SCATTERING OF WAVES BY HALF PLANES AND SLITS WITH MIXED BOUNDARY CONDITIONS

BY

TASAWAR HAYAT

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CERTIFICATE

Certified that the work contained in this thesis was carried out by MR. TASAWAR HAYAT under my Supervision.

(PROFESSOR SALEEM ASGHAR)
DEPARTMENT OF MATHEMATICS
QUAID-I-AZAM UNIVERSITY
ISLAMABAD, PAKISTAN

DEDICATED

To my ***father*** (late) who nurtured me well,
inculcated in me the spirit to learn more and more,
and encouraged me to achieve higher ideals of life

To my ***mother*** who has always given me
love, care and cheer. Whose prayers have always
been a source of great inspiration for me and
whose sustained hope in me led me to where I
stand today

and beyond them to my devoted **supervisor**
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still do year after year.

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ABSTRACT

Considerable attention has been given to the acoustic scattering from half planes. The aim of this thesis is to contribute something by studying some acoustic diffraction problems from the absorbing half plane. Although, the acoustic wave diffraction from an absorbing half plane with similar absorbing parameter has been studied in the literature, nevertheless no attempt has been made to consider the spherical wave diffraction from a bi-impedant half plane in still air. The absorption in the half plane is introduced through different absorbing parameters. The problem is formulated in terms of matrix Wiener-Hopf (W. H.) functional equation. Physically, it corresponds to a mathematical model for a noise barrier whose surface is treated with two different acoustically absorbent materials. The modified W.H. method is used to arrive at the solution.

We further calculate the diffraction of a spherical acoustic wave from a porous barrier using a simple theory of porous materials [77] in still air. Our model assumes the barrier is made from a rigid material that is riddled with small pores that are approximately normal to the plane of the barrier. We take limited account of the compressibility of the gas in the pores. However, the gas in each pore behaves primarily as an incompressible cylinder, driven back and forth by the harmonic wavefield, but opposed by the frictional force generated at the pore walls (the flow resistance). The barrier is thin enough (with respect to wavelength) that sound is communicated from one side to the other by the motion of numerous incompressible cylinders. The approximate boundary conditions for such a situation are derived. A formal analytic solution to the complete problem is given, for the diffracted wavefield in the farfield region of the slit. The dependence on the barrier parameters of the power removed from the reflected wavefield by the diffraction at the slit is exhibited .

In the case of noise radiated by aero engines and inside wind tunnels, it is necessary to discuss acoustic diffraction in the presence of a moving fluid . Therefore, the theory of acoustic diffraction is further extended to include the case of moving fluid

and the following two problems are addressed in this direction. (1) "The diffraction of spherical wave by a half plane in a moving fluid". A finite region in the vicinity of the edge of a half plane has an impedance boundary conditions; the remaining part of the half plane is taken as rigid. It is found that the field is increased in case of moving fluid when compared with still air case. The field is also independent of the direction of the flow. This model has potential application in engine noise shielding by aircraft wings. (2) "The diffraction of a spherical Gaussian pulse by an absorbing half plane in a moving fluid for trailing edge (situation)". The trailing edge adds the complications of a trailing vortex sheet to the absorbing half plane. The motivation of this problem comes from a desire to understand the transient nature of the wavefields—since these can be expressed as linear combination of Gaussian pulses. The time dependence of the field is tackled by the use of temporal Fourier transform. It is found that field ratio of no wake to wake situation is independent of the type of acoustic sources. Also near the edge of absorbing half plane, the field of a spherical pulse caused by the Kutta-Joukowski condition is in excess of that in its absence.

Chapter 0 is devoted to the brief history of the problems of acoustic scattering. This chapter also contains the motivation for the work presented in this thesis. In chapter 1, we calculate the diffracted field by a slit in an infinite porous barrier. Chapter 2 deals with the problems of scattering by a bi-impedant half plane. Chapter 3 is devoted to the scattering of a spherical wave by a rigid screen with an absorbent edge in a moving fluid. In chapter 4, we present the scattering of a spherical Gaussian pulse near an absorbing half plane in a moving fluid.

Contents

Chapter 0

INTRODUCTION

Although mathematical analyses of light scattering by faceted objects occupied the time of many medieval scientists, perhaps the most notable being Ibn-al-Haitham of Basra who flourished in the 10th century AD. The reintroduction of the subject of scattering of sound and electromagnetic waves, commence with the analysis by Poincare [88], followed soon afterwards by Sommerfeld[104]. The problem of calculating an exact analytical solution for the scattering of sound or electromagnetic waves by an arbitrarily shaped body is generally intractable. However, for particular geometries (e.g. half plane, slit, wedge etc.) and certain restricted classes of problems it is possible to obtain explicit solutions to the associated mathematical boundary value problems.

HALF PLANES: The scattering of sound and electromagnetic waves by a half plane has been investigated by a number of authors. Sommerfeld [104] was the first who obtained the solution of the diffraction of plane waves from a half plane by using image waves. These solutions are valid in the far field, although they become unbounded at the incident or reflected shadow boundaries. MacDonald [75] and Carslaw [14] obtained the diffraction of a line source and a point source field by an ideal half plane. Clemmow [17] and Senior [102] obtained solutions in terms of Fresnel functions which are bounded at the shadow boundaries. A further improvement on the solution for the half plane with ideal boundaries was obtained by Pathak and Kouyoumjian [84] employing a uniform geometrical theory of diffraction. A useful collection of diffraction formulae are given by Bowman et al. [11], and an extensive review is given by Pierce [85].

The diffraction of waves by half planes with nonideal boundary conditions was discussed by Williams [108]. His solution, pertaining to surfaces of the half plane with identical point reacting impedance is in the form of an infinite product. A closed form solution for the diffraction of a plane wave by a rigid-soft half plane was obtained by Rawlins [90]. The scattering by half plane surfaces with more complicated boundary conditions has been studied as new physical applications have occurred, for instance, impedance surfaces have found application as absorbent liners in aero engine exhausts. The method of solution for these half plane problems involves the use of the Wiener-Hopf technique.

THE WIENER-HOPF (W.H.) TECHNIQUE: This technique provides an extremely powerful tool for attacking two and three dimensional diffraction problems. Out of all possible approaches towards the reduction of the physical problem at hand to a Wiener-Hopf problem, "Jones method" is the most suitable option as it is direct and straightforward. A beautiful account of Jones method can be found in the books of Jones [46] and Noble

[82]. The W. H. technique has many applications to problems outside the field of acoustics and electromagnetism, see, for example, Wickham [111], Kuiken [61], Shaw [103], Davis[19] and Soward [105]. The literature also includes some interesting new modifications of the classical method [58] for treating problems with axial symmetry. In scattering theory, the diffraction by arrays of parallel plates and diffraction by a plate with different absorbing conditions on either face is interesting from both a physical and a mathematical point of view. Apart from some special cases (Heins [40, 41], Lewin and Schwinger [63], Carlson and Heins [16] and Rawlins [91]) these diffraction problems give rise to matrix W. H. equations. There is, as yet, no general method of solution of such equations.

To solve a W. H. functional equation it is necessary to decompose the kernel (in general this is known function of a complex variable with a number of isolated singularities that characterize the underlying physical processes) into a product of functions, one analytic and of atmost algebraic growth in the upper half of the complex plane, whereas the other is analytic and of atmost of algebraic growth in the lower half plane. In the scalar case, this may be achieved by the use of Cauchy's integral theorem, but, for a matrix kernel, the formal extension of this technique is only successful for a restricted class of matrices. Thus a considerable effort has been made in recent years both to extend this class and to find new constructive methods for calculating matrix decompositions, especially with reference to particular physical problems. Examples of specific problems may be found in the papers by Rawlins [90], Hurd and Przezdziecki [44] and Przezdziecki and Hurd [89], and Abrahams [2], Hurd [45], Daniele [20], Khrapkor [57] and Jones[47].

Extension of these techniques, and other relevant material, can be found in Rawlins and Williams [92], Williams [110], Daniele[21] and Rawlins [93]. The abstract theory of matrix factorization, in the sense described above, has also received a great deal of attention following the existence theorem of GÖhberg and Krein [28], see, for example Speck [106], Meister [76] and GÖhberg and Kashock [29].

POINT SOURCES: In scattering, plane wave theory is generally assumed to be sufficiently accurate to predict wave scattering phenomena when a sound source is far away from a scatterer. The approach, however, becomes unavailable when, for practical reasons, a large source-to-scatterer separation cannot be achieved. Actually, it is difficult to answer the problem of how large the separation must be sufficient. At large distances from a sound source with a finite aperture, the wave front of the incident wave always becomes spherical. If the scatterer extends over a fairly large size, the sphericity of the wave front may cause significant bending stress, resulting the change in frequency and spatial characteristics. Keeping in view the importance of a spherical wave emanating from a point source, the diffraction of a spherical wave by an absorbing half plane is considered in this thesis. This consideration is important because point sources are regarded as better substitutes for real sources than line sources/plane waves. In many experimental situations, the insonifying source is more appropriately modeled as a point source or an array of point sources. If the point source is close to the scatterer, the scattered field could be significantly different compared to that of a plane incident wave due to the following two reasons:

(1) Geometrically, the surface area of the scatterer 'insonified' by a spher-

ical wavefront is smaller than that of a plane wave and therefore the scattered field contributed by the specular reflection is smaller.

(2) Structurally, the pressure loading on the scatterer is different due to the spherical spreading and therefore the scatterer's dynamic response and radiation pressure are different from those due to a plane incident wave.

WAKES: The interaction between acoustic sources and wave supporting structures is an important area of flow generated noise theory. Lighthill [65, 66] in his theory of aerodynamic sound, modeled the problem of sound generation by turbulence in an exact analogy with sound radiated by a volume distribution of acoustic quadrupoles embedded in an ideal acoustic medium. The strength density of the equivalent quadrupoles is Lighthill's stress tensor which is essentially the unsteady component of the Reynolds stress in low Mach number flows. Curle[18] showed how the presence of boundary surfaces could be accounted for by additional surface distributions of dipole and monopole sources. A dimensional analysis based on the idea that the only velocity and length scale in the problem are set by those in the turbulence yield the well-known laws that the intensity of sound generated by free turbulence increases with the eighth power of flow velocity, while that induced by unsteady surface forces increases in proportion to the sixth power of flow velocity. In 1970, Ffowcs Williams and Hall [23] have shown that the aerodynamic sound scattered by a sharp edge is proportional in intensity to the fifth power of flow velocity and to the inverse cube of the distance of the source from the edge. However, it is observed that the analysis of Ffowcs Williams and Hall was based upon the assumption of a potential flow near the sharp edge with velocity becoming infinite there. Instead of that if one wishes to prescribe that the velocity is finite, there are two possible points of view. One way is to abandon the Lighthill theory and use linearized Navier-Stokes equations with source terms. This would lead to analysis of the type employed by Alblas [3] who showed that, in the absence of the main flow, small viscosity removed the singularity in the velocity at the edge without appreciably affecting the far field pressure. However, in order to make the procedure satisfactory in the presence of a flow one would need to feel convinced that the linearization of the Navier-Stokes equation was still legitimate, and that the flow did not cause any shedding of vortices from the edge.

The other point of view is to retain the equation of small amplitude sound waves and attempt to apply a Kutta-Joukowski condition at the edge, as in aerofoil theory. It was due to Jones [48] ,who introduced the wake condition to examine the effect of the Kutta-Joukowski condition at the edge of the half plane which generates noise in the turbulent fluid at low Mach number. He calculated the field scattered from a line source in still air as well as for a moving fluid. It was observed that when the sound field is convected the orders of magnitude of the acoustic far field are the same whether or not the Kutta-Joukowski condition is applied, provided that the point of observation is not near the wake. Thus the wake acts as a convenient transmission channel for carrying intense sound away from the source. This problem was further extended to the point source excitation by Balasubramanyam [10] and to diffraction of a cylindrical impulse by Rienstra [101]. Keeping this in mind, the diffraction of a spherical Gaussian pulse by an absorbing half plane with fluid flow and in the presence of a wake is considered in chapter 4.

SLITS: Scattering from a slit is a well studied problem in diffraction theory from both theoretical and engineering points of view. The diffraction from a slit in a screen of finite thickness has been treated by several authors by using different analytical and numerical approaches. Asvestas and Kleinman [4] summarize and review much of the work done on it. Jones [46] and Nobel [82] discusses diffraction from a slit using the Wiener-Hopf method. Keller [56] treates diffraction of a wave from a slit of any shape in a thin screen by using the "geometrical theory of diffraction". Further, the most extensive investigation is that of Lehman [64], who used the analytical properties of finite Fourier transform. The solution of Kashyap and Hamid [52] used the Wiener-Hopf formulation in conjuction with the generalized scattering matrix technique. The solution of Hongo [42] and of Neerhof and Mur [80] are obtained by a numerical treatment of coupled integral equations. The transmission characteristics of a slit in a conducting screen of finite thickness placed between two different media is obtained by Auckland and Harrington [9] by formulating the problem in terms of a pair of coupled integral equations.

Keeping in view the importance of the scattering from a slit, the diffraction of an acoustic wave by a slit in an infinite, plane, porous barrier is investigated in chapter 1. The aim of this work is to calculate the scattered wavefield excited by a spherical wave incident to a slit in a barrier exhibiting both absorption and transmission. The source is assumed to be sufficiently far from the slit that its wavefront is locally plane. Throughout we assume that field is harmonic in time. In this chapter we give a formal solution to the complete problem and demonstrate that, in the limit of a rigid barrier, the solution reduces to that calculated by the geometrical theory of diffraction. The asymptotic analysis of the resulting integrals is only carried far enough to permit the calculation of the diffracted wavefields far from the slit as well as the power removed from the reflected wavefield by interference with the diffracted one. We anticipate extending the analysis of these integrals, so that expressions for the wavefield in the slit and close to the barrier can be obtained, and have therefore given more details of solution than is necessary to calculate only the farfield results. To calculate the diffracted wavefield from the interaction between the edges we assume that the slit is large, with respect to wavelength, and asymptotically approximate several integrals using this assumption. Karp and Keller [53] calculate this interaction term for diffraction from a slit in a perfectly rigid barrier using the geometrical theory of diffraction (this theory also assumes that the slit is large with respect to wavelength). Their work is a limiting case for ours and we show that, in this limit, the power removed from the reflected wavefield by interference with the diffracted one, that we calculate, agrees with theirs. Lastly, the same overall approach used here has been taken by Asghar [5] in his study of scattering from an absorbing strip in a moving fluid.

Rawlins [95], continuing his earlier work on diffraction from an absorbing barrier [96], presented a model of an acoustically penetrable but absorbing half plane barrier, and calculated the diffraction from its edge. He used a boundary condition, having two parameters, that mixes the pressure and its normal derivative at each side of the barrier. The boundary condition produces discontinuities are set by the two parameters. They are chosen to give approximately the same reflection and transmission coefficients as those found for the case of a plane wave incident to a thin layer, whose governing equation is a scalar wave equation. Adopting the same form of boundary condition in chapter 1, we identify the parameters in a different way. Using a simple theory of porous materials described in [77], our model assumes the barrier is made from a rigid material that is riddled with small pores that are approximately normal to the plane of the barrier. No particle velocity in the barrier parallel to its plane is permitted (a kinematic constraint). We take limited account of the compressibility of the gas in the pores. However, the gas in each pore behaves primarily as an incompressible cylinder, driven back and forth by the harmonic wavefield, but opposed by the frictional force generated at the pore walls (the flow resistance). The barrier is thin enough (with respect to wavelength) that sound is communicated from one side to the other by the motion of the numerous incompressible cylinders. The model is accurate provided $\frac{h\Phi}{\rho c} = O(1)$, where h is half the thickness, Φ the flow resistance and ρc the specific acoustic impedance of the surrounding gas. There have been other attempts do derive approximate boundary conditions that model thin layers, though, unlike the one discussed in chapter 1, they have not involved a kinematic constraint. Bovik [12] derives approximate boundary conditions for thin fluid and elastic layers in a differential form, using Taylor expansions as the basis of the approximation procedure. Wickham [111] takes a different approach and reduces the approximate boundary condition to an integral formulation that avoids the need to approximate the boundary conditions pointwise, but imposes instead a condition averaged over the boundary. Though our approach lies somewhat mid-way between the two, we end with a differential form because the boundary conditions are locally reacting. The gas in each pore responds only to the wavefield in its immediate neighborhood.

The final results are presented in the form of the power removed from the reflected wavefield by interference with the diffracted one. To make this calculation we adopt an argument given by Newton [81]. Normalized with respect to the reflected intensity times twice the width of the slit, this gives a measure of the effectiveness of the barrier, with the slit, at reducing sound transmission. This term is a function of the slit width and the properties of the barrier. These contents of chapter 1 have been accepted for publication in **"Contribution to the issue of Wave Motion honouring Gerry Wickham".**

NOISE REDUCTION: The diffraction theory has found its applications in the problem of noise reduction by means of barriers. Highway noise in urban areas is one area of noise pollution that can be rectified. The primary mechanism for traffic noise reduction in existing highways is through the use of barriers. Noise barriers are physical barriers that can reduce the noise transmitted directly from the vehicles in the traffic. They can be constructed from natural materials, such as earth and wooden barriers or from man-made materials such as fiberglass, aluminum, concrete, etc.

To define the noise problem first, one needs to talk about traffic noise. Vehicle emit noise due to different causes. For example, an automobile has engine noise and tire generated noise. While some of these noise sources are pure tone, the overall radiated noise can be treated as a random noise with a known spectrum. Because the dominant noise from an automobile comes from the vicinity of the tires, the overall noise source height is located at the ground, with a typical normalized noise spectrum. Similar arguments are advanced for the noise emitted from light trucks. On the other hand, heavy trucks have a variety of noise sources. Among the most important are exhaust noise, engine noise, tire noise, radiate noise from side panels and flow noise from the truck itself. Traffic noise is a composite of a mixture of automobiles, light and heavy trucks. The total noise level received at a point is the sum of the noise in each frequency band of all the vehicles travelling in all the lanes of a highway, and then added incoherently in space and overall frequency bands. This overall traffic noise level is the basis for evaluating the need for, and effectiveness of a noise barrier. A noise barrier erected between the traffic and the receiver acts to reduce the directly transmitted noise from each vehicle. If the transmission loss of the barrier is high enough then the only noise that can propagate to the receiver is through the mechanism of acoustic diffraction over the barrier's top or side edge(s).

Noise barrier come in a variety of shapes. A noise control highway engineer has to take into consideration the cost of materials and construction, the esthetic appearance of the barrier, its acoustic absorbency, its resistance to environmental conditions, durability, ease of repair and maintenance, as well as its acoustic performance. These factors influence the choice of the barrier's shape, height, length and its location in relation to the pavement. The first attempt at modeling the excess attenuation of a thin wall barrier was made by Maekawa [70, 71, 72] in terms of a Fresnel number. This was followed by Kurze and Anderson [60] and Maekawa et al. [73] for finite length screens and Kurze [59] for accurate modelling of thin barriers. These models used primarily the expression for diffraction by a half plane given by Sommerfeld

[104], Carslaw [14], MacDonald [75] and Redfearn [100] for edges with ideal boundary conditions.

The theory for wedges having ideal boundary conditions or having surfaces with impedance coverages were developed by a few authors including Carslaw [15], Malyuzhinets [74], Williams [110], Jonasson [50], Marson [69] and Hayek [30]. Models for barriers of other shapes such as thick barriers, double walls, cylindrical beams and trapezoids were generated by Jones [49], Fujiwara et al.[26, 27], Pierce [85], Foss [24], May and Osman [68], Lohmann [67], Hayek and Nobile [31] and Hayek [32].

The diffraction theory for half planes with non-ideal boundary conditions including half planes with impedance covered surfaces and for absorbent thin wall barriers were developed by Senior [102], Rawlins [98], Hayek [33, 34, 35], Pierce and Hadden [86, 87], Yuzawa [113], Kendig [55], Nobile et al. [83], Kawai et al. [51] and Kendig and Hayek [54]. In continuation to the study of absorbing half planes, Rawlins [91] obtained the diffraction of a cylindrical sound wave by a bi-impedant half plane. In another paper [99], he discussed the diffraction of a plane wave by a rigid screen with a soft or perfectly absorbing edge. In the case of noise radiated from aero engines and inside wind tunnels it is necessary to discuss acoustic diffraction in the presence of a moving fluid. Taking into account the importance of moving fluid, Rawlins [96, 97] established the solution for the diffraction of a line source of sound by an absorbing half plane. This analysis was further extended to a finite plate by Asghar [5] and to point source excitation by Asghar et al.[6, 8].

However, the diffraction of an acoustic field from absorbing half planes needs further attention. It is observed that the diffraction of a spherical sound wave by a half plane with different face impedances has not been attempted so far. Therefore, it seems to be a worthwhile attempt to consider point source diffraction and determine the effect of different face absorption on the diffracted field. With this in mind, the problem of diffraction of a spherical wave from bi-impedant half plane is investigated in chapter 2. The different absorption in the half plane is introduced through the absorption parameters β_1 and β_2. The modified W. H. procedure has been employed to calculate the diffracted field. It is observed that the problem pertaining to rigid boundary condition can be obtained as a special case of this problem by equating the absorption parameters to zero.

Now, in the situation when noise is shielded by a barrier which intercepts the line of hearing from the noise source to the receiver, the acoustic field in the shadow region of a barrier (when transmission through the barrier is negligible) is due to diffraction at the edge alone. For this reason Butler. [13] suggested that the region in the immediate vicinity of the edge should be lined with absorbent material to reduced the sound level in the shadow region. This technique has potential applications in engine noise shielding by aircraft wings. Keeping this in mind, the diffraction of a spherical acoustic wave by a rigid screen with an absorbent edge in a moving fluid is discussed in chapter 3. A finite region in the vicinity of the edge has an impedance boundary condition; the remaining part of the half plane is rigid. The problem which is solved is a mathematical model for a rigid barrier with an absorbent edge. The analysis presented here is concerned with the more general and practical case where absorbing parameter β is finite. It is also noted that the limiting case [99] when the surface is ideally soft (the pressure fluctuation vanishing on the surface) is given by $|\beta| \to \infty$. These observations have been published in **Canadian Applied Mathematics Quarterly 5(2), 105-129 (1997).**

TRANSIENT PHENOMENA: Another important feature in the theory of acoustic diffraction is the transient nature of the field. In the literature, there has been a strong emphasis on time harmonic wave propagation. however, there has emerged interest in transient wave phenomena due to the present ability to produce short pulses with a broad frequency spectrum. The subject of a wave propagation is usually introduced via time harmonic regime. This practice is based on the assumption that time harmonic wave processes can be described more readily than transient processes, the later being derivable from the former by the additional computation of the Fourier or Laplace transform. However, the basic phenomena of wave propagation that travels from a source to a receiver through an ambient environment are more easily understandable in the transient state, which permits direct signal tracking. The time harmonic field then emerges as the special case of continuously emitted excitation at constant frequency. It may also be remarked that the Fourier or Laplace transform route from the time harmonic regime does not provide the only approach to transient solutions. Sometimes a direct analysis of a transient problem is considerably simpler and even easier than the solution for the time harmonic case. Hence, there arose strong interest in the transient wave phenomena, stimulated by various applications that require the explicit treatment of time dependent effects, see Friedlander[25], Harris [36, 37], Rienstra [101]. deHoop [43], Jones [46] and Asghar [7]. Further, one such application is due to present ability to produce short electromagnetic pulses and consequently requires the development of new time dependent techniques. Also short pulses for higher power, especially in the optical frequency range, are finding application as diagnostic tools for the study of implosion and other wave material applications. Impulsively excited electromagnetic bursts on electronic equipment, devices and installations have become a matter of concern also. Such bursts may range from naturally caused lightening discharge to man made nuclear explosions. The security and reliability of communication channels under the influence of such bursts has motivated extensive investigations by military agencies and by private, public organizations, concerned with communication. In the areas of underwater acoustics and of supersonic aircraft noise, acoustic transients have been studied extensively and much attention has been paid to transient structure-fluid interaction, for example, between the "Sonic boom" generated by supersonic aircraft and a windowpane or between an underwater shock wave and a submersible.

Keeping in view the importance of the transient nature of wave phenomena, the Gaussian pulse response of the absorbing half plane for trailing edge situation in a moving fluid is studied in chapter 4. This investigation is important since the wavefield due to any transient source can be expressed as a linear combination of Gaussian pulses. The wavefield for the impulsive source can be easily obtained by using the representation of the Dirac delta function in terms of Gaussian pulse. In terms of the boundary conditions the trailing edge requires a trailing vortex sheet or wake attached to the absorbing half plane. The time dependence of the field requires a temporal Fourier transform in addition to spatial Fourier transforms. The spatial integrals appearing in the solution for the diffracted field are solved asymptotically in the far field approximation. It is once again found that the field due to a spherical Gaussian pulse with the Kutta-Joukowski condition is substantially in excess of that in its absence when the source is near the edge. We also observe that results for the diffraction of a spherical Gaussian pulse from a rigid barrier can be obtained as a special case of this problem by taking the absorption parameter equal to zero. These observations have been published in **Applied Acoustics 54(4), 323-338 (1998).**

Chapter 1

SCATTERING OF A SPHERICAL ACOUSTIC WAVE BY A SLIT IN AN INFINITE POROUS BARRIER

In this chapter, the problem of scattering of an acoustic wave by a slit in an infinite, plane, porous barrier is studied. The barrier is modeled as a rigid material with narrow pores, normal to the plane of the barrier, that provide sound damping. However, the barrier is thin enough that sound transmission takes place. An approximate boundary condition is derived that models both these effects. The source point is assumed far from the slit so that the incident spherical wave is locally plane. The slit is wide and the barrier thin, both with respect to wavelength. The principal purpose of the barrier is to reduce the reflected and transmitted sound so that we assume that the flow resistance of the pores is large. The diffracted field is calculated using integral transforms, the Wiener-Hopf technique and asymptotic methods. While a formal solution to the complete problem is given, only the diffracted wavefield is studied, and that only in the farfield zone. The diffracted field is the sum of the wavefields produced by the two edges of the slit and an interaction wavefield. The dependence on the barrier parameters of the power removed from the reflected wavefield by the diffraction at the slit is exhibited.

1.1 FORMULATION

We consider the diffraction of an acoustic wave excited by a point source located at (x_0, y_0, z_0) or (r_0, θ_0, z_0) by a slit in the plane $y = 0$ of the width $2a$, $-a \leq x \leq a$. We shall also ask that $0 < \theta_0 < \frac{\pi}{2}$. The geometry of the problem is shown in Fig.1. Throughout, the time harmonic factor $\exp(-i\omega t)$ is understood. We shall work with the velocity potential σ_t, where the particle velocity $\mathbf{u}$ is given by $\mathbf{u} = -\nabla\sigma_t$. The total velocity potential satisfies

$$\left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} + k^2\right]\sigma_t = \delta(x - x_0)\delta(y - y_0)\delta(z - z_0), \tag{1.1}$$

where

$$k = \frac{\omega}{c} = k_1 + ik_2 \tag{1.2}$$

is the free space wave number. The wave number k is assumed to have a small positive imaginary part whenever this is needed to ensure the convergence (regularity) of the Fourier transform integrals defined subsequently. The term k_2 is otherwise set to zero. The boundary conditions satisfied by σ_t on $(-\infty < x \leq -a) \sqcup (a \leq x < \infty)$, $\quad y = 0^{\pm}$ are

$$\pm\frac{\partial}{\partial y}\sigma_t(x, 0^{\pm}, z) + ik\alpha\sigma_t(x, 0^{\pm}, z) + ik\beta\sigma_t(x, 0^{\mp}, z) = 0, \tag{1.3}$$

where $0^{\pm}$ means that the field term is to be evaluated as $y \to 0$ through positive or negative value of y. The parameters α and β will be identified shortly. For the case of line source, Eq. (1.3) takes the form of Rawlins boundary condition [Eq.(10); 95]. The boundary conditions on $-a < x < a$, $\quad y = 0^{\pm}$ are

$$\sigma_t(x, 0^+, z) = \sigma_t(x, 0^-, z), \tag{1.4}$$

$$\frac{\partial}{\partial y}\sigma_t(x, 0^+, z) = \frac{\partial}{\partial y}\sigma_t(x, 0^-, z). \tag{1.5}$$

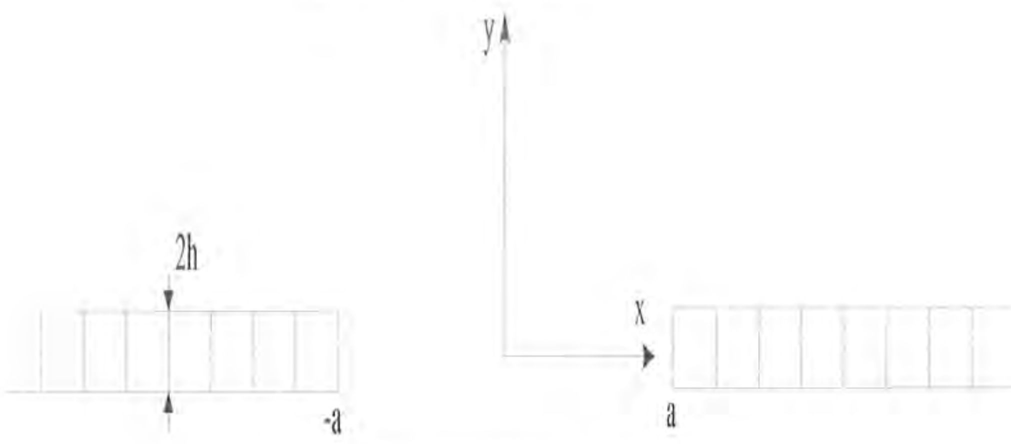

Fig.1: The geometry of the problem

In addition, we insist that σ_t satisfy the edge condition as $x \to -a^+, a^-$,

$$\sigma_t(x,0,z) = O(1), \tag{1.6}$$

$$\frac{\partial}{\partial y}\sigma_t(x,0,z) = O(x^{\frac{-1}{2}}). \tag{1.7}$$

The order $x^{\frac{-1}{2}}$ remains the same as in the case of rigid/soft or absorbing barrier (Rawlins [94-99]). The plus sign indicates a limit taken from the left and the minus sign one taken from the right.

It is useful to split the total field σ_t in two ways. To discuss the boundary condition we write

$$\sigma_t = \sigma_i + \sigma_s, \tag{1.8}$$

where σ_i is the incident wave and σ_s represent an outward radiating wavefield. However, to discuss the diffraction problem, it is more useful to write σ_t as

$$\begin{aligned} \sigma_t &= \sigma_i + \sigma_r + \sigma, \quad y \geq 0^+, \\ &= \sigma \qquad\qquad\quad , y \leq 0^-, \end{aligned} \tag{1.9}$$

where σ_r is the wave reflected from a perfectly rigid barrier and σ is the scattered wavefield. It is comprised of the diffracted wave, a correction to the reflected wave and a transmitted wave.

1.2 THE BOUNDARY CONDITION

Fig. 2. shows a porous barrier of thickness $2h$ extending to infinity in the $\pm x$ directions. No slit is present. The space is divided into three regions. The region V^+ and V^- are those above and below the barrier and are occupied by a gas having density ρ and sound speed c. The region V_0 is that occupied by the porous barrier. Following a formulation that is identical to that given in I.B of Harris et al. [39], the velocity potential σ_s scattered from this barrier is represented by

$$\sigma_s(\mathbf{x}) = -\int_s \left[\sigma_g(\tilde{x},\mathbf{x})\nabla\sigma_t(\tilde{x}) - \sigma_t(\tilde{x})\widetilde{\nabla}\sigma_g(\tilde{x},\mathbf{x})\right].\hat{n}\ dS(\tilde{x}), \mathbf{x} \in V^+ \sqcup V^-, \tag{1.10}$$

where σ_t is the total potential given by Eq. (1.8) and σ_g is the three-dimensional, free-space Greens' function. The surface S is comprised of the upper and lower surfaces of the barrier, $\hat{n}$ is a unit normal vector pointing out of the barrier and $\widetilde{\nabla}$ indicates that the gradient is taken with respect to the argument $\tilde{x}$. The vector $\mathbf{x}$ indicates the observation point and lies outside the barrier, while the vector $\tilde{x}$ indicates the source point and lies on the surface S.

Asking that the unit normal $\hat{n}$ now point only in the positive y direction, we define the discontinuities

$$[\nabla\sigma_t.\hat{n}] = \nabla\sigma_t(x,h,z).\hat{n} - \nabla\sigma_t(x,-h,z).\hat{n}, \tag{1.11}$$

and

$$[\sigma_t] = \sigma_t(x,h,z) - \sigma_t(x,-h,z). \tag{1.12}$$

These are the sources of the scattered sound as can be seen by noting that, provided the discontinuities in Eqs.(1.11)and (1.12) are no large than $O(1)$, then the integral Eq. (1.10) can be approximated to $O(kh)$ by evaluating the Green's terms at $\tilde{y} = 0$. This leaves us with

$$\sigma_s(\mathbf{x}) = -\iint\limits_{S} \left[\sigma_g(\tilde{x}, 0, \tilde{z}, x)\left[\nabla\sigma_t.\hat{n}\right] - [\sigma_t]\,\widetilde{\nabla}\sigma_g.\hat{n}\right] d\tilde{x}\, d\tilde{z} + O(kh). \quad (1.13)$$

where $\mathbf{x}$ lies outside the volume enclosed by S. Note that we have approximated a function that we know and whose length scale is set by the wavenumber k and not by the wavenumber of the porous material. It is therefore the discontinuities, Eqs. (1.11) and (1.12), that Eq. (1.3) must mimic.

