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BEST APPROXIMATION IN FUNCTION SPACES

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TO MY

MOTHER

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"And if all the trees on earth were pen and the oceans (were ink), with seven oceans behind it to add to its (supply). Yet would not the words of God be exhausted (in the writing) for God is exalted in power. Fall of wisdom. "

(AL QURAN)

"He thanks not God, who thanks not the people."

Holy Prophet Muhammad
(peace be upon him)

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PREFACE

In this dissertation we are primarily concerned with presenting recent results on Best Approximation in metric linear spaces, locally convex spaces, and spaces of continuous vector-valued functions. The dissertation consists of three chapters.

In chapter 1, we introduce basic terminology such as proximinal, Chebyshev and semi-Chebyshev sets, and consider best approximation for normed spaces and, in particular, for Hilbert, uniformly convex, strictly convex, and reflexive spaces. The proofs of the results are mostly omitted as they can be found in standard books on Functional Analysis and Approximation Theory. In this chapter, we also mention an open problem which is still not resolved: Is every Chebyshev set in a Hilbert space convex ? Some partial answers and their generalizations are given.

Second chapter deals with generalizations of some results of chapter 1 to metric linear spaces and locally convex spaces. The results and their proofs are taken from recent research papers.

In the last chapter, we are concerned with approximation and best approximation in function spaces. In particular, we present the statements and proofs of Stone-Weierstrass type theorems for vector-valued functions, both in the locally convex and non-locally convex settings. An important result on best approximation in continuous real valued function spaces is the Haar criteria of uniqueness. This was established by A.Haar in 1918 for functions on a closed interval and subsequently extended by R.Phelps in 1960 for functions on a compact topological space. Other forms of approximation such as the least-square approximation and the rational approximation are also mentioned. Finally, two results on best approximation in spaces of bounded functions on paracompact spaces and of continuous linear mappings on Hilbert spaces, by Holmes and Kripke, are presented. As a counter example, it is mentioned that the last result does not hold for linear mappings on certain reflexive Banach spaces.

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CHAPTER 1

BEST APPROXIMATION
IN
NORMED SPACES

§ 1.1 INTRODUCTION

In this chapter we shall give some definitions and a survey of result on Best approximation in normed spaces and, in particular, in inner product, uniformly convex, strictly convex and reflexive spaces. The proofs of most of these results are omitted as they may be found in standard Text books. (e.g., E.W.Cheney [6], Heuser [10], R.B.Holmes [12], G.Köthe [20], E.Kreyszig [22], J.R.Rice [28], I.Singer [33]). Some of these results will be generalized in chapter 2 to metric linear spaces and locally convex spaces.

§ 1.2. BASIC DEFINITIONS AND EXAMPLES

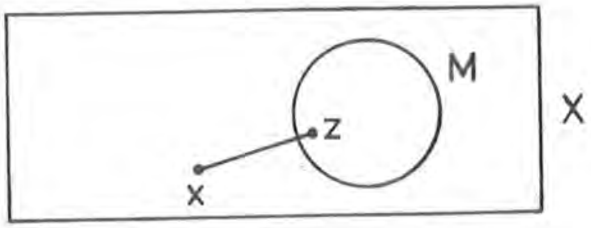
We begin with the following definitions.

DEFINITION 1.2.1.

Let (X,d) be a metric space and $M \subseteq X$. For each $x \in X$, let $d(x,M) = \inf_{y \in M} d(x,y)$ and $P_M(x) = \{z \in M : d(x,z) = d(x,M)\}$

i.e., $z \in P_M(x)$ iff $d(x,z) = d(x,M)$ (1)

Any $z \in M$ wich satisfy (1) is called a *point of best approximation* from x to M. (z is also called the nearest point of M from x)



Note that

$$P_M(x) = \begin{cases} \{x\} & \text{if} \quad x \in M \\ \phi & \text{if} \quad x \in \bar{M} - M \end{cases}$$

Consequently, we may always assume in the sequel that $x \in X - \bar{M}$.

DEFINITION 1.2.2.

If $P_M(x) \neq \phi$ for each $x \in X$, then M is called a *proximinal set* (or *existance set*); if $P_M(x)$ = singleton i.e., if each $x \in X$ has a unique point of best approximation in M, then M is called a *Chebyshev set*; if $P_M(x)$ = singleton or empty, then M is called a *semi-Chebyshev* (or *unicity*) set.

Clearly every Chebyshev set is proximinal and semi-Chebyshev but converse is not true as we shall see by examples.

Note that if $(X, \| \|)$ is a normed space and M a subset of X, then for any $x \in X$, $d(x,M) = \inf_{y \in M} \|x-y\|$ and

$$P_M(x) = \{z \in M : \|x-z\| = d(x,M)\}$$

EXAMPLE 1.2.3.

Let $X = R^2$ with norm $\|x\| = \sqrt{x_1^2 + x_2^2}$

where $x = (x_1, x_2) \in R^2$

Let $M = \{(\alpha,0) : \alpha \in R\}$, the X-axis.

Let $x = (2,1) \in R^2$

Then clearly $z = (2,0)$ is the unique best approximation in M for $x = (2,1)$; i.e., $P_M(x) = \{(2,0)\}$.

Note that if $x = (1,1)$, then $P_M(x) = \{(1,0)\}$

In this case M is a Chebyshev set.

EXAMPLE 1.2.4.

Let $X = R^2$, with norm $\|x\|' = \max \{ |x_1|, |x_2| \}$

Let $M = \{ (\alpha,0) : \alpha \in R \}$, the x-axis.

Let $x = (1,1) \in R^2$.

Then, for any $y = (\alpha,0) \in M$,

$$\|x-y\|' = \| (1,1) - (\alpha,0) \|' = \max \{ |1-\alpha| , |1-0| \}$$
$$= \max \{ |1-\alpha|, 1 \}.$$

This has minimum value if $|1-\alpha| \leq 1$ i.e., if $0 \leq \alpha \leq 2$.

Therefore if $y = (\alpha, 0)$, where $0 \le \alpha \le 2$ then y is a point of best approximation. Hence, in this case,

$$P_M(x) = \{ (\alpha, 0) : 0 \le \alpha \le 2 \},$$

the line segment joining $(0,0)$ and $(2,0)$. Therefore M is Proximinal but not Chebyshev.

EXAMPLE 1.2.5.

Let $X = R$ and $M = (1,2)$, an open interval. Then, for any $x \in X-M$, $P_M(x) = \phi$. So M is semi-Chebyshev but not proximinal or Chebyshev.

As a sort of another example, we have the following result.

THEOREM 1.2.6. [6]

Any Compact subset K of a metric space (X,d) is proximinal.

PROOF

Let $x \in X$, and let $\gamma = \inf_{y \in K} d(x,y)$.

We show that there exists a point $z \in K$ such that $d(x,z) = \gamma$. From the definition of infimum, it follows that there exists a sequence $\{y_m\}_{m=1}^{\infty} \subseteq K$, (usually called the minimizing sequence) such that $\lim_{m \to \infty} d(x, y_m) = \gamma$. (1)

Since K is Compact, every sequence in K has a subsequence with limit in K. So there exists a subsequence $\{y_{m_i}\}_{i=1}^{\infty}$ of $\{y_m\}_{m=1}^{\infty}$ such that $\lim_{i \to \infty} y_{m_i} = z \in K$.

Then, by continuity of d, $\lim_{i\to\infty} d(x,y_{m_i}) = d(x,z)$ (2)

Now $\{d(x,y_{m_i})\}_{i=1}^{\infty}$ is a convergent subsequence of $\{d(x,y_m)\}_{m=1}^{\infty}$.

Since $\{d(x,y_m)\}_{m=1}^{\infty}$ is convergent and hence a Cauchy sequence in R, it follows from (1) that

$$\lim_{i\to\infty} d(x,y_{m_i}) = \lim_{m\to\infty} d(x,y_m) = \gamma \qquad (3)$$

By uniqueness of limit, it follows from (2) and (3) that

$$d(x,z) = \gamma$$

Hence $z \in P_K(x)$, and so K is proximinal. ∎

§ 1.3. BEST APPROXIMATION IN NORMED AND HILBERT SPACES.

In this section, we will give the statement of the results on best approximation in normed spaces and inner product spaces. We recall that a subset A of a vector space X is called *convex* if $rx + (1-r)y \in A$ for all $x, y \in A$ and $0 \le r \le 1$.

THEOREM 1.3.1. [6]

Let $(X, \| \ \|)$ be a normed space over K. Then any finite dimensional vector subspace M of X is Proximinal. ∎

THEOREM 1.3.2. [33]

If M is a non-empty convex subset of a normed space X, then, for any $x \in X$, $P_M(x)$ is a convex, closed and bounded subset of X. ∎

THEOREM 1.3.3. [6]

Every convex subset K of an inner product space X is semi-Chebyshev. ■

THEOREM 1.3.4. [6]

Let $\{e_1, e_2, \ldots, e_n\}$ be an orthonormal system in an inner product space X. Then

$M = sp\{e_1, e_2, \ldots, e_n\}$ is a Chebyshev set in X. ■

THEOREM 1.3.5. [10]

Every non-empty convex complete subset K of an inner product space X is Chebyshev. ■

THEOREM 1.3.6. [10]

Let M be a vector subspace of an inner product space X, and let $x \in X$ and $z \in M$. Then

$$z \in P_M(x) \quad \Leftrightarrow \quad x-z \perp M. \blacksquare$$

§ 1.4. BEST APPROXIMATION IN UNIFORMLY AND STRICTLY CONVEX SPACES.

DEFINITION 1.4.1. [20]

A normed space $(X, \|\ \|)$ is called *uniformly convex* if for each $0 < \varepsilon \leq 2$ there exists $\delta = \delta(\varepsilon) > 0$ such that $\|x\| \leq 1$, $\|y\| \leq 1$ and $\|x-y\| \geq \varepsilon$ implies

$$\|(x+y)/2\| \leq 1-\delta.$$

NOTE:

Uniformly Convexity is a geometric property of the unit sphere of the space: if the "mid point" of a line segment with end points on the surface of the sphere approaches the surface, then the end points must come closer together.

Uniformly Convex

Not Uniformly Convex

EXAMPLE 1.4.2.

Any inner product space $(X, \langle , \rangle)$ is Uniformly convex.

EXAMPLE 1.4.3. [20]

The spaces l_p and L_p are uniformly convex for $1 < p < \infty$.

THEOREM 1.4.4. [20]

The following are equivalent.

(a) The normed space $(X, \| \ \|)$ is uniformly convex.

(b) If $\{x_n\}$, $\{y_n\}$ are sequences in $(X, \| \ \|)$ with $\|x_n\| \le 1$, $\|y_n\| \le 1$, then $\|(x_n+y_n)/2\| \to 1$ implies that

$$\|x_n - y_n\| \to 0. \blacksquare$$

THEOREM 1.4.5. [6]

Any non-empty convex and complete subset K of a uniformly convex space X is Chebyshev. ■

DEFINITION 1.4.6. [6]

A normed space X is called *strictly convex* if for any $x, y \in X$ with $\|x\| = 1$, $\|y\| = 1$, $x \neq y$, implies $\|x+y\| < 2$.

