ON THE EQUATIONS OF MOTION OF A NONHOLONOMIC DYNAMICAL SYSTEM IN POINCARÉ-ČETAEV VARIABLES

by

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ABSTRACT

Consider a dynamical system whose position is defined by n Poincaré-Četaev variables. Let the system be subject to holonomic as well as nonlinear nonholonomic constraints. The nonlinear nonholonomic constraints are assumed to be of the type of N. G. Četaev [14].

The author has obtained many a different form of the equations of motion for a nonlinear nonholonomic system. The starting point is the construction of infinitesimal displacement operators of the nonholonomic system in terms of the operators of the associated holonomic system obtained by ignoring nonholonomic constraints. The equations of motion, involving these operators, are obtained by the direct method based on the use of fundamental equation of dynamics with simultaneous allowance for all the imposed constraints. The equations of motion are also derived by the methods of Chaplygin, Hamel and Appell. It is shown that the equations of motion in Poincaré-Četaev variables obtained by these different methods are equivalent.

Employing the derivatives of the kinetic energy of the system or the energy of acceleration, the equations of motion are transformed to several new forms. Also using the definition of cyclic displacement operators due to N. G. Četaev [15], the equations of motion are transformed so as to obtain a

eneralisation of Chaplygin equations of motion.

It is shown that a nonholonomic system can be made quivalent to a holonomic system by adjunction of supple- entary forces which depend on the parameters of real isplacement. This allows to derive the canonical equations nd Routhian equations of motion. It is proved that under uitable conditions the equations of motion admit certain irst integrals.

With a view to integrate the equations of motion, lamilton-Jacobi method is discussed for the associated olonomic system. It is shown that this method when suitably odified is applicable to integrate the equations of a linear onholonomic system when the constraints are of Chaplygin ype. In the general case of a nonlinear nonholonomic system ecessary and sufficient conditions are obtained for the pplicability of the Hamilton-Jacobi method.

The various new forms for the equations of motion re applied to solve a few problems.

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INTRODUCTION

The idea of applying group theoretic methods to the
olution of physical problems originated with Felix Klein [26]
ho first applied it to geometry. However, problems outside
e field of geometry do arise in which certain facts can be
ormulated in terms of the invariants of some group of trans-
ormations. In fact, a wide class of problems in dynamics
nds itself to the group theoretic treatment.

In his investigations of the motion of a rigid body
aving a cavity filled with liquid, H.Poincaré [36] applied
e groups of continuous transformations to obtain a general
orm of the equations of motion of a holonomic dynamical
ystem. Let the position of the system, having n degrees of
reedom, be defined by the so-called Poincaré variables
$_1, x_2, \dots, x_n$. Let T be the kinetic energy and U the force
unction of the system, and let $X_1, X_2, \dots, X_n$ be the infinite-
imal displacement operators of a transitive group of possible
splacements. The Poincaré equations of motion of the system
n be written, using summation convention, in the form

$$.) \quad \frac{d}{dt}\left(\frac{\partial T}{\partial \eta_e}\right) - C_{efh}\eta_f \frac{\partial T}{\partial \eta_h} - X_e(T+U) = 0, \quad (e,f,h = 1,2,\dots,n),$$

ere the η's are parameters of real displacement and
$_{fh}$ are the constants of structure of the group.

In 1941, N.G.Četaev [15] extended the equations of Poincaré to the case when the variables $x_1, x_2, \ldots, x_n$ are dependent due to the presence of holonomic constraints. Constructing an intransitive group of possible displacements with the help of holonomic constraints, he derives the equations of motion in the form of Poincaré, in canonical form and in the form of equations in partial derivatives of first order. He also investigates the general properties of cyclic displacements and properties of Hamilton's function of action.

Recently Fam Guen [21; 22] has studied the problem of formulating the Poincaré equations for a nonholonomic dynamical system when the imposed constraints are partly holonomic and partly linear nonholonomic. Using the method of Poincaré-Četaev variables, it is shown [21] that such a general formulation of the equations of motion contains, as particular cases, the equations of Chaplygin and the equations of Volterra. In [22] the equations of motion of a linear nonholonomic dynamical system in Poincaré-Četaev variables are derived directly from the fundamental equation of dynamics with simultaneous allowance for all the imposed constraints. In the same paper the problem of the equivalence of the equations of motion obtained by different methods is also discussed.

The present thesis is concerned with extending the results of Poincaré, Četaev and Fam Guen to the case when the dynamical system moves subject to nonlinear nonholonomic constraints. But one is confronted with a serious difficulty while dealing with nonlinear nonholonomic constraints. Considering such constraints from the analytical point of view, the two fundamental principles of dynamics - the principle of d'Alembert-Lagrange and the principle of least constraint of Gauss - are found to be inconsistent. However, this difficulty can be surmounted by introducing the definition of possible displacements in the manner of N.G.Četaev [14].

Though a natural system with nonlinear nonholonomic constraints has not been encountered so far, a general theory of nonlinear nonholonomic constraints is not entirely useless. In fact, this general theory is analytically equivalent to the problem of determining with the greatest possible accuracy the motion of a holonomic system in which the forces are not completely known though certain first integrals are known.

The nonholonomic dynamical system considered in this work is of a very general type. Some of the imposed constraints are holonomic and the others are nonlinear nonholonomic. The holonomic constraints are expressed by n-m distinct equations of the form

$$(2) \quad A_{se}(x_1, x_2, \ldots, x_n; t)\dot{x}_e + A_s(x_1, x_2, \ldots, x_n; t) = 0,$$

$$(s = m+1, \ldots, n; \; e = 1, 2, \ldots, n),$$

the nonholonomic constraints by m-ℓ distinct equations the form

$$F_\alpha(x_1,x_2,\ldots,x_n;\ \dot{x}_1,\ldots,\dot{x}_n;\ t) = 0, \qquad (\alpha = \ell+1,\ldots,m),$$

re F_α are not necessarily linear in the $\dot{x}$'s. A brief résumé the different aspects of the present work is given below:

(i) The starting point is the construction of the finitesimal displacement operators $Y_0, Y_1, \ldots, Y_\ell$ for the linear nonholonomic system moving with constraints of the m (2) and (3). In Sec. 2.2 the operators $Y_0, Y_1, \ldots, Y_\ell$ are pressed in terms of the displacement operators $X_0, X_1, \ldots, X_m$ the associated holonomic system obtained by ignoring the holonomic constraints (3).

Defining the functions K_{oij}, K_{ijk}, $\overset{*}{K}_{oi\alpha}$ and $\overset{*}{K}_{ij\alpha}$ as in . 2.3, the equations of motion of the nonholonomic system derived in the form (Sec. 2.4)

$$\frac{d}{dt}\left(\frac{\partial\overline{T}}{\partial\eta_i}\right) - \frac{\partial\overline{T}}{\partial\eta_k}\{K_{oik}+\eta_j K_{jik}\} - \frac{\partial\overline{T}}{\partial\eta_\beta}\{\overset{*}{K}_{oi\beta}+\eta_j\overset{*}{K}_{ji\beta}+\frac{\partial c_{\beta i}}{\partial\eta_j}\dot{\eta}_j\} -$$

$$- Y_i(\overline{T}+U) = 0, \qquad (i,j,k = 1,2,\ldots,\ell;\ \beta=\ell+1,\ldots,m).$$

e $c_{\beta i}$ stands for $\frac{\partial\eta_\beta}{\partial\eta_i}$ and $\overline{T}$ denotes the kinetic energy T ained by taking into account the equations of the nonholonomic constraints (3) expressed in terms of the η's. The equations (4) are derived by using the fundamental equation of dynamics in which the possible displacements are defined according to N.G.Četaev.

In constrast to this direct method, we employ the methods of Chaplygin and of Hamel to obtain the equations of motion in the form (Sec. 2.5 and 2.6)

$$(5)\quad \frac{d}{dt}\left(\frac{\partial T}{\partial \eta_i}\right) - C_{oiq}\frac{\partial T}{\partial \eta_q} - \eta_q C_{qir}\frac{\partial T}{\partial \eta_r} - X_i(T+U) +$$

$$+c_{\alpha i}\left[\frac{d}{dt}\left(\frac{\partial T}{\partial \eta_\alpha}\right) - C_{o\alpha q}\frac{\partial T}{\partial \eta_q} - \eta_q C_{q\alpha r}\frac{\partial T}{\partial \eta_r} - X_\alpha(T+U)\right] = o,$$

$$(i = 1,2,\ldots,\ell;\ \alpha = \ell+1,\ldots,m;\ q,r = 1,2,\ldots,m).$$

Finally, Appell's method is used in Sec. 2.7 to derive the equations of motion in the form

$$(6)\qquad \frac{\partial \overline{S}}{\partial \dot{\eta}_i} = Y_i(U), \qquad (i = 1,2,\ldots,\ell),$$

where $\overline{S}$ denotes the energy of acceleration of the nonholonomic system calculated by using the equations of nonholonomic constraints. If S denotes the energy of acceleration of the associated holonomic system, the equations (6) are shown to assume the symmetric form

$$(7)\quad \frac{\partial S}{\partial \dot{\eta}_i} - X_i(U) + c_{\alpha i}\left[\frac{\partial S}{\partial \dot{\eta}_\alpha} - X_\alpha(U)\right] = 0, \quad (i=1,2,\ldots,\ell;\alpha=\ell+1,\ldots,m).$$

The problem of the equivalence of the equations of motion obtained by these different methods is discussed in Sec.2.8.

(ii) Let $T^{(\sigma)}$ denote the σth derivative with respect to the time t of the kinetic energy T. In Sec. 3.3 the equations of motion (5) are transformed to the form

$$(8)\quad \sigma\left[\frac{\partial T^{(\sigma)}}{\partial \eta_i^{(\sigma-1)}} + c_{\alpha i}\frac{\partial T^{(\sigma)}}{\partial \eta_\alpha^{(\sigma-1)}}\right] - (\sigma+1)\left[\frac{\partial T^{(\sigma-1)}}{\partial \eta_i^{(\sigma-2)}} + c_{\alpha i}\frac{\partial T^{(\sigma-1)}}{\partial \eta_\alpha^{(\sigma-2)}}\right] =$$

$$Y_i(U) + \bar{P}_{oi}, \quad (i = 1,2,\ldots,\ell;\ \alpha=\ell+1,\ldots,m;\ \sigma=2,3,\ldots),$$

where

$$\bar{P}_{oi} = P_{oi} + c_{\alpha i} P_{o\alpha},$$

and

$$P_{op} = C_{opq}\frac{\partial T}{\partial \eta_q} + \eta_q C_{qpr}\frac{\partial T}{\partial \eta_r}, \quad (p,q,r = 1,2,\ldots,m).$$

It is also shown that the equations (8) are equivalent to

$$(9)\quad \sigma\frac{\partial \tilde{T}^{(\sigma)}}{\partial \eta_i^{(\sigma-1)}} - (\sigma+1)\frac{\partial \tilde{T}^{(\sigma-1)}}{\partial \eta_i^{(\sigma-2)}} = Y_i(U) + \bar{P}_{oi},$$

$$(i = 1,2,\ldots,\ell;\sigma = 2,3,\ldots),$$

where $\tilde{T}^{(\sigma)}$ denotes the function which is obtained from $T^{(\sigma)}$

by first considering it as a function of $\eta^{(\sigma-1)}$ and then using the nonholonomic constraints to eliminate the dependent $\eta^{(\sigma-1)}$.

Let $\bar{T}^{(\sigma)}$ denote the function $T^{(\sigma)}$ after using the nonholonomic constraints and let $T_i^{(\sigma)}$ denote the function obtained from $T^{(\sigma)}$ by first considering it as a function of $\eta^{(\sigma)}$ and then taking nonholonomic constraints into account. The equations (9) are shown to assume the form (Sec.3.3)

$$(10)\quad \sigma\left[\frac{\partial\bar{T}^{(\sigma)}}{\partial\eta_i^{(\sigma-1)}} - \frac{\partial T_i^{(\sigma)}}{\partial\eta_i^{(\sigma-1)}}\right] - (\sigma+1)\left[\frac{\partial\bar{T}^{(\sigma-1)}}{\partial\eta_i^{(\sigma-2)}} - \frac{\partial T_i^{(\sigma-1)}}{\partial\eta_i^{(\sigma-2)}}\right] = Y_i(U) + \bar{F}_{oi},$$

$$(i = 1,2,\ldots,\ell;\ \sigma = 2,3,\ldots).$$

With the help of the identity

$$(11)\quad \frac{\partial\bar{T}^{(\sigma)}}{\partial\eta_i^{(\sigma-1)}} - \frac{\partial\bar{T}^{(\sigma-1)}}{\partial\eta_i^{(\sigma-2)}} = \frac{d}{dt}\left(\frac{\partial\bar{T}}{\partial\eta_i}\right), \quad (i=1,2,\ldots,\ell;\ \sigma=2,3,\ldots),$$

the equations (10) are transformed to the form

$$(12)\quad \sigma\frac{d}{dt}\left(\frac{\partial\bar{T}}{\partial\eta_i}\right) - \frac{\partial\bar{T}^{(\sigma-1)}}{\partial\eta_i^{(\sigma-2)}} - \sigma\frac{\partial T_i^{(\sigma)}}{\partial\eta_i^{(\sigma-1)}} + (\sigma+1)\frac{\partial T_i^{(\sigma-1)}}{\partial\eta_i^{(\sigma-2)}} = Y_i(U) + \bar{F}_{oi},$$

$$(i = 1,2,\ldots,\ell;\ \sigma = 2,3,\ldots).$$

In the same section we discuss the transformation of the equations (5) to the form

$$(13)\quad \frac{1}{\sigma}[\frac{\partial T^{(\sigma)}}{\partial \eta_i^{(\sigma-1)}}-(\sigma+1)X_i(T)]-P_{oi}+c_{\alpha i}[\frac{1}{\sigma}(\frac{\partial T^{(\sigma)}}{\partial \eta_\alpha^{(\sigma-1)}}-(\sigma+1)X_\alpha(T))-P_{o\alpha}] =$$

$$Y_i(U),\quad (i = 1,2,\ldots,\ell;\ \alpha = \ell+1,\ldots m;\ \sigma = 1,2,3,\ldots),$$

which are the generalised Mangeron-Deleanu equations [28] in Poincaré-Četaev variables. The equations (13) include as special cases the equations in the forms due to Nielsen and Cenov for $\sigma=1$ and 2 respectively.

If T_o denotes the function T for fixed η's, we introduce (Sec.3.4) the function R_σ by the relation

$$R_\sigma = \frac{1}{\sigma}(T^{(\sigma)} - (\sigma+1)T_o^{(\sigma)}] - P_{op}\eta_p^{(\sigma-1)},$$

$$(\sigma = 2,3,\ldots;p=1,2,\ldots,m).$$

In terms of R_σ the equations (13) assume the form

$$(14)\quad \frac{\partial R_\sigma}{\partial \eta_i^{(\sigma-1)}} + c_{\alpha i}\frac{\partial R_\sigma}{\partial \eta_\alpha^{(\sigma-1)}} = Y_i(U),$$

$$(i = 1,2,\ldots\ell;\ \alpha= \ell+1,\ldots,m;\ \sigma = 2,3,\ldots).$$

Let $\bar{R}_\sigma$ denote the function which is obtained from R_σ by first considering it as a function of $\eta^{(\sigma-1)}$ and then using the nonholonomic constraints. Then the equations (14) are shown to take the form

$$\frac{\partial \tilde{R}_\sigma}{\partial \eta_i^{(\sigma-1)}} = Y_i(U), \quad (i = 1,2,\ldots,\ell;\ \sigma = 2,3,\ldots). \tag{15}$$

e equations (15) are a generalisation of the Appell-Cenov uations [17] in Poincaré-Četaev variables.

Let $\bar{S}^{(\sigma-2)}$ denote $S^{(\sigma-2)}$ when the dependent $\eta^{(\sigma-1)}$ e eliminated with the help of the equations of nonholonomic nstraints. A generalisation of Appell's equations (6) is tained in the form (Sec.3.5)

$$\frac{\partial \bar{S}^{(\sigma-2)}}{\partial \eta_i^{(\sigma-1)}} = Y_i(U), \quad (i = 1,2,\ldots,\ell;\ \sigma = 2,3,\ldots). \tag{16}$$

An application of Appell's transformation

$$\tilde{K}_\sigma = \tilde{R}_\sigma - \eta_i^{(\sigma-1)} Y_i(U), \quad (i=1,2,\ldots,\ell;\sigma=2,3,\ldots),$$

lows to write the equations (15) in the form (Sec.3.6)

$$\frac{\partial \tilde{K}_\sigma}{\partial \eta_i^{(\sigma-1)}} = 0, \quad (i = 1,2,\ldots,\ell;\ \sigma = 2,3,\ldots). \tag{17}$$

In Sec.3.7 it is shown that the function $\tilde{K}_s$ and the ussian constraint coincide as far as the terms of $\ddot{\eta}$'s are ncerned.

In Sec.3.8 cyclic displacement operators are defined cording to N.G.Četaev [15]. Using these operators and ing some additional assumptions, a generalisation of haplygin equations is obtained in the form

$$
18) \quad \frac{d}{dt}\left(\frac{\partial \overline{T}}{\partial \eta_i}\right) - X_i(\overline{T}+U) - \frac{\partial \overline{T}}{\partial \eta_k}[C_{oik} + \eta_j C_{jik}] -
$$

$$
-\frac{\partial T}{\partial \eta_\beta}\{C_{oi\beta} - c_{\beta k} C_{oik} + X_o(c_{\beta i}) - X_i(\eta_\beta - \eta_j c_{\beta j}) +
$$

$$
+\eta_j\{C_{ji\beta} - c_{\beta k} C_{jik} + X_j(c_{\beta i}) - X_i(c_{\beta j})\} + \frac{\partial c_{\beta i}}{\partial \eta_j}\dot{\eta}_j] = 0,
$$

$$
(i,j,k = 1,2,\ldots,\ell;\ \beta = \ell+1,\ldots,m).
$$

(iii) The theory developed in chapters 2 and 3 is applied to solve four well known examples of nonholonomic dynamical systems. In the first example, due to Appell, the equation of nonlinear nonholonomic constraint arises as a natural consequence of the mechanism employed. In the other examples the equations of constraint, though essentially linear, are mathematically transformed to nonlinear forms with a view to illustrate the theory.

(iv) In Sec.5.2 it is shown that the nonholonomic system with time dependent constraints of the form (2) and (3) is reducible to an associated holonomic system by adjunction of certain supplementary forces depending on the η's and admitting as integrals the equations of nonholonomic constraints. If the nonlinear nonholonomic constraints (3) are expressed by the equations

$$f_\alpha(x_1, x_2, \ldots, x_n; \eta_1, \ldots, \eta_m; t) = 0, \qquad (\alpha = \ell+1, \ldots, m),$$

the equations of motion are derived in the form (Sec.5.2)

$$(19) \quad \frac{d}{dt}\left(\frac{\partial T}{\partial \eta_p}\right) - C_{opq}\frac{\partial T}{\partial \eta_q} - \eta_q C_{qpr}\frac{\partial T}{\partial \eta_r} - X_p(T+U) - \lambda_\alpha \frac{\partial f_\alpha}{\partial \eta_p} = 0,$$

$$(\alpha = \ell+1, \ldots, m; \; p,q,r = 1,2,\ldots,m),$$

where λ's are the undetermined multipliers.

By means of (19) the equations of motion are obtained in the canonical form (Sec.5.3)

$$(20) \quad \begin{cases} \eta_p = \dfrac{\partial H}{\partial y_p}, \\ \dfrac{dy_p}{dt} = C_{opq}y_q + \eta_q C_{qpr}y_r - X_p(H) + \lambda_\alpha \dfrac{\partial f_\alpha}{\partial \eta_p}, \end{cases}$$

$$(\alpha = \ell+1, \ldots, m; \; p,q,r = 1,2,\ldots,m),$$

where H is the Hamiltonian and the variables y_p are obtained from the Lagrangian L by the formulae

$$y_p = \frac{\partial L}{\partial \eta_p}, \qquad (p = 1,2,\ldots,m).$$

In Sec. 5.4 the Routhian function R is introduced and the equations of motion are obtained. Some of these equations are in the Hamiltonian form and the others are in the Lagrangian form.

Sec. 5.5 is concerned with the investigation of certain first integrals of the equations of motion.

(v) The function of action V is introduced in Sec.6.2 for the associated holonomic system and it is shown that V satisfies the partial differential equation

$$X_0(V) + H(x_1,x_2,\ldots,x_n; X_1(V),\ldots,X_m(V);t) = 0.$$

Sec.6.3 is concerned with the Hamilton-Jacobi theorem for associated holonomic systems.

For constraints of Chaplygin type it is shown (Sec.6.4) that the equations of motion can be integrated by the Hamilton-Jacobi method.

In Sec.6.5 a general theorem is proved which furnishes necessary and sufficient conditions in order that the Hamilton-Jacobi theorem may be applied to a nonlinear nonholonomic system.

CHAPTER I

PRELIMINARIES

1.1. The Infinitesimal Displacement Operators

Consider the motion of a dynamical system with n degrees of freedom. Let $x_1, x_2, \ldots, x_n$ be the parameters which specify the position of the system at the time t.

As we know [18], a group of continuous transformations can be defined on the space of variables $x_1, x_2, \ldots, x_n$ in which the infinitesimal operators are

$$X_e = \xi_e^f \frac{\partial}{\partial x_f}, \qquad (e, f = 1, 2, \ldots, n), \tag{1.1.1}$$

where ξ_e^f are functions of $x_1, x_2, \ldots, x_n$.

Here summation over a repeated suffix is understood whereas a suffix within parenthesis will not imply summation.

Since the operators (1.1.1) form a group, the commutators

$$(X_e, X_f) = X_e X_f - X_f X_e,$$

satisfy the relations

$$(X_e, X_f) = C_{efh} X_h, \qquad (e, f, h = 1, 2, \ldots, n), \tag{1.1.2}$$

which serve to define the structure constants C_{efh}.

Let $f(x_1, x_2, \ldots, x_n)$ be an arbitrary function of the position of the dynamical system. According to H. Poincare [36], we introduce the following definitions.

<u>Definition 1.</u> As the system undergoes an infinitesimal displacement dx_1, $dx_2, \ldots, dx_n$ in the time dt, the change df in the

function f is defined by the relation

(1.1.3) $df = \eta_e X_e(f)dt$, $(e = 1,2,\ldots,n)$.

The η's are called the parameters of real displacement.

Definition 2. In a possible displacement $\delta x_1, \delta x_2, \ldots, \delta x_n$ of the system, the change δf in the function f is defined by the relation

(1.1.4) $\delta f = \omega_e X_e(f)$, $(e = 1,2,\ldots,n)$.

The ω's are called the parameters of possible displacement.

It is known that the structure constants C_{efh}, occuring in (1.1.2), depend on the choice of the displacement parameters η's and ω's. Furthermore, if we take $f = x_e$, the formulae (1.1.3) gives the derivatives $\dot{x}_e$ of x_e with respect to the time t. Similarly the formulae (1.1.4) yield the possible displacement δx_e corresponding to the parameter x_e.

1.2. Constraints

The parameters $x_1, x_2, \ldots, x_n$ chosen to specify the position of the system will be independent if we have taken into consideration all the equations of constraint. But in many cases, for reasons of convenience or of necessity, we may leave out certain equations of constraint and therefore all the parameters are not independent.

If n-m holonomic constraints are not taken into consideration, the x's are connected by integrable equations of the form

(1.2.1) $A_{se}\dot{x}_e + A_s = 0$, $(s = m+1,\ldots,n;\ e=1,2,\ldots,n)$,

here A_{se} and A_s are functions of $x_1,x_2,\ldots,x_n$ and t. This ituation gives rise to the presence of redundant coordinates. n this case the parameters $x_1,x_2,\ldots,x_n$ are called Poincaré-etaev variables.

In addition to holonomic constraints (1.2.1) the system ay be subject to nonholonomic constraints expressed by m-ℓ on-intergrable equations of the form

(1.2.2) $F_\alpha(x_1,x_2,\ldots,x_n;\ \dot{x}_1,\dot{x}_2,\ldots,\dot{x}_n;\ t) = 0,$

$(\alpha = \ell+1,\ldots,m)$.

n case the equations (1.2.2) are linear in the $\dot{x}$'s, the onstraints are said to be linear nonholonomic; otherwise onlinear nonholonomic. An airplane, rising at a given angle, s an example of a nonlinear nonholonomic constraint.

The presence of holonomic constraints of the form (1.2.1) mplies that the possible displacements $\delta x_1, \delta x_2,\ldots, \delta x_n$ atisfy the relations

(1.2.3) $A_{se}\ \delta x_e = 0, (s = m+1,\ldots,n;\ e = 1,2,\ldots,n)$.

For nonlinear nonholonomic constraints of the type (1.2.2), he two fundamental principles of analytical dynamics - the rinciple of d'Alembert-Lagrange and the least-constraint rinciple of Gauss - are found to be inconsistent. However, he inconsistency can be removed by defining carefully the ossible displacements δx_e. Following N.G.Četaev [14], we have

efinition 3. δx_e are said to be possible displacements con-istent with the constraints (1.2.2), provided that the relations

(1.2.4) $\frac{\partial F_\alpha}{\partial \dot{x}_e}\ \delta x_e = 0$, $(\alpha = \ell+1,\ldots,m;\ e = 1,2,\ldots,n)$, ld where δx_e are infinitely small arbitrary quantities.

The constraints for which relations (1.2.4) hold are lled constraints of the type of N.G.Četaev. In the case of lonomic or linear nonholonomic constraints, the relations .2.4) reduce to the usual definition of possible displacents.

It must be remarked, however, that the relations (1.2.4) e not satisfied by every nonlinear nonholonomic constraint. .S.Novoselov [31] has given an example of such a constraint ich is not of Četaev's type.

In the sequel, we consider the system subject to lonomic constraints (1.2.1) and also to nonholonomic contraints (1.2.2). However, it will be convenient to allow the ystem to have a larger number of degrees of freedom by ignoring ll the nonholonomic constraints. The system is then referred as the associated holonomic system.

