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SOME CHARACTERIZATIONS AND SHEAF REPRESENTATIONS OF REGULAR AND WEAKLY REGULAR MONOIDS AND SEMIRINGS

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To

My Dear Mother

Whose prayers have always been a source

of great inspiration to me

and

My wife

For her understanding of the inevitable

neglect of my obligation during the

course of my research work.

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INTRODUCTION AND ABSTRACT

A ring R is called *regular* if for each $a \in R$, there exists an element $x \in R$ such that $axa = a$. Regular rings were introduced by von Neumann in 1936, in order to clarify certain aspects of operator algebras. Since then regular rings have been very extensively studied both for their own sake, as well as for the sake of their links with operator algebras. In this thesis, we will be concerned with this important notion and some of its generalizations, from a purely algebraic point of view, in the contexts of semigroups and semirings. We will determine new characterizations of regular, weakly regular and some of the other related classes of semigroups and semirings, using algebraic and homological techniques. We will also initiate the study of sheafs for certain classes of semigroups and semirings.

Throughout this thesis, which contains five chapters, S will denote a semigroup and S-systems are representations of S. Moreover, R will denote a semiring and R-semimodules are non-subtractive generalizations of modules over rings. Chapter 1 is of an introductory nature which provides basic definitions and reviews some of the background material which is needed for reading the subsequent chapters. In chapter 2, we introduce *P-injective* and *divisible S-systems.* We use these notions to construct an S-divisible S-system, Q(A), for an S-system A under some conditions. We also define and characterize von Neumann regular S-systems, and deduce several new characterizations of (von Neumann) regular monoids. In this chapter, we also study weakly regular monoids, and as a generalization of these monoids, we introduce the notion of normal S-systems. We show that an arbitrary monoid S is weakly regular if and only if each S-system is normal. In chapter 3, we introduce the notion of a regular semimodule, which is analogous to the notion of (von Neumann) regular S-systems studied in chapter 2. We characterize regular semimodules in terms of certain restricted injectivity properties, and use this characterization to obtain new characterizations of regular semirings. We also examine semiring analogs of the notions of hereditary, semihereditary and PP-rings. As an application of our results in this chapter, we obtain a homological characterization of PP-semirings. We also establish a characterization theorem for projective semimodules, which is analogous to the Classical

Projective Basis Theorem for projective (ring) modules. In chapter 4, we define and characterize weakly regular semirings and study some properties of their prime ideal space. In chapter 5, we construct sheafs for classes of monoids and semirings, which include regular and weakly regular monoids and semirings.

CONTENTS

CHAPTER 1

FUNDAMENTAL CONCEPTS

In this introductory chapter we shall define basic concepts of semigroups and semirings and review some of the background material that will be of value for our later persuits. For undefined terms and notations of semigroups, we refer to [11] and [25]. We also refer to [23] for basic terminology and results in semirings.

1.1 Basic concepts in semigroups

A system $(S,*)$ consisting of a nonempty set S, together with an associative binary operation $*$ on S is called a *semigroup*. Hence forth we shall write $x*y$ simply as xy, and usually refer to the binary operation as multiplication $\cdot$ on S. If $(S,\cdot)$ or more simply S is a semigroup with the additional property that multiplication is commutative, then S is called a *commutative semigroup*. S is called a *monoid* if S is a semigroup which contains an identity element. If S has no identity element then it is very easy to adjoin an identity element 1 to the set by defining $1\cdot s = s\cdot 1 = s$, for all $s \in S$, and $1\cdot 1 = 1$. Then

$S \cup \{1\}$ becomes a semigroup with an identity element 1. We shall use the notation S^1 with the following meaning:

$$S^1 = \begin{cases} S, \text{ if } S \text{ has an identity element} \\ S \cup \{1\} \text{ otherwise} \end{cases}$$

and call S^1 the semigroup obtained from S by adjoining an identity element. If a semigroup with at least two elements contains a zero element 0 then S is called a *semigroup with zero*. If S has no zero element then it is easy to adjoin an extra element 0 to the set S, by defining $0 \cdot s = s \cdot 0 = 0$ and $0 \cdot 0 = 0$, for all $s \in S$. This makes the set $S \cup \{0\}$ a semigroup with zero element 0. We shall use the notation S^0 with the following meaning:

$$S^0 = \begin{cases} S, \text{ if } S \text{ has a zero element} \\ S \cup \{0\} \text{ otherwise} \end{cases}$$

and call S^0 the semigroup obtained from S by adjoining a zero (if necessary). An element a of a semigroup S is called *idempotent* if $a^2 = a \cdot a = a$. S is called an *idempotent semigroup* (also called a band) if each element of S is idempotent. If $(E,\leq)$ is a lower semilattice, then E may be characterized as a commutative idempotent semigroup by defining the product of two elements to be their greatest lower bound. Thus for $e,f \in E$, $e \leq f$ if and only if $ef = fe = e$. A semigroup S is called *right* (*left*) *cancellative* if for all a,b,c in S, $ac = bc \Rightarrow a = b$ ($ca =$

$cb \Rightarrow a = b$); S is called *cancellative* if it is both left and right cancellative. If A and B are subsets of a semigroup S, we write $AB = \{ab: a \in A, b \in B\} = \cup \{Ab: b \in B\} = \cup \{aB: a \in A\}$. If a is an element of a semigroup S without an identity element, then aS or Sa will not, in general, contain a. In this situation, we use the notations S^1a for $Sa \cup \{a\}$, aS^1 for $aS \cup \{a\}$, and S^1aS^1 for $SaS \cup Sa \cup aS \cup \{a\}$. Note that S^1a, aS^1 and S^1aS^1 are all subsets of S (which do not contain 1). A non empty subset T of a semigroup S is called a *subsemigroup* of S if for all $x,y \in T$, $xy \in T$. Thus T is a subsemigroup if $T^2 = T \cdot T \subseteq T$. A subsemigroup T of a semigroup S is called a *subgroup* of S if T is a group. Recall that a semigroup S which has the property: $aS = S$ and $Sa = S$, for all $a \in S$ then S is a group in the usual sense. Thus a nonempty subset T of a semigroup S is a subgroup of S if and only if $aT = Ta = T$, for all $a \in T$. A semigroup S is called a *union of groups* if each element of S is contained in some subgroup of S. If a is an element of such a semigroup S, then $a \in G$, where G is a subgroup of S. An element of a semigroup S which commutes with every element of S is called a *central* element of S. The set of all central elements of S is either empty or a subsemigroup of S, and in the latter case, is called the *center* of S. Let A be a subset of a semigroup S. The intersection of all subsemigroups of S containing A is a subsemigroup of S denoted by $\langle A\rangle$. Clearly $\langle A\rangle$ contains A and is contained in every other subsemigroup of S containing A; it is called the *subsemigroup of S generated by A.* $\langle A\rangle$ may also be described as the set of all elements of S which are expressible as finite products of elements of A. If $\langle A\rangle=S$ then A is called the set of *generators* of S or a *generating set* of S. If A is finite, say $A = \{a_1,a_2,\ldots a_n\}$ then $\langle A\rangle = \langle a_1,a_2,\ldots a_n\rangle$. In particular, if $A = \{a\}$, then $\langle A\rangle = \langle a\rangle = \{a,a^2,a^3,\ldots\}$. $\langle a\rangle$ is called the *cyclic subsemigroup* of S generated by the element a. S is called *cyclic* if $S = \langle a\rangle$ for some $a \in S$.

A nonempty subset A of a semigroup S is called a *right* (*left*) *ideal* of S if $AS \subseteq A$ $(SA \subseteq A)$; A is a *two-sided ideal*, or simply, an ideal of S if A is both a right and left ideal. Clearly S is an ideal of S, and if S has a zero element, then (0) is an ideal of S. An ideal I of S different from these two ideals is called *proper.* The definitions of right (left) and two-sided ideals of S generated by a nonempty subset A of S are given in the usual manner. Note that the right ideal of S generated by A is $A \cup AS = AS^1$ and the two-sided ideal of S generated by A is $A \cup AS \cup SA \cup SAS = S^1AS^1$. If A is a finite subset of S such that $I = S^1AS^1$, then I is a *finitely generated ideal* of S. A right (left or two-sided) ideal of S generated by one element set $\{a\}$ is called a *principal right* (*left* or *two-sided*) *ideal* generated by a, and are denoted , respectively by $R(a)$, $L(a)$ and $J(a)$. Thus $R(a) = \{a\} \cup aS = aS^1$, $L(a) = \{a\} \cup Sa = S^1a$ and $J(a) = \{a\} \cup aS \cup Sa \cup SaS = S^1aS^1$. A semigroup S is called a *principal right* (*left* or *two-sided*) *ideal semigroup* if every right (left or two-sided) ideal in S is principal.

Let S and T be two semigroups with operation $\cdot$ and $*$. A function $f: S \longrightarrow T$ is called a *semigroup homomorphism* if $f(a \cdot b) = f(a)*f(b)$, for all $a,b \in S$. Semigroup monomorphisms, epimorphisms, isomorphisms and automorphisms are defined as usual. A relation ρ on a semigroup S is said to be *right* (*left*) *compatible* if for a,b in S, $a \rho b$ implies that $as \rho bs$ ($sa \rho sb$) for all $s \in S$. A *congruence* on S is an equivalence relation that is both right and left compatible. If ρ is a congruence on S then S/ρ denotes the set of all equivalence classes of S determined by ρ. If $a\rho$ denotes the equivalence class of S containing the element a ($a \in S$), then S/ρ can be made into a semigroup by defining $(a\rho)(b\rho) = (ab)\rho$; S/ρ is called the *factor semigroup* of S modulo ρ. The function $\rho^{\#}: S \longrightarrow S/\rho$ defined by $\rho^{\#}(a) = a\rho$ ($a \in S$) is a (semigroup) homomorphism. Let I be an ideal of a semigroup S. Define a relation ρ on S by a ρ b (a,b $\in$ S) to mean that either a=b or else both a and b belong to I. Clearly ρ is a congruence on S, called the *Rees congruence modulo I*. The equivalence classes of S modulo ρ are I itself and every one element set {a} with a $\in$ S\I. We shall write S/I instead of S/ρ, and call S/I the *Rees factor semigroup* of S modulo I.

Let S be a semigroup without zero. Then S is called *simple* if it has no proper ideals. A semigroup S with zero is called *O-simple* if (0) and S are the only ideals of S, and $S^2 \neq (0)$. A simple semigroup can be converted to a 0-simple semigroup by adjoining a zero element. However, not all 0-simple semigroups arise from simple semigroups in this way. It can be shown that a semigroup S is 0-simple if and only if SaS = S, for every a $\in$ S\{0}. Equivalently, for every a,b $\in$ S\{0}, there exist x,y $\in$ S such that xay = b (see [25,p.58]). Hence it follows that a semigroup S is simple if and only if SaS = S for all a $\in$ S. Equivalently, S is simple if and only if for all a,b $\in$ S, there exist x,y $\in$ S such that xay = b. A semigroup S is *right simple* if and only if aS = S, for all a $\in$ S. *Left simple semigroups* are defined analogously. Thus a semigroup is a group if and only if it is both right and left simple.

An element x of a semigroup S is said to be *regular* if there exists an element $x' \in S$ such that $xx'x = x$; S is called a *regular semigroup* if every element of S is regular (cf.[11]). An element $x' \in S$ is said to be an *inverse* of $x \in S$ if and only if $xx'x = x$ and $x'xx' = x'$; S is called an *inverse semigroup* if every element of S has a unique inverse. A semigroup S is an inverse semigroup if and only if S is a regular semigroup and any two idempotent elements of S commute with each other (cf.[11,p.28]).

1.2 S-systems and S-homomorphisms

Let S be a semigroup. A *right S-system* M over S is a nonempty set M together with a map $M \times S \longrightarrow M$, such that if ms denotes the image of (m,s), then for all $m \in M$ and $s,t \in S$, we have $m(st) = (ms)t$. We write M_S to indicate that M is a right S-system. Analogously, we define a left S-system M, written as ${}_SM$. A right S-system M_S is said to be *unitary* if S is a semigroup with an identity 1, such that $m1 = m$, for all $m \in M$. An element $d \in M_S$ is called a *fixed element* of M if $ds = d$ for all $s \in S$. An S-system may have several fixed elements, and it may also have no fixed element. Let D denote the set of all fixed elements of M. A right S-system M is called *centered* if $S = S^0$ and $|D| = 1$. Thus m is centered if and only if there is a fixed element (necessarily unique) denoted by θ such that:

(i) $\theta s = \theta$, for all $s \in S$

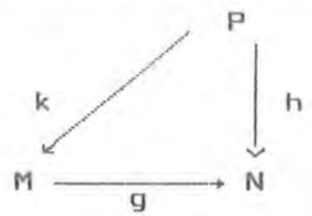
(ii) $m0 = \theta$, for all $m \in M$ and 0 is the zero of S

θ will be called the *zero* of M. A nonempty subset N of a right S-system M is called an *S-subsystem* of M if $NS \subseteq N$, that is, $ns \in N$, for all $n \in N$ and $s \in S$. An equivalence relation ρ on an S-system is called a (right) *congruence* if $a \rho b$ $(a,b \in M)$ implies $as \rho bs$ for all $s \in S$, that is, $(a,b) \in \rho$ implies $(as,bs) \in \rho$. The set of all congruences on M_S form a lattice with universal congruence denoted by ω_M and identity congruence ι_M. Let ρ be a congruence on M_S then the set of equivalence classes of M determined by ρ is denoted by M/ρ. Then M/ρ is a right S-system if we define $(m\rho)s = (ms)\rho$ for $m \in M$ and $s \in S$; M/ρ is called the *factor* S-system of M by ρ. If M_S is centered, the zero of M/ρ is $\theta\rho$.

A function $f\colon M_S \longrightarrow N_S$ between right S-systems M and N is called an *S-homomorphism* if for each $m \in M$ and $s \in S$, $f(ms) = f(m)s$. S-monomorphisms, S-epimorphisms, S-isomorphisms and S-endomorphisms are defined as usual. The class of right (left) S-systems together with S-homomorphisms form a category which will be denoted by ACT-S (S-ACT). An S-system B_S is a *retract* of an S-system A_S if there exist S-homomorphisms $\alpha: A \longrightarrow B$ and $\beta: B \longrightarrow A$ such that $\alpha\beta = 1_B$. If this is the case then α is necessarily epic and β is necessarily monic. If A_S and B_S are S-systems then the set of all S-homomorphisms from A to B is denoted by $Hom_S(A,B)$. Let $H = Hom_S(A,A)$. Then H is a monoid and A is an (H,S)-bisystem, that is A is a right S-system which is also a left H-system in which the elements of H are regarded as left operators. If B is an S-subsystem of an S-system A then B determines a congruence ρ on A as follows: For $a,b \in A$, $a \rho b$ if and only if $a = b$ or both a and b belong to B. In this case we write A/B instead of A/ρ and call it the *Rees factor* S-system of A by B. A right S-system M is called *totally irreducible* if $M_S \neq \theta$ and the only right S-congruences are the universal congruence ω_M and the identity congruence ι_M. Thus if M_S is totally irreducible, then M_S has no proper S-subsystem. An S-system M is called *cyclic* if there exists $x \in M$ such that $M = xS \cup \{x\}$ where $xS = \{xs: s \in S\}$; x is called a *generator* of M_S. M is called *strictly cyclic* if there exists $x \in M$ such that $M = xS$ and in this case x is a *strict generator* of M_S. If $S = S^1$ then of course the difference between the strictly cyclic and cyclic disappears. On the other hand, if M is cyclic but not strictly, say, $M = x \cup xS$ with x not in xS, then x is the only generator of M. If M is any cyclic S-system then it is easy to show that M is isomorphic to S/ρ, where ρ is a right congruence on M. The definitions of a finitely generated S-system or more generally, a generating set for an arbitrary S-system are given in the usual way (cf. [11, Chapter 11]). Let S be a semigroup and let $\left\{ M_i : i \in I \right\}$ be a family of right S-systems, then the *product* $\pi_{i \in I} M_i$ and the *coproduct* $\amalg_{i \in I} M_i$ are isomorphic, respectively, to the cartesian product and the disjoint union of the sets M_i with a suitable action of S. Moreover, in the category ACT-S, epimorphisms are surjective and monomorphisms are injective. Let $\left\{ M_i : i \in I \right\}$ be a family of right S-systems and let every M_i $(i \in I)$ contain a fixed one element subsystem (that is, the zero element) θ_i. By their *direct sum* $\oplus M_i$, we mean the subset of $\pi_{i \in I} M_i$ consisting of all $(m_i) \in \pi_{i \in I} M_i$ for which $\left\{ i : m_i \neq \theta_i \right\}$ is finite and the respective zeros of the M_i's are identified. Then $\oplus_{i \in I} M_i$ is a right S-system under the componentwise multiplication.

1.3 Free, projective and injective S-systems

In 1967, Berthiaume [8] introduced the concept of an injective S-system by generalizing the notion of an injective module over a ring (cf. Rotman [42]). He proved that the category of S-systems has enough injectives. Injective S-systems and their various generalizations were later investigated by many authors (see, for example, [1,3 15,16,17,18,19,21,44], among others). On the other hand, following Berthiaume's paper on injective S-systems, many papers have appeared extending other homological notions from the category of modules to the category of S-systems. Thus, for example, the concepts of free, projective and flat S-systems have been investigated among others. In this section we define the concepts of free, projective and injective S-systems and review some of their basic properties. An S-system F is said to be *free* provided there exists a subset X of F such that each element y in F has a unique representation $y = xs$, $x \in X$, $s \in S$; X is called a *basis* for F. A (right) S-system P is called *projective* if for every S-epimorphism $g: M \longrightarrow N$ and every S-homomorphism $h: P \longrightarrow N$, there exists an S-homomorphism $k: P \longrightarrow M$ such that $gk = h$. Diagrammatically, P is projective if and only if the diagram is commutative, that is, $gk = h$.



We list some of the basic properties of free and projective S-systems [cf. 31].

(1) Every free S-system is projective, and every retract of a projective S-system is projective.

(2) Every S-system is the epimorphic image of a free S-system.

(3) An S-system is projective if and only if it is a retract of a free S-system.

(4) A coproduct $\coprod_{i \in I} P_i$ of S-systems is projective if and only if P_i is projective for each $i \in I$.

(5) Every right ideal of S generated by an idempotent is projective.