THEOREM 1.4.7. [6]

Every uniformly convex normed space (in particular every inner product space) is strictly convex. ■

EXAMPLE 1.4.8.

The space C[a,b] with sup norm is not strictly convex.

SOLUTION

We consider $f, g : [a,b] \to R$ defined by:

$f(t)=1$, $g(t)=\frac{t-a}{b-a}$, where $a \leq t \leq b$, then clearly $f, g \in C[a,b]$.

We also have $\|f\| = 1$, $\|g\| = \sup\limits_{a\leq t\leq b} \left|\frac{t-a}{b-a}\right| = \left|\frac{b-a}{b-a}\right| = 1$.

but $\|f+g\| = \sup\limits_{a\leq t\leq b} \left| 1+ \frac{t-a}{b-a} \right| = \left| 1+ \frac{b-a}{b-a}\right| = 2$.

This shows that C[a,b] is not strictly convex.

We mention that the spaces C_0, l_1, l_∞, L_1, L_∞ are also not strictly convex ([20], p. 343). A partial converse of Theorem 1.4.7 is

THEOREM 1.4.9. [6]

Every finite-dimensional strictly convex normed space is uniformly convex. ■

THEOREM 1.4.10. ([20], p. 343)

Any convex subset K of a strictly convex normed space X is semi-Chebyshev. ■

The converse of the above result (or Theorem 1.3.3) is a hard problem which is still open.

OPEN PROBLEM: (see [19])

Is every Chebyshev set in a Hilbert space convex?

We now state some known partial solutions to this problem and their subsequent generalizations.

THEOREM 1.4.11. (Motzkin, 1935)

Every Chebyshev set in a finite dimensional space is convex. ■

THEOREM 1.4.12. (Ficken, 1951)

Every compact Chebyshev set in a Hilbert space is convex. ■

For further generalizations we need the following terminology (see [8], [35]). A normed space X is called *Uniformly smooth* if, for each $\varepsilon > 0$, there exists $\delta(\varepsilon) > 0$, such that

$\|x-y\| < \delta$ implies $\|x+y\| \geq \|x\| + \|y\| - \varepsilon \|x-y\|$.

A subset K of a normed space X is called *boundedly compact* if $K \cap B_r(x)$ is empty or convex for each closed ball $B_r(x) = \{y \in X : \|x-y\| \leq r\}$, ($x \in X$, $r > 0$); K is called *approximatively compact* if for any $x \in K$ each minimizing sequence $\{y_n\}$ in K has a subsequence converging to an element of K. Every compact set is boundedly compact and every boundedly cmpact set is approximatively compact. For any $f \in X'$ and $r \in R$, the sets $\{x \in X : f(x) \geq r\}$ and $\{x \in X : f(x) \leq r\}$ are called *half spaces*.

THEOREM 1.4.13. [9]

Every boundedly compact Chebyshev set in a uniformly smooth and uniformly convex Banach space is convex. ■

THEOREM 1.4.14. [19]

Every weakly closed Chebyshev set in a uniformly smooth and uniformly convex Banach space is convex. ■

THEOREM 1.4.15. [9]

Every approximatively compact Chebyshev set in a uniformly smooth and uniformly convex Banach space is convex. ■

THEOREM 1.4.16. [2]

(a) If the metric projection P_K onto a Chebyshev set K in a Hilbert space is norm continuous, then K is convex.

(b) If K is Chebyshev set in a Hilbert space such that its intersection with each closed half-space is proximinal set, then K is convex.

§ 1.5 BEST APPROXIMATION IN REFLEXIVE SPACES

In this section we show that every closed linear subspace of a reflexive Banach space is proximinal. To prove this important result we need some definitions and results, which we state as follows.

DEFINITION 1.5.1.

Let X be a normed space. Then its continuous dual X' is a Banach space w.r.t the norm $\|f\| = \sup_{\substack{x \in X \\ \|x\| \le 1}} |f(x)|$, $f \in X'$. Let $X'' = (X', \|\ \|)'$. Then X'' is a Banach space w.r.t the norm $\|G\| = \sup_{\substack{f \in X' \\ \|f\| \le 1}} |G(f)|$. X'' is called the bidual of X. For each $x \in X$, define $\hat{x}(f) = f(x)$ for all $f \in X'$. It is well known that $\hat{x} \in (X')'$ with $\|\hat{x}\| = \|x\|$. Let $\hat{X} = \{\hat{x} : x \in X\}$. Then X is isometrically isomorphic to $\hat{X}$ and we may identify $X = \hat{X} \subseteq X''$.

DEFINITION 1.5.2.

A normed space $(X, \|\ \|)$ is called reflexive if $X = X''$.

Note that, since X'' is always complete, a reflexive normed space is necessarily a Banach space.

EXAMPLES 1.5.3.

(1) Every Hilbert space (e.g., l_2) is reflexive.

since $H'' = (H')' = (H)' = H$.

(2) For $1 < p < \infty$, l_p is reflexive.

since $(l_p)'' = (l_p')' = (l_q)' = l_p$.

where $\frac{1}{q} + \frac{1}{p} = 1$.

(3) C_o, C, l_1, l_∞, $C[a,b]$, l_p ($o < p \le 1$) are not reflexive.

(4) Every uniformly convex Banach space is reflexive. (Köthe [20], p.354)

(5) The space $l_p(E_n)$, $E_n = l_n^\infty$, $p > 1$, is reflexive but not uniformly convex (Köthe [20] p 361).

THEOREM 1.5.4.

A closed vector subspace M of a reflexive Banach space $(E, \| \|)$ is reflexive. ■

Recall that a sequence $\langle x_n \rangle$ in $(X, \| \|)$ is said to *converge weakly* to $x \in X$ if $|f(x_n)-f(x)| \to 0$ for all $f \in X'$

THEOREM 1.5.5. ([10], p.252)

Every bounded sequence in a reflexive Banach space has a weakly convergent subsequence. ■

We now state and prove the main result of this section.

THEOREM 1.5.6. (Klee 1948; see [10], [20])

Let M be a closed linear subspace of a reflexive Banach space E. Then M is proximinal.

PROOF

Let $x \in X$ and $\gamma = \inf_{y \in M} \|x-y\|$. Let $\{y_n\}$ be a minimizing sequence in M such that $\lim_{n\to\infty} \|x-y_n\| = \gamma$. Then it follows that $\{y_n\}$ is a bounded sequence in M.

Now M being a closed subspace of a reflexive Banach space is itself a Banach space (Theorem 1.5.4.).

So, by Theorem 1.5.5, $\{y_n\}$ has a convergent subsequence $\{y_{nk}\}_{k=1}^{\infty}$ with $y_{nk} \xrightarrow{\text{weakly}} y_o$ for some $y_o \in M$. Then $x-y_{nk} \xrightarrow{\text{weakly}} x-y_o$. Now $\|x-y_o\| = \|\widehat{x-y_o}\| = \sup_{\substack{f \in X' \\ \|f\| \le 1}} |f(x-y_o)|$

$$= \sup_{\substack{f \in X' \\ \|f\| \le 1}} \lim_{K\to\infty} |f(x-y_{nk})|$$

$$\le \sup_{\substack{f \in X' \\ \|f\| \le 1}} \lim_{m\to\infty} \inf_{K \ge m} \|f\| \; \|x-y_{nk}\|$$

$$= \lim_{m\to\infty} \inf_{K \ge m} \|x-y_{nk}\|.$$

Hence

$$\gamma \le \|x-y_o\| \le \lim_{m\to\infty} \inf_{K\ge m} \|x-y_{nk}\|$$

$$= \lim_{n\to\infty} \|x-y_n\|$$

$$\text{or } \|x-y_o\| = \gamma.$$

Thus $y_o \in P_M(x)$ and so M is proximinal. ∎

CHAPTER 2

BEST APPROXIMATION IN TOPOLOGICAL VECTOR SPACES

§ 2.1 INTRODUCTION

In this chapter, we extend some of results of chapter 1 to metric linear spaces and locally convex spaces. These results are mostly taken from the papers [1,23,24,25,34].

§ 2.2 BEST APPROXIMATION IN METRIC LINEAR SPACES.

The problem of best approximation has been extensively studied in normed spaces. The same problem in metric linear spaces has been studied by G.Albinus [see 1], Ivan Singer [33] and a few others. Ahuja-Narang-Trehan [1] extended the notion of uniform convexity and strict convexity from normed spaces to metric linear spaces. In this section we present the generalizations obtained in [1]

We first recall some basic definitions and facts about metric linear spaces.

Let E be a vector space over the field K (R or C). A subset u of E is called

(a) *absorbing* if , for each $x \in E$, there exists a number $r = r(x) > 0$ such that

$$x \in \lambda u \qquad \text{for all } |\lambda| \geq r$$

(b) *balanced* if $\lambda x \in u$ for all $x \in u$ and $|\lambda| \leq 1$.

A vector space E over K together with a topology τ on E is called a *topological vector space* (TVS) if the operation of addition $(x,y) \to x + y$ of $E\text{x}E \to E$ and the operation of scalar multiplication $(\lambda,x) \to \lambda x$ of $K\text{x}E \to E$ are both jointly continuous. It is a well-known result that every TVS E has a base of neighbourhoods of 0 consisting of absorbing and balanced sets. A TVS (E,τ) is called a *metric linear space* if there exists a metric d on E such that

(i) the topology induced by the metric d on E coincides with τ.

(ii) $d(x+z, y+z) = d(x,y) = d(x-y, 0)$

for all $x, y, z \in E$ (translation invariant).

(iii) $d(\lambda x, 0) \leq d(x, 0)$ for all $x \in E$ and $|\lambda| \leq 1$.

All the normed spaces are metric linear spaces.

Further the space l_p ($0 < p < 1$) of all sequence $\{x_n\} \subseteq K$ such that $\sum_{n=1}^{\infty}|x_n|^p < \infty$ with metric defined by $d(x,y) = \sum_{n=1}^{\infty}|x_n-y_n|^p$ [$x = \{x_n\}$, $y = \{y_n\} \in l_p$] is a metric linear space (but not a normed or locally convex space). For the detail of the above facts, the reader is refered to any standard book on topological spaces (e.g., A.& W. Robertson [29], H.Schaefer [32], G.Köthe [20])

DEFINITION 2.2.1.

A metric linear space (X,d) is said to be *Uniformly Convex* if there corresponds to each pair of positive number (ε,r), a positive number δ such that if x and y lie in X with $d(x,y) \geq \varepsilon$, $d(x,0) < r+\delta$, $d(y,0) < r+\delta$, then

$$d(\tfrac{x+y}{2},0) < r.$$

EXAMPLE 2.2.2.

The Set R of real numbers with metric $d(x,y) = \frac{|x-y|}{1+|x-y|}$ is a uniformly convex metric linear space.

SOLUTION

First it is easy to verify that (R,| |) is uniformly convex. Now, let (ε,r) be given and $x,y \in R$ be such that $d(x,y) \geq \varepsilon$, $d(x,0) \leq r$, $d(y,0) \leq r \Rightarrow d(\frac{x+y}{2},0) < r$.
i.e., $|x-y| \geq \varepsilon$, $|x| \leq \frac{r}{1-r}$, $|y| \leq \frac{r}{1-r}$, $\Rightarrow |\frac{x+y}{2}| < \frac{r}{1-r}$.