1.3. Displacement operators for the Associated Holonomic ystem

Let the system whose position is defined by Poincaré-Cetaev ariables $x_1, x_2, \ldots, x_n$ be subject to n-m holonomic constraints the form (1.2.1).

Following the point of view of N.G.Četaev [15], we construct isplacement operators for the system under consideration. Let us troduce n-m independent parameters $\eta_{m+1}, \eta_{m+2}, \ldots, \eta_n$ by the nstraint equations (1.2.1):

..3.1) $\eta_s = A_{se}\,\dot{x}_e + A_s = 0,$ $(s = m+1,\dots,n;\ e = 1,2,\dots,n).$

now choose any m parameters $\eta_1, \eta_2, \dots, \eta_m$ which are independent mong themselves and also independent with respect to $\eta_{m+1}, \eta_{m+2}, \dots,$ $_n$. Let these parameters be given by the equations

..3.2) $\eta_p = A_{pe}\,\dot{x}_e + A_p,$ $(p = 1,2,\dots,m;\ e = 1,2,\dots,n),$

here A_{pe} and A_p are known functions of $x_1, x_2, \dots, x_n$ and t. quations (1.3.1) and (1.3.2) furnish n independent differential orms $\eta_1 dt, \eta_2 dt, \dots, \eta_n dt$.

Since any linear form relative to $dx_1, dx_2, \dots, dx_n$, dt can e expressed as a linear function of $\eta_1 dt, \eta_2 dt, \dots, \eta_n dt$, dt, the otal differential of the arbitrary function $f(x_1, x_2, \dots, x_n, t)$ an be expressed in the form

1.3.3) $df = [X_0(f) + \eta_e X_e(f)]dt,$ $(e = 1,2,\dots,n)$

s we are interested in the explicit expressions of the operators n (1.3.3), we substitute for the η's in (1.3.3) to obtain

1.3.4) $df = X_0(f)dt + X_e(f)A_{eh}dx_h + X_e(f)A_e dt,$

$(e,h = 1,2,\dots,n).$

ut

1.3.5) $df = \frac{\partial f}{\partial t}\,dt + \frac{\partial f}{\partial x_h}\,dx_h,$ $(h = 1,2,\dots,n).$

omparing (1.3.4) and (1.3.5), we have

1.3.6) $$\begin{cases} A_{eh}X_e = \dfrac{\partial}{\partial x_h}, \\ X_0 = \dfrac{\partial}{\partial t} - A_e X_e, \end{cases}$$

$(e,h = 1,2,\dots,n).$

From (1.3.6) it follows that

$$X_e = \xi_e^h \frac{\partial}{\partial x_h}, \qquad (e,h = 1,2,\dots,n), \tag{1.3.7}$$

where ξ_e^h are functions of $x_1, x_2, \dots, x_n, t$. In view of the relations (1.3.6) and (1.3.7) we obtain

$$X_o = \frac{\partial}{\partial t} + \xi_o^h \frac{\partial}{\partial x_h}, \qquad (h = 1,2,\dots,n), \tag{1.3.8}$$

where

$$\xi_o^h = - \xi_e^h A_e, \qquad (e,h = 1,2,\dots,n),$$

Let us now make use of equations (1.2.1), so that the relation (1.3.3) reduces to

$$df = [X_o(f) + \eta_p X_p(f)]dt, \qquad (p = 1,2,\dots,m). \tag{1.3.9}$$

Analogously we introduce the parameters $\omega_1, \omega_2, \dots, \omega_n$ of which $\omega_{m+1}, \omega_{m+2}, \dots, \omega_n$ vanish by virtue of equations of constraint (1.2.3). The variation δf can, therefore, be expressed in the form

$$\delta f = \omega_p X_p(f), \qquad (p = 1,2,\dots,m). \tag{1.3.10}$$

Finally we obtain the infinitesimal displacement operators

$$X_o = \frac{\partial}{\partial t} + \xi_o^h \frac{\partial}{\partial x_h}, \quad X_p = \xi_p^h \frac{\partial}{\partial x_h}, \tag{1.3.11}$$

$$(p = 1,2,\dots,m;\ h = 1,2,\dots,n),$$

where ξ_o^h and ξ_p^h are functions of $x_1, x_2, \dots, x_n, t$. It is evident that the operators X_o, X_p depend on the choice of the displacement parameters η_p and ω_p.

N.G.Čataev [15] has shown that the operators X_o, X_p form closed system. Therefore we have

(.3.12) $(X_o, X_p) = C_{opq} X_q, (X_p, X_q) = C_{pqr} X_r$,

$(p,q,r = 1,2,\ldots,n)$,

ere C_{opq} and C_{pqr} are functions of $x_1, x_2, \ldots, x_n, t$.

1.4. The Fundamental Equations

Consider a dynamical system consisting of N particles. denote the coordinates of the particles, referred to some xed set of rectangular axes, by $u_1, u_2, \ldots, u_{3N}$. The coordites of the ρth particle are $u_{3\rho-2}, u_{3\rho-1}, u_{3\rho}$. The mass this particle is denoted indifferently by $m_{3\rho-2}, m_{3\rho-1}$ $m_{3\rho}$. Let $\dot{u}_\rho, \ddot{u}_\rho$ denote the velocity and acceleration, espectively, corresponding to the coordinate u_ρ of the article of mass m_ρ.

We assume that the external forces admit a force nction U depending on the coordinates $u_1, u_2, \ldots, u_{3N}$ and e time t; so that the external force corresponding to u_ρ $\frac{\partial U}{\partial u_\rho}$.

Let $x_1, x_2, \ldots, x_n$ be the Poincaré-Četaev variables ecifying the position of the system at the time t. Then e Cartesian coordinates $u_1, u_2, \ldots, u_{3N}$ and the force funcion U will be functions of $x_1, x_2, \ldots, x_n$ and t.

The equations of motion for each of the N particles f the system are

(1.4.1) $$m_{(\rho)}\ddot{u}_\rho = \frac{\partial U}{\partial u_\rho} + \Phi_\rho, \qquad (\rho = 1,2,\ldots,3N),$$

where Φ_ρ are the forces of constraint. Assuming that the constraints are ideal (which are supposed to do no work), Φ_ρ satisfy the condition

(1.4.2) $$\Phi_\rho \delta u_\rho = 0,$$

for an arbitrary possible displacement $\delta u_1, \delta u_2, \ldots, \delta u_{3N}$. From (1.4.1) and (1.4.2) we derive the <u>fundamental equation</u> for a dynamical system

(1.4.3) $$\left(m_{(\rho)}\ddot{u}_\rho - \frac{\partial U}{\partial u_\rho}\right)\delta u_\rho = 0,$$

valid for an arbitrary possible displacement. It is a unification of the principles of virtual work in statics and of d'Alembert for a single rigid body. The equation (1.4.3) was discovered by Lagrange in or about 1760 and embodies the <u>principle of d'Alembert-Lagrange.</u>

The fundamental equation is the basis of the succeeding theory. We shall need to express it in a form due to K.E.Shurova [38], involving the displacement parameters ω_p, η_p and displacement operators X_p.

In (1.3.10) we put $f = u_\rho$ to obtain

(1.4.4) $$\delta u_\rho = \omega_p X_p(u_\rho), \qquad (\rho = 1,2,\ldots,3N; p = 1,2,\ldots,m).$$

By means of (1.4.3) and (1.4.4) we get

$$\omega_p \left[m_{(\rho)}\ddot{u}_\rho X_p(u_\rho) - \frac{\partial U}{\partial u_\rho} X_p(u_\rho)\right] = 0,$$

or

(1.4.5) $$\omega_p \left[\frac{d}{dt}\left(m_{(\rho)}\dot{u}_\rho X_p(u_\rho)\right) - m_{(\rho)}\dot{u}_\rho \frac{d}{dt}\left(X_p(u_\rho)\right) - X_p(U)\right] = 0,$$
$$(\rho = 1,2,\ldots,3N;\ p = 1,2,\ldots,m).$$

CHAPTER II

EQUATIONS OF MOTION

2.1. General Considerations

In this chapter we discuss the construction of infinite-mal displacement operators for a dynamical system which ves subject to holonomic as well as nonlinear nonholonomic nstraints of Cetaev's type. Using these operators, different ethods are employed to derive the general equations of motion the system. To begin with, we use direct method based on e use of the fundamental equation with simultaneous allowance r the constraints. Then we make use of the methods due to aplygin, Hamel and Appell. We also show the equivalence of e equations of motion obtained by these different methods.

2.2. Construction of Infinitesimal Displacement Operators

Let the position of a dynamical system be defined by the oincaré-Četaev variables $x_1, x_2, \ldots, x_n$, and let the system move ubject to n-m holonomic constraints of the form (1.2.1) and -ℓ nonlinear nonholonomic constraints of the form (1.2.2).

Taking $f(x_1, x_2, \ldots, x_n, t) = x_e$ the relations (1.3.9) and (1.3.10), n view of (1.3.11), yield

$$\dot{x}_e = \xi_o^e + \eta_p \xi_p^e, \qquad (e = 1,2,\ldots,n;\ p=1,2,\ldots,m), \tag{2.2.1}$$

nd

$$\delta x_e = \omega_p \xi_p^e, \qquad (e = 1,2,\ldots,n;\ p = 1,2,\ldots,m). \tag{2.2.2}$$

virtue of (2.2.1) the equations of constraint (1.2.2) take

form

2.3) $f_\alpha(x_1,x_2,\ldots,x_n;\eta_1,\eta_2,\ldots,\eta_m;t) = 0$, $(\alpha = \ell+1,\ldots,m)$,

re

$$f_\alpha(x_1,x_2,\ldots,x_n;\eta_1,\eta_2,\ldots,\eta_m;t)$$
$$\equiv F_\alpha(x_1,x_2,\ldots,x_n;\dot{x}_1,\ldots,\dot{x}_n;t).$$

m the relations (2.2.2) we have

$$\frac{\partial F_\alpha}{\partial \dot{x}_e}\,\delta x_e = \frac{\partial F_\alpha}{\partial \dot{x}_e}\,\omega_p \xi_p^e ,$$

ich, in view of (2.2.1), becomes

$$\frac{\partial F_\alpha}{\partial \dot{x}_e}\,\delta x_e = \frac{\partial F_\alpha}{\partial \dot{x}_e}\,\frac{\partial \dot{x}_e}{\partial \eta_p}\,\omega_p = \frac{\partial f_\alpha}{\partial \eta_p}\,\omega_p .$$

nsequently the equations (1.2.4), defining the possible dis-

acements, assume the form

.2.4) $\frac{\partial f_\alpha}{\partial \eta_p}\,\omega_p = 0$, $(\alpha = \ell+1,\ldots,m;\, p=1,2,\ldots,m)$.

If the rank of the matrix $\left\| \frac{\partial f_\alpha}{\partial \eta_p} \right\|$ is $m - \ell$, we can

lve the equations (2.2.3) for $\eta_{\ell+1},\eta_{\ell+2},\ldots,\eta_m$ in terms of

e independent parameters $\eta_1, \eta_2,\ldots,\eta_\ell$. Thus, the equations of

nholonomic constraints in terms of the parameters of real

isplacement become

.2.5) $\eta_\alpha = \eta_\alpha(x_1,x_2,\ldots,x_n;\eta_1,\eta_2,\ldots,\eta_\ell;t)$, $(\alpha=\ell+1,\ldots,m)$.

rresponding to the constraint equations (2.2.5), the relations

.2.4) become

$$\text{.6)}\qquad \omega_\alpha = c_{\alpha i}\omega_i, \qquad (i=1,2,\ldots,\ell;\ \alpha=\ell+1,\ldots,m),$$

s

$$c_{\alpha i} = \frac{\partial \eta_\alpha}{\partial \eta_i}.$$

ollows from the last equation that $c_{\alpha i}$ are functions of $_2,\ldots,x_n;\ \eta_1,\eta_2,\ldots,\eta_\ell$ and t.

In order to construct displacement operators, we use .5) to eliminate the dependent parameters η_α from (1.3.9). , the change df in an arbitrary function $f(x_1,x_2,\ldots,x_n;t)$ esponding to the real displacement of the nonholonomic em is expressed by the relation

$$\text{.7)}\qquad df = \{X_0(f) + \eta_i X_i(f) + \eta_\alpha X_\alpha(f)\}dt,$$

$$(i=1,2,\ldots,\ell;\ \alpha=\ell+1,\ldots,m).$$

larly the relations (1.3.10) and (2.2.6) lead to

$$\text{.8)}\qquad \delta f = \omega_i X_i(f) + c_{\alpha i}\omega_i X_\alpha(f), \quad (i=1,2,\ldots,\ell;\ \alpha=\ell+1,\ldots,m).$$

us put

$$\text{.9)}\qquad \begin{cases} Y_0 = X_0 + (\eta_\alpha - c_{\alpha i}\eta_i)X_\alpha, \\ Y_i = X_i + c_{\alpha i}X_\alpha, \\ (i=1,2,\ldots,\ell;\ \alpha=\ell+1,\ldots,m). \end{cases}$$

the relation (2.2.7) becomes

$$\text{.10)}\qquad df = \{Y_0(f) + \eta_i Y_i(f)\}dt, \qquad (i=1,2,\ldots,\ell),$$

(2.2.8) becomes

$$\text{.11)}\qquad \delta f = \omega_i Y_i(f), \qquad (i=1,2,\ldots,\ell),$$

operators $Y_0, Y_1,\ldots,Y_\ell$ are the infinitesimal displacement ators for the nonlinear nonholonomic system for constraints Chetaev's type.

2.3. Computation of the Commutators

We use the relations (2.2.9) to compute the commutators (Y_o,Y_i) and (Y_i,Y_j). We have

$$(Y_o,Y_i) = Y_oY_i - Y_iY_o$$

$$= [X_o+(\eta_\alpha-\eta_j c_{\alpha j})X_\alpha][X_i+c_{\beta i}X_\beta]-[X_i+c_{\beta i}X_\beta][X_o+(\eta_\alpha-\eta_j c_{\alpha j})X_\alpha],$$

$$(i,j = 1,2,\ldots,\ell; \quad \alpha,\beta = \ell+1,\ldots,m),$$

which reduces to

$$(Y_o,Y_i) = (X_o,X_i)+c_{\beta i}(X_o,X_\beta)+(\eta_\alpha-\eta_j c_{\alpha j})(X_\alpha,X_i)+c_{\beta i}(\eta_\alpha-\eta_j c_{\alpha j})(X_\alpha,X$$

$$[illegible]+Y_o(c_{\beta i})X_\beta-Y_i(\eta_\alpha-\eta_j c_{\alpha j})X_\alpha.$$

In view of (1.3.12), the last relation is equivalent to

$$(Y_o,Y_i) = C_{oip}X_p+c_{\beta i}C_{o\beta p}X_p+(\eta_\alpha-\eta_j c_{\alpha j})C_{\alpha ip}X_p+c_{\beta i}(\eta_\alpha-\eta_j c_{\alpha j})C_{\alpha\beta p}X_p +$$

$$[illegible]+Y_o(c_{\beta i})X_\beta-Y_i(\eta_\alpha-\eta_j c_{\alpha j})X_\alpha,$$

$$(i,j=1,2,\ldots,\ell;\ \alpha,\beta=\ell+1,\ldots,m;\ p=1,2,\ldots,m).$$

Separating the sums with respect to the index p into separate sums from 1 to ℓ and $\ell+1$ to m, we obtain

$$(2.3.1)\quad (Y_o,Y_i) = [C_{oik}+c_{\beta i}C_{o\beta k}+(\eta_\alpha-\eta_j c_{\alpha j})(C_{\alpha ik}+c_{\beta i}C_{\alpha\beta k})]X_k+$$

$$+ [C_{oi\beta}+c_{\alpha i}C_{o\alpha\beta}+(\eta_\alpha-\eta_j c_{\alpha j})(C_{\alpha i\beta}+c_{\gamma i}C_{\alpha\gamma\beta})+Y_o(c_{\beta i})$$

$$- Y_i(\eta_\beta-c_{\beta j}\eta_j)]X_\beta, \qquad (i,j,k=1,2,\ldots,\ell;\alpha,\beta,\gamma=\ell+1,\ldots,m).$$

We introduce

$$(2.3.2)\ K_{oik} = C_{oik}+c_{\beta i}C_{o\beta k}+(\eta_\alpha-\eta_j c_{\alpha j})(C_{\alpha ik}+c_{\beta i}C_{\alpha\beta k}),$$

$$(i,j,k=1,2,\ldots,\ell;\ \alpha,\beta=\ell+1,\ldots,m),$$

and

(2.2.3) $K^{*}_{oi\beta} = K_{oi\beta} + Y_o(c_{\beta i}) - Y_i(\eta_\beta - \eta_j c_{\beta j}) - c_{\beta k}K_{oik}.$

The term $c_{\beta k}K_{oik}$, occurring in the expression for $K^{*}_{oi\beta}$, is introduced so that (2.3.1) may be expressed in terms of $Y_1, Y_2, \ldots, Y_\ell$. Finally the relation (2.3.1) becomes

$$(2.3.4) \qquad (Y_o, Y_i) = K_{oik}Y_k + K^{*}_{oi\beta}X_\beta, \quad (i,k=1,2,\ldots,\ell;\beta=\ell+1,\ldots,m).$$

Again, with the help of (2.2.9), we have

$$(Y_i, Y_j) = Y_iY_j - Y_jY_i$$
$$= (X_i + c_{\alpha i}X_\alpha)(X_j + c_{\beta j}X_\beta) - (X_j + c_{\beta j}X_\beta)(X_i + c_{\alpha i}X_\alpha),$$

which is equivalent to

$$(Y_i, Y_j) = (X_i, X_j) + c_{\beta j}(X_i, X_\beta) + c_{\alpha i}(X_\alpha, X_j) + c_{\alpha i}c_{\beta j}(X_\alpha, X_\beta) +$$
$$+ Y_i(c_{\beta j})X_\beta - Y_j(c_{\alpha i})X_\alpha.$$

By virtue of (1.3.12), the last relation becomes

$$(Y_i, Y_j) = C_{ijp}X_p + c_{\beta j}C_{i\beta p}X_p + c_{\alpha i}C_{\alpha jp}X_p + c_{\alpha i}c_{\beta j}C_{\alpha\beta p}X_p +$$
$$+ Y_i(c_{\beta j})X_\beta - Y_j(c_{\alpha i})X_\alpha$$
$$= [C_{ijk} + c_{\beta j}C_{i\beta k} + c_{\alpha i}(C_{\alpha jk} + c_{\beta j}C_{\alpha\beta k})]X_k +$$
$$+ [C_{ij\beta} + c_{\alpha j}C_{i\alpha\beta} + c_{\alpha i}(C_{\alpha j\beta} + c_{\gamma j}C_{\alpha\gamma\beta}) + Y_i(c_{\beta j}) - Y_j(c_{\beta i})]X_\beta$$
$$(i,j,k = 1,2,\ldots,\ell;\ \alpha,\beta,\gamma = \ell+1,\ldots,m).$$

Let us put

(2.3.5) $$K_{ijk} = C_{ijk} + c_{\beta j}C_{i\beta k} + c_{\alpha i}(C_{\alpha jk} + c_{\beta j}C_{\alpha\beta k}),$$

and

(2.3.6) $$K^{*}_{ij\beta} = K_{ij\beta} + Y_i(c_{\beta j}) - Y_j(c_{\beta i}) - c_{\beta k}K_{ijk},$$

the last term in (2.3.6) is introduced so as to express (Y_i, Y_j) in terms of $Y_1, Y_2, \ldots, Y_\ell$. Hence we get

(2.3.7) $$(Y_i, Y_j) = K_{ijk}Y_k + K^{*}_{ij\beta}X_\beta, \quad (i,j=1,2,\ldots,\ell;\beta=\ell+1,\ldots,m).$$

2.4. Equations of Motion in Poincaré-Četaev Variables

In this section we obtain the equations of motion of the nonholonomic system by using the displacement operators (2.2.9) and the fundamental equation of dynamics with simultaneous allowance for all the constraints imposed on the system.

Let u_ρ typify any one of the three rectangular coordinates of any particle of mass m_ρ and let U be the force function so that $\frac{\partial U}{\partial u_\rho}$ typifies a component of force corresponding to u_ρ. According to the fundamental equation of dynamics, we have

(2.4.1) $$(m_{(\rho)}\ddot{u}_\rho - \frac{\partial U}{\partial u_\rho})\delta u_\rho = 0, \qquad (\rho = 1,2,\ldots,3N).$$

Since the system is subject to constraints (1.2.1) and (1.2.2), the equation (2.2.11) shows that δu_ρ satisfy the relations

(2.4.2) $$\delta u_\rho = \omega_i Y_i(u_\rho), \qquad (i = 1,2,\ldots,\ell).$$

Substituting from (2.4.2) in (2.4.1) and taking into account the independence of $\omega_1, \omega_2, \ldots, \omega_\ell$, we get the equations

$$m_{(\rho)}\ddot{u}_\rho Y_i(u_\rho) - \frac{\partial U}{\partial u_\rho} Y_i(u_\rho) = 0, \quad (i = 1,2,\ldots,\ell),$$

which are equivalent to the equations

$$(2.4.3) \quad \frac{d}{dt}[m_{(\rho)}\dot{u}_\rho Y_i(u_\rho)] - m_{(\rho)}\dot{u}_\rho \frac{d}{dt}[Y_i(u_\rho)] = Y_i(U), \quad (i=1,2,\ldots,\ell).$$

Since $Y_i(u_\rho)$ is a function of $x_1, x_2, \ldots, x_n, \eta_1, \eta_2, \ldots, \eta_\ell, t$, the derivatives $\dot{u}_\rho$ and $\frac{d}{dt}[Y_i(u_\rho)]$, in view of (2.2.10), are given by

$$(2.4.4) \quad \dot{u}_\rho = Y_0(u_\rho) + \eta_i Y_i(u_\rho), \qquad (i = 1,2,\ldots,\ell),$$

and

$$\frac{d}{dt}[Y_i(u_\rho)] = Y_0 Y_i(u_\rho) + \eta_j Y_j Y_i(u_\rho) + \frac{\partial Y_i(u_\rho)}{\partial \eta_j}\dot{\eta}_j.$$

$$(i,j = 1,2,\ldots,\ell)$$

Using equations (2.2.9) and (2.4.4), the last relation becomes

$$(2.4.5) \quad \frac{d}{dt}[Y_i(u_\rho)] = (Y_0, Y_i)u_\rho + \eta_j (Y_j, Y_i)u_\rho + Y_i(\dot{u}_\rho) + \dot{\eta}_j \frac{\partial c_{\alpha i}}{\partial \eta_j} X_\alpha(u_\rho),$$

$$(i,j = 1,2,\ldots,\ell; \; \alpha = \ell+1,\ldots,m).$$

Again from (2.4.4) we have

$$(2.4.6) \quad \frac{\partial \dot{u}_\rho}{\partial \eta_i} = Y_i(u_\rho).$$

The last equation together with (2.3.4) and (2.3.7) allows us to write the equation (2.4.5) in the form

$$(2.4.7) \quad \frac{d}{dt}[Y_i(u_\rho)] = K_{0ik}\frac{\partial \dot{u}_\rho}{\partial \eta_k} + K^*_{0i\beta}X_\beta(u_\rho) + \eta_j K_{jik}\frac{\partial \dot{u}_\rho}{\partial \eta_k} + \eta_j K^*_{ji\beta}X_\beta(u_\rho)$$

$$+ Y_i(\dot{u}_\rho) + \dot{\eta}_j \frac{\partial c_{\beta i}}{\partial \eta_j} X_\beta(u_\rho),$$

$$(i,j,k = 1,2,\ldots,\ell; \; \beta = \ell+1,\ldots,m).$$

Substituting from (2.4.6) and (2.4.7) in (2.4.3), we obtain the equations

$$\text{(2.4.8)}\quad \frac{d}{dt}[m_{(\rho)}\dot{u}_\rho\frac{\partial\dot{u}_\rho}{\partial\eta_i}]-m_{(\rho)}\dot{u}_\rho\frac{\partial\dot{u}_\rho}{\partial\eta_k}[K_{oik}+\eta_j K_{jik}]-m_{(\rho)}\dot{u}_\rho X_\beta(u_\rho)\times$$

$$[K^*_{oi\beta}+\eta_j K^*_{ji\beta}+\frac{\partial c_{\beta i}}{\partial\eta_j}\dot{\eta}_j]-m_{(\rho)}\dot{u}_\rho Y_i(\dot{u}_\rho)-Y_i(U)=0,$$

$$(i,j,k=1,2,\ldots,\ell;\ \beta=\ell+1,\ldots,m).$$

If T is the kinetic energy of the system, we have

$$T=\frac{1}{2}m_{(\rho)}\dot{u}^2_\rho,\qquad (\rho=1,2,\ldots,3N).$$

By means of equations (2.4.4) we may assume $\dot{u}_\rho$ to be expressed in terms of the independent parameters $\eta_1,\eta_2,\ldots,\eta_\ell$, so that T becomes a function of $x_1,x_2,\ldots,x_n$, $\eta_1,\ldots,\eta_\ell$ and t. Let this form of T be denoted by $\overline{T}$. Then we have

$$\text{(2.4.9)}\quad \frac{\partial\overline{T}}{\partial\eta_k}=m_{(\rho)}\dot{u}_\rho\frac{\partial\dot{u}_\rho}{\partial\eta_k},\quad Y_i(\overline{T})=m_{(\rho)}\dot{u}_\rho Y_i(\dot{u}_\rho).$$

Let us put

$$\text{(2.4.10)}\quad \frac{\partial\overline{T}}{\partial\eta_\beta}=m_{(\rho)}\dot{u}_\rho X_\beta(u_\rho).$$

By means of (2.4.9) and (2.4.10), the equations (2.4.8) finally assume the form

2.4.11) $$\frac{d}{dt}\left(\frac{\partial \bar{T}}{\partial \eta_i}\right) - \frac{\partial \bar{T}}{\partial \eta_k}[K_{oik}+\eta_j K_{jik}] - \frac{\partial T}{\partial \eta_\beta}[K^*_{oi\beta}+\eta_j K^*_{ji\beta}+\frac{\partial c_{\beta i}}{\partial \eta_j}\dot{\eta}_j] -$$

$$- Y_i(\bar{T}+U) = 0, \qquad (i,j,k=1,2,\ldots,\ell;\beta=\ell+1,\ldots,m).$$

These are the equations of motion of the nonholonomic ystem in Poincaré-Četaev variables.

