POINT SOURCE DIFFRACTION BY AN ACOUSTICALLY PENETRABLE OR ELECTROMAGNATICALLY DIELECTRIC HALF PLANE

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IN THE NAME OF ALLAH
THE MOST GRACIOUS
THE MOST MERCIFUL

DEDICATED TO

MY PARENTS

BROTHER

AND

SISTER

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CERTIFICATE

We accept the work contained in this dissertation as conforming to the required standard for the partial fulfilment of the degree of

MASTER OF PHILOSOPHY

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CONTENTS

PREFACE

In recent years, noise has become a serious issue of environmental protection in heavily built up areas, Kurze [1]. Noise abatement has, therefore, attracted the attention of many scientists. Traffic noise from motorways, railways and airports,and other outdoor noises from heavy construction machinery or stationary installation (transformers or plants), can be shielded by a barrier which intercepts the line of sight from source to receiver. In most of the calculations with noise barriers the field in the shadow region of the barrier is assumed to be solely due to diffraction at the edge. This assumption supposes that the barrier is perfectly rigid and therefore does not transmit sound. However, most practical barrier are made of wood or plastic and will consequently transmit some of the noise through the barrier.

Yeh [2] has considered the problem of diffraction by penetrable parabolic cylinder, and as a limiting case approximates to a penetrable half plane. An other approximate approach using parabolic cylinder coordinates has been used by Shmoys [3]. His results are much simpler than Yeh [2] and are expressed in terms of Fresnels integrals. However, his approach is not rigorous and does not elaborate on the penetrable half plane solution.

Pistol'kors et al. [4] use the Kirchoff-Huygens integral equation approach to solve the more general problem of diffraction by a penetrable strip when shadow face of the strip is assumed to be the same as for an infinite penetrable sheet. The same approximate boundary condition is used by Khrebet [5] in conjuction with Kontrovich-Lebedev integral transforms to obtain a solution for a dielectric half plane. The approximate boundary condition [4 and 5] is only good in describing a perfectly penetrable half plane.

A.D. Rawlins [6] has used an alternative boundary condition which gives a smooth transition from a perfectly penetrable half plane to a non penetrable half plane. This boundary condition is slightly more complicated than that used in [4 and 5], but symmetrical. He solved the problem of diffraction by an acoustically penetrable or an electromagnetically dielectric half plane due to line source.

The present work is a natural and important extension of the above to consider the diffraction of acoustic waves in three dimensions (Point source) by an acoustically penetrable or an electromagnetically dielectric half plane. This consideration will help understand acoustic diffraction from the more general case of special wave and will go a step further to complete the discussion for penetrable half planes. Introduction of the point source introduces another variable. The difficulty that arises is the solution of the integrals occurring in the inverse transforms. These integrals are normally difficult to handle because of the presence of branch points and are only enable to solve using asymptotic approximations. The analytic solution of these integrals is obtained and the far field is presented.

In chapter one, basic relations of acoustic waves are derived. A few asymptotic methods are presented to calculate the integrals approximately for large frequency.

Chapter two consists of diffraction by an acoustically penetrable or an electromagnetically dielectric half plane due to line source.

In chapter three, we present the far field solution of the problem of point source diffraction by an acoustically penetrable or electromagnetically dielectric half plane. Also the integrals appeared in the diffracted field solution are solved.

CHAPTER ONE

BASIC ACOUSTICS AND ASYMPTOTIC METHODS

In this chapter we present the basic equations of Fluid Dynamics and Acoustic Waves. We also discuss standard methods usually adopted to calculate asymptotically certain integrals appearing in diffraction problems. General procedure for solving cylindrical and spherical wave equation is also presented. The contents of this chapter are taken from "Theoratical Acoustics By Philip M. Morse And K. UNO Ingard.

BASIC ACOUSTICS

1.1 — ACOUSTICS

Acoustics may be defined as the study of the generation, transmission and reception of energy in the form of vibrational waves in matter. As the atoms or the molecules of a fluid or solid are displaced from their normal configuration an inertial elastic restoring force arises. Examples include the tensile force produced when a spring is stretched, the increase in pressure produced when a fluid is compressed and the transverse restoring force produced when a point on a stretched wire is displaced in a direction normal to its length. It is this elastic restoring force, coupled with the inertia of the system, that enables matter to participate in oscillatory vibrations and there by generate and transmit acoustic wave.

The most familiar acoustic phenomenon is that associated with the sensation of sound. For an average young person, a vibrational disturbance is interpreted as sound if its frequency lies in the range of about 20 to 20,000 Hertz. However, in a broader sense, acoustics also include the ultrasonic frequencies above 20,000 Hertz and infrasonic frequencies below 20 Hertz. The nature of vibration associated with acoustics are for example, the simple sinusoidal vibrations produced by a tuning fork and non periodic motions associated with an explosion.

1.2 — ACOUSTIC WAVE MOTION

Acoustic waves that produce the sensation of sound are one of a variety of pressure disturbances that can propagate through a compressible fluid. There are also ultrasonic and infrasonic waves whose frequencies are beyond the audible limits, e.g. high intensity waves generated by air crafts and explosions.

It will be convenient to start with the simpler case of plane waves. The characteristic property of the plane wave is that each acoustic variable has constant amplitude on any given plane perpendicular to the direction of wave propagation. The propagation of sound is always associated with some medium. Sound does not propagate in vaccum. Sound is generated when the medium is dynamically disturbed. Such disturbance of the medium affects its pressure, density, particle velocity and temperature. Most known fluids and solids have relatively small heat conductivity and sound propagation is nearly adiabatic, even at very low frequencies. Therefore the temperature is of little significance. The effects of gravitational forces will also be neglected so that constant equilibrium density (ρ_0) and constant equilibrium pressure (p_0) have uniform values throughout the fluid. The fluid is also assumed to be homogeneous, isotropic and perfectly elastic; no dissipative effects such as those arising from viscosity or heat conduction are present. Finally, the analysis will be limited to waves of relatively small amplitude so that changes in density of the medium will be small as compared with its equilibrium value. These assumptions are necessary to arrive at the simplest theory for sound in fluids. Fortunately, experiments have shown that this simplest theory adequately describes most common phenomena.

1.3 — THE BASIC EQUATIONS

In this section, we develop some equations of basic equations of acoustics. These definitions are taken from [22].

1.3.1 — THE EQUATION OF STATE

A relation between pressure and density is the adiabatic equation of state is

$$P = f(\rho). \tag{1.1}$$

Since the change in pressure and density is very small, this equation can be expanded in a Taylor series:

$$P = P_o + \left(\frac{\partial P}{\partial \rho}\right)_{\rho_o}(\rho - \rho_o) + \frac{1}{2}\left(\frac{\partial^2 P}{\partial \rho 2}\right)_{\rho_o}(\rho - \rho_o)^2, \tag{1.2}$$

where the partial derivatives are constants, determined for adiabatic compression and expansion of the fluid about its equilibrium density. If the fluctuations are small, only the lowest order term $(\rho - \rho_o)$ need be retained. This gives a linear relationship between pressure fluctuation and change in density

$$P - P_o = B(\rho - \rho_o)/\rho_o, \tag{1.3}$$

where $B = \rho_o\left(\frac{\partial p}{\partial \rho}\right)_{\rho_o}$ is the adiabatic bulk modulus. In terms of acoustic pressure and condensation $\bar{S}$, (1.3) can be expressed as

$$P = B\bar{S}, \tag{1.4}$$

where $\bar{S} = (\rho - \rho_o)/\rho_o$ and $|\ \bar{S}\ | \ll 1$.

1.3.2 — THE EQUATION OF CONTINUITY

To relate the motion of the fluid to the compression or dilation we need a functional relationship between the particle velocity V and instantaneous density ρ. In other words we want to observe what happens if one part of the fluid affects the other part. We represent this effect in equation form, known as the equation of conservation of matter. The equation of conservation of matter in the Eulerian approach is derived as follows:

The total mass in volume T bounded by the surface s at time t is $\int_T \rho \, dx$, where ρ is the volume density of the fluid and dx is the volume element. Let us keep this volume fixed in space. Then the increase in mass in a small time δt is $\delta t \int_T (\partial\rho/\partial t) dx$. Since the mass is conserved the increase must be due to flow across the boundary s of T. Now the fluid crosses s is only on account of the velocity component along the normal to s. If $\hat{n}$ is a unit normal vector directed out of T, the mass which is transferred across the small element ds is that obtained in a cylinder of volume $n.v \, \delta t ds$, where the vector v is the velocity of the fluid. Hence

$$\int_T (\partial\rho/\partial t) \, dx = - \int_S \rho \, n.v \, ds. \qquad (1.5)$$

Using the divergence theorem, (1.5) can be written as

$$\int_T (\partial\rho/\partial t) \, dx = -\int_T \operatorname{div}(\rho v) \, dx. \qquad (1.6)$$

Because this holds for an arbitrary volume T, we conclude that

$$\partial\rho/\partial t + \operatorname{div}(\rho v) = 0, \qquad (1.7)$$

which is the Eulerian form for conservation of mass.

Notice that it is non-linear equation. If we write $\rho = \rho_o(1 + \bar{S})$ and use the fact that ρ_o is constant in both space and time and assuming that $\bar{S}$ is very small, equation

(1.7) becomes

$$\partial \bar{S}/\partial t + \mathrm{div}(v) = 0, \qquad (1.8)$$

which is known as the linearised equation of continuity.

1.3.3 — FUNDAMENTAL EQUATIONS OF MOTION

In fluids the existence of viscosity and the failure of acoustic processes to be perfectly adiabatic introduce dissipative terms. Since we have already neglected the effects of thermal conductivity in the equation of state, we also ignore the effects of viscosity and consider the fluid to be inviscid. The equation of motion comes from the consideration of the forces in a fluid. Let P be the pressure, then the total surface force will be

$$-\int_s P.n\,ds,$$

where s is the closed surface bounding the volume T of the fluid.

Let f represents the acceleration of the fluid particle, then the total inertial force will be

$$-\int_T \rho\, f\, dx.$$

According to De'Alembert's principle,

Total surface force + Total body force + Total inertial force = 0.

