ACOUSTIC FIELD DUE TO A MOVING POINT SOURCE

BY

NAEEM AHMED

SUPERVISED BY

DR. SALEEM ASGHAR

DEPARTMENT OF MATHEMATICS
QUAID-I-AZAM UNIVERSITY
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CERTIFICATE

This dissertation by NAEEM AHMED is accepted in its present form by the Department of Mathematics, Quaid-i-Azam University, Islamabad, as satisfying the dissertation requirements for the degree of Master of Philosophy in Mathematics.

1. ______________________

CHAIRMAN
Dr. Saleem Asghar

2. ______________________

SUPERVISOR
Dr. Saleem Asghar

3. ______________________

EXTERNAL EXAMINER

DEPARTMENT OF MATHEMATICS

QUAID-I-AZAM UNIVERSITY

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DEDICATED TO:

*My Parents, Brother and Sisters,
for their many sacrifices and
especially to my Uncle who always
encouraged me at every stage.*

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To begin with the name of Almighty allah, who provided me a golden opportunity to complete this dissertation through His special blessing.

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PREFACE

Although the work in literature on the problems of basic formulation of linear acoustics for a moving source [see Warren 1] and the related problems of a stationary source in a moving medium [see Lighthill's theory 2] exists. However, in recent years considerable attention has been given to the problem of solving the resulting wave equation for a source moving in the acoustic waveguide formed by the ocean and its boundaries. Very little work has been reported in this case, notable exceptions are the work of Clark, Flanagan, and Weinberg [3] and Guthrie et al. [4].

Clark, Flanagan, and Weinberg [3] studied the moving source problem in terms of ray theory. Guthrie et al.[4] discussed the problem when the source is moving radially with respect to the stationary receiver and obtained the expression for the acoustic field using normal mode theory. It was further extended by Hawker [5] with the assumption that source motion is uniform (unaccelerated) and is not restricted to a path radial to the receiver. He presented the field generated by a slowly moving point source in a waveguide in a horizontally stratified medium. The acoustic field for source moving with arbitrary but small horizontal velocity in terms of retarded time was first considered by Lim et al. [6].

Chapter one introduces basic concepts of acoustics and Fourier transform. A few asymptotic methods are also presented to calculate the integrals approximately for large frequency. We also deal with Green's function technique in some detail.

In chapter two we have discussed the paper by Paul H.Lim and Ozard with titled "On The Underwater Acoutics field Of A Moving Point Source In Range-independent Environment." A temporal Green's function approach is used to find the acoustic field of a range-independent ocean.

In chapter three the acoustic field generated by a moving point source with an arbitrary velocity in a stratified medium in the presence of moving fluid is investigated. The velocity of sound in the medium is assumed to be depth dependent. The acoustic field due to a source moving with arbitrary small velocity in still fluid (Lim's et al.result) can be recorded as a special case by taking the Mach number to be zero.

CHAPTER 1

INTRODUCTION

In this chapter, the basic equations of Fluid Dynamics and Acoustic Waves are presented. Some definitions of generalized functions (Dirac delta function, Hankel function etc.) are also a part of this chapter. After the above definitions the mathematical techniques used in solving the problems presented in chapter 2 and chapter 3 are given. These methods includes the transformation techniques, asymptotic methods, Green's function method etc.

A. DEFINITION AND DEVELOPMENT IN THEORY OF ACOUSTICS

1. ACOUSTICS

Acoustics may be defined as the study of the generation, transmission and reception of energy in the form of vibrational waves in matter. As the atoms or the molecules of a fluid or solid are displaced from their normal configuration an inertial elastic restoring force arises. Examples include the tensile force produced when a spring is stretched, the increase in pressure produced when a fluid is compressed and the transverse restoring force produced when a point on a stretched wire is displaced in a direction normal to its length. It is this elastic restoring force, coupled with the inertia of the system, that enables matter to participate in oscillatory vibrations and there by generate and transmit acoustic wave.

The most familiar acoustic phenomenon is that associated with the sensation of sound. For an average young person, a vibrational disturbance is interpreted as sound if its frequency lies in the range of about 20 to 20,000 Hertz. However, in a broader sense, acoustics also include the ultrasonic frequencies above 20,000 Hertz and infrasonic frequencies below 20 Hertz. The nature of vibration associated with acoustics are for example, the simple sinusoidal vibrations produced by a tuning fork and non periodic motions associated with an explosion.

2. ACOUSTIC WAVE MOTION

Acoustic waves that produce the sensation of sound are one of a variety of pressure disturbances that can propagate through a compressible fluid. There are also ultrasonic and infrasonic waves whose frequencies are beyond the audible limits, e.g. high intensity waves generated by air crafts and explosions.

It will be convenient to start with the simpler case of plane waves. The characteristic property of the plane wave is that each acoustic variable has constant amplitude on any given plane perpendicular to the direction of wave propagation. The propagation of sound is always associated with some medium. Sound does not propagate in vacuum. Sound is generated when the medium is dynamically disturbed. Such disturbance of the medium affects its pressure, density, particle velocity and temperature. Most known fluids and solids have relatively small heat conductivity and sound propagation is nearly adiabatic, even at very low frequencies. Therefore the temperature is of little significance. The effects of gravitational forces will also be neglected so that constant equilibrium density (ρ_o) and constant equilibrium pressure (p_o) have uniform values throughout the fluid. The fluid is also assumed to be homogeneous, isotropic and perfectly elastic; no dissipative effects such as those arising from viscosity or heat conduction are present. Finally, the analysis will be limited to waves of relatively small amplitude so that changes in density of the medium will be small as compared with its equilibrium value. These assumptions are necessary to arrive at the simplest theory for sound in fluids. Fortunately, experiments have shown that this simplest theory adequately describes most common phenomena.

3. THE BASIC EQUATIONS

In this section, we develop some equations of basic equations of acoustics. These definitions are taken from [15].

3.1 THE EQUATION OF STATE

A relation between pressure and density is the adiabatic equation of state is

$$P = f(\rho). \tag{1}$$

Since the change in pressure and density is very small, this equation can be expanded in a Taylor series:

$$P = P_o + \left(\frac{\partial P}{\partial \rho}\right)_{\rho_o}(\rho - \rho_o) + \frac{1}{2}\left(\frac{\partial^2 P}{\partial \rho 2}\right)_{\rho_o}(\rho - \rho_o)^2, \tag{2}$$

where the partial derivatives are constants, determined for adiabatic compression and expansion of the fluid about its equilibrium density. If the fluctuations are small, only the lowest order term $(\rho - \rho_o)$ need be retained. This gives a linear relationship between pressure fluctuation and change in density

$$P - P_o = B(\rho - \rho_o)/\rho_o, \tag{3}$$

where $B = \rho_o\left(\frac{\partial p}{\partial \rho}\right)_{\rho_o}$ is the adiabatic bulk modulus. In terms of acoustic pressure and condensation $\bar{S}$, (3) can be expressed as

$$P = B\bar{S}, \tag{4}$$

where

$$\bar{S} = (\rho - \rho_o)/\rho_o \text{ and } |\bar{S}| \ll 1.$$

3.2 THE EQUATION OF CONTINUITY

To relate the motion of the fluid to the compression or dilation we need a functional relationship between the particle velocity V and instantaneous density ρ. In other words we want to observe what happens if one part of the fluid affects the other part. We represent this effect in equation form, known as the equation of conservation of matter. The equation of conservation of matter in the Eulerian approach is derived as follows:

The total mass in volume T bounded by the surface "s" at time t is $\int_T \rho \, dx$, where ρ is the volume density of the fluid and dx is the volume element. Let us keep this volume fixed in space. Then the increase in mass in a small time δt is $\delta t \int_T (\partial\rho/\partial t) dx$. Since the mass is conserved the increase must be due to flow across the boundary s of T. Now the fluid crosses s is only on account of the velocity component along the normal to s. If $\hat{n}$ is a unit normal vector directed out of T, the mass which is transferred across the small element ds is that obtained in a cylinder of volume n.v δtds, where the vector v is the velocity of the fluid. Hence

$$\int_T (\partial\rho/\partial t) \, dx = - \int_s \rho \, n.v \, ds. \tag{5}$$

Using the divergence theorem, (1.5) can be written as

$$\int_T (\partial\rho/\partial t) \, dx = -\int_T \mathrm{div}(\rho v) \, dx. \tag{6}$$

Because this holds for an arbitrary volume T, we conclude that

$$\partial\rho/\partial t + \mathrm{div}(\rho v) = 0, \qquad (7)$$

which is the Eulerian form for conservation of mass.
Notice that it is non-linear equation. If we write $\rho = \rho_o(1 + \bar{S})$ and use the fact that ρ_o is constant in both space and time and assuming that $\bar{S}$ is very small, equation (7) becomes

$$\partial\bar{S}/\partial t + \mathrm{div}(v) = 0, \qquad (8)$$

which is known as the linearised equation of continuity.

3.3 FUNDAMENTAL EQUATIONS OF MOTION

In fluids the existence of viscosity and the failure of acoustic processes to be perfectly adiabatic introduce dissipative terms. Since we have already neglected the effects of thermal conductivity in the equation of state, we also ignore the effects of viscosity and consider the fluid to be inviscid. The equation of motion comes from the consideration of the forces in a fluid. Let P be the pressure, then the total surface force will be

$$-\int_s P.n ds,$$

where s is the closed surface bounding the volume T of the fluid.

Let f represents the acceleration of the fluid particle, then the total inertial force will be

$$-\int_T \rho\, f\, dx.$$

According to De'Alembert's principle,

Total surface force + Total body force + Total inertial force = 0

Since we are neglecting the body forces like gravity, the above principle takes the form:

$$\int_S P.n \, ds + \int_T \rho \, f \, dx = 0.$$

Using the divergence theorem in this equation we have

$$\int_T \nabla P \, dx + \int_T \rho \, f \, dx = 0.$$

Because this equation holds for an arbitrary volume, we conclude that

$$f = -\frac{1}{\rho} \nabla P.$$

Now using the relationship

$$f = \frac{DV}{Dt} = \frac{\partial V}{\partial t} + (V \cdot \nabla)V,$$

we have, from the last equation

$$\frac{\partial V}{\partial t} + (V \cdot \nabla) V = -\frac{1}{\rho} \nabla P. \tag{9}$$

This non-linear, inviscid force equation is called Euler's equation of motion. It can be simplified if we require $|\bar{S}| \ll 1$ and $|(V \cdot \nabla)V| \ll |\frac{\partial V}{\partial t}|$. Then replacing ρ with ρ_o and dropping the term $(V \cdot \nabla)V$ in equation (9), we obtain

$$\rho_o \frac{\partial V}{\partial t} = -\nabla P. \tag{10}$$

This is the linear inviscid force equation, valid for acoustic processes of small amplitude.