If we take $f = x_e$ in equation (2.2.7) we obtain

2.4.12) $$\frac{dx_e}{dt} = \xi^e_o + \eta_i \xi^e_i + \eta_\alpha \xi^e_\alpha,$$

$$(i = 1,2,\ldots,\ell;\ \alpha= \ell+1,\ldots,m;\ e=1,2,\ldots,n)$$

he equations (2.4.11) and (2.4.12) furnish n+ℓ equations to etermine $x_1, x_2, \ldots, x_n$ and $\eta_1, \eta_2, \ldots, \eta_\ell$ as functions of the ime t.

To give a mechanical interpretation of the terms $\frac{\partial T}{\partial \eta_\beta}$, hich occur in the equations of motion (2.4.11), we consider he associated holonomic system. Differentiating the kinetic nergy T of the associated holonomic system, we obtain

$$\frac{\partial T}{\partial \eta_\beta} = m_{(\rho)}\dot{a}_\rho \frac{\partial \dot{a}_\rho}{\partial \eta_\beta}, \qquad (\beta = \ell+1,\ldots,m).$$

ubstituting from (1.4.8) we have

$$\frac{\partial T}{\partial \eta_\beta} = m_{(\rho)}\dot{a}_\rho X_\beta(a_\rho).$$

t follows, therefore, that $\frac{\partial T}{\partial \eta_\beta}$ represents the momenta orresponding to the dependent parameters η_β as defined by e equations (2.2.5).

Particular case: Let the nonholonomic constraints (1.2.2) be linear in $\dot{x}_1, \dot{x}_2, \ldots, \dot{x}_n$. Then the equations (2.2.5) and (2.2.6) reduce to

$$(2.4.13) \qquad \eta_\alpha = \eta_i c_{\alpha i} + c_{\alpha o}, \qquad (i=1,2,\ldots,\ell;\ \alpha=\ell+1,\ldots,m),$$

and

$$(2.4.14) \qquad \omega_\alpha = \omega_i c_{\alpha i}.$$

Here $c_{\alpha i}$ and $c_{\alpha o}$ are functions of $x_1, x_2, \ldots, x_n$ and t, so that

$$(2.4.15) \qquad \frac{\partial c_{\beta i}}{\partial \eta_j} = 0, \qquad (i,j=1,2,\ldots,\ell;\beta=\ell+1,\ldots,m).$$

Using (2.2.9), the displacement operators are given by the relations

$$Y_o = X_o + c_{\alpha o} X_\alpha, \quad Y_i = X_i + c_{\alpha i} X_\alpha,$$
$$(\alpha = \ell+1,\ldots,m;\ i=1,2,\ldots,\ell).$$

Furthermore, the equations (2.3.2), (2.3.3), (2.3.5) and (2.3.6) reduce to the equations

$$(2.4.16) \quad \begin{cases} K_{ijk} = C_{ijk} + c_{\beta j} C_{i\beta k} + c_{\alpha i}(C_{\alpha jk} + c_{\beta j} C_{\alpha\beta k}), \\ \overset{*}{K}_{ij\beta} = K_{ij\beta} - c_{\beta k} K_{ijk} + Y_i(c_{\beta j}) - Y_j(c_{\beta i}), \\ (i=o,1,\ldots,\ell;j,k=1,2,\ldots,\ell;\alpha,\beta=\ell+1,\ldots,m). \end{cases}$$

As a consequence of (2.4.15) and (2.4.16) the equations (2.4.11) become

$$(2.4.17)\quad \frac{d}{dt}\left(\frac{\partial \overline{T}}{\partial \eta_i}\right) - Y_i(\overline{T}+U) - \frac{\partial \overline{T}}{\partial \eta_k}\left[K_{oik} + \eta_j K_{jik}\right] - \frac{\partial T}{\partial \eta_\beta}\left[K^*_{oi\beta} + \eta_j K^*_{ji\beta}\right] = 0,$$

$$(i, j, k = 1, 2, \ldots, \ell;\ \beta = \ell+1, \ldots, m).$$

ese are the equations of motion obtained by Fam Guen [22].

2.5. Equations of Motion by Chaplygin's Method

In the formulation of the equations of motion of the nlinear nonholonomic system in the form (2.4.11) we take to consideration the nonholonomic constraints from the ry beginning. In this section we follow the point of view S.A.Chaplygin [16] and use the equations of nonholonomic nstraints after the fundamental equation has been transformed the form (1.4.11).

The ω_p's occurring in the fundamental equation (1.4.11) e not all independent but are connected by the equations .2.6). Eliminating the dependent ω_α's from (1.4.11), we tain

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \eta_i}\right) - C_{oiq}\frac{\partial T}{\partial \eta_q} - \eta_q C_{qir}\frac{\partial T}{\partial \eta_r} - X_i(T+U) + c_{\alpha i}\left\{\frac{d}{dt}\left(\frac{\partial T}{\partial \eta_\alpha}\right) - \right.$$

$$\left. C_{o\alpha q}\frac{\partial T}{\partial \eta_q} - \eta_q C_{q\alpha r}\frac{\partial T}{\partial \eta_r} - X_\alpha(T+U)\right\}\omega_i = 0.$$

view of the independence of ω_i's we have the equations of tion in the form

$$(2.5.1)\quad \frac{d}{dt}\left(\frac{\partial T}{\partial \eta_i}\right) - C_{oiq}\frac{\partial T}{\partial \eta_q} - \eta_q C_{qir}\frac{\partial T}{\partial \eta_r} - X_i(T+U) + c_{ai}\left[\frac{d}{dt}\left(\frac{\partial T}{\partial \eta_a}\right) - C_{oaq}\frac{\partial T}{\partial \eta_q} - \eta_q C_{qar}\frac{\partial T}{\partial \eta_r} - X_a(T+U)\right] = 0,$$

$$(i=1,2,\ldots,\ell;\, a=\ell+1,\ldots,m;\, q,r=1,2,\ldots,m).$$

2.6. Equations of Motion by Hamel's Method

The formulation of the equations of motion by the method of Hamel [24] consists in using the rule

$$(2.6.1)\qquad d\delta f = \delta df,$$

to transform the fundamental equation (1.4.3). It is of course to be noted that the presence of nonholonomic constraints is to be taken into consideration after the transformation of the fundamental equation. This is necessary because for nonholonomic constraints, the rule given by (2.6.1) does not, in general, hold [5], though it does hold for holonomic systems [27].

From the relations (1.3.9) and (1.3.10) it follows that

$$d\delta f = d\omega_p X_p(f) + \omega_p [X_o X_p(f) + \eta_q X_q X_p(f)]dt, \quad (p,q=1,2,\ldots,m),$$

and

$$\delta df = \omega_p X_p X_o(f)dt + \delta\eta_p X_p(f)dt + \eta_p \omega_q X_q X_p(f)dt.$$

As a consequence of (2.6.1), we have

$$\delta\eta_p X_p(f) = \frac{d\omega_p}{dt} X_p(f) + \omega_p (X_o, X_p) f + \eta_p \omega_q (X_p, X_q) f, \quad (p,q=1,2,\ldots,m).$$

Since $X_o, X_1, \ldots, X_m$ form a closed system, the relations (1.3.12) yield

$$\delta\eta_p X_p(f) = \frac{d\omega_p}{dt} X_p(f) + \omega_p C_{opq} X_q(f) + \eta_p \omega_q C_{pqr} X_r(f),$$

$$(p,q,r = 1,2,\ldots,m).$$

Equating on both sides the coefficients of $X_p(f)$, we have

(2.6.2) $$\delta\eta_p = \frac{d\omega_p}{dt} + \omega_q C_{oqp} + \eta_q \omega_r C_{qrp}.$$

We now use the rule (2.6.1) to transform fundamental equation (1.4.3). Thus we obtain

$$\frac{d}{dt}(m_{(\rho)} \dot{u}_\rho \delta u_\rho) - m_{(\rho)} \dot{u}_\rho \delta \dot{u}_\rho - \frac{\partial U}{\partial u_\rho} \delta u_\rho = 0.$$

Substituting for δu_ρ from (1.4.4), the last equation becomes

$$\frac{d}{dt}(\omega_p m_{(\rho)} \dot{u}_\rho X_p(u_\rho)) - \delta(\frac{1}{2} m_{(\rho)} \dot{u}_\rho^2) - \delta U = 0,$$

which is equivalent to

(2.6.3) $$\frac{d}{dt}(\omega_p \frac{\partial T}{\partial \eta_p}) - \delta(T+U) = 0,$$

where T is the kinetic energy of the associated holonomic system, expressed as a function of $x_1, x_2, \ldots, x_n; \eta_1, \ldots, \eta_m$ and t. The equation (2.6.3) is the fundamental equation in terms of the Poincaré-Četaev variables provided that the relation (2.6.1) holds.

The equation (2.6.3) can be written as

$$\frac{d\omega_p}{dt}\frac{\partial T}{\partial \eta_p} + \omega_p \frac{d}{dt}\left(\frac{\partial T}{\partial \eta_p}\right) - \frac{\partial T}{\partial \eta_p}\delta\eta_p - \omega_p X_p(T+U) = 0,$$

ch, by the help of (2.6.2), reduces to

6.4) $$\omega_p\left[\frac{d}{dt}\left(\frac{\partial T}{\partial \eta_p}\right) - C_{opq}\frac{\partial T}{\partial \eta_q} - \eta_q C_{qpr}\frac{\partial T}{\partial \eta_r} - X_p(T+U)\right] = 0.$$

The equation (2.6.4) is the same as the equation (1.4.11). m this point, the derivation of the equations of motion ceeds as in Chaplygin's method explained in section 2.5. ce the equations of motion are

6.5) $$\frac{d}{dt}\left(\frac{\partial T}{\partial \eta_i}\right) - C_{oiq}\frac{\partial T}{\partial \eta_q} - \eta_q C_{qir}\frac{\partial T}{\partial \eta_r} - X_i(T+U) +$$

$$+\sigma_{\alpha i}\left[\frac{d}{dt}\left(\frac{\partial T}{\partial \eta_\alpha}\right) - C_{o\alpha q}\frac{\partial T}{\partial \eta_q} - \eta_q C_{q\alpha r}\frac{\partial T}{\partial \eta_r} - X_\alpha(T+U)\right] = 0,$$

$$(i=1,2,\ldots,\ell;\ \alpha=\ell+1,\ldots,m;\ \ q,r=1,2,\ldots,m).$$

Particular Case:- Let us take $x_1, x_2, \ldots, x_n$ as the rangian coordinates and suppose that the system is subject y to n-ℓ nonlinear nonholonomic constraints of the form

$$f_\alpha(x_1, x_2, \ldots, x_n;\ \dot{x}_1, \ldots, \dot{x}_n; t) = 0,\ (\alpha=\ell+1,\ldots,n).$$

this case we take $\eta_1, \eta_2, \ldots, \eta_n$ as $\dot{x}_1, \dot{x}_2, \ldots, \dot{x}_n$ and $\omega_2, \ldots, \omega_n$ as $\delta x_1, \delta x_2, \ldots, \delta x_n$; so that the displacement rators (1.3.11) are given by

$$X_o = \frac{\partial}{\partial t},\quad X_e = \frac{\partial}{\partial x_e},\quad (e = 1,2,\ldots,n).$$

ese operators commute and hence all the C's in (2.6.5) vanish. Moreover, let Q_a be the generalised forces:

$$Q_a = \frac{\partial U}{\partial x_a}.$$

In view of these considerations, the equations of motion (2.6.5) take the form

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{x}_i}\right) - \frac{\partial T}{\partial x_i} - Q_i + c_{ai}\left[\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{x}_a}\right) - \frac{\partial T}{\partial x_a} - Q_a\right] = 0,$$

$$(i = 1,2,\ldots,\ell \; ; \; a = \ell+1,\ldots,n),$$

in Lagrangian coordinates. These are the equations of motion obtained by G.S.Pogosov [35].

2.7. Appell's Equations of Motion

In 1900, P. Appell [4] derived a general form of the equations of motion of a system in Lagrangian coordinates. These equations involve a function S, called the energy of acceleration of the system. The importance of these equations lies in the fact that they are applicable to every system whether holonomic or nonholonomic. In this section we derive Appell's equations for a nonholonomic system in Poincaré-Četaev variables.

Consider a dynamical system which moves subject to n-m holonomic constraints of the form (1.2.3) and m-ℓ nonlinear nonholonomic constraints of the form (1.2.4), which express the variations $\delta x_1, \delta x_2, \ldots, \delta x_n$ of the Poincaré-Četaev variables $x_1, x_2, \ldots, x_n$.

Let m_ρ typify the mass of a particle of the system, one of whose rectangular coordinates at time t is u_ρ.

The fundamental equation (2.3.1) in conjunction with equation (2.4.2) gives

$$\{m_{(\rho)}\ddot{u}_\rho Y_i(u_\rho) - Y_i(U)\}\omega_i = 0, \quad (i=1,2,\ldots,\ell;\ \rho=1,2,\ldots,3N).$$

Taking into account the independence of $\omega_1, \omega_2, \ldots, \omega_\ell$, the last relation leads to the equations

$$(2.7.1) \qquad m_{(\rho)}\ddot{u}_\rho Y_i(u_\rho) = Y_i(U) \qquad (i=1,2,\ldots,\ell;\ \rho=1,2,\ldots,3N).$$

Differentiating with respect to the time t the expression (2.4.4) for $\dot{u}_\rho$, we obtain

$$(2.7.2) \qquad \ddot{u}_\rho = \ddot{\eta}_i Y_i(u_\rho) + \text{terms not containing } \ddot{\eta}\text{'s}.$$

Therefore

$$(2.7.3) \qquad \frac{\partial \ddot{u}_\rho}{\partial \ddot{\eta}_i} = Y_i(u_\rho).$$

In view of (2.7.3) the equations (2.7.1) become

$$(2.7.4) \qquad m_{(\rho)}\ddot{u}_\rho \frac{\partial \ddot{u}_\rho}{\partial \ddot{\eta}_i} = Y_i(U), \qquad (i = 1,2,\ldots,\ell).$$

Let us introduce the energy of acceleration $\bar{S}$ of the system:

$$(2.7.5) \qquad \bar{S} = \frac{1}{2} m_{(\rho)}\ddot{u}_\rho^2,$$

where $\ddot{u}_\rho$ are given by (2.7.2). Then

$$\frac{\partial \bar{S}}{\partial \ddot{\eta}_i} = m_{(\rho)}\ddot{u}_\rho \frac{\partial \ddot{u}_\rho}{\partial \ddot{\eta}_i}.$$

...paring the last relation with (2.7.4), we obtain the ...ations of motion

$$(2.7.6) \qquad \frac{\partial \bar{S}}{\partial \dot{\eta}_i} = Y_i(U), \qquad (i = 1,2,\ldots,\ell).$$

These are Appell's equations of motion in Poincaré-...taev variables.

If instead of the function $\bar{S}$ we employ S which is the ...ergy of acceleration of the associated holonomic system, we ...e

$$(2.7.7) \qquad \frac{\partial \bar{S}}{\partial \dot{\eta}_i} = \frac{\partial S}{\partial \dot{\eta}_i} + \frac{\partial S}{\partial \dot{\eta}_\alpha}\frac{\partial \dot{\eta}_\alpha}{\partial \dot{\eta}_i}.$$

...fferentiation of the relations (2.2.5) with respect to the ...e t, yields

$$\dot{\eta}_\alpha = \frac{\partial \eta_\alpha}{\partial \eta_i}\dot{\eta}_i + \text{terms not containing } \ddot{\eta}\text{'s}.$$

...sequently

$$(2.7.8) \qquad \frac{\partial \dot{\eta}_\alpha}{\partial \dot{\eta}_i} = \frac{\partial \eta_\alpha}{\partial \eta_i} = c_{\alpha i}.$$

...means of (2.7.7) and (2.7.8) we obtain

$$\frac{\partial \bar{S}}{\partial \dot{\eta}_i} = \frac{\partial S}{\partial \dot{\eta}_i} + c_{\alpha i}\frac{\partial S}{\partial \dot{\eta}_\alpha}.$$

...equations (2.7.6) we substitute for $\frac{\partial \bar{S}}{\partial \dot{\eta}_i}$ from the last rela-...n and for Y_i from (2.2.9). We finally obtain the equations ...motion in the symmetric form

$$(2.7.9) \qquad \frac{\partial S}{\partial \dot{\eta}_i} - X_i(U) + c_{\alpha i}\left[\frac{\partial S}{\partial \dot{\eta}_\alpha} - X_\alpha(U)\right] = 0, \quad (i=1,2,\ldots,\ell;\ \alpha=\ell+1,\ldots,m).$$

2.8. Equivalence of the Equations of Motion

In the preceding sections we used the direct method as
ll as the methods due to Chaplygin, Hamel and Appell to
ormulate the general equations of motion of a nonlinear
onholonomic dynamical system in Poincaré-Četaev variables.
his naturally raises the problem of their equivalence. In
he present section we take up this problem.

The methods of Chaplygin and Hamel lead to the same
quations of motion. It therefore remains to prove their
quivalence to the equations of motion obtained by the direct
ethod or by the method due to Appell.

Let us consider the equations of motion

(2.8.1) $$\frac{d}{dt}\left(\frac{\partial T}{\partial \eta_i}\right) - C_{oiq}\frac{\partial T}{\partial \eta_q} - \eta_q C_{qir}\frac{\partial T}{\partial \eta_r} - X_i(T+U) + c_{\alpha i}\left[\frac{d}{dt}\left(\frac{\partial T}{\partial \eta_\alpha}\right) - C_{o\alpha q}\frac{\partial T}{\partial \eta_q} - \eta_q C_{q\alpha r}\frac{\partial T}{\partial \eta_r} - X_\alpha(T+U)\right] = 0,$$

$$(i=1,2,\ldots,\ell;\ \alpha=\ell+1,\ldots,m;\ q,r=1,2,\ldots,m),$$

hich we obtain by the method of Chaplygin. We transform these
quations to a form which does not contain the dependent para-
eters of real displacement η_α. To this end, we separate in
quations (2.8.1) the sums with respect to the indices q and
 into separate sums from 1 to ℓ and from $\ell+1$ to m, and then
bstitute from (2.2.5) the expressions for the dependent para-
eters η_α. Thus we obtain

$$(2.8.2)\quad \frac{d}{dt}\left(\frac{\partial T}{\partial \eta_i}\right)+c_{\beta i}\frac{d}{dt}\left(\frac{\partial T}{\partial \eta_\beta}\right)-\frac{\partial T}{\partial \eta_k}[C_{oik}+c_{\beta i}C_{o\beta k}+\eta_\alpha(C_{\alpha ik}+c_{\beta i}C_{\alpha\beta k})+$$

$$+\eta_j(C_{jik}+c_{\beta i}C_{j\beta k})]-\frac{\partial T}{\partial \eta_\beta}[C_{oi\beta}+c_{\alpha i}C_{o\alpha\beta}+\eta_\alpha(C_{\alpha i\beta}+c_{\gamma i}C_{\alpha\gamma\beta})+$$

$$+\eta_j(C_{ji\beta}+c_{\alpha i}C_{j\alpha\beta})]-(X_i+c_{\beta i}X_\beta)(T+U) = o,$$

$$(i,j,k = 1,2,\ldots,\ell;\ \alpha,\beta,\gamma = \ell+1,\ldots,m).$$

In view of the relations (2.2.9), the last term on the left side of (2.8.2) is $Y_i(T+U)$. We now replace each η_α by $\eta_\alpha-\eta_j c_{\alpha j}$ in (2.8.2) and make adjustments to obtain

$$(2.8.3)\quad \frac{d}{dt}\left(\frac{\partial T}{\partial \eta_i}\right)+c_{\beta i}\frac{d}{dt}\left(\frac{\partial T}{\partial \eta_\beta}\right)-\frac{\partial T}{\partial \eta_k}[C_{oik}+c_{\beta i}C_{o\beta k}+(\eta_\alpha-\eta_j c_{\alpha j})$$

$$(C_{\alpha ik}+c_{\beta i}C_{\alpha\beta k})+\eta_j(C_{jik}+c_{\beta i}C_{j\beta k}+c_{\alpha j}(C_{\alpha ik}+c_{\beta i}C_{\alpha\beta k}))]-$$

$$-\frac{\partial T}{\partial \eta_\beta}[C_{oi\beta}+c_{\alpha i}C_{o\alpha\beta}+(\eta_\alpha-\eta_j c_{\alpha j})(C_{\alpha i\beta}+c_{\gamma i}C_{\alpha\gamma\beta})+$$

$$+\eta_j(C_{ji\beta}+c_{\alpha i}C_{j\alpha\beta}+c_{\alpha j}(C_{\alpha i\beta}+c_{\gamma i}C_{\alpha\gamma\beta}))]-Y_i(T+U) = o.$$

By virtue of the relations (2.3.2) and (2.3.5), the last equations become

$$(2.8.4)\quad \frac{d}{dt}\left(\frac{\partial T}{\partial \eta_i}\right)+c_{\beta i}\frac{d}{dt}\left(\frac{\partial T}{\partial \eta_\beta}\right)-\frac{\partial T}{\partial \eta_k}[K_{oik}+\eta_j K_{jik}]-\frac{\partial T}{\partial \eta_\beta}[K_{oi\beta}+\eta_j K_{ji\beta}\}-$$

$$-Y_i(T+U) = o,\qquad (i,j,k=1,2,\ldots,\ell;\ \beta=\ell+1,\ldots,m).$$

In the function T, we replace the quantities $\eta_{\ell+1},\ldots,\eta_m$ by their expressions given by (2.2.5), and denote the resulting unction by $\bar{T}(x_1,\ldots,x_n;\eta_1,\ldots,\eta_\ell;t)$. Then differentiating the equation

$$T(x_1,\ldots,x_n;\eta_1,\ldots,\eta_m;t) = \bar{T}(x_1,\ldots,x_n;\eta_1,\ldots,\eta_\ell;t),$$

nd using (2.7.8), we have

$$(2.8.5)\quad \begin{cases} \frac{d}{dt}\left(\frac{\partial \bar{T}}{\partial \eta_i}\right) = \frac{d}{dt}\left(\frac{\partial T}{\partial \eta_i}\right) + c_{\beta i}\,\frac{d}{dt}\left(\frac{\partial T}{\partial \eta_\beta}\right) + \frac{d}{dt}(c_{\beta i})\frac{\partial T}{\partial \eta_\beta}, \\ \frac{\partial \bar{T}}{\partial \eta_k} = \frac{\partial T}{\partial \eta_k} + \frac{\partial T}{\partial \eta_\beta}\,\frac{\partial \eta_\beta}{\partial \eta_k} = \frac{\partial T}{\partial \eta_k} + c_{\beta k}\,\frac{\partial T}{\partial \eta_\beta}, \\ Y_i(\bar{T}) = Y_i(T) + \frac{\partial T}{\partial \eta_\beta}\,Y_i(\eta_\beta). \end{cases}$$

ince $c_{\beta i}$ are functions of $x_1,x_2,\ldots,x_n;\ \eta_1,\ldots,\eta_\ell$ and t, he derivative $\frac{d}{dt}(c_{\beta i})$, occurring in the first relation of (2.8.5), can be calculated by means of the relation (2.2.10). hus, we get

$$(2.8.6)\quad \frac{d}{dt}(c_{\beta i}) = \frac{\partial c_{\beta i}}{\partial \eta_j}\,\dot{\eta}_j + Y_o(c_{\beta i}) + \eta_j Y_j(c_{\beta i}).$$

ubstituting from (2.8.5) and (2.8.6) in (2.8.4), we obtain

$$\frac{d}{dt}\left(\frac{\partial \bar{T}}{\partial \eta_i}\right) - \frac{\partial \bar{T}}{\partial \eta_k}(K_{oik} + \eta_j K_{jik}) - \frac{\partial T}{\partial \eta_\beta}\Big[K_{oi\beta} + \eta_j K_{ji\beta} + \frac{\partial c_{\beta i}}{\partial \eta_j}\,\dot{\eta}_j +$$

$$Y_o(c_{\beta i}) + \eta_j Y_j(c_{\beta i}) - c_{\beta k}K_{oik} - \eta_j c_{\beta k}K_{jik} - Y_i(\eta_\beta)\Big] - Y_i(\bar{T}+U) = 0,$$

hich is equivalent to

$$\frac{d}{dt}\left(\frac{\partial \overline{T}}{\partial \eta_i}\right) - \frac{\partial \overline{T}}{\partial \eta_k}(K_{oik} + \eta_j K_{jik}) - \frac{\partial T}{\partial \eta_\beta}[K_{oi\beta} - c_{\beta k}K_{oik} + Y_o(c_{\beta i}) -$$

$$- Y_i(\eta_\beta - \eta_j c_{\beta j}) + \eta_j(K_{ji\beta} - c_{\beta k}K_{jik} + Y_j(c_{\beta i}) - Y_i(c_{\beta j})) + \dot{\eta}_j \frac{\partial c_{\beta i}}{\partial \eta_j}]$$

$$- Y_i(\overline{T} + U) = o.$$

Using the relations (2.3.3) and (2.3.6) in the last equations, we finally obtain the equations of motion of the system in the form

$$(2.8.7) \quad \frac{d}{dt}\left(\frac{\partial \overline{T}}{\partial \eta_i}\right) - \frac{\partial \overline{T}}{\partial \eta_k}(K_{oik} + \eta_j K_{jik}) - \frac{\partial T}{\partial \eta_\beta}[K^*_{oi\beta} + \eta_j K^*_{ji\beta} + \dot{\eta}_j \frac{\partial c_{\beta i}}{\partial \eta_j}]$$

$$- Y_i(\overline{T} + U) = o, \quad (i,j,k = 1,2,\ldots,\ell;\ \beta = \ell+1,\ldots,m),$$

which are the equations (2.4.11).

This establishes the equivalence of the equations of motion obtained by the direct method and by the methods of Chaplygin and Hamel.