Since we are neglecting the body forces like gravity, the above principle takes the form:

$$\int_s P.n\, ds + \int_T \rho\, f\, dx = 0.$$

Using the divergence theorem in this equation we have

$$\int_T \nabla P \; dx + \int_T \rho \; f \; dx = 0.$$

Because this equation holds for an arbitrary volume, we conclude that

$$f = - \frac{1}{\rho} \nabla P.$$

Now using the relationship

$$f = \frac{DV}{Dt} = \frac{\partial V}{\partial t} + (V \cdot \nabla) V,$$

we have, from the last equation

$$\frac{\partial V}{\partial t} + (V \cdot \nabla) \; V = - \frac{1}{\rho} \nabla P. \tag{1.9}$$

This non-linear, inviscid force equation is called Euler's equation of motion. It can be simplified if we require $|\bar{S}| \ll 1$ and $|(V \cdot \nabla) V| \ll |\frac{\partial V}{\partial t}|$. Then replacing ρ with ρ_o and dropping the term $(V \cdot \nabla) V$ in equation (1.9), we obtain

$$\rho_o \frac{\partial V}{\partial t} = - \nabla P. \tag{1.10}$$

This is the linear inviscid force equation, valid for acoustic processes of small amplitude.

1.3.4 — THE LINEARISED WAVE EQUATION

The two equations (1.8) and (1.10) can be combined to yield a single differential equation with one dependent variable. Taking the divergence of equation (1.10), we have

$$\rho_o \; \nabla \cdot \frac{\partial V}{\partial t} = - \nabla^2 P \; , \tag{1.11}$$

where ∇^2 is the three dimensional Laplacian operator. Now taking the time derivative of (1.8), we have

$$\frac{\partial^2 \bar{S}}{\partial t^2} + \frac{\partial}{\partial t} \; (\nabla \cdot V) = 0. \tag{1.12}$$

Combining equations (1.11) and (1.12) and using $\frac{\partial}{\partial t}(\nabla \cdot V) = \nabla \cdot \frac{\partial V}{\partial t}$, we obtain

$$\rho_o \frac{\partial^2 S}{\partial t^2} = \nabla^2 P.$$

Using the equation of state (1.4), we get

$$\frac{\rho_o}{B} \frac{\partial^2 P}{\partial t^2} = \nabla^2 P.$$

or

$$\nabla^2 P = \frac{1}{c^2} \frac{\partial^2 P}{\partial t^2}, \qquad (1.13)$$

where

$$c = \sqrt{B/\rho_o}. \qquad (1.14)$$

Equation (1.13) is the linearised, loss less wave equation for the propagation of sound in fluids. In equation (1.13), c is the phase speed for acoustic waves in fluids. Use of (1.14) allows the equation of state to be written in a more convenient form

$$c^2 \rho_o \bar{S} = P.$$

Thus the condensation also satisfies the wave equation.

Since the curl of the gradient of a function f must vanish, i.e. $\nabla x \nabla f = 0$, from (1.10), $\nabla x(\partial V/\partial t) = 0$. This implies that $\partial V/\partial t$ can be expressed as the gradient of a scalar function ϕ. For the purposes of dealing with transient effects we can write $V = \nabla\phi$. The physical meaning of this important result is that the acoustical excitation of an inviscid fluid involves no transient rotational flow; there are no effects such as boundary layers, shear waves, or turbulence. In real fluids, for which there is finite viscosity, the particle velocity is not curl-free everywhere but for most acoustic processes the presence of small rotational effects is confined to the vicinity of boundaries and exerts little influence on the propagation of sound. Thus, substituting $V = \nabla\phi$ in (1.10) we have

$$\rho_o \frac{\partial}{\partial t} \nabla\phi = - \nabla P,$$

or

$$\nabla(\rho_o \frac{\partial\phi}{\partial t} + P) = 0.$$

The quantity in the parenthesis can be chosen to vanish identically if there is no acoustic excitation. This gives

$$P = - \rho_o \frac{\partial\phi}{\partial t} .$$

1.3.5 — GENERAL WAVE EQUATION

A general equation of wave equation can be obtained by taking into account the small perturbations made by the sound waves and dropping the requirement that the flow be insentropic i.e.the entropy S is constant throughout the medium then we have the equations

$$D\rho/Dt + \rho \text{ div}.V = 0, \quad (1.15)$$

$$DV/Dt = -1/\rho\nabla P, \quad (1.16)$$

$$DS/Dt = 0, \quad (1.17)$$

$$P = f(\rho,S). \quad (1.18)$$

Suppose that there is steady flow in which $V = U$, $P = P_o$, $\rho = \rho_o$ and $S = S_o$ then the equations (1.15)—(1.18) take the form

$$U\ .\nabla\rho + \rho_0\ \mathrm{div}.U = 0, \tag{1.19}$$

$$\rho_0(U\ .\nabla)U = -\nabla P_0 \tag{1.20}$$

$$U\ .\nabla S_0 = 0, \tag{1.21}$$

$$\nabla P_0 = (\partial f/\partial\rho)\nabla\rho_0 + (\partial f/\partial S)\nabla S_0. \tag{1.22}$$

Where $\partial f/\partial\rho$ and $\partial f/\partial S$ are calculated at $\rho = \rho_0$ and $S = S_0$. Now the speed of sound is given by $C^2 = \partial f/\partial S$. If we write $h = \partial f/\partial S$, then C and h will be known at every point once (1.19)—(1.22) have been solved for ρ_0 and S_0 as function of position.

Let the sound waves make small perturbation so that

$\rho = \rho_0 + \rho_1$, $V = U + U_1$ and $S = S_0 + S_1$.

Neglecting the product of small quantities, we arrive at the following equations:

$$\partial\rho_1/\partial t + U\ .\ \nabla\rho_1 + U_1.\nabla\rho_0 + \rho_0\mathrm{div}.U_1 + \rho_1\mathrm{div}.U = 0, \tag{1.23}$$

$$\left.\begin{aligned} &\rho_0\partial U_1/\partial t + \rho_0(U\ .\ \nabla)U_1 + \rho_0(U_1.\nabla)U + \rho_1(U.\nabla)U = -C^2\nabla\rho_1 \\ &-h\ \mathrm{grad}.S_1 - (\rho_1\partial^2 f/\partial\rho^2 + S_1\partial^2 f/\partial\rho\partial S)\mathrm{grad}.\rho_0 \\ &-(\rho_1\partial^2 f/\partial\rho\partial S + S_1\partial^2 f/\partial S^2)\mathrm{grad}.S_0 \end{aligned}\right], \tag{1.24}$$

$$\partial S_1/\partial t + U\ \mathrm{grad}.S_1 + U_1\mathrm{grad}.S_0 = 0. \tag{1.25}$$

It is immediately evident that a background flow complicates the analysis which has to be under taken in order to determine the acoustic disturbance. Some simplification can be achieved for particular cases. For example,suppose that the basic flow consists of a steady velocity $\bar{U}$ parallel to x—axis so that $\bar{U} = \mu i$, where i is a unit vector along the x—axis. Assume further that μ and ρ_0 are constant and that flow is insentropic. Then equations (1.19)—(1.22) are certainly satisfied with $C = C_0 =$ constant and $h = 0$. Therefore equations.(1.23) and (1.24) become

$$\partial\rho_1/\partial t + \mu\ \partial\rho_1/\partial x + \rho_0 \mathrm{div}.U_1 = 0, \tag{1.26}$$

$$\rho_0\partial U_1/\partial t + \rho_0\mu\ \partial U_1/\partial x = -C^2\nabla\rho_1. \tag{1.27}$$

To eliminate U_1 from equations.(1.26) and (1.27), we take divergence of equation.(1.27) to give

$$\rho_0\nabla.(\partial U_1/\partial t) + \rho_0\mu\ \nabla.(\partial U_1/\partial x) = -C^2\nabla.\nabla\rho_1,$$

and using

$$\nabla.\partial U_1/\partial t = \partial/\partial t(\nabla.U_1),$$

we obtain

$$\rho_0\partial/\partial t(\nabla.U_1) + \rho_0\mu\ \partial/\partial x(\nabla.U_1) = -C^2\nabla^2\rho_1. \tag{1.28}$$

Substituting the value of $\mathrm{div}.U_1$ from equation.(1.26) into (1.28),we have

$$\partial^2\rho_1/\partial t^2 + 2\mu(\partial^2\rho_1/\partial t\partial x) + \mu^2(\partial^2\rho_1/\partial x^2) = C^2\nabla^2\rho_1. \tag{1.29}$$

The ratio μ/C is known as Mach number M. If $M < 1$, the flow is said to be subsonic where as if $M > 1$, it is super sonic. Equation.(1.29) in terms M can be written as

$$\nabla^2\rho_1 = 1/C^2\ \partial^2\rho_1/\partial t^2 + (2M/C)\partial^2\rho_1/\partial t\partial x + M^2\partial^2\rho_1/\partial x^2. \tag{1.30}$$

or

$$(1-M^2)\left\{\partial^2/\partial^2 + \partial^2/\partial y^2 + \partial^2/\partial z^2 - 2M/C\ \partial^2/\partial t\partial x - 1/C^2\partial^2/\partial t^2\right\}\rho_1 = 0. \tag{1.31}$$

1.4 — THE LINE SOURCE

Suppose that there is a line source at (x_o,y_o). The time dependent of the field is taken to be harmonic. Then the partial differential equation satisfied by the potential ϕ is given by

$$\partial^2\phi/\partial x^2 + \partial^2\phi/\partial y^2 + k^2\phi = -4\pi\delta(x-x_o)\delta(y-y_o), \qquad (1.32)$$

where the right hand side is a forcing term due to the line source at (x_o,y_o). We determine the solution of (1.32) in free space, such that ϕ represents an outgoing wave at infinity.