3.4 THE LINEARISED WAVE EQUATION

The two equations (8) and (10) can be combined to yield a single differential equation with one dependent variable. Taking the divergence of equation (10), we have

$$\rho_o \nabla \cdot \frac{\partial V}{\partial t} = - \nabla^2 P , \tag{11}$$

where ∇^2 is the three dimensional Laplacian operator. Now taking the time derivative of (8), we have

$$\frac{\partial^2 \bar{S}}{\partial t^2} + \frac{\partial}{\partial t} (\nabla \cdot V) = 0. \tag{12}$$

Combining equations (11) and (12) and using $\frac{\partial}{\partial t}(\nabla \cdot V) = \nabla \cdot \frac{\partial V}{\partial t}$, we obtain

$$\rho_o \frac{\partial^2 S}{\partial t^2} = \nabla^2 P.$$

Using the equation of state (4), we get

$$\frac{\rho_o}{B} \frac{\partial^2 P}{\partial t^2} = \nabla^2 P.$$

or

$$\nabla^2 P = \frac{1}{c^2} \frac{\partial^2 P}{\partial t^2} , \tag{13}$$

where

$$c = \sqrt{B/\rho_o}. \tag{14}$$

Equation (13) is the linearised, loss less wave equation for the propagation of sound in fluids. In equation (13), c is the phase speed for acoustic waves in fluids. Use of (14) allows the equation of state to be written in a more convenient form

$$c^2 \rho_o \bar{S} = P.$$

Thus the condensation also satisfies the wave equation. Since the curl of the gradient of a function f must vanish, i.e. $\nabla \times \nabla f = 0$, from (10), $\nabla \times (\partial V/\partial t) = 0$. This implies that

$\partial V/\partial t$ can be expressed as the gradient of a scalar function ϕ. For the purposes of dealing with transient effects we can write $V = \nabla\phi$. The physical meaning of this important result is that the acoustical excitation of an inviscid fluid involves no transient rotational flow; there are no effects such as boundary layers, shear waves, or turbulence. In real fluids, for which there is finite viscosity, the particle velocity is not curl-free everywhere but for most acoustic processes the presence of small rotational effects is confined to the vicinity of boundaries and exerts little influence on the propagation of sound. Thus, substituting $V = \nabla\phi$ in (10) we have

$$\rho_o \frac{\partial}{\partial t} \nabla\phi = - \nabla P,$$

or

$$\nabla(\rho_o \frac{\partial \phi}{\partial t} + P) = 0.$$

The quantity in the parenthesis can be chosen to vanish identically if there is no acoustic excitation. This gives

$$P = - \rho_o \frac{\partial \phi}{\partial t} .$$

3.5 GENERAL WAVE EQUATION

A general equation of wave equation can be obtained by taking into account the small perturbations made by the sound waves and dropping the requirement that the flow be insentropic i.e.the entropy S is constant throughout the medium then we have the equations

$$D\rho/Dt + \rho \, div.V = 0, \quad (15)$$

$$DV/Dt = -1/\rho \nabla P, \quad (16)$$

$$DS/Dt = 0, \quad (17)$$

$$P = f(\rho, S). \quad (18)$$

Suppose that there is steady flow in which $V = U$, $P = P_o$, $\rho = \rho_o$ and $S = S_o$ then the equations (15)—(18) take the form

$$U . \nabla\rho + \rho_o \, div.U = 0, \quad (19)$$

$$\rho_o (U . \nabla) U = -\nabla P_o \quad (20)$$

$$U . \nabla S_o = 0, \quad (21)$$

$$\nabla P_o = (\partial f/\partial\rho) \nabla\rho_o + (\partial f/\partial S) \nabla S_o. \quad (22)$$

Where $\partial f/\partial\rho$ and $\partial f/\partial S$ are calculated at $\rho = \rho_o$ and $S = S_o$. Now the speed of sound is given by $C^2 = \partial f/\partial S$. If we write $h = \partial f/\partial S$, then C and h will be known at every point once (19)—(22) have been solved for ρ_o and S_o as function of position.

Let the sound waves make small perturbation so that

$$\rho = \rho_o + \rho_1, \; V = U + U_1 \text{ and } S = S_o + S_1.$$

Neglecting the product of small quantities, we arrive at the following equations:

$$\partial\rho_1/\partial t + U . \nabla\rho_1 + U_1 . \nabla\rho_o + \rho_o did.U_1 + \rho_1 div.U = 0, \quad (23)$$

$$\left.\begin{array}{l} \rho_o \partial U_1/\partial t + \rho_o (U . \nabla) U_1 + \rho_o (U_1 . \nabla) U + \rho_1 (U.\nabla) U = -C^2 \nabla\rho_1 \\ -h \, grad.S_1 - (\rho_1 \partial^2 f/\partial\rho^2 + S_1 \partial^2 f/\partial\rho\partial S) grad.\rho_o \\ -(\rho_1 \partial^2 f/\partial\rho\partial S + S_1 \partial^2 f/\partial S^2) grad.S_o \end{array}\right], \quad (24)$$

$$\partial S_1/\partial t + U \, grad.S_1 + U_1 grad.S_o = 0. \quad (25)$$

It is immediately evident that a background flow complicates the analysis which has to be under taken in order to determine the acoustic disturbance. Some simplification can be achieved for particular cases. For example,suppose that the basic flow consists of a steady velocity $\bar{U}$ parallel to x-axis so that $\bar{U} = \mu i$, where i is a unit vector along the x-axis. Assume further that μ and ρ_0 are constant and that flow is insentropic. Then equations (19)-(22) are certainly satisfied with $C = C_0$ = constant and $h = 0$. Therefore equations.(23) and (24) become

$$\partial\rho_1/\partial t + \mu\ \partial\rho_1/\partial x + \rho_0 \mathrm{div}.U_1 = 0, \tag{26}$$

$$\rho_0\partial U_1/\partial t + \rho_0\mu\ \partial U_1/\partial x = -C^2\nabla\rho_1. \tag{27}$$

To eliminate U_1 from equations.(26) and (27), we take divergence of equation.(27) to give

$$\rho_0\nabla.(\partial U_1/\partial t) + \rho_0\mu\ \nabla.(\partial U_1/\partial x) = -C^2\nabla.\nabla\rho_1,$$

and using

$$\nabla.\partial U_1/\partial t = \partial/\partial t(\nabla.U_1),$$

we obtain

$$\rho_0\partial/\partial t(\nabla.U_1) + \rho_0\mu\ \partial/\partial x(\nabla.U_1) = -C^2\nabla^2\rho_1. \tag{28}$$

Substituting the value of $\mathrm{div}.U_1$ from equation.(26) into (28), we have

$$\partial^2\rho_1/\partial t^2 + 2\mu(\partial^2\rho_1/\partial t\partial x) + \mu^2(\partial^2\rho_1/\partial x^2) = C^2\nabla^2\rho_1. \tag{29}$$

The ratio μ/C is known as Mach number M. If $M < 1$, the flow is said to be subsonic where as if $M > 1$, it is super sonic. equation. Differential equation in terms M can be written as

$$\nabla^2\rho_1 = 1/C^2\ \partial^2\rho_1/\partial t^2 + (2M/C)\partial^2\rho_1/\partial t\partial x + M^2\partial^2\rho_1/\partial x^2.$$

or

$$(1-M^2)\left\{\partial^2/\partial^2+\partial^2/\partial y^2+\partial^2/\partial z^2-2M/C\ \partial^2/\partial t\partial x-1/C^2\partial^2/\partial t^2\right\} = 0. \quad (30)$$

B.MATHEMATICAL PRELIMINARIES

1. ASYMPTOTIC EVALUATION OF INTEGRALS

Here we discuss the method usually adopted to write down the asymptotic form of certain integrals appearing in diffraction problems.

1.1 THE METHOD OF STATIONARY PHASE

In many problems we have to deal with integrals of the form

$$\int_a^b e^{it\phi(\mu)} g(\mu)\ g(\mu), \quad (31)$$

where ϕ is a real valued function, called the phase function, while g may be either real or complex valued. In contrast to Laplace's method, the exponent is now purely imaginary; hence the integrand is an oscillatory function of t. As long as $\phi'(\mu) \neq 0$, we may integrate by parts and conclude that the integral is O(1/t) when $t \longrightarrow \infty$. The main contribution comes from the points (μ_j), where $\phi'(\mu_j) = 0$. These are called stationary points. We assume a finite number of stationary points (μ_j) with $a < \mu_j < b$, $\phi''(\mu_j) \neq 0$, and $\int_a^b |g(\mu)|d\mu < \infty$.

Then, when $t \longrightarrow \infty$,

$$\int_a^b e^{it\phi(\mu)} g(\mu)\, d\mu = \sum_{j:\phi''(\mu_j)>0} \left[\frac{2\pi}{t\phi''(\mu_j)}\right]^{1/2} e^{i\phi(\mu_j)+i\pi/4} g(\mu_j)$$

$$+\sum_{j:\phi''(\mu_j)<0} \left[\frac{2\pi}{t|\phi''(\mu_j)}\right]^{1/2} e^{it\phi(\mu_j)+i\pi/4} g(\mu_j) + O(1/t \qquad (32)$$

In contrast to Laplace's method, we must sum over all stationary points of ϕ not simply those where ϕ is maximum. If the end points $\mu = a$ or $\mu = b$ are stationary points, they contribute to Eq.(28) with a factor of 1/2, just as in Laplace's method.

This complicated-looking formula becomes easier to remember if we restate it in the following fashion: replace $\phi(\mu)$ by its second-order Taylor expansion and replace $g(\mu)$ by its value at the stationary point. Do the resulting integrals, one for each stationary point, and sum over all stationary points.

1.2 THE METHOD OF STEEPEST DESCENT

Consider the integral of the form

$$I_1 = \int_C e^{sg(z)} f(z)\, dz \qquad (33)$$

where c is a contour in the complex z-plane. We assume that s to be large complex variable, g and f to be analytic functions of the complex variable z and the integral to be taken along some path in the complex z-plane. This integral may be evaluated asymptotically by the method of steepest descents, which was originated by Debye. Copson(1946) gives a detailed description of this method.

