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# FUZZY ALMOST LINEAR SPACES

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# CERTIFICATE

We accept the work contained in this dissertation conforming the required standard for the partial fulfillment of the degree of MASTER OF PHILOSOPHY in MATHEMATICS.

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# PREFACE

The hard sciences construct exact mathematical models of empirical phenomena and these models are then used to make predictions. However, some aspects of the real world always escape such precise mathematical models, and thus there is an elusive inexactness as part of the original model. The purpose of fuzziness is to introduce a unifying point of view to the notion of inexactness. The FUZZY in the sense used here was first introduced by Zadeh.

Essentially, fuzziness is a type of imprecision that stems from grouping of elements into classes that do not have sharply defined boundaries. Such classes called fuzzy sets arise, whenever we describe ambiguity, vagueness, and ambivalence in mathematical models of empirical phenomena. The theory of fuzzy sets has as one of its aims the development of a methodology for the formulation and solution of problems that are too complex or too ill defined to be susceptible to analysis by conventional techniques. Because of its unorthodoxy, it has been and will continue to be controversial for some time to come.

The main aim of this dissertation is to define and study the notion of fuzzy almost linear spaces. This dissertation consists of two chapters, each beginning with brief introductions which summarize the material presented in that chapter. Chapter ONE is a survey aimed at clarifying the terminology to be used and recalling basic definitions and facts. this chapter is also devoted to the study of fuzzy vector spaces and almost linear spaces. In chapter TWO we present the notion of fuzzy almost linear spaces, sums and products of fuzzy sets, almost fuzzy subspaces, convex, balanced and absorbing fuzzy sets and results regarding fuzzy basis.

ISLAMABAD, MUHAMMAD ARSHAD

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# CONTENTS

# CHAPTER ONE
# DEFINITION AND BASIC CONCEPTS

The aim of this chapter is to present some basic concepts and to explain the terminology used throughout in this dissertation. In section 1.1, we discuss the concept of fuzzy sets and the membership function. In sections 1.2 and 1.3, a brief discussion on fuzzy linear spaces and almost linear spaces is given.

## 1.1 FUZZY SETS

Russel [10] gave the concept of Vagueness, subsequently Zadeh [13] achieved a remarkable success by introducing the notion of fuzzy set and some related mathematical concepts.

1.1.1 DEFINITION [13];

Let X be a nonempty set. A fuzzy set A in X is characterized by a membership function $\mu_A$ which associates to each $x \in X$ with its grade of membership $\mu_A(x) \in [0,1]$; $\mu_A$ is called the membership function of A.

For the sake of simplicity, we shall often consider a fuzzy set in X as a function $\mu : X \longrightarrow [0,1]$, or equivalently, as the set

$$\mu = \{(x,\mu(x)) : x \in X\}$$

of ordered pairs, where $\mu(x) \in [0,1]$ for each $x \in X$.

NOTE;

If A is an ordinary set, then $\mu_A$ can take only two values 0 and 1, with

$$\mu_A(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A, \end{cases}$$

in this case $\mu_A = \chi_A$, the usual characteristic function of A.

1.1.2 EXAMPLE [7];

Let X = N, the set of all natural numbers, and A be the set of small natural numbers. Then A is a fuzzy set. Here $\mu_A$ may be given subjectively so that

$$\mu_A = \{(1,1),(2,0.8),(3,0.6),(4,0.4),(5,0.2),(6,0),(7,0),(8,0),\ldots\ldots\ldots\}.$$

We may write

$$1 \in_1 A,\ 2 \in_{0.8} A,\ 3 \in_{0.6} A,\ 4 \in_{0.4} A,\ 5 \in_{0.2} A,$$

and $n \in_0 A$ for all $n \geq 6$.

1.1.3 EXAMPLE [7];

Let X = R (The set of all real numbers), and let A = $\{x \in R : x \gg 0\}$ (the set of real numbers which are much bigger than zero). Then A is a fuzzy set in X. A membership function $\mu_A$ might be subjectively defined by, say

$$\mu_A(x) = \begin{cases} \dfrac{x^2}{x^2+1} & \text{if } x \geq 0 \\ 0 & \text{if } x < 0. \end{cases}$$

Then $\mu_A(0.5) = 0.2, \mu_A(1) = 0.5, \mu_A(2) = 0.8, \ldots$.
The membership function has some resemblance to a probability function when X is a countable set. But there are essential differences between these concepts.

We next define some notions which are analogous to those in the usual set theory.

1.1.4 DEFINITION [7];

Let $\mu$ and $\lambda$ be two fuzzy sets in X then,

(a) $\mu$ is called empty if $\mu = \chi_\phi$ i.e. if $\mu(x) = 0$ for all $x \in X$.

(b) $\mu$ is called subset of $\lambda$ if $\mu \leq \lambda$ i.e. if $\mu(x) \leq \lambda(x)$ for all $x \in X$.

(c) $\mu$ is equal to $\lambda$ if $\mu \equiv \lambda$ i.e. if $\mu(x) = \lambda(x)$ for all $x \in X$.

(d) The pseudo complement $\mu^c$ of $\mu$ is defined by

$\mu^c(x) = 1-\mu(x)$ for all $x \in X$.

(e) The union (or supremum) $\mu \vee \lambda$ is defined by

$(\mu \vee \lambda)(x) = \mu(x) \vee \lambda(x) = \max\{\mu(x), \lambda(x)\}$, $x \in X$.

(f) The intersection (or infimum) $\mu \wedge \lambda$ is defined by

$(\mu \wedge \lambda)(x) = \mu(x) \wedge \lambda(x) = \min\{\mu(x), \lambda(x)\}$, $x \in X$.

1.1.5 EXAMPLE [7];

Let $X = \{x_1, x_2, x_3, x_4\}$. Consider the fuzzy sets

$\mu = \{(x_1, 0.4), (x_2, 0.2), (x_3, 0), (x_4, 1)\}$.

$\lambda = \{(x_1, 0.3) (x_2, 0.1), (x_3, 0), (x_4, 0.5)\}$.

Then $\lambda \le \mu$ since $0.3 < 0.4$, $0.1 < 0.2$, $0 = 0$, $0.5 < 1$. By writing

$\chi_x = \{(x_1, 1), (x_2, 1), (x_3, 1), (x_4, 1)\}$, X is a subset of

itself in the sense of the theory of fuzzy sets and in this case

$$\chi_\phi < \lambda < \mu < \chi_x, \text{ where}$$

$$\chi_\phi = \{(x_1,0), (x_2,0), (x_3,0), (x_4,0)\}.$$

The next example is an illustration for pseudo-complement, union and intersection, this also shows that in general

$$\mu \vee \mu^c \neq \chi_x, \ \mu \wedge \mu^c \neq \chi_\phi$$

1.1.6 EXAMPLE [7];

Let $X = \{x_1, x_2, x_3, x_4\}$. Consider

$\mu = \{(x_1, 0.2), (x_2, 0.5), (x_3, 0), (x_4, 1)\}$. Then,

$\mu^c = \{(x_1, 0.8), (x_2, 0.5), (x_3, 1), (x_4, 0)\}$,

$\mu \vee \mu^c = \{(x_1, 0.8), (x_2, 0.5), (x_3, 1), (x_4, 1)\} \neq \chi_x$,

$\mu \wedge \mu^c = \{(x_1, 0.2), (x_2, 0.5), (x_3, 0), (x_4, 0)\} \neq \chi_\phi$.

