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DEDICATED TO

MY MOTHER AND GRAND MOTHER(LATES),

MY FATHER, BROTHERS, SISTERS

AND NEPHEW USAMA

CONTENTS

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ABSRTACT

The diffraction of a line-source by an acoustically penetrable half-plane is studied. The problem solved is an approximate model for a noise barrier which is not perfectly rigid and therefore transmits sound. Expressions for the total far field for the leading edge (no wake present) and the trailing edge (wake present) situation are given. It is found that the field produced by Kutta Joukowski condition will be substantially large than the field in the absence when the source is near the edge.

PREFACE

In 1931, Wiener And Hopf [1] developed a useful technique to solve an integral equation of a spherical type. This technique proved to be a powerful tool for solving the problems involving diffraction by semi-infinite/finite planes. The technique is based on the application of integral transforms and the theory of analytic continuation of the complex valued functions [1].

In this procedure, the associated mathematical boundary value problem is transformed to the Wiener-Hopf functional equation. An important step in the solution of the functional equation is to decompose the Kernel (in general this is a known function of a complex variable with a number of poles characterizing the underlying physical processes) into a product or sum of two functions, one analytic in the upper/right of the complex plane whereas the other is analytic in the lower/left half-plane. The procedure of the decomposition is relatively simple for a scalar Kernel. However in the case of system of Wiener-Hopf equations, one has to work with a matrix equation involving a matrix kernel and the success has only been achieved for a limited class of matrices. Thus a considerable attempt has been made in recent years not only to extend this class but also to develop new constructive methods for calculation matrix decompositions, especially with reference to the diffraction problems. An appreciable account of the problems based on the Wiener-Hopf technique in the wave propagation theory can be found in the literature, for example Copson [2], Noble [3], Jones [9,10,11] and Brekhovskikh [7].

Noise is deafening! Stop it now. In recent years, noise has become a serious issue of environmental concern. Noise abatement has therefore attracted the attention of many scientists. Traffic noise from motorways, railways, airports and other outdoor noises from heavy construction machinery or stationary installation, such as large transformers and plants, can be shielded by a barrier. Noise in an open plan office can also be reduced by means of barrier partitions. In most of the calculations with noise barriers the fields in the shadow region of the barrier Kurze [8] is assumed to be solely due to diffraction at the edge. This assumption supposes that the barrier is perfectly rigid and therefore does not transmit sound. However the most practical barriers which are made of wood and plastic will consequently transmit some of the noise through the barrier. The object of present work is to make some allowance for the transmitted field, and to investigate its effect, together with the edge diffracted field. It has also applications in electromagnetism when considering diffraction by a dielectric half-plane.

The results of Yeh [9] for the problem of diffraction by penetrable parabolic cylinder are of some what complicated in the form of infinite series of parabolic cylinder functions. Limiting case of diffraction by penetrable parabolic cylinder approximates to a penetrable half-plane. Difficulty arises in relating the dimensions of a practical barrier with a thin parabolic cylinder. Shmoys [10] has used different approximate approach using parabolic cylinder coordinates and has given results which are expressed in the form of Fresnel integrals, and simpler than Yeh [9]. His approach is not rigorous and he does not elaborate on the penetrable half-plane solution but merely quoted results obtained heuristically.

Pistol'kors et. al [11] and Khrebet [12] have both used same approximate boundary condition. The approximate boundary condition used by Pistol'kors et. al [11] and Khrebet [12] is only good in describing a perfectly half-plane, no loss within the material which comprises the half-plane A. D. Rawlins [13] uses an alternate boundary condition which gives a smooth transition from a perfectly penetrable half-plane to a nonpenetrable half-plane, with an absorption type of boundary condition. By this boundary condition which is slightly more complicated than that used in [16,17] and its symmetry amens the boundary value problem to the Wiener-Hopf technique.

In Chapter One, some theoretical as well as mathematical preliminaries are discussed. These help in calculations which will come in Chapter Two and Three.

Chapter Two gives the diffraction of acoustically penetrable or electromagnetically dielectric half-plane. This diffraction problem provides a method to solve the cylindrical problem of diffraction in the trailing edge situation.

Chapter Three is devoted to the diffraction of a cylindrical wave scattering from a penetrable half-plane in the trailing edge (wake present) situation.

CHAPTER ONE

ACOUSTICS

INTRODUCTION

In this chapter some definitions including also the different phenomena in wave motion are discussed. Some definitions of generalized functions (Dirac delta function, Hankel function etc) are also part of it. Inspite of these definitions the mathematical techniques used for solving the problems presented in Chapter Two and Three are given. These methods are consist of transformation techniques, asymptotic methods etc.

A BASIC ACOUSTICS

1.1 ACOUSTICS

Acoustics may be defined as the study of the generation, transmission and reception of energy in the form of vibration waves in matter. As the atoms and molecules of a fluid or solid are displaced from their normal configuration an inertial elastic restoring force arises. Examples include the tensile force produced when a fluid is compressed and the transverse restoring force produced when a point on a stretched wire is displaced in a direction normal to its length. It is this elastic restoring force, coupled with the inertia of the system, that enables matter to participate in oscillatory vibrations and there by generating and transmitting acoustic wave.

The most familiar acoustic phenomena is that associated with the sensation of sound. For an average young person, a vibrational disturbance is interpreted as sound if its frequency lies in the range of about 20 to 20,000 Hertz. However in a broader sense, acoustic also include the ultrasonic frequencies above, 20,000 Hertz, and infrasonic frequencies below 20 Hertz. The nature of vibration associated with acoustic are for example, the simple sinu soidal vibrations produced by a tuning fork and non-periodic motions associated with an explosion.

1. The most important type of wave motion studied in acoustics is wave motion in air. (e.g. sound waves)

2. Sound waves (3D-waves) differ from waves on a string or membrane (which are transverse in nature) because these are longitudinal in nature.

3. The restoring force responsible for keeping the wave going is the opposition offered by fluid.

4. Since in the sound waves the molecules of air, move in the

direction of propagation of wave so that there are no alternate crests and troughs (as with waves on surface of water), but alternate compressions and rarefactions.

1.2 PLANE WAVE

The type of waves having the same direction of propagation everywhere in space whose 'crests' are in planes perpendicular to the direction of propagation are called plane waves. Waves travelling along the inside of tubes of uniform cross-section will usually be plane waves. Waves that have travelled unimpeded along distance from their source will be very nearly, plane waves.

1.3 DIFFERENT PHENOMENA INVOLVED IN ACOUSTIC WAVE MOTION

i) Transmission Phenomena

When an acoustic wave travelling in one medium encounters the boundary of the second medium, reflected and transmitted waves are generated [17]. For simplification, it is assumed that both the incident wave and the boundary between the two media are planar and all media are fluids. The ratios of the pressure amplitudes and intensities of the reflected and transmitted waves to those of incident wave depend both on the characteristic impedance and speed of sound in two media and on the angle the incident wave makes with normal to the interface. If the complex pressure wave p_r, and that of transmitted wave P_i, that of reflected wave P_r, and that of transmitted wave P_t, then we define the transmission and reflection co-efficients as

$$T = P_t/P_i \ ,$$

$$R = P_r/P_i \ .$$

Since the intensity of plane harmonic wave is $P^2/2\rho_o c$; where ρ_o is the density and c is the speed of sound, the intensity transmission (I_t) and reflection (I_r) co-efficients are real and are defined by

$$T_I = I_t/I_i \ ,$$

$$R_I = I_r/I_i \ .$$

ii) <u>Absorption and Attenuation of Sound Waves in Fluids</u>

So far we have not considered the dissipation of acoustic energy. In many situations, dissipation takes place so slowly that it can be ignored for small disturbances. The sources of these dissipation may be divided into two general categories: those due to losses in the medium and those associated with losses at the boundaries of the medium. The first is important when the volume of the fluid is large e.g. the transmission of sound in the earth's atmosphere and oceans. The second is important in the opposite sense porous- material and small rooms. Losses in the medium may be classified into three basic types:

i) Viscous losses result when there is relative motion between adjacent portions of the medium, such as during the compressions and expansions.

ii) Heat conduction losses result from the conduction of thermal energy (heat) between higher temperature condensations and lower temperature rarefactions.

iii) Losses associated with molecular exchange of energy, that can lead to the absorption, include the conversion of kinetic energy of molecules into (a) stored potential energy (b) internal rotational and vibrational energies (c) energies of association and dissociation.

Each of these absorption processes is characterized by a relaxation time, which measures amount of time for the particular process to be nearly completed.

So far we have assumed that the fluid is a continuum with observable properties such as pressure, density, compressibility, specific heat and temperature without being concerned with its molecular structure. Under the same assumptions, by use of viscosity, Stokes developed the first theory of sound absorption. Subsequently, Kirchoff utilized the property of thermal conductivity to generate what is called classical sound absorption in fluids. In more recent times, as more accurate sound absorption measurements were made, it became evident that explanations of sound absorption from this view point were inadequate in some fluids. Consequently it became necessary to adopt a microscopic view and consider such phenomena as the binding energies within and between molecules to develop an additional absorption mechanisms. These Mechanisms are commonly referred to as molecular or relaxation types of sound absorption.

Attenuation is the loss of acoustic energy from a sound beam. Attenuation can be divided into two parts:

(a) Absorption mechanism that convert acoustic energy into thermal energy

(b) Other mechanisms that deflect or scatter acoustic energy out of the beam.

When a fluid contains inhomogeneities such as suspended particles, microcells or regions of turbulence, acoustic energy is lost from a sound beam faster than in a homogeneous medium. Fog and smoke particles produce a decided effect on sound propagation through atmosphere. Extremely high attenuations are also produced in water containing suspended gas bubbles. For instance viscous forces and heat conduction losses associated with the compression and expansion of small gas bubbles by a passing sound result in a loss of energy by the sound wave. A further effect of such inhomogeneities, of particular importance in the transmission of directed sonar beams of sound energy is scattering i.e. the removal of small amount of energy from the directed beam by each bubble and its subsequent radiations in all directions.

1.4. WAKE

If we consider the case of a non-viscous flow, the boundary condition for the flow about a body is simply the the normal velocity component of the surface vanishes. The proper boundary condition in a viscous fluid is that the fluid adheres to the bounding surface. Thus both the normal and tangential velocity relative to the body must vanish. At a small distance from the surface, the velocity reaches a value of the order of the free-stream value and the influence of viscosity is restricted to a small boundary layer with strong vorticity near the surface. However for thin wings the vortex layer is also thin. The vortices are also carried along with the flow and form a thin vortex wake behind the wing. The strength of this wake can be determined approximately by what is called the Kutta-Joukowski condition. This condition consists in requiring that the fluid velocity does not become infinite at the sharp trailing edge of the wing, in simple, in this connection we may recall that when an ideal fluid flows round an angle, the fluid velocity in general becomes infinite, according to a power law, at the vertex of the angle. We can say that the condition stated implies that the jets coming from the two sides of the wing must meet smoothly without turning through an angle. When this condition is fulfilled, the solution of the problem of potential flow gives a pattern very like to the true one, where the velocity is everywhere finite and separation occurs only at the trailing edge. The solution will now become unique and in particular the circulation Γ needed to calculate the lift force has a definite value. Stagnation point at the front of the body (which may stretch into a stagnation region if the body is very blunt) and there is the flow region behind the body called a wake (shown in Fig.1). When thickness of the wake increases pressure decreases and if thickness decreases the pressure increases. Discontinuous distribution of pressure is called drag.

1.5 The Radiation of Sound

Sound waves are generated by the vibration of any solid body in contact with the fluid medium, or by vibratory forces acting directly on the fluid, or by the violent motion of the fluid itself, as from a jet or by oscillatory thermal effects, as would be produced by a modulated laser beam. In each case, the energy is transferred from the source to the fluid. From the point of view of the acoustic, a SOURCE is a region of space, in contact with the fluid medium, where new acoustic energy is being generated, to be radiated outward as sound waves. Discussing the generated sound waves as they move outward from the source, we will assume that the fluid medium outside the source region is initially uniform and at rest.

1.6 The Scattering of Sound

When a sound wave encounters an obstacle, some of the wave is deflected from its original course. It is useful to define the difference between the actual wave and the undisturbed wave, which would be present if the obstacle were not there, as the scattered wave. When a plane wave, for instance, strikes a body in its path, in addition to the undisturbed plane wave, there is a scattered wave, spreading out in all directions from the obstacle. If the obstacle is very large compared with wave length (as it usually is for light waves and very seldom is for sound), half of the scattered wave spreads out more or less uniformly in all directions from the scatterer, and the other half is concentrated behind the obstacle in such a manner as to interfere destructively with the unchanged plane wave behind the obstacle, creating a sharp-edged shadow there. This is the case of geometrical optics; in this case, half of the scattered wave spreading out uniformly is called the reflected wave, and the half responsible for shadow is called the interfering wave. If the obstacle is very small as compared with the wave length (as it often is for sound directions and there exists no sharp-edged shadow. In the intermediate case, where the obstacle is about the same size as the wavelength, a variety of curious interference phenomena can occur.