It will be assumed that f and g are independent of "s" and suitably regular. It will be sufficient to consider the case $s \longrightarrow \infty$ for if $s = |s|\ e^{i\theta}$ we can split sg into $|s|$ and $ge^{i\theta}$. Let $g(z) = U(x,y) + iV(x,y)$ where U and V are real. When s is large, a small displacement causing a small change in V will produce a rapid oscillation of the sinusoidal terms in e^{sg}. In general, the contribution from any one part of the path of integration will be about the same as that from any other part. However, if a path is chosen on which V is constant the rapid oscillation will disappear. Then the contribution will come from the neighborhood of the point s, where U is the greatest. The essence of the method, therefore, consists in deforming the contour, as far as this is possible, into a curve V = constant passing through the point where U = constant. Now, a point where $g'(z) = 0$ is called a saddle point. Let $g'(z) = 0$ at $z = z_o = x_o + iy_o$ Now, at $z = z_o$,

$$U_x = U_y = V_x = V_y = 0$$

because g(z) is analytic. Further,

$$U_{xx} + U_{yy} = V_{xx} + V_{yy} = 0 \Rightarrow U_{xx}\ U_{yy} = -\ U^2_{xx}$$

which further gives

$$\begin{vmatrix} U_{xx} & U_{xy} \\ U_{yx} & U_{yy} \end{vmatrix} = U_{xx}U_{yy} - U^2_{xy} = - [U^2_{xx} + U^2_{xy}] < 0,$$

this implies that $U = U_o$ is neither a maximum nor a minimum. That is why every stationary point is called a saddle point. Near the saddle point $z = z_o$

$$g(z) - g(z_o) = 1/2\ g''(z_o)\ (z-z_o)^2 = 1/2\ Ar^2 e^{i(\alpha+2\theta)}$$

If $z-z_o = re^{i\theta}$ and $g''(z_o) = Ae^{i\alpha}$ where A, α are real with A positive if $g(z_o) = U_o + iV_o$,

$$U - U_o = \frac{A}{2} r^2 \cos(2\theta+\alpha),\ V - V_o = \frac{A}{2} r^2 \sin(2\theta+\alpha).$$

Now, $U-U_o$ is negative if θ is such that its cosine is negative and drops fastest with r if $\theta = (\pi-\alpha)/2$ or $\theta = (3\pi-\alpha)/2$. These are desirable directions because they force exponential decay on the integrand as one moves away from the saddle point. They are called paths of steepest descent. Note that on a path of steepest descent $V = V_o$. Having seen that $V = V_o$ is a good route to start on from a saddle point. Let us see what happens if we stay on it. Suppose we reach a point z_1 and that $z_1 + \rho e^{i\phi}$ is a nearby point on $V = V_o$. Since V does not change in the move from one point to the other, ϕ must satisfy $V_x \cos\phi + V_y \sin\phi = 0$. By means of the Cauchy-Riemann relation this may be expressed as

$$-U_y \cos\phi + U_x \sin\phi = 0. \qquad (34)$$

The change of U in the move is $\rho(U_x\cos\phi + U_y \sin\phi)$. This quantity is known to be negative at z_o. It will therefore remain negative until a point z_1 is arrived at where it is zero. But, on account of (38), this is impossible unless $U_x = U_y = 0$, i.e. z_1 is a saddle point. Hence, on a path of steepest descent, U decreases steadily from a saddle point until another saddle point is reached. Should $g'(z)$ have a singularity on the path, that can upset the apple cart too. It will now be assumed that the path of steepest descent goes off to infinity without encountering another saddle point or singularity of $g'(z)$. Consider I_1 taken along the path of steepest descent that begins from z_o along $\theta = (\pi-\alpha)/2$. Convergence of the integral at infinity is assumed because of the property of U demonstrated above. Indeed, that property guarantees that the main contribution to the integral comes from a neighborhood of the saddle point. In this vicinity

$$g(z) - g(z_o) = -\tfrac{1}{2} Ar^2 \quad \text{and} \quad z-z_o = re^{i(\pi-\alpha)/2}.$$

Hence the integral is essentially

$$f(z_o) \int_0^{\infty} e^{\sigma\{g(z_o)-i/2\, Ar^2\}+i(\pi-\alpha)/2} dr.$$

or

$$\int_{z_o}^{\infty} f(z)\, e^{\sigma g(z)} dz \cong [\pi/(2\sigma A e^{i\alpha})]^{1/2} \; f(z_o) e^{\sigma g(z_o)+\pi i/2}.$$

(35)

Going from z_o to infinity via the path of steepest descent on which $\theta = (3\pi-\alpha)/2$ merely reverses the sign of the right-hand side of (35). The strategy for dealing with I_1 therefore is to deform the contour C as far as possible into a path or paths of steepest descent. Then the contribution of each saddle point is calculated by means of (35). However, we bear in mind that in the deformation of C poles or other singularities of the integrand may be captured; their contribution may be as significant as that from the saddle point. If $g''(z_o) = 0$ there will be more than two paths of steepest descent from the saddle point but on each one the argument about U is unaffected. The mode of calculation is still effective though the asymptotic formula will differ from (35).

2. Generalized Functions

In order to facilitate a variety of operations in mathematical physics, Dirac proposed the introduction of the so called delta function $\delta(x)$. Dirac delta function define to be representative of very high, very narrow peak, centred at $x = 0$, having unit area under the peak and is symbolically given by

$$\delta(x) = \begin{cases} 0 & x \neq 0 \\ \infty & x = 0, \end{cases} \tag{36}$$

such that the integral of $\delta(x)$ is normalized to unity

$$\int_{-\infty}^{\infty} \delta(x)dx = 1 . \tag{37}$$

The first and basic operation to which Dirac sought to subject $\delta(x)$ is the integral

$$\int_{-\infty}^{\infty} \delta(x)f(x)dx,$$

where f(x) is any continuous function. This integral can be evaluated by the following arguments:

Since $\delta(x)$ is zero for $x \neq 0$, the limits of integration may be changed to $-\epsilon$ and $+\epsilon$ where ϵ is a small positive number. Moreover, since f(x) is continuous at x=0, its value within the interval $(-\epsilon, +\epsilon)$ will not differ from f(0) and approximately, we have

$$\int_{-\infty}^{\infty} \delta(x)f(x)dx = \int_{-\epsilon}^{\epsilon} \delta(x)f(x)dx \simeq f(0) \int_{-\epsilon}^{\epsilon} \delta(x)dx.$$

But

$$\int_{-\epsilon}^{\epsilon} \delta(x)dx = 1,$$

for all values of ϵ, because $\delta(x) = 0$ for $x \neq 0$ and $\delta(x)$ is normalized. It appears then letting $\epsilon \longrightarrow 0^{+}$ we have exactly

$$\int_{-\infty}^{\infty} \delta(x)f(x)dx = f(0) \quad . \tag{38}$$

Note that $-\infty$ and $+\infty$ may be replaced by an two numbers 'a' and 'b' provided $a < 0 < b$.

The above integral is sometimes referred to as the shifting property of the delta function. This property may also be expressed for a continuous function f(x), the δ-function which picks out the value of that function at the origin. It turns out that the δ-funciton can be handled algebrically as if it were an ordinary function. Howver, any equation involving δ-function must be understood in the following sense:- If the equatin is multiplied by an arbitrary continuous function f(x) and then integrated from $-\infty$ to ∞, with equation (3.3) used to evaluate the integral involving δ-function, the result will be correct equation involving ordinary functions.

It is also assumed that the usual techniques of integration such as substitution and integration by parts may also be applied to integrals involving δ-function. As an illustration of this, we can easily show that if $\phi(x)$ is a monotonic function of x which vanishes for $x = x_o$, then

$$\delta(\phi(x)) = \frac{\delta(x-x_o)}{|\phi'(x_o)|} \tag{39}$$

Consider the integral

$$\int_{-\infty}^{\infty} \delta(\phi(x)f(x)dx \ , \tag{40}$$

put $\phi(x) = y$ and $\psi(y) = \dfrac{f(x)}{|\phi'(x_o)|}$.

Then the above integral becomes

$$\int_{-\infty}^{\infty} \delta(\phi(x)\psi(y)dy = \psi(0) = \frac{(x-x_o)}{|\phi'(x_o)|} ,$$

$$= \int_{-\infty}^{\infty} \frac{\delta(x-x_o)}{|\phi'(x_o)|} f(x)dx \tag{41}$$

Comparing equations (40) and (41), we obtain

$$\delta(\phi(x)) = \frac{\delta(x-x_o)}{|\phi'(x_o)|} \tag{42}$$

By substituting $\phi(x) = ax-b$ in equation (1.43) we find that

$$\int_{-\infty}^{\infty} \delta(ax-b)f(x)dx = \frac{1}{|a|} f(b/a) \tag{43}$$

If we put $\phi(x) = -x$, we conclude that $\delta(-x) = \delta(x)$, which shows that δ-function is an even function.

Thus we notice thta δ-function can be treated as an ordinary function with the drawback that we talk about the values of integral involving $\delta(x)$ instead of the values of $\delta(x)$.

2.1 THE RELATION BETWEEN δ-CALCULUS AND HEAVISIDE FUNCTION

Consider the integral

$$\int_{-\infty}^{\infty} \delta'(x)f(x)dx ,$$

where $f(x)$ is differentiable. Integrating by parts and assuming that $f(x) \longrightarrow 0$ as $x \longrightarrow \pm\infty$, we obtain,

$$\int_{-\infty}^{\infty} \delta'(x)f(x)dx = -f'(0) \tag{44}$$

It appears that $\delta'(x)$ is associated with the derivative shifting property i.e., $\delta'(x)$ assigns the value $-f'(0)$ to the testing function $f(x)$.

This idea can be developed further, giving rise to the higher order derivative of $\delta(x)$, possessing the property

$$\int_{-\infty}^{\infty} \delta^{m}(\phi(x)f(x)dx = (-1)^{m}f^{m}(0) .$$

It is assumed that the function involved are differentiable m times and $f, f^{(1)}, f^{(2)}, \ldots, f^{(m-1)}$ approach zero as $x \rightarrow \pm\infty$.