1.1.7 THEOREM [7, Theorem 2.7];

Let $\lambda, \mu, \nu$ be fuzzy sets in X. Then

(i) $(\lambda \vee \mu)^c = \lambda^c \wedge \mu^c$.

(ii) $(\lambda \wedge \mu)^c = \lambda^c \vee \mu^c$.

(iii) $\nu \vee (\lambda \wedge \mu) = (\nu \vee \lambda) \wedge (\nu \vee \mu)$.

(iv) $\nu \wedge (\lambda \vee \mu) = (\nu \wedge \lambda) \vee (\nu \wedge \mu)$.

(v) $(\mu^c)^c = \mu$.

PROOF;

(i) For any $x \in X$,

$(\lambda \vee \mu)^c (x) = 1 - (\lambda \vee \mu)(x) = 1 - (\lambda(x) \vee \mu(x))$

$= (1-\lambda(x)) \wedge (1-\mu(x)) = \lambda^c(x) \wedge \mu^c(x)$.

Thus $(\lambda \vee \mu)^c = \lambda^c \wedge \mu^c$.

(ii) The proof is similar to the above part, it uses the relation

$1-(\lambda(x) \wedge \mu(x)) = (1-\lambda(x)) \vee (1-\mu(x))$.

(iii) This follows from the relation

$\nu(x) \vee (\lambda(x) \wedge \mu(x)) = (\nu(x) \vee \lambda(x)) \wedge ((\nu(x) \vee \mu(x))$.

(iv) This follows from the relation

$\nu(x) \wedge (\lambda(x) \vee \mu(x)) = (\nu(x) \wedge \lambda(x)) \vee ((\nu(x) \wedge \mu(x))$.

(v) This is trivial.

## 1.2 FUZZY LINEAR SPACES

Throughout this article E will denote a vector space over K, where Kisthe field of real or complex numbers.

1.2.1 DEFINITION [6];

Let $A_1, A_2, \ldots\ldots, A_n$ be fuzzy sets in E. We define $A_1 \times A_2 \times A_3 \times \ldots\ldots\ldots \times A_n$ to be the fuzzy set A in E, whose membership function is given by

$$\mu_A(x_1, x_2, \ldots\ldots, x_n) = \min\{\mu_{A_1}(x_1), \mu_{A_2}(x_2), \ldots\ldots, \mu_{A_n}(x_n)\}.$$

Let $\mu: E^n \longrightarrow E$ , $\mu(x_1, x_2, \ldots\ldots, x_n) = x_1 + x_2 + \ldots. + x_n$.

We define $A_1 + A_2 + \ldots\ldots + A_n = \mu(A)$. For $\lambda$ a scalar and B a fuzzy set in E, we define $\lambda(B) = g(B)$, where $g: E \longrightarrow E$, defined by $g(x) = \lambda x$.

Let T be a mapping from X to Y. Let B be a fuzzy set in Y with membership function $\mu_B$. The inverse mapping $T^{-1}$ induces a fuzzy set A whose membership function is defined by $\mu_A(x) = \mu_B(y) = \mu_B(Tx)$.

If A is a fuzzy set in X with membership function $\mu_A$, then the membership function for the fuzzy set B in Y which is induced by this mapping T is

$$\mu_B(y) = \max_{-1} \mu_A(x).$$

$x \in T^{-1}(y)$

If T is one-one then $T^{-1}(y)$ is singleton and

$\mu_B(y) = \mu_A(T^{-1}(y))$.

For a fuzzy set A of X, the set $\{x \in X : \mu_A(x) > 0\}$ is called support of A and is denoted by supp A or $A_0$.

1.2.2 LEMMA [6, Lemma 2.1];

For $\lambda \neq 0$ $\mu_{\lambda B}(x) = \mu_B((1/\lambda)x)$ for all $x \in E$. For $\lambda = 0$,

$$\mu_{\lambda B}(x) = \begin{cases} 0, & \text{when } x \neq 0 \\ \sup_y \mu_B(y), & \text{when } x = 0. \end{cases}$$

PROOF;

We know that

$\mu_{\lambda B}(x) = \mu_B(g^{-1}(x))$,

where $g(x) = \lambda x$,

$g^{-1}(x) = (1/\lambda)x$ if $\lambda \neq 0$, and $x \neq 0$.

$\therefore \mu_{\lambda B}(x) = \mu_B((1/\lambda)x)$.

If $x = 0$, then

$g(0) = \lambda.0 = 0$,

$g^{-1}(0) = 0$ and

$\mu_{\lambda B}(0) = \mu_B(0) = 0$.

If $\lambda = 0$ then

$g(x) = 0$ and

$g(y) = 0$ for all $y \in E$.

$$\therefore \mu_{\lambda B}(x) = \mu_B(g^{-1}(0)$$

$$= \mu_B(y)$$

$$= \sup_y \mu_B(y).$$

1.2.3 COROLLARY [6, Corollary 2.3];

$\lambda(A + B) = \lambda A + \lambda B$ for all fuzzy sets A, Bin E, and all scalars $\lambda$.

1.2.4 LEMMA [6 Lemma 2.4];

Let $A_1, \ldots\ldots, A_n$, $B_1, \ldots\ldots, B_m$ be fuzzy sets in E and put,

$A = A_1 + A_2 + \ldots\ldots + A_n$, $B = B_1 + B_2 + \ldots\ldots\ldots + B_m$,

$F = A_1 + A_2 + \ldots\ldots + A_n + B_1 + B_2 + \ldots\ldots + B_m$.

Then $F = A + B$.

An ordinary subset A of E can be considered as a fuzzy set with membership function equal to characteristic function of A. In this way we may consider the sums of the form A+B where one(or both) of A, B is an ordinary subset of E. For $x \in X$ and a fuzzy set, We define $x+B=\{x\}+B$. Let $f_x : E \longrightarrow E$, $f_x(y)=x+y$.

1.2.5 LEMMA [6, Lemma 2.5];

Let A be an Ordinary subset of E and B be a fuzzy set. Then

(i) $x+B=f_x(B)$;

(ii) $\mu_{x+B}(z) = \mu_B(z-x)$;

(iii) $A+B=\bigcup_{x\in B}(x+B)$.

1.2.6 PROPOSITION [6,Proposition2.6];

Let $A_1, A_2, \ldots, A_n$ be fuzzy sets in E and $\lambda_1, \lambda_2, \ldots, \lambda_n$ scalar. The following assertion are equivalent:

(i) $\lambda_1 A_1 + \lambda_2 A_2 + \ldots\ldots + \lambda_n A_n \subset A$.