Since (R,| |) is uniformly convex, there exists

$\delta > 0$ such that $|x-y| \geq \varepsilon$, $|x| < \frac{r}{1-r} + \delta$, $|y| < \frac{r}{1-r} + \delta$

$$\Rightarrow \left|\frac{x+y}{2}\right| < \frac{r}{1-r}$$

Now, $d(x,0) = \frac{|x|}{1+|x|} < \frac{\frac{r}{1-r}+\delta}{1+\frac{r}{1-r}+\delta} = \frac{r+(1-r)\delta}{1+(1-r)\delta}$.

Let $\frac{r+(1-r)\delta}{r+(1-r)\delta} = r+D$

$\Rightarrow D = \frac{r+(1-r)\delta}{1+(1-r)\delta} - r = \frac{r+(1-r)\delta - r - r(1-r)\delta}{r+(1-r)\delta} = \frac{\delta(1-r)(1-r)}{1+(1-r)\delta}$

$= \frac{\delta(r-1)^2}{\delta - r\delta + 1} > 0$. $\Rightarrow D > 0$. With this value of D we have $d(x,y) \geq \varepsilon$ $d(x,0) < r+D$, $d(y,0) < r+D$ and

$$d\left(\frac{x+y}{2},0\right) = \frac{|(x+y)/2|}{1+\left|\frac{x+y}{2}\right|} < \frac{\frac{r}{1-r}}{1+\frac{r}{1-r}} = \frac{r}{1-r+r} = r.$$

$$\Rightarrow d\left(\frac{x+y}{2},0\right) < r.$$

Hence (R,d) is uniformly convex.

NOTE:

Finite dimensional metric linear space need not be uniformly convex, as is shown by the following example.

EXAMPLE 2.2.3.

Consider (R^2,d) where d is defined as

$d(x,y) = \max\{|x_1 - y_1|, |x_2 - y_2|\}$;

$x = (x_1,x_2)$, $y = (y_1,y_2)$

Let $x = (1,1)$, $y = (1,0)$. Then

$d(x,y) = \max\{|1-0|,|1-0|\} = 1$

$d(x,0) = 1$, $d(y,0) = 1$, $\Rightarrow d(\frac{x+y}{2},0) = d\{(1,\frac{1}{2}),(0,0)\} = 1$.

DEFINITION 2.2.4.

A metric linear space (X,d) is said to be *Strictly Convex* if $d(x,0) \leq r$, $d(y,0) \leq r \Rightarrow d(\frac{x+y}{2},0) < r$, $x \neq y$, where $x,y \in X$ and r is any positive real number.

DEFINITION 2.2.5.

A metric linear space (X,d) is said to be *totally complete* if it has the property that its d-bounded closed sets are compact.

THEOREM 2.2.6.

(a) Every uniformly convex metric linear space is strictly convex.

(b) Every totally complete and strictly convex metric linear space is uniformly convex.

PROOF

(a) Let (X,d) be uniformly convex metric space. Let $x \neq y$, $d(x,0) \leq r$ and $d(y,0) \leq r$. Take $\varepsilon = d(x,y) > 0$. We choose a $\delta > 0$. So we have

$$d(x,y) \geq \varepsilon,\ d(x,0) < r + \delta,\ d(y,0) < r + \delta.$$

By uniform convexity, we have $d((x+y)/2,\ 0) < r$.

Hense (X,d) is strictly convex.

(b) Let (X,d) be a totally complete and strictly convex metric linear space and (ε,r) be given. Define

$$S = \{ \langle x,y \rangle : x,y \in X ;\ d(x,0) \leq r,\ d(y,0) \leq r \text{ and } d(x,y) \geq \varepsilon \}$$

It can be shown that S is a closed and bounded subset of X x X, the metric on X x X being

$$d_1(\langle x_1, y_1\rangle, \langle x_2, y_2\rangle) = [\{d(x_1,x_2)\}^2 + \{d(y_1,y_2)\}^2]^{1/2}$$

S, being a closed and bounded subset of a totally complete metric linear space, is compact. Define

$$\phi : S \to R \text{ as } \phi(\langle x,y\rangle) = r-d((x+y)/2,\ 0)$$

ϕ, by the strict convexity of X, is a positive continuous real valued function on a compact set S. It will attain its positive infimum, say δ, on S.

Therefore, for $\langle x,y\rangle \in S$, we have

$$r-d((x+y)/2,\ 0) \geq \delta, \text{ i.e., } d((x+y)/2,\ 0) \leq r-\delta < r.$$

Therefore, $d(x,0) < r+\delta$, $d(y,0) < r+\delta$, and $d(x,y) \geq \varepsilon$ imply $d((x+y)/2,\ 0) < r$. Hence (X,d) is uniformly convex.■

NOTE:

Finite dimensional metric linear space need not be strictly convex. For example,

Consider (R^2,d), where $d(x,y) = \max\{|x_1-y_1|, |x_2-y_2|\}$ and $x = (x_1,x_2)$, $y = (y_1,y_2)$

Let $x = (1,1)$, $y = (1,0)$. Then $d(x,0) = 1, d(y,0) = 1$ and also $d(\frac{x+y}{2},0) = d((1,\frac{1}{2}),(0,0)) = \max\{|1-0|, |\frac{1}{2}-0|\} = 1$. Hence a finite dimensional metric linear space need not be strictly convex. We recall the following definitions.

A set K in a metric space (X,d) is said to be *Proximinal* if for each point $x \in X$ there is a point $z \in K$

such that $d(x,z) = d(x,K)$; K is called *Chebyshev* (resp *semi-Chebyshev*) if for each $x \in X$ there exists exactly one (resp at most one) z such that $d(x,z) = d(x,K)$.

THEOREM 2.2.7.

Every convex set in a strictly convex metric linear space is semi-Chebyshev.

PROOF

Let K be a convex set in a strictly convex metric space (X,d). For a given $x \in X$, if possible, let $z_1, z_2 \in K$ be such that $z_1 \neq z_2$ and

$$d(z_1,x) = d(z_2,x) = \gamma = d(x,K) \text{ i.e.,}$$

$$d(z_1-x,0) = d(z_2-x,0) = \gamma.$$

X being strictly convex so

$$d\left(\frac{(z_1-X) + (z_2-X)}{2},0\right) < \gamma, \quad \text{since} \quad z_1-x \neq z_2-x,$$

$$\text{i.e., } d\left(\frac{z_1+z_2}{2}, x\right) < \gamma. \tag{1}$$

Since K is convex, $\frac{z_1+z_2}{2} \in K$ and so by definition of γ,

$$\gamma \leq d\left(\frac{z_1+z_2}{2}, x\right), \text{ which contradicts (1).}$$

Hence $z_1 = z_2$. ■

DEFINITION 2.2.8.

A set K in a metric space (X,d) is said to be *approximatively compact* if for every x in X and every sequence $\{y_n\}$ in K with $\lim_{n\to\infty} d(x,y_n) = d(x,K)$ there exists a subsequence $\{y_{nk}\}$ converging to an element y in K.

LEMMA 2.2.9.

In a uniformly convex metric linear space every complete convex set is approximatively compact.

PROOF

Let K be a complete convex set in a uniformly convex metric linear space (X,d) and $\{y_n\}$ be a sequence in K satisfying $\lim_{n\to\infty} d(x,y_n) = d(x,K) = r$ (say).

Now let $\varepsilon > 0$ be given. Let δ be taken as in the definition of uniform convexity. Choose N such that $d(x,y_n) < r + \delta$, whenever $n \geq N$.

Let $n,m \geq N$. Then $d(x,y_n) < r + \delta$, $d(x,y_m) < r + \delta$. Since K is convex so $\frac{y_n+y_m}{2} \in K$. Therefore $d(x, \frac{y_n+y_m}{2}) \geq r$ and so, by uniform convexity $d(y_n,y_m) < \varepsilon$, for all $n,m \geq N$. This implies $\{y_n\}$ is a cauchy sequence in K. K being complete , $\{y_n\}$ will converge to a point of K. Hence K is approximatively compact.■

DEFINITION 2.2.10.

A metric linear space (X,d) is said to satisfy *Property* P if any sequence $\{y_n\}$ in a convex subset K of X satisfying $\lim_{n\to\infty} d(y_n,x) = d(x,K)$ has a cauchy subsequence.

THEOREM 2.2.11.

A complete convex set K in a metric linear space (X,d) satisfying the property p is chebyshev.

PROOF

Let $x \in X$ and $r = d(x,K)$. By the definition of infimum there is a sequence $\{y_n\}$ in K such that $\lim_{n\to\infty} d(y_n,x) = r$. By property P $\{y_n\}$ has a cauchy subsequence $\{y_{nk}\}$ in K. K being complete $\{y_{nk}\} \to z \in K$ and consequently $d(z,x) \geq r$. Also $d(z,x) \leq d(z,y_{nk}) + d(y_{nk},x)$ implies $d(z,x) \leq r$. Hence $d(z,x) = r$. Now, if possible $z_1, z_2 \in K$ be such that $d(z_1,x) = d(z_2,x) = r$. Consider the sequence $\{y_n\}$ defined by

$$y_n = \begin{cases} z_1 & \text{if } n \text{ is odd} \\ z_2 & \text{if } n \text{ is even} \end{cases}$$

Then $\lim_{n\to\infty} d(y_n,x) = d(z_1,x) = d(z_2,x) = r$

By the property P, $\{y_n\}$ has a Cauchy subsequence $\{y_{nk}\}$ and therefore for a given $\varepsilon > 0$, there exists a positive integer N such that

$$d(x_{nk},x_{nj}) < \varepsilon \text{ for all } k,j \geq N$$

i.e., $d(z_1,z_2) < \varepsilon$. Since ε is arbitrary so $z_1 = z_2$.■

§ 2.3. BEST APPROXIMATION IN LOCALLY CONVEX SPACES

In this section we present some results due to Rao-Elumalai [27], Narang [23-25] and Thaheem [34]. We first recall that a topological vector space E is said to be *locally convex* if its topology has a base of convex (and absorbing, balanced) neighbourhoods of 0. Every locally convex topology on E can alternatively be defined by a family $\mathcal{P} = \{p_\alpha : \alpha \in I\}$ (say) of continuous seminorms on E. A locally convex space need not be a metrizable linear space, and vice-versa.

HAHN-BANACH THEOREM (Extension form).

Let E be a locally convex space, M a subspace of E, and f_o a continuous linear functional on M with $f_o(x) \le p(x)$ $(x \in M)$ for some continuous semi-norm p on E. Then there exists an $f \in E'$ such that $f = f_o$ on M and $|f(x)| \le p(x)$ for all $x \in E$.

For the detail of the above, the reader is referred to standard books on topological vector spaces (e.g., A & W Robertson [29], Shaefer [32], Köthe [20]).