Calculation of $\delta(x)$ and its derivatives as functions in ordinary sense is short cut method for obtaining results depending upon certain limits. This procedure may be called δ-calculus. Now we show that δ function itself is the derivative of the function H(x), defined by the relation

$$\int_{-\infty}^{\infty} H(x)f(x)dx = \int_{-\infty}^{\infty} f(x)dx \ . \qquad (45)$$

To see this, we consider the integral

$$\int_{-\infty}^{\infty} H'(x)f(x)dx \ .$$

Integrating by parts, we have

$$\int_{-\infty}^{\infty} H'(x)f(x)dx = - \int_{0}^{\infty} f'(x)dx$$

$$= f(0) = \int_{-\infty}^{\infty} \delta(x)f(x)dx \ .$$

Consequently, H'(x) defined by the equation Eq.(45) is equal to the ordinary function define by

$$H(x) = 1 \quad x > 0$$
$$= 0 \quad x < 0. \qquad (48)$$

This function is well known as the Heaviside unit function. Note that the derivative of the function H(x) is zero for $x < 0$, zero for $x > 0$ and undefined for $x = 0$.

2.2 Delta Function in General Coordinate System

To explain the meaning of such a δ-function, we first consider the case of three dimensional space. Let the three dimensional δ-function $\delta(x,y,z;\ \xi,\eta,\zeta)$ be defined by following equation

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \delta(x,y,z;\ \xi,\eta,\zeta)f(\xi,\eta,\zeta)d\xi d\eta d\zeta = f(x,y,z), \qquad (49)$$

for all continuous functions f(x,y,z) which vanish outside a finite region of xyz space.

If $\delta(x-\xi)$, $\delta(y-\eta)$, $\delta(z-\zeta)$, be the one dimensional δ-function, then

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(\xi,\eta,\zeta)\delta(x-\xi)\delta(y-\eta)\delta(z-\zeta)d\xi d\eta d\zeta = f(x,y,z) \quad (50)$$

Comparing equations (49) and (50) we see that

$$\delta(x,y,z;\ \xi,\eta,\zeta) = \delta(x-\xi)\delta(y-\eta)\delta(z-\zeta) \qquad (51)$$

A similar argument holds for n-dimensional space, therefore we conclude that the n-dimensional δ-function is the product of n one dimensional δ-functions.

3. Transform Techniques

Transform techniques play an important role in obtaining the solution of partial differential equations, especially when the boundary conditions include the infinite or semi-infinite domain. In this section our main stress will be on the fourier transform, which is widely applicable in the diffraction problems.

Fourier series can be used to analyze the periodic motion into its various component frequencies. But the non periodic motion cannot be represented in terms of a fundamental component plus sequence of harmonics; it must be considered as superposition of motion of all frequencies.

Thus the series of harmonic terms must be generalized into an integral over all values of 'ω'. Using the complex exponential form for the component simple harmonic terms, the basic equations for the Fourier integral representation are:

If

$$\left.\begin{aligned} \hat{f}(\omega) &= \int_{-\infty}^{\infty} f(t)\, e^{i\omega t} dt \\ f(t) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{f}(\omega)\, e^{-i\omega t} d\omega \end{aligned}\right\} \qquad (52)$$

Function $\hat{f}(\omega)$ is called the Fourier transform of $f(t)$, and $f(t)$ is the inverse transform of $\hat{f}(\omega)$. The first equation shows that any function of 't' can be expressed in term of a superposition of simple harmonic motion of all possible frequencies $\omega/2\pi$.

Mathematically speaking, both 't' and 'ω' can be considered to be complex quantities and thus the integrals of the equation (52) are counter integrals. It is worth noting that the imaginary part of ω must be positive in order that the integrals in the equation (52) converge. In each case, $\hat{f}(\omega)$ has a pole or poles on the real axis of ω plane. Setting $\omega = \omega + i\epsilon$, the integrand $e^{i\omega t} f(t)$ in equation (52) for $\hat{f}(\omega)$ has an exponential factor $e^{+i\omega t - \epsilon t}$, which ensures convergence of the integral as long as ϵ is positive.

If we are only interested in representing functions f(t) which are zero for negative values of t (i.e, $f(t) = 0$, $t < 0$) it may be more convenient to rotate real-imaginary axis of ω, setting $\omega = is$ $(s > 0)$, so that

$$\hat{f}_L(s) = \hat{f}(s) = \int_0^{\infty} f(t)e^{-st}dt \ , \tag{53}$$

$$f(t) = \frac{1}{2\pi i}\int_{-i\infty+\tau}^{i\infty+\tau} \hat{f}_L(s)e^{st}ds \ , \tag{54}$$

function $\hat{f}_L(s)$ is called the Laplace transform of f(t). It is just a different way of writing the fourier transform, which is useful for the functions f(t) which are zero for $t < 0$.

4. Hankel Functions

Hankel functions of first and second kind are defined respectively by

$$H_n^{(1)}(x) = J_n(x) + iY_n(x), \tag{55}$$

$$H_n^{(2)}(x) = J_n(x) + iY_n(x), \tag{56}$$

where $J_n(x)$ is the Bessel function of first kind of order n and $Y_n(x)$ is the Bessel function of second kind of order n, and both $J_n(x)$ and $Y_n(x)$ are the solutions of the differential equation

$$x^2Y'' + xY' + (x^2 - n^2)Y = 0 \quad n \geq 0, \tag{57}$$

which is called Bessel's differential equation of order n and prime denotes the differentiation w.r.t. `x'. The integral representations of the Bessel functions of order zero are given by

$$J_o(x) = \frac{2}{\pi} \int_o^{\infty} \sin(x\cosh s) ds \ , \tag{58}$$

$$J_o(x) = \frac{-2}{\pi} \int_o^{\infty} \cos(x\cosh s) ds \ , \tag{59}$$

so that

$$\begin{aligned} H_o^{(1)}(x) &= J_o(x) + iY_o(x) \\ &= \frac{-2}{\pi} \int_o^{\infty} e^{ix\cosh s} ds \ . \end{aligned} \tag{60}$$

Now, for the large values of x, the asymptotic formulas for the Bessel functions of order n are given by

$$J_n(x) \sim \sqrt{\frac{2}{\pi x}} \cos(x-\pi/4 - \frac{n\pi}{2}), \tag{61}$$

$$Y_n(x) \sim \sqrt{\frac{2}{\pi x}} \sin(x-\pi/4 - \frac{n\pi}{2}), \tag{62}$$

so the asymptotic formulas for the Hankel functions of first and second kinds of order zero are given by

$$H_o^{(1)}(x) \sim \sqrt{\frac{2}{\pi x}} e^{i(x-\pi 4)} \ , \tag{63}$$

$$H_o^{(2)}(x) \sim \sqrt{\frac{2}{\pi x}} e^{i(x-\pi 4)} \ , \tag{64}$$

5. GREEN'S FUNCTION TECHNIQUE

The green's function technique may be found in many standard books (see e.g stakgold [36]). To illustrate this technique, let us consider the one dimensional boundary vlaue problem

$$-\frac{d^2u}{dx^2} = f(x). \quad 0 < x < 1; \quad u(0) = a, \quad u(1) = b, \tag{65}$$

where f(x) is the source density. The three quantities $\{f(x);a,b\}$ are known collectively as the data for the problem. The data consists of the boundary data a,b and of the forcing function f(x). We are concerned not only with the solution of system (65) for the specific data but also with the finding a suitable form for the solution that exhibits its dependence on the data. Thus as we change the data our expression for the solution remains useful. The feature of system (65) that enables us to achieve this goal is its linearity, as reflected in the superposition principle: If $u_1(x)$ is the solution for the data $\{f_1(x),a_1,b_1\}$ and $u_2(x)$ for the data $\{f_2(x),a_2,b_2\}$ then $Au_1(x) + Bu_2(x)$ is a solution for the data $\{Af_1(x) + Bf_2(x); Aa_1,Ba_1, Ab_1+Bb_2\}$. Thus the superposition principle permits us to decompose complicated data into possibly simpler parts, to solve each of the simpler boundary value problems and to reassemble these solutions to find the solution of the original problem. One decomposition of the data which is often used is

$$\{f(x);a,b\} = \{f(x);0,0\} + \{0;a,b\} \tag{66}$$

The problem with data $\{f(x);0,0\}$ is an inhomogeneous equation with homogeneous boundary conditions; the problem with data $\{0;a,b\}$ is a homogeneous equation with inhomogeneous boundary conditions.

Since we want to solve the problem as compactly as possible for arbitrary data $\{f(x);a,b\}$; the differential operators appearing on the left side of the equality sign in the system (65) are kept fixed; no one is proposing to solve all differential equations with arbitrary conditions at one stroke.

To solve the system (65) for arbitrary data, we introduce an accessory problem where, instead of a distributed density of sources, there is only a concentrated source of unit strength at $x = \zeta$ and where the boundary data vanishes. This solution of the accessory problem is known as the Green's function and is denoted by $G(x,\zeta)$. Here ζ is the position of the source and x is the observation point. We usually regard ζ as a parameter and x as the running variable.