We next consider the Appell's equations in the symmetric form

$$(2.8.8) \qquad \frac{\partial S}{\partial \ddot{\eta}_i} - X_i(U) + c_{\alpha i}\left[\frac{\partial S}{\partial \ddot{\eta}_\alpha} - X_\alpha(U)\right] = o.$$

To prove their equivalence to equations (2.8.1) we establish the identity

$$(2.8.9) \qquad \frac{\partial S}{\partial \ddot{\eta}_p} = \frac{d}{dt}\left(\frac{\partial T}{\partial \eta_p}\right) - C_{opq}\frac{\partial T}{\partial \eta_q} - \eta_q C_{qpr}\frac{\partial T}{\partial \eta_r} - X_p(T),$$

$$(p,q,r = 1,2,\ldots,m).$$

We differentiate S partially with respect to $\ddot{\eta}_p$ to obtain

$$(2.8.10) \qquad \frac{\partial S}{\partial \ddot{\eta}_p} = m_{(\rho)}\ddot{u}_\rho \frac{\partial \ddot{u}_\rho}{\partial \ddot{\eta}_p}.$$

Differentiating with respect to the time t the expression (1.4.6) for $\dot{u}_\rho$, we have

$$\ddot{u}_\rho = \ddot{\eta}_p X_p(u_\rho) + \text{terms not containing } \ddot{\eta}\text{'s}.$$

Consequently

$$\frac{\partial \dot{u}_\rho}{\partial \eta_p} = \frac{\partial \ddot{u}_\rho}{\partial \ddot{\eta}_p} = X_p(u_\rho).$$

The relation (2.8.10), in view of the last result, becomes

$$\frac{\partial S}{\partial \ddot{\eta}_p} = \frac{d}{dt}[m_{(\rho)}\dot{u}_\rho \frac{\partial \dot{u}_\rho}{\partial \eta_p}] - m_{(\rho)}\dot{u}_\rho \frac{d}{dt}[X_p(u_\rho)],$$

which, by virtue of the equation (1.4.9), is equivalent to

$$\frac{\partial S}{\partial \ddot{\eta}_p} = \frac{d}{dt}[m_{(\rho)}\dot{u}_\rho \frac{\partial \dot{u}_\rho}{\partial \eta_p}] - m_{(\rho)}\dot{u}_\rho [X_p(\dot{u}_\rho) + c_{opq}\frac{\partial \dot{u}_\rho}{\partial \eta_q} + \eta_q C_{qpr}\frac{\partial \dot{u}_\rho}{\partial \eta_r}].$$

The last result when expressed in terms of T leads to the identity (2.8.9).

Substitution from the equation (2.8.9) into (2.8.8) immediately leads to the equations (2.8.1).

This establishes the equivalence of Appell's equations to the equations obtained by the method of Chaplygin.

Thus, the proof of the equivalence of the various forms of the equations of motion is complete.

CHAPTER III

TRANSFORMATION OF THE EQUATIONS OF MOTION

3.1. General Considerations

In this chapter we shall transform the equations (2.5.1) to obtain various other forms of the equations of motion of a nonlinear nonholonomic system which moves subject to constraints of the forms (1.2.1) and (1.2.2). In terms of the Poincaré-Četaev variables, the different forms of the equations of motion involve either the derivatives of the kinetic energy or the derivatives of the energy of acceleration of the system. Besides, we use Appell's transformation to introduce a function K_σ and write the equations of motion involving this function. We also introduce cyclic displacement operators in the manner of N.G.Četaev and derive a generalised form of the Chaplygin's equations of motion.

3.2. Important Identities

Let $\phi^{(\sigma)}$ denote the σth derivative with respect to the time t of an arbitrary function $\phi(x_1, x_2, \ldots, x_n; \eta_1, \ldots, \eta_m; t)$. We prove the following identities:

(3.2.1) $$\frac{d}{dt}\left(\frac{\partial\phi}{\partial\eta_p}\right) = \frac{1}{\sigma}\left[\frac{\partial\phi^{(\sigma)}}{\partial\eta_p^{(\sigma-1)}} - X_p(\phi)\right],$$

(3.2.2 $$\frac{\partial\phi_o^{(\sigma)}}{\partial\eta_p^{(\sigma-1)}} = X_p(\phi), \qquad (p = 1,2,\ldots,m;\ \sigma=1,2,3,\ldots).$$

where ϕ_o is the function ϕ regarded as a function of the x's and t, i.e. for fixed values of the η's.

Proof of (3.2.1). Differentiating ϕ with respect to t, we have

$$\dot{\phi} = X_o(\phi) + \eta_p X_p(\phi) + \frac{\partial\phi}{\partial\eta_p}\dot{\eta}_p.$$

Differentiating again and noting that $\frac{\partial}{\partial\eta_p}$ and the X's commute, we get

$$\ddot{\phi} = 2X_o(\frac{\partial\phi}{\partial\eta_p})\dot{\eta}_p + 2X_q(\frac{\partial\phi}{\partial\eta_p})\dot{\eta}_p\eta_q + \frac{\partial^2\phi}{\partial\eta_p\partial\eta_q}\dot{\eta}_p\dot{\eta}_q + X_p(\phi)\dot{\eta}_p + \frac{\partial\phi}{\partial\eta_p}\ddot{\eta}_p +$$

+ terms not containing either $\dot{\eta}$'s or $\ddot{\eta}$'s, $(p,q=1,2,\ldots,m)$.

Similarly

$$\dddot{\phi} = 3[X_o(\frac{\partial\phi}{\partial\eta_p}) + \eta_q X_q(\frac{\partial\phi}{\partial\eta_p}) + \dot{\eta}_q\frac{\partial^2\phi}{\partial\eta_p\partial\eta_q}]\ddot{\eta}_p + \ddot{\eta}_p X_p(\phi) + \frac{\partial\phi}{\partial\eta_p}\dddot{\eta}_p + \ldots,$$

and, more generally,

(3.2.3) $$\phi^{(\sigma)} = \sigma[X_o(\frac{\partial\phi}{\partial\eta_p}) + \eta_q X_q(\frac{\partial\phi}{\partial\eta_p}) + \dot{\eta}_q\frac{\partial^2\phi}{\partial\eta_p\partial\eta_q}]\eta_p^{(\sigma-1)} + X_p(\phi)\eta_p^{(\sigma-1)} + \frac{\partial\phi}{\partial\eta_p}\eta_p^{(\sigma)} + \ldots .$$

From (3.2.3) we obtain

(3.2.4) $$\frac{\partial\phi^{(\sigma)}}{\partial\eta_p^{(\sigma-1)}} = \sigma[X_o(\frac{\partial\phi}{\partial\eta_p}) + \eta_q X_q(\frac{\partial\phi}{\partial\eta_p}) + \dot{\eta}_q\frac{\partial^2\phi}{\partial\eta_p\partial\eta_q}] + X_p(\phi), \qquad (p,q=1,2,\ldots,m;\ \sigma=1,2,3,\ldots).$$

But

$$\frac{d}{dt}\left(\frac{\partial\phi}{\partial\eta_p}\right) = X_o\left(\frac{\partial\phi}{\partial\eta_p}\right) + \eta_q X_q\left(\frac{\partial\phi}{\partial\eta_p}\right) + \dot{\eta}_q \frac{\partial^2\phi}{\partial\eta_p\partial\eta_q}, \tag{3.2.5}$$

$$(p,q = 1,2,\ldots,m).$$

Comparing (3.2.4) and (3.2.5), we obtain the identity (3.2.1).

Proof of (3.2.2). Since ϕ_o is the function ϕ considered as a function of x's and t only, we keep η's fixed. Then (3.2.3) allows us to write

$$\phi_o^{(\sigma)} = \eta_p^{(\sigma-1)} X_p(\phi) + \ldots, \quad (p=1,2,\ldots,m; \sigma=1,2,3,\ldots).$$

From the last relation we have

$$\frac{\partial\phi_o^{(\sigma)}}{\partial\eta_p^{(\sigma-1)}} = X_p(\phi),$$

which establishes the identity (3.2.2).

3.3. Transformations of the Equations of Motion

Here we use the identities (3.2.1) and (3.2.2) to transform the equations of motion (2.5.1) to four other forms which depend on the derivatives of kinetic energy of the system. Although these forms of the equations of motion involve derivatives of η's of order higher than one, yet a physical interpretation of these derivatives is possible. For Lagrangian coordinates $q_1, q_2, \ldots, q_n$ when η's become $\dot{q}$'s, D. Rašković [37] has given an example contributing to the physical reality of the super accelerations $\dddot{q}$'s.

(i) For a nonholonomic system moving subject to constraints of the forms (1.2.1) and (1.2.2), the equations of motion obtained in section 2.5 are

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \eta_i}\right) - C_{oiq}\frac{\partial T}{\partial \eta_q} - \eta_q C_{qir}\frac{\partial T}{\partial \eta_r} - X_i(T+U) + c_{\alpha i}\left[\frac{d}{dt}\left(\frac{\partial T}{\partial \eta_\alpha}\right) - C_{o\alpha q}\frac{\partial T}{\partial \eta_q} - \eta_q C_{q\alpha r}\frac{\partial T}{\partial \eta_r} - X_\alpha(T+U)\right] = 0, \tag{3.3.1}$$

$$(i = 1,2,\ldots,\ell; \alpha = \ell+1,\ldots m; p,q,r = 1,2,\ldots,m).$$

In the identity (3.2.1) we put $\phi = T$ to obtain

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \eta_p}\right) = \frac{1}{\sigma}\left[\frac{\partial T^{(\sigma)}}{\partial \eta_p^{(\sigma-1)}} - X_p(T)\right], \tag{3.3.2}$$

$$(p = 1,2,\ldots,m; \sigma=1,2,3,\ldots),$$

and replacing σ by $(\sigma-1)$ in (3.3.2), we have

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \eta_p}\right) = \frac{1}{\sigma-1}\left[\frac{\partial T^{(\sigma-1)}}{\partial \eta_p^{(\sigma-2)}} - X_p(T)\right], \tag{3.3.3}$$

$$(p = 1,2,\ldots,m; \sigma = 2,3,\ldots).$$

In view of (3.3.2) and (3.3.3), we obtain

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \eta_p}\right) = \frac{\partial T^{(\sigma)}}{\partial \eta_p^{(\sigma-1)}} - \frac{\partial T^{(\sigma-1)}}{\partial \eta_p^{(\sigma-2)}}, \tag{3.3.4}$$

and

$$(3.3.5) \qquad X_p(T) = \sigma\frac{\partial T^{(\sigma-1)}}{\partial \eta_p^{(\sigma-2)}} - (\sigma-1)\frac{\partial T^{(\sigma)}}{\partial \eta_p^{(\sigma-1)}},$$

$$(p = 1,2,\ldots,m;\ \sigma = 2,3,\ldots).$$

Employing (3.3.4) and (3.3.5), the equations (3.3.1) assume the form

$$(3.3.6) \qquad \sigma\left[\frac{\partial T^{(\sigma)}}{\partial \eta_i^{(\sigma-1)}} + c_{\alpha i}\frac{\partial T^{(\sigma)}}{\partial \eta_\alpha^{(\sigma-1)}}\right] - (\sigma+1)\left[\frac{\partial T^{(\sigma-1)}}{\partial \eta_i^{(\sigma-2)}} + c_{\alpha i}\frac{\partial T^{(\sigma-1)}}{\partial \eta_\alpha^{(\sigma-2)}}\right] = Y_i(U) + \bar{F}_{oi},$$

where

$$(3.3.7) \qquad \bar{F}_{oi} = F_{oi} + c_{\alpha i}F_{o\alpha}, \qquad (i=1,2,\ldots,\ell;\ \alpha=\ell+1,\ldots,m),$$

and

$$(3.3.8) \qquad F_{op} = C_{opq}\frac{\partial T}{\partial \eta_q} + \eta_q C_{qpr}\frac{\partial T}{\partial \eta_r}, \qquad (p,q,r = 1,2,\ldots,m).$$

Let us introduce a function $\tilde{T}^{(\sigma)}$ which is obtained from $T^{(\sigma)}$ by first considering it as a function of $\eta^{(\sigma-1)}$'s and then using the nonholonomic constraints (2.2.5). We have

$$(3.3.9) \qquad \frac{\partial \tilde{T}^{(\sigma)}}{\partial \eta_i^{(\sigma-1)}} = \frac{\partial T^{(\sigma)}}{\partial \eta_i^{(\sigma-1)}} + \frac{\partial T^{(\sigma)}}{\partial \eta_\alpha^{(\sigma-1)}}\frac{\partial \eta_\alpha^{(\sigma-1)}}{\partial \eta_i^{(\sigma-1)}},$$

$$(i=1,2,\ldots,\ell;\ \alpha=\ell+1,\ldots,m;\ \sigma=2,3,\ldots).$$

We now consider the constraint equations (2.2.5) and differentiate them σ-1 times with respect to the time t. Then

$$\eta_\alpha^{(\sigma-1)} = \frac{\partial\eta_\alpha}{\partial\eta_i}\eta_i^{(\sigma-1)} + \text{terms not containing } \eta_i^{(\sigma-1)}\text{'s.}$$

Consequently

$$(3.3.10) \qquad \frac{\partial\eta_\alpha^{(\sigma-1)}}{\partial\eta_i^{(\sigma-1)}} = \frac{\partial\eta_\alpha}{\partial\eta_i} = c_{\alpha i}.$$

From (3.3.9) and (3.3.10) it follows that

$$\frac{\partial T^{(\sigma)}}{\partial\eta_i^{(\sigma-1)}} + c_{\alpha i}\frac{\partial T^{(\sigma)}}{\partial\eta_\alpha^{(\sigma-1)}} = \frac{\partial\tilde{T}^{(\sigma)}}{\partial\eta_i^{(\sigma-1)}}.$$

As a consequence of the last relations the equations of motion (3.3.6) assume the form

$$(3.3.11) \qquad \sigma\frac{\partial\tilde{T}^{(\sigma)}}{\partial\eta_i^{(\sigma-1)}} - (\sigma+1)\frac{\partial\tilde{T}^{(\sigma-1)}}{\partial\eta_i^{(\sigma-2)}} = Y_i(U) + \overline{P_{oi}},$$

$$(i = 1,2,\ldots,\ell;\ \sigma = 2,3,\ldots).$$

(ii) Let $\bar{T}^{(\sigma)}$ denote the function $T^{(\sigma)}$ after using the nonholonomic constraints (2.2.5) and $T_1^{(\sigma)}$ denote the function obtained from $T_1^{(\sigma)}$ by first considering it as a function of $\eta^{(\sigma)}$'s and then taking constraints (2.2.5) into account. Then we have

$$(3.3.12) \qquad \frac{\partial\bar{T}^{(\sigma)}}{\partial\eta_i^{(\sigma-1)}} = \frac{\partial T^{(\sigma)}}{\partial\eta_i^{(\sigma-1)}} + c_{\alpha i}\frac{\partial T^{(\sigma)}}{\partial\eta_\alpha^{(\sigma-1)}} + \frac{\partial T^{(\sigma)}}{\partial\eta_\alpha^{(\sigma)}}\frac{\partial\eta_\alpha^{(\sigma)}}{\partial\eta_i^{(\sigma-1)}},$$

and

$$\frac{\partial T_i^{(\sigma)}}{\partial \eta_i^{(\sigma-1)}} = \frac{\partial T^{(\sigma)}}{\partial \eta_\alpha^{(\sigma)}} \frac{\partial \eta_\alpha^{(\sigma)}}{\partial \eta_i^{(\sigma-1)}} .$$

From the last two relations it follows that

$$\frac{\partial \overline{T}^{(\sigma)}}{\partial \eta_i^{(\sigma-1)}} - \frac{\partial T_i^{(\sigma)}}{\partial \eta_i^{(\sigma-1)}} = \frac{\partial T^{(\sigma)}}{\partial \eta_i^{(\sigma-1)}} + c_{\alpha i} \frac{\partial T^{(\sigma)}}{\partial \eta_\alpha^{(\sigma-1)}} .$$

Consequently the equations (3.3.6) assume the form

$$(3.3.13) \quad \sigma\left[\frac{\partial \overline{T}^{(\sigma)}}{\partial \eta_i^{(\sigma-1)}} - \frac{\partial T_i^{(\sigma)}}{\partial \eta_i^{(\sigma-1)}}\right] - (\sigma+1)\left[\frac{\partial \overline{T}^{(\sigma-1)}}{\partial \eta_i^{(\sigma-2)}} - \frac{\partial T_i^{(\sigma-1)}}{\partial \eta_i^{(\sigma-2)}}\right] =$$

$$= Y_i(U) + \overline{F_{oi}} , \quad (i = 1,2,\ldots,\ell;\sigma = 2,3,\ldots).$$

This is the desired transformation of the equations of motion.

(iii) From (3.3.12) we have

$$\frac{\partial \overline{T}^{(\sigma)}}{\partial \eta_i^{(\sigma-1)}} - \frac{\partial \overline{T}^{(\sigma-1)}}{\partial \eta_i^{(\sigma-2)}} = \left[\frac{\partial T^{(\sigma)}}{\partial \eta_i^{(\sigma-1)}} - \frac{\partial T^{(\sigma-1)}}{\partial \eta_i^{(\sigma-2)}}\right] + c_{\alpha i}\left[\frac{\partial T^{(\sigma)}}{\partial \eta_\alpha^{(\sigma-1)}} - \frac{\partial T^{(\sigma-1)}}{\partial \eta_\alpha^{(\sigma-2)}}\right] +$$

$$+ \frac{\partial T}{\partial \eta_\alpha}\left[\frac{\partial \eta_\alpha^{(\sigma)}}{\partial \eta_i^{(\sigma-1)}} - \frac{\partial \eta_\alpha^{(\sigma-1)}}{\partial \eta_i^{(\sigma-2)}}\right] ,$$

which, in view of (3.3.4), is equivalent to

$$(3.3.14) \quad \frac{\partial \overline{T}^{(\sigma)}}{\partial \eta_i^{(\sigma-1)}} - \frac{\partial \overline{T}^{(\sigma-1)}}{\partial \eta_i^{(\sigma-2)}} = \frac{d}{dt}\left(\frac{\partial T}{\partial \eta_i}\right) + c_{\alpha i} \frac{d}{dt}\left(\frac{\partial T}{\partial \eta_\alpha}\right) +$$

$$+ \frac{\partial T}{\partial \eta_\alpha}\left(\frac{\partial \eta_\alpha^{(\sigma)}}{\partial \eta_i^{(\sigma-1)}} - \frac{\partial \eta_\alpha^{(\sigma-1)}}{\partial \eta_i^{(\sigma-2)}}\right).$$

Consequently the equations of motion (3.3.13) are transformed into the equations

$$\sigma\frac{d}{dt}\left(\frac{\partial \overline{T}}{\partial \eta_i}\right) - \frac{\partial \overline{T}^{(\sigma-1)}}{\partial \eta_i^{(\sigma-2)}} - \sigma\frac{\partial T^{(\sigma)}}{\partial \eta_i^{(\sigma-1)}} + (\sigma+1)\frac{\partial T_1^{(\sigma-1)}}{\partial \eta_i^{(\sigma-2)}} \tag{3.3.16}$$

$$= Y_i(U) + \overline{F_{oi}}\,, \qquad (i = 1,2,\ldots,l;\ \sigma=2,3,\ldots).$$

If we take $\sigma = 2$ in the equations of motion of the form (3.3.11), (3.3.13) or (3.3.16) and consider the x's to be the Langrangian coordinates we obtain as a special case the equations of motion due to I. Cenov [8], who claims these equations to be novel forms though, in fact, they represent only a mathematical transformation of Lagrangian equations of motion.

(iv) To obtain another transformation of the equations of motion we substitute from (3.3.2) in (3.3.1) and take (3.3.8) into account. Then we get

$$\frac{1}{\sigma}\left\{\frac{\partial T^{(\sigma)}}{\partial \eta_i^{(\sigma-1)}} - (\sigma+1)X_i(T)\right\} - P_{oi} + c_{\alpha i}\left[\frac{1}{\sigma}\left\{\frac{\partial T^{(\sigma)}}{\partial \eta_\alpha^{(\sigma-1)}} - (\sigma+1)X_\alpha(T)\right\} - P_{o\alpha}\right] \tag{3.3.17}$$

$$= Y_i(U)\,, \qquad (i=1,2,\ldots,l;\ \alpha=l+1,\ldots,m;\ \sigma=1,2,3,\ldots).$$

These are the generalised Mangeron-Deleanu equations[28] in Poincaré-Četaev variables.

Taking $\sigma = 1$ in (3.3.17), we obtain the Kielsen form of the equations of motion [30]:

$$(3.3.18) \quad \frac{\partial \dot{T}}{\partial \eta_i} - 2X_i(T) - P_{oi} + c_{\alpha i}[\frac{\partial \dot{T}}{\partial \eta_\alpha} - 2X_\alpha(T) - P_{o\alpha}] = Y_i(U),$$

$$(i = 1,2,\ldots \ell; \alpha = \ell+1,\ldots,m).$$

For $\sigma = 2$, the equations (3.3.17) yield the Canov's equations [11]:

$$(3.3.19) \quad \frac{1}{2}[\frac{\partial \ddot{T}}{\partial \eta_i} - 3X_i(T)] - P_{oi} + c_{\alpha i}[\frac{1}{2}(\frac{\partial \ddot{T}}{\partial \eta_\alpha} - 3X_\alpha(T)) - P_{o\alpha}] = Y_i(U),$$

$$(i = 1,2,\ldots,\ell; \alpha = \ell+1,\ldots m).$$

3.4. The Function R_σ and the Equations of Motion

In the identity (3.2.2) we put $\phi = T$ to obtain

$$(3.4.1) \quad \frac{\partial T_o^{(\sigma)}}{\partial \eta_p^{(\sigma-1)}} = X_p(T), \quad (p=1,2,\ldots m; \sigma=1,2,3,\ldots).$$

By means of (3.4.1) the equations of motion (3.3.17) become

$$(3.4.2) \quad \frac{1}{\sigma}[\frac{\partial T^{(\sigma)}}{\partial \eta_i^{(\sigma-1)}} - (\sigma+1)\frac{\partial T_o^{(\sigma)}}{\partial \eta_i^{(\sigma-1)}}] - P_{oi} + c_{\alpha i}[\frac{1}{\sigma}(\frac{\partial T^{(\sigma)}}{\partial \eta_\alpha^{(\sigma-1)}} - (\sigma+1)\frac{\partial T_o^{(\sigma)}}{\partial \eta_\alpha^{(\sigma-1)}}) - P_{o\alpha}]$$

$$= Y_i(U) \qquad (i = 1,2,\ldots,\ell; \alpha=\ell+1,\ldots,m; \sigma= 1,2,3,\ldots).$$

Let us consider the function R_σ:

$$R_\sigma = \frac{1}{\sigma}[T^{(\sigma)} - (\sigma+1)T_o^{(\sigma)}] - P_{op}\eta_p^{(\sigma-1)},$$

$$(p = 1,2,\ldots m).$$

Then we have

$$(3.4.3)\qquad \frac{\partial R_\sigma}{\partial \eta_p^{(\sigma-1)}} = \frac{1}{\sigma}\{\frac{\partial T^{(\sigma)}}{\partial \eta_p^{(\sigma-1)}} - (\sigma+1)\frac{\partial T_o^{(\sigma)}}{\partial \eta_p^{(\sigma-1)}}\} - P_{op},$$

which is valid for $\sigma = 2,3,\ldots$ and not for $\sigma = 1$ because P_{op} itself depends upon η's. In view of (3.4.3), the equations (3.4.2) take the form

$$(3.4.4)\qquad \frac{\partial R_\sigma}{\partial \eta_i^{(\sigma-1)}} + c_{\alpha i}\frac{\partial R_\sigma}{\partial \eta_\alpha^{(\sigma-1)}} = Y_i(U),$$

$$(i=1,2,\ldots,\ell;\alpha=\ell+1,\ldots,m;\sigma=2,3,\ldots).$$

We now introduce a function $\tilde{R}_\sigma$ which is obtained from R_σ by first considering it as a function of $\eta^{(\sigma-1)}$'s and then using the constraints (2.2.5). Therefore we have

$$(3.4.5)\qquad \frac{\partial \tilde{R}_\sigma}{\partial \eta_i^{(\sigma-1)}} = \frac{\partial R_\sigma}{\partial \eta_i^{(\sigma-1)}} + c_{\alpha i}\frac{\partial R_\sigma}{\partial \eta_\alpha^{(\sigma-1)}} .$$

Comparing (3.4.4) and (3.4.5), we finally get the equations of motion in the form

$$(3.4.6)\qquad \frac{\partial \tilde{R}_\sigma}{\partial \eta_i^{(\sigma-1)}} = Y_i(U), \qquad (i = 1,2,\ldots,\ell;\ \sigma=2,3,\ldots).$$

These equations are a generalisation of the Appell-Canov equations [17] in Poincaré-Četaev variables.

