problem, the Love-type waves are generated in a layer due to a line source whose density varies linearly (see figure 2.2) with the depth. The effect of variable density on the propagation of the Love-type waves is discussed.

2.3.1 Formulation and solution of the Problem

The formulation is the same as that of first problem, except that the equation of motion for the inhomogeneous layer is

$$\frac{\partial^2 v_z}{\partial x^2} + \frac{\partial^2 v_z}{\partial z^2} + \frac{\rho_z \omega^2 v_z}{\mu_z} = \frac{4\pi}{\mu_z}\,\delta(x)\,\delta(z-h), \qquad (2.39)$$

where v_z is the displacement in the layer and ω is the angular frequency. Due to the inhomogeneity in the layer, ρ_z is taken to be $\rho_z+\varepsilon z$, ε being a small parameter. We can rewrite equation (2.39) as

$$\frac{\partial^2 v_z}{\partial x^2} + \frac{\partial^2 v_z}{\partial z^2} + k_z^2 v_z = \frac{4\pi}{\mu_z}\,\delta(x)\,\delta(z-h) - \frac{\varepsilon z \omega^2}{\mu_z}\, v_z. \qquad (2.40)$$

The equations of motion for the upper and lower substratum are

$$\frac{\partial^2 v_1}{\partial x^2} + \frac{\partial^2 v_1}{\partial z^2} + k_1^2 v_1 = 0, \tag{2.41}$$

$$\frac{\partial^2 v_3}{\partial x^2} + \frac{\partial^2 v_3}{\partial z^2} + k_3^2 v_3 = 0, \tag{2.42}$$

where in the above equations k_i^2 (i=1,2,3) are the same as given by equation (2.7). The boundary conditions of the problem are same as that of first problem.

Transforming the equations of motion (2.40) to (2.42), we have

$$\frac{d^2 V_2}{dz^2} - \alpha_2^2 V_2 = \frac{2}{\mu_2} \delta(z-h) - \frac{\varepsilon z \omega^2}{\mu_2} V_2 = \tau_1(z), \tag{2.43}$$

where

$$\tau_1(z) = \frac{2}{\mu_2} \delta(z-h) - \frac{\varepsilon z \omega^2}{\mu_2} V_2,$$

for the layer and

$$\frac{d^2 V_{1,3}}{dz^2} - \alpha_{1,3}^2 V_{1,3} = 0, \tag{2.44}$$

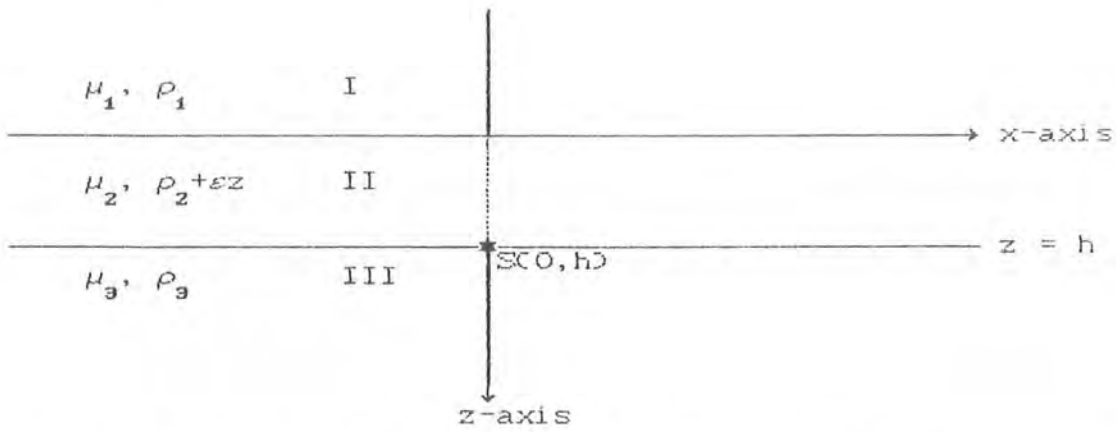

Figure 2.2. Geometry of the problem with an inhomogeneous layer.

for the upper and lower substratum. Were α_i^2 are the same as given by equation (2.12).

As the boundary conditions of this problem are the same as that of problem one, therefore by same procedure, we arrive at

$$V_2(z) = G_2(z,0)[V_2']_{z=o} - G_2(z,h)[V_2']_{z=h}$$

$$+ \int_o^h \tau_1(z_o)G_2(z,z_o)dz_o, \qquad (2.45)$$

$$V_1(z) = - G_1(z,0)[V_1']_{z=o}, \qquad (2.46)$$

and

$$V_3(z) = G_3(z,h)[V_3']_{z=h}. \qquad (2.47)$$

Now the boundary conditions (2.8a) give us

$$[V_2']_{z=o} = \frac{1}{A}\left\{G_2(0,h)[V_2']_{z=h} - \int_o^h \tau_1(z_o)G_2(0,z_o)dz_o\right\}, \qquad (2.48)$$

where A is given by equation (2.21). Similarly the boundary conditions (2.8b) yield

$$[V_2']_{z=h} = \frac{-G_2(h,0)}{\{AB-G_2^2(h,0)\}}\int_o^h \tau_1(z_o)G_2(0,z_o)dz_o$$

$$+ \frac{A}{\{AB-G_2^2(h,0)\}}\int_o^h \tau_1(z_o)G_2(h,z_o)dz_o, \qquad (2.49)$$

where B is given by equation (2.23). Using equations (2.48) and (2.49) in equation (2.45), substituting back value of $\tau_1(z)$ and using the property of the delta function, we get

$$V_z(z) = \frac{2}{\mu_3}\left\{\frac{G_z(z,h)D - G_z(z,0)E}{AB - G_z^2(h,0)}\right\} - \left\{\frac{G_z(z,h)G_z(h,0)}{AB - G_z^2(h,0)}\right.$$

$$\left.\frac{- G_z(z,0)B}{}\right\}\frac{\omega^2\varepsilon}{\mu_z}\int_0^h z_o V_z(z_o)G_z(0,z_o)dz_o$$

$$- \left\{\frac{G_z(z,0)G_z(h,0) - G_z(z,h)A}{AB - G_z^2(h,0)}\right\}\frac{\omega^2\varepsilon}{\mu_z}\int_0^h z_o V_z(z_o)$$

$$\times G_z(h,z_o)dz_o - \frac{\omega^2\varepsilon}{\mu_z}\int_0^h z_o V_z(z_o)G_z(z,z_o)dz_o, \tag{2.50}$$

where D and E are given by equations (2.26) and (2.27) respectively.

The series so obtained can easily be shown to be converge to $V_z(z)$. Now equation (2.50) is an integral equation and $V_z(z)$ may be determined from this equation using successive substitutions. As a first approximation, we neglect terms involving ε to obtain

$$V_z(z) = \frac{2}{\mu_3}\left\{\frac{G_z(z,h)D - G_z(z,0)E}{AB - G_z^2(h,0)}\right\}. \tag{2.51}$$

Putting this value back in the right hand side of equation (2.50), we obtain $V_z(z)$ in the form

$$V_z(z) = \frac{2}{\mu_3}\left\{\frac{G_z(z,h)D - G_z(z,0)E}{AB - G_z^2(h,0)}\right\} - \frac{2\omega^2\varepsilon}{\mu_z\mu_3}\left\{\frac{G_z(z,h)}{AB -}\right.$$

$$\left.\frac{\times G_z(h,0)-G_z(z,0)B}{G_z^2(h,0)}\right\}\int_0^h z_o M_1(z_o)G_z(0,z_o)dz_o$$

$$- \frac{2\omega^2 \varepsilon}{\mu_2 \mu_3} \left\{ \frac{G_2(z,0) G_2(h,0) - G_2(z,h) A}{AB - G_2^2(h,0)} \right\} \int_0^h z_o M_1(z_o)$$

$$\times \; G_2(h, z_o) dz_o \; - \; \frac{2\omega^2 \varepsilon}{\mu_2 \mu_3} \int_0^h z_o M_1(z_o) G_2(z, z_o) dz_o , \tag{2.52}$$

where

$$M_1(z_o) \; = \; \left\{ \frac{G_2(z_o, h) D \; - \; G_2(z_o, 0) E}{AB \; - \; G_2^2(h, 0)} \right\}.$$

We note that $V_2(z)$ is completely determined through equation (2.52) provided that G_1, G_2 and G_3 are known.

The Green's functions G_1, G_2 and G_3 are the same as given by equations (2.32) to (2.34), using their values and after some manipulations, we have

$$V_2(z) \; = \; \frac{-2(\mu_2 \alpha_2 \cosh\alpha_2 z \; + \; \mu_1 \alpha_1 \sinh\alpha_2 z)}{\mu_1 \mu_3 \alpha_1 \alpha_2^2 \alpha_3 \{AB - G_2^2(h,0)\}(1 - \varepsilon J_2) \sinh\alpha_2 h}, \tag{2.53}$$

or

$$V_2(z) \; = \; \frac{-2(\mu_2 \alpha_2 \cosh\alpha_2 z \; + \; \mu_1 \alpha_1 \sinh\alpha_2 z)}{F_2(\xi, h)}, \tag{2.54}$$

where

$$F_2(\xi, h) \; = \; \mu_1 \mu_3 \alpha_1 \alpha_2^2 \alpha_3 \{AB - G_2^2(h,0)\}(1 - \varepsilon J_2) \sinh\alpha_2 h,$$

and

$$J_2 \; = \; \frac{\omega^2}{4\{AB - G_2^2(h,0)\}} \left[\frac{\mu_3 \alpha_3 \; - \; \mu_1 \alpha_1}{\mu_1 \mu_3 \alpha_2^4 \alpha_3} \; + \; \frac{h\left\{(\mu_2^2 \alpha_2^2 \; - \; \mu_1 \alpha_1 \mu_3 \alpha_3)\right.}{\mu_1 \mu_2 \mu_3 \alpha_1 \alpha_2^4 \alpha_3} \right.$$

$$\frac{\left. + \; \mu_2 \alpha_2 (\mu_1 \alpha_1 \; - \; \mu_3 \alpha_3) \coth\alpha_2 h \right\}}{} \; + \; \frac{h^2 \left\{ \mu_2 \alpha_2 (\mu_1 \alpha_1 \; + \; \mu_3 \alpha_3) \right.}{\mu_1 \mu_2 \mu_3 \alpha_1 \alpha_2^3 \alpha_3}$$

$$\left. \frac{\left. + \; (\mu_1 \alpha_1 \mu_3 \alpha_3 \; + \; \mu_2^2 \alpha_2^2) \coth\alpha_2 h \right\}}{} \right]. \tag{2.55}$$

Now, we determine the transmitted waves in the inhomogeneous layer trapped between two homogeneous half-spaces. The Fourier inversion formula, when applied to equation (2.54) gives

$$v_2(x,z) = -2\int_{-\infty}^{\infty} \frac{(\mu_2\alpha_2\cosh\alpha_2 z + \mu_1\alpha_1\sinh\alpha_2 z)\, e^{-i\xi x}}{F_2(\xi,h)}\, d\xi. \tag{2.56}$$

The expression in the denominator equated to zero gives the dispersion equation of Love-type waves for a trapped layer due to the presence of the inhomogeneity in the layer. It is

$$\begin{aligned}\tan\hat{\alpha}_2 h = {} & \frac{\mu_2\hat{\alpha}_2(\mu_1\alpha_1 + \mu_3\alpha_3)}{\mu_2^2\hat{\alpha}_2^2 - \mu_1\mu_3\alpha_1\alpha_3} - \frac{\omega^2\varepsilon}{4(\mu_2^2\hat{\alpha}_2^2 - \mu_1\mu_3\alpha_1\alpha_3)} \\ & \times \Bigg[\frac{(\mu_1\alpha_1 - \mu_3\alpha_3)\tan\hat{\alpha}_2 h}{\hat{\alpha}_2^2} + \frac{h}{\mu_2\hat{\alpha}_2^2}\Big\{(\mu_2^2\hat{\alpha}_2^2 + \mu_1\mu_3\alpha_1\alpha_3) \\ & \times \tan\hat{\alpha}_2 h - \mu_2\hat{\alpha}_2(\mu_1\alpha_1 - \mu_3\alpha_3)\Big\} + \frac{h^2}{\mu_2\hat{\alpha}_2}\Big\{\mu_2\hat{\alpha}_2(\mu_1\alpha_1 \\ & + \mu_3\alpha_3)\tan\hat{\alpha}_2 h + (\mu_2^2\hat{\alpha}_2^2 - \mu_1\mu_3\alpha_1\alpha_3)\Big\}\Bigg],\end{aligned} \tag{2.57}$$

where

$$\hat{\alpha}_2 = (k_2^2 - \xi^2)^{1/2}.$$

It is imperative to note that in the case of a homogeneous medium $\varepsilon = 0$, and this dispersion relation reduces to the dispersion relation obtained by Ewing [17] for the homogeneous case.

In order to obtain the transmitted wave, we need to calculate the integral in equation (2.56). Note that the poles of the integrand are the roots of the equation $F_2(\xi, h) = 0$. This equation yields the dispersion relation given by the equation (2.57). The solutions of this equation are in fact the poles of the integrand, which can only be calculated numerically and are denoted by $\xi = p_{2,n}$. Calculating the pole contribution at these poles, we find that

$$v_2(x,z) = 4\pi \sum_{n=1}^{\infty} \frac{\left\{\mu_2 \hat{\alpha}_{2,2,n} \cos\hat{\alpha}_{2,2,n} z + \mu_1 \alpha_{1,2,n} \times \sin\hat{\alpha}_{2,2,n} z\right\}}{\left.\dfrac{dF_2(\xi, h)}{d\xi}\right|_{\xi = p_{2,n}}} \exp(-ip_{2,n} x), \tag{2.58}$$

where

$$\left.\hat{\alpha}_2\right|_{\xi = p_{2,n}} = \hat{\alpha}_{2,2,n}, \qquad \left.\alpha_1\right|_{\xi = p_{2,n}} = \alpha_{1,2,n}.$$

Equation (2.58) represents the travelling wave in the layer in the direction of x-axis.

In order to provide a feel for the problem in real life we present the values of the parameters involved. This includes the rigidities, densities and shear velocities in the three regions. It is further noted that $\mu_3 > \mu_2$ which is essential for the propagation of Love-type waves in the layer. The typical value of the depth h of the intermediate layer is 6.0 km.

TABLE

Medium	Rigidity dyn/cm^2	Density gm/cm^3	shear velocity km/s
I	4.67×10^{11}	5.14	3.104
II	2.12×10^{11}	4.52	2.165
III	5.32×10^{11}	3.29	4.021

CHAPTER THREE

EFFECT OF RIGIDITY VARIATION FOR A TRAPPED LAYER MODEL

3.1 INTRODUCTION

The effect of rigidity variation on propagation of Love-type waves for a trapped layer model is considered in this chapter. The Love-type wave is generated due to a line source lying at the interface between layer and lower half-space and the rigidity varies with depth. Two problems are addressed corresponding to:

(a) rigidity variation in the lower half-space;

(b) rigidity variation in the layer.

The dispersion relations for the Love-type waves are obtained and the transmitted wave in the layer is calculated analytically for both the problems.

3.2 INHOMOGENEOUS HALF-SPACE

For the inhomogeneous half-space problem, the rigidity of the lower half-space varies linearly with the depth (see figure 3.1). The effect of this rigidity variation on the propagation of Love-type waves due to a line source in the trapped layer is presented here.

The formulation of the problem is the same as that given in the first problem of chapter two, except that the equation of motion for the lower inhomogeneous medium is replaced by:

$$\frac{\partial}{\partial x}\left(\mu_3 \frac{\partial v_3}{\partial x}\right) + \frac{\partial}{\partial z}\left(\mu_3 \frac{\partial v_3}{\partial z}\right) + \rho_3 \omega^2 v_3 = 0. \tag{3.1}$$

For the inhomogeneity in the lower medium, the coefficient of rigidity μ_3 is replaced by $\mu_3+\varepsilon(z-h)$, ε being a small parameter. The wave equation (3.1) for this case then takes the form

$$\frac{\partial^2 v_3}{\partial x^2} + \frac{\partial^2 v_3}{\partial z^2} + k_3^2 v_3 = - \frac{\varepsilon(z-h)}{\mu_3} \frac{\partial^2 v_3}{\partial z^2} - \frac{\varepsilon}{\mu_3} \frac{\partial v_3}{\partial z}$$

$$- \frac{\varepsilon(z-h)}{\mu_3} \frac{\partial^2 v_3}{\partial x^2}, \tag{3.2}$$

subject to the boundary conditions (2.8a) to (2.8c), where

$$k_3^2 = \frac{\rho_3 \omega^2}{\mu_3} ,$$

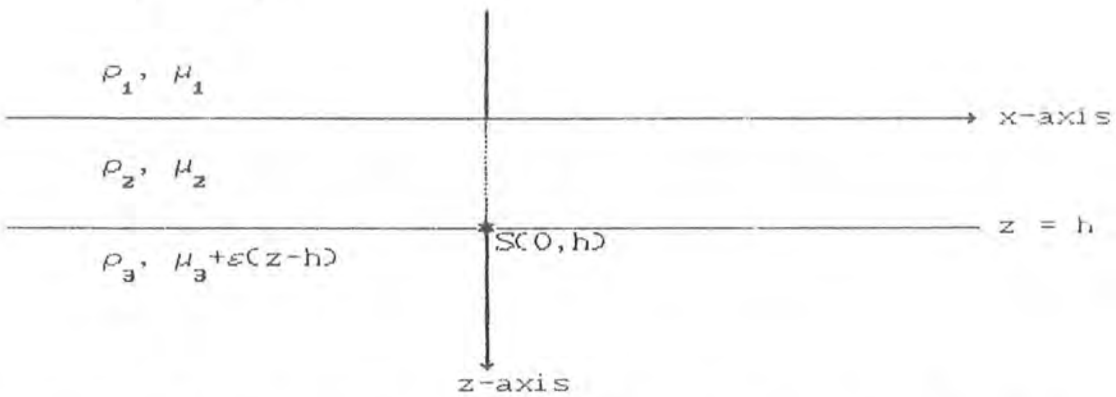

Figure 3.1. Geometry of the problem with an inhomogeneous lower half-space.

Using the Fourier transform, the equations of motion (2.4), (2.5) and (3.2) for the upper medium, intermediate layer and for the lower inhomogeneous medium take the form

$$\frac{d^2V_1}{dz^2} - \alpha_1^2V_1 = 0, \tag{3.3}$$

$$\frac{d^2V_2}{dz^2} - \alpha_2^2V_2 = \frac{2}{\mu_2}\,\delta(z-h), \tag{3.4}$$

$$\frac{d^2V_3}{dz^2} - \alpha_3^2V_3 = -\frac{\varepsilon(z-h)}{\mu_3}\frac{d^2V_3}{dz^2} - \frac{\varepsilon}{\mu_3}\frac{dV_3}{dz} + \frac{\varepsilon(z-h)\xi^2}{\mu_3}V_3$$

$$= \tau_2(z) \text{ say}, \tag{3.5}$$

where α_i^2 (i=1,2,3) are the same as given by equation (2.12).