Dual to that of projective S-system is the notion of injective S-system. A right S-system A is called *injective* if any S-homomorphism $C_S \longrightarrow A_S$ can be extended to B_S for any B_S containing C_S. Thus an S-system A_S is injective if and only if for any S-monomorphism α: $C_S \longrightarrow B_S$ and S-homomorphism β: $C_S \longrightarrow A_S$, there is an S-homomorphism μ: $B_S \longrightarrow A_S$ such that $\mu\alpha = \beta$. Clearly a retract of an injective S-system is injective. An S-system A_S is called *weakly injective* if for any right ideal K of S and any S-homomorphism ϕ: K $\longrightarrow$ A, there exists an element a $\in$ A such that $\phi(s)$ = as for all s $\in$ K. In ring theory, it is well-known that weakly injective R-modules over a ring R are injective in the usual sense. Weakly injective S-systems, however, need not be injective (see Berthiaume [8]). It is however true that injective S-systems are weakly injective [Berthiaume 8]. Let A_S be an S-subsystem of a right S-system B. Then A_S is *large* (or *essential*) in B_S and written as $A_S \leq' B_S$ if and only if for any S-system C_S and any S-homomorphism ϕ: $B_S \longrightarrow C_S$ with restriction to A injection, is itself an injection. If $A_S \leq' B_S$, then B_S is also said to be an *essential extension* of A_S. Berthiaume showed that an S-system A is injective if and only if A has no proper essential extension [8, Thm. 9]. He also showed that every S-system A has a maximal essential extension which is injective and unique up to isomorphism over A_S [8, Thm. 10]. Any maximal essential extension of an S-systyem A_S is called an *injective envelope* (or *injective hull*) of A_S. It is unique up to isomorphism over A_S and is denoted by $E_S = E(A_S)$. Furthermore, E_S is the injective envelope of A_S if and only if E_S is a maximal injective extension of A_S.

Berthiaume [8] also showed that every S-system can be embedded into an injective S-system and that an S-system A is injective if and only if it is a retract of every extension.

1.4 Semirings: Basic definitions and examples

A semiring is a set R together with two binary operations + (addition) and $\cdot$ (multiplication) such that (R,+) is a commutative semigroup, and $(R,\cdot)$ is a (generally) non commutative monoid with 1 as its identity element: connecting the two algebraic structures are the distributive laws, $a(b+c) = ab + ac$ and $(a+b)c = ac + bc$, for all $a,b,c \in R$. We shall always assume that $(R,+,\cdot)$ has an absorbing zero 0, that is, $a + 0 = 0 + a = a$ and $a\cdot 0 = 0\cdot a = 0$ hold for all $a \in R$ (cf. [23]). Thus all rings with identity elements are semirings. A natural example of a semiring which is not a ring is the set $\mathbb{N}_o$ of non negative integers with usual addition and multiplication. Moreover, if $(L,\vee,\wedge)$ is a distributive lattice with 0 and 1, then L is a semiring with $+ = \vee$, and $\cdot = \wedge$. In particular, the unit interval [0,1] of real numbers is a semiring with + = max and $\cdot$ = min or, with + = min and $\cdot$ = max or, even with + = max and $\cdot$ = usual product of real numbers. A semiring

R is *commutative* if multiplication in R is commutative; R is called *right* (*left*) *cancellative* if multiplication in R is right (left) cancellative, that is, ax = bx (xa = xb) implies a = b , for all $a,b,x \in R$. A nonzero element a of a semiring R is called *right zero divisor* if there exists a nonzero element b of R satisfying ba = 0. Left zero divisors are defined similarly. By a zero divisor we shall mean one which is both a right and a left zero divisor. A commutative semiring in which each nonzero element has a multiplicative inverse is called a *semifield*. Following a classical construction, it can be shown that a commutative cancellative semiring can be embedded in a semifield [46]. A function $\phi\colon R \longrightarrow R'$ between two semirings R and R' is a (semiring) homomorphism if: $\phi(x + y) = \phi(x) + \phi(y)$ and $\phi(xy) = \phi(x)\phi(y)$ for all $x,y \in R$. The concepts of monomorphisms, epimorphisms, isomorphisms and endomorphisms are defined as usual. A subset I of a semiring R is a *subsemiring* of R if I is a semiring under the operations of R; I is a *right* (*left*) *ideal* of R if for $a,b \in I$ and $r \in R$, $a + b \in I$ and $ar\ (ra) \in I$; I is a (two-sided) *ideal* if I is both a right and left ideal of R. A right (left or two-sided) ideal of R is *principal* (*finitely generated*) if it is generated by a single (finitely many) element(s). A principal right ideal of R generated by an element a will be denoted by $\langle a \rangle = aR = \{ar: r \in R\}$. The sum and product of ideals of a semiring are defined as in the case of rings. A direct *summand* of a semiring R is a (two-sided) ideal J for which there exists an ideal K, called a *cosummand* of J such that each $x \in R$ can be written uniquely in the form $x = a + b$, with $a \in J$, $b \in K$. An ideal J is called *complemented* (*uniquely complemented*) if there exists an (a unique) ideal K such that $J \cap K = (0)$ and $J + K = R$. It was shown by Cornish [12, Thm. 2.5] that for semirings with 1 and an absorbing, the notions of direct summand, complemented and unique complemented ideals are all equivalent. A right (left or two-sided) ideal I of a semiring R is called a right (left or two-sided) *k-ideal* provided that $a, a+x \in I$ implies $x \in I$ (cf. [34]).The multiples of 2 and the multiples of 3 are k-ideals of the semiring $\mathbb{N}_o$. If I is an ideal of a semiring R, and $I^* = \{a \in R: a + x \in I \text{ for some } x \in I\}$, then I^* is a k-ideal generated by I. Moreover, if $\phi: R \longrightarrow R'$ is an epimorphism between semirings R and R' then for each right (left or two-sided) k-ideal I of R', $\phi^{-1}(I)$ is a right (left or two-sided) k-ideal of R, and $\ker \phi = \{a \in R: \phi(a) = 0\}$ is a k-ideal of R [34].

Let I be a two sided ideal of a semiring R. We define a relation ~ on R as follows: for a, b in R, a ~ b if and only if there exist x_1, x_2 in I such that $a + x_1 = b + x_2$. Then ~ is a congruence on R. The relation ~ is called Bourne's congruence relation on R. The set of all congruence classes determined by ~ will be denoted by R/I. The set R/I admits the structure of a semiring under the following rules of addition and multiplication:

$$[a] + [b] = [a+b]$$

$$[a] \cdot [b] = [ab]$$

The congruence class [x], where $x \in I$, is the zero element of R/I and [1] is the identity of R/I. R/I is called the Bourne factor semiring of R modulo I [23]. Generalizing the notion of regular rings (cf. Rotman [42]), a semiring R is called *regular* if for each $x \in R$, there exists $a \in R$ such that $xax = x$. This class of semirings have been investigated by many authors (see, for example, [23,26,47, 56,57], among others).

1.5 R-semimodules and R-homomorphisms

Let R be a semiring with an identity element 1 and an absorbing zero 0. An additively written commutative semigroup M with a neutral 0 is a *right* R-*semimodule*, M_R, if there is a function α: MxR ⟶ M such that if $\alpha(m,r)$

is denoted by mr, then the following conditions hold:

(1) $(m + m')r = mr + m'r$

(2) $m(r + r') = mr + mr'$

(3) $m(rr') = (mr)r'$

(4) $m \cdot 1 = m$

(5) $0 \cdot r = m \cdot 0 = 0$, for all $r, r' \in R$ and $m, m' \in M$ (cf. [23,p.138], [54]).

One can similaly define a left R-semimodule $_RM$. A semiring R is a right semimodule over itself which will be denoted by R_R. A *subsemimodule* N of a right R-semimodule M is a subsemigroup of M such that $nr \in N$ for all $n \in N$ and $r \in R$. Thus subsemimodules of R_R ($_RR$) are the right (left) ideals of the semiring R. Let S be a subset of a right R-semimodule M. By M_o we denote the set of all elements of the form $\sum_{s \in S} sr_s$ ($r_s \in R$) such that all but a finite number of terms in the sum are zero i.e. $r_s = 0$ except for a finite number of $s \in S$. Then M_o is an R-subsemimodule of M, containing S. M_o is called the *subsemimodule generated* by S. Symbolically, we write $M_o = \langle S \rangle$. If $\langle S \rangle = M$, then M is said to be generated by S. If S is a finite subset of M such that $M = \langle S \rangle$, then M is called *finitely generated*. In particular, if $|S| = 1$ and $M = \langle S \rangle$ then M is called *cyclic*. It can be easily verified that M is cyclic if and only if $M = xR = \langle x \rangle$ for some $x \in M$.

A function $f: M \longrightarrow M'$ between right R-semimodules M and M' is a *right R-homomorphism* if:

(1) $f(m + m') = f(m) + f(m')$,

(2) $f(mr) = f(m)r$, for $m,m' \in M$ and $r \in R$.

Let A and B be right R-semimodules. A is called a *retract* of B if there exist R-homomorphisms $g: A \longrightarrow B$ and $p: B \longrightarrow A$ such that $pog = 1$. The set of all R-homomorphisms from M_R to M'_R is denoted by $Hom_R(M,M')$. By $End_R(M)$ we shall mean the set of R-endomorphisms of M. Using standard arguments, it can be shown that for each right R-semimodules M, $End_R(M)$ is a semiring.

1.6 Free, projective and injective semimodules

Let R be a semiring and let M be a right R-semimodule. A subset S of M is called *linearly independent* if $\sum_{s\in S} s\lambda_s = \sum_{s\in S} s\mu_s$ implies $\lambda_s = \mu_s$ for all $s \in S$ and $\lambda_s,\mu_s \in R$. M is called a *free* R-semimodule if M has a linearly independent generating subset S. In this case, S is said to be a *basis* of M. If M is a free R-semimodule with a basis S, then every element of M is uniquely written as $\sum_{s\in S} s\lambda_s$. For any set S, there exists a free right R-semimodule with S as a basis [48]. Let A and B be R-semimodules and let $f:A\longrightarrow B$

be an R-homomorphism. Then ker f = $\{a \in A: f(a) = 0\}$, Im f = $\{b \in B: b + f(a) = f(a'), \text{ for some } a, a' \in A\}$ and f(A) = $\{b \in B: b = f(a) \text{ for some } a \in A\}$. We call them *kernel*, *image* and *proper image* of f, respectively. In general, $f(A) \subseteq \text{Im } f \subseteq B$. We shall say that f is i-*regular* (*image-regular*) if f(A) = Im f; f is called k-*regular* (*kernel-regular*) if $f(a) = f(a')$ implies $a + k = a' + k'$ for some k,k' in ker f; f is called *regular* if f is both i-regular and k-regular [48]. Furthermore, f is an *injection* if $f(a) = f(a')$ implies $a = a'$; a *surjection* if $b \in B$ implies $b = f(a)$ for some $a \in A$; and a *bijection* if f is both an injection and a surjection. For an R-homomorphism $f: A \longrightarrow B$ and a right R-semimodule P, the induced R-homomorphism: $f_*: \text{Hom}_R(P,A) \longrightarrow \text{Hom}_R(P,B)$ is defined by $f_*(\phi) = f \circ \phi$, where $\phi \in \text{Hom}_R(P,A)$. A right R-semimodule P is called *projective* if

(i) for each surjective R-homomorphism $f: A \longrightarrow B$, the induced map $f_*: \text{Hom}_R(P,A) \longrightarrow \text{Hom}_R(P,B)$ is surjective,

(ii) f_* is k-regular whenever f is k-regular [49].

The following results are due to M.Takahashi [49].

Proposition 1.6.1 ([49], Proposition 1.3, p. 84)

Every free right R-semimodule is projective. In particular, R_R is projective.

Proposition 1.6.2 ([49], Theorem 1.9, p. 86)

A right R-semimodule P is projective if and only if P is a retract of a free R-semimodule. A retract of a projective R-semimodule is projective.

A right R-semimodule E is *injective* if and only if given a right R-semimodule M and a subsemimodule N, any R-homomorphism from N to E can be extended to an R-homomorphism from M to E. In ring theory it is well known that if R is a ring then any right R-module is contained in an injective right R-module [42]. This, however, is not true for arbitrary semirings. It was shown by B.Banascewski that if R is an entire, cancellative, zero sum free semiring then the only injective R-semimodules is {0} [23, Proposition 15.17, p. 178]. In particular, there are no non zero injective semimodules over the semiring of non-negative integers. Injective semimodules over special classes of semirings have been, however, investigated by some authors (see Golan [23], for references to this subject)

CHAPTER 2

CHARACTERIZATIONS OF MONOIDS BY P-INJECTIVE AND NORMAL S-SYSTEMS

In this chapter we introduce the notions of P-injective and divisible S-systems and use these notions to construct an S-divisible S-system Q(A) from an S-system A under some conditions. We define and characterize regular and von Neumann regular S-systems in terms of certain relative injectivity properties. As an application of our result, we obtain characterizations of PP monoids and von Neumann regular monoids defined in the sequel. These characterizations are similar to those found in [15] for hereditary and semihereditary monoids. We also study weakly regular monoids. Moreover, as a generalization of these types of monoids, we introduce the notion of normal S-systems and characterize weakly regular monoids by the property that each S-system is normal. Throughout this chapter S will denote a monoid and all S-systems are unitary right S-systems.

2.1 P-injective and divisible S-systems

We begin with some definitions.

Definition 2.1.1 Let M be a fixed right S-system. An S-system Q is called *PM-injective* if each S-homomorphism (that is, right S-homomorphism) from a cyclic S-subsystem aS ($a \in M$) of M to Q extends to an S-homomorphism from M to Q. Thus, Q is called a *P-injective* S-system if Q is PS-injective [35]. An S-system all of whose quotient S-systems are PM-injective will be called a *completely PM-injective* S-system. *Completely P-injective* S-systems are defined analogously.

Definition 2.1.2 Let S be a monoid and Q an S-system. An element x of Q is said to be *S-divisible* in Q if, for every $a \in S$, there exists $y \in Q$ such that $x = ya$. Q is *S-divisible* if $Qa = Q$ for all $a \in S$. From this definition it follows that a monoid S is a group in the usual sense if and only if S is S-divisible. An S-system Q will be called *completely S-divisible* if and only if every quotient S-system $\bar{Q}$ of Q is S-divisible.

Proposition 2.1.3 If Q is (completely) S-divisible then Q is (completely) P-injective.

Proof Suppose that Q is S-divisible. To show the P-injectivity, let aS ($a \in S$) be any principal right ideal of S and ϕ: aS $\longrightarrow$ Q be an S-homomorphism. Then ϕ is determined by the element $\phi(a) = x \in Q$, that is, $\phi(as) = xs$ for all $s \in S$. Since Q is S-divisible, there exists, an element $y \in Q$ such that $x = ya$. Define ψ: S $\longrightarrow$ Q by $\psi(1) = y$, that is, $\psi(s) = ys$ for all $s \in S$. Then, we have $\psi(as) = yas = xs = \phi(as)$ for $s \in S$. This shows that ψ is an extension of ϕ. Thus Q is P-injective. The proof of the parenthetical version is now immediate.

Proposition 2.1.4 If A is a retract of an S-divisible S-system Q, then A is S-divisible.

Proof Let p be the retraction and q the coretraction such that $poq = 1_A$. To show that A is S-divisible, let $x \in A$ and $a \in S$. Then $q(x) \in Q$. Since Q is S-divisible, there exists, $y \in Q$ such that $q(x) = ya$. Then

$$x = poq(x) = p(ya) = p(y)a \text{ and } p(y) \in A.$$

This shows that A is S-divisible.

Definition 2.1.5 Let A be a right S-system. A is *right S-cancellative* if A has the following property:

$$xs = x's \text{ for } x,x' \in A \text{ and } s \in S \Rightarrow x = x'.$$

Thus, S is right cancellative if S_S is right S-cancellative. Dually, A is *left A-cancellative* if A has the following property:

$$xs = xs' \text{ for } x \in A \text{ and } s,s' \in S \Rightarrow s = s'.$$

Thus, S is left cancellative if S is left S-cancellative, that is, S is left cancellative as a left S-system.

Proposition 2.1.6 If A is a retract of a right S-cancellative (left B-cancellative) S-system B, then A is right S-cancellative (left A-cancellative).

Proof Let p be the retraction and q the coretraction such that $poq = 1_A$. Let $xs = x's$ for $x,x' \in A$ and $s \in S$. Then $q(xs) = q(x's)$. This implies that $q(x)s = q(x')s$. Thus, $q(x) = q(x')$, since B is S-cancellative. As $poq = 1_A$, therefore, $p(q(x)) = p(q(x')) \Rightarrow x = x'$. Hence, A is S-cancellative. Similarly, if $xs = xs'$, then $q(xs) = q(xs')$. This implies that $q(x)s = q(x)s'$. But, B is left B-cancellative, therefore, $s = s'$. Hence, A is left A-cancellative.

Proposition 2.1.7 For a left cancellative monoid S, the following assertions are equivalent:

(1) Q is a (completely) P-injective right S-system,.

(2) Q is a (completely) S-divisible right S-system.

Proof (2) $\Rightarrow$ (1): This follows from Proposition 2.1.3.
(1) $\Rightarrow$ (2): Let $x \in Q$ and $a \in S$. Define a map $\phi: aS \longrightarrow Q$ by $\phi(as) = xs$ for all $s \in S$. Since S is left cancellative, ϕ is a well-defined S-homomorphism. Since Q is P-injective, there exists, an extension ψ from S to Q. Then $x = \phi(a) = \psi(a) = \psi(1a) = \psi(1)a$ and $\psi(1) \in Q$. This shows that Q is S-divisible.

Proposition 2.1.8 The following assertions are equivalent.

(1) All right S-systems are S-divisible,

(2) All right ideals of S are S-divisible,

(3) S is divisible,

(4) S is a group ,

(5) All right S-systems are P-injective,

(6) S is P-injective.

Proof (1) $\Rightarrow$ (2) $\Rightarrow$ (3) are clear. (3) $\Rightarrow$ (4): Let a be an element of S. Then, as S_S is divisible, there exists, an element b of S with $1 = ba$. Thus, a is left invertible. Hence a is invertible. This shows that S is a group.

$(4) \Rightarrow (1)$: Let Q be a right S-system. Let $x \in Q$ and a be a non zero element of S. From (4), there is an element $b \in S$ with $1 = ba$. Then, $x = x1 = x(ba) = (xb)a$. Hence, $Q = Qa$. Hence, Q is S-divisible. Thus, (1) if and only if (2) if and only if (3) if and only if (4).

Now, suppose that S is a group, and so, in particular, cancellative. Hence, by Proposition 2.1.7, (1) if and only if (5) and (3) if and only if (6). This proves the proposition.

Next, we will construct an S-divisible S-system Q(A) from a right S-system A under some conditions.

Consider the set $A \times S = \left\{(x,a) : x \in A \text{ and } a \in S\right\}$. On this set define S-action by $(x,a)s = (xs,a)$ for all $s \in S$. Then, $A \times S$, together with this S-action, is a right S-system which we shall denote by Q(A). Consider a relation $\equiv$ on Q(A) defined by

$$(x,a) \equiv (x',a') \text{ if and only if } xa' = x'a.$$

Then, we have,

Lemma 2.1.9 If S is a commutative monoid and A is a right S-cancellative S-system, then, the above relation $\equiv$ is an S-congruence on Q(A).