Returning to the Rawlins boundary condition, we note that if we take the limit $kh \to 0^{\pm}$ of the following

$$\left[\nabla\sigma_t.\hat{n}\right] = -ik\,(\alpha + \beta)\left[\sigma_t(x, h, z) + \sigma_t(x, -h, z)\right], \quad (1.14)$$

and

$$[\sigma_t] = -\left(ik\,(\alpha - \beta)\right)^{-1}\left[\nabla\sigma_t(x, h, z).\hat{n} - \nabla\sigma_t(x, -h, z).\hat{n}\right], \quad (1.15)$$

then, by adding and subtracting Eqs. (1.14) and (1.15), we recover Eq. (1.3). Accordingly, by estimating the discontinuities, Eqs. (1.11) and (1.12), we may use Eqs. (1.14) and (1.15) to determine the parameters α and β.

Adapting a simple theory of porous materials given in [77], the equations governing the acoustical behavior of the porous barrier are

$$i\omega\kappa_p\Omega_p = \frac{du_2}{dy}, \tag{1.16}$$

$$\frac{dp}{dy} = i\omega\rho_p\left[1 + \left(\frac{i\Phi}{\omega\rho_p}\right)\right]u_2. \tag{1.17}$$

The particle velocity in the barrier u_2 is restricted to be in the normal direction only, the particle velocity in the tangential direction must be zero, and the acoustic pressure in the barrier is p. The parameters of the model are κ_p the compressibility of the gas in the pores, Ω the porosity or fraction of the volume occupied by the pores and hence by the gas, ρ_p the effective density of the gas in the pores and Φ the flow resistance. This last parameter determines the effective sound absorbing properties of the barrier. At the boundaries of the barrier the pressure and normal components of the particle velocity are continuous. No condition is placed on the tangential particle velocity components immediately outside the barrier. Integrating Eqs. (1.16) and (1.17), noting that p and u_2 are the total fields in the barrier and using the boundary conditions at the barrier walls gives

$$[\nabla\sigma_t.\hat{n}] = -\omega^2\rho\kappa_p\Omega\left(-i\omega\rho\right)^{-1}\int_{-h}^{h} p\,dy, \tag{1.18}$$

and

$$[\sigma_t] = i\omega\rho_p\left[1 + \left(\frac{i\Phi}{\omega\rho_p}\right)\right]\left(-i\omega\rho\right)^{-1}\int_{-h}^{h} u_2\,dy. \tag{1.19}$$

The barrier is both thin and absorbing. We wish to capture both these features. Defining $\kappa_e = \kappa_p\Omega$, $\rho_e = \rho_p\left[1 + \left(\frac{i\Phi}{\omega\rho_p}\right)\right]$ and $c_e = (\rho_e\kappa_e)^{\frac{-1}{2}}$, the effective wavenumber in the barrier is $k_e = \frac{\omega}{c_e}$. We assume that p and u_2 vary slowly enough through the barrier to be approximated accurately by the first two terms of a Taylor series in the scaled thickness variable $k_e h(y/h)$. This assumes that the flow resistance is not so strong as to cause the wavefield in the barrier to very rapidly decay. We are therefore able to relate Eqs. (1.14) and (1.15) to the porous barrier model by noting that

$$\frac{1}{(-i\omega\rho)2h}\int_{-h}^{h} p\,dy = \frac{[\sigma_t(x,h,z)+\sigma_t(x,-h,z)]}{2} + O(k_e h)^2. \qquad (1.20)$$

and

$$-\frac{1}{2h}\int_{-h}^{h} u_2\,dy = \frac{[\nabla\sigma_t(x,h,z).\widehat{n}+\nabla\sigma_t(x,-h,z).\widehat{n}]}{2} + O(k_e h)^2. \qquad (1.21)$$

Assuming that $(k_e h)^2$ is small, we find that

$$\alpha+\beta = -i\rho c^2\kappa_p\Omega kh, \qquad (1.22)$$

and

$$\alpha-\beta = \frac{i\rho}{kh}\left(\rho_p\left[1+\left(\frac{i\Phi}{\omega\rho_p}\right)\right]\right)^{-1}. \qquad (1.23)$$

Note that only $(\alpha-\beta)$ contains the flow resistance term. To estimate the sizes of these terms assume that κ_p and ρ_p are equal to the compressibility κ and density ρ of the surrounding gas, so that $\kappa_p\rho_p c^2 = 1$. This is not quite the case because ρ_p can be larger than ρ, and κ_p can be the isothermal compressibility rather than the adiabatic compressibility κ. Nevertheless, if the barrier is to absorb the incident sound then $\frac{\Phi}{\rho\omega}$ must be moderately large. Morse and Ingard [77] suggest a value as high as 10 at 1000 Hz. We are therefore left with the following estimates.

$$\alpha + \beta = -i\Omega kh, \tag{1.24}$$

and

$$(\alpha - \beta)^{-1} = \frac{kh\Phi}{\rho\omega}. \tag{1.25}$$

For kh small $(\alpha + \beta)$ is small because $\Omega < 1$, but $(\alpha - \beta)^{-1}$ need not be because, for effective sound absorption, $\frac{\Phi}{\rho\omega} > 1$. Moreover, $|k_e h| = kh\sqrt{\frac{\Phi\Omega}{\rho\omega}}$. Examining the approximation in Eqs. (1.20) and (1.21), we note that, provided $kh\Phi/\rho\omega = O(1)$ or equivalently $h\Phi/\rho c = O(1)$, then the error leading to the approximate equivalence between Eqs. (1.14) and (1.15), and Eqs. (1.20) and (1.21) is $O(kh)$ throughout. As we continue with the calculation we shall find that some terms are proportional to $(\alpha + \beta)$ and can be dropped, while other terms contain $(\alpha - \beta)$or $(\alpha - \beta)^{-1}$ and cannot. We could just set $(\alpha + \beta)$ to zero at this point, but, by carrying it through the calculation the different roles of the barrier thickness and absorption become clearer. Moreover, though we are assuming that $(\alpha - \beta)$ is not small, it can be set to zero to recover the case of a rigid barrier.

The reflection R and transmission T coefficients for the velocity potential using the boundary condition (1.3) are given in Rawlins [95, *Eq.*(38)]. Neglecting the $(\alpha + \beta)$, they are

$$R(\theta) = \sin\theta\,[\sin\theta + (\alpha + \beta)]^{-1}, \tag{1.26}$$

$$T(\theta) = -2\beta\,[\sin\theta + (\alpha - \beta)]^{-1}. \tag{1.27}$$

We note that when $\alpha = -\beta$ and thus $-2\beta \approx (\alpha - \beta)$, equations (1.26) and (1.27) are identical to equation (38) of Rawlins [95]. The parameter

β clearly controls transmission. For normal incidence, using the previous estimates $\Gamma(\frac{\pi}{2})$ is approximately $-\left(\frac{\rho c}{2h\Phi}\right)$ so that the barrier allows weak transmission of sound. The coefficients have no poles on the real θ axis $(0 < \theta < \pi)$.

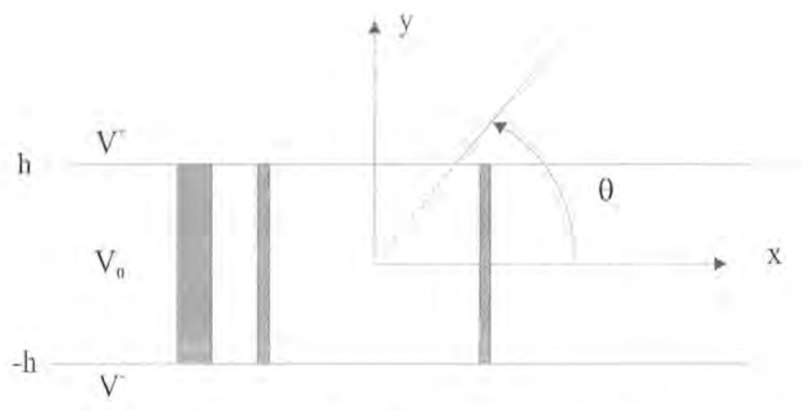

Fig. 2: The geometry of the barrier. The hatched regions are intended to suggest the presence of pores in an otherwise rigid material. In practice the pores are unlikely to be so regular.

1.3 THE WIENER-HOPF PROBLEM

We now proceed with the calculation of the diffraction by the slit. The Fourier transform over z and its inverse are defined, respectively, as

$$\psi_t(x,y,\mu) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \sigma_t(x,y,z)\, e^{-i\mu z} dz, \tag{1.28}$$

and

$$\sigma_t(x,y,z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \psi_t(x,y,\mu)\, e^{i\mu z} d\mu \tag{1.29}$$

with identical definitions for the other potentials σ_i, σ_r and σ. The problem now becomes

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\gamma^2\right)^{\psi_i(x,y,\mu)}_{\psi_r(x,y,\mu)} = \frac{e^{-i\mu z_0}}{2\pi}\delta(x-x_0)\delta(y\mp y_0), \tag{1.30}$$

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\gamma^2\right)\psi(x,y,\mu)=0. \tag{1.31}$$

where

$$\gamma^2=\sqrt{k^2-\mu^2}, \operatorname{Im}\gamma>0. \tag{1.32}$$

The boundary conditions at $y=0$ are

$$\frac{\partial}{\partial y}(\psi_i+\psi_r)=0, \tag{1.33}$$

for $(-\infty<x<\infty)$

$$\left(\frac{\partial}{\partial y}+ik\alpha\right)\left[\psi_i(x,0,\mu)+\psi_r(x,0,\mu)+\psi(x,0^+,\mu)\right]+ik\beta\psi(x,0^-,\mu)=0, \tag{1.34}$$

$$\left(\frac{\partial}{\partial y}-ik\alpha\right)\psi(x,0^-,\mu)-ik\beta\left[\psi_i(x,0,\mu)+\psi_r(x,0,\mu)+\psi(x,0^+,\mu)\right]=0, \tag{1.35}$$

for $(-\infty<x\leq -a)\sqcup(a\leq x<\infty)$ and

$$\psi(x,0^+,\mu)-\psi(x,0^-,\mu)=-\left[\psi_i(x,0,\mu)+\psi_r(x,0,\mu)\right], \tag{1.36}$$

$$\frac{\partial}{\partial y}\psi(x,0^+,\mu) - \frac{\partial}{\partial y}\psi(x,0^-,\mu) = -\frac{\partial}{\partial y}\left[\psi_i(x,0,\mu) + \psi_r(x,0,\mu)\right], \quad (1.37)$$

for $(-a < x < a)$.

The solution to Eq. (1.30), giving the incident wave ψ_i, is

$$\psi_i(x,y,\mu) = -\frac{e^{-i\mu z_0}}{\sqrt{2\pi}4i} H_0^{(1)}\left[\gamma\sqrt{(x-x_0)^2 + (y-y_0)^2}\right]. \quad (1.38)$$

The reflected wave ψ_r has the same form with the source point replaced by its reflected image source $(x_0, -y_0, z_0)$ and $H_0^{(1)}(.)$ is the cylindrical Hankel function of order zero and of first kind. As we are interested in a situation where the source point is far from the slit. Accordingly, we may use the asymptotic approximation to the Hankel function, assuming that $|\gamma r_0| \to \infty$, to obtain

$$\psi_i = b(\mu)\, e^{-i\gamma(x\cos\theta_0 + y\sin\theta_0)}, \quad (1.39)$$

$$\psi_r = b(\mu)\, e^{-i\gamma(x\cos\theta_0 - y\sin\theta_0)}, \quad (1.40)$$

where $x_0 = r_0\cos\theta_0$ and $y_0 = r_0\sin\theta_0$ $\left(0 < \theta_0 < \frac{\pi}{2}\right)$ and $x = r\cos\theta$ and $|y| = r\sin\theta$. The possibility that γ is near 0 can always be avoided. The term $b(\mu)$ is given by

$$b(\mu) = -\frac{e^{-i\mu z_0}}{\sqrt{2\pi}4i}\sqrt{\frac{2}{\pi\gamma r_0}}\, e^{i(\gamma r_0 - \frac{\pi}{4})}. \quad (1.41)$$

Note that by asking that $\operatorname{Im}\gamma > 0$, we have succeeded only in causing the incident and reflected disturbance to be damped in the negative x direction.

We next define the Fourier transform pair

$$\overline{\psi}(\nu, y, \mu) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \psi(x, y, \mu)\, e^{i\nu x} dx, \tag{1.42}$$

and

$$\psi(x, y, \mu) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \overline{\psi}(\nu, y, \mu)\, e^{-i\nu x} d\nu \tag{1.43}$$

with identical definitions for the other wavefield terms. We split $\overline{\psi}(\nu, y, \mu)$ as

$$\overline{\psi}(\nu, y, \mu) = \overline{\psi}_+(\nu, y, \mu)\, e^{i\nu a} + \overline{\psi}_-(\nu, y, \mu)\, e^{-i\nu a} + \overline{\psi}_1(\nu, y, \mu). \tag{1.44}$$

where

$$\overline{\psi}_\pm(\nu, y, \mu) = \frac{1}{\sqrt{2\pi}} \int_{a,-\infty}^{\infty,-a} \psi(x, y, \mu)\, e^{i\nu(x \mp a)} dx, \tag{1.45}$$

and

$$\overline{\psi}_1(\nu, y, \mu) = \frac{1}{\sqrt{2\pi}} \int_{-a}^{a} \psi(x, y, \mu)\, e^{i\nu x} dx. \tag{1.46}$$

In Eq. (1.45) the first (reading from left to right) set of limits accompany the plus sign and the second minus sign. In calculating the partial transforms, Eq. (1.45), of ψ_i and ψ_r, Eqs. (1.39) and (1.40) care needs to be taken as $|x| \to \infty$. Accordingly, we shall assume that ψ_i and ψ_r are multiplied by $H(x-a)e^{-\varepsilon(x-a)}$ for $x > 0$ and by $H(x+a)e^{\varepsilon(x+a)}$ for $x < 0$. Later we shall let $\varepsilon \to 0$. This device allows us to sort out the regions of analyticity. Because ψ_r is that for a rigid rather than a porous barrier, the wavefield ψ will contain a transmitted term that behaves as $e^{-i\gamma(x\cos\theta_0)-\varepsilon(x-a)}$ for $x > a, e^{-i\gamma(x\cos\theta_0)}$ for $-a < x < a$, and

$e^{-i\gamma(x\cos\theta_0)+\varepsilon(x+a)}$ for $x < -a$. This fact will dominate the regions of analyticity. The term $\overline{\psi}_+(\nu, y, \mu)$ is regular for $\mathrm{Im}(\nu) > [\mathrm{Im}(\gamma\cos\theta_0) - i\varepsilon]$ and $\overline{\psi}_-(\nu, y, \mu)$ for $\mathrm{Im}(\nu) < [\mathrm{Im}(\gamma\cos\theta_0) + i\varepsilon]$. The function $\overline{\psi}_1(\nu, y, \mu)$ is an integral function. We shall end with two Wiener-Hopf problems one with the common region $[\mathrm{Im}(\gamma\cos\theta_0) - i\varepsilon] < \mathrm{Im}(\nu) < [\mathrm{Im}(\gamma\cos\theta_0) + i\varepsilon]$.

Taking the Fourier transform over x of Eq. (1.31) and solving the resulting differential equation, so that the radiation condition is satisfied, gives

$$\begin{aligned} \overline{\psi}(\nu, y, \mu) &= A_1(\nu)e^{-\tilde{\gamma}y}, y \geq 0^+, \\ &= A_2(\nu)e^{\tilde{\gamma}y}, y \leq 0^-, \end{aligned} \tag{1.47}$$

where

$$\tilde{\gamma} = \sqrt{\nu^2 - \gamma^2}, \quad \mathrm{Re}\,\tilde{\gamma} > 0. \tag{1.48}$$

Transforming boundary conditions (1.34) to (1.37), and using Eqs. (1.39) to (1.41) we get

$$\begin{aligned} &\frac{d}{dy}\overline{\psi}_-(\nu, 0^\pm, \mu) \pm ik \begin{bmatrix} \alpha\overline{\psi}_-(\nu, 0^\pm, \mu) \\ +\beta\overline{\psi}_-(\nu, 0^\mp, \mu) \end{bmatrix} \\ &\pm \frac{2k^\alpha_\beta\ e^{i\gamma\cos\theta_0 a}b(\mu)}{\sqrt{2\pi}\,[\nu - (\gamma\cos\theta_0 + i\varepsilon)]} = 0, \end{aligned} \tag{1.49}$$

$$\begin{aligned} &\frac{d}{dy}\overline{\psi}_+(\nu, 0^\pm, \mu) \pm ik \begin{bmatrix} \alpha\overline{\psi}_+(\nu, 0^\pm, \mu) \\ +\beta\overline{\psi}_-(\nu, 0^\mp, \mu) \end{bmatrix} \\ &\mp \frac{2k^\alpha_\beta\ e^{-i\gamma\cos\theta_0 a}b(\mu)}{\sqrt{2\pi}\,[\nu - (\gamma\cos\theta_0 - i\varepsilon)]} = 0, \end{aligned} \tag{1.50}$$

$$\overline{\psi}_1(\nu, 0^+, \mu) - \overline{\psi}_1(\nu, 0^-, \mu) = 2iG(\nu)b(\mu), \tag{1.51}$$

$$\frac{d}{dy}\overline{\psi}_1(\nu, 0^+, \mu) = \frac{d}{dy}\overline{\psi}_1(\nu, 0^-, \mu). \tag{1.52}$$

In Eqs. (1.49) and (1.50) the term α goes with the upper sign and β with the lower sign. The term $G(\nu)$ is given by

$$G(\nu) = \frac{1}{\sqrt{2\pi}\,[\nu - \gamma\cos\theta_0]}\left\{e^{i(\nu-\gamma\cos\theta_0)a} - e^{-i(\nu-\gamma\cos\theta_0)a}\right\}. \tag{1.53}$$

From Eq. (1.47) and using the boundary conditions (1.49) to (1.52), we eliminate $\frac{d}{dy}\overline{\psi}_+$ and $\frac{d}{dy}\overline{\psi}_-$ to get

$$\begin{aligned}
& e^{i\nu a}\overline{\eta}_+(\nu, 0, \mu)\left[\tilde{\gamma} - ik(\alpha - \beta)\right] \\
& + e^{-i\nu a}\overline{\eta}_-(\nu, 0, \mu)\left[\tilde{\gamma} - ik(\alpha - \beta)\right] \\
& + \frac{d}{dy}\overline{\psi}_1(\nu, 0, \mu) + iG(\nu)\tilde{\gamma}b(\mu) \\
& + \frac{kb(\mu)(\alpha - \beta)\; e^{i(\nu-\gamma\cos\theta_0)a}}{\sqrt{2\pi}\,[\nu - (\gamma\cos\theta_0 - i\varepsilon)]} \\
& - \frac{kb(\mu)(\alpha - \beta)\; e^{-i(\nu-\gamma\cos\theta_0)a}}{\sqrt{2\pi}\,[\nu - (\gamma\cos\theta_0 + i\varepsilon)]} \\
= \;& 0,
\end{aligned} \tag{1.54}$$

where

$$2\overline{\eta}_\pm(\nu, 0, \mu) = \overline{\psi}_\pm(\nu, 0^+, \mu) - \overline{\psi}_\pm(\nu, 0^-, \mu). \tag{1.55}$$

Equation (1.54) is the Wiener-Hopf functional equation.

1.4 THE SOLUTION TO THE WIENER – HOPF PROBLEM

To solve Eq. (1.54), we make the following factorizations:

$$\tilde{\gamma} = K_+(\nu)K_-(\nu) = \left[e^{\frac{-i\pi}{4}}(\nu+\gamma)^{\frac{1}{2}}\right]\left[e^{\frac{-i\pi}{4}}(\gamma-\nu)^{\frac{1}{2}}\right], \tag{1.56}$$

and

$$\left[1 - \frac{ik(\alpha-\beta)}{\tilde{\gamma}}\right] = L_+(\nu)L_-(\nu), \tag{1.57}$$

where $L_+(\nu)$ and $K_+(\nu)$ are regular for $\mathrm{Im}(\nu) > -\mathrm{Im}\,\gamma$ and $L_-(\nu)$ and $K_-(\nu)$ are regular for $\mathrm{Im}(\nu) < \mathrm{Im}\,\gamma$. Rawlins gives the exact factorization of Eq. (1.57) in both [95] and [96]. The $L_\pm(0)$are given by

$$L_+(0) = L_-(0) = \left[1 + \frac{k(\alpha-\beta)}{\gamma}\right]^{\frac{1}{2}}$$

and

$$L_+(\gamma) = L_-(-\gamma) = \sqrt{\frac{1+\cos\chi}{2}}\exp\left[-\frac{1}{2\pi}\int_{-\chi}^{\chi}\frac{u}{\sin u}du\right], \tag{1.59}$$

where $\sin\chi = \frac{-(\alpha-\beta)k}{\gamma}$.

Using Eqs. (1.56) and (1.57), we rewrite Eq.(1.54) as

$$\begin{aligned}
& e^{i\nu a}\overline{\eta}_+(\nu,0,\mu) + e^{-i\nu a}\overline{\eta}_-(\nu,0,\mu) + iG(\nu)b(\mu) \\
& +\frac{d}{dy}\psi_1(\nu,0,\mu)\left[S_+(\nu)S_-(\nu)\right]^{-1} \\
& -\frac{kb(\mu)(\alpha-\beta)\,e^{i(\nu-\gamma\cos\theta_0)a}}{\sqrt{2\pi}\left[S_+(\nu)S_-(\nu)\right]}\left[\frac{1}{[\nu-\gamma\cos\theta_0]} - \frac{1}{[\nu-(\gamma\cos\theta_0 - i\varepsilon)]}\right] \\
& +\frac{kb(\mu)(\alpha-\beta)\,e^{-i(\nu-\gamma\cos\theta_0)a}}{\sqrt{2\pi}\left[S_+(\nu)S_-(\nu)\right]}\left[\frac{1}{[\nu-\gamma\cos\theta_0]} - \frac{1}{[\nu-(\gamma\cos\theta_0 + i\varepsilon)]}\right] \\
= {} & 0,
\end{aligned} \tag{1.60}$$

where

$$S_{\pm}(\nu) = K_{\pm}(\nu)L_{\pm}(\nu). \tag{1.61}$$

With the help of Eqs. (1.44), (1.47) and (1.49) to (1.52), the unknown functions $A_1(\nu)$ and $A_2(\nu)$ are given by

$$\begin{aligned}
\pm 2A_{1,2}(\nu) &= e^{i\nu a}\left[\overline{\psi}_+(\nu,0^+,\mu) - \overline{\psi}_+(\nu,0^-,\mu)\right] \\
&\quad + e^{-i\nu a}\left[\overline{\psi}_-(\nu,0^+,\mu) - \overline{\psi}_-(\nu,0^-,\mu)\right] + 2iG(\nu)b(\mu) \\
&\quad \pm \frac{ik(\alpha-\beta)}{\widetilde{\gamma}}\left\{e^{i\nu a}\left[\overline{\psi}_+(\nu,0^+,\mu) + \overline{\psi}_+(\nu,0^-,\mu)\right]\right. \\
&\qquad + e^{-i\nu a}\left[\overline{\psi}_-(\nu,0^+,\mu) + \overline{\psi}_-(\nu,0^-,\mu)\right] \\
&\qquad + \frac{2ib(\mu)\, e^{i(\nu-\gamma\cos\theta_0)a}}{\sqrt{2\pi}\left[\nu - (\gamma\cos\theta_0 - i\varepsilon)\right]} \\
&\qquad \left. - \frac{2ib(\mu)\, e^{-i(\nu-\gamma\cos\theta_0)a}}{\sqrt{2\pi}\left[\nu - (\gamma\cos\theta_0 + i\varepsilon)\right]}\right\}.
\end{aligned} \tag{1.62}$$

The + sign is used with the subscript 1 and the − sign with the subscript 2. As we indicated in our discussion of the boundary conditions, terms multiplied by $(\alpha+\beta)$ are $O(kh)$ (after the inverse transforms are taken) and are dropped, but terms containing $(\alpha-\beta)$, that appear in $L_{\pm}(\nu)$, need not be small and are retained. Thus, using this approximation, Eq. (1.62) becomes

$$A_1(\nu) = -A_2(\nu) = e^{i\nu a}\overline{\eta}_+(\nu,0,\mu) + e^{-i\nu a}\overline{\eta}_-(\nu,0,\mu) + iG(\nu)b(\mu). \tag{1.63}$$

By multiplying Eq. (1.60) by $S_+(\nu)e^{-i\nu a}$ and using the general decomposition theorem, we obtain

$$
\begin{aligned}
&S_+(\nu)\overline{\eta}_+(\nu,0,\mu)+\frac{ie^{-i\gamma\cos\theta_0 a}b(\mu)}{\sqrt{2\pi}\,[\nu-\gamma\cos\theta_0]}\left[S_+(\nu)-S_+(\gamma\cos\theta_0)\right] \qquad (1.64)\\
&+U_+(\nu)+V_+(\nu)+\frac{kb(\mu)(\alpha-\beta)\;e^{-i\gamma\cos\theta_0 a}}{\sqrt{2\pi}S_-(\gamma\cos\theta_0-i\varepsilon)\left[\nu-(\gamma\cos\theta_0-i\varepsilon)\right]}\\
=\;&\frac{-ie^{-i\gamma\cos\theta_0 a}S_+(\gamma\cos\theta_0)b(\mu)}{\sqrt{2\pi}\,[\nu-\gamma\cos\theta_0]}-U_-(\nu)-V_-(\nu)-\frac{e^{-i\nu a}}{S_-(\nu)}\frac{d}{dy}\overline{\psi}_1(\nu,0,\mu)\\
&+\frac{ie^{-i(2\nu-\gamma\cos\theta_0)a}S_+(\gamma\cos\theta_0)b(\mu)}{\sqrt{2\pi}\,[\nu-\gamma\cos\theta_0]}+\frac{kb(\mu)(\alpha-\beta)\;e^{-i\gamma\cos\theta_0 a}}{\sqrt{2\pi}S_-(\nu)\left[\nu-\gamma\cos\theta_0\right]}\\
&-\frac{kb(\mu)(\alpha-\beta)\;e^{-i\gamma\cos\theta_0 a}}{\sqrt{2\pi}\left[\nu-(\gamma\cos\theta_0-i\varepsilon)\right]}\left[\frac{1}{S_-(\nu)}-\frac{1}{S_-(\gamma\cos\theta_0-i\varepsilon)}\right]\\
&-\frac{kb(\mu)(\alpha-\beta)\;e^{-i(2\nu-\gamma\cos\theta_0)a}}{\sqrt{2\pi}S_-(\nu)}\left[\frac{1}{[\nu-\gamma\cos\theta_0]}-\frac{1}{[\nu-(\gamma\cos\theta_0-i\varepsilon)]}\right].
\end{aligned}
$$

The functions $U_\pm(\nu)$ and $V_\pm(\nu)$ are the decomposition of

$$
S_+(\nu)\overline{\eta}_-(\nu,0,\mu)e^{-2i\nu a}=U(\nu). \qquad (1.65)
$$

and

$$
\frac{-ie^{-i(2\nu-\gamma\cos\theta_0)a}b(\mu)}{\sqrt{2\pi}\,[\nu-\gamma\cos\theta_0]}\left[S_+(\nu)-S_+(\gamma\cos\theta_0)\right]=V(\nu). \qquad (1.66)
$$

Similarly, multiplying Eq. (1.60) by $S_-(\nu)e^{i\nu a}$, we obtain

$$
\begin{aligned}
&S_-(\nu)\overline{\eta}_-(\nu,0,\mu)-\frac{ie^{i\gamma\cos\theta_0 a}b(\mu)}{\sqrt{2\pi}\,[\nu-\gamma\cos\theta_0]}\left[S_-(\nu)-S_-(\gamma\cos\theta_0)\right] \qquad (1.67)\\
&+P_-(\nu)-Q_-(\nu)-\frac{kb(\mu)(\alpha-\beta)\;e^{i\gamma\cos\theta_0 a}}{\sqrt{2\pi}S_+(\gamma\cos\theta_0+i\varepsilon)\left[\nu-(\gamma\cos\theta_0+i\varepsilon)\right]}\\
=\;&\frac{ie^{i\gamma\cos\theta_0 a}S_-(\gamma\cos\theta_0)b(\mu)}{\sqrt{2\pi}\,[\nu-\gamma\cos\theta_0]}-P_+(\nu)+Q_+(\nu)-\frac{e^{i\nu a}}{S_+(\nu)}\frac{d}{dy}\overline{\psi}_1(\nu,0,\mu)\\
&-\frac{ie^{i(2\nu-\gamma\cos\theta_0)a}S_-(\gamma\cos\theta_0)b(\mu)}{\sqrt{2\pi}\,[\nu-\gamma\cos\theta_0]}-\frac{kb(\mu)(\alpha-\beta)\;e^{i\gamma\cos\theta_0 a}}{\sqrt{2\pi}S_+(\nu)\left[\nu-\gamma\cos\theta_0\right]}
\end{aligned}
$$

$$+\frac{kb(\mu)(\alpha-\beta)\ e^{i\gamma\cos\theta_0 a}}{\sqrt{2\pi}\left[\nu-(\gamma\cos\theta_0-i\varepsilon)\right]}\left[\frac{1}{S_+(\nu)}-\frac{1}{S_+(\gamma\cos\theta_0+i\varepsilon)}\right]$$
$$+\frac{kb(\mu)(\alpha-\beta)\ e^{i(2\nu-\gamma\cos\theta_0)a}}{\sqrt{2\pi}S_+(\nu)}\left[\frac{1}{[\nu-\gamma\cos\theta_0]}-\frac{1}{[\nu-(\gamma\cos\theta_0-i\varepsilon)]}\right].$$

The functions $P_\pm(\nu)$ and $Q_\pm(\nu)$ are the decomposition of

$$S_-(\nu)\overline{\eta}_+(\nu,0,\mu)e^{2i\nu a}=P(\nu). \tag{1.68}$$

and

$$\frac{-ie^{i(2\nu-\gamma\cos\theta_0)a}b(\mu)}{\sqrt{2\pi}\left[\nu-\gamma\cos\theta_0\right]}\left[S_-(\nu)-S_-(\gamma\cos\theta_0)\right]=Q(\nu). \tag{1.69}$$

Let $\widetilde{f}_1(\nu)$ define a function equal to both sides of Eq. (1.64). The left hand side is regular for $\mathrm{Im}(\nu)>\mathrm{Im}(\gamma\cos\theta_0-i\varepsilon)$ and the right hand side is regular for $\mathrm{Im}(\nu)<\mathrm{Im}(\gamma\cos\theta_0)$. Therefore, by analytic continuation, the definition of $\widetilde{f}_1(\nu)$ can be extended throughout the complex ν plane. The form of $\widetilde{f}_1(\nu)$ is ascertained by examining the asymptotic behavior of the terms in Eq. (1.64) as $|\nu|\to\infty$. We note that $|L_\pm(\nu)|\sim O(1)$ as $|\nu|\to\infty$ and with the help of the edge conditions, we find that $\overline{\eta}_+(\nu,0,\mu)$ and $\overline{\eta}_-(\nu,0,\mu)$ must be atmost of $O(|\nu|^{\frac{-1}{2}})$ as $|\nu|\to\infty$. Using the extended form of Liouville's theorem, it can be seen that $\widetilde{f}_1(\nu)$ can only be a constant equal to zero. Hence, from Eq. (1.64), we obtain

$$\begin{aligned}
&S_+(\nu)\overline{\eta}^*_+(\nu,0,\mu)+\frac{1}{2\pi i}\int_{-\infty+ic}^{\infty+ic}\frac{K_+(\zeta)\overline{\eta}^*_-(\zeta,0,\mu)e^{-2i\zeta a}}{(\zeta-\nu)L_-(\zeta)}d\zeta\\
&-\frac{ie^{-i\gamma\cos\theta_0 a}b(\mu)S_+(\gamma\cos\theta_0)}{\sqrt{2\pi}\left[\nu-\gamma\cos\theta_0\right]}\\
&+\frac{kb(\mu)(\alpha-\beta)\ e^{-i\gamma\cos\theta_0 a}}{\sqrt{2\pi}S_-(\gamma\cos\theta_0-i\varepsilon)\left[\nu-(\gamma\cos\theta_0-i\varepsilon)\right]}\\
=&\ 0,
\end{aligned} \tag{1.70}$$

where

$$\overline{\eta}^*_{\pm}(\nu,0,\mu) = \overline{\eta}_{\pm}(\nu,0,\mu) \pm \frac{ie^{\mp i\gamma\cos\theta_0 a}b(\mu)}{\sqrt{2\pi}\,[\nu-\gamma\cos\theta_0]}. \tag{1.71}$$

Similarly, from the equality of both sides of Eq. (1.67) in the strip $[\mathrm{Im}(\gamma\cos\theta_0)] < \mathrm{Im}(\nu) < [\mathrm{Im}(\gamma\cos\theta_0) + i\varepsilon]$, we have

$$S_-(\nu)\overline{\eta}^*_-(\nu,0,\mu) - \frac{kb(\mu)(\alpha-\beta)\,e^{i\gamma\cos\theta_0 a}}{\sqrt{2\pi}S_+(\gamma\cos\theta_0+i\varepsilon)\,[\nu-(\gamma\cos\theta_0+i\varepsilon)]} \tag{1.72}$$
$$-\frac{1}{2\pi i}\int_{-\infty+id}^{\infty+id}\frac{K_-(\zeta)\overline{\eta}^*_+(\zeta,0,\mu)e^{2i\zeta a}}{(\zeta-\nu)L_+(\zeta)}d\zeta$$
$$= \frac{-ie^{i\gamma\cos\theta_0 a}b(\mu)S_-(\gamma\cos\theta_0)}{\sqrt{2\pi}\,[\nu-\gamma\cos\theta_0]}.$$

1.5 THE DIFFRACTED WAVEFIELD

The unknown functions $\overline{\eta}_+$ and $\overline{\eta}_-$ have been determined by using the procedure discussed by Noble [82]. Since the terms multiplied by $(\alpha+\beta)$ are $O(kh)$ and are dropped, but terms containing $(\alpha-\beta)$ (that appear in $L(\nu)$ and $L_{\pm}(\nu)$ need not be small and are retained. Moreover, the procedure includes asymptotically evaluating the integrals appearing in Eqs.(1.70) and (1.72) for large ζa, where ζ scales with k. That is, we have taken ka to be large. With these approximations the functions as given by Eqs. (1.55) are given by

$$\overline{\eta}_{\pm}(\nu,0,\mu) = -\frac{ib(\mu)}{\sqrt{2\pi}S_{\pm}(\nu)}\left[G_{1,2}(\pm\nu) + C_{1,2}(\gamma)T(\pm\nu)\right]. \tag{1.73}$$

where the subscript 1 accompanies the upper sign and the subscript 2 the lower. The $C_{1,2}$ are

$$C_{1,2}(\gamma) = \left[S_+(\gamma)G_{2,1}(\gamma) + T(\gamma)G_{1,2}(\gamma)\right]\left[S_+^2(\gamma) - T^2(\gamma)\right]^{-1}, \tag{1.74}$$

where

$$G_{1,2}(\nu) = P_{1,2}(\nu)e^{\mp i\gamma\cos\theta_0 a} - R_{1,2}(\nu)e^{\pm i\gamma\cos\theta_0 a}. \tag{1.75}$$

$$\begin{aligned} P_{1,2}(\nu) &= \frac{S_+(\nu) - S_\pm(\gamma\cos\theta_0)}{(\nu \mp \gamma\cos\theta_0)} \\ &\quad - \frac{ik(\alpha-\beta)}{S_\mp(\gamma\cos\theta_0 \mp i\varepsilon)\left[\nu - (\gamma\cos\theta_0 \mp i\varepsilon)\right]}, \end{aligned} \tag{1.76}$$

$$R_{1,2}(\nu) = \frac{E_0\left[W_0\left(-i\left(\gamma \pm \gamma\cos\theta_0\right)2a\right) - W_0\left(-i\left(\gamma+\nu\right)2a\right)\right]}{2\pi i(\nu \mp \gamma\cos\theta_0)}, \tag{1.77}$$

$$T(\nu) = \frac{1}{2\pi i L_+(\gamma)}E_0\sqrt{\gamma}W_0\left(-i\left(\gamma+\nu\right)2a\right), \tag{1.78}$$

and

$$E_0 = 2e^{\frac{i\pi}{2}}\frac{e^{2i\gamma a}}{\sqrt{2\gamma a}}. \tag{1.79}$$

The definition of $W_0(\tilde{z})$ is

$$W_0(-i\tilde{y}) = \sqrt{\pi}\left\{1 + \sqrt{\pi}e^{-i\tilde{y}}\sqrt{-i\tilde{y}}\,\mathrm{erf}\,c\left[\sqrt{-i\tilde{y}}\right]\right\}, \tag{1.80}$$

where $\tilde{y}$ is real and positive, and $\mathrm{erf}\,c(\tilde{z})$ is the complementary error function. It is easily related to the Fresnel integral.