Taking the Fourier transform of (1.32), we get

$$\frac{d^2\Phi}{dy^2} - \gamma^2\Phi = \frac{-4\pi}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \delta(x-x_o)\delta(y-y_o)e^{i\alpha x}dx, \qquad (1.33)$$

where $\gamma^2 = (\alpha^2 - k^2)$. Using the property of the δ-function, we obtain

$$\frac{d^2\Phi}{dy^2} - \gamma^2\Phi = -2(2\pi)^{1/2}\, e^{i\alpha x_o}\, \delta(y-y_o). \qquad (1.34)$$

We know that if $\dfrac{d^2\Phi}{dy^2} - \gamma^2\Phi = f(y)$ then the solution in $-\infty < y < \infty$ such that $\Phi \to 0$ as $y \to \pm\infty$ is given by

$$\Phi(y) = -\frac{1}{2\gamma} \int_{-\infty}^{\infty} f(\eta)\, e^{-\gamma|y-\eta|}\, d\eta. \qquad (1.35)$$

Using (1.35) the solution of (1.34) can be written as

$$\Phi(\alpha,y) = \frac{\sqrt{2\pi}}{\gamma} \int_{-\infty}^{\infty} e^{i\alpha x_o}\, \delta(\eta-y_o)\, e^{-\gamma|\eta-y_o|}\, d\eta,$$

or

$$\Phi(\alpha,y) = \frac{\sqrt{2\pi}}{\gamma}\, e^{i\alpha x_o}\, e^{-\gamma|y-y_o|}. \qquad (1.36)$$

Now taking the inverse Fourier transform of (1.36), we obtain

$$\phi(x,y) = \int_{-\infty}^{\infty} \frac{e^{-i\alpha(x-x_o)-\gamma|y-y_o|}}{\gamma} d\alpha. \tag{1.37}$$

To solve (1.37), let us define the following substitution:

$$x-x_o = r\cos\theta\upsilon, \quad |y-y_o| = r\sin\theta,$$

$$\alpha = -k\cos(\theta + it), \quad -\infty < t < \infty,$$

$$\gamma = -ik\sin(\theta + it).$$

Then (1.37) takes the form

$$\phi(x,y) = \int_{-\infty}^{\infty} e^{ikr\cosh t} dt = \pi i\, H_o^{(1)}(kr), \tag{1.38}$$

where $r = \sqrt{(x-x_o)^2 + (y-y_o)^2}$ and the integral representation of the Hankel function has been used. Using the asymptotic behaviour of the Hankel function the field given by $\phi(k,r)$ in (1.38) can finally be written as:

$$\phi(k,r) = i\left(\frac{2\pi}{kr}\right)^{1/2} e^{i(kr - \pi/4)}. \tag{1.39}$$

1.5 — THE POINT SOURCE

Suppose that there is a point source at the position (x_o,y_o,z_o). Then the partial differential equation satisfied by the potential is given by

$$\partial^2\phi/\partial x^2+\partial^2\phi/\partial y^2+\partial^2\phi/\partial z^2+k=-4\pi\delta(x-x_o)\delta(y-y_o)\delta(z-z_o), \tag{1.40}$$

where the right hand side is a forcing term due to the point source at (x_o,y_o,z_o). We determine the solution of equation (1.40) in free space, such that ϕ represents an outgoing wave at infinity. The fourier transform and its inverse over the variable z is defined as

$$\Phi(x,y,\omega)= \int_{-\infty}^{\infty} \phi(x,y,z)e^{ik\omega z}\, dz, \tag{1.41}$$

$$\phi(x,y,z)= \frac{k}{2\pi} \int_{-\infty}^{\infty} \Phi(x,y,\omega)e^{-ik\omega z}\, d\omega. \tag{1.42}$$

Taking the Fourier transform of (1.40), we have

$$(\partial^2/\partial x^2 + \partial^2/\partial y^2 + k^2\gamma^2)\Phi(x,y,\omega)= -4\pi e^{-ik\omega z_o} \delta(x-x_o)\delta(y-y_o). \tag{1.43}$$

We see that (1.43) is same as (1.32) except that $k^2\gamma^2$ replaces k^2 and a multiplicative factor $e^{-ik\omega z_o}$ is extra on the right hand side of the equation (1.43). Following the procedure adopted in the previous article and omitting the details of calculation we get

$$\Phi(x,y,\omega)=\pi i e^{-ik\omega z_o} H_o^{(1)}(k\gamma r), \tag{1.44}$$

where $r = \sqrt{(x-x_o)^2 + (y-y_o)^2}$ and the integral representation of the Hankel function has been used.

1.6 — ASYMPTOTIC EVALUATION OF INTEGRALS

Here we discuss the method usually adopted to write down the asymptotic form of certain integrals appearing in diffraction problems.

1.6.1 — THE METHOD OF STATIONARY PHASE

In many problems we have to deal with integrals of the form

$$\int_a^b e^{it\phi(\mu)} g(\mu)\, g(\mu), \tag{1.45}$$

where ϕ is a real valued function, called the phase function, while g may be either real or complex - valued. In contrast to Laplace's method, the exponent is now purely imaginary; hence the integrand is an oscillatory function of t. As long as $\phi'(\mu) \neq 0$, we may integrate by parts and conclude that the integral is $O(1/t)$ when $t \longrightarrow \infty$. The main contribution comes from the points (μ_j), where $\phi'(\mu_j) = 0$. These are called stationary points. We assume a finite number of stationary points (μ_j) with $a < \mu_j < b$, $\phi''(\mu_j) \neq 0$, and $\int^b |g(\mu)| d\mu < \infty$. Then, when $t \longrightarrow \infty$,

$$\int_a^b e^{it\phi(\mu)} g(\mu)\, d\mu = \sum_{j:\phi''(\mu_j)>0} \left[\frac{2\pi}{t\phi''(\mu_j)}\right]^{1/2} e^{it\phi(\mu_j)+i\pi/4} g(\mu_j)$$

$$+\sum_{j:\phi''(\mu_j)<0} \left[\frac{2\pi}{t|\phi''(\mu_j)}\right]^{1/2} e^{it\phi(\mu_j)+i\pi/4} g(\mu_j) + O(1/t) \tag{1.46}$$

In contrast to Laplace's method, we must sum over all stationary points of ϕ not simply those where ϕ is maximum.

If the end points $\mu = a$ or $\mu = b$ are stationary points, they contribute to Eq.(1.28) with a factor of 1/2, just asin Laplace's method.

This complicated-looking formula becomes easier to remember if we restate it in the following fashion: replace $\phi(\mu)$ by its second-order Taylor expansion and replace $g(\mu)$ by its value at the stationary point. Do the resulting integrals, one for each stationary point, and sum over all stationary points.

1.6.2 — THE METHOD OF STEEPEST DESCENT

We consider the integral

$$I_1 = \int_c e^{sg(z)} f(z)\, dz \tag{1.47}$$

where c is a contour in the complex z-plane. We assume that s to be large complex variable, g and f to be analytic functions of the complex variable z and the integral to be taken along some path in the complex z-plane. This integral may be evaluated asymptotically by the method of steepest descents, which was originated by Debye. Copson(1946) gives a detailed description of this method.

It will be assumed that f and g are independent of s and suitably regular. It will be sufficient to consider the case $s \longrightarrow \infty$ for if $s = |s|\, e^{i\theta}$ we can split sg into $|s|$ and $ge^{i\theta}$. Let $g(z) = U(x,y) + iV(x,y)$ where U and V are real. When s is large, a small displacement causing a small change in V will produce a rapid oscillation of the sinusoidal terms in e^{sg}. In general, the contribution from any one part of the path of integration will be about the same as that from any other

$I_1 = \int_c e^{sg(z)} f$

part. However, if a path is chosen on which V is constant the rapid oscillation will disappear. Then the contribution will come from the neighbourhood of the point s, where U is the greatest. The essence of the method, therefore, consists in deforming the contour, as far as this is possible, into a curve V = constant passing through the point where U = constant. Now, a point where $g'(z) = 0$ is called a saddle point. Let $g'(z) = 0$ at $z = z_o = x_o + i y_o$

Now, at $z = z_o$,

$$U_x = U_y = V_x = V_y = 0$$

because $g(z)$ is analytic. Further,

$$U_{xx} + U_{yy} = V_{xx} + V_{yy} = 0 \Rightarrow U_{xx} U_{yy} = - U^2_{xx}$$

which further gives

$$\begin{vmatrix} U_{xx} & U_{xy} \\ U_{yx} & U_{yy} \end{vmatrix} = U_{xx}U_{yy} - U^2_{xy} = - [U^2_{xx} + U^2_{xy}] < 0,$$

this implies that $U = U_o$ is neither a maximum nor a minimum. That is why every stationary point is called a saddle point. Near the saddle point $z = z_o$

$$g(z) - g(z_o) = 1/2\, g''(z_o)\,(z-z_o)^2 = 1/2\, Ar^2 e^{i(\alpha+2\theta)}$$

If $z-z_o = re^{i\theta}$ and $g''(z_o) = Ae^{i\alpha}$ where A, α are real with A positive if $g(z_o) = U_o + iV_o$,

$$U - U_o = \frac{A}{2}\, r^2 \cos(2\theta+\alpha),\quad V - V_o = \frac{A}{2}\, r^2 \sin(2\theta+\alpha).$$

Now, $U-U_o$ is negative if θ is such that its cosine is negative and drops fastest with r if $\theta = (\pi-\alpha)/2$ or $\theta = (3\pi-\alpha)/2$. These are desirable directions because they force exponential decay on the integrand as one moves away from the saddle point. They are called paths of steepest descent. Note that on a path of steepest descent $V = V_o$. Having seen that $V = V_o$ is a good route to start on from a saddle point. Let us see what happens if we stay on it. Suppose we reach a point z_1 and that $z_1 + \rho e^{i\phi}$ is a nearby point on $V = V_o$. Since V does not change in the move from one point to the other, ϕ must satisfy $V_x \cos\phi + V_y \sin\phi = 0$. By means of the Cauchy-Riemann relation this may be expressed as

$$-U_y \cos\phi + U_x \sin\phi = 0. \qquad (1.48)$$

The change of U in the move is $\rho(U_x \cos\phi + U_y \sin\phi)$. This quantity is known to be negative at z_o. It will therefore remain negative until a point z_1 is arrived at where it is zero. But, on account of (1.38), this is impossible unless $U_x = U_y = 0$, i.e. z_1 is a saddle point. Hence, on a path of steepest descent, U decreases steadily from a saddle point until another saddle point is reached. Should $g'(z)$ have a singularity on the path, that can upset the apple cart too.