Since the boundary conditions in this problem are the same as those of the problems of chapter two, using the same procedure we arrive at

$$V_2(z) = \frac{2}{\mu_3}\left\{\frac{G_2(z,h)D - G_2(z,0)E}{AB - G_2^2(h,0)}\right\} - \left\{\frac{G_2(z,0)G_2(h,0)}{AB - G_2^2(h,0)}\right.$$

$$\left.\frac{- G_2(z,h)A}{}\right\}\int_h^\infty \tau_2(z_o)G_3(h,z_o)dz_o, \tag{3.6}$$

$$V_3(z) = \frac{2}{\mu_3}\left\{\frac{G_3(z,h)C}{AB-G_2^2(h,0)}\right\} - \frac{\mu_2}{\mu_3}\frac{AG_3(z,h)}{(AB-G_2^2(h,0))}\int_h^\infty \tau_2(z_o)$$

$$\times G_3(h,z_o)dz_o + \int_h^\infty \tau_2(z_o)G_3(z,z_o)dz_o, \tag{3.7}$$

where A, B, C, D and E are given by equations (2.21), (2.23), (2.24), (2.26) and (2.27) respectively. After substitution of the value of $\tau_z(z_o)$ in equations (3.6) and (3.7), we have

$$V_z(z) = \frac{2}{\mu_3}\left\{\frac{G_z(z,h)D - G_z(z,0)E}{AB - G_z^2(h,0)}\right\} + \frac{\varepsilon}{\mu_3}\left\{\frac{G_z(z,0)G_z(h,0)}{AB - G_z^2(h,0)}\right.$$

$$\left.\frac{- G_z(z,h)A}{}\right\}\int_h^\infty\left\{(z_o-h)\frac{d^2V_3(z_o)}{dz_o^2} + \frac{dV_3(z_o)}{dz_o}\right.$$

$$\left. - (z_o-h)\xi^2V_3(z_o)\right\} G_3(h,z_o)dz_o, \qquad (3.8)$$

and

$$V_3(z) = \frac{2}{\mu_3}\left\{\frac{G_3(z,h)C}{AB-G_z^2(h,0)}\right\} + \frac{\varepsilon\mu_z}{\mu_3^2}\frac{AG_3(z,h)}{(AB-G_z^2(h,0))}$$

$$\times\int_h^\infty\left\{(z_o-h)\frac{d^2V_3(z_o)}{dz_o^2} + \frac{dV_3(z_o)}{dz_o} - (z_o-h)\xi^2V_3(z_o)\right\}$$

$$\times\, G_3(h,z_o)dz_o - \frac{\varepsilon}{\mu_3}\int_h^\infty\left\{(z_o-h)\frac{d^2V_3(z_o)}{dz_o^2} + \frac{dV_3(z_o)}{dz_o}\right.$$

$$\left. - (z_o-h)\xi^2V_3(z_o)\right\} G_3(z,z_o)dz_o. \qquad (3.9)$$

The field $V_z(z)$ in the layer can be determined from the equation (3.8), if we substitute the value of $V_3(z_o)$ from equation (3.9) in equation (3.8). Equation (3.9) is an integral equation and $V_3(z)$ may be determined from this equation using successive approximations. As a first approximation, we neglect terms involving ε to obtain

$$V_3(z) = \frac{2}{\mu_3}\left\{\frac{G_3(z,h)C}{AB-G_z^2(h,0)}\right\}. \qquad (3.10)$$

Putting this value back in the right hand side of equation

(3.8), we obtain $V_2(z)$ in the following form:

$$V_2(z) = \frac{2}{\mu_3}\left\{\frac{G_2(z,h)D - G_2(z,0)E}{AB - G_2^2(h,0)}\right\} + \frac{2\varepsilon}{\mu_3^2}\left\{\frac{G_2(z,0)G_2(h,0)}{(AB - G_2^2(h,0))^2}\right.$$

$$\left.\frac{- G_2(z,h)A}{}\right\}\int_h^{\infty}\left\{(z_o-h)\frac{d^2M_2(z_o)}{dz_o^2} + \frac{dM_2(z_o)}{dz_o}\right.$$

$$\left. - (z_o-h)\xi^2M_2(z_o)\right\} G_3(h,z_o)dz_o, \qquad (3.11)$$

where

$$M_2(z_o) = G_3(z_o,h)C.$$

To calculate the value of $V_2(z)$ from equation (3.11), the expressions for G_1, G_2 and G_3 are required.

The Green's functions G_1, G_2 and G_3 are the same as given by equations (2.32) to (2.34), so we insert them in to equation (3.11), and after some calculations obtain

$$V_2(z) = \frac{-2(\mu_2\alpha_2\cosh\alpha_2 z + \mu_1\alpha_1\sinh\alpha_2 z)}{F_3(\xi,h)}, \qquad (3.12)$$

where

$$F_3(\xi,h) = \mu_1\mu_3\alpha_1\alpha_2^2\alpha_3\{AB-G_2^2(h,0)\}(1+\varepsilon J_3)\sinh\alpha_2 h$$

and

$$J_3 = \frac{(\mu_2\alpha_2\cosh\alpha_2 h + \mu_1\alpha_1\sinh\alpha_2 h)(2\alpha_3^2 + k_3^2)}{4\mu_1\mu_3\alpha_1\alpha_2^2\alpha_3^3\{AB-G_2^2(h,0)\}\sinh\alpha_2 h}.$$

We determine the transmitted waves in the homogeneous layer. This can be done by taking the inverse Fourier transform of equation (3.12). Thus, the displacement in the layer is given by

$$v_2(x,z) = -2 \int_{-\infty}^{\infty} \frac{(\mu_2\alpha_2\cosh\alpha_2 z + \mu_1\alpha_1\sinh\alpha_2 z)e^{-i\xi x}}{F_3(\xi,h)}\, d\xi. \tag{3.13}$$

The dispersion equation for the propagation of Love-type waves for a homogeneous trapped layer between homogeneous and inhomogeneous half-spaces is obtained by equating the expression in the denominator with zero and is

$$\tan\hat{\alpha}_2 h = \frac{\mu_2\hat{\alpha}_2(\mu_1\alpha_1 + \mu_3\alpha_3)}{\mu_2^2\hat{\alpha}_2^2 - \mu_1\mu_3\alpha_1\alpha_3} + \frac{\varepsilon}{2}\left\{\frac{\mu_2\hat{\alpha}_2 + \mu_1\alpha_1\tan\hat{\alpha}_2 h}{\mu_2^2\hat{\alpha}_2^2 - \mu_1\mu_3\alpha_1\alpha_3}\right\} \times \left\{\frac{2\alpha_3^2 + k_3^2}{2\alpha_3^2}\right\}, \tag{3.14}$$

where

$$\hat{\alpha}_2 = \sqrt{k_2^2 - \xi^2}.$$

We note that the dispersion relation given by equation (3.14) reduces to the dispersion relation of Ghosh [20], and Ewing et al. [17], in the limits μ_1 and ε equal to zero respectively.

To find the transmitted waves in the layer we have to solve the integral in the equation (3.13). We find that the poles of the integrand are roots $p_{3,n}$ (n=1,2,3,...) of equation $F_3(\xi,h) = 0$. In addition, there are branch points which give rise to body waves in which we are not interested. Thus, calculating the integral at these poles, the transmitted Love-type waves are given by

$$v_z(x,z) = 4\pi \sum_{n=1}^{\infty} \frac{(\mu_2 \hat{\alpha}_{2,3,n} \cos\hat{\alpha}_{2,3,n} z + \mu_1 \alpha_{1,3,n}}{\left. \frac{dF_3(\xi,h)}{d\xi} \right|_{\xi = p_{3,n}}}$$

$$\frac{\times \sin\hat{\alpha}_{2,3,n} z)}{} \exp(-ip_{3,n} x), \qquad (3.15)$$

where

$$\hat{\alpha}_2 \Big|_{\xi = p_{3,n}} = \hat{\alpha}_{2,3,n}, \qquad \alpha_1 \Big|_{\xi = p_{3,n}} = \alpha_{1,3,n}.$$

Equation (3.15) represents the transmitted waves in the layer.

3.3 INHOMOGENEOUS LAYER

In this section, we calculate the transmitted field due to a line source in an inhomogeneous layer whose rigidity is variable (see figure 3.2). The inhomogeneous layer is trapped between two homogeneous half-spaces.

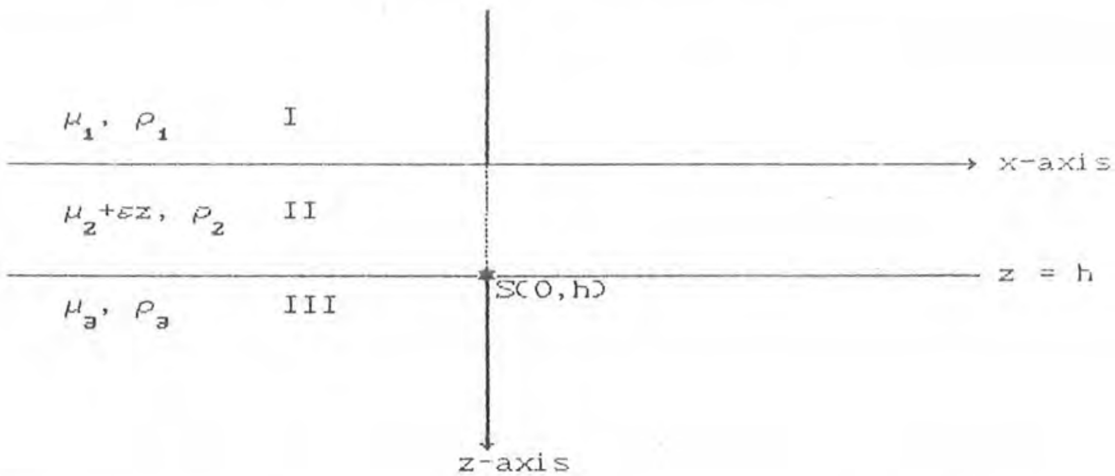

Figure 3.2. Geometry of the problem with an inhomogeneous layer.

The formulation of the problem is the same as that given in chapter two, except that the equation of motion for the inhomogeneous layer is of the form

$$\frac{\partial p_{xy}}{\partial x} + \frac{\partial p_{yz}}{\partial z} + \rho_2 \omega^2 v_2 = 4\pi\delta(x)\delta(z-h), \tag{3.16}$$

where v_2 is the displacement component, p_{xy}, p_{yz} are stress components, ρ_2 is the density of the layer and ω is the angular frequency. We suppose that in the inhomogeneous layer the coefficient of rigidity μ_2 varies as $\mu_2+\varepsilon z$, ε being a small parameter. Then equation (3.16) becomes

$$\mu_2\frac{\partial^2 v_2}{\partial x^2} + \varepsilon z\frac{\partial^2 v_2}{\partial x^2} + \mu_2\frac{\partial^2 v_2}{\partial z^2} + \varepsilon z\frac{\partial^2 v_2}{\partial z^2} + \varepsilon\frac{\partial v_2}{\partial z} + \rho_2\omega^2 v_2$$

$$= 4\pi\delta(x)\delta(z-h). \tag{3.17}$$

Dividing equation (3.17) throughout by μ_2 and re-arranging, we have

$$\frac{\partial^2 v_2}{\partial x^2} + \frac{\partial^2 v_2}{\partial z^2} + k_2^2 v_2 = \frac{4\pi}{\mu_2}\,\delta(x)\delta(z-h) - \frac{\varepsilon z}{\mu_2}\,\frac{\partial^2 v_2}{\partial z^2}$$

$$- \frac{\varepsilon}{\mu_2}\,\frac{\partial v_2}{\partial z} - \frac{\varepsilon z}{\mu_2}\,\frac{\partial^2 v_2}{\partial x^2}. \tag{3.18}$$

The equations of motion for the upper and lower semi-infinite media are

$$\frac{\partial^2 v_1}{\partial x^2} + \frac{\partial^2 v_1}{\partial z^2} + k_1^2 v_1 = 0, \tag{3.19}$$

$$\frac{\partial^2 v_3}{\partial x^2} + \frac{\partial^2 v_3}{\partial z^2} + k_3^2 v_3 = 0, \qquad (3.20)$$

where k_i^2 (i=1,2,3) are the same as given by equation (2.7). The boundary conditions of the problem are same as that, first problem of chapter two, except that the boundary conditions (2.8b) are replaced by the conditions at $z = h, \quad -\infty < x < \infty$:

$$v_2 = v_3,$$

$$(\mu_2+\varepsilon h) \frac{\partial v_2}{\partial z} = \mu_3 \frac{\partial v_3}{\partial z}. \qquad (3.21)$$

Using the Fourier transform, the equations of motion (3.18)—(3.20) reduce to

$$\frac{d^2 V_2}{dz^2} - \alpha_2^2 V_2 = \frac{2}{\mu_2}\delta(z-h) - \frac{\varepsilon z}{\mu_2}\frac{d^2 V_2}{dz^2} - \frac{\varepsilon}{\mu_2}\frac{dV_2}{dz} + \frac{\varepsilon z}{\mu_2}\xi^2 V_2$$

$$= \tau_3(z) \text{ (say)}, \qquad (3.22)$$

for the layer and

$$\frac{d^2 V_{1,3}}{dz^2} - \alpha_{1,3}^2 V_{1,3} = 0, \qquad (3.23)$$

for the upper and lower substrata. In equations (3.22) and (3.23) α_i^2 (i=1,2,3) are the same as given by equation (2.12). Using the boundary conditions (2.8a), (2.8c) and (3.21), adopting the procedure of chapter two, we get

$$V_2(z) = G_2(z,0)[V_2']_{z=0} - G_2(z,h)[V_2']_{z=h}$$

$$+ \int_0^h \tau_3(z_0) G_2(z,z_0) dz_0, \qquad (3.24)$$

$$V_1(z) = -G_1(z,0)[V_1']_{z=0}, \tag{3.25}$$

$$V_3(z) = G_3(z,h)[V_3']_{z=h}. \tag{3.26}$$

Applying the boundary conditions (2.8a), we obtain

$$[V_2']_{z=0} = \frac{1}{A}\left\{G_2(0,h)[V_2']_{z=h} - \int_0^h \tau_3(z_0)G_2(0,z_0)dz_0\right\}, \tag{3.27}$$

where A is given by equation (2.21). Similarly the boundary conditions (3.21) yield

$$[V_2']_{z=h} = \frac{1}{(AB-G_2^2(h,0)+\frac{\varepsilon h}{\mu_3}AG_3(h,h))}\left\{-G_2(h,0)\int_0^h \tau_3(z_0)\right.$$

$$\left.\times\, G_2(0,z_0)dz_0 + A\int_0^h \tau_3(z_0)G_2(h,z_0)dz_0\right\}, \tag{3.28}$$

where B is given by equation (2.23). Using equations (3.27) and (3.28) in equation (3.24), substituting back value of $\tau_3(z_0)$ and using the property of the delta function, we get

$$V_2(z) = \frac{2(\mu_2+\varepsilon h)}{\mu_2\mu_3}\left\{\frac{G_2(z,h)D - G_2(z,0)E}{AB - G_2^2(h,0) + \frac{\varepsilon h}{\mu_3}AG_3(h,h)}\right\}$$

$$-\left\{\frac{G_2(z,h)G_2(h,0) - (B+\frac{\varepsilon h}{\mu_3}G_3(h,h))G_2(z,0)}{AB - G_2^2(h,0) + \frac{\varepsilon h}{\mu_3}AG_3(h,h)}\right\}$$

$$\times\,\frac{\varepsilon}{\mu_2}\int_0^h\left\{z_0\frac{d^2V_2(z_0)}{dz_0^2} + \frac{dV_2(z_0)}{dz_0} - z_0\xi^2V_2(z_0)\right\}$$

$$\times\ G_z(0,z_o)dz_o - \left\{\frac{G_z(z,0)G_z(h,0) - G_z(z,h)A}{AB - G_z^2(h,0) + \frac{\varepsilon h}{\mu_3} AG_3(h,h)}\right\}$$

$$\times\ \frac{\varepsilon}{\mu_2} \int_o^h \left\{z_o \frac{d^2V_z(z_o)}{dz_o^2} + \frac{dV_z(z_o)}{dz_o} - z_o\xi^2 V_z(z_o)\right\}$$

$$\times\ G_z(h,z_o)dz_o - \frac{\varepsilon}{\mu_2} \int_o^h \left\{z_o \frac{d^2V_z(z_o)}{dz_o^2} + \frac{dV_z(z_o)}{dz_o}\right.$$

$$\left. - z_o\xi^2 V_z(z_o)\right\} G_z(z,z_o)dz_o, \qquad (3.29)$$

where D and E are given by equations (2.26) and (2.27) respectively.

Equation (3.29) is an integral equation and $V_z(z)$ may be determined from this equation using successive substitutions. As a first approximation, we neglect terms involving ε to obtain

$$V_z(z) = \frac{2}{\mu_3}\left\{\frac{G_z(z,h)D - G_z(z,0)E}{AB - G_z^2(h,0)}\right\}. \qquad (3.30)$$

Substituting this value back in the right side of equation (3.29), we obtain $V_z(z)$ in the following form

$$V_z(z) = \frac{2(\mu_2+\varepsilon h)}{\mu_2\mu_3}\left\{\frac{G_z(z,h)D - G_z(z,0)E}{AB - G_z^2(h,0) + \frac{\varepsilon h}{\mu_3} AG_3(h,h)}\right\}$$

$$- \frac{2\varepsilon}{\mu_2\mu_3}\left\{\frac{G_z(z,h)G_z(h,0) - (B+\frac{\varepsilon h}{\mu_3} G_3(h,h))G_z(z,0)}{AB - G_z^2(h,0) + \frac{\varepsilon h}{\mu_3} AG_3(h,h)}\right\}$$

$$\times \int_o^h \left\{z_o \frac{d^2M_3(z_o)}{dz_o^2} + \frac{dM_3(z_o)}{dz_o} - z_o\xi^2 M_3(z_o)\right\}$$

$$\times G_2(0,z_o)dz_o - \frac{2\varepsilon}{\mu_2\mu_3}\left\{\frac{G_2(z,0)G_2(h,0) - G_2(z,h)A}{AB - G_2^2(h,0) + \frac{\varepsilon h}{\mu_3} AG_3(h,h)}\right\}$$

$$\times \int_0^h \left\{z_o \frac{d^2M_3(z_o)}{dz_o^2} + \frac{dM_3(z_o)}{dz_o} - z_o\xi^2M_3(z_o)\right\}$$

$$\times G_2(h,z_o)dz_o - \frac{2\varepsilon}{\mu_2\mu_3}\int_0^h \left\{z_o \frac{d^2M_3(z_o)}{dz_o^2} + \frac{dM_3(z_o)}{dz_o}\right.$$

$$\left. - z_o\xi^2M_3(z_o)\right\} G_2(z,z_o)dz, \qquad (3.31)$$

where

$$M_3(z_o) = \left\{\frac{G_2(z_o,h)D - G_2(z_o,0)E}{AB - G_2^2(h,0)}\right\}.$$

We note that $V_2(z)$ is completely determined through equation (3.31) provided G_1, G_2 and G_3 are known. Using the Green's functions given by equations (2.32) to (2.34) in equation (3.31), simplifying and neglecting square and higher powers of ε, we arrive at

$$V_2(z) = \frac{-2(\mu_2\alpha_2\cosh\alpha_2 z + \mu_1\alpha_1\sinh\alpha_2 z)}{\mu_1\mu_3\alpha_1\alpha_2^2\alpha_3(AB-G_2^2(h,0))(1+\varepsilon J_4)\sinh\alpha_2 h}, \qquad (3.32)$$

where

$$J_4 = \frac{1}{4(AB-G_2^2(h,0))}\left[\frac{(\mu_3\alpha_3-\mu_1\alpha_1)}{\mu_1\mu_3\alpha_1\alpha_2^2\alpha_3} + \frac{\xi^2(\mu_3\alpha_3-\mu_1\alpha_1)}{\mu_1\mu_3\alpha_1\alpha_2^4\alpha_3}\right.$$

$$- \frac{h\left\{(5\mu_1\mu_3\alpha_1\alpha_3+3\mu_2^2\alpha_2^2)+\mu_2\alpha_2(3\mu_1\alpha_1+5\mu_3\alpha_3)\coth\alpha_2 h\right\}}{\mu_1\mu_2\mu_3\alpha_1\alpha_2^2\alpha_3}$$

$$- \frac{h^2\left\{\mu_2\alpha_2(\mu_1\alpha_1+\mu_3\alpha_3)+(\mu_1\mu_3\alpha_1\alpha_3+\mu_2^2\alpha_2^2)\coth\alpha_2 h\right\}}{\mu_1\mu_2\mu_3\alpha_1\alpha_2\alpha_3}$$

$$- \frac{h\xi^2\left\{(\mu_1\mu_3\alpha_1\alpha_3-\mu_2^2\alpha_2^2)+\mu_2\alpha_2(\mu_3\alpha_3-\mu_1\alpha_1)\coth\alpha_2 h\right\}}{\mu_1\mu_2\mu_3\alpha_1\alpha_2^4\alpha_3}$$

$$+ \frac{h^2\xi^2\left\{\mu_2\alpha_2(\mu_1\alpha_1+\mu_3\alpha_3)+(\mu_1\mu_3\alpha_1\alpha_3+\mu_2^2\alpha_2^2)\coth\alpha_2 h\right\}}{\mu_1\mu_2\mu_3\alpha_1\alpha_2^3\alpha_3}\Bigg].$$

Taking the inverse Fourier transform of equation (3.32), the displacement in the intermediate layer is

$$v_2(x,z) = \int_{-\infty}^{\infty} \frac{-2(\mu_2\alpha_2\cosh\alpha_2 z + \mu_1\alpha_1\sinh\alpha_2 z)e^{-i\xi x}}{F_4(\xi,h)}\, d\xi, \tag{3.33}$$

where

$$F_4(\xi,h) = \mu_1\mu_3\alpha_1\alpha_2^2\alpha_3\{AB-G_2^2(h,0)\}(1+\varepsilon J_4)\sinh\alpha_2 h.$$

In the expression for $v_2(x,z)$ contour integration is to be performed. The poles of the integrand are obtained by equating the expression in the denominator to zero. The resultant relation gives us the dispersion relation that the Love-type wave propagating in the inhomogeneous layer must satisfy. Replacing α_2 by $i\hat{\alpha}_2$, the dispersion relation in our case can be reduced to the form

$$\tan\hat{\alpha}_2 h = \frac{\mu_2\hat{\alpha}_2(\mu_1\alpha_1+\mu_3\alpha_3)}{\mu_2^2\hat{\alpha}_2^2-\mu_1\mu_3\alpha_1\alpha_3} + \frac{\varepsilon}{4(\mu_2^2\hat{\alpha}_2^2-\mu_1\mu_3\alpha_1\alpha_3)}\Bigg[(\mu_3\alpha_3$$

$$-\mu_1\alpha_1)\tan\hat{\alpha}_2 h - \frac{\xi^2}{\hat{\alpha}_2^2}(\mu_3\alpha_3-\mu_1\alpha_1)\tan\hat{\alpha}_2 h$$

$$- \frac{h}{\mu_2}\left\{(5\mu_1\mu_3\alpha_1\alpha_3-3\mu_2^2\hat{\alpha}_2^2)\tan\hat{\alpha}_2 h+\mu_2\hat{\alpha}_2(3\mu_1\alpha_1+5\mu_3\alpha_3)\right\}$$

$$- \frac{h^2\hat{\alpha}_2}{\mu_2}\left\{(\mu_1\mu_3\alpha_1\alpha_3-\mu_2^2\hat{\alpha}_2^2)-\mu_2\hat{\alpha}_2(\mu_1\alpha_1+\mu_3\alpha_3)\tan\hat{\alpha}_2 h\right\}$$

$$+ \frac{h\xi^2}{\mu_2\hat{\alpha}_2^2}\left\{(\mu_1\mu_3\alpha_1\alpha_3+\mu_2^2\hat{\alpha}_2^2)\tan\hat{\alpha}_2 h+\mu_2\hat{\alpha}_2(\mu_3\alpha_3-\mu_1\alpha_1)\right\}$$

$$- \frac{h^2\xi^2}{\mu_2\hat{\alpha}_2}\left\{(\mu_1\mu_3\alpha_1\alpha_3-\mu_2^2\hat{\alpha}_2^2)-\mu_2\hat{\alpha}_2(\mu_1\alpha_1+\mu_3\alpha_3)\tan\hat{\alpha}_2 h\right\}\Bigg]. \tag{3.34}$$

It is important to note that in the case of a homogeneous medium $\varepsilon = 0$ and this dispersion relation reduces to the dispersion relation obtained by Ewing [17] for the homogeneous case.