A sound wave is scattered not only by a solid object, but also by a region in which the acoustic properties differ from their values in the rest of the medium. Turbulent air scatters, as well as generates sound. Fog particles in air, and bubbles in water scatter sound.

When an object or region scatters sound, some of the energy carried by the incident wave is dispersed. The energy lost to the incident wave may be absorbed by the scatterer or it may simply be deflected from its original course. In any case, the incident plane wave is reduced in intensity because of the loss.

1.7 Diffraction of Sound

When the scattering object is large compared with the wave-length of the scattered sound, we usually say the sound is reflected and diffracted, rather than scattered. The effects are really the same but the relative magnitudes differ enough so that there seems to be a qualitative difference. Behind the object, there is a shadow, where the pressure amplitude is vanishing small, in front or to the side, in the "illuminated" region, there is a combination of the incident wave and the wave reflected from the surface of the scattering object. At the edge of the shadow, the wave amplitude does not drop discontinuously from its value in the illuminated region to zero, the amplitude oscillates about its illuminated value, reaching its maximum just before the edge of the shadow and then dropping monotonically, approaching zero well inside the shadow. These fluctuations of amplitude, near the shadow edge, are called diffraction bands. Their angular spacing depends on the ratio between the wavelength of incident sound and the distance from the observation point to the line on the scattering object separating "light" from "shadow".

(B) MATHEMATICAL PRELIMINARIES

1.1 FOURIOR TRANSFORMS

If f(u) is continuous function, for real u and if

$$F(\alpha) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(u) e^{i\alpha u} du,$$

exists, then the function $F(\alpha)$ is called the Fourior transform of f(u) and is sometimes written as

$$F(\alpha) = f(\alpha) f(u)\}$$

where α may be a real or a complex variable. And $\alpha = \sigma + i\tau$, called the Fourior parameter.

Similarly, if α is real variable and $F(\alpha)$ is a continuous function of α and if the integral

$$f(\alpha) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} F(\alpha) e^{i\alpha u} d\alpha,$$

exists, then the function f(u) is called the inverse Fourior transform of $F(\alpha)$. It is written as

$$f(u) = f^{-1}\{F(\alpha)\}.$$

Now let f'(u) represents the derivative of f(u), also f'(u) is continuous function of the real variable u, then if

$$F'(\alpha) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f'(\alpha) e^{i\alpha u} du,$$

exists then $F'(\alpha)$ is called the Fourior transform of the derivative of f(u). Similarly for the inverse Fourior transform. More details can be seen in [3].

1.2 The Wiener-Hopf Technique

The Fourier transform technique can be used in solving the integral equations or boundary value problems if the domain of the problem is from $-\infty$ to ∞. If this is from 0 to ∞ or in case of the mixed boundary value problem we use the Wiener-Hopf technique. N.Wiener and E.Hopf discovered in 1931 this useful technique to solve an integral equation of special type. Problems involving diffraction by semi infinite planes can be formulated in terms of integral equations which can be solved by Wiener-Hopf technique. The general method of solving functional equations which became know as the Wiener-Hopf method or factorization method has been successfully employed in the solution of many problems of diffraction, in the theory of elasticity, the boundary value problems dealing with the heat transfer and many other problems of mathematical physics. This technique as discussed by Noble [4] will be described and used to solve different boundary value problems.

Procedure of the Technique

To illustrate the method we consider the following type of functional equation.

$$A(k)\ \psi_+(k) + B(k)\ \psi_-(k) + C(k) = 0, \tag{1.1}$$

Here A(k), B(k) and C(k) are given functions of complex variable k, analytic in the strip $\tau_- < \mathrm{Im}(k) < \tau < \tau_+$. Also A(k) and B(k) are non zero in the strip. Our first step is to re-cast the boundary value problems in the form (1.1). The fundamental step in the Wiener-Hopf procedure is based on the possibility of factorizing the expression A(k)/B(k) i.e. possibility of factorizing the expression:

$$\frac{A(k)}{B(k)} = \frac{L_+(k)}{L_-(k)}\ , \tag{1.2}$$

where $L_+(k)$ is regular and free of zeros in $\tau > \tau'_-$ and $L_-(k)$ is regular and free of zeros in $\tau < \tau'_+$. The strip $\tau_- < \tau < \tau_+$ and

$\tau'_{-}<\tau<\tau'_{+}$ have a common portion. Equation (1.1) can be written as

$$\frac{A(k)}{B(k)}\,\psi_{+}(k) + \psi_{-}(k)\,\frac{C(k)}{B(k)} = 0. \tag{1.3}$$

Using Eq.(1.2), Equation (1.3) becomes

$$\frac{L_{+}(k)}{L_{-}(k)}\,\psi_{+}(k) + \psi_{-}(k)\,\frac{C(k)}{B(k)} = 0,$$

$$L_{+}(k)\psi_{+}(k) + \psi_{-}(k)L_{-}(k) + \frac{C(k)}{B(k)}\,L_{-}(k) = 0. \tag{1.4}$$

Decompose

$$L_{-}(k)\,\frac{C(k)}{B(k)} = D_{+}(k) + D_{-}(k),$$

where the functions $D_{+}(k)$ and $D_{-}(k)$ are analytic in the half plane $Im(k)>\tau''_{-}$, $Im(k)>\tau''_{+}$ respectively. All the three strips $\tau_{-}\tau<\tau_{+}$, $\tau'_{-}\tau<\tau'_{+}$, $\tau''_{-}\tau<\tau''_{+}$ have a common portion $\tau^{\circ}_{-}<\tau<\tau^{\circ}_{+}$, then in this strip the following functional equation is true

$$J(k) = L_{+}(k)\psi_{+}(k) + D_{+}(k) = -\,L_{-}(k)\psi_{-}(k) - D_{-}(k). \tag{1.5}$$

Left hand side of the above equation is analytic in half plane $\tau^{\circ}_{-}<Im(k)$. Right hand side of the above equation is analytic in the half plane $Im(k)<\tau^{\circ}_{-}$. Hence by analytic continuation, we can define J(k) over the whole $\varkappa$-plane, so that J(k) is regular in the whole $\varkappa$-plane. Now it can be shown that the function J(k) has algebraic behaviour as $|k| \longrightarrow \infty$. i.e.

$$|L_{+}(k)\psi_{+}(k) + D_{+}(k)| < (k)^{p} \text{ as } \alpha \longrightarrow \infty\ \tau>\tau^{\circ}_{-}, \tag{1.6}$$

$$|L_{-}(k)\psi_{-}(k) + D_{-}(k)| < (k)^{q} \text{ as } \alpha \longrightarrow \infty\ \tau>\tau^{\circ}_{+}, \tag{1.7}$$

Then from the extended form of Liouville's Theorem $J(k)$ is a polynomial $P(k)$ of degree less than or equal to the integral part of $\min(p,q)$, i.e.

$$L_+(k)\psi_+(k) + D_+(k) = P(k), \tag{1.8}$$

$$L_-(k)\psi_-(k) + D_-(k) = -P(k). \tag{1.9}$$

Eq.(1.8) gives

$$\psi_-(k) = \frac{P(k) - D_+(k)}{L_+(k)} . \tag{1.10}$$

Eq.(1.9) gives

$$\psi_+(k) = \frac{-(D_-(k) + P(k)}{L_+(k)} . \tag{1.11}$$

Eq.(1.10) and Eq.(1.11) gives values of $\psi_+(k)$ and $\psi_-(k)$ within the arbitrary polynomial i.e. within a finite number of arbitrary constants which must be determined otherwise. The solution can thus be obtained which is valid throughout. We note that the decomposition of functions into additive and multiplicative parts is imperative here. We, therefore give the conditions under which this can be done Noble [3].

Theorem 1

Let $F(k)$ be an analytic function of $k = \sigma + i\tau$ regular in the strip $\tau_- < \tau < \tau_+$ such that

$$|F(\sigma+i\tau)| < C|k|^{-p},\ p > 0 \text{ for } |\sigma| \longrightarrow \infty,$$

the inequality holding uniformly for all τ in the strip

$\tau_-+\varepsilon\leq\tau\leq\tau_+-\varepsilon$, $\varepsilon>0$, then for $\tau_-+c<\tau<d<\tau_+$

$$F(k) = F_+(k) + F_-(k),$$

where $F_+(k)$ is regular for all $\tau>\tau_-$ and $F_-(k)$ is regular for all $\tau<\tau_+$.

We note that the additive decomposition is a generalization of the Laurent theorem. We know that if a function is analytic in an annular region then it can be written as the sum of two functions one of which is analytic inside a circle while the other is analytic outside another circle both being analytic in the common annular region.

Theorem 2

In $\ell n[L(k)]$ satisfies the conditions of Theorem 1 which impose in particular that L(k) is regular and non-zero in the strip $\tau_-<\tau<\tau_+$, $-\infty<\sigma<\infty$ and $L(k) \longrightarrow +1$ as $\sigma \longrightarrow \pm\infty$ in the strip, then we can write

$$L(k) = L_+(k)L_-(k),$$

where $L_+(k)$ and $L_-(k)$ are regular, bounded and non-zero in $\tau>\tau_-$ and $\tau<\tau_+$ respectively.

1.3 HANKEL FUNCTIONS

Hankel functions of first and second kinds are respectively defined by

$$H_n^{(1)}(x) = J_n(x) + iY_n(x),\ H_n^{(2)}(x) = J_n(x) - iY_n(x) \qquad (1.12)$$

where $J_n(x)$ and $Y_n(x)$ are the solutions of the differential equation

$$x^2y'' + xy' + (x^2-n^2)y = 0, \quad n \geq 0 \tag{1.13}$$

$J_n(x)$ and $Y_n(x)$ are Bessel functions of first and second kind respectively of order n. Eq.(1.13) is called Bessel's differential equation, primes are the derivatives with respect to x. Bessel functions of order zero have the following integral representation

$$J_o(x) = \frac{2}{\pi}\int_o^\infty \sin(x\cosh s)ds, \quad Y_o(x) = \frac{-2}{\pi}\int_o^\infty \cos(x\cosh s)ds, \tag{1.14}$$

so that

$$H_o^{(1)}(x) = J_o(x) + iY_o(x) = \frac{-2}{i\pi}\int_o^\infty e^{ix\cosh s}ds. \tag{1.15}$$

Now for the large values of x, the asymptotic formulas for the Bessel functions of order n are given by

$$\left.\begin{aligned} J_n(x) &\sim \sqrt{\frac{2}{\pi x}}\cos(x-\pi/4 - n\pi/2), \\ J_n(x) &\sim \sqrt{\frac{2}{\pi x}}\sin(x-\pi/4 - n\pi/2), \end{aligned}\right\} \tag{1.16}$$

so that asymptotic formulae for the Hankel functions of first and second kinds (of order zero) are given by

$$H_o^{(1)}(x) \sim \sqrt{\frac{2}{\pi x}}\, e^{(x-\pi/4)},$$

$$H_o^{(2)}(x) \sim \sqrt{\frac{2}{\pi x}}\, e^{-i(x-\pi/4)}.$$

1.4 THE DIRAC DELTA FUNCTION

In mathematical physics we often encounter functions which have non-zero values in very short intervals. If we consider the function

$$\delta_a(x) = \begin{cases} \frac{1}{2a} & , \quad |x| < a, \\ 0 & , \quad |x| > a. \end{cases} \tag{1.17}$$

It can be easily shown that

$$\int_{-\infty}^{+\infty} \delta_a(x)dx = 1. \tag{1.18}$$

Also, if f(x) is any function which is integrable in the interval (-a,a), then by using the mean value theorem of integral calculus, we see that

$$\int_{-\infty}^{+\infty} f(x)\delta_a(x)dx = \frac{1}{2a}\int_{-a}^{a} f(x)dx = f(\vartheta a), \tag{1.19}$$

$$\text{where } |\vartheta| \le 1.$$

We now define

$$\delta(x) = \lim_{a\to 0} \delta_a(x). \tag{1.20}$$

Letting a tends to zero in eqs.(1.17) and (1.19) it is clear that $\delta(x)$ satisfies the following

$$\delta(x) = 0, \quad \text{if} \quad x \neq 0, \tag{1.21}$$

$$\int_{-\infty}^{+\infty} \delta(x)dx = 1. \tag{1.22}$$

The function defined by Eqs.(1.21) and (1.22) is called Diract delta function.

