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RADICAL THEORY FOR HEMIRINGS

by

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PREFACE

We discuss radical classes for hemirings and it is infact a generalization of radical classes of a ring.

The first chapter contains some definitions and basic results that are needed for the subsequent development of the subject. Generalization of isomorphism theorems is also given in this chapter.

The second chapter begins with the definition of a radical class. Some equivalent definitions for radical classes have been given. It is shown that every radical class satisfies the extension property. Semisimple classes for hemirings are defined. Construction of semisimple class from a regular class corresponding to a radical class is also defined. - iven a regular class M, an upper radical class is defined which is denoted by $\mathcal{U}M$. It is shown that every semisimple class is the semisimple class of its upper radical class i.e., $S = S(\mathcal{U}S)$ and every radical class P is the upper radical class of the semisimple class SP. Accessible sub-semirings are defined. Two equivalent definitions of semisimple classes have been given. It is proved that if I is a k-ideal of a hemiring A such that A/I and I are in S, then $A \in S$. The second chapter concludes with the definition of a lower radical class $\mathcal{L}$L and Yu-Lee Lee's radical class YL. It is proved that YL = $\mathcal{L}$L.

The third chapter contains some examples of radical classes.

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CHAPTER-1

DEFINITIONS AND BASIC CONCEPTS

Definition: 1.1

An ordered triple $(S,+,\cdot)$ consisting of a non-empty set S, two binary operations '+' and '·' defined on S is called a semiring provided

i) $(S,+)$ is a semigroup,

ii) $(S,\cdot)$ is a semigroup,

iii) '·' is both left and right distributive over '+' i.e., $a\cdot(b+c) = a\cdot b + a\cdot c \quad a,b,c \in S$ and $(b+c)\cdot a = b\cdot a + c\cdot a \quad a,b,c \in S$

Definition: 1.2

A semiring $(S,+,\cdot)$ is called a hemiring if

i) '+' is commutative,

ii) There exist $0 \in S$ such that 0 is the identity of $(S,+)$ and 0 is the zero element of $(S,\cdot)$, i.e., $0s = s0 = 0$ for all $s \in S$

We shall refer to 0 as the zero element of the hemiring $(S,+,\cdot)$.

Definition: 1.3

A non-empty subset A of a semiring S is called a sub semiring of the semiring S if

$$x,y \in A \Rightarrow x + y \in A, \quad xy \in A$$

In case S is a hemiring, a subsemiring of S containing 0 is referred to as a sub hemiring of S.

Definition: 1.4

A non-empty subset I of a semiring S is called a left semi-ideal iff

i) $a,b \in I \Rightarrow a + b \in I$,

ii) $a \in I$ and $s \in S \Rightarrow sa \in I$.

If we replace ii) by ii)` $a \in I$ and $s \in S \Rightarrow as \in I$, then I is called a right semi-ideal of S. If we require both ii) and ii)` then I is called a two-sided semi-ideal or simply a semi-ideal.

Lemma: 1.5

If I and J are semi-ideals of a hemiring S then I + J is a semi-ideal of S generated by I U J.

Proof

Let $i_1 + j_1, i_2 + j_2 \in I + J$, then

$$\begin{aligned} \text{i) } (i_1 + j_1) + (i_2 + j_2) &= i_1 + j_1 + i_2 + j_2 \\ &= i_1 + i_2 + j_1 + j_2 \\ &= (i_1 + i_2) + (j_1 + j_2) \in I + J \end{aligned}$$

ii) Let $i + j \in I + J$ and $s \in S$, then

$s(i + j) = si + sj \in I + J$

$(i + j)s = is + js \in I + J$

Since $0 \in J$, $i + 0 = i \in I + J$. Thus $I \subseteq I + J$. Similarly $J \subseteq I + J$. Hence $I \cup J \subseteq I + J$.

Now if K is any other semi-ideal containing $I \cup J$ then $i + j \in K$ for all $i \in I$ and $j \in J$. Thus $I + J \subseteq K$. Hence $I + J$ is a semi-ideal of S generated by $I \cup J$.

Definition: 1.6

A left [right, two-sided] semi-ideal of a hemiring S is called a left [right, two-sided] k-ideal]if $a \in I$, $s \in S$ and $a + s \in I$ imply $s \in I$.

A k-subhemiring is similarly defined.

Lemma: 1.7

If I is any arbitrary semi-ideal of a hemiring S, then $I^* = \{x \in S : x + a \in I$ for some $a \in I\}$ is the k-ideal generated by I.

Proof

Let $x, y \in I^*$. Then $x + a_1,\ y + a_2 \in I$ for some $a_1, a_2 \in I$. Hence $(x + a_1) + (y + a_2) \in I \Rightarrow x + y + a_1 + a_2 \in I$. As $a_1, a_2 \in I$, $a_1 + a_2 \in I$. Thus $x + y \in I^*$.

Let $s \in S$ and $x \in I^*$. Then $x + a \in I$ for some $a \in I$. Hence $s(x + a) \in I \Rightarrow sx + sa \in I$. As $sa \in I$, we have $sx \in I^*$. Similarly $xs \in I^*$. Thus I^* is a two-sided semi-ideal. Now if $a \in I$ then $a + a \in I$. Hence $a \in I^*$. Thus I^* contains I. Let $y \in S$, $x \in I^*$ and $y + x \in I^*$. Then $x + a_1$ and $y + x + a_2 \in I$ for some $a_1, a_2 \in I$. Therefore, $y + x + a_2 + a_1 \in I \Rightarrow y + x + a_1 + a_2 \in I$. As $x + a_1, a_2 \in I \Rightarrow x + a_1 + a_2 \in I$. Hence $y \in I^*$. Thus I^* is a k-ideal. Let K be any k-ideal containing I. If $x \in I^*$. Then $x + a \in I$ for some $a \in I \Rightarrow x + a \in K \Rightarrow x \in K \Rightarrow I^* \subseteq K$. Thus I^* is the smallest k-ideal containing I.

Definition: 1.8

A mapping ϕ of a semiring $(S,+,\cdot)$ into a semiring $(T,\oplus,\odot)$ is called a homomorphism provided that for all $a,b \in S$ we have

$$\phi(a + b) = \phi(a) \oplus \phi(b)$$

$$\phi(ab) = \phi(a) \odot \phi(b)$$

Definition: 1.9

Let ϕ be a homomorphism of a hemiring S onto a hemiring T. The set of all elements of S which are mapped onto the zero element of T is called the kernel of ϕ. Clearly $\phi(0) = 0$. If ker $\phi = \{0\}$, ϕ is called semi-isomorphism from S onto T. Symbolically, it is written

$$S \xrightarrow{\simeq} T.$$

Theorem: 1.10

Let ϕ be a homomorphism of a hemiring S onto a hemiring T. If I is a left [right, two-sided] semi-ideal of S, then $\phi(I)$ is a left [right, two-sided] semi-ideal of T.

Proof

Let I be a left semi-ideal of S. Let $a_1, a_2 \in \phi(I)$. Then there exist $i_1, i_2 \in I$ such that $\phi(i_1) = a_1$, $\phi(i_2) = a_2$. Now $a_1 + a_2 = \phi(i_1) + \phi(i_2) = \phi(i_1 + i_2) \in \phi(I)$. If $t \in T$, then there exist $s \in S$ such that $\phi(s) = t$. Then $ta_1 = \phi(s)\phi(i_1) = \phi(si_1) \in \phi(I)$. Since $si_1 \in I$. Thus $\phi(I)$ is a left semi-ideal of T. Similarly if I is right or two-sided semi-ideal of S, then $\phi(I)$ is right or two-sided semi-ideal of T respectively.

Theorem: 1.11

Let ϕ be a homomorphism of a hemiring S onto a hemiring T. If I is a left [right, two-sided] semi-ideal (k-ideal) of T, then $\phi^{-1}(I)$ is a left [right, two-sided] semi-ideal (k-ideal) of S.

Proof

Suppose I is a left semi-ideal of T. Let $A = \phi^{-1}(I)$. If $a,b \in A$, then $\phi(a), \phi(b) \in I \Rightarrow \phi(a) + \phi(b) \in I \Rightarrow \phi(a + b) \in I \Rightarrow a + b \in \phi^{-1}(I) = A$.

If $a \in A$ and $s \in S$, then $\phi(a) \in I$ and $\phi(s) \in T$. Thus $\phi(sa) = \phi(s)\phi(a) \in TI \subseteq I \Rightarrow sa \in \phi^{-1}(I) = A$. Thus A is a left semi-ideal of S. Similarly, if I is right or two-sided semi-ideal of T, then $\phi^{-1}(I)$ is right or two-sided semi-ideal of S respectively.

Suppose that I is a left k-ideal of T, $a \in A$, $s \in S$ and $s + a \in A$. Then $\phi(a) \in I$, $\phi(s) \in T$ and $\phi(s + a) \in I$. Now $\phi(s) + \phi(a) = \phi(s + a) \in I$, $\phi(a) \in I$. Therefore, $\phi(s) \in I$ as I is a left k-ideal. Hence $s \in \phi^{-1}(I) = A$. Thus A is a left k-ideal of S. Similarly, if I is a right or two-sided k-ideal of T, then $\phi^{-1}(I)$ is right or two-sided k-ideal of S respectively.

Corollary: 1.12

If ϕ is a homomorphism of a hemiring S into a hemiring T *then* ker ϕ *is a k-ideal of* S.

Proof

Since $\{0\}$ is a k-ideal of T. Thus by Theorem 1.11 $\phi^{-1}\{0\}$ is a k-ideal of S. But $\phi^{-1}\{0\} = \ker \phi$. Thus ker ϕ is a k-ideal of S.

Definition: 1.13

Let I be a two-sided semi-ideal in a hemiring S. If $a,b \in S$, $a \equiv b(I)$ if and only if there exist elements $i_1, i_2 \in I$ such that $a + i_1 = b + i_2$.

Lemma: 1.14

The relation '$\equiv$' defined above is a congruence relation. (It is called Bourne Congruence relation).