Proof By definition, the relation $\equiv$ is reflexive and symmetric. To show the transitivity, suppose that

$(x,a) \equiv (x',a')$ and $(x',a') \equiv (x'',a'')$ for $x,x',x'' \in A$ and $a,a',a'' \in S$. Since, by assumption, $xa' = x'a$ and $x'a'' = x''a'$ and S is being commutative, we have

$$xa''a' = xa'a'' = x'aa'' = x'a''a = x''a'a = x''aa'$$

Thus $xa''a' = x''aa'$. Since A is right S-cancellative, we have $xa'' = x''a$. This shows that $(x,a) \equiv (x'',a'')$.

Finally, compatibility with S follows directly from the definition and commutativity of S. Thus, the relation $\equiv$ is an S-congruence on Q(A).

By the above lemma, we may construct a quotient right S-system $Q(A)/\equiv$ which will be denoted by Q(A). For each element $(x,a) \in Q(A)$, we shall denote by $\overline{(x,a)}$ the corresponding element of Q(A). Moreover, the S-action on Q(A) is defined by

$$\overline{(x,a)}s = \overline{(x,a)s} = \overline{(xs,a)} \quad \text{for, } s \in S.$$

Proposition 2.1.10 Let S be a commutative monoid and A a right S-cancellative S-system. Then Q(A) has following properties:

(1) Q(A) is S-divisible with A considered as an S-subsystem of Q(A).

(2) Q(A) is right S-cancellative.

(3) For every $\overline{(x,a)} \in Q(A)$, $\overline{(x,a)}a = \overline{(x,1)}$.

Proof (1) Define $q:A \longrightarrow Q(A)$ by $q(x) = \overline{(x,1)}$. Then q is injective. Thus, we may consider A as an S-subsystem of Q(A). Let $\overline{(x,a)} \in Q(A)$ and $s \in S$. Since S is commutative, $xas = xsa$. This shows that $(x,a) \equiv (xs,as)$ and $\overline{(x,a)} = \overline{(x,as)}s$. This means that Q(A) is S-divisible.

(2) Suppose that $\overline{(x,a)}s = \overline{(x',a')}s$. Then, $(xs,a) \equiv (x's,a')$. Thus $xsa' = x'sa$. Since S is commutative, $xa's = x'as$. Since A is right S-cancellative, we have $xa' = x'a$. This means that $\overline{(x,a)} = \overline{(x',a')}$. Hence, Q(A) is right S-cancellative.

(3) For every $\overline{(x,a)} \in Q(A)$, $\overline{(x,a)}a = \overline{(xa,a)} = \overline{(x,1)}$.

Corollary 2.1.11 Let S be a commutative and cancellative monoid. Then, Q(S) is S-divisible and $S \leq Q(S)$.

In this case, Q(S) is a commutative group with the following multiplication:

$$\overline{(b,a)}.\overline{(b',a')} = \overline{(bb',aa')}$$

Remark In this case, Q(S) is the well-known classical construction from a commutative and cancellative monoid S.

Proposition 2.1.12 Let S be a commutative monoid and A a right S-cancellative S-system. Then, the following assertions are equivalent:

(1) A is S-divisible,

(2) A is a retract of Q(A).

Proof (2) $\Rightarrow$ (1): This follows from Proposition 2.1.4, since Q(A) is S-divisible.

(1) $\Rightarrow$ (2): To define a retraction $p:Q(A)\longrightarrow A$, let $\overline{(x,a)} \in Q(A)$. Since A is S-divisible, there exists, $y \in A$, such that, $x = ya$. Since A is right S-cancellative, y is unique. Then, we define p by $p\overline{(x,a)} = y$ for all $\overline{(x,a)} \in Q(A)$. Now, suppose that $(x,a) \equiv (x',a')$ with $x = ya$ and $x'= y'a'$. Since, $xa'= x'a$ and $yaa'= y'a'a$, therefore, $yaa'= y'aa'$, by the commutativity of S. Also, since A is right S-cancellative, therefore, $y = y'$. Thus, the map p is well-defined. To show that p is an S-homomorphism, let $p\overline{(x,a)} = y$ with $x = ya$, and $p\overline{(xs,a)} = y'$ with $xs = y'a$. Then, $yas = y'a$. Since S is commutative, we have $ysa = y'a$. Since A is right S-cancellative, it follows that $y'= ys$. Hence,

$$p(\overline{(x,a)}s) = p\overline{(xs,a)} = y'= ys = p\overline{(x,a)}s.$$

This shows that p is an S-homomorphism. Let $q:A\longrightarrow Q(A)$ be the inclusion defined by $q(x) = \overline{(x,1)}$. Then, $poq(x) =$

$p\overline{(x,1)} = x$ because $x = x1$. This shows that A is a retraction of Q(A).

Corollary 2.1.13 Let S be a commutative and cancellative monoid. Then the following assertions are equivalent:

(1) S is a commutative group.

(2) S, considered as an S-system, is P-injective.

(3) S, considered as an S-system, is S-divisible.

(4) S is a retract of Q(S).

Finally, we will prove the following universal property for Q(A).

Theorem 2.1.14 Let S be a commutative monoid and A a right S-cancellative S-system. Then, there exist an S-system $\bar{A}$ and an S-homomorphism $f: A \longrightarrow \bar{A}$ satisfying the following four conditions.

(1) f is injective.

(2) Each element of f(A) is S-divisible in $\bar{A}$.

(3) $\bar{A}$ is right S-cancellative.

(4) For each $y \in \bar{A}$, there exist, $a \in S$ and $x \in A$, such that, $ya = f(x)$.

If $\bar{A}'$ and f' satisfy conditions (1) through (4), then, there exists a unique S-isomorphism $\phi: \bar{A} \longrightarrow \bar{A}'$, such that, $f' = \phi o f$.

Proof Since $Q(A)$ satisfies conditions (1) through (4), we need only to prove the last part. To define a map $\phi: \bar{A} \longrightarrow \bar{A}'$, let y be any element of $\bar{A}$. By condition (4), there exist, $a \in S$ and $x \in A$, such that, $ya = f(x)$. For $f'(x) \in \bar{A}'$ and $a \in S$, there exists $y' \in \bar{A}'$, such that, $y'a = f'(x)$, by condition (2).

Now, let $ya' = f(x')$ and $y''a' = f'(x')$ be another expression. Then, $yaa' = f(x)a' = f(xa')$ and $ya'a = f(x')a = f(x'a)$. Hence, we have, $xa' = x'a$ by commutativity of S and injectivity of f. It follows that

$$y'aa' = f'(x)a' = f'(xa') = f'(x'a) = f'(x')a = y''a'a.$$

By commutativity of S and right S-cancellativity of $\bar{A}'$ we have $y' = y''$. This shows that $y' \in \bar{A}'$ is uniquely determined from $y \in \bar{A}$ by the rule: $ya = f(x)$ and $y'a = f'(x)$

Thus, we may define a map $\phi: \bar{A} \longrightarrow \bar{A}'$ by $\phi(y)a = f'(x)$, for all $y \in \bar{A}$.

To show that $\phi(ys) = \phi(y)s$, let $\phi(y) = y'$ and $\phi(ys) = y''$. Since $ya = f(x)$, $ysa = yas = f(xs)$. Therefore, $y''a = f'(xs) = f'(x)s = y'sa$. Since $\bar{A}'$ is right S-cancellative, $y'' = y's$. Thus, we have an S-homomorphism $\phi: \bar{A} \longrightarrow \bar{A}'$. By the definition of ϕ we may easily check that ϕ is an S-isomorphism such that $f' = \phi o f$.

Finally, suppose that $f' = \phi' \circ f$ and $ya = f(x)$. Then, $\phi'(y)a = f'(x) = \phi(y)a$. Since $\bar{A}'$ is right S-cancellative, we have $\phi'(y) = \phi(y)$. This establishes the uniqueness of ϕ with the property that $f' = \phi \circ f$.

Remark If $\bar{A}$ satisfies conditions (1) through (4), then $\bar{A}$ is S-divisible and, therefore, P-injective.

To see this, suppose $y \in \bar{A}$ and $a \in S$. By condition (4), there exist, $b \in S$ and $x \in A$, such that, $yb = f(x)$. By condition (2), for $f(x) \in f(A)$ and $ab \in S$, there exists, $z \in \bar{A}$, such that, $f(x) = zab$. Hence, $yb = zab$. By condition (3), it follows that $y = za$. This shows that $\bar{A}$ is S-divisible.

2.2 Characterizations of monoids by P-injective S-systems

Definition 2.2.1 An S-system M is called *regular* if, for each $a \in M$, there exists, an S-homomorphism $f \in Hom_S(aS,S)$ such that, $a = af(a)$. A monoid S is called regular if S_S is regular as an S-system [24]. An S-system M is called von Neumann regular if, for each $a \in M$, there exists an S-homomorphism $g \in Hom_S(S,S)$, such that, $a = ag(a)$ [58].

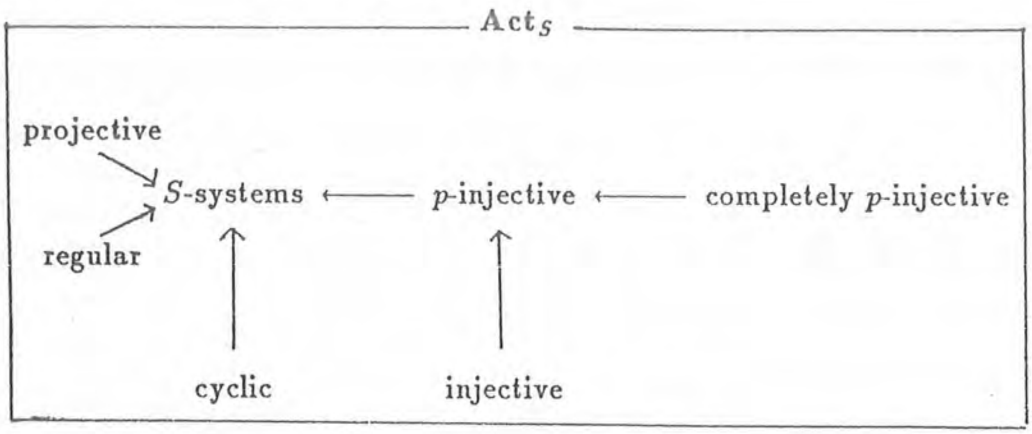
Thus, if S_S is von Neumann regular, then, for each $a \in S$ there exists, $g \in Hom_S(S,S)$, such that, $a = ag(a) = ag(1)a$ and $g(1) \in S$. Hence, S is von Neumann regular in the familiar sense.

Definition 2.2.2 Let M and Q be right S-systems. Q is *M-projective* if, for each S-epimorphism $g: M \longrightarrow \bar{M}$ and each S-homomorphism $h: Q \longrightarrow \bar{M}$, there exists, an S-homomorphism $k: Q \longrightarrow M$, such that, $gok = h$. Thus, Q is projective if Q is M-projective for each S-system M. We notice that every monoid S is always projective.

Dually, Q is *M-injective* if, for each S-monomorphism $g: N \longrightarrow M$ and each S-homomorphism $h: N \longrightarrow Q$, there exists, an S-homomorphism $k: M \longrightarrow Q$, such that, $kog = h$. Thus, Q is injective if Q is M-injective for each S-system M.

Definition 2.2.3 A right S-system M is called a right PP S-system if each cyclic S-subsystem aS of M with $a \in M$ is projective. S is called a right PP-monoid if all its principal right ideals are projective as right S-systems.

From the above definitions we have the following diagram in the category Act S:



The following result is due to T.L.Hách. [24]

Proposition 2.2.4 For an S-system M the following are equivalent:

(1) M is a regular S-system,

(2) M is a PP S-system.

Corollary 2.2.5 For a monoid S the following are equivalent:

(1) S is regular,

(2) S is a PP-monoid.

(3) Every projective S-system is regular.

For a right S-system M and $a \in M$, we may always define an S-epimorphism $\pi: S \longrightarrow aS$ by $\pi(s) = as$ for all $s \in S$, also we have an inclusion $k: aS \longrightarrow M$.

Note that the following are equivalent:

(1) M is regular,

(2) for each $a \in M$, aS is a retract of S,

(3) for each $a \in M$, π is a retraction.

For a monoid S, S is von Neumann regular if and only if the inclusion $k: aS \longrightarrow S$ is a coretraction for each $a \in S$.

Proposition 2.2.6 For a projective S-system M the following assertions are equivalent:

(1) M is a regular S-system.

(2) Each PM-injective S-system is completely PM-injective.

(3) Each injective S-system is completely PM-injective.

Proof (1) $\Rightarrow$ (2) Let A be a PM-injective S-system and $\bar{A}$ a quotient S-system of A. Hence, there is an S-epimorphism $p: A \longrightarrow \bar{A}$. In order to prove that $\bar{A}$ is PM-injective, consider a cyclic S-subsystem aS with $a \in M$ and an S-homomorphism $f: aS \longrightarrow \bar{A}$. Since M is a regular S-system, it follows that aS is projective, by Proposition 2.2.4. Hence, there exists an S-homomorphism $h: aS \longrightarrow A$, such that, $p \circ h = f$. Now, since A is PM-injective, there exists, an S-homomorphism $g: M \longrightarrow A$, such that, $g \circ k = h$, where $k: aS \longrightarrow M$ is the inclusion map. Let $\mu = p \circ g$. Then, $\mu: M \longrightarrow \bar{A}$ is an S-homomorphism, such that, $\mu \circ k = p \circ g \circ k =$

$p\circ h = f$. Hence, is PM-injective.

(2) $\Rightarrow$ (3): Clear.

(3) $\Rightarrow$ (1): Let aS be a cyclic S-subsystem of M and let $k\colon aS \longrightarrow M$ be the inclusion map. Now, consider an epimorphism $p\colon A \longrightarrow \bar{A}$ and an S-homomorphism $f\colon aS \longrightarrow \bar{A}$. In order to prove that aS is projective, we may assume,(without loss of generality) by Lemma 4 of [15], that A is injective. The rest of the proof is dual to that of (1) $\Rightarrow$ (2). This proves the proposition.

Proposition 2.2.7 For an S-system M the following assertions are equivalent:

(1) M is von Neumann regular,

(2) M is regular and S is PM-injective.

Proof (1) $\Rightarrow$ (2): Suppose that M is von Neumann regular. It follows easily that M is regular. We show that S is PM-injective. Let aS ($a \in M$) be a cyclic S-subsystem of M and let $f\colon aS \longrightarrow S$ be an S-homomorphism. Since M is von Neumann regular and $a \in M$, there exists an S-homomorphism $g\colon M \longrightarrow S$, such that, $a = ag(a)$. Define $\bar{f}\colon M \longrightarrow S$ by $\bar{f}(x) = f(a)g(x)$, for all $x \in M$. Clearly, $\bar{f}$ is an S-homomorphism which extends f. Hence, S is PM-injective.

(2) $\Rightarrow$ (1): Suppose that M is regular and S

PM-injective. Then, for every $a \in M$, there exists an S-homomorphism $f: aS \longrightarrow S$, such that, $a = af(a)$. Since S is PM-injective, so there exists an S-homomorphism $g: M \longrightarrow S$, extending f. Hence, $a = ag(a)$ and therefore M is von Neumann regular.

Lemma 2.2.8 Let Q be an M-projective S-system. If M_o is either an S-homomorphic image or an S-subsystem of M, then Q is M_o-projective.

Proof The result is almost obvious in case M_o is an S-homomorphic image of M. Thus, we assume that M_o is an S-subsystem of M. In order to show that Q is M_o-projective, consider an S-epimorphism $\phi_o: M_o \longrightarrow \bar{M}_o$ and an S-homomorphism $g: Q \longrightarrow \bar{M}_o$. Let ρ be the relation on M defined by $\rho = \ker\phi_o \cup i_M$, where $\ker\phi_o$ is the usual kernel S-congruence and i_M the identity relation on M. Set $\bar{M} = M/\rho$ and let $\phi: M \longrightarrow \bar{M}$ be the natural map. We can identify $\bar{M}_o$ with the S-subsystem $M_o/\ker\phi_o$ of the S-system $\bar{M}$. Thus, the natural map $\phi: M \longrightarrow \bar{M}$ is an extension of ϕ_o. By the M-projectivity of Q, there exists, an S-homomorphism $f: Q \longrightarrow M$, such that, $\phi o f = g$. But $\phi(f(Q)) = g(Q) \subseteq \bar{M}_o = M_o/\ker\phi_o$. Hence $f(Q) \subseteq M_o$. Thus, f can be regarded as an S-homomorphism from Q to M_o. This proves that Q is M_o-projective.

Lemma 2.2.9 Let M be a projective S-system and E = E(M) the injective hull of M. If E is completely PM-injective then, each cyclic S-subsystem of M is M-projective.

Proof Let aS ($a \in M$) be a cyclic S-subsystem of M and let $k: aS \longrightarrow M$ be the inclusion map. Consider an epimorphism $p: E \longrightarrow \bar{E}$ and an S-homomorphism $\alpha: aS \longrightarrow \bar{E}$. Since E is completely PM-injective, $\bar{E}$ is PM-injective. Hence, there exists an S-homomorphism $\beta: M \longrightarrow \bar{E}$, such that, $\beta o k = \alpha$. Since M is a projective S-system, there exists, an s-homomorphism $\phi: M \longrightarrow E$, such that, $p o \phi = \beta$. Let $\theta = \phi o k$. Then $p o \theta = p o \phi o k = \beta o k = \alpha$. Thus, aS is E-projective. Hence, by Lemma 2.2.8, aS is M-projective.

To summarize, we may now state the following characterization theorem for a monoid.

Theorem 2.2.10 For a monoid S, the following assertions are equivalent:

(1) S is regular.

(2) S is a PP-monoid.

(3) E = E(S) is completely P-injective.

(4) Every P-injective S-system is completely P-injective.

(5) Every injective S-system is completely P-injective.

Proof From corollary 2.2.5 and Proposition 2.2.6 we have (1) if and only if (2) if and only if (4) if and only if (5), since S_S is projective. We need only to prove (2) if and only if (3). Necessity follows from Proposition 2.2.6 as a corollary. For sufficiency, let aS be a principal right ideal of S ($a \in S$). From Lemma 2.2.8 it follows that aS is S-projective and, from this, it follows that aS is a retract of S. Hence, aS is projective, since S is always projective. Therefore, S is a PP-monoid. This completes the proof.

Theorem 2.2.11 For a monoid S, the following assertions are equivalent:

(1) S is von Neumann regular.

(2) S is regular and P-injective.

(3) S is completely P-injective.

(4) Every S-system is P-injective.

(5) Every cyclic S-system is P-injective.

Proof As a corollary of Proposition 2.2.7 , we have (1) if and only if (2). The proof of (1) if and only if (4) if and only if (5) is found in [36]. We need only to prove (2) if and only if (3).

(2) $\Rightarrow$ (3): This follows as a corollary of Proposition 2.2.6.

(3) $\Rightarrow$ (2): Suppose that S is completely P-injective. For every $a \in S$, consider a surjection $\pi: S \longrightarrow aS$ defined by $\pi(s) = as$, for all $s \in S$. Since S is completely P-injective, aS is P-injective. Consider the inclusion map $k: aS \longrightarrow S$ and the identity map $1_{aS}: aS \longrightarrow aS$. Since aS is P-injective, we have an S-homomorphism $g: S \longrightarrow aS$, such that, $a = g(a) = g(1)a$. Since π is surjective, there exists, $x \in S$ such that $g(1) = ax$. Thus, we have $a = axa$ with $x \in S$. This shows that S is von Neumann regular. Since (1) if and only if (2), this completes the proof.