Now we construct $G(x,\zeta)$ on the basis of the information available so far. Since there are no sources in $0 < x < \zeta$ and in $\zeta < x < 1$, we have

$$-\frac{d^2G(x,\zeta)}{dx^2} = 0 , \qquad (68)$$

in both intervals. Therefore, we can write

$$\begin{aligned} G(x,\zeta) &= A_1x + B_1 , \qquad 0 \le x < \zeta , \qquad (69)\\ &= A_2x + B_2 , \qquad \zeta < x \le 1. \end{aligned}$$

Here A_1,A_2, B_1,B_2 are constants, which are independent of x; they may however depend on the parameter ζ. Taking into account the fact that $G(x,\zeta)$ vanishes at $x = 0$ and $x = 1$, we find that

$$G(x,\zeta) = A_1 x, \quad , \quad 0 \le x < \zeta \;, \qquad (70)$$
$$= B_2(1-x) \quad , \quad \zeta < x \le 1.$$

The jump condition for $G'(x,\zeta)$ i.e,

$$G'(x,\zeta)\Big|_{x=\zeta^+} - G'(x,\zeta)\Big|_{x=\zeta^-} = -1. \qquad (71)$$

and continuity of $G(x,\zeta)$ at $x = \zeta$ enable us to calculate A_1 and B_2 in equation (70) from simultaneous equations $-B_2-A_1 = -1$ and $A_1\zeta = B_2(1-\zeta)$. Thus $B_2 = \zeta$ and $A_1 = 1-\zeta$, so that

$$G(x,\zeta) = (1-\zeta)x \quad , \quad 0 \le x < \zeta \;, \qquad (70)$$
$$= (1-x)\xi \quad , \quad \zeta < x \le 1.$$

6. Delta Function Formulation

The Green's function $G(x,\zeta)$ associated with the problem

$$-\frac{d^2u}{dx^2} = f(x). \quad 0 < x < 1; \quad u(0) = a, \quad u(1) = b.$$

satisfies the following equation

$$-\frac{d^2G(x,\zeta)}{dx^2} = \delta(x-\zeta), \; 0 < x < 1, \quad 0 < \zeta < 1;$$
$$G(0,\zeta) = G(1,\zeta) = 0. \qquad (74)$$

Multiplying the equation (73) by $G(x,\zeta)$, equation (74) by u, subtracting and integrating from 0 to 1, we obtain

$$\int_2^1 (Gu'' - uG'')dx = -\int_2^1 f(x)G(x,\xi)dx + \int_2^1 \delta(x-\xi)u(x)dx \;,$$

which reduces to

$$u(\xi) = \int_{2}^{1} f(x)\ G(x,\xi)dx + (Gu' - uG')\Big|_{0}^{1} .$$

Since $G(x,\xi)$ vanishes at the end points, and

$$G'(x,\xi)\big|_{x=0} = 1-\xi, \qquad G'(x,\xi)\big|_{x=1} = 1-\xi ,$$

we have

$$u(\xi) = \int_{2}^{1} f(x)\ G(x,\xi)dx + (1-\xi)a + \xi b.$$

Interchanging the lables x and ξ and using the symmetry of Green's function

$$u(\xi) = \int_{z}^{1} f(\xi)\ G(x,\xi)dx + (1-x)a + xb. \tag{75}$$

Now equation (75) satisfies the system (74) with data $\{f(x);a,b\}$.

CHAPTER 2

THE THEORY FOR THE EVALUATION OF A ACOUSTIC FIELD OF A MOVING SOURCE

INTRODUCTION .

In this chapter, we are going to discuss the paper by Paul H.Lim and Ozard with titled "On The Underwater Acoustics field Of A Moving Point Source In Range-independent Environment."

A temporal Green's function approach is developed to find the acoustic field of a range-independent ocean. The temporal Green's function naturally leads to a solution for the acoustics field in retarded time. This contrasts sharply with conventional solutions in acoustics that are expressed in terms of contemporary time. For a given source motion, the solution may be recast entirely in terms of contemporary time.

1. CONTEMPORARY AND RETARDED TIMES IN ACOUSTICS

We consider an idealized problem of an arbitrary source distribution for which the sound speed c, relative to an inertial frame, is constant. In that frame, the velocity potential ψ satisfies the inhomogeneous wave equation

$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\psi(r,t) = -4\pi\ f(r,t), \tag{1}$$

where $\mathbf{r}$ is the field point position and function f describes the source distribution.The essential solution to this equation is the retarded integeral

$$\psi(r,t) = \int \frac{f(r',t-|R|/c)}{R} d^9r', \tag{2}$$

where

$R = r - r'$,

t is the contemporary time and t' is the retarded time at which the signal is emitted by the source, and the volume integral extends over all space. Retardation in this integral is a reflection of causality: effects at the field point at time t are caused by the source at the retarded (i,e., earlier) time t'. The relation between contemporary time t and retarded time t' is given by

$t' = t + |R|/c.$

The problem of interest in this chapter is that of finding the acoustic field, ψ , of a point source moving with arbitrary motion in a range-independent ocean. The source is taken to be monochromatic with frequency ω_o relative to an inertial frame. Thus wave equation satisfied by the velocity potential ψ in inertial frame can be written as

$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\psi(r,t) = -4\pi\, e^{-\omega_o t}\,\delta(r - r_s(t)), \tag{3}$$

where r and $r_s(t)$ describes the positions of field and the source points respectively.Eq. (2) in this case yields the Lienard-Wiechart potential

$$\psi(r,t) = \frac{e^{-i\omega_o(t-R/c)}}{[R - (v.R)/c]}, \tag{4}$$

where v is the source velocity and square brackets denotes retardation. This potential clearly exhibits the causal connection between the source velocity and the observed field at receiver. Retarded fields are difficult to manipulate, and for such purposes one normally reverts to fields in contemporary time. In principle, one can always recast the retarded potential in such a way that the right side of Eq. (4) is an explicit function of contemporary time t without reference to the retarded time t′.

2. DEVELOPMENT OF THE TEMPORAL GREEN'S FUNCTION

In this section, Green's function approach is used to solve Eq. (3). However, the space time Green's function does not have simple solution if c is z (depth) dependent. To allow for spatially dependent sound-speed profile, consider a temporal Green's function for which ψ in Eq. (3) is an integral over time

$$\psi(r,t) = \int_{-\infty}^{\infty} G(r,t/t')\, e^{-\omega_o t}\, dt' \;, \tag{5}$$

and the temporal Green's function $G(r,t/t')$ satisfies the equation

$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right) G(r,t/t') = -4\pi\delta(r-r_s(t'))\delta(t-t'). \tag{6}$$

Eq. (5) comprises a superposition of all possible waves emanating from the source (at various times and arriving at the field point at same time t). The phase-front trajectories of these waves are not necessarily straight lines. Thus Eq.(6) is a superior solution to that of conventional ray-tracing techniques in that the former takes into account all possible rays connecting the source and field point. To ensure that the Green's function is causal, $G(r,t/t')$ vanishes for time $t' > t$. We define the temporal Fourier transform G_F by the equation

$$G_F(r,w\ /t') = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} G(r,t/t')\, e^{i\omega t}\, dt. \tag{7}$$

Thus,taking Fourier transform of Eq. (6), we get

$$\left[\nabla^2 + k^2(z)\right] G_F = -2\sqrt{2\pi}\;\; \delta(r - r_s(t'))e^{i\omega t'}, \tag{8}$$

where

$$k(z) = \frac{\omega}{c(z)}$$

(ω being the temporal frequency). This equation for G_F can be solved using standard techniques, even for arbitrary time dependence in $r(t')$.

3. SOLUTION FOR THE GREEN'S FUNCTION

In this section, G_F which appears in Eq. (8) is calculated using a normal mode approach. For a range independent ocean the propagating sound can be presented as a linear combination of the normal modes $U_n(z)$. These functions form a complete orthonormal set and satisfy the eigenvalue equation [6]

$$\left(\frac{d^2}{dz^2} + k^2(z) - k_n^2 \right) U_n(z) = 0, \tag{9}$$

where

$$k_n = \frac{\omega}{c_n}$$

(c_n is a real positive constant [6]).

Thus G_F can be expanded as a normal mode sum and is expressed as

$$G_F(r) = \sum_n a_n(x,y)\, U_n(z), \tag{10}$$

where sum extends over all number n and x,y are horizontal cartesian coordinates. With the help of Eqs. (6) and (8) and by using the orthonormality condition the exact equation for the coefficients $a_n(x,y)$ is given by

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + k_n^2 \right) a_n(x,y) = -2\sqrt{2\pi}\; e^{-i\omega t'}\, U_n(z_s)\, \delta(\rho - \rho_s), \tag{11}$$

where subscript 's' indicates the moving source coordinates and ρ is the horizontal vector i.e,

$\rho = xi + yj$.

Now we define Fourier transform and its inverse on the variable x as

$$\left.\begin{aligned} \bar{a}_n(\nu,y) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} a_n(x,y)e^{i\nu x}\, dx, \\ a_n(x,y) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \bar{a}_n(\nu,y)e^{-i\nu x}\, d\nu. \end{aligned}\right\} \qquad (12)$$

Using Eq.(12), Eq.(11) becomes

$$\left[\frac{d^2}{dy^2} - (\nu^2 - k^2)\right]\bar{a}_n(\nu,y) = -2e^{i\omega t'} e^{i\nu x_s} U_n(z_s)\delta(y - y_s) \qquad (13)$$

To solve Eq.(13) we use the method given by Noble [7] P-44. For this we put

$\gamma^2 = \nu^2 - k_s^2$

Then Eq. (13) takes the form

$$\left[\frac{d^2}{dy^2} - \gamma^2\right] \bar{a}_n(\nu,y) = -2e^{i\omega t'} e^{i\nu x_s} U_n(z_s)\delta(y - y_s).$$

Then the solution of $\bar{a}_n(\nu,y)$ is given by

$$\bar{a}_n(\nu,y) = \frac{2e^{i\omega t'} e^{i\nu x_s} U_n(z_s)}{\sqrt{\nu^2 - k_n^2}} \int_{-\infty}^{\infty} e^{-\gamma\,|y - \eta|}\,\delta(y - y_s)\, d\eta.$$

Using the property of Dirac delta function, we obtain

$$\bar{a}_n(\nu,y) = - \frac{e^{i\omega t'} e^{i\nu x_s - \gamma |y - \eta|} U_n(z_s)}{\sqrt{\nu^2 - k^2}}$$

Thus, we have

$$\bar{a}_n(\nu,y) = \frac{e^{i\omega t'} e^{i\nu x_s - \sqrt{\nu^2 - k_n^2}\, |y - y_s|} U_n(z_s)}{\sqrt{\nu^2 - k_n^2}}. \tag{14}$$

If we take the inverse Fourier transform of the Eq.(14), we get

$$a_n(x,y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{e^{i\omega t' + i\nu x_s - \sqrt{\nu^2 - k_n^2}\, |y - y_s| - i\nu x} U_n(z_s)}{\sqrt{\nu^2 - k_n^2}} d\nu. \tag{15}$$

To calculate the co-efficients a_n, the integral appearing in Eq. (15) can be solved asymptotically using steepest descent method [9]. For that, making use of the substitution $\nu = - k_n \cos(\vartheta + ip)$, $x - x_s = D\cos\vartheta$, $|y - y_s| = D\sin\vartheta$, $(0 < \vartheta < \pi)$, $-\infty < p < \infty$ in Eq. (15), assuming $k_n D \gg 1$ and using the standard procedure, discussed in chapter one, we arrive at

$$a_n(x,y) = i\, (\pi/2)^{1/2}\, e^{i\omega t'} U_n\, (z_s)\, H_0^{(1)}(k_n D), \tag{16}$$

where

$$D = |\rho - \rho_s|.$$

Eqs. (10) and (16) together give

$$G_F(r) = \sum_n i\,(\pi/2)^{1/2}\, e^{i\omega t'} U_n(z_s)\, H_o^{(1)}(k_n D)\, U_n(z\ ), \tag{17}$$

where

$H_o^{(1)}$ is the zeroth order Hankel function with the asymptotic representation

$$H_o^{(1)}(k_n\ D) = \sqrt{\frac{2}{\pi k_n\ D}}\ e^{i(k_n D\ -\ \pi/4)}\ . \tag{18}$$