(ii) For all $x_1, x_2, \ldots, x_n$ in E, and we have

$$\mu_A(\lambda_1 x_1 + \lambda_2 x_2 + \ldots + \lambda_n x_n) \geq \min\{\mu_{A_1}(x_1) \ldots \mu_{A_n}(x_n)\}.$$

1.2.7 LEMMA [6, LEmma 2.7];

Let A, B be fuzzy subsets of E. Then,

(i) $A + 0B \subset A$;

(ii) $A + 0B = A$ iff $\sup_{x\in E} \mu_A(x) \leq \sup_{x\in E} \mu_B(x)$.

1.2.8 DEFINITION [6];

A fuzzy set F in E is called a fuzzy subspace if,

(i) $F + F \subset F$,

(ii) $\lambda F \subset F$, for every scalar $\lambda$.

1.2.9 PROPOSITION [6, Proposition 3.1]:

Let F be a fuzzy set in E. Then,the following are equivalent:

(i) F is a subspace of E.

(ii) For all scalars k, m, we have $kF + mF \subset F$.

(iii) For all scalars k, m and all $x, y \in E$, we have

$$\mu_F(kx + my) \geq \min\{\mu_F(x), \mu_F(y)\}.$$

1.2.10 PROPOSITION [6,Proposition 3.3];

If A, B are fuzzy subspaces of E and k is a scalar, then A + B and kA are fuzzy subspaces.

1.2.11 DEFINITION [6];

A fuzzy set A in E is said to be:

(a) Convex if $kA + (1-k)A \subset A$ for all $k \in [0,1]$.

(b) Balanced if $kA \subset A$ for all scalars k with $|k| \leq 1$.

(c) Absorbing if $E = \bigcup_{k>0} kA$.

1.2.12 PROPOSITION [6, Proposition 4.1];

Let A be a fuzzy set in E. Then the following assertions are equivalent:

(i) A is convex.

(ii) $\mu_A(kx+(1-k)y) \geq \min\{\mu_A(x), \mu_A(y)\}$ for all $x,y \in X$ and all $k \in [0,1]$.

(iii) For each $d \in [0,1]$, the set $A_d = \{x \in X: \mu_A(x) \geq d\}$ is convex.

1.2.13 PROPOSITION [6, Proposition 4.1];

Let A be a fuzzy set in E. Then the following assertions are equivalent:

(i) A is balanced.

(ii) $\mu_A(kx) \geq \mu_A(x)$ for all k with $|k| \leq 1$.

(iii) $A_d = \{x \in X: \mu_A(x) \geq d\}$ for each $d \in [0,1]$ is balanced.

1.2.14 PROPOSITION [6, Proposition 4.4];

For a fuzzy set A of E, the following are equivalent:

(i) A is absorbing.

(ii) For each $x \in X$, $\sup_{k>0} \mu_A(kx) = 1$.

(iii) For each $d \in [0,1]$, the set $A_d = \{x \in E: \mu_A(x) \geq d\}$ is absorbing.

## 1.3 ALMOST LINEAR SPACES

An almost linear space is a nonempty set X together with two mappings s: $X \times X \longrightarrow X$ and $m: R \times X \longrightarrow X$ satisfying the conditions, given below: For $x,y \in X$ and $\lambda \in R$ we denote $s(x,y)$ by $x+y$ and $m(\lambda,x)$ by $\lambda x$. Let $x,y,z \in X$ and $\lambda, \mu \in R$,

$L_1$) $(x+y)+z = x+(y+z)$;

$L_2$) $x+y = y+x$;

$L_3$) there exists an element $0 \in X$ such that $x+0 = x$;

$L_4$) $1x = x$;

$L_5$) $\lambda(x+y) = \lambda x+\lambda y$;

$L_6$) $0x = 0$;

$L_7$) $\lambda(\mu x) = (\lambda\mu)x$;

$L_8$) $(\lambda+\mu)x = \lambda x+\mu x$ for $\lambda, \mu \geq 0$.

We denote $-1x$ by $-x$ when this will not lead to misunderstanding, and in the sequel $x-y$ means $x+(-y)$. A non empty set Y of an almost linear space X is called an almost linear subspace of X, if for each $y_1, y_2 \in Y$ and $\lambda \in R$, $y_1 + y_2 \in Y$ and $\lambda y_1 \in Y$. For an almost linear space X, the sets $V_X$ and $W_X$ are defined as:

$$V_X = \{x \in X : x - x = 0\},$$

$$W_X = \{x \in X : x = -x\}.$$

NOTE;

Throughout in section (1.3), and chapter 2, X will denote an almost linear space (als).

1.3.1 DEFINITION [2 ];

A subset B of X is called a basis of X if for each $x \in X\setminus\{0\}$ there exists unique sets $\{b_1, \dots, b_n\} \subset B$, $\{\lambda_1, \dots, \lambda_n\} \in R\setminus\{0\}$ (n depending on x) such that,

$$x = \sum_{i=1}^{n} \lambda_i b_i, \text{ where } \lambda_i > 0 \text{ for } b_i \notin V_X.$$

Clearly, if B is a basis of X then $0 \notin B$.

1.3.2 LEMMA [2, Lemma 2.2];

If the als X has a basis B, then the sets $\{\alpha_b b : b \in B, \alpha_b > 0 \text{ for } b \notin V_X\}$ are also basis of X.

1.3.3 LEMMA [2, Lemma 2.3];

Let X be an als with a basis and $x_1, x_2 \in X$. If $x_1 + x_2 \in$

$V_x$ then $x_i \in V_x$.

PROOF;

Suppose $x_1 + x_2 \in V_x$ and let $x_3 = -x_1$ and $x_4 = -x_2$. since X has a basis B, there exist $b_1, \ldots, b_n \in B$, $b_i \neq b_j$ for $i \neq j$, such that $x_i = \sum_{j=1}^{n} \alpha_{ij} b_j$, where $\alpha_{ij} > 0$ if $b_j \notin V_x$, $1 \leq i \leq 4$. By hypothesis we get that $\sum_{i=1}^{4} x_i = 0$ and so $\sum_{j=1}^{n} (\sum_{i=1}^{4} \alpha_{ij}) b_j = 0$.

Suppose $b_1 \notin V_x$. Then,

$b_1 = (1 + \sum_{i=1}^{4} \alpha_{i1}) b_1 + \sum_{j=2}^{n} (\sum_{i=1}^{4} \alpha_{ij}) b_j = 0$. Since $b_1 \in B$, it follows that $1 + \sum_{i=1}^{4} \alpha_{i1} = 1$. But $\alpha_{i1} \geq 0$, $1 \leq i \leq n$, and so $\alpha_{i1} = 0$, $1 \leq i \leq 4$. Consequently for each $b_j \notin V_x$, $1 \leq j \leq 1$, we get $\alpha_{ij} = 0$, $1 \leq i \leq 4$, which shows that $x_i \in V_x$, $1 \leq i \leq 4$.