It will now be assumed that the path of steepest descent goes off to infinity without encountering another saddle point or singularity of $g'(z)$. Consider I_1 taken along the path of steepest descent that begins from z_o along $\theta=(\pi-\alpha)/2$. Convergence of the integral at infinity is assumed because of the property of U demonstrated above. Indeed, that property guarantees that the main contribution to the integral comes from a neighbourhood of the saddle point. In this vicinity

$$g(z) - g(z_o) = -\tfrac{1}{2} Ar^2 \quad \text{and} \quad z-z_o = re^{i(\pi-\alpha)/2}.$$

Hence the integral is essentially

$$f(z_o) \int_0^{\infty} e^{\sigma\{g(z_o)-1/2\ Ar^2\}+i(\pi-\alpha)/2} dr.$$

or

$$\int_{z_o}^{\infty} f(z)\ e^{\sigma g(z)} dz \cong [\pi/(2\sigma A e^{i\alpha})]^{1/2} \quad f(z_o) e^{\sigma g(z_o)+\pi i/2}. \tag{1.49}$$

Going from z_o to infinity via the path of steepest descent on which $\theta = (3\pi-\alpha)/2$ merely reverses the sign of the right-hand side of (1.49). The strategy for dealing with I_1 therefore is to deform the contour C as far as possible into a path or paths of steepest descent. Then the contribution of each saddle point is calculated by means of (1.49). However, we bear in mind that in the deformation of C poles or other singularities of the integrand may be captured; their contribution may be as significant as that from the saddle point. If $g''(z_o) = 0$ there will be more than two paths of steepest descent from the saddle point but on each one the argument about U is unaffected. The mode of calculation is still effective though the asymptotic formula will differ from (1.49).

1.6.3 — LAPLACE'S METHOD

We take the integral of the form

$$f(t) = \int_a^b g(x)e^{th(x)}dx \qquad (1.50)$$

with the possibility that $h'(x) = 0$ at one or more points. In this case it is still true that $f(t) \sim e^{tH}$ $t \to \infty$. where H is the maximum of $h(x)$, $a \le x \le b$. The feature results from the possibility of points x_i, where $h(x_i) = H$ and $h'(x_i) = 0$. We assume that $h''(x_i) \neq 0$ at each of these points. [Of course it follows that $h''(x_i) < 0$ since we are at maximum of h]. These points fall into two groups: (1) interior global maximum of h and (2) boundary maxima where $h'(x_i) = 0$. The exact contribution of the second type of point is one- half of the first type of contribution. We now state the result of Laplace's method.

$$f(t) = \frac{e^{tH}}{\sqrt{t}}\left[C + O\left(1/\sqrt{t}\right)\right] \qquad t \to \infty \qquad (1.51)$$

where

$$C = \sqrt{2\pi}\left[\sum_{\substack{a<x_i<b \\ h(x_i)=H}} \frac{g(x_i)}{[-h''(x_i)]^{1/2}} + \sum_{\substack{x_i=a \text{ or } b \\ h(x_i)=H \\ h'(x_i)=0}} \frac{g(x_i)}{[-h''(x_i)]^{1/2}}\right]$$

(1.52)

CHAPTER 2

DIFFRACTION BY AN ACOUTICALLY PENETRABLE OR ELECTROMAGNETICALLY DIELECTRIC HALF PLANE

In this chapter we present diffraction by an acoustically penetrable or electromagnetically dielectric half plane due to a line source. This problem has been addressed by Rawlins [9]. The problem is formulated in terms of two Wiener Hopf equation. The method of stationary phase is used to calculate the diffracted field.

2.1 — FORMULATION OF THE PROBLEM

We consider the situation where a penetrable half plane occupies $x \le 0$, $y = 0$. The half plane is assumed to be thin compared with the wave length of the incident line source. A line source is situated at (x_0, y_0), $y_0 > 0$, and has time harmonic variation $e^{-i\omega t}$. The factor $e^{-i\omega t}$ will be suppressed in the following work. The problem is solved by finding a solution of the wave equation

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + k^2 \right) u(x,y) = \delta(x-x_0)\,\delta(y-y_0); \qquad |y| > h, \tag{2.1}$$

subject to the boundary conditions

$$\frac{\partial u(x,0^{\pm})}{\partial y} \pm ik\,\{\alpha u(x,0^{\pm}) + \beta u(x,0^{\mp})\} = 0; \; x < 0, \tag{2.2}$$

$$\left.\begin{aligned} u(x,0^+) &= u(x,0^-), \\ \frac{\partial}{\partial y} u(x,0^+) &= \frac{\partial}{\partial y} u(x,0^-) \end{aligned}\right\} ; \; x > 0, \tag{2.3}$$

where α and β are constants and are given as under

$$\alpha = \left\{ \frac{T^2 e^{2ikh\sin\theta_o} + (e^{-2ikh\sin\theta_o} - R^2 e^{2ikh\sin\theta_o})}{(e^{-ikh\sin\theta_o} + Re^{ikh\sin\theta_o}) T^2 e^{2ikh\sin\theta_o}} \right\} \sin\theta_o, \tag{2.4}$$

$$\beta = \left\{ \frac{2T\sin\theta_o}{(e^{-ikh\sin\theta_o} + Re^{ikh\sin\theta_o})^2 - T^2 e^{2ikh\sin\theta_o}} \right\}, \tag{2.5}$$

and 2h is the width of the half plane and R, T are the reflection and transmission coefficients.

For a unique solution of the problem, we also require the radiation conditions:

$$\sqrt{r}\left(\frac{\partial}{\partial r} - ik\right)u \longrightarrow 0 \text{ as } r = \sqrt{x^2 + y^2} \longrightarrow \infty, \tag{2.6}$$

and the edge conditions

$$u(x,0) = O(1), \; \frac{\partial}{\partial y} u(x,0\;) = O(x^{-1/2}); \text{ as } x \longrightarrow 0^+. \tag{2.7}$$

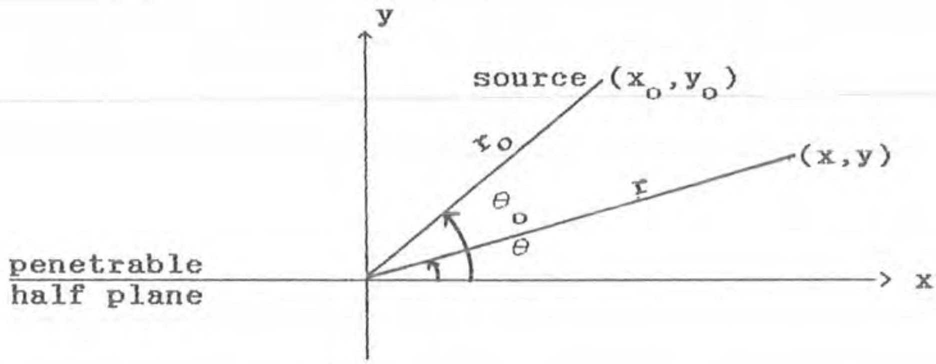
2.2 — SOLUTION OF THE BOUNDARY VALUE PROBLEM

A solution to the boundary value problem (2.1)—(2.5) can be written in the form:

$$u(x,y) = \phi_o(x,y) + \phi(x,y), \tag{2.8}$$

where $\phi_o(x,y)$ accounts for the source in all space in the absence of the half plane, and $\phi(x,y)$is the perturbation field due to the presence of the half plane.

A suitable representation for $\phi_o(x,y)$ and $\phi(x,y)$, which satisfies the radiation condition (2.4) is



$$\phi_o(x,y) = (1/4i)H_o^{(1)}\left\{k\sqrt{(x-x_o)^2+(y-y_o)^2}\right\}$$

$$= (1/4\pi i)\int_{-\infty+i\gamma}^{\infty+i\gamma} \frac{e^{i[\nu(x-x_o)+\varkappa(y-y_o)]}}{\varkappa}d\nu, \qquad (2.9)$$

$$\phi(x,y) = (1/2\pi i)\int_{-\infty+i\gamma}^{\infty+i\gamma} \frac{A(\nu)}{\varkappa}\, e^{i[\nu x+\varkappa y]}\, d\nu;\ (y>0), \qquad (2.10)$$

$$= (1/2\pi i)\int_{-\infty+i\gamma}^{\infty+i\gamma} \frac{B(\nu)}{\varkappa}\, e^{i[\nu x-\varkappa y]}d\nu;\quad (y<0). \qquad (2.11)$$

Where $A(\nu)$,$B(\nu)$ are functions of ν and we shall assume that $k = k_r+k_i$ $(k_r,k_i>0)$ and the ν—plane is cut such that $Im(\varkappa)>0$, where $\varkappa = \sqrt{k^2-\nu^2}$.