3.5. Equations of Motion Involving Derivatives of the Energy of Acceleration

In section 2.7, we obtained the Appell's equations of motion in the form

.1) $$\frac{\partial S}{\partial \ddot{\eta}_i} + c_{\alpha i}\frac{\partial S}{\partial \ddot{\eta}_\alpha} = Y_i(U) , \quad (i=1,2,\dots,\ell;\alpha=\ell+1,\dots,m).$$

S is the energy of acceleration of the associated holonomic
em. Differentiating S successively (σ-2) times with respect
and noting that S is a function of $x_1, x_2, \dots, x_n; \eta_1, \dots, \eta_m;$
$\dots, \dot{\eta}_m$ and t, we have

$$\dot{S} = X_o(S) + \eta_p X_p(S) + \frac{\partial S}{\partial \dot{\eta}_p}\ddot{\eta}_p + \frac{\partial S}{\partial \ddot{\eta}_p}\dddot{\eta}_p,$$

$$\ddot{S} = \frac{\partial S}{\partial \ddot{\eta}_p}\ddddot{\eta}_p + \dots,$$

....

$$S^{(\sigma-2)} = \frac{\partial S}{\partial \ddot{\eta}_p}\eta_p^{(\sigma-1)} + \dots .$$

equently

.2) $$\frac{\partial S^{(\sigma-2)}}{\partial \eta_p^{(\sigma-1)}} = \frac{\partial S}{\partial \ddot{\eta}_p} , \quad (p = 1,2,\dots,m;\sigma = 2,3,\dots).$$

ccount of (3.5.2) the equations (3.5.1) become

.3) $$\frac{\partial S^{(\sigma-2)}}{\partial \eta_i^{(\sigma-1)}} + c_{\alpha i}\frac{\partial S^{(\sigma-2)}}{\partial \eta_\alpha^{(\sigma-1)}} = Y_i(U).$$

$\bar{S}^{(\sigma-2)}$ denote the function $S^{(\sigma-2)}$ when the constraints are
n into account. Then we have

$$\frac{\partial \bar{S}^{(\sigma-2)}}{\partial \eta_i^{(\sigma-1)}} = \frac{\partial S^{(\sigma-2)}}{\partial \eta_i^{(\sigma-1)}} + c_{\alpha i}\frac{\partial S^{(\sigma-2)}}{\partial \eta_\alpha^{(\sigma-1)}} .$$

last relation together with (3.5.3) leads to the equations
tion

$$(3.5.4)\qquad \frac{\partial \bar{S}^{(\sigma-2)}}{\partial \eta_i^{(\sigma-1)}} = Y_i(U), \qquad (i = 1,2,\ldots\ell;\sigma = 2,3,\ldots).$$

A comparison of (3.4.6) with (3.5.4) shows that the functions $\tilde{R}_\sigma$ and $\bar{S}^{(\sigma-2)}$ are the same as far as the terms in $\eta^{(\sigma-1)}$'s are concerned. In particular, $\tilde{R}_2$ coincides with the energy of acceleration $\bar{S}$ of the nonholonomic system.

3.6. Appell's Transformation

Let us define a function K_σ :

$$K_\sigma = R_\sigma - \eta_p^{(\sigma-1)} X_p(U), \quad (p = 1,2,\ldots,m; \sigma=2,3,\ldots).$$

Then

$$\frac{\partial K_\sigma}{\partial \eta_p^{(\sigma-1)}} = \frac{\partial R_\sigma}{\partial \eta_p^{(\sigma-1)}} - X_p(U), \quad (p=1,2,\ldots,m;\sigma=2,3,\ldots),$$

and the equations (3.4.4) become

$$(3.6.1)\qquad \frac{\partial K_\sigma}{\partial \eta_i^{(\sigma-1)}} + c_{\alpha i}\frac{\partial K_\sigma}{\partial \eta_\alpha^{(\sigma-1)}} = 0, \quad (i=1,2,\ldots\ell;\alpha=\ell+1,\ldots,m;\sigma=2,3,\ldots).$$

If we use the function $\tilde{K}_\sigma$ which is obtained from K_σ by taking constraints (2.2.5) into account, we have

$$\frac{\partial \tilde{K}_\sigma}{\partial \eta_i^{(\sigma-1)}} = \frac{\partial K_\sigma}{\partial \eta_i^{(\sigma-1)}} + c_{\alpha i}\frac{\partial K_\sigma}{\partial \eta_\alpha^{(\sigma-1)}}.$$

Consequently the equations (3.6.1) assume the form

$$(3.6.2)\qquad \frac{\partial \tilde{K}_\sigma}{\partial \eta_i^{(\sigma-1)}} = 0, \qquad (i=1,2,\ldots,\ell;\ \sigma= 2,3,\ldots).$$

The equations of motion in the form (3.6.2) show that the function $\tilde{K}_\sigma$ assumes the stationary value in the actual motion as compared with any concievable motion (consistent with the constraints) obtained by varying the $\eta^{(\sigma-1)}$'s in $\tilde{K}_\sigma$.

3.7. Identification of $\tilde{K}_2$ with the Gaussian Constraint

Let m_ρ denote the mass of a particle of the system one of whose rectangular coordinates is u_ρ. Let $\ddot{u}_\rho$ be the acceleration and $\frac{\partial U}{\partial u_\rho}$ the external force corresponding to u_ρ. Then the Gaussian constraint G is defined by

$$G = \frac{1}{2} m_{(\rho)} [\ddot{u}_\rho - \frac{\partial U}{\partial u_\rho} \Big/ m_{(\rho)}]^2, \qquad (\rho = 1,2,\ldots,3N).$$

The expression for G can be written in the form

$$(3.7.1) \quad G = \frac{1}{2} m_{(\rho)} \ddot{u}_\rho^2 - \frac{\partial U}{\partial u_\rho} \ddot{u}_\rho + \text{terms not containing } \ddot{u}_\rho\text{'s}.$$

The first term on the right-hand side of (3.7.1) is the energy of acceleration $\overline{S}$ obtained by taking the constraints (2.2.5) into account. If in the second term we substitute for $\ddot{u}_\rho$ its expression given by (2.7.2), we have

$$G = \overline{S} - \frac{\partial U}{\partial u_\rho} [\ddot{\eta}_i Y_i(u_\rho) + \text{terms not containing } \ddot{\eta}\text{'s}]$$

$$= \overline{S} - \ddot{\eta}_i Y_i(U) + \ldots, \qquad (i = 1,2,\ldots,\ell).$$

As shown in section 3.6, the functions $\overline{S}$ and $\tilde{R}_2$ coincide as far as the terms in $\ddot{\eta}$'s are concerned. Therefore we may write

$$G = \bar{K}_2 - \dot{\eta}_i Y_i(U) + \ldots$$
$$= \tilde{K}_2 + \ldots .$$

follows that the function $\tilde{K}_2$ and the Gaussian constraint G e identical as far as the terms in $\dot{\eta}$'s are concerned.

3.8. Cyclic Displacement Operators

Let the m - ℓ displacement operators $X_{\ell+1},\ldots,X_m$ of the sociated holonomic system corresponding to the dependent splacement parameters η_α and θ_α be cyclic in the sense of G.Čataev [15], and let X_o commute with $X_{\ell+1},\ldots,X_m$. Then then e following conditions are satisfied:

3.8.1) $(X_p,X_\alpha) = o,\ X_\alpha(T+U) = o,\quad (X_o,X_\alpha) = o,$

$(\alpha = \ell+1,\ldots,m;\ p = 1,2,\ldots,m).$

urthermore, for the nonholonomic constraints (2.2.5) we assume e relations:

3.8.2) $X_\alpha(\eta_\beta) = 0,\quad X_\alpha(c_{\beta i}) = 0,\ (i=1,2,\ldots,\ell;\alpha,\beta=\ell+1,\ldots,m).$

With the help of the relations (3.8.1) and (3.8.2) we ransform the equations of motion in the form (2.4.11). To this nd we notice that in the present case the following conditions re satisfied:

$$(3.8.3)\quad \begin{cases} Y_i(\bar{T}+U) = X_i(\bar{T}+U), \\ K_{oik} = C_{oik},\ K_{jik} = C_{jik}, \\ K^*_{oi\beta} = C_{oi\beta} - c_{\beta k}C_{oik} + X_o(c_{\beta i}) - X_i(n_\beta - {}_j c_{\beta j}), \\ K^*_{ji\beta} = C_{ji\beta} - c_{\beta k}C_{jik} + X_j(\sigma_{\beta i}) - X_i(\sigma_{\beta j}). \end{cases}$$

view of the relations (3.8.3), the equations (2.4.11) become

$$(3.8.4)\quad \frac{d}{dt}\left(\frac{\partial \bar{T}}{\partial \eta_i}\right) - X_i(\bar{T}+U) - \frac{\partial \bar{T}}{\partial \eta_k}[C_{oik} + \eta_j C_{jik}] - \frac{\partial T}{\partial \eta_\beta}[C_{oi\beta} - c_{\beta k}C_{oik} +$$

$$+ X_o(c_{\beta i}) - X_i(n_\beta - \eta_j c_{\beta j}) + \eta_j(C_{ji\beta} - \sigma_{\beta k}C_{jik} + X_j(c_{\beta i}) - X_i(c_{\beta j})) +$$

$$+ \frac{\partial \sigma_{\beta i}}{\partial \eta_j}\dot{\eta}_j] = o, \qquad (i,j,k=1,2,\ldots,\ell;\beta=\ell+1,\ldots,m).$$

se equations give a generalisation of the Chaplygin equations Poincaré-C... variables for the nonlinear nonholonomic tem.

We consider the following particular cases of the ations (3.8.4):

(i) If the nonholonomic constraints are linear, they are en by equations of the form (2.4.13). Consequently the ditions (3.8.2) become

$$X_\alpha(\sigma_{\beta i}) = o, \quad X_\alpha(\sigma_{\beta o}) = o, \quad (\alpha,\beta=\ell+1,\ldots,m).$$

...ever, the quantities $\frac{\partial \sigma_{\beta i}}{\partial \eta_j}$ vanish, so that the equations

(3.8.4) reduces to the form

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \eta_i}\right) - X_i(T+U) - \frac{\partial T}{\partial \eta_k}[C_{oik} + \eta_j C_{jik}] - \tag{3.8.5}$$

$$- \frac{\partial T}{\partial \eta_\beta}[C_{oi\beta} - \sigma_{\beta k}C_{oik} + X_o(\sigma_{\beta i}) - X_i(\sigma_{\beta o}) +$$

$$+ \eta_j(C_{ji\beta} - \sigma_{\beta k}C_{jik} + X_j(\sigma_{\beta i}) - X_i(\sigma_{\beta j}))] = o,$$

$$(i,j,k = 1,2,\ldots,\ell;\ \beta = \ell+1,\ldots,m).$$

These are the equations of motion obtained by Fam Guen [21].

(ii) Let the parameters $x_1, x_2, \ldots, x_n$ be the Lagrangian coordinates, and let the constraints imposed on the system be only nonholonomic constraints of Cetaev's type:

$$f_\alpha(x_1, x_2, \ldots, x_\ell;\ \dot{x}_1, \ldots, \dot{x}_n;\ t) = o, \qquad (\alpha=\ell+1,\ldots,n). \tag{3.8.6}$$

Here

$$X_o = \frac{\partial}{\partial t},\quad X_e = \frac{\partial}{\partial x_e},\quad \eta_e = \dot{x}_e \qquad (e=1,2,\ldots,n).$$

The system has $x_{\ell+1}, \ldots, x_n$ as cyclic coordinates and hence the conditions (3.8.1) and (3.8.2) become

$$(X_e, X_\alpha) = o,\quad X_\alpha(T+U) = o,\quad (X_o, X_\alpha) = 0,\ (e=1,2,\ldots,n;\ \alpha=\ell+1,\ldots,n)$$

Consequently the equations (3.8.4) assume the form

CHAPTER IV

APPLICATIONS

4.1. General Considerations

The equations of motion derived in chapter 2 and 3 are applied to solve some examples of the motion of nonholonomic dynamical systems. In all, four examples have been solved to illustrate the theory.

Though the important case in practice is that in which the nonholonomic constraints are linear in the velocity components, there do arise problems in analytical dynamics in which the nonholonomic constraints are nonlinear in the velocities. In 1911, Appell gave an example of a nonholonomic constraint which is expressed by an equation which is nonlinear with respect to the velocities.

Nonlinear nonholonomic constraints can be introduced analytically in problems of dynamics. Such constraints can also be realised in problems concerning the regulation of the motion, or in other problems of technical interest where the constraints between the moving parts are realised by means of electromagnetic devices. It is expected that with technical development the use of nonlinear nonholonomic constraints will also increase.

4.2. Example of P. Appell

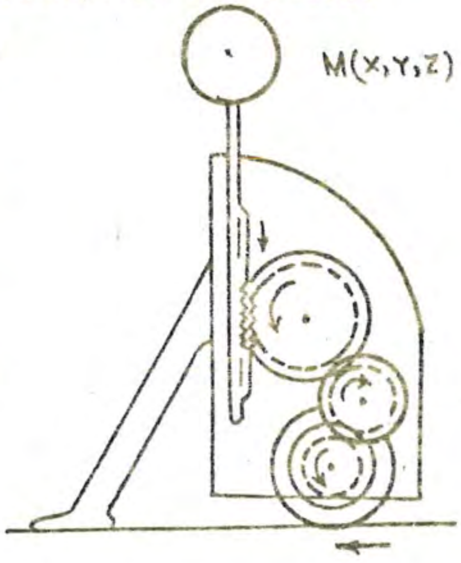
In 1911, P.Appell [3] gave the following example of a nonlinear nonholonomic constraint. Suppose that a leg of an armchair is propped by a small round wheel which rolls without sliding on a smooth horizontal plane. The wheel is connected with the leg through a fork. The collar in which the fork terminates, surrounds the leg and can easily revolve round it, therefore the plane of the wheel can turn. Besides, the wheel can revolve round a horizontal axle fixed in the fork.

Let there be some mechanism within the fork so that the wheel is connected with a vertical pivot which is fixed in the leg and carries at its end a particle M.

Let x,y,z be the coordinates of M referred to a fixed set of rectangular axes in which the z-axis is vertical. Let the mechanism be such that the relation $dz = bd\phi$, is satisfied where $b\,(>0)$ is a constant and ϕ is the angle of rotation of the wheel. We may, therefore, suppose that the z coordinate of the particle M is proportional to the angle of rotation of the wheel.

If the shoulder of the fork is small then the coordinates of the point of contact of the wheel will be equal to the coordinates x,y of the particle M.

A sketch of the model of this example is shown in the adjoining figure.



The condition of no sliding of the wheel demands that

$$\dot{x}^2 + \dot{y}^2 = r^2\dot{\phi}^2,$$

where r is the radius of the wheel.

We assume that the transmission mechanism is ideal. Consequently we impose on the motion of the particle M a nonlinear nonholonomic constraint expressed by the equation

$$\dot{z} = a\sqrt{(\dot{x}^2+\dot{y}^2)}, \tag{4.2.1}$$

where

$$a = \frac{b}{r} = \text{constant } (>0).$$

The problem is to find the path of the particle M in space. To solve this problem, we apply the equations of motion in the form (2.4.11).

Let us choose x,y,z as the Poincaré-Četaev variables. Due to the absence of holonomic constraints, the parameters of real displacement of the associated holonomic system can be taken as

$$\eta_1 = \dot{x}, \quad \eta_2 = \dot{y}, \quad \eta_3 = \dot{z}. \tag{4.2.2}$$

Consequently the displacement operators are

$$X_0 = \frac{\partial}{\partial t}, \quad X_1 = \frac{\partial}{\partial x}, \quad X_2 = \frac{\partial}{\partial y}, \quad X_3 = \frac{\partial}{\partial z}. \tag{4.2.3}$$

The equation of constraint (4.2.1) can be expressed in the form

$$\eta_3 = a\sqrt{(\eta_1^2+\eta_2^2)}, \tag{4.2.4}$$

which yields

$$(4.2.5)\qquad \begin{cases} c_{31} = \dfrac{\partial \eta_3}{\partial \eta_1} = \dfrac{a\eta_1}{\sqrt{(\eta_1^2+\eta_2^2)}}\,, \\[2ex] c_{32} = \dfrac{\partial \eta_3}{\partial \eta_2} = \dfrac{a\eta_2}{\sqrt{(\eta_1^2+\eta_2^2)}}\,. \end{cases}$$

With the help of the relation (2.2.9) and (4.2.5) the displacement operators Y_o, Y_1, Y_2 for the nonholonomic system are given by

$$(4.2.6)\qquad \begin{cases} Y_o = X_o = \dfrac{\partial}{\partial t} \\[2ex] Y_1 = X_1 + \dfrac{a\eta_1}{\sqrt{(\eta_1^2+\eta_2^2)}}\,X_3 = \dfrac{\partial}{\partial x} + \dfrac{a\eta_1}{\sqrt{(\eta_1^2+\eta_2^2)}}\,\dfrac{\partial}{\partial z}\,, \\[2ex] Y_2 = X_2 + \dfrac{a\eta_2}{\sqrt{(\eta_1^2+\eta_2^2)}}\,X_3 = \dfrac{\partial}{\partial y} + \dfrac{a\eta_2}{\sqrt{(\eta_1^2+\eta_2^2)}}\,\dfrac{\partial}{\partial z}\,. \end{cases}$$

Since all the operators X_o, X_1, X_2, X_3 commute, the C's vanish. This implies that all the K's given by (2.3.2) and (2.3.5) are zero. Moreover, the constraint (4.2.4) is independent of the parameters x,y,z and the time t, it follows that all the K^*'s given by (2.3.3) and (2.3.6) also vanish.

The kinetic energy T of the associated holonomic system is

$$(4.2.7)\qquad T = \tfrac{1}{2}m(\eta_1^2+\eta_2^2+\eta_3^2)\,,$$

where m is the mass of the particle M. Substituting for η_3 from (4.2.4), the kinetic energy $\overline{T}$ of the nonholonomic system is

(4.2.8) $\qquad \overline{T} = \frac{1}{2} m(a^2+1)(\eta_1^2+\eta_2^2).$

From (4.2.6) and (4.2.8) it follows that

(4.2.9) $\qquad Y_1(\overline{T}) = Y_2(\overline{T}) = 0.$

Again, from (4.2.7) and (4.2.8) we obtain

$$(4.2.10) \qquad \begin{cases} \dfrac{\partial \overline{T}}{\partial \eta_3} = m\eta_3, \\ \dfrac{\partial \overline{T}}{\partial \eta_1} = m(a^2+1)\eta_1, \\ \dfrac{\partial \overline{T}}{\partial \eta_2} = m(a^2+1)\eta_2. \end{cases}$$

Differentiating the relations (4.2.5) with respect to η_1 and η_2, we get

$$(4.2.11) \qquad \begin{cases} \dfrac{\partial c_{11}}{\partial \eta_1} = \dfrac{a\eta_2^2}{(\eta_1^2+\eta_2^2)^{3/2}}, & \dfrac{\partial c_{21}}{\partial \eta_2} = \dfrac{-a\eta_1\eta_2}{(\eta_1^2+\eta_2^2)^{3/2}}, \\ \dfrac{\partial c_{12}}{\partial \eta_1} = \dfrac{-a\eta_1\eta_2}{(\eta_1^2+\eta_2^2)^{3/2}}, & \dfrac{\partial c_{22}}{\partial \eta_2} = \dfrac{a\eta_1^2}{(\eta_1^2+\eta_2^2)^{3/2}}. \end{cases}$$

The only force acting on the particle is the force of gravity. Therefore the force function U is given by

$$U = -mgz.$$

Consequently

(4.2.12) $\qquad Y_1(U) = \dfrac{-mga\eta_1}{\sqrt{(\eta_1^2+\eta_2^2)}}, \qquad Y_2(U) = \dfrac{-mga\eta_2}{\sqrt{(\eta_1^2+\eta_2^2)}}.$

In view of (4.2.9), (4.2.10), (4.2.11) and (4.2.12) the equations of motion (2.4.11) give

$$m(a^2+1)\ddot{\eta}_1 - ma\eta_2\eta_3\left[\frac{\dot{\eta}_1\eta_2-\eta_1\dot{\eta}_2}{(\eta_1^2+\eta_2^2)^{3/2}}\right] = \frac{-mga\eta_1}{\sqrt{(\eta_1^2+\eta_2^2)}},$$

$$m(a^2+1)\ddot{\eta}_2 + ma\eta_1\eta_3\left[\frac{\dot{\eta}_1\eta_2-\eta_1\dot{\eta}_2}{(\eta_1^2+\eta_2^2)^{3/2}}\right] = \frac{-mga\eta_2}{\sqrt{(\eta_1^2+\eta_2^2)}}.$$

Putting the value of η_3 from (4.2.4), the last equations becomes

$$(a^2+1)\ddot{\eta}_1 - a^2\eta_2\left[\frac{\dot{\eta}_1\eta_2-\eta_1\dot{\eta}_2}{\eta_1^2+\eta_2^2}\right] = \frac{-ag\eta_1}{\sqrt{(\eta_1^2+\eta_2^2)}},$$

$$(a^2+1)\ddot{\eta}_2 + a^2\eta_1\left[\frac{\dot{\eta}_1\eta_2-\eta_1\dot{\eta}_2}{\eta_1^2+\eta_2^2}\right] = \frac{-ag\eta_2}{\sqrt{(\eta_1^2+\eta_2^2)}}.$$

These equations yield

$$\ddot{\eta}_1\eta_2 - \eta_1\ddot{\eta}_2 = 0, \quad \eta_1\ddot{\eta}_1 + \eta_2\ddot{\eta}_2 = \frac{-ga}{1+a^2}\sqrt{(\eta_1^2+\eta_2^2)}.$$

Solving the last equations, we get

$$\ddot{\eta}_1 = \frac{-ga}{1+a^2}\,\frac{\eta_1}{\sqrt{(\eta_1^2+\eta_2^2)}}, \quad \ddot{\eta}_2 = \frac{-ga}{1+a^2}\,\frac{\eta_2}{\sqrt{(\eta_1^2+\eta_2^2)}}.$$

Integrating, we have

(4.2.13) $$\eta_2 = c\eta_1, \quad v = \sqrt{(\eta_1^2+\eta_2^2)} = v_0 - \frac{ga}{1+a^2}t,$$

where c is an arbitrary constant and v_0 is the initial value of v. Solving the equations (4.2.13) and using (4.2.4), we obtain

$$\eta_1 = \dot{x} = \frac{1}{\sqrt{(1+c^2)}}\,(v_o - \frac{ga}{1+a^2}\,t),$$

$$\eta_2 = \dot{y} = \frac{c}{\sqrt{(1+c^2)}}\,(v_o - \frac{ga}{1+a^2}\,t),$$

$$\eta_3 = \dot{z} = a(v_o - \frac{ga}{1+a^2}\,t).$$

ntegrating we have

(4.2.14)
$$\begin{cases} x - x_o = \dfrac{t}{\sqrt{(1+c^2)}}\,(v_o - \dfrac{ga}{2(1+a^2)}\,t), \\ y - y_o = \dfrac{c}{\sqrt{(1+c^2)}}\,t\,(v_o - \dfrac{ga}{2(1+a^2)}\,t), \\ z - z_o = at(v_o - \dfrac{ga}{2(1+a^2)}\,t), \end{cases}$$

here x_o, y_o, z_o are the coordinates of the particle at the ime $t = o$. Eliminating t from the relations (4.2.14), the path f the particle in space is given by

$$(x - x_o)\sqrt{(1+c^2)} = (y - y_o)\,\frac{\sqrt{(1+c^2)}}{c} = \frac{z - z_o}{a}.$$

4.3. Sphere on Turntable

A sphere of mass M and radius a rolls on a rough horizontal lane which turns about a fixed point O of itself with prescribed ngular velocity Ω. The rotation is not necessarily uniform, being a prescribed function of t of class C_1. The sphere is uniform solid sphere, or a uniform spherical shell, or any phere whose centre of gravity G is at its centre and whose llipsoid of inertia at G is a sphere. We use axes Oxyz in fixed irections, with O as origin and Oz normal to the plane. The arameters defining the position of the system are x,y,θ,ψ,ϕ, here x,y,a are the coordinates of G and θ,ψ,ϕ the Eulerian ngles specifying the orientation of the sphere about G. Let p,q,r be the components of the angular velocity of the sphere bout axes through G and parallel to the fixed axes. The components of the velocity of P, the point of contact of the sphere with the plane, are $\dot{x}-aq$ and $\dot{y}+ap$ when regarded as a point of the sphere and $-\Omega y, \Omega x$ when regarded as a point of the plane. Therefore the rolling conditions give

$$(4.3.1)\quad \begin{cases} (\dot{x}-aq)^2+(\dot{y}+ap)^2=\Omega^2(x^2+y^2), \\ \dfrac{\dot{y}+ap}{\dot{x}-aq}=\dfrac{-x}{y}. \end{cases}$$

These are the equations of constraint of which the first is nonlinear in the velocities.

Let us choose x,y,θ,ψ,ϕ as the Poincaré-Četaev variables and $\dot{x},\dot{y},r,p,q,$ as the parameters of real displacement of the associated holonomic system. Then

$$\eta_1 = \dot{x},\ \eta_2 = \dot{y},\ \eta_3 = r,\ \eta_4 = p,\ \eta_5 = q.$$

In terms of the η's the equations of constraint (4.3.1) become

$$(\eta_1 - a\eta_5)^2 + (\eta_2 + a\eta_4)^2 = \Omega^2(x^2+y^2),$$

$$\frac{\eta_2 + a\eta_4}{\eta_1 - a\eta_5} = \frac{-x}{y},$$

which when solved for η_4 and η_5 yield

$$(4.3.2) \quad \begin{cases} \eta_4 = \dfrac{\Omega x - \eta_2}{a}, \\ \eta_5 = \dfrac{\Omega y + \eta_1}{a}. \end{cases}$$

The energy of acceleration of the system is

$$S = \frac{1}{2}M(\dot{\eta}_1^2 + \dot{\eta}_2^2) + \frac{1}{2}A(\dot{\eta}_3^2 + \dot{\eta}_4^2 + \dot{\eta}_5^2),$$

where A is the moment of inertia of the sphere about a diameter. Differentiating (4.3.2) with respect to time t, we get

$$(4.3.3) \quad \begin{cases} \dot{\eta}_4 = \dfrac{\Omega\eta_1 + \dot{\Omega}x - \dot{\eta}_2}{a}, \\ \dot{\eta}_5 = \dfrac{\Omega\eta_2 + \dot{\Omega}y + \dot{\eta}_1}{a}. \end{cases}$$

With the help of (4.3.3) we eliminate $\dot{\eta}_4$ and $\dot{\eta}_5$ from S to obtain

$$\bar{S} = \frac{1}{2}M(\mathring{\eta}_1^2+\mathring{\eta}_2^2)+\frac{1}{2}A\mathring{\eta}_3^2+\frac{A}{2a^2}(\mathring{\eta}_1+\Omega\eta_2+\mathring{\Omega}y)^2+\frac{A}{2a^2}(\mathring{\eta}_2-\Omega\eta_1-\mathring{\Omega}x)^2.$$

erefore

$$(4.3.4)\quad \begin{cases} \dfrac{\partial\bar{S}}{\partial\mathring{\eta}_1} = M\mathring{\eta}_1 + \dfrac{A}{a^2}(\mathring{\eta}_1+\Omega\eta_2+\mathring{\Omega}y), \\ \dfrac{\partial\bar{S}}{\partial\mathring{\eta}_2} = M\mathring{\eta}_2 + \dfrac{A}{a^2}(\mathring{\eta}_2-\Omega\eta_1-\mathring{\Omega}x), \\ \dfrac{\partial S}{\partial\mathring{\eta}_3} = A\mathring{\eta}_3. \end{cases}$$

view of (4.3.4) the Appell's equations (2.7.6) become

$$(4.3.5)\quad \begin{cases} M\mathring{\eta}_1 + \dfrac{A}{a^2}(\mathring{\eta}_1+\Omega\eta_2+\mathring{\Omega}y) = Y_1(U), \\ M\mathring{\eta}_2 + \dfrac{A}{a^2}(\mathring{\eta}_2-\Omega\eta_1-\mathring{\Omega}x) = Y_2(U), \\ A\mathring{\eta}_3 = Y_3(U), \end{cases}$$

re U is the force function.