DEFINITION 2.3.1. [27]

Let E be a locally convex space with a family $\mathcal{P}$ of seminorm, M a subset of E, $x \in E$ and $z \in M$. Then z is called an element of *best approximation* of x by elements of M if for every $p \in \mathcal{P}$, $p(x-z) = \inf_{y \in M} p(x-y) = d_p(x,M)$, i.e.,

$$p(x-z) \le p(x-y) \qquad \text{for all } y \in M$$

We denote $P_M(x) = \{z \in M : p(x-z) = d_p(x,M) \ \forall\, p \in \mathcal{P}\}$

DEFINITION 2.3.2. [24]

E is said to be *Strictly convex* if for every $p \in \mathcal{P}$ and $x,y \in E$ with $p(x) \leq r$, $p(y) \leq r$, $x \neq y$ implies $p(\frac{x+y}{2}) < r$.

LEMMA 2.3.3. [24]

E is strictly convex iff for all $p \in \mathcal{P}$, $x,y \in E$ with $p(x) \leq r$, $p(y) \leq r$, $x \neq y$ implies $p(tx + (1-t)y) < r$ for all $0 < t < 1$.

PROOF

Suppose E is strictly convex, and let $p \in \mathcal{P}$, $x,y \in E$ with $p(x) \leq r$, $p(y) \leq r$, $x \neq y$ and $0<t<1$. There exists $s > 0$ such that $0 < t-s < t < t+s < 1$.

Let $z = (t-s)x + (1-t+s)y$ and $w = (t+s)x + (1-t-s)y$. Then $p(z) \leq (t-s)p(x) + (1-t+s)p(y)$

$$\leq (t-s)r + (1-t+s)r = r$$

and similarly $p(w) \leq r$. By strict convexity of E, $p(\frac{z+w}{2}) < r$, i.e., $p(tx + (1-t)y) < r$.

Conversely,

If we take $t = \frac{1}{2}$, then it follows that E is strictly convex.■

THEOREM 2.3.4. [24]

Let K be a convex subset of a strictly convex LCS E. Then K is semi-Chebyshev.

PROOF

Let $x \in E$ and $p \in \mathcal{P}$. Suppose $z_1, z_2 \in P_K(x)$. Then $p(x-z_1) = p(x-z_2) = \gamma = \inf_{y\in K} p(x-y)$.

If $z_1 \neq z_2$, then $x-z_1 \neq x-z_2$ and so by strict convexity,

$$p\left(\frac{x-z_1 + x-z_2}{2}\right) < \gamma$$

or $$p\left(x - \frac{z_1+z_2}{2}\right) < \gamma \qquad (1)$$

Since K is convex, $\frac{z_1+z_2}{2} \in K$ and so by definition of γ $\gamma \le p\left(x - \frac{z_1+z_2}{2}\right)$, which contradicts (1). Hence $z_1 = z_2$, and so K is semi-Chebyshev.■

DEFINITION 2.3.5. [24]

A subset K of a LCS E is called P-*inf-compact* if for every $x \in E$, each minimizing net $\{y_\alpha\}$ in K (i.e. $p(x-y_\alpha) \to d_p(x,K)$) has a convergent subnet converging in K for all $p \in \mathcal{P}$.

THEOREM 2.3.6. [24]

Let K be a non-empty P-inf-compact subset of a LCS E. Then K is proximinal.

PROOF

Let $x \in E$, $p \in \mathcal{P}$ and $r = d_p(x,K)$. Then there exists a net $\{y_\alpha\}$ in K such that $p(x-y_\alpha) \to r$. Since K is P-inf-compact, there exists a subnet $\{y_\beta\}$ of $\{y_\alpha\}$ such that $y_\beta \to z \in K$. Now

$$p(x-z) \le p(x-y_\beta) + p(y_\beta - z)$$

Taking limit, we have $p(x-z) \le r$. But $r \le p(x-z)$. Hence $p(x-z) = r$ and so $z \in P_K(x)$. Thus K is proximinal.■

The following classical result for normed spaces is due to Singer ([33], p.18)

THEOREM 2.3.7.

Let X be a normed space, M a vector subspace, $x \in X \backslash \bar{M}$ and $z \in M$. Then $z \in P_M(x)$ iff there exists some $f \in X'$ such that $f = 0$ on M, $\|f\| = 1$ and $f(x-z) = \|x-z\|$.■

THEOREM 2.3.8. ([27],p.15)

Let E be a locally convex space with a family $\mathcal{P}$ of seminorms, M a linear subspace of E, $x \in E \backslash M$ and $z \in M$. Then $z \in P_M(x)$ iff for every $p \in \mathcal{P}$ there exists $f \in E'$, depending on p, such that

$$f = 0 \text{ on } M, \tag{1}$$

$$|f(x-z)| = p(x-z), \tag{2}$$

$$|f(x-y)| \le p(x-y). \ \forall\, y \in M. \tag{3}$$

PROOF

Let $z \in p_M(x)$ and $p \in \mathcal{P}$. Then for every $y \in M$,

$p(x-z) \le p(x-y)$. In particular

if $\alpha \neq 0$, then

$$p(x-z) \le p(x-z+\frac{y}{\alpha}), \qquad \forall\, y \in M$$

Let $M_o = \{y + \alpha(x-z) : y \in M, \alpha \in F\}$. Clearly $M \subseteq M_o$. We define f_o on M_o such that

$$f_o(y + \alpha(x-z)) = \alpha\, p(x-z)$$

for each $y \in M$ and all $\alpha \in F$. It is easy to verify that f_o is linear. Therefore, $f_o(y) = 0$ and $| f_o(x-z)| = p(x-z)$. For $\alpha \neq 0$

$$| f_o(y + \alpha(x-z))| = |\alpha|\, p(x-z) \leq |\alpha|\, p(x-z + \frac{y}{\alpha})$$

$$= p(y + \alpha(x-z)), \qquad \forall\, y \in M.$$

For $\alpha = 0$ and for each $y \in M$,

$$|f_o(y + \alpha(x-z))| = 0 \leq p(y + \alpha(x-z))$$

Thus, for every $m \in M_o$, and for every $p \in \mathcal{P}$

$$|f_o(m)| \leq p(m)$$

Then, by Hahn-Banach theorem, f_o can be extended to a continuous linear functional f on E such that $|f(x)| \leq p(x)$ for every $x \in E$ and $|f(m)| = | f_o(m)|$ for every $m \in M_o$. So $f(y) = 0$, $|f(x-z)| = p(x-z)$ and $|f(x-y)| \leq p(x-y)$ for every $p \in \mathcal{P}$ and $y \in M$.

Conversely,

Let the given condition be satisfied then

$$p(x-z) = |f\ (x-z)| = |f\ (x-y+(y-z))|$$

$$= |f\ (x-y) + f\ (y-z)| = |f\ (x-y) + 0|$$

$$= |f\ (x-y)| \leq p(x-y).$$

for every $y \in M$ and $p \in \mathcal{P}$. Hence the proof.■

In [25], T.D.Narang considered the above result for simultaneous characterization of a set of elements of best approximation: Given E, M, and x as above and a subset S of

M, what are the necessary and sufficient conditions in order that every element $y \in S$ be an element of best approximation of x (by means of elements of M)?

THEOREM 2.3.9. ([25],p.60)

Let E be a LCS with a family $\mathcal{P}$ of seminorms, M a subspace of E, $x \in E\backslash M$ and $S \subseteq M$. Then $S \subseteq P_M(x)$ iff for each $p \in \mathcal{P}$, there exists an $f \in E'$, depending on p, such that

$$f = 0 \text{ on } M \tag{4}$$

$$|f(x-z)| = p(x-z) \quad \text{for all } z \in S \tag{5}$$

$$|f(x-y)| \leq p(x-y) \quad \text{for all } y \in M \tag{6}$$

PROOF

Suppose $S \subseteq P_M(x)$ and $z \in S$. Then $z \in P_M(x)$ and so by theorem (2.3.8) there exists $f \in E'$ such that conditions (1), (2) and (3) hold. Now let $y_1 \in S$, then $y_1 \in P_M(x)$ and so $p(x-y_1) = p(x-z) = \inf_{y \in M} p(x-y)$ for all $p \in \mathcal{P}$.

Consider

$$\begin{aligned} |f(x-y_1)| &= |f(x-z+z-y_1)| \\ &= |f(x-z)| \\ &= p(x-z) \\ &= p(x-y_1) \end{aligned}$$

Thus (4), (5) and (6) are satisfied.

Conversely,

Let us suppose that there exists a $f \in E'$ satisfying

(4), (5) and (6) and let $z \in S$. Then by theorem (2.3.8) $z \in P_M(x)$. Hence $S \subseteq P_M(x)$. ■

Next we consider the problem of characterization of proximinal subsets K of E.

DEFINITION 2.3.10. [23]

A set K in a locally convex metric linear space (X,d) is said to be *boundedly weakly compact* if every bounded sequence in K contains a subsequence wich converges weakly to an element of K.

THEOREM 2.3.11. [23]

A convex boundedly weakly compact set in a strictly convex metric linear space is Chebyshev.

PROOF

Let K be a bounded weakly compact set in a strictly convex metric space (X,d) and x be an arbitrary point in X. Then we can find a sequence $\{y_n\}$ in K such that the sequence $\{d(x,y_n)\}$ converges to $r = d(x,K)$. The sequence $\{y_n\}$, being a bounded sequence in K, has a subsequence $\{y_{nk}\}$ converging weakly to an element z of K. Since the space is strictly convex, the closed ball $S_n(x) = \{y \in X : d(x,y) \leq r + (\frac{1}{n})\}$ is convex. Since a closed convex set in a locally convex linear space is weakly closed, the element z is in $S_n(x)$ for each n, and so $d(x,z) \leq r + \frac{1}{n}$ for each n.

Hence $d(x,z) \leq r$. Thus $d(x,z) = r$, establishing that K is proximinal. Now we show that K is Chebyshev. Let, if possible, $z, z_1 \in K$ be such that $d(x,z) = r = d(x,z_1)$ and $z \neq z_1$. Then strict convexity of the space implies $d(x,(z+z_1)/2) < r$, which contradicts the fact that $(z+z_1)/2$ is in K.■

Finally we present a generalization due to A.B.Thaheem [34].

DEFINITION 2.3.12. [34]

Let (E,d) be a locally convex metrizable linear space. A ball $B_r(0) = \{x \in E: d(x,0) \leq r\}$, $r > 0$ is said to be *compressible* if for every $s > 1$, there is $t > r$ such that $B_t(0) \subseteq sB_r(0)$. If every ball $B_r(0)$ in (E,d) is compressible (resp convex), then we say d is compressible (resp convex).

We shall require the following result from [31].

THEOREM 2.3.13. ([31],p.65)

Let (E,d) be a locally convex metrizable linear space. If $\{x_n\}$ is a sequence in E that converges weakly to some $x \in E$, then there exists a sequence $\{y_i\}$ in X such that

(a) each y_i is a convex combination of finitely many x_n.

(b) $y_i \to x$ in the original topology.

THEOREM 2.3.14. [34]

Let (E,d) be a locally convex metrizable linear space. If d is convex and compressible then every weak sequentially compact subset K of E is proximinal.

PROOF

Let $x \in E$. Suppose that $d(x,K) = r$, $r > 0$. Since K is weak sequentially compact and hence closed in the strong (or original) topology, there exists a sequence $\{y_n\}$ in K such that $d(x,y_n) \to d(x,K) = r$. Further there exists a subsequence $\{y'_n\}$ of $\{y_n\}$ which converges weakly to y_o in K. Then $(x-y'_n)$ also converges weakly to $(x-y_o)$. We show that $y_o \in P_K(x)$.