Proposition 2.2.12 If a right S-system M is left M-cancellative, then M is regular.

Proof Let $a \in M$ be any element. Then, by the assumption,we may define an S-homomorphism $g: aS \longrightarrow S$ by $g(as) = s$, for all $s \in S$. Since $g(a) = 1$, $a = a1 = ag(a)$. Hence, M is regular.

Corollary 2.2.13 If S is left cancellative, then S is a

PP-monoid.

Proposition 2.2.14 For a monoid S, the following assertions are equivalent:

(1) S is a PP-monoid with a unique idempotent element.

(2) S is left cancellative.

Proof From the above corollary, we have (2) $\Rightarrow$ (1).

(1) $\Rightarrow$ (2): Suppose we have $as = as'$ for a,s,s' in S. Since S is regular, there exists, an S-homomorphism $f: aS \longrightarrow S$, such that, $a = af(a)$. Then, $f(a) = f(a)f(a)$. By the uniqueness of the idempotent element, it follows that $f(a) = 1$. Then, the equation $as = as'$ implies that $s = 1s = f(a)s = f(as) = f(as') = f(a)s' = 1s' = s'$, that is, $s = s'$. Therefore, S is left cancellative.

Proposition 2.2.15 For a monoid S, the following assertions are equivalent:

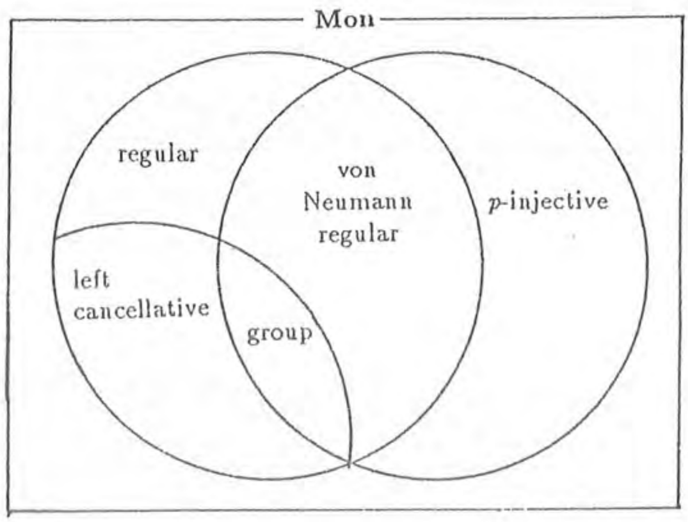
(1) S is left cancellative and S is P-injective.

(2) S is a group.

Proof (2) $\Rightarrow$ (1): Suppose that S is a group. Then, S is left cancellative and is S-divisible. It follows that S is P-injective, by Proposition 2.1.3.

(1) $\Rightarrow$ (2): Let $a \in S$. Since S is left cancellative, S is S-divisible by Proposition 2.1.7. It follows that S is a group.

To summarize, we may now describe the following figure of the category Mon:



2.3 Weakly regular monoids and normal S-systems

In [10], Brown and McCoy considered the notion of weakly regular rings. These rings were later studied by Ramamurthy [41], [52] and others. Adopting this notion we have the following definitions.

Definition 2.3.1 A semigroup S is right *weakly regular* if, for all $x \in S$, $x \in (xS)^2$.

Thus, if S is commutative and weakly regular, then, S is (von Neumann) regular.

We shall now define normal homomorphisms and normal S-systems, and use these notions to characterize weakly regular monoids. In particular, we will prove that each S-system is normal if and only if S is right weakly regular.

First, we introduce some notation.

Let S be a monoid, I a two sided ideal of S and A a right S-system. Then $AI = \left\{ax : a \in A \text{ and } x \in I\right\}$ is a right S-subsystem of A. By A_I we shall denote the Rees factor of A by AI, that is,

$$A/AI = (A\setminus AI) \cup \{AI\}.$$

Let $\alpha: A \longrightarrow B$ be an S-homomorphism. Then, we define the map $\alpha_I: A_I \longrightarrow B_I$ by

$$\alpha_I(a) = \begin{cases} \alpha(a) & \text{if } a \in A\setminus AI \\ \{BI\} & \text{if } a=\{AI\}. \end{cases}$$

Clearly, α_I is an S-homomorphism. Moreover, if α is an epimorphism then so is α_I. We also note that if $\alpha: A \longrightarrow B$ is an S-homomorphism and I is a two-sided ideal of S, then, we always have the inclusion $\alpha(AI) \subseteq BI \cap \alpha(A)$. When equality holds then α is of special interest and motivates the following.

Definition 2.3.2 An S-homomorphism $\alpha: A \longrightarrow B$ is called *normal if* $\alpha(AI) = BI \cap \alpha(A)$ for every two-sided ideal I of S.

Proposition 2.3.3 Let $\alpha: A \longrightarrow B$ be an S-monomorphism. and I a two-sided ideal of S. Then, the following assertions are equivalent:

(1) $\alpha_I: A_I \longrightarrow B_I$ is an S-monomorphism,

(2) $\alpha(AI) = BI \cap \alpha(A)$.

Proof (1) $\Rightarrow$ (2): Suppose that α_I is an S-monomorphism. We verify that $\alpha(AI) = BI \cap \alpha(A)$. Since we always have the inclusion $\alpha(AI) \subseteq BI \cap \alpha(A)$, we only need to show that $BI \cap \alpha(A) \subseteq \alpha(AI)$. Let $\alpha(x) \in BI$, for some $x \in A$. Then, $\alpha_I(x) = BI = \alpha_I(AI)$. Since α_I is an S-monomorphism, the coset x and AI are same, that is, $x \in AI$. Hence, $\alpha(x) \in \alpha(AI)$. Thus, it follows that $\alpha(AI) = BI \cap \alpha(A)$.

(2) $\Rightarrow$ (1): Assume that $\alpha(AI) = BI \cap \alpha(A)$. Suppose we have $\alpha_I(a) = \alpha_I(a')$ for $a,a' \in A_I$. Assume that $a,a' \notin AI$. Then, $\alpha_I(a) = \alpha_I(a')$ implies that $\alpha(a) = \alpha(a')$. Since α is an S-monomorphism, it follows that $a = a'$.

If $a \in AI$, then a' also belongs to AI. For, otherwise, $\alpha_I(a) = BI = \alpha_I(a') = \alpha(a')$. This means that $\alpha(a') \in BI$, that is, $\alpha(a') \in BI \cap \alpha(A) = \alpha(AI)$. Since α is a monomorphism, it follows that $a' \in AI$, which is absurd. Also, since a and a' are both in AI, they represent the same coset in A_I, that is, $a = a'$.

Corollary 2.3.4 Let $\alpha: A \longrightarrow B$ be an S-monomorphism. Then α is normal if and only if α_I is an S-monomorphism for every two-sided ideal I of S.

Definition 2.3.5 An S-subsystem N of an S-system M is *normal* in M if the inclusion $k: N \longrightarrow M$ is normal. M is called *normal* if every S-subsystem N of M is normal in M.

From the above definition and Proposition 2.3.3, it follows that M is normal if and only if $NI = MI \cap N$ for every S-subsystem N of M and every two-sided ideal I of S.

Definition 2.3.6 A two-sided ideal I of a semigroup S is called *right (left) pure* if, for each $x \in I$, there exists $y \in I$, such that, $x = xy$ ($x = yx$). In other words, I is right pure if and only if, for every $a \in I$, the equation $a = ax$ has a solution in I.

Lemma 2.3.7 Each two-sided ideal of a right weakly regular semigroup is right weakly regular as a semigroup.

Proof Let I be a two-sided ideal of a right weakly regular semigroup S and let $x \in I$. Then, $x \in (xS)(xS)$. Hence, $x \in xSxSxSxS \subseteq x(SxS)x(SxS) \subseteq (xI)(xI)$. This means that I is right weakly regular.

Lemma 2.3.8 Let S be a right weakly regular semigroup. Then each S-monomorphism $\alpha: A \longrightarrow B$ is normal.

Proof By the Corollary 2.3.4, in order to prove that α is normal, we must show that $\alpha_I: A_I \longrightarrow B_I$ is an S-monomorphism for every two-sided ideal I of S. By Proposition 2.3.3, it is sufficient to prove that, for every two-sided ideal I of S, we have $BI \cap \alpha(A) \subseteq \alpha(AI)$. Let $x \in BI \cap \alpha(A)$. Since $x \in BI$, we can write $x = bt$, where $b \in B$ and $t \in I$. By Lemma 2.3.7, I (considered as a semigroup) is right weakly regular. Hence $t \in I$ implies that there exists an element $s \in I$ such that $t = ts$. Hence, $x = bt = bts = xs$. Since $x \in \alpha(A)$, it follows that $x = xs \in \alpha(A)s \subseteq \alpha(As) \subseteq \alpha(AI)$. Hence, $BI \cap \alpha(A) \subseteq \alpha(AI)$. This proves the lemma.

Proposition 2.3.9 For a monoid S the following assertions are equivalent:

(1) S is right weakly regular.

(2) $B^2 = B$ for all right ideals B of S.

(3) $BA = B \cap A$ for all right ideals B and all two-sided ideals A of S.

(4) Every two-sided ideal of S is right pure.

Proof (1) $\Rightarrow$ (2): Let B be a right ideal of S. Clearly, $B^2 \subseteq B$. Let $x \in B$. Then, $x \in (xS)(xS) \subseteq BB = B^2$. This proves that $B = B^2$.

(2) $\Rightarrow$ (3): Let B be a right ideal and A a two-sided ideal of S. Clearly, $BA \subseteq B \cap A$. To prove the reverse inclusion, let $x \in B \cap A$. Since $x \in xS = (xS)(xS) = x(SxS) \subseteq xA \subseteq BA$, we have $B \cap A \subseteq BA$ and so $B \cap A = BA$.

(3) $\Rightarrow$ (1): Let $x \in S$ and let $A = SxS$ be the two-sided ideal generated by x. If B is the right ideal xS generated by x then, $x \in B \cap A = BA = (xS)(SxS) \subseteq xS^2xS \subseteq (xS)(xS)$. This implies that S is right weakly regular.

(1) $\Rightarrow$ (4): Suppose that S is right weakly regular. Let A be a two-sided ideal of S and $a \in A$. Since S is right weakly regular, $a \in (aS)(aS) = a(SaS) \subseteq aA$. Hence, there exists an element $x \in A$ such that $a = ax$. Thus, A is right pure.

(4) $\Rightarrow$ (1): Assume that each two-sided ideal of S is right pure. In order to show that S is right weakly regular, let $x \in S$ and $A = SxS$ be the two-sided ideal of S generated by x. By the hypothesis, $x \in xA = x(SxS) = (xS)(xS)$. Hence, S is right weakly regular. This completes the proof.

Theorem 2.3.10 For a monoid S the following assertions are equivalent:

(1) S is right weakly regular.

(2) S is normal.

(3) Every S-monomorphism is normal.

(4) Every S-system is normal.

Proof (1) $\Rightarrow$ (2): Suppose that S is right weakly regular. If B is a right ideal of S then the inclusion map is normal by Lemma 2.3.8. Hence B is normal in S. Thus S is normal.

(2) $\Rightarrow$ (1): Assume that S is normal. If B is any right ideal of S, then B, considered as a right S-system, is normal in S. This means that the inclusion map $\alpha: B \longrightarrow S$ is a normal monomorphism. Hence, it follows from Definition 2.3.2 that, for any two-sided ideal A of S, $BA = B \cap A$. Thus, by Proposition 2.3.9, S is right weakly

regular.

(1) ⇒ (3): This follows from Lemma 2.3.8.

(3) ⇒ (4): This is immediate.

(4) ⇒ (1): Since S is normal by hypothesis, S is right weakly regular by the argument (1) if and only if (2) proved as above.

(1) ⇒ (4): This follows from Lemma 2.3.8. This completes the proof.

Corollary 2.3.11 For a commutative monoid S the following assertions are equivalent:

(1) S is von Neumann regular.

(2) Every ideal of S is pure.

(3) S is normal.

(4) Every S-system is normal.

Proof Since for a commutative monoid S, S is von Neumann regular if and only if S is weakly regular, so the above proposition follows from Proposition 2.3.9 and Theorem 2.3.10 as a corollary.

Proposition 2.3.12 Let S be a PP-monoid (not necessarily commutative). If A is a two-sided ideal of S such that the Rees factor S/A is P-injective as an S-system then, A is right pure in S.

Proof Let $a \in A$. Then, aS is a principal right ideal of S. Consider the Rees factor aS/aA of the right S-system aS. Let $g: aS \longrightarrow aS/aA$ be the natural map defined by

$$g(as) = \begin{cases} as & \text{if } as \notin aA \\ aA & \text{if } as \in aA. \end{cases}$$

Also define $f: S/A \longrightarrow aS/aA$ by

$$f(s) = \begin{cases} as & \text{if } s \in S \setminus A \\ aA & \text{if } s = \{A\}. \end{cases}$$

Clearly, f is an S-epimorphism. Since S is a right PP-monoid, the principal ideal aS is projective as an S-system. It follows that there exists an S-homomorphism $h: aS \longrightarrow S/A$ such that $foh = g$. Hence $foh = g$.

Let us now consider the inclusion map $k: aS \longrightarrow S$. Since S/A is P-injective, for the S-homomorphism h there exists an S-homomorphism $\bar{h}: S \longrightarrow S/A$ such that $\bar{h}ok = h$. Note that $\bar{h}$ extends h: $\bar{h}ok = h$. Let $\bar{h}(1) = s$ for $s \in S/A$. Let $\bar{g} = fo\bar{h}$. Then $\bar{g}: S \longrightarrow aS/aA$ is an S-homomorphism.

We now verify that $\bar{g}$ is an extension of g. Let $x \in aS$. Then $\bar{g}(x) = fo\bar{h}(x) = fo\bar{h}ok(x) = foh(x) = g(x)$.

Now, if $a \in aA$ then we are done. If $a \notin aA$ then $g(a)=a$. Since $\bar{g}$ is an extension of g, therefore, $a = g(a) = \bar{g}(a) = \bar{g}(1a) = \bar{g}(1)a = (fo\bar{h}(1))a$

$$(f(s))a = \begin{cases} asa & \text{if } s \notin A \\ aAa & \text{if } s \in A. \end{cases}$$

This implies that $a \in aA$, in any case. Hence A is right pure.

Corollary 2.3.13 Let S be a right PP-monoid. If, for each two-sided ideal A of S, the Rees factor S/A is P-injective, as an S-system, then S is right weakly regular.

Proof By the above Proposition, each two-sided ideal of S is right pure. Hence by Proposition 2.3.9, S is right weakly regular.

Corollary 2.3.14 A commutative monoid S is von Neumann regular if and only if S is a PP-monoid such that, for each two-sided ideal A of S, the Rees factor S/A is P-injective.

Recall that a right S-system M ($\neq\theta$) is called *simple* if M has no proper non zero S-subsystem.

Proposition 2.3.15 If S is a monoid for which, every simple S-system is P-injective then, S is right weakly regular.

Proof We prove that each right ideal A of S is idempotent, i.e., $A^2 = A$. Suppose that $A \neq A^2$. Let $a \in A$ be such that $a \notin A^2$. Then, $aS \neq (aS)^2$. By Zorn's Lemma, the set of right ideals I, such that, $(aS)^2 \subseteq I \subset aS$ has a maximal element B (say). Then, aS/B is simple and, hence, P-injective by hypothesis. Let $f: aS \longrightarrow aS/B$ be the natural S-homomorphism defined by

$$f(as) = \begin{cases} as & \text{if } as \in aS\backslash B \\ \{B\} & \text{if } as \in B. \end{cases}$$

By the P-injectivity of aS/B, there exists, an S-homomorphism $g: S \longrightarrow aS/B$ which extends f. Let $g(1) = at$ $(t \in S)$. Then $f(as) = g(as) = g(1)as = (at)(as)$. Since $f(a) = a$, we have $a = f(a) = g(a) = (at)a = (at)(a1) \in (aS)^2 \subseteq B$. Hence, $a \in B$. Thus, $B = aS$. This is a contradiction. Hence, $A = A^2$. From Proposition 2.3.9, it follows that S is weakly regular.

Corollary 2.3.16 ([35]) A commutative monoid S is von Neumann regular if and only if each simple S-system is P-injective.

CHAPTER 3

REGULAR AND PP-SEMIRINGS

In this chapter we will be concerned with certain classes of semirings and their semimodules. In particular we will investigate certain aspects of regular semirings, taking a homological approach. We extend the usual elementwise definition of a regular semiring to arbitrary semimodules, and introduce the notion of a von Neumann regular semimodule. We characterize von Neumann regular semimodules in terms of certain restricted injectivity properties (Theorem 3.1.8). Using this characterizations, we obtain new characterizations of (von Neumann) regular semirings. We will also examine the semiring analogs of hereditary, semihereditary and PP-rings. Recall that a ring R is right hereditary (semihereditary; PP) if every right (right finitely generated; right principal) ideal of R is projective (cf. [42]). We will also define and characterize the notion of a PP-semimodule. As an application of our results, we obtain characterization theorem for projective semimodules which is analogous to the Classical Projective Basis Theorem for projective modules over rings (Theorem 3.4.12). Throughout this chapter, R will denote a semiring as defined in chapter 1.

3.1. R-divisible and P-injective semimodules and regular semirings

We begin with the following definition.

Definition 3.1.1 Let R be a semiring and Q a right R-semimodule. An element $x \in Q$ is R-*divisible* in Q if for each (nonzero) $\lambda \in R$, there exists $y \in Q$ such that $x = y\lambda$; Q is R-*divisible* if each element of Q is R-divisible in Q. Thus Q is R-divisible if and only if $Q\lambda = Q$ for all (nonzero) $\lambda \in R$. If every quotient R-semimodule $\bar{Q}$ of Q is R-divisible, Q will be called a *completely R-divisible* R-semimodule.

Example 3.1.2 Let $\mathbb{Z}_o^+$ denote the semiring of non-negative integers with usual addition and multiplication. Then the semigroup $(\mathbb{Q}_o^+,+)$ of non-negative rationals is a $\mathbb{Z}_o^+$-divisible semimodule.

Proposition 3.1.3 Let Q be an R-divisible semimodule. Then each R-homomorphism $\phi\colon I \longrightarrow Q$, where I is a principal right ideal of R, extends to an R-homomorphism

$\overline{\phi}$: R $\longrightarrow$ Q.

Proof Let I = aR (a $\in$ R). Suppose ϕ(a) = x (x $\in$ Q). Then for each $\lambda \in$ R, $\phi(a\lambda) = \phi(a)\lambda = x\lambda$. Since Q is R-divisible, there exists y $\in$ Q such that x = ya. Define $\overline{\phi}$: R $\longrightarrow$ Q by $\overline{\phi}(\lambda) = y\lambda$, ($\lambda \in$R). In particular, $\overline{\phi}(1)$ = y Hence $\overline{\phi}(a\lambda) = ya\lambda = x\lambda = \phi(a\lambda)$, Thus $\overline{\phi}$ is an extension of ϕ.