Dirac delta function and its derivatives play such a useful role in the formulation and solution of boundary value problems in classical mathematical physics as well as in quantum mechanics that is important to derive the formal properties of Dirac delta function.

If we let $a \longrightarrow 0$ in Eq.(1.19) we obtain the relation

$$\int_{-\infty}^{+\infty} f(x)\delta(x)dx = f(0), \tag{1.23}$$

with a simple change of variable transforms to

$$\int_{-\infty}^{+\infty} f(x)\delta(x-a)dx = f(a). \tag{1.24}$$

Let us c now consider the interpretation, we must put upon the "derivatives" of $\delta(x)$.

If we assume $\delta'(x)$ exists and that both it and $\delta(x)$ can be regarded as ordinary functions in the role for integration by parts we see that

$$\int_{-\infty}^{+\infty} f(x)\delta'(x)dx = \Big[f(x)\delta(x)\Big]_{-\infty}^{+\infty} + \int_{-\infty}^{+\infty} f'(x).\delta(x)dx = -f'(0).$$

Repeating the above process, we find that

$$\int_{-\infty}^{+\infty} f(x)\delta^{(n)}(x)dx = (-1)^{n}f^{(n)}(0).$$

1.5 SOLUTION OF INHOMOGENEOUS WAVE EQUATION

Suppose that there is a line source at (x_0,y_0). The time dependent of the field is taken to be harmonic. Then the partial differential equation satisfied by the potential ϕ is given by

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + k^2\phi = -4\pi\ \delta(x,x_o)(y,y_o) \tag{1.25}$$

where the right hand term is a forcing term due to the line source at (x_o, y_o). We determine the solution of (1.25) in free space, such that ϕ represents an outgoing wave at infinity. Taking the Fourior transform of (1.25) we get

$$\frac{d^2\Phi}{dy^2} - \varkappa^2\Phi = \frac{-4\pi}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \delta(x-x_o)\delta(y-y_o)e^{i\alpha x}dx, \tag{1.26}$$

where $\varkappa^2 = \alpha^2 - k^2$.

Using the property of δ-function, we obtain

$$\frac{d^2\Phi}{dy^2} - \varkappa^2\Phi = -2(2\pi)^{1/2}e^{i\alpha x}\delta(y-y_o) \tag{1.27}$$

we know that if $\frac{d^2\Phi}{dy^2} - \varkappa^2\Phi = f(y)$ then the solution in $-\infty<y<\infty$

such that $\Phi \longrightarrow 0$ as $y \longrightarrow \pm\infty$ is given by

$$\Phi(y) = \frac{-1}{2\varkappa} \int_{-\infty}^{\infty} f(\eta)\ e^{-\varkappa|y-\eta|}d\eta. \tag{1.28}$$

Using (1.28) the solution of (1.27) can be written as

$$\Phi(\alpha,y) = \frac{\sqrt{2\pi}}{\varkappa} \int_{-\infty}^{\infty} e^{i\alpha x_o}\ \delta(\eta-y_o)e^{-\varkappa|\eta-y_o|}d\eta\ ,$$

or

$$\Phi(\alpha,y) = \frac{\sqrt{2\pi}}{\varkappa} \int_{-\infty}^{\infty} e^{i\alpha x_o}\ e^{-\varkappa|y-y_o|}\ . \tag{1.29}$$

Now taking the inverse Fourior transform of (1.29), we obtain,

$$\phi(x,y) = \int_{-\infty}^{\infty} e^{-i\alpha(x-x_o)-\varkappa|y-y_o|} \; . \tag{1.30}$$

To solve (1.30). Let us define the following substitutions

$$x - x_o = r\cos\vartheta, \quad |y-y_o| = r\sin\vartheta,$$

$$\alpha = -k\cos(\vartheta+it), \; -\infty<t<\infty$$

Then (1.30) takes the form

$$\phi(x,y) = \int_{-\infty}^{\infty} e^{ikr\cosh t} dt = \pi i \; H_o^{(1)}(kr). \tag{1.31}$$

where $r = \sqrt{(x-x_o)^2 + (y-y_o)^2}$ and the integral representation of the Hankel function has been used. Using the asymptotic behaviour of the Hankel function the field given by $\phi(k,r)$ in (1.31) can finally be written as

$$\phi(k,r) \simeq i\left(\frac{2}{\pi kr}\right)^{1/2} e^{i(kr-\pi/4)}.$$

1.6 ASYMPTOTIC EVALUATION OF INTEGRALS

Here we discuss the methods usually adopted to calculate asymptotically the integrals appearing in certain diffraction problems. We discuss the following two methods;

i) The Method of Stationary Phase

ii) The Laplace Method

we are not going to discuss the steepest descent method:

i) The Method of Stationary Phase:

It is one of the methods usually adopted to write down asymptotically form of certain integrals appearing in diffraction problems. In many problems we have to deal with integrals of the form

$$I = \int_a^b e^{it\phi(\mu)} g(\mu) d\mu \tag{1.32}$$

where ϕ is a real valued function, called the phase function, while g may be either real or complex valued. In contrast to the Laplace Method which is the other method used to solve the integrals asymptotically, the exponent is purely imaginary; hence the integrand is an oscillatory function of t. As long as $\phi'(\mu) \neq 0$, we may integrate by parts and conclude that the integral is $O(1/t)$ when $t \longrightarrow \infty$. The main contribution comes from the points (μ_j), where $\phi'(\mu_j) = 0$. These are called the stationary points (the points where the maxima or minima lies). We assume a finite number of stationary points (μ_j) with $a<\mu_j<b$, $\phi''(\mu_j) \neq 0$ and $\int_a^b |g(\mu)| d\mu<\infty$. Then, when $t \longrightarrow \infty$.

$$I = \int_a^b e^{it\phi(\mu)} g(\mu) d\mu = \sum_{j=\phi''(\mu_j)>0} \left[\frac{2\pi}{t\phi''(\mu_j)}\right]^{1/2} e^{it\phi(\mu_j)+i\pi/4} g(\mu_j)$$

$$= \sum_{j=\phi''(\mu_j)>0} \left[\frac{2\pi}{t\phi''(\mu_j)}\right]^{1/2} e^{it\phi(\mu_j)+i\pi/4} g(\mu_j)+O(1/t). \tag{1.33}$$

In contrast to Laplace's method, we must sum over all stationary points of ϕ not simply those where ϕ is maximum.
If the end points $\mu = a$ or $\mu = b$ are stationary points, then the expression for I must be halved.

ii) Laplace's Method

We take the integral of the form

$$f(t) = \int_a^b g(x).e^{th(x)}.dx, \qquad (1.34)$$

with the possibility that $h'(x) = 0$ at one or more points. In this case it is still true that $f(t) \sim e^{tH}$, $t \to \infty$, where H is the maximum of $h(x)$, $a \le x \le b$. The feature results from the possibility of points x_i, where $h(x_i) = H$ and $h'(x_i)$. We assume that $h''(x_i) \neq 0$ at each of these points [of course it follows that $h''(x_i) < 0$, since we are at maximum of h]. These points fall into two groups:

(1) interior global maximum of h and

(2) boundary maxima where $h(x_i) = 0$.

The exact contribution of the second type of point is one-half the first type of contribution. We now state the result of Laplace's method

$$f(t) = \frac{e^{tH}}{\sqrt{t}} [c+O(t/\sqrt{t})] \; t \longrightarrow \infty ,$$

where

$$c = \sqrt{2\pi} \left[\sum_{\substack{a<x_i<b \\ h(x_i) = H}} \frac{g(x_i)}{[-h''(x_i)]^{1/2}} + \sum_{\substack{x_i=a \text{ or } x_i=b \\ h(x_i) = H}} \frac{g(x_i)}{[-h_i''(x\ )]^{1/2}} \right]. \qquad (1.35)$$

CHAPTER TWO

DIFFRACTION BY AN ACOUSTICALLY PENETRABLE OR ELECTROMAGNETICALLY DIELECTRIC HALF-PLANE

2.1 INTRODUCTION

This chapter is devoted to the problem of diffraction by an acoustically penetrable or an electromagnetically dielectric half-plane addressed by Rawlin [13]. We have reproduced this problem here to understand the problem that we have considered in chapter three.

2.2 FORMULATION OF THE PROBLEM:

We consider the scattering of an acoustic wave due to a line source by a penetrable half-plane. The penetrable half- plane is assumed to be thin and occupies the position $x \le 0$, $y = 0$ as shown in the Fig. 2. We consider a line source to be located at $(x_\circ, y_\circ)$, $y_\circ > 0$. The time harmonic factor $e^{-i\omega t}$ (ω is the angular frequency) is understood and is suppressed throughout. Thus the wave equation in presence of a line source reduces to.

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + k^2\right)u(x,y) = \delta(x-x_\circ)\delta(y-y_\circ), \tag{2.1}$$

where $k = \omega/c = k_r + ik_i$ is the wave number and c is the velocity of sound. For analytic convenience $k_i > 0$.

On the penetrable half-plane we have the boundary condition [13]

$$\frac{\partial}{\partial y} u(x,0^{\pm}) \pm ik\{\alpha U(x,0^{\pm}) + \beta u(x,0^{\mp})\} = 0, \quad x<0. \tag{2.2}$$

Also we impose the conditions of continuity,

$$\left.\begin{aligned} u(x,0^{+}) &= u(x,0^{-}), \\ \frac{\partial u(x,0^{+})}{\partial y} &= \frac{\partial u(x,0^{-})}{\partial y}, \end{aligned}\right\} \quad x > 0; \tag{2.3}$$

In Eq.(2.2), the parameters α and β are given by

$$\alpha = \left[\frac{T^2 e^{2ikh\sin\vartheta_o} + (e^{-ikh\sin\vartheta_o} - R^2 e^{2ikh\sin\vartheta_o})}{(e^{-ikh\sin\vartheta_o} + Re^{ikh\sin\vartheta_o})^2 - T^2 e^{2ikh\sin\vartheta_o}}\right]\sin\vartheta_o, \tag{2.4}$$

$$\beta = \left[\frac{-2T\sin\vartheta_o}{(e^{-ikh\sin\vartheta_o} + Re^{ikh\sin\vartheta_o})^2 - T^2 e^{2ikh\sin\vartheta_o}}\right], \tag{2.5}$$

where R and T are the reflection and transmission coefficients respectively and 2h is the width of the half-plane. We decompose the total field u into

$$U(x,y) = \phi_o(x,y) + \phi(x,y), \tag{2.6}$$

where ϕ_o is the solution of inhomogeneous wave equation that corresponds to the incident wave and ϕ is the solution of homogeneous wave Eq.(2.1) that gives the diffracted field. Thus the wave equations satisfied by ϕ_o and ϕ are

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + k^2\right)\phi_o(x,y) = \delta(x-x_o)\delta(y-y_o), \tag{2.7}$$

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + k^2\right)\phi(x,y) = 0, \tag{2.8}$$

we also assume that the field requires the outgoing waves at infinity i.e.

$$r^{1/2}\left(\frac{\partial}{\partial r} - ik\right)u \longrightarrow 0 \quad \text{as} \quad r = \sqrt{x^2+y^2} \longrightarrow \infty, \tag{2.9}$$

and satisfies the edge condition [3]

$$\begin{aligned} &u(x,0) = O(1), \\ &\frac{\partial u(x,0)}{\partial y} = O(x^{-1/2}); \quad \text{as } x \longrightarrow 0^+. \end{aligned} \tag{2.10}$$

2.3 SOLUTION OF THE BOUNDARY VALUE PROBLEM:

A solution of Eqs.(2.7) and (2.8) is given by

$$\left.\begin{aligned} \phi_o(x,y) &= \frac{1}{4i} H_o^{(1)}\{k\sqrt{(x-x_o)^2 + (y-y_o)^2}, \\ &= \frac{1}{4\pi i}\int_{-\infty+i\tau}^{\infty+i\tau} \frac{e^{i[\nu(x-x_o)+\varkappa|y-y_o|]}}{\varkappa}\, d\nu, \end{aligned}\right\} \tag{2.11}$$

where $\varkappa = \sqrt{k^2-\nu^2}$ and $\nu = \sigma+i\tau$. The plane ν is cut in such a way that Imk > 0. The solution of Eq.(2.8) satisfying the radiation conditions is given by

$$\phi(x,y) = \frac{1}{2\pi i}\int_{-\infty+i\tau}^{\infty+i\tau} \frac{A(\nu)}{\varkappa}\, e^{i[\nu x+\varkappa y]}d\nu; \; y > 0, \tag{2.12}$$

$$= \frac{1}{2\pi i} \int_{-\infty+i\tau}^{\infty+i\tau} \frac{B(\nu)}{\varkappa} e^{i[\nu x - \varkappa y]} d\nu ; \ y < 0. \qquad (2.13)$$

For a unique solution of the problem we must have

$A(\nu) \sim |\nu|^{-1/2}$ and $B(\nu) \sim |\nu|^{-1/2}$ as $|\nu| \longrightarrow \infty$.