Let q_B denote the Minkowski functional associated to $B = B_r(0)$. Since $B_r(0)$ is a convex neighbourhood of 0 in E, q_B is a semi-norm on E with $B_r(0) = \{y \in E : q_B(y) \le 1\}$. The main part of the remaining proof is to show that $q_B(x-y_o) \le 1$. Now, let I be a subset of the natural numbers N, and consider the following set of finite combinations

$$L_I = \{\sum_{i=n}^{m} c_i y'_i,\ c_i \ge 0,\ \sum_{i=n}^{m} c_i = 1,\ i \in (N/I)\}$$

Let $\bar{L}_I$ denote the closure of L_I in the space (X,q_B). Obviously $\bar{L}_I$ is convex. $\bar{L}_I$ is also closed in (E,d) because the topology generated by q_B is weaker then the strong topology. Thus $\bar{L}_I$ is also weakly closed in (E,d) and hence by convexity, $y_o \in \bar{L}_I$ by Theorem 2.3.13.

Therefore, for an $\varepsilon > 0$ there exists a convex combination $\sum_{i=n}^{m} c_i y'_i \in L_I$ (with n, m, c_i depending on ε) such that

$$q_B\left(\sum_{i=n}^{m} c_i(y'_i - y_0)\right) < \varepsilon$$

or $$q_B\left(\sum_{i=n}^{m} (c_i(y'_i - x) - (y_0 - x)\right) < \varepsilon.$$

Since I is arbitrary, therefore for $\varepsilon > 0$ small enough and $i \in (N\setminus I)$, we have

$$q_B(x-y'_i) < \liminf_n (q_B(x-y'_n)) + \varepsilon.$$

It follows that $q_B(y_0-x) \le \liminf_n q_B(x-y'_n)$ (1)

Now if $q_B(x-y_0) \le 1$ does not hold, then we have

$$q_B(x-y_0) > 1. \quad (2)$$

Put $s = (\frac{1}{2})(1 + q_B(x-y_0))$. Thus $s > 1$. Since $B_r(0)$ is compressible, therefore there exists $t > r$ such that

$$B_t(0) \subset sB_r(0) \quad (3)$$

Also $\lim_n d(x,y'_n) = r$, therefore for sufficiently large values of n, $(x-y'_n) \in B_t(0)$. By (3), $(x-y'_n) \in sB_r(0)$, and so

$$q_{sB}(x-y'_n) \le 1 \quad \text{or} \quad \frac{1}{s} q_B(x-y'_n) \le 1$$

$$\text{or } q_B(x-y'_n) \le s = \frac{1}{2}[1 + q_B(x-y_0)]$$

$$< \frac{1}{2}[q_B(x-y_0) + q_B(x-y_0)], \text{ by (2)}$$

$$= q_B(x-y_0),$$

which contradicts (1). Thus $q_B(x-y_0) \le 1$ or equivalently $d(x,y_0) \le r$. Since $y_0 \in K$, so

$$r = d(x,K) \le d(x,y_0)$$

Hence $d(x,y_0) = d(x,K)$, and so $y_0 \in P_K(x)$, as required.■

The following is a partial generalization of Theorem 1.5.6.

COROLLARY 2.3.15. [34]

Let (E,d) be a reflexive locally convex metrizable linear space with d as a convex and compressible metric. Then every bounded, closed and convex subset K of E is proximinal.

PROOF

This follows immediately from the above theorem and the Eberein-Smulian Theorem (see [13],p.227-228) which states that a locally convex metrizable linear space (E,d) is reflexive iff every bounded, closed and convex subset of E is weak sequentially compact.∎

CHAPTER 3

APPROXIMATION AND BEST APPROXIMATION IN FUNCTION SPACES

§ 3.1. INTRODUCTION

In this chapter, we obtain some approximation results of the Stone-Weierstrass type theorem for the uniform, compact-open, and weighted topologies. We also present results on best approximation in spaces of continuous function and of continuous linear mappings.

§ 3.2. STONE-WEIERSTRASS THEOREM FOR SCALAR AND VECTOR VALUED FUNCTION.

We begin with a brief discussion on the Stone-Weierstrass theorem, a generalization of the Weierstrass approximation theorem by M.H.Stone in 1938, which may be stated as follows.

3.2.1. STONE-WEIERSTRASS THEOREM

(for scalar valued function)

Let X be a compact Housdorff space and C(X) the algebra of all real-valued continuous function on X with the uniform (sup norm) topology. Let $\mathscr{A}$ be a subalgebra of C(X) such that

(1) $\mathscr{A}$ separate the points of X (i.e.,if $x \neq y$, there exists $g \in \mathscr{A}$ such that $g(x) \neq g(y)$).

(2) $\mathscr{A}$ contains constant functions (or $1 \in \mathscr{A}$, where $1(x)=1$ for all $x \in X$). Then $\mathscr{A}$ is uniformly dense in C(X) (i.e.,given any $f \in C(X)$ and $\varepsilon > 0$, there exists some $g \in \mathscr{A}$ such that $\|g-f\| \leq \varepsilon$).

NOTE:

If C(X) consists of all complex valued continuous function on X, then the above theorem requires the additional hypothesis that if $f \in \mathscr{A}$, then also $\bar{f} \in \mathscr{A}$.

Many earlier proofs of this theorem depend on the following facts.

(a) The classical Weierstrass theorem (or its special case of uniformly approximation $f(t) = |t|$ on $[-1,1]$ by polynomials).

(b) The closure of a subalgebra is a subalgebra.

(c) The closure of a subalgebra is a sublattice.

In 1981, Brosowski and Deutsch [4] gave a simple and elegant proof of this theorem. Their proof does not appeal to any of the above facts. It is simple in the sense that it depends only on the definitions of "Compactness" (each open cover has a finite subcover), "Continuity" (the inverse images of open sets are open) and the elementry Bernoulli inequality

$$(1 + h)^n \geq 1 + nh \qquad (n = 1,2,3,\ldots) \text{ if } h \geq 1$$

We now consider some generalizations of the Stone-Weierstrass Theorem to vector-valued functions, in particular, we present some results established in ([15]–[17])

DEFINITION 3.2.2.

Let X be a locally compact Hausdorff space and E a topological vector space. Then a function $f : X \to E$ (resp $f : X \to \mathbb{C}$) is said to *Vanish at infinity* if given any neighbourhood $\mathcal{U}$ of 0 in E (resp $\varepsilon > 0$) there exists a compact set $K \subseteq X$ such that $f(x) \in \mathcal{U}$ (resp $|f(x)| < \varepsilon$) $\forall\ x \in X-K$.

DEFINITION 3.2.3.

A function $f : X \to E$ is said to have *compact support* if there exists a compact subset $K \subseteq X$ such that $f(x)= 0$ for all $x \in X-K$.

Let $C(X,E)$ (resp $C_b(X,E)$) denote the vector space of all continuous (resp and bounded) E-valued function on X and let $C_o(X,E)$ (resp $C_{oo}(X,E)$) denote the vector space of all those $f \in C(X,E)$ which vanish at infinity (resp have compact support). Clearly $C_{oo}(X,E) \subseteq C_o(X,E) \subseteq C_b(X,E) \subseteq C(X,E)$ and it is easy to see that $C_{oo}(X,E) = C(X,E)$ iff X is compact. When E is the scalar field (R or C), these spaces are denoted by $C(X)$, $C_b(X)$, $C_o(X)$ and $C_{oo}(X)$.

DEFINITION 3.2.4.

The *Uniform topology* σ on $C_b(X,E)$ is defined as the linear topology σ which has a base of neighbourhoods of 0 all sets of the form $\{f \in C_b(X,E) : f(x) \in W \quad \forall \; x \in X\}$. Where W varies over a base of neighbourhoods of 0 in E.

When E is a locally convex space whose topology is given by $\{p_\alpha : \alpha \in I\}$, a family of continuous semi-norms on E, then σ *topology* can be defined by the family $\{\| \, \|_\alpha : \alpha \in I\}$ of semi-norms, where $\|f\|_\alpha = \sup_{x \in X} p_\alpha(f(x))$ $(f \in C_b(X,E))$.

REMARK:

If C(X,E) consists of an unbounded function, then we cannot define uniform topology on it. However, a useful topology on C(X,E) is the *Compact-open topology* κ which has a base of neighbourhoods of 0 consisting of all sets of the form

$$\{ f \in C(X,E) : f(K) \subseteq W \}$$

where K is a compact subset of X and W is a neighbourhood of 0 in E. On $C_b(X,E)$, we have $\kappa \le \sigma$.

THEOREM 3.2.5. (Rudin [30], p.40)

Let X be a locally compact Housdorff space, K a compact subset of X, and $\{u_1, u_2, \ldots, u_n\}$ an open cover of K. Then there exist function $\phi_1, \phi_2, \ldots, \phi_n \in C_b(X)$ such that $0 \le \phi_i \le 1$, $\phi_i = 0$ outside u_i, $\sum_{i=1}^{n} \phi_i = 1$ on K, and $\sum_{i=1}^{n} \phi_i \le 1$ on X ■

We now consider some Stone-Weierstrass type theorems for $C_o(X,E)$. It is clear that $C_o(X,E)$ is not in general an algebra (since E is a vector space) but it possess the algebric structure of $C_b(X)$-module. (i.e., if $\phi \in C_b(X)$ and $f \in C_o(X,E)$, then $\phi f \in C_o(X,E)$).

We note that the condition "separates points of X" would not be enough ; because if M is a proper closed subspace of E then for any $x_o \in X$, $A_{x_o} = \{f \in C_o(X,E) : f(x_o) \in M\}$ separates points of X but is a proper σ-closed $C_b(X)$-submodule of $C_o(X,E)$. (i.e., A_{x_o} is not σ-dense in $C_o(X,E)$). Therefore we have to assume an alternate hypothesis on a $C_b(X)$-submodule A of $C_o(X,E)$, so that A is σ-dense in $C_o(X,E)$.

STONE-WEIERSTRASS THEOREM IN LOCALLY CONVEX SETTING.

THEOREM 3.2.6. [15, 16]

Let X be a locally compact Housdorff topological space and E a Housdorff locally convex space. If A is a

$C_b(X)$-submodule of $C_o(X,E)$ such that, for each $x \in X$, $A(x) = \{g(x) : g \in A\}$ is dense in E, then A is σ-dense in $C_o(X,E)$.

PROOF

Let $f \in C_o(X,E)$, and let p be a continuous semi-norm on E and $\varepsilon > 0$. We show that there exists a function g in A such that $p(g(x)-f(x)) < \varepsilon$ for all $x \in X$. Since $f \in C_o(X,E)$ there exists a compact set K in X such that $p(f(x)) < \frac{\varepsilon}{2}$ if $x \notin K$. For each $x \in X$, there exists by hypothesis a function g_x in A such that $p(g_x(x)- f(x)) < \frac{\varepsilon}{2}$.

Now, $p \circ (g_x-f) : X \to R$ is continuous and so there exists an open neighbourhood N(x) of x in X such that $p(g_x(y) - f(y)) < \frac{\varepsilon}{2}$, for all $y \in N(x)$.