For a unique solution the edge condition (2.7) requires that $A(\nu)$, $B(\nu)$ ~ $|\nu|^{-1/2}$ as $|\nu| \longrightarrow \infty$. Substituting equations (2.9)——(2.11) into the boundary conditions (2.2) and (2.3) and carrying out some simple manipulation gives

$$\int_{-\infty+i\gamma}^{\infty+i\gamma} C(\nu)\quad e^{i\nu x}\, d\nu = 0; \qquad (x > 0)\ , \qquad (2.12)$$

$$\int_{-\infty+i\gamma}^{\infty+i\gamma} \frac{D(\nu)e^{i\nu x}}{\varkappa}\, d\nu = 0; \qquad (x > 0), \qquad (2.13)$$

$$\int_{-\infty+i\gamma}^{\infty+i\gamma} \left[C(\nu)\ K(\nu)\ \frac{+\ k(\alpha+\beta)}{\varkappa} e^{-i[\nu x_o - \varkappa y_o]}\right] e^{i\nu x}\, d\nu = 0;$$

$$(x < 0), \quad (2.14)$$

$$\int_{-\infty+i\gamma}^{\infty+i\gamma} \left[D(\nu)L(\nu) + e^{i[\nu x_o - \varkappa y_o]}\right] e^{i\nu x}\, d\nu = 0;$$

$$(x < 0), \quad (2.15)$$

where

$$\left.\begin{aligned} K(\nu) &= 1 + k(\alpha + \beta)/\varkappa \\ L(\nu) &= 1 + k(\alpha - \beta)/\varkappa \end{aligned}\right\}, \quad (2.16)$$

$$\left.\begin{aligned} C(\nu) &= A(\nu) + B(\nu) \\ D(\nu) &= A(\nu) - B(\nu) \end{aligned}\right\}. \quad (2.17)$$

A solution of the equations (2.12)—(2.15) can be written in the form

$$C(\nu) = \phi_+(\nu), \quad (2.18)$$

$$C(\nu)\ K(\nu) + \frac{k(\alpha + \beta)}{\varkappa}\ e^{-i[\nu x_o - \varkappa y_o]} = \phi_-(\nu), \quad (2.19)$$

$$\frac{D(\nu)}{\varkappa} = \psi_+(\nu), \quad (2.20)$$

$$D(\nu)\ L(\nu) + e^{-i[\nu x_o - \varkappa y_o]} = \psi_-(\nu). \quad (2.21)$$

Where the ± subscript denotes a regular functions in the +ve and -ve half planes respectively. The positive subscript denotes that the function is regular in the domain $Im(\nu) > -\mathcal{T}_o$ and the negative subscript denotes that the function is regular in the domain $Im(\nu) < \mathcal{T}_o$, $-k_i < \mathcal{T}_o < k_i$. These two domains have the intersection $|Im(\nu)| < \mathcal{T}_o$, and $\mathcal{T}_o$ is assumed to be such that no singularities occur in this common region of intersection.

We now split $K(\nu)$ and $L(\nu)$ in the form

$$K(\nu) = K_+(\nu)\, K_-(\nu), \tag{2.22}$$

$$L(\nu) = L_+(\nu)\, L_-(\nu). \tag{2.23}$$

Then eliminating $C(\nu)$ and $D(\nu)$ from the equations (2.18)—(2.21) gives

$$\phi_+(\nu)\, K_+(\nu) + \frac{k(\alpha + \beta)\, e^{-i[\nu x_o - \varkappa y_o]}}{K_-(\nu)\sqrt{k^2 - \nu^2}} = \frac{\phi_-(\nu)}{K_-(\nu)} \tag{2.24}$$

$$\psi_+(\nu)\sqrt{k + \nu}\quad L_+(\nu)\, \frac{e^{-i[\nu x_o - \varkappa y_o]}}{\sqrt{k - \nu}\, L_-(\nu)} = \frac{\psi_-(\nu)}{\sqrt{k - \nu}\, L_-(\nu)}. \tag{2.25}$$

Let

$$\Lambda(\nu) = \frac{k(\alpha + \beta)\, e^{-i[\nu x_o - \varkappa y_o]}}{\sqrt{k^2 - \nu^2}\, K_-(\nu)}, \tag{2.26}$$

and

$$\Delta(\nu) = \frac{e^{-i[\nu x_o - \varkappa y_o]}}{\sqrt{k - \nu}\quad L_-(\nu)}, \tag{2.27}$$

then, from the theory of complex variable, we can split these terms by means of Cauchy integrals.

$$\Lambda(\nu) = \Lambda_+(\nu) - \Lambda_-(\nu), \tag{2.28}$$

$$\Delta(\nu) = \Delta_+(\nu) - \Delta_-(\nu). \tag{2.29}$$

Where

$$\left.\begin{aligned} \Lambda_+(\nu) &= (-1/2\pi i)\int_{-\infty}^{\infty} \frac{\Lambda(t)}{(\nu - t)}\, dt; \ \operatorname{Im}(\nu) > 0, \\ \Lambda_-(\nu) &= (-1/2\pi i)\int_{-\infty}^{\infty} \frac{\Lambda(t)}{(\nu - t)}\, dt; \ \operatorname{Im}(\nu) < 0, \end{aligned}\right\} \tag{2.30}$$

$$\left.\begin{aligned} \Delta_+(\nu) &= (-1/2\pi i)\int_{-\infty}^{\infty} \frac{\Delta(t)}{(\nu - t)}\, dt; \ \operatorname{Im}(\nu) > 0, \\ \Delta_-(\nu) &= (-1/2\pi i)\int_{-\infty}^{\infty} \frac{\Delta(t)}{(\nu - t)}\, dt; \ \operatorname{Im}(\nu) < 0. \end{aligned}\right\} \tag{2.31}$$

From equations (2.26) and (2.27) it can be seen that the integrands (2.30) and (2.31) are exponentially bounded as $|t| \longrightarrow \infty$ so that the integrals exist. For real ν the contour of integration is indented so that for $\Lambda_+(\nu)$ and $\Delta_+(\nu)$the point ν lies above the contour of integration and for $\Lambda_-(\nu)$ and $\Delta_-(\nu)$ the point ν lies below the contour of integration.It is also worth noting that $|\Lambda_\pm(\nu)|$ and $|\Delta_\pm(\nu)|$ are at the least O(0) as $|\nu| \longrightarrow \infty$. Thus equations. (2.24) and (2.23) can be written in the form

$$\phi_+(\nu)\ K_+(\nu) + \Lambda_+(\nu) = \frac{\phi_-(\nu)}{K_-(\nu)} + \Lambda_-(\nu) = J_1(\nu) \tag{2.32}$$

and

$$\psi_+(\nu)\ \sqrt{k+\nu}\ L_+(\nu) + \Delta_+(\nu) = \frac{\psi_-(\nu)}{\sqrt{k-\nu}\ L_-(\nu)} + \Delta\ (\nu) = J_2(\nu), \tag{2.33}$$

It can be seen that from equation (2.32) that this equation holds only in the common strip of regularity of both sides. However,the left hand side defines $J_1(\nu)$ throughout the lower ν—plane. By means of the common strip of regularity, each side of equation.(2.26) provides the analytic continuation of the other hand, the composite function defined by equation (2.32) is regular in the entire ν — plane. Provided that we can show that both sides of equation (2.32) have only algebraic growth as $|\nu| \longrightarrow \infty$,then it follows from the extension of Liouville's theorem that $J_1(\nu)$ must be polynomial in ν. Now

$$L(\nu) \longrightarrow 1, \quad \text{as } |\nu| \longrightarrow \infty,$$

and

$$K(\nu) \longrightarrow 1, \quad \text{as } |\nu| \longrightarrow \infty.$$

$$L_{\pm}(\nu) \sim O(1), \quad \text{as } |\nu| \longrightarrow \infty,$$

also

$$K_{\pm}(\nu) \sim O(1), \quad \text{as} |\nu| \longrightarrow \infty.$$

From the edge condition we know that $A(\nu)$, $B(\nu) \sim O(|\nu|^{-1/2})$ and therefore, $C(\nu)$, $D(\nu)$ must be at least of $O(|\nu|^{-1/2})$ as $|\nu| \longrightarrow \infty$.

Using this asymptotic estimate it can be seen from equation (2.32) that $J_1(\nu) \sim O(|\nu|^{-1/2})$ and therefore, the polynomial representing $J_1(\nu)$ can only be a constant which equals zero. Hence from equation (2.32)

$$\phi_{+}(\nu)\, K_{+}(\nu) = -\, \Lambda_{+}(\nu), \tag{2.34}$$

and equation (2.34) combined with equations (2.19) and (2.30) gives

$$C(\nu) = \frac{k(\alpha + \beta)}{2\pi i K_{+}(\nu)} \int_{-\infty}^{\infty} \frac{e^{-i[\nu x_0 - \sqrt{k^2 - t^2}\, y_0]}}{(\sqrt{k^2 - t^2})(\nu - t)K_{-}(t)}\, dt. \tag{2.35}$$

Applying exactly the same argument as above to equation (2.33) gives that $J_2(\nu) \sim O(|\nu|^{-1})$ so that the constant $J_2 = 0$. Hence

$$\psi_{+}(\nu) \sqrt{k + \nu}\, L_{+}(\nu) = -\, \Delta_{+}(\nu), \tag{2.36}$$

and thus from equations (2.20), (2.31) and (2.36)

$$D(\nu) = -\frac{\sqrt{k-\nu}}{2\pi i L_+(\nu)} \int_{-\infty}^{\infty} \frac{e^{-i[\nu x_o - \sqrt{k^2 - t^2}\, y_o]}}{(\sqrt{k-t})(\nu - t)L_-(t)} dt. \qquad (2.37)$$

Now adding and subtracting equations(2.35) and (2.37) gives $A(\nu)$ and $B(\nu)$, which on substitution into equations (2.9) and (2.10) gives the total field as

$$u(x, y) = (1/4i)H_o^{(1)}\left\{k\sqrt{(x-x_o)^2+(y-y_o)^2}\right\} + (1/8\pi^2)\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F(\nu,t)\, e^{i[\nu x + \varkappa|y| - (\nu x_o - \sqrt{k^2 - t^2}\, y_o]}\, d\nu\, dt, \qquad (2.38)$$

where

$$F(\nu,t) = \frac{1}{(\sqrt{k^2-t^2})(\sqrt{k^2-\nu^2})(t-\nu)}\left\{\frac{k(\alpha+\beta)}{K_+(\nu)\, K_-(t)} + \frac{(\sqrt{k+t})(\sqrt{k-\nu})\, \mathrm{sgn}(y)}{L_+(\nu)\, L_-(t)}\right\}. \qquad (2.39)$$

For k real the t— path of integration is indented below $t = 0$ and the ν—path of integration indented above $\nu = 0$. By removing the source to infinitely i.e. $kr_o \longrightarrow \infty$ and asymptotically evaluating the integral integrated with respect to the plane wave solution is given by [6]

$$u(x,y)=e^{-i(x\cos\theta_o + y\sin\theta_o)} + (1/2\pi i)\int_{-\infty}^{\infty} G(\nu,\theta_o)e^{i[\nu x + \varkappa|y|]}\, d\nu, \qquad (2.40)$$

where

$$G(\nu,\theta_o) = \frac{1}{(\nu + k\cos\theta_o)(\sqrt{k^2-\nu^2})}\left\{\frac{k(\alpha+\beta)}{K_+(\nu)\, K_+(k\cos\theta_o)} + \right.$$

$$\left.\frac{(\sqrt{k-\nu})(\sqrt{k-k\cos\theta_o})\,\mathrm{sgn}(y)}{L_+(\nu)\,L_+(k\cos\theta_o)}\right\}. \qquad (2.41)$$