In order to obtain the transmitted wave, we need to calculate the integral in equation (3.33). Note that the poles of the integrand are roots $P_{4,n}$ (n=1,2,3, ...) of

$$F_4(\xi,h) = \mu_1\mu_3\alpha_1\alpha_2^2\alpha_3(AB-G_2^2(h,0))(1+\varepsilon J_4)\sinh\alpha_2 h.$$

Calculating the contribution by these poles, we find that

$$v_2(x,z) = 4\pi \sum_{n=1}^{\infty} \frac{\left\{\mu_2\hat{\alpha}_{2,4,n}\cos\hat{\alpha}_{2,4,n}z + \mu_1\alpha_{1,4,n}\sin\hat{\alpha}_{2,4,n}z\right\}}{\left.\frac{dF_4(\xi,h)}{d\xi}\right|_{\xi=p_{4,n}}} \exp(-ip_{4,n}x), \tag{3.35}$$

where

$$\hat{\alpha}_2\Big|_{\xi=p_{4,n}} = \hat{\alpha}_{2,4,n}, \qquad \alpha_1\Big|_{\xi=p_{4,n}} = \alpha_{1,4,n}.$$

Equation (3.35) represents a travelling wave in the layer in the direction of the x-axis.

Besides the poles, we have branch points which give rise to body waves and are of no interest for the present study. However, these calculations can be easily made with the help of the saddle point method considering the asymptotic behaviour of the integrals.

CHAPTER FOUR

FIELD DUE TO A POINT SOURCE IN A LAYER OVER AN INHOMOGENEOUS MEDIUM

4.1 INTRODUCTION

In this chapter, we first consider a line source situated at the interface between a homogeneous layer and an inhomogeneous half-space. The upper surface of the layer is stress free. The transmitted wave and the dispersion relation for the Love waves are calculated analytically. Further, the problem is extended to the case of point source excitations. The wave equation governing the point source problem consists of three spatial coordinates. The Fourier transform is used to reduce the dependence on one variable. The transformed equations then correspond to the line source problem and the point source field is recovered by application of the inverse transform. This procedure can be applied to other configurations, once the line source excitation is known.

4.2 LINE SOURCE PROBLEM

Suppose that a horizontal homogeneous elastic medium of thickness h is overlying a semi-infinite inhomogeneous substratum. The origin of the coordinates is chosen along the free surface with the z-axis vertically downwards (see figure 4.1.). A line source disturbance has been taken as S(0,h), where the z-axis intersects the interface. Subscripts 1 and 2 will refer to the layer and the semi-infinite substratum respectively. If $\sigma(r,t)$ be the source density distribution in the layer, the general equations of motion in the layer and the lower inhomogeneous medium are given by:

$$\frac{\partial^2 v_1}{\partial x^2} + \frac{\partial^2 v_1}{\partial z^2} - \frac{\rho_1}{\mu_1}\frac{\partial^2 v_1}{\partial t^2} = \frac{4\pi}{\mu_1}\sigma(r,t), \tag{4.1}$$

$$\frac{\partial^2 v_2}{\partial x^2} + \frac{\partial^2 v_2}{\partial z^2} - \frac{\rho_2}{\mu_2}\frac{\partial^2 v_2}{\partial^2 t} = 0. \tag{4.2}$$

Considering the time harmonic dependence and the line source, we can replace v_i by $v_i e^{i\omega t}$ (i =1,2) and $\sigma(r,t)$ by

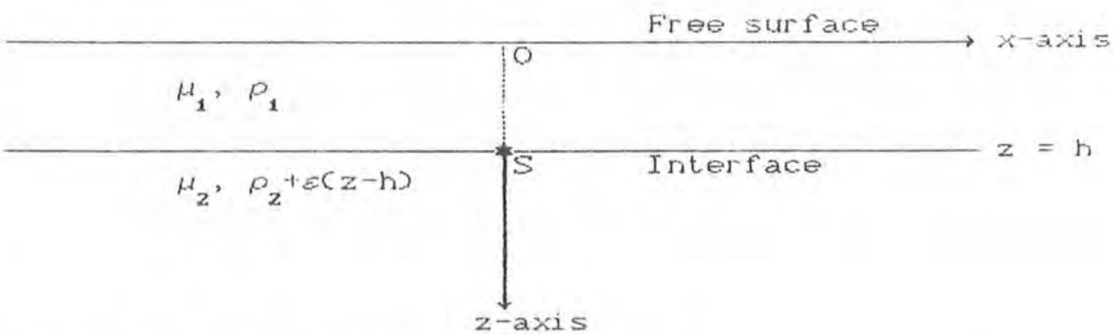

Figure 4.1. Geometry of the problem.

$\delta(x)\delta(z-h)e^{i\omega t}$. From equations (4.1) and (4.2) this gives

$$\frac{\partial^2 v_1}{\partial x^2} + \frac{\partial^2 v_1}{\partial z^2} + \frac{\rho_1\omega^2}{\mu_1} v_1 = \frac{4\pi}{\mu_1}\delta(x)\delta(z-h), \tag{4.3}$$

$$\frac{\partial^2 v_2}{\partial x^2} + \frac{\partial^2 v_2}{\partial z^2} + \frac{\rho_2\omega^2}{\mu_2} v_2 = 0, \tag{4.4}$$

where ω is the angular frequency and δ is the usual Dirac delta function. In order to study the inhomogeneity effect, we replace ρ_2 by $\rho_2 + \varepsilon(z-h)$, ε being a small parameter. Thus, we can rewrite equations (4.3) and (4.4) as

$$\frac{\partial^2 v_1}{\partial x^2} + \frac{\partial^2 v_1}{\partial z^2} + k_1^2 v_1 = \frac{4\pi}{\mu_1}\delta(x)\delta(z-h), \tag{4.5}$$

$$\frac{\partial^2 v_2}{\partial x^2} + \frac{\partial^2 v_2}{\partial z^2} + k_2^2 v_2 = -\frac{\varepsilon\omega^2}{\mu_2}(z-h)v_2, \tag{4.6}$$

where

$$k_i^2 = \frac{\rho_i\omega^2}{\mu_i}, \qquad i = 1, 2. \tag{4.7}$$

The boundary conditions for the determination of v_1 and v_2 are given by:

(a) At $z = 0$, $-\infty < x < \infty$,

$$\frac{\partial v_1}{\partial z} = 0. \tag{4.8a}$$

(b) At $z = h$, $-\infty < x < \infty$,

$$v_1 = v_2,$$

$$\mu_1 \frac{\partial v_1}{\partial z} = \mu_2 \frac{\partial v_2}{\partial z}. \tag{4.8b}$$

(c) $v_2 \longrightarrow 0$ as $z \longrightarrow \infty$. (4.8c)

Taking the Fourier transform of equations (4.5) and (4.6) with respect to x, we get

$$\frac{d^2V_1}{dz^2} - \gamma^2 V_1 = \frac{2}{\mu_1}\,\delta(z-h), \tag{4.9}$$

$$\frac{d^2V_2}{dz^2} - \eta^2 V_2 = -\,\frac{\varepsilon\omega^2}{\mu_2}\,(z-h)V_2, \tag{4.10}$$

where

$$\gamma^2 = \xi^2 - k_1^2, \qquad \eta^2 = \xi^2 - k_2^2.$$

Let the Green's function for the inhomogeneous equation (4.9) be $G_1(z,z_o)$, where z_o is an arbitrary point in the medium 1. The equation satisfied by $G_1(z,z_o)$ is

$$G_1''(z,z_o) - \gamma^2 G_1(z,z_o) = \delta(z-z_o), \tag{4.11a}$$

together with the homogeneous boundary conditions

$$G_1'(z,z_o) = 0 \quad \text{at} \quad z = 0,\ h. \tag{4.11b}$$

Here prime denotes differentiation with respect to z. Multiplying equation (4.9) by $G_1(z,z_o)$, equation (4.11a) by $V_1(\xi,z)$, subtracting and integrating over $0 \le z \le h$, we obtain

$$G_1(h,z_o)[V_1']_{z=h} = \frac{2}{\mu_1}\,G_1(h,z_o) - V_1(z_o). \tag{4.12}$$

Similarly, we suppose that $G_2(z,z_o)$ be the Green's function for the lower substratum satisfying

$$G_2'(z,z_o) = 0 \quad \text{at} \quad z = h,$$

and

$$G_2(z,z_o) \longrightarrow 0, \quad \text{as} \quad z \longrightarrow \infty.$$

A similar procedure to the above leads to

$$G_2(h,z_o)[V_2']_{z=h} = V_2(z_o) + \frac{\varepsilon\omega^2}{\mu_2}\int_h^{\infty}(z-h)V_2(z)G_2(z,z_o)dz. \tag{4.13}$$

Now, interchanging z and z_o and using the symmetry property of Green's functions in equations (4.12) and (4.13), we obtain

$$V_1(z) = \frac{2}{\mu_1}G_1(z,h) - G_1(z,h)[V_1']_{z=h}, \tag{4.14}$$

$$V_2(z) = G_2(z,h)[V_2']_{z=h} - \frac{\varepsilon\omega^2}{\mu_2}\int_h^{\infty}(z_o-h)V_2(z_o)G_2(z,z_o)dz_o. \tag{4.15}$$

The boundary conditions (4.8b) yield

$$[V_1']_{z=h} = \frac{2\mu_2 G_1(h,h)}{\mu_1\{\mu_2 G_1(h,h)+\mu_1 G_2(h,h)\}} + \frac{\varepsilon\omega^2}{\{\mu_2 G_1(h,h)+\mu_1 G_2(h,h)\}}\int_h^{\infty}(z_o-h)V_2(z_o)G_2(h,z_o)dz_o. \tag{4.16}$$

Substituting the value of $[V_1']_{z=h}$ from equation (4.16) in equation (4.14), we obtain

$$V_1(z) = \frac{2G_1(z,h)G_2(h,h)}{\{\mu_2 G_1(h,h)+\mu_1 G_2(h,h)\}} - \frac{\varepsilon\omega^2 G_1(z,h)}{\{\mu_2 G_1(h,h)+\mu_1 G_2(h,h)\}} \times \int_h^{\infty}(z_o-h)V_2(z_o)G_2(h,z_o)dz_o. \tag{4.17}$$

The continuity condition and the value of $[V_1']_{z=h}$, gives $V_2(z)$, which can be written as

$$V_2(z) = \frac{2G_1(h,h)G_2(z,h)}{\{\mu_2 G_1(h,h)+\mu_1 G_2(h,h)\}} + \frac{\varepsilon\omega^2\mu_1 G_2(z,h)}{\mu_2\{\mu_2 G_1(h,h)+\mu_1 G_2(h,h)\}}$$

$$\times \int_h^\infty (z_o - h) V_2(z_o) G_2(h, z_o) dz_o - \frac{\varepsilon\omega^2}{\mu_2} \int_h^\infty (z_o - h) V_2(z_o)$$

$$\times G_2(z, z_o) dz_o. \tag{4.18}$$

The field $V_1(z)$ in the layer can be determined from equation (4.17), if we substitute the value of $V_2(z)$ given by equation (4.18). Since equation (4.18) is an integral equation and $V_2(z)$ may be determined from this equation by the method of successive approximations. As a first approximation, we neglect terms involving ε to obtain

$$V_2(z) = \frac{2G_1(h,h)G_2(z,h)}{\{\mu_2 G_1(h,h) + \mu_1 G_2(h,h)\}}. \tag{4.19}$$

Substituting the value of $V_2(z)$ from equation (4.19) in the expression (4.17) for $V_1(z)$, we get

$$V_1(z) = \frac{2G_1(z,h)G_2(h,h)}{\{\mu_2 G_1(h,h) + \mu_1 G_2(h,h)\}} - \frac{2\varepsilon\omega^2 G_1(z,h)G_1(h,h)}{\{\mu_2 G_1(h,h) + \mu_1 G_2(h,h)\}^2}$$

$$\times \int_h^\infty (z_o - h) G_2(z_o, h) G_2(h, z_o) dz_o. \tag{4.20}$$

The value of $V_1(z)$ is fully specified by equation (4.20) provided we know the values of G_1 and G_2.

To find these Green's functions, we adopt the procedure used in chapter 2. Thus, equations (4.11a,b) would give us

$$G_1(z, z_o) = -\frac{1}{2\gamma}\left[e^{-\gamma|z-z_o|} + e^{\gamma z}\left\{\frac{e^{-\gamma(h+z_o)} + e^{-\gamma(h-z_o)}}{e^{\gamma h} - e^{-\gamma h}}\right\}\right.$$

$$\left. + e^{-\gamma z}\left\{\frac{e^{\gamma(h-z_o)} + e^{-\gamma(h-z_o)}}{e^{\gamma h} - e^{-\gamma h}}\right\}\right]. \tag{4.21}$$

Similarly

$$G_2(z,z_0) = -\frac{1}{2\eta}\left[e^{-\eta|z-z_0|} + e^{-\eta(z+z_0-2h)}\right]. \qquad (4.22)$$

Using the values of G_1 and G_2 in equation (4.20) and after some manipulations, we arrive at

$$V_1(\xi,z) = \frac{-2(e^{\gamma z} + e^{-\gamma z})}{B^* - \dfrac{\varepsilon\omega^2(e^{\gamma h} + e^{-\gamma h})}{4\eta^2}}, \qquad (4.23)$$

where

$$B^* = \mu_2\eta(e^{\gamma h} + e^{-\gamma h}) + \mu_1\gamma(e^{\gamma h} - e^{-\gamma h}).$$

The displacement $v_1(x,z)$ in the layer is obtained by taking the inverse Fourier transform of equation (4.23). This gives

$$v_1(x,z) = -2\int_{-\infty}^{\infty} V_1(\xi,z)e^{-i\xi x}d\xi. \qquad (4.24)$$

The dispersion equation for the propagation of Love waves in the layered structure consisting of a semi-infinite inhomogeneous medium of rigidity μ_2 covered by a homogeneous surface layer of uniform thickness h, rigidity μ_1 and with a free upper surface is given by setting the denominator of $V_1(\xi,z)$ in equation (4.23) equal to zero. It is

$$\tan\hat{\gamma}h = \frac{1}{\hat{\gamma}}\left\{\nu\eta - \frac{\varepsilon\omega^2}{4\mu_1\eta^2}\right\}, \qquad (4.25)$$

where

$$\hat{\gamma}^2 = (k_1^2 - \xi^2).$$

The equation (4.24) can be rewritten as

$$v_1(x,z) = \frac{-2}{\mu_1}\int_{-\infty}^{\infty}\frac{\cosh\gamma z\ e^{-i\xi x}}{F_5(\xi,h)}\,d\xi, \tag{4.26}$$

where

$$F_5(\xi,h) = \gamma\sinh\gamma h + (\nu\eta - \frac{\varepsilon\omega^2}{4\mu_1\eta^2})\cosh\gamma h, \qquad \nu = \frac{\mu_2}{\mu_1}.$$

The poles of the integrand in equation (4.26) are the real roots λ_m of the equation $F_5(\xi,h) = 0$. In addition, there are branch points which give rise to body waves in which we are not interested. However, we remark that the integral can be solved at the branch points using asymptotic methods. Thus, calculating this integral at the poles, by indenting the contour and using the residue method, the field in the layer can be easily written in the form:

$$v_1(x,z) = \frac{-2\pi i}{\mu_1}\frac{\left\{\cos\gamma_m z\ e^{-i\lambda_m x}\right\}}{\left.\frac{dF_5(\xi,h)}{d\xi}\right|_{\xi=\lambda_m}}, \tag{4.27}$$

where

$$\gamma_m^2 = (k_1^2 - \lambda_m^2).$$

Equation (4.27) can also be written as

$$v_1(x,z) = -8\pi i\sum_{m=1}^{\infty}\frac{\eta_m^4\,\gamma_m\cos\gamma_m h\,\cos\gamma_m z\ e^{-i\lambda_m x}}{\lambda_m\left\{4\mu_1 h\eta_m^2\gamma_m^3 + 3\varepsilon\omega^2\gamma_m\cos^2\gamma_m h\right\}}. \tag{4.28}$$

The above equation represents the transmitted field in the layer due to a line source excitation.

4.3 POINT SOURCE PROBLEM

We consider a point source occupying the position (0,0,h). The wave equations in this case take the form

$$\frac{\partial^2 v_1}{\partial x^2} + \frac{\partial^2 v_1}{\partial y^2} + \frac{\partial^2 v_1}{\partial z^2} + \frac{\rho_1 \omega^2}{\mu_1} v_1 = \frac{4\pi}{\mu_1} \delta(x)\delta(y)\delta(z-h), \tag{4.29}$$

$$\frac{\partial^2 v_2}{\partial x^2} + \frac{\partial^2 v_2}{\partial y^2} + \frac{\partial^2 v_2}{\partial z^2} + \frac{\rho_2 \omega^2}{\mu_2} v_2 = 0. \tag{4.30}$$

By introducing the inhomogeneity effect, equation (4.30) becomes

$$\frac{\partial^2 v_2}{\partial x^2} + \frac{\partial^2 v_2}{\partial y^2} + \frac{\partial^2 v_2}{\partial z^2} + \frac{\rho_2 \omega^2}{\mu_2} v_2 = -\frac{\varepsilon\omega^2}{\mu_2}(z-h)v_2. \tag{4.31}$$

Let us define the Fourier transform pair with respect to y:

$$\tilde{v}_i(x,w,z) = \int_{-\infty}^{\infty} v_i(x,y,z)\exp(\imath k_i wy)\,dy,$$

$$v_i(x,y,z) = \frac{k_i}{2\pi}\int_{-\infty}^{\infty} \tilde{v}_i(x,w,z)\exp(-\imath k_i wy)\,dw, \qquad i = 1,\ 2.$$

Using the above transform, equations (4.29) and (4.31) reduce to

$$\frac{\partial^2 \tilde{v}_1}{\partial x^2} + \frac{\partial^2 \tilde{v}_1}{\partial z^2} + k_1^2(1-w^2)\tilde{v}_1 = \frac{4\pi}{\mu_1}\delta(x)\delta(z-h), \tag{4.32}$$

$$\frac{\partial^2 \tilde{v}_2}{\partial x^2} + \frac{\partial^2 \tilde{v}_2}{\partial z^2} + k_2^2(1-w^2)\tilde{v}_2 = -\frac{\varepsilon\omega^2}{\mu_2}(z-h)\tilde{v}_2, \tag{4.33}$$

where k_i^2 (i=1,2) are the same as given by equation (4.7).