1.3.4 LEMMA [2, Lemma 2.4];

Let X be an als with a basis B. Then $B \cap V_x$ is a basis of $V_x$.

PROOF;

Let $x \in V_x$, then $x \in X$. Since X has a basis B ,so for each $x \in X$ there exists unique $\{b_1, \ldots, b_n\} \subset B$ , $\{\lambda_1, \ldots\ldots, \lambda_n\} \subset R\backslash\{0\}$ such that,

$$x = \sum_{i=1}^{n} \lambda_i b_i, \text{ where } \lambda_i > 0 \text{ for } b_i \notin V_x.$$

So, $\sum_{i=1}^{n} \lambda_i b_i \in V_x$ , $\Rightarrow$ $\lambda_i b_i \in V_x$, by Lemma 1.3.3. Further

$1/\lambda_i (\lambda_i b_i) \in V_x$ (Because $V_x$ is a linear space).

$\Rightarrow$ $b_i \in V_x$ , so $b_i \in B \cap V_x$.

Hence $B \cap V_x$ is a basis of $V_x$.

1.3.5 LEMMA [2, Lemma 2.5];

Let X be an almost linear space. The set $B \subset X$ is a basis of X iff $B \cap V_X$ is a basis of $V_X$ and for each $x \in X\backslash V_x$ there exists unique $\{ b_1, \ldots, b_n\} \in B\backslash V_X$, $v \in V_X$ and $\lambda_1, \ldots\ldots, \lambda_n > 0$, such that

$$x = \sum_{i=1}^{n} \lambda_i b_i + v.$$

PROOF;

Suppose that $B \subset X$ is a basis of X . By Lemma 1.3.4 $B \cap V_x$ is a basis of $V_x$. Now suppose that $x \in X\backslash V_x$ and by definition of basis for each $x \in X$ there exists unique $\{b_1,$

$\ldots\ldots, b_n \} \subset B\backslash V_x$ , $\{\lambda_1, \ldots\ldots, \lambda_n\} \subset R\backslash\{0\}$ such that

$$x = \sum_{i=1}^{n} \lambda_i b_i \text{ ,where } \lambda_i > 0 \text{ for } b_i \notin V_x .$$

Now suppose for $i = 1, \ldots., m$, $b_i \notin V_x$, and $b_i \in V_x$ for $i = m+1, \ldots.., n$. For $b_i \in V_x$, we may have $\lambda_i < 0$.

$$x = \sum_{i=1}^{m} \lambda_i b_i + \sum_{i=1+m}^{n} \lambda_i b_i .$$

As $b_i \in V_x$, then $\lambda_i b_i$ and $\sum_{i=1+m}^{n} \lambda_i b_i \in V_x$ ( $\therefore V_x$ is a linear

spaces), put $v = \sum_{i=1+m}^{n} \lambda_i b_i$, then $x = \sum_{i=1}^{n} \lambda_i b_i + v$.

Conversely;

Suppose that $x \in X$, then either $x \in V_x$ or $x \notin V_x$.

If $x \in V_x$, then $B \cap V_x$ is a basis of $V_x$, by given condition. If $x \notin V_x$, then $x \in X\backslash V_x$ and again by given condition for each $x \in X\backslash V_x$ there exists unique $\{b_1, \ldots., b_n\} \subset B\backslash V_x$ $\{\lambda_1,$

$\ldots, \lambda_n\} \in R\setminus\{0\}$ such that

$$x = \sum_{i=1}^{n} \lambda_i b_i + v$$

As $v \in V_x$ so $v = \sum_{i=1}^{m} \lambda_i b_i$

Hence B is a basis of X.

1.3.6 LEMMA [2, Lemma 2.6];

Let B be a basis of the als X. Then for each $b \in B\setminus V_x$ there exist unique $\psi(b) \in B\setminus V_x$, $V(b) \in V_x$ and $\lambda(b) > 0$ such that $-b = \lambda(b)\ \psi(b) + V(b)$.

PROOF;

Let $b \in B\setminus V_x$. Then $-b \notin V_x$ and by Lemma 1.3.5 we get

$$-b = \sum_{i=1}^{k} \lambda_i b_i + V \qquad \text{(i)}$$

where $b_1, \ldots, b_k \in B\setminus V_x$, $k \geq 1$, $b_i \neq b_j$ for $i \neq j$, $v \in V_x$ and $\lambda_i > 0$, $1 \leq i \leq k$, are uniquely determined. Clearly the Lemma is proved if we show that $k = 1$. Let $e_1, \ldots, e_m \in B\setminus V_x$, $e_i \neq e_j$ for $i \neq j$, $v_i \in V_x$, $U_{ij} \geq 0$ $1 \leq i \leq k$, $1 \leq j \leq m$, s.t

$$-b_i = \sum_{j=1}^{m} U_{ij}\ e_j + v_i \qquad (1 \leq i \leq k) \qquad \text{(ii)}$$

Multiplying (i) by -1 and using (ii)we get

$$b = \sum_{j=1}^{m} (\sum_{i=1}^{k} \lambda_i u_{ij}) e_j + \sum_{i=1}^{k} \lambda_i v_i + v \qquad \text{(iii)}$$

Since $b \in B\backslash V_x$, there exists an index $j_0 \in \{1,\ldots,m\}$ - say $j_0 = 1$ such that $b = e_1$ and we must have

$$\sum_{i=1}^{k} \lambda_i u_{ij} = 0, \quad 2 \le j \le m.$$

Since $\lambda_i > 0$ and $u_{ij} \ge 0$ it follows that $u_{ij} = 0$ for each $1 \le i \le k$ and each $2 \le j \le m$ . Consequently , we get by (ii)

$$-b_i = u_{i1}\, e_1 + v_i \qquad (1 \le i \le k) \qquad \text{(iv)}$$

and $u_{ij} \ge 0$, since $-b_i \notin V_x$, $1 \le i \le k$. Suppose $k > 1$. By (iv) for i = 1, 2 we get that $e = ( -b_1 - v_1)/u_{11} = ( -b_2 - v_2)/u_{21}$ and so $b_1 = (u_{11}/u_{21}) b_2 + (( v_2/u_{21}) - (v_1/u_{11}))$, contradicting Lemma,1.3.5.

Let $\psi : B\backslash V_x \longrightarrow B\backslash V_x$ be defined as in Lemma 1.3.6.Then $\psi$ is well-defined and we have :

1.3.7 LEMMA [2, Lemma 2.7];

The mapping $\psi : B\backslash V_x \longrightarrow B\backslash V_x$ defined as above is injective and $\psi(\psi(b)) = b$ for each $b \in B\backslash V_x$.in particular $\psi$ is surjective.