2.3 — ASYMPTOTIC EXPRESSIONS FOR THE FAR FIELD FOR PLANE WAVE INCIDENCE:

For the purpose of obtaining the far field expression (2.38) is asymptotically evaluated for kr large. By making the substitutions

$$x = r\cos\theta,\ y = r\sin\theta,$$

the second term in the expression (2.38) reduces to the form

$$(1/2\pi i)\int_{-\infty}^{\infty} \frac{e^{i\,rg(\nu)}}{(\sqrt{k^2-\nu^2})(\nu-\nu_p)}\,d\nu, \qquad (2.42)$$

where

$$\nu_p = -k\cos\theta_o, \qquad (2.43)$$

$$g(\nu) = \nu\cos\theta - \sqrt{k^2-\nu^2}\,|\sin\theta|, \qquad (2.44)$$

$$f(\nu) = \left\{\frac{k(\alpha+\beta)}{K_+(\nu)\,K_+(k\cos\theta_o)} + \frac{(\sqrt{k-\nu})(\sqrt{k-k\cos\theta_o})\,\mathrm{sgn}(y)}{L_+(\nu)\,L_+(k\cos\theta_o)}\right\} \qquad (2.45)$$

This integral can be asymptotically evaluated for large r by modification of the method of stationary phase. The modification is required because the pole ν_p can come close to the point of stationary phase $\nu_s = k\cos\theta$. Rawlins [6] has described a method for obtaining a uniformly valid asymptotic approximation for such integrals. Without going through the details it can be shown that

$$(1/2\pi i)\int_{-\infty}^{\infty} \frac{e^{irg(\nu)}}{(\sqrt{k^2 - \nu^2})(\nu - \nu_p)} d\nu \sim \frac{e^{ikr - i\pi/4}}{i\sqrt{2\pi kr}} f(\nu_s)$$

$$\times \left[\frac{2|Q|}{(\nu_p - \nu_s)} \right] F(|Q|) + \text{Pole contribution}, \qquad (2.46)$$

where

$$Q = (\sqrt{kr/2})\, \frac{\cos\theta + \cos\theta_o}{\sin\theta}, \qquad (2.47)$$

$$F(Q) = e^{-iQ^2} \int_Q^{\infty} e^{it^2} dt. \qquad (2.48)$$

The pole contributions account for geometrical acoustic field terms. The total field can be represented in terms geometrical acoustic field terms and the diffracted field. The diffracted field is given by

$$u_{diff.} = \frac{e^{ikr - i\pi/4}}{i\sqrt{2\pi kr}} f(\nu_s) \left[\frac{2|Q|}{(\nu_p - \nu_s)} \right] F(|Q|), \qquad (2.49)$$

where

$$f(\nu_s) = k \left\{ \frac{(\alpha + \beta)}{K_+(k\cos\theta)K_+(k\cos\theta_o)} + \frac{2\sin\theta/2\,\sin\theta_o/2}{L_+(k\cos\theta)\,L_+(k\cos\theta_o)} \right\}. \qquad (2.50)$$

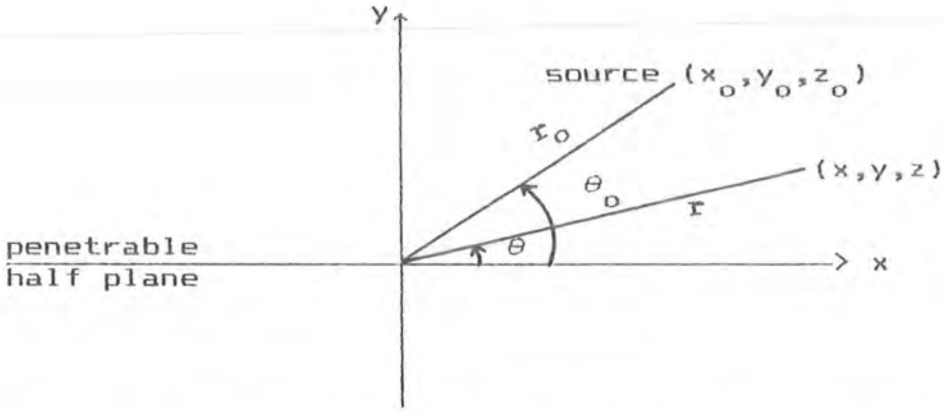
CHAPTER THREE

POINT SOURCE DIFFRACTION BY AN ACOUSTICALLY PENETRABLE OR AN ELECTROMAGNETICALLY DIELECTRIC HALF PLANE

In this chapter we consider diffraction of acoustic waves by an acoustically penetrable or an electromagnetically dielectric half plane due to a point source. The problem is formulated in terms of boundary value problem in three dimensions. The integral transform and asymptotic methods are used to complete the diffracted field.

3.1 — FORMULATION OF THE PROBLEM

We consider the scattering of a small amplitude sound wave from a semi-infinite penetrable plane occupying a position $y = 0$, $x \le 0$. The half plane is assumed to be of negligible thickness. The point source is situated at (x_0, y_0, z_0) having the time harmonic variation $e^{-i\omega t}$.



Thus, the wave equation satisfied by velocity potential u in the presence of a point source is

$$\left(\partial^2/\partial x^2 + \partial^2/\partial y^2 + \partial^2/\partial z^2 + k^2\right)u(x,y,z)$$
$$= \delta(x - x_0)\delta(y - y_0)\delta(z - z_0). \qquad (3.1)$$

The boundary conditions of the problem are given as

$$\left.\begin{aligned}(\partial/\partial y)u(x,0^+,z)+ik\{\alpha u(x,0^+,z)+\beta u(x,0^-,z)\}=0,\\ (\partial/\partial y)u(x,0^-,z)-ik\{\alpha u(x,0^-,z)+\beta u(x,0^+,z)\}=0,\end{aligned}\right] ;x<0, \qquad (3.2)$$

$$\left.\begin{aligned}(\partial/\partial y)u(x,0^+,z) &= (\partial/\partial y)u(x,0^-,z)\\ u(x,0^+,z) &= u(x,0^-,z)\end{aligned}\right] ;\ x>0. \qquad (3.3)$$

Where α, β are given by equations (2.4) and (2.5). The radiation and edge conditions are given by equations (2.6) and (2.7).

3.2 — SOLUTION OF THE PROBLEM :

We define the Fourier Transform and its inverse over the variable z by

$$\left.\begin{aligned}\psi(x,y,w) &= \int_{-\infty}^{\infty} \psi(x,y,z)\, \bar{e}^{ikwz}\, dz \\ \psi(x,y,z) &= (k/2\pi)\int_{-\infty}^{\infty} \psi(x,y,w)\, \bar{e}^{ikwz}\, dw\, ,\end{aligned}\right] \tag{3.4}$$

where

$$k = k_r + ik_i .$$

Transforming equations (3.1)—(3.3), with respect to z by using (3.4) gives

$$[\partial^2/\partial x^2 + \partial^2/\partial y^2 + \gamma^2 k^2]u(x,y,w) = e^{-ikwz_o}\, \delta(x-x_o)\delta(y-y_o), \tag{3.5}$$

where

$$\gamma^2 = (1-w^2),$$

and

$$\left.\begin{aligned}&\partial/\partial y\, u(x,o^+,w) + ik\{\alpha u(x,o^+,w) + \beta u(x,o^-,w)\} = o \\ &\partial/\partial y\, u(x,o^-,w) - ik\{\alpha u(x,o^-,w) + \beta u(x,o^+,w)\} = o\, ,\end{aligned}\right\} \tag{3.6}$$

$$\left.\begin{aligned}&\partial/\partial y\, u(x,o^+,w) = \partial/\partial y\, u(x,o^-,w), \\ &u(x,o^-,w) = u(x,o^-,w).\end{aligned}\right\} \tag{3.7}$$

A solution to the boundary value problem defined by the Equations (3.5)–(3.7) can be written in the form

$$u(x,y,w) = \phi_o(x,y,w) + \phi(x,y,w)\ , \tag{3.8}$$

where $\phi_o(x,y,w)$ accounts for the inhomogeneous source term and $\phi(x,y,w)$ is a solution of the homogeneous wave equation (3.7). A suitable representation for ϕ_o and ϕ which satisfies the radiation condition is given by

$$\begin{aligned}\phi_o &= (a/4i)H_o^{(1)}\{\gamma k \sqrt{(x-x_o)^2 + (y-\ _o)^2}\} \\ &= (a/4\pi i)\int_{-\infty}^{\infty} (1/\varkappa) e^{i[\nu(x-x_o) + \varkappa|y-y_o|]}\, d\nu\end{aligned} \tag{3.9}$$

$$\phi = (1/2\pi i)\int_{-\infty}^{\infty}(1/\varkappa)A(\nu)e^{i[\nu x+\varkappa y]}\,d\nu; \quad y>0 \qquad (3.10)$$

$$= (1/2\pi i)\int_{-\infty}^{\infty}(1/\varkappa)B(\nu)e^{i[\nu x-\varkappa y]}\,d\nu; \quad y<0 \qquad (3.11)$$

where $a = e^{-ikwz_o}$ and $\varkappa = \sqrt{\gamma^2k^2 - \nu^2}$ is choosen such that $\mathrm{Im}\left(\sqrt{\gamma^2k^2 - \nu^2}\right) > 0$ for $|\mathrm{Im}\ \nu| < \mathrm{Im}\ (\gamma k)$. For a unique solution the edge condition (3.5) requires that $A(\nu)$, $B(\nu) \sim |\nu|^{-1/2}$ as $|\nu| \longrightarrow \infty$. Substituting (3.11) — (3.13) into the boundary conditions (3.6) and (3.7) and carrying out simple manipulation gives

$$\int_{-\infty}^{\infty}[A(\nu) + B(\nu)]e^{i\nu x}\,d\nu = 0; \; x > 0, \qquad (3.12)$$

$$\int_{-\infty}^{\infty}[\{A(\nu) - B(\nu)\}/\varkappa]\,e^{i\nu x}\,d\nu = 0; \; x > 0, \qquad (3.13)$$

$$\int_{\infty}^{\infty}[A(\nu) + (k/\varkappa)\{\alpha A(\nu) + \beta B(\nu)\}]e^{i\nu x}\,d\nu$$

$$+ (a/2)\int_{\infty}^{\infty}\{1 + (k/\varkappa)(\alpha+\beta)\}e^{i[\nu(x-x_o)+\varkappa y_o]}\,d\nu, = 0, \qquad (3.14)$$

$$\int_{-\infty}^{\infty}[- B(\nu) - (k/\varkappa)\{\alpha B(\nu) + \beta A(\nu)\}]e^{i\nu x}\,d\nu$$

$$+ (a/2)\int_{-\infty}^{\infty}\{1 - (k/\varkappa)(\alpha+\beta)\}e^{i[\nu(x-x_o)+\varkappa y_o]}\,d\nu, = 0. \qquad (3.15)$$