We observe that mathematical problem given by equations (4.32) and (4.33) is the same as in the two dimensional case given by equations (4.5) and (4.6), except that $\tilde{k}_i^2 = k_i^2(1-w^2)$ replaces k_i^2 and $\ddot{v}_i$ replaces v_i (i=1,2). Thus, the field in the transformed plane is given by (see equation 4.24)

$$\ddot{v}_1(x,w,z) = -2\int_{-\infty}^{\infty} \tilde{V}_1(\xi,w,z)e^{-i\xi x}\,d\xi, \tag{4.34}$$

where

$$\tilde{V}_1(\xi,w,z) = \frac{(e^{\tilde{\gamma}z} + e^{-\tilde{\gamma}z})}{\tilde{B}^* - \dfrac{\varepsilon\omega^2(e^{\tilde{\gamma}h} + e^{-\tilde{\gamma}h})}{4\tilde{\eta}^2}},$$

$$\tilde{\gamma}^2 = (\xi^2 - \tilde{k}_1^2), \qquad \tilde{\eta}^2 = (\xi^2 - \tilde{k}_2^2),$$

and

$$\tilde{B}^* = \mu_2\tilde{\eta}(e^{\tilde{\gamma}h} + e^{-\tilde{\gamma}h}) + \mu_1\tilde{\gamma}(e^{\tilde{\gamma}h} - e^{-\tilde{\gamma}h}).$$

The dispersion relations for the point source in the layer is given by

$$\tan\hat{\tilde{\gamma}}h = \frac{1}{\hat{\tilde{\gamma}}}\left\{\mu\tilde{\eta} - \frac{\varepsilon\omega^2}{4\mu_1\tilde{\eta}^2}\right\}, \qquad \hat{\tilde{\gamma}}^2 = (\tilde{k}_1^2 - \xi^2). \tag{4.35}$$

The results of section 4.2 for the line source problem will be used giving the field in the transformed plane. Thus the field due to a point source in the transformed plane will be

$$\tilde{v}_1(x,w,z) = -8\pi i\sum_{m=1}^{\infty} \frac{\tilde{\eta}_m^4\,\tilde{\gamma}_m\cos\tilde{\gamma}_m h\,\cos\tilde{\gamma}_m z\,e^{-i\tilde{\lambda}_m x}}{\tilde{\lambda}_m\left\{4\mu_1 h\,\tilde{\eta}_m^2\,\tilde{\gamma}_m^3 + 3\varepsilon\omega^2\,\tilde{\gamma}_m\cos^2\tilde{\gamma}_m h\right\}}. \tag{4.36}$$

It is interesting to note at this point that if we put $w = 0$, the expression (4.35) reduces to the dispersion relation obtained by Chattopadhyay [13] and equation (4.36) gives the transmitted field for the case of line source excitation (see equation 4.28).

The field $v_1(x,y,z)$, can be finally obtained by taking inverse transform of equation (4.36). This gives

$$v_1(x,y,z) = \frac{k_1}{2\pi} \int_{-\infty}^{\infty} \sum_{m=1}^{\infty} \frac{\left\{-8\pi i \tilde{\eta}_m^4 \tilde{\gamma}_m \cos\tilde{\gamma}_m h \cos\tilde{\gamma}_m z \, e^{-i\tilde{\lambda}_m x}\right\}}{\tilde{I}(w,h)}$$

$$\times e^{-ik_1 wy} dw, \qquad (4.37)$$

where

$$\tilde{I}(w,h) = \tilde{\lambda}_m (4\mu_1 h \tilde{\eta}_m^2 \tilde{\gamma}_m^3 + 3\varepsilon\omega^2 \tilde{\gamma}_m \cos^2\tilde{\gamma}_m h).$$

The poles of the integral in equation (4.37) are the roots of the equation $\tilde{I}(w,h) = 0$. If we denote these poles by p_n, the transmitted field $v_1(x,y,z)$ in the layer is given by

$$v_1(x,y,z) = 4\pi \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{k_1 \eta_{m,n}^4 \gamma_{m,n} \cos\gamma_{m,n} h \cos\gamma_{m,n} z}{\left.\frac{d\tilde{I}(w,h)}{dw}\right|_{w=p_n}}$$

$$\times e^{-i\lambda_{m,n} x} e^{-ik_1 p_n y}, \qquad (4.38)$$

where

$$\left.\tilde{\gamma}_m\right|_{w=p_n} = \gamma_{m,n}, \quad \left.\tilde{\eta}_m\right|_{w=p_n} = \eta_{m,n}, \quad \left.\tilde{\lambda}_m\right|_{w=p_n} = \lambda_{m,n}.$$

This completes the solution for the propagation of a spherical wave, i.e. for a point source excitation.

CHAPTER FIVE

SPECTRAL REPRESENTATION OF A LOVE-TYPE OPERATOR

5.1 INTRODUCTION

This chapter is devoted to the spectral representation of the two dimensional Love-type wave operator, associated with the propagation of monochromatic SH-waves in a layer trapped between two half-spaces. It is found that the spectrum of eigenvalues for a trapped layer model is the disjoint union of the discrete and the continuous spectrum, giving rise to proper and improper eigenfunctions. These eigenfunctions (proper as well as improper) may be generated from a Green's function. The essential step is the construction of a Green's function G as a function of a certain parameter λ and integration around a large circle $|\lambda| = R$ in the complex λ-plane. For a trapped layer, we have in addition to poles, branch point singularities. The sum of residues at the poles and the contribution from the branch-cuts yields the representation of the delta function in terms of proper and improper eigenfunctions.

5.2 EQUATION OF MOTION

The elastic layer $0 \le z \le h$ with uniform thickness h is assumed to be sandwiched between two homogeneous half-spaces $z < 0$ and $z > h$. The rigidity, shear velocity and density of the respective medium are denoted by μ_i, β_i $(\beta_1 < \beta_2 < \beta_3)$ and ρ_i for i = 1, 2, 3. The subscripts 1, 2 and 3 refer to the upper medium, the intermediate layer and the lower medium respectively. The geometry of the problem is shown in the figure 5.1. Let $v(x,z,t)$ be the horizontal component of displacement, then the equation of motion for horizontally polarized shear waves is

$$\frac{\partial}{\partial x}\left[\mu(z)\frac{\partial v(x,z,t)}{\partial x}\right] + \frac{\partial}{\partial z}\left[\mu(z)\frac{\partial v(x,z,t)}{\partial z}\right]$$
$$= \rho(z)\frac{\partial^2 v(x,z,t)}{\partial t^2}, \qquad (5.1)$$

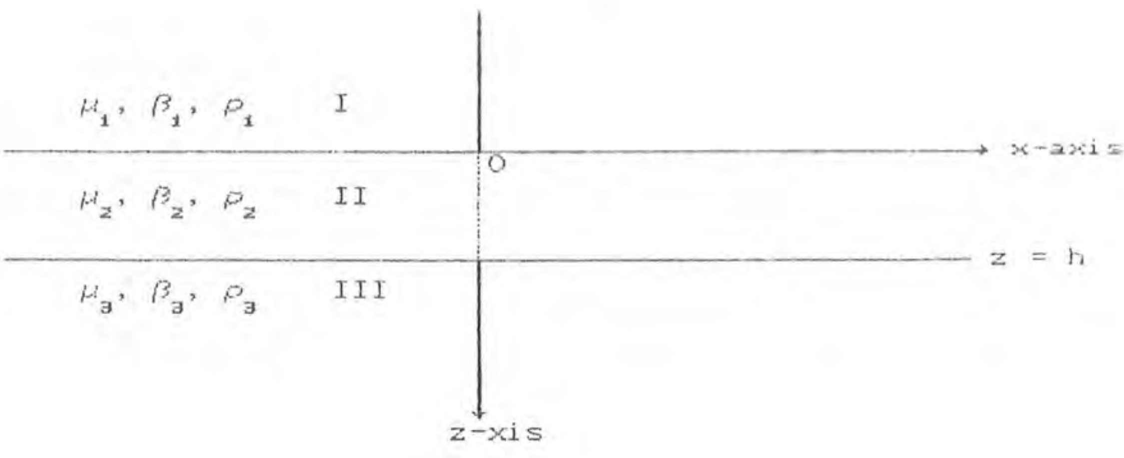

Figure 5.1. Geometry of the problem.

where

$$\left.\begin{aligned} \mu(z) &= \mu_1, \quad & z < 0, \\ &= \mu_2, \quad & 0 \le z \le h, \\ &= \mu_3, \quad & z > h, \end{aligned}\right\} \tag{5.2}$$

and

$$\left.\begin{aligned} \rho(z) &= \rho_1, \quad & z < 0, \\ &= \rho_2, \quad & 0 \le z \le h, \\ &= \rho_3, \quad & z > h, \end{aligned}\right\} \tag{5.3}$$

are piecewise constant functions and $v(x,z,t) = v_i(x,z,t)$ denotes the displacement in the respective medium for $i = 1, 2, 3$. We assume the motion to be time harmonic and write

$$v(x,z,t) = \bar{v}(z)\exp[i(\omega t - kx)], \tag{5.4}$$

where ω is the angular frequency and k is wave number. Omitting the time factor $e^{i\omega t}$, we can write the equation of motion as

$$\mathscr{L}(\bar{v}) = \frac{d}{dz}\left(\mu \frac{d\bar{v}}{dz}\right) + (\omega^2\rho - k^2\mu)\bar{v} = 0, \tag{5.5}$$

where

$$\bar{v}(z) = \bar{v}_i(z), \quad i = 1, 2, 3.$$

$v_i(z)$ satisfy the following equations

$$\frac{d^2\bar{v}_1}{dz^2} - \sigma_1^2\bar{v}_1 = 0, \quad \sigma_1^2 = (\lambda - \frac{\omega^2}{\beta_1^2}), \quad \lambda = k^2,$$

$$\beta_1^2 = \frac{\mu_1}{\rho_1}, \quad z < 0, \tag{5.6}$$

$$\frac{d^2\bar{v}_2}{dz^2} + \sigma_2^2\bar{v}_2 = 0, \quad \sigma_2^2 = (\frac{\omega^2}{\beta_2^2} - \lambda), \quad \beta_2^2 = \frac{\mu_2}{\rho_2},$$

$$0 \le z \le h, \tag{5.7}$$

$$\frac{d^2\bar{v}_3}{dz^2} - \sigma_3^2\bar{v}_3 = 0, \quad \sigma_3^2 = (\lambda - \frac{\omega^2}{\beta_3^2}), \quad \beta_3^2 = \frac{\mu_3}{\rho_3},$$

$$z > h. \qquad (5.8)$$

The interface conditions are

$$\bar{v}_1(0) = \bar{v}_2(0), \qquad (5.9a)$$

$$\mu_1\bar{v}_1'(0) = \mu_2\bar{v}_2'(0), \qquad (5.9b)$$

and

$$\bar{v}_2(h) = \bar{v}_3(h), \qquad (5.10a)$$

$$\mu_2\bar{v}_2'(h) = \mu_3\bar{v}_3'(h). \qquad (5.10b)$$

Here prime denotes differentiation with respect to z. The conditions at $\pm\infty$ are

$$\int_{-\infty}^{0}\mu|\bar{v}(z)|^2 dz < \infty, \qquad (5.11)$$

and

$$\int_{0}^{\infty}\mu|\bar{v}(z)|^2 dz < \infty. \qquad (5.12)$$

We thus have a self-adjoint Sturm-Liouville's system that is singular in the half-spaces $z < 0$ and $z > h$ and is regular in the layer $0 \le z \le h$. As noted by Kazi [27], the problem in the half-spaces is in the limit point case at infinity and so the requirement that the solution must be of finite μ-norm is adequate for defining the solution precisely.

5.3 THE GREEN'S FUNCTION

Let $G(z,\zeta;\lambda) = G_{ij}$, $i,j = 1, 2, 3$ be the Green's function associated with the problem. Here the first subscript refers to the z-interval and the second subscript refers to the ζ-interval. Thus G_{13}, for example, would refer to the Green's function $G(z,\zeta;\lambda)$ for z in the half-space $z < 0$ and ζ lying in the half-space $z > h$. G_{ij} satisfy the following conditions:

(G_1) $G_{ij}(z,\zeta;\lambda)$ is a continuous function of z;

(G_2) G_{ij} possesses a continuous first order derivative at each point z of the ith medium except that when $i = j$, at $z = \zeta$ it has a jump discontinuity given by

$$G'_{ii}(\zeta^+,\zeta;\lambda) - G'_{ii}(\zeta^-,\zeta;\lambda) = {}^1/\mu_i(\zeta);$$

(G_3) $\mathscr{L}(G_{ij}) = \delta(z-\zeta)\delta_{ij}$;

(G_4) $G(z,\zeta;\lambda)$ satisfy the interface conditions (5.9a) to (5.10b).

We determine G_{ij} as follows.

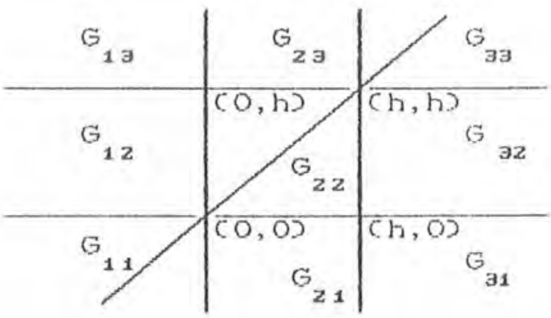

Figure 5.2. The Character of Green's Function.

(1) For $j = 1$, ζ is in the half-space $z < 0$ and G_{11}, G_{21} and G_{31} satisfy the differential equations

$$\frac{d^2G_{11}}{dz^2} - \sigma_1^2 G_{11} = \delta(z-\zeta), \tag{5.13}$$

$$\frac{d^2G_{21}}{dz^2} + \sigma_2^2 G_{21} = 0, \tag{5.14}$$

and

$$\frac{d^2G_{31}}{dz^2} - \sigma_3^2 G_{31} = 0, \tag{5.15}$$

together with the following conditions

$$\int_{-\infty}^{0} \mu(z) |G_{11}(z)|^2 dz < \infty, \tag{5.16a}$$

$$G_{11} = G_{21}, \qquad \text{at} \quad z = 0, \tag{5.16b}$$

$$\mu_1 G'_{11} = \mu_2 G'_{21}, \qquad \text{at} \quad z = 0, \tag{5.16c}$$

$$G_{21} = G_{31}, \qquad \text{at} \quad z = h, \tag{5.16d}$$

$$\mu_2 G'_{21} = \mu_3 G'_{31}, \qquad \text{at} \quad z = h, \tag{5.16e}$$

$$G_{11}(\zeta^+,\zeta;\lambda) = G_{11}(\zeta^-,\zeta;\lambda), \tag{5.16f}$$

$$G'_{11}(\zeta^+,\zeta;\lambda) - G'_{11}(\zeta^-,\zeta;\lambda) = {}^1\!/\!\mu_1, \tag{5.16g}$$

and

$$\int_{0}^{\infty} \mu(z) |G_{31}(z)|^2 dz < \infty. \tag{5.16h}$$

We find

$$G_{11}(z,\zeta;\lambda) = \frac{\square}{2\mu_1\sigma_1\Delta} e^{\sigma_1(z+\zeta)} - \frac{1}{2\mu_1\sigma_1}\Big[e^{\sigma_1(z-\zeta)}H(\zeta-z) + e^{-\sigma_1(z-\zeta)}H(z-\zeta)\Big], \tag{5.17}$$

where

$$\square = \mu_2\sigma_2(\mu_2\sigma_2\tan\sigma_2 h - \mu_3\sigma_3) + \mu_1\sigma_1(\mu_3\sigma_3\tan\sigma_2 h + \mu_2\sigma_2), \tag{5.18}$$

$$\Delta = \mu_2\sigma_2(\mu_2\sigma_2\tan\sigma_2 h - \mu_3\sigma_3) - \mu_1\sigma_1(\mu_3\sigma_3\tan\sigma_2 h + \mu_2\sigma_2), \tag{5.19}$$

and $H(x)$ is a usual Heaviside function.

$$G_{21}(z,\zeta;\lambda) = \frac{e^{\sigma_1\zeta}\,P(z)}{\Delta\cos\sigma_2 h}, \tag{5.20}$$

where

$$P(z) = \mu_2\sigma_2\cos\sigma_2(h-z) + \mu_3\sigma_3\sin\sigma_2(h-z), \tag{5.21}$$

$$G_{31}(z,\zeta;\lambda) = \frac{\mu_2\sigma_2\, e^{\sigma_1\zeta}\, e^{-\sigma_3(z-h)}}{\Delta\cos\sigma_2 h}. \tag{5.22}$$

(2) If ζ is in the layer $0 \le z \le h$, G_{12}, G_{22} and G_{32} satisfy the differential equations

$$\frac{d^2G_{12}}{dz^2} - \sigma_1^2 G_{12} = 0, \tag{5.23}$$

$$\frac{d^2G_{22}}{dz^2} + \sigma_2^2 G_{22} = \delta(z-\zeta), \tag{5.24}$$

$$\frac{d^2G_{32}}{dz^2} - \sigma_3^2 G_{32} = 0, \tag{5.25}$$

together with the conditions

$$\int_{-\infty}^{0}\mu(z)\,|G_{12}(z)|^2 dz < \infty, \tag{5.26a}$$

$$G_{12} = G_{22}, \qquad \text{at} \quad z = 0, \tag{5.26b}$$

$$\mu_1 G_{12}' = \mu_2 G_{22}', \qquad \text{at} \quad z = 0, \tag{5.26c}$$

$$G_{22} = G_{32}, \qquad \text{at} \quad z = h, \tag{5.26d}$$

$$\mu_2 G'_{2z} = \mu_3 G'_{3z}, \quad \text{at} \quad z = h, \tag{5.26e}$$

$$\int_0^{\infty} \mu(z) \left| G_{3z}(z) \right|^2 dz < \infty, \tag{5.26f}$$

$$G_{22}(\zeta^+, \zeta; \lambda) = G_{22}(\zeta^-, \zeta; \lambda), \tag{5.26g}$$

and

$$G'_{22}(\zeta^+, \zeta; \lambda) - G'_{22}(\zeta^-, \zeta; \lambda) = 1/\mu_2. \tag{5.26h}$$

we find that

$$G_{1z}(z, \zeta; \lambda) = \frac{P(\zeta) e^{\sigma_1 z}}{\Delta \cos\sigma_2 h}, \tag{5.27}$$

where

$$P(\zeta) = \mu_2 \sigma_2 \cos\sigma_2 (h-\zeta) + \mu_3 \sigma_3 \sin\sigma_2 (h-\zeta). \tag{5.28}$$

$$G_{22}(z, \zeta; \lambda) = \frac{K\mu_1 \sigma_1 \sin\sigma_2 \zeta \sin\sigma_2 z + L\mu_2 \sigma_2 \cos\sigma_2 \zeta \cos\sigma_2 z}{\mu_2 \sigma_2 \Delta \cos\sigma_2 h}$$

$$+ \frac{1}{\mu_2 \sigma_2 \Delta \cos\sigma_2 h} \left[\left\{ K\mu_2 \sigma_2 \sin\sigma_2 \zeta \cos\sigma_2 z + L\mu_1 \sigma_1 \right. \right.$$

$$\left. \times \cos\sigma_2 \zeta \sin\sigma_2 z \right\} H(\zeta - z) + \left\{ K\mu_2 \sigma_2 \sin\sigma_2 z \cos\sigma_2 \zeta \right.$$

$$\left. \left. + L\mu_1 \sigma_1 \sin\sigma_2 \zeta \cos\sigma_2 z \right\} H(z - \zeta) \right], \tag{5.29}$$

here

$$K = \mu_2 \sigma_2 \sin\sigma_2 h - \mu_3 \sigma_3 \cos\sigma_2 h,$$

$$L = \mu_2 \sigma_2 \cos\sigma_2 h + \mu_3 \sigma_3 \sin\sigma_2 h,$$

$$G_{3z}(z, \zeta; \lambda) = \frac{(\mu_2 \sigma_2 \cos\sigma_2 \zeta + \mu_1 \sigma_1 \sin\sigma_2 \zeta) e^{-\sigma_3 (z-h)}}{\Delta \cos\sigma_2 h}. \tag{5.30}$$

(3) If ζ lies in the lower half-space $z > h$, then G_{13}, G_{23} and G_{33} satisfy the differential equations

$$\frac{d^2G_{13}}{dz^2} - \sigma_1^2G_{13} = 0, \tag{5.31}$$

$$\frac{d^2G_{23}}{dz^2} + \sigma_2^2G_{23} = 0, \tag{5.32}$$

$$\frac{d^2G_{33}}{dz^2} - \sigma_3^2G_{33} = \delta(z-\zeta), \tag{5.33}$$

together with the conditions

$$\int_{-\infty}^{0}\mu(z)\,|G_{13}(z)|^2dz < \infty, \tag{5.34a}$$

$$G_{13} = G_{23}, \qquad \text{at} \quad z = 0, \tag{5.34b}$$

$$\mu_1G_{13}' = \mu_2G_{23}', \qquad \text{at} \quad z = 0, \tag{5.34c}$$

$$G_{23} = G_{33}, \qquad \text{at} \quad z = h, \tag{5.34d}$$

$$\mu_2G_{23}' = \mu_3G_{33}', \qquad \text{at} \quad z = h, \tag{5.34e}$$

$$\int_{0}^{\infty}\mu(z)\,|G_{33}(z)|^2dz < \infty, \tag{5.34f}$$

$$G_{33}(\zeta^+,\zeta;\lambda) = G_{33}(\zeta^-,\zeta;\lambda), \tag{5.34g}$$

and the jump condition

$$G_{33}'(\zeta^+,\zeta;\lambda) - G_{33}'(\zeta^-,\zeta;\lambda) = {}^1/\mu_3. \tag{5.34h}$$

It follows that

$$G_{13}(z,\zeta;\lambda) = \frac{\mu_2\sigma_2e^{-\sigma_3(\zeta-h)}e^{\sigma_1z}}{\Delta\cos\sigma_2h}, \tag{5.35}$$

$$G_{23}(z,\zeta;\lambda) = \frac{(\mu_2\sigma_2\cos\sigma_2 z + \mu_1\sigma_1\sin\sigma_2 z)e^{-\sigma_3(\zeta-h)}}{\Delta\cos\sigma_2 h},$$

(5.36)

and

$$G_{33}(z,\zeta;\lambda) = \frac{W}{2\mu_3\sigma_3\Delta} e^{-\sigma_3(z+\zeta-2h)} - \frac{1}{2\mu_3\sigma_3}\Big[e^{\sigma_3(z-\zeta)}$$

$$\times H(\zeta-z) + e^{-\sigma_3(z-\zeta)}H(z-\zeta)\Big], \qquad (5.37)$$

where

$$W = \mu_2\sigma_2(\mu_2\sigma_2\tan\sigma_2 h + \mu_3\sigma_3) + \mu_1\sigma_1(\mu_3\sigma_3\tan\sigma_2 h - \mu_2\sigma_2).$$

(5.38)

We have thus determined $G_{ij}(z,\zeta;\lambda)$ and therefore $G(z,\zeta;\lambda)$. We note that $G_{ij} = G_{ji}$ so that the Green's function is symmetric as should be expected.