PROOF;

Let $b_1, b_2 \in V_x$ such that $\psi(b_1) = \psi(b_2) = b \in B\backslash V_x$. Then $-b_i = \lambda_i b + v_i$, $\lambda_i > 0$, $v_i \in V_x$, $i = 1,2$, and similarly with the proof given at the end of Lemma 1.3.6 this contradicts Lemma 1.3.5,

Let now $b \in B\backslash V_x$. Then $-b = \lambda\ \psi(b) + v$, where $\lambda > 0$, $v \in V_x$ and $\psi(b) \in B\backslash V_x$ are given by Lemma 1.3.6. Then $-\ \psi(b) = (b/\lambda) + (v/\lambda)$, and so, again by Lemma 1.3.6 we get $\psi(\psi(b)) = b$.

1.3.8 THEOREM [2, Lemma 2.8];

Let B be a basis of the als X. Then there exiists a basis $B'$ of X with the property that for each $b' \in B' \setminus V_x$, we have $-b' \in B' \setminus V_x$.

# CHAPTER TWO
# FUZZY ALMOST LINEAR SPACES

Godini [2] gave the notion of almost linear spaces. The aim of this chapter is to define and study the fuzzy almost linear spaces. Section 2.1 contains sums and products of fuzzy sets in almost linear spaces. Section 2.2 deals with the almost fuzzy subspaces. Section 2.3 contains convex, balanced, absorbing fuzzy sets, in almost linear spaces. Section 2.4 includes the definition of fuzzy basis and its relevant results. Throughout this chapter $F$ will denote a fuzzy subset of X.

## 2.1 SUMS AND PRODUCTS OF FUZZY SETS IN ALMOST LINEAR SPACES

2.1.1 DEFINITION;

Let $A_1, \ldots, A_n$ be fuzzy sets in $X^n$. We define $A_1 x A_2 x \ldots x A_n$ to be the fuzzy set A in X whose membership function is given by

$$\mu_A(x_1, \ldots, x_n) = \min\{\mu_{A_1}(x_1), \ldots, \mu_{A_n}(x_n)\}.$$

Let f: $X^n \longrightarrow X$, be defined by $f(x_1, \ldots, x_n) =$

$x_1 + \ldots + x_n$.

Define $A_1 + \ldots + A_n = f(A)$. If A and B are fuzzy sets then the membership function of A+B is given by

$$\mu_{A+B}(x) = \sup_{x=a+b} \{\min\{\mu_A(a), \mu_B(b)\}\}.$$

For $\lambda$ a scalar and B a fuzzy set in X, define $\lambda B = g(B)$ where $g: X \longrightarrow X$, $g(x) = \lambda x$ and

$$\mu_{\lambda B}(x) = \begin{cases} \mu_B(g^{-1}(x)) = \mu_B((1/\lambda)x) & \text{if } \lambda \neq 0 \text{ and all } x \in E \\ 0 & \text{if } \lambda = 0 \text{ and } x \neq 0, \\ \sup_y \mu_B(y) & \text{if } \lambda = 0 \text{ and } x = 0. \end{cases}$$

Let $f_x: X \longrightarrow X$ be defined by $f_x(y) = x + y$.

2.1.2 LEMMA;

Let A be any set and B a fuzzy set in X. Then for $x \in X$,

(i) $x + B = f_x(B)$,

(ii) $\mu_{x+B}(a) = \begin{cases} \sup_{a=x+y} \mu_B(y) , \\ 0 & \text{if } a \neq x + y, \end{cases}$

(iii) $A + B = \bigcup_{x \in A} x + B$ .

PROOF;

(i).As $x + B = \{x\} + B$, therefore by definition of sum of two fuzzy sets

$$\mu_{x+B}(a) = \sup_{a=z+y} \{\min\{\mu_{\{x\}}(z),\ \mu_B(y)\}\}$$

$$= \begin{cases} \sup\limits_{a=z+y} \{\mu_B(y)\} & \text{if } \mu_{\{x\}}(z)=1, \\ 0 & \text{if } \mu_{\{x\}}(z)=0, \text{and } a \neq x+y \text{ for } y \in X \end{cases}$$

On the other hand

$$\mu_{f_x(B)}(a) = \sup_{y \in f_x^{-1}(B)} \mu_B(y)$$

$$= \begin{cases} \sup\limits_{a=x+y} \mu_B(y) & \text{if } a = x+y \text{ for some } y \in X, \\ 0 & \text{if } a \neq x+y \text{ for any } y \in X. \end{cases}$$

Thus $\{x\} + B = f_x(B)$.

(ii) It follows from the proof of (i).

(iii) Since $\mu_{A+B}(x) = \sup\limits_{x=a+b} \{\min\{\mu_A(a),\ \mu_B(b)\}\}$

$$= \begin{cases} \sup\limits_{x=a+b} \mu_B(b) \\ 0 \end{cases}$$

Hence $A + B = \bigcup\limits_{x \in A} x + B$ .

2.1.3 PROPOSITION;

Let $A$, $A_1$, ..., $A_n$ be fuzzy sets in $X$ and $\lambda_1$, $\lambda_2$,...

..., $\lambda_n \in R$.

Then the following are equivalent:

(i) $\lambda_1 A_1 + \ldots + \lambda_n A_n \subset A$.

(ii) For all $x_1, \ldots, x_n \in X$ we have

$$\mu_A(\lambda_1 x_1 + \ldots + \lambda_n x_n) \geq \min\{\mu_{A_1}(x_1), \ldots, \mu_{A_n}(x_n)\}.$$

PROOF;

(i) $\Rightarrow$ (ii):

$$\mu_A(\lambda_1 A_1 + \lambda_2 A_2 + \ldots + \lambda_n A_n) \geq \mu_{\lambda_1 A_1 + \ldots + \lambda_n A_n}(\lambda_1 x_1 + \ldots + \lambda_n x_n)$$

$$\geq \min\{\mu_{\lambda_1 A_1}(\lambda_1 x_1, \ldots, \mu_{\lambda_n A_n}(\lambda_n x_n)\}$$

$$\geq \min\{\mu_{A_1}(x_1), \ldots, \mu_{A_n}(x_n).$$

Now (ii) $\Rightarrow$ (i):