Let the external forces acting on the system be equivalent a force (X,Y,Z) through the centre of the sphere and a couple Ω,R). In order to calculate the values of $Y_1(U)$, $Y_2(U)$ and U) we have

$$(4.3.6)\qquad \delta U = \omega_1 Y_1(U) + \omega_2 Y_2(U) + \omega_3 Y_3(U),$$

e ω's are the independent parameters of possible displacement.

But

$$\delta U = \omega_1 X + \omega_2 Y + \omega_3 R + \omega_4 P + \omega_5 Q.$$

With a view to eliminate ω_4 and ω_5 from the last result, the constraint equations (4.3.2) give

$$\omega_4 = -\frac{\omega_2}{a}, \quad \omega_5 = \frac{\omega_1}{a}.$$

Therefore

$$\delta U = \omega_1 (X + \frac{Q}{a}) + \omega_2 (Y - \frac{P}{a}) + \omega_3 R. \tag{4.3.7}$$

Comparing (4.3.6) and (4.3.7) we have

$$Y_1 (U) = X + \frac{Q}{a}, \quad Y_2 (U) = Y - \frac{P}{a}, \quad Y_3 (U) = R.$$

Thus the equations of motion (4.3.5) become

$$M\mathring{\eta}_1 + \frac{A}{a^2}(\mathring{\eta}_1 + \Omega\eta_2 + \dot{\Omega}y) = X + \frac{Q}{a}, \tag{4.3.8}$$

$$M\mathring{\eta}_2 + \frac{A}{a^2}(\mathring{\eta}_2 - \Omega\eta_1 - \dot{\Omega}x) = Y - \frac{P}{a}, \tag{4.3.9}$$

$$A\mathring{\eta}_3 = R. \tag{4.3.10}$$

Consider as a particular application the case where the rotation of the plane is uniform ($\dot{\Omega} = o$), and where the external force system is equivalent to a force $(M\xi, M\eta, M\zeta)$ through the

centre of the sphere ($P = Q = R = o$). From (4.3.10) η_3=r=consta
and the equations for the motion of the centre are

$$(4.3.11) \qquad \begin{cases} B\dot{\eta}_1 + A\Omega\eta_2 = Ma^2\xi, \\ B\dot{\eta}_2 - A\Omega\eta_1 = Ma^2\eta, \end{cases}$$

where $B(= A + Ma^2)$ is the moment of inertia about a tangent to the sphere. For a uniform solid sphere

$$\frac{A}{2} = \frac{B}{7} = \frac{Ma^2}{5},$$

giving

$$(4.3.12) \qquad \begin{cases} \dot{\eta}_1 + \frac{2}{7}\Omega\eta_2 = \frac{5}{7}\xi, \\ \dot{\eta}_2 - \frac{2}{7}\Omega\eta_1 = \frac{5}{7}\eta. \end{cases}$$

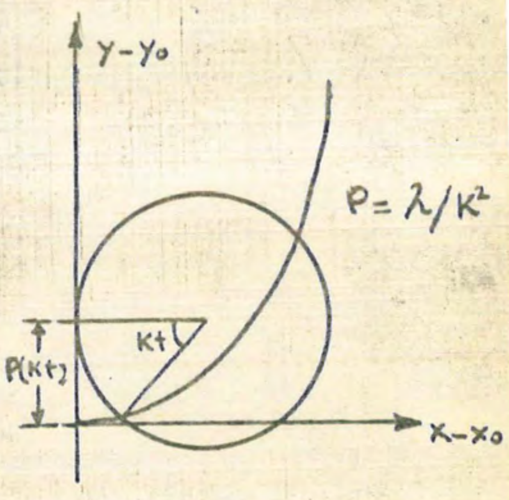
Take the case of a uniform field, where ξ is a positive constant and $\eta = o$; if the sphere is heavy and the turntable is not horizontal but inclined at an angle α to the horizonta and if we take the axis Ox down the line of greatest slope, $\xi = g \sin \alpha$. Putting $x + \iota y = z$, the equations (4.3.12) lead t

$$(4.3.13) \qquad \ddot{z} - \iota\kappa\dot{z} = \lambda,$$

where κ and λ are real constants, $\kappa = \frac{2}{7}\Omega$ and $\lambda = \frac{5}{7}\xi$. The solution is

$$(4.3.14) \qquad z-z_o = \frac{1}{\kappa^2}(\lambda+i\kappa\omega_o)(1-e^{i\kappa t})+\frac{i\lambda}{\kappa^2}(\kappa t),$$

where $z = z_o$ and $\dot{z} = \omega_o$ at $t = o$. The curve is a trachoid, generated by the rolling of a circle on a line at right angles to the field; in the problem of the inclined turntable, this line is horizontal . We observe that the value of z_o is unimportant; the motion relative to the initial point depends only on ω_o. For the particular case $\omega_o = o$ we get cycloid

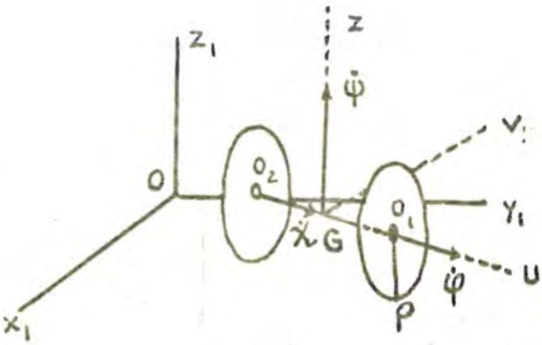


$$(4.3.15) \quad z-z_o=\rho(1-e^{i\kappa t})+i\rho(\kappa t),$$

where $\rho = \frac{\lambda}{\kappa^2}$ as shown in the adjoining figure. The radius of the rolling circle is ρ, and for the problem of the inclined turntable this is $\frac{35}{4}\frac{g}{\Omega^2}\sin\alpha$.

4.4. Motion of a System of Two Wheels and Their Axle on a Horizontal Plane

Let the axle O_1O_2 be a homogeneous rod of length 2a and mass m_1 and wheels be two homogeneous discs, each of radius a and mass m_2, which are fixed normally to the rod at the centre O_1 and O_2 and free to turn about it. Let $O\,x_1y_1z_1$ be a referen system fixed in space and let the wheels move on the plane $z_1 = o$ which is horizontal. The wheel with centre O_2 having a contact without friction and that with centre O_1 having perfectly rough contact.



Suppose we introduce an intermediate trihedron Guvz at the centre G of the rod with Gu along the rod, Gv horizontal and perpendicular to Gu and Gz vertical. The parameters characterising the system are the coordinates (x_1, y_1) of G, the angle ψ which Gu makes with Ox_1, and the angles of rotation ϕ and χ of the two discs with the centres O_1 and O_2 respectively.

For the rolling of the wheel with centre O_1 we have the condition that the point of contact P has no velocity. But the velocity of P is the resultant of the velocity of G and of $a(\dot\phi+\dot\psi)$, parallel to GV, due to rotations $\dot\phi$ and $\dot\psi$. Hence we have

$$(4.4.1) \quad \begin{cases} \dot x_1^2 + \dot y_1^2 = a^2(\dot\phi+\dot\psi)^2, \\ \dfrac{\dot x_1}{\dot y_1} = -\tan\psi. \end{cases}$$

These are the equations of constraint of which the first is nonlinear in $\dot x_1, \dot y_1, \dot\phi, \dot\psi$.

We choose $\phi, \psi, \chi, x_1, y_1$ as the Poincaré-Četaev variables and take

$$\eta_1 = \dot\phi, \eta_2 = \dot\psi, \eta_3 = \dot\chi, \eta_4 = \dot x_1, \eta_5 = \dot y_1. \tag{4.4.2}$$

We immediately obtain

$$X_0 = \frac{\partial}{\partial t}, X_1 = \frac{\partial}{\partial \phi}, X_2 = \frac{\partial}{\partial \psi}, X_3 = \frac{\partial}{\partial \chi}, X_4 = \frac{\partial}{\partial x_1}, X_5 = \frac{\partial}{\partial y_1} \tag{4.4.3}$$

Since these operators commute all the C's vanish.

The equations of constraint (4.4.1) when expressed in terms of η's become

$$\eta_4^2 + \eta_5^2 = a^2(\eta_1 + \eta_2)^2,$$

$$\frac{\eta_4}{\eta_5} = -\tan\psi,$$

which, on solving for η_4 and η_5, yield

$$\begin{cases} \eta_4 = -a(\eta_1 + \eta_2)\sin\psi, \\ \eta_5 = a(\eta_1 + \eta_2)\cos\psi. \end{cases} \tag{4.4.4}$$

In order to obtain the solution of the problem we use equations (3.3.11) for σ = 2, so that we have

$$2\frac{\partial \bar{\ddot T}}{\partial \ddot\eta_i} - 3\frac{\partial \bar{\dot T}}{\partial \eta_i} = Y_i(U) + \overline{P_{oi}}, \quad (i = 1,2,3). \tag{4.4.5}$$

Here T is the kinetic energy of the associated holonomic system:

$$T = \frac{1}{2}(m_1+2m_2)(\dot{x}_1^2+\dot{y}_1^2)+\frac{1}{2}(\frac{m_1}{3}+\frac{5}{2}m_2)a^2\dot{\psi}^2+\frac{m_2a^2}{4}(\dot{\phi}^2+\dot{\chi}^2).$$

From (3.3.7) and (3.3.8) and the fact that all the C's vanish we have

$$\overline{F}_{oi} = o, \qquad (i = 1,2,3).$$

The kinetic energy T, in terms of η's, take the form

$$T = \frac{1}{2}(m_1+2m_2)(\eta_4^2+\eta_5^2)+\frac{1}{2}(\frac{m_1}{3}+\frac{5}{2}m_2)a^2\eta_2^2+\frac{m_2a^2}{4}(\eta_1^2+\eta_3^2).$$

Now

(4.4.6) $$\begin{cases}\overset{\circ}{T} = (m_1+2m_2)(\eta_4\dot{\eta}_4+\eta_5\dot{\eta}_5)+(\frac{m_1}{3}+\frac{5}{2}m_2)a^2\eta_2\dot{\eta}_2+\frac{m_2a^2}{2}(\eta_1\dot{\eta}_1+\eta_3\dot{\eta}_3),\\ \overset{\circ\circ}{T} = (m_1+2m_2)(\dot{\eta}_4^2+\dot{\eta}_5^2)+(\frac{m_1}{3}+\frac{5}{2}m_2)a^2\dot{\eta}_2^2+\frac{m_2a^2}{2}(\dot{\eta}_1^2+\dot{\eta}_3^2) + \text{terms}\\ \text{not containing } \dot{\eta}\text{'s}.\end{cases}$$

Differentiating (4.4.4) and making use of (4.4.2), we get

(4.4.7) $$\begin{cases}\dot{\eta}_4 = - a(\dot{\eta}_1+\dot{\eta}_2)\sin\psi-a\eta_2(\eta_1+\eta_2)\cos\psi,\\ \dot{\eta}_5 = a(\dot{\eta}_1+\dot{\eta}_2)\cos\psi-a\eta_2(\eta_1+\eta_2)\sin\psi.\end{cases}$$

With the help of (4.4.4), (4.4.6) and (4.4.7) we obtain

.4.8)

$$\left\{\begin{aligned}
\frac{\partial \dot{\tilde{T}}}{\partial \eta_1} &= \frac{m_2 a^2}{2}\dot{\eta}_1 + a^2(m_1 + 2m_2)(\dot{\eta}_1+\dot{\eta}_2),\\
\frac{\partial \dot{\tilde{T}}}{\partial \eta_2} &= (\frac{m_1}{3} + \frac{5}{2} m_2)a^2\dot{\eta}_2 + a^2(m_1 + 2m_2)(\dot{\eta}_1+\dot{\eta}_2),\\
\frac{\partial \dot{\tilde{T}}}{\partial \eta_3} &= \frac{m_2 a^2}{2}\dot{\eta}_3 ,\\
\frac{\partial \tilde{\tilde{T}}}{\partial \dot{\eta}_1} &= m_2 a^2\dot{\eta}_1 + 2a^2(m_1 + 2m_2)(\dot{\eta}_1+\dot{\eta}_2),\\
\frac{\partial \tilde{\tilde{T}}}{\partial \dot{\eta}_2} &= 2(\frac{m_1}{3} + \frac{5}{2} m_2)a^2\dot{\eta}_2 + 2a^2(m_1+2m_2)(\dot{\eta}_1+\dot{\eta}_2),\\
\frac{\partial \tilde{\tilde{T}}}{\partial \dot{\eta}_3} &= m_2 a^2\dot{\eta}_3 .
\end{aligned}\right.$$

nce the force function U is zero, we have

$$Y_i(U) = 0, \qquad (i = 1,2,3).$$

nally the equations (4.4.5) become

$$(\dot{\eta}_1+\dot{\eta}_2)(m_1+2m_2) + \frac{m_2}{2}\dot{\eta}_1 = 0,$$

$$(\dot{\eta}_1+\dot{\eta}_2)(m_1+2m_2) + (\frac{m_1}{3} + \frac{5}{2} m_2)\dot{\eta}_2 = 0,$$

$$\dot{\eta}_3 = 0.$$

Integrating these equations, we get

$${}_1+2m_2)(\eta_1+\eta_2)+\frac{m_2}{2}\eta_1 = (m_1+2m_2)(\eta_1^0+\eta_2^0)+\frac{m_2}{2}\eta_1^0,$$

$${}_1+2m_2)(\eta_1+\eta_2)+(\frac{m_1}{3}+\frac{5}{2}m_2)\eta_2 = (m_1+2m_2)(\eta_1^0+\eta_2^0)+(\frac{m_1}{3}+\frac{5}{2}m_2)\eta_2^0,$$

$$= \eta_3^0,$$

where η_1^o, η_2^o and η_3^o are the initial values of η_1, η_2 and η_3 respectively. The last equations give

$$(4.4.9) \qquad \eta_1^o = \eta_1^o, \quad \eta_2 = \eta_2^o, \quad \eta_3 = \eta_3^o.$$

Substituting for the η's from (4.4.2) in (4.4.9), we have

$$\dot{\phi} = \eta_1^o, \quad \dot{\psi} = \eta_2^o, \quad \dot{\chi} = \eta_3^o .$$

Integrating the last relations, we obtain

$$\phi = \eta_1^o t, \; \psi = \eta_2^o t, \; \chi = \eta_3^o t,$$

if for $t = o$, $\phi = \psi = \chi = o$. From the relations,

$$\dot{x}_1 = -\, a(\dot{\phi} + \dot{\psi}) \sin \psi, \; \dot{y}_1 = a(\dot{\phi} + \dot{\psi}) \cos \psi,$$

which are obtained from (4.4.4), we get on substituting for $\dot{\phi}$, $\dot{\psi}$ and ψ

$$\dot{x}_1 = -\, a(\eta_1^o + \eta_2^o) \sin(\eta_2^o t),$$

$$\dot{y}_1 = a(\eta_1^o + \eta_2^o) \cos(\eta_2^o t).$$

Consequently

$$x_1 = \frac{a(\eta_1^o + \eta_2^o)\cos(\eta_2^o t)}{\eta_2^o} .$$

$$y_1 = \frac{a(\eta_1^o + \eta_2^o)}{\eta_2^o} \sin(\eta_2^o t).$$

Squaring and adding the last two relations, we get

$$x_1^2 + y_1^2 = \frac{a^2 (\eta_1^o + \eta_2^o)^2}{(\eta_2^o)^2} .$$

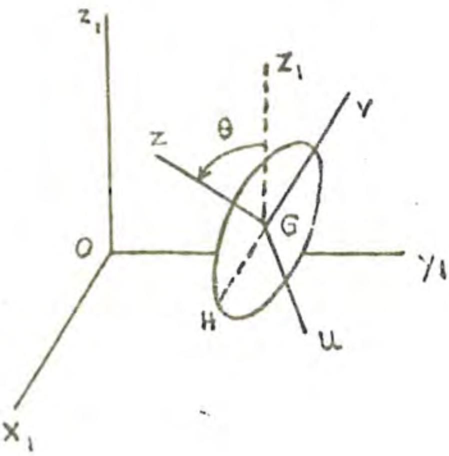
This shows that the trajectory of the centre G is a circle of radius $a \left|\frac{\eta_1^0 + \eta_2^0}{\eta_2^0}\right|$, described with the uniform velocity

$$\sqrt{\dot{x}_1^2 + \dot{y}_1^2} = | a(\eta_1^0 + \eta_2^0)|.$$

4.5. Rolling Hoop.

Let us study the motion of a heavy circular hoop, of unit mass and radius a, which rolls without sliding on a fixed horizontal plane Ox_1y_1. The centre of inertia G of the hoop is the centre of the figure and the central ellipsoid of inertia is a surface of revolution about the axis Gz of the hoop. As shown in the figure, let H be the point of contact of the hoop with the fixed plane.

The parameters are : the coordinates x_1, y_1 of the point G relative to the space fixed system $Ox_1y_1z_1$, and the Euler angles θ, ψ, ϕ. The coordinate z_1 of the point G is given as a function of θ by the relation



(4.5.1) $\quad z_1 = a \sin \theta.$

In addition the requirement of no sliding at H gives rise to two conditions of constraint.

We shall use Nielsen's equations of motion in the form (3.3.18) to solve this problem. Let p,q,r be the components of the instantaneous rotation vector $\overline{\Omega}$ of the hoop, referred to a semi-moving rectangular trihedral Guvz where the axis Gu is perpendicular to the plane zGz$_1$ and the axis Gv is directed upwards along the line of greatest slope of the plane of the hoop. Then we have

$$p = \dot{\theta},\ q = \dot{\psi} \sin \theta,\ \ r = \dot{\psi} \cos \theta + \dot{\phi}.$$

The instantaneous motion of the hoop is a rotation about the point H and is represented by the vector $\overline{\omega}$. Consequently the components of the velocity $\overline{\omega} \wedge \overline{HG}$ of the point G in the trihedral Guvz are

$$- ar,\ o,\ ap.$$

Projecting these velocities on Ox_1 and Oy_1 we obtain

$$\dot{x}_1^2 + \dot{y}_1^2 = a^2(p^2 \sin^2\theta + r^2),$$

$$\frac{\dot{y}_1}{\dot{x}_1} = \frac{p \sin \theta \cos \psi + r \sin \psi}{r \cos \psi - p \sin \theta \sin \psi},$$

which are equivalent to

$$(4.5.2) \qquad \begin{cases} \dot{x}_1 = ap \sin \psi \sin \theta - ar \cos \psi, \\ \dot{y}_1 = - ap \cos \psi \sin \theta - ar \sin \psi. \end{cases}$$

Let us choose θ, ψ, ϕ, x_1, y_1, z_1 as the Poincaré-Četaev variables. Due to the holonomic constraint (4.5.1) the associated holonomic system has five degrees of freedom and therefore five real displacement parameters are needed. We take

$$\text{(4.5.3)} \qquad \eta_1 = p = \dot{\theta},\ \eta_2 = q = \dot{\psi}\sin\theta,\ \eta_3 = r = \dot{\psi}\cos\theta + \dot{\phi},\ \eta_4 = \dot{x}_1,\ \eta_5 = \dot{y}_1 .$$

To find the displacement operators of the associated holonomic system we consider an arbitrary function $f(t,\theta,\psi,\phi,x_1,y_1,z_1)$ for which we have

$$df = [\frac{\partial f}{\partial t} + \frac{\partial f}{\partial \theta}\dot{\theta} + \frac{\partial f}{\partial \psi}\dot{\psi} + \frac{\partial f}{\partial \phi}\dot{\phi} + \frac{\partial f}{\partial x_1}\dot{x}_1 + \frac{\partial f}{\partial y_1}\dot{y}_1 + \frac{\partial f}{\partial z_1}\dot{z}_1]dt.$$

With the help of (4.5.1) the last relation becomes

$$\text{(4.5.4)} \quad df = [\frac{\partial f}{\partial t} + (\frac{\partial f}{\partial \theta} + a\cos\theta\frac{\partial f}{\partial z_1})\dot{\theta} + \frac{\partial f}{\partial \psi}\dot{\psi} + \frac{\partial f}{\partial \phi}\dot{\phi} + \frac{\partial f}{\partial x_1}\dot{x}_1 + \frac{\partial f}{\partial y_1}\dot{y}_1]dt.$$

But:

$$\text{(4.5.5)} \quad df = [X_0(f) + \eta_1 X_1(f) + \eta_2 X_2(f) + \eta_3 X_3(f) + \eta_4 X_4(f) + \eta_5 X_5(f)]dt.$$

In the last result we substitute the values of the η's from (4.5.3) and compare with (4.5.4). Then

$$
(4.5.6)\qquad
\begin{cases}
X_0 = \dfrac{\partial}{\partial t}\,, \\[2ex]
X_1 = \dfrac{\partial}{\partial \theta} + a \cos \theta \dfrac{\partial}{\partial z_1}\,, \\[2ex]
X_2 = \dfrac{1}{\sin \theta}\dfrac{\partial}{\partial \psi} - \cot \theta \dfrac{\partial}{\partial \phi}\,, \\[2ex]
X_3 = \dfrac{\partial}{\partial \phi}\,, \quad X_4 = \dfrac{\partial}{\partial x_1}\,, \quad X_5 = \dfrac{\partial}{\partial y_1}\,.
\end{cases}
$$

An easy calculation shows that the commutators of all the operators except (X_1,X_2) vanish. The commutators (X_1,X_2) satisfies the relation

$$(X_1,X_2) = -\cot\theta \operatorname{cosec}\theta \frac{\partial}{\partial\psi} + \operatorname{cosec}^2\theta \frac{\partial}{\partial\phi}\,,$$

which, in view of (4.5.6), can be written in the form

$$(X_1,X_2) = -\cot\theta\, X_2 + X_3\,.$$

It follows that the non-vanishing C's are given by

$$
(4.5.7)\qquad
\begin{cases}
C_{122} = -C_{212} = -\cot\theta, \\[2ex]
C_{123} = -C_{213} = 1\,.
\end{cases}
$$

The equations of constraint (4.5.2) when expressed in terms of the η's become

$$(4.5.8)\qquad \begin{cases} \eta_4 = a\,\eta_1 \sin\psi \sin\theta - a\,\eta_3 \cos\psi\,, \\ \eta_5 = -a\,\eta_1 \cos\psi \sin\theta - a\,\eta_3 \sin\psi\,. \end{cases}$$

Hence the non-vanishing c's are given by

$$c_{41} = \frac{\partial\eta_4}{\partial\eta_1} = a \sin\psi \sin\theta,$$

$$c_{43} = \frac{\partial\eta_4}{\partial\eta_3} = -\,a \cos\psi,$$

$$c_{51} = \frac{\partial\eta_5}{\partial\eta_1} = -\,a \cos\psi \sin\theta\,,$$

$$c_{53} = \frac{\partial\eta_5}{\partial\eta_3} = -\,a \sin\psi\,.$$

With the help of (2.2.9) we obtain

$$(4.5.9)\qquad \begin{cases} Y_0 = \dfrac{\partial}{\partial t}\,, \\ Y_1 = \dfrac{\partial}{\partial\theta} + a \sin\psi\sin\theta \dfrac{\partial}{\partial x_1} - a \cos\psi\sin\theta \dfrac{\partial}{\partial y_1} + a \cos\theta \dfrac{\partial}{\partial z_1} \\ Y_2 = \dfrac{1}{\sin\theta}\dfrac{\partial}{\partial\psi} - \cot\theta \dfrac{\partial}{\partial\phi}\,, \\ Y_3 = \dfrac{\partial}{\partial\phi} - a \cos\phi \dfrac{\partial}{\partial x_1} - a \sin\phi \dfrac{\partial}{\partial y_1}\,. \end{cases}$$

The kinetic energy of the associated holonomic system is

$$(4.5.10) \qquad T = \frac{1}{2}\left[\left(\frac{1}{2} + \cos^2\theta\right)a^2\eta_1^2 + \frac{a^2\eta_2^2}{2} + a^2\eta_3^2 + \eta_4^2 + \eta_5^2\right].$$

With the help of (4.5.3) and (4.5.10) we have

$$(4.5.11) \qquad \dot{T} = -a^2\sin\theta\cos\theta\,\eta_1^3 + \left(\frac{1}{2}+\cos^2\theta\right)a^2\eta_1\dot{\eta}_1 + \frac{a^2}{2}\eta_2\dot{\eta}_2 + a^2\eta_3\dot{\eta}_3 + \eta_4\dot{\eta}_4 + \eta_5\dot{\eta}_5.$$

Similarly from (4.5.3) and (4.5.8) we get

$$(4.5.12) \begin{cases} \dot{\eta}_4 = a\eta_1\eta_2\cos\psi + a\eta_1^2\sin\psi\cos\theta + \dfrac{a\eta_2\eta_3\sin\psi}{\sin\theta} + a\dot{\eta}_1\sin\psi\sin\theta - a\dot{\eta}_3\cos\psi, \\ \dot{\eta}_5 = a\eta_1\eta_2\sin\psi - a\eta_1^2\cos\psi\cos\theta - \dfrac{a\eta_2\eta_3\cos\psi}{\sin\theta} - a\dot{\eta}_1\cos\psi\sin\theta - a\dot{\eta}_3\sin\psi. \end{cases}$$