Proof

Let $a \in S$, then $a + i = a + i \;\; \forall i \in I$. Hence $a \equiv a(I)$. Thus '$\equiv$' is reflexive. Suppose $a \equiv b(I)$. Then there exist $i_1, i_2 \in I$ such that $a + i_1 = b + i_2$. This implies $b + i_2 = a + i_1$ whence $b \equiv a(I)$. Thus '$\equiv$' is symmetric. If $a \equiv b(I)$ and $b \equiv c(I)$, then there exist $i_1, i_2, i_3, i_4 \in I$ such that $a + i_1 = b + i_2$ and $b + i_3 =$

$c + i_4$. Now $a + i_1 = b + i_2 \Rightarrow a + i_1 + i_3 = b + i_2 + i_3$ $\Rightarrow a + i_1 + i_3 = b + i_3 + i_2 \Rightarrow a + i_1 + i_3 = c + i_4 + i_2$ $\Rightarrow a + i_5 = c + i_6$ where $i_5 = i_1 + i_3$ and $i_6 = i_4 + i_2$ $\Rightarrow a \equiv c(I)$. Thus '$\equiv$' is transitive. Hence the relation is an equivalence relation. Now suppose $a \equiv b(I)$ and $c \equiv d(I)$, then there exist $i_1, i_2, i_3, i_4 \in I$ such that $a + i_1 = b + i_2$ and $c + i_3 = d + i_4$.

Thus $\quad (a + i_1)(c + i_3) = (b + i_2)(d + i_4)$

$\Rightarrow \quad a(c + i_3) + i_1(c + i_3) = b(d + i_4) + i_2(d + i_4)$

$\Rightarrow \quad ac + ai_3 + i_1c + i_1i_3 = bd + bi_4 + i_2d + i_2i_4$

$\Rightarrow \quad ac + (ai_3 + i_1c + i_1i_3) = bd + (bi_4 + i_2d + i_2i_4)$

$\Rightarrow \quad ac + i_5 = bd + i_6$

$\Rightarrow \quad ac \equiv bd(I)$. Thus the relation '$\equiv$' is a congruence relation. Hence the relation '$\equiv$' partions S into congruence classes [a], [b], [c], $\cdots$. The class [a] consist of all elements $x \in S$ such that $x \equiv a(I)$.

If the addition and multiplication of these classes are defined by $[a] \oplus [b] = [a + b]$ and $[a] \odot [b] = [ab]$ respectively, then the congruence classes form a hemiring S/I. We now prove that the congruence class [i] where $i \in I$ is the zero element of S/I and [i] is a k-ideal of S generated by I, i.e. $[i] = I^*$. If $i_1, i_2 \in I$ then $i_1 + i_2 = i_2 + i_1$ $\Rightarrow i_1 \equiv i_2(I) \Rightarrow [i_1] = [i_2]$. Thus $[i_1] = [i_2] = [i]$, $i \in I$.

Now $[a] \oplus [i] = [a + i]$.

Since $(a + i) + 0 = a + i$, $(a + i) \equiv a(I)$.

Hence $[a + i] = [a]$. Thus $[a] \oplus [i] = [a]$.

Also $[a] \odot [i] = [ai] = [i]$.

Thus $[i]$ is the zero element of S/I. As $i \in [i]$, $I \subseteq [i]$. It is very easy to see that $[i]$ is a semi-ideal of S. If $a \in S$, $x \in [i]$ and $a + x \in [i]$. Then $a + x + i_2 = i + i_3$ and $x + i_4 = i + i_5$ for some $i_2, i_3, i_4, i_5 \in I$. Now

$$a + x + i_2 = i + i_3$$
$$\Rightarrow \quad a + x + i_2 + 2_4 = i + i_3 + i_4$$
$$\Rightarrow \quad a + x + i_4 + i_2 = i + i_3 + i_4$$
$$\Rightarrow \quad a + i + i_5 + i_2 = i + i_3 + i_4$$
$$\Rightarrow \quad a + (i + i_5 + i_2) = i + (i_3 + i_4)$$

$\Rightarrow$ $a \equiv i(I)$ i.e. $a \in [i]$. Thus $[i]$ is a k-ideal of S containing I. This implies $I^* \subseteq [i]$. Now, if $x \in [i]$ then $x + i_1 = i + i_2$ for some $i_1, i_2 \in I$. Thus $x + i_1 \in I \Rightarrow x \in I^*$. Therefore $[i] \subseteq I^*$. Hence $[i] = I^*$.

Theorem: 1.15

Let S be a hemiring and I be a semi-ideal of S. Then the mapping $\nu : S \longrightarrow S/I$ defined by $\nu(x) = [x]$ is a homomorphism and $\ker \nu = [i] = I^*$ where $i \in I$.

Proof

Let $x, y \in S$. Then

$$\nu(x + y) = [x + y] = [x] \oplus [y] = \nu(x) \oplus \nu(y)$$

$$\nu(xy) = [xy] = [x] \odot [y] = \nu(x) \odot \nu(y)$$

Thus ν is a homomorphism.

Let $x \varepsilon \ker \nu$. Then $\nu(x) = [i]$. But $\nu(x) = [x]$ $\Rightarrow$ $[x] = [i] \Rightarrow x \varepsilon [i] \Rightarrow \ker \nu \subseteq [i]$.

If $y \varepsilon [i]$, then $[y] = [i] \Rightarrow \nu(y) = [i] \Rightarrow y \varepsilon \ker \nu$ $\Rightarrow [i] \subseteq \ker \nu$.

Thus $\ker \nu = [i] = I^*$.

<u>Theorem:</u> 1.16

<u>If</u> ϕ <u>is</u> <u>a</u> <u>homomorphism</u> <u>from</u> <u>a</u> <u>hemiring</u> S <u>onto</u> <u>a</u> <u>hemiring</u> T <u>with</u> kernel K, <u>then</u> $S/K \cong T$.

<u>Proof</u>

Define $\psi : S/K \longrightarrow T$ by $\psi([a]) = \phi(a)$, ψ is well defined, for let $b \varepsilon [a] \Rightarrow a \equiv b(K)$. Hence there exist $k_1, k_2 \varepsilon K$ such that $a + k_1 = b + k_2 \Rightarrow \phi(a + k_1) = \phi(b + k_2)$ $\Rightarrow \phi(a) + \phi(k_1) = \phi(b) + \phi(k_2) \Rightarrow \phi(a) = \phi(b)$ because $\phi(k_1) = \phi(k_2) = 0$.

$$\psi([a] \oplus [b]) = \psi([a + b]) = \phi(a + b) = \phi(a) + \phi(b)$$
$$= \psi([a]) + \psi([b]).$$

$$\psi([a] \odot [b]) = \psi([a \cdot b]) = \phi(ab) = \phi(a)\phi(b)$$
$$= \psi([a]) \cdot \psi([b]).$$

Thus ψ is a homomorphism. Clearly ψ is onto, as ϕ is onto mapping. Now let $[a] \in \ker \psi$. Then $\psi([a]) = 0 \Rightarrow \phi(x) = 0$ $\forall x \in [a] \Rightarrow [a] \subseteq K$. Thus $\ker \psi \subseteq K$. Conversely, if $k \in K$ then $\phi(k) = 0$. Hence $K \subseteq \ker \psi$ and so $\ker \psi = K$. Hence ψ is a semi-isomorphism from S/K onto T i.e., $S/K \simeq T$.

Corollary: 1.17

If S is a hemiring and I is a semi-ideal of S then $S/I \cong S/I^*$.

Proof

By Theorem 1.15 $\nu: S \longrightarrow S/I$ defined by $\nu(x) = [x]$ is a homomorphism from S onto S/I with $\ker \nu = I^*$. By Theorem 1.16 there exist a semi-isomorphism $\psi: S/I^* \longrightarrow S/I$. Now we show that ψ is one-one. Let $C_a, C_b \in S/I^*$ such that $\psi(C_a) = \psi(C_b)$. {We denote by $C_a, C_b, \cdots$ the congruence classes mod (I^*) and by $[a], [b], \cdots$ the congruence classes mod (I)}. Then $\nu(a) = \nu(b) \Rightarrow [a] = [b] \Rightarrow a \equiv b(I) \Rightarrow a + i_1 = b + i_2$ for some $i_1, i_2 \in I$. Since $I \subseteq I^* \Rightarrow a \equiv b(I^*) \Rightarrow C_a = C_b$. Thus ψ is one-one and so ψ is an isomorphism.

Theorem: 1.18

If I and J are semi-ideals of a hemiring S, then

$I/I \cap J^* \xrightarrow{\simeq} I+J/J$.

Proof

Since I + J is a semi-ideal and J is a semi-ideal of I + J, therefore $(I+J)/J$ is meaningfull. Let us define a mapping $\phi: I \longrightarrow (I+J)/J$ by $\phi(x) = [x]$ for all $x \in I$. Let $x,y \in I$, then $\phi(x + y) = [x + y] = [x] \oplus [y] = \phi(x) \oplus \phi(y)$

$$\phi(xy) = [xy] = [x] \odot [y] = \phi(x) \odot \phi(y)$$

Thus ϕ is a homomorphism. Let $[i + j] \in (I+J)/J$, then $[i + j] = [i] + [j] = [i]$.

Thus there exist $i \in I$ such that $\phi(i) = [i] = [i + j]$. This implies ϕ is onto. Now we show that $\ker \phi = I \cap J^*$. Let $x \in \ker \phi$, then $\phi(x) = [x] = J^* \Rightarrow x \in J^*$. Thus $x \in I \cap J^*$ and so $\ker \phi \subseteq I \cap J^*$.

Conversly, suppose that $y \in I \cap J^*$, then $\phi(y) = [y] = J^*$. Therefore $y \in \ker \phi \Rightarrow I \cap J^* \subseteq \ker \phi$. Hence $\ker \phi = I \cap J^*$. Thus by Theorem 1.16

$$I/I \cap J^* \xrightarrow{\simeq} (I+J)/J.$$

Lemma: 1.19

If ϕ is a homomorphism from S onto S/I and if A is a k-ideal of S containing $\ker \phi$ then $A = \phi^{-1}(\phi(A))$.

Proof

Clearly $A \subseteq \phi^{-1}(\phi(A))$. Let $x \in \phi^{-1}(\phi(A))$, then $\phi(x) \in \phi(A) \Rightarrow \phi(x) = \phi(a)$ for some $a \in A$. This implies $[x] = [a]$. Therefore $x \equiv a(I) \Rightarrow x + i_1 = a + i_2$ for some $i_1, i_2 \in I$. As $\ker \phi = I^* \subseteq A$. Therefore $a + i_2 \in A$ $\Rightarrow x + i_1 \in A$, $i_1 \in A$ implies $x \in A$. Hence $\phi^{-1}(\phi(A)) \subseteq A$. Consequently $A = \phi^{-1}(\phi(A))$.

Theorem: 1.20

If ϕ is the natural homomorphism from S onto S/I then there is a one-one correspondence between the k-ideals of S containing $\ker \phi$ and k-ideals of S/I.