By rearranging the order if necessary, we may assume that $\lambda_i \neq 0$ for $i = 1, 2, \ldots, k$, and $\lambda_i = 0$ for $k < i \leq n$. Let $x_1, x_2, \ldots, x_n$ be elements of E. For all $y_1, y_2, \ldots, y_{n-k}$ in E we have

$$\mu_A(\lambda_1 x_1 + \ldots + \lambda_n x_n) \geq \{\mu_{A_1}(x_1), \ldots, \mu_{A_k}(x_k), \mu_{A_1}(y_1), \ldots, \mu_A(y_{n-k})\}.$$

n

Since $\mu_{0A_i}(0) = \sup_{y\in E} \mu_{A_i}(y)$, we get

$$\mu_A(\lambda_1 x_1+\ldots+\lambda_n x_n) \geq \min\{\mu_{A_1}(x_1), \ldots , \mu_{A_k}(x_k), \ldots..$$

$$\mu_{0A_{k+1}}(0), \ldots\ldots, \mu_{0A_n}(0)\}.$$

Now,

$$\mu_{\lambda_1 A_1+\ldots.+\lambda_n A_n}(z)$$

$$= \sup_{x_1+x_2++x_k=z} [\min\{\mu_{\lambda_1 A_1}(x_1), \ldots\ldots, \mu_{\lambda_n A_n}(x_n)\}]$$

$$= \sup_{x_1+\ldots.x_k=z} [\min\{\mu_{\lambda_1 A_1}(x_1), \ldots\ldots\ldots, \mu_{\lambda_k A_k}(x_k),$$

$$\mu_{0A_{k+1}}(0), \ldots.., \mu_{0A_n}(0)\}]$$

$$= \sup_{x_1+\ldots+x_k= z} [\min\{\mu_{A_1}((1/\lambda_1)x_1), \ldots,$$

$$\mu_{A_k}((1/\lambda_k)x_k), \mu_{0A_{k+1}}(0), \ldots, \mu_{0A_n}(0)\}]$$

$$\leq \sup_{x_1+\ldots+x_k= z} \mu_{A_1}(\lambda_1(1/\lambda_1)x_1+\ldots.$$

$$+\lambda_k(1/\lambda_k)x_k) = \mu_A(z)$$

2.1.4 LEMMA;

Let A and B be two fuzzy sets in X. Then

(i) $A + 0B \subset A$;

(ii) $A + 0B = A$ iff $\sup_{x \in X} \mu_A(x) \leq \sup_{x \in X} \mu_B(x)$.

PROOF;

(i) $\mu_A(x + 0Y) = \mu_A(x) \geq \min\{\mu_A(x), \mu_B(y)\}$. Hence (i) follows from proposition 2.1.3.

(ii) Suppose that $\sup \mu_A(x) \leq \sup \mu_B(x) = \mu_{0B}(0)$.

Then $\mu_{A+0B}(z) = \sup_{x+y=z} [\min\{\mu_A(x), \mu_{0B}(y)\}] =$

$\min\{\mu_A(z), \mu_{0B}(0)\} = \mu_A(z)$.

On the other hand , if $\mu_A(z) > \sup \mu_B(x) =$

$\mu_{0B}(0)$ for some z, then

$\mu_{A+0B}(z) = \min\{\mu_A(z), \mu_{0B}(0)\} < \mu_A(z)$,

and hence $A + 0B \neq A$.

## 2.2 ALMOST FUZZY SUBSPACES

2.2.1 DEFINITION;

A nonempty fuzzy set F in X is called a fuzzy almost linear subspace of X if

(i) $F + F \subset F$,

(ii) $\lambda F \subset F$ for all $\lambda \in R$.

2.2.2 PROPOSITION;

Let F be a fuzzy set in X. Then the following are equivalent:

(i) F is a fuzzy almost linear subspace of X.

(ii) For all $k, m \in R$, $kF + mF \subset F$.

(iii) For all $k, m \in R$ and $x, y \in X$,

$$\mu_F(kx + my) \geq \min\{\mu_F(x), F_F(y)\}.$$

PROOF;

(i) $\Rightarrow$ (ii): $\mu_{kF+mF}(x) = \sup_{x=a+b} \{\min\{\mu_{kF}(a), \mu_{mF}(b)\}\}$

$$\leq \sup_{x=a+b} \{\min\{\mu_F(a), \mu_F(b)\}\}$$

$$= \mu_{F+F}(x) \leq \mu_F(x) .$$

Therefore $kF + mF \subseteq F$.

(ii) $\Rightarrow$ (i): Take $k = 1 = m$, then $F + F \subset F$. Also if $m = 0$, then by Lemma 2.1.4, $kF = kF + oF \subset F$.

(ii) ⇔ (iii): Follows from Proposition 2.1.3.

2.2.3 LEMMA;

If A is an almost linear subspace of X then $\mu_A$ (characteristic function of A) is a fuzzy almost linear subspace of X.

PROOF;

Let x, y $\in$ X and k, m $\in$ R. If kx + my $\in$ A, then

$$\mu_A(kx + my) = 1 \geq \min\{\mu_A(x), \mu_A(y)\} .$$

If kx + my $\notin$ A then either x or y does not belong to A and so $\min\{\mu_A(x), \mu_A(y)\} = 0$. Therefore

$$\mu_A(kx + my) \geq \min\{\mu_A(x), \mu_A(y)\} .$$

Thus by Proposition **2.2.2**, A is a fuzzy almost linear subspace of X.

2.2.4 LEMMA;

If F is a fuzzy almost linear subspace of X, then the following sets are almost linear subspaces of X:

(i) $F_1 = \{x \in X: \mu_F(x) = \mu_F(0)\}$.

(ii) $F_o = \{x \in X: \mu_F(x) > 0\}$.

PROOF;

(i).Obviously, $F_1$ is nonempty. If $x, y \in F_1$ then $\mu_F(x) = \mu_F(y) = \mu_F(o)$ and $\mu_F(x+y) \geq \min\{\mu_F(x), \mu_F(y)\}$ $= \mu_F(o)$.

But $\mu_F(o) \geq \mu_F(x)$ for all $x \in X$. Therefore $\mu_F(x+y) = \mu_F(o)$. Thus $x+y \in F_1$. Also $\mu_F(\lambda x) \geq \mu_F(x) = \mu_F(o)$. It implies that $\mu_F(\lambda x) = \mu_F(o)$. Thus $\lambda x \in F_1$. Hence $F_1$ is an almost linear subspace of X.

(ii).Obviously $F_o$ is nonempty. Let $x, y \in F_o$ then $\mu_F(x) > 0$ and $\mu_F(y) > 0$. Thus $\mu_F(x+y) \geq \min\{\mu_F(x), \mu_F(y)\}$ $> 0$.

Therefore $x+y \in F_o$. Also $\mu_F(\lambda x) \geq \mu_F(x) > o$. Thus $\lambda x \in F_o$.

Hence $F_o$ is an almost linear subspace of X.

2.2.5 PROPOSITION;

If A and B are fuzzy almost linear subspaces of X and k is a non zero real number, then A+B and kA are fuzzy almost linear subspaces.

PROOF;

Let x, y $\in$ X, then

$$\mu_{A+B}(x) = \sup_{x=a+b} \{\mu_A(a) \wedge \mu_B(b)\},$$

and

$$\mu_{A+B}(y) = \sup_{y=c+d} \{\mu_A(c) \wedge \mu_B(d)\} .$$

It further implies that,

$$\mu_{A+B}(x) \wedge \mu_{A+B}(y) = \inf\{ \sup_{x=a+b} \{\mu_A(a) \wedge \mu_B(b)\} , \sup_{y=c+d} \{\mu_A(c) \wedge \mu_B(d)\}\}.$$

$$= \sup_{x=a+b} \sup_{y=c+d} \{\mu_A(a) \wedge \mu_B(b)\} \wedge \{\mu_A(c) \wedge \mu_B(d)\}\}.$$

$$\leq \sup_{x+y=e+f} \{\mu_A(e) \wedge \mu_B(f)\}.$$

$$= \mu_{A+B}(x+y) \ . \tag{1}$$

Also,

$$\mu_{A+B}(x) = \sup_{x=a+b} \{\mu_A(a) \wedge \mu_B(b)\}$$

$$\leq \sup_{x=a+b} \{\mu_A(ka) \wedge \mu_B(kb)\}$$

$$\leq \sup_{kx=ka+kb} \{\mu_A(ka) \wedge \mu_B(kb)\}$$

$$= \mu_{A+B}(kx) \ . \tag{2}$$

From (1) and (2)

$$\mu_A(kx+my) \geq \min\{\mu_{A+B}(x),\ \mu_{A+B}(y)\} \ .$$

Thus A+B is a fuzzy almost linear subspace of X.