5.4 SPECTRAL REPRESENTATION

We now use the following formula (Stakgold [36]) to obtain the discrete and continuous spectrum along with the corresponding eigenfunctions $\{\phi_n(z)\}$, improper eigenfunctions $\{\psi(z,\lambda)\}$ and $\{\chi(z,\lambda)\}$.

$$\lim_{R\to\infty} \frac{-1}{2\pi i} \oint_{|\lambda|=R} G(z,\zeta;\lambda)d\lambda = \sum_n \phi^n(z)\overline{\phi^n(\zeta)} + \int \psi(z,\lambda)\overline{\psi(\zeta,\lambda)}d\lambda$$

$$+ \int \chi(z,\lambda)\overline{\chi(\zeta,\lambda)}d\lambda = \frac{\delta(z-\zeta)}{\mu(\zeta)}.$$

(5.39)

We shall use this formula, step by step, for each $G_{ij}(z,\zeta;\lambda)$.

(i) First, We consider

$$I_{11} = \lim_{R\to\infty} \frac{-1}{2\pi i} \oint_{|\lambda|=R} G_{11}(z,\zeta;\lambda)d\lambda = \lim_{R\to\infty} \frac{-1}{2\pi i} \oint_{|\lambda|=R} \frac{1}{2\mu_1\sigma_1}$$

$$\times \left\{\frac{\square e^{\sigma_1(z+\zeta)}}{\Delta} - e^{-\sigma_1|z-\zeta|}\right\}d\lambda, \qquad (5.40)$$

where $\square$ and Δ are given by equations (5.18) and (5.19) respectively. We notice that $\sigma_1 = 0$ gives rise to the branch point $\lambda = \frac{\omega^2}{\beta_1^2}$, $\sigma_3 = 0$ gives the branch point $\lambda = \frac{\omega^2}{\beta_3^2}$ and the poles of the integrand are roots of the equation

$$\Delta = \mu_2\sigma_2(\mu_2\sigma_2\tan\sigma_2 h-\mu_3\sigma_3) - \mu_1\sigma_1(\mu_3\sigma_3\tan\sigma_2 h+\mu_2\sigma_2) = 0,$$

which is the dispersion relation for Love-type waves propagating in a layer of uniform thickness sandwiched between two half-spaces. The poles are all simple, finite in number, and are located in the open interval $(\frac{\omega^2}{\beta_3^2}, \frac{\omega^2}{\beta_1^2})$.

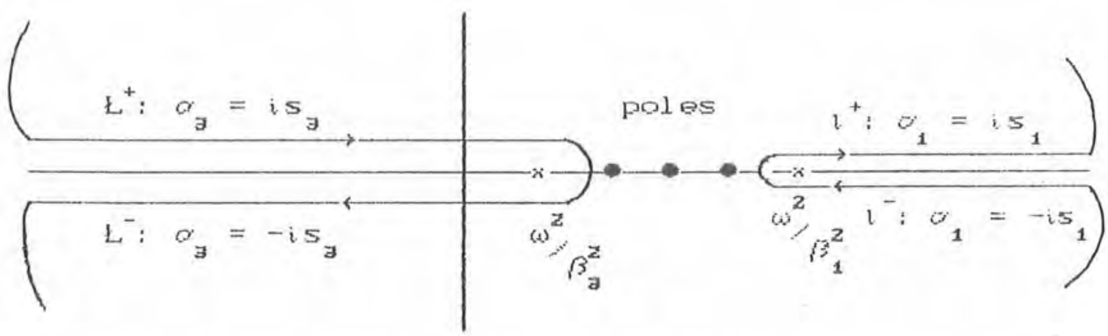

Figure 5.3. The contour of integration in the complex λ-plane.

The set of these poles constitutes the discrete spectrum. The continuous spectrum arises from the integral over the branch cuts at $\frac{\omega^2}{\beta_3^2}$ and $\frac{\omega^2}{\beta_1^2}$. The sum of the residues at the poles $\{\lambda_n\}$ is given by

$$-\sum_{n=1}^{N}\frac{(\square)_{\lambda=\lambda_n}\exp\{\sigma_1^{(n)}(z+\zeta)\}}{2\mu_1\sigma_1^{(n)}\;\frac{\partial}{\partial\lambda}[\Delta]_{\lambda=\lambda_n}} = \sum_{n=1}^{N}\phi_1^{(n)}(z)\phi_1^{(n)}(\zeta), \qquad (5.41)$$

where

$$\sigma_1^{(n)} = \left(\lambda_n - \frac{\omega^2}{\beta_1^2}\right)^{1/2}, \qquad \sigma_2^{(n)} = \left(\frac{\omega^2}{\beta_2^2} - \lambda_n\right)^{1/2},$$

$$\sigma_3^{(n)} = \left(\lambda_n - \frac{\omega^2}{\beta_3^2}\right)^{1/2},$$

and

$$\phi_1^{(n)}(z) = \left[\frac{1}{2\mu_1\sigma_1^{(n)}}\left\{\frac{\square}{\frac{\partial}{\partial\lambda}(-\Delta)}\right\}_{\lambda=\lambda_n}\right]^{1/2}\exp(\sigma_1^{(n)}z). \qquad (5.42)$$

The contribution from the branch cuts for the two branch points $\frac{\omega^2}{\beta_1^2}$ (or $\frac{\omega^2}{\beta_3^2}$) can be evaluated by taking $\mathrm{Re}(\sigma_1) > 0$ $\{\mathrm{Re}(\sigma_3) > 0\}$ on l^+, putting $\sigma_1 = is_1$ on l^+ and $\sigma_1 = -is_1$ on l^- (or $\sigma_3 = is_3$ on L^+ and $\sigma_3 = -is_3$ on L^-), where $s_1 = \left(\frac{\omega^2}{\beta_1^2} - \lambda\right)^{1/2}$ is real and positive for $\lambda < \frac{\omega^2}{\beta_1^2}$ $\left[\text{or } s_3 = \left(\frac{\omega^2}{\beta_3^2} - \lambda\right)^{1/2} \text{ is real and positive for } \lambda < \frac{\omega^2}{\beta_3^2}\right]$.

These contributions respectively, can be shown to be

$$I'_{11} = \int (G^+_{11} - G^-_{11}) d\lambda = -\frac{1}{\pi} \int_{\omega^2/\beta_1^2}^{\infty} \frac{1}{\mu_1 s_1} \sin(\theta + s_1 z)$$

$$\times \sin(\theta + s_1 \zeta) d\lambda = - \int_{\omega^2/\beta_1^2}^{\infty} \psi_1(z,\lambda) \psi_1(\zeta,\lambda) d\lambda, \qquad (5.43)$$

where

$$\theta = \tan^{-1}\left\{ \frac{\mu_1 s_1 (\mu_3 \sigma_3 \tan\sigma_2 h + \mu_2 \sigma_2)}{\mu_2 \sigma_2 (\mu_2 \sigma_2 \tan\sigma_2 h - \mu_3 \sigma_3)} \right\},$$

$$\psi_1(z,\lambda) = G_\lambda \sin(\theta + s_1 z), \qquad G_\lambda = \frac{1}{\sqrt{\pi \mu_1 s_1}},$$

and

$$I''_{11} = -\frac{1}{\pi} \int_{-\infty}^{\omega^2/\beta_3^2} \frac{\mu_2^2 \sigma_2^2 \mu_3 s_3 \exp(\sigma_1 (z+\zeta))}{(p^2+q^2)\cos^2\sigma_2 h} d\lambda,$$

$$= - \int_{-\infty}^{\omega^2/\beta_3^2} \chi_1(z,\lambda) \chi_1(\zeta,\lambda) d\lambda, \qquad (5.44)$$

where

$$p = \cos\varphi = \mu_2 \sigma_2 (\mu_2 \sigma_2 \tan\sigma_2 h - \mu_1 \sigma_1),$$

$$q = \sin\varphi = \mu_3 s_3 (\mu_1 \sigma_1 \tan\sigma_2 h + \mu_2 \sigma_2),$$

$$\chi_1(z,\lambda) = \frac{\mu_2 \sigma_2 \mu_3 s_3 \; e^{\sigma_1 z}}{\sqrt{\pi \mu_3 s_3} \; \cos\sigma_2 h}.$$

Therefore

$$I_{11} = \sum_{n=1}^{N} \phi_1^{(n)}(z)\phi_1^{(n)}(\zeta) - \int_{-\infty}^{\omega^2/\beta_3^2} \chi_1(z,\lambda)\chi_1(\zeta,\lambda)d\lambda$$

$$- \int_{\omega^2/\beta_1^2}^{\infty} \psi_1(z,\lambda)\psi_1(\zeta,\lambda)d\lambda. \tag{5.45}$$

(ii) Next, we consider

$$I_{21} = \lim_{R\to\infty} \frac{-1}{2\pi i} \oint_{|\lambda|=R} G_{21}(z,\zeta;\lambda)d\lambda$$

$$= \lim_{R\to\infty} \frac{-1}{2\pi i} \oint_{|\lambda|=R} \frac{e^{\sigma_1\zeta}\, P(z)}{\Delta\cos\sigma_2 h} d\lambda, \tag{5.46}$$

where P(z) is given by equation (5.21). The branch line contribution at $\lambda = \dfrac{\omega^2}{\beta_1^2}$ (or $\lambda = \dfrac{\omega^2}{\beta_3^2}$) is given by putting $\sigma_1 = is_1$ on l^+ and $\sigma_1 = -is_1$ on l^- (or $\sigma_3 = is_3$ on L^+ and $\sigma_3 = -is_3$ on L^-), so that for $\lambda = \dfrac{\omega^2}{\beta_1^2}$,

$$G_{21}^{+} - G_{21}^{-} = \frac{2i(m\sin s_1\zeta + n\cos s_1\zeta)P(z)}{m^2 + n^2},$$

where

$$m = \mu_2\sigma_2(\mu_2\sigma_2\tan\sigma_2 h - \mu_3\sigma_3), \quad n = \mu_1 s_1(\mu_3\sigma_3\tan\sigma_2 h + \mu_2\sigma_2).$$

Hence

$$I_{21}' = -\frac{1}{\pi}\int_{\omega^2/\beta_1^2}^{\infty} \frac{\sin(\theta+s_1\zeta)P(z)}{\cos\sigma_2 h} d\lambda,$$

$$= -\int_{\omega^2/\beta_1^2}^{\infty} \frac{\sin(\theta+s_1\zeta)}{\sqrt{\pi\mu_1 s_1}} \; \frac{\mu_1 s_1 \; P(z)}{\sqrt{\pi\mu_1 s_1} \; \cos\sigma_2 h} \, d\lambda,$$

$$= -\int_{\omega^2/\beta_1^2}^{\infty} \psi_1(\zeta,\lambda)\psi_2(z,\lambda)d\lambda, \qquad (5.47)$$

here

$$\psi_2(z,\lambda) = G_\lambda \frac{\mu_1 s_1 P(z)}{\cos\sigma_2 h}.$$

For the branch line contribution at $\lambda = \frac{\omega^2}{\beta_3^2}$, we get

$$I''_{21} = -\int_{-\infty}^{\omega^2/\beta_3^2} \left\{\frac{\mu_2\sigma_2\mu_3 s_3}{\sqrt{\pi\mu_3 s_3}} \; \frac{e^{\sigma_1\zeta}}{\cos\sigma_2 h}\right\}\left\{\frac{\mu_3 s_3}{\sqrt{\pi\mu_3 s_3}} \; \frac{Q(z)}{\cos\sigma_2 h}\right\} d\lambda,$$

$$= -\int_{-\infty}^{\omega^2/\beta_3^2} \chi_1(\zeta,\lambda)\chi_2(z,\lambda)d\lambda, \qquad (5.48)$$

where

$$\chi_2(z,\lambda) = \frac{\mu_3 s_3 Q(z)}{\sqrt{\pi\mu_3 s_3} \; \cos\sigma_2 h},$$

$$Q(z) = \mu_2\sigma_2\cos\sigma_2 z + \mu_1\sigma_1\sin\sigma_2 z.$$

The contribution from the poles is given by

$$-\sum_{n=1}^{N} \frac{\exp(\sigma_1^{(n)}\zeta)[P(z)]_{\lambda=\lambda_n}}{\cos\sigma_2^{(n)}h \; \frac{\partial}{\partial\lambda}[\Delta]_{\lambda=\lambda_n}} = \left[\frac{1}{2\mu_1\sigma_1^{(n)}}\left\{\frac{\square}{\frac{\partial}{\partial\lambda}(-\Delta)}\right\}_{\lambda=\lambda_n}\right]^{1/2} e^{\sigma_1^{(n)}\zeta}$$

$$\times \left[2\mu_1 \sigma_1^{(n)} \left\{ \frac{1}{\frac{\partial}{\partial\lambda}(-\Delta)} \right\}_{\lambda=\lambda_n} \right]^{1/2} \frac{[P(z)]_{\lambda=\lambda_n}}{\cos\sigma_2^{(n)} h},$$

$$= \sum_{n=1}^{N} \phi_1^{(n)}(\zeta)\phi_2^{(n)}(z), \tag{5.49}$$

where

$$\phi_2^{(n)}(z) = \left[2\mu_1 \sigma_1^{(n)} \left\{ \frac{1}{\frac{\partial}{\partial\lambda}(-\Delta)} \right\}_{\lambda=\lambda_n} \right]^{1/2} \frac{[P(z)]_{\lambda=\lambda_n}}{\cos\sigma_2^{(n)} h}.$$

Hence

$$I_{21} = \sum_{n=1}^{N} \phi_2^{(n)}(z)\phi_1^{(n)}(\zeta) - \int_{-\infty}^{\omega^2/\beta_3^2} \chi_2(z,\lambda)\chi_1(\zeta,\lambda)d\lambda$$

$$- \int_{\omega^2/\beta_1^2}^{\infty} \psi_2(z,\lambda)\psi_1(\zeta,\lambda)d\lambda. \tag{5.50}$$

(iii) Following a similar procedure we obtain

$$I_{31} = \lim_{R\to\infty} \frac{-1}{2\pi i} \oint_{|\lambda|=R} G_{31}(z,\zeta;\lambda)d\lambda,$$

$$= \lim_{R\to\infty} \frac{-1}{2\pi i} \oint_{|\lambda|=R} \frac{\mu_2\sigma_2 \; e^{\sigma_1\zeta} \; e^{-\sigma_3(z-h)}}{\Delta\cos\sigma_2 h} d\lambda. \tag{5.51}$$

The poles are zeros of Δ as above while the branch points are also the same. We can calculate the residues at poles and the branch points contribution to write

$$I_{31} = \sum_{n=1}^{N} \phi_3^{(n)}(z)\phi_1^{(n)}(\zeta) - \int_{-\infty}^{\omega^2/\beta_3^2} \chi_3(z,\lambda)\chi_1(\zeta,\lambda)d\lambda$$

$$- \int_{\omega^2/\beta_1^2}^{\infty} \psi_3(z,\lambda)\psi_1(\zeta,\lambda)d\lambda, \tag{5.52}$$

where

$$\phi_3(z) = \left[2\mu_1\sigma_1^{(n)}\left\{\frac{1}{\square\,\frac{\partial}{\partial\lambda}(-\Delta)}\right\}_{\lambda=\lambda_n}\right]^{1/2} \frac{\mu_2\sigma_2^{(n)}\ e^{-\sigma_3^{(n)}(z-h)}}{\cos\sigma_2^{(n)}h},$$

$$\chi_3(z,\lambda) = \frac{\sin\{\varphi - s_3(z-h)\}}{\sqrt{\pi\mu_3 s_3}},$$

$$\psi_3(z,\lambda) = G_\lambda \frac{\mu_2\sigma_2\mu_1 s_1 e^{-\sigma_3(z-h)}}{\cos\sigma_2 h},$$

$$\varphi = \tan^{-1}(q/p).$$

All the other integrals can be manipulated in the same manner as (i) to (iii). Thus we can write

$$I_{ij} = \lim_{R\to\infty} \frac{-1}{2\pi i}\oint_{|\lambda|=R} G_{ij}(z,\zeta;\lambda)d\lambda = \sum_{n=1}^{N}\phi_i^{(n)}(z)\phi_j^{(n)}(\zeta)$$

$$- \int_{-\infty}^{\omega^2/\beta_3^2} \chi_i(z,\lambda)\chi_j(\zeta,\lambda)d\lambda - \int_{\omega^2/\beta_1^2}^{\infty} \psi_i(z,\lambda)\psi_j(\zeta,\lambda)d\lambda, \tag{5.53}$$

$(i,j=1,2,3)$ with G_{ij} given by equations (5.27), (5.29), (5.30), (5.35), (5.36) and (5.37).

integrating with respect to ζ from $-\infty$ to ∞

$$\int_{-\infty}^{\infty} f(\zeta)\delta(z-\zeta)d\zeta = \sum_{n=1}^{N} \phi^{(n)}(z)\int_{-\infty}^{\infty}\mu(\zeta)f(\zeta)\phi^{(n)}(\zeta)d\zeta$$

$$- \int_{-\infty}^{\omega^2/\beta_3^2} \chi(z,\lambda)d\lambda \int_{-\infty}^{\infty}\mu(\zeta)f(\zeta)\chi(\zeta,\lambda)d\zeta$$

$$- \int_{\omega^2/\beta_1^2}^{\infty} \psi(z,\lambda)d\lambda \int_{-\infty}^{\infty}\mu(\zeta)f(\zeta)\psi(\zeta,\lambda)d\zeta. \tag{5.58}$$

If we write

$$f_n = \langle f, \phi^{(n)}\rangle = \int_{-\infty}^{\infty}\mu(\zeta)f(\zeta)\phi^{(n)}(\zeta)d\zeta, \tag{5.59}$$

$$f_\alpha = \langle f, \chi(\zeta,\lambda)\rangle = \int_{-\infty}^{\infty}\mu(\zeta)f(\zeta)\chi(\zeta,\lambda)d\zeta, \tag{5.60}$$

$$f_\beta = \langle f, \psi(\zeta,\lambda)\rangle = \int_{-\infty}^{\infty}\mu(\zeta)f(\zeta)\psi(\zeta,\lambda)d\zeta, \tag{5.61}$$

then equation (5.58) becomes

$$f(z) = \sum_{n=1}^{N} f_n \, \phi^{(n)}(z) - \int_{-\infty}^{\omega^2/\beta_3^2} f_\alpha \, \chi(z,\lambda)dz - \int_{\omega^2/\beta_1^2}^{\infty} f_\beta \, \psi(z,\lambda)dz. \tag{5.62}$$