1.6 FAR FIELD ASYMPTOTIC APPROXIMATIONS TO THE DIFFRACTED FIELD

Substitutions of Eqs. (1.73) in Eq. (1.63) yields

$$\begin{aligned} A_{1,2}(\nu) &= -sgn(y)\frac{ib(\mu)}{\sqrt{2\pi}}\left\{\frac{e^{i\nu a}}{S_+(\nu)}\left[G_1(\nu)+C_1(\gamma)T(\nu)\right]\right. \\ &\quad \left.+\frac{e^{-i\nu a}}{S_-(\nu)}\left[G_2(-\nu)+C_2(\gamma)T(-\nu)\right]\right\} \\ &\quad +isgn(y)G(\nu)b(\mu), \end{aligned} \tag{1.81}$$

where the first subscript corresponds to $y > 0$ and the second to $y < 0$. and therefore the wavefield $\psi(x,y,\mu)$. We divide ψ as $\psi = \psi_1(x,y,\mu) + \psi_2(x,y,\mu)$. Each part is given by

$$\begin{aligned} \psi_1(x,y,\mu) &= sgn(y)\frac{ib(\mu)}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-i\nu x-\tilde{\gamma}|y|}d\nu \\ &\quad \left\{\frac{S_+(\gamma\cos\theta_0)e^{i(\nu-\gamma\cos\theta_0)a}}{S_+(\nu)(\nu-\gamma\cos\theta_0)} - \frac{S_-(\gamma\cos\theta_0)e^{-i(\nu-\gamma\cos\theta_0)a}}{S_-(\nu)(\nu-\gamma\cos\theta_0)}\right. \\ &\quad -\frac{ik(\alpha-\beta)\, e^{-i(\nu-\gamma\cos\theta_0)a}}{S_-(\nu)S_+(\gamma\cos\theta_0+i\varepsilon)\left[\nu-(\gamma\cos\theta_0+i\varepsilon)\right]} \\ &\quad \left.+\frac{ik(\alpha-\beta)\, e^{i(\nu-\gamma\cos\theta_0)a}}{S_+(\nu)S_-(\gamma\cos\theta_0-i\varepsilon)\left[\nu-(\gamma\cos\theta_0-i\varepsilon)\right]}\right\}, \end{aligned} \tag{1.82}$$

and

$$\begin{aligned} \psi_2(x,y,\mu) &= sgn(y)\frac{ib(\mu)}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-i\nu x-\tilde{\gamma}|y|}d\nu \\ &\quad \left\{\left[R_1(\nu)e^{i\gamma\cos\theta_0 a} - C_1(\gamma)T(\nu)\right]\frac{e^{i\nu a}}{S_+(\nu)}\right. \\ &\quad \left.+\left[R_2(-\nu)e^{-i\gamma\cos\theta_0 a} - C_2(\gamma)T(-\nu)\right]\frac{e^{-i\nu a}}{S_-(\nu)}\right\} \end{aligned} \tag{1.83}$$

The first term $\psi_1(x, y, \mu)$ represents the field diffracted by the edges at $x = \pm a$, plus the geometrical wavefield. Note that there is one pole above the contour and a second below it. These terms are the transmitted wavefield. Once these pole contributions are captured we can let $\varepsilon \rightarrow 0$. The second term $\psi_2(x, y, \mu)$ gives the interaction of one edge with the other.

The integrals appearing in Eqs. (1.82) and (1.83) can be evaluated asymptotically by using the method of steepest descents. Harris [38] shows that, beyond the Fresnel distance, $k(2a)^2/2\pi$, the exponential phase terms in the braces need not be considered and only the exponential with x and $|y|$ needs to be considered in making the steepest descents calculation. In other words we evaluate the diffracted wavefield at points sufficiently distant from the slit that it has evolved into a cylindrical wavefield (a spheroidal wavefield after the inversion in μ) with a radiation pattern. For that, we put $x = r\cos\theta, y = r\sin\theta$ and deform the contour by the Sommerfeld transformation $\nu = -\gamma\cos(\tau)$. Hence, for large γr, the diffracted wavefields are

$$\psi_1(x, y, \mu) = sgn(y)\frac{i\sin\theta b(\mu)}{\sqrt{2\pi\gamma r}}F_1(-\cos\theta)e^{i(\gamma r - \frac{\pi}{4})}. \tag{1.84}$$

$$\psi_2(x, y, \mu) = sgn(y)\frac{i\gamma\sin\theta b(\mu)}{\sqrt{2\pi\gamma r}}F_2(-\cos\theta)e^{i(\gamma r - \frac{\pi}{4})}. \tag{1.85}$$

The radiation patterns are given by

$$F_1(-\cos\theta) = \Bigg\{ \frac{S_+(\gamma\cos\theta_0)e^{-i\gamma(\cos\theta+\cos\theta_0)a}}{S_+(-\gamma\cos\theta)(\cos\theta+\cos\theta_0)} - \frac{S_-(\gamma\cos\theta_0)e^{i\gamma(\cos\theta+\cos\theta_0)a}}{S_+(\gamma\cos\theta)(\cos\theta+\cos\theta_0)}$$

$$+\frac{ik(\alpha-\beta)e^{-i\gamma(\cos\theta+\cos\theta_0)a}}{S_+(-\gamma\cos\theta)(\cos\theta+\cos\theta_0)S_-(\gamma\cos\theta_0)}$$
$$\left.-\frac{ik(\alpha-\beta)e^{-i\gamma(\cos\theta+\cos\theta_0)a}}{S-(-\gamma\cos\theta)(\cos\theta+\cos\theta_0)S_+(\gamma\cos\theta_0)}\right\}. \quad (1.86)$$

$$F_2(-\cos\theta) = -\left\{\left[R_1(-\gamma\cos\theta)e^{-i\gamma\cos\theta_0 a} - C_1(\gamma)T(-\gamma\cos\theta)\right]\frac{e^{-i\gamma\cos\theta a}}{S_+(-\cos\theta)}\right.$$
$$\left.-\left[R_2(\gamma\cos\theta)e^{i\gamma\cos\theta_0 a} - C_2(\gamma)T(\gamma\cos\theta)\right]\frac{e^{i\gamma\cos\theta a}}{S_-(-\cos\theta)}\right\} \quad (1.87)$$

Next we take the inverse transform over μ using Eqs. (1.84) and (1.85) in Eq. (1.29) i.e.

$$\sigma_{d1}(x,y,z) = sgn(y)\frac{i\sin\theta}{8\pi^2\sqrt{rr_0}}\int_{-\infty}^{\infty}F_1(-\cos\theta)\frac{e^{i[\gamma(r+r_0)+\mu(z-z_0)]}}{\gamma}d\mu, \quad (1.88)$$

$$\sigma_{d2}(x,y,z) = sgn(y)\frac{i\sin\theta}{8\pi^2\sqrt{rr_0}}\int_{-\infty}^{\infty}F_2(-\cos\theta)\frac{e^{i[\gamma(r+r_0)+\mu(z-z_0)]}}{\gamma}d\mu. \quad (1.89)$$

In order to solve the integrals appearing in Eqs. (1.88) and (1.89) we introduce $r+r_0 = r_{12}\sin\theta_{12}$ and $z-z_0 = r_{12}\cos\theta_{12}$, with $0 < \theta_{12} < \pi$. Using the transformation $\mu = k\cos(\tau)$, Eqs.(1.88) and (1.89) are approximated as

$$\sigma_{d1}(x,y,z) = sgn(y)\frac{i\sin\theta F_1(-k\cos\theta\sin\theta_{12})}{4\pi\sqrt{2rr_0r_{12}\pi k}}e^{i(kr_{12}-\frac{\pi}{4})}, \quad (1.90)$$

$$\sigma_{d2}(x,y,z) = sgn(y)\frac{ik\sin\theta\sin\theta_{12}F_2(-k\cos\theta\sin\theta_{12})}{4\pi\sqrt{2rr_0r_{12}\pi k}}e^{i(kr_{12}-\frac{\pi}{4})}. \quad (1.91)$$

where $F_1(-k\cos\theta\sin\theta_{12})$ and $F_2(-k\cos\theta\sin\theta_{12})$ are given by Eqs. (1.86) and (1.87), respectively.

1.7 DISCUSSION

We are concerned with understanding how successfully the barrier reduces the sound transmission despite the presence of the slit. Moreover, we want to understand how the absorption of the barrier makes its presence felt. To do so we imagine that source lies on the positive y−axis far from the slit and that the reflected sound is measured at a point on the y−axis, also far from the slit. We take $b(\mu) = 1$ and $\mu = 0$, so that $\gamma = k$, in Eqs. (1.84) and (1.85). Moreover we set $\theta_0 = \frac{\pi}{2}$. The power both carried by the reflected wavefield and by wavefield diffracted from the slit is then calculated in the far field. The term resulting from their interference is then extracted. This term is the power removed from the reflected wavefield by that scattered by and transmitted through the slit, and by that absorbed by the barrier. It is then normalized by dividing by the reflected intensity times twice the width of the slit. This quantity is given by

$$\frac{\Gamma(ka, \alpha - \beta)}{4a} = G = \frac{1}{2ka} \operatorname{Im} \left\{ [F_1(0) + kF_2(0)] \left[1 + (\alpha - \beta)\right] \right\}. \qquad (1.92)$$

The term $F_1(0)$ is given by

$$F_1(0) = \frac{2ika}{[1 + (\alpha - \beta)]} - \left\{ 1 + \left[\frac{2(\alpha - \beta)}{1 + (\alpha - \beta)} \right] \right\}, \qquad (1.93)$$

and $F_2(0)$ is given by

$$kF_2(0) = -\frac{e^{i(2ka - \frac{\pi}{4})}}{\sqrt{2\pi}(2ka)^{\frac{3}{2}}} \frac{1}{[L_+(k)]^2 \left[1 + (\alpha - \beta)\right]}. \qquad (1.94)$$

The interesting behavior is largely confined to the second term in Eq. (1.92). Setting

$$F(ka, \alpha - \beta) = \frac{1}{2ka} \operatorname{Im} \{kF_2(0) [1 + (\alpha - \beta)]\}. \tag{1.95}$$

Fig. 3. shows a plot of F against ka, for values of $(\alpha - \beta)$ from 0 to 1. The increasing effect of the absorption is apparent. The form $kF_2(0)$ suggests that the interaction between the edges is affected more strongly by the properties of the barrier than are singly diffracted rays. However, because $2ka$ is large in our approximation, the interaction term is always small.

Note that the case $(\alpha - \beta) = 0$ corresponds to a rigid barrier. In this case our expression for $\frac{\Gamma(ka,\alpha-\beta)}{4a}$ corresponds to the transmission cross-section given by Karp and Keller [56, $Eq.$(16)], namely,

$$\frac{\Gamma(ka, \alpha - \beta)}{4a} = 1 - \frac{\sin(2ka - \frac{\pi}{4})}{\sqrt{2\pi}(2ka)^{\frac{5}{2}}}. \tag{1.96}$$

It is of interest to note how the parameters $(\alpha \pm \beta)$ enter the calculation. The parameter $(\alpha + \beta)$ represents essentially the thickness of the barrier and appears in the calculation separated from the other terms, while $(\alpha - \beta)$ represents the absorption of the barrier and is intimately included in the calculation through its role in the terms $L_\pm$ and L. We believe that the Rawlins boundary condition more adequately represents the mechanical response of a thin absorbing barrier than would a boundary condition with $\beta = 0$.

While we have not explored our expressions in any completeness, we conjecture that they are more accurate than those calculated using the geometrical theory of diffraction and hence that they permit us to approximate the wavefields both near the slit and near the barrier itself.

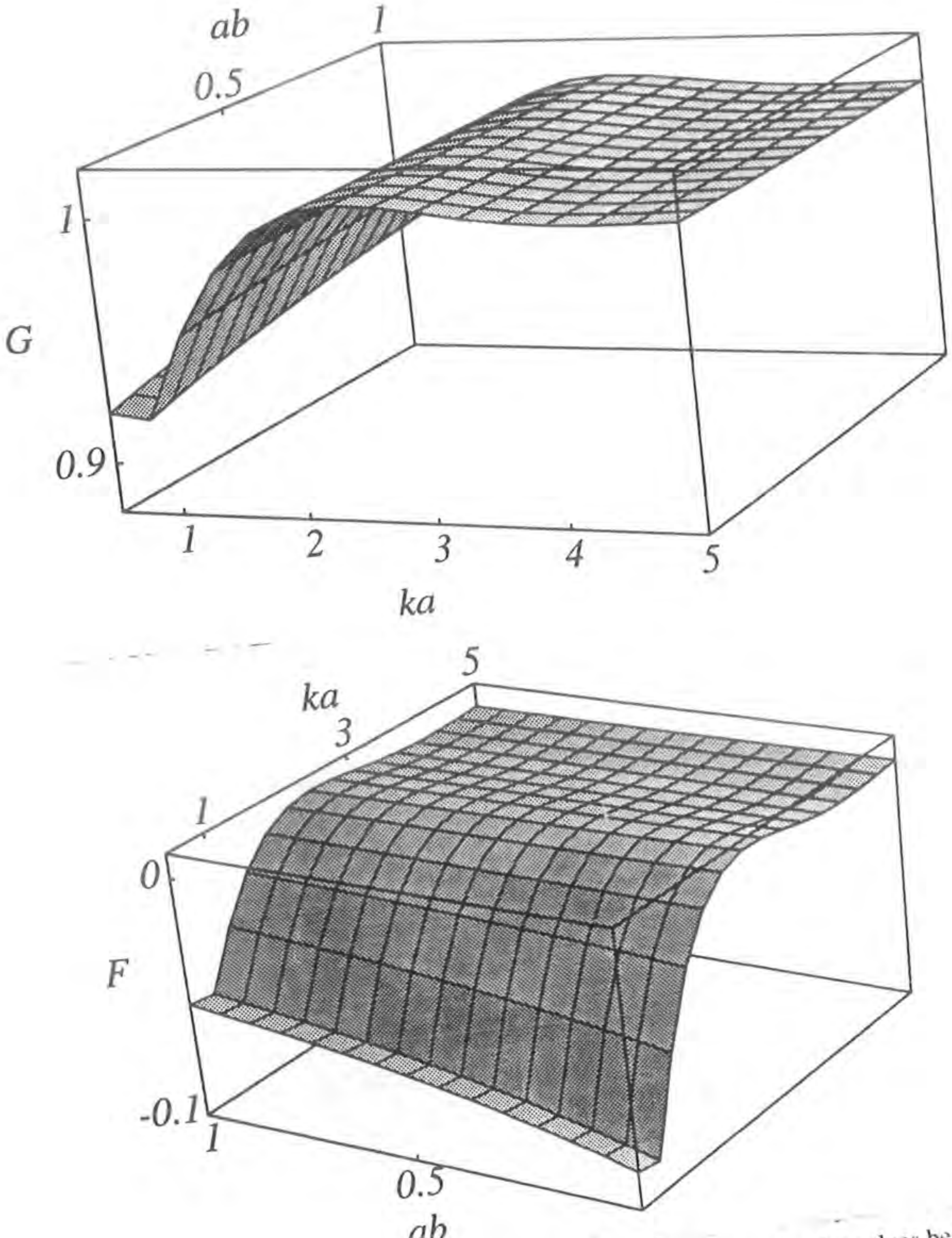

Figure 3 A three-dimensional graph of $F(ka, \alpha - \beta)$ against ka, for values between 0.5 and 10, and against $(\alpha - \beta)$ for values from 0 to 1.0. The $(\alpha - \beta)$ axis is labeled ab.

Chapter 2

SCATTERING OF A SPHERICAL SOUND WAVE BY A BI-IMPEDANT HALF PLANE

An exact solution for the problem of diffraction of a spherical sound wave by an absorbent half plane is obtained in this chapter. The two faces of the half plane have different impedance boundary conditions. The problem which is solved is a mathematical model for a noise barrier whose surface is treated with two different acoustically absorbent materials. The problem is formulated into a matrix W. H. functional equation for which no general method of solution has yet been found. However, studies have been made for restricted classes of these equations, and contents of this chapter show that an exact solution is available in the present case. The exact solution is found using a technique whereby the W. H. functional equation is converted into a scalar Hilbert problem. Comparisons are made with existing solutions for barriers that have a rigid half plane.

2.1 PROBLEM DESCRIPTION

Consider a small amplitude sound wave diffracted by the half plane $x \leq 0, y = 0$. The half plane is assumed to be infinitely thin and rigid with its surfaces treated with acoustically absorbent materials (*see Fig.*4). On the upper ($x \leq 0, y = 0^+$) and the lower ($x < 0, y = 0^-$) surfaces, respectively, we have the following absorbing boundary conditions:

$$\begin{aligned} p - u_n z_1 &= 0, \\ p - u_n z_2 &= 0. \end{aligned} \tag{2.1}$$

Here p is the acoustic pressure, u_n is the normal component of the perturbation velocity at a point on the surface of the half plane and $z_1(z_2)$ is the acoustic impedance of the upper(lower) surface. The perturbation velocity $\mathbf{u}$ of the irrotational sound waves can be expressed in terms of the total velocity potential $\eta(x, y, z)$ by $\mathbf{u} = \mathbf{grad}\ \eta$. The resulting pressure in the sound field is given by $p = i\omega\rho_0\eta(x, y, z)$ where ρ_0 is the density of the initially undisturbed ambient medium. Cartesian coordinates (x, y, z) are chosen so that a point source is located at $(x_0, y_0, z_0), y_0 > 0$. The total velocity potential $\eta(x, y, z)$ satisfies the wave equation

$$\left[\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}+k^2\right]\eta(x,y,z)=\delta(x-x_0)\delta(y-y_0)\delta(z-z_0). \quad (2.2)$$

The boundary conditions satisfied by η on $x \leq 0$, $y = 0^{\pm}$ are

$$\begin{aligned}\left(\frac{\partial}{\partial y}+ik\beta_1\right)\eta(x,0^+,z) &= 0, \quad x<0, \\ \left(\frac{\partial}{\partial y}-ik\beta_2\right)\eta(x,0^-,z) &= 0, \quad x<0,\end{aligned} \quad (2.3)$$

where $\beta_1(=\frac{\rho_0 c}{z_1})$ and $\beta_2(=\frac{\rho_0 c}{z_2})$ are the specific admittances of the absorbent surfaces, and for acoustic absorption $\mathrm{Re}(\beta_1)>0$, $\mathrm{Re}(\beta_2)>0$.

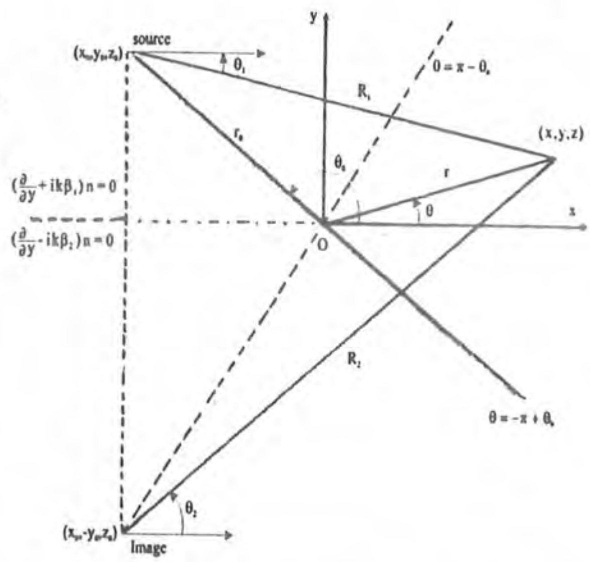

Fig. 4: The geometry of the problem.

In order to ensure the unique solution of the boundary value problem, we also require that the field be continuous and that the edge does not radiate any energy; also that the field should be radiating outwards at infinity. The condition that the edge does not behave like a source, and therefore radiates energy, requires that the field near the edge behave like

$$\begin{aligned} \eta &= O(1), \quad \mathbf{grad}\eta = O(r^{\frac{-1}{2}}), \qquad (2.4) \\ as \quad r &= \sqrt{x^2+y^2+z^2} \to 0, \quad |\beta_1| < \infty, |\beta_2| < \infty. \end{aligned}$$

The behavior of the edge field as given in the above expression is different to that given in Rawlins [94] where $|\beta_1| \to \infty$, $\beta_2 = 0$. We have here excluded the latter case and also $|\beta_2| \to \infty$, $\beta_1 = 0$, the reason being that our solution obtained is not uniformly continuous in either the limit $|\beta_1| \to \infty$ or $|\beta_2| \to \infty$.

2.2 SOLUTION OF THE PROBLEM

The spatial Fourier transform over z and its inverse are defined as

$$\chi(x,y,\mu) = \int_{-\infty}^{\infty} \eta(x,y,z)\, e^{-ik\mu z} dz, \qquad (2.5)$$

$$\eta(x,y,z) = \frac{k}{2\pi}\int_{-\infty}^{\infty} \chi(x,y,\mu)\, e^{ik\mu z} d\mu. \qquad (2.6)$$

In above equations, the transform parameter is taken conveniently to be $k\mu, \mu$ being non-dimensional. The decomposition (2.5) is common in other field theories as well, for example, Fourier Optics [62, 78]. Transforming Eq. (2.2) and the boundary conditions (2.3) by using Eq.(2.5), we obtain

$$\left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + k^2\lambda^2\right] \chi(x,y,\mu) = e^{-ik\mu z_0}\delta(x-x_0)\delta(y-y_0), \qquad (2.7)$$

$$\left(\frac{\partial}{\partial y}+ik\beta_1\right)\chi(x,0^+,\mu)=0, \quad x<0, \tag{2.8}$$

$$\left(\frac{\partial}{\partial y}-ik\beta_2\right)\chi(x,0^-,\mu)=0, \quad x<0, \tag{2.9}$$

where

$$\lambda^2=(1-\mu^2). \tag{2.10}$$

Define λ to be that branch which reduces to 1 when $\mu=0$ and $-i(\mu^2-1)^{\frac{1}{2}}$ when $|\mu|>1$. This is feasible as long as μ is not near ± 1 which can be arranged by adjusting the μ contour first. In fact ± 1 are the branch points associated with the function $\lambda=(1-\mu^2)^{\frac{1}{2}}$ and we take the branch cut from $+1$ just below the positive real axis to $+\infty$ and the cut from -1 just above the negative real axis to $-\infty$. Here the integration path in the μ plane is the real axis, indented above any poles on the positive real axis and below on the negative half ($Fig.5$).

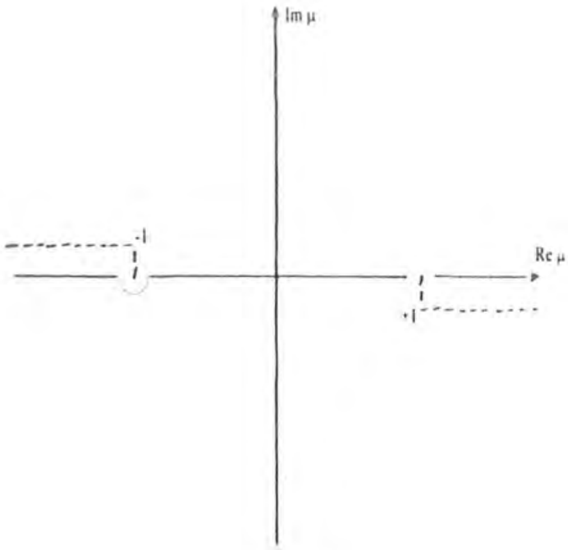

Fig. 5:

We next define the spatial Fourier transform pair on x as

$$\widehat{\chi}(\nu, y, \mu) = \int_{-\infty}^{\infty} \chi(x, y, \mu)\, e^{i\nu x} dx, \tag{2.11}$$

$$\chi(x, y, \mu) = \frac{1}{2\pi} \int_{-\infty+i\tau}^{\infty+i\tau} \widehat{\chi}(\nu, y, \mu) e^{-i\nu x} d\nu. \tag{2.12}$$

The transform (2.11) and its inverse (2.12) exist provided $-\text{Im}(k\lambda) < \tau < \text{Im}(k\lambda)$; this follows from the radiation condition.

Applying the transform (2.11) to Eq. (2.7) gives

$$\left[\frac{d^2}{dy^2} + \vartheta^2\right] \widehat{\chi}(\nu, y, \mu) = e^{i\nu x_0 - ik\mu z_0} \delta(y - y_0), \; y_0 > 0, \tag{2.13}$$

where $\vartheta = \sqrt{k^2\lambda^2 - \nu^2}$ is defined on the cut sheet for which $\text{Im}(\vartheta) > 0$ when $|\text{Im}(\nu)| < \text{Im}(k\lambda)$.

A solution of Eq. (2.13) for ν lying in the strip $|\text{Im}(\nu)| < \text{Im}(k\lambda)$, and which decays as $|y| \to \infty$, is given by

$$\widehat{\chi}(\nu, y, \mu) = A_3(\nu)e^{i\vartheta y} + \frac{e^{i(\nu x_0 - k\mu z_0 + \vartheta|y-y_0|)}}{2i\vartheta}, \quad y > 0, \tag{2.14}$$

$$\widehat{\chi}(\nu, y, \mu) = A_4(\nu)e^{-i\vartheta y}, \quad y < 0. \tag{2.15}$$

where $A_3(\nu)$ and $A_4(\nu)$ are unknown functions to be determined. Let

$$\Phi_1^-(\nu) = \int_{-\infty}^{0} \left[\chi(x, 0^+, \mu) - \chi(x, 0^-, \mu)\right] e^{i\nu x} dx, \tag{2.16}$$

$$\Phi_2^-(\nu) = \int_{-\infty}^{0} \left[\frac{\partial}{\partial y}\chi(x, 0^+, \mu) - \frac{\partial}{\partial y}\chi(x, 0^-, \mu)\right] e^{i\nu x} dx, \tag{2.17}$$

$$\Psi_1^+(\nu) = \int_{0}^{\infty} \left[\left(\frac{\partial}{\partial y} + ik\beta_1\right)\chi(x, 0^+, \mu)\right] e^{i\nu x} dx, \tag{2.18}$$

$$\Psi_2^+(\nu) = \int_{0}^{\infty} \left[\left(\frac{\partial}{\partial y} - ik\beta_2\right)\chi(x, 0^-, \mu)\right] e^{i\nu x} dx. \tag{2.19}$$

In Eqs. (2.16) to (2.19), $\Phi_{1,2}^-(\nu)$ are analytic for $\text{Im}(\nu) < \text{Im}(k\lambda)$, and $\Psi_{1,2}^+(\nu)$ are analytic for $\text{Im}(\nu) > -\text{Im}(k\lambda)$. Throughout this chapter a superscript (or subscript) plus or minus attached to any function will denote that the function is analytic in $\text{Im}(\nu) > -\text{Im}(k\lambda)$ or $\text{Im}(\nu) < \text{Im}(k\lambda)$, respectively. After using Eqs. (2.8), (2.9), (2.11), (2.14) and (2.15) in the Eqs. (2.16) to (2.19), we get

$$\Phi_1^-(\nu) = A_3(\nu) - A_4(\nu) + \frac{e^{i(\nu x_0 - k\mu z_0 + \vartheta y_0)}}{2i\vartheta}, \tag{2.20}$$

$$\Phi_2^-(\nu) = i\vartheta\left[A_3(\nu) + A_4(\nu)\right] - \frac{e^{i(\nu x_0 - k\mu z_0 + \vartheta y_0)}}{2}. \tag{2.21}$$

$$\Psi_1^+(\nu) = A_3(\nu)\left(i\vartheta + ik\beta_1\right) + \left(-i\vartheta + ik\beta_1\right)\frac{e^{i(\nu x_0 - k\mu z_0 + \vartheta y_0)}}{2i\vartheta}. \tag{2.22}$$

$$\Psi_2^+(\nu) = -A_4(\nu)\left(i\vartheta + ik\beta_2\right). \tag{2.23}$$

It is now straightforward to eliminate $A_3(\nu)$ and $A_4(\nu)$ from Eqs.. (2.20) to (2.23) and obtain the matrix W. H. equation

$$\Psi_+(\nu) = K(\nu)\Phi_-(\nu) + D(\nu), \tag{2.24}$$

which holds in the strip $|\mathrm{Im}(\nu)| < \mathrm{Im}(k\lambda)$, with

$$\Psi_+(\nu) = \begin{bmatrix} \Psi_1^+(\nu) \\ \Psi_2^+(\nu) \end{bmatrix}, \quad \Phi_-(\nu) = \begin{bmatrix} \Phi_1^-(\nu) \\ \Phi_2^-(\nu) \end{bmatrix}, \tag{2.25}$$

$$K(\nu) = \frac{1}{2}\begin{bmatrix} i(\vartheta + k\beta_1) & (\vartheta + k\beta_1)/\vartheta \\ i(\vartheta + k\beta_2) & -(\vartheta + k\beta_2)/\vartheta \end{bmatrix}, \tag{2.26}$$

$$D(\nu) = \begin{bmatrix} (-\vartheta + k\beta_1)\exp[i\left((\nu x_0 - k\mu z_0 + \vartheta y_0)\right)]/2\vartheta \\ -(\vartheta + k\beta_2)\exp[i\left((\nu x_0 - k\mu z_0 + \vartheta y_0)\right)]/2\vartheta \end{bmatrix}. \tag{2.27}$$

Expression (2.24) constitutes a coupled system of W. H. equations.