Now we show that kA is a fuzzy almost linear subspace of X. Let $x, y \in X$ and $m, n \in R$, then

$$\mu_{kA}(mx+ny) = \mu_A(\frac{1}{k}\ (mx+ny))$$

$$= \mu_A(\frac{m}{k}\ x + \frac{n}{k}\ y)$$

$$\geq \min\{\mu_A(\frac{x}{k}),\ \mu_A(\frac{y}{k})\}$$

$$= \min\{\mu_{kA}(x),\ \mu_{kA}(y)\}.$$

Thus by Proposition 2.2.2 kA is a fuzzy almost linear subspace of X.

## 2.3 CONVEX, BALANCED AND ABSORBING FUZZY SETS IN ALMOST LINEAR SUBSPACES

### 2.3.1 DEFINITION;

A fuzzy set A in X is said to be:

(a) Convex if $kA + (1-k)A \subset A$ for all $k \in [0,1]$.

(b) Balanced if $kA \subset A$ for all scalars k with $|k| \leq 1$

(c) Absorbing if $X = \bigcup_{k>o} kA$.

### 2.3.2 PROPOSITION;

Let A be a fuzzy set in X. Then the following assertions are equivalent:

(i) A is convex.

(ii) $\mu_A(kx+(1-k)y) \geq \min\{\mu_A(x), \mu_A(y)\}$ for all $x, y \in X$ and all $k \in [0,1]$.

(iii). For each $d \in [0,1]$, the set $A_d = \{x \in X: \mu_A(x) \geq d\}$

is convex.

PROOF;

(i) $\Rightarrow$ (ii): From the definition of convex fuzzy set, $kA+(1-k)A \subset A$ for all $k \in [0,1]$. Proposition 2.1.3 further implies

$$\mu_A(kx+(1-k)y) \geq \min\{\mu_A(x), \mu_A(y)\} .$$

(ii) $\Rightarrow$ (i): Follows from Proposition 2.1.3.

(ii) $\Rightarrow$ (iii): Let $x, y \in A_d$ then $\mu_A(x)$ and $\mu_A(y) \geq d$ . By hypothesis $\mu_A(kx+(1-k)y) \geq \min\{\mu_A(x), \mu_A(y)\} \geq d$. Therefore

$$kx + (1-k)y \in A_d.$$

Hence $A_d$ is a convex set.

(iii) $\Rightarrow$ (ii): Let $x, y \in X$ and $\mu_A(x) \leq \mu_A(y)$. Take $\mu_A(x) = d$ then $x, y \in A_d$ and so by (iii) $kx+(1-k)y \in A_d$. Thus $\mu_A(kx+(1-k)y) \geq d$.

Hence $\mu_A(kx+(1-k)y) \geq \min\{\mu_A(x), \mu_A(y)\}$.

2.3.3 PROPOSITION;

Let A be a fuzzy set in X. Then the following assertions are equivalent:

(i). A is balanced.

(ii). $\mu_A(kx) \geq \mu_A(x)$ for all k with $|k| \leq 1$.

(iii). $A_d = \{x \in X: \mu_A(x) \geq d\}$ for each $d \in [0,1]$ is balanced.

PROOF;

(i) $\Rightarrow$ (ii): By the definition of balanced fuzzy set $kA \subset A$. Therefore $\mu_A(kx) \geq \mu_{kA}(kx)$ for all $x \in X$ and scalars k with $|k| \leq 1$. Hence $\mu_A(kx) \geq \mu_A(\frac{1}{k}(kx)) = \mu_A(x)$.

(ii) $\Rightarrow$ (i): By hypothesis, $\mu_A(kx) \geq \mu_A(x)$ for all k with $|k| \leq 1$. Therefore $\mu_A(x) \geq \mu_A((1/k)x) = \mu_{kA}(x)$. Hence $kA \subset A$ for all scalars k with $|k| \leq 1$.

(ii) $\Rightarrow$ (iii): Let $x \in A_d$ then $\mu_A(x) \geq d$. By hypothesis, $\mu_A(kx) \geq \mu_A(x)$. Therefore $\mu_A(kx) \geq d$ for $|k| \leq 1$. Thus $kx \in A_d$ for $|k| \leq 1$. Hence $A_d$ is balanced.

(iii) $\Rightarrow$ (ii): Let $x \in X$ and take $d = \mu_A(x)$ then $A_d$ is balanced and $x \in A_d$. Therefore $kx \in A_d$ for all scalar k with $|k| \leq 1$. Thus $\mu_A(kx) \geq d$. Hence $\mu_A(kx) \geq \mu_A(x)$.

2.3.4 PROPOSITION;

For a fuzzy set A of X, the following are equivalent:

(i) A is absorbing.

(ii) For each $x \in X$, $\sup_{k>o} \mu_A(kx) = 1$.

(iii) For each $d \in [0,1]$, the set $A_d = \{x \in X: \mu_A(x) \geq d\}$ is absorbing.

PROOF;

(i) $\Rightarrow$ (ii): Let A be an absorbing fuzzy set, then by definition $\bigcup_{k>o} kA = X$. Therefore for each $x \in X$

$$\mu_{\bigcup_{k>o} kA}(x) = \mu_X(x).$$

Thus $\sup_{k>o} \mu_{kA}(x) = \mu_X(x) = 1$.In particular $\sup_{k>o} \mu_{kA}(k^2x) = 1$.

Hence $\sup_{k>o} \mu_A(kx) = 1$.

(ii) $\Rightarrow$ (i): By hypothesis for each $x \in X$, $\sup_{k>o} \mu_A(kx) = 1$

Therefore $\sup_{k>o} \mu_A((1/k)x) = 1$. Thus $\sup_{k>o} \mu_{kA}(x) = 1$.

Hence $\bigcup_{k>o} kA = X$ .

(ii) $\Rightarrow$ (iii): Clearly $\bigcup_{k>o} kA_d \subseteq X$. Let $x \in X$ then $\sup_{k>o}$

$\mu_A(kx) = 1$. Therefore there exists $k_o > 0$ such that $\mu_A((1/k_o)x) = 1$. It further implies that $\frac{1}{k_o} x \in A_d$. Thus $x \in k_o A_d$. Hence $X = \bigcup_{k>o} kA_d$.