In particular, we can write the following orthonormality relations:

$$\int_{-\infty}^{\infty}\mu(z)\phi^{(m)}(z)\phi^{(n)}(z)dz = \langle \phi^{(m)}, \phi^{(n)}\rangle = \delta_{mn},$$

$$1 \le m,\ n \le N. \tag{5.63}$$

From equations (5.39) and (5.53), we obtain the following representation of the delta function:

$$\delta(z-\zeta) = \sum_{n=1}^{N} \mu(\zeta)\phi^{(n)}(z)\phi^{(n)}(\zeta) - \int_{-\infty}^{\omega^2/\beta_3^2} \mu(\zeta)\chi(z,\lambda)\chi(\zeta,\lambda)d\lambda$$

$$- \int_{\omega^2/\beta_1^2}^{\infty} \mu(\zeta)\psi(z,\lambda)\psi(\zeta,\lambda)d\lambda, \qquad (5.54)$$

where $\mu(\zeta)$ is given by equation (5.2) and

$$\left.\begin{aligned} \phi^{(n)}(z) &= \phi_1^{(n)}(z), && z < 0, \\ &= \phi_2^{(n)}(z), && 0 \le z \le h, \\ &= \phi_3^{(n)}(z), && z > h, \end{aligned}\right\} \qquad (5.55)$$

are the normalized eigenfunctions. The improper eigenfunctions are

$$\left.\begin{aligned} \chi(z,\lambda) &= \chi_1(z,\lambda), && z < 0, \\ &= \chi_2(z,\lambda), && 0 \le z \le h, \\ &= \chi_3(z,\lambda), && z > h, \end{aligned}\right\} \qquad (5.56)$$

and

$$\left.\begin{aligned} \psi(z,\lambda) &= \psi_1(z,\lambda), && z < 0, \\ &= \psi_2(z,\lambda), && 0 \le z \le h, \\ &= \psi_3(z,\lambda), && z > h. \end{aligned}\right\} \qquad (5.57)$$

If f(z) is of finite μ-norm over the integral $(-\infty,\infty)$, then the representation of f(z) in terms of eigenfunctions $\{\phi^{(n)}(z)\}$ and improper eigenfunctions $\chi(z,\lambda)$ and $\psi(z,\lambda)$ can be obtained by multiplying equation (5.54) by $f(\zeta)$ and

$$\int_{-\infty}^{\infty}\mu(z)\,\chi(z,\lambda)\,\chi(z,\lambda')\,dz = \langle\chi(z,\lambda),\chi(z,\lambda')\rangle = \delta(\lambda-\lambda'),$$

$$-\infty < \lambda,\ \lambda' < \omega^2/\beta_3^2. \qquad (5.64)$$

$$\int_{-\infty}^{\infty}\mu(z)\,\psi(z,\lambda)\,\psi(z,\lambda')\,dz = \langle\psi(z,\lambda),\ \psi(z,\lambda')\rangle = \delta(\lambda-\lambda'),$$

$$\omega^2/\beta_1^2 \le \lambda,\ \lambda' < \infty. \qquad (5.65)$$

$$\int_{-\infty}^{\infty}\mu(z)\,\phi^m(z)\,\chi(z,\lambda)\,dz = 0 = \langle\phi^m,\chi\rangle,\quad 1 \le m \le N,$$

$$-\infty < \lambda,\ \lambda' < \omega^2/\beta_3^2. \qquad (5.66)$$

$$\int_{-\infty}^{\infty}\mu(z)\,\phi^m(z)\,\psi(z,\lambda)\,dz = 0 = \langle\phi^m,\psi\rangle,\quad 1 \le m \le N,$$

$$\omega^2/\beta_1^2 \le \lambda,\ \lambda' < \infty. \qquad (5.67)$$

$$\int_{-\infty}^{\infty}\mu(z)\,\chi(z,\lambda)\,\psi(z,\lambda)\,dz = 0 = \langle\chi,\psi\rangle,\ -\infty < \lambda < \omega^2/\beta_3^2,$$

$$\omega^2/\beta_1^2 < \lambda < \infty. \qquad (5.68)$$

CONCLUSION

I have discussed some problems in the theory of Love-type wave propagation in a trapped layer. The trapped layer exists between two half-spaces. The Love wave is assumed to be generated by a line source (Cylindrical wave) at an interface between a lower half-space and a layer. The dispersion relations are calculated by introducing inhomogeneity in the lower half-space and the layer respectively. These inhomogeneities are introduced in terms of density variation and rigidity variation. Further, the transmitted wave has also been obtained in the layer. The discussion is then carried over to the most general case of the point source (Spherical wave) excitation of Love waves. For this a layer over an inhomogeneous half-space model is considered. A point source is assumed at the interface between the half-space and the layer. A simple procedure is devised to construct the solution for the point source from the line source solution obtained earlier. The dispersion relation and the transmitted wave are then calculated in the layer. This method can be employed to other configurations such as trapped layer model and the wave propagation phenomena in acoustics. In this way a whole range of problems starting from a plane wave to line source and point source are presented systematically for a trapped layer model introducing inhomogeneity in the trapped layer and the half-space through density and rigidity variations.

In many problems, we need to express the displacement of field functions on either side of discontinuity in the earth crust in terms of complete set of eigenfunctions (proper or improper). This can be achieved through the spectral representation of Love-type operator associated with monochromatic SH wave. Thus, the spectral representation enables to tackle the problems associated with transmission and reflection of Love waves at a horizontally discontinuous change either in elevation or in material properties of the trapped material. This in mind, the spectral representation is obtained for the two dimensional Love-type operator associated with monochromatic SH waves for the trapped layer model.

In a nutshell, the following observations have been made:

(a) The dispersion relations and the transmitted Love wave generated by a line source at the interface between the half-space and the layer are obtained for a trapped layer model, when the lower half-space and the layer are taken to be inhomogeneous in terms of density and rigidity variations.

(b) The dispersion relations and transmitted wave are obtained when a point source is situated at the interface between an inhomogeneous half-space and a layer. The analysis can be extended to a trapped layer model.

(c) The spectral representation of a Love-type operator for a trapped layer model is obtained.

Thus, it is hoped that a complete picture of wave propagation phenomena for a trapped layer model for line source/point source excitation of Love-type waves introducing inhomogeneities has been achieved in this thesis.

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Dispersion of Love-Type Waves in a Vertically Inhomogeneous Intermediate Layer

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The problem of excitation of Love-type waves in an inhomogeneous layer lying between two half-spaces is studied. Using the Fourier transform and Green's function method, the dispersion relation for propagation of such waves is derived. Finally, the transmitted wave in the layer is presented.

1. Introduction

The propagation of Love waves in an inhomogeneous layer is of considerable importance in earthquake engineering and seismology because of occurrence of inhomogeneities in the crust of the earth. Many authors have studied propagation of Love waves in different media. A few interesting papers are by Sezawa (1935), Lapwood (1948), Takahashi (1955), Harkrider (1964). Hudson (1962) considered different models of a layer changing either density or rigidity and established existence of Love waves in each case. Ghosh (1970) showed that a point source can produce Love waves in each case. Ghosh (1970) showed that a point source can produce Love waves in a homogeneous layer over an inhomogeneous half-space. Chattopadhyay *et al.* (1984) employed Ghosh's method (1963, 1970) to study Love waves excited by a source in a layer overlying an inhomogeneous half-space. The problem of propagation of Love-type waves in a layer deep in the earth, where the layer is either porous or inhomogeneous, has been studied by Paul (1964), Chattopadhyay (1975), and Kar (1977) among other. Sinah (1966) also considered this problem for variable rigidity in the layer but he reduced it to a Bessel's equation. The analysis, however, in this case becomes a little complicated.

In this paper, we have solved the problem of propagation of Love-type waves in a layer of variable rigidity lying between two homogeneous half-spaces. The Love-type waves are excited in the layer due to the presence of a line source at the interface of the layer and the lower half-space. We use the Fourier transform and Green's function method to derive dispersion equation for Love-type waves excited by the source. In the end, the transmitted wave in the layer is calculated.

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2. Formulation of the Problem

We consider an elastic layer $0 \leq z \leq H$ of uniform thickness H, with upper and lower homogeneous half-spaces given by $z<0$ and $z>H$, respectively. A line source of disturbance is situated at $x=0$ and $z=H$. The geometry of the problem is shown in Fig. 1. Subscripts 1, 2, and 3 refer, respectively, to the upper medium, intermediate layer, and the lower substratum. Assuming that the source is time harmonic and taking the time dependence $e^{i\omega t}$ to be understood throughout, we can write the equation of motion for the inhomogeneous layer as

$$\frac{\partial p_{xy}}{\partial x}+\frac{\partial p_{yz}}{\partial z}+\rho_2\omega^2 v_2=4\pi\delta(x)\delta(z-H)\,, \tag{1}$$

where v_2 is displacement component, P_{xy}, P_{yz} are stress components, ρ_2 is the density of the layer, and ω is the angular frequency. We suppose that in the inhomogeneous layer, the coefficient of rigidity μ_2 varies as $\mu_2+\varepsilon z$, ε being a small parameter. Then Eq. (1) becomes

$$\mu_2\frac{\partial^2 v_2}{\partial x^2}+\varepsilon z\frac{\partial^2 v_2}{\partial x^2}+\mu_2\frac{\partial^2 v_2}{\partial z^2}+\varepsilon z\frac{\partial^2 v_2}{\partial z^2}+\varepsilon\frac{\partial v_2}{\partial z}+\rho_2\omega^2 v_2=4\pi\delta(x)\delta(z-H)\,. \tag{2}$$

Dividing Eq. (2) throughout by μ_2 and re-arranging, we have

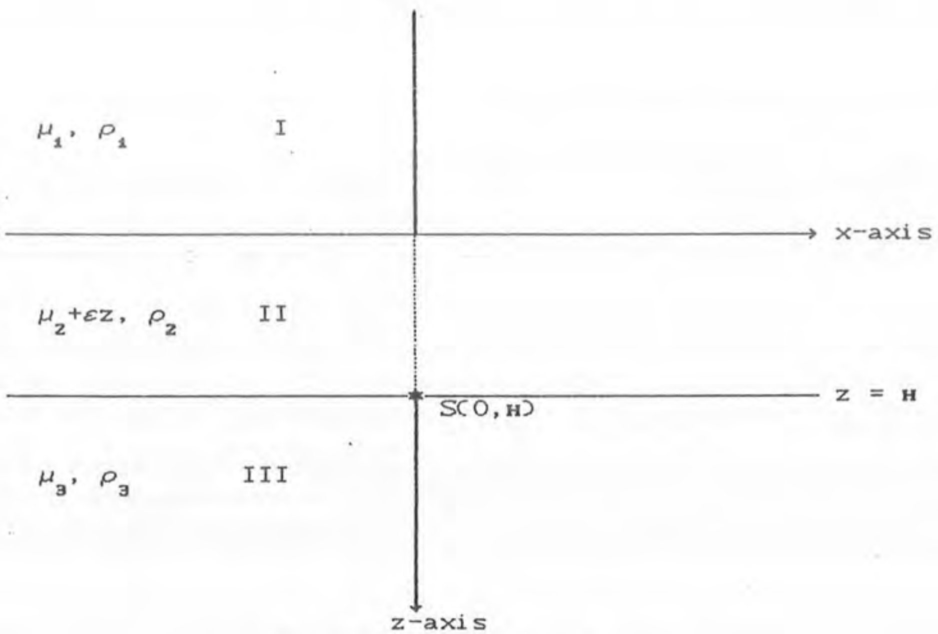

Fig. 1. Geometry of the problem.

$$\frac{\partial^2 v_2}{\partial x^2}+\frac{\partial^2 v_2}{\partial z^2}+k_2^2 v_2=\frac{4\pi}{\mu_2}\delta(x)\delta(z-H)-\frac{\varepsilon z}{\mu_2}\frac{\partial^2 v_2}{\partial z^2}-\frac{\varepsilon}{\mu_2}\frac{\partial v_2}{\partial z}-\frac{\varepsilon z}{\mu_2}\frac{\partial^2 v_2}{\partial x^2}. \tag{3}$$

The equation of motion for the upper and lower semi-infinite media are

$$\frac{\partial^2 v_1}{\partial x^2}+\frac{\partial^2 v_1}{\partial z^2}+k_1^2 v_1=0, \tag{4}$$

$$\frac{\partial^2 v_3}{\partial x^2}+\frac{\partial^2 v_3}{\partial z^2}+k_3^2 v_3=0, \tag{5}$$

where

$$k_i^2=\frac{\rho_{i\omega}^2}{\mu_i}, \qquad i=1,2,3.$$

The geometry of the problem leads to the following boundary conditions:

(a) at $z=0$, $-\infty<x<\infty$,

$$v_1=v_2,$$

$$\mu_1\frac{\partial v_1}{\partial z}=\mu_2\frac{\partial v_2}{\partial z}. \tag{6a}$$

(b) at $z=H$, $-\infty<x<\infty$,

$$v_2=v_3,$$

$$(\mu_2+\varepsilon H)\frac{\partial v_2}{\partial z}=\mu_3\frac{\partial v_3}{\partial z}. \tag{6b}$$

(c) $v_1\to 0$ as $z\to-\infty$,

$$v_3\to 0 \text{ as } z\to\infty. \tag{6c}$$

3. Solution of the Problem

In order to find the solution of Eqs. (3) to (5), the following transforms are used:

$$\left.\begin{aligned} V(\xi,z)&=\frac{1}{2\pi}\int_{-\infty}^{\infty} v(x,z)e^{i\xi x}dx,\\ v(x,z)&=\int_{-\infty}^{\infty} V(\xi,z)e^{-i\xi x}d\xi. \end{aligned}\right\} \tag{7}$$

Using these transform, the equations of motion—Eqs. (3), (4), and (5)—reduce to

$$\frac{d^2V_2}{dz^2}-\alpha_2^2 V_2=\frac{2}{\mu_2}\delta(z-H)-\frac{\varepsilon z}{\mu_2}\frac{d^2V_2}{dz^2}-\frac{\varepsilon}{\mu_2}\frac{dV_2}{dz}+\frac{\varepsilon z}{\mu_2}\xi^2 V_2=\sigma_2(z) \text{ (say)} \tag{8}$$

for the layer and

$$\frac{d^2V_{1,3}}{dz^2}-\alpha_{1,3}^2V_{1,3}=0\,, \tag{9}$$

for the upper and lower substrata. In Eqs. (8) and (9)

$$\alpha_i^2=\xi^2-k_i^2\,. \tag{10}$$

We suppose that Green's function for the inhomogeneous Eq. (8) is $G_2(z/z_0)$. The equation satisfied by $G_2(z/z_0)$ is

$$\frac{d^2G_2(z/z_0)}{dz^2}-\alpha_2^2G_2(z/z_0)=\delta(z-z_0)\,, \tag{11a}$$

together with homogeneous boundary conditions

$$\frac{dG_2(z/z_0)}{dz}=0\,,\qquad \text{at}\quad z=0\,,\quad H\,. \tag{11b}$$

Here z_0 is an arbitrary point in the medium 2. Multiplying Eq. (8) by $G_2(z/z_0)$, Eq. (11a) by $V_2(\xi, z)$, subtracting and integrating over $0\leq z\leq H$, we obtain

$$G_2(H/z_0)\left[\frac{dV_2}{dz}\right]_{z=H}-G_2(0/z_0)\left[\frac{dV_2}{dz}\right]_{z=0}=\int_0^H\sigma_2(z)G_2(z/z_0)dz-V_2(z_0)\,. \tag{12}$$

Similarly, we suppose that $G_1(z/z_0)$ and $G_3(z/z_0)$ are Green's functions corresponding to upper and lower media satisfying

$$\frac{dG_1(z/z_0)}{dz}=0\quad \text{at}\quad z=0\,;\qquad \frac{dG_1(z/z_0)}{dz}\to 0\quad \text{as}\quad z\to-\infty\,,$$

and

$$\frac{dG_3(z/z_0)}{dz}=0\quad \text{at}\quad z=H\,;\quad \frac{dG_3(z/z_0)}{dz}\to 0\quad \text{as}\quad z\to\infty\,.$$

Adopting the same procedure as above, we get

$$G_1(0/z_0)\left[\frac{dV_1}{dz}\right]_{z=0}=-V_1(z_0)\,, \tag{13}$$

$$G_3(H/z_0)\left[\frac{dV_3}{dz}\right]_{z=H}=V_3(z_0)\,. \tag{14}$$

Replacing by z by z_0 and using symmetry of Green's function, Eqs. (12) to (14) take the form

$$V_2(z)=G_2(z/0)\left[\frac{dV_2}{dz}\right]_{z=0}-G_2(z/H)\left[\frac{dV_2}{dz}\right]_{z=H}+\int_0^H\sigma_2(z_0)G_2(z/z_0)dz_0\,, \tag{15}$$

$$V_1(z) = -G_1(z/0)\left[\frac{dV_1}{dz}\right]_{z=0}, \tag{16}$$

$$V_3(z) = G_3(z/H)\left[\frac{dV_3}{dz}\right]_{z=H}. \tag{17}$$

Using the boundary conditions Eq. (6a) in Eqs. (15) and (16), we obtain

$$\left[\frac{dV_2}{dz}\right]_{z=0} = \frac{1}{A}\left\{G_2(0/H)\left[\frac{dV_2}{dz}\right]_{z=H} - \int_0^H \sigma_2(z_0)G_2(0/z_0)dz_0\right\}, \tag{18}$$

where

$$A = G_2(0/0) + \frac{\mu_2}{\mu_1}G_1(0/0).$$

Similarly Eq. (6b) yields

$$\left[\frac{dV_2}{dz}\right]_{z=H} = \frac{1}{\{AB - G_2^2(H/0) + \frac{\varepsilon H}{\mu_3}AG_3(H/H)\}}\left\{-G_2(\mathrm{H}/0)\int_0^H \sigma_2(z_0)G_2(0/z_0)dz_0 + A\int_0^H \sigma_2(z_0)G_2(H/z_0)dz_0\right\}, \tag{19}$$

where

$$B = G_2(H/H) + \frac{\mu_2}{\mu_3}G_3(H/H).$$

Using Eqs. (18) and (19) in Eq. (15), substituting back value of $\sigma_2(z_0)$ and using property of the delta function, we get

$$V_2(z) = \frac{2(\mu_2 + \varepsilon H)}{\mu_2\mu_3}\left\{\frac{G_2(z/H)C - G_2(z/0)D}{AB - G_2^2(H/0) + \frac{\varepsilon H}{\mu_3}AG_3(H/H)}\right\}$$
$$-\left\{\frac{G_2(z/H)G_2(H/0) - \left\{B + \frac{\varepsilon H}{\mu_3}G_3(H/H)\right\}G_2(z/0)}{AB - G_2^2(H/0) + \frac{\varepsilon H}{\mu_3}AG_3(H/H)}\right\}\frac{\varepsilon}{\mu_2}$$
$$\times \int_0^H \left\{z_0\frac{d^2V_2(z_0)}{dz_0^2} + \frac{dV_2(z_0)}{dz_0} - z_0\xi^2V_2(z_0)\right\}G_2(0/z_0)dz_0$$

$$-\left\{\frac{G_2(z/0)G_2(H/0)-G_2(z/H)A}{AB-G_2^2(H/0)+\frac{\varepsilon H}{\mu_3}AG_3(H/H)}\right\}\frac{\varepsilon}{\mu_2}\int_0^H\left\{z_0\frac{\mathrm{d}^2V_2(z_0)}{\mathrm{d}z_0^2}\right.$$

$$\left.+\frac{\mathrm{d}V_2(z_0)}{\mathrm{d}z_0}-z_0\xi^2V_2(z_0)\right\}G_2(H/z_0)\mathrm{d}z_0-\frac{\varepsilon}{\mu_2}\int_0^H\left\{z_0\frac{\mathrm{d}^2V_2(z_0)}{\mathrm{d}z_0^2}\right.$$

$$\left.+\frac{\mathrm{d}V_2(z_0)}{\mathrm{d}z_0}-z_0\xi^2V_2(z_0)\right\}G_2(z/z_0)\mathrm{d}z_0\,, \tag{20}$$

where

$$C=G_3(H/H)A\,,\qquad D=G_3(H/H)G_2(H/0)\,.$$

Equation (20) is an integral equation and $V_2(z)$ may be determined from this equation using successive substitution. As a first approximation, we neglect terms involving ε to obtain

$$V_2(z)=\frac{2}{\mu_3}\left\{\frac{G_2(z/H)C-G_2(z/0)D}{AB-G_2^2(H/0)}\right\}. \tag{21}$$

Substituting this value back in the right side of Eq. (20), we obtain $V_2(z)$ in the following form:

$$V_2(z)=\frac{2(\mu_2+\varepsilon H)}{\mu_2\mu_3}\left\{\frac{G_2(z/H)C-G_2(z/0)D}{AB-G_2^2(H/0)+\frac{\varepsilon H}{\mu_3}AG_3(H/H)}\right\}$$

$$-\frac{2\varepsilon}{\mu_2\mu_3}\left\{\frac{G_2(z/H)G_2(H/0)-\left\{B+\frac{\varepsilon H}{\mu_3}G_3(H/H)\right\}G_2(z/0)}{AB-G_2^2(H/0)+\frac{\varepsilon H}{\mu_3}AG_3(H/H)}\right\}$$

$$\times\int_0^H\left\{z_0\frac{\mathrm{d}^2M(z_0)}{\mathrm{d}z_0^2}+\frac{\mathrm{d}M(z_0)}{\mathrm{d}z_0}-z_0\xi^2M(z_0)\right\}G_2(0/z_0)\mathrm{d}z_0$$

$$-\frac{2\varepsilon}{\mu_2\mu_3}\left\{\frac{G_2(z/0)G_2(H/0)-G_2(z/H)A}{AB-G_2^2(H/0)+\frac{\varepsilon H}{\mu_3}AG_3(H/H)}\right\}\int_0^H\left\{z_0\frac{\mathrm{d}^2M(z_0)}{\mathrm{d}z_0^2}+\frac{\mathrm{d}M(z_0)}{\mathrm{d}z_0}\right.$$

$$\left.-z_0\xi^2M(z_0)\right\}G_2(H/z_0)\mathrm{d}z_0-\frac{2\varepsilon}{\mu_2\mu_3}\int_0^H\left\{z_0\frac{\mathrm{d}^2M(z_0)}{\mathrm{d}z_0^2}+\frac{\mathrm{d}M(z_0)}{\mathrm{d}z_0}\right.$$

$$\left.-z_0\xi^2M(z_0)\right\}G_2(z/z_0)\mathrm{d}z\,, \tag{22}$$

where

$$M(z_0)=\left\{\frac{G_2(z_0/H)C-G_2(z_0/0)D}{AB-G_2^2(H/0)}\right\}.$$

We note that $V_2(z)$ is completely detrermined through Eq. (22), provided G_1, G_2, and G_3 are known.