Substitution of Eqs.(2.9) to (2.11) in Eqs.(2.2) to (2.3) yields

$$\frac{1}{4\pi} \int_{-\infty+i\tau}^{\infty+i\tau} (1 + \frac{\alpha k}{\varkappa}) e^{i[\nu(x-x_o)-\varkappa y_o]} d\nu + \frac{1}{2\pi} \int_{-\infty+i\tau}^{\infty+i\tau} A(\nu).e^{i\nu x}(1 + \frac{\alpha k}{\varkappa}) d\nu$$

$$+ \frac{1}{4\pi} \int_{-\infty+i\tau}^{\infty+i\tau} B(\nu) \frac{\beta k}{\varkappa} e^{i\nu x}.d\nu + \frac{\beta k}{\varkappa} \int_{-\infty+i\tau}^{\infty+i\tau} \frac{e^{i[\nu(x-x_o)+\varkappa y_o]}}{\varkappa} d\nu = 0, \qquad (2.14)$$

$$- \frac{1}{4\pi} \int_{-\infty+i\tau}^{\infty+i\tau} (1+ \frac{\alpha k}{\varkappa}) e^{i[\nu(x-x_o)-\varkappa y_o]} d\nu - \frac{1}{2\pi} \int_{-\infty+i\tau}^{\infty+i\tau} B(\nu)(1+ \frac{\alpha k}{\varkappa}) e^{i\nu x} d\nu$$

$$- \frac{\beta k}{2\pi} \int_{-\infty+i\tau}^{\infty+i\tau} \frac{A(\nu)}{\varkappa} e^{i\nu x}.d\nu - \frac{\beta k}{4\pi} \int_{-\infty+i\tau}^{\infty+i\tau} \frac{e^{i[\nu(x-x_o)+\varkappa y_o]}}{\varkappa} d\nu = 0, \qquad (2.15)$$

$$\int_{-\infty+i\tau}^{\infty+i\tau} \{A(\nu)+B(\nu)\} e^{i\nu x}.d\nu = 0, \qquad (2.16)$$

$$\int_{-\infty+i\tau}^{\infty+i\tau} \frac{\{A(\nu)-B(\nu)\}}{\varkappa} e^{i\nu x}.d\nu = 0. \qquad (2.17)$$

Subtracting Eqs.(2.15) from (2.14), we obtain

$$\Rightarrow \int_{-\infty+i\tau}^{\infty+i\tau} \left\{ -\mathrm{Sgn} y_o \ e^{i[\nu(x-x_o)+\varkappa|y_o|]} + \mathrm{Sgn} y_o \ e^{i[\nu(x-x_o)+\varkappa|y_o|]} \right\} d\nu = 0$$

where

$$\mathrm{Sgny}_o = \begin{cases} 1, & \text{if } y_o > 0, \\ -1, & \text{if } y_o < 0. \end{cases}$$

$$\int_{-\infty+i\tau}^{\infty+i\tau} \left\{ C(\nu)K(\nu) + \frac{k(\alpha+\beta)}{\varkappa}\, e^{-i[\nu x_o - \varkappa y_o]} \right\} e^{i\nu x}.d\nu = 0, \tag{2.18}$$

Addition of Eqs.(2.14) and (2.15) gives

$$\Rightarrow \int_{-\infty+i\tau}^{\infty+i\tau} \left\{ D(\nu)L(\nu) - e^{-i[\nu x_o - \varkappa y_o]} \right\} e^{i\nu x}.d\nu = 0. \tag{2.19}$$

Eqs.(2.16) and (2.17) can also be rewritten as

$$\int_{-\infty+i\tau}^{\infty+i\tau} C(\nu)e^{i\nu x}.d\nu = 0, \quad x > 0 \tag{2.20}$$

$$\int_{-\infty+i\tau}^{\infty+i\tau} \frac{D(\nu)}{\varkappa}\, e^{i\nu x}.d\nu = 0, \quad x < 0 \tag{2.21}$$

where

$$\left.\begin{aligned} C(\nu) &= A(\nu) + B(\nu), \\ D(\nu) &= A(\nu) - B(\nu), \end{aligned}\right\} \tag{2.22}$$

$$K(\nu) = 1 + \frac{k(\alpha+\beta)}{\varkappa}, \quad L(\nu) = 1 + \frac{k(\alpha-\beta)}{\varkappa}. \tag{2.23}$$

We shall solve the Eqs.(2.18) to (2.21) by the Wiener-Hopf technique. We write the solution in the form

$$C(\nu) = \Phi_+(\nu), \tag{2.24}$$

$$\frac{D(\nu)}{\varkappa} = \Phi_+(\nu), \tag{2.25}$$

$$C(\nu)K(\nu) + \frac{k(\alpha+\beta)}{\varkappa}\, e^{-i[\nu x_o - \varkappa y_o]} = \Phi_-(\nu) \tag{2.26}$$

$$D(\nu)L(\nu) - e^{-i[\nu x_o - \varkappa y_o]} = \Phi_-(\nu). \tag{2.27}$$

In order to solve by the Wiener-Hopf technique, functional Eqs.(2.24) to (2.27), we need to factorize the functions $K(\nu)$ and $L(\nu)$. The factorizations of $K(\nu)$ and $L(\nu)$ are discussed by using the procedure in Noble [3, p-164] and are given by

$$K(\nu) = K_\pm(\nu) = 1 + \frac{k(\alpha+\beta)}{\pi\ \varkappa} \cos^{-1}(\pm\nu/k), \tag{2.28}$$

$$L(\nu) = L_\pm(\nu) = 1 + \frac{k(\alpha-\beta)}{\pi\ \varkappa} \cos^{-1}(\pm\nu/k). \tag{2.29}$$

In Eqs.(2.24) to (2.29), $\Phi_+(\nu)$, $K_+(\nu)$ and $\Phi_-(\nu)$, $K_-(\nu)$, $L_-(\nu)$ are regular analytic functions in the domains $\mathrm{Im}(\nu) > -\tau_o$ and $\mathrm{Im}(\nu) < \tau_o$, respectively. These two domains have the intersection $|\mathrm{Im}(\nu)| < \tau_o$ and τ_o is assumed that no singularities occur in this common region of intersection. Eliminating $C(\nu)$ and $D(\nu)$ from Eqs.(2.24) to (2.27), we have

$$\Phi_+(\nu)K_+(\nu) + \frac{k(\alpha+\beta)}{\sqrt{k^2-\nu^2}\ K_-(\nu)} e^{-i[\nu x_o - \varkappa y_o]} = \frac{\Phi_-(\nu)}{K_-(\nu)} \tag{2.30}$$

$$\psi_+(\nu)\sqrt{k+\nu}\ L_+(\nu) + \Delta_+(\nu) = \frac{\psi_-(\nu)}{\sqrt{k-\nu}} L_-(\nu) + \Delta_-(\nu), \tag{2.31}$$

where we have used

$$\Lambda(\nu) = \frac{k(\alpha+\beta)}{K_-(\nu)\sqrt{k^2-\nu^2}} e^{-i[\nu x_o - \sqrt{k^2-\nu^2}\, y_o]} = \Lambda(\nu) = \Lambda_+(\nu) - \Lambda_-(\nu), \tag{2.32}$$

$$\Delta(\nu) = e^{-i[\nu x_o - \varkappa y_o]} = \Delta(\nu) = \Delta_+(\nu) - \Delta_-(\nu), \tag{2.33}$$

$$\Lambda_+(\nu) = \frac{-1}{2\pi i}\int_{-\infty}^{\infty} \frac{\Lambda(t)}{\nu - t}\,dt; \quad \text{Im}(\nu) > 0, \tag{2.34}$$

$$\Delta_+(\nu) = \frac{-1}{2\pi i}\int_{-\infty}^{\infty} \frac{\Delta(t)}{\nu - t}\,dt; \quad \text{Im}(\nu) > 0, \tag{2.35}$$

$$\Lambda_-(\nu) = \frac{-1}{2\pi i}\int_{-\infty}^{\infty} \frac{\Lambda(t)}{\nu - t}\,dt; \quad \text{Im}(\nu) < 0, \tag{2.36}$$

$$\Delta_-(\nu) = \frac{-1}{2\pi i}\int_{-\infty}^{\infty} \frac{\Delta(t)}{\nu - t}\,dt; \quad \text{Im}(\nu) < 0. \tag{2.37}$$

With the help of Eqs.(3.32) and (2.33), Eqs.(2.30) and (2.31) become

$$\Phi_+(\nu)K_+(\nu)+\Lambda_+(\nu) = \frac{\Phi_-(\nu)}{K_-(\nu)} + \Lambda_-(\nu) = J_1(\nu), \tag{2.38}$$

$$\psi_+(\nu)L_+(\nu)+\Delta_+(\nu) = \frac{\psi_-(\nu)}{K_-(\nu)\sqrt{k-\nu}} + \Delta_-(\nu) = J_2(\nu). \tag{2.39}$$

The integrals in (2.34) - (2.37) exist as these are exponentially bounded as $|t| \longrightarrow \infty$.

For $\Lambda_+(\nu)$, $\Delta_+(\nu)$ and $\Lambda_-(\nu)$, $\Delta_-(\nu)$ the point lies above and below the contour of integration which is indented for real ν respectively. Also $|\Lambda_\pm(\nu)|$ and $|\Lambda_\pm(\nu)|$ are of $O(|\nu|^{-1})$ and $|\nu| \longrightarrow \infty$ these are of $O(0)$. Consequently, we get (2.39) and (2.40) which hold in the common strip of regularity of both sides. However, the left hand sides and right hand sides define $J_1(\nu)$ and $J_2(\nu)$ in (2.39) and (2.40) throughout the upper and lower ν-plane respectively.

Common strip of regularities provide the analytic continuation of the other and so the composite functions defined in (2.38) and (2.39) are regular in the entire plane. The both sides of the Eqs.(2.38) and (2.39) have only the algebraic growth as $|\nu| \longrightarrow \infty$.

Therefore from the extended form of Liouvill's theorem [3]. Thus $J_1(\nu)$ and $J_2(\nu)$ must be polynomials in ν. Using the asymptotic estimate used in Rawlin [14],

$$L(\nu) \longrightarrow 1, \quad \text{as} \quad |\nu| \longrightarrow \infty,$$

$$K(\nu) \longrightarrow 1, \quad \text{as} \quad |\nu| \longrightarrow \infty,$$

$$L_{\pm}(\nu) \sim O(1), \quad \text{as} \quad |\nu| \longrightarrow \infty,$$

$$K_{\pm}(\nu) \sim O(1), \quad \text{as} \quad |\nu| \longrightarrow \infty.$$

The edge condition (2.7) gives $A(\nu) \sim O(|\nu|^{-1/2})$, $B(\nu) \sim O(|\nu|^{-1/2})$. This implies that

$$C(\nu) = A(\nu) + B(\nu) \sim O(|\nu|^{-1/2}), \quad \text{as} \quad |\nu| \longrightarrow \infty.$$

and $$D(\nu) = A(\nu) - B(\nu) \sim O(|\nu|^{-1/2}), \quad \text{as} \quad |\nu| \longrightarrow \infty.$$

Thus

$$J_1(\nu) \sim O(|\nu|^{-1/2}), \quad \text{as} \quad |\nu| \longrightarrow \infty.$$

and $$J_2(\nu) \sim O(|\nu|^{-1/2}), \quad \text{as} \quad |\nu| \longrightarrow \infty.$$

Hence the polynomials representing $J_1(\nu)$ and $J_2(\nu)$ can only be constants which equal to zero. With the help of this argument, Eqs.(2.38) and (2.39) implies

$$\Phi_+(\nu)K_+(\nu) = -\Lambda_+(\nu) \qquad (2.40)$$

and $\quad \psi_+(\nu)(K+\nu)^{1/2}L_+(\nu) = -\Delta_+(\nu)$ (2.41)