Proof

First we show that if A is a k-ideal of S containing $\ker \phi$ then $\phi(A)$ is a k-ideal of S/I. Let $[s] \in S/I$, $[a] \in \phi(A)$ and $[s] + [a] \in \phi(A)$. Thus $[s + a] \in \phi(A) \Rightarrow [s + a] = [a_1]$ for some $[a_1] \in \phi(A) \Rightarrow s + a \equiv a_1(I) \Rightarrow s + a + i_1 = a_1 + i_2$ for some $i_1, i_2 \in I$. As $\ker \phi = I^* \subseteq A$, $a_1 + i_2 \in A$ and so $s + a + i_1 \in A$. Since $a + i_1 \in A \Rightarrow s \in A \Rightarrow [s] = \phi(s) \in \phi(A)$. Thus $\phi(A)$ is a k-ideal of S/I.

By Theorem 1.11 $\phi^{-1}(\bar{A})$ is a k-ideal of S containing ker ϕ provided $\bar{A}$ is a k-ideal of S/I. Let f be a function from the set of all k-ideals of S containing ker ϕ and k-ideals of S/I defined by $f(A) = \phi(A)$. Clearly f is onto. If A_1, A_2 are k-ideals of S containing ker ϕ such that

$$f(A_1) = f(A_2) \Rightarrow \phi(A_1) = \phi(A_2)$$
$$\Rightarrow \quad A_1 = \phi^{-1}(\phi(A_1)) = \phi^{-1}(\phi(A_2)) = A_2$$

Thus f is one-one and so f is a bijection.

Corollary: 1.21

If A and B are semi-ideals of S with $A \subset B$, then B^*/A is a k-ideal of S/A.

Proof

As B^* is a k-ideal of S and contains A^*, which is kernel of $\nu: S \longrightarrow S/A$. Hence the result follows from Theorem 1.20.

Theorem: 1.22

If A and B are semi-ideals of a hemiring S with $A \subset B$, then $S/B \cong S/A \big/ B^*/A$.

Proof

By Lemma 1.21 B^*/A is a k-ideal of S/A. Thus

$S/A \diagup B^*/A$ is meaningful.

Let $\phi : S \longrightarrow S/A$ and $\psi : S/A \longrightarrow S/A \diagup B^*/A$ be natural homomorphisms. Then $\psi \circ \phi$ is a homomorphism from S onto $S/A \diagup B^*/A$. We show that $\ker \psi \circ \phi = B^*$. Let $x \in \ker \psi \circ \phi$, then $\psi \circ \phi(x) = B^*/A$. But $\psi \circ \phi(x) = \psi([x]) = C_{[x]} = B^*/A = C_{[b]}$ say (where $C_{[x]}$, $C_{[b]}$ are the congruence classes of $S/A \diagup B^*/A$ mod B^*/A). This implies

$$[x] + [b_1] = [b] + [b_2] \text{ for some } [b_1],\ [b_2] \in B^*/A$$
$$\Rightarrow [x + b_1] = [b + b_2] \Rightarrow x + b_1 \equiv b + b_2 (A)$$
$$\Rightarrow x + b_1 + a_1 = b + b_2 + a_2 \text{ for some } a_1, a_2 \in A.$$

As $A \subseteq B$, $b + b_2 + a_2 \in B^*$. Hence $x + b_1 + a_1 \in B^*$. Since $b_1 + a_1 \in B^*$, $x \in B^*$. Thus $\ker \psi \circ \phi \subseteq B^*$. If $y \in B^*$, then $\psi \circ \phi(y) = \psi([y]) = C_{[y]} = B^*/A \Rightarrow y \in \ker \psi \circ \phi$. Hence $\ker \psi \circ \phi = B^*$. Thus by Theorem 1.16 $S/B^* \cong S/A \diagup B^*/A$.

Let f be the given semi isomorphism, then $f([s]_{B^*}) = \psi \circ \phi(s)$ where $[s]_{B^*} \in S/B^*$. Let $[s_1]_{B^*}, [s_2]_{B^*} \in S/B^*$ such that $f([s_1]_{B^*}) = f([s_2]_{B^*})$. Then $\psi \circ \phi(s_1) = \psi \circ \phi(s_2)$. Hence $C_{[s_1]} = C_{[s_2]}$. This implies

$$[s_1] + [b_1] = [s_2] + [b_2] \text{ for some } [b_1],\ [b_2] \in B^*/A$$

Hence $[s_1 + b_1] = [s_2 + b_2] \Rightarrow s_1 + b_1 + a_1 = s_2 + b_2 + a_2$ for some a_1, a_2 A. As $A \subseteq B$, $a_1 + b_1$, $a_2 + b_2 \in B^*$. Hence $s_1 \equiv s_2 (B^*)$, which implies $[s_i]_{B^*} = [s_2]_{B^*}$. Hence f is one-one and so f is an isomorphism. Thus

$$S/B^* \cong S/A \big/ B^*/A$$

But $$S/B \cong S/B^*$$ by Corollary 1.17

Thus $$S/B \cong S/A \big/ B^*/A$$

CHAPTER-2

RADICAL CLASSES AND SEMISIMPLE CLASSES

We assume that all hemirings mentioned belong to a class μ of hemirings with the following two properties:

(1) <u>If</u> S <u>is a hemiring in</u> μ, <u>then every homomorphic image of</u> S <u>is in</u> μ.

(2) <u>If</u> S <u>is a hemiring in</u> μ <u>and</u> I <u>is a semi-ideal of</u> S <u>then</u> I <u>is a hemiring in</u> μ.

We say a class is <u>homomorphically closed</u> if it has property (1) above.

<u>Definition</u>: 2.1

A non-empty sub class P of hemirings μ is called a <u>radical class</u> if:

R_1) P is homomorphically closed.

R_2) If $A \not\in P$, then A contains a proper k-ideal K such that A/K has no non-zero P-semi-ideals (semi-ideals which as hemirings are in the class P).

<u>Theorem</u>: 2.2

A non-empty sub class P of hemirings μ is a radical class iff it satisfies:

R_1') If A is a P-hemiring, then every non-zero homomorphic image of A has a non-zero P-semi-ideal.

R_2') If every non-zero homomorphic image of a hemiring B has a non-zero P-semi-ideal, then B is a P-hemiring.

Proof

We assume that P is a radical class. Let A be a P-hemiring and let $\phi(A)$ be a non-zero homomorphic image of A. Then by (R_1), $\phi(A) \in P$. Therefore $\phi(A)$ has a non-zero P-semi-ideal (i.e., $\phi(A)$ itself). Thus (R_1') is proved.

Suppose $B \notin P$. Then by (R_2), B contains a proper k-ideal K such that B/K has no non-zero P-semi ideal. Now B/K is a homomorphic image of B and has no non-zero P-semi ideal. Thus if $B \notin P$ then there exist a non-zero homomorphic image of B which has no non-zero P-semi ideal. This proves (R_2').

Conversely, suppose that a sub class P of hemi-rings satisfies the given two conditions (R_1') and (R_2'). We prove that P is a radical class. Let $A \in P$ and $\phi(A)$ be a non-zero homomorphic image of A. Then every homomorphic image of $\phi(A)$

is also a homomorphic image of A. By (R_1') every non-zero homomorphic image of $\phi(A)$ (which is also a homomorphic image of A) has a non-zero P-semi ideal. Hence by (R_2') $\phi(A) \in P$. So P is homomorphically closed. Thus (R_1) is satisfied.

Now, if $A \not\in P$, then by (R_2') there exist a non-zero homomorphic image of A which has no non-zero P-semi ideal, i.e., A has homomorphic image A/K where K is a proper k-ideal, such that A/K has no non-zero P semi-ideal.

Lemma: 2.3

If I *and* J *are* P-*semi ideals of a hemiring* S *then* I + J *is also a* P-*semi ideal of* S.

Proof

Assume $(I+J) \not\in P$. Then by (R_2) there exist a proper k-ideal K of I + J such that I + J/K has no non-zero P-semi ideal. Let $\nu: I + J \longrightarrow (I+J)/K$ be the natural homomorphism. As $I \in P$ and I is a semi-ideal of I + J, $\nu(I) \in P$. But $\nu(I)$ is a semi-ideal of $(I+J)/K$, therefore $\nu(I) = K$ (zero of $(I+J)/K$). This implies $I \subseteq K$.

Hence $(I+J)/K \cong (I+J)/I\big/K/I$ by Theorem 1.22 i.e., (I+J)/K is a homomorphic image (I+J)/I. But $J/I^* \cap J \cong (I+J)/I$. As $J \in P$, $(I+J)/I \in P$. This implies $(I+J)/K \in P$, which is a contradiction. Hence $I + J \in P$.

Lemma: 2.4

If $\{A_\alpha\}_{\alpha \in I}$ *is a chain of P-semi ideals of S, then* $A = \bigcup_{\alpha \in I} A_\alpha \in P$.

Proof

Assume $A \notin P$. By (R_2) there exist a proper k-ideal K of A such that A/K has no non-zero P-semi ideal. Let $\nu : A \longrightarrow A/K$ be the natural homomorphism. Since $K \neq A$, there is some index α_o for which $A_{\alpha_o} \not\subseteq K$. As $A_{\alpha_o} \in P \Rightarrow \nu(A_{\alpha_o}) \in P$ and $\nu(A_{\alpha_o})$ is a semi-ideal of A/K. Thus $\nu(A_{\alpha_o}) = K$ $\Rightarrow A_{\alpha_o} \subseteq K$ which is a contradiction. Hence $A \in P$.

Lemma: 2.5

If I is a semi-ideal of S such that S/I and I are in P, then $S \in P$.

Proof

Assume that $S \notin P$. Then by (R_2) there exist a proper k-ideal K of S such that S/K has no non-zero P-semi ideals. Let $\nu: S \longrightarrow S/K$ be the natural homomorphism. As I is a semi-ideal of S, therefore $\nu(I)$ is a semi-ideal of S/K. Since $I \in P \Rightarrow \nu(I) \in P$. Hence $\nu(I) = K$. This implies $I \subseteq K$. By Theorem 1.22.

$S/K \cong S/I \big/ K/I$. As $S/I \in P$, S/K (being homomorphic image of S/I) belongs to P, which is a contradiction. Hence $S \in P$.

Theorem: 2.6

Every hemiring S contains a unique maximal P-semi ideal which contains all other P-semi ideal of S.

Proof

If S is a member of P, the result is obvious. So suppose $S \notin P$. Then the collection of all P-semi ideals of S is non-empty since 0 is a member of this collection by (R_1). This collection is partially ordered by inclusion, and if $\{T_\alpha\}$ is any chain of P-semi ideals. Thus $\cup T_\alpha$ is a P-semi ideal by Lemma 2.4 and is clearly an upper bound of the chain. Thus by Zorn's lemma, there exist a maximal element M among the P-semi ideals of S. If I is any P-semi ideal of S, then M + I is a P-semi ideal by Lemma 2.3 and since M is a maximal P-semi-ideal, M + I = M, which implies

$I \subseteq M$. This establishes that M is unique and that it contains all P-semi ideals of S.