In order to determine these Green's functions, we follow the procedure outlined by Stakgold (1979). Thus Eq. (11a, b) would give us

$$G_2(z/z_0)=-\frac{1}{2\alpha_2}\left[e^{-\alpha_2|z-z_0|}+e^{\alpha_2 z}\left\{\frac{e^{-\alpha_2(H+z_0)}+e^{-\alpha_2(H-z_0)}}{e^{\alpha_2 H}-e^{-\alpha_2 H}}\right\}\right.$$
$$\left.+e^{-\alpha_2 z}\left\{\frac{e^{\alpha_2(H-z_0)}+e^{-\alpha_2(H-z_0)}}{e^{\alpha_2 H}-e^{-\alpha_2 H}}\right\}\right]. \tag{23}$$

Similarly

$$G_1(z/z_0)=-\frac{1}{2\alpha_1}\left[e^{-\alpha_1|z-z_0|}+e^{\alpha_1(z+z_0)}\right], \tag{24}$$

and

$$G_3(z/z_0)=-\frac{1}{2\alpha_3}\left[e^{-\alpha_3|z-z_0|}+e^{-\alpha_3(z+z_0-2H)}\right]. \tag{25}$$

Using Eqs. (23), (24), and (25) in Eq. (22), simplifying and neglecting square and higher powers of ε, we arrive at

$$V_2(z)=\frac{-2(\mu_2\alpha_2\cosh\alpha_2 z+\mu_1\alpha_1\sinh\alpha_2 z)}{\mu_1\mu_3\alpha_1\alpha_2^2\alpha_3\{AB-G_2^2(H/0)\}E(\varepsilon)\sinh\alpha_2 H}, \tag{26}$$

where

$$E(\varepsilon)=1+\frac{\varepsilon}{4\{AB-G_2^2(H/0)\}}\left[\frac{(\mu_3\alpha_3-\mu_1\alpha_1)}{\mu_1\mu_3\alpha_1\alpha_2^2\alpha_3}\right.$$
$$-\frac{H\{(5\mu_1\mu_3\alpha_1\alpha_3+3\mu_2^2\alpha_2^2)+\mu_2\alpha_2(3\mu_1\alpha_1+5\mu_3\alpha_3)\coth\alpha_2 H\}}{\mu_1\mu_2\mu_3\alpha_1\alpha_2^2\alpha_3}$$
$$-\frac{H^2\{\mu_2\alpha_2(\mu_1\alpha_1+\mu_3\alpha_3)+(\mu_1\mu_3\alpha_1\alpha_3+\mu_2^2\alpha_2^2)\coth\alpha_2 H\}}{\mu_1\mu_2\mu_3\alpha_1\alpha_2\alpha_3}$$
$$-\frac{H\xi^2\{(\mu_1\mu_3\alpha_1\alpha_3-\mu_2^2\alpha_2^2)+\mu_2\alpha_2(\mu_3\alpha_3-\mu_1\alpha_1)\coth\alpha_2 H\}}{\mu_1\mu_2\mu_3\alpha_1\alpha_2^4\alpha_3}$$
$$+\frac{H^2\xi^2\{\mu_2\alpha_2(\mu_1\alpha_1+\mu_3\alpha_3)+(\mu_1\mu_3\alpha_1\alpha_3+\mu_2^2\alpha_2^2)\coth\alpha_2 H\}}{\mu_1\mu_2\mu_3\alpha_1\alpha_2^3\alpha_3}$$
$$\left.+\frac{\xi^2(\mu_3\alpha_3-\mu_1\alpha_1)}{\mu_1\mu_3\alpha_1\alpha_2^4\alpha_3}\right].$$

4. Transmitted Waves

Taking inverse Fourier transform of Eq. (26), the displacement in the intermediate layer is

$$v_2(x, z)=\int_{ic-\infty}^{ic+\infty} \frac{-2(\mu_2\alpha_2 \cosh \alpha_2 z+\mu_1\alpha_1 \sinh \alpha_2 z)e^{-i\xi x}}{F(\xi, H)} d\xi , \tag{27}$$

where

$$F(\xi, H)=\mu_1\mu_3\alpha_1\alpha_2^2\alpha_3\{AB-G_2^2(H/0)\}E(\varepsilon) \sinh \alpha_2 H .$$

In the expression of $v_2(x, z)$, the contour integration is to be performed. The poles of the integrand are obtained by equating the expression in the denominator to zero. The resultant relation gives us the dispersion relation that the Lover-type wave propagating in the inhomogeneous layer must satisfy. Replacing α_2 by $i\hat{\alpha}_2$, the dispersion relation in our case can be reduced to the form

$$\begin{aligned}\tan \hat{\alpha}_2 H=&\frac{\mu_2\hat{\alpha}_2(\mu_1\alpha_1+\mu_3\alpha_3)}{\mu_2^2\hat{\alpha}_2^2-\mu_1\mu_3\alpha_1\alpha_3}+\frac{\varepsilon}{4(\mu_2^2\hat{\alpha}_2^2-\mu_1\mu_3\alpha_1\alpha_3)}\Big[(\mu_3\alpha_3-\mu_1\alpha_1)\tan \hat{\alpha}_2 H\\
&-\frac{H}{\mu_2}\{(5\mu_1\mu_3\alpha_1\alpha_3-3\mu_2^2\hat{\alpha}_2^2)\tan \hat{\alpha}_2 H+\mu_2\hat{\alpha}_2(3\mu_1\alpha_1+5\mu_3\alpha_3)\}\\
&-\frac{H^2\hat{\alpha}_2}{\mu_2}\{(\mu_1\mu_3\alpha_1\alpha_3-\mu_2^2\hat{\alpha}_2^2)-\mu_2\hat{\alpha}_2(\mu_1\alpha_1+\mu_3\alpha_3)\tan \hat{\alpha}_2 H\}\\
&+\frac{H\xi^2}{\mu_2\hat{\alpha}_2^2}\{(\mu_1\mu_3\alpha_1\alpha_3+\mu_2^2\hat{\alpha}_2^2)\tan \hat{\alpha}_2 H+\mu_2\hat{\alpha}_2(\mu_3\alpha_3-\mu_1\alpha_1)\}\\
&-\frac{H^2\xi^2}{\mu_2\hat{\alpha}_2}\{(\mu_1\mu_3\alpha_1\alpha_3-\mu_2^2\hat{\alpha}_2^2)-\mu_2\hat{\alpha}_2(\mu_1\alpha_1+\mu_3\alpha_3)\tan \hat{\alpha}_2 H\}\\
&-\frac{\xi^2}{\hat{\alpha}_2^2}(\mu_3\alpha_3-\mu_1\alpha_1)\tan \hat{\alpha}_2 H\Big] .\end{aligned} \tag{28}$$

It is imperative to note that in the case of homogeneous medium $\varepsilon=0$ and this dispersion relation reduces to the dispersion relation obtained by Ewing *et al.* (1957) for the homogeneous case.

In order to obtain the transmitted wave, we need to calculate the integral in Eq. (27). For that, we note that the poles of the integrand are roots $P_{2,n}$ $(n=1, 2, 3, \cdots)$ of

$$F(\xi, H)=\mu_1\mu_3\alpha_1\alpha_2^2\alpha_3\{AB-G_2^2(H/0)\}E(\varepsilon) \sinh \alpha_2 H .$$

Calculating the pole contribution at these poles, we find that

$$v_2(x, z)=2\pi \sum_{n=1}^{\infty} \frac{\exp(-ip_{2,n}x)\{\mu_2\hat{\alpha}_{2,n}\cos \hat{\alpha}_{2,n}z+\mu_1\alpha_{1,n}\sin \hat{\alpha}_{2,n}z\}}{\left.\dfrac{dF(\xi, H)}{d\xi}\right|_{\xi=p_{2,n}}} \tag{29}$$

where

$$\hat{\alpha}_2 \big|_{\xi=p_{2,n}} = \hat{\alpha}_{2,n}, \qquad \alpha_1 \big|_{\xi=p_{2,n}} = \alpha_{1,n}.$$

Equation (29) represents the travelling wave in the layer in the direction of x-axis.

Besides the poles, we have the branch points which give rise to the body waves and are of no interest to us for the present study. However, these calculations can be easily made with the help of the saddle point method considering the asymptotic behavior of the integrals.

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Field Due to a Point Source in a Layer over an Inhomogeneous Medium (*).

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Summary. — A point source at an interface between a homogeneous layer of finite depth and an inhomogeneous half-space is considered. The transmitted wave and the dispersion relation for the Love waves is calculated analytically.

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1. - Introduction.

Shear and explosive point sources have been used extensively in geophysics [1-4]. The behaviour of the medium in the neighbourhood of an underground explosion can be regarded as perfectly elastic outside a certain sphere surrounding the source. A thorough understanding of motion in an elastic medium subject to an applied force is an essential part of wave propagation theory needed to interpret seismic waves.

The study of propagation of SH waves due to a line source in a homogeneous layer overlying an inhomogeneous half-space has been discussed by Chattopadhyay [5]. He used the method developed by Ghosh [6] and calculated the dispersion relation for Love waves due to the presence of the inhomogeneity in the lower medium. Ghosh [7] solved the problem of propagation of Love waves excited by a line source lying at the interface of the two media by allowing either the layer or the lower substratum to be inhomogeneous. Sato [8] solved the problem of Love waves in case the surface layer is variable in thickness. Kazi [9] discussed the problem of diffraction of Love waves by perfectly rigid and perfectly weak half-plane located in a layer overlying an elastic half-space.

We observe that Chattopadhyay [5] has discussed dispersion relation in the layer for the Love waves without calculating the transmitted field. In this paper, Chattopadhyay's [5] problem has been completed in the sense that the analytic solution of the transmitted wave is presented. Using the results so obtained, the problem is further extended to the case of a point source excitation, which is the main aim of this

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paper. Thus the corresponding theory for three-dimensional problems (a point source in a layer over a half-space) is developed. This will help understand the propagation of Love waves in the layer from a point source, which is very important for the description of seismic sources and seismic effects.

2. – Formulation of the problem.

We consider a homogeneous and isotropic elastic layer $0 \leq z \leq H$ of uniform thickness H, overlying a semi-infinite inhomogeneous medium. A point source is assumed to be situated at $(0, 0, H)$. The geometry of the problem is shown in fig. 1. The subscripts 1 and 2 refer, respectively, to the layer and the lower inhomogeneous medium. The rigidity and density of media 1 and 2 are denoted by μ_i and ρ_i $(i = 1, 2)$ respectively. The time harmonic variation is taken as $\exp[i\omega t]$ and can be suppressed throughout. We can write the equations of motion in the layer and the lower inhomogeneous medium as

$$\frac{\partial^2 v_1}{\partial x^2} + \frac{\partial^2 v_1}{\partial y^2} + \frac{\partial^2 v_1}{\partial z^2} + \frac{\rho_1 \omega^2}{\mu_1} v_1 = \frac{4\pi}{\mu_1} \delta(x)\, \delta(y)\, \delta(z - H), \tag{1}$$

$$\frac{\partial^2 v_2}{\partial x^2} + \frac{\partial^2 v_2}{\partial y^2} + \frac{\partial^2 v_2}{\partial z^2} + \frac{\rho_2 \omega^2}{\mu_2} v_2 = 0, \tag{2}$$

where ω is the angular frequency and δ is the usual Dirac delta function. To study the inhomogeneity effect, we replace ρ_2 by $\rho_2 + \varepsilon(z - H)$, ε being a small parameter. Thus, we can rewrite eq. (2) as

$$\frac{\partial^2 v_2}{\partial x^2} + \frac{\partial^2 v_2}{\partial y^2} + \frac{\partial^2 v_2}{\partial z^2} + \frac{\rho_2 \omega^2}{\mu_2} v_2 = -\frac{\varepsilon w^2}{\mu_2} (z - H)\, v_2 . \tag{3}$$

Let us define the Fourier transform pair with respect to y:

$$\begin{cases} \bar{v}_i(x, w, z) = \displaystyle\int_{-\infty}^{\infty} v_i(x, y, z) \exp[ik_i wy]\, \mathrm{d}y, \\ v_i(x, y, z) = \dfrac{k_i}{2\pi} \displaystyle\int_{-\infty}^{\infty} \bar{v}_i(x, w, z) \exp[-ik_i wy]\, \mathrm{d}w, \end{cases} \qquad i = 1, 2 .$$

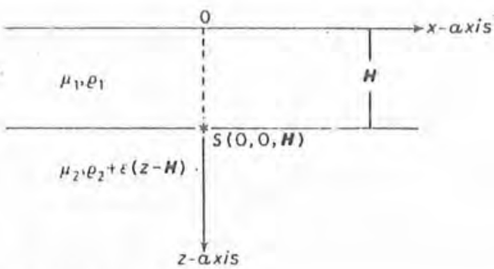

Fig. 1. – Geometry of the problem.

Using the above transform, eqs. (1) and (3) reduce to

$$\frac{\partial^2 \tilde{v}_1}{\partial x^2} + \frac{\partial^2 \tilde{v}_1}{\partial z^2} + k_1^2(1-w^2)\,\tilde{v}_1 = \frac{4\pi}{\mu_1}\,\delta(x)\,\delta(z-H)\,, \tag{4}$$

$$\frac{\partial^2 \tilde{v}_2}{\partial x^2} + \frac{\partial^2 \tilde{v}_2}{\partial z^2} + k_2^2(1-w^2)\,\tilde{v}_2 = -\frac{\varepsilon\omega^2}{\mu_2}(z-H)\,\tilde{v}_2\,, \tag{5}$$

where

$$k_i^2 = \frac{\rho_i\,\omega^2}{\mu_i}\,, \qquad i=1,2\,.$$

3. – Solution of the problem.

We observe that mathematical problem given by eqs. (4) and (5) is the same as in the two-dimensional case [5] except that $\tilde{k}_i^2 = k_i^2(1-w^2)$ replaces k_i^2 and $\tilde{v}_i$ replaces v_i $(i=1,2)$. Thus, we can apply the results of [5] directly to write the field as

$$\tilde{v}_1(x,w,z) = -2\int\limits_{-\infty}^{\infty} \frac{(\exp[\alpha z]+\exp[-\alpha z])\exp[-ifx]\,\mathrm{d}f}{B-(\varepsilon\omega^2(\exp[\alpha H]+\exp[-\alpha H]))/4\beta^2}\,, \tag{6}$$

where

$$\alpha^2 = f^2-\tilde{k}_1^2\,, \qquad \beta^2 = f^2-\tilde{k}_2^2\,,$$

$$B = \mu_2\beta(\exp[\alpha H]+\exp[-\alpha H]) + \mu_1\alpha(\exp[\alpha H]-\exp[-\alpha H])\,,$$

and f (real) is the Fourier transform variable with respect to x. Equation (6) can be rewritten in the form

$$\tilde{v}_1(x,w,z) = \frac{-2}{\mu_1}\int\limits_{-\infty}^{\infty} \frac{\cosh\alpha z\exp[-ifx]}{F(f,w,H)}\,\mathrm{d}f\,, \tag{7}$$

with

$$F(f,w,H) = \alpha\,\sinh\alpha H + \left(\nu\beta - \frac{\varepsilon\omega^2}{4\mu_1\beta^2}\right)\cosh\alpha H\,, \qquad \nu = \frac{\mu_2}{\mu_1}\,.$$

The poles of the integrand in (7) are the real roots λ_m of the equation $F(f,w,H)=0$. In addition, there are branch points which give rise to body waves in which we are not interested. However. we remark that the integral can be solved at the branch points using the asymptotic methods. Thus, calculating this integral at these poles, by indenting these poles and using the residue method, the transformed field can be easily written as

$$\tilde{v}_1(x,w,z) = \frac{-2\pi i}{\mu_1}\,\frac{\{\cos\alpha_m z\,\exp[-i\lambda_m x]\}}{\mathrm{d}F(f,w,H)/\mathrm{d}f|_{f=\lambda_m}}\,, \tag{8}$$

where $\alpha_m^2 = (\bar{k}_1^2 - \lambda_m^2)$ and λ_m is a function of the transformed variable w. If we simplify the relation $F(\lambda_m) = 0$ we arrive at

$$\tan \alpha_m H = \frac{1}{\alpha_m}\left\{\nu\beta_m - \frac{\varepsilon\omega^2}{4\mu_1\beta_m^2}\right\}, \qquad \beta_m^2 = \lambda_m^2 - \bar{k}_2^2 . \tag{9}$$

Equation (9) gives the dispersion equation for the propagation of Love waves in the layered structure consisting of a semi-infinite inhomogeneous medium of rigidity μ_2 covered by a homogeneous surface layer of uniform thickness H, rigidity μ_1 and with a free upper surface. After some effort, eq. (8) can be written as

$$\bar{v}_1(x, w, z) = -8\pi i \sum_{m=1}^{\infty} \frac{\beta_m^4 \alpha_m \cos\alpha_m H \cos\alpha_m z \exp[-i\lambda_m x]}{\lambda_m\{4\mu_1 H\beta_m^2\alpha_m^3 + 3\varepsilon\omega^2\alpha_m\cos^2\alpha_m H\}} . \tag{10}$$

If we put $w = 0$, expression (9) reduces to the dispersion relation obtained by Chattopadhyay [5] and eq. (10) gives the transmitted field for the case of line source excitation.

The field $v_1(x, y, z)$, corresponding to the point source excitation, can be finally obtained by taking inverse transform of eq. (10). This gives

$$v_1(x, y, z) = \frac{k_1}{2\pi}\int_{-\infty}^{\infty}\sum_{m=1}^{\infty}\frac{\{-8\pi i\beta_m^4\alpha_m\cos\alpha_m H\cos\alpha_m z\exp[-i\lambda_m x]\}\exp[-ik_1 wy]}{E(w, H)}\,\mathrm{d}w , \tag{11}$$

where

$$E(w, H) = \lambda_m(4\mu_1 H\beta_m^2\alpha_m^3 + 3\varepsilon\omega^2\alpha_m\cos^2\alpha_m H) .$$

The poles of the integral in (11) are the roots of the equation $E(w, H) = 0$. If we denote these poles by p_n, the transmitted field $v_1(x, y, z)$ is given by

$$v_1(x, y, z) = 4\pi\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{k_1\beta_{m,n}^4\alpha_{m,n}\cos\alpha_{m,n}H\cos\alpha_{m,n}z}{\mathrm{d}E(w, H)/\mathrm{d}w|_{w=p_n}} \cdot \exp[-i\lambda_m x]\exp[-ik_1 p_n y] , \tag{12}$$

where

$$\alpha_m|_{w=p_n} = \alpha_{m,n} , \qquad \beta_m|_{w=p_n} = \beta_{m,n} .$$

4. – Conclusion.

First, we would like to compare our method of solution with earlier works. For example, Vlaar [10] considered a point source in a heterogeneous layer of finite depth. The field is calculated using classic Sturm-Liouville theory and Green's function working in cylindrical polar coordinates. We, on the other hand, consider two-regions problem consisting of a layer overlying a half-space. This represents a more realistic model in the Earth crust and needs a separate discussion. The half-space is assumed to be inhomogeneous along the depth. The field is calculated due to a point source lying on the interface.

The wave equation governing this problem consists of three spatial coordinates. The Fourier transform is used to reduce the dependence on one variable. The transformed equation then corresponds to the line source problem discussed by Chattopadhyay [5] and the point source field is then recovered using inverse transform. The integral appearing in the process is solved analytically using asymptotic methods. This procedure can be fairly easily extendable to the problems of other configurations once the solution for the line source [11-13] incidence is known. It is further remarked that the solution of wave problems for a line source incidence differs from that of plane-wave incidence [14-16] by a multiplicative factor [17], in case the line source is taken to a far off distance. Therefore, we conclude that a whole range of problems starting from a plane wave to spherical wave (point source) can be tackled in a systematic way for realistic models such as [14-16].