From Eqs.(2.24), (2.25), (2.32), (2.34) and (2.35), we have

$$C(\nu) = \frac{k(\alpha+\beta)}{K_+(\nu)2\pi i}\int_{-\infty}^{\infty}\frac{e^{-i[tx_o-(k^2-t^2)^{1/2}y_o]}}{(k^2-t^2)^{1/2}K_-(t)(\nu-t)}\,dt, \tag{2.42}$$

$$D(\nu) = \frac{k(\alpha+\beta)}{L_+(\nu)2\pi i}\int_{-\infty}^{\infty}\frac{e^{-i[tx_o-(k^2-t^2)^{1/2}y_o]}}{K_-(t)(k^2-t^2)^{1/2}(\nu-t)}\,dt. \tag{2.43}$$

Adding and subtracting Eqs.(2.22) and using (2.42) and (2.43), we arrive at

$$\left.\begin{aligned} A(\nu) &= \frac{1}{4\pi i}\,\frac{k(\alpha+\beta)}{K_+(\nu)}\int_{-\infty}^{\infty}\frac{e^{-i[tx_o-(k^2-t^2)^{1/2}y_o]}}{K_-(t)(k^2-t^2)^{1/2}(\nu-t)}\,dt + \frac{(k-\nu)^{1/2}}{L_+(\nu)} \\ &\qquad\times\int_{-\infty}^{\infty}\frac{e^{-i[tx_o-(k^2-t^2)^{1/2}y_o]}}{L_-(t)(k-t)^{1/2}(\nu-t)}\,dt \\ B(\nu) &= \frac{-1}{4\pi i}\,\frac{k(\alpha+\beta)}{K_+(\nu)}\int_{-\infty}^{\infty}\frac{e^{-i[tx_o-(k^2-t^2)^{1/2}y_o]}}{K_-(t)(k^2-t^2)^{1/2}(\nu-t)}\,dt \\ &\qquad\times\int_{-\infty}^{\infty}\frac{(k-\nu)^{1/2}(k+t)^{1/2}\,e^{-i[tx_o-(k^2-t^2)^{1/2}y_o]}}{L_+(\nu)L_-(t)(k^2-t^2)^{1/2}(\nu-t)}\,dt. \end{aligned}\right\} \tag{2.44}$$

In view of Eqs.(2.11), (2.12), (2.13) and (2.44), total field becomes

$$U(x,y) = \frac{1}{4i}\,H_o^{(1)}(k\sqrt{(x-x_o)^2+(y-y_o)^2}$$

$$+\frac{1}{8\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F(\nu,t)e^{i[\nu x+\varkappa|y|-(\nu x_o-\sqrt{k^2-t^2}y_o)]}d\nu.dt \tag{2.45}$$

where

$$F(\nu,t)=\frac{1}{\sqrt{k^2-t^2}\,\sqrt{k^2-\nu^2}(t-\nu)}\left\{\frac{k(\alpha+\beta)}{K_+(\nu)K_-(t)}+\frac{\sqrt{k+t}\,\sqrt{k-\nu}}{K_+(\nu)K_-(t)}\,\mathrm{Sgn}(y)\right\}. \tag{2.46}$$

When k is real, the path of integration is indented below $t = 0$ and the ν-path of integration indented above $\nu = 0$.

It is well known that results for line source are the same as for plane wave incidence except a constant factor

$$\sqrt{\frac{2\pi}{kr_o}}\;e^{i(kr_o-\pi/4)} \quad .$$

So the plane wave solution (given by [13]) obtained by evaluating the integral asymptotically as the source goes to infinity i.e. as $kr_o \longrightarrow \infty$ is given by

$$U(x,y)=e^{-i(x\cos\vartheta_o+y\sin\vartheta_o)}+\frac{1}{2\pi i}\int_{-\infty}^{\infty} G(\nu,\vartheta_o)e^{i[\nu x+\varkappa|y|]}d\nu \tag{2.47}$$

where

$$G(\nu,\vartheta_o)=\frac{1}{(\nu+k\cos\vartheta_o)\sqrt{k^2-\nu^2}}\left\{\frac{k(\alpha+\beta)}{K_+(\nu)K_+(k\cos\vartheta_o)}+\right.$$

$$\left.\frac{\sqrt{k-\nu}\,\sqrt{k-k\cos\vartheta_o}}{L_+(\nu)L_+(k\cos\vartheta_o)}\,\mathrm{Sgn}(y)\right\}. \tag{2.48}$$

The far field can be obtained by solving the integral appearing in Eq.(2.47) asymptotically. For that we put

$$x = r\cos\vartheta, \quad y = r\sin\vartheta.$$

Hence for large kr i.e. $kr \longrightarrow \infty$, we obtain,

2.4 ASYMPTOTIC EXPRESSIONS FOR THE FAR FIELD INCIDENCE

Let us consider second term appearing in Eq. (2.45)

$$I = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{e^{irg(\nu,\vartheta_o)}}{\sqrt{k^2-\nu^2}(\nu-\nu_p)} f(\nu,\vartheta_o)d\nu, \tag{2.50}$$

where

$$\nu_p = -k\cos\vartheta_o ,$$

$$g(\nu,\vartheta_o) = \nu\cos\vartheta = \sqrt{k^2-\nu^2}\ |\sin\vartheta|, \tag{2.51}$$

$$f(\nu,\vartheta_o) = \left\{\frac{k(\alpha+\beta)}{K_+(\nu)K_+(k\cos\vartheta_o)} + \frac{(\sqrt{k-\nu})(\sqrt{k-k\cos\vartheta_o})\mathrm{Sgn}(y)}{L_+(\nu)L_+(k\cos\vartheta_o)}\right\}. \tag{2.52}$$

The integral (2.50) can be evaluated by asymptotic method for large r by modification of the method of stationary phase. The modification is required because the pole ν_p can come close to the point of stationary phase $\nu_s = k\cos\vartheta$. The method used is similar to that in Rawlins [13]. In order to avoid repetition details of calculations are omitted and I becomes

$$I = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{e^{irg(\nu,\vartheta_o)} f(\nu,\vartheta_o)d\nu}{\sqrt{k^2-\nu^2}\ (\nu-\nu_p)}$$

$$= \frac{e^{+ikr-i\pi/4}}{i\sqrt{2\pi kr}} f(\nu_s).\left[\frac{2|Q|}{(\nu_p-\nu_s)}\right].F(|Q|) + \text{pole contribution}, \tag{2.53}$$

where

$$|Q| = (\sqrt{kr/2})\,\frac{\cos\vartheta+\cos\vartheta_o}{\sin\vartheta}, \tag{2.54}$$

$$F(|Q|) = e^{-iQ^2}\int_{|Q|}^{\infty} e^{it^2}dt. \tag{2.55}$$

Note:- if the source is far from half-plane then $F(z) \sim \frac{i}{2|Q|}$. Otherwise $F(z) \sim 2i|Q|$. Eq.(2.54) gives the complete asymptotic evaluation. If we do not take asymptotic evaluation, then we have some other terms as well.

The pole contribution account for geometrical acoustic field terms. The total field can be represented in terms of geometrical acoustic field terms and the diffracted field

$$U_{diff} = \frac{e^{i(kr-\pi/4)}}{i\sqrt{2\pi kr}} f(\nu_s).\left[\frac{2|Q|}{(\nu_p-\nu_s)}\right]F(|Q|), \tag{2.56}$$

where

$$f(\nu_s) = k\left\{\frac{(\alpha+\beta)}{K_+(\nu\cos\vartheta)K_+(k\cos\vartheta_o)} + \frac{2\sin\vartheta/2\,\sin\vartheta_o/2}{L_+(k\cos\vartheta)L_+(k\cos\vartheta_o)}\right\}. \tag{2.57}$$

For the purposes of obtaining the far field in terms of the geometrical acoustic and diffracted field the expression (2.46) is asymptotically evaluated for large kr. Thus using the modified saddle point method [4], we have

By Rawlin [13], for $0<\vartheta_o<\pi$, $x = r\cos\vartheta$, $y = r\sin\vartheta$, $-\pi<\vartheta<\pi$.

$$
\begin{aligned}
u(r,\vartheta) &= I + RF + D_+ && \text{for } \pi-\vartheta_o < \vartheta < \pi, \\
&= I + D_+ && \text{for } 0 < \vartheta < \pi-\vartheta_o, \\
&= I + D_- && \text{for } \vartheta_o - \pi < \vartheta < 0, \\
&= TR + D_- && \text{for } \pi-\vartheta < \vartheta_o < \pi. \qquad (2.58)
\end{aligned}
$$

where $0<\vartheta_o<\pi$,

$$I = \text{Incident wave} = e^{-ikr\cos(\vartheta-\vartheta_o)},$$

$$RF = \text{reflected wave} = \frac{(\sin^2\vartheta_o-(\alpha^2-\beta^2))e^{-ikr\cos(\vartheta+\vartheta_o)}}{(|\sin\vartheta_o|+(\alpha+\beta))(|\sin\vartheta_o|+(\alpha-\beta))},$$

$$= R\, e^{ik}[2h\sin\vartheta_o - r\cos(\vartheta+\vartheta_o)],$$

$$D_\pm = \text{diffracted field} = \frac{-e^{i(kr-\pi/4)}}{\sqrt{2\pi kr}} \cdot \frac{2(|Q|F(|Q|)}{(\cos\vartheta+\cos\vartheta_o)}$$

$$\times \left\{\frac{(\alpha+\beta)}{K_+(\nu\cos\vartheta)K_+(k\cos\vartheta_o)} \mp \frac{\sqrt{1-\cos\vartheta}\ \sqrt{1-\cos\vartheta_o}}{L_+(k\cos\vartheta)L_+(k\cos\vartheta_o)}\right\},$$

$$TR = \text{transmitted field} = \frac{-2\beta\sin\vartheta_o e^{-ikr\cos(\vartheta-\vartheta_o)}}{(\sin\vartheta_o+(\alpha+\beta))(\sin\vartheta_o+(\alpha-\beta)},$$

$$= Te^{ik[2h\sin\vartheta_o-r\cos(\vartheta-\vartheta_o)]},$$

where R & T are given by Eqs.(24) and (25) respectively.

2.5 Concluding Remarks:

The physical interpretation of Eq.(2.58) is now obvious. I represents the incident wave directly coming from the line source, RF is the reflected wave and TR is the transmitted field. These represent geometrical acoustic field. $D_{\pm}$ are the diffracted waves in illuminated region $(0<\vartheta<\pi)$ and shadow region $(-\pi<\vartheta<0)$ respectively. Further, the results for a rigid barrier can be obtained by putting $\alpha = 0 = \beta$. In addition, the results for an absorbing half-plane can be obtained as a special case of this problem by taking $\alpha = \rho_o c/z$ and $\beta = 0$, where ρ_o is the density of the ambient medium and z is the acoustic impedance of the surface.

CHAPTER THREE

CYLINDRICAL WAVE SCATTERING FROM A PENETRABLE HALF-PLANE

3.1 INTRODUCTION

We have discussed the problem of scattering of a cylindrical wave from a penetrable half-plane in the presence of a wake in the present chapter. Thus the problem corresponds to the trailing edge situation. The integral transforms and the Wiener-Hopf procedure are employed to obtain the approximate mathematical model for the problem in terms of integrals representing the diffracted field. These integrals are normally difficult to handle because of the presence of branch points and are only amenable to solution using asymptotic approximations. The analytic solution is thus obtained using asymptotic methods (saddle point method) and the diffracted field is presented.

3.2 FORMULATION OF THE PROBLEM:

We consider the diffraction by a semi-infinite plane. The semi-infinite plane is assumed to be penetrable occupying the position $x \leq 0$, $y = 0$. We consider a line source situated at (x_o, y_o), $y_o > 0$, as shown in the Fig.3. On suppressing the time harmonic factor $e^{-i\omega t}$ (ω is the angular frequency) the equation governing the total velocity potential U reduces to

$$\left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + k^2\right] U = \delta(x-x_o)\delta(y-y_o), \tag{3.1}$$

where $k = \omega/c$ is the wave number and c, is the velocity of sound. On the penetrable half-plane of negligible thickness the boundary conditions

$$\frac{\partial}{\partial y} U(x,0^{\pm}) \pm ik\alpha\, U(x,0^{\pm}) \pm ik\beta\, U(x,0^{\mp}) = 0, \tag{3.2}$$

where α and β are parameters to be shortly identified. We also require that field be radiating outwards at infinity. If we assume that the pressure and velocity are continuous we obtain the condition

$$U \text{ and } \frac{\partial U}{\partial y} \text{ are continuous when } y = 0,\ x>0, \tag{3.2a}$$

In this instance the field does not satisfy the Kutta-Joukowski condition of finite velocity at the edge of the semi-infinite plane. Therefore to find a solution of (3.1) which satisfies the Kutta-Joukowski condition the only possibility is to abandon the continuity of the field. As discussed by Jones [16] the most natural way to introduce a discontinuity in the field is to postulate the existance of a wake across which U is discontinuous whilst $\frac{\partial U}{\partial y}$ remains continuous. The wake occupies $y = 0$, $x>0$ and should be similar to that in steady flow but modified to account for oscillatory field. Thus we take as the boundary condition on $y = 0,\ x>0$

$$\left.\begin{aligned} &U(x,0^+) - U(x,0^-) = \gamma\ e^{i\mu x}\ , \\ &\frac{\partial U}{\partial y}(x,0^+) = \frac{\partial U}{\partial y}(x,0^-), \end{aligned}\right\} \qquad (3.3)$$

where γ and μ are constants. The constant μ is regarded as known and γ is to be determined by the condition imposed at the edge.