Adding and subtracting equations (3.14) and (3.15) and putting

$$C(\nu) = A(\nu) + B(\nu), \qquad (3.16)$$

$$D(\nu) = A(\nu) - B(\nu), \qquad (3.17)$$

the resulting expression combined with equations (3.12) and (3.13) give the pair of coupled equations

$$\int_{-\infty}^{\infty} C(\nu)e^{i\nu x}\, d\nu = 0; \; x > 0, \tag{3.18}$$

$$\int_{-\infty}^{\infty} [D(\nu)/\varkappa]e^{i\nu x}\, d\nu = 0; \; x > 0, \tag{3.19}$$

$$\int_{-\infty}^{\infty} \left[C(\nu)K(\nu) + (ak/\varkappa)(\alpha+\beta)e^{-i(\nu x_o - \varkappa y_o)}\right]e^{i\nu x}\, d\nu = 0;$$

$$x < 0, \tag{3.20}$$

$$\int_{-\infty}^{\infty} \left[D(\nu)L(\nu) + ae^{-i(\nu x_o - \varkappa y_o)}\right]e^{i\nu x}\, d\nu = 0; \quad x < 0, \tag{3.21}$$

where

$$K(\nu) = 1 + k(\alpha + \beta)/\varkappa \;, \tag{3.22}$$

$$L(\nu) = 1 + k(\alpha - \beta)/\varkappa \;. \tag{3.23}$$

A solution of the equations (3.18)-(3.21) can be written in the form

$$C(\nu) = \phi_+(\nu), \tag{3.24}$$

$$C(\nu)K(\nu) + (ak/\varkappa)(\alpha + \beta)e^{-i(\nu x_o - \varkappa y_o)} = \phi_-(\nu), \tag{3.25}$$

$$D(\nu)/\varkappa = \psi_+(\nu), \tag{3.26}$$

$$D(\nu)L(\nu) + ae^{-i(\nu x_o - \varkappa y_o)} = \psi_-(\nu), \tag{3.27}$$

where the ± subscript denotes a regular analytic function as defined in chapter 2.

The expressions (3.24)-(3.27) constitute two separate Wiener-Hopf equations. A similar set of equations have been discussed in chapter 2. Thus by means of the technique given in the chapter 2 these equations can be solved to give

$$C(\nu) = \frac{ak(\alpha + \beta)}{2\pi i K_+(\nu)} \int_{-\infty}^{\infty} \frac{e^{-i[\nu x_o - (\sqrt{\gamma^2 k^2 - t^2})y_o]}}{K_-(t)(\sqrt{\gamma^2 k^2 - t^2})(\nu - t)} dt, \tag{3.28}$$

$$D(\nu) = \frac{a\sqrt{\gamma k - \nu}}{2\pi i L_+(\nu)} \int_{-\infty}^{\infty} \frac{e^{-i[\nu x_o - (\sqrt{\gamma^2 k^2 - t^2}) y_o]}}{L_-(t)(\sqrt{\gamma k - t})(\nu - t)} dt. \tag{3.29}$$

Adding and subtracting equations (3.28) and (3.29) give $A(\nu)$ and $B(\nu)$

$$A(\nu) = -(a/4\pi i)\int_{-\infty}^{\infty} \left[\frac{k(\alpha + \beta)}{K_+(\nu)K_-(t)} + \frac{(\sqrt{\gamma k - \nu})(\sqrt{\gamma k + t})}{L_+(\nu)L_-(t)}\right]$$

$$\times \frac{e^{-i[\nu x_o - (\sqrt{\gamma^2 k^2 - t^2}) y_o]}}{(\sqrt{\gamma^2 k^2 - t^2})(t - \nu)} dt, \tag{3.30}$$

$$B(\nu) = -(a/4\pi i)\int_{-\infty}^{\infty} \left[\frac{k(\alpha + \beta)}{K_+(\nu)K_-(t)} - \frac{(\sqrt{\gamma k - \nu})(\sqrt{\gamma k + t})}{L_+(\nu)L_-(t)}\right]$$

$$\times \frac{e^{-i[\nu x_o - (\sqrt{\gamma^2 k^2 - t^2}) y_o]}}{(\sqrt{\gamma^2 k^2 - t^2})(t - \nu)} dt. \tag{3.31}$$

Substituting these values into equations (3.10) and (3.11) gives the total field as

$$u(x,y,w) = (a/4i)H_o^{(1)}\left\{\gamma k\sqrt{(x - x_o)^2 + (y - y_o)^2}\right\}$$

$$+ (a/8\pi^2)\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{i[\nu x + (\sqrt{\gamma^2 k^2 - \nu^2})|y| - \{\nu x_o - (\sqrt{\gamma^2 k^2 - t^2}) y_o\}]}$$

$$\times F(\nu, t) dt d\nu, \tag{3.32}$$

where

$$F(\nu,t) = \frac{1}{\sqrt{(\gamma^2 k^2 - \nu^2)}\ \sqrt{(\gamma^2 k^2 - t^2)}(t - \nu)}\left[\frac{k(\alpha + \beta)}{K_+(\nu)K_-(t)}\right.$$

$$\left. + \frac{(\sqrt{\gamma k - \nu})(\sqrt{\gamma k + t})\mathrm{Sgn}(y)}{L_+(\nu)L_-(t)}\right]. \tag{3.33}$$

3.3 — SOLUTION FOR PLANE WAVE INCIDENCE

Without loss of generality we shall assume $y_o > 0$. In the expression (3.32) make the change of variables

$$x_o = r_o \cos\theta_o,$$

$$y_o = r_o \sin\theta_o.$$

The first term of the expression (3.32) represents ϕ_o the incident point source. If we let $r_o \longrightarrow \infty$, we obtain using the asymptotic form of Hankel function (see Rawlins [6])

$$\phi_o \sim \left\{ (a/4i)\sqrt{(2/\pi k r_o)}\, e^{i(kr_o - \pi/4)} \right\} e^{-ik(x\cos\theta_o + y\sin\theta_o)}, \quad (3.34)$$

where

$$a = e^{-ikwz} .$$

Thus we obtain the solution for plane wave incidence by removing the source to infinity i.e. $r_o \longrightarrow \infty$ (for k real) and evaluating the integral integrated in the point source (3.32) by a straight forward application of the method of stationary phase see Rawlins [23]. Hence for an incident plane wave

$$u(x,y,w) = \frac{e^{-ikwz_o}}{2\pi i} \int_{-\infty}^{\infty} G(\nu,\theta_o) e^{i[\nu x + \sqrt{k^2\gamma^2 - \nu^2}\, |y|]} d\nu$$

$$+ e^{-ik\gamma(x\cos\theta_o + y\sin\theta_o)} \quad (3.35)$$

where

$$G(\nu,\theta_o) = \left\{ 1/(\nu + \gamma k \cos\theta)\sqrt{\gamma^2 k^2 - \nu^2} \right\} \left\{ \frac{k(\alpha + \beta)}{K_+(\nu) K_+(\gamma k \cos\theta_o)} \right.$$

$$\left. + \frac{(\sqrt{\gamma k - \nu})(\sqrt{\gamma k - \gamma k \cos\theta_o})\,\mathrm{Sgn}(y)}{L_+(\nu) L_+(\gamma k \cos\theta_o)} \right\}. \quad (3.36)$$

Thus the expression for diffracted field (as discussed in chapter 2) is given by

$$u(x,y,w) = -\frac{i e^{ik\gamma(r + r_0) - ikwz_0} F(|Q|)}{2\pi\sqrt{r_0}\,\sin\theta}$$

$$\times \left[\frac{\alpha + \beta}{\gamma\sqrt{k\gamma}\; K_+(\gamma k \cos\theta)K_+(\gamma k \cos\theta_0)} + \frac{2(\sin\theta/2)(\sin\theta_0/2)}{\sqrt{k\gamma}\; L_+(\gamma k \cos\theta)L_+(\gamma k \cos\theta_0)}\right], \qquad (3.37)$$

where

$$Q = \sqrt{\gamma k r/2}\,\frac{\cos\theta + \cos\theta_0}{\sin\theta}, \qquad F(Q) = e^{-iQ^2}\int_Q^\infty e^{it^2}\,dt,$$

$$\gamma = \sqrt{1 - w^2}. \qquad (3.38)$$

equation (3.37) can be written as

$$u(x,y,w) = C_1\, e^{ik\gamma(r + r_0) - ikwz_0}\, F(|Q|)$$

$$\times \left[\frac{\alpha + \beta}{\gamma\sqrt{k\gamma}\; K_+(\gamma k \cos\theta)K_+(\gamma k \cos\theta_0)} + \frac{2(\sin\theta/2)(\sin\theta_0/2)}{\sqrt{k\gamma}\; L_+(\gamma k \cos\theta)L_+(\gamma k \cos\theta_0)}\right], \qquad (3.39)$$

where

$$C_1 = -(i/2\pi)\,\sin\theta\,\sqrt{r_0}\;. \qquad (3.40)$$

The field u(x,y,z) can now be calculated by taking inverse Fourier transformation of (3.39). Thus the inverse Fourier transform of (3.39) gives