For the solution of expression (2.23) we need to factorize the matrix function $K(\nu)$. This is not a trivial operation and it is not always obvious that one can in fact factorize the matrix. The factorization of the matrix $K(\nu)$ is given in *Appendix A*, i.e.,

$$K(\nu) = U(\nu)L^{-1}(\nu), \tag{2.28}$$

where

$$L(\nu) = \begin{bmatrix} l_{11}(\nu) & l_{12}(\nu) \\ l_{21}(\nu) & l_{22}(\nu) \end{bmatrix}, \qquad U(\nu) = \begin{bmatrix} u_{11}(\nu) & u_{12}(\nu) \\ u_{21}(\nu) & u_{22}(\nu) \end{bmatrix}, \tag{2.29}$$

$$u_{11} = -\left[\frac{\left(\sqrt{k\lambda+\nu} + \sqrt{k\aleph_1(+)}\right)\left(\sqrt{k\lambda+\nu} + \sqrt{k\aleph_1(-)}\right)}{\left(\sqrt{k\lambda+\nu} + \sqrt{k\aleph_2(+)}\right)\left(\sqrt{k\lambda+\nu} + \sqrt{k\aleph_2(-)}\right)}\right]^{\frac{1}{2}} \exp\left[\frac{1}{2}\int_{\infty}^{\nu} Q(u)du\right]. \tag{2.30}$$

$$u_{21} = \left[\frac{\left(\sqrt{k\lambda+\nu} + \sqrt{k\aleph_2(+)}\right)\left(\sqrt{k\lambda+\nu} + \sqrt{k\aleph_2(-)}\right)}{\left(\sqrt{k\lambda+\nu} + \sqrt{k\aleph_1(+)}\right)\left(\sqrt{k\lambda+\nu} + \sqrt{k\aleph_1(-)}\right)}\right]^{\frac{1}{2}} \exp\left[\frac{1}{2}\int_{\infty}^{\nu} Q(u)du\right]. \tag{2.31}$$

$$u_{12} = \sqrt{k\lambda+\nu}\, u_{11}(\nu), \tag{2.32}$$

$$u_{22} = -\sqrt{k\lambda+\nu}\, u_{21}(\nu), \tag{2.33}$$

$$Q(u) \;=\; \frac{-1}{u+k\lambda} + \frac{\cos^{-1}\left(-\sqrt{\left(\lambda^2-\beta_1^2\right)}(\lambda)^{-1}\right)}{2\pi\left(u+k\sqrt{\left(\lambda^2-\beta_1^2\right)}\right)} + \frac{\cos^{-1}\left(\sqrt{\left(\lambda^2-\beta_2^2\right)}(\lambda)^{-1}\right)}{2\pi\left(u-k\sqrt{\left(\lambda^2-\beta_2^2\right)}\right)}$$

$$+\frac{\cos^{-1}\left(\sqrt{\left(\lambda^2-\beta_1^2\right)}(\lambda)^{-1}\right)}{2\pi\left(u-k\sqrt{\left(\lambda^2-\beta_1^2\right)}\right)}+\frac{\cos^{-1}\left(-\sqrt{\left(\lambda^2-\beta_2^2\right)}(\lambda)^{-1}\right)}{2\pi\left(u+k\sqrt{\left(\lambda^2-\beta_2^2\right)}\right)}$$

$$-\frac{k\beta_1\cos^{-1}(u/k\lambda)}{2\pi\sqrt{k^2\lambda^2-u^2}}\left[\frac{1}{u+k\sqrt{\left(\lambda^2-\beta_1^2\right)}}+\frac{1}{u-k\sqrt{\left(\lambda^2-\beta_1^2\right)}}\right]$$

$$-\frac{k\beta_2\cos^{-1}(u/k\lambda)}{2\pi\sqrt{k^2\lambda^2-u^2}}\left[\frac{1}{u+k\sqrt{\left(\lambda^2-\beta_2^2\right)}}+\frac{1}{u-k\sqrt{\left(\lambda^2-\beta_2^2\right)}}\right]. \quad (2.34)$$

$$\aleph_{1,2}(\pm)=\lambda\pm\sqrt{\lambda^2-\beta_{1,2}^2} \quad (2.35)$$

and elements of $U(\nu); u_{ij}, i,j=1,2$ and $L(\nu)$ are analytic in $\mathrm{Im}(\nu)>-\mathrm{Im}(k\lambda)$ and $\mathrm{Im}(\nu)<\mathrm{Im}(k\lambda)$, respectively. The elements of $L(\nu)$ in terms of $u_{11}(\nu)$ and $u_{21}(\nu)$ are given by

$$l_{11}(\nu)=\frac{u_{11}}{i(\vartheta+k\beta_1)}+\frac{u_{21}}{i(\vartheta+k\beta_2)}. \quad (2.36)$$

$$l_{12}(\nu)=\frac{u_{11}\sqrt{k\lambda+\nu}}{i(\vartheta+k\beta_1)}-\frac{u_{21}\sqrt{k\lambda+\nu}}{i(\vartheta+k\beta_2)}, \quad (2.37)$$

$$l_{21}(\nu)=\frac{\vartheta u_{11}}{(\vartheta+k\beta_1)}-\frac{\vartheta u_{21}}{(\vartheta+k\beta_2)}. \quad (2.38)$$

$$l_{22}(\nu)=\frac{\vartheta\, u_{11}\sqrt{k\lambda+\nu}}{(\vartheta+k\beta_1)}+\frac{\vartheta\, u_{21}\sqrt{k\lambda+\nu}}{(\vartheta+k\beta_2)}. \quad (2.39)$$

Now substituting Eq. (2.28) into Eq. (2.24) and then carrying out the matrix multiplication in the resulting expression we get

$$(Det\ U)^{-1}\left[u_{22}\Psi_1^+ - u_{12}\Psi_2^+\right] - G_1^+ \tag{2.40}$$
$$= (Det\ L)^{-1}\left[l_{22}\Phi_1^- - l_{12}\Phi_2^-\right] + G_1^-,$$

$$(Det\ U)^{-1}\left[-u_{21}\Psi_1^+ + u_{11}\Psi_2^+\right] - G_2^+ \tag{2.41}$$
$$= (Det\ L)^{-1}\left[-l_{21}\Phi_1^- + l_{11}\Phi_2^-\right] + G_2^-.$$

In Eqs. (2.40) and (2.41)

$$\begin{aligned} G_1(\nu) &= (Det\ U)^{-1}\left[u_{22}(k\beta_1 - \vartheta) + u_{12}(k\beta_2 + \vartheta)\right]\left[\frac{e^{i(\nu x_0 - k\mu z_0 + \vartheta y_0)}}{2\vartheta}\right] \\ &= G_1^+ + G_1^-, \end{aligned} \tag{2.42}$$

$$\begin{aligned} G_2(\nu) &= (Det\ U)^{-1}\left[-u_{21}(k\beta_1 - \vartheta) - u_{11}(k\beta_2 + \vartheta)\right]\left[\frac{e^{i(\nu x_0 - k\mu z_0 + \vartheta y_0)}}{2\vartheta}\right] \\ &= G_2^+ + G_2^-, \end{aligned} \tag{2.43}$$

$$Det\ U = u_{11}(\nu)u_{22}(\nu) - u_{12}(\nu)u_{21}(\nu), \tag{2.44}$$

$$Det\ L = l_{11}(\nu)l_{22}(\nu) - l_{12}(\nu)l_{21}(\nu), \tag{2.45}$$

$$G_1^{\pm}(\nu) = \pm\frac{1}{2\pi i}\int_{-\infty\mp i\tau_1}^{\infty\mp i\tau_1}\frac{G_1(t)}{t-\nu}dt, \tag{2.46}$$

$$G_2^{\pm}(\nu) = \pm\frac{1}{2\pi i}\int_{-\infty\mp i\tau_1}^{\infty\mp i\tau_1} \frac{G_2(t)}{t-\nu}dt, \quad 0 < \tau_1 < \mathrm{Im}(k\lambda). \tag{2.47}$$

The representations (2.46) and (2.47) with the upper (lower) sign are valid when $\mathrm{Im}(\nu) > -\tau_1(\mathrm{Im}(\nu) < \tau_1)$ and define $G_{1,2}^{+}(\nu)$ $(G_{1,2}^{-}(\nu))$ as analytic functions in $\mathrm{Im}(\nu) > -\tau_1(\mathrm{Im}(\nu) < \tau_1)$. We note that in the limiting case of $\tau_1 = \mathrm{Im}(k\lambda) = 0$ the above integrands have an integrable singularity at $t = -k\lambda$. This follows from (A 47) of *Appendix A*, which gives

$$G_1(t) = O(1), G_2(t) = O((k\lambda + t)^{-\frac{1}{2}}).$$

Standard asymptotic analysis in their regions of regularity also shows that

$$G_{1,2}^{\pm}(\nu) = O(\nu^{-1}), \text{ as } |\nu| \to \infty. \tag{2.48}$$

Before proceeding further with Eqs. (2.40) and (2.41), it is necessary to know how the various quantities behave as $|\nu| \to \infty$. The edge condition (2.4) means that the transformed functions satisfy the following growth estimates as $|\nu| \to \infty$:

$$\begin{aligned} \Phi_1^-(\nu) &= O(\nu^{-1}), \qquad \Phi_2^-(\nu) = O(\nu^{\frac{-1}{2}}) \text{ for } \mathrm{Im}(\nu) < \mathrm{Im}(k\lambda), |\nu| \to \infty; \\ \Psi_1^+(\nu) &= O(\nu^{\frac{-1}{2}}), \qquad \Psi_2^+(\nu) = O(\nu^{\frac{-1}{2}}) \text{ for } \mathrm{Im}(\nu) > \mathrm{Im}(k\lambda), |\nu| \to \infty. \end{aligned} \tag{2.49}$$

Using the asymptotic estimates (2.49) and (A45) and (A46) of *Appendix A*, we find that:

For $\mathrm{Im}(\nu) > -\mathrm{Im}(k\lambda)$ as $|\nu| \to \infty$,

$$\begin{aligned}(Det\ U)^{-1}\, u_{22}\Psi_1^+ &= O(\nu^{\frac{-1}{2}}), \quad (Det\ U)^{-1}\, u_{12}\Psi_2^+ = O(\nu^{\frac{-1}{2}}), \qquad (2.50)\\ G_1^+ &= O(\nu^{-1}), \qquad G_2^+ = O(\nu^{-1}),\\ (Det\ U)^{-1}\, u_{21}\Psi_1^+ &= O(\nu^{-1}), \qquad (Det\ U)^{-1}\, u_{11}\Psi_2^+ = O(\nu^{-1}).\end{aligned}$$

For $\text{Im}(\nu) < \text{Im}(k\lambda)$ as $|\nu| \to \infty$,

$$\begin{aligned}(Det\ L)^{-1}\, l_{22}\Phi_1^- &= O(1), \quad (Det\ L)^{-1}\, l_{12}\Phi_2^- = O(\nu^{\frac{-1}{2}}), \qquad (2.51)\\ G_1^- &= O(\nu^{-1}), \qquad G_2^- = O(\nu^{-1}),\\ (Det\ L)^{-1}\, l_{21}\Phi_1^- &= O(\nu^{\frac{-1}{2}}), \qquad (Det\ L)^{-1}\, l_{11}\Phi_2^- = O(\nu^{-1}).\end{aligned}$$

The results (2.50) and (2.51) show that the left side and right side of the Eq. (2.41) are analytic and asymptotic to $o(1)$ as $|\nu| \to \infty$ in $\text{Im}(\nu) > -\text{Im}(k\lambda)$ and $\text{Im}(\nu) < \text{Im}(k\lambda)$, respectively. Similarly the left side of Eq. (2.40) is analytic and asymptotic to $o(1)$ in $\text{Im}(\nu) > -\text{Im}(k\lambda)$ as $|\nu| \to \infty$, whereas the right side is analytic and asymptotic to $O(1)$ as $|\nu| \to \infty$ in $\text{Im}(\nu) < \text{Im}(k\lambda)$. Thus, by virtue of Liouville's theorem, the analytic function which is a continuation of both sides of these equations in the entire ν plane is a constant; the constant being zero. Hence

$$U^{-1}\Psi_+ = G_+ = \begin{bmatrix} G_1^+ \\ G_2^+ \end{bmatrix} \Rightarrow \Psi_+ = UG_+$$

or

$$\Psi_1^+ = G_1^+\, u_{11} + G_2^+\, u_{12}, \qquad (2.52)$$

$$\Psi_2^+ = G_1^+ u_{21} + G_2^+ u_{22}. \tag{2.53}$$

The expressions (2.52) and (2.53) in conjunction with the expressions (2.22) and (2.23) yield explicit expressions for $A_3(\nu)$ and $A_4(\nu)$. Using the resulting expressions in (2.14) and (2.15) and then taking Fourier inversion over x, we have

$$\chi(x, y > 0, \mu) = \frac{e^{-ik\mu z_0}}{4i} H_0^{(1)}\left(k\lambda\sqrt{(x-x_0)^2 + (y-y_0)^2}\right) + \frac{1}{2\pi}\int_{-\infty+i\tau}^{\infty+i\tau} e^{-i\nu x + i\vartheta y} d\nu \left\{ \begin{array}{c} \frac{G_1^+ u_{11} + G_2^+ u_{12}}{i(\vartheta + k\beta_1)} \\ + \left(\frac{\vartheta - k\beta_1}{\vartheta + k\beta_1}\right)\left[\frac{e^{i(\nu x_0 - k\mu z_0 + \vartheta y_0)}}{2i\vartheta}\right] \end{array} \right\}, \tag{2.54}$$

$$\chi(x, y < 0, \mu) = \frac{1}{2\pi}\int_{-\infty+i\tau}^{\infty+i\tau} e^{-i\nu x - i\vartheta y} d\nu \left\{ \frac{G_1^+ u_{21} + G_2^+ u_{22}}{-i(\vartheta + k\beta_2)} \right\}. \tag{2.55}$$

$-\operatorname{Im}(k\lambda) < -\tau_1 < \tau < \operatorname{Im}(k\lambda)$.

2.3 ASYMPTOTIC EXPRESSIONS FOR THE FAR FIELD

For an approximate solution of Eqs. (2.54) and (2.55) we first consider $G_{1,2}^+(\nu)$ as given by Eqs. (2.46) and (2.47) : Let $k\lambda$ be real; then $\tau_1 = 0$ and the integration path along the real axis is indented below the point $t = \nu$. We note that the integrand has a saddle point at $\xi = \theta_0$ after putting

$x_0 = r_0\cos\theta_0, y_0 = r_0\sin\theta_0, 0 < \theta_0 < \pi, t = k\lambda\cos\xi, 0 < \mathrm{Re}\,\xi < \pi$. The integration path is now deformed into the steepest descent path $S(\theta_0)$ described by $\mathrm{Re}[\cos(\xi-\theta_0)] = 1$, $\mathrm{Im}[\cos(\xi-\theta_0)] \geq 0$. In the deformation the pole at $k\lambda\cos\xi = \nu$ is intercepted if $\nu < k\lambda\cos\theta_0$. Thus, for $k\lambda r_0 \to \infty$, we have

$$\begin{aligned} G_1^+(\nu) \sim\ & \frac{e^{-ik\mu z_0}}{4\pi i}\sqrt{\frac{2\pi}{k\lambda r_0}}e^{i(k\lambda r_0-\frac{\pi}{4})} \\ & \times\left[\frac{(\beta_1-\lambda\sin\theta_0)\,u_{22}(k\lambda\cos\theta_0)+(\beta_2+\lambda\sin\theta_0)\,u_{12}(k\lambda\cos\theta_0)}{[Det\ U(k\lambda\cos\theta_0)]\,(\lambda\cos\theta_0-\nu/k)}\right] \\ & +\frac{(k\beta_1-\vartheta)u_{22}(\nu)+(k\beta_2+\vartheta)u_{12}(\nu)}{2\vartheta\,[Det\ U(\nu)]} \\ & \times H[k\lambda\cos\theta_0-\nu]\ e^{i(\nu x_0+\vartheta y_0-k\mu z_0)}, \end{aligned} \tag{2.56}$$

$$\begin{aligned} G_2^+(\nu) \sim\ & -\frac{e^{-ik\mu z_0}}{4\pi i}\sqrt{\frac{2\pi}{k\lambda r_0}}e^{i(k\lambda r_0-\frac{\pi}{4})} \\ & \times\left[\frac{(\beta_1-\lambda\sin\theta_0)\,u_{21}(k\lambda\cos\theta_0)+(\beta_2+\lambda\sin\theta_0)\,u_{11}(k\lambda\cos\theta_0)}{[Det\ U(k\lambda\cos\theta_0)]\,(\lambda\cos\theta_0-\nu/k)}\right] \\ & -\frac{(k\beta_1-\vartheta)u_{21}(\nu)+(k\beta_2+\vartheta)u_{11}(\nu)}{2\vartheta\,[Det\ U(\nu)]} \\ & \times H[k\lambda\cos\theta_0-\nu]e^{i(\nu x_0+\vartheta y_0-k\mu z_0)}, \end{aligned} \tag{2.57}$$

where $H(x_1) = 1$ for $x_1 > 0, H(x_1) = 0$ for $x_1 < 0$. The result is valid for $k\lambda r_0 \to \infty, -k\lambda < \nu < k\lambda$; and the second terms arise from the residue contributions. After inserting Eqs. (2.56) and (2.57) into Eqs. (2.54) and (2.55) we decompose the total field

$$\chi(x,y,\mu) = \chi_D(x,y,\mu) + \chi_{GA}(x,y,\mu). \tag{2.58}$$

where

$$\chi_D(x, y\ >\ 0, \mu) = \sqrt{\frac{2\pi}{k\lambda r_0}}\frac{e^{ik\lambda r_0-\frac{i\pi}{4}-ik\mu z_0}}{4\pi i} \tag{2.59}$$

$$\times \int_{-\infty+i\tau}^{\infty+i\tau} e^{-i\nu x+i\vartheta y} d\nu \begin{bmatrix} (\beta_1 - \lambda \sin\theta_0)\, u_{22}(k\lambda\cos\theta_0) u_{11}(\nu) \\ +(\beta_2 + \lambda \sin\theta_0)\, u_{12}(k\lambda\cos\theta_0) u_{11}(\nu) \\ -(\beta_1 - \lambda \sin\theta_0)\, u_{21}(k\lambda\cos\theta_0) u_{12}(\nu) \\ -(\beta_2 + \lambda \sin\theta_0)\, u_{11}(k\lambda\cos\theta_0) u_{12}(\nu) \end{bmatrix}$$
$$\times \frac{[Det\ U(k\lambda\cos\theta_0)(\lambda\cos\theta_0 - \nu/k) i(\vartheta + k\beta_1)]^{-1}}{2\pi}$$

$$\chi_D(x, y < 0, \mu) = \sqrt{\frac{2\pi}{k\lambda r_0}} \frac{e^{ik\lambda r_0 - \frac{i\pi}{4} - ik\mu z_0}}{4\pi i} \tag{2.60}$$
$$\times \int_{-\infty+i\tau}^{\infty+i\tau} e^{-i\nu x-i\vartheta y} d\nu \begin{bmatrix} (\beta_1 - \lambda \sin\theta_0)\, u_{22}(k\lambda\cos\theta_0) u_{21}(\nu) \\ +(\beta_2 + \lambda \sin\theta_0)\, u_{12}(k\lambda\cos\theta_0) u_{21}(\nu) \\ -(\beta_1 - \lambda \sin\theta_0)\, u_{21}(k\lambda\cos\theta_0) u_{22}(\nu) \\ -(\beta_2 + \lambda \sin\theta_0)\, u_{11}(k\lambda\cos\theta_0) u_{22}(\nu) \end{bmatrix}$$
$$\times \frac{[Det\ U(k\lambda\cos\theta_0)(\lambda\cos\theta_0 - \nu/k)(-i)(\vartheta + k\beta_2)]^{-1}}{2\pi};$$

$$\chi_{GA}(x, y > 0, \mu) = \frac{e^{-ik\mu z_0}}{4i} H_0^{(1)}\left[k\lambda\sqrt{(x - x_0)^2 + (y - y_0)^2}\right] \tag{2.61}$$
$$+ \frac{e^{-ik\mu z_0}}{2\pi} \int_{-\infty+i\tau}^{\infty+i\tau} \left(\frac{k\beta_1 - \vartheta}{k\beta_1 + \vartheta}\right) \{H[k\lambda\cos\theta_0 - \nu] - 1\}$$
$$\times \frac{e^{-i\nu(x-x_0)+i\vartheta(y+y_0)}}{2i\vartheta} d\nu.$$

$$\chi_{GA}(x, y < 0, \mu) = \frac{e^{-ik\mu z_0}}{2\pi} \int_{-\infty+i\tau}^{\infty+i\tau} H[k\lambda\cos\theta_0 - \nu] \frac{e^{-i\nu(x-x_0)+i\vartheta(y_0-y)}}{2i\vartheta} d\nu. \tag{2.62}$$

The integrals appearing in Eqs.(2.59) and (2.60) can be solved asymptotically by using the saddle point method. For that we substitute $x =$

$r\cos\theta, y = r\sin\theta, -\pi < \theta < \pi; \nu = k\lambda\cos\xi$, $0 < \mathrm{Re}\,\xi < \pi$, then the integration has a saddle point at $\xi = \pi - \theta$ and $\xi = \pi + \theta$, respectively. Hence, for large $k\lambda r$

$$\chi_D(r\cos\theta, r\sin\theta, \mu) \sim \sqrt{\frac{2\pi}{k\lambda r r_0}} \frac{e^{ik\lambda r_0 - \frac{i\pi}{4} - ik\mu z_0}}{4i} D(\theta, \theta_0, \mu), \tag{2.63}$$

where

$$\begin{aligned}
D(\theta,\theta_0,\mu) \;=\; & -\frac{e^{\frac{i\pi}{4}}}{2\sqrt{2\pi k\lambda}} \frac{\exp\left[\frac{1}{2}\int_{k\lambda\cos\theta_0}^{-k\lambda\cos\theta_0} Q(u)du\right]}{(\cos\theta+\cos\theta_0)\cos(\theta_0/2)} \qquad (2.64)\\
& \times \frac{|\sin\theta|}{\{(\lambda\sin\theta+\beta_1)(\lambda\sin\theta-\beta_2)\}^{\frac{1}{2}}} \left[\frac{|\lambda\sin\theta|-\beta_2}{|\lambda\sin\theta|+\beta_1}\right]^{\frac{1}{2}} \\
& \times \left[\frac{\left(\sqrt{2\lambda}\sin(\theta/2)+\sqrt{\aleph_1(+)}\right)\left(\sqrt{2\lambda}\sin(\theta/2)+\sqrt{\aleph_1(-)}\right)}{\left(\sqrt{2\lambda}\sin(\theta/2)+\sqrt{\aleph_2(+)}\right)\left(\sqrt{2\lambda}\sin(\theta/2)+\sqrt{\aleph_2(-)}\right)}\right]^{\frac{1}{2}} \\
& \times\{(\beta_1-\lambda\sin\theta)(\sin(\theta/2)+\cos(\theta_0/2)) \\
& \times \left[\frac{\left(\sqrt{2\lambda}\cos(\theta_0/2)+\sqrt{\aleph_2(+)}\right)\left(\sqrt{2\lambda}\cos(\theta_0/2)+\sqrt{\aleph_2(-)}\right)}{\left(\sqrt{2\lambda}\cos(\theta_0/2)+\sqrt{\aleph_1(+)}\right)\left(\sqrt{2\lambda}\cos(\theta_0/2)+\sqrt{\aleph_1(-)}\right)}\right]^{\frac{1}{2}} \\
& +\{(\beta_2+\lambda\sin\theta_0)(-\sin(\theta/2)+\cos(\theta_0/2))\} \\
& \times \left[\frac{\left(\sqrt{2\lambda}\cos(\theta_0/2)+\sqrt{\aleph_1(+)}\right)\left(\sqrt{2\lambda}\cos(\theta_0/2)+\sqrt{\aleph_1(-)}\right)}{\left(\sqrt{2\lambda}\cos(\theta_0/2)+\sqrt{\aleph_2(+)}\right)\left(\sqrt{2\lambda}\cos(\theta_0/2)+\sqrt{\aleph_2(-)}\right)}\right]^{\frac{1}{2}} \},
\end{aligned}$$

$k\lambda r \to \infty$, $k\lambda r_0 \to \infty, \cos\theta + \cos\theta_0 \neq 0$.

In order to solve the integrals in Eqs.. (2.61) and (2.62), we put $x - x_0 = r_2\cos\theta_2$, $y + y_0 = r_2\sin\theta_2, \nu = k\lambda\cos\xi$, $0 < \theta_2 < \pi$; in Eq. (2.61) and in Eq. (2.62) let $x - x_0 = r_1\cos\theta_1$, $y - y_0 = r_1\sin\theta_1, \nu = k\lambda\cos\xi$, $0 < \theta_1 < \pi$ and deform the path of integration into $S(\pi - \theta_2)$ and $S(\pi - \theta_1)$, respectively. Hence, for large $k\lambda r_1$ and $k\lambda r_2$, we obtain

$$\begin{aligned}\eta_{GA}(r\cos\theta, r\sin\theta,\mu) &\sim \frac{e^{-i\mu k z_0}}{4i}\sqrt{\frac{2}{\pi k r_1 \lambda}} e^{i(k\lambda r_1-\frac{\pi}{4})} + \frac{e^{-i\mu k z_0}}{4i}\sqrt{\frac{2}{\pi k r_2 \lambda}} e^{i(k\lambda r_2-\frac{\pi}{4})} \\ &\times \left[\frac{(\beta_1-\lambda\sin\theta_2)}{(\beta_1+\lambda\sin\theta_2)}\right] H[\lambda(\cos\theta_2+\cos\theta_0)-1], \\ 0 &< \theta < \pi, \end{aligned} \tag{2.65}$$

$$\begin{aligned}&\eta_{GA}(r\cos\theta, r\sin\theta,\mu) \\ \sim\ & \frac{e^{-i\mu k z_0}}{4i}\sqrt{\frac{2}{\pi k r_1 \lambda}} e^{i(k\lambda r_1-\frac{\pi}{4})} H[\lambda(\cos\theta_1+\cos\theta_0)], -\pi<\theta<0. \end{aligned} \tag{2.66}$$

Now taking the inverse Fourier transform of Eqs. (2.63),(2.65) and (2.66) over z we get

$$\eta_D(x,y,z) = \frac{e^{\frac{-3i\pi}{4}}}{4\pi}\sqrt{\frac{k}{2\pi r_0 r}}\int_{-\infty}^{\infty}\frac{e^{ik\lambda(r+r_0)+ik\mu(z-z_0)}D(\theta,\theta_0,\mu)}{\sqrt{\lambda}}d\mu, -\pi<\theta<\pi, \tag{2.67}$$

$$\begin{aligned}\eta_{GA}(x,y,z) &\sim \frac{e^{\frac{-3i\pi}{4}}}{4i}\sqrt{\frac{k}{2\pi r_1}}\int_{-\infty}^{\infty}\frac{e^{ik\lambda r_1+ik\mu(z-z_0)}}{\sqrt{\lambda}}d\mu \\ &+\frac{e^{\frac{-3i\pi}{4}}}{4\pi}\sqrt{\frac{k}{2\pi r_2}}\int_{-\infty}^{\infty}\frac{e^{ik\lambda r_2+ik\mu(z-z_0)}}{\sqrt{\lambda}} \\ &\times\left[\frac{(\beta_1-\lambda\sin\theta_2)}{(\beta_1+\lambda\sin\theta_2)}\right] H[\lambda(\cos\theta_2+\cos\theta_0)-1]d\mu, 0<\theta<\pi, \end{aligned} \tag{2.68}$$

$$\begin{aligned}\eta_{GA}(x,y,z) &\sim \frac{e^{\frac{-3i\pi}{4}}}{4\pi}\sqrt{\frac{k}{2\pi r_1}}\int_{-\infty}^{\infty}\frac{e^{ik\lambda r_1+ik\mu(z-z_0)}}{\sqrt{\lambda}}H[\lambda(\cos\theta_1+\cos\theta_0)]d\mu, \\ -\pi &< \theta<0. \end{aligned} \tag{2.69}$$

The integrals appearing in Eqs. (2.67), (2.68) and (2.69) are solved by following the same method of solution as in Eqs. (2.60) to (2.62) and the resulting expression is given by

$$\eta(x,y,z) = \eta_D(x,y,z) + \eta_{GA}(x,y,z), \tag{2.70}$$

where

$$\begin{aligned}
\eta_D(x,y,z) &= \frac{e^{\frac{-3i\pi}{4}}}{4}\sqrt{\frac{k\sin\theta_{12}}{2\pi r r_0}}D(\theta,\theta_0,\theta_{12})H_0^{(1)}(kr_{12}), & (2.71)\\
\eta_{GA}(x,y,z) &\sim \frac{e^{\frac{-3i\pi}{4}}}{4}\sqrt{\frac{k}{2\pi r_{121}}}H_0^{(1)}(kr_{121}) + \frac{e^{\frac{-3i\pi}{4}}}{4}\sqrt{\frac{k}{2\pi r_{122}}}H_0^{(1)}(kr_{122}) & \\
& & (2.72)\\
&\times\left[\frac{(\beta_1 - \sin\theta_{122}\sin\theta_2)}{(\beta_1 + \sin\theta_{22}\sin\theta_2)}\right]H[\sin\theta_{122}(\cos\theta_2 + \cos\theta_0) - 1], & \\
0 &< \theta < \pi, &
\end{aligned}$$

$$\begin{aligned}
\eta_{GA}(x,y,z) &\sim -\frac{e^{\frac{-3i\pi}{4}}}{4}\sqrt{\frac{k}{2\pi r_{121}}}H_0^{(1)}(kr_{121})H[\sin\theta_{121}(\cos\theta_1 + \cos\theta_0)], & \\
-\pi &< \theta < 0, & (2.73)
\end{aligned}$$

and

$$\begin{aligned}
r_{12}^2 &= (r+r_0)^2 + (z-z_0)^2, r_{121}^2 = {r_1}^2 + (z-z_0)^2,\\
r_{122}^2 &= {r_2}^2 + (z-z_0)^2,\\
kr_{12} &\to \infty, kr_{121} \to \infty, kr_{122} \to \infty,\\
-\pi &< \theta < \pi, 0 < \theta_0 < \pi, \theta \neq \pm(\pi - \theta_0).
\end{aligned}$$

2.4 CONCLUDING REMARKS

The physical interpretation of the result (2.70) in conjuction with *Fig.4.* is now obvious. The first term in Eq. (2.72) represents the incident spherical wave due to a point source at (x_0, y_0, z_0). The second term in Eq. (2.72) is the wave reflected from the upper impedance face of the half plane. This reflected wave appears to radiate from an image point source at $(x_0, -y_0, z_0)$, the reflection coefficient being $(\beta_1 - \sin\theta_{122}\sin\theta_2)(\beta_1 + \sin\theta_{122}\sin\theta_2)^{-1}$. These two terms show the geometrical acoustic field and they will not exist everywhere. The regions where they are present are governed by the Heaviside step functions which multiply the Hankel functions. Physically these regions correspond to the shadow region behind the screen, and the insonified regions. Eq. (2.71) gives the diffracted field, which is a spherical wave which appears to radiate from the edge of the half plane to all points in space. It is worth mentioning that the strength of the diffracted field dies down as $\frac{1}{\sqrt{r_0}}$. It is also of interest to note that the reflection coefficient vanishes identically if $\beta_1 = \sin\theta_{122}\sin\theta_2$. The criterion that the reflection coefficient should vanish means physically that the half plane absorbs all the energy incident upon it and does not reflect any. Finally, the results for the diffraction of a spherical field by a rigid half plane can be obtained as a special case by choosing $\beta_1 = \beta_2 = 0$.

Chapter 3

SCATTERING OF A SPHERICAL SOUND WAVE BY A RIGID SCREEN WITH AN ABSORBENT EDGE

In this chapter, we present the scattering of a spherical wave by a half plane in the presence of a moving fluid. A finite region in the vicinity of the edge has an impedance boundary condition; the remaining part of the half plane is rigid. The problem which is solved is a mathematical model for a rigid barrier with an absorbent edge in the presence of a moving fluid. Mathematical route which is used to solve the problem is the spatial Fourier transform, asymptotic methods and the W. H. technique. It is found that the absorbing material that comprises the edge need only be the order of a wavelength long to have approximately the same effect on the sound attenuation in the shadow region of the barrier as a semi-infinite absorbent barrier.