The collection $\{N(x) : x \in K\}$ form an open covering of K, and so, since K is compact, there exists a finite open subcovering, $\{N(x_i) : i = 1,2,\dots,m\}$ say.

By Theorem 3.2.5 there exists a collection $\{ \phi_i : i = 1,2,\dots,m \}$ of functions in $C_b(X)$ such that $0 \le \phi_i \le 1$, $\phi_i = 0$ outside of $N(x_i)$, $\sum_{i=1}^{m} \phi_i(x) = 1$ for $x \in K$, and $\sum_{i=1}^{m} \phi_i(x) \le 1$, for $x \in X$.

Define a E-valued function g on X by the equation $g(x) = \sum_{i=1}^{m} \phi_i(x)\, g_{x_i}(x)$ $(x \in X)$. Then it is clear that $g \in A$.

Let y be any point in X. If $y \in K$, then

$$p(g(y) - f(y)) = p\left[\sum_{i=1}^{m} \phi_i(y)\, g_{xi}(y) - f(y) \right]$$

$$= p\left[\sum_{i=1}^{m}\phi_i(y)\,(\,g_{xi}(y) - f(y)\,\right]$$
$$= \sum_{i=1}^{m}\phi_i(y)\ \ p\left[\,g_{xi}(y) - f(y)\,\right]$$
$$< \sum_{i=1}^{m}\phi_i(y)\,\frac{\varepsilon}{2} < \varepsilon.$$

If $y \in X-K$, then $p(g\,(y) - f(y))$

$$= p\left[\sum_{i=1}^{m}\phi_i(y)\,g_{xi}(y) - f(y)\right] + \left[\sum_{i=1}^{m}\phi_i(y) - 1\right] f(y)$$
$$\le \sum_{i=1}^{m}\phi_i(y)\ p\left[\,g_{xi}(y) - f(y)\,\right] + \left[\sum_{i=1}^{m}\phi_i(y) - 1\right] p(f(y))$$
$$\le \sum_{i=1}^{m}\phi_i(y)\,\frac{\varepsilon}{2} + \left[\sum_{i=1}^{m}\phi_i(y) - 1\right]\frac{\varepsilon}{2} < \varepsilon.$$

This implies that f belongs to the σ-closure of A. Hence A is σ-dense in $C_o(X,E)$. ∎

NOTATION:

$C_o(X) \otimes E$ is the span of all functions of the form $\phi \otimes a$, where $\phi \in C_o(X)$, $a \in E$. $(\phi \otimes a)(x) = \phi(x)a$ $(x \in X)$. Clearly $C_o(X) \otimes E \subseteq C_o(X,E)$; in fact $C_o(X) \otimes E$ is a $C_b(X)$-submodule of $C_o(X,E)$.

COROLLARY 3.2.7. [16]

Let X and E be as in the theorem, then $C_o(X) \otimes E$ is σ-dence in $C_o(X,E)$.

PROOF

By the theorem, it is sufficient to show that, for each $x \in X$, $(C_o(x) \otimes E)(x) = E$. Let $a \in E$, and let $y \in X$, since X is locally compact Housdorff, there exists (by a theorem) a function ψ in $C_o(X)$ such that $0 \le \psi \le 1$ and $\psi(y) = 1$. Then $\psi \otimes a \in C_o(X) \otimes E$ and $(\psi \otimes a)(y) = a$. Since $a \in E$ and $y \in X$ were chosen arbitrary, the result follows. ■

In order to establish the Stone-Weierstrass theorem for general topological vector space E, we proceed as follows.

DEFINITION 3.2.8.([21], p.9)

Let X be a topological space and $\mathcal{U}$ a collection of subsets of X. For any $x \in X$, we define $ord_x\mathcal{U}$, the order of $\mathcal{U}$ at x, as the number of members of $\mathcal{U}$ which contain x. The order of $\mathcal{U}$ is defined as

$$\text{ord } \mathcal{U} = \sup_{x \in X} \{ \text{ord}_x \mathcal{U} \}$$

The *covering dimension* of X is defined as the least positive integer n such that every finite covering of X has a refinement of order $\le n + 1$. We shall briefly write it as $\dim X = n$. If no such finite n exists, then we say that $\dim X = \infty$.

STONE-WEIERSTRASS THEOREM
IN THE NON LOCALLY CONVEX SETTING

THEOREM 3.2.9. [15, 16]

Let X be a locally compact Housdorff topological space of finite covering dimension, and let E be a Housdorff topological vector space. If A is a C(X)-submodule of $C_o(X,E)$ such that, for each $x \in X$, $A(x)$ is dense in E, then A is σ-dense in $C_o(X,E)$.

PROOF

Suppose dim $X = n$, and let $f \in C_o(X,E)$. For any neighbourhood W of 0 in E, Choose a balanced neighbourhood V of 0 in E such that $V+V+\ldots+V$ (n + 2-times) $\subseteq W$.

Since $f \in C_o(X,E)$, there exists a compact set K in X such that $f(x) \in V$ if $x \notin K$. For each $x \in X$ there exists a function h_x in A and an open neighbourhood $N(x)$ of x in X such that $h_x(y) - f(y) \in V$ for all $y \in N(x)$

Since K is compact, the open covering $\{N(x) : x \in K\}$ of K has a finite subcovering $\{N(x_i) : i = 1,2,\ldots,m\}$ say. Since dim $X = n$, dim $K \le n$ and so $\{N(x_i) : i = 1,2,\ldots,m\}$ has an open refinement $\{N'(x\ ) : j = 1,2,\ldots,r\}$ (say). Where $r \le n + 1$. Let $\{\phi_j : j = 1,2,\ldots,r\}$ be funtions in $C(X)$ such that $0 \le \phi_j \le 1$, $\phi_j = 0$ outside of $N'(x_j)$, $\sum_{j=1}^{r} \phi_j(x) = 1$ for $x \in K$, and $\sum_{j=1}^{r} \phi_j(x) \le 1$ for $x \in X$.

Let h be the E-valued function on X defined by $h(x) = \sum_{j=1}^{r} \phi_j(x)\, h_{xj}(x)$ $(x \in X)$, then $h \in A$. Let $y \in K$, then

$$h(y)-f(y) = \sum_{j=1}^{r} \phi_j(y)(h_{xj}(y)-f(y)) \in \sum_{j=1}^{r} \phi_j(y)V \subseteq V+V+\ldots+V \subseteq W$$

If $y \notin K$, then

$$h(y)-f(y) = \sum_{j=1}^{r} \phi_j(y)\ (h_{xj}(y)-f(y)) + [\ \sum_{j=1}^{r} \phi_j(y) - 1\ ]\ f(y)$$

$$\in V+V+\ldots+V\ (r\text{-terms})+V \subseteq W$$

Thus f belongs to the σ-closure of A and so A is σ-dense in $C_o(X,E)$.■

COROLLARY 3.2.10. [16]

Let X and E be as in the Theorem. Then $C_o(X) \otimes E$ is σ-dense in $C_o(X,E)$.

PROOF

It is similar to that of corollary 3.2.6.■

REMARK:

With slight modification of argument, Theorems 3.2.6 and 3.2.8 and their corollaries hold easily for C(X,E) in place of $C_o(X,E)$ and the uniform topology in place of the compact open topology.

STRONG STONE-WEIERSTRASS THEOREM

The following result is due to R.C.Buck [5].

THEOREM 3.2.11.

Let X be a compact Housdorff space, $(E,\|\ \|)$ a normed space and let $\mathscr{A}$ a C(X)-submodule of C(X,E). Then for any $g \in C(X,E)$, $\inf_{f\in\mathscr{A}} \|g-f\| = \sup_{x\in X} \inf_{u\in\mathscr{A}_x} \|g(x)-u\|$ where $\mathscr{A}_x = \overline{\mathscr{A}(x)}$

i.e., $d(g,\mathscr{A}) = \sup_{x\in X} d(g(x),\mathscr{A}_x) = \lambda$ (say).

PROOF

Since $\mathscr{A}_x = \overline{\mathscr{A}(x)}$, we have

$$\lambda = \sup_{x\in X} d(g(x), \mathscr{A}_x) = \sup_{x\in X} \inf_{f\in\mathscr{A}} \|g(x)-f(x)\|$$

$$\leq \inf_{f\in\mathscr{A}} \|g-f\| = d(g,\mathscr{A})$$

To prove the reverse relation, let $\varepsilon > 0$. For $x \in X$, there exists some $f_x \in \mathcal{A}$ such that

$$\left| \|g(x) - f_x(x)\| - \lambda \right| < \varepsilon$$

or $$\|g(x) - f_x(x)\| < \lambda + \varepsilon.$$

By continuity of $\|g-f\| : X \to R$ at x_o, there exists an open neighbourhood $\mathcal{U}(x)$ of x in X such that

$$\|g(y) - f_x(y)\| < \lambda + \varepsilon \qquad \forall\ y \in \mathcal{U}(x)$$

Now $\{\mathcal{U}(x) : x \in X\}$ is an open cover of the compact space X, so there exists $x_1, x_2, \ldots, x_n \in X$, such that

$$X = \bigcup_{i=1}^{n} \mathcal{U}(x_i)$$

Choose $\phi_i \in C(X)$ such that $\phi_i(x) \geq 0$, $\sum_{i=1}^{n} \phi_i(x) = 1$, for all $x \in X$. Set $f = \sum_{i=1}^{n} \phi_i f_{x_i} \in \mathcal{A}$. Then,

$$\begin{aligned} |g(x) - f(x)| &\leq \left| \sum_{i=1}^{n} \phi_i(x) g(x) - \sum_{i=1}^{n} \phi_i(x) f_{x_i}(x) \right| \\ &\leq \sum |g(x) - f_{x_i}(x)| \phi_i(x) \\ &\leq \sum (\lambda + \varepsilon)\ \phi_i(x) = \lambda + \varepsilon. \end{aligned}$$

Since this holds for all $x \in X$, we have $\|g-f\| \leq \lambda + \varepsilon$. Since ε is arbitrary, we have $d(g, \mathcal{A}) \leq \lambda$. This completes the proof.■

§ 3.3. WEIGHTED APPROXIMATION

DEFINITION AND TERMINOLOGY 3.3.1.

Let X be a topological space.

(1) A function $\phi : X \to R$ is called *upper semi continuous* at $x_o \in X$ if, for every $r \in R$ with $\phi(x) < r$,

there exists an open neighbourhood $\mathcal{U}(x_o)$ of x_o in X such that $\phi(x) < r$ for all $x \in \mathcal{U}(x_o)$.

(2) A function $\phi : X \to R$ is called *lower semi continuous* at $x_o \in X$ if, for every $r \in R$ with $\phi(x_o) > r$, there exists an open neighbourhood $\mathcal{U}(x_o)$ of x_o in X such that $\phi(x) > r$ for all $x \in \mathcal{U}(x_o)$.

(3) $\phi : X \to R$ is called *upper(lower) semi continuous* on X if ϕ is upper(lower) semi continuous at each point of X.

NOTE:

(i) Clearly $\phi : X \to R$ is continuous iff ϕ is both upper and lower semi-continuous on X.