$$u(x,y,z) = C_{11}\int_{-\infty}^{\infty}\frac{e^{ik[\gamma(r + r_0) + w(z - z_0)]} F(|Q|)}{\gamma\sqrt{k\gamma}\; K_+(\gamma k \cos\theta)K_+(\gamma k \cos\theta_0)}\,dw$$

$$+ C_{12}\int_{-\infty}^{\infty} \frac{e^{ik[\gamma(r + r_o) + w(z - z_o)]}F(|Q|)}{\sqrt{k\gamma}\; L_+(\gamma k \cos\theta)L_+(\gamma k \cos\theta_o)}\, dw, \tag{3.41}$$

$$u(x,y,z) = C_{11}I_1 + C_{12}I_2 \tag{3.42}$$

where

$$\gamma = \sqrt{1 - w^2}, \qquad C_{11} = \frac{k(\alpha + \beta)}{2\pi} C_1,$$

$$C_{12} = \frac{ke^{-i\pi/2}[(2\sin\theta/2)(\sin\theta_o/2)]}{2\pi} C_1, \tag{3.43}$$

$$I_1 = \int_{\infty}^{\infty} \frac{e^{ik[\gamma(r + r_o) + w(z - z_o)]}F(|Q|)}{\gamma\sqrt{k\gamma}\; K_+(\gamma k \cos\theta)K_+(\gamma k \cos\theta_o)}\, dw, \tag{3.44}$$

and

$$I_2 = \int_{-\infty}^{\infty} \frac{e^{ik[\gamma(r + r_o) + w(z - z_o)]}F(|Q|)}{\sqrt{k\gamma}\; L_+(\gamma k \cos\theta)L_+(\gamma k \cos\theta_o)}\, dw. \tag{3.45}$$

Equation (3.42) gives the required field. To complete the problem we only need to calculate the integrals in (3.42) which is accomplished in the Appendix.

APPENDIX

$$I_2 = \int_{-\infty}^{\infty} \frac{e^{ik[\sqrt{1 - w^2}\,(r + r_o) + w(z - z_o)]}F(|Q|)}{\sqrt{k\sqrt{1 - w^2}}\; L_+(k\sqrt{1 - w^2}\cos\theta)L_+(k\sqrt{1 - w^2}\cos\theta_o)}\, dw \tag{A_1}$$

where

$$F(|Q|) = F\left[\mu\sqrt{k\sqrt{1 - w^2}}\right],$$

$$\mu = \sqrt{r/2}\,\frac{\cos\theta + \cos\theta_o}{\sin\theta}.$$

Making use of the result

$$\int_{\nu}^{\infty} e^{i\lambda t^2}\, dt = e^{i\lambda\nu^2} F(\sqrt{\lambda}\nu)/\sqrt{\lambda}\ , \qquad (A_2)$$

the equation (A_1) can be written as

$$I_2 = \int_{\mu}^{\infty}\int_{-\infty}^{\infty} \frac{e^{ik[\sqrt{1-w^2}\,(r + r_o + t^2 - \mu^2) + w(z - z_o)]}}{L_+(k\sqrt{1-w^2}\,\cos\theta)L_+(k\sqrt{1-w^2}\,\cos\theta_o)} F(|Q|)$$

$$\times\ dw\ dt. \qquad (A_3)$$

Consider the integral

$$I_2' = \int_{-\infty}^{\infty} \frac{e^{ik[\sqrt{1-w^2}\ P + w(z - z_o)]}}{L_+(k\sqrt{1-w^2}\,\cos\theta)L_+(k\sqrt{1-w^2}\,\cos\theta_o)}\, dw. \qquad (A_4)$$

By the substitutions

$$\left.\begin{aligned} &w = \cos\xi, \quad z - z_o = R_1\cos\eta, \\ &P = R_1\sin\eta, \quad \sqrt{1-w^2} = \sin\xi, \\ &R_1 = \sqrt{(z - z_o)^2 + P^2} \end{aligned}\right] \qquad (A_5)$$

I_2' takes the form

$$I_2' = \int_{-\infty}^{\infty} f(\xi)\ e^{ikR_1\cos(\xi - \eta)}\ d\xi, \qquad (A_6)$$

where

$$f(\xi) = -\sin\xi/L_+(k\sin\xi\ \cos\theta)L_+(k\sin\xi\ \cos\theta_o). \qquad (A_7)$$

We apply the method of steepest decent to solve the integral I_2'. For that we deform the contour of integration to pass through the point of steepest decent, $\xi = \eta$ such that the major part of the integrand is given by the integration over the part of the deformed contour near η, with $f(\xi)$ slowly varying around it. Hence we can write

$$I_2' = \pi f(\eta)\, H_o^{(1)}(kR_1)$$

$$\cong - \frac{\pi\, H_o^{(1)}\{k\sqrt{(z - z_o)^2 + P^2}\}.P}{\{\sqrt{(z - z_o)^2 + P^2}\}L_+(k\xi\ \cos\theta)L_+(k\xi\ \cos\theta_o)}, \qquad (A_8)$$

where

$$\xi = P/\{\sqrt{(z - z_o)^2 + P^2}\}. \qquad (A_9)$$

Using (A_8), (A_4) can be rewritten as

$$I_2 = -\pi\int_{\mu}^{\infty} \frac{H_o^{(1)}\{k\sqrt{(z - z_o)^2 + (t^2 + r + r_o - \mu^2)^2}}{L_+(k\xi\ \cos\theta)L_+(k\xi\ \cos\theta_o)}$$

$$\times \frac{(t^2 + r + r_o - \mu^2)}{\{\sqrt{(z - z_o)^2 + (t^2 + r + r_o - \mu^2)^2}} dt. \qquad (A_{10})$$

If we make the substitutions

$$t^2 = - A_1 + \sqrt{A_1^{\ 2} + R_{11}^{\ 2}\ \sinh^2 u}\ ,\ R_{11}^{\ 2} = (z - z_o)^2 + A_1^{\ 2}$$

$$A_1 = r + r_o - \mu^2,$$

in A_{10}, we obtain

$$I_2 = -\frac{\pi}{2}\int_{\varepsilon}^{\infty} \frac{H_o^{(1)}\{kR_{11}\cosh u\}\sqrt{R_{11}^{\ 2}\sinh^2 u + A_1^{\ 2}}}{L_+(k\tilde{\zeta}\ \cos\theta)L_+(k\tilde{\zeta}\ \cos\theta_o)} du, \qquad (A_{11})$$

where

$$\left.\begin{aligned} \zeta &= \sqrt{R_{11}^{\ 2}\sinh^2 u + A_1^{\ 2}}/R_{11}\cosh u, \\ \varepsilon &= \sinh^{-1}(\sqrt{\mu^2 + 2A_1^{\ 2}})\mu/R_{11}. \end{aligned}\right] \qquad (A_{12})$$

The integral A_{11} can be solved asymptotically by taking

$kR_{11}\cosh u >> 1$. Therefore, we can replace the Hankel function by the first term of its asymptotic expression to give

$$I_2 = \frac{\sqrt{\pi}\ e^{-r\pi/4}}{\sqrt{2kR_{11}}} \int_{\varepsilon}^{\infty} \frac{e^{ikR_{11}\cosh u} \left\{(\sqrt{R_{11}^{\ 2}\sinh^2 u + A_1^{\ 2}}) + A_1\right\}^{1/2}}{L_+(k\tilde{\zeta}\cos\theta)L_+(k\tilde{\zeta}\cos\theta_0)\sqrt{\cosh u}}\, du$$

(A_{12})

If we take

$$\tau = \sqrt{2kR_{11}}\ \sinh(u/2),$$

then

$$I_2 = -\sqrt{2\pi}\ e^{ikR_{11} - i\pi/4} \int_{\tau_{R_{12}}}^{\infty} e^{i\tau^2}\ f_2(\tau)\ d\tau, \qquad (A_{14})$$

where

$$f_2(\tau) = \left[\frac{\sqrt{\tau^2(\tau^2 + 2kR_{11}) + A_1^{\ 2}k^2} + A_1 k}{(\tau^2 + kR_{11})(\tau^2 + 2kR_{11})}\right]^{1/2}$$

$$\times \frac{1}{L_+(k\tilde{\zeta}\cos\theta)L_+(k\tilde{\zeta}\cos\theta_0)},$$

$$\tilde{\zeta} = \left[\frac{\sqrt{\tau^2(\tau^2 + 2kR_{11}) + A_1^{\ 2}k^2}}{(\tau^2 + kR_{11})}\right],$$

$$\tau_{R_{12}} = \sqrt{k(R_{12} - R_{11})},$$

and

$$R_{12} = \sqrt{(z - z_0)^2 + (r + r_0)^2}\ .$$

An asymptotic expansion of I_2 then follows by putting τ equal to the lower limit value in the non-exponential factor of the integrand plus the contribution from $\tau = 0$, depending it zero lies in the interval of integration. Hence

$$I_2 = -\frac{\sqrt{2\pi}\ e^{ikR_{11}}}{\sqrt{k}} I_o H(-\varepsilon) - \frac{\varepsilon_1\ e^{ikR_{12}-i\pi/4}}{\sqrt{k}} F(\tau_{R_{12}})$$

$$\times \frac{\sqrt{2\pi(A_1 + r + r_o)}}{\sqrt{k(R_{12} + R_{11})R_{12}}\ \ L_+(k\zeta_2\cos\theta)L_+(k\zeta_2\cos\theta_o)}, \qquad (A_{15})$$

where

$$\varepsilon_1 = \text{sgn}\ \tau_{R_{12}} ,$$

$$H(-\varepsilon) = \begin{cases} 1 & \text{if } \varepsilon_1 < 0 \\ 0 & \text{if } \varepsilon_1 > 0 \end{cases}$$

and

$$I_o = \frac{\sqrt{\pi A_1}}{2\sqrt{k}\ R_{11}} \times \frac{1}{L_+(k\zeta_1\cos\theta)L_+(k\zeta_1\cos\theta_o)},$$

$$\zeta_1 = A_1/R_{11}, \qquad \zeta_2 = \frac{r + r_o}{R_{12}} .$$

The integral I_1 can be evaluated similarly.

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