3.1 FORMULATION

We consider a small amplitude sound wave due to a point source on a main stream moving with a velocity U parallel to the x−axis. The source is assumed to be located at (x_0, y_0, z_0), $y_0 > 0$. A semi-infinite plane is assumed to occupy $y = 0, x \leq 0$ as shown in the $Fig.6$. The half plane is assumed to be infinitely thin, and over the interval $-l < x < 0$ there is an absorbing substance satisfying

$$p - u_n z_a = 0, \tag{3.1}$$

on both sides of the surface and the remainder $-\infty < x < -l$, of the half plane is rigid. In Eq. (3.1) u_n is the normal component of the perturbation velocity, z_a is the acoustic impedance of the plane. $\mathbf{n}$ is normal vector pointing from the fluid into the surface. The perturbation velocity $\mathbf{u}_1$ of the irrotational sound wave and the resulting pressure into the sound field are, respectively, given by

$$\mathbf{u}_1 = \mathbf{grad}\ \eta_t(x, y, z). \tag{3.2}$$

$$p_1 = -\rho_0 \left(\frac{\partial}{\partial t} + \mathrm{U}\frac{\partial}{\partial x} \right) \eta_t(x, y, z). \tag{3.3}$$

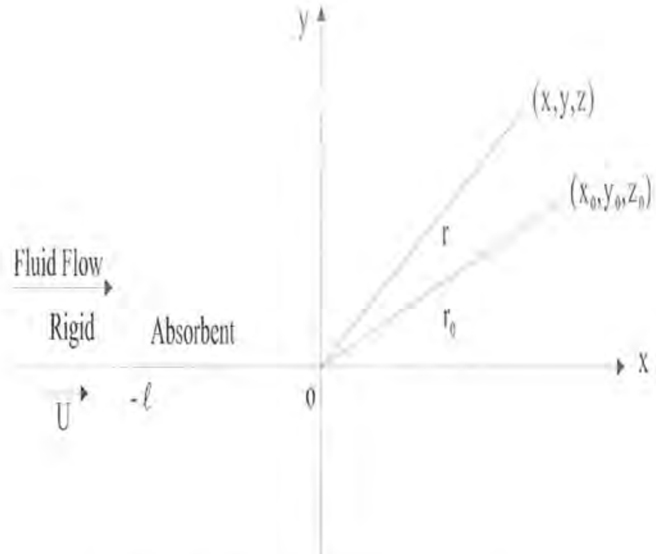

Fig. 6: Sketch of the geometry.

The governing problem becomes one of solving the convective Helmholtz equation

$$\begin{aligned}&\left[\left(1-M^2\right)\frac{\partial^2}{\partial x^2}+2ikM\frac{\partial}{\partial x}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}+k^2\right]\eta_t(x,y,z) \\ =\ & \delta(x-x_0)\delta(y-y_0)\delta(z-z_0),\end{aligned} \tag{3.4}$$

subject to the following boundary conditions:

$$\frac{\partial}{\partial y}\eta_t(x,0^{\pm},z)=0,\ x<-l, \tag{3.5}$$

$$\left[\frac{\partial}{\partial y}\mp\beta M\frac{\partial}{\partial x}\pm ik\beta\right]\eta_t(x,0^{\pm},z)=0,\ \ -l\leq x\leq 0, \tag{3.6}$$

$$\eta_t(x,0^+,z)\ =\ \eta_t(x,0^-,z),\ x>0, \tag{3.7}$$

$$\frac{\partial}{\partial y}\eta_t(x,0^+,z) = \frac{\partial}{\partial y}\eta_t(x,0^-,z), \; x > 0,$$

where $\beta(=\frac{\rho_0 c}{z_a})$ is the specific admittance of the absorbent surface, and $M = \mathrm{U}/c$ is the Mach number. For subsonic flow $|M| < 1$ and for acoustic absorption $\mathrm{Re}(z_a) > 0$.

It is assumed that a solution can be written in the form

$$\eta_t(x,y,z) = \eta_0(x,y,z) + \eta(x,y,z), \tag{3.8}$$

where $\eta_0(x,y,z)$ and $\eta(x,y,z)$ are the incident wave and the diffracted field.

In addition for a unique solution of the problem Eqs. (3.4) to (3.8), we insist that η_t represents an outward travelling wave as $r = \sqrt{x^2+y^2+z^2} \to \infty$ and satisfies the edge condition

$$\begin{aligned} \eta_t(x,0,z) &= O(1) \text{ and } \frac{\partial}{\partial y}\eta_t(x,0,z) = O(x^{\frac{-1}{2}}) \text{ as } x \to 0^+. \\ \eta_t(x,0,z) &= O(1) \text{ and } \frac{\partial}{\partial y}\eta_t(x,0,z) = O((x+l)^{\frac{-1}{2}}) \text{ as } x \to -l. \end{aligned}$$

3.2 SOLUTION OF THE PROBLEM

Transforming Eqs. (3.4) to (3.7) through Eq. (2.5) and making use of the subsonic substitutions

$$\begin{aligned} x &= \sqrt{1-M^2}X, \; x_0 = \sqrt{1-M^2}X_0, \; y = Y, \; y_0 = Y_0, \qquad (3.9) \\ z &= Z, \; z_0 = Z_0, \; k = \sqrt{1-M^2}\mathrm{K}, \; \beta = \sqrt{1-M^2}B, \\ l &= \sqrt{1-M^2}L, \chi_t(x,y,\mu) = \varphi_t(X,Y,\mu)e^{-i\mathrm{K}MX}. \end{aligned}$$

the boundary value problem takes the following form

$$\left[\frac{\partial^2}{\partial X^2} + \frac{\partial^2}{\partial Y^2} + K^2 s^2\right] \phi_0(X, Y, \mu) = \tilde{a}\delta(X - X_0)\delta(Y - Y_0). \tag{3.10}$$

$$\left[\frac{\partial^2}{\partial X^2} + \frac{\partial^2}{\partial Y^2} + K^2 s^2\right] \phi(X, Y, \mu) = 0, \tag{3.11}$$

$$\frac{\partial}{\partial Y}\phi(X, 0^{\pm}, \mu) = -\frac{\partial}{\partial Y}\phi_0(X, 0^{\pm}, \mu), X < -L. \tag{3.12}$$

$$\begin{aligned} &\left[\frac{\partial}{\partial Y} \mp BM\frac{\partial}{\partial X} \pm iKB\right] \phi(X, 0^{\pm}, \mu) \\ = &-\left[\frac{\partial}{\partial Y} \mp BM\frac{\partial}{\partial X} \pm iKB\right] \phi_0(X, 0^{\pm}, \mu), -L \leq X \leq 0. \end{aligned} \tag{3.13}$$

$$\begin{aligned} \phi(X, 0^+, \mu) - \phi(X, 0^-, \mu) &= \left[\phi_0(X, 0^-, \mu) - \phi_0(X, 0^+, \mu)\right], X > 0, \\ \frac{\partial}{\partial Y}\left[\phi(X, 0^+, \mu) - \phi(X, 0^-, \mu)\right] &= \frac{\partial}{\partial Y}\left[\phi_0(X, 0^-, \mu) - \phi_0(X, 0^+, \mu)\right], X > 0. \end{aligned} \tag{3.14}$$

where

$$\tilde{a} = \frac{e^{iKMX_0 - iK\sqrt{1-M^2}\mu Z_0}}{\sqrt{1-M^2}}, \; s^2 = \left[1 - \mu^2(1 - M^2)\right]. \tag{3.15}$$

Now we define the Fourier transform pair by

$$\overline{\phi}(\nu, Y, \mu) = \int_{-\infty}^{\infty} \phi(X, Y, \mu) e^{i\nu X} dX, \tag{3.16}$$

$$\phi(X, Y, \mu) = \frac{1}{\sqrt{2\pi}} \int_{-\infty+i\tau}^{\infty+i\tau} \overline{\phi}(\nu, Y, \mu)\, e^{-i\nu X} dX. \tag{3.17}$$

The transform (3.16) and its inverse (3.17) will exist provided $-\operatorname{Im}(Ks) < \tau < \operatorname{Im}(Ks)$. In order to accommodate three-part boundary conditions on $Y = 0$, we split $\overline{\phi}(\nu, Y, \mu)$ as

$$\overline{\phi}(\nu, Y, \mu) = e^{-i\nu L}\overline{\phi}_{-}(\nu, Y, \mu) + \overline{\phi}_{1}(\nu, Y, \mu) + \overline{\phi}_{+}(\nu, Y, \mu). \tag{3.18}$$

where

$$\begin{aligned}
\overline{\phi}_{-}(\nu, Y, \mu) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{-L} \phi(X, Y, \mu) e^{i\nu(X+L)} dX, \\
\overline{\phi}_{1}(\nu, Y, \mu) &= \frac{1}{\sqrt{2\pi}} \int_{-L}^{0} \phi(X, Y, \mu) e^{i\nu X} dX, \\
\overline{\phi}_{+}(\nu, Y, \mu) &= \frac{1}{\sqrt{2\pi}} \int_{0}^{\infty} \phi(X, Y, \mu) e^{i\nu X} dX.
\end{aligned} \tag{3.19}$$

In Eq. (3.19), $\overline{\phi}_{-}(\nu, Y, \mu)$ is regular for $\operatorname{Im}(\nu) < \operatorname{Im}(Ks)$, $\overline{\phi}_{+}(\nu, Y, \mu)$ is regular for $\operatorname{Im}(\nu) > -\operatorname{Im}(Ks)$, and $\overline{\phi}_{1}(\nu, Y, \mu)$ is an integral function. With the help of Eq. (3.16) and (3.17), the solutions of Eqs. (3.10) and (3.11) satisfying the radiation condition can be written as

$$\phi_0(X, Y, \mu) = b_1(\mu)\, e^{-iKs(X\cos_0 + Y\sin\theta_0)}, \tag{3.20}$$

$$\overline{\phi}(\nu, Y, \mu) = A_5(\nu) e^{i\vartheta_1 Y}, \;\; Y > 0, \tag{3.21}$$

$$\overline{\phi}(\nu, Y, \mu) = A_6(\nu) e^{-i\vartheta_1 Y}, \;\; Y < 0, \tag{3.22}$$

where $\vartheta_1 = \sqrt{K^2 s^2 - \nu^2}$ is defined on the cut sheet for which $\operatorname{Im}(\vartheta_1) > 0$ when $|\operatorname{Im}(\vartheta_1)| < \operatorname{Im} Ks$ and

$$b_1(\mu) = -\frac{\tilde{a}}{4i}\sqrt{\frac{2}{\pi K s R_0}}e^{i(KsR_0-\frac{\pi}{4})}, \tag{3.23}$$

$X_0 = R_0\cos\theta_0,\ Y_0 = R_0\sin\theta_0.$

Transforming the boundary conditions (3.12) to (3.14) we get

$$\frac{d}{dY}\overline{\phi}_-(\nu,0^\pm,\mu) = -\frac{d}{dY}\zeta_-(\nu,0,\mu), \tag{3.24}$$

$$\begin{aligned} &\frac{d}{dY}\overline{\phi}_1(\nu,0^+,\mu) + i(K+M\nu)B\overline{\phi}_1(\nu,0^+,\mu) \\ = &-\left[\frac{d}{dY}\zeta_1(\nu,0,\mu) + i(K+M\nu)B\zeta_1(\nu,0,\mu)\right], \end{aligned} \tag{3.25}$$

$$\begin{aligned} &\frac{d}{dY}\overline{\phi}_1(\nu,0^-,\mu) - i(K+M\nu)B\overline{\phi}_1(\nu,0^-,\mu) \\ = &-\left[\frac{d}{dY}\zeta_1(\nu,0,\mu) - i(K+M\nu)B\zeta_1(\nu,0,\mu)\right], \end{aligned} \tag{3.26}$$

$$\begin{aligned} \overline{\phi}_+(\nu,0^+,\mu) &= \overline{\phi}_+(\nu,0^-,\mu) = \overline{\phi}_+(\nu,0,\mu), \\ \frac{d}{dY}\overline{\phi}_+(\nu,0^+,\mu) &= \frac{d}{dY}\overline{\phi}_+(\nu,0^-,\mu) = \frac{d}{dY}\overline{\phi}_+(\nu,0,\mu), \end{aligned} \tag{3.27}$$

where

$$\zeta_1(\nu,Y,\mu) = \int_{-L}^{0}\phi_0(X,Y,\mu)e^{i\nu X}dX, \tag{3.28}$$

$$\frac{d}{dY}\zeta_-(\nu,Y,\mu) = \int_{-\infty}^{-L}\frac{\partial}{\partial Y}\phi_0(X,Y,\mu)e^{i\nu(X+L)}dX. \tag{3.29}$$

From Eqs. (3.18), (3.21), (3.22), (3.24) and (3.27) we arrive at

$$
\begin{aligned}
& -e^{-i\nu L}\frac{d}{dY}\zeta_-(\nu,0,\mu) + \frac{d}{dY}\overline{\phi}_1(\nu,0^+,\mu) + \frac{d}{dY}\overline{\phi}_+(\nu,0,\mu) \qquad (3.30) \\
= \; & i\vartheta_1\left[e^{-i\nu L}\overline{\phi}_-(\nu,0^+,\mu) + \overline{\phi}_1(\nu,0^+,\mu) + \overline{\phi}_+(\nu,0,\mu)\right].
\end{aligned}
$$

$$
\begin{aligned}
& -e^{-i\nu L}\frac{d}{dY}\zeta_-(\nu,0,\mu) + \frac{d}{dY}\overline{\phi}_1(\nu,0^-,\mu) + \frac{d}{dY}\overline{\phi}_+(\nu,0,\mu) \qquad (3.31) \\
= \; & -i\vartheta_1\left[e^{-i\nu L}\overline{\phi}_-(\nu,0^-,\mu) + \overline{\phi}_1(\nu,0^-,\mu) + \overline{\phi}_+(\nu,0,\mu)\right].
\end{aligned}
$$

Eliminating $\overline{\phi}_1(\nu,0^+,\mu)$ from Eqs. (3.25) and (3.30) and $\overline{\phi}_1(\nu,0^-,\mu)$ from Eqs. (3.26) and (3.31) gives

$$
\begin{aligned}
& i\vartheta_1\Bigg[e^{-i\nu L}\overline{\phi}_-(\nu,0^+,\mu) - \frac{1}{iB(K+M\nu)}\Bigg\{\frac{d}{dY}\overline{\phi}_1(\nu,0^+,\mu) \\
& + \left[\frac{d}{dY}\zeta_1(\nu,0^+,\mu)\right] + i(K+M\nu)B\zeta_1(\nu,0,\mu)\Bigg\} + \overline{\phi}_+(\nu,0,\mu)\Bigg] \\
= \; & -e^{-i\nu L}\frac{d}{dY}\zeta_-(\nu,0,\mu) + \frac{d}{dY}\overline{\phi}_1(\nu,0^+,\mu) + \frac{d}{dY}\overline{\phi}_+(\nu,0,\mu), \qquad (3.32) \\
& i\vartheta_1\Bigg[e^{-i\nu L}\overline{\phi}_-(\nu,0^-,\mu) + \frac{1}{iB(K+M\nu)}\Bigg\{\frac{d}{dY}\overline{\phi}_1(\nu,0^-,\mu) \\
& + \left[\frac{d}{dY}\zeta_1(\nu,0^+,\mu)\right] - i(K+M\nu)B\zeta_1(\nu,0,\mu)\Bigg\} + \overline{\phi}_+(\nu,0,\mu)\Bigg] \\
= \; & -e^{-i\nu L}\frac{d}{dY}\zeta_-(\nu,0,\mu) + \frac{d}{dY}\overline{\phi}_1(\nu,0^-,\mu) + \frac{d}{dY}\overline{\phi}_+(\nu,0,\mu). \qquad (3.33)
\end{aligned}
$$

In a similar way, elimination of $\frac{d}{dY}\overline{\phi}_1(\nu, 0^+, \mu)$ from Eqs. (3.25) and (3.30) and $\frac{d}{dY}\overline{\phi}_1(\nu, 0^-, \mu)$ from Eqs. (3.26) and (3.31) yields

$$\begin{aligned} & -e^{-i\nu L}\frac{d}{dY}\zeta_-(\nu, 0, \mu) - iB(K + M\nu)\overline{\phi}_1(\nu, 0^+, \mu) \\ & - \left[\begin{array}{c} \frac{d}{dY}\zeta_1(\nu, 0, \mu) \\ +iB(K + M\nu)\zeta_1(\nu, 0, \mu) \end{array} \right] + \frac{d}{dY}\overline{\phi}_+(\nu, 0, \mu) \\ = & \left[e^{-i\nu L}\overline{\phi}_-(\nu, 0^+, \mu) + \overline{\phi}_1(\nu, 0^+, \mu) + \overline{\phi}_+(\nu, 0, \mu)\right]. \end{aligned} \tag{3.34}$$

$$\begin{aligned} & -e^{-i\nu L}\frac{d}{dY}\zeta_-(\nu, 0, \mu) + iB(K + M\nu)\overline{\phi}_1(\nu, 0^-, \mu) \\ & - \left[\begin{array}{c} \frac{d}{dY}\zeta_1(\nu, 0, \mu) \\ -iB(K + M\nu)\zeta_1(\nu, 0, \mu) \end{array} \right] + \frac{d}{dY}\overline{\phi}_+(\nu, 0, \mu) \\ = & -\left[e^{-i\nu L}\overline{\phi}_-(\nu, 0^-, \mu) + \overline{\phi}_1(\nu, 0^-, \mu) + \overline{\phi}_+(\nu, 0, \mu)\right]. \end{aligned} \tag{3.35}$$

Subtracting Eq. (3.32) from Eq. (3.33) and adding Eq. (3.34) to Eq. (3.35) gives

$$e^{-i\nu L}\varphi_-(\nu) + Q(\nu)\varphi_1(\nu) + \varphi_+(\nu) = S(\nu). \tag{3.36}$$

$$e^{-i\nu L}\Lambda_-(\nu) + \sqrt{Ks - \nu}\; Q(\nu)\Lambda_1(\nu) + \Lambda_+(\nu) = N(\nu), \tag{3.37}$$

where

$$\begin{aligned} \varphi_-(\nu) &= \frac{1}{2}\left[\overline{\phi}_-(\nu, 0^+, \mu) + \overline{\phi}_-(\nu, 0^-, \mu)\right], \\ \varphi_1(\nu) &= \frac{i}{2}\left[\frac{d}{dY}\overline{\phi}_1(\nu, 0^+, \mu) - \frac{d}{dY}\overline{\phi}_1(\nu, 0^-, \mu)\right]. \end{aligned} \tag{3.38}$$

$$
\begin{aligned}
\varphi_+(\nu) &= \overline{\phi}_+(\nu,0,\mu),\ S(\nu)=\zeta_1(\nu,0,\mu),\\
\Lambda_-(\nu) &= -\frac{i\sqrt{Ks-\nu}}{2}\left[\overline{\phi}_-(\nu,0^+,\mu)-\overline{\phi}_-(\nu,0^-,\mu)\right],\\
\Lambda_1(\nu) &= -\frac{iB(K+M\nu)}{2}\left[\overline{\phi}_1(\nu,0^+,\mu)-\overline{\phi}_-(\nu,0^-,\mu)\right],\\
N(\nu) &= \frac{\left[\frac{d}{dY}\zeta_1(\nu,0,\mu)+e^{-i\nu L}\frac{d}{dY}\zeta_-(\nu,0,\mu)\right]}{\sqrt{Ks+\nu}}.\\
\overline{Q}(\nu) &= \left[\frac{1}{\vartheta}+\frac{1}{B(K+M\nu)}\right],
\end{aligned}
$$

$$
Q(\nu)=\left[1+\frac{B(K+M\nu)}{\vartheta}\right]=Q_+(\nu)Q_-(\nu). \tag{3.39}
$$

Explicit expressions for the functions $Q_\pm(\nu)$ has been discussed by Rawlins [96]. Following a similar procedure the final results are given by

$$
Q_\pm(\nu)=Q_\pm(0)\ \exp\int_0^\nu \Delta_\pm(\nu)d\nu,
$$

$$
Q_+(0)=Q_-(0)=\sqrt{1+B/s},
$$

$$
\begin{aligned}
\Delta_+(\nu) &= -\frac{1}{2(\nu+Ks)}+\frac{BK(Ms^2-\nu_1)\tilde{F}(\nu,\nu_1)}{\pi(1+B^2M^2)(\nu_1-\nu_2)}\\
&\quad -\frac{BK(Ms^2-\nu_2)\tilde{F}(\nu,\nu_2)}{\pi(1+B^2M^2)(\nu_1-\nu_2)},
\end{aligned}
$$

$$
|\Delta_-(-\nu)|_{M=-M}=-\Delta_+(\nu)
$$

$$
\tilde{F}(\nu,\nu_0)=\frac{1}{(\nu_1-\nu_0)}\left[f_1(\nu)-f_1(\nu_0)\right],
$$

$$
\nu_1=\frac{1}{(1+B^2M^2)}\left\{MB^2+\sqrt{(s^2-B^2+M^2B^2s^2)}\right\}.
$$

$$\nu_2 = \frac{1}{(1+B^2M^2)}\left\{MB^2 - \sqrt{(s^2 - B^2 + M^2B^2s^2)}\right\}.$$

$$f_1(\nu) = \frac{\arccos(\frac{\nu}{K})}{\sqrt{K^2s^2 - \nu^2}}.$$

With the help of Eqs.(3.18), (3.21), (3.22) and (3.27) we obtain the unknown functions

$$A_{5,6}(\nu) = \frac{1}{\vartheta Q(\nu)}\left[\begin{array}{c} e^{-i\nu L}\left\{\overline{\phi}_-(\nu,0^\pm,\mu) \pm \frac{1}{i(K+M\nu)B}\frac{d}{dY}\overline{\phi}_-(\nu,0^\pm,\mu)\right\} \\ +\left\{\overline{\phi}_1(\nu,0^\pm,\mu) \pm \frac{1}{i(K+M\nu)B}\frac{d}{dY}\overline{\phi}_1(\nu,0^\pm,\mu)\right\} \\ +\left\{\overline{\phi}_+(\nu,0,\mu) \pm \frac{1}{i(K+M\nu)B}\frac{d}{dY}\overline{\phi}_+(\nu,0,\mu)\right\} \end{array}\right]. \tag{3.40}$$

where + sign is used with the subscript 5 and the − sign with the subscript 6. The solution of the W. H. equations (3.36) and (3.37) can be obtained by employing the procedure of chapter 1 to arrive at

$$\begin{aligned} \frac{\Gamma_+(\nu)}{Q_+(\nu)} &= \frac{1}{2\pi i}\int_{-\infty+ia}^{\infty+ia}\frac{e^{i\zeta L}\Gamma_+(\zeta)}{Q_-(\zeta)(\zeta+\nu)}d\zeta \\ &\quad -\frac{1}{2\pi i}\int_{-\infty+ia}^{\infty+ia}\frac{[e^{i\zeta L}S(\zeta)+S(-\zeta)]}{Q_-(\zeta)(\zeta+\nu)}d\zeta, \end{aligned} \tag{3.41}$$

$$\begin{aligned} \frac{\gamma_+(\nu)}{Q_+(\nu)} &= -\frac{1}{2\pi i}\int_{-\infty+ia}^{\infty+ia}\frac{e^{i\zeta L}\gamma_+(\zeta)}{Q_-(\zeta)(\zeta+\nu)}d\zeta \\ &\quad -\frac{1}{2\pi i}\int_{-\infty+ia}^{\infty+ia}\frac{[-e^{i\zeta L}S(\zeta)+S(-\zeta)]}{Q_-(\zeta)(\zeta+\nu)}d\zeta. \end{aligned} \tag{3.42}$$

$$\frac{\Lambda_+(\nu)}{Q_+(\nu)} = \frac{1}{2\pi i}\int_{-\infty+ic}^{\infty+ic}\frac{e^{-i\zeta L}\Lambda_-(\zeta)}{Q_+(\zeta)(\zeta-\nu)}d\zeta - \frac{1}{2\pi i}\int_{-\infty+ic}^{\infty+ic}\frac{N(\zeta)}{Q_+(\zeta)(\nu-\zeta)}d\zeta. \tag{3.43}$$

$$\begin{aligned}\frac{\Lambda_-(\nu)}{Q_-(\nu)\sqrt{Ks-\nu}} &= -\frac{1}{2\pi i}\int_{-\infty+id}^{\infty+id}\frac{e^{i\zeta L}\Lambda_+(\zeta)}{Q_-(\zeta)(\nu-\zeta)\sqrt{Ks-\zeta}}d\zeta \qquad (3.44)\\ &\quad -\frac{1}{2\pi i}\int_{-\infty+id}^{\infty+id}\frac{e^{i\zeta L}N(\zeta)}{Q_-(\zeta)(\zeta-\nu)\sqrt{Ks-\zeta}}d\zeta.\end{aligned}$$

$$\Gamma_+(\nu) = \varphi_+(\nu) + \varphi_-(-\nu), \tag{3.45}$$

$$\gamma_+(\nu) = \varphi_+(\nu) - \varphi_-(-\nu), \tag{3.46}$$

$$\mathrm{Im}(\nu) > -a > -\mathrm{Im}(Ks),\ \mathrm{Im}(\nu) > c > -\mathrm{Im}(Ks),\ \mathrm{Im}(\nu) < d < \mathrm{Im}(Ks).$$

The solution of Eqs. (3.41) to (3.44) ultimately gives the solution of the boundary value problem. An exact solution of these integral equations is too difficult, and therefore approximate solutions will be obtained by asymptotic methods. From Eqs. (3.20), (3.28), (3.29) and (3.38) we get

$$\frac{d}{dY}\zeta_-(\nu,0,\mu) = \frac{-Ks\ \sin\theta_0 b_1(\mu)}{(\nu - Ks\cos\theta_0)}e^{iKsL\cos\theta_0}, \tag{3.47}$$

$$\zeta_1(\nu,0,\mu) = \frac{ib_1(\mu)}{(\nu - Ks\cos\theta_0)}\left(e^{-i(\nu-Ks\cos\theta_0)L} - 1\right), \tag{3.48}$$

$$\frac{d}{dY}\zeta_1(\nu,0,\mu) = \frac{Ks\ \sin\theta_0 b_1(\mu)}{(\nu - Ks\cos\theta_0)}\left(e^{-i(\nu-Ks\cos\theta_0)L} - 1\right). \tag{3.49}$$

$$S(\nu) = \frac{ib_1(\mu)}{(\nu - Ks\cos\theta_0)}\left(e^{-i(\nu - Ks\cos\theta_0)L} - 1\right). \tag{3.50}$$

$$N(\nu) = -\frac{b_1(\mu)Ks\sin\theta_0}{(\nu - Ks\cos\theta_0)\sqrt{\nu + Ks}}. \tag{3.51}$$

3.3 APPROXIMATE SOLUTION OF EQUATIONS (3.41) AND (3.42) FOR $KsL \geq 1$

Restricting the path of integration in expression (3.41) to the band $\mathrm{Im}(Ks\cos\theta_0) < a < \mathrm{Im}(Ks)$ (see *Fig.*7.) and then using Eq. (3.50), into Eq. (3.41) and making the further substitution

$$\Gamma_+(\nu) = G_+(\nu) - \frac{ib_1 e^{iKsL\cos\theta_0}}{(\nu + Ks\cos\theta_0)} - \frac{ib_1}{(\nu - Ks\cos\theta_0)}. \tag{3.52}$$

gives

$$\begin{aligned}
\frac{G_+(\nu)}{Q_+(\nu)} &= \frac{ib_1 e^{iKsL\cos\theta_0}}{(\nu + Ks\cos\theta_0)Q_+(\nu)} + \frac{ib_1}{(\nu - Ks\cos\theta_0)Q_+(\nu)} \\
&+\frac{1}{2\pi i}\int_{-\infty+ia}^{\infty+ia} \frac{e^{i\zeta L}G_+(\zeta)}{Q_-(\zeta)(\zeta+\nu)}d\zeta \\
&-\frac{b_1}{2\pi}\int_{-\infty+ia}^{\infty+ia} \frac{1}{Q_-(\zeta)(\zeta+\nu)}\left\{\frac{1}{(\zeta + Ks\cos\theta_0)} + \frac{e^{iKsL\cos\theta_0}}{(\nu - Ks\cos\theta_0)}\right\}. \\
&\mathrm{Im}(Ks\cos\theta_0) < a < \mathrm{Im}(Ks).
\end{aligned} \tag{3.53}$$

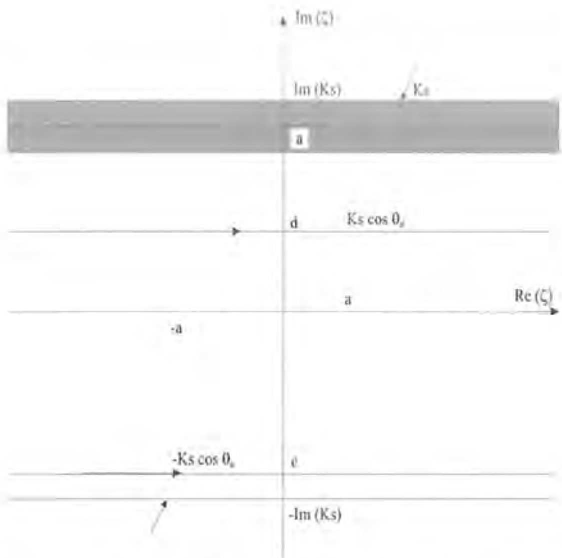

Fig. 7: Illustration of location of the poles in the complex plane.

The last two integrals appearing in Eq. (3.53) can be evaluated by distorting the path of integration into the lower half of the ζ plane. The only poles captured will be $\zeta = -\nu$ and $\zeta = -Ks\cos\theta_0$. Thus

$$
\begin{aligned}
\frac{G_+(\nu)}{Q_+(\nu)} &= \frac{ib_1 e^{iKsL\cos\theta_0}}{(\nu + Ks\cos\theta_0)Q_-(Ks\cos\theta_0)} \\
&\quad + \frac{ib_1}{(\nu - Ks\cos\theta_0)Q_+(Ks\cos\theta_0)} \\
&\quad + \frac{1}{2\pi i}\int_{-\infty+ia}^{\infty+ia} \frac{e^{i\zeta L} G_+(\zeta) Q_+(\zeta)}{Q(\zeta)(\zeta+\nu)} d\zeta, \\
&\quad \mathrm{Im}(Ks\cos\theta_0) < a < \mathrm{Im}(Ks).
\end{aligned}
\tag{3.54}
$$

Equation (3.42) can be dealt in a similar manner by substituting Eq. (3.50) for $S(\nu)$ into Eq. (3.42), where $\mathrm{Im}(Ks\cos\theta_0) < a < \mathrm{Im}(Ks)$, and making the substitution

$$\gamma_+(\nu) = g_+(\nu) + \frac{ib_1 e^{iKsL\cos\theta_0}}{(\nu + Ks\cos\theta_0)} - \frac{ib_1}{(\nu - Ks\cos\theta_0)}. \tag{3.55}$$

Thus one obtains eventually

$$\begin{aligned}\frac{g_+(\nu)}{Q_+(\nu)} &= -\frac{ib_1 e^{iKsL\cos\theta_0}}{(\nu + Ks\cos\theta_0)Q_-(Ks\cos\theta_0)} \\ &+\frac{ib_1}{(\nu - Ks\cos\theta_0)Q_+(Ks\cos\theta_0)} \\ &-\frac{1}{2\pi i}\int_{-\infty+ia}^{\infty+ia} \frac{e^{i\zeta L} g_+(\zeta)Q_+(\zeta)}{Q(\zeta)(\zeta+\nu)} d\zeta. \\ &\mathrm{Im}(Ks\cos\theta_0) < a < \mathrm{Im}(Ks).\end{aligned} \tag{3.56}$$

For the solutions of Eqs. (3.54) and (3.56), we use an asymptotic technique given by Jones [46] and the approximate solutions for $KsL \geq 1$, are respectively, given by

$$G_+(\nu) = (b_1 S_1(\nu) + \mathrm{G}_+(\nu))\, Q_+(\nu). \tag{3.57}$$

$$g_+(\nu) = (b_1 S_2(\nu) - \mathrm{g}_+(\nu))\, Q_+(\nu). \tag{3.58}$$

where

$$S_1(\nu) = \frac{ie^{iKsL\cos\theta_0}}{(\nu + Ks\cos\theta_0)Q_-(Ks\cos\theta_0)} + \frac{i}{(\nu - Ks\cos\theta_0)Q_+(Ks\cos\theta_0)}. \tag{3.59}$$

$$S_2(\nu) = -\frac{ie^{iKsL\cos\theta_0}}{(\nu + Ks\cos\theta_0)Q_-(Ks\cos\theta_0)} + \frac{i}{(\nu - Ks\cos\theta_0)Q_+(Ks\cos\theta_0)}. \tag{3.60}$$

$$\mathrm{G}_+(\nu) = \frac{1}{2\pi i}\int_{-\infty+ia}^{\infty+ia} \frac{e^{i\zeta L} G_+(\zeta) Q_+(\zeta)}{Q(\zeta)(\zeta+\nu)} d\zeta. \tag{3.61}$$

$$\mathrm{g}_+(\nu) = \frac{1}{2\pi i}\int_{-\infty+ia}^{\infty+ia} \frac{e^{i\zeta L} g_+(\zeta) Q_+(\zeta)}{Q(\zeta)(\zeta+\nu)} d\zeta. \tag{3.62}$$

Putting the values of $G_+(\zeta)$ from Eq. (3.57) in Eq. (3.61) and $g_+(\zeta)$ from Eq. (3.58) into Eq. (3.62), we have

$$\begin{aligned} \mathrm{G}_+(\nu) &= \frac{1}{2\pi i}\int_{-\infty+ia}^{\infty+ia} \frac{e^{i\zeta L} Q_+^2(\zeta)\left\{b_1 S_1(\zeta) + \mathrm{G}_+(\zeta)\right\}}{Q(\zeta)(\zeta+\nu)} d\zeta. \\ & \mathrm{Im}(Ks\cos\theta_0) < a < \mathrm{Im}(Ks). \end{aligned} \tag{3.63}$$

$$\mathrm{g}_+(\nu) = \frac{1}{2\pi i}\int_{-\infty+ia}^{\infty+ia} \frac{e^{i\zeta L} Q_+^2(\zeta)\left\{b_1 S_2(\zeta) - \mathrm{g}_+(\zeta)\right\}}{Q(\zeta)(\zeta+\nu)} d\zeta. \tag{3.64}$$

If the contour of integration in expressions(3.63) and (3.64) is distorted into the region $\mathrm{Im}(\nu) > a$, then the integrals can be asymptotically approximated, for $KsL \geq 1$, by the integrals with its path of integration wrapped around the branch cut $\zeta = Ks$. The part of the integrands of expressions (3.63) and (3.64) within the curly brackets are regular and analytic in this region and provided $\theta_0 \neq 0, \pi$; this term will vary slowly in the vicinity of $\zeta = Ks$. Thus since the dominant part of the integrands come from the region $\zeta = Ks$ the terms in the curly brackets can be removed under the integral sign and ζ can be replaced by Ks. The remaining integrals can be replaced by the asymptotic approximation $(C2)$ of the *Appendix C*. Hence

$$\mathrm{G}_+(\nu) \approx Q_+^2(Ks)\left(b_1 S_1(Ks) + \mathrm{G}_+(Ks)\right) W(\nu). \tag{3.65}$$

$$g_+(\nu) \approx Q_+^2(Ks)\,(b_1 S_2(Ks) - g_+(Ks))\,W(\nu), \tag{3.66}$$

where

$$\begin{aligned} W(\nu) &= \frac{\sqrt{2Ks}}{\left[1-\lambda_1^2(\nu+Ks)\right]}\left\{W_0\left[\left(L\sqrt{\nu+Ks}\right)\right] - W_0\left[\left(\sqrt{L}/\lambda_1\right)\right]\right\}. \\ \lambda_1 &= \frac{\sqrt{2Ks}}{B(K+M\nu)} \end{aligned}$$

and $G_+(Ks)$ and $g_+(Ks)$ are obtained by putting $\nu = Ks$ in Eqs. (3.65) and (3.66) respectively and solving the resulting equations for $G_+(Ks)$ and $g_+(Ks)$.