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LOVE-TYPE WAVES DUE TO A LINE SOURCE IN AN INHOMOGENEOUS LAYER TRAPPED BETWEEN TWO HALF SPACES

Abstract. The propagation of Love-type guided SH-waves due to a line source in an inhomogeneous layer with variable density sandwiched between two half-spaces has been studied. Using the Fourier transform and Green's function method, the dispersion relations are obtained. The field in the layer is also presented.

INTRODUCTION

Love-waves are known to propagate as horizontally polarised shear (SH) waves in a layer overlying a half-space. Various authors have studied propagation of Love-waves in different heterogeneous media. Hudson (1962) considered different models consisting of a layer with changing density or rigidity, and established the existence of Love-waves in each case. Later, Sinha (1967), Chattopadhyay (1975), Paul (1970) and Kar (1977), among others, studied problems of propagation of Love-type waves propagating in a layer lying deep in the earth, where the layer is either inhomogeneous or porous. Ghosh (1963) showed that a point source could excite Love-waves in a homogeneous layer lying over an inhomogeneous half-space. His method was used by himself (1970) and Chattopadhyay et al. (1984) to study Love-waves excited by a source lying at the interface of the two by allowing either the layer or the substratum to be inhomogeneous.

In this paper, we have considered a layer with variable density sandwiched between two half-spaces. The propagation of Love-type waves is studied in the presence of a source lying at the interface of the layer and lower substratum. Using the Fourier transform and Green's function method, we derive dispersion equations of the Love-type waves excited by the source. The dispersion relations are found to be in agreement with the homogeneous case. Finally, the wave in the layer is calculated analytically.

FORMULATION OF THE PROBLEM

Consider an elastic layer $0 \leq z \leq H$ with uniform thickness H sandwiched between two homogeneous half-spaces $z<0$ and $z>H$. A line source is assumed to be situated at $x=0$ and $z=H$. The geometry of the problem is shown in the Figure. The subscripts 1, 2 and 3 refer to the upper medium, intermediate layer and lower medium respectively. The rigidity, shear velocity and density are denoted by μ_i, β_i, ϱ_i $(i=1, 2, 3)$, respectively. Assuming that the source is time harmonic, and taking the time dependence $e^{i\omega t}$ to be understood throughout, we can write the equation of motion in the layer $0 \leq z \leq H$ as

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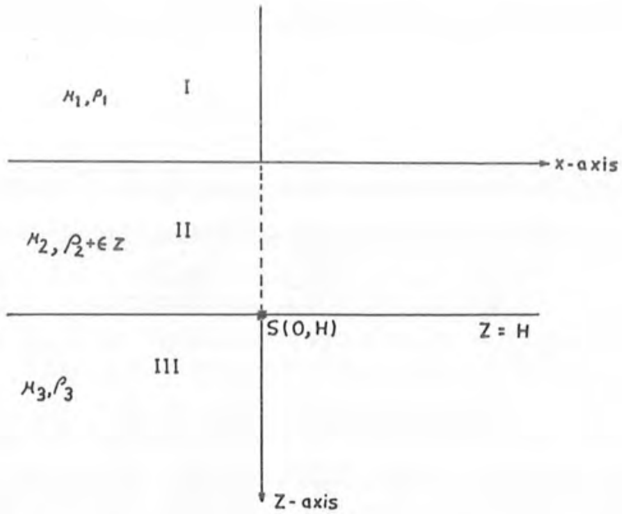

Figure — Geometry of the problem.

$$\frac{\partial^2 v_2}{\partial x^2} + \frac{\partial^2 v_2}{\partial z^2} + \frac{\varrho_2 \omega^2 v_2}{\mu_2} = \frac{4\pi}{\mu_2} \delta(x)\, \delta(z-H), \tag{1}$$

where v_2 is the displacement in the layer and ω is the angular frequency. Due to the inhomogeneity in the layer, ϱ_2 is taken to be $\varrho_2 + \epsilon z$, ϵ being a small parameter. We can rewrite equation (1) as

$$\frac{\partial^2 v_2}{\partial x^2} + \frac{\partial^2 v_2}{\partial x^2} + k_2^2 v_2 = \frac{4\pi}{\mu_2} \delta(x)\, \delta(z-H) - \epsilon z \frac{\omega^2}{\mu_2} v_2 . \tag{2}$$

The equations of motion of the upper and lower substratum are

$$\frac{\partial^2 v_1}{\partial x^2} + \frac{\partial^2 v_1}{\partial z^2} + k_1^2 v_1 = 0, \tag{3}$$

$$\frac{\partial^2 v_3}{\partial x^2} + \frac{\partial^2 v_3}{\partial z^2} + k_3^2 v_3 = 0, \tag{4}$$

where in the above equations

$$k_i^2 = \frac{\varrho_i\, \omega^2}{\mu_i}, \quad i = 1, 2, 3. \tag{5}$$

The physical considerations of the problem lead to the following boundary conditions:

(a) At $z = 0$, $-\infty < x < \infty$,

$$v_1 = v_2 ,$$

$$\mu_1 \frac{\partial v_1}{\partial z} = \mu_2 \frac{\partial v_2}{\partial z} . \tag{6a}$$

(b) At $z=H, -\infty<x<\infty,$

$$v_2=v_3 ,$$

$$\mu_2 \frac{\partial v_2}{\partial z} = \mu_3 \frac{\partial v_3}{\partial z} . \tag{6b}$$

(c) $v_1 \to 0$ as $z - \infty$,

$$v_3 \to 0 \text{ as } z \to \infty . \tag{6c}$$

SOLUTION OF THE PROBLEM

Let us introduce the Fourier transform pair

$$V_i (\zeta, z) = \frac{1}{2\pi} \int_{-\infty}^{\infty} v_i (x, z)\, e^{i\zeta x}\, dx,$$

$$v_i (x, z) = \int_{-\infty}^{\infty} V_i (\zeta, z)\, e^{-i\zeta x}\, d\zeta , \tag{7}$$

where $i=1, 2, 3$. The equations of motion (2), (3) and (4) then transform into

$$\frac{d^2 V_2}{dz^2} - \alpha_2^2 V_2 = \frac{2}{\mu_2} \delta (z-H) - \epsilon z \frac{\omega^2}{\mu_2} V_2 = \sigma_2 (z), \tag{8}$$

where

$$\sigma_2 (z) = \frac{2}{\mu_2} \delta (z-H) - \epsilon z \frac{\omega^2}{\mu_2} V_2 ,$$

for the layer and

$$\frac{d^2 V_{1,3}}{dz^2} - \alpha_{1,3}^2 V_{1,3} = 0, \tag{9}$$

for the upper and lower substratum. In (8) and (9)

$$\alpha_i^2 = \zeta^2 - k_i^2 . \tag{10}$$

We suppose that the Green's function for the inhomogeneous equation (8) is $G_2 (z/z_0)$, where z_0 is an arbitrary point in medium 2. Since we choose the source to be at the interface between the layer and the lower half-space, the value of the Green's function so introduced will depend upon the depth of the layer H (see equation (23)). Also this choice of the position of the source guarantees that the field produced will travel in the layer only and not in the lower half-space. The equation satisfied by $G_2 (z/z_0)$ is

$$G''_2 (z/z_0) - \alpha_2^2 G_2 (z/z_0) = \delta (z-z_0) \tag{11a}$$

together with homogeneous boundary conditions

$$G'_2\ (z/z_0)=0 \text{ at } z=0,\ H. \tag{11b}$$

Here prime denotes differentiation with respect to z. As the boundary conditions in our problem are of an inhomogeneous nature, we follow Stakgold (1979) and multiply equation (8) by $G_2\ (z/z_0)$, equation (11a) by $V_2\ (\zeta,\ z)$, and substract and integrate over $0 \leq z \leq H$ to obtain

$$G_2\ (H/z_0)\ [V'_2]_{z=H} - G_2\ (0/z_0)\ [V'_2]_{z=o} = \int_0^H \sigma_2\ (z)\ G_2\ (z/z_0)\ dz - V_2\ (z_0). \tag{12}$$

Similarly, we suppose that $G_1\ (z/z_0)$ and $G_3\ (z/z_0)$ are Green's functions corresponding to upper and lower substratum satisfying

$$G'_1\ (z/z_0)=0 \quad \text{at} \quad z=0; \qquad G_1\ (z/z_0) \to 0 \text{ as } z \to -\infty$$

and

$$G'_3\ (z/z_0)=0 \quad \text{at} \quad z=H; \qquad G_3\ (z/z_0) \to 0 \text{ as } z \to \infty$$

A similar procedure as above leads to

$$G_1\ (0/z_0)\ [V'_1]_{z=0} = -V_1\ (z_0), \tag{13}$$

$$G_3\ (H/z_0)\ [V'_3]_{z=H} = V_3\ (z_0). \tag{14}$$

Interchanging z and z_0 and using the symmetry of Green's function in (12), (13) and (14), we get

$$V_2\ (z) = G_2\ (z/0)\ [V_2]_{z=0} - G_2\ (z/H)\ [V'_2]_{z=H} + \int_0^H \sigma_2\ (z_0)\ G_2\ (z/z_0)\ dz_0\,, \tag{15}$$

$$V_1\ (z) = -G_1\ (z/0)\ [V'_1]_{z=0}\,, \tag{16}$$

and

$$V_3\ (z) = G_3\ (z/H)\ [V'_3]_{z=H}\,. \tag{17}$$

Now the boundary conditions (6a) give us

$$[V'_2]_{z=0} = \frac{1}{A}\ \{G_2\ (0/H)\ [V'_2]_{z=H} - \int_0^H \sigma_2\ (z_0)\ G_2\ (0/z_0)\ dz_0\}\,, \tag{18}$$

where

$$A = G_2\ (0/0) + \frac{\mu_2}{\mu_1}\ G_1\ (0/0)\,.$$

Similarly (6b) yields

$$[V'_2]_{z=H} = \frac{-G_2\ (H/0)}{AB - G_2^2\ (H/0)} \int_0^H \sigma_2\ (z_0)\ G_2\ (0/z_0)\ dz_0$$

$$+\frac{A}{AB-G_2^2\,(H/0)}\int_0^H \sigma_2\,(z_0)\,G_2\,(H/z_0)\,dz_0\,, \tag{19}$$

where

$$B=G_2\,(H/H)+\frac{\mu_2}{\mu_3}\,G_3\,(H/H)\,.$$

Using (18) and (19) in (15), substituting back the value of $\sigma_2\,(z)$ and using the property of the delta function, we get

$$V_2\,(z)=\frac{2}{\mu_2}\left\{\frac{G_2\,(z/H)\,C-G_2\,(z/0)\,D}{AB-G_2^2\,(H/0)}\right\}-\left\{\frac{G_2\,(z/H)\,G_2\,(H/0)-G_2\,(z/0)\,B}{AB-G_2^2\,(H/0)}\right\}$$

$$\times\frac{\omega^2\,\epsilon}{\mu_2}\int_0^H z_0\,V_2\,(z_0),\,G_2\,(0/z_0)\,dz_0-\left\{\frac{G_2\,(z/0)\,G_2\,(H/0)-G_2\,(z/H)\,A}{AB-G_2^2\,(H/0)}\right\}$$

$$\times\frac{\omega^2\,\epsilon}{\mu_2}\int_0^H z_0\,V_2\,(z_0)\,G_2\,(H/z_0)\,dz_0-\frac{\omega^2\,\epsilon}{\mu_2}\int_0^H z_0\,V_2\,(z_0)\,G_2\,(z/z_0)\,dz_0\,, \tag{20}$$

where

$$C=\frac{\mu_2}{\mu_3}\,G_3\,(H/H)\,A,\; D=\frac{\mu_2}{\mu_3}\,G_3\,(H/H)\,G_2\,(H/0).$$

The series so obtained can easily be shown to be a convergent series which converges to $V_2\,(z)$. Now equation (20) is an integral equation and $V_2\,(z)$ may be determined from this equation using successive substitution. As a first approximation, we neglect terms involving ϵ to obtain

$$V_2\,(z)=\frac{2}{\mu_2}\left\{\frac{G_2\,(z/H)\,C-G_2\,(z/0)\,D}{AB-G_2^2\,(H/0)}\right\}. \tag{21}$$

Putting this value back in the right hand side of (20), we obtain $V_2\,(z)$ in the following form

$$V_2\,(z)=\frac{2}{\mu_2}\left\{\frac{G_2\,(z/H)\,C-G_2\,(z/0)\,D}{AB-G_2^2\,(H/0)}\right\}-\frac{2\omega^2\,\epsilon}{\mu_2^2}\left\{\frac{G_2\,(z/H)\,G_2\,(H/0)-}{AB-G_2^2\,(H/0)}\right.$$

$$\left.\frac{G_2\,(z/0)\,B}{}\right\}\int_0^H\left\{\frac{G_2\,(z_0/H)\,C-G_2\,(z_0/0)\,D}{AB-G_2^2\,(H/0)}\right\}z_0\,G_2\,(0/z_0)\,dz_0$$

$$-\frac{2\omega^2\,\epsilon}{\mu_2^2}\left\{\frac{G_2\,(z/0)\,G_2\,(H/0)-G_2\,(z/H)\,A}{AB-G_2^2\,(H/0)}\right\}\int_0^H\left\{\frac{G_2\,(z_0/H)\,C-}{AB-G_2^2\,(H/0)}\right.$$

$$\left.\frac{G_2\,(z_0/0)\,D}{}\right\}z_0\,G_2\,(H/z_0)\,dz_0-\frac{2\omega^2\,\epsilon}{\mu_2^2}$$

$$\times \int_0^H \{ \frac{G_2 (z_0/H) C - G_2 (z_0/0) D}{AB - G_2^2 (H/0)} \} z_0 G_2 (z/z_0) dz_0 . \tag{22}$$

We note that $V_2 (z)$ is completely determined through (22) provided that G_1 , G_2 and G_3 are known.

In order to determine these Green's functions, we follow the procedure outlined by Stakgold (1979). Thus (11a, b) would give us

$$G_2 (z/z_0) = -\frac{1}{2\alpha_2} [e^{-\alpha_2 |z-z_0|} + \frac{e^{\alpha_2 z} \{ e^{-\alpha_2 (H+z_0)} + e^{-\alpha_2 (H-z_0)} \}}{e^{\alpha_2 H} - e^{-\alpha_2 H}}$$

$$+ \frac{e^{-\alpha_2 z} \{ e^{\alpha_2 (H-z_0)} + e^{-\alpha_2 (H-z_0)} \}}{e^{\alpha_2 H} - e^{-\alpha_2 H}}] . \tag{23}$$

Similarly

$$G_1 (z/z_0) = -\frac{1}{2\alpha_1} [e^{-\alpha_1 |z-z_0|} + e^{\alpha_1 (z+z_0)}] , \tag{24}$$

and

$$G_3 (z/z_0) = -\frac{1}{2\alpha_3} [e^{-\alpha_3 |z-z_0|} + e^{\alpha_3 (z+z_0-2H)}] . \tag{25}$$

Using (23), (24) and (25) in (22), we obtain after some manipulations

$$V_2 (z) = \frac{-2 (\mu_2 \alpha_2 \cosh\alpha_2 z + \mu_1 \alpha_1 \sinh\alpha_2 z)}{\mu_1 \mu_3 \alpha_1 \alpha_2^2 \alpha_3 \sinh\alpha_2 H (AB - G_2^2 (H/0)) (1 + \epsilon E)} . \tag{26}$$

or

$$V_2 (z) = \frac{-2 (\mu_2 \alpha_2 \cosh\alpha_2 z + \mu_1 \alpha_1 \sinh\alpha_2 z)}{F (\zeta, H)} , \tag{27}$$

where

$$F (\zeta, H) = \mu_1 \mu_3 \alpha_1 \alpha_2^2 \alpha_3 \sinh\alpha_2 H (AB - G_2^2 (H/0)) (1 + \epsilon E) ,$$

and

$$E = \frac{\omega^2}{4 (AB - G_2^2 (H/0))} [\frac{\mu_1 \alpha_1 - \mu_3 \alpha_3}{\mu_1 \mu_3 \alpha_2^4 \alpha_3} - \frac{H \{ (\mu_2^2 \alpha_2^2 - \mu_1 \alpha_1 \mu_3 \alpha_3)}{\mu_1 \mu_2 \mu_3 \alpha_1 \alpha_2^4 \alpha_3}$$

$$\frac{+ \mu_2 \alpha_2 (\mu_1 \alpha_1 - \mu_3 \alpha_3) \coth\alpha_2 H \}}{} - \frac{H^2 \{ \mu_2 \alpha_2 (\mu_1 \alpha_1 + \mu_3 \alpha_3)}{\mu_1 \mu_2 \mu_3 \alpha_1 \alpha_2^3 \alpha_3}$$

$$\frac{+ (\mu_1 \alpha_1 \mu_3 \alpha_3 + \mu_2^2 \alpha_2^2) \coth\alpha_2 H \}}{}] . \tag{28}$$

TRANSMITTED WAVES

In this section, we determine the transmitted waves in the inhomogeneous layer trapped between two half-spaces. The Fourier inversion formula, when applied to (27), gives

$$v_2(x, z) = -2 \int_{ic-\infty}^{ic+\infty} \frac{(\mu_2 \alpha_2 \cosh\alpha_2 z + \mu_1 \alpha_1 \sinh\alpha_2 z)\, e^{-i\zeta x}}{F(\zeta, H)} d\zeta . \qquad (29)$$

The expression in the denominator equated to zero gives the dispersion equation of Love waves for a trapped layer due to the presence of the inhomogeneity in the layer. It is

$$\tan\tilde{\alpha}_2 H = \frac{\mu_2 \tilde{\alpha}_2 (\mu_1 \alpha_1 + \mu_3 \alpha_3)}{\mu_2^2 \tilde{\alpha}_2^2 - \mu_1 \mu_3 \alpha_1 \alpha_3} + \frac{\omega^2 \epsilon}{4(\mu_2^2 \tilde{\alpha}_2^2 - \mu_1 \mu_3 \alpha_1 \alpha_3)} \Big[\frac{(\mu_3 \alpha_3 - \mu_1 \alpha_1)}{\tilde{\alpha}_2^2}$$

$$\times \tan\tilde{\alpha}_2 H - \frac{H}{\mu_2 \tilde{\alpha}_2^2} \{ (\mu_2^2 \tilde{\alpha}_2^2 + \mu_1 \mu_3 \alpha_1 \alpha_3) \tan\tilde{\alpha}_2 H - \mu_2 \tilde{\alpha}_2 (\mu_1 \alpha_1 - \mu_3 \alpha_3) \}$$

$$- \frac{H^2}{\mu_2 \tilde{\alpha}_2} \{ \mu_2 \tilde{\alpha}_2 (\mu_1 \alpha_1 + \mu_3 \alpha_3) \tan\tilde{\alpha}_2 H + (\mu_2 \tilde{\alpha}_2^2 - \mu_1 \mu_3 \alpha_1 \alpha_3) \} \Big], \qquad (30)$$

where

$$\tilde{\alpha}_2 = (k_2^2 - \zeta^2)^{1/2} .$$

It is imperative to note that in the case of a homogeneous medium, $\epsilon = 0$ and this dispersion relation reduces to the dispersion relation obtained by Ewing, et al. (1957) for the homogeneous case.

In order to obtain the transmitted wave, we need to calculate the integral in (29). For that, we note that the poles of the integrand are the roots of the equation $F(\zeta, H) = 0$. This equation yields the dispersion relation given by the equation (30). The solutions of this equation are in fact the poles of the integrand, which can only be calculated numerically and are denoted by $\zeta = P_{2,n}$. Calculating the pole contribution at these poles, we find that

$$v_2(x, z) = 4\pi \sum_{n=1}^{\infty} \frac{e^{-ip_{2,n} x} \{ \mu_2 \tilde{\alpha}_{2,n} \cos \tilde{\alpha}_{2,n} z + \mu_1 \alpha_{1,n} \sin \tilde{\alpha}_{2,n} z \}}{\left[\frac{dF(\zeta, H)}{d\zeta} \right]_{\zeta = p_{2,n}}} , \qquad (31)$$

where

$$\hat{\alpha}_2 \Big|_{\zeta = p_{2n}} = \hat{\alpha}_{2,n} , \quad \alpha_1 \Big|_{\zeta = p_{2,n}} = \alpha_{1,n}$$

Equation (31) represents the travelling wave in the layer in the x-axis direction.

Besides the poles, we have the branch points which give rise to the body waves and which are of no interest to us for the present study. However, these waves can be easily calculated with the help of the saddle point method considering the asymptotic behaviour of the integrals.

In order to have a feeling for the problem in concrete terms real life, we would like to quote the realistic values of the parameters involved (see the Table). This will include the values of the rigidities, densities and shear velocities in the three regions. It is further noted that $\mu_3 > \mu_2$, which is essential for the propagation of Love-wave in the layer.

Table — Parameters of the model

Medium	Rigidity dynes/cm^2	Density gm/cm^3	Shear velocity km/s
I	4.6×10^{11}	5.14	3.104
II	2.12×10^{11}	4.52	2.165
III	5.32×10^{11}	3.29	4.021

We assume that the depth of the intermediate layer H is 6.0 km.

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