Definition: 2.7

The maximal P-semi ideal of a hemiring S will be called the P-radical of S and designated P(S). If P(S) = 0, S is called semi-simple.

Lemma: 2.8

If P is a radical class and I is a P-semi ideal of S then I is a P-semi ideal of S.*

Proof

Since $I^*/I = (0) \in P$ and $I \in P$, so by Lemma 2.5 $I^* \in P$.

Theorem: 2.9

If P is a radical class, then for any hemiring S, P(S) is a k-ideal of S.

Proof

If P(S) = S, the result is obvious. If $P(S) \neq S$ then by Lemma 2.8 $P(S)^*$ is a P-semi ideal. But $P(S)$ is a unique maximal P-semi ideal which contains all other P-semi ideals of S, the $P(S)^* \subseteq P(S)$ and so $P(S) = P(S)^*$. Thus $P(S)$ is a k-ideal.

Corollary: 2.10

If S is a ring, P(S) is a ring ideal of S.

Proof

If S is a ring, a k-ideal of S is a ring ideal. Hence in the present case, $P(S)$ is a ring ideal of S.

Theorem: 2.11

For any hemiring S, $P(S/P(S)) = 0$.

Proof

Let $P(S/P(S)) = I$, then I is a k-ideal of $S/P(S)$ by Theorem 2.9. But by Theorem 1.20, $I = A/P(S)$ where A is a k-ideal of S. As $A/P(S)$ and $P(S)$ are in P so by Lemma 2.5 $A \in P$. This implies $A \subseteq P(S)$. Thus $I = A/P(S) = 0$. Thus $P(S/P(S)) = 0$.

Theorem: 2.12

A subclass P of hemirings μ is a radical class iff P has properties:

(R_1) P is homomorphically closed.

(R_3) Every hemiring S has a unique maximal P-semi ideal P(S) which is a k-ideal of S and contains every other P-semi ideal of S.

(R_4) S/P(S) has no non-zero P-semi ideal.

Proof

If P is a radical class then Theorems 2.6, 2.9 and 2.11 implies (R_1), (R_3) and (R_4).

Conversely, assume that (R_1), (R_3) and (R_4) are satisfied by P. Now let $A \notin P$, (R_3) gives us a proper k-ideal P(A) and A/P(A) has no non-zero P-semi ideal by (R_4). Thus (R_2) is satisfied. Hence P is a radical class.

Theorem: 2.13

A subclass P of hemirings μ is a radical class iff P has properties:

(R_1) P is homomorphically closed.

(R_3) Every hemiring S has a maximal P-semi ideal

P(S) <u>which is a k-ideal of</u> S <u>and contains every other</u> P-<u>semi ideal of</u> S.

(R_5) <u>If</u> I <u>is a semi-ideal of</u> S <u>such that</u> S/I <u>and</u> I <u>are in</u> P, <u>then</u> S ε P.

<u>Proof</u>

If P is a radical class then by Theorem 2.12 (R_1) and (R_3) are satisfied by P and by Lemma 2.5 (R_5) is satisfied.

Now assume that (R_1), (R_3) and (R_5) are satisfied by a class P. We show that (R_4) is also satisfied by the class P. Suppose $P(S/P(S)) \neq 0$ then $P(S/P(S)) = A/P(S)$ where A is a k-ideal of S. As A/P(S) and P(S) are in P therefore $A \varepsilon P$ by (R_5). Thus $A \subset P(S) \Rightarrow A/P(S) = 0$. So $P(S/P(S)) = 0$ which satisfies (R_4). Thus P is a radical class.

<u>Lemma</u>: 2.14

<u>If</u> $\phi(S)$ <u>is a homomorphic image of</u> S <u>then</u> $\phi(P(S)) \subseteq P(\phi(S))$.

<u>Proof</u>

Since P(S) is a P-semi ideal of S, therefore $\phi(P(S))$ is a P-semi ideal of $\phi(S)$. Thus $\phi(P(S)) \subseteq P(\phi(S))$.

Lemma: 2.15

Let P be a radical class. Then for any hemiring S, $P(S) = \bigcap_{I \in C} I$ where $C = \{$k-ideals I of S such that $P(S/I) = 0\}$.

Proof

Since $P(S) \in C$, we have $\bigcap_{I \in C} I \subseteq P(S)$. Also if I is a k-ideal of S such that $P(S/I) = 0$ then for the natural homomorphism $\nu : S \longrightarrow S/I$

$$\nu\big(P(S)\big) \subseteq P\big(\nu(S)\big) \qquad \text{by Lemma 2.14}$$

Now $P\big(\nu(S)\big) = P(S/I) = 0$. So $P(S) \subseteq I^* = I$ for each $I \in C$. Consequently $P(S) = \bigcap_{I \in C} I$. Thus $P(S) = \bigcap_{I \in C} I$.

Proposition: 2.16

Let I be a semi-ideal of a hemi-ring S with K a semi-ideal of I. For any element $a \in S$ the set $aK + K$ is a semi ideal of I, and the mapping $\phi(x) = [ax]$, $a \in K$ maps K homomorphically onto the factor hemiring $(aK + K)/K$.

Proof

Let $ak_1 + k_2$, $ak_3 + k_4 \in aK + K$, then

$$(ak_1 + k_2) + (ak_3 + k_4) = a(k_1 + k_3) + (k_2 + k_4)$$
$$= ak_5 + k_6 \in aK + K$$

where $k_5 = k_1 + k_3$ and $k_6 = k_2 + k_4$

Now let $i \in I$ then $i(aK + K) = iaK + iK \subseteq i_1K + k$

$$\subseteq K + K = K \subseteq aK + K$$

$$(aK + K)i = aKi + Ki \subseteq aK + K$$

Thus $aK + K$ is a semi-ideal of I. Now we show that ϕ is a homomorphism. Let $x,y \in K$ then

$$\phi(x + y) = [a(x + y)] = [ax + ay]$$

$$= [ax] \oplus [ay] = \phi(x) \oplus \phi(y)$$

Again $\phi(xy) = [axy] = K^*$

$$\phi(x) \odot \phi(y) = [ax] \odot [ay] = [axay]$$

Now $ax \in I$ thus $axa \in I \Rightarrow axay \in IK \subseteq K$

$$\phi(x) \odot \phi(y) = [axay] = K^*$$

Hence $\phi(xy) = \phi(x) \odot \phi(y)$. Thus ϕ is a homomorphism.

Let $[ak_1 + k_2] \in (aK + K)/K$. Then $[ak_1 + k_2] = [ak_1] \oplus [k_2]$ $= [ak_1]$. Thus there exist $k_1 \in K$ such that

$$\phi(k_1) = [ak_1] = [ak_1 + k_2]$$

Therefore ϕ maps K homomorphically onto the factor hemiring $(aK + K)/K$.

<u>Theorem</u>: 2.17

Let P be a radical class. If I is a semi-ideal of a hemiring S then P(I) is also a semi-ideal of S.

<u>Proof</u>

Suppose P(I) is not a semi-ideal of S. Then there is an element $a \in S$ such that either $aP(I)$ or $P(I)a$ is not contained in $P(I)$. Without loss of generality we can suppose $a P(I) \not\subseteq P(I)$.

Now we have $P(I) \subset aP(I) + P(I) \subseteq I$. Applying Proposition 2.16, there exist a homomorphism $\phi: P(I) \longrightarrow (aP(I) + P(I))/P(I)$. Since $P(I) \in P$, therefore $(aP(I) + P(I))/P(I) \in P$ being the homomorphic image of $P(I)$. Thus $aP(I) + P(I) \in P$ by Lemma 2.5. Hence $aP(I) + P(I) \subseteq P(I)$ by Lemma 2.6. Thus, $aP(I) \subseteq P(I)$.
Which contradicts our assumption. Thus $P(I)$ is a semi-ideal of S.

Corollary: 2.18

Let P be a radical class. If I is a semi-ideal of a hemiring S, then $P(I) \subseteq P(S) \cap I$.

Proof

Obviously $P(I) \subseteq I$. Since Theorem 2.17, $P(I)$ is a semi-ideal of S therefore $P(I) \subseteq P(S)$. Hence $P(I) \subseteq P(S) \cap I$.

Corollary: 2.19

Let I be a semi-ideal of a hemiring S. Then

$P(S) = 0 \Rightarrow P(I) = 0$.

Proof

By Corollary 2.18, $P(I) \subseteq P(S) \cap I = 0 \cap I = 0$.

Definition: 2.20

A radical class P is called hereditary if P is closed under taking semi-ideals, i.e., if $A \in P$ and I is a semi-ideal of A, then $I \in P$.

Lemma: 2.21

If a radical class P is hereditary, then $P(I) = I \cap P(S)$ *for any semi-ideal I of S.*

Proof

By Corollary 2.18, $P(I) \subseteq P(S) \cap I$. Now $P(S) \in P$ and since $P(S) \cap I$ is a semi-ideal of $P(S)$ and P is hereditary, therefore $P(S) \cap I \in P$. But $P(S) \cap I$ is also a semi-ideal of I, therefore $P(S) \cap I \subseteq P(I)$. Hence $P(I) = P(S) \cap I$.

Definition: 2.22

A subclass S of hemirings μ will be called a semisimple class, wherever the following two axioms are fulfilled:

(S_1) If A is an S-hemiring, then every non-zero semi-ideal of A has a non-zero homomorphic image in S.

(S_2) If every non-zero semi-ideal of a hemiring B has a non-zero homomorphic image in S, then B is an S-hemiring.

Definition: 2.23

A subclass of hemirings μ satisfying the condition (S_1) is called a regular class.

Theorem: 2.24

Let M be a regular class. Define $\bar{M} = \{A \in \mu$: every non-zero semi-ideal of A can be mapped homomorphically onto some non-zero M-hemiring$\}$. The class $\bar{M}$ is semisimple class. Moreover if S is any semisimple class containing the class M, then S contains also the class $\bar{M}$. Thus $\bar{M}$ can be called the semisimple closure of the class M.