Using Fourier inversion given by Eq. (7). Eq.(17) gives

$$G(r,t/t') = i/2\sum_n \int_{-\infty}^{\infty} U_n(z_s) U_n(z) H_o^{(1)}(k_n D(t'))e^{-i\omega(t\ -\ t')}d\omega. \tag{19}$$

4. CAUSALITY CONDITION AND THE TEMPORAL GREEN'S FUNCTION

In Eq. (19), the function $k_n(\omega)$ has a branch cut in the complex ω plane. This cut may be chosen to lie on the straight line running from $\omega = \infty$ to $\omega = A_n$ [A_n is the largest zero of $k_n(\omega)$]. The causality condition $G(r,t/t') = 0$ for $t' > t$, determines that the contour of integration in Eq. (16) runs above the cut in the ω plane. Thus, closing the contour of integration from above [6], we have the following stronger causality condition

$$G(r,t/t') = 0 \text{ for } t - t' < D(t')\,\frac{dk_n}{d\omega}\,(\infty), \tag{20}$$

where

$$\frac{1}{c_n} = \frac{dk_n}{d\omega},$$

the reciprocal modal group speed, is evaluated at infinite frequency. Note that the branch cuts are required for self-consistency of theory; for times t′ for which G(r, t/t′) exists, Eq. (19) may be recasted as an integral around the branch cut (BC)

$$G(r,t/t') = -i/2\sum_n \int_{BC} U_n(z_s)U_n(z)H_o^{(1)}(k_nD(t'))e^{-i\omega(t - t')}d\omega.$$

(The path of the BC integral is as follows: ω varies from −∞ to A_n below the cut, and then varies from A_n to −∞ above the cut.) From Eqs. (5) and (19), we arrive at

$$\psi(r,t) = i/2 \int_{-\infty}^{\infty}dt' \; e^{-i\omega_o t'} \sum_n \int_{-\infty}^{\infty} U_n(z_s)\; U_n(z)\; H_o^{(1)}(k_nD(t'))$$

$$\times e^{-i\omega(t - t')}\; d\omega. \tag{21}$$

Alternatively, Eq. (19) could be employed to obtian the following equivalent representation for the acoustic field as an integral around BC;

$$\psi(r,t) = i/2 \int_{-\infty}^{t-D(dk_n/d\omega)}dt' \; e^{-i\omega_o t'} \sum_n \int_{-\infty}^{\infty} U_n(z_s)\; U_n(z)$$

$$\times H_o^{(1)}(k_nD(t'))e^{-i\omega(t - t')}d\omega. \tag{22}$$

These expressions represent the acoustic field at time t as a superposition of contributions at frequency ω and at retarded time t'. For a given frequency ω, the causalty condition is that contributing retarded time t' lies on or inside the past sound cone at the field point in space-time. The sound cone is defined by signals traveling with group speed

$$c_n(\omega) = \frac{d\omega}{dk_n} \tag{23}$$

Hence the causality condition for the nth mode takes the form

$$t - t' > D(t')/c_n(\omega). \tag{24}$$

Two features of Eqs. (21)and for the acoustic field are to be noted. First, the terms D and z_n appearing in the summation are generally t' dependent. Second, in this formal solution, t' integration is made after the ω integration. In this chapter, it is now taken that the source moves only in the horizontal plane, thus z_n is henceforth independent of t'.

5. SOLUTION FOR THE ACOUSTIC FIELD

With the original order of integration, the integrals of Eq. (21) appear to be extremely difficult to evaluate analytically. In order to evaluate the integrals in Eq. (21) we reverse the order of integration which is ensured since both the integrals appearing in Eq. (21) are absolutely convergent [6]. Thus, a change of order of integration in Eq. (21) yields

$$\psi(r,t) = i/2 \sum_n \int_{-\infty}^{\infty} U_n(z_s) U_n(z) e^{-i\omega t}\, d\omega \int_{-\infty}^{\infty} dt'\, e^{-i\omega_o t'} H_o\, e^{i\omega t'}. \tag{25}$$

In order to calculate the integrals in Eq. (24), let us denote

$$I = \int_{-\infty}^{\infty} e^{-i\omega(t - t')} H_o\, dt'.$$

Using integration by parts, we have

$$I = \left| \frac{e^{i(\omega - \omega_o)t'}}{i(\omega - \omega_o)} H_o \right|_{-\infty}^{\infty} - \frac{1}{i(\omega - \omega_o)} \int_{-\infty}^{\infty} \frac{dH_o}{dt'} e^{i(\omega - \omega_o)t'} dt'. \tag{26}$$

Now this term could be inserted into Eq. (25), with the order of integration reversed yet again. The asymptotic boundary term, in square brackets, vanishes on integration over ω, since the term $e^{i(\omega-\omega_o)t'}$ oscillates with infinite frequency. Thus the acoustic field is given by Eq. (25) and Eq.(26) in which the boundary term is set to zero. Another reversal of integration order results in

$$\psi(r,t) = -1/2 \sum_n \int_{-\infty}^{\infty} e^{-i\omega_o t'} dt' \int_{-\infty}^{\infty} \frac{U_n(z_s) U_n(z) e^{i(\omega - \omega_o)t'}}{(\omega - \omega_o)} \times \frac{dH_o}{dt'}\, d\omega. \tag{27}$$

The contour of ω integration must run over the pole at ω_o otherwise, the condition of Eq. (20)is contradicted. Now we consider the ω integral of Eq.(27). Application of residue theorem to this integral with causality condition for nth reduces Eq. (27)to the form [Pennsi]

$$\psi(r,t) = i\pi \sum_n \int_{-\infty}^{t-D/c_n(\infty)} e^{-\omega_o t'} \left\{ U_n(z_s)U_n(z)e^{-i\omega\tau} \frac{dH_o}{dt'} \times S(\tau - D/c_n) \right\}_{\omega = \omega_o} dt' + 1/2 \sum_n \int_{-\infty}^{t-D/c_n(\infty)} e^{-\omega_o t'}$$

$$\int_{-\infty}^{\infty} \frac{U_n(z_s)U_n(z)e^{-i\omega\tau}}{(\omega - \omega_o)} \frac{dH_o}{dt'} d\omega\, dt', \qquad (28)$$

where $c_n(\infty)$ is the model group speed at infinite frequency and $S(\tau - D/c_n)$ is Heaviside step function given by

$$S(\tau - D/c_n) = \begin{cases} 0, & \tau - D/c_n < 0 \\ 1. & \text{otherwise.} \end{cases}$$

where

$$\tau = t - t'.$$

Now, we consider only the first integral of Eq. (28)

$$I = i\pi \sum_n \int_{-\infty}^{t-D/c_n(\infty)} e^{-\omega_o t'} \left\{ U_n(z_s)U_n(z)e^{-i\omega\tau} \frac{dH_o}{dt'} \times S(\tau - D/c_n) \right\}_{\omega = \omega_o} dt',$$

evaluting the values $U(z_n)$ and $U(z)$ for $\omega = \omega_o$, we obtain

$$I = i\pi \sum_n \int_{-\infty}^{t-D/c_n(\infty)} e^{-\omega_o t'} \left\{ U_n^o(z_s) U_n^o(z) e^{-i\omega\tau} \frac{dH_o}{dt'} \right.$$

$$\left. \times S(\tau - D/c\ ) \right\} t',$$

using the defination of Heavisde step function, we get

$$I = i\pi \sum_n \int_{-\infty}^{t-D/c_n(\infty)} \left[e^{-\omega_o t'} \quad U_n^o(z_s) U_n^o(z)\ H_o \right] dt',$$

on integration the above equation takes the form

$$I = i\pi \sum_n \left[e^{-\omega_o t} U_n^o(z_s) U_n^o(z)\ H_o(t - D/c_n) \right.$$

$$\left. - U_n^o(z_s) U_n^o(z)\ H_o(-\infty\ ) \right].$$

Now the first contribution, from the pole at ω_o, takes a particularly simple form in the radiation zone. This term is reduced by Eq. (18), to yield the acoustic field

$$\psi(r,t) = i\sqrt{2\pi}\, e^{-i(\pi/4 + \omega_o t)} \sum_n U_n^o(z_s)\ U_n^o(z)\ \frac{e^{ik_n^o D(t')}}{\sqrt{k_n^o D(t')}},$$

$$1/2 \sum_n \int_{-\infty}^{t-D/c_n(\infty)} e^{-\omega_o t'} \int_{-\infty}^{\infty} \frac{U_n(z_s) U_n(z) e^{-i\omega\tau}}{(\omega - \omega_o)} \frac{dH_o}{dt'}\, d\omega\, dt'$$

$$+ \psi_{nr}, \qquad (29)$$

where ψ_{nr} is a non radiative contribution arising from H_o at retarded time $t' = -\infty$. This contribution is negligible as compared to other contribution in Eq.(29). Moreover, Eq. (29) has a contribution from the pole at ω_o plus a contribution from branch cut. For the case of a slowly moving source with the field point in the radiation zone, the branch cut contribution may be neglected resulting in the solution

$$\psi(r,t) = i\sqrt{2\pi}\ e^{-i(\pi/4\ +\omega_o t)} \sum_n U_n^o(z_s)\ U_n^o(z)\ \frac{e^{ik_n^o D(t')}}{\sqrt{k_n^o D(t')}},$$

where

$$t' = t - \frac{D(t')}{c_n^o} \text{ is the retarded time.}$$

c_n^o being the modal group speed of Eq. (23), and superscripts o indicate functions (U_n, k_n, and c_n) evaluated at frequency ω_o.