For analytic convenience we assume that k has a small positive imaginary part which we place equal to zero at the end of the analysis. This assumption corresponds to an absorption of sound so that the waves decay at infinity. We also write

$$\mu = k\cos\vartheta_i\ , \qquad (3.4)$$

where $0 \le \mathrm{Re}\vartheta_i < \pi$, $\mathrm{Im}\vartheta_i \ge 0$. While k has a positive imaginary part we shall take $0 \le \mathrm{Re}\vartheta_i < \pi$, $\mathrm{Im}\vartheta_i > 0$; eventually we shall be concerned primarily with the case $\mathrm{Re}\vartheta_i = 0$, $\mathrm{Im}\vartheta_i > 0$. We decompose the total field into the incident wave ϕ_o and the diffracted field ϕ as

$$U(x,y) = \phi_o(x,y) + \phi(x,y), \qquad (3.5)$$

where ϕ_o is solution of inhomogeneous wave equation that corresponds to the incident wave and ϕ is the solution of homogeneous wave equation (3.1) which gives the diffracted field.

3.3 SOLUTION OF THE PROBLEM

The Fourior transform and its inverse over the variable x is defined as

$$\left.\begin{aligned} \psi(\nu,y) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} U(x,y)e^{-i\nu x}dx, \\ U(\nu,y) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \psi(\nu,y)e^{-i\nu x}d\nu, \end{aligned}\right\} \tag{3.6}$$

where $\nu = \sigma + i\tau$.

Transforming Eqs.(3.1), (3.3),(3.5) and (2.2) with respect to x by (3.6), we obtain,

$$\left[\frac{\partial^2}{\partial y^2} + \varkappa^2\right]\psi_o(\nu,y) = e^{i\nu x_o}\delta(y-y_o), \tag{3.7}$$

$$\left[\frac{\partial^2}{\partial y^2} + \varkappa^2\right]\psi(\nu,y) = 0, \tag{3.8}$$

$$\frac{\partial}{\partial y}\psi(\nu,0^{\pm}) \pm ik\{\alpha\psi(\nu,0^{\pm})+\beta\psi(\nu,0^{\mp})\} = 0, \quad x<0, \tag{3.9}$$

$$\left.\begin{aligned} \psi(\nu,0^{\pm}) - \psi(\nu,0^{\pm}) &= \frac{\gamma}{\mu-\nu}, \\ \partial\psi/\partial y(\nu,0^{\pm}) &= \partial\psi/\partial y(\nu,0^{-}), \end{aligned}\right\} \tag{3.10}$$

$$\psi(\nu,y) = \psi_o(\nu,y) + \psi(\nu,y), \tag{3.11}$$

where $\varkappa = (k^2-\nu^2)^{1/2}$.

Eq.(3.7) can be solved in a straightforward manner to give

$$\phi_o = \frac{1}{4i} H_o^{(1)}\left\{k\sqrt{(x-x_o)^2+(y-y_o)^2}\right\}$$

$$= \frac{1}{4\pi i}\int_{-\infty+i\tau}^{\infty+i\tau} \frac{e^{i[\nu x+\varkappa|y-y_o|]}}{\varkappa}\, d\nu. \tag{3.12}$$

Fourier transform of (3.8) gives

$$\left[\frac{\partial^2}{\partial y^2} + \varkappa^2\right]\psi(\nu, y) = 0,$$

$$\phi = \frac{1}{2\pi i}\int_{-\infty+i\tau}^{\infty+i\tau} \frac{A(\nu)}{\varkappa}\, e^{i[\nu x+\varkappa y]} d\nu, \quad y>0. \tag{3.13}$$

$$= \frac{1}{2\pi i}\int_{-\infty+i\tau}^{\infty+i\tau} \frac{B(\nu)}{\varkappa}\, e^{i[\nu x+\varkappa y]} d\nu, \quad y<0. \tag{3.14}$$

In order to obtain the unique solution of the problem, the edge condition requires that $A(\nu)$, $B(\nu) \sim |\nu|^{-\varepsilon}$ $\nu \longrightarrow \infty$ where $0<\varepsilon<1$, ($\varepsilon = \frac{1}{2} - \frac{1}{2\pi}$ arg(1)) [15]) substituting Eqs.(3.11), (3.12), (3.13) and (3.14) into Eqs.(3.2) and (3.3) give

$$\int_{-\infty+i\tau}^{\infty+i\tau} \left[\frac{A(\nu) + B(\nu)}{\varkappa}\right] e^{i\nu x} d\nu = 0, \quad x>0, \tag{3.15}$$

$$\int_{-\infty+i\tau}^{\infty+i\tau} \left\{\frac{[A(\nu) + B(\nu)]}{\varkappa} - \frac{\gamma}{\nu-\mu}\right\} e^{i\nu x} d\nu = 0, \quad x>0 \tag{3.16}$$

$$\int_{-\infty+i\tau}^{\infty+i\tau} A(\nu)\left[1 + \frac{k(\alpha+\beta)}{\varkappa}\right] e^{i\nu x} d\nu - \frac{1}{2}\int_{-\infty+i\tau}^{\infty+i\tau}\left[\mathrm{Sgn}(y_o) - \frac{k(\alpha+\beta)}{\varkappa}\right] \times e^{i[\nu(x-x_o)+\varkappa|y_o|]} d\nu = 0, \quad x<0 \tag{3.17}$$

$$\int_{-\infty+i\tau}^{\infty+i\tau} B(\nu)\left[1 + \frac{k(\alpha+\beta)}{\varkappa}\right] e^{i\nu x} d\nu + \frac{1}{2}\int_{-\infty+i\tau}^{\infty+i\tau}\left[\mathrm{Sgn}(y_o) + \frac{k(\alpha+\beta)}{\varkappa}\right] \times e^{i[\nu(x-x_o)+\varkappa|y_o|]} d\nu = 0, \quad x<0, \tag{3.18}$$

where

$$\mathrm{Sgn}(y_o) = \begin{cases} 1, & \text{if } y_o > 0, \\ -1, & \text{if } y_o \le 0. \end{cases} \tag{3.18a}$$

Addition and subtraction of Eqs.(3.17) and (3.18) with (3.15) and (3.16) yield

$$\int_{-\infty+i\tau}^{\infty+i\tau} C(\nu)\, e^{i\nu x}\, d\nu = 0, \quad x>0, \tag{3.19}$$

$$\int_{-\infty+i\tau}^{\infty+i\tau} \left[\frac{D(\nu)}{\varkappa} - \frac{\gamma}{\nu-\mu}\right] e^{i\nu x}\, d\nu = 0, \quad x>0, \tag{3.20}$$

$$\int_{-\infty+i\tau}^{\infty+i\tau} \left\{C(\nu)K(\nu) + \frac{k(\alpha+\beta)}{\varkappa}\, e^{-i[\nu x_o - \varkappa y_o]}\right\} e^{i\nu x}\, d\nu = 0, \quad x<0, \tag{3.21}$$

$$\int_{-\infty+i\tau}^{\infty+i\tau} \left\{D(\nu)L(\nu) - e^{-i[\nu x_o - \varkappa y_o]}\right\} e^{i\nu x}\, d\nu = 0, \quad x<0, \tag{3.22}$$

where

$$\left.\begin{aligned} C(\nu) &= A(\nu) + B(\nu), \\ D(\nu) &= A(\nu) - B(\nu), \end{aligned}\right\} \tag{3.23}$$

$$\left.\begin{aligned} K(\nu) &= 1 + \frac{k(\alpha+\beta)}{\varkappa}, \\ L(\nu) &= 1 + \frac{k(\alpha-\beta)}{\varkappa}. \end{aligned}\right\} \tag{3.24}$$

A solútion of Eqs.(3.19), (3.20), (3.21), (3.22) and (3.23) can be written by keeping in view the Wiener-Hopf method as

$$C(\nu) = \phi_+(\nu), \tag{3.25}$$

$$C(\nu)K(\nu) + \frac{k(\alpha+\beta)}{\varkappa} e^{-i[\nu x_o - \varkappa y_o]} = \phi_-(\nu), \tag{3.26}$$

$$\frac{D(\nu)}{\varkappa} - \frac{\gamma}{\nu-\mu} = \psi_+(\nu), \tag{3.27}$$

$$D(\nu)L(\nu) - e^{-i[\nu x_o - \varkappa y_o]} = \psi_-(\nu), \tag{3.28}$$

where the positive subscript denotes that the function is regular and analytic in the domain $\mathrm{Im}(\nu)>\mathrm{Im}(k)$. The negative subscript denotes that the function is regular and analytic in the domain $\mathrm{Im}(\nu)<\mathrm{Im}(k)$. These two domains have the intersection $|\mathrm{Im}(\nu)| < \mathrm{Im}(k)$. Now eliminating $C(\nu)$ and $D(\nu)$ from the Eqs.(3.25) – (3.28) gives,

$$\phi_+(\nu)K(\nu) + \frac{k(\alpha+\beta)}{\varkappa} e^{-i[\nu x_o - \varkappa y_o]} = \phi_-(\nu), \tag{3.29}$$

$$\psi_+(\nu)L(\nu) - e^{-i[\nu x_o - \varkappa y_o]} = \psi_-(\nu). \tag{3.30}$$

For the solutions of Eqs.(3.29) and (3.30), we need to factorize the kernel functions $K(\nu)$ and $L(\nu)$. The factorization of these function is given in chapter 2. According to this

$$\phi_+(\nu)K_+(\nu) + \frac{k(\alpha+\beta)}{(k^2-\nu^2)^{1/2}K_-(\nu)} e^{-i[\nu x_o - (k^2-\nu^2)^{1/2} y_o]} = \frac{\phi_-(\nu)}{K_-(\nu)} \tag{3.31}$$

with the help of Eq.(3.31), Eqs.(3.29) and (3.30) can be rewritten as

$$\psi_+(\nu)L_+(\nu) - \frac{e^{-i[\nu x_o-(k^2-\nu^2)^{1/2}y_o]}}{(k^2-\nu^2)^{1/2}L_-(\nu)} = \frac{\psi_-(\nu)}{L_-(\nu)} . \tag{3.32}$$

Now Eqs.(3.31) and (3.32) are the usual Wiener-Hopf functional equations. For the solution, we have to split these equations into the functions which are regular in the upper and lower half-planes. For that we need to factorize

$$\Lambda(\nu) = \frac{k(\alpha+\beta)e^{-i[\nu x_o-(k^2-\nu^2)^{1/2}y_o]}}{(k^2-\nu^2)^{1/2}K_-(\nu)} , \tag{3.33}$$

$$\Delta(\nu) = \frac{1.\, e^{-i[\nu x_o-(k^2-\nu^2)^{1/2}y_o]}}{(k^2-\nu^2)^{1/2}L_-(\nu)} . \tag{3.34}$$

We can split $\Lambda(\nu)$ and $\Delta(\nu)$ by means of Cauchy integrals [3] into the form

$$\left.\begin{aligned} \Lambda(\nu) &= \Lambda_+(\nu) + \Lambda_-(\nu), \\ \Delta(\nu) &= \Delta_+(\nu) + \Delta_-(\nu), \end{aligned}\right\} \tag{3.35}$$

where

$$\Lambda_+(\nu) = \frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{\Lambda(t)}{(t-\nu)}\, dt, \tag{3.36}$$

$$\Lambda_-(\nu) = -\frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{\Lambda(t)}{(t-\nu)}\, dt, \tag{3.37}$$

$$\Delta_+(\nu) = \frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{\Delta(t)}{(t-\nu)}\, dt, \tag{3.38}$$

$$\Delta_-(\nu) = -\frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{\Delta(t)}{(t-\nu)}\, dt. \tag{3.39}$$