Proof

Clearly $M \subseteq \bar{M}$. Now if $A \in \bar{M}$ then every non-zero

semi-ideal of A can be mapped homomorphically onto some non-zero M-hemiring and so on $\bar{M}$-hemiring. Thus (S_1) is satisfied. Let I be any arbitrary non-zero semi-ideal of a hemiring B which has a non-zero homomorphic image $\phi(I)$ in $\bar{M}$. Then $\phi(I)$ can be mapped homomorphically onto some non-zero M-hemiring (say J) i.e., $I \longrightarrow \phi(I) \longrightarrow J$.
This implies I has a non-zero homomorphic image J in M. Thus by the definition of $\bar{M}$, $B \in \bar{M}$. Thus (S_2) is satisfied.

Let S be any semisimple class such that $M \subseteq S$. Now every M-hemiring is also an S-hemiring. Hence if $A \in \bar{M}$, then every non-zero semi-ideal of A has a non-zero homomorphic image in M and so in S. Since S satisfies (S_2), therefore $A \in S$. Hence $\bar{M} \subseteq S$.

Proposition: 2.25

For every radical class P. $P \cap SP = 0$ where $SP = \{A \in \mu : P(A) = 0\}$.

Proof

Let $A \in P \cap SP$ then $A \in P$ and $A \in SP$, $A \in P \Rightarrow P(A) = A$, $A \in SP \Rightarrow P(A) = 0$. Thus $A = 0$.

Proposition: 2.26

Let P be a radical class and let SP be the class of

<u>all hemirings having zero</u> P-<u>radical</u> i.e., $SP = \{A \in \mu: P(A) = 0\}$ <u>is a semisimple class</u>.

<u>Proof</u>

Let $A \in SP$ then $P(A) = 0$. Let I be a non-zero semi-ideal of A which cannot be mapped homomorphically onto some non-zero SP-hemiring. Since $P(I/P(I)) = 0$ therefore $I/P(I) \in SP$. Since I cannot be mapped homomorphically onto some non-zero SP-hemiring, therefore $I/P(I) = 0 \Rightarrow I \subseteq P(I)$ so $0 \neq I \subseteq P(I) = 0$, which is a contradiction. Thus (S_1) is satisfied.

In order to prove (S_2), we show that if $A \notin SP$ then there exist a non-zero semi-ideal of A which has no non-zero homomorphic image in SP. Since $A \notin SP \Rightarrow P(A) \neq 0$. Since P is homomorphically closed, therefore every homomorphic image of P(A) is in P. Since $P \cap SP = 0$ by Proposition 2.25. Therefore P(A) has no non-zero homomorphic image in SP.

<u>Definition</u>: 2.27

Let M be a regular class. $UM = \{A \in \mu$: A has no non-zero homomorphic image in M$\}$. The class UM is called the <u>upper radical class</u> determined by the class M.

Theorem: 2.28

$\mathcal{U}M$ is a radical class. Moreover if P is radical class such that $M \subseteq SP$ then $P \subseteq \mathcal{U}M$.

Proof

First we show that $\mathcal{U}M$ is homomorphically closed. Let $A \in \mathcal{U}M$ and let $\phi(A)$ be a homomorphic image of A. Now every homomorphic image of $\phi(A)$ is also a homomorphic image of A. Since A has no non-zero homomorphic image in M, therefore $\phi(A)$ has no non-zero homomorphic image in M, thus $\phi(A) \in \mathcal{U}M$.

Now let $B \notin \mathcal{U}M$. This implies B has a non-zero homomorphic image $\phi(B)$ in M. By Theorem 1.16 there exist a semi isomorphism $\psi: B/K \longrightarrow \phi(B)$ where $K = \ker \phi$. We show that B/K has no non-zero $\mathcal{U}M$-semi-ideal. Let N be a non-zero $\mathcal{U}M$-semi-ideal of B/K. Then $\psi(N) \neq 0$ because if $\psi(N) = 0$ $\Rightarrow N \subseteq K \Rightarrow N = 0$. Therefore $\psi(N)$ is a non-zero semi-ideal of $\phi(B)$. As $\phi(B) \in M$ and M is a regular class this implies $\psi(N)$ has a non-zero homomorphic image in $M \Rightarrow N$ has a non-zero homomorphic image in M. This contradicts that N is $\mathcal{U}M$-semi-ideal of B/K. Thus B/K has no non-zero $\mathcal{U}M$-semi-ideal. Therefore $\mathcal{U}M$ is a radical class.

Finally, let P be a radical class such that the semisimple class $\mathcal{S}P$ contains the class M. Let A ε P. Then every homomorphic image of A is in P. Now suppose A $\not\in \mathcal{U}M$. This implies A has a non-zero homomorphic image $\phi(A)$ in M

$\Rightarrow \phi(A) \in P \cap M \subseteq P \cap \mathcal{S}P = 0$

$\Rightarrow \phi(A) = 0$ which contradicts that A $\not\in \mathcal{U}M$. Hence A ε $\mathcal{U}M$.

Proposition: 2.29

Let M be a regular class and $\bar{M}$ its semisimple closure. Then $\mathcal{U}M = \mathcal{U}\bar{M}$ holds.

Proof

The $\mathcal{U}\bar{M}$-hemirings cannot be mapped homomorphically onto any non-zero $\bar{M}$-hemiring. Since $M \subseteq \bar{M}$, $\mathcal{U}\bar{M}$ hemirings cannot be mapped homomorphically onto any non-zero $\bar{M}$-hemiring. Thus $\mathcal{U}\bar{M} \subseteq \mathcal{U}M$. Let A ε $\mathcal{U}M$ and $A \not\in \mathcal{U}\bar{M}$. Then A has a non-zero homomorphic image $\phi(A) \in \bar{M}$. This implies $\phi(A)$ and hence A has a non-zero homomorphic image in M. This contradicts that A ε $\mathcal{U}M$. Thus A ε $\mathcal{U}\bar{M}$, i.e., $\mathcal{U}M \subseteq \mathcal{U}\bar{M}$. Hence $\mathcal{U}M = \mathcal{U}\bar{M}$.

Theorem: 2.30

Every semisimple class $\mathcal{S}$ is the semisimple class of its upper radical class, that is

$\mathcal{S} = \mathcal{S}(\mathcal{U}\mathcal{S})$ holds.

Proof

By definition $\mathcal{U}\mathcal{S}$ is the class of hemirings which cannot be mapped onto any non-zero $\mathcal{S}$-hemiring, and $\mathcal{S}(\mathcal{U}\mathcal{S})$ is the class of all hemiring with zero $\mathcal{U}\mathcal{S}$-radical.

Let A be an $\mathcal{S}$-hemiring. Then every non-zero semi-ideal of A has a non-zero homomorphic image in $\mathcal{S}$. Suppose A has a non-zero $\mathcal{U}\mathcal{S}$-semi-ideal. Such a semi-ideal has no non-zero homomorphic image in $\mathcal{S}$. This contradicts the earlier statement. Thus A has no non-zero $\mathcal{U}\mathcal{S}$-semi-ideal i.e., $\mathcal{U}\mathcal{S}(A) = 0$. This implies $A \in \mathcal{S}(\mathcal{U}\mathcal{S}) \Rightarrow \mathcal{S} \subseteq \mathcal{S}(\mathcal{U}\mathcal{S})$.

Conversly, suppose $A \in \mathcal{S}(\mathcal{U}\mathcal{S}) \Rightarrow \mathcal{U}\mathcal{S}(A) = 0$ i.e., A has non non-zero $\mathcal{U}\mathcal{S}$-semi-ideal. Hence every non-zero semi-ideal of A can be mapped homomorphically onto some non-zero $\mathcal{S}$-hemiring and so by (S_2) $A \in \mathcal{S} \Rightarrow \mathcal{S}(\mathcal{U}\mathcal{S}) \subseteq \mathcal{S}$. Hence $\mathcal{S} = \mathcal{S}(\mathcal{U}\mathcal{S})$

Theorem: 2.31

Every radical class P is the upper radical class of the semisimple class $\mathcal{S}P$, i.e., $P = \mathcal{U}(\mathcal{S}P)$.

Proof

Let $A \in P$. Then every homomorphic image of A is in P.

Since $P \cap SP = 0$, A has no non-zero homomorphic image in SP. Therefore $A \in \mathcal{U}(SP) \Rightarrow P \subseteq \mathcal{U}(SP)$.

Suppose $A \notin P$, then $P(A) \neq A$. Since $P(A/P(A)) = 0$ $\Rightarrow A/P(A) \in SP$ and $A/P(A) \neq 0$. This implies that A has a non-zero homomorphic image in SP. Therefore $A \notin \mathcal{U}(SP)$. Hence $\mathcal{U}(SP) \subseteq P$. Hence $P = \mathcal{U}(SP)$.

Theorem: 2.32

Every semi-simple class is hereditary.

Proof

By Theorem 2.30, $S = S(\mathcal{U}S)$. Let $A \in S$, then $\mathcal{U}S(A) = 0$. Suppose I is a semi-ideal of A. By Corollary 2.19 $\mathcal{U}S(A) = 0 \Rightarrow \mathcal{U}S(I) = 0$, i.e., $I \in S$. So S is hereditary.

Definition: 2.33

A subhemiring B of a hemiring A is called an accessible subhemiring of A, if there exist a finite sequence $C_1, C_2, \cdots, C_n$ of subhemirings of A such that $C_1 = B$, $C_n = A$ and each C_i is a semi-ideal of C_{i+1} for $i = 1, 2, \cdots, n-1$.

By definition, a hemiring A is an accessible subhemiring of itself. Starting from a given abstract class M_o (i.e., if $A \varepsilon M_o$ and $A \cong B$ then $B \varepsilon M_o$) let us define the class M by

$$M = \{B \varepsilon \mu: B \text{ is an accessible subhemiring of some } M_o\text{-hemiring}\}.$$

As B is an accessible subhemiring of an M_o-hemiring, say A, there exists a finite sequence $C_1, C_2, \cdots, C_n$ of subhemirings of A such that $B = C_1$, $A = C_n$ and each C_i is a semi-ideal of C_{i+1} for $i = 1,2,\cdots,n-1$. Let I be a semi-ideal of B then $C_0, C_1, \cdots, C_n$ is a sequence of subhemirings of A such that $I = C_0$, $A = C_n$ and each C_i is a semi-ideal of C_{i+1} for $i = 0,1,2,\cdots,n-1$. Thus I is an accessible subhemiring of an M_o-hemiring A. This shows that M is a regular class, because every semi-ideal of an accessible subhemiring of an M_o-hemiring is again an accessible subhemiring of an M_o-hemiring. Furthermore, from M we can form its semisimple closure $\bar{M}$.