(iii) $\Rightarrow$ (ii): Since $A_d$ is absorbing therefore $\bigcup_{k>o} kA_d =$ X. If $d = 1$, then $\bigcup_{k>o} kA_1 = X$. Let $x \in X$ then $x \in \bigcup_{k>o} kA_1$. Thus there exists $k_o > 0$ such that $x \in k_o A_1$. It follows that $x = k_o y$ where $y \in A_1$. Therefore $\mu_A(\frac{1}{k_o} x) = \mu_A(y) = 1$. Hence $\sup_{k>o} \mu_A(kx) = 1$.

## 2.4 FUZZY BASIS

### 2.4.1 DEFINITION;

Let F be a fuzzy almost linear subspace of X. A fuzzy subset B of X is said to be a fuzzy basis of F if for each $0 \neq x \in X$, with $\mu_F(x) > 0$ there exist unique sets $\{b_1, b_2, \ldots, b_n\} \subset X$, $\{\lambda_1, \ldots, \lambda_n\} \subset R\setminus\{0\}$ ($b_i$ depending on x) with $\mu_B(b_i) > 0$ such that

$$x = \sum_{i=1}^{n} \lambda_i b_i \qquad \text{where } \lambda_j > 0 \text{ for } b_j \notin V_X .$$

Clearly if B is a fuzzy basis of F then $B_o$ is a basis for $F_o$. Conversely, if for some fuzzy set B, $B_o$ is a basis for

$F_o$ then B is a fuzzy basis for F.

2.4.2 LEMMA;

If a fuzzy set B is a fuzzy basis for a fuzzy almost linear subspace F of X then aB is also a fuzzy basis for F, where a > 0.

PROOF;

Let $0 \neq x \in X$ such that $\mu_F(x) > 0$ then there exists $\{\lambda_1, \lambda_2, \ldots, \lambda_n\} \subseteq R\backslash\{0\}$ and $\{b_1, b_2, \ldots, b_n\} \subset X$ with $\mu_B(b_i) > 0$ such that $x = \sum_{i=1}^{n} \lambda_i b_i$ and $\lambda_j > 0$ for $b_j \notin V_X$. Take $C_i = ab_i$. If $b_j \notin V_X$ then $ab_j = C_j \notin V_X$. Also $\mu_{aB}(C_i) = \mu_B(\frac{1}{a} C_i) = \mu_B(b_i) > 0$. Thus $x = \sum_{i=1}^{n} \lambda_i b_i = \sum_{i=1}^{n} \lambda_i (\frac{1}{a} C_i) = \sum_{i=1}^{n} \frac{\lambda_i}{a} C_i = \sum_{i=1}^{n} \eta_i C_i$ where $\eta_i = \lambda_i/a$. Hence, there exists $\{\eta_1, \ldots, \eta_n\} \subseteq R\backslash\{0\}$ and $\{C_1, C_2, \ldots, C_n\} \subset X$ with $\mu_{aB}(C_i) > 0$ and

$x = \sum_{i=1}^{n} \eta_i C_i$ and $\eta_j > 0$ for $C_j \notin V_X$.

2.4.3 PROPOSITION;

Let B be a fuzzy set and F be a fuzzy almost linear subspace of X. If B is a fuzzy basis of F and $\mu_F(x) \leq \min\{\mu_B(b_i)\}_{i=1}^{n}$ where $b_i$ are elements which satisfy $x = \sum_{i=1}^{n} \lambda_i b_i$. Then $B_d$ is a basis for $F_d$ where $0 < d \leq \mu_F(0)$.

PROOF;

If $0 < d \leq \mu_F(0)$ then $F_d$ is an almost linear subspace of X. Suppose $x \in F_d$ then $\mu_F(x) \geq d$. Then there exist $\{\lambda_1, \ldots, \lambda_n\}$ and $\{b_1, \ldots, b_n\}$ with $\mu_B(b_i) > 0$ such that $x = \sum_{i=1}^{n} \lambda_i b_i$ where $\lambda_j > 0$ for $b_j \not\in V_X$. As $\mu_F(x) \leq \min\{\mu_B(b_i)\}_{i=1}^{n}$, therefore $\mu_B(b_i) \geq d$. Hence $b_i \in B_d$. Thus $B_d$ is a basis for $F_d$.

Let F be a fuzzy almost linear subspace of X. Define $V_{F_o} = V_X \wedge F_o = \{x \in F_o : x-x = 0\}$. Then $V_{F_o}$ is an almost linear subspace of $F_o$.

2.4.4 LEMMA;

If a fuzzy set B of X is a fuzzy basis for a fuzzy almost linear subspace F of X then $B \cap V_{F_o}$ is a fuzzy basis for $V_{F_o}$.

PROOF;

If B is a fuzzy basis for F then $B_o$ is a basis for $F_o$. Hence by Lemma 1.3.4 $B_o \cap V_{F_o}$ is a basis for $V_{F_o}$. Thus B $\cap V_{F_o}$ is a fuzzy basis for $V_{F_o}$ (considered as a fuzzy almost linear subspace).

2.4.5 LEMMA;

Let B be a fuzzy set of X and F be a fuzzy almost linear subspace of X. Then B is a fuzzy basis for F if and only if $B_o \cap V_{F_o}$ is a basis for $V_{F_o}$ and for each x $\notin V_{F_o}$ and $\mu_F(x) > 0$, there exist $b_1, \ldots, b_n \in B_o \backslash V_{F_o}$, $\lambda_i > 0$, $v \in V_{F_o}$ and $\lambda_1, \ldots, \lambda_n$ such that $x = \sum_{i=1}^{n} \lambda_i b_i + v$.

PROOF;

The fuzzy set B is a fuzzy basis for F if and only if $B_o$ is a basis for $F_o$. Thus by Lemma 1.3.4 the Lemma follows.

2.4.6 LEMMA;

Let B be a fuzzy basis of F.Then for each $b \notin V_{F_o}$, there exists unique $\psi(b) \notin V_{F_o}$, $v(b) \in V_{F_o}$ and $\lambda(b) > 0$

such that $-b = \lambda(b)\ \psi(b) + v(b)$.

PROOF;

This is a consequence of Lemma 2.4.5.

2.4.7 THEOREM;

Let B be a fuzzy basis of F, then there exists a fuzzy basis B′ of F such that for each $b' \in B'_o \backslash V_{F_o}$, $-b' \in B'_o \backslash V_{F_o}$.

PROOF;

If B is a fuzzy basis of F then $B_o$ is a basis of $F_o$. By Theorem 1.3.8 there exists a basis $B'_o$ of X with the property that for each $b' \in B'_o \backslash V_{F_o}$, we have $-b' \in B'_O \backslash V_{F_O}$ Hence the fuzzy set B′ is a basis of F such that for each $b' \in B'_o \backslash V_{F_o}$ we have $-b' \in b'_o \backslash V_{F_o}$.