6. SOLUTION FOR A SLOW MOVING SOURCE IN TERMS OF CONTEMPORARY TIME

The source receiver horizontal retarded distance $D(t')$ is related to that of contemporary distance $D(t)$ as

$$D(t') = \Delta + D(t), \tag{30}$$

where Δ is the horizontal displacement of the source from its position at t' to that of at time t. For a slowly moving source the velocity v is much smaller than all asymptotic modal group velocities c_n ($|\Delta| \ll D(t)$) and Eq. (30) takes the form

$$D(t') = D(t) \left[1 + \frac{\Delta.\hat{D}(t)}{D(t)} + O(\frac{\Delta}{D(t)})^2 \right]. \tag{31}$$

For constant horizontal velocity, the source displacement is given by

$$\Delta = \frac{vD(t')}{c_n},$$

We define

$$\sin\sigma(t) = - \hat{v}.\hat{D}(t), \quad \sigma(t) = \frac{\pi}{2} - \vartheta, \tag{32}$$

where ϑ is the angle between $-\hat{v}$ and $\hat{D}(t)$. Using this result, Eq. (31) takes the form

$$D(t') = D(t) \left[1 - \frac{v \sin\sigma(t)}{c_n} + O(\frac{v}{c_n})^2 \right]. \tag{33}$$

For slow moving source the terms of $O(\frac{v}{c_n})^2$ in Eq. (33) can be ignored and the wave function $\psi(r,t)$ (Eq.(25)) can be finally written as

$$\psi(r,t) = i\sqrt{2\pi}\, e^{-i\pi/4 \, -\omega_o t} \sum_n U_n^o(z_s)\, U_n^o(z) \frac{e^{ik_n^o D(t')}}{\sqrt{k_n^o D(t')}}, \tag{34}$$

where

$$D(t') = D(t) \left[1 - \frac{v \sin\sigma(t)}{c_n} \right]. \tag{35}$$

CHAPTER 3

PROPAGATION OF A MOVING POINT SOURCE IN A STRATIFIED MEDIUM WITH FLUID FLOW

1. INTRODUCTION

In this chapter, we investigate the acoustic field for moving source having small arbitrary horizontal velocity in the presence of a moving fluid. This consideration will help to understand the effects of fluid on the acoustic field and will go a step further to complete the discussion for moving source with arbitrary small velocity. The velocity of sound in the medium is assumed to be depth dependent. The acoustic field is evaluated using temporal Green's function method. The acoustic field due to a source moving with arbitrary small velocity in still fluid (Lim's et al.result) can be recorded as a special case by taking the Mach number to be zero. It is also shown that Hawker's result for moving source having constant horizontal velocity is also recovered by considering the source motion to be uniform in still fluid.

2. THE TEMPORAL GREEN'S FUNCTION OF THE PROBLEM

In this chapter, we find the acoustic field of an underwater source moving with arbitrary velocity v in a range independent ocean. The acoustic velocity [c = c(z)] of an under water source is considered to be depth dependent in a stratified medium. The whole system is assumed to be in a fluid moving with subsonic velocity U parallel to the x-axis. The perturbation velocity u of the irrotational sound wave can be written as u = gradψ. The resulting pressure in the sound field is given by

$$p = -\rho_o\left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x}\right)\psi,$$

where ρ_o is the density of the undisturbed stream. The wave equation satisfied by the velocity potential ψ in case of a moving point source with fixed frequency ω_o relative to inertial frame in the presence of a moving fluid, takes the form

$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \frac{2M}{c}\frac{\partial^2}{\partial t\partial x} - \frac{M^2\partial^2}{\partial x^2}\right)\psi(r,t)$$

$$= -4\pi e^{-i\omega_o t}\,\delta(r - r_s(t)), \qquad (1)$$

where ∇^2 is usual Laplacian, r and $r_s(t)$ describes the positions of field and source points respectively. In Eq.(1) $M = \frac{U}{c}$ is the Mach number and for subsonic flow $|M| < 1$. Also

$$t = t' + \frac{|R|}{c}, \tag{2}$$

where $R = r - r'$,

t is the contemporary time and t' is the retarded time at which the signal is emitted by the source. The Green's function approach which is used to solve Eq. (1) does not have simple solution if c is z (depth) dependent. We, therefore consider a temporal Green's function for which ψ in Eq. (1) is an integral over time

$$\psi\ (r,t) = \int_{-\infty}^{\infty} G(r,t/t')\ e^{-i\omega_o t'}\ dt' , \tag{3}$$

where the temporal Green's function G(r,t/t') satisfies

$$\left[\nabla^2 - \frac{1}{c^2}\ \frac{\partial^2}{\partial t^2} - \frac{2M\ \partial^2}{c\ \partial t \partial x} - \frac{M^2 \partial^2}{\partial x^2}\right] G(r,t/t')$$

$$= -4\pi\delta(r-r_s(t'))\delta(t-t'). \tag{4}$$

It is noted that Eq. (3) comprises a superposition of all possible waves emanating from the source (at various times and arriving at the field point at same time t). The phase-front trajectories of these waves are not necessarily straight lines. To ensure that the Green's function is casual, G(r,t/t') vanishes for time t' > t. For finding the

Green's function, we define the temporal Fourier transform G_F as

$$G_F(r,\omega\ /t') = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} G(r,t/t')\ e^{i\omega t}\ dt, \tag{5}$$

Taking Fourier transform of Eq. (4), we get

$$\left[\nabla^2 + k^2(z) + 2iMk(z)\ \frac{\partial}{\partial x} - \frac{M^2\partial^2}{\partial x^2}\right]\ G_F(r,\omega/t')$$

$$= -2\sqrt{2\pi}\ \ \delta(r - r_s(t'))e^{i\omega t'}, \tag{6}$$

where $k(z) = \frac{\omega}{c(z)}$ (ω being temporal frequency). Since we are dealing with subsonic flow, therefore we can make the following substitutions

$$x = (1 - M^2)^{1/2}X,\quad y = Y,\quad z = Z,\quad k(z) = (1 - M^2)^{1/2}K(Z),$$

$$x_s = (1 - M^2)^{1/2}X_s,\quad y_s = Y_s,\quad z_s = Z_s,$$

$$G_F(r,\omega/t') = \xi(X,Y,Z,\omega/t')\ e^{-iK(Z)MX}. \tag{7}$$

Using these substitutions, Eq. (6) takes the form

$$\left(\frac{\partial^2}{\partial X^2} + \frac{\partial^2}{\partial Y^2} + \frac{\partial^2}{\partial Z^2} + K^2(Z)\right]\xi(X,Y,Z,\omega/t')$$

$$= \frac{-2\sqrt{2\pi}\ e^{i\omega t' + iK(Z)MX}}{(1 - M^2)^{1/2}}\ \delta(X-X_s)\delta(Y-Y_s)\delta(Z-Z_s). \tag{8}$$

$\xi(X,Y,Z,\omega/t')$ can be found from Eq. (8) by using standard techniques, even for arbitrary time dependence in $(X_s(t'), Y_s(t'), Z_s(t'))$. In the next section, ξ is calculated using a normal mode approach.

3. SOLUTION FOR THE GREEN'S FUNCTION

For a range independent ocean the propagating sound can be a linear combination of the normal modes $u_n(Z)$. These functions form a complete orthonormal set and satisfy the eigen value equation [8]

$$\left[\frac{d^2}{dZ^2} + K^2(Z) - K_n^2 \right] u_n(Z) = 0, \qquad (9)$$

where

$$K_n = \frac{\omega}{c_n} \text{ (c_n is a real positive constant [6]).}$$

Thus ξ which appears in Eq. (8) can be expanded as a normal mode sum,

$$\xi(X,Y,Z) = \sum_n a_n(X,Y) u_n(Z), \qquad (10)$$

where sum extends over all number n and X,Y are horizontal cartesian coordinates. With the help of Eqs. (8) and (10) by using the orthonormality condition the exact equation for the coefficients $a_n(X,Y)$ is given by

$$\left[\frac{\partial^2}{\partial X^2} + \frac{\partial^2}{\partial Y^2} + K_n^2 \right] a_n(X,Y) = -2\sqrt{2\pi}\, e^{-i\omega t' + iK(Zs\qquad)MX}$$

$$(1 - M^2)^{1/2}\, u\ (Zs)\delta(\rho - \rho(t')), \qquad (11)$$

where a subscript "s" indicates the moving source coordinate and ρ is the horizontal vector i.e.,

$\rho = Xi + Yj.$

Now, we define the Fourier transform and its inverse on X as

$$\left.\begin{aligned} \bar{a}_n(\alpha,Y) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} a_n(X,Y) e^{i\alpha X}\, dX, \\ a_n(X,Y) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \bar{a}_n(\alpha,Y) e^{-i\alpha X}\, d\alpha. \end{aligned}\right\} \qquad (12)$$

Taking Fourier transform of Eq. (11) by using Eq. (12), we have

$$\left[\frac{d^2}{dY^2} - (\alpha^2 - K_n^2) \right] \bar{a}_n(\alpha,Y) = -2e^{i\omega t'} e^{i\alpha X_s + iK(Z_s)MX_s} U_n(Z_s)\delta(Y-Y_s). \qquad (13)$$

The solution of Eq. (13) can be written in a straight forward manner as [8]

$$\bar{a}_n(\alpha,Y) = \frac{e^{i\omega t'} e^{i\alpha X_s + iK(Z_s)MX_s - \sqrt{\alpha^2 - K_n^2}\,|Y - Y_s|}\, U_n(Z_s)}{\sqrt{\alpha^2 - K_n^2}}. \qquad (14)$$

The inverse Fourier transform of Eq.(14) yields

$$a_n(X,Y) = U_n(Z_s)\, e^{i\omega t' + iK(Z_s)MX_s}$$

$$\times \int_{-\infty}^{\infty} \frac{e^{-i\alpha(X-X_s) - \sqrt{\alpha^2 - K_n^2}\,|Y - Y_s|}}{\sqrt{\alpha^2 - K_n^2}} \, d\alpha. \tag{15}$$

The integral appearing in Eq.(15) can be evaluated asymptotically by using the steepest descent method [9]. For that we put, $X-X_s = D\cos\vartheta$, $|Y-Y_s| = D\sin\vartheta$ and deform the contour by the transformation $\alpha = -K_n\cos(\vartheta + i\nu)$, $(0 < \vartheta < \pi,$ $-\infty < \nu < \infty)$. Hence for large $K_n D$

$$a_n(X,Y) = i\,(\pi/2)^{1/2}\, U_n(Z_s)\, H_o^{(1)}(K_n D) e^{i\omega t' + iK(Z_s)MX_s}, \tag{16}$$

where $D = |\rho - \rho_s|$. Now substitution of Eq. (16) in Eq. (10), yields

$$\xi(X,Y,Z) = \sum_n i\,(\pi/2)^{1/2}\, U_n(Z_s)\, U_n(Z) H_o^{(1)}(K_n D)$$

$$\times\, e^{i\omega t' + iK(Z_s)MX_s}, \tag{17}$$

where $H_o^{(1)}$ is the zeroth order Hankel function with the following asymptotic representation

$$H_o^{(1)}(K_n D) = (2/\pi)^{1/2} \frac{1}{(K_n D)^{1/2}}\, e^{i(K_n D - \pi/4)}. \tag{18}$$