Definition 3.1.4 A right R-semimodule Q is *p-injective* if each R-homomorphism ϕ: I $\longrightarrow$ Q, where I is a principal right ideal of R, extends to an R-homomorphism $\overline{\phi}$: R $\longrightarrow$ Q. More generally, for an arbitrary but fixed R-semimodule M, Q is *PM-injective* if each R-homomorphism from a cyclic subsemimodule of M to Q extends to an R-homomorphism from M to Q. An R-semimodule all of whose quotient R-semimodules are PM-injective will be called *completely PM-injective* R-semimodule. Completely P-injective R-semimodule are defined analogously.

Proposition 3.1.5 Let R be a right cancellative semiring. Then the following assertions are equivalent:

(1) Q is a (completely) P-injective right R-semimodule;

(2) Q is a (completely) R-divisible right R-semimodule.

Proof (1) ⇒ (2): Let $x \in Q$ and λ be any nonzero element of R. Define ϕ: aR $\longrightarrow$ Q by $\phi(a\lambda) = x\lambda$, ($\lambda \in R$ and $a \in R$) ϕ is a well defined R-homomorphism, since R is right cancellative. Moreover, ϕ extends to an R-homomorphism $\bar{\phi}$: R $\longrightarrow$ Q, since Q is P-injective. Hence $x = \phi(a) = \bar{\phi}(a) = \bar{\phi}(1.a) = \bar{\phi}(1)a$. since $\bar{\phi}(1) \in Q$, x is R-divisible. Hence Q is R-divisible.

(2) ⇒ (1): This is proposition 3.1.3.

Proposition 3.1.6 If M is a retract of an R-divisible R-semimodule Q, then M is R-divisible.

Proof Let p be the retraction and q the coretraction such that $poq = 1_M$. To show that M is R-divisible, let $x \in M$ and $a \in R$. Then $q(x) \in Q$. Since Q is R-divisible, there exists $y \in Q$ such that $q(x) = ya$. Then $x = poq(x) = p(ya) = p(y)a$ and $p(y) \in R$. This shows that M is R-divisible.

Proposition 3.1.7 The following conditions are equivalent for a semiring R.

(1) All right R-semimodules are R-divisible;

(2) All right ideals of R are R-divisible;

(3) R is divisible;

(4) All non zero elements of R are invertible;

Proof It is clear that (1) implies (2) and (2) implies (3). (3) $\Rightarrow$ (4): Let a be a non zero element of R. Then as R is divisible there exists a non zero element b of R with 1 = ba. Thus, a is left invertible. Hence, all non-zero elements are invertible.

(4) $\Rightarrow$ (1): Let x be an element of an R-semimodule Q and let a be a non zero element of R. From (4), there is an element $b \in R$ with $1 = ba$. Then $x = x1 = x(ba) = (xb)a$. Hence Q = QR. So Q is R-divisible.

Theorem 3.1.8 For a semiring R, the following are equivalent:

(1) R is von Neumann regular;

(2) Every R-semimodule is P-injective;

(3) Every cyclic R-semimodule is P-injective.

Proof (1) $\Rightarrow$ (2): Let Ra be a principal right ideal of R and M be an R-semimodule. Let $f: aR \longrightarrow M$ be a R-homomorphism. As $a \in R$ and R is regular, there exists $x \in R$ such that $a = axa$ and $ax \in aR$. Let $f(ax) = m$. Define $\bar{f}:R \longrightarrow M$ by $\bar{f}(1) = m$ and $\bar{f}(s) = \bar{f}(1)s = ms$ then $\bar{f}$ is an extension of f. Thus M is P-injective.

(2) $\Rightarrow$ (3): obvious.

(3) $\Rightarrow$ (1): Let $a \in R$. Consider the right ideal generated by a and the identity R-homomorphism ι, i.e. ι: aR $\longrightarrow$ aR As aR is P-injective, this mapping is extendable to $\bar{\iota}$: R $\longrightarrow$ aR. Let $\bar{\iota}(1) = ax \in aR$ then $\bar{\iota}(a) = \iota(a)$. This implies that $a = \bar{\iota}(a) = \bar{\iota}(1)a = axa$. Thus a is regular.

Definition 3.1.9 A right R-semimodule M is totally irreducible if the only right R-congruences are the universal congruence and the identity congruence and $M \neq 0$.

Theorem 3.1.10 Let R be a semiring with no non-zero zero divisors such that every ideal of R is a K-ideal. Then the following assertions are equivalent:

(a) R is von Neumann regular and $Ra \subseteq aR$ for all $a \in R$;

(b) Every totally irreducible R-semimodule is P-injective and every right ideal is two-sided.

Proof (a) $\Rightarrow$ (b): If R is von Neumann regular then every R-semimodule is P-injective by Theorem 3.1.8. Moreover, if I is a right ideal of R then $Ra \subseteq aR \subseteq I$ for all $a \in I$. Hence I is two-sided.

(b) $\Rightarrow$ (a): Let $0 \neq a \in R$ where a is not regular. Then consider the right ideal aR which is two-sided. Let $\rho(aR)$

be the right R-linear equivalence relation corresponding to right ideal aR. (An equivalence relation ρ on M is said to be right R-linear if and only if for all $a,b,c,d \in M$ and $r \in R$, we have $(a,b), (c,d) \in \rho \Rightarrow (a+c,b+d) \in \rho$ and $(a,b) \in \rho \Rightarrow (ar,br) \in \rho$. The relation $\rho(aR)$ is defined by $(x,y) \in \rho(aR)$, for all $x,y \in R$, if and only if there exist $s,t \in aR$ such that $x + c = y + d$).

Let $f: R \longrightarrow aR$ be defined by $f(\lambda) = a\lambda$ and ker f be the right R-linear equivalence relation corresponding to R-homomorphism f (that is $(a,b) \in \ker f$ if and only if $f(a) = f(b)$ for all $a,b \in R$). Then $(1,0) \notin \rho(aR)$, because if $(1,0) \in \rho(aR)$ then there exist $x,y \in aR$ such that $1+x = y+0 \Rightarrow 1+x = y \in aR \Rightarrow 1 \in aR$ because aR is K-ideal implies that a is regular. Which is a contradiction.

Also $(1,0) \notin \ker f$, because if $(1,0) \in \ker f$ then $f(1) = f(0) \Rightarrow a = 0$ which is again a contradiction.

Let α be the right R-linear equivalence relation generated by $\rho(aR)$ and ker f. Then $(1,0) \notin \alpha$, because, if $(1,0) \in \alpha$, then there exists $c \in R$ such that either $(1,c) \in \rho(aR)$ and $(c,0) \in \ker f$ or $(1,c) \in \ker f$ and $(c,0) \in \rho(aR)$. If $(1,c) \in \rho(aR)$ and $(c,0) \in \ker f$ then $f(c) = f(0) \Rightarrow ac = 0 \Rightarrow c = 0$. and $(1,0) \in \rho(aR) \Rightarrow$ a is regular. Which is a contradiction. If $(1,c) \in \ker f$ and $(c,0) \in \rho(aR)$ then $f(1) = f(c) \Rightarrow a = ac$ and there exist

$x,y \in aR$ such that $x+c = y+0 \Rightarrow x+c = y \in aR$ and $c \in aR$ because aR is a K-ideal. This implies that $c = \lambda a$ because aR is two sided. Hence $a = a\lambda a$, that is, a is regular. Which is a contradiction.

Let β be the right R-linear equivalence relation maximal with respect to condition $\alpha \subseteq \beta$ and $(1,0) \notin \beta$. Then β is a maximal right R-linear equivalence relation, because if $\beta \subseteq \gamma$ then $(1,0) \in \gamma$ and so $(0,1) \in \gamma$. This implies that $\gamma = \omega$ the universal relation. Thus R/β is totally irreducible. Define ψ: $aR \longrightarrow R/\beta$ by $\psi(a\lambda) = [\lambda]$. Then ψ is well defined because if $a\lambda_1 = a\lambda_2$ then $(\lambda_1,\lambda_2) \in \ker f \subseteq \beta \Rightarrow [\lambda_1] = [\lambda_2]$. ψ is an R-homomorphism from $aR \longrightarrow R/\beta$. As R/β is P-injective, this R-homomorphism is extendable from R to R/β i.e. $\bar{\psi}:R \longrightarrow R/\beta$. Let $\bar{\psi}(1) = [x]$ then $[1] = \psi(a) = \bar{\psi}(a) = \bar{\psi}(1)a = [x]a = [xa] \Rightarrow (1,xa) \in \beta$. But $ax \in aR$ and so $(xa,0) \in \rho(aR) \subseteq \beta \Rightarrow (1,0) \in \beta$ which is a contradiction. Hence a is regular, that is, R is regular.

Corollary 3.1.11 If R is a commutative semiring in which every ideal is a K-ideal and has no non-zero zero divisor, then R is von Neumann regular if and only if every totally irreducible R-semimodule is P-injective.

3.2 PP semirings and PP R-semimodules

Definition 3.2.1 Let M be a right R-semimodule. We call M a *PP R-semimodule* if each cyclic subsemimodule of M is projective. In particular, R is a *right PP semiring* if R, considered as a right R-semimodule, is a PP R-semimodule. Hence R is a right PP semiring if each principal right ideal of R is projective (as an R-semimodule).

Definition 3.2.2 An element $a \in R$ (semiring) is called a left e-cancellative if $ae = a$ and, from $ax = ay$, $x,y \in R$, it follows that $ex = ey$.

Proposition 3.2.3 Every right ideal of a semiring R generated by an idempotent is projective.

Proof Let e be an idempotent of R. Consider the maps $f: R \longrightarrow eR$ defined by $f(\lambda) = e\lambda$ and $g: eR \longrightarrow R$ be the identity map, i.e $g(e\lambda) = e\lambda$. Then we have

$$fg(e\lambda) = f(e\lambda) = e(e\lambda) = e^2\lambda = e\lambda$$

Therefore $fg = i_{eR}$. So eR is a retract of R. Thus eR is projective.

Proposition 3.2.4 A cyclic R-semimodule P is projective if and only if $P \cong eR$ for some idempotent e in R.

Proof Suppose P is a cyclic projective R-semimodule, then $P = aR$ for some $a \in P$. Let $f: R \longrightarrow aR$ (defined by $f(1) = a$) be an R-epimorphism. Since aR is projective, there exists $g: aR \longrightarrow R$ such that $fg = 1_{aR}$. Let $g(a) = e$.Then $a = fg(a) = f(e) = f(1)e = ae$, implies that $a = ae \Rightarrow g(a) = g(ae) \Rightarrow e = g(a)e \Rightarrow e = ee \Rightarrow e$ is idempotent.

Now we show that $g: aR \longrightarrow eR$ is an isomorphism. Let $g(a\lambda) = g(a\mu)$. This implies that $g(a)\lambda = g(a)\mu \Rightarrow e\lambda = e\mu \Rightarrow a(e\lambda) = a(e\mu) \Rightarrow (ae)\lambda = (ae)\mu \Rightarrow a\lambda = a\mu$. Thus g is one one. Clearly, g is onto, therefore g is an isomorphism.

Conversely, if $P \cong eR$ for some idempotent $e \in R$ then P is cyclic and as eR is projective, by Proposition 3.2.3. Therefore P is projective.

Corollary 3.2.5 A principal right ideal aR is projective if and only if a is left e-cancelable for some $e \in R$.

Corollary 3.2.6 A semiring R is right PP if every principal right ideal of R is generated by a left e-cancelable element for some $e \in R$.

Lemma 3.2.7 Let A be a right R-semimodule and $\phi \in \text{End } A$.

If $\phi(A)$ is projective then ϕ is left $\mu\phi$-cancelable where $\mu: \phi(A) \longrightarrow A$ is a monomorphism.

Proof As $\phi: A \longrightarrow \phi(A)$ is an epimorphism and $\phi(A)$ is projective, therefore, there exists $\mu: \phi(A) \longrightarrow A$ such that $\phi\mu = 1_{\phi(A)}$. We have $\phi(\mu\phi) = (\phi\mu)\phi = 1_{\phi(A)}\phi = \phi$. Now, let $\phi\alpha = \phi\beta$ for some $\alpha,\beta \in$ End A. Then $(\mu\phi)\alpha = \mu(\phi\alpha) = \mu(\phi\beta) = (\mu\phi)\beta$. Thus ϕ is left $\mu\phi$-cancelable.

Theorem 3.2.8 A semiring R is a PP semiring if and only if End P is PP for every cyclic projective R-semimodule P.

Proof Let R be a PP semiring and P = aR, where $a \in P$, be a cyclic projective R-semimodule. Let $\phi \in$ End P. First, we show that $\phi(P)$ is projective. Clearly, $\phi(P)$ is a cyclic R-subsemimodule of P.

Let $f: R \longrightarrow aR = P$ defined by $f(1) = a$ be an epimorphism. Then there exists $g: aR \longrightarrow R$ such that, $fg = 1_{aR}$. Thus g is a monomorphism from P to R. Hence g maps $\phi(P)$ onto a principal right ideal of R, isomorphically. Thus, $\phi(P)$ is projective because R is PP. Hence by Lemma 3.2.7, ϕ is e-cancelable for some $e \in$ End P. By Corollary. 3.2.6 End P is PP.

Conversely, since R is a cyclic, projective

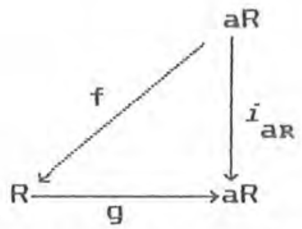
R-semimodule, therefore End R is PP by hypothesis. Also f: End R $\longrightarrow$ R defined by $f(\phi) = \phi(1)$ is an isomorphism therefore, R is PP.

Proposition 3.2.9 For a right R-semimodule M, the following assertions are equivalent:

(1) M is a PP R-semimodule;

(2) For each $a \in M$, there exists an R-homomorphism $f \in Hom_R(aR,R)$ such that $a = a.f(a)$.

Proof (1) $\Rightarrow$ (2): Let $a \in M$. Then $aR = \left\{a\lambda : \lambda \in R\right\}$ is a cyclic subsemimodule of M, and is projective by the hypothesis. Define $g: R \longrightarrow aR$ by setting $g(\lambda) = a\lambda$ ($\lambda \in R$). Clearly, g is a surjective R-homomorphism. Hence, there exists $f \in Hom_R(aR,R)$ such that the following diagram commutes:



Thus $gf = 1_{aR}$. This implies that $gf(a) = a$. Since $g(f(a)) = af(a)$ it follows that $a = a.f(a)$.

(2) $\Rightarrow$ (1): In order to prove that M is a PP R-semimodule, we show that, for each $a \in M$, the cyclic subsemimodule,

aR, of M is projective. Let $g: R \longrightarrow aR$ be defined by $g(\lambda) = a\lambda$, $(\lambda \in R)$. Then, by the hypothesis, there exists $f \in Hom_R(aR,R)$ such that $a.f(a) = a$. Since $a.f(a\lambda) = a.f(a).\lambda = a\lambda$, it follows that $g(f(a\lambda)) = a.f(a\lambda) = a\lambda$. This implies that $g \circ f = 1_{aR}$. Thus, aR is a retract of R and is thus projective

3.3. Von Neumann regular semimodules

Definition 3.3.1 A right R-semimodule M is called *von Neumann regular* if, for each $a \in M$, there exists an R-homomorphism $g \in Hom_R(M,R)$ such that $a = a.g(a)$ ([58]). Thus, if R, considered as a right R-semimodule, is *von Neumann regular* then, for each $\lambda \in R$, there exists $g \in Hom_R(R,R)$ such that $a = a.g(a) = a.g(1.a) = a.g(1).a$ and $g(1) \in R$. Hence R is von Neumann regular in the usual sense.

Lemma 3.3.2 Every R-subsemimodule of a von Neumann regular R-semimodule is von Neumann regular.

Proof Let M be a von Neumann regular R-semimodule and N be a Subsemimodule of M. Let $a \in N$. Then $a \in M$. Thus, there exists an R-homomorphism $g \in Hom_R(M,R)$ such that

$a = a.g(a)$. Let $\bar{g}$ be the restriction of g to N then $\bar{g} \in Hom_R(N,R)$ and $g(a) = \bar{g}(a)$. Thus, $a = a.\bar{g}(a)$. Hence, N is von Neumann regular.

Lemma 3.3.3 Every retract of a von Neumann regular R-semimodule is von Neumann regular.

Proof Let M be a von Neumann regular R-semimodule and N be a retract of M. Then, there exist R-homomorphisms $f: N \longrightarrow M$ and $g: M \longrightarrow N$ such that $gof = 1_N$. Let $a \in N$. Then $f(a) \in M$. Hence, there exists an R-homomorphism $\phi \in Hom_R(M,R)$ such that

$$f(a) = f(a)\phi(f(a)) = f(a).\phi f(a)$$

and $g(f(a)) = g(f(a).\phi f(a)) = gf(a).\phi f(a) \Rightarrow a = a.\phi f(a)$. Where $\phi f: N \longrightarrow R$. Thus, N is a von Neumann regular.

Proposition 3.3.4 For a right R-semimodule M the following assertions are equivalent:

(1) M is (von Neumann) regular;

(2) M is a PP R-semimodule and R is PM-injective.

Proof (1) $\Rightarrow$ (2): Suppose M is von Neumann regular. Then it follows easily that, for each $a \in M$, there exists an R-homomorphism $f \in Hom_R(aR,R)$ such that $a = a.f(a)$. Hence, by Proposition 3.2.9, M is a PP R-semimodule. We now show that R is PM-injective. Let aR ($a \in M$) be a cyclic R-sub-semimodule of M and let $f: aR \longrightarrow R$ be an R-homomorphism. Since M is von Neumann regular and $a \in M$, there exists an R-homomorphism $g: M \longrightarrow R$, such that, $a = a.g(a)$. Define $\bar{f}: M \longrightarrow R$ by $\bar{f}(m) = f(a).g(m)$, for all $m \in M$. Clearly, $\bar{f}$ is an R-homomorphism which extends f. Hence, R is PM-injective.

(2) $\Rightarrow$ (1): Suppose M is a PP R-semimodule, and R is PM-injective. Let $a \in M$. By Proposition 3.2.9, there exists an R-homomorphism $f: aR \longrightarrow R$ such that $a = a.f(a)$. Since R is PM- injective, there exists an R-homomorphism $g: M \longrightarrow R$, which extends f. Hence $a = a.g(a)$. This implies that M is von Neumann regular.

Corollary 3.3.5 A semiring R is von Neumann regular if and only if R is a PP semiring which is P-injective (as a right R-semimodule).

Proposition 3.3.6 For a semiring R the following assertions are equivalent:

(1) R is a PP semiring with a unique idempotent ($x \in R$ is idempotent if $x^2 = x$);

(2) R is left cancellative.