It is worth noting that $|\Lambda_{\pm}(\nu)|$ and $|\Delta_{\pm}(\nu)|$ are at least of $O(|\nu|^{-1})$ as $|\nu| \longrightarrow \infty$. Thus Eqs.(3.31) and (3.32) can now be written as

$$\phi_{+}(\nu)K_{+}(\nu) + \Lambda_{+}(\nu) = J_{1}(\nu) = \frac{\phi_{-}(\nu)}{K_{-}(\nu)} - \Lambda_{-}(\nu), \tag{3.40}$$

$$\psi_{+}(\nu)L_{+}(\nu) + \Delta_{+}(\nu) = J_{2}(\nu) = \frac{\psi_{-}(\nu)}{L_{-}(\nu)} - \Delta_{-}(\nu). \tag{3.41}$$

In Eqs.(3.40) and (3.41), the left-hand side is regular in the upper half-plane and the right hand-side is regular in the lower half-plane. Therefore by analytic continuation, both sides of Eqs.(3.40) and (3.41) define entire functions $J_1(\nu)$ and $J_2(\nu)$ because of the strip common to both the domains. Now, $K(\nu) \rightarrow 1$ as $|\nu| \rightarrow \infty$ and $K(\nu) \rightarrow 1$ as $|\nu| \rightarrow \infty$, $|K_{\pm}(\nu)|$ and $|L_{\pm}(\nu)| \sim O(|\nu|^{\mp\delta})$ as $|\nu| \rightarrow \infty$, $\delta = \frac{1}{2\pi}$ arg (1). Also from the edge conditions, we conclude that $\phi_{+}(\nu)$ and $\phi_{-}(\nu)$ must be of order $O(|\nu|^{-\varepsilon})$ as $|\nu| \rightarrow \infty$. Using this asymptotic estimate, it can be seen from Eq.(3.40) that $J_1(\nu) \sim O(|\nu|^{-\varepsilon-\delta})$ and therefore the polynomial representing $J_1(\nu)$ can only be a constant which equals zero since $\varepsilon+\delta = 1/2$ (Abelian Theorem [3]). Equating left hand side of Eq.(3.40) to zero, we obtain

$$\phi_{+}(\nu)K_{+}(\nu) = -\Lambda_{+}(\nu). \tag{3.42}$$

Eq.(3.42) together with Eqs.(3.25) and (3.36) gives

$$C(\nu) = \frac{-k(\alpha+\beta)}{2\pi i \; K_+(\nu)} \int_{-\infty}^{\infty} \frac{e^{-i[\nu x_o - (k^2-\nu^2)^{1/2} y_o]}}{(k^2-\nu^2)^{1/2} K_-(\nu)(t-\nu)} \, dt. \tag{3.43}$$

Similarly, from Eqs.(3.41), (3.38) and (3.27)

$$D(\nu) = \frac{(k-\nu)^{1/2}}{2\pi i \; L_+(\nu)} \int_{-\infty}^{\infty} \frac{e^{-i[\nu x_o - (k^2-\nu^2)^{1/2} y_o]}}{(k^2-\nu^2)^{1/2} L_-(\nu)(t-\nu)} \, dt. \tag{3.44}$$

The value of $A(\nu)$ and $B(\nu)$ can be determined by adding and subtracting the Eqs.(3.43) and (3.44) and are given by

$$A(\nu) = \frac{-1}{4\pi i \; L_+(\nu)}$$
$$\times \int_{-\infty}^{\infty} \frac{e^{-i[\nu x_o - (k^2-\nu^2)^{1/2} y_o]} [k(\alpha+\beta) - (k-\nu)^{1/2}(k+t)^{1/2}]}{(k^2-\nu^2)^{1/2} L_-(\nu)(t-\nu)} \, dt$$
$$+ \frac{\gamma (k-\nu)^{1/2} (k+\mu)^{1/2} L_+(\mu)}{2L_+(\nu)(\nu-\mu)}, \tag{3.45}$$

$$B(\nu) = \frac{-1}{4\pi i \; L_+(\nu)}$$
$$\times \int_{-\infty}^{\infty} \frac{e^{-i[\nu x_o - (k^2-\iota^2)^{1/2} y_o]} [k(\alpha+\beta) + (k-\nu)^{1/2}(k+t)^{1/2}]}{(k^2-\nu^2)^{1/2} L_-(\nu)(t-\nu)} \, dt$$
$$- \frac{\gamma (k-\nu)^{1/2} (k+\mu)^{1/2} L_+(\mu)}{2L_+(\nu)(\nu-\mu)}. \tag{3.46}$$

The requirement of Kutta-Joukowski condition velocity must be finite at the origin. This infact means that in the above expression for $A(\nu)$, $B(\nu)$, the term of $O(|\nu|^{-1/2+\delta})$ as

$|\nu| \longrightarrow \infty$ must vanish. So from (3.45), γ is chosen such that

$$\gamma = \lim_{\nu\to o} \frac{-(\nu-\mu)}{2\pi i(k+\mu)^{1/2}L_+(\mu)(k-\nu)^{1/2}}$$

$$\times \int_{-\infty}^{\infty} \frac{k(\alpha+\beta)L_+(\nu)L_-(t)-(k-\nu)^{1/2}(k+t)^{1/2}K_+(\nu)L_-(t)}{K_+(\nu)K_-(t)L_+(\nu)L_-(t)(k^2-t^2)(t-\nu)}$$

$$\times e^{-i[tx_o-(k^2-t^2)^{1/2}y_o]}dt \tag{3.47}$$

$$= \frac{1}{2\pi i(k+\mu)^{1/2}L_+(\mu)} \int_{-\infty}^{\infty} \frac{e^{-i[tx_o-(k^2-t^2)^{1/2}y_o]}}{L_-(\nu)(k-t)^{1/2}}\, dt. \tag{3.48}$$

Substitution of Eqs.(3.48) is to Eqs.(3.45) and (3.46)

$$A(\nu) = \frac{-1}{4\pi i\ L_+(\nu)}$$

$$\times \int_{-\infty}^{\infty} \frac{e^{-i[\nu x_o-(k^2-t^2)^{1/2}y_o]}[k(\alpha+\beta)-(k-\nu)^{1/2}(k+t)^{1/2}]}{(k^2-\nu^2)^{1/2}L_-(t)(t-\nu)}\, dt$$

$$+ \frac{(k-\nu)^{1/2}}{4\pi i\ L_+(\nu)} \int_{-\infty}^{\infty} \frac{e^{-i[\nu x_o-(k^2-t^2)^{1/2}y_o]}}{L_-(\nu)(k-t)^{1/2}}\, dt, \tag{3.49}$$

$$B(\nu) = \frac{-1}{4\pi i\ L_+(\nu)} \int_{-\infty}^{\infty} \frac{k(\alpha+\beta)\_+\ (k+t)^{1/2}(k-\nu)^{1/2}}{(k^2-\nu^2)^{1/2}L_-(t)(t-\nu)}\, dt$$

$$+ \frac{(k-\nu)^{1/2}}{4\pi i\ L_+(\nu)(\nu-\mu)} \int_{-\infty}^{\infty} \frac{e^{-i[\nu x_o-(k^2-t^2)^{1/2}y_o]}}{L_-(\nu)(k-t)^{1/2}}\, dt. \tag{3.50}$$

Substituting the values of $A(\nu)$ and $B(\nu)$ from Eqs.(3.49) and (3.50) into Eqs.(3.13) and (3.14) give the total field

$$U(x,y) = \frac{1}{4i} H_o^{(1)}\left\{k\sqrt{(x-x_o)^2+(y-y_o)^2}\right\}$$

$$+ \frac{i}{8\pi^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} F(\nu,t) e^{i[\nu x+(k^2-\nu^2)^{1/2}|y|-tx_o+(k^2-t^2)^{1/2}y_o]} d\nu dt$$

$$- \frac{Sgn(y)}{8\pi^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} G(\nu,t) e^{i[\nu x+(k^2-\nu^2)^{1/2}|y|-tx_o+(k^2-t^2)^{1/2}y_o]} d\nu dt, \tag{3.51}$$

where

$$F(\nu,t) = \frac{k(\alpha+\beta) - \sqrt{k-\nu}\ \sqrt{k+t}\ Sgn(y)}{L_+(\nu)L_-(t)(t-\nu)(k^2-t^2)^{1/2}(k^2-\nu^2)^{1/2}}, \tag{3.52}$$

$$G(\nu,t) = \frac{\sqrt{k+t}\ \sqrt{k-\nu}}{L_+(\nu)L_-(t)(t-\nu)\sqrt{k^2-\nu^2}(k^2-t^2)^{1/2}}. \tag{3.53}$$

3.4 THE FAR FIELD

In order to obtain the field $U(x,y)$ in spatial coordinates, we have to solve the double integral appearing in Eq.(3.53). WE solve the integral over the variables t and ν in sequel. For that we substitute

$$x_o = r_o\cos\vartheta_o, \quad y_o = r_o\sin\vartheta_o,$$

and introduce the transformation $t = -k\ \cos(\vartheta_o+ip_1)$ $(0\le\vartheta_o<\pi$, $-\infty<p_1<\infty)$, which changes the contour integration over t into a hyperbola passing through the point $-k\ \cos\vartheta_o$, the integral with respect to t is then solved asymptotically for $r_o \longrightarrow \infty$

by the method of stationary phase and the resulting expression for the diffracted field is given by

$$\phi(x,y) = -\frac{1}{8\pi}\int_{-\infty}^{\infty} e^{i[\nu x+(k^2-\nu^2)^{1/2}|y|]} F(\nu,\vartheta_o)d\nu$$

$$+\frac{Sgn(y)}{8\pi}\int_{-\infty}^{\infty} e^{i[\nu x+(k^2-\nu^2)^{1/2}|y|]} G(\nu,\vartheta_o)d\nu, \tag{3.54}$$

where

$$F(\nu,\vartheta_o) = \frac{H_o^{(1)}(kr_o)Sgn(y)}{K_+(\nu)K_-(-k\cos\vartheta_o)(\nu+k\cos\vartheta_o)(k^2-\nu^2)^{1/2}}, \tag{3.55}$$

$$G(\nu,\vartheta_o) = \frac{H_o^{(1)}(kr_o)(k-k\cos\vartheta_o)^{1/2}(k-\nu)^{1/2}}{L_+(\nu)L_-(-k\cos\vartheta_o)(\nu-\mu)(k^2-\nu^2)^{1/2}}, \tag{3.56}$$

The first integral appearing in Eq.(3.54) in the variable ν can be evaluated using a modified method of stationary phase since the pole at $\nu = -k\cos\vartheta_o$ may come close to the saddle point. For that if we put

$$F(\nu,\vartheta_o) = \frac{F_1(\nu,\vartheta_o)}{\nu-\nu_p}, \quad \nu_p = -k\cos\vartheta_o,$$

$$g(\nu) = i\nu\cos\vartheta + i(k^2-\nu^2)\sin\vartheta,$$

$$x = r\cos\vartheta, \quad y = r\sin\vartheta,$$

Eq.(3.54) becomes

$$I_1^* = -\frac{1}{8\pi^2}\int_{-\infty}^{\infty}\frac{F_1(\nu,\vartheta_o)e^{rg(\nu)}}{\nu-\nu_p}d\nu, \tag{3.57}$$

where $F(\nu,\vartheta_o)$ has no pole at $\nu = \nu_p$ let $\nu = \nu_s$ be the saddle point of $g(\nu)$ and r is large and positive. The contour of integration has to be deformed over the pole in being moved to the path of steepest descent through ν_s. On the path of steepest descent $F_1(\nu,\vartheta_o)$ is approximated by $F_1(\nu,\vartheta_o)$ and $g(\nu)$ is approximated by its Taylor series upto the second derivative. Then (3.57) takes the form

$$I_1^* = -\frac{1}{8\pi^2}\left[2\pi i\, F(\nu_p,\vartheta_o)e^{rg(\nu_p)} + I_2\right], \tag{3.58}$$

where

$$I_2 = \int_{-\infty}^{\infty} \frac{F(\nu_p,\vartheta_o)e^{rg(\nu)-\frac{1}{2}q^2}}{q + i r^{1/2}A}\, dq, \tag{3.59}$$

$$A = (\nu_p - \nu_s)\,[g''(\nu_s)]^{1/2}.$$

The first term in expression (3.58) is present only if $\mathbb{R}(A)>0$. When $\mathbb{R}(A)>0$, the integral (3.59) can be written as

$$I_2 = -i\, F(\nu_s,\vartheta_o)e^{rg(\nu_s)}\int_{-\infty}^{\infty} e^{-1/2q^2}\, dq \int_{0}^{\infty} e^{ip(q+ir^{1/2}A)}\, dp. \tag{3.60}$$