Using the expressions (3.45), (3.46), (3.52), (3.55), (3.57), (3.58), (3.65) and (3.66) and some simple manipulation one obtains

$$\begin{aligned} \varphi_+(\nu) &= -\frac{ib_1}{(\nu - Ks\cos\theta_0)}\left[1 - \frac{Q_+(\nu)}{Q_+(Ks\cos\theta_0)}\right] + \frac{b_1 Q_+^2(Ks)W(\nu)Q_+(\nu)}{2} \\ &\quad \times \left\{S_1(Ks) - S_2(Ks) + G_+(Ks)/b_1 + g_+(Ks)/b_1\right\}, \qquad (3.67) \\ \varphi_-(\nu) &= \frac{ib_1 e^{iKsL\cos\theta_0}}{(\nu - Ks\cos\theta_0)}\left[1 - \frac{Q_-(\nu)}{Q_-(Ks\cos\theta_0)}\right] + \frac{b_1 Q_+^2(Ks)W(-\nu)Q_-(\nu)}{2} \\ &\quad \times \left\{S_1(Ks) + S_2(Ks) + G_+(Ks)/b_1 - g_+(Ks)/b_1\right\}. \qquad (3.68) \end{aligned}$$

3.4 APPROXIMATE SOLUTION OF EQUATIONS (3.43) AND (3.44) FOR $KsL \geq 1$

The method of obtaining an approximate solution for equations (3.43) and (3.44) is slightly different from the method used in the last section. Substituting the expression for $N(\nu)$, Eq. (3.51), into expressions (3.43) and (3.44) gives

$$\begin{aligned}\frac{\Lambda_+(\nu)}{Q_+(\nu)} &= \frac{Ksb_1\sin\theta_0}{2\pi i}\int_{-\infty+ic}^{\infty+ic}\frac{d\zeta}{Q_+(\zeta)(\nu-\zeta)(\zeta-Ks\cos\theta_0)\sqrt{\zeta+Ks}} \\ &-\frac{1}{2\pi i}\int_{-\infty+ic}^{\infty+ic}\frac{e^{-i\zeta L}\Lambda_-(\zeta)}{Q_+(\zeta)(\zeta-\nu)}d\zeta, \text{Im }(\nu)>c>-\text{Im}(Ks).\end{aligned} \tag{3.69}$$

$$\begin{aligned}\frac{\Lambda_-(\nu)}{Q_-(\nu)\sqrt{Ks-\nu}} &= -\frac{1}{2\pi i}\int_{-\infty+id}^{\infty+id}\frac{e^{i\zeta L}\Lambda_+(\zeta)}{Q_-(\zeta)(\nu-\zeta)\sqrt{Ks-\zeta}}d\zeta+\frac{Ksb_1\sin\theta_0}{2\pi i} \\ &\times\int_{-\infty+id}^{\infty+id}\frac{e^{i\zeta L}\,d\zeta}{Q_-(\zeta)(\zeta-\nu)(\zeta-Ks\cos\theta_0)\sqrt{K^2s^2-\zeta^2}}, \\ &\text{Im }(\nu)<d<\text{Im}(Ks\cos\theta_0).\end{aligned} \tag{3.70}$$

The first integral of expression (3.69) can be evaluated by distorting the path of integration into the upper ζ plane. Since Im $(\nu) > c > -\text{Im}(Ks)$ then the pole at $\zeta=\nu$ and $\zeta = Ks\cos\theta_0$ will give rise to residue contributions (see $Fig.7$). Hence

$$\begin{aligned}\frac{\Lambda_+(\nu)}{Q_+(\nu)} &= -\frac{Ks\,b_1\sin\theta_0}{Q_+(\nu)(\nu-Ks\cos\theta_0)\sqrt{Ks+\nu}} \\ &+\frac{b_1\left[Ks\,(1-\cos\theta_0)\right]^{\frac{1}{2}}}{Q_+(Ks\cos\theta_0)(\nu-Ks\cos\theta_0)} \\ &-\frac{1}{2\pi i}\int_{-\infty+ic}^{\infty+ic}\frac{e^{-i\zeta L}\left\{\Lambda_-(\zeta)\sqrt{Ks-\zeta}Q_-(\zeta)\right\}}{Q(\zeta)(\zeta-\nu)\sqrt{Ks-\zeta}}d\zeta. \\ \text{Im }(\nu) &> c>-\text{Im}(Ks).\end{aligned} \tag{3.71}$$

The evaluation of the second integral in Eq. (3.70) is best achieved by distorting the path of integration into the upper half of the ζ plane. However this requires a knowledge of the singularities of $Q_-(\zeta)$ in $\mathrm{Im}(\zeta) > -\mathrm{Im}(Ks)$. Since the only singularities of $Q(\zeta)$ are the branch points at $\zeta = \pm Ks$; no poles occur in the cut plane. Hence moving the path of integration vertically until it crosses the pole $\zeta = Ks\cos\theta_0$ (see *Fig.*7), but not the branch point $\zeta = Ks$; gives

$$\begin{aligned}
&\frac{\Lambda_-(\nu)}{Q_-(\nu)\sqrt{Ks-\nu}} \\
= & -\frac{b_1 e^{iKsL\cos\theta_0}}{Q_-(Ks\cos\theta_0)(\nu - Ks\cos\theta_0)} \\
&+\frac{1}{2\pi i}\int_{-\infty+id}^{\infty+id} d\zeta \left\{\frac{Ks\, b_1 \sin\theta_0 Q_+(\zeta)}{\sqrt{Ks+\zeta}(\zeta - Ks\cos\theta_0)} + \Lambda_+(\zeta)Q_+(\zeta)\right\} \\
&\times\frac{e^{i\zeta L}}{Q(\zeta)\sqrt{Ks-\zeta}(\zeta-\nu)}, \mathrm{Im}(Ks\cos\theta_0) < d < \mathrm{Im}(Ks). \qquad (3.72)
\end{aligned}$$

For $KsL > 1$, the dominant contribution of the integral in expression (3.71) comes from the region $\zeta = -Ks$; and of the integral in expression (3.72) from the region $\zeta = Ks$. Provided $\theta_0 \neq 0$ the term in the curly bracket of the integrands in expressions (3.71) and (3.72) are slowly varying in the vicinity of $\zeta = \pm Ks$. One can therefore replace ζ by $-Ks$ in this part of the integrand in expression (3.71) and remove it from under the integral sign. Similarly one can replace ζ by Ks in the curly part of the integrand in expression (3.72) and remove it from under the integral sign. The integrals remaining can be replaced by the asymptotic approximation (C'6) and (C'8) of *Appendix C*. Thus

$$\frac{\Lambda_+(\nu)}{Q_+(\nu)} = -\frac{Ks\, b_1 \sin\theta_0}{Q_+(\nu)(\nu - Ks\cos\theta_0)\sqrt{Ks+\nu}} \qquad (3.73)$$

$$+\frac{b_1\left[Ks\left(1-\cos\theta_0\right)\right]^{\frac{1}{2}}}{Q_+(Ks\cos\theta_0)(\nu-Ks\cos\theta_0)}+W(\nu)Q_+(Ks)\Lambda_-(-Ks),$$

$$\begin{aligned}\frac{\Lambda_-(\nu)}{Q_-(\nu)\sqrt{Ks-\nu}} &= -\frac{b_1 e^{iKsL\cos\theta_0}}{Q_-(Ks\cos\theta_0)(\nu-Ks\cos\theta_0)} \\ &\quad -\frac{W(-\nu)Q_+(Ks)}{B(K+M\nu)}\left\{\frac{\sin\theta_0}{(1-\cos\theta_0)}+\sqrt{2Ks}\Lambda_+(Ks)\right\}.\end{aligned} \tag{3.74}$$

The constants $\Lambda_\pm(\pm Ks)$ are obtained by putting $\nu = Ks$ in expression (3.73) and $\nu = -Ks$ in expression (3.74) and solving the resulting two equations for the two unknowns $\Lambda_\pm(\pm Ks)$.

The expression (3.38) in conjunction with the expressions (3.67), (3.68), (3.73) and (3.74) will now give explicit expressions for $\overline{\phi}_+(\nu,0,\mu)$, $\frac{d}{dY}\overline{\phi}_+(\nu,0,\mu)$, $\overline{\phi}_-(\nu,0^+,\mu)$ and $\overline{\phi}_-(\nu,0^-,\mu)$. Thus, using these values and the boundary conditions (3.24) to (3.26) in equations (3.40) it is not difficult to show that

$$A_5(\nu) = b_1(\nu)A_7(\nu), \tag{3.75}$$

$$A_6(\nu) = b_1(\nu)A_8(\nu), \tag{3.76}$$

where

$$\begin{aligned}A_7(\nu) &= \frac{1}{\vartheta_1 Q(\nu)}\left[\frac{i}{(\nu-Ks\cos\theta_0)}\left\{-\frac{2Q_-(\nu)e^{-i(\nu-Ks\cos\theta_0)L}}{Q_-(Ks\cos\theta_0)}\right.\right. \\ &\quad \left.+\frac{Q_+(\nu)}{Q_+(Ks\cos\theta_0)}-\frac{\left[Ks\left(1-\cos\theta_0\right)\right]^{\frac{1}{2}}\sqrt{\nu+Ks}\,Q_+(\nu)}{Q_+(Ks\cos\theta_0)B(K+M\nu)}\right\} \\ &\quad +e^{-i\nu L}Q_+(Ks)Q_-(\nu)\left\{\frac{Q_+(Ks)}{2}\left[\begin{array}{c} S_1(Ks)+S_2(Ks) \\ +\mathrm{G}_+(Ks)/b_1-\mathrm{g}_+(Ks)/b_1\end{array}\right]\right.\end{aligned}$$

$$-\frac{i}{B(K+M\nu)}\left[\frac{\sin\theta_0}{1-\cos\theta_0}+\frac{\sqrt{2Ks}}{b_1}\Lambda_+(Ks)\right]\Bigg\}W(-\nu)$$
$$+\frac{Q_+^2(Ks)W(\nu)Q_+(\nu)}{2}\left\{\begin{array}{c} S_1(Ks)-S_2(Ks) \\ +G_+(Ks)/b_1+g_+(Ks)/b_1 \end{array}\right\}$$
$$+\frac{W(\nu)Q_+(Ks)\Lambda_-(-Ks)Q_+(\nu)\sqrt{\nu+Ks}}{ib_1B(K+M\nu)}\Bigg]. \quad (3.77)$$

$$\begin{aligned} A_8(\nu) &= \frac{1}{\vartheta_1 Q(\nu)}\left[\frac{i}{(\nu-Ks\cos\theta_0)}\left\{\frac{Q_+(\nu)}{Q_+(Ks\cos\theta_0)}\right.\right. \\ &+\left.\frac{[Ks(1-\cos\theta_0)]^{\frac{1}{2}}\sqrt{\nu+Ks}\,Q_+(\nu)}{Q_+(Ks\cos\theta_0)B(K+M\nu)}\right\}+e^{-i\nu L}Q_+(Ks)Q_-(\nu) \\ &\times\left\{\frac{Q_+(Ks)}{2}\left[S_1(Ks)+S_2(Ks)+G_+(Ks)/b_1-g_+(Ks)/b_1\right]\right. \\ &+\left.\frac{i}{B(K+M\nu)}\left[\frac{\sin\theta_0}{1-\cos\theta_0}+\frac{\sqrt{2Ks}}{b_1}\Lambda_+(Ks)\right]\right\}W(-\nu) \\ &+\frac{Q_+^2(Ks)W(\nu)Q_+(\nu)}{2}\left\{\begin{array}{c} S_1(Ks)-S_2(Ks) \\ +G_+(Ks)/b_1+g_+(Ks)/b_1 \end{array}\right\} \\ &-\left.\frac{W(\nu)Q_+(Ks)\Lambda_-(-Ks)Q_+(\nu)\sqrt{\nu+Ks}}{ib_1B(K+M\nu)}\right]. \quad (3.78) \end{aligned}$$

Now using Eqs. (3.21), (3.22), (3.77) and (3.78) in Eq. (2.6) and the employing the procedure of chapter 1 for the solution of integrals in inverse transforms we obtain

$$\begin{aligned} &\eta(x,y,z) \quad (3.79) \\ &= \frac{K\sin\theta\,\sin\Theta_{12}A_7(-K\cos\theta\sin\Theta_{12})e^{-iKM(X-X_0)}e^{i(KR_{12}+\frac{3\pi}{4})}}{4\pi\sqrt{KRR_0R_{12}(1-M^2)}}, Y>0, \\ &\eta(x,y,z) \quad (3.80) \\ &= \frac{K\sin\theta\,\sin\Theta_{12}A_8(-K\cos\theta\sin\Theta_{12})e^{-iKM(X-X_0)}e^{i(KR_{12}+\frac{3\pi}{4})}}{4\pi\sqrt{KRR_0R_{12}(1-M^2)}}, Y<0, \end{aligned}$$

where

$$
\begin{aligned}
(Z - Z_0)^2 + (R + R_0)^2 &= R_{12}^2,\ X^2 + Y^2 = R^2, \\
R &= r\left[\frac{1 - M^2\sin^2\theta}{1 - M^2}\right]^{\frac{1}{2}},\cos\Theta_{12} = \frac{\cos\theta}{\sqrt{1 - M^2\sin^2\theta}}.
\end{aligned}
$$

and $A_7(-K\cos\theta\sin\Theta)$ and $A_8(-K\cos\theta\sin\Theta)$ are given by Eqs. (3.77) and (3.78) respectively.

3.5 CONCLUSIONS

We have solved a new canonical diffraction problem of a **spherical** wave in the presence of a **moving fluid**. From Eqs.(3.79) and (3.80), we observe that as a result of fluid motion the field is increased by the factor $\left(\sqrt{1 - M^2}\right)^{-1}$ in comparison to still fluid. Also, the field is independent of the direction of flow since the fluid velocity U appears as $|\mathrm{U}|^2$ in the factor $\left(\sqrt{1 - M^2}\right)^{-1}$. The results for still air case can be obtained by putting $M = 0$. It is also interesting to note that Eqs. (3.79) and (3.80) represent fields diffracted from the edges $x = 0$ and $x = -l$. The radiated sound intensity in the illuminated region $0 < \theta < \pi$ is due to constructive/destructive interference between the incident wave; the diffracted fields from the edges $(0, 0)$ and the joint $(0, -l)$ between the absorptive strip and the rigid region of the screen.

Chapter 4

THE TRANSIENT RESPONSE OF A SPHERICAL GAUSSIAN PULSE BY AN ABSORBING HALF PLANE

In this chapter, we discuss the acoustic wave diffraction due to a spherical **pulse** near an absorbing half plane introducing the Kutta-Joukowski condition (wake condition). The whole system is assumed to be in a moving fluid. The temporal Fourier transform is used to calculate the diffracted field. It is once again found [48] that the field produced by the Kutta-Joukowski condition will be substantially in excess of that in its absence when the source is near the edge even in the case of spherical Gaussian pulse. It is also found that ratio of the diffracted fields when the wake is absent to that due to the wake for the spherical Gaussian pulse is the same as for the cylindrical Gaussian pulse. Thus the ratio of no wake to wake situation is independent of the type of acoustic sources. We also note that the ratio of the diffracted field corresponding to no wake to wake situation for rigid half plane is the same as calculated by Balasubramanyam [10]. This ratio for rigid half plane can be recorded by equating the absorption parameter to zero.

4.1 FORMULATION OF THE PROBLEM

We consider a small amplitude sound wave on a main stream moving with subsonic velocity U. An absorbing plane is assumed to occupy $y = 0, x \leq 0$ as shown in the Fig.8. We consider a spherical Gaussian pulse from a source parallel to the edge at (x_0, y_0, z_0). The convective wave equation satisfied by ψ_1 in the presence of a point source is

$$\begin{aligned} \nabla^2\psi_1(x,y,z;t) - \left[\frac{1}{c}\frac{\partial}{\partial t} + M\frac{\partial}{\partial x}\right]^2 \psi_1(x,y,z;t) \qquad (4.1) \\ = \delta(x-x_0)\delta(y-y_0)\delta(z-z_0)\frac{v}{\sqrt{\pi}}e^{-v^2t^2}, \end{aligned}$$

subject to the boundary conditions:

$$\left[\frac{\partial}{\partial y} \mp \beta M\frac{\partial}{\partial x} \mp \frac{\beta}{c}\frac{\partial}{\partial t}\right]\psi_1(x,0^{\pm},z;t) = 0, x \leq 0, \qquad (4.2)$$

where ∇^2 is the usual Laplacian. We choose the coefficients of the Gaussian pulse to be $\frac{v}{\sqrt{\pi}}$ so that the strength of the pulse $\int_{-\infty}^{\infty} \frac{v}{\sqrt{\pi}}e^{-v^2t^2}dt$, is unity. We shall assume that flow is subsonic, $-1 < M < 1$ (for a leading edge situation $-1 < M \leq 0$ and for a trailing edge $0 < M < 1$). The trailing edge problem adds the complication of a trailing vortex sheet or wake attached to the absorbing half plane. The usual edge conditions give rise to a field which is singular at the origin for the trailing edge situation. Therefore, the Kutta-Joukowski condition is imposed to obtain a unique solution of the problem. In order to satisfy this condition, we introduce a discontinuity in the field at the aperture $(0 < x < \infty)$ and postulate the existence of a wake condition [48]. According to this, ψ_1 is discontinuous whilst $\frac{\partial}{\partial y}\psi_1$ and pressure remain continuous for $y = 0, x > 0$

$$\begin{aligned} \frac{\partial}{\partial y}\psi_1(x, 0^+, z; t) &= \frac{\partial}{\partial y}\psi_1(x, 0^-, z; t), x > 0, \\ \left[\frac{\partial}{\partial t} + \mathrm{U}\frac{\partial}{\partial x}\right]\psi_1(x, 0^+, z; t) &= \left[\frac{\partial}{\partial t} + \mathrm{U}\frac{\partial}{\partial x}\right]\psi_1(x, 0^-, z; t), x > 0. \end{aligned} \tag{4.3}$$

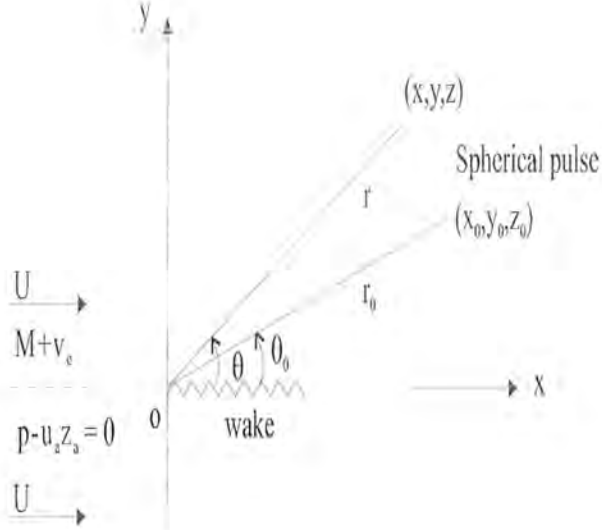

Fig. 8: The scattering geometry.

4.2 SOLUTION OF THE PROBLEM

We define the temporal Fourier transform and its inverse by

$$\widehat{\psi}(x, y, z; \omega) = \int_{-\infty}^{\infty} \psi_1(x, y, z; t) e^{i\omega t} dt, \tag{4.4}$$

$$\psi_1(x, y, z; t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \widehat{\psi}(x, y, z; \omega) e^{-i\omega t} d\omega. \tag{4.5}$$

By analogy to the time harmonic problem, we use ω as the variable of the Fourier transform. Transforming Eqs. (4.1) to (4.3), we obtain

$$\begin{aligned} & \left[(1 - M^2)\frac{\partial^2}{\partial x^2} + 2ikM\frac{\partial}{\partial x} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} + k^2\right] \widehat{\psi}(x, y, z; \omega) \quad (4.6) \\ = \;& \delta(x - x_0)\delta(y - y_0)\delta(z - z_0) e^{-\omega^2/4v^2}, \end{aligned}$$

$$\left[\frac{\partial}{\partial y} \mp \beta M \frac{\partial}{\partial X} \pm ik\beta\right] \widehat{\psi}(x, 0^{\pm}, z; \omega) = 0, x \leq 0. \tag{4.7}$$

$$\frac{\partial}{\partial y}\widehat{\psi}(x, 0^{+}, z; \omega) = \frac{\partial}{\partial y}\widehat{\psi}(x, 0^{-}, z; \omega), x > 0. \tag{4.8}$$

$$\left[-ik + M\frac{\partial}{\partial x}\right] \widehat{\psi}(x, 0^{+}, z; \omega) = \left[-ik + M\frac{\partial}{\partial x}\right] \widehat{\psi}(x, 0^{-}, z; \omega), x > 0. \tag{4.9}$$

The boundary condition (4.9) can be written in the alternative form as

$$\widehat{\psi}(x, 0^{+}, z; \omega) - \widehat{\psi}(x, 0^{-}, z; \omega) = \alpha_1(z)\, e^{ikx/M}, \tag{4.10}$$

where $\alpha_1(z)$ can be determined by means of the Kutta-Joukowski condition. We note that $\alpha_1(z) = 0$ corresponds to the no wake situation.

The solution of the boundary value problem consisting of Eqs. (4.6) to (4.10) can be obtained using the standard W. H. technique and asymptotic methods. The detail of calculations for $\widehat{\psi}_d$ is given in *Appendix D*. Thus, the diffracted field $\widehat{\psi}_d$ for the spherical Gaussian pulse is given by

$$\begin{aligned}
\widehat{\psi}_d(x, y, z; \omega) &= \widetilde{C}_{11} \int_{-\infty}^{\infty} \frac{e^{iKs(R+R_0)+iK\sqrt{1-M^2}\mu(Z-Z_0)} F(\left|\widetilde{Q}\right|) d\mu}{\sqrt{1-(1-M^2)\mu^2}\left[K\sqrt{1-(1-M^2)\mu^2}\right]^{\frac{1}{2}} N_1(s)} \\
&\quad + \widetilde{C}_{22} \int_{-\infty}^{\infty} \frac{e^{iKs(R+R_0)+iK\sqrt{1-M^2}\mu(Z-Z_0)} F(\left|\widetilde{Q}\right|)}{\left[K\sqrt{1-(1-M^2)\mu^2}\right]^{\frac{1}{2}} N_1(s)} d\mu \\
&\quad - \widetilde{C}_{33} \int_{-\infty}^{\infty} \frac{e^{iKs(R+R_0)+iK\sqrt{1-M^2}\mu(Z-Z_0)} F(\left|\widetilde{Q}_1\right|)}{\left[K\sqrt{1-(1-M^2)\mu^2}\right]^{\frac{1}{2}} N_1(s)} d\mu. \qquad (4.11)
\end{aligned}$$

where

$$\widetilde{C}_{11} = \frac{BK\sqrt{1-M^2}}{2\pi}\widetilde{C}_1 e^{-\omega^2/4v^2},$$

$$\widetilde{C}_{22} = \frac{K\sqrt{1-M^2}}{2\pi}\left[BM\cos\theta_0 - 2\sin(\theta/2)\sin(\theta_0/2)\right]\widetilde{C}_1 e^{-\omega^2/4v^2},$$

$$\widetilde{C}_{33} = \frac{K\sqrt{1-M^2}}{\pi}\sin(\theta/2)\sin(\theta_0/2)\widetilde{C}_1 e^{-\omega^2/4v^2}.$$

$$N_1(s) = Q_+(Ks\cos\theta)Q_-(-Ks\cos\theta_0).$$

The integrals appearing on the right side of Eq. (4.11) can be evaluated asymptotically using the method of steepest descent (see Appendix E) and the diffracted field $\widehat{\psi}_d$ is written as

$$\widehat{\psi}_d = \widehat{\psi}_{dA} + \widehat{\psi}_{dW}, \tag{4.12}$$

where $\widehat{\psi}_{dA}$ denotes that part of $\widehat{\psi}_d$ which arises when there is no wake and $\widehat{\psi}_{dW}$ when there is a wake and are explicitly given by

$$\begin{aligned}\widehat{\psi}_{dA} &= \left[\frac{BR_{12}}{R+R_0} + BM\cos\theta_0 - 2\sin(\theta/2)\sin(\theta_0/2)\right] e^{-\omega^2/4v^2} \\ &\times \frac{e^{-iKM(X-X_0)}e^{iKR_{12}-\frac{i\pi}{4}}\sqrt{R+R_0+A_1}F(\tau_{R_{12}})}{4\pi\sqrt{\pi R_0}\sqrt{1-M^2}\sin\theta\sqrt{R_{12}(R_{12}+R_{11})}N(\zeta_2)},\end{aligned} \tag{4.13}$$

$$\begin{aligned}\widehat{\psi}_{dW} &= -\frac{\sin(\theta_0/2)e^{-iKM(X-X_0)}e^{-\omega^2/4v^2}}{4\pi\sqrt{\pi R_0}\sqrt{1-M^2}\cos(\theta/2)} \\ &\times \left\{\frac{\sqrt{\pi A_1'}e^{iKR_{11}'}}{2R_{11}'N(\zeta_1')}H(-\epsilon_1') + \epsilon_1' e^{iKR_{12}-\frac{i\pi}{4}}\left[\frac{\sqrt{R+R_0+A_1'}}{\sqrt{R_{12}(R_{12}+R_{11})}}\right]\frac{F(\tau'_{R_{12}})}{N(\zeta_2)}\right\}.\end{aligned} \tag{4.14}$$

where

$$
\begin{aligned}
R_{11} &= \sqrt{(Z-Z_0)^2 + A_1^2},\ R_{11}' = \sqrt{(Z-Z_0)^2 + A_1'^2}, \\
A_1 &= R + R_0 - \mu_1^2,\ A_1 = R + R_0 - (\mu_1')^2, \\
\mu_1 &= \frac{\cos\theta + \cos\theta_0}{\sin\theta}\sqrt{\frac{R}{2}},\ \mu_1' = \frac{\frac{1}{M} - \cos\theta}{\sin\theta}\sqrt{\frac{R}{2}}, \\
\tau_{R_{12}} &= \left[\frac{\sqrt{K(R+R_0+A_1)}}{\sqrt{(R_{12}+R_{11})}}\right]\mu_1, \tau_{R_{12}'} = \left[\frac{\sqrt{K(R+R_0+A_1')}}{\sqrt{(R_{12}'+R_{11}')}}\right]\mu_1', \\
\zeta_2 &= \frac{R+R_0}{R_{12}},\ \zeta_1' = \frac{A_1'}{R_{11}'},\ \epsilon_1' = sgn\tau_{R_{12}'}.
\end{aligned}
$$

When the source is very close to the edge ($KR_0 << 1$) and the point of observation is at a large distance from the source but not near the wake, the dominant part of $\widehat{\psi}_d$ denoted by $\widehat{\psi}_{d1}$ is given by

$$
\widehat{\psi}_{d1} = \widehat{\psi}_{dA1} + \widehat{\psi}_{dW1}. \tag{4.15}
$$

where

$$
\begin{aligned}
\widehat{\psi}_{dA1} &= -\left[\frac{B\widetilde{R}_{12}}{R+R_0} + BM\cos\theta_0 - 2\sin(\theta/2)\sin(\theta_0/2)\right] e^{-\omega^2/4v^2} \qquad (4.16) \\
&\left[\frac{KRR_0}{2\pi\widetilde{R}_{12}(1-M^2)}\right]^{\frac{1}{2}} \frac{e^{-iKM(X-X_0)}e^{iK\widetilde{R}_{12}-\frac{i\pi}{4}}}{2\pi\widetilde{R}_{12}N(\zeta_2)},
\end{aligned}
$$

$$
\widehat{\psi}_{dW1} = -\frac{ie^{-iKM(X-X_0)}e^{iK\widetilde{R}_{12}-\frac{i\pi}{4}}}{2\pi\sqrt{2\pi KRR_0\widetilde{R}_{12}(1-M^2)}N(\zeta_2)(\cos\theta - \frac{1}{M})}\sin(\theta/2)\sin(\theta_0/2)e^{-\omega^2/4v^2}. \tag{4.17}
$$

$$
\widetilde{R}_{12} = \sqrt{R^2 + (Z-Z_0)^2}.
$$

Now using $k = \omega/c$, $F(\nu) \approx \frac{i}{2\nu}$ (when $\nu \to \infty$) and $N(\zeta_2)$ in Eq. (4.15), we obtain

$$\widehat{\psi}_d(x,y,z;\omega) \approx (\mathcal{F} + \mathcal{F}_1)\,\frac{e^{-i\omega[M(X-X_0)-R_{12}]/\Im}}{\sqrt{\omega}}e^{-\omega^2/4v^2}, \tag{4.18}$$

where

$$\begin{aligned}\mathcal{F} &= \left[\frac{BR_{12}}{R+R_0} + BM\cos\theta_0 - 2\sin(\theta/2)\sin(\theta_0/2)\right]\sqrt{\Im/(1-M^2)} \\ &\times \frac{1}{4\pi\sqrt{2\pi RR_0R_{12}}(\cos\theta+\cos\theta_0)Q_+(K\zeta_2\cos\theta)Q_-(-K\zeta_2\cos\theta)},\end{aligned} \tag{4.19}$$

$$\mathcal{F}_1 = \sqrt{\Im/(1-M^2)}\frac{\sin(\theta/2)\sin(\theta_0/2)e^{-iKM(X-X_0)}(\frac{1}{M}-\cos\theta)^{-1}}{2\pi\sqrt{2\pi RR_0R_{12}}(\cos\theta+\cos\theta_0)Q_+(K\zeta_2\cos\theta)Q_-(-K\zeta_2\cos\theta)}, \tag{4.20}$$

where

$$\Im = c\sqrt{1-M^2}.$$

It is important to note that $Q_+(K\zeta_2\cos\theta)$ and $Q_-(-K\zeta_2\cos\theta)$, appearing in Eqs. (4.19) and (4.20) are independent of ω.