<u>Theorem</u>: 2.34

<u>$\bar{M}$ is the minimal semisimple class containing the class M_o, so $\bar{M}$ is the semisimple closure of M_o.</u>

<u>Proof</u>

Clearly $M_o \subseteq M \subseteq \bar{M}$. Let S be a semisimple class containing M_o. Let $B \in M$, then B is an accessible subhemiring of an M_o-hemiring. Since $M_o \subseteq S \Rightarrow B$ is an accessible subhemirings of an S-hemiring. By the hereditariness of S, $B \in S$. This implies $M \subseteq S$ and by Theorem 2.24, $\bar{M} \subseteq S$.

Lemma: 2.35

Let A be a hemiring and J and K are semi-ideals of A then A/K can be mapped onto A/J naturally iff $K \subseteq J^*$.

Proof

Let $[x]_K, [x]_J$ be the congruence classes corresponding to semi-ideals K and J respectively. Suppose A/K is mapped homomorphically onto A/J. Let $x \in K$ then $[x]_K = K^*$ is mapped onto the zero element of A/J i.e., $[x]_J = J^* \Rightarrow x \in J^*$ $\Rightarrow K \subseteq J^*$.

Conversly, if $K \subseteq J^*$ then by Theorem 1.22

$S/K \big/ J^*/K \cong S/J^* \cong S/J$

$\Rightarrow$ S/K is mapped onto S/J.

If $\{I_\alpha\}_{\alpha \in \Lambda}$ be a family of semi-ideals. Then the semi-ideal $\sum_{\alpha \in \Lambda} I_\alpha$ generated by $\bigcup_{\alpha \in \Lambda} I_\alpha$, has the following characteristic properties:

(i) Every I_α is contained in the semi-ideal $\sum_{\alpha \in \Lambda} I_\alpha$.

(ii) If every I_α is contained in a semi-ideal A then also $\sum_{\alpha \in \Lambda} I_\alpha$ ___ A holds.

Now we can define the union of factor hemirings dually to the ideals $\sum_{\alpha \in \Lambda} I_\alpha$ as follows:

The union of factor hemiring A/J_α of a hemiring A is defined as the factor hemiring $A/(\bigcap_{\alpha \in \Lambda} J^*_\alpha)$. The hemiring $A/(\bigcap_{\alpha \in \Lambda} J^*_\alpha)$ is obviously characterized by the properties:

(i) $A/(\bigcap J^*_\alpha)$ can be mapped onto every A/J_α by natural homomorphism ($\because \bigcap J^*_\alpha \subseteq J^*_\alpha \Rightarrow A/\bigcap J^*_\alpha$ is mapped onto A/J^*_α. As $A/J_\alpha \cong A/J^*_\alpha$, therefore $A/\bigcap_{\alpha \in \Lambda} J^*_\alpha$ is mapped onto A/J_α).

(ii) If a factor hemiring A/K can be mapped onto every factor hemiring A/J_α by natural homomorphism then it can also be mapped onto $A/(\bigcap_{\alpha \in \Lambda} J^*_\alpha)$. (If A/K is mapped onto A/J_α then $K \subseteq J^*_\alpha$. As $K \subseteq \bigcap_{\alpha \in \Lambda} J^*_\alpha$, therefore A/K_α is mapped onto $A/\bigcap_{\alpha \in \Lambda} J^*_\alpha)$.

<u>Definition</u>: 2.36

We shall call a factor hemiring as S-<u>factor hemiring</u> if it is an S-hemiring.

Theorem: 2.37

A subclass S of μ is a semisimple class iff S satisfies:

S_1') *S is hereditary.*

S_2') *Every hemiring A has an S-factor hemiring (A)S which can be mapped onto every S-factor hemiring of A by natural homomorphism.*
{(A)S is actually the union of all S-factor hemirings of A and so by definition we have $(A)S = A/(\bigcap_\alpha J^*_\alpha)$ where J_α runs over all the semi-ideals of A with $A/J_\alpha \in S$. The factor hemiring (A)S will be called the S-semisimple image of A}.

S_3') *The kernel of the mapping A ⟶ (A)S has no* non-zero S-factor hemiring, that is $(\bigcap J^*_\alpha)S = 0$.

Proof

Let S be a semisimple class. By Theorem 2.32 S is hereditary and so (S_1') is satisfied.

Let A be a hemiring and consider its factor hemiring $(A)S = A/(\bigcap J^*_\alpha)$ (for all $A/J_\alpha \in S$). In order to prove (S_2') we have to show only that $(A)S \in S$. Suppose $(A)S \notin S$. Now by (S_2), $A(S) = A/(\bigcap J^*_\alpha)$ has a non-zero semi-ideal, say K, which cannot be mapped onto any non-zero S-hemiring. Let $\phi: A \longrightarrow A/\bigcap J^*_\alpha$ be the natural homomorphism, then $\phi^{-1}(K)$ is a semi-ideal of A and $\bigcap J^*_\alpha \subset \phi^{-1}(K)$. Thus there exist an idex α_o such that $\phi^{-1}(K) \not\subseteq J^*_{\alpha_o}$

Further $\bigcap J^*_\alpha \subseteq J^*_\alpha \Rightarrow$ there exist a homomorphism $f: A/\bigcap J^* \longrightarrow A/J^*_{\alpha_o}$. Now $f(K)$ is a semi-ideal of $A/J^*_{\alpha_o}$ and as $A/J^*_{\alpha_o} \in S$ and S is hereditary therefore $f(K)$ belongs to S and $f(K) \neq 0$. {Because $\phi: A \longrightarrow A/\ J^*_\alpha$ and $f: A/\bigcap J^*_\alpha \longrightarrow A/J^*_{\alpha_o}$ therefore $f(K) \neq 0$}. This contradicts our assumption. Thus $(A)S \in S$.

By Theorem 2.30, $S(US) = S$. Also for any hemiring A, $US(A/US(A)) = 0 \Rightarrow A/US(A) \in S$. Hence $US(A)$ is one of the terms in the intersection $\bigcap_\alpha J^*_\alpha$ where J_α runs over all the semi-ideals J_α with $A/J_\alpha \in S$. Moreover, $US(A)/\bigcap_\alpha J^*_\alpha$ is a k-ideal of $A/\bigcap_\alpha J^*_\alpha$. Hence $US(A)/\bigcap_\alpha J^*_\alpha \in S$ by hereditariness of S. On the other hand being a homomorphic image of US-hemiring belongs to US. Therefore, $US(A)/\bigcap_\alpha J^*_\alpha \in S \cap US = 0$

$$\Rightarrow US(A) = \bigcap_\alpha J^*_\alpha \qquad \text{———— (A)}$$

Being a US-hemiring $\bigcap J^*_\alpha$ cannot be mapped onto a non-zero S-hemiring $\Rightarrow (\bigcap_\alpha J^*_\alpha)S = 0$ and condition (S_3') is fulfilled.

Conversly, let us consider a class S satisfying given conditions. Condition (S_1') trivially implies condition (S_1).

For (S_2) let A be a hemiring such that every non-zero semi-ideal of A has a non-zero S-factor hemiring. Consider the semisimple image $(A)S = A/\bigcap J^*_\alpha$. By condition (S_3) $(\bigcap_\alpha J^*_\alpha)S = 0$ so by the assumption we have $\bigcap J^*_\alpha = 0 \Rightarrow (A)S = A$ and so by (S_2^*), $A \in S$. Hence S is a semisimple class.

Theorem: 2.38

Let P be a radical class. The P-radical P(A) of a hemiring A has always an intersection representation $P(A) = \bigcap_\alpha J^*_\alpha$ where J_α runs over all the semi-ideals of A such that $A/J_\alpha \in SP$. Furthermore, for every semisimple class S we have $(A)S = A/\mathcal{U}S(A)$.

Proof

By Theorem 2.31, $P = \mathcal{U}(SP)$ and by (A) given in the last theorem, we have $\mathcal{U}(SP)A = \bigcap J^*_\alpha$ thus $P(A) = \bigcap_\alpha J^*_\alpha$.

As $(A)S = A/\bigcap J^*_\alpha$ and $\bigcap J^*_\alpha = \mathcal{U}S(A)$. Therefore, $(A)S = A/\mathcal{U}S(A)$.

Proposition: 2.39

Let S be a semisimple class. If I is a k-ideal of a hemiring A such that A/I and I are in S, then $A \in S$.

Proof

Suppose there exist a hemiring $A \notin S$ which has a k-ideal I such that I and A/I are in S and $A \notin S$. Since $A \notin S \Rightarrow US(A) \neq 0$. Also by Theorem 2.38, $(A)S = A/US(A)$. Since $A/I \in S \Rightarrow US(A) \subseteq I$. By the hereditariness of S, $US(A) \in S$. This implies $US(A) \in S \cap US = 0$. Thus $US(A) = 0$ which is a contradiction. Hence $A \in S$.

Let L be a homomorphically closed class of hemirings and let us define the class L_α as follows. Define $L_1 = L$. Assuming L_β has been defined already for every ordinal $\beta < \alpha$, define $L_\alpha = \{A \in \mu$: every non-zero homomorphic image of A has a non-zero L_β semi-ideal for some $\beta < \alpha\}$.

Proposition: 2.40

If $\beta < \alpha$, then $L_\beta \subseteq L_\alpha$.

Proposition: 2.41

Every class L_α is homomorphically closed.

Proof

For $\alpha = 1$ the statement is trivial. If $\alpha > 1$ and

$\phi(A)$ is any homomorphic image of a hemiring $A \in L_\alpha$ then every non-zero homomorphic image of $\phi(A)$ is a homomorphic image of A, so has a non-zero L_β-semi ideal for some ordinal $\beta < \alpha$ and so $\phi(A) \in L_\alpha$.

Theorem: 2.42

$LL = \underset{\alpha}{U} L_\alpha$ is a radical class and it is the smallest radical class containing L.

Proof

By Proposition 2.41 every L_α and so also LL is homomorphically closed. So $(R_1^{\backprime})$ is satisfied.

In order to prove $(R_2^{\backprime})$ suppose that every non-zero homomorphic image of a hemiring A has a non-zero semi-ideal J_α in $LL = \underset{\alpha}{U} L_\alpha$. Then $A/I_\sigma \rightsquigarrow A_\alpha$ for some semi-ideal I_σ of A. Let τ be an ordinal greater than every ordinal σ. (Since the cardinality of the set of all semi-ideals of A is less than 2^m where m denotes the cardinality of A, therefore such an ordinal τ does exist). $J_\alpha \in \underset{\alpha}{U} L_\alpha$ implies the existance of an ordinal α_σ such that $J_\sigma \in L_{\alpha_\sigma}$. If α_τ is an ordinal such that $\alpha_\sigma < \alpha_\tau$ for every σ, then obviously every semi-ideal J_α is contained in L_{α_τ}. Hence every non-zero homomorphic image of A contains a non-zero semi-ideal in L_{α_τ}. Thus

$A \in L_{\alpha_{\tau+1}} \subseteq \mathcal{L}L$ holds. Thus (R_2^{*}) is satisfied. Hence $\mathcal{L}L$ is a radical class.