Proof (1) $\Rightarrow$ (2): Suppose that $\lambda a = \lambda b$ for $a, b, \lambda \in R$. Since R is a PP semiring, it follows, from Proposition 3.2.9, that there exists an R-homomorphism $f:\lambda R \longrightarrow R$ such that $\lambda = \lambda.f(\lambda)$. Hence,

$$f(\lambda) = f(\lambda.f(\lambda)) = f(\lambda).f(\lambda).$$

Hence $f(\lambda) = 1$ by the uniqueness of the idempotent. Hence $\lambda a = \lambda b$ implies that $a = 1.a = f(\lambda).a = f(\lambda a) = f(\lambda b) = f(\lambda)b = 1.b = b$. That is, $a = b$. Hence R is left cancellative.

(2) $\Rightarrow$ (1): Suppose R is left cancellative. Then, obviously, R has a unique idempotent. Let $a \in R$ and define $g: aR \longrightarrow R$ by $g(a\lambda) = \lambda$, for all $\lambda \in R$. Then $g(a) = 1$ and we have $a = a.1 = a.g(a)$. Hence, by Proposition 3.2.9, R is a PP semiring.

3.4. Projective Basis Theorem for R-semimodules

Definition 3.4.1 Let R be a commutative cancellative semiring and Q be the semifield of quotients of R (cf.[46]). An ideal I of R is called *invertible* if there exist elements $a_1,\ldots,a_n \in I$, $q_1,\ldots,q_n \in Q$ such that :

(i) $q_i I \subseteq R$ $(i = 1,\ldots,n)$

(ii) $\sum_{i=1}^{n} q_i a_i = 1$

Remark If R is a commutative cancellative semiring then each nonzero principal ideal aR ($a \in R$) of R is invertible (by choosing $a_1 = a$, $q_1 = (a)^{-1} \in Q$).

Proposition 3.4.2 If A is an invertible ideal of a commutative cancellative semiring R then A is finitely generated.

Proof As A is an invertible ideal of R, therefore there exist $a_1, \ldots, a_n \in A$, and $q_1, \ldots, q_n \in Q$ such that $q_i A \subseteq R$ and $\sum_{i=1}^{n} q_i a_i = 1$. We claim that $A = \langle a_1, \ldots, a_n \rangle$. Clearly $\langle a_1, \ldots, a_n \rangle \subseteq A$. If $x \in A$ then $x = x1 = x(\sum_{i=1}^{n} q_i a_i) = \sum_{i=1}^{n} (xq_i)a_i = \sum_{i=1}^{n} r_i a_i$ where $r_i = xq_i \in R$. Thus $x \in \langle a_1, \ldots, a_n \rangle$. Hence $A = \langle a_1, \ldots, a_n \rangle$. So A is finitely generated.

Definition 3.4.3 If A is an invertible ideal of a commutative cancellative semiring R, then, we define A^{-1} to be the R-subsemimodule of Q generated by $q_1, q_2, \ldots q_n$.

Proposition 3.4.4 If A is an invertible ideal of a commutative cancellative semiring R then $AA^{-1} = A^{-1}A = R$

where

$$AA^{-1} = \left\{ \sum_{finite} a_i b_i : a_i \in A \text{ and } b_i \in A^{-1} \right\}$$

Proof As $1 = \sum_{i=1}^{n} q_i a_i \in AA^{-1}$. Thus $AA^{-1} = R$.

Definition 3.4.5 Let R be a commutative cancellative semiring and Q be its semifield of quotients. Every R-subsemimodule A of Q such that there exist $0 \neq \lambda \in R$ for which $\lambda A \subseteq R$ is called a *fractional ideal* of R.

Proposition 3.4.6 Every finitely generated R-subsemimodule of Q is a fractional ideal.

Proof Let $A = \langle a_1, \ldots, a_n \rangle$, where $a_i \in Q$, be a finitely generated R-subsemimodule of Q. As $a_i \in Q$ therefore $a_i = b_i(d_i)^{-1}$ where $b_i, d_i \in R$ and $d_i \neq 0$ for $i = 1, \ldots, n$. Let $d = d_1 d_2 \ldots d_n$, then $d \in R$. Now if $x \in A$ then $x = \sum_{i=1}^{n} \lambda_i a_i = \sum_{i=1}^{n} \lambda_i b_i (d_i)^{-1}$. Thus $dx = (d_1 d_2 \ldots d_n)(\sum_{i=1}^{n} \lambda_i b_i (d_i)^{-1} = \sum_{i=1}^{n} \lambda_i b_i d_1 \ldots d_{i-1} d_{i+1} \ldots d_n \in R$. Thus A is a fractional ideal of R.

Corollary 3.4.7 If A is an invertible ideal of R then A and A^{-1} are fractional ideals.

Proposition 3.4.8 An ideal A of a commutative cancellative semiring R is invertible if and only if there exists a fractional ideal B such that AB = BA = R.

Proof If A is an invertible ideal of R then by above Corollary there exists a fractional ideal A^{-1}. By Proposition 3.4.4 $AA^{-1} = R$.

Conversely, suppose that there exists a fractional ideal B such that AB = BA = R. As $1 \in R$ therefore $1 \in AB$. Hence $1 = \sum_{i=1}^{n} a_i b_i$ where $a_i \in A$ and $b_i \in B$. Also $Ab_i \subseteq R$. Thus by Definition 3.4.1 A is invertible.

Definition 3.4.9 A fractional ideal A of a commutative cancellative semiring R is invertible if there exist a fractional ideal B such that AB = BA = R.

Definition 3.4.10 Let R be a commutative cancellative semiring with semifield of quotients Q and A , B are fractional ideals of R then $(A:B) = \left\{x \in Q: xB \subseteq A\right\}$.

Proposition 3.4.11 If A is an invertible fractional ideal of R, then A has a unique inverse and this inverse is equal to $(R:A)$. Hence, a necessary and sufficient condition for A to be invertible is that $A.(R:A) = R$.

Proof As A is invertible, therefore there exists a fractional ideal A^{-1} such that $AA^{-1} = A^{-1}A = R$. This implies that $A^{-1} \subseteq (R:A)$. On the other hand $A(R:A) \subseteq R$. Now $(R:A) = A^{-1}A(R:A) \subseteq A^{-1}R \subseteq A^{-1}$. Thus $A^{-1} = (R:A)$.

Theorem 3.4.12 Let M be a right R-semimodule. Then the following assertions are equivalent:

(1) M is projective;

(2) There exists elements $\{a_k \in M : (k \in K)\}$ and R-homomorphism $\{\phi_k : M \longrightarrow R \ (k \in K)\}$ such that:

(a) if $x \in M$ then $\phi_k(x) = 0$ for almost all $k \in K$;

(b) if $x \in M$ then $x = \sum_{k\in K} a_k \phi_k(x)$. Moreover, M is then generated by $\{a_k : k \in K\}$.

Proof (1) $\Rightarrow$ (2): Suppose M is projective. By Proposition 1.6.2, there exists a free right R-semimodule F and an R-epimorphism $\psi: F \longrightarrow M$. Since M is projective, there exists an R-homomorphism $\phi: M \longrightarrow F$ such that $\psi\,\phi = 1_M$. Let $\{e_k : k \in K\}$ be a basis for F. If $x \in M$ then $\phi(x) \in F$. Hence, we can write $\phi(x) = \sum_{k\in K} e_k.\lambda_k$, where $\lambda_k \in R$ and $\lambda_k=0$

for almost all k. Define $\phi_k(x) = \lambda_k$. Then ϕ_k is an R-homomorphism. Since ψ is an epimorphism, therefore $\{a_k : \psi(e_k) = a_k, k \in K\}$ generate M. Moreover, if $x \in M$ then $x = \psi\,\phi(x) = \psi(\sum e_k\lambda_k) = \sum (\psi(e_k))\lambda_k = \sum a_k\lambda_k = \sum a_k\phi_k(x)$.

(2) $\Rightarrow$ (1): Assume the existence of $\{a_k : k \in K\}$ and R-homomorphism $\{\phi_k : M \longrightarrow R\}$. Let F be a free R-semimodule with basis $\{e_k : k \in K\}$. Define $\psi: F \longrightarrow M$ by setting $\psi(e_k) = a_k$ and extending ψ to F. Then, ψ is an R-homomorphism which is surjective. Now, define $\phi : M \longrightarrow F$ by $\phi(x) = \sum_{k\in K} e_k.\phi_k(x)$. As this sum is finite, ϕ is well-defined. Then we have

$$\psi\phi(x) = \psi(\sum e_k.\phi_k(x)) = \sum \psi(e_k)\phi_k(x) = \sum a_k\ \phi_k(x) = x.$$

Hence $\psi\phi = 1_M$. This implies that M is projective by Proposition 1.6.2.

Proposition 3.4.13 Let R be a commutative cancellative semiring. Then each nonzero ideal I of R is projective if and only if I is invertible.

Proof (1) Suppose I is a (nonzero) projective ideal of R. Then I has a projective basis by the above Theorem. Hence, there are elements $\{a_k : k \in K\} \subseteq I$ and R-homomorphisms $\phi_k : I \longrightarrow R$ such that: (i) if $a \in I$, then $\phi_k(a) = 0$ for almost all $k \in K$; (ii) if $a \in I$, then $a = \sum_{k\in K} a_k(\phi_k(a))$. If $b \in I$ and $b \neq 0$, then define $q_k = \phi_k(b)/b$, where Q is the semifield of quotients of R. Note that q_k does not depend on the choice of b. For, if $b' \in I$ and $b' \neq 0$, then $b'\phi_k(b) = \phi_k(b'b) = \phi_k(bb') = b\phi_k(b') \Rightarrow \phi_k(b)/b = \phi_k(b')/b'$. We now verify that $q_k I \subseteq R$ for all $k \in K$. Let $b \in I$ ($b \neq 0$). Then $q_k b = [\phi_k(b)/b]b = \phi_k(b) \in R$. By condition (i), if $b \in I$ and $b \neq 0$, then $\phi_k(b) = 0$ for almost all k. Since $q_k = \phi_k(b)/b$, there are only finitely many nonzero q_k. Finally, by condition (ii), if $b \in I$ then

$$b = \sum a_k(\phi_k(b)) = \sum a_k(q_k b) = (\sum q_k a_k)b.$$

If we discard all those a_k for which $q_k = 0$, then there remain finitely many $a_k \in I$. Furthermore, if $b \neq 0$, we may cancel b from both sides of the above equation to get $1 = \sum q_k a_k$. This proves I is invertible.

(2) Suppose now that I is invertible and let $a_1, \ldots, a_n \in I$, and $q_1, \ldots, q_n \in Q$ be as in the definition. Define $\phi_k: I \longrightarrow R$ by $\phi_k(a) = q_k a$ ($q_k I \subseteq R$). Let $a \in I$. Then $\sum (\phi_k(a))a_k = \sum q_k a a_k = a \sum q_k a_k = a$. This implies that I has a projective basis. Hence I is projective by Theorem 3.4.12.

Following the terminology in ring theory, we call a semiring R *right hereditary* (*semihereditary*) if each right ideal (finitely generated) of R is projective (as a right R-semimodule). The following corollaries follow from the above proposition and the remark stated before Lemma 3.4.2.

Corollary 3.4.14 Let R be a commutative cancellative semiring. Then R is a semihereditary semiring if and only if every finitely generated ideal of R is invertible.

Corollary 3.4.15 Let R be a commutative cancellative semiring. Then R is hereditary if and only if each ideal of R is invertible.

Corollary 3.4.16 Each ideal of a commutative cancellative hereditary semiring is finitely generated

CHAPTER 4

WEAKLY REGULAR SEMIRINGS AND THEIR PRIME IDEAL SPACES

Analogous to von Neumann regular rings, a ring R is called right weakly regular if $x \in (xR)^2$, for each $x \in$ R. These rings were introduced by Brown and McCoy [10], later investigated by Ramamurthy [41], [52] and others. In this chapter we define and characterize weakly regular semirings and study some properties of the space of their prime ideals.

4.1 Weakly regular semirings

A semiring R is called *right weakly regular* if $a \in (aR)^2$, for each $a \in$ R. Thus, if R is commutative then R is weakly regular if and only if R is regular. In general, however, regular semirings form a proper subclass of weakly regular semirings.

Theorem 4.1.1 The following assertions for a semiring R are equivalent:

1. R is right weakly regular;

2. $J^2 = J$ for each right ideal J of R;

3. For each (two-sided) ideal I of R; $J \cap I = JI$, for any right ideal J of R.

Proof (1) $\Rightarrow$ (2): Let J be a right ideal of R. Clearly, $J^2 \subseteq J$. For the reverse inclusion, let $x \in J$; so $x \in (xR)^2$ Hence $x \in J^2$, so $J = J^2$.

(2) $\Rightarrow$ (3): Let I be any ideal of R and let $x \in I$. Since $x \in (xR) = (xR)^2$, it follows that $x = xy$, for some $y \in I$. Let J be a right ideal of R. Clearly, $JI \subseteq J \cap I$. Let $x \in J \cap I$. Then there exists $y \in I$ such that $x = xy$. Thus $x \in JI$ i.e., $J \cap I \subseteq JI$, so $J \cap I = JI$.

(3) $\Rightarrow$ (1): Let $x \in R$. Then $x \in (xR) \cap (RxR) = (xR)(RxR) \subseteq (xR^2)(xR) \subseteq (xR)(xR)$. Hence R is right weakly regular.

Proposition 4.1.2 ([41], Prop. 5, p. 318). Each ideal of a right weakly regular semiring is (right) weakly regular (as a semiring).

Definition 4.1.3 A two sided ideal I of a semiring R is called *right (left) pure* if, for each $x \in I$, there exists $y \in I$ such that $x = xy$ ($x = yx$); in other words, I is right pure if and only if for every $a \in I$ the equation $a = ax$ has a solution in I.

Definition 4.1.4 An R-subsemimodule N of an R-semimodule M is *normal* if and only if $NI = MI \cap N$ for every ideal I of R. M is called *normal* if every R-subsemimodule N of M is normal in M.

We characterize right weakly regular semirings in terms of pure ideals and normal semimodules.

Proposition 4.1.5 A semiring R is right weakly regular if and only if every two sided ideal of R is right pure.

Proof Suppose R is a right weakly regular semiring and I is a two sided ideal of R. Let $a \in I$. Since R is right weakly regular, $a \in (aR)(aR)$ i.e. $a = \sum_{finite} (ax_i)(ay_j) = \sum a(x_i ay_j) = a \sum x_i ay_j = ax$ where $x = \sum x_i ay_j \in I$. Thus I is right pure.

Conversely, suppose that each two sided ideal of R is right pure. Let $x \in R$ and I be a two sided ideal generated by x. Then, by hypothesis, $x \in xI$ i.e. $x = x \sum a_i xb_i = \sum xa_i xb_i \in (xR)(xR)$. Hence R is right weakly regular.

Proposition 4.1.6 A semiring R is right weakly regular if and only if each cyclic R-semimodule is normal.

Proof Let R be a right weakly regular semiring and M=xR be a cyclic R-semimodule. Clearly, $NI \subseteq N \cap MI$ (where N is an R-subsemimodule of M). For the reverse inclusion, let $a \in N \cap MI$ implies that $a \in N$ and $a \in MI$. Then $a \in MI$ implies that $a = \sum_{finite} xa_j i_j = x \sum a_j i_j = xi$ where $i = \sum a_j i_j \in I$. As R is a weakly regular semiring and $i \in I$, therefore, there exists $j \in I$ such that $ij = i$. Hence, $a = xi = x(ij) = (xi)j = aj \in NI$. Thus, $N \cap MI \subseteq NI$.

Conversely, suppose that each cyclic R-semimodule is normal. As R is a cyclic R-semimodule, therefore, $JI = RI \cap J = I \cap J$ for every ideal I and every right ideal J of R. Thus, R is a right weakly regular semiring.

We shall now examine some properties of the lattice of ideals of a right weakly regular semiring R. In the sequel we shall denote this lattice by $\mathscr{L}_R$. First we show that the lattice $\mathscr{L}_R$ is a complete Brouwerian and, hence, distributive lattice. Recall that a lattice $\mathscr{L}$ is called *Brouwerian* if for any $a,b \in \mathscr{L}$, the set of all $x \in \mathscr{L}$ satisfying $a \wedge x \leq b$ contains a greatest element c, the pseudo-complement of a relative to b (cf. [9])

Proposition 4.1.7 Let R be a right weakly regular semiring. Then, the lattice $\mathscr{L}_R$ is a complete Brouwerian lattice under the sum and intersection of ideals.

Proof Clearly, $\mathscr{L}_R$ is a complete lattice under the sum and intersection of ideals. Let B and C be ideals of R. By Zorn's Lemma, there exists an ideal M of R which is maximal in the family of ideals I satisfying $B \cap I \subseteq C$. Thus, for any such ideal I we have $BI \subseteq C$ by Theorem 4.1.1 Again, by Theorem 4.1.1 $B(I + M) = B \cap (I + M) \subseteq C$. By the maximality of M, we get $I + M \subseteq M$ and, therefore, $I \subseteq M$, as required. This proves that $\mathscr{L}_R$ is a Brouwerian lattice. Since $\mathscr{L}_R$ is also a complete lattice, therefore it follows from ([9] II.11) that $\mathscr{L}_R$ is distributive.

Recall that an ideal P of a semiring R is *prime (irreducible; strongly irreducible)* if $IJ \subseteq P \Rightarrow I \subseteq P$ or $J \subseteq P$ ($I \cap J = P \Rightarrow I = P$ or $J = P$; $I \cap J \subseteq P \Rightarrow I \subseteq P$ or $J \subseteq P$) holds for all ideals I,J of R. Thus, any prime ideal is strongly irreducible and any strongly irreducible ideal is irreducible (cf. [23]). The notions of prime, irreducible and strongly irreducible ideals coincide for right weakly regular semirings, as shown below.

Proposition 4.1.8 Let R be a right weakly regular semiring. Then the following assertions for an ideal P of R are equivalent:

(1) P is irreducible.

(2) P is prime.

Proof It is clear that (2) implies (1), thus, it suffices to show that (1) $\Rightarrow$ (2). Suppose that $IJ \subseteq P$ for ideals I and J of R. Hence $I \cap J \subseteq P$, by Theorem 4.1.1. Thus, it follows that $(I \cap J) + P = P$. Since the ideal lattice of R is distributive , we have $P = (I \cap J) + P = (I+P) \cap (J+P)$. Since P is irreducible, therefore, $I+P = P$ or $J+P = P$. This implies that $I \subseteq P$ or $J \subseteq P$. Hence P is a prime ideal.

As an application of the above Proposition, we prove the following result.

Theorem 4.1.9 Let R be a right weakly regular semiring. Then each proper ideal of R is the intersection of prime ideals which contain it.