Now from Eqs. (7) and (17) we have

$$G_F = \sum_n i\,(\pi/2)^{1/2}\, U_n(Z_s)\, U_n(Z) H_o^{(1)}(K_n D)$$

$$\times\ e^{i\omega t' + iK(Z_s)MX_s - iK(Z)MX} . \qquad (19)$$

The original Green's function can now be recovered from Eq. (19) using Fourier inversion of Eq. (5). This gives

$$G(r,t/t') = \frac{i}{2}\sum_n \int_{-\infty}^{\infty} H_0^{(1)}(K_n D(t'))\, e^{iK(Z_s)MX_s - iK(Z)MX}$$

$$(1 - M^2)^{-1/2}\, U_n(Z_s)U_n(Z)e^{-i\omega(t - t')}d\omega. \qquad (20)$$

4. CAUSALITY AND THE TEMPORAL GREEN'S FUNCTION

In Eq. (20), the function $K_n(\omega)$ has a branch cut in the complex ω plane. This cut may be chosen to lie on the straight line running from $\omega = \infty$ to $\omega = A_n$ [A_n is the largest zero of $K_n(\omega)$]. The causality condition $G(r,t/t') = 0$ for $t' > t$ then determines that the contour of integration in Eq. (20) runs above the cut in the ω plane. Thus, closing the contour from above, we have the stronger causality condition

$$G(r,t/t') = 0 \text{ for } t - t' < D(t')\frac{dK_n}{d\omega}(\infty), \qquad (21)$$

where $\frac{1}{c_n} = \frac{dK_n}{d\omega}$, the reciprocal model group speed, is evaluated at infinite frequency. Now from Eqs. (3) and (20), we have

$$\psi(r,t) = i/2 \int_{-\infty}^{\infty} dt' \; e^{-i\omega_o t'} \sum_n \int_{-\infty}^{\infty} U_n(Z_s)\; U_n(Z)\; H_o^{(1)}(K_n D(t'))$$

$$(1 - M^2)^{-1/2} e^{iK(Z_s)MX_s - iK(Z)MX} \; e^{-i\omega(t - t')} \; d\omega. \qquad (21)$$

This expression represents the acoustic field at time t as a superposition of contributions at frequency ω and at retarded times t'. A change of order of integration of Eq. (21) yields

$$\psi(r,t) = i/2 \sum_n \int_{-\infty}^{\infty} U_n(Z_s) U_n(Z) e^{iK(Z_s)MX_s - iK(Z)MX} \; e^{-i\omega t} \; d\omega$$

$$\times \int_{-\infty}^{\infty} e^{i(\omega - \omega_o)t'} H_o \; dt'. \qquad (22)$$

In order to calculate the integrals in Eq. (22), let us denote

$$I = \int_{-\infty}^{\infty} e^{i(\omega - \omega_o)t'} H_o \; dt'.$$

Using integration by parts we have

$$I = \left[\frac{e^{i(\omega - \omega_o)t'}}{i(\omega - \omega_o)} H_o \right]_{-\infty}^{\infty} - \frac{1}{i(\omega - \omega_o)} \int_{-\infty}^{\infty} e^{i(\omega - \omega_o)t'} \frac{dH_o}{dt'} dt'. \qquad (23)$$

Since the term $e^{i(\omega - \omega_o)t'}$ oscillates with infinite frequency, so the term in square brackets vanishes on integration over ω. Thus the acoustic field is given by Eqs.

(22) and (23) in which the boundary term is set to zero. Another reversal of integration order results

$$\psi(r,t) = i(\pi/2)^{1/2} \frac{e^{iK(Z_s)MX_s - iK(Z)MX}}{(1-M^2)^{1/2}} \sum_n \int_{-\infty}^{\infty} e^{-i\omega_o t'} dt'$$

$$\times \int_{\infty}^{\infty} U_n(Z_s)U_n(Z) \frac{e^{-i\omega(t-t')}}{(\omega-\omega_o)} \frac{dH_o}{dt'} d\omega, \qquad (24)$$

The required field is that of Eq. (24). Consider the ω integral in this expression. Application of the residue theorem to this integral with $t - t' > D(t')/c_n(\omega)$ $[c_n(\omega) = \frac{dK_n}{d\omega}]$, for the causality condition in the nth mode, reduces Eq. (24) to the form

$$\psi(r,t) = i\sqrt{2\pi} \frac{e^{iK(Z_s)MX_s - iK(Z)MX}}{(1-M^2)^{1/2}} \sum_n \int_{-\infty}^{t-D/c_n(\infty)} \left\{ e^{-i\omega_o t'} \right.$$

$$\left. \times U_n(Z_s)U_n(Z)e^{-i\omega(t-t')} S(\tau - D/c_n) \frac{dH_o}{dt'} \right\}_{\omega=\omega_o} dt'$$

$$+ 1/2 \frac{e^{iK(Z_s)MX_s - iK(Z)MX}}{(1-M^2)^{1/2}} \sum_n \int_{-\infty}^{t-D/c_n(\infty)} e^{-i\omega_o t'} \int_{BC} U_n(Z_s)U_n(Z)$$

$$\times e^{-i\omega(t-t')} S(\tau - D/c_n) \frac{dH_o}{dt'} d\omega\, dt'. \qquad (25)$$

In Eq. (25), $c_n(\infty)$ is the modal group speed at infinite frequency, $S(\tau - D/c_n)$ is the Heaviside step function given by

$$S(\tau - D/c_n) = \begin{cases} 0, & \tau - D/c_n < 0 \\ 1, & \text{otherwise}, \end{cases}$$

where

$\tau = t - t'$.

Now taking contribution from the pole at ω_o in the radiation zone from Eq. (25) and using Eq (18) we have

$$\psi(r,t) = i\sqrt{2\pi}\ \frac{e^{iK(Z_s)MX_s - iK(Z)MX}}{(1-M^2)^{1/2}}\ e^{-i\pi/4\ -\omega_o t}$$

$$\times \sum_n U_n^o(Z_s)\ U_n^o(Z)\ \frac{e^{iK_n^o D(t')}}{\sqrt{K_n^o D(t')}} + 1/2\ \frac{e^{iK(Z_s)MX_s - iK(Z)MX}}{(1-M^2)^{1/2}}$$

$$\times \sum_n \int_{-\infty}^{t-D/c_n(\infty)} e^{-i\omega_o t'} \int_{BC} U_n(Z_s)U_n(Z)\ e^{-i\ \omega(t-t')}$$

$$S(\tau - D/c_n)\ \frac{dH_o}{dt'}\ d\omega\ dt' + \psi_{nr}\ . \qquad (26)$$

In Eq. (26), ψ_{nr} is a non radiative contribution arising from Eq (18) at retarded time $t' = -\infty$. For the case of a slowly moving source with the field point in the radiation zone, and with the assumption that $K_n(\omega_o)$ be real, the branch cut contribution may be neglected resulting in the solution from Eq. (26) as

$$\psi(r,t) = i\sqrt{2\pi}\ \frac{e^{iK(Z_s)MX_s - iK(Z)MX}}{(1-M^2)^{1/2}}\ e^{-i\pi/4\ -\omega_o t}$$

$$\times \sum_n U_n^o(Z_s)\, U_n^o(Z)\, \frac{e^{iK_n^o D(t')}}{\sqrt{K_n^o D(t')}} (1 - M^2)^{-1/2}, \tag{27}$$

where

$t' = t - D(t')/c_n^o$, is the retarded time, c_n being the modal group speed and superscript "o" indicate functions (U_n, K_n, and c_n) evaluated at frequency ω_o . The term D is the horizontal source-receiver distance.

5. SOLUTION FOR A SLOW MOVING SOURCE IN TERMS OF CONTEMPORARY TIME

The source receiver horizontal retarded distance D(t') is related to that of contemporary distance D(t) as

$$D(t') = \Delta + D(t), \tag{28}$$

where Δ is the horizontal displacement of the source from its position at t' to that of at time t. For a slowly moving source the velocity v is much smaller than all asymptotic modal velocities c_n ($|\Delta| \ll D(t)$) and Eq. (28) takes the form

$$D(t') = D(t) \left[1 + \frac{\Delta.\hat{D}(t)}{D(t)} + O(\frac{\Delta}{D(t)})^2 \right]. \tag{29}$$

For constant horizontal velocity, the source displacement is

$$\Delta = \frac{vD(t')}{c_n}.$$

We define

$$\sin\sigma(t) = -\hat{v}.\hat{D}(t), \quad \sigma(t) = \frac{\pi}{2} - \vartheta, \tag{30}$$

where ϑ is the angle between $-\hat{v}$ and $\hat{D}(t)$. Then using above results, the Eq. (29) takes the form

$$D(t') = D(t)\left[1 - \frac{v\sin\sigma(t)}{c_n} + O\left(\frac{v}{c_n}\right)^2\right]. \tag{31}$$

For slow moving source the terms of $O(\frac{v}{c_n})^2$ in Eq. (31) can be ignored and the acoustic field $\psi(r,t)$ (Eq. (27)) can be finally written as

$$\psi(r,t) = i\sqrt{2\pi}\, e^{-i(\pi/4 + \omega_o t)} \frac{e^{iK(Z_s)MX_s - iK(Z)MX}}{(1 - M^2)^{1/2}} \sum_n U_n(Z_s)$$

$$\times U_n(Z) \frac{e^{iK_n D(t')}}{\sqrt{K_n D(t')}} (1 - M^2)^{-1/2}, \tag{32}$$

where

$$D(t') = D(t)\left[1 - \frac{v\sin\sigma(t)}{c}\right]. \tag{33}$$

CONCLUSION

The problem of calculating the acoustic field generated by a moving point source with an arbitrary velocity in a stratified medium in the presence of moving fluid is investigated. It is observed that the field is increased and is independent of the flow direction . Still air case is recovered as a special case by taking the Mach number M to be zero.

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