Assume that P is a radical class containing the class $L \Rightarrow L_1 \subseteq P$. Suppose $L_\beta \subseteq P$ for every $\beta < \alpha$, and let $A \in L_\alpha$. Then every non-zero homomorphic image of A contains a non-zero semi-ideal I in L_β for some $\beta < \alpha$, and so by hypothesis I belongs to P. Since P is a radical class, by (R_2^{*}) $A \in P$. Thus $\mathcal{L}L \subseteq P$.

Definition: 2.43

The class $\mathcal{L}L = \bigcup_\alpha L_\alpha$ is called the Lower radical class determined by the homomorphically closed class L. The assumption that L should be homomorphically closed, is not an essential restriction; if we start from any class L_o of hemirings, then we can form its homomorphic closure L consisting of all homomorphic images of L_o-hemirings. Now L is homomorphically closed and the procedure can be performed. In this construction we start from a class L satisfying (R_1^{*}). We enlarge it step by step to a class satisfying also (R_2^{*}). Putting $L = L_1$ and performing the first step we get L_2. L_2 satisfies axiom (R_1^{*}) but may be it is not a radical class yet, so we have to continue the procedure to L_3, etc., transfinitely.

Starting from a homomorphically closed class L, we can reach a radical class in one step, if in the above construction we replace semi-ideals by accessible sub-hemirings. Consider YL = {A ε μ: every non-zero homomorphic image of A has a non-zero accessible L-subhemiring}.

Theorem: 2.44

YL is a radical class.

Proof

Suppose A ε YL, and let $\phi(A)$ be any homomorphic image of A. If $\phi(A) = 0$, then clearly $\phi(A)$ ε YL. If $\phi(A) \neq 0$, then every homomorphic image of $\phi(A)$ is also a homomorphic image of A and so has a non-zero accessible L-subhemiring. Thus $\phi(A)$ belongs to YL and so YL is homomorphically closed. Hence $(R_1^{\hat{}})$ is satisfied.

Now assume that every non-zero homomorphic image $\phi(A)$ of A has a non-zero YL-semi-ideal I. Since I ε YL, there exist a non-zero accessible L subhemiring of I, and so of $\phi(A)$. Hence every non-zero homomorphic image of A has a non-zero accessible sub-hemiring in L and consequently A ε YL. Thus YL satisfies $(R_2^{\hat{}})$. So YL is a radical class.

Definition: 2.45

The radical class YL is called Yu-Lee Lee's radical class.

Theorem: 2.46

If P is a radical class, then YP = P.

Proof

Clearly P $\subseteq$ YP. Let A $\in$ YP but A $\notin$ P, then A/P(A) has zero P-radical, that is, it is SP semisimple. Since semisimplicity is hereditary, so every non-zero semi-ideal and also every non-zero accessible subhemiring of A/P(A) is in SP and so has zero radical. Hence the non-zero homomorphic image A/P(A) of A has no non-zero accessible P-subhemiring, and therefore A $\notin$ YP. This contradiction proves the statement.

Theorem: 2.47

YL = LL.

Proof

As L $\subseteq$ YL, by Theorem 2.42 LL $\subseteq$ YL. By Theorem 2.46, Y(LL) = LL. As L $\subseteq$ LL => YL $\subseteq$ Y(LL). Thus

YL $\subseteq$ Y(LL) = LL $\subseteq$ YL => YL = LL.

CHAPTER-3

EXAMPLES OF RADICAL CLASSES

Definition: 3.1

Let S be a hemiring. An element $a \in S$ is called nilpotent provided there is a positive integer n such that $a^n = 0$.

Definition: 3.2

Let S be a hemiring. A complex M of S is called a nil subset provided every element of M is nilpotent.

Definition: 3.3

Let S be a hemiring. A semi-ideal I of S is called nilpotent provided there is a positive integer n such that semi-ideal product $I^n = 0$.

Lemma: 3.4

Let I be a semi-ideal of a hemiring S. Then semi-

ideal product $\mathit{I}^n = 0$ *iff the complex product* $\mathrm{I}^n = 0$, n *being a positive integer.*

Proof

If $\mathit{I}^n = 0$ then obviously $\mathrm{I}^n = 0$. Conversely, if $\mathrm{I}^n = 0$ then, by Definition of I^n, $\mathit{I}^n = 0$.

Lemma: 3.5

If S *is a nilpotent hemiring then every homomorphic image of* S *is nilpotent.*

Proof

As S is nilpotent there exist a positive integer n such that complex product $S^n = 0$. If $\phi(S)$ is any homomorphic image of S then $(\phi(S))^n = \phi(S^n) = \phi(0) = 0$. Thus $\phi(S)$ is nilpotent.

Lemma: 3.6

If S *is semi-isomorphic to* T *i.e.,* $\phi: S \xrightarrow{\sim} T$ *and* I *is a nilpotent semi-ideal of* T *then* $\overline{\phi^{-1}}(\mathrm{I})$ *is also nilpotent semi-ideal of* S.

Proof

Since I is nilpotent, there exist a positive integer n such that $I^n = 0$. Suppose $A = \phi^{-1}(I)$. Then $\phi(A) = I$. Now $(\phi(A))^n = I^n = 0$. Hence $\phi(A^n) = 0$ that is $A^n \subset \ker\phi = \{0\}$. Hence $A^n = 0$.

Lemma: 3.7

Let S be a hemiring then the sum of two nilpotent left [two-sided] semi-ideals of S is nilpotent.

Proof

Let A_1 and A_2 be nilpotent left semi-ideals of S and let n,m be positive intgers such that $A_1^n = 0$, $A_2^m = 0$. Let $B = A_1 + A_2$, we shall show that $B^{n+m-1} = 0$. Let $x \varepsilon B^{n+m-1}$

$B^{n+m-1} = (A_1 + A_2)(A_1 + A_2) \cdots (A_1 + A_2)$, n+m-1 factors.

Then $x = x_1 x_2 \cdots x_{n+m-1}$ where $x_i \varepsilon A_1 + A_2$. Hence

$$x = (a_1^1 + a_2^1)(a_1^2 + a_2^2) \cdots (a_1^{n+m-1} + a_2^{n+m-1})$$
$$= \sum a_{j_1}^{i_1} a_{j_2}^{i_2} a_{j_3}^{i_3} \cdots a_{j_{n+m-1}}^{n+m-1}$$

where $j_k = 1$ or 2 and $i_1, i_2, \cdots, i_{n+m+1}$ is a permutation of $1, 2, \cdots, n+m-1$. Thus x is a finite sum whose summands are products of (n+m-1) factors from A_1 or A_2. Now in each summand appearing in x there must be at least n factors from A_1 or at least m factors from A_2. Suppose s is a summand appearing in x which has at least n factors from A_1. Then s has the form

$$s = \cdots a_1^1 \cdots a_1^2 \cdots a_1^n \cdots \text{ where the dots}$$

indicate factors from A_2. Since $SA_1 \subseteq A_1$, we have $s \in A^n S = 0\ S = \{0\}$, whence $s = 0$. Similarly, if s is a summand appearing in x which has at least m factors from A_2, $s = 0$. Therefore, it follows that $x = 0$. Hence $B^{n+m-1} = 0$. Thus $A_1 + A_2$ is nilpotent.

Theorem: 3.8

If A is a nilpotent left semi-ideal of a hemiring S then the semi-ideal product AS of A and S is a nilpotent two-sided semi-ideal of S.

Proof

If $x,y \in AS$ then $x = \sum_{i=1}^{n} a_i s_i$ and $y = \sum_{j=1}^{n} a_j s_j$ then clearly $x + y \in AS$. If $s \in S$ then $sx = s \sum_{i=1}^{n} (a_i s_i)$ $= \sum_{i=1}^{n} (sa_i)\, si \Rightarrow sx \in AS$, since $sa_i \in A$.

$$xs = \left(\sum_{i=1}^{n} a_i s_i\right) s = \sum_{i=1}^{n} a_i (s_i s) \in AS$$

Thus AS is a two-sided semi-ideal. Let n be a positive integer such that $A^n = 0$. Using the generalized associative law.

$$\begin{aligned}(AS)^n &= (AS)\cdot(AS)\cdots(AS)\\ &= A(SAS\cdots A)S \subseteq A(AA\cdots A)S\\ &\subseteq A^nS = 0\end{aligned}$$

Thus AS is nilpotent.

Theorem: 3.9

Let A be a nilpotent left semi-ideal of a hemiring S. Then A + AS is a nilpotent two-sided semi-ideal of S (and hence is the two-sided semi-ideal generated by A).

Proof

By Theorem 3.8, AS is a nilpotent two-sided semi-ideal of S. By Lemma 3.7, A + AS is the nilpotent left semi-ideal of S. Let $x \in A + AS$, then $x = a + \sum_{i=1}^{n} a_i s_i$. Let $s \in S$, then

$$xs = \left(a + \sum_{i=1}^{n} a_i s_i\right)s = as + \sum_{i=1}^{n} a_i(s_i s)$$

$$= 0 + \left(as + \sum_{i=1}^{n} a_i(s_i s)\right) \in A + AS$$

Thus A + AS is right semi-ideal of S.

If I is any semi-ideal of S containing A then I contains all products $a_i s_i$ where $a_i \in A$ and $s_i \in S$. Hence I contains all finite sums of such products, i.e., $I \supseteq AS$.

Since $I \supseteq A$ and $I \supseteq AS \Rightarrow I \supseteq A + AS$.
It follows that $A + AS$ is the semi-ideal of S generated by A.

On the same lines we can show that if A is a right nilpotent semi-ideal of a hemiring S. Then SA is two-sided nilpotent semi-ideal of S. Also A + SA is a two-sided nilpotent semi-ideal of S generated by A.

Lemma: 3.10

Let S be a hemiring and I a semi-ideal of S. Then for any semi-ideal A of I, $(A')^3 \subseteq A$, where A' is the semi-ideal of S generated by A.

Proof

Clearly the ideal of S generated by A will be of the form $A + AS + SA + SAS$. Now

$$\begin{aligned}(A') &\subseteq I(A + AS + SA + SAS)I \quad \text{since } A' \subseteq I \\ &\subseteq IAI + IASI + ISAI + SASI \\ &\subseteq IAI + IAI + IAI + IAI \\ &\quad \text{since I is a semi-ideal of S.}\end{aligned}$$

As A is a semi-ideal of I, therefore $(A')^3 \subseteq A$.