Proof Let I be a proper ideal of R and let $\left\{P_\alpha : \alpha \in \Lambda\right\}$ be a family of prime ideals of R which contain I. Clearly, $I \subseteq \cap P_\alpha$. To prove the converse, suppose that $a \notin I$. By

Zorn's Lemma, there exists an ideal P_α such that P_α is proper, $I \subseteq P_\alpha$, $a \notin P_\alpha$, and P_α is maximal with these properties. Then P_α is irreducible. For, suppose on the contrary, that $P_\alpha = K \cap L$, and both K and L properly contain P_α. Then K and L both contain a. Hence $a \in K \cap L$. This contradicts the assumption that $P_\alpha = K \cap L$. Hence, P_α is irreducible, and, therefore prime by Proposition 4.1.8. This establishes the existence of a prime ideal P_α such that $a \notin P_\alpha$ and $I \subseteq P_\alpha$. Hence, $a \notin \cap P_\alpha$. As this is true for every $a \notin I$, the desired result follows.

Remark The above property doesnot fully characterize weakly regular semirings. We refer to [2] for semirings which are precisely characterized by the property that each proper ideal is the intersection of prime ideals.

Proposition 4.1.10 The set of direct summands of a right weakly regular semiring R is a Boolean sub lattice of the lattice of ideals of R.

Proof The proposition will follow if we show that $A_1 + A_2$ and $A_1 \cap A_2$ are direct summands of R if A_1 and A_2 are direct summands of R. Let B_1 and B_2 be the cosumands of A_1 and A_2, respectively, that is, $A_1 + B_1 = R$, $A_1 \cap B_1 = (0)$,

$A_2 + B_2 = R$ and $A_2 \cap B_2 = (0)$. We show that $A_1 + A_2$ is a summand of R with $B_1 \cap B_2$ as the cosummand of $A_1 + A_2$. It is easily seen that every ideal is contained in a maximal ideal which is irreducible. If there exists a maximal ideal M containing $B_1 \cap B_2$, then M contains B_1 or B_2 by the irreducibility of M. Suppose M contains B_1. Since $A_1 + B_1 = R$, M can not contain A_1. Hence there is no maximal ideal bigger than $(B_1 \cap B_2) + (A_1 + A_2)$, i.e., $(B_1 \cap B_2) + (A_1 + A_2) = R$. Now, if $x \in B_1 \cap B_2$ and $y_k \in A_k$ $(k = 1,2)$, then $x(y_1 + y_2) = xy_1 + xy_2 = 0$, since $A_k \cap B_k = 0$ $(k = 1,2)$. Thus, $(B_1 \cap B_2)(A_1 + A_2) = (0)$. Hence by Theorem 4.1.1 $(B_1 \cap B_2) \cap (A_1 + A_2) = (0)$. Thus $(B_1 \cap B_2)$ is the cosummand of $A_1 + A_2$. An exactly similar proof can be given for the intersection.

4.2. Prime spectrum of a weakly regular semiring

We continue to let $\mathscr{L}_R$ denote the lattice of ideals of R and P(R) will denote the set of proper prime ideals of R. For any ideal I of R, we define $\Theta_I = \left\{J \in P(R) : I \nsubseteq J\right\}$, and $\tau(P(R)) = \left\{\Theta_I : I \in \mathscr{L}_R\right\}$. In the rest of this section, R will denote a right weakly regular semiring.

Theorem 4.2.1 The set $\tau(P(R))$ forms a topology on the set P(R). Moreover, the assignment $I \longrightarrow \Theta_I$ is an isomorphism between the lattice $\mathscr{L}_R$ of ideals of R and the lattice of open subsets of P(R).

Proof First we show that $\tau(P(R))$ forms a topology on the set P(R). Note that $\Theta_{(O)} = \left\{J \in P(R): (O) \nsubseteq J\right\} = \phi$, since (O) is contained in every (prime) ideal. Thus $\Theta_{(O)}$ is the empty subset of $\tau(P(R))$. On the other hand, $\Theta_R = \left\{J \in P(R): R \nsubseteq J\right\} = P(R)$. This is true, since prime ideals are proper. So Θ_R (= P(R)) is an element of $\tau(P(R))$. Now, let $\Theta_{I_1}, \Theta_{I_2} \in \tau(P(R))$ with I_1, I_2 in $\mathscr{L}_R$. Then $\Theta_{I_1} \cap \Theta_{I_2} = \left\{J \in P(R): I_1 \nsubseteq J \text{ and } I_2 \nsubseteq J\right\} = \left\{J \in P(R): I_1 \cap I_2 \nsubseteq J\right\}$. This follows from Proposition 4.1.8. Next, let us consider an arbitrary family $(I_\lambda)_{\lambda\in\Lambda}$ of ideals of R. Since $\cup\, \Theta_{I_\lambda} = \cup \left\{J \in P(R) : I_\lambda \nsubseteq J\right\} = \left\{J \in P(R) : \exists\lambda \in \Lambda \text{ such that } I_\lambda \nsubseteq J\right\} = \left\{J \in P(R) : \sum I_\lambda \nsubseteq J\right\} = \Theta_{\sum_{\lambda\in\Lambda} I_\lambda}$. Since $\sum_{\lambda\in\Lambda} I_\lambda \in \mathscr{L}_R$, it follows that $\bigcup_\lambda \Theta_{I_\lambda} \in \tau(P(R))$. Thus, the set $\tau(P(R))$ of subsets Θ_I with $I \in \mathscr{L}_R$ constitutes a topology on the set P(R). Let $\phi: \mathscr{L}_R \longrightarrow \tau(P(R))$ be the mapping defined by $I \longrightarrow \Theta_I$. It follows from the above that the prescription $\phi(I) = \Theta_I$ preserves finite intersections and arbitrary unions. Thus, ϕ is a lattice homomorphism. To conclude the proof, we must show that ϕ is bijective. In fact, we need to prove the equivalence $I_1 = I_2$ if and only if $\Theta_{I_1} = \Theta_{I_2}$ for I_1, I_2 in $\mathscr{L}_R$. Suppose that $\Theta_{I_1} = \Theta_{I_2}$. If $I_1 \neq I_2$, then there exists $x \in I_1$ such that $x \notin I_2$. Then there exists a prime ideal J such that $I_2 \subseteq J$ and $x \notin J$. Hence, $I_1 \nsubseteq J$, therefore, $J \in \Theta_{I_1}$. By the assumption $\Theta_{I_1} = \Theta_{I_2}$; we have $J \in \Theta_{I_2}$. Hence, $I_2 \nsubseteq J$. But this is a contradiction. Hence, $I_1 = I_2$.

Definition 4.2.2 The set P(R) of prime ideals of R will be called *prime spectrum* of R. The topology $\tau(P(R))$ in the above theorem will be called the *prime spectral topology* on P(R). We shall denote by $\mathcal{P}(R)$ the *prime ideal space* of R.

Proposition 4.2.3 For a right weakly regular semiring R, the following hold:

(1) For $I \in \mathscr{L}_R$, Θ_I is open and closed in $\mathcal{P}(R)$ if and only if I is a direct summand of R.

(2) $\mathcal{P}(R)$ is a compact space. (but not Hausdorff, in general).

Proof (1) Suppose that Θ_I $(I \in \mathscr{L}_R) \in \tau(P(R))$ is both open and closed. Then there exists Θ_J with $J \in \mathscr{L}_R$ such that $\Theta_I \cup \Theta_J = P(R)$ and $\Theta_I \cap \Theta_J = \phi$. This implies that $I + J = R$ and $I \cap J = (0)$. Therefore, I is a direct summand of R.

(2) Suppose that $\bigcup_\lambda \Theta_{I_\lambda} = P(R)$ is an open covering of P(R). Then $\sum_\lambda I_\lambda = R$. Since $1 \in R$, there exist $I_{\lambda_1},\ldots,I_{\lambda_n}$ such that $1 \in \sum_{i=1}^{n} I_{\lambda_i}$. Hence, $R = \sum_{i=1}^{n} I_{\lambda_i}$. Thus, P(R) $= \bigcup_{i=1}^{n} \Theta_{I_{\lambda_i}}$. Hence $\mathscr{P}(R)$ is compact.

Proposition 4.2.4 A right weakly regular semiring R is directly indecomposable if and only if $\mathscr{P}(R)$ is a connected space.

Proof A topological space is connected if and only if it has no nonempty proper open and closed subsets. Hence, the proof follows from part (1) of the above proposition.

CHAPTER 5

SHEAFS FOR CLASSES OF MONOIDS AND SEMIRINGS

A classical result in ring theory asserts that any commutative ring with identity is isomorphic to the full ring of global sections in a sheaf of local rings. Following this result, proved by A.Grothendieck in the late 1950's, several authors have established representations of rings and other algebraic structures by sections in sheafs. In 1966, J.Dauns and K.H.Hofmann [13] obtained a representation of (not necessarily commutative) biregular rings. In 1967, R.S.Pierce [40] proposed a different kind of sheaf representation for rings. On the other hand, Dauns and Hofmann [14] extended their representation theory of biregular rings to weakly biregular rings. They proved that a weakly biregular ring with identity is isomorphic to the ring of all continuous sections in a sheaf of local rings over a zero dimensional compact Hausdorff space ([14], 3.2, Thm. XI, P. 154). In 1969, S.Teleman developed a functional representation theory for harmonic rings by sheafs (see the bibliography in [51] for several references to Teleman's work). In 1971 J.Lambek obtained a representation theorem for modules by sheafs of factor modules. For a survey of results dealing with representations of rings and modules, we refer to Mulvey [37]. We also refer to K.Keimel [27] in which he developed a representation theory for lattice ordered rings which also applies to abelian lattice ordered groups and to vector lattices. The aim of this chapter is to initiate an analogous study of sheafs for monoids and semirings. In section 1, we construct sheafs of regular monoids with zero. In section 2, we establish a representation theorem for weakly regular semirings by sections in a presheaf. As an application of our results, we obtain a sheaf representation of weakly regular rings.

5.1 Sheafs of regular monoids with zero

Throughout this section, S will denote a monoid with a two-sided zero 0. The word ideal will always mean a two-sided ideal. Let I be an ideal of S; I is called *prime* if for any $a,b \in S$, $aSb \subseteq I$ implies that either $a \in I$ or $b \in I$. Equivalently, I is prime if and only if for any ideals A and B of S, $AB \subseteq I$ implies that $A \subseteq I$ or $B \subseteq I$. Let P(S) denote the set of proper prime ideals of S. For any ideal I of S, we define the sets: $\Theta_I = \left\{ J \in P(S) : I \nsubseteq J \right\}$, and $\tau(P(S)) = \left\{ \Theta_I : \text{I is an ideal of S} \right\}$.

First we prove some preliminary lemmas.

Lemma 5.1.1 Let S be a regular semigroup. Then for each pair I,J of ideals of S, $I \cap J = IJ$.

Proof Always $IJ \subseteq I \cap J$. To prove the converse, let $x \in I \cap J$. Since S is regular, there exists $y \in S$ such that $xyx = x$. Hence $I \cap J \subseteq IJ$, and therefore $IJ = I \cap J$.

The following lemma can be proved by using the usual arguments.

Lemma 5.1.2 Let S be a monoid (with a zero 0) and let A be a right S-system. Then the set $End_S(A)$ of all S-endomorphisms of A is a monoid with zero.

Lemma 5.1.3 Let S be a regular semigroup with zero. For each pair I,J of ideals of S with $J \subseteq I$, any S-homomorphism from J to I factors through J.

Proof Let $f: J \longrightarrow I$ be an S-homomorphism. Let $a \in J$. Since S is regular, there exists $b \in S$ such that $a = aba$. Hence $f(a) = f(aba) = f(ab)a \in J$.

Lemma 5.1.4 Let S be a monoid with zero. For each ideal

I of S, $End_S(I)$ is a monoid with zero which admits the structure of a right S-system.

Proof Let $f \in Hom_S(I,I)$ and $s \in S$. we define fs by $(fs)(x) = f(sx)$, for all $x \in I$. Note that $sx \in I$, since I is both a left and right ideal of S. Hence $Hom_S(I,I)$ is a right S-system.

We will now topologize the set of proper prime ideals of a regular monoid.

Theorem 5.1.5 Let S be a regular monoid. The set $\tau(P(S))$ constitutes a topology on the set P(S) and the assignment $I \longmapsto \Theta_I$ is a lattice isomorphism between the lattice $\mathcal{L}_S$ of ideals of S and the lattice of open subsets of P(S).

Proof First we show that the set $\tau(P(S))$ forms a topology on the set P(S). Since the zero ideal (O) of S is contained in every prime ideal, therefore $\Theta_{(O)} = \left\{J \in P(S):(O) \nsubseteq J\right\} = \phi$. Thus $\Theta_{(O)}$ is the empty subset of $\tau(P(S))$. Moreover $\Theta_S = \left\{J \in P(S):S \nsubseteq J\right\} = P(S)$. This is true since prime ideals in P(S) are proper. Thus P(S) (= Θ_S) is an element of the family $\tau(P(S))$. Now let $\Theta_{I_1}, \Theta_{I_2} \in$

$\tau(P(S))$ where $I_1, I_2 \in \mathscr{L}_S$. Then $\Theta_{I_1} \cap \Theta_{I_2} = \{J \in P(S): I_1 \nsubseteq J$ and $I_2 \nsubseteq J\} = \{J \in P(S): I_1 \cap I_2 \nsubseteq J\}$ This follows from Lemma 5.1.1. Now consider an arbitrary family $(I_k)_{k \in K}$ of ideals of S. Since $\cup\, \Theta_{I_k} = \cup \{J \in P(S): I_k \nsubseteq J\} = \{J \in P(S)$: there exists $k \in K$ such that $I_k \nsubseteq J\} = \{J \in P(S): \cup I_k \nsubseteq J\} = \Theta_{\cup I_k}$. Since $\cup I_k$ is an ideal of S, it follows that $\bigcup_{k \in K} \Theta_{I_k} \in \tau(P(S))$. Thus the set $\tau(P(S))$ of subsets Θ_I $(I \in \mathscr{L}_S)$ is a topology on the set $P(S)$. Define $\phi: \mathscr{L}_S \longrightarrow \tau(P(S))$ by $\phi(I) = \Theta_I$. It is easy to verify that ϕ preserves finite intersection and arbitrary union. Hence ϕ is a lattice homomorphism. Finally we show that ϕ is an isomorphism. For this purpose we show that $I_1 = I_2$ if and only if $\Theta_{I_1} = \Theta_{I_2}$ for I_1, I_2 in $\mathscr{L}_S$. Suppose $\Theta_{I_1} = \Theta_{I_2}$. If $I_1 \neq I_2$, then there exists $x \in I_1$ such that $x \notin I_2$. Then by Zorn's Lemma, there exists an ideal J of R which is maximal with respect to the property that J is proper, $I_2 \subseteq J$ and $x \notin J$. Then J is an irreducible ideal of S (in the sense that $J = P \cap L$ for ideals P and L, implies either $J = P$ or $J = L$). For if $J = P \cap L$ and both P and L properly contain J, then P and L both contain x. Hence $x \in P \cap L = J$, which is a contradiction. Since $x \notin J$, therefore $I_1 \nsubseteq J$. Hence $J \in \Theta_{I_1}$. But $\Theta_{I_1} = \Theta_{I_2}$. Hence

$J \in \Theta_{I_2}$. This means that $I_2 \notin J$. But this is a contradiction. Hence $I_1 = I_2$.

Definition 5.1.6 The set $P(S)$ is called the *prime spectrum* of S and the topology $\tau(P(S))$ will be called the *spectral topology* on $P(S)$. The corresponding space is called the *spectral space* of S.

We now formulate a definition of a sheaf of monoids with zero as follows:

Definition 5.1.7 Let X be a topological space and let $\tau(X)$ be the category of open subsets of X and inclusion maps. A *presheaf* $\mathcal{P}$ of monoids with zero on X is a contravariant functor from the category $\tau(X)$ to the category Mon of monoids with zero, that is, it consists of the data:

(a) For every open set $U \subseteq X$, there exists a monoid with zero $\mathcal{P}(U)$, and

(b) For every inclusion $V \subseteq U$ of open sets, there exists a semigroup homomorphism $\mathcal{P}_{\rho_{UV}} : \mathcal{P}(U) \longrightarrow \mathcal{P}(V)$ subject to the following conditions:

(1) $\mathcal{P}(\phi) = (0)$, where ϕ is the empty set of X;

(2) $\mathcal{P}_{\rho_{UU}} : \mathcal{P}(U) \longrightarrow \mathcal{P}(U)$ is the identity map, and

(3) if $W \subseteq V \subseteq U$ are three open sets, then $\mathcal{P}_{\rho_{UW}} = \mathcal{P}_{\rho_{VW}} \circ \mathcal{P}_{\rho_{UV}}$

If $\mathcal{P}$ is a presheaf on X, then $\mathcal{P}(U)$ is called a *section* of the presheaf $\mathcal{P}$ on the set U and the maps $\mathcal{P}_{\rho_{UV}}$ are called the *restriction maps* for which the notation $\alpha|_V$ is also used instead of $\mathcal{P}_{\rho_{UV}}(\alpha)$ where $\alpha \in \mathcal{P}(U)$.

The presheaf $\mathcal{P}$ is called a *sheaf* if the following additional conditions are satisfied:

(4) If U is an open set and $(V_\lambda)_{\lambda \in \Lambda}$ is an open covering of U and if $\alpha|_{V_\lambda} = \beta|_{V_\lambda}$ for $\alpha, \beta \in \mathcal{P}(U)$ and for all V_λ, then $\alpha = \beta$;

(5) If U is an open set and $(V_\lambda)_{\lambda \in \Lambda}$ is an open covering of U and if there are elements $\alpha_\lambda \in \mathcal{P}(V_\lambda)$ for each $\lambda \in \Lambda$ such that for each pair $\lambda, \mu \in \Lambda$, $\alpha_\lambda|_{V_\lambda \cap V_\mu} = \alpha_\mu|_{V_\lambda \cap V_\mu}$, then there exists $\alpha \in \mathcal{P}(U)$ such that $\alpha|_{V_\lambda} = \alpha_\lambda$ for each $\lambda \in \Lambda$.

If a presheaf satisfies condition (4) only, it is called *separated* [cf. G.Berdan, I.R.Shafarevich].

We now describe a sheaf of monoids with zero on the prime spectrum of a regular monoid with zero.

Theorem 5.1.8 Let S be a regular monoid with zero. For every ideal I of S, the assignment $\Theta_I \longrightarrow End_S(I) = \mathcal{F}_S(I)$

defines a sheaf $\mathscr{F}_S$ of monoids with zero on the prime spectrum of S.