In order to calculate the field $\psi_{1d}(x,y,z;t)$, we need to find the inverse temporal Fourier transform of Eq. (4.18). This gives, on using Eq. (4.5)

$$\psi_{1d}(x,y,z;t) \approx \frac{(\mathcal{F}+\mathcal{F}_1)}{2\pi}\int_{-\infty}^{\infty}\frac{e^{-i\omega[MR'\cos\Theta_{12}-R_{12}]/\Im}}{\sqrt{\omega}}e^{-\omega^2/4v^2-i\omega t}d\omega, \tag{4.21}$$

where

$$R' = \sqrt{(X-X_0)^2+(Y-Y_0)^2}.$$

The integral appearing in Eq. (4.21) can be evaluated with the help of Mathematica [1]. Thus,

$$\begin{aligned}\psi_{1d}(x,y,z;t) \;\approx\; & \frac{(\mathcal{F}+\mathcal{F}_1)}{2}(1-i)^{\frac{3}{4}}e^{\frac{i\pi}{8}}ve^{-v^2(b_2-t)^2/2} \\ & \times(b_2-t)^{\frac{-5}{8}}\left[(1-i)^{\frac{1}{4}}(b_2-t)^{\frac{3}{4}}Bessel\ I\left(\frac{-1}{4},\frac{-v^2(b_2-t)^2}{2}\right)\right. \\ & \left.+2^{\frac{3}{4}}e^{\frac{i\pi}{4}}(\pi)^{-\frac{1}{4}}Bessel\ I\left(\frac{1}{4},\frac{-v^2(b_2-t)^2}{2}\right)\right], \qquad (4.22)\end{aligned}$$

where $b_2 = \left[R_{12} - MR^{'}\cos\Theta_{12}\right]/\Im$ and $Bessel\ I[n_1,t]$ is the modified Bessel function of the first kind of order n_1 in t.

4.3 DISCUSSION

A complete analytical description has been provided for the scattering of a spherical Gaussian pulse for trailing edge (wake present) situation. Of particular significance are the following points:

(i). It is good to note that the wave profile at $y = y_0$, $z = z_0$ moves along the direction of $x - axis$ with the velocity $c +$ U, which is due to the fact that the fluid is moving in the x-direction.

(ii). The ratio of $\widehat{\psi}_{dA1}$ to $\widehat{\psi}_{dW1}$ is found to be

$$\frac{\widehat{\psi}_{dA1}}{\widehat{\psi}_{dW1}} \approx i\frac{[B(1+M\cos\theta_0) - 2\sin(\theta/2)\sin(\theta_0/2)]KR_0(\cos\theta - \frac{1}{M})}{\sin(\theta/2)\sin(\theta_0/2)}. \qquad (4.23)$$

Eq. (4.23) gives the ratio of the diffracted wave when the wake is absent to that due to the wake for the point source. If we calculate this ratio for the line source situation [96] , who has not explicitly shown it), we find that both the ratios are exactly the same. Thus the ratio of no wake to wake situation is independent of the type of acoustic sources.

(*iii*). For the rigid half plane if we put $\beta = 0$ in Eq.. (4.23), this ratio becomes

$$\frac{\widehat{\psi}_{dA1}}{\widehat{\psi}_{dW1}} = -2iKR_0(\cos\theta - \frac{1}{M}). \tag{4.24}$$

We note that ratio is the same as calculated by Balasubramanyam [10]. For small Mach number, the ratio (4.24) is effectively $2iKR_0/M$ and is independent of angle. If KR_0 is of the order of πM this ratio is of the order of 2π. Consequently, the dependence of the intensity on Mach number would be M^5 whether the Kutta-Joukowski condition were imposed or not.

At any rate, observations of the sound intensity at low Mach number in a moving medium would fail to detect whether or not a Kutta-Joukowski condition has been imposed., if the observations are not near the wake and are limited to the dependence on angle and Mach number. This conclusion remains unmodified for quadrupoles since the ratio of the two terms is not essentially altered by derivatives with respect to either R_0 or θ_0.

(*iv*). We also conclude from Eq. (4.23) that for point sources near the edge of absorbing half plane ($R_0 \to 0$), the field caused by the Kutta-Joukowski condition will be substantially in excess of that in its absence as discussed by Jones [48]. Also, the imposition of the Kutta-Joukowski condition and the associated wake has the effect of producing a stronger scattered field away from the wake than that in the neighborhood of the wake an intense sound is created; it is much stronger than the scattered field away from the wake and does not decay downstream. This is true whether or not the sound be near the edge.

(v). Near the wake θ is small and an additional term is required in Eq. (4.12). This extra term is given by

$$\frac{\sin(\theta_0/2)e^{-iKM(X-X_0)+iKR_{12}}\sqrt{2M}sgn(Y)}{2\pi\widetilde{R}_2\sqrt{1-M^2}\sqrt{1+M}Q_-(-K\widetilde{l}\cos\theta_0/\widetilde{R}_2)Q_+(K\widetilde{l}/M\widetilde{R}_2)}$$
$$\times\left[1+\frac{X+\sqrt{M^2-1}\,|Y|}{MR_0}\right]^{\frac{1}{2}},$$

where

$$\widetilde{R}_2=\sqrt{(Z-Z_0)^2+R_0^2},\ \widetilde{l}=R_0+\frac{X}{M}+\frac{\sqrt{M^2-1}\,|Y|}{M}..$$

It is imperative to note the smaller that M becomes the more closely is the surface wave confined to the wake. It is the pressure of this wave which is the main distinguishing feature in the radiated-sound between the absence or otherwise of the Kutta-Joukowski condition. It is good to note that the surface wave disappears from the pressure, but remains in velocity. Therefore, measurements of the pressure fluctuations alone will not indicate the existence of the surface wave. However, if the product of pressure and velocity is taken as a measure of energy, differences in the energy due to the surface wave will manifest.

(vi) The results of leading edge situation for spherical Gaussian pulse can be obtained easily by taking $\mathcal{F}_1 = 0$ in Eq. (4.22).

Appendix A

FACTORIZATION OF THE MATRIX $K(\nu)$

The purpose of this appendix is to present the complete factorization of the matrix $K(\nu)$ given by the expression (2.28). We assume throughout $k\lambda$ is real, i.e. $\text{Im}(k\lambda) = 0$. The end results are analytic functions of $k\lambda$ which will be valid for $\text{Im}(k\lambda) \geq 0$. We shall reduce the problem of factorization to the solution of a set of Hilbert problems. These Hilbert problems are then solved using Muskhelishvili's approach [79].

REDUCTION OF MATRIX FACTORIZATION PROBLEM TO HILBERT PROBLEMS

We assume a factorization of the form

$$K(\nu) = U(\nu)L^{-1}(\nu), \tag{A1}$$

where

$$L(\nu) = \begin{bmatrix} l_{11}(\nu) & l_{12}(\nu) \\ l_{21}(\nu) & l_{22}(\nu) \end{bmatrix}, \tag{A2}$$

$$U(\nu) = \begin{bmatrix} u_{11}(\nu) & u_{12}(\nu) \\ u_{21}(\nu) & u_{22}(\nu) \end{bmatrix}. \tag{A3}$$

The elements l_{ij}, of $L(\nu)$ are assumed to be analytic in the cut ν plane $|\arg(k\lambda - \nu)| < \pi$. The elements of U_{ij} are analytic in the cut ν plane $|\arg(k\lambda + \nu)| < \pi$. This means that $L(\nu)$ is analytic everywhere except along the branch cut $k\lambda \leq \nu < \infty, \mathrm{Im}(\nu) = 0$; and $U(\nu)$ is analytic everywhere except along the branch cut $-\infty < \nu \leq -k\lambda, \mathrm{Im}(\nu) = 0$.

From Eq. (2.28) we note that

$$Det\, K(\nu) = -i\frac{(\vartheta + k\beta_1)(\vartheta + k\beta_2)}{2\vartheta} \neq 0, \tag{A4}$$

in the cut ν plane, since $-\frac{\pi}{2} < \arg(\vartheta) < \frac{\pi}{2}, \mathrm{Re}(k\beta_1) \geq 0$, and $\mathrm{Re}(k\beta_2) \geq 0$. Hence $K(\nu)$ and, consequently, $U(\nu)$ as well as $L^{-1}(\nu)$ are non-singular matrices in the cut ν plane.

We now analytically evaluate the left side, and consequently the right side of $(A1)$, about the branch cut at $\nu = -k\lambda$. This gives

$$\begin{aligned} K^+(\zeta) &= U^+(\zeta)L^{-1}(\zeta), -\infty < \zeta < -k\lambda, \\ K^-(\zeta) &= U^-(\zeta)L^{-1}(\zeta), -\infty < \zeta < -k\lambda, \end{aligned} \tag{A5}$$

where in this appendix only we use the notation $F^+(\zeta) = F(|\zeta|\, e^{i\pi})$ to denote values of F on the upper side of the cut, and $F^-(\zeta) = F(|\zeta|\, e^{-i\pi})$ to denote values of F on the lower side of the cut. We remark that, in ($A5$), $L^{-1}(\nu)$ does not jump in value on crossing this cut because it is analytic at $\nu = \zeta$, $-\infty < \zeta \leq -k\lambda$.

Eliminating $L^{-1}(\zeta)$ in Eq. (A5), we get

$$U^+(\zeta) = K^+(\zeta)\left[K^{-1}(\zeta)\right]^- U^-(\zeta), -\infty < \zeta < -k\lambda, \tag{A6}$$

where

$$K^+(\zeta) = \frac{1}{2}\begin{bmatrix} (-|\vartheta| + ik\beta_1) & -\frac{(-|\vartheta|+ik\beta_1)}{|\vartheta|} \\ (-|\vartheta| + ik\beta_2) & \frac{(-|\vartheta|+ik\beta_2)}{|\vartheta|} \end{bmatrix}, \tag{A7}$$

$$\left[K^{-1}(\zeta)\right]^- = \begin{bmatrix} (|\vartheta| + ik\beta_1)^{-1} & (|\vartheta| + ik\beta_2)^{-1} \\ |\vartheta|\left(|\vartheta| + ik\beta_1\right)^{-1} & -|\vartheta|\left(|\vartheta| + ik\beta_2\right)^{-1} \end{bmatrix}, \tag{A8}$$

$$K^+(\zeta)\left[K^{-1}(\zeta)\right]^- = \begin{bmatrix} 0 & \frac{(-|\vartheta|+ik\beta_1)}{(|\vartheta|+ik\beta_2)} \\ \frac{(-|\vartheta|+ik\beta_2)}{(|\vartheta|+ik\beta_1)} & 0 \end{bmatrix}, \tag{A9}$$

and

$$\vartheta = \pm i\sqrt{\zeta^2 - k^2\lambda^2} = \pm i\,|\vartheta| \text{ for } \nu = -\zeta e^{\pm i\pi}, -\infty < \zeta < -k\lambda. \tag{A10}$$

From Eqs. ($A6$) and ($A9$), we obtain

$$u_{11}^+(\zeta) = \left[\frac{-|\vartheta| + ik\beta_1}{|\vartheta| + ik\beta_2}\right] u_{21}^-(\zeta), \tag{A11}$$

$$u_{21}^+(\zeta) = \left[\frac{-|\vartheta| + ik\beta_2}{|\vartheta| + ik\beta_1}\right] u_{11}^-(\zeta), \tag{A12}$$

$$u_{12}^+(\zeta) = \left[\frac{-|\vartheta| + ik\beta_1}{|\vartheta| + ik\beta_2}\right] u_{22}^-(\zeta), \tag{A13}$$

$$u_{22}^{+}(\zeta) = \left[\frac{-|\vartheta| + ik\beta_2}{|\vartheta| + ik\beta_1}\right] u_{12}^{-}(\zeta), -\infty < \zeta < -k\lambda. \quad (A14)$$

Eqs. $(A11)$ and $(A12)$ form a coupled system of Hilbert problems for u_{11} and u_{21}. Similarly equations $(A13)$ and $(A14)$ form a coupled system of Hilbert problems for u_{12} and u_{22}. If we can solve the coupled Hilbert problems

$$u_{1}^{+}(\zeta) = \left[\frac{-|\vartheta| + ik\beta_1}{|\vartheta| + ik\beta_2}\right] u_{2}^{-}(\zeta), \quad (A15)$$

$$u_{2}^{+}(\zeta) = \left[\frac{-|\vartheta| + ik\beta_2}{|\vartheta| + ik\beta_1}\right] u_{1}^{-}(\zeta), -\infty < \zeta < -k\lambda, \quad (A16)$$

then we can also solve equations $(A11)$ to $(A14)$.

SOLUTION OF THE HILBERT PROBLEMS (A15) AND (A16)

Taking logarithms of Eqs. $(A15)$ and $(A16)$ and then adding and subtracting the resulting equations, we get the two uncoupled equations

$$[\log V(\zeta)]^{+} - [\log V(\zeta)]^{-} = \log\left[\left(\frac{-|\vartheta| + ik\beta_1}{|\vartheta| + ik\beta_1}\right)\left(\frac{-|\vartheta| + ik\beta_2}{|\vartheta| + ik\beta_2}\right)\right]. \quad (A17)$$

$$[\log W(\zeta)]^{+} + [\log W(\zeta)]^{-} = \log\left(\frac{|\vartheta|^2 + k^2\beta_1^2}{|\vartheta|^2 + k^2\beta_2^2}\right), -\infty < \zeta < -k\lambda; \quad (A18)$$

where

$$V(\zeta) = u_1(\zeta)u_2(\zeta), \quad (A19)$$

$$W(\zeta) = \frac{u_1(\zeta)}{u_2(\zeta)}, \quad (A20)$$

and $\left(\sqrt{k\lambda+\nu}\right)^{\pm} = \pm i\,|k\lambda+\zeta|^{\frac{1}{2}}$.

Equations ($A17$) and ($A18$) are standard Hilbert problems whose solution is given by

$$V(\nu) = \exp\left[\frac{1}{2\pi i}\int_{-\infty}^{-k\lambda} \log\left\{\left(\frac{|\vartheta| - ik\beta_1}{|\vartheta| + ik\beta_1}\right)\left(\frac{|\vartheta| - ik\beta_2}{|\vartheta| + ik\beta_2}\right)\right\}\frac{dt}{t-\nu}\right]. \quad (A21)$$

$$W(\nu) = \exp\left[-\frac{\sqrt{k\lambda+\nu}}{2\pi}\int_{-\infty}^{-k\lambda} \frac{1}{|k\lambda+t|^{\frac{1}{2}}}\log\left\{\left(\frac{|\vartheta|^2 + k^2\beta_1^2}{|\vartheta|^2 + k^2\beta_2^2}\right)\right\}\frac{dt}{t-\nu}\right]. \quad (A22)$$

Obviously the exponents of $V(\nu)$ and $W(\nu)$, and consequently $V(\nu)$ and $W(\nu)$, are analytic in $|\arg(k\lambda+\nu)| < \pi$; furthermore $V(\nu) \neq 0$ and $W(\nu) \neq 0$ in $|\arg(k\lambda+\nu)| < \pi$. Eqs. ($A21$) and ($A22$) can be simplified further by carrying out the integrations, see *Appendix B*. In particular it is shown there that

$$W(\nu) = O(1), \text{ and } V(\nu) = O(1), \text{ as } |\nu| \to \infty, |\arg(k\lambda+\nu)| < \pi; \quad (A23)$$

$$W(\nu) = O(1), \text{ and } V(\nu) = O((k\lambda+\nu)^{-1}), \text{ as } |\nu| \to -k\lambda, \mathrm{Re}(\beta_{1,2}) > 0. \quad (A24)$$

Thus particular solutions of ($A15$) and ($A16$) are given, from ($A19$) and ($A20$), as follows:

$$u_1(\nu) = -\sqrt{V(\nu)W(\nu)}. \quad (A25)$$

$$u_2(\nu) = \sqrt{\frac{V(\nu)}{W(\nu)}}, \tag{A26}$$

where

$$\sqrt{V(\nu)} = \exp\left[\frac{1}{2}J(\nu)\right], \tag{A27}$$

$$\sqrt{W(\nu)} = \left[\frac{\left(\sqrt{k\lambda+\nu} + \sqrt{k\aleph_1(+)}\right)\left(\sqrt{k\lambda+\nu} + \sqrt{k\aleph_1(-)}\right)}{\left(\sqrt{k\lambda+\nu} + \sqrt{k\aleph_2(+)}\right)\left(\sqrt{k\lambda+\nu} + \sqrt{k\aleph_2(-)}\right)}\right]^{\frac{1}{2}}. \tag{A28}$$

and $J(\nu)$ is given by the expression $(B13)$ of *Appendix B*.

The choice of sign, on taking the square roots, for $u_1(\nu)$ and $u_2(\nu)$ in $(A25)$ to $(A28)$ is justified as follows. With the signs given by $(A25)$ to $(A28)$ we have

$$\frac{u_1^+(\zeta)}{u_2^-(\zeta)} = -\sqrt{\frac{V^+(\zeta)W^+(\zeta)W^-(\zeta)}{V^-(\zeta)}}. \tag{A29}$$

By means of Plemelj's formula [79], $(A21)$ gives

$$\frac{V^+(\zeta)}{V^-(\zeta)} = \left(\frac{|\vartheta| - ik\beta_1}{|\vartheta| + ik\beta_1}\right)\left(\frac{|\vartheta| - ik\beta_2}{|\vartheta| + ik\beta_2}\right), -\infty < \zeta < -k\lambda; \tag{A30}$$

and from $(A28)$

$$\begin{aligned}\sqrt{W^+(\zeta)W^-(\zeta)} &= \left[\frac{\left(\sqrt{k\lambda+\zeta} + \sqrt{k\aleph_1(+)}\right)\left(\sqrt{k\lambda+\zeta} + \sqrt{k\aleph_1(-)}\right)}{\left(\sqrt{k\lambda+\zeta} + \sqrt{k\aleph_2(+)}\right)\left(\sqrt{k\lambda+\zeta} + \sqrt{k\aleph_2(-)}\right)}\right]^{\frac{1}{2}} \\ &= \left[\left(\frac{|\vartheta| - ik\beta_1}{|\vartheta| + ik\beta_2}\right)\left(\frac{|\vartheta| + ik\beta_1}{|\vartheta| - ik\beta_2}\right)\right]^{\frac{1}{2}}, -\infty < \zeta < -k\lambda.\end{aligned} \tag{A31}$$

Hence

$$-\sqrt{\frac{V^+(\zeta)}{V^-(\zeta)}}\sqrt{W^+(\zeta)W^-(\zeta)} = \left(\frac{-|\vartheta| + ik\beta_1}{|\vartheta| + ik\beta_2}\right), \tag{A32}$$

and therefore

$$\frac{u_1^+(\zeta)}{u_2^-(\zeta)} = \left(\frac{-|\vartheta| + ik\beta_1}{|\vartheta| + ik\beta_2}\right), \tag{A33}$$

which is clearly consistent with $(A15)$.

It is emphasized that the above result for $u_1(\nu)$ and $u_2(\nu)$ is just a particular solution, and not the general solution. To obtain the general solution we must impose further conditions on the functions $u_1(\nu)$ and $u_2(\nu)$ that we are interested in. First, we require that

$$u_1(\nu) = O\left[(k\lambda+\nu)^{\delta_1}\right],\ u_2(\nu) = O\left[(k\lambda+\nu)^{\frac{1}{2}+\delta_1}\right], as\ \nu \to -k\lambda. \tag{A34}$$

for some $\delta_{1,2} > -1$, in order to guarantee the convergence of the integrals (46) and (47), the singularity at $t = -k\lambda$ being integrable. Second, it is customary for the Hilbert problem to require that $u_1(\nu)$ and $u_2(\nu)$ have finite degrees at infinity, that is, $u_1(\nu)$ and $u_2(\nu)$ have polynomial growth as $|\nu| \to \infty$.

To determine the general solution for $u_1(\nu)$ and $u_2(\nu)$ under these conditions, substitute

$$\begin{aligned} u_1(\nu) &= -\sqrt{V(\nu)W(\nu)}u_1^*(\nu), \\ u_2(\nu) &= -\sqrt{\frac{V(\nu)}{W(\nu)}}u_2^*(\nu), \end{aligned} \tag{A35}$$

into the Eqs.($A15$) and ($A16$), leading to the vector Hilbert problem

$$[u_1^*(\zeta)]^+ = [u_2^*(\zeta)]^- \,,\ [u_2^*(\zeta)]^+ = [u_1^*(\zeta)]^- \,, -\infty < \zeta < -k\lambda. \tag{A36}$$

under the conditions

$$u_1^*(\nu) = O\left[(k\lambda+\nu)^{\delta_1+\frac{1}{2}}\right],\ u_2^*(\nu) = O\left[(k\lambda+\nu)^{\delta_2+1}\right], as\ \nu \to -k\lambda. \tag{A37}$$

and $u_1^*(\nu)$ and $u_2^*(\nu)$ have finite degree at infinity.

After addition and subtraction, the Hilbert problem becomes uncoupled. i.e.

$$[u_1^*(\zeta)+u_2^*(\zeta)]^+ = [u_1^*(\zeta)+u_2^*(\zeta)]^- \,, -\infty < \zeta < -k\lambda. \tag{A38}$$

$$\left[\frac{u_1^*(\zeta)-u_2^*(\zeta)}{\sqrt{k\lambda+\zeta}}\right]^+ = \left[\frac{u_1^*(\zeta)-u_2^*(\zeta)}{\sqrt{k\lambda+\zeta}}\right]^- , -\infty < \zeta < -k\lambda. \tag{A39}$$

where $\left[\sqrt{k\lambda+\zeta}\right]^\pm = \pm i\,|k\lambda+\zeta|^{\frac{1}{2}}$. As the functions $u_1^*(\nu)+u_2^*(\nu)$ and $\frac{u_1^*(\nu)-u_2^*(\nu)}{\sqrt{k\lambda+\nu}}$ are continuous across the branch cut, they are analytic in the entire ν plane except possibly at $\nu = -k\lambda$. Such a possibility is ruled out by the requirement $\delta_{1,2} > 1$, which ensures that there can be no pole at $\nu = -k\lambda$. In conclusion, $u_1^*(\nu)+u_2^*(\nu)$ and $\frac{u_1^*(\nu)-u_2^*(\nu)}{\sqrt{k\lambda+\nu}}$ are entire functions.

The second requirement of $u_1^*(\nu)$ and $u_2^*(\nu)$ having finite degree at infinity, combined with Liouville's theorem, then yields

$$\begin{aligned} u_1^*(\nu)+u_2^*(\nu) &= 2P_1(\nu), \\ \frac{u_1^*(\nu)-u_2^*(\nu)}{\sqrt{k\lambda+\nu}} &= 2P_2(\nu), \end{aligned} \tag{A40}$$

$$
\begin{aligned}
u_1^*(\nu) &= P_1(\nu) + P_2(\nu)\sqrt{k\lambda + \nu}, \\
u_2^*(\nu) &= P_1(\nu) - P_2(\nu)\sqrt{k\lambda + \nu},
\end{aligned} \tag{A41}
$$

where $P_1(\nu)$ and $P_2(\nu)$ are arbitrary polynomials. With the help of Eqs. ($A35$) and ($A41$), the general solution for $u_1(\nu)$ and $u_2(\nu)$ is given by

$$
u_1(\nu) = -\sqrt{V(\nu)W(\nu)}\left[P_1(\nu) + P_2(\nu)\sqrt{k\lambda + \nu}\right], \tag{A42}
$$

$$
u_2(\nu) = \sqrt{\frac{V(\nu)}{W(\nu)}}\left[P_1(\nu) - P_2(\nu)\sqrt{k\lambda + \nu}\right]. \tag{A43}
$$

SOLUTION OF THE EQUATION (A11) TO (A14)

After using Eqs. ($A42$) and ($A43$), the matrix elements $u_{ij}(\nu)$, satisfying ($A11$) **to** ($A14$), are given by

$$
u_{11}(\nu) = -\sqrt{V(\nu)W(\nu)}\left[P_{11}(\nu) + P_{21}(\nu)\sqrt{k\lambda + \nu}\right]. \tag{A44}
$$

$$
u_{21}(\nu) = \sqrt{\frac{V(\nu)}{W(\nu)}}\left[P_{11}(\nu) - P_{21}(\nu)\sqrt{k\lambda + \nu}\right], \tag{A45}
$$

$$
u_{12}(\nu) = -\sqrt{V(\nu)W(\nu)}\left[P_{12}(\nu) + P_{22}(\nu)\sqrt{k\lambda + \nu}\right]. \tag{A46}
$$

$$
u_{22}(\nu) = \sqrt{\frac{V(\nu)}{W(\nu)}}\left[P_{12}(\nu) - P_{22}(\nu)\sqrt{k\lambda + \nu}\right], \tag{A47}
$$

where $P_{ij}(\nu)$ $(i, j = 1, 2)$ are as yet arbitrary polynomials.

The matrix $U(\nu)$ can be written more compactly as

$$U(\nu) = U^{(0)}(\nu)P(\nu), \tag{A48}$$

where

$$U^{(0)}(\nu) = \begin{bmatrix} -\sqrt{V(\nu)W(\nu)} & -\sqrt{V(\nu)W(\nu)(k\lambda+\nu)} \\ \sqrt{\frac{V(\nu)}{W(\nu)}} & -\sqrt{\frac{V(\nu)}{W(\nu)}(k\lambda+\nu)} \end{bmatrix}. \tag{A49}$$

Finally we must ensure that $U(\nu)$ and $L(\nu)$ are non-singular in the cut ν plane. This puts some restrictions on P_{ij}. The exact restrictions are determined by looking at $Det\ U(\nu)$ and $Det\ L(\nu)$. Thus,

$$Det\,U(\nu) = Det\,U^{(0)}(\nu)\ Det P(\nu) = 2V(\nu) Det P(\nu)\sqrt{k\lambda+\nu}. \tag{A50}$$

$$\begin{aligned} Det L(\nu) &= Det K^{-1}(\nu)\ Det\,U(\nu) \\ &= \frac{2i\vartheta}{(\vartheta+k\beta_1)(\vartheta+k\beta_2)} 2V(\nu) Det P(\nu)\sqrt{k\lambda+\nu}. \end{aligned} \tag{A51}$$

Therefore U and L are non-singular in the cut ν plane if $Det P(\nu) \neq 0$ for all ν. Since $Det P(\nu)$ is a polynomial, one must have $Det P(\nu)$ = constant, i.e. a polynomial of zero degree.

The matrix factorization is not unique, and it is desirable that the polynomials P_{ij} have lowest possible degree, in order that both sides of the split Eqs. (2.40) and (2,41), also have lowest possible degree at infinity. Then the best choice for $P(\nu)$ is

$$P(\nu) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Thus

$$u_{11}(\nu) = -\sqrt{V(\nu)W(\nu)}, \tag{A52}$$

$$u_{21}(\nu) = \sqrt{\frac{V(\nu)}{W(\nu)}}, \tag{A53}$$

$$u_{12}(\nu) = -\sqrt{V(\nu)W(\nu)(k\lambda + \nu)}, \tag{A54}$$

$$u_{22}(\nu) = -\sqrt{\frac{V(\nu)}{W(\nu)}(k\lambda + \nu)}, \tag{A55}$$

where $\sqrt{V(\nu)}$ and $\sqrt{W(\nu)}$ are given by Eqs. ($A27$) and ($A28$). The elements of $L(\nu)$ after using Eqs. ($A52$) to ($A55$) into $L(\nu) = K^{-1}(\nu)U(\nu)$ are given by

$$l_{11}(\nu) = \frac{i\sqrt{V(\nu)W(\nu)}}{\vartheta + k\beta_1} - \frac{i}{\vartheta + k\beta_2}\sqrt{\frac{V(\nu)}{W(\nu)}}, \tag{A56}$$

$$l_{12}(\nu) = \frac{i\sqrt{V(\nu)W(\nu)(k\lambda + \nu)}}{\vartheta + k\beta_1} + \frac{i}{\vartheta + k\beta_2}\sqrt{\frac{V(\nu)(k\lambda + \nu)}{W(\nu)}}, \tag{A57}$$

$$l_{21}(\nu) = \frac{-\vartheta\sqrt{V(\nu)W(\nu)}}{\vartheta + k\beta_1} - \frac{\vartheta}{\vartheta + k\beta_2}\sqrt{\frac{V(\nu)}{W(\nu)}}, \tag{A58}$$

$$l_{22}(\nu) = \frac{-\vartheta\sqrt{V(\nu)W(\nu)(k\lambda + \nu)}}{\vartheta + k\beta_1} + \frac{\vartheta}{\vartheta + k\beta_2}\sqrt{\frac{V(\nu)(k\lambda + \nu)}{W(\nu)}}. \tag{A59}$$

ASYMPTOTIC GROWTH ESTIMATES

From the expressions $(A52)$ to $(A59)$, and the results $(B15)$, $(B17)$ of *Appendix B*, we obtain the following growth estimates for large $|\nu|$,

$$\begin{aligned} u_{11}(\nu) &= O(1), \qquad u_{12}(\nu) = O(\nu^{\frac{1}{2}}), \\ u_{21}(\nu) &= O(1), \qquad u_{22}(\nu) = O(\nu^{\frac{1}{2}}), \end{aligned} \tag{A60}$$

Det $U(\nu) = O(\nu^{\frac{1}{2}})$, as $|\nu| \to \infty$ in $|\arg(k\lambda + \nu)| < \pi$;

$$\begin{aligned} l_{11}(\nu) &= O(\nu^{-1}), \qquad l_{12}(\nu) = O(\nu^{-\frac{1}{2}}), \\ l_{21}(\nu) &= O(1), \qquad l_{22}(\nu) = O(\nu^{\frac{1}{2}}), \end{aligned} \tag{A61}$$

Det $L(\nu) = O(\nu^{-\frac{1}{2}})$, as $|\nu| \to \infty$ in $|\arg(k\lambda + \nu)| < \pi$;

When $\nu \to -k\lambda$, expressions $(A52)$ to $(A59)$ with $(B19)$ and $(B20)$ of *Appendix B* give

$$\begin{aligned} u_{11}(\nu) &= O\left[(k\lambda + \nu)^{\frac{-1}{2}}\right], \qquad u_{12}(\nu) = O(1), \\ u_{21}(\nu) &= O\left[(k\lambda + \nu)^{\frac{-1}{2}}\right], \qquad u_{22}(\nu) = O(1), \end{aligned} \tag{A62}$$

Det $U(\nu) = O\left[(k\lambda + \nu)^{\frac{-1}{2}}\right]$;

$$\begin{aligned} l_{11}(\nu) &= O\left[(k\lambda + \nu)^{\frac{-1}{2}}\right], \qquad l_{12}(\nu) = O(1), \\ l_{21}(\nu) &= O(1), \qquad l_{22}(\nu) = O\left[(k\lambda + \nu)^{\frac{1}{2}}\right], \end{aligned} \tag{A63}$$

Det $L(\nu) = O(1)$; as $\nu \to -k\lambda$, $\mathrm{Re}(\beta_{1,2}) > 0$.

Appendix B

EVALUATION OF THE INTEGRALS $I(\nu)$ AND $J(\nu)$

Here, we present the evaluation of the integrals appearing in *Appendix A*, namely

$$I(\nu) = -\frac{1}{2\pi}\int_{-\infty}^{-k\lambda} \frac{1}{|k\lambda + t|^{\frac{1}{2}}} \log\left\{\left(\frac{|\vartheta|^2 + k^2\beta_1^2}{|\vartheta|^2 + k^2\beta_2^2}\right)\right\}\frac{dt}{t-\nu}, \qquad \text{(B1)}$$

$$J(\nu) = \frac{1}{2\pi i}\int_{-\infty}^{-k\lambda} \log\left\{\left(\frac{|\vartheta| - ik\beta_1}{|\vartheta| + ik\beta_1}\right)\left(\frac{|\vartheta| - ik\beta_2}{|\vartheta| + ik\beta_2}\right)\right\}\frac{dt}{t-\nu}, \qquad \text{(B2)}$$

where $|\vartheta| = \sqrt{t^2 - k^2\lambda^2}$ for $-\infty < \zeta < -k\lambda$.

Equation $(B1)$ can also be written as

$$I(\nu) = \frac{1}{2\pi}\int_{k\lambda}^{\infty} \frac{\left\{\log\left[t^2 - k^2\left(\lambda^2 - \beta_1^2\right)\right] - \log\left[t^2 - k^2\left(\lambda^2 - \beta_2^2\right)\right]\right\}}{\sqrt{t-k\lambda}\,(t+\nu)}dt \qquad \text{(B3)}$$

$$
\begin{aligned}
&= \frac{1}{2\pi}\int_{k\lambda}^{\infty}\left\{\begin{array}{l}\log\left[t+k\left(\lambda^2-\beta_1^2\right)^{\frac{1}{2}}\right]+\log\left[t-k\left(\lambda^2-\beta_1^2\right)^{\frac{1}{2}}\right]\\ -\log\left[t+k\left(\lambda^2-\beta_2^2\right)^{\frac{1}{2}}\right]-\log\left[t-k\left(\lambda^2-\beta_2^2\right)^{\frac{1}{2}}\right]\end{array}\right\}\\
&\qquad\times\frac{dt}{\sqrt{t-k\lambda}(t+\nu)}\\
&= \frac{1}{2\pi}\int_{0}^{\infty}\left\{\begin{array}{l}\log\left[t+k\aleph_1(+)\right]+\log\left[t+k\aleph_1(-)\right]\\ -\log\left[t+k\aleph_2(+)\right]-\log\left[t+k\aleph_2(-)\right]\end{array}\right\}\frac{dt}{\sqrt{t-k\lambda}(t+\nu)},
\end{aligned}
$$

where $\aleph_{1,2}(\pm)=\lambda\pm\sqrt{\lambda^2-\beta_{1,2}^2}$.