(ii) The characteristic function χ_A is upper(lower) semi continuous on X iff A is closed (open) subset of X.

(iii) Every non-negative upper semi continuous function on compact subset K of X is bounded.

X is called *Completely regular* if given any closed subset $F \subseteq X$ and $x \notin F$, there exists $\phi \in C_b(X)$ such that $0 \le \phi \le 1$, $\phi(x) = 0$, $\phi = 1$ on F[14].

A *Nachbin Family* V on X is a set of non-negative upper semi continuous functions on X, called *weights*, such that, given $u, v \in V$ and $t \ge 0$, there exists a $w \in V$ such that $tu, tv \le w$[17]

In the sequel, X denotes a completely regular

Hausdorff space, V a Nachbin family on X, and E a Hausdorff TVS. Let $CV_b(X,E)[CV_o(X,E)]$ denote the subspace consisting of those $f \in C(X,E)$ such that vf is bounded (vanish at infinity) for all $v \in V$. The *Weighted topology* w_v on $CV_b(X,E)$ is defined as the linear topology which has a base of neighbourhood of 0 consisting of all sets of the form $N(v,W) = \{f \in CV_b(X,E) : (vf)(X) \subseteq W\}$. where $v \in V$ and W is a neighbourhood of 0 in E. $(CV_b(X,E),w_V)$ is called a *weighted space*[17].

The following are some intances of weighted spaces.

(i) If $V = K^+(X)$, the set of all non-negative constant functions on X, then $CV_b(X,E) = C_b(X,E)$ and w_v is the uniform topology σ.

(ii) If $V = \{t\chi_K : t \geq 0$ and K is compact subset of X$\}$, then $CV_b(X,E) = CV_o(X,E) = C(X,E)$ and w_v is the compact-open topology k.

A neighbourhood W of 0 in E is called *Shrinkable* [18] if $r\bar{W} \subseteq \text{int } W$, for $0 \leq r \leq 1$. By [18], every Housdorff topological vector space has a base of Shrinkable neighbourhoods of 0 and also the Minkowski functionals of such neighbourhood are continuous. E is said to be *admissible* if the identity map on E can be approximated uniformly on compact sets by continuous maps with range contained in finite dimensional subspaces of E. By [18] locally convex spaces, topological vector spaces having the approximation property, and ultrabarrelled topological vector spaces with a Shauder basis (in particular, F-spaces with a base) are admissible. A TVS E is called *locally bounded* if there exists a bounded neighbourhood of 0 in E.

THEOREM 3.3.2. [17]

(1) $CV_o(X,E)$ is w_V-closed in $CV_b(X,E)$.

(2) If X is locally compact, then $C_{oo}(X,E)$ is w_v-dense in $CV_o(X,E)$.

PROOF

(1) Let f belongs to the w_V-closure of $CV_o(X,E)$ in $CV_b(X,E)$, and let $v \in V$ and W a neighbourhood of 0 in E. Choose a balanced neighbourhood G of 0 in E with $G+G \subseteq W$. There exists a function $g \in CV_o(X,E)$ such that $g-f \in N(v,G)$.

Let $F = \{ x \in X : v(x)f(x) \notin W \}$ and

$K = \{ x \in X : v(x)g(x) \notin G \}$

By hypothesis K is compact (since $g \in CV_o(X,E)$). To show that $f \in CV_o(X,E)$ it suffices to show that $F \subseteq K$ or equivalently $K' \subseteq F'$. Now $x \in K' \Rightarrow v(x)g(x) \in G \Rightarrow v(x)f(x) \subset v(x)g(x) + G \subseteq G + G \subseteq W \Rightarrow x \in F'$. Thus $f \in CV_o(X,E)$.

(2) Let $f \in CV_o(X,E)$, and let $v \in V$ and W a balanced neighbourhood of 0 in E. Then $K = \{ x \in X : v(x)f(x) \notin W \}$ is compact. Since X is locally compact, there exists a $\phi \in C_{oo}(X)$ such that $0 \le \phi \le 1$ and $\phi = 1$ on K. Then $\phi f \in C_{oo}(X,E)$ For any $x \in X$,

$$v(x)[\phi(x)f(x)-f(x)] = v(x)f(x)[\phi(x)-1]$$

$$= \begin{cases} v(x)f(x)[1-1] = 0 \in W & \text{if } x \in K \\ \varepsilon(\phi(x)-1)\, W \subseteq W & \text{if } x \notin K \end{cases}$$

Therefore $\phi f - f \in N(v,W)$ and so $C_{oo}(X,E)$ is w_v-dense in $CV_o(X,E)$. ■

THEOREM 3.3.3. [17]

Suppose E is a locally bounded topological vector space. Then $C_b(X,E) \cap CV_o(X,E)$ is w_v-dense in $CV_o(X,E)$.

PROOF

Let $f \in CV_o(X,E)$, and let $v \in V$ and W a balanced neighbourhood of 0 in E. Let H be a boundecd neighbourhood of 0 in E. There exists a closed Shrinkable neighbourhood S of 0 in E with $S \subseteq H$. Choose $r \geq 1$ such that $H \subseteq rW$ and $H \subseteq rS$. The set $K = \{ x \in X : v(x)f(x) \notin (\frac{1}{r})S \}$ is a compact and so we choose $t \geq 1$ with $f(K) \subseteq (t/r)H$. The Minkowski functional ρ of S is continuous and positively homogeneous and, consequently, the function $h_t : E \to E$ defined by

$$h_t(a) = \begin{cases} a & \text{if } a \in tS \\ (\frac{t}{\rho(a)})a & \text{if } a \notin tS \end{cases}$$

is continuous with $h_t(E) \subseteq tS$. Let $g = h_t o f$. Clearly, $g \in C_b(X,E)$. Further $g \in CV_o(X,E)$ as follows, let $v_1 \in V$ and $G \in W$. We show that $A = \{ x \in X : v_1(x)g(x) \notin G \}$ is compact.

By hypothesis $B = \{ x \in X : v_1(x)f(x) \notin G \}$ is compact, so it suffices to show that $A \subseteq B$ or equivalently $B' \subseteq A'$.

Let $x \in B'$. Then $v_1(x)f(x) \in G$ and so

$$v_1(x)g(x) = v_1(x)h_t(f(x))$$

$$= \begin{cases} v_1(x)f(x) & \text{if } f(x) \in tS \\ v_1(\frac{t}{\rho(f(x))})\ f(x) = \frac{t}{\rho(f(x))}\ v_1(x)f(x) & \text{if } f(x) \notin tS \end{cases}$$

$$\in \begin{cases} G & \text{if } f(x) \in tS \\ \frac{t}{\rho(f(x))}\ G \subseteq G & \text{if } f(x) \notin tS \end{cases}$$

Hence $x \in A'$. Let $x \in X$, then, since $f(K) \subseteq tS$, we have

$$v(x)(g(x)-f(x)) = \begin{cases} 0 \in W & \text{if } f(x) \in tS \\ \frac{t}{\rho(f(x)-1)}\ v(x)f(x) \in (\frac{1}{r})S \subseteq W & \text{if } f(x) \notin tS \end{cases}$$

Thus $g-f \in N(v,W)$, as required.∎

THEOREM 3.3.4. [17]

Suppose E is an admissible T.V.S. and $V \subseteq S_o^+(X)$. Then $C_b(X) \otimes E$ is w_V-dense in $C_b(X,E)$.

PROOF

Let $f \in C_b(X,E)$, and let $v \in V$ and W a neighbourhood of 0 in E. Choose an open balanced neighbourhood G of 0 in E such that $G+G+G \subseteq W$. Choose $r > \|v\|$ with $f(x) \subseteq rG$, and let $K = \{ x \in X : v(x) \geq 1/r \}$. Then $f(K)$ is a compact subset of E and so, by hypothesis, there exists a continuous map $\phi : f(K) \to E$ with range contained in a finite dimensional subspace of E such that

$$\phi(f(x))-f(x) \in (\tfrac{1}{r})G \qquad \forall\ x \in K$$

We can write $\phi \circ f = \sum_{i=1}^{n} (\phi_i \circ f) \otimes a_i$, where $\phi_i \circ f \in C(K)$ and $a_i \in E$. By Tietz extension theorem, there exists $\psi_i (1 \leq i \leq n)$ in $C_b(X)$ such that $\psi_i = \phi_i \circ f$ on K. Let $h = \sum_{i=1}^{n} \psi_i \otimes a_i$. Then $k \subseteq h^{-1}(rG+rG) = F$, which is open in X, and so there exists a $\psi \in C_b(X)$ with $0 \leq \psi \leq 1$, $\psi = 1$ on K and $\psi = 0$ on $X \backslash F$. Let $g = \psi h$, then $g \in C_b(X) \otimes E$ and $g = h = \phi \circ f$ on K. Further, $g(x) \subseteq rG + rG$ as follows, Let $y \in X$. Then,

$$g(y) = \psi(y)h(y) \in \begin{cases} \psi(y)(rG+rG) & \text{if } y \in F \\ \{0\} & \text{if } y \notin F \end{cases}$$

$$\subseteq rG + rG$$

We next show that $g-f \in N(v,W)$, let $x \in X$. Then,

$$v(x)(g(x)-f(x)) = v(x)(\phi(x)f(x)-f(x))$$

$$\in \begin{cases} \dfrac{v(x)}{r} \; G & \text{if } x \in K \\ v(x)[\; rG+rG-rG \;] & \text{if } x \notin K \end{cases}$$

$$\subseteq \begin{cases} G \subseteq W & \text{if } x \in K \\ v(x)rW \subseteq W & \text{if } x \notin K \end{cases}$$

This completes the the proof.■

THEOREM 3.3.5. [17]

Let X be a locally compact space of finite covering dimension. Then $C_{oo}(X) \otimes E$ is w_v-dense in $CV_o(X,E)$.

PROOF

In view of Theorem 3.3.2.(2) ,it suffices to show that $C_{oo}(X) \otimes E$ is w_v-dense in $C_{oo}(X,E)$. Let $f \in C_{oo}(X,E)$, let $v \in V$ and w a balanced neighbourhood of 0 in E. There exists a compact set $K \subseteq X$ such that $f(x) = 0$ for $x \notin K$. Since X is of finite covering dimension, it follows that there exists a function $g \in C_{oo}(X) \otimes E$, with $g = 0$ outside K, such that $g(x)-f(x) \in N(v,W)$, as required.■

REMARK:

If E is assumed to be locally convex, then theorem 3.2.5. holds without restricting X to have a finite covering dimension.

§ 3.4. BEST APPROXIMATION IN SPACES OF CONTINUOUS FUNCTIONS

Consider C[a,b] with the sup norm $\|f\| = \sup_{x\in[a,b]}|f(x)|$ ($f \in C[a,b]$). If M is a finite dimensional subspace of C[a,b], then, by Theorem 1.3.1, M is proximinal. However, since $(C[a,b], \| \|)$ is not strictly convex, M need not be Chebyshev. Hence, for the uniqueness of best approximation, we need to impose Haar condition on M which we define as follows.

DEFINITION 3.4.1. ([11],p.91-92)

Let X be a locally compact Housdorff space and M an n-dimensional subspace of $C_o(X)$. Then M is said to satisfy *Haar condition* if every non-zero $f \in M$ has at most n-1 zeros.