Proof First we prepare the data for the existence of a presheaf. By Lemma 5.1.2, $\mathscr{F}_S(I) = End_S(I)$ is a monoid with zero for every ideal I of S. We now define a restriction map: $\mathscr{F}_{\rho_{IJ}} : End_S(I) \longrightarrow End_S(J)$, whenever $\Theta_J \subseteq \Theta_I$, that is, when $J \subseteq I$ for each pair of ideals I,J of S. For $\alpha \in End_S(I)$, we define $\mathscr{F}_{\rho_{IJ}}(\alpha) = \alpha|_J$. Note that $\alpha|_J \in End_S(J)$ by Lemma 5.1.3. Clearly $\mathscr{F}_{\rho_{IJ}}$ is a semigroup homomorphism. Thus $\mathscr{F}_S$ satisfies the conditions of a presheaf. Hence we have described the desired presheaf $\mathscr{F}_S$. We now show that $\mathscr{F}_S$ is separated. Let $(I_k)_{k\in K}$ be a family of ideals of S and let $I = \cup_{k\in K} I_k$. Suppose $f,g \in \mathscr{F}_S(I)$ such that $f|_{I_k} = g|_{I_k}$ for all $k \in K$. Then for each $x \in I$, we have $x \in I_k$ for some k. Thus $f(x) = f|_{I_k}(x) = g|_{I_k}(x) = g(x)$. Hence $f = g$ and $\mathscr{F}_S$ is separated. Finally we check condition (5). Let $(I_k)_{k\in K}$ be a family of ideals of S and let $I = \cup_{k\in K} I_k$, and let $(f_k)_{k\in K}$ be a family of maps with $f_k \in End_S(I_k)$ such that $f_k|_{I_k\cap I_l} = f_l|_{I_k\cap I_l}$ for $k,l \in K$. For S-endomorphisms $f_k: I_k \longrightarrow I_k$ and $f_l: I_l \longrightarrow I_l$ which coincide on $I_k \cap I_l$ we define a map $f: I_k \cup I_l \longrightarrow I_k \cup I_l$ by

$$f(x) = \begin{cases} f_k(x) & \text{if } x \in I_k \\ f_l(x) & \text{if } x \in I_l \end{cases}$$

Since f_k and f_l coincide on $I_k \cap I_l$, f is an S-homomorphism extension of f_k and f_l. Now if I_m is any ideal in the family, then $I_m \cap (I_k \cup I_l) = (I_m \cap I_k) \cup (I_m \cap I_l)$ Thus if $x \in I_m \cap (I_k \cup I_l)$, then $x \in (I_m \cap I_k)$ or $x \in (I_m \cap I_l)$. Hence

$$f(x) = \begin{cases} f_k(x) = f_m(x) & \text{or} \\ f_l(x) = f_m(x) & \end{cases}$$

Hence $f(x) = f_m(x)$ for $x \in I_m \cap (I_k \cup I_l)$. This implies that the family $(I_k)_{k \in K}$ is stable with respect to finite unions. Now if $x \in \bigcup_{k \in K} I_k$, then $x \in I_k$ for some k in K. So we may define with no ambiguity, the map $f\colon \bigcup_{k \in K} I_k \longrightarrow \bigcup_{k \in K} I_k$ by $f(x) = f_k(x)$, since two different f_l agree on x when $f_l(x)$ make sense. The map f is clearly an S-homomorphism extending each f_k for all $k \in K$. This proves that $\mathscr{F}_S$ is a sheaf of monoids with zero.

Corollary The monoid $\mathscr{F}_S(S)$ (called the monoid of the global section of $\mathscr{F}_S$) is S-isomorphic to S, as an S-system.

Proof Note that $End_S(S)$ is an S-system by Lemma 5.1.4. We show that $End_S(S) \cong S$ as S-systems. For this purpose, we define $h\colon End_S(S) \longrightarrow S$ by $h(\alpha) = \alpha(1)$ for $\alpha \in End_S(S)$

It is easy to verify that h is an S-homomorphism. Suppose $h(\alpha) = h(\beta)$. Then $\alpha(1) = \beta(1)$. Hence for all $s \in S$, $\alpha(s) = \alpha(1s) = \alpha(1)s = \beta(1)s = \beta(1s) = \beta(s)$. Therefore $\alpha = \beta$. Hence h is injective. To show that h is surjective, let $t \in S$ and let $\alpha_t : S \longrightarrow S$ be defined by $\alpha_t(s) = ts$ for all $s \in S$. Evidently, $\alpha_t \in End_S(S)$ and $h(\alpha_t) = \alpha_t(1) = t1 = t$. Therefore h is surjective.

Finally it is remarked that the sheaf representation of regular monoids given above (Theorem 5.1.8) can actually be proved, with some minor modifications, for more general classes of monoids including weakly regular and semisimple monoids. Recall that a semigroup S (not necessarily with identity or zero element) is called *semisimple* if all ideals of S are idempotent (an ideal I is called idempotent if $I = I^2$). These semigroups admit many interesting characterizations (see [14, vol. I, p. 76], see also [2] for a recent characterizations of these semigroups in terms of their prime ideals). Semisimple semigroups contain regular and weakly regular semigroups as proper subclasses.

5.2 Representations of weakly regular semirings by sections in a presheaf

Throughout this section, R will denote a semiring with a zero 0 and an identity 1 and all R-semimodules M are right unital (that is, $m \cdot 1 = m$, for all $m \in M$). Let R and L be semirings. We shall say that L is an *R-semiring* if L has the structure of an R-semimodule so that $(xy)r = x(yr)$, for $x,y \in L$ and $r \in R$. For two such R-semirings L_1 and L_2, a semiring homomorphism $f\colon L_1 \longrightarrow L_2$ is a *homomorphism of R-semirings* if f is an R-homomorphism. If R is a semiring and L is an R-semiring, then an R-semimodule M is called an *L-R-semimodule* if M is an L-semimodule such that $(mx)r = m(xr)$, for all $m \in M$, $x \in L$ and $r \in R$. We begin with some preliminary lemmas.

Lemma 5.2.1 Let R be a semiring and M a right R-semimodule. Then the following hold:

(1) For each ideal I of R, $End_R(I)$ is an R-semiring, and $Hom_R(I,M)$ is an $End_R(I)$-R semimodule.

(2) If R is commutative, and I is an ideal of R such that for each $x \in I$, there exists $y \in I$ with $x = xy$, then $End_R(I)$ is a commutative semiring.

Proof The proof is similar to that of the corresponding result in rings and hence omitted.

Lemma 5.2.2 Let I and J be ideals of a right weakly regular semiring R with $J \subseteq I$. Then any R-homomorphism from J to I factors through J.

Proof Let $f: J \longrightarrow I$ be an R-homomorphism. Since each ideal of a right weakly regular semiring is a right weakly regular semiring (see [41], Proposition 5, p.318), therefore J, considered as a semiring, is right weakly regular. If $a \in J$, then we can write $a = ax_1ay_1 + ax_2ay_2 + \ldots + ax_nay_n$, for $x_1, \ldots, x_n$ and $y_1, \ldots, y_n \in J$. Therefore, $f(a) = f(ax_1ay_1) + \ldots + f(ax_nay_n) = f(ax_1a)y_1 + \ldots + f(ax_na)y_n \in J$.

We now define the concept of a sheaf for semirings in the following way:

Definition 5.2.3 Let X be a topological space and $\tau(X)$ be the category of open subsets of X and inclusion maps. A *presheaf* $\mathcal{P}$ of R-semimodules on X is a contravariant functor from the category $\tau(X)$ to the category $\mathcal{M}_R$ of R-semimodules, that is, it consists of the following data:

(a) For every open set $U \subseteq X$, there exists an R-semimodule $\mathcal{P}(U)$, and

(b) For every inclusion $V \subseteq U$ of open sets, there exists an R-homomorphism $\mathcal{P}_{\rho_{UV}} : \mathcal{P}(U) \longrightarrow : \mathcal{P}(V)$ satisfying:

(1) $\mathcal{P}(\phi) = (0)$, where ϕ is the empty set of X;

(2) $\mathcal{P}_{\rho_{UU}} : \mathcal{P}(U) \longrightarrow \mathcal{P}(U)$ is the identity map, and

(3) if $W \subseteq V \subseteq U$ are three open sets, then $\mathcal{P}_{\rho_{UW}} = \mathcal{P}_{\rho_{VW}} \circ \mathcal{P}_{\rho_{UV}}$

If $\mathcal{P}$ is a presheaf on X, then $\mathcal{P}(U)$ is called a *section* of the presheaf $\mathcal{P}$ on the open set U and the maps $\mathcal{P}_{\rho_{UV}}$ are called the *restriction maps*, and often the notation $\alpha|_V$ is used instead of $\mathcal{P}_{\rho_{UV}}(\alpha)$ if $\alpha \in \mathcal{P}(U)$.

A presheaf $\mathcal{F}$ on a topological space X is called a *sheaf* if the following additional conditions are satisfied:

(4) If U is an open set and $(V_\lambda)_{\lambda\in\Lambda}$ is an open covering of U, and if $\alpha|_{V_\lambda} = \beta|_{V_\lambda}$ for $\alpha, \beta \in \mathcal{F}(U)$ and for all V_λ, then $\alpha = \beta$;

(5) If U is an open set and $(V_\lambda)_{\lambda\in\Lambda}$ is an open covering of U and if there are elements $\alpha_\lambda \in \mathcal{F}(V_\lambda)$ for each $\lambda \in \Lambda$, with the properties that for each $\lambda, \mu \in \Lambda$, $\alpha_\lambda|_{V_\lambda \cap V_\mu} = \alpha_\mu|_{V_\lambda \cap V_\mu}$, then there exists $\alpha \in \mathcal{F}(U)$ such that $\alpha|_{V_\lambda} = \alpha_\lambda$ for each $\lambda \in \Lambda$.

If a presheaf satisfies condition (4) only it is called *separated* [cf. G.Berdan, I.R.Shafarevich].

Definition 5.2.4 Let $\mathcal{P}$ be a presheaf (sheaf) of R-semimodules on a topological space X. If each $\mathcal{P}(U)$ is an R-semiring and $\rho^{\mathcal{P}}_{UV}$ are homomorphisms of R-semirings, then $\mathcal{P}$ is called a *presheaf* (*sheaf*) of R-semirings.

As defined in Chapter 4, we shall use the notations $\mathcal{L}_R$ and P(R) for the lattice of ideals of R and the set of proper prime ideals of R, respectively. Moreover, for any ideal I of R, $\Theta_I = \left\{ J \in P(R): I \nsubseteq J \right\}$, and $\tau(\mathcal{P}_R) = \left\{ \Theta_I: I \in \mathcal{L}_R \right\}$. As shown in Chapter 4 (Theorem 4.2.1), the set $\tau(P(R))$ constitutes a topology on the prime spectrum P(R) of R. On this prime spectrum P(R) we now describe a presheaf $\mathcal{F}_R$ of R-semirings.

Theorem 5.2.5 Let R be a right weakly regular semiring. For every ideal I of R, the assignment $\Theta_I \longrightarrow \mathrm{End}_R(I) = \mathcal{F}_R(I)$ defines a separated presheaf $\mathcal{F}_R$ of R-semirings on $\tau(P(R))$. The semiring of the global section of this presheaf is isomorphic to R. If R is commutative, then $\mathcal{F}_R$ is a presheaf of commutative semirings.

Proof First we prepare the data for the existence of a presheaf. By Lemma 5.2.1, $\mathcal{F}_R(I) = \mathrm{End}_R(I)$ is an R-semiring for every ideal I of R. We need to define a restriction map $\mathscr{F}_{\rho_{IJ}}$: $\text{End}_R(I) \longrightarrow \text{End}_R(J)$, whenever $\Theta_J \subseteq \Theta_I$, that is, when $J \subseteq I$. By Lemma 5.2.2, this is just the usual restriction of an R-endomorphism $f\colon I \longrightarrow I$ to the R-subsemimodule J, that is, $\mathscr{F}_{\rho_{IJ}}(f) = f|_J$. By the definition, $\mathscr{F}_{\rho_{IJ}}$ is a homomorphism of R-semirings. Thus $\mathscr{F}_R$ satisfies the conditions of a presheaf. Thus we have described the presheaf $\mathscr{F}_R$. In order to show that $\mathscr{F}_R$ is separated, we verify condition (4) in Definition 5.2.3. Let $I = \sum_{\lambda\in\Lambda} I_\lambda \in \mathscr{L}_R$, and suppose $f,g \in \mathscr{F}_R(I)$ such that $f|_{I_\lambda} = g|_{I_\lambda}$ for all $\lambda \in \Lambda$. For each $x \in I$, we have $x = x_1+\dots+x_n$, where $x_\lambda \in I_\lambda$. Then $f(x) = f(x_1) +\dots+ f(x_n) = g(x_1) +\dots+ g(x_n) = g(x_1+\dots+x_n) = g(x)$. Hence $f = g$, and so $\mathscr{F}_R$ is separated. Now we show that $\mathscr{F}_R(R) = \text{End}_R(R) \cong R$. Define $h\colon \text{End}_R(R) \longrightarrow R$ by $h(\alpha) = \alpha(1)$, for $\alpha \in \text{End}_R(R)$. Clearly h is a homomorphism of R-semirings. Suppose $h(\alpha) = h(\beta)$. Then $\alpha(1) = \beta(1)$. Hence, for all $r \in R$, $\alpha(r) = \alpha(1r) = \alpha(1)r = \beta(1)r = \beta(1r) = \beta(r)$. Hence $\alpha = \beta$; showing that h is injective. To show that h is surjective, let $t \in R$, and define $\alpha_t\colon R \longrightarrow R$ by $\alpha_t(r) = tr$ for all $r \in R$. Clearly, α_t is an R-homomorphism. Hence $\alpha_t \in \text{End}_R(R)$, and $h(\alpha_t) = t1 = t$. Thus h is surjective, and hence bijective. Finally, if R is commutative, then $\text{End}_R(I)$ is a commutative R-semiring by Lemma 5.2.1. This follows, since

R is right weakly regular, therefore for each $x \in I$, we have $x \in (xR) = (xR)^2$. Hence $x = xy$ for each $y \in I$. This completes the proof of the theorem.

Let us now assume that the weakly regular semiring R in the above theorem is actually a ring. Then the presheaf $\mathscr{F}_R$ defined in the above theorem is in fact, a sheaf. To show this we check condition (5) in Definition 5.2.3. Let $I = \sum_{\lambda\in\Lambda} I_\lambda \in \mathscr{L}_R$ and suppose $f_\lambda \in End_R(I)$ such that $f_\lambda|_{I_\mu} = f_\mu|_{I_\lambda}$. Consider $f_\lambda: I_\lambda \longrightarrow I_\lambda$ and $f_\mu: I_\mu \longrightarrow I_\mu$ which coincide on $I_\lambda \cap I_\mu$. Let $x \in I_\lambda + I_\mu$. Then $x = x_\lambda + x_\mu$; $x_\lambda \in I_\lambda$ and $x_\mu \in I_\mu$. Define $f: I_\lambda + I_\mu \longrightarrow I_\lambda + I_\mu$ by $f(x) = f_\lambda(x_\lambda) + f_\mu(x_\mu)$. We show that f is well-defined. Suppose $x = x_\lambda + x_\mu = x'_\lambda + x'_\mu$. Then $x_\lambda - x'_\lambda = x'_\mu - x_\mu \in I_\lambda \cap I_\mu$. Hence $f_\lambda(x_\lambda - x'_\lambda) = f_\mu(x'_\mu - x_\mu)$. Hence $f_\lambda(x_\lambda) + f_\mu(x_\mu) = f_\lambda(x'_\lambda) + f_\mu(x'_\mu)$. Thus f is a correctly defined extension of f_λ and f_μ. Now if I_ν is any ideal of R, then $I_\nu \cap (I_\lambda + I_\mu) = (I_\nu \cap I_\lambda) + (I_\nu \cap I_\mu)$. Note that this follows since the lattice of ideals of a right weakly regular semiring is distributive (see Chapter 4, Proposition 4.1.7). Hence if $x \in I_\nu \cap (I_\lambda + I_\mu)$, then we can write $x = x_\lambda + x_\mu$, where $x_\lambda \in I_\nu \cap I_\lambda$ and $x_\mu \in I_\nu \cap I_\mu$. Hence $f(x) = f_\lambda(x_\lambda) + f_\mu(x_\mu) = f_\nu(x_\lambda) + f_\nu(x_\mu) = f_\nu(x_\lambda + x_\mu) = f_\nu(x)$. This proves that the family $(I_\lambda)_{\lambda\in\Lambda}$

is stable under finite sums. Let $x \in \sum_{\lambda\in\Lambda} I_\lambda$. Then we can write $x = x_1 + \ldots + x_n$, where $x_\lambda \in I_\lambda$. Thus x belongs to a finite sum of I_λ's and hence by the first part of the proof, we can suppose that $x \in I_\mu$ for some μ. Thus we define $f(x) = f_\mu(x)$ with no ambiguity in the definition of $f: \sum_{\lambda\in\lambda} I_\lambda \longrightarrow \sum_{\lambda\in\Lambda} I_\lambda$ because two different f_μ agree on x as soon as $f_\mu(x)$ make sense. Finally, f is evidently an R-homomorphism extending each f_λ. Hence $\mathcal{F}_R$ is a sheaf. Thus we have proved:

Theorem 5.2.6 Let R be a right weakly regular ring. For every ideal I of R, the assignment $\Theta_I \longmapsto \mathrm{End}_R(I) = \mathcal{F}_R(I)$ defines a sheaf $\mathcal{F}_R$ of R-semirings on P(R). The (semi-)ring of the global sections of this sheaf is isomorphic to R. If R is commutative, then $\mathcal{F}_R$ is a sheaf of commutative (semi-)rings.

Finally, we prove:

Theorem 5.2.7 Let R be a right weakly regular semiring all of whose ideals are linearly ordered. For every ideal I of R, the assignment $\Theta_I \longrightarrow \mathrm{End}_R(I) = \mathcal{F}_R(I)$ defines a sheaf $\mathcal{F}_R$ of R-semirings on P(R). The semiring of the global sections of this sheaf is isomorphic to R. If R is commutative, then $\mathscr{F}_R$ is a sheaf of commutative semirings.

Proof We need only to check condition (5) in Definition 5.2.3. Let $I \sum_{\lambda\in\Lambda} I_\lambda \in \mathscr{L}_R$. Suppose $f_\lambda \in End_R(I)$ such that $f_\lambda|_{I_\mu} = f_\mu|_{I_\mu}$. Consider $f_\lambda: I_\lambda \longrightarrow I_\lambda$ and $f_\mu: I_\mu \longrightarrow I_\mu$ which coincide on $I_\lambda \cap I_\mu$. Since ideals of R are linearly ordered, therefore $I_\lambda \subseteq I_\mu$ or $I_\mu \subseteq I_\lambda$. Hence $I_\lambda + I_\mu = I_\mu$ or I_λ (respectively). We now define f: $I_\lambda + I_\mu$ by

$$f(x) = \begin{cases} f_\mu(x) & \text{if } I_\lambda + I_\mu = I_\mu. \\ f_\lambda(x) & \text{if } I_\lambda + I_\mu = I_\lambda. \end{cases}$$

Obviously, f is an extension of f_λ and f_μ. Hence the family $(I_\lambda)_{\lambda\in\Lambda}$ is stable under finite sums. Therefore, $f: \sum_{\lambda\in\Lambda} I_\lambda \longrightarrow \sum_{\lambda\in\Lambda} I_\lambda$ can be correctly defined in such a way that f extends each f_λ. Hence $\mathscr{F}_R$ is a sheaf.

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