Theorem: 3.11

Let $\mathcal{B}$ = {A ε μ: every non-zero homomorphic image of A contains a non-zero nilpotent semi-ideal}. Then $\mathcal{B}$ is a radical class and is called Baer's Lower radical class.

Proof

First we show that $\mathcal{B}$ is homomorphically closed. Let $A \in \mathcal{B}$ and $\phi(A)$ be a homomorphic image of A. If $\phi(A) = 0$ then $\phi(A) \in \mathcal{B}$. So suppose $\phi(A) \neq 0$. Every non-zero homomorphic image of $\phi(A)$ is also a homomorphic image of A and so contains a non-zero nilpotent semi-ideal. Thus $\phi(A) \in \mathcal{B}$ and we have (R_1).

Now assume $A \notin \mathcal{B}$. Then there is a non-zero homomorphic image $\phi(A)$ which has no non-zero nilpotent semi-ideal. By Theorem 1.16 there exist a semi-isomorphism $\psi: A/K \longrightarrow \phi(A)$ where $K = \ker \phi$. Suppose N is a non-zero $\mathcal{B}$-ideal of A/K. Then $\psi(N)$ is a non-zero $\mathcal{B}$-ideal of $\phi(A)$. For $\psi(N) = 0 \Rightarrow N = K$ zero element of A/K. Also, $\psi(N)$ has a non-zero nilpotent semi-ideal say I. Note that I is a semi-ideal of $\psi(N)$ and $\psi(N)$ is a semi-ideal of $\phi(A)$. Then by virtue of Lemma 3.10, I', the semi-ideal of $\phi(A)$ generated by I, is also nilpotent and non-zero. But $\phi(A)$ has no non-zero nilpotent semi ideals. This contradiction establishes that $N = 0$ and as result A/K has no non-zero $\mathcal{B}$-ideals, so (R_2) is satisfied.

Theorem: 3.12

$\mathcal{B}$ is the smallest radical class containing all nilpotent hemirings.

Proof

If S is a nilpotent hemiring then every homomorphic image of S is nilpotent, and so $S \in \mathcal{B}$. Thus $\mathcal{B}$ is a radical class containing all nilpotent hemirings. Let P be a radical class containing all nilpotent hemirings. Suppose a hemiring $A \notin P$. Then there exist a proper k-ideal K such that A/K has no non-zero P-semi-ideal. So A/K has no non-zero nilpotent semi-ideal and thus $A \notin \mathcal{B}$. That is $\mathcal{B} \subseteq P$.

Lemma: 3.13

The semisimple class $S\mathcal{B}$ of Baer's lower radical class $\mathcal{B}$ consists of all hemirings which have no non-zero nilpotent semi-ideals.

Proof

If $A \in SB$ then A has no non-zero nilpotent semi-ideal, because if A has some non-zeron nilpotent semi-ideal then every non-zero homomorphic image of A has a non-zero nilpotent semi-ideal. Hence $A \in B$ which is a contradiction to $B \cap SB = 0$.

Now if $A \notin SB$ then $B(A) \neq 0$. Hence $B(A)$ has a non-zero nilpotent semi-ideal, say, N. Then by Lemma 3.10, $N^!$ is a non-zero nilpotent semi-ideal of A. Thus $SB = \{A \in \mu$:A has no non-zero nilpotent semi-ideal.

Theorem: 3.14

$B(S)$ contains every nilpotent left or right semi-ideal of S.

Proof

Let A be a nilpotent left semi-ideal of a hemiring S. Then by Theorem 3.19, A + AS is a nilpotent two-sided semi-ideal of S generated by A. Let ν be a natural homomorphism from S onto $S/B(S)$. Then

$$\nu(A + AS) = 0 \Rightarrow A + AS \subseteq B(S)$$
$$\Rightarrow A \subseteq B(S).$$

Similarly if A is right nilpotent semi-ideal of S then $A \subseteq B(S)$.

Definition: 3.15

A hemiring S is called locally nilpotent if every finite subset F of S generates a nilpotent sub-hemiring of S, or equivalently, if for each finite subset F of S there exist a positive integer n such that every product of n elements from F is zero i.e., $F^n = 0$.

Lemma: 3.16

If A and B are locally nilpotent semi-ideal of a hemiring S, then A + B is a locally nilpotent semi-ideal of S.

Proof

Let $F = \{x_1, \cdots, x_n\}$ be a finite subset of A + B. Then $x_i = a_i + b_i$ where $a_i \in A$, $b_i \in B$, for $i = 1, \cdots, n$.

Let $G = \{a_1, \cdots, a_n\}$, $H = \{b_1, \cdots, b_n\}$, $K = \{a_{i1}, \cdots, a_{ik}b_j: j = 1, \cdots, n\}$ where $a_{i1}, \cdots, a_{ik}$ are all products from G with $k \leq n_1$ where $G^{n_1} = 0$. $L = \{b_{i1}, \cdots, b_{i\ell}a_j: j = 1, \cdots, m\}$ where $b_{i1}, \cdots, b_{i\ell}$ are all products from H with

$\ell \leq n_2$ where $H^{n_2} = 0$.

Now $K \cup L \subseteq A \cap B$ since A and B are assumed to be two-sided semi-ideals. Suppose that n_3 has been determined so that $(G \cup L)^{n_3} = 0$. Also let n_4 be determined so that $(H \cup K)^{n_4} = 0$. Now let $N = 2 \max(n_3, n_4)$. In any monomial occuring in the product, $(a_{i1} + b_{i1}) \cdots (a_{iN} + b_{iN})$ the number of a_{ij}'s plus the number of b_{ik}'s must equal N.

Hence $|a| + |b| = N$ where $|a|$ denotes the number of a_{ij}'s occuring in a nomonial and likewise for $|b|$.

Thus $|a| \geq \frac{1}{2}N = \max(n_3, n_4)$ are $|b| \geq \frac{1}{2}N$. If $|a| \geq \max(n_3, n_4) \geq n_3$, then the monomial is zero by the choice of n_3. Similarly if $|b| \geq \max(n_3, n_4) \geq n_4$ then the monomial is again zero and thus $F^N = 0$. Hence A + B is locally nilpotent.

Lemma: 3.17

If I is a locally nilpotent semi-ideal of a hemiring S then I is also a locally nilpotent ideal of S.*

Proof

We must show that for every finite subset F of I* we can find a positive integer p such that any product of p elements from F is zero. We do so by induction on the number, n, of elements in F which are in $I^* - I$.

Any finite subset F of I^* which has $n = 0$ elements from $I^* - I$ is a subset of I which is locally nilpotent. Now assume for induction that for any finite subset of I^* with $n = k$ elements from $I^* - I$ that there exists a positive integer p such that any product of p elements from this subset is zero.

Let F be any finite subset of I^* with $n = k+1$ elements from $I^* - I$. Choose $x_1 \in F$ such that $x_1 \in I^* - I$ so that by definition of I^*, $x_1 + a = b$ for some $a, b \in I$. Then $F_1 = \{F - \{x_1\}\} \cup \{a,b\}$ is a finite subset of I^* with only $n = k$ elements from $I^* - I$. By induction, there exist a positive integer p_o such that any product of p_o elements from F_1 is zero.

Let $F_2 = F_1 \cup \{x_1\} = F \cup \{a,b\}$. We show that any product of p_o elements from F_2 is zero by induction on the number, m, of times x_1 occurs in the product.

Consider any product of p_o elements from F_2. If x_1 occurs $m = 0$ times the product is entirely of elements from F_1 and thus must be zero. Now assume that any product of p_o elements from F_2 in which x_1 occurs $m = k$ times is zero. Any product of p_o elements from F_2 in which x_1 occurs $m = k + 1$ times can be written $A \cdot x_1 \cdot B$ where A and B are products of elements from F_2. But $x_1 + a = b$ for some $a_1 b \in I$ so

$$A \cdot x_1 \cdot B + A \cdot a \cdot B = A \cdot b \cdot B$$

However, $A \cdot a \cdot B$ and $A \cdot b \cdot B$ are products of p_o elements from F_2 in which x_1 occurs $m = k$ times and as a result must be zero. Consequently $A\ x_1\ B$ is zero.

By induction, then, any proudct of p_o elements from F_2 is zero and since $F \subseteq F_2$, any product of p_o elements from F is zero. Thus we have completed our original induction and we conclude that I* is locally nilpotent.

Theorem: 3.18

$L = \{$all locally nilpotent hemirings$\}$. Then L is a radical class.

Proof

Let $S \in L$ and $\phi(S)$ be a homomorphic image of S. Let $\bar{F}$ be a finite subset of $\phi(S)$. Choose a finite subset F of S such that $F \subseteq \phi^{-1}(\bar{F})$ and $\phi(F) = \bar{F}$. Then there exists n such that $F^n = 0$. But then $\phi(F^n) = (\phi(F))^n = \bar{F}^n = 0$ and $\phi(S) \in L$. This shows (R_1).

To show (R_3), consider the collection of all locally nilpotent semi-ideals of a hemiring S. This collection is non-empty since (0) is locally nilpotent. If T_α is any chain of locally nilpotent semi-ideals of S, then UT_α is an upper bound of the chain, and since any finite subset of UT_α is in some T_β, we have UT_α locally nilpotent. Thus there is a maximal locally nilpotent semi-ideal M of S. By Lemma 3.17 M* is also locally nilpotent ideal, so M must be a k-ideal. Also if I is any other locally nilpotent semi-ideal, then by Lemma 3.16, I + M is locally nilpotent, so $I \subseteq M$. This proves (R_3).

Finally, if K is locally nilpotent semi-ideal of S/M then $\nu^{-1}(K)$ is a locally nilpotent semi-ideal of S containing M. Where ν is the natural homomorphism from S onto S/M. Let $\nu^{-1}(K) = A$. Let F be a finite subset of A. Then $\nu(F)$ ___ $\nu(A) = K$. As K is locally nilpotent, there exist a positive integer n such that $(\nu(F))^n = 0 \Rightarrow \nu(F^n) = 0$ $\Rightarrow F^n$ ___ $\ker \nu = M^*$. Now M is locally nilpotent, M* is locally nilpotent. F^n is a finite subset of M* => there exist a positive integer m such that $(F^n)^m = 0 \Rightarrow F^{nm} = 0$.

Thus $\nu^{-1}(K)$ is locally nilpotent . Since M is maximal locally nilpotent ideal of S, therefore $\nu^{-1}(K) = M \Rightarrow K = 0$. This proves (R_4). Thus L is a radical class.

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