From (4.5.11) and (4.5.12) it follows that

$$(4.5.13) \begin{cases} \dfrac{\partial\dot{T}}{\partial\eta_1} = -\,3a^2\eta_1^2\sin\theta\cos\theta + \left(\dfrac{1}{2} + \cos^2\theta\right)a^2\dot{\eta}_1, \\ \dfrac{\partial\dot{T}}{\partial\dot{\eta}_2} = \dfrac{a^2}{2}\dot{\eta}_2, \quad \dfrac{\partial\dot{T}}{\partial\eta_3} = a^2\dot{\eta}_3, \\ \dfrac{\partial\dot{T}}{\partial\eta_4} = a\eta_1\eta_2\cos\psi + a\eta_1^2\sin\psi\cos\theta + \dfrac{a\eta_2\eta_3\sin\psi}{\sin\theta} + a\dot{\eta}_1\sin\psi\sin\theta - a\dot{\eta}_3\cos\psi, \\ \dfrac{\partial\dot{T}}{\partial\eta_5} = a\eta_1\eta_2\sin\psi - a\eta_1^2\cos\psi\cos\theta - \dfrac{a\eta_2\eta_3\cos\psi}{\sin\theta} - a\dot{\eta}_1\cos\psi\sin\theta - a\dot{\eta}_3\sin\psi. \end{cases}$$

In view of (4.5.6) and (4.5.10), we have

$$(4.5.14)\quad \begin{cases} X_1(T) = -\ a^2\eta_1^2\sin\theta\cos\theta, \\ X_2(T) = X_3(T) = X_4(T) = X_5(T) = o. \end{cases}$$

With the help of (4.5.7) the values of $P_{O1}, P_{O2}, P_{O3}, P_{O4}$ and P_{O5} as given by (3.3.8), are

$$(4.5.15)\quad \begin{cases} P_{O1} = \eta_2\cot\theta\ \frac{\partial T}{\partial\eta_2} - \eta_2\frac{\partial T}{\partial\eta_3} = \frac{1}{2}a^2\eta_2^2\cot\theta - a^2\eta_2\eta_3, \\ P_{O2} = \eta_1(\frac{\partial T}{\partial\eta_3} - \cot\theta\ \frac{\partial T}{\partial\eta_2}) = -\ \frac{1}{2}\ a^2\eta_1\eta_2\cot\theta + a^2\eta_1\eta_3, \\ P_{O3} = P_{O4} = P_{O5} = o. \end{cases}$$

The force function is

$$U = -gz_1 = -ga\sin\theta.$$

The last relation together with (4.5.9) yields

$$(4.5.16)\quad Y_1(U) = -ga\cos\theta,\quad Y_2(U) = Y_3(U) = o.$$

Thus the equations (3.3.18) become

$$(4.5.17)\quad \begin{cases} \frac{3}{2}\ \dot{\eta}_1 - \frac{1}{2}\eta_2^2\cot\theta + 2\eta_2\eta_3 = -\ \frac{g}{a}\cos\theta, \\ \frac{1}{2}\ \dot{\eta}_2 + \frac{1}{2}\eta_1\eta_2\ \cot\theta - \eta_1\eta_3 = o\ , \\ 2\dot{\eta}_3 - \eta_1\eta_2 = o\ . \end{cases}$$

If in place of t we take θ as the independent variable we obtain

$$\dot{\eta}_i = \eta_i' \eta_3 \qquad (i = 1,2,3),$$

where the prime denotes differentiation with respect to θ. The second and third of equations (4.5.17) become

$$(4.5.18) \qquad \eta_2' + \eta_2 \cot\theta - 2\eta_3 = 0, \qquad 2\eta_3' = \eta_2 .$$

The elimination of η_2 from (4.5.18) leads to the relation

$$(4.5.19) \qquad \eta_3'' + \eta_3' \cot\theta - \eta_3 = 0,$$

with the initial condition

$$(4.5.20) \qquad t = 0,\ \theta = \theta_0,\ \eta_1 = \eta_1^0,\ \eta_2 = \eta_2^0 = 2\eta_3'^0,\ \eta_3 = \eta_3^0 .$$

The general solution of the differential equation (4.5.19) , integrated by means of a hypergeometric series, will be of the form

$$(4.5.21) \qquad \eta_3 = F(\theta,\ \theta_0,\ \eta_2^0,\ \eta_3^0),$$

and η_2 can be found by differentiation. The first equation of (4.5.17) can be reduced to the form

$$\frac{3}{4}\frac{d}{d\theta}(\eta_1^2) = -\frac{g}{a}\cos\theta + \frac{1}{2}\eta_2^2 \cot\theta - 2\eta_2\eta_3,$$

therefore $\eta_1(\theta)$ is given by simple quadrature. A further integration determines θ as a function of the time t.

CHAPTER V

CANONICAL EQUATIONS OF MOTION

5.1. General Considerations

In this chapter we discuss the possibility of reducing nonholonomic system to a holonomic one by adjunction of ertain supplementary forces. Then we derive the Hamilton's anonical equations of motion for the nonlinear nonholonomic ystem. Finally the Routhian function R is introduced and the quations of motion are obtained in terms of this function. sing these equations, sufficient conditions are established or the existence of certain first integrals of the equations f motion.

5.2. Equivalence of a Nonholonomic System to a olonomic System

Let us consider a nonholonomic dynamical system whose osition at time t is defined by the Poincaré-Četaev variables $_1, x_2, \ldots, x_n$. Let the constraints imposed on the system be epresented by eqtions of the forms (1.2.1) and (1.2.2).

To find the equations of motion of the nonholonomic ystem we use the equations (1.4.11). Since ω's in (1.4.11) re not independent and have to satisfy the equations (2.2.4), e introduce the undetermined multipliers $\lambda_{\ell+1}, \ldots, \lambda_m$ to obtain he equation

$$(5.2.1)\quad \omega_p\left[\frac{d}{dt}\left(\frac{\partial T}{\partial \eta_p}\right) - C_{opq}\frac{\partial T}{\partial \eta_q} - \eta_q C_{qpr}\frac{\partial T}{\partial \eta_r} - X_p(T+U) - \lambda_\alpha \frac{\partial f_\alpha}{\partial \eta_p}\right] = 0,$$

$$(\alpha = \ell+1,\ldots,m;\ p,q,r = 1,2,\ldots,m).$$

From (5.2.1) we obtain the equations of motion in the form

$$(5.2.2)\quad \frac{d}{dt}\left(\frac{\partial T}{\partial \eta_p}\right) - C_{opq}\frac{\partial T}{\partial \eta_q} - \eta_q C_{qpr}\frac{\partial T}{\partial \eta_r} - X_p(T+U) - \lambda_\alpha \frac{\partial f_\alpha}{\partial \eta_p} = 0,$$

$$(\alpha = \ell+1,\ldots,m;\ p,q,r = 1,2,\ldots,m).$$

These equations together with (2.2.3) and (2.4.12) form a system of $n+2m-\ell$ equations to determine $x_1, x_2,\ldots,x_n; \eta_1,\ldots,\eta_m;$ and $\lambda_{\ell+1},\ldots,\lambda_m$ as functions of the time t.

We now demonstrate the important fact it is possible to determine the undetermined multipliers λ_α as functions of $x_1, x_2,\ldots,x_n;\ \eta_1,\ldots,\eta_m$ and t. For this purpose we note that the kinetic energy T is of the form

$$T = \frac{1}{2} a_{pq}\eta_p\eta_q + a_{op}\eta_p + \frac{1}{2} a_{oo}, \qquad (a_{pq} = a_{qp}),$$

$$(p,q = 1,2,\ldots,m),$$

where a_{pq}, a_{op} and a_{oo} are functions of the x's and t. Consequently the equations (5.2.2) can be written in the form

$$(5.2.3)\quad a_{pq}\dot{\eta}_q - \lambda_\alpha b_{\alpha p} = \Phi_p, \qquad (\alpha = \ell+1,\ldots,m;\ p,q = 1,2,\ldots,m),$$

where

$$b_{\alpha p} = \frac{\partial f_\alpha}{\partial \eta_p},$$

$$\Phi_p = X_p(T+U) + C_{opq}\frac{\partial T}{\partial \eta_q} + \eta_q C_{qpr}\frac{\partial T}{\partial \eta_r} - X_o\left(\frac{\partial T}{\partial \eta_p}\right) - \eta_q X_q\left(\frac{\partial T}{\partial \eta_p}\right),$$

$$(\alpha = \ell+1,\ldots,m;\quad p,q,r = 1,2,\ldots,m).$$

Differentiating the equations of constraint (2.2.3) with respect to the time t, we have

$$(5.2.4)\qquad b_{\alpha q}\dot{\eta}_q = \psi_\alpha,\qquad (\alpha = \ell+1,\ldots,m;\ q = 1,2,\ldots,m),$$

where

$$\psi_\alpha = X_o(f_\alpha) + \eta_q X_q(f_\alpha).$$

The equations (5.2.3) and (5.2.4) form a system of $2m-\ell$ linear equations in the $2m-\ell$ unknowns: $\dot{\eta}_1, \dot{\eta}_2, \ldots, \dot{\eta}_m, \lambda_{\ell+1}, \ldots, \lambda_m$. Their determinant:

$$\Delta = \pm \begin{vmatrix} a_{11} \ldots a_{1m} & b_{\ell+1,1} \ldots b_{m1} \\ \cdots\cdots\cdots & \cdots\cdots\cdots \\ a_{m1} \ldots a_{mm} & b_{\ell+1,m} \ldots b_{mm} \\ b_{\ell+1,1} \ldots b_{\ell+1,m} & 0 \ \ldots\ 0 \\ \cdots\cdots\cdots & \cdots\cdots\cdots \\ b_{m1} \ldots b_{mm} & 0 \ \ldots\ 0 \end{vmatrix}$$

does not vanish. For otherwise the $2m-\ell$ linear homogeneous equations:

$$(5.2.5)\quad \begin{cases} a_{11}y_1 + \dots + a_{1m}y_m + b_{\ell+1,1}z_{\ell+1} + \dots + b_{m1}z_m = 0, \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ a_{m1}y_1 + \dots + a_{mm}y_m + b_{\ell+1,m}z_{\ell+1} + \dots + b_{mm}z_m = 0, \\ b_{\ell+1,1}y_1 + \dots + b_{\ell+1,m}y_m = 0, \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ b_{m1}y_1 + \dots + b_{mm}y_m = 0, \end{cases}$$

would admit a nontrivial solution $(y_1,y_2,\dots,y_m;z_{\ell+1},\dots,z_m)$. Moreover, not all the $y_1,y_2,\dots,y_m$ in this solution could vanish; for then we should have

$$\begin{aligned} b_{\ell+1,1}\, z_{\ell+1} + \dots + b_{m1}z_m &= 0 \\ \dots\dots\dots\dots\dots\dots\dots\dots \\ b_{\ell+1,m}\, z_{\ell+1} + \dots + b_{mm}z_m &= 0. \end{aligned}$$

But the rank of the matrix of these equations, namely the matrix $\left\|\frac{\partial f_\alpha}{\partial \eta_p}\right\|$, is $m-\ell$. Hence all the $z_{\ell+1},\dots,z_m$ of the solution vanish which is a contradiction.

Next, we multiply the pth equation (5.2.5) by y_p, $p = 1,2,\dots,m$, and add. The terms in $z_{\ell+1},\dots,z_m$ drop out because of the last $m-\ell$ of the equations (5.2.5), and so here results the equation:

$$a_{pq}\, y_p\, y_q = o, \qquad (p,q = 1,2,\ldots,m),$$

here not all the $y_1, y_2, \ldots, y_m$ are zero. This is impossible, ince the quadratic form $a_{pq} y_p y_q$ is positive definite.

Since Φ_p and ψ_α depend on the x's, η's and t, the uations (5.2.3) and (5.2.4) admit, therefore, a solution:

$$\text{(5.2.6)} \begin{cases} \dot{\eta}_q = F_q(x_1, x_2, \ldots, x_n; \eta_1, \ldots, \eta_m; t), (q = 1,2,\ldots,m), \\ \lambda_\alpha = F'_\alpha(x_1, x_2, \ldots, x_n; \eta_1, \ldots, \eta_m; t), (\alpha = \ell+1,\ldots,m). \end{cases}$$

Thus the equations (2.4.12) and (5.2.2), in which λ's re supposed to have been replaced by known functions of the 's, η's and t, as given by (5.2.6), constitute a system to etermine the unknown x's and η's.

The equations (5.2.2) can be considered as the equations f motion of the associated holonomic system whose kinetic nergy is T and which is acted upon by forces $X_p(U) + \lambda_\alpha \frac{\partial f_\alpha}{\partial \eta_p}$. ndependently of the considerations of the nonholonomic consraints of the form (2.2.3), the equations (5.2.2) admit m-ℓ irst integrals.

$$f_\alpha(x_1, x_2, \ldots, x_n; \eta_1, \ldots, \eta_m; t) = \text{constant}, (\alpha = \ell+1, \ldots, m).$$

f the initial conditions are such as to satisfy the equations

$$f_\alpha(x_1^0, x_2^0, \ldots, x_n^0;\ \eta_1^0, \ldots, \eta_m^0;\ t) = 0, \quad (\alpha = \ell+1, \ldots, m),$$

hen the solutions of the system (5.2.2) identically satisfy he constraint equations (2.2.3). We have therefore established he following

heorem. Every nonholonomic system with time dependent constraints is reducible to a holonomic system by adjunction of ertain supplementary forces depending on η's and admitting s integrals the equations of nonholonomic constraints.

This theorem constitutes an extension of analogous heorems established by C. Agostinelli [2] and I. Grindei [23] or the nonholonomic systems in Lagrangian coordinates.

5.3. Canonical Equations

If L = T+U is the kinetic potential of the system the quations (5.2.2) take the form

$$\text{(5.3.1)} \qquad \frac{d}{dt}\left(\frac{\partial L}{\partial \eta_p}\right) - C_{opq}\frac{\partial L}{\partial \eta_q} - \eta_q C_{qpr}\frac{\partial L}{\partial \eta_r} - X_p(L) - \lambda_\alpha \frac{\partial f_\alpha}{\partial \eta_p} = 0,$$

$$(\alpha = \ell+1, \ldots, m;\ p, q, r = 1, 2, \ldots, m).$$

et us introduce for the parameters of real displacement 's the new variables y's defined by the equations

$$\text{(5.3.2)} \qquad y_p = \frac{\partial L}{\partial \eta_p}, \qquad (p = 1, 2, \ldots, m),$$

which when solved yield η's as functions of the x's, y's and t.

In order to obtain the equations of motion in terms of new variables we consider the Hamiltonian

$$(5.3.3)\qquad H(x_1,x_2,\ldots,x_n;y_1,\ldots,y_m;t)=y_p\eta_p-L,\quad (p=1,2,\ldots,m).$$

The relation (1.3.10) allows us to write

$$\delta H = \delta y_p\,\eta_p + y_p\delta\eta_p - \omega_p X_p(L) - \frac{\partial L}{\partial \eta_p}\,\delta\eta_p,$$

which, in view of (5.3.2), is equivalent to

$$(5.3.4)\qquad \delta H = \eta_p\delta y_p - \omega_p X_p(L).$$

But

$$(5.3.5)\qquad \delta H = \frac{\partial H}{\partial y_p}\,\delta y_p + \omega_p X_p(H).$$

Since H is an arbitrary function of the x's and y's, we compare the expressions (5.3.4) and (5.3.5) to obtain

$$(5.3.6)\qquad \eta_p = \frac{\partial H}{\partial y_p},$$

$$(5.3.7)\qquad X_p(L) = -X_p(H),\qquad (p=1,2,\ldots,m).$$

Substituting for $X_p(L)$ from (5.3.1) in (5.3.7) and using (5.3.2), we get the canonical equations in the form

$$(5.3.8)\qquad \eta_p = \frac{\partial H}{\partial y_p},$$

$$\frac{dy_p}{dt} = C_{opq}y_q + \eta_q C_{qpr}y_r - X_p(H) + \lambda_\alpha \frac{\partial f_\alpha}{\partial \eta_p}, \tag{5.3.9}$$

$$(\alpha = \ell+1,\ldots,m;\ p,q,r = 1,2,\ldots,m).$$

In the particular case of the associated holonomic system moving subject to time dependent constraints of the form (1.2.1) we have all the λ's equal to zero. Consequently the canonical equations of motion are

$$\eta_p = \frac{\partial H}{\partial y_p}, \tag{5.3.10}$$

$$\frac{dy_p}{dt} = C_{opq}y_q + \eta_q C_{qpr}y_r - X_p(H), \tag{5.3.11}$$

$$(p,q,r = 1,2,\ldots,m).$$

If the holonomic constraints (1.2.1) are time independent all the C_{opq} vanish and we obtain the equations of motion, as obtained by N.G.Četaev [15], in the form

$$\eta_p = \frac{\partial H}{\partial y_p},$$

$$\frac{dy_p}{dt} = \eta_q C_{qpr}y_r - X_p(H),$$

$$(p,q,r = 1,2,\ldots,m).$$

5.4. Routhian Function and the Equations of Motion

In this section we find a descriptive function R which enables us to write some of the equations, say the first m' pairs, in the Hamiltonian form (5.3.8) and (5.3.9) and the rest in the form (5.3.1).

We put

$$(5.4.1) \qquad y_\sigma = \frac{\partial L}{\partial \eta_\sigma} \qquad (\sigma = 1,2,\ldots,m' < m),$$

and consider the function $R(x_1, x_2, \ldots, x_n; \eta_{m'+1}, \ldots, \eta_m; y_1, \ldots, y_{m'}; t)$ defined by

$$(5.4.2) \qquad R = L - \eta_\sigma \frac{\partial L}{\partial \eta_\sigma}.$$

With the help of the relation (1.3.10) and (5.4.1) we get

$$(5.4.3) \quad \delta R = \omega_p X_p(L) + \frac{\partial L}{\partial \eta_\rho}\delta\eta_\rho - \eta_\sigma \delta y_\sigma, \quad (\sigma=1,2,\ldots,m';\ \rho=m'+1,\ldots,m).$$

Also

$$(5.4.4) \qquad \delta R = \omega_p X_p(R) + \frac{\partial R}{\partial \eta_\rho}\delta\eta_\rho + \frac{\partial R}{\partial y_\sigma}\delta y_\sigma.$$

Comparing the coefficients of $\omega_1, \omega_2, \ldots, \omega_m; \delta\eta_{m'+1}, \ldots, \delta\eta_m;$ $\delta y_1, \ldots, \delta y_{m'}$ in the last two relations, we have

$$(5.4.5)\quad \begin{cases} X_p(L) = X_p(R), \\ \frac{\partial L}{\partial \eta_\rho} = \frac{\partial R}{\partial \eta_\rho}, \\ \eta_\sigma = -\frac{\partial R}{\partial y_\sigma}, \end{cases}$$

$(\sigma=1,2,\ldots,\acute{m};\rho=\acute{m}+1,\ldots,m;p=1,2,\ldots,m)$.

The first $\acute{m}$ equations (5.3.1) together with (5.4.1) and (5.4.5) yield

$$(5.4.6)\quad \eta_\sigma = -\frac{\partial R}{\partial y_\sigma},$$

$$(5.4.7)\quad \frac{dy_\sigma}{dt} = y_\mu P_{o\sigma\mu} + \frac{\partial R}{\partial \eta_\rho} P_{o\sigma\rho} + X_\sigma(R) + \lambda_\alpha \frac{\partial f_\alpha}{\partial \eta_\sigma},$$

$(\mu,\sigma = 1,2,\ldots,\acute{m};\ \rho= \acute{m}+1,\ldots,m;\alpha=\ell+1,\ldots,m)$,

where

$$P_{opq} = C_{opq} + \eta_r C_{rpq}, \quad (p,q,r = 1,2,\ldots,m).$$

In the same manner the last $m-\acute{m}$ equations (5.3.1) give

$$(5.4.8)\quad \frac{d}{dt}\left(\frac{\partial R}{\partial \eta_\rho}\right) - y_\mu P_{o\rho\mu} - \frac{\partial R}{\partial \eta_\nu} P_{o\rho\nu} - X_\rho(R) - \lambda_\alpha \frac{\partial f_\alpha}{\partial \eta_\rho} = 0,$$

$(\mu = 1,2,\ldots,\acute{m};\rho,\nu=\acute{m}+1,\ldots,m;\alpha=\ell+1,\ldots,m)$.

The set of equations (5.4.6), (5.4.7) and (5.4.8) together with (2.2.3) and (2.4.12) form a system of n+2m-ℓ equations to determine $x_1, x_2, \ldots, x_n$; $y_1, \ldots, y_m$; $\eta_{m'+1}, \ldots, \eta_m$; $\lambda_{\ell+1}, \ldots \lambda_m$ as functions of the time t. The equations (5.4.6) and (5.4.7) form a system of m' pairs of equations in the Hamiltonian form with -R in place of the Hamiltonian H and the equations (5.4.8) are in the form (5.3.1) with R in place of the Lagrangian L.

5.5. First Integrals and the reduction of the order of the system

Let the displacement operators $X_1, X_2, \ldots, X_{m'}$ corresponding to the parameters of real displacement $\eta_1, \eta_2, \ldots, \eta_{m'}$ be cyclic according to N.G.Četaev [15] and let X_0 commute with each of them.

It follows that

$$X_\sigma(L) = 0, \; C_{0\sigma q} = 0, \; C_{r\sigma q} = 0,$$

$$(\sigma=1,2,\ldots,m'; \; q,r=1,2,\ldots,m).$$

Consequently

$$(5.5.1) \quad \begin{cases} X_\sigma(R) = -X_\sigma(L) = 0, \\ P_{0\sigma\mu} = P_{0\sigma\rho} = 0, \end{cases}$$

$$(\mu,\sigma=1,2,\ldots,m'; \rho=m'+1,\ldots,m).$$

We also assume that the constraints (2.2.3) are independent of the parameters $\eta_1, \eta_2, \ldots, \eta_{m'}$, so that

$$5.5.2) \qquad \frac{\partial f_\alpha}{\partial \eta_\sigma} = 0, \qquad (\sigma = 1,2,\ldots,m'; \alpha = \ell+1,\ldots,m).$$

ith the help of (5.5.1) and (5.5.2) the equations (5.4.7) ield

$$\frac{dy_\sigma}{dt} = 0,$$

r

$$5.5.3) \qquad y_\sigma = y_\sigma^0 \text{ (constant)}, \qquad (\sigma = 1,2,\ldots m').$$

his leads to the following theorem:

f a dynamical system moves subject to constraints of the orm (1.2.1) and (1.2.2) such that the operators $X_1, X_2, \ldots, X_{m'}$ re cyclic and X_0 commutes with each of them and the equations 2.2.3) are independent of the parameters $\eta_1, \ldots, \eta_{m'}$ then the quations of motion admit

$$y_\sigma = \text{constant}, \qquad (\sigma = 1,2,\ldots,m').$$

s first integrals.

As a particular case of the preceding theorem we deduce result proved by C. Agostinelli [/] when the x's are the agrangian coordinates and the system moves subject only to $-\ell$ nonholonomic constraints

$$5.5.4) \qquad \dot{x}_\alpha = c_{\alpha i}\,\dot{x}_i + c_\alpha, \qquad (i = 1,2,\ldots,\ell;\ \alpha=\ell+1,\ldots,n).$$

e take $\dot{x}_1, \dot{x}_2, \ldots, \dot{x}_n$ as the parameters of real displacement, that $y_1, y_2, \ldots, y_{m'}$ are the momenta corresponding to the coordinates $x_1, x_2, \ldots, x_{m'}$. In view of (5.5.1) and (5.5.2) it follows that the sufficient conditions for the existence of momentum integrals (5.5.3) are: (i) L is independent of the coordinates $x_1, x_2, \ldots, x_{m'}$, and (ii) the constraints (5.5.4) do not involve the velocity components $\dot{x}_1, \dot{x}_2, \ldots, \dot{x}_{m'}$.

Let us now consider the equations of motion (5.4.6), (5.4.7) and (5.4.8). In view of the conditions (5.5.1) and (5.5.2) these equations reduce to

$$(5.5.5) \qquad \eta_\sigma = -\frac{\partial R}{\partial y^o_\sigma}, \qquad (\sigma = 1,2,\ldots,m'),$$

$$(5.5.6) \qquad \frac{d}{dt}\left(\frac{\partial R}{\partial \eta_\rho}\right) - y^o_\mu [C_{o\rho\mu} + \eta_\nu C_{\nu\rho\mu}] - \frac{\partial R}{\partial \eta_\nu}[C_{o\rho\nu} + \eta_\omega C_{\omega\rho\nu}] -$$

$$- X_\rho(R) - \lambda_\alpha \frac{\partial f_\alpha}{\partial \eta_\rho} = o,$$

$$(\mu = 1,2,\ldots,m'; \rho,\nu,\omega = m'+1,\ldots,m; \alpha = \ell+1,\ldots,m).$$

The equations (5.5.5) and (5.5.6), together with (2.2.3) and (2.4.12), constitute a system of $n+2m-\ell-m'$ equations to determine $x_1, x_2, \ldots, x_n$; $\eta_{m'+1}, \ldots, \eta_m$; $\lambda_{\ell+1}, \ldots, \lambda_m$ as functions of the time. Also $\eta_1, \eta_2, \ldots, \eta_{m'}$ are determined from (5.5.5). Thus, a knowledge of the m' first integrals (5.5.3) reduce the system of equations (5.4.7) and (5.4.8) of order m to another system (5.5.6) of order $m-m'$.