On inverting the order of integration, Eq.(3.60) can be written as

$$I_2 = -i(2\pi)^{1/2}\, F(\nu_s,\vartheta_o)e^{rg(\nu_s)}\int_{0}^{\infty} e^{-1/2p^2-r^{1/2}}\, dp. \tag{3.61}$$

This can be written in the following alternate forms in terms of Error and Fresenal functions such that

$$I_2 = -i\pi\, F(\nu_s,\vartheta_o)e^{rg(\nu_s)}F\left\{(\tfrac{1}{2}\, r)^{1/2}A\, e^{-i\pi/4}\right\}. \tag{3.62}$$

In particular if we introduce the hyperbolic path $\nu = k\cos(\vartheta_o+iq)(0\leq\vartheta<\pi,\ -\infty<q<\infty)$ in Eq.(3.57) the corresponding saddle point appears at $\nu_s = k\cos\vartheta$. Thus using Eq.(3.62) in Eq.(3.58), the first integral appearing in Eq.(3.54) gives

$$-\frac{1}{8\pi^2}\int_{-\infty}^{\infty} e^{i[\nu x+(k^2-\nu^2)^{1/2}|y|]} F(\nu,\vartheta_o)d\nu$$

$$= \frac{1}{2\pi\ \sin\vartheta\ \sqrt{2r_o}}\left[\frac{(\alpha+\beta)}{\sqrt{k}} - \frac{2\sin\vartheta_o/2\ \sin\vartheta/2}{\sqrt{k}}\right]$$

$$\times\ \frac{e^{ik(r+r_o)}\qquad F(|Q|)}{K_+(k\cos\vartheta)K_-(-k\cos\vartheta_o)} + \text{ Pole contributions,} \qquad (3.63)$$

where

$$|Q| = \sqrt{\frac{kr}{2}}\left(\frac{\cos\vartheta-\cos\vartheta_o}{\sin\vartheta}\right), \qquad (3.63a)$$

$$F(|Q|) = e^{-iQ^2}\int_{|Q|}^{\infty} e^{it^2}dt. \qquad (3.63b)$$

Adopting the same procedure for the second integral appearing in Eq.(3.54) we obtain

$$\frac{\text{Sgn}(y)}{8\pi^2}\int_{-\infty}^{\infty} e^{i[\nu x+(k^2-\nu^2)^{1/2}|y|]} G(\nu,\vartheta_o)d\nu$$

$$= -\frac{1}{2\pi\ \sin\vartheta\ \sqrt{2r_o}}\ \frac{\sin(\vartheta_o/2)\sin(\vartheta/2)F(|Q'|)}{\sqrt{k}\ L_+(k\cos\vartheta)L_-(k\cos\vartheta)}\ e^{ik(r+r_o)}, \qquad (3.64)$$

where

$$|Q'| = (\frac{kr}{2})^{1/2}\ \frac{(\mu-k\cos\vartheta)}{\sin\vartheta}\ (\frac{kr}{2})^{1/2}\left(\frac{k\cos\vartheta_1-k\cos\vartheta}{\sin\vartheta}\right), \qquad (3.65a)$$

$$F(|Q'|) = e^{-iQ'^2}\int_{|Q'|}^{\infty} e^{it^2}dt. \qquad (3.65b)$$

Thus the total diffracted field can be obtained by Eqs.(3.63) and (3.64) as

$$\phi(x,y) = \frac{1}{2\pi\sin\vartheta}\left[\frac{(\alpha+\beta)}{\sqrt{k}} - \frac{2\sin(\vartheta_o/2)(\sin\vartheta/2)}{\sqrt{k}}\right]$$

$$\times\frac{e^{ik(r+r_o)}\quad F(|Q|)}{K_+(k\cos\vartheta)K_-(-k\cos\vartheta_o)} - \frac{2\quad \sin(\vartheta_o/2)\sin(\vartheta/2)F(|Q'|)}{2\pi\sin\vartheta\quad \sqrt{k}\, L_+(k\cos\vartheta)L_-(k\cos\vartheta)}\,. \tag{3.66}$$

Solving Eq.(3.54) for $y>0$, and taking pole contribution, and $x = r\cos\vartheta$, $y = r\sin\vartheta$, we get the reflected wave and solution of Eq.(3.54) for $y<0$ with pole contribution yields the transmitted wave. Therefore the acoustic far field for leading and trailing edge situation is given by

$$\begin{aligned}\phi(x,y) &= I + D_+\,, & 0<\vartheta<\pi-\vartheta_o,\\ &= I + RF + D_+, & \pi-\vartheta_o<\vartheta<\pi,\\ &= I + D_-\,, & \vartheta_o-\pi<\vartheta<0,\\ &= TR + D_-, & -\pi<\vartheta<\vartheta_o-\pi.\end{aligned} \tag{3.67}$$

where

$$I = \text{incident wave} = e^{-ikr\cos(\vartheta-\vartheta_o)}$$

$$RF = \text{reflected wave} = \frac{(\sin^2\vartheta_o)-(\alpha^2-\beta^2))e^{-ikr\cos(\vartheta-\vartheta_o)}}{(|\sin\vartheta_o|)+(\alpha+\beta)(|\sin\vartheta_o|)+(\alpha-\beta)}$$

$$D_+ = \text{diffracted field} = \left[-\frac{e^{i(kr-\pi/4)}}{\sqrt{2\pi kr}}\frac{2|Q|\ F(|Q|)}{(\cos\vartheta+\cos\vartheta_o)}\right.$$

$$\left\{\frac{(\alpha+\beta)}{K_+(k\cos\vartheta)K_+(k\cos\vartheta_o)} \mp \frac{2\sin(\vartheta_o/2)(\sin\vartheta/2)}{K_+(k\cos\vartheta)L_+(k\cos\vartheta_o)}\right\}$$

$$\mp \varepsilon \frac{2\sin(\vartheta_o/2)(\sin\vartheta/2)}{K_+(k\cos\vartheta)L_+(k\cos\vartheta_o)} \frac{2|\Omega'|F(|\Omega'|)}{(\cos\vartheta_1+\cos\vartheta_o)}\Bigg],$$

where

$$\varepsilon = \begin{cases} 1 & \text{for trailing edge situation,} \\ 0 & \text{for leading edge situation.} \end{cases}$$

3.5 CONCLUDING REMARKS

The problem of diffraction of a line source by a semi-infinite penetrable plane is studied in this chapter. Expression for the trailing edge (wake present) is given. It is found that near the wake the field is strengthened and weakend elsewhere even when the source is near the edge. It is further observed that field for trailing edge situation will be greater than the field in its absence. It is worth noting that Rawlins [14] results for an absorbing half-plane can be obtained as a special case of this problem by taking $\alpha = \frac{\rho_0 c}{z}$ and $\beta = 0$, where z is the acoustic impedance of the surface. Also Rawlins [13] results in case of penetrable half-plane for no wake situation can be obtained by putting $\gamma = 0$. These agreements provide the useful check.

WAKE:

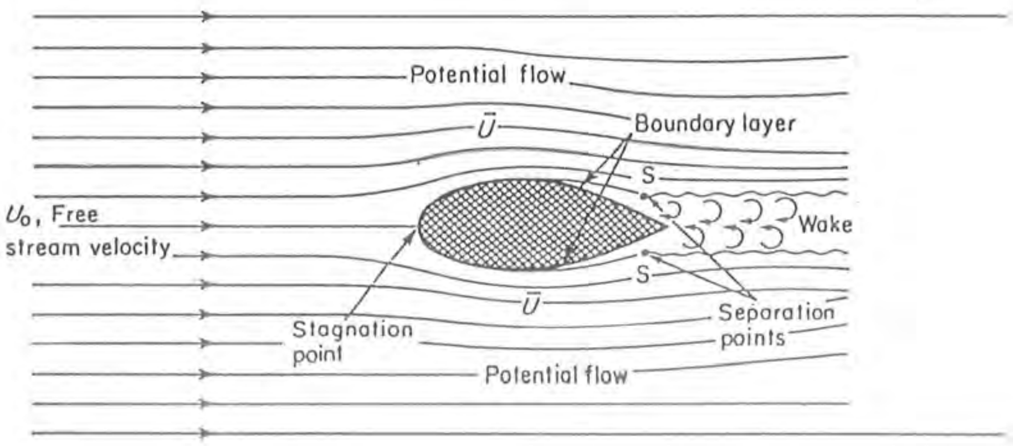
Potential flow
$\bar{U}$
Boundary layer
S
U_0, Free stream velocity
Wake
S
$\bar{U}$
Separation points
Stagnation point
Potential flow

Fig. 1.

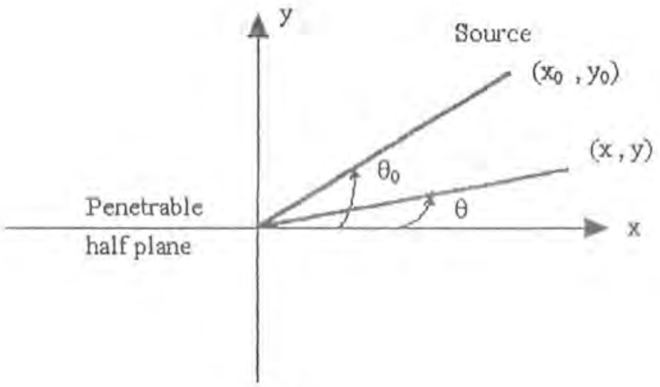

Fig. 2

Cylindrical Wave Scattering from A Penetrable Half-Plane

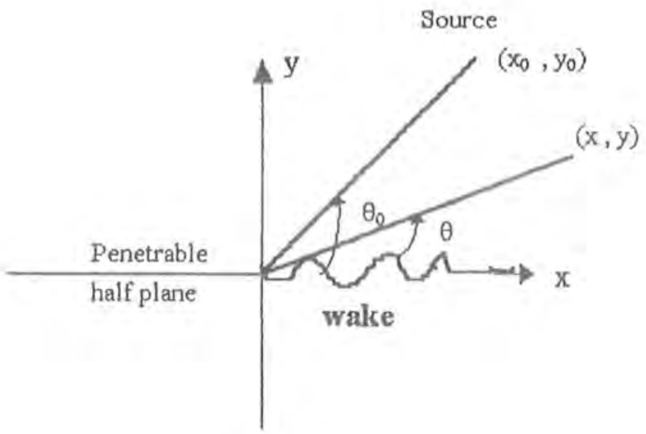

Fig. 3

References

[1] N. Wiener and E. Hopf, S. B. Preuss. Akad. Wiss., *696*, (1931).

[2] E. T. Copson, "Asymptotic Expansion", London Cambridge Univ. Press (1967).

[3] B. Noble, "Mthods bassed on the Wiener-Hopf Techniques",London: Pergamon Press (1958).

[4] D. S. Jones, "The Theory of Electromagnetism", London: Pergamon Press (1964).

[5] D. S. Jones, Proc. Roy. Soc. Lond., Ser. *A 404, 299,* (1986).

[6] D. S. Jones, "Acoustic and Electromagnetic Waves",Oxford: Clarendon Press (1986).

[7] L. M. Brekhovskikh, "Waves in Layred Media", Acadamic Press, New York, (1960).

[8] U. J. Kurze, Noise reduction by barriers, J. Acoust., Soc. Amer., *55, 504,* (1974).

[9] C. Yeh, Diffration of Waves by a dielectric parabolic cylinder., J. Opt. Soc. Amer., *57, 195,* (1967).

[10] J. Shmoys, Diffraction by a half-plane with a special impedane variation, I. E. E. E. Trans. on ant. & Prop., *588,* (1969).

[11] A. A. Pistol'kors, V. A. Kaplun and L. V. Kniazeva, On diffraction of electomagnetic waves at dielectric or semi conductiong sheets, Rad. Engng. and Elet. Phys., *4, 1,* (1959).

[12] N. G. Khrebet, Diffraction of plane electromagnetic waves on the edge of a dielectric half-plane, Rad. Engng. and Elet. Phys., *13, 331*, (1968).

[13] A. D. Rawlins, Int. J. Engng. Sci., *15, 569,* (1977).

[14] A. D. Rawlins, Proc. R. S. E. (A) *Vol. 75, 2,* (1975-76).

[15] A. D. Rawlins, Proc. R. S. E. (A) *Vol. 72, 4,* (1973-74).

[16] D. S. Jones, J. Inst. Maths. Applic., *9, 114,* (1972).

[17] P. M. Morse and K. U. Ingard, "Theoretical Acoustics", McGraw-Hill Book Co., (1968).