The following result was initially proved by A.Haar in 1918 for the case $X = [a,b]$ (see[22], p.340). Its extension in the present form is due to R.R.Phelps ([26], Theorem 3.6). (See also [11], p.115)

THEOREM 3.4.2.

Let X be a (locally) compact Hausdorff space and M is n-dimensional subspace of C(X) (of $C_o(X)$). Then M is Chebyshev iff M satisfies the Haar condition.■

The following is a recent result which generalizes the above theorem to vector-valued functions.

THEOREM 3.4.3. [7]

Let X be a locally compact Hausdorff space and E a real normed space. Let K be a convex subset of an n-dimensional linear space of $C_o(X,E)$, and let $f \in C_o(X,E)\backslash K$ and $g_o \in K$. Then $g_o \in P_K(f)$ iff there exists $h_1, h_2, \ldots, h_m$, $m \le n+1$, extreme points of the unit ball of E', m points $x_i \in X$ and m scalars $t_i > 0$ with $\sum_{i=1}^{m} t_i = 1$ such that

(i) $\sum_{i=1}^{m} t_i h_i[g(x_i)-g_o(x_i)] \le 0$ for all $g \in K$.

(ii) $h_i[f(x_i)-g_o(x_i)] = \|f-g_o\|$, $(i=1,2,\ldots,m)$.

We next describe two other types of approximations. Least-square approximation and the rational approximation.

First consider C[a,b] with the quadratic norm

$$\|f\|_2 = \left(\int_a^b |f(x)|^2 dx \right)^{1/2}, \qquad f \in C[a,b]$$

C[a,b] becomes an inner product space with respect to the inner product defined by

$$\langle f,g \rangle = \int_a^b f(x)g(x)dx$$

THEOREM 3.4.4. ([6],p.102-103)

Let $\{g_1,g_2,\ldots,g_m\}$ be any linearly independent subset of C[a,b] and let $\sum_{i=1}^{m}\alpha_i g_i \in M = sp\{g_1,g_2,\ldots,g_m\}$. Then for any $f \in C[a,b]$, the least-square approximation problem of minimizing $\| f- \sum_{i=1}^{m}\alpha_i g_i \|_2$ has a solution if $\{g_1,g_2,\ldots,g_m\}$ is an orthonormal base for M.■

As regards the rational approximation, a function of the form $R(x) = \frac{P(x)}{Q(x)}$, $x \in [a,b]$, where P(x) and Q(x) are polynomials of finite degree with $Q(x) > 0$, is called a *rational function*. For any integer $n,m \geq 0$, let

$R_m^n [a,b] = \{ \frac{P}{Q} : \deg P \leq n, \deg Q \leq m, Q(x) > 0 \text{ on } [a,b]\}$

Clearly $R_m^n [a,b] \subseteq C[a,b]$.

THEOREM 3.4.5. ([6], p.154)

With the above notations, for any $n,m \geq 0$, $R_m^n [a,b]$ is a proximinal subset of C[a,b].■

Finally, we present with proof an interesting result due to Holmes-Kripke ([11],p.125). We recall some definitions. An open cover $\mathcal{U}$ of a topological space is said to be *locally finite*, if each $x \in X$ has a neighbiurhood meeting only finitely many members of $\mathcal{U}$. X is said to be *paracompact*, if each open cover of X has a refinement which is locally-finite. A classical result of J.Dieudonne (1944), known as the Interposition theorem, states that if u(x) and v(x) are respectively the lower and upper semi-continuous functions on a paracompact space X with $v \leq u$, then there exists $g \in C_b(X)$ such that $v \leq g \leq u$ ([14],p.172).

THEOREM 3.4.6. ([11],p.125)

Let X be a paracompact Housdorff space and B(X) the space of all real-valued bounded functions on X with the sup norm topology. Then $M = C_b(X)$ is a proximinal subset of B(X).

PROOF

For any $f \in B(X)$, let $d = d(f,M)$. Define

$$f_1(y) = \lim\inf \{f(x) : x \to y\}$$

$$f_2(y) = \lim\sup \{f(x) : x \to y\}$$

Then f_1 and f_2 are lower and upper semi-continuous functions, respectively, on X. Further

$f_1(y) \leq f(y) \leq f_2(y)$ for all $y \in X$.

Let $r = r(f) = \frac{1}{2} \| f_2 - f_1 \|$,

$u = f_1 + r$ and $v = f_2 - r$. Then

$u-v = (f_1-f_2) + 2r \geqslant 0$ and so $v \leq u$ on X. We claim that $r = d$ and that $g \in P_M(f)$ iff $v \leq g \leq u$ on X.

We first show that $r \leq d$. Let $\varepsilon > 0$. Then there exists $g \in C_b(X)$ such that

$$\|f-g\| \leq d + \varepsilon$$

or $\quad f - d - \varepsilon \leq g \leq f + d + \varepsilon$ on X.

Taking lim sup of the left hand inequality and lim inf of the right hand inequality, we have

$$f_2 - d - \varepsilon \leq g \leq f_1 + d + \varepsilon \quad \text{on } X.$$

Hence $0 \leq f_2 - f_1 \leq 2d + 2\varepsilon$ on X, and so $r \leq d + \varepsilon$. Since ε is arbitrary, we obtain $r \leq d$.

Next, since u and v are lower and upper semi-continuous functions, respectively, on paracompact space X with $v \leq u$, it follows from the Dieudonne's Interposition Theorem, there exists $g \in C_b(X)$ such that $v \leq g \leq u$ on X. Then

$$f - r \leq f_2 - r = v \leq g \leq u = f_1 + r \leq f + r,$$

and so $d(f,M) \leq \| f-g \| \leq r$. Then $r = d$, and this completes the proof.■

To conclude, we present without proof another interesting and deep result of Holmes and Kripke [12] on best approximation in spaces of bounded linear mappings. Let X and Y be Hilbert spaces and $CL(X,Y)$ the space of all continuous (=bounded) linear mappings, with the operator norm $\|T\| = \sup\{ \|Tx\| : x \in X, \|x\| \leq 1 \}$. Let $KL(X,Y)$ be the subspace of $CL(X,Y)$ consisting of compact mappings. (A linear mapping $T : X \to Y$ is called compact if, for every bounded subset A of X, T(A) is relatively compact subset of Y; or equivalently if, for any sequence $\{x_n\}$ in X with $x_n \xrightarrow{weakly} 0$, we have $Tx_n \xrightarrow{norm} 0$).

THEOREM 3.4.7. ([12],p.257)

With the above notations, KL(X,Y) is proximinal subspace of CL(X,Y).■

REMARK

The above result may not hold when X and Y are not Hilbert spaces. In [3], it is shown that if $1 \le p \le \infty$, $p \ne 2$, then $KL(L_p,L_p)$ is not proximinal in $CL(L_p,L_p)$. Thus the above theorem need not hold even if X and Y are reflexive Banach spaces.

REFERENCES

[1] G.C.Ahuja, T.D.Narang and Swaran Trehan: "Best Approximation on convex sets in metric linear spaces", Math. Nachr. 17(1977), 125-130.

[2] E.Asplund: "Chebyshev sets in Hilbert spaces", Trans. Amer. Math. Soc. 144(1969), 235-240.

[3] Y.Benyamini and P.K.Lin: "An operator on L^P without best compact approximation", Isarel J.Math. 51(1985).

[4] B.Brosowski and F.Deutsch : "An elementary proof of the Stone-Weierstrass theorem", Proc. Amer. Math. Soc. 81(1981), 89-92.

[5] R.C.Buck: "Approximation properties of vector valued functions", Pacific J. Math. 53(1974), 85-94.

[6] C.W.Cheney: "Introduction to Approximation Theory", McGraw-Hill, New York, 1966.

[7] F.Deutsch: "Best approximation in the space of continuous vector-valued functions", J.Approx. Th. 53(1988), 112-116.

[8] M.M.Day: "Normed Linear Spaces", Springer-Verlag. Berlin, 1959.

[9] N.V.Efimov and S.B.Stečkin: "Approximative compactness and Chebyshev sets", Transl. Soviet Math. Dokl. 2(1961), 1226-1228.

[10] H.G.Heuser: "Functional Analysis", John Wiley and sons, New York, 1982.

[11] R.B.Holmes: "A Course on Optimization and Best Approximation", LNM No. 257, Springer-Verlag, Berlin, 1971.

[12] R.B.Holmes and B.R.Kripke: "Best approximation by compact operators", Indiana Univ. Math. J. 21(1971), 255-263.

[13] H.Jarchow: "Locally convex spaces", B.G.Teubner, Stuttgart, 1981.

[14] J.L.Kelley: "General Topology ", D.Van Nostrand New York, 1955.

[15] L.A.Khan: "The strict topology on a space of vector-valued functions", Proc. Edinburgh Math. Soc. 22(1979), 35-41.

[16] L.A.Khan: "On the Stone-Weierstrass theorem for vector-valued functions", Punjab Univ. J.Math. 11/12(1979-80), 11-14.

[17] L.A.Khan: "On approximation in weighted spaces of continuous vector-valued functions", Glasgow Math.J. 29(1987), 65-68.

[18] V.Klee: "Shrinkable neighbourhood in Housdorff linear spaces", Math. Ann. 141(1960), 281-285.

[19] V.Klee: "Convexity of Chebyshev sets", Math. Ann. 142(1961), 292-304.

[20] G.Köthe: "Topological vector spaces", Springer Verlag, Berlin(1969).

[21] J.Nagata: "Modern Dimension Theory", Interscience, 1965.

[22] E.Kryeszig: "Introductory Functional Analysis with application", John Wiley & Sons, New York, 1978.

[23] T.D.Narang: "On characterization of sets in metric linear spaces", Math. Seminar Notes, Kobe Univ. 7(1979), 593-596.

[24] T.D.Narang: "Best approximation in locally convex spaces", Pure and applied Mathematika sciences, 22(1985), 33-35.

[25] T.D.Narang: "Some results on approximation LCS", Pure and applied Mathematika sciences, 25(1987), 59-65.

[26] R.R.Phelps: "Uniqueness of Hahn-Banach extension and unique best approximation", Trans. Amer. Math. Soc. 95(1960), 238-255.

[27] G.S.Rao and S.Elumalai: "Approximation and strong approximation in locally convex spaces", Pure and applied Mathematika sciences, 19(1984), 13-26.

[28] J.R.Rice: "The approximation of Functions", Addison-Wesley, London, 1964.

[29] A.Robertson and W.Robertson: "Topological vector spaces", Cambridge Univ. press, Cambridge, 1964.

[30] W.Rudin: "Real and Complex Analysis", McGraww Hill New York, 1966.

[31] W.Rudin: "Functional Analysis", McGraw Hill, New York, 1973.

[32] H.H.Schaefer: "Topological vector spaces", Macmillan, New York, 1966.

[33] I.Singer: "Best approximation in normed spaces by elements of linear subspaces", Springer-Verlag, New York, 1970.

[34] A.B.Thaheem: "Existence of best approximation", Portugallae Math. 42(1983-84), 435-440.

[35] L.P.Vlasov: "Approximative properties of sets in normed linear spaces", Russian Math. Surveys, 28(1973),1-66.