It is interesting to note that the dynamical system does not represent an independent nonholonomic system in the noncyclic operators $X_{m'+1},\dots,X_m$. However, if the set of operators $X_o, X_{m'+1},\dots,X_m$ forms a subgroup of the group of operators $X_o, X_1,\dots,X_m$, we have

$$C_{o\rho\mu} = C_{\nu\rho\mu} = o, \quad (\mu=1,2,\dots,m';\ \rho,\nu=m'+1,\dots,m).$$

Consequently the dynamical system under consideration behaves like an independent nonholonomic system with respect to the noncyclic displacement operators $X_{m'+1},\dots,X_m$, and the role of the Lagrangian L is performed by the Routhian function R in the equations

$$\frac{d}{dt}\left(\frac{\partial R}{\partial \eta_\rho}\right) - C_{o\rho\nu}\frac{\partial R}{\partial \eta_\nu} - \eta_\omega C_{\omega\rho\nu}\frac{\partial R}{\partial \eta_\nu} - X_\rho(R) - \lambda_\alpha \frac{\partial f_\alpha}{\partial \eta_\rho} = o,$$

$$(\omega,\rho,\nu = m'+1,\dots,m;\ \alpha=\ell+1,\dots,m).$$

5.6. <u>Canonical Form of Chaplygin Equations in a Particular Case</u>

Let all the constraints, holonomic as well as nonholonomic, imposed on the linear nonholonomic system be time independent. Then in the equations of motion (3.8.5) we have

$$C_{oip} = o, \quad (i = 1,2,\dots,\ell;\ p = 1,2,\dots,m),$$

$$X_o(c_{\beta i}) = o,\ X_i(c_{\beta o}) = o,\ (i=1,2,\dots,\ell;\ \beta=\ell+1,\dots m).$$

Consequently the equations of motion (3.8.5) can be written in the form

$$(5.6.1)\quad \frac{d}{dt}\left(\frac{\partial \bar{T}}{\partial \eta_i}\right)-\eta_j C_{jik}\frac{\partial \bar{T}}{\partial \eta_k}-\eta_j\frac{\partial T}{\partial \eta_\beta}[c_{ji\beta}-c_{\beta k}C_{jik}+X_j(c_{\beta i})-X_i(c_{\beta j})]-$$

$$-X_i(\bar{T}+U)=o,\quad (i,j,k=1,2,\ldots,\ell;\ \beta=\ell+1,\ldots,m).$$

Let L be the Lagrangian of the system and let $\bar{L}$ denote the value of L after the nonholonomic constraints are taken into account. Thus

$$L = T+U,\quad \bar{L} = \bar{T}+U.$$

We also put

$$\Omega^\beta_{ji} = C_{ji\beta} - c_{\beta k}C_{jik} + X_j(c_{\beta i}) - X_i(c_{\beta j}).$$

Then the equations (5.6.1) become

$$(5.6.2)\quad \frac{d}{dt}\left(\frac{\partial \bar{L}}{\partial \eta_i}\right) -\eta_j C_{jik}\frac{\partial \bar{L}}{\partial \eta_k} -\eta_j\frac{\partial L}{\partial \eta_\beta}\Omega^\beta_{ji} - X_i(\bar{L}) = o,$$

$$(i,j,k = 1,2,\ldots,\ell;\ \beta=\ell+1,\ldots m).$$

Now we proceed to derive the canonical form of the equations (5.6.2). To this end, we put

$$(5.6.3)\quad Y_i = \frac{\partial \bar{L}}{\partial \eta_i}, \qquad (i = 1,2,\ldots,\ell),$$

(5.6.4) $$y_\beta = \frac{\partial L}{\partial \eta_\beta}, \qquad (\beta = \ell+1,\dots,m).$$

Let

$$\bar{H} = \bar{H}(x_1, x_2, \dots, x_n;\ y_1, \dots, y_\ell;\ t)$$

be the Hamiltonian H after eliminating from it the dependent y_β with the help of the equations of nonholonomic constraints, so that

(5.6.5) $$\bar{H}(x_1, x_2, \dots, x_n; y_1, \dots, y_\ell; t) = \eta_i y_i - \bar{L}, \quad (i = 1, 2, \dots, \ell).$$

Then, in view of (2.2.11), we have

$$\delta\bar{H} = \delta\eta_i y_i + \eta_i \delta y_i - \omega_i Y_i(\bar{L}) - \frac{\partial \bar{L}}{\partial \eta_i}\delta\eta_i,$$

which, by virtue of (5.6.3), takes the form

(5.6.6) $$\delta\bar{H} = \eta_i \delta y_i - \omega_i Y_i(\bar{L}), \qquad (i = 1, 2, \dots, \ell).$$

From the conditions (3.8.3) it follows that

(5.6.7) $$Y_i(\bar{L}) = X_i(\bar{L}).$$

Consequently (5.6.6) assumes the form

(5.6.8) $$\delta\bar{H} = \eta_i \delta y_i - \omega_i X_i(\bar{L}).$$

But

$$\delta\bar{H} = \omega_i X_i(\bar{H}) + \frac{\partial \bar{H}}{\partial y_i}\delta y_i,$$

which because of (5.6.7) becomes

$$(5.6.9) \qquad \delta\bar{H} = \omega_i X_i(\bar{H}) + \frac{\partial\bar{H}}{\partial y_i}\delta y_i .$$

Comparing (5.6.8) and (5.6.9), we get

$$(5.6.10) \qquad \begin{cases} \eta_i = \dfrac{\partial\bar{H}}{\partial y_i}, \\ X_i(\bar{L}) = -X_i(\bar{H}), \end{cases}$$

$$(i=1,2,\ldots,\ell).$$

With the help of (5.6.3) and (5.6.4), the equations (5.6.2) yield

$$X_i(\bar{L}) = \frac{dy_i}{dt} - \eta_j C_{jik} y_k - \eta_j y_\beta \Omega^\beta_{ji} .$$

As a consequence of the last relation the equations of motion (5.6.10) reduce to the canonical form

$$(5.6.11) \qquad \begin{cases} \eta_i = \dfrac{\partial\bar{H}}{\partial y_i}, \\ \dfrac{dy_i}{dt} = \eta_j C_{jik} y_k + \eta_j y_\beta \Omega^\beta_{ji} - X_i(\bar{H}), \end{cases}$$

$$(i,j,k = 1,2,\ldots,\ell;\ \beta=\ell+1,\ldots,m).$$

CHAPTER VI

HAMILTON-JACOBI THEOREM

6.1. General Considerations

In 1941, N.G.Četaev derived the partial differential equation satisfied by the Hamilton's function of action for a holonomic system which moves subject to time independent constraints. We extend this result to the case when the holonomic constraints are time dependent. We also consider the applicability of the Hamilton-Jacobi theorem for the integration of the equations of motion of a nonholonomic system.

6.2. Hamilton's Function of Action

In this attempts to establish an analogy between wave optics of Huygens and the motions of a holonomic dynamical system, Hamilton introduced the so-called function of action. This function is the integral which appears in the Hamilton's principle provided the upper limit in the integral is kept variable. Thus, if L is the kinetic potential of the system whose position is defined by the Poincaré-Cetaev variables $x_1, x_2, \ldots, x_n$ and which moves subject to holonomic constraints of the form (1.2.1), the Hamilton's function of action is

$$(6.2.1) \qquad V(x_1, x_2, \ldots, x_n; x_1^0, \ldots, x_n^0; t) = \int_{t_0}^{t} L\,dt,$$

where $x_1^0, x_2^0, \ldots, x_n^0$ are the values of $x_1, x_2, \ldots, x_n$ at the time $t = t_0$. Let $\omega_1, \omega_2, \ldots, \omega_p$ be the parameters of possible dis-

displacement and $X_1, X_2, \ldots, X_m$ the corresponding operators and let $\omega_1^o, \omega_2^o, \ldots, \omega_m^o$ and $X_1^o, X_2^o, \ldots, X_m^o$ correspond respectively to the ω's and X's at the time $t = t_o$. The variation of the function V, in view of (1.3.10), is given by

$$(6.2.2) \qquad \delta V = \omega_p X_p(V) + \omega_p^o X_p^o(V), \qquad (p = 1,2,\ldots,m).$$

Again, expressing L in terms of the Hamiltonian H, we have

$$\delta\int_{t_o}^{t} L\, dt = \delta\int_{t_o}^{t} (y_p\eta_p - H)dt = \int_{t_o}^{t} [\delta y_p\eta_p + y_p\delta\eta_p - \frac{\partial H}{\partial y_p}\delta y_p - \omega_p X_p(H)]dt,$$

$$(p = 1,2,\ldots,m).$$

Substituting for $\delta\eta_p$ from (2.6.2) and integrating, the last relation gives

$$\delta\int_{t_o}^{t} L\, dt = \omega_p y_p - \omega_p^o y_p^o +$$

$$+\int_{t_o}^{t} [\delta y_p(\eta_p - \frac{\partial H}{\partial y_p}) - \omega_p(\frac{dy_p}{dt} - C_{opq}y_q - \eta_q C_{qpr}y_r + X_p(H)]dt,$$

$$(p,q,r = 1,2,\ldots,m).$$

Since the integral is taken along the actual trajectory, the canonical equations (5.3.10) and (5.3.11) hold at each stant in the interval (t_o, t). Therefore the coefficients δy_p and ω_p under the integral sign on the right-hand the last relation vanish and we have

$$.2.3) \qquad \delta \int_{t_o}^{t} L\, dt = \omega_p y_p - \omega_p^o y_p^o, \qquad (p = 1,2,\ldots,m).$$

mparing (6.2.2) and (6.2.3), we get

$$.2.4) \qquad y_p = X_p(V), \quad y_p^o = - X_p^o(V), \qquad (p = 1,2,\ldots,m).$$

the function V is known the relations (6.2.4) together ith n-m equations of the holonomic constraints of the form .2.1) provided a solution of the Hamiltonian problem. owever, the solution of the Lagrangian problem is furnished the second group of relations (6.2.4) and the equations .2.1).

To derive the differential equation satisfied by V, differentiate (6.2.1) with respect to the time t to obtain

$$\frac{dV}{dt} = X_o(V) + \eta_p X_p(V) = L, \qquad (p = 1,2,\ldots,m).$$

view of (6.2.4), the last result gives

$$.2.5) \qquad X_o(V) + y_p \eta_p - L = 0.$$

the Hamiltonian H is defined by the relation

(6.2.6) $H(x_1, x_2, \ldots, x_n; y_1, \ldots, y_m; t) = y_p \eta_p - L$

The relations (6.2.5) and (6.2.6), together with (6.2.4), yield a differential equation which is satisfied by V:

$$X_o(V) + H(x_1, x_2, \ldots, x_n; X_1(V), \ldots, X_m(V); t) = o.$$

This is Hamilton's partial differential equation. The function V, expressed in terms of the x's and t and the n parameters x^o's is a complete integral because without loss of generality we can take $t_o = o$. It is to be noted here that the x^o's are not all independent and have to satisfy the holonomic constraints (1.2.1) at the initial instant.

Now there exists a great variety of complete integrals of a partial differential equation, and, if we started from Hamilton's differential equation and found a complete integral of it, we should have no guarantee that this integral would be an expression for the function V which we seek. But the question suggests itself, will any complete integral serve our purpose? The answer is affirmative, and this fact is the heart of the Hamilton-Jacobi theorem which is discussed below.

6.3. Hamilton-Jacobi Theorem for a Holonomic System

We prove the following

Theorem. If

(6.3.1) $V = V(x_1, x_2, \ldots, x_n; a_1, a_2, \ldots, a_n; t)$

is a complete integral of Hamilton's partial differential equation

(6.3.2) $X_o(V) + H(x_1, x_2, \ldots, x_n; X_1(V), \ldots, X_m(V); t) = o,$

then for a system subject to holonomic constraints

(6.3.3) $B_{se}(x_1, \ldots, x_n; t)\dot{x}_e + B_s(x_1, \ldots, x_n; t) = o$

$$(s = m+1, \ldots, n;\ e = 1, \ldots, n),$$

the integrals of the Hamilton's equations

(6.3.4) $$\begin{cases} \eta_p = \dfrac{\partial H}{\partial y_p}, \\ \dfrac{dy_p}{dt} = C_{opq}y_q + \eta_q C_{qpr}y_r - X_p(H), \end{cases}$$

$$(p, q, r = 1, 2, \ldots, m),$$

are given by the equations

(6.3.5) $y_p = X_p(V),\ b_p = -A_p(V), \quad (p = 1, 2, \ldots, m),$

where A_p define the group of infinitesimal displacement operators for the a's and the b's are m new arbitrary constants.

The equations (6.3.5), together with the equations (6.3.3), determine the x's and y's as functions of the a's, b's and t.

Proof. Since a complete integral is a function of class C_2

containing n arbitrary constants $a_1, a_2, \ldots, a_n$ and the displacement operators $X_1, X_2, \ldots, X_m$ are independent, the determinant

$$| X_p A_q (V) | , \qquad (p,q = 1,2,\ldots,m),$$

is nowhere zero in the relevant domain of the x's and a's.

We have to show that the functions

$$(6.3.6)\begin{cases} x_e = x_e(a_1,a_2,\ldots,a_n;\ b_1,b_2,\ldots,b_m;t), (e = 1,2,\ldots,n), \\ y_p = y_p(a_1,a_2,\ldots,a_n;\ b_1,b_2,\ldots,b_m;t), (p = 1,2,\ldots,m), \end{cases}$$

determined from (6.3.3) and (6.3.5) satisfy the equations (6.3.4) for arbitrary values of a's and b's.

Now V satisfies (6.3.2) for all values of x's and a's and t in the appropriate domain; so substituting the complete integral in (6.3.2) and applying the operator A_p, we have

$$(6.3.7) \qquad A_p X_o (V) + \frac{\partial H}{\partial y_q} A_p X_q (V) = o, \quad (p,q = 1,2,\ldots,m).$$

Also the equation

$$b_p = - A_p(V),$$

is satisfied identically if we substitute for each x from (6.3.6), so substituting these values and differentiating with respect to t, we get

$$(6.3.8) \qquad X_o A_p (V) + \eta_q X_q A_p (V) = o.$$

these values, and differentiating with respect to t, we have

$$(6.3.12)\qquad \frac{dy_p}{dt} = X_o X_p(V) + \eta_q X_q X_p(V), \qquad (q = 1,2,\ldots,m).$$

Now the relations (6.3.11) and (6.3.12), in view of (6.3.10) give

$$\frac{dy_p}{dt} = (X_o, X_p)V + \eta_q (X_q, X_p)V - X_p(H), \quad (p,q=1,2,\ldots,m).$$

With the help of (1.3.12) and (6.3.5), the last equations become

$$(6.3.13)\qquad \frac{dy_p}{dt} = c_{opq} y_q + \eta_q c_{qpr} y_r - X_p(H), \quad (p,q,r=1,2,\ldots,m).$$

The equations (6.3.10) and (6.3.13) prove that the functions x_e and y_p given by (6.3.6) satisfy Hamilton's equations (6.3.4) for arbitrary values of a's and b's, and the theorem is proved.

6.4. Integration of the Chaplygin Equations

S.A. Chaplygin [16] has shown that in certain cases the Hamilton-Jacobi method can be applied to integrate Hamilton's equations of motion of a nonholonomic dynamical system. It follows that Hamilton-Jacobi method may not always be applicable in the case of nonholonomic systems. In the sequel we consider the modifications of the Hamilton-Jacobi theorem when applied to integrate the dynamical equations (5.6.11) of a linear nonholonomic system.

For the nonholonomic dynamical system considered in section 5.6 the equations of motion are

Now since $V \varepsilon C_s$ we have

$$X_o A_p(V) = A_p X_o(V),$$

$$X_q A_p(V) = A_p X_q(V),$$

so from (6.3.7) and (6.3.8) we obtain

$$(6.3.9) \qquad (\eta_q - \frac{\partial H}{\partial y_q}) X_q A_p(V) = o.$$

Moreover, there are m such equations, one corresponding to each A_p, and the determinant

$$|X_q A_p(V)|$$

of the coefficients is non-vanishing, whence

$$(6.3.10) \qquad \eta_p = \frac{\partial H}{\partial y_p}, \qquad (p = 1,2,\ldots,m).$$

Next we again substitute the complete integral in (6.3.2), and now we apply the operator X_p to obtain

$$(6.3.11) \qquad X_p X_o(V) + X_p(H) + \frac{\partial H}{\partial y_q} X_p X_q(V) = o.$$

Now the equation

$$y_p = X_p(V),$$

is satisfied identically if we substitute for x's and y's their values in terms of a's, b's and t; so substituting

$$(6.4.1)\quad \begin{cases} \eta_i = \dfrac{\partial \bar{H}}{\partial y_i}, \\ \dfrac{dy_i}{dt} = \eta_j C_{jik} y_k + \eta_j y_\beta \Omega^\beta_{ji} - X_i(\bar{H}), \end{cases}$$

$$(i,j,k = 1,2,\dots,\ell;\ \beta = \ell+1,\dots,m).$$

Let us construct the first order partial differential equation

$$(6.4.2)\qquad Y_o(V) + \bar{H}(x_1, x_2, \dots, x_n; Y_1(V), \dots, Y_m(V); t) = o,$$

where we have substituted $Y_i(V)$ for y_i in $\bar{H}$. We have the following

Theorem. If $V(x\ , x\ , \dots, x_n;\ a\ , \dots, a_n;\ t)$ is a complete integral of equation (6.4.2) then the integrals of the equations of motion (6.4.1) of the nonholonomic dynamical system are given by the equations

$$(6.4.3)\qquad y_i = Y_i(V) = X_i(V) + c_{\alpha i} X_\alpha(V),\quad (i=1,2,\dots,\ell,\ \alpha=\ell+1,\dots,m)$$

$$(6.4.4)\qquad b_p = \Lambda_p(V),\qquad (p = 1,2,\dots,m),$$

where $a_1, a_2, \dots, a_n$, $b_1, \dots, b_m$ are arbitrary constants compatible with the constraint equations and Λ's are the operators X's corresponding to $a_1, a_2, \dots, a_n$.

Proof. In the equation (6.4.2) we substitute the complete integra V and operate by Λ_p. Then we have

$$(6.4.5) \qquad A_pY_o(V) + \frac{\partial\overline{H}}{\partial y_i} A_pY_i(V) = o.$$

Differentiating (6.4.4) with respect to the time t, we get

$$(6.4.6) \qquad Y_oA_p(V) + \eta_i Y_iA_p(V) = o.$$

Since the complete integral V is of class C_2, we have

$$Y_oA_p(V) = A_pY_o(V),$$
$$Y_iA_p(V) = A_pY_i(V),$$

and so the relations (6.4.5) and (6.4.6) yield

$$(6.4.7) \qquad (\eta_i - \frac{\partial\overline{H}}{\partial y_i}) Y_iA_p(V) = o, \qquad (i=1,2,\ldots,\ell;p=1,2,\ldots,m).$$

The equations (6.4.7) form a system of m equations and since the matrix $||Y_iA_p(V)||$ is of rank ℓ, the equations (6.4.7) give

give

$$(6.4.8) \qquad \eta_i = \frac{\partial\overline{H}}{\partial y_i}, \qquad (i = 1,2,\ldots,\ell)$$

Again substituting the complete integral for V in (6.4.2) and operating by Y_i, we get

$$Y_iY_o(V) + Y_i(\overline{H}) + \frac{\partial\overline{H}}{\partial y_j} Y_iY_j(V) = o,$$

which, in view of (6.4.8), becomes

$$(6.4.9) \qquad Y_iY_o(V) + Y_i(\overline{H}) + \eta_j Y_iY_j(V) = o, \qquad (j = 1,2,\ldots,\ell).$$

ifferentiating (6.4.3) with respect to the time t, we have

(6.4.10) $$\frac{dy_i}{dt} = Y_o Y_i(V) + \eta_j Y_j Y_i(V).$$

Since all the constraints are time independent, we have

$$X_o = \frac{\partial}{\partial t} = Y_o.$$

Moreover, X_o commutes with $X_1, X_2, \ldots, X_m$ it follows that Y_o commutes with $Y_1, Y_2, \ldots, Y_\ell$. This fact enables us to obtain from (6.4.9) and (6.4.10) the relations

(6.4.11) $$\frac{dy_i}{dt} = \eta_j (Y_j, Y_i)V - Y_i(\bar{H}), \quad (i,j = 1,2,\ldots,\ell).$$

From the relations (2.2.9) we have

$$(Y_j, Y_i) = (X_j + c_{\beta j} X_\beta)(X_i + c_{\alpha i} X_\alpha) - (X_i + c_{\alpha i} X_\alpha)(X_\beta + c_{\beta j} X_\beta),$$
$$(\alpha, \beta = \ell+1, \ldots, m).$$

But for the system under consideration we have, as explained in section 3.8, the following relations

$$(X_\alpha, X_p) = 0, \; X_\alpha(c_{\beta i}) = 0, \; (i=1,2,\ldots,\ell; \alpha,\beta=\ell+1,\ldots,m; p=1,2,\ldots,$$

In view of the last relations, the commutator (Y_j, Y_i) can be expressed in the form

(6.4.12) $$(Y_j, Y_i) = C_{jik} Y_k + \Omega^\beta_{ji} X_\beta.$$

As a consequence of (6.4.3) and (6.4.12) the equations (6.4.11) become

$$\frac{dy_i}{dt} = \eta_j C_{jik} y_k + \eta_j \Omega^{\beta}_{ji} X_\beta(V) - Y_i(\bar{H}), \tag{6.4.13}$$

$$(i,j,k = 1,2,\ldots,\ell;\ \beta = \ell+1,\ldots,m).$$

Also, by virtue of the definition of $\bar{H}$ and the relation (5.6.7), we have

$$Y_i(\bar{H}) = X_i(\bar{H}), \qquad (i = 1,2,\ldots,\ell).$$

Consequently (6.4.13) is equivalent to

$$\frac{dy_i}{dt} = \eta_j C_{jik} y_k + \eta_j \Omega^{\beta}_{ji} X_\beta(V) - X_i(\bar{H}), \tag{6.4.14}$$

$$(i,j,k = 1,2,\ldots,\ell;\ \beta = \ell+1,\ldots,m).$$

Since

$$Y_\beta = X_\beta(V), \qquad (\beta = \ell+1,\ldots,m).$$

the equations (6.4.14) lead to the second half of equations (6.4.1). This completes the proof.

If the x's are the Lagrangian coordinates, the theorem discussed here includes as a special case an analogous theorem proved by M.F.Sul'gin [39].

6.5. Hamilton-Jacobi Theorem for Nonholonomic Systems

In the last section we considered the modifications in the Hamilton-Jacobi theorem so as to apply to a linear nonholonomic system of a special type. Here we investigate the necessary and sufficient conditions for the applicability of the Hamilton-Jacobi theorem to nonholonomic systems moving with nonlinear nonholonomic constraints of Četaev's type.

Let us consider a nonholonomic dynamical system moving with constraints of the form (1.2.1) and (1.2.2). The canonical equations of motion (5.3.8) and (5.3.9) of the system are:

$$(6.5.1)\quad \begin{cases} \eta_p = \dfrac{\partial H}{\partial y_p} \\ \dfrac{dy_p}{dt} = C_{opq}y_q + \eta_q C_{qpr}y_r - X_p(H) + \lambda_\alpha \dfrac{\partial f_\alpha}{\partial \eta_p}, \end{cases}$$

$$(\alpha = \ell+1,\ldots,m;\ p,q,r = 1,2,\ldots,m).$$

By introducing a function ϕ which may be assumed to depend on $x_1, x_2, \ldots, x_n, y_1, \ldots, y_m$ and t, we consider in place of the Hamilton's partial differential equation (6.3.2) the modified equation

$$(6.5.2)\quad X_o(V) + H(x_1, x_2, \ldots, x_n;\ X_1(V), \ldots, X_m(V);\ t) + \phi = 0.$$

The function ϕ is determined in such a way that if $V(x$ $V(x_1, x_2, \ldots, x_n;\ a_1, \ldots, a_n;\ t)$ is a complete integral of (6.5.2), the integrals of the canonical equations (6.5.1) are given by the equations

$$(6.5.3)\quad y_p = X_p(V),$$

$$(6.5.4) \qquad b_p = - A_p(V), \qquad (p = 1,2,\ldots,m).$$

Differentiating (6.5.4) with respect to t, we get

$$(6.5.5) \qquad X_o A_p(V) + \eta_q X_q A_p(V) = o, \qquad (q = 1,2,\ldots,m).$$

Substituting the complete integral for V in (6.5.2), and applying the operator A_p we have

$$(6.5.6) \qquad A_p X_o(V) + \frac{\partial H}{\partial y_q} A_p X_q(V) + \frac{\partial \phi}{\partial y_q} A_p X_q(V) = o.$$

Since the function $V(x_1, x_2, \ldots, x_n;\ a_1, \ldots, a_n;\ t)$ is of class C_2 we have

$$X_o A_p(V) = A_p X_o(V), \quad A_p X_q(V) = X_q A_p(V),$$

and the determinant $|X_p A_q(V)| \neq o$.

Therefore the relations (6.5.5) and (6.5.6) give

$$(6.5.7) \qquad \frac{\partial \phi}{\partial y_q} = o, \qquad (q = 1,2,\ldots,m).$$

This shows that the function ϕ is independent of $y_1, y_2, \ldots, y_m$.

Again differentiating (6.5.3) with respect to t, we get

$$(6.5.8) \qquad \frac{dy_p}{dt} = X_o X_p(V) + \eta_q X_q X_p(V).$$

Application of the operator X_p to (6.5.2) gives

$$(6.5.9) \qquad X_p X_o(V) + X_p(H) + \frac{\partial H}{\partial y_q} X_p X_q(V) + X_p(\phi) = 0.$$

Now the relations (6.5.8) and (6.5.9), with the help of (1.3.12) and (6.5.3), yield

$$(6.5.10) \qquad \frac{dy_p}{dt} = C_{opq} y_q + \eta_q C_{qpr} y_r - X_p(H) - X_p(\phi).$$

Comparing (6.5.10) with the second of equations (6.5.1), we get

$$(6.5.11) \qquad X_p(\phi) = -\lambda_\alpha \frac{\partial f_\alpha}{\partial \eta_p}, \qquad (\alpha = \ell+1,\ldots,m;\ p=1,2,\ldots,m).$$

Thus we have the generalised Hamilton-Jacobi theorem:
In order that Hamilton-Jacobi theorem be applicable to nonholonomic systems subject to constraints of the form (1.2.1) and (1.2.2), it is necessary and sufficient to modify the original partial differential equation (6.3.2) by means of a function ϕ which satisfies the conditions (6.5.7) and (6.5.11).

The theorem proved here is an extension of an analogous theorem discussed by Q.K.Ghori [20] when the x's are the Lagrangian coordinates.

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