Making use of the result [22]

$$
\int_0^{\infty}\frac{\log(t+\delta)}{\sqrt{t}(t+\lambda_1)}dt=\frac{2\pi}{\sqrt{\lambda_1}}\log\left(\sqrt{\lambda_1}+\sqrt{\delta}\right),\ |\arg\lambda_1|<\pi,\ |\arg\delta|<\pi.
$$

giving

$$
\begin{aligned}
I(\nu) &= \log\left[\frac{\left(\sqrt{k\lambda+\nu}+\sqrt{k\aleph_1(+)}\right)\left(\sqrt{k\lambda+\nu}+\sqrt{k\aleph_1(-)}\right)}{\left(\sqrt{k\lambda+\nu}+\sqrt{k\aleph_2(+)}\right)\left(\sqrt{k\lambda+\nu}+\sqrt{k\aleph_2(-)}\right)}\right] \qquad \text{(B4)}\\
&\quad\times(k\lambda+\nu)^{\frac{-1}{2}}, \qquad\qquad |\arg(k\lambda+\nu)|<\pi,\ \mathrm{Re}(\beta_{1,2})>0.
\end{aligned}
$$

$J(\nu)$ given by the expression $(B2)$ can be recasted as

$$
J(\nu)=\int_{\infty}^{\nu}\left[\frac{1}{2\pi i}\int_{-\infty}^{-k\lambda}\log\left\{\left(\frac{|\vartheta|-ik\beta_1}{|\vartheta|+ik\beta_1}\right)\left(\frac{|\vartheta|-ik\beta_2}{|\vartheta|+ik\beta_2}\right)\right\}\frac{dt}{(t-u)^2}\right]du \qquad \text{(B5)}
$$

$$
J(\nu)=J_1(\nu)+J_2(\nu), \qquad \text{(B6)}
$$

where

$$
J_1(\nu)=\int_{\infty}^{\nu}\left[\frac{1}{2\pi i}\int_{-\infty}^{-k\lambda}\log\left(\frac{|\vartheta|-ik\beta_1}{|\vartheta|+ik\beta_1}\right)\frac{dt}{(t-u)^2}\right]du, \qquad \text{(B7)}
$$

$$J_2(\nu) = \int_\infty^\nu \left[\frac{1}{2\pi i}\int_{-\infty}^{-k\lambda} \log\left(\frac{|\vartheta| - ik\beta_2}{|\vartheta| + ik\beta_2}\right)\frac{dt}{(t-u)^2}\right]du. \tag{B8}$$

After using integration by parts, Eq. $(B7)$ yields

$$J_1(\nu) = \int_\infty^\nu \left[-\frac{1}{2(u+k\lambda)} + Q_1(u)\right]du, \tag{B9}$$

where

$$\begin{aligned} Q_1(u) &= \frac{1}{2\pi i}\int_{k\lambda}^\infty \frac{\frac{d}{dt}\log\left(\frac{|\vartheta|-ik\beta_1}{|\vartheta|+ik\beta_1}\right)}{(t+u)}dt \\ &= \frac{k\beta_1}{\pi}\int_{k\lambda}^\infty \frac{tdt}{\sqrt{t^2-k^2\lambda^2}\left[t-k\sqrt{\lambda^2-\beta_1^2}\right](t+u)\left[t+k\sqrt{\lambda^2-\beta_1^2}\right]}. \end{aligned} \tag{B10}$$

Now making use of the result

$$\int_{k\lambda}^\infty \frac{dt}{\sqrt{t^2-k^2\lambda_1^2}(t+\delta)} = \frac{\cos^{-1}\left(\frac{\delta}{k\lambda_1}\right)}{\sqrt{k^2\lambda_1^2-\delta^2}}, |\arg(k\lambda_1+\delta)| < \pi,$$

$\cos^{-1}(0) = \frac{\pi}{2}$, and the partial fractions in Eq. $(B10)$, we obtain

$$\begin{aligned} Q_1(u) &= \frac{k\beta_1}{2\pi\left[u+k\sqrt{\lambda^2-\beta_1^2}\right]}\left[\frac{\cos^{-1}\left(-\frac{\sqrt{\lambda^2-\beta_1^2}}{\lambda}\right)}{\sqrt{k^2\lambda^2-k^2(\lambda^2-\beta_1^2)}} - \frac{\cos^{-1}\left(-\frac{u}{k\lambda}\right)}{\sqrt{k^2\lambda^2-u^2}}\right] \\ &+ \frac{k\beta_1}{2\pi\left[u-k\sqrt{\lambda^2-\beta_1^2}\right]}\left[\frac{\cos^{-1}\left(\frac{\sqrt{\lambda^2-\beta_1^2}}{\lambda}\right)}{\sqrt{k^2\lambda^2-k^2(\lambda^2-\beta_1^2)}} - \frac{\cos^{-1}\left(\frac{u}{k\lambda}\right)}{\sqrt{k^2\lambda^2-u^2}}\right], \end{aligned} \tag{B11}$$

$0 < \text{Re}\,\cos^{-1}(u/k\lambda) \leq \pi$, $\text{Re}\left(\sqrt{k^2\lambda^2-u^2}\right) \geq 0, |\arg(k\lambda+u)| < \pi$, $\text{Re}(\beta_1) > 0$.

In Eq. $(B11)$ both the functions $\cos^{-1}(u/k\lambda)$ and $\sqrt{k^2\lambda^2 - u^2}$ have branch cut at $-\infty < u \leq -k\lambda$ and $k\lambda \leq u < \infty$, hence, this branch cut can be omitted. Then $Q_1(u)$ is indeed analytic in $|\arg(k\lambda + u)| < \pi$ with a single branch cut $-\infty < u \leq -k\lambda$. We also note that $Q_1(u)$ is analytic at $u = \pm k\sqrt{\lambda^2 - \beta_1^2}$, since the singularities cancel. Substituting $(B11)$ into $(B9)$ gives

$$J_1(\nu) = \int_{\infty}^{\nu} \left[\begin{array}{c} \frac{-1}{2(u+K\lambda)} + \frac{k\beta_1}{2\pi\left[u+k\sqrt{\lambda^2-\beta_1^2}\right]} \left[\frac{\cos^{-1}\left(-\frac{\sqrt{\lambda^2-\beta_1^2}}{\lambda}\right)}{k\beta_1} - \frac{\cos^{-1}\left(\frac{u}{k\lambda}\right)}{\sqrt{k^2\lambda^2-u^2}} \right] \\ + \frac{k\beta_1}{2\pi\left[u-k\sqrt{\lambda^2-\beta_1^2}\right]} \left[\frac{\cos^{-1}\left(\frac{\sqrt{\lambda^2-\beta_1^2}}{\lambda}\right)}{k\beta_1} - \frac{\cos^{-1}\left(\frac{u}{k\lambda}\right)}{\sqrt{k^2\lambda^2-u^2}} \right] \end{array} \right] du. \quad (B12)$$

Similarly from Eqs. $(B7)$ and $(B8)$

$$J_2(\nu) = \int_{\infty}^{\nu} \left[\begin{array}{c} \frac{-1}{2(u+K\lambda)} + \frac{k\beta_2}{2\pi\left[u+k\sqrt{\lambda^2-\beta_2^2}\right]} \left[\frac{\cos^{-1}\left(-\frac{\sqrt{\lambda^2-\beta_2^2}}{\lambda}\right)}{k\beta_2} - \frac{\cos^{-1}\left(\frac{u}{k\lambda}\right)}{\sqrt{k^2\lambda^2-u^2}} \right] \\ + \frac{k\beta_2}{2\pi\left[u-k\sqrt{\lambda^2-\beta_2^2}\right]} \left[\frac{\cos^{-1}\left(\frac{\sqrt{\lambda^2-\beta_2^2}}{\lambda}\right)}{k\beta_2} - \frac{\cos^{-1}\left(\frac{u}{k\lambda}\right)}{\sqrt{k^2\lambda^2-u^2}} \right] \end{array} \right] du. \quad (B13)$$

Combining Eqs.. $(B12)$ and $(B13)$ into $(B5)$ gives

$$J(\nu) = \int_{\infty}^{\nu} Q_1(u)du, \quad (B14)$$

where

$$
\begin{aligned}
Q_1(u) &= \frac{-1}{(u+K\lambda)} + \frac{1}{2\pi\left[u+k\sqrt{\lambda^2-\beta_1^2}\right]}\cos^{-1}\left(-\frac{\sqrt{\lambda^2-\beta_1^2}}{\lambda}\right) \quad \text{(B15)}\\
&+\frac{1}{2\pi\left[u-k\sqrt{\lambda^2-\beta_1^2}\right]}\cos^{-1}\left(\frac{\sqrt{\lambda^2-\beta_1^2}}{\lambda}\right)\\
&+\frac{1}{2\pi\left[u+k\sqrt{\lambda^2-\beta_2^2}\right]}\cos^{-1}\left(-\frac{\sqrt{\lambda^2-\beta_2^2}}{\lambda}\right)\\
&+\frac{1}{2\pi\left[u-k\sqrt{\lambda^2-\beta_2^2}\right]}\cos^{-1}\left(\frac{\sqrt{\lambda^2-\beta_2^2}}{\lambda}\right)\\
&-\frac{k\beta_1\cos^{-1}\left(\frac{u}{k\lambda}\right)}{2\pi\sqrt{k^2\lambda^2-u^2}}\left[\frac{1}{u+k\sqrt{\lambda^2-\beta_1^2}}+\frac{1}{u-k\sqrt{\lambda^2-\beta_1^2}}\right]\\
&-\frac{k\beta_2\cos^{-1}\left(\frac{u}{k\lambda}\right)}{2\pi\sqrt{k^2\lambda^2-u^2}}\left[\frac{1}{u+k\sqrt{\lambda^2-\beta_2^2}}+\frac{1}{u-k\sqrt{\lambda^2-\beta_2^2}}\right].
\end{aligned}
$$

We note in particular that

$$
\begin{aligned}
&V(\nu)\\
&= \exp[J(\nu)] = \frac{\left\{\sqrt{\nu+k\sqrt{\lambda^2-\beta_1^2}}\right\}\left\{\sqrt{\nu+k\sqrt{\lambda^2-\beta_2^2}}\right\}}{(\nu+k\lambda)}\\
&\times\left[\frac{\nu-k\sqrt{\lambda^2-\beta_1^2}}{\nu+k\sqrt{\lambda^2-\beta_1^2}}\right]^{\cos^{-1}\left(\sqrt{1-\beta_1^2/\lambda^2}\right)(2\pi)^{-1}}\\
&\times\left[\frac{\nu-k\sqrt{\lambda^2-\beta_2^2}}{\nu+k\sqrt{\lambda^2-\beta_2^2}}\right]^{\cos^{-1}\left(\sqrt{1-\beta_2^2/\lambda^2}\right)(2\pi)^{-1}}
\end{aligned}
$$

$$\times \exp\left[-\frac{k\beta_1}{2\pi}\int_\infty^\nu \frac{\cos^{-1}\left(\frac{u}{k\lambda}\right)}{\sqrt{k^2\lambda^2-u^2}}\left[\frac{1}{u+k\sqrt{\lambda^2-\beta_1^2}}+\frac{1}{u-k\sqrt{\lambda^2-\beta_1^2}}\right]du\right]$$
$$\times \exp\left[-\frac{k\beta_2}{2\pi}\int_\infty^\nu \frac{\cos^{-1}\left(\frac{u}{k\lambda}\right)}{\sqrt{k^2\lambda^2-u^2}}\left[\frac{1}{u+k\sqrt{\lambda^2-\beta_2^2}}+\frac{1}{u-k\sqrt{\lambda^2-\beta_2^2}}\right]du\right] \tag{B16}$$

$$V(\nu) = O(1), \text{ as } |\nu| \to \infty, |\arg(k\lambda+\nu)| < \pi, \mathrm{Re}(\beta_{1,2}) > 0. \tag{B17}$$

In addition,

$$\begin{aligned} W(\nu) &= \exp\left[\sqrt{k\lambda+\nu}\,I(\nu)\right] \quad \text{(B18)} \\ &= \left[\frac{\left(\sqrt{k\lambda+\nu}+\sqrt{k\aleph_1(+)}\right)\left(\sqrt{k\lambda+\nu}+\sqrt{k\aleph_1(-)}\right)}{\left(\sqrt{k\lambda+\nu}+\sqrt{k\aleph_2(+)}\right)\left(\sqrt{k\lambda+\nu}+\sqrt{k\aleph_2(-)}\right)}\right] \end{aligned}$$

or

$$W(\nu) = O(1), \text{ as } |\nu| \to \infty, |\arg(k\lambda+\nu)| < \pi, \mathrm{Re}(\beta_{1,2}) > 0; \tag{B19}$$

furthermore, as $\nu \to -k\lambda$,

$$W(\nu) = O(1), \mathrm{Re}(\beta_{1,2}) > 0, \tag{B20}$$

$$V(\nu) = O(k\lambda+\nu)^{-1}, \mathrm{Re}(\beta_{1,2}) > 0. \tag{B21}$$

The result (B21) follows from [79]

$$\begin{aligned} &\frac{1}{2\pi i}\int_{-\infty}^{-k\lambda} \log\left\{\left(\frac{|\vartheta|-ik\beta_1}{|\vartheta|+ik\beta_1}\right)\left(\frac{|\vartheta|-ik\beta_2}{|\vartheta|+ik\beta_2}\right)\right\}\frac{dt}{t-\nu} \quad \text{(B22)} \\ = &-\log(k\lambda+\nu) + \text{ bounded function, as } \nu \to -k\lambda. \end{aligned}$$

Appendix C

SOLUTION OF INTEGRALS

In this appendix, we present the solution of the integrals which appears in chapter 3

$$I_1 = \frac{1}{2\pi i}\int_{-\infty+ia}^{\infty+ia}\frac{e^{i\zeta L}}{Q(\zeta)(\zeta+\nu)}d\zeta, \mathrm{Im}(\nu) > -a, \ \mathrm{Im}(Ks\cos\theta_0) < a < \mathrm{Im}(Ks). \tag{C1}$$

From the way ϑ has been defined that for $\mathrm{Re}(B) > 0$, $Q(\zeta)$ has no poles or zeros in the cut plane. Thus in the region $a < \mathrm{Im}(Ks)$ the only singularity is a branch cut at $\zeta = Ks$. Distorting the path of integration in Eq. (C1) into the upper ζ plane until it runs around the branch cut $\zeta = Ks$, gives

$$I_1 = \frac{\sqrt{2Ks}}{2\pi i}\left[\begin{array}{c} -i\int_{\infty Ks}^{Ks}\frac{e^{i\zeta L}\sqrt{\zeta-Ks}}{(\zeta+\nu)\left[1-i\lambda_1\sqrt{\zeta-Ks}\right]}d\zeta \\ +i\int_{Ks}^{\infty Ks}\frac{e^{i\zeta L}\sqrt{\zeta-Ks}}{(\zeta+\nu)\left[1+i\lambda_1\sqrt{\zeta-Ks}\right]}d\zeta \end{array}\right],$$

where $\lambda_1 = \sqrt{2Ks}/(K+M\nu)B$ is obtained by replacing the smoothly varying function $\sqrt{\nu+Ks}$ by $\sqrt{2Ks}$. Making an obvious change of variable one obtains

$$I_1 = \frac{\sqrt{2Ks}e^{iKsL}}{\pi\left[1-\lambda_1^2(\nu+Ks)\right]}\int_0^\infty e^{iuL}\sqrt{u}\left\{\frac{1}{u+Ks+\nu}-\frac{1}{u+\lambda_1^{-2}}\right\}du \quad (C2)$$

$$= \frac{\sqrt{2Ks}}{\left[1-\lambda_1^2(\nu+Ks)\right]}\left\{W_0\left[\sqrt{L(\nu+Ks)}\right]-W_0\left[\sqrt{L}\lambda_1^{-1}\right]\right\} = \widetilde{W}(\nu).$$

W_0 can be expressed in terms of the Fresnel integral $F(z_1)$ by

$$W_0\left[\sqrt{z_1 L}\right] = \frac{e^{i(KsL+\frac{\pi}{4})}}{\sqrt{L\pi}}\left\{1+2i\sqrt{z_1 L}F(\sqrt{z_1 L})\right\}, L>0, |\arg(z_1)|<\pi. \quad (C3)$$

Also note the asymptotic expansion

$$W_0\left[\sqrt{z_1 L}\right] \approx -\frac{e^{i(KsL-\frac{\pi}{4})}}{2\sqrt{\pi}L^{\frac{3}{2}}z_1}. \quad (C4)$$

Now consider the integral

$$I_2 = \frac{1}{2\pi i}\int_{-\infty+id}^{\infty+id}\frac{e^{i\zeta L}}{Q(\zeta)(\zeta-\nu)\sqrt{-\zeta+Ks}}d\zeta, \operatorname{Im}(\nu)<d<\operatorname{Im}(Ks). \quad (C5)$$

Anticipating that the major contribution from the smooth functions under the integral sign occurs at $\zeta = Ks$, one obtains, on distorting the path of integration round the branch cut $\zeta = Ks$,

$$I_2 = \frac{\sqrt{2Ks}}{2\pi i}\left[\begin{array}{c}\int_{\infty Ks}^{Ks}\frac{e^{i\zeta L}}{(\zeta-\nu)\left[1-i\lambda_1\sqrt{\zeta-Ks}\right]}d\zeta \\ +\int_{Ks}^{\infty Ks}\frac{e^{i\zeta L}}{(\zeta-\nu)\left[1+i\lambda_1\sqrt{\zeta-Ks}\right]}d\zeta\end{array}\right],$$

or

$$
\begin{aligned}
I_2 &= -\frac{\sqrt{2Ks}e^{iKsL}\lambda_1}{\left[1+\lambda_1^2(\nu-Ks)\right]}\int_0^\infty e^{iuL}\sqrt{u}\left\{\frac{1}{u+Ks-\nu}-\frac{1}{u+\lambda_1^{-2}}\right\}du \\
&= -\frac{\lambda_1\sqrt{2Ks}}{\left[1+\lambda_1^2(\nu-Ks)\right]}\left\{W_0\left[\sqrt{L(-\nu+Ks)}\right]-W_0\left[\sqrt{L}\lambda_1^{-1}\right]\right\} \\
&= -\lambda_1\widetilde{W}(\nu),\ L>0, |\arg(Ks-\nu)|<\pi, \left|\arg(\lambda_1^{-1})\right|<\pi. \qquad (C6)
\end{aligned}
$$

Consider the integral

$$
I_3=\frac{1}{2\pi i}\int_{-\infty+ic}^{\infty+ic}\frac{e^{-i\zeta L}}{Q(\zeta)(\zeta-\nu)\sqrt{\zeta+Ks}}d\zeta, \mathrm{Im}(\nu)>c>\mathrm{Im}(Ks). \qquad (C7)
$$

Letting ζ be replaced by $(-\zeta)$, c by $-a$, and using the fact that $Q(-\zeta)=Q(\zeta)$, $\sqrt{\zeta+Ks}\approx\sqrt{2Ks}$, gives

$$
I_3=-\frac{1}{2\pi i\sqrt{2Ks}}\int_{-\infty+ia}^{\infty+ia}\frac{e^{i\zeta L}}{Q(\zeta)(\zeta+\nu)}d\zeta=-\frac{I_1}{\sqrt{2Ks}}.
$$

Appendix D

DETAIL OF CALCULATIONS OF Eq. (4.11)

In this appendix, we give the detail of calculations to arrive at Eq. (4.11). Transforming Eqs. (4.6) to (4.10) through Eq. (2.5) and making use of the subsonic substitutions (3.9), the boundary value problem takes the following form

$$\left(\frac{\partial^2}{\partial X^2} + \frac{\partial^2}{\partial Y^2} + K^2 s^2\right) \phi_t(X, Y, \mu; \omega) = \tilde{a} e^{-\omega^2/4v^2} \delta(X - X_0) \delta(Y - Y_0), \quad \text{(D1)}$$

$$\left(\frac{\partial}{\partial Y} \mp BM \frac{\partial}{\partial X} \pm iKB\right) \phi_t(X, 0^\pm, \mu; \omega) = 0, X < 0, \quad \text{(D2)}$$

$$\begin{aligned} \frac{\partial}{\partial Y} \phi_t(X, 0^+, \mu; \omega) &= \frac{\partial}{\partial Y} \phi_t(X, 0^-, \mu; \omega), X > 0, \quad \text{(D3)} \\ \phi_t(X, 0^+, \mu; \omega) - \phi_t(X, 0^-, \mu; \omega) &= \tilde{\alpha}_1(\mu) e^{iKsX/M}, X > 0, \end{aligned}$$

where s and $\tilde{a}$ are defined by Eq. (3.15). The field satisfying Eq. (D1) can conveniently be written as

$$\phi_t(X, Y, \mu; \omega) = \phi_0(X, Y, \mu; \omega) + \phi(X, Y, \mu; \omega), \tag{D4}$$

where

$$\left(\frac{\partial^2}{\partial X^2} + \frac{\partial^2}{\partial Y^2} + K^2 s^2\right) \phi_0(X, Y, \mu; \omega) = \tilde{a} e^{-\omega^2/4v^2} \delta(X - X_0)\delta(Y - Y_0), \tag{D5}$$

$$\left(\frac{\partial^2}{\partial X^2} + \frac{\partial^2}{\partial Y^2} + K^2 s^2\right) \phi(X, Y, \mu; \omega) = 0. \tag{D6}$$

Using Eq. (3.16) the solution of Eq. (D5) is given by

$$\begin{aligned} \phi_0(X, Y, \mu; \omega) &= \frac{\tilde{a} e^{-\omega^2/4v^2}}{4i} \int_{-\infty}^{\infty} \frac{e^{i\nu(X - X_0) + i\vartheta_1(Y - Y_0)}}{\vartheta_1} d\nu \\ &= b_1(\mu) e^{-iKs(X\cos\theta_0 + Y\sin\theta_0) - \omega^2/4v^2}, \end{aligned} \tag{D7}$$

where $b_1(\mu)$ is given by Eq. (3.23) and $\vartheta_1 = \sqrt{K^2 s^2 - \nu^2}$.

After using Eq. (3.16) the solution of Eq. (D6) satisfying radiation condition can be written as

$$\overline{\phi}(\nu, Y, \mu; \omega) = \frac{e^{-\omega^2/4v^2}}{2\pi i} \int_{-\infty}^{\infty} \frac{A_9(\nu)}{\vartheta_1} e^{i\nu X + i\vartheta_1 Y} d\nu, Y > 0, \tag{D8}$$

$$\overline{\phi}(\nu, Y, \mu; \omega) = \frac{e^{-\omega^2/4v^2}}{2\pi i} \int_{-\infty}^{\infty} \frac{A_{10}(\nu)}{\vartheta_1} e^{i\nu X - i\vartheta_1 Y} d\nu, Y < 0. \tag{D9}$$

Substituting Eqs. (D4), (D7), (D8) and (D9) into Eqs. (D2) and (D3) gives

$$\int_{-\infty}^{\infty} \frac{A_9(\nu) - A_{10}(\nu)}{\vartheta_1} e^{i\nu X} d\nu = \tilde{\alpha}_1 e^{iKsX/M}, X > 0. \tag{D10}$$

$$\int_{-\infty}^{\infty} \left[A_9(\nu) + A_{10}(\nu)\right] e^{i\nu X} d\nu = 0, X > 0, \tag{D11}$$

$$\begin{aligned} &\int_{-\infty}^{\infty} A_9(\nu) \left[1 - \frac{B(K - M\nu)}{\vartheta_1}\right] e^{i\nu X} d\nu - \frac{\tilde{a}e^{-\omega^2/4v^2}}{2} \\ &\times \int_{-\infty}^{\infty} \left[Sgn(Y_0) + \frac{B(M\nu - K)}{\vartheta_1}\right] e^{i\nu(X - X_0) + i\vartheta_1|Y_0|} d\nu \\ = &\ 0, \qquad X < 0, \end{aligned} \tag{D12}$$

$$\begin{aligned} &\int_{-\infty}^{\infty} A_{10}(\nu) \left[1 - \frac{B(K - M\nu)}{\vartheta_1}\right] e^{i\nu X} d\nu + \frac{\tilde{a}e^{-\omega^2/4v^2}}{2} \\ &\times \int_{-\infty}^{\infty} \left[Sgn(Y_0) - \frac{B(M\nu - K)}{\vartheta_1}\right] e^{i\nu(X - X_0) + i\vartheta_1|-Y_0|} d\nu \\ = &\ 0, \qquad X < 0, \end{aligned} \tag{D13}$$

where

$$\begin{aligned} Sgn(Y_0) &= 1, Y_0 > 0, \\ &= -1, Y_0 \leq 0. \end{aligned}$$

Adding and subtracting Eqs. (D12) and (D13) and substituting

$$C_1(\nu) = A_9(\nu) + A_{10}(\nu), C_2(\nu) = A_9(\nu) - A_{10}(\nu), \tag{D14}$$

the resulting equations together with Eqs. (D10) and (D11) give a pair of coupled equations

$$\int_{-\infty}^{\infty} C_1(\nu)e^{i\nu X}d\nu = 0, X > 0, \tag{D15}$$

$$\int_{-\infty}^{\infty}\left[C_1(\nu)Q(\nu) + \tilde{a}e^{-\omega^2/4v^2}\frac{B(K-M\nu)}{\vartheta_1}e^{-i\nu X_0)+i\vartheta_1|Y_0|}\right]e^{i\nu X}d\nu = 0, X < 0. \tag{D16}$$

$$\int_{-\infty}^{\infty}\left[\frac{C_2(\nu)}{\vartheta_1} - \frac{\tilde{\alpha}_1}{\nu - \frac{Ks}{M}}\right]e^{i\nu X}d\nu = 0, X > 0, \tag{D17}$$

$$\int_{-\infty}^{\infty}\left[\begin{array}{c} C_2(\nu)Q(\nu) \\ -\tilde{a}Sgn(Y_0)e^{-\omega^2/4v^2}\frac{B(K-M\nu)}{\vartheta_1}e^{-i\nu X_0)+i\vartheta_1|Y_0|}\end{array}\right]e^{i\nu X}d\nu = 0, X < 0. \tag{D18}$$

A solution of the Eqs. (D15) to (D18) can be written as

$$C_1(\nu) = P_+(\nu), \tag{D19}$$

$$C_1(\nu)Q(\nu) + \tilde{a}e^{-\omega^2/4v^2}\frac{B(K-M\nu)}{\vartheta_1}e^{-i\nu X_0)+i\vartheta_1|Y_0|} = P_-(\nu), \tag{D20}$$

$$\frac{C_2(\nu)}{\vartheta_1} - \frac{\tilde{\alpha}_1}{\nu - \frac{Ks}{M}} = T_+(\nu), \tag{D21}$$

$$\begin{array}{c} C_2(\nu)Q(\nu) \\ -\tilde{a}Sgn(Y_0)e^{-\omega^2/4v^2}\frac{B(K-M\nu)}{\vartheta_1}e^{-i\nu X_0)+i\vartheta_1|Y_0|}\end{array} = T_-(\nu), \tag{D22}$$

where the positive subscript denotes that the function is regular in the domain $\mathrm{Im}(\nu) > -\mathrm{Im}(Ks)$ and the negative subscript denotes the function which is regular in the domain $\mathrm{Im}(\nu) < \mathrm{Im}(Ks)$. These two domains have the intersection $-\mathrm{Im}(Ks) < \mathrm{Im}(\nu) < \mathrm{Im}(Ks)$.

Now eliminating $C_1(\nu)$ and $C_2(\nu)$ from Eqs. (D19) to (D22) and then writing $Q(\nu) = Q_+(\nu)Q_-(\nu)$ in the resulting equations, we obtain

$$P_+(\nu)\sqrt{Ks+\nu}Q_+(\nu)+\frac{\tilde{a}B(K-M\nu)}{\sqrt{Ks-\nu}Q_-(\nu)}e^{-i\nu X_0)+i\vartheta_1|Y_0|-\omega^2/4v^2} = \frac{P_-(\nu)\sqrt{Ks+\nu}}{Q_-(\nu)}. \quad \text{(D23)}$$

$$\begin{aligned} & T_+(\nu)\sqrt{Ks+\nu}Q_+(\nu) \qquad \text{(D24)} \\ & -Sgn(Y_0)\frac{\tilde{a}B(K-M\nu)}{\sqrt{Ks-\nu}Q_-(\nu)}e^{-i\nu X_0)+i\vartheta_1|Y_0|-\omega^2/4v^2} + \frac{\tilde{\alpha}_1\sqrt{Ks+\nu}Q_+(\nu)}{\nu-Ks/M} \\ = & \ \frac{T_-(\nu)}{Q_-(\nu)\sqrt{Ks-\nu}}. \end{aligned}$$

Equations (D23) and (D24) are the usual Wiener-Hopf equations. Now following the same method of solution as in Eqs. (1.64) and (1.67) and then using (D14) we get

$$\begin{aligned} A_9(\nu) &= -\frac{\tilde{a}e^{-\omega^2/4v^2}}{4\pi i Q_+(\nu)}\int_{-\infty}^{\infty}\frac{e^{-i\zeta X_0+i\sqrt{K^2s^2-\zeta^2}|Y_0|}d\zeta}{Q_-(\zeta)\sqrt{K^2s^2-\zeta^2}(\zeta-\nu)}\left[\begin{array}{c} B(K-M\zeta)-Sgn(Y_0) \\ \times\sqrt{Ks+\zeta}\sqrt{Ks-\nu}\end{array}\right] \\ & +\tilde{\alpha}_1\sqrt{Ks-\nu}\sqrt{Ks+\frac{Ks}{M}}Q_+(Ks/M)\left[2Q_+(\nu)(\nu-\frac{Ks}{M})\right]^{-1}, \qquad \text{(D25)} \end{aligned}$$

$$\begin{aligned} A_{10}(\nu) &= -\frac{\tilde{a}e^{-\omega^2/4v^2}}{4\pi i Q_+(\nu)}\int_{-\infty}^{\infty}\frac{e^{-i\zeta X_0+i\sqrt{K^2s^2-\zeta^2}|Y_0|}d\zeta}{Q_-(\zeta)\sqrt{K^2s^2-\zeta^2}(\zeta-\nu)}\left[\begin{array}{c} B(K-M\zeta)+Sgn(Y_0) \\ \times\sqrt{Ks+\zeta}\sqrt{Ks-\nu}\end{array}\right] \\ & -\tilde{\alpha}_1\sqrt{Ks-\nu}\sqrt{Ks+\frac{Ks}{M}}Q_+(\frac{Ks}{M})\left[2Q_+(\nu)(\nu-\frac{Ks}{M})\right]^{-1}. \qquad \text{(D26)} \end{aligned}$$

In order to satisfy the Kutta-Joukowski condition, the expressions for $A_9(\nu)$ and $A_{10}(\nu)$, which is of $O(|\nu|^{-\frac{1}{2}+\delta})$ as $\nu \to \infty$, must vanish [97]. Therefore

$$\widetilde{\alpha}_1 = \frac{\widetilde{a}e^{-\omega^2/4v^2}Sgn(Y_0)}{2\pi i\sqrt{Ks+\frac{Ks}{M}}Q_+(\frac{Ks}{M})}\int_{-\infty}^{\infty}\frac{e^{-i\zeta X_0+i\sqrt{K^2s^2-\zeta^2}|Y_0|}}{Q_-(\zeta)\sqrt{Ks-\zeta}}d\zeta. \tag{D27}$$

Substituting Eq. (D27) into Eqs. (D25) and (D26) we get $A_9(\nu)$ and $A_{10}(\nu)$. Using these resulting expressions of $A_9(\nu)$ and $A_{10}(\nu)$ in Eqs. (D8) and (D9) and then using modified method of stationary phase we have

$$\begin{aligned}\phi(X,Y,\mu;\omega) &= \widetilde{C}_1e^{-\omega^2/4v^2}\left[\frac{B}{s\sqrt{Ks}}+\frac{BM\cos\theta_0-2\sin(\theta/2)\sin(\theta_0/2)}{\sqrt{Ks}}\right]\\ &\times\frac{e^{iK\left[s(R+R_0)-\sqrt{1-M^2}\mu Z_0\right]}F(|\widetilde{Q}|)}{Q_+(Ks\cos\theta)Q_-(-Ks\cos\theta_0)}\\ &-\frac{2\widetilde{C}_1e^{-\omega^2/4v^2}\sin(\theta/2)\sin(\theta_0/2)F(|\widetilde{Q}_1|)e^{iK\left[s(R+R_0)-\sqrt{1-M^2}\mu Z_0\right]}}{\sqrt{Ks}Q_+(Ks\cos\theta)Q_-(-Ks\cos\theta_0)}.\end{aligned} \tag{D28}$$

where

$$\widetilde{C}_1 = -\frac{e^{-iKM(X-X_0)}}{2\pi\sin\theta\sqrt{2R(1-M^2)}},$$

$$F(\widetilde{Q}_1) = e^{-i\widetilde{Q}_1^2}\int_{\widetilde{Q}_1}^{\infty}e^{it^2}dt,$$

$$|\widetilde{Q}| = \left[\frac{KR}{2}\sqrt{1-(1-M^2)\mu^2}\right]^{\frac{1}{2}}\frac{\cos\theta-\cos\theta_0}{\sin\theta},$$

$$|\widetilde{Q}_1| = \left[\frac{KR}{2}\sqrt{1-(1-M^2)\mu^2}\right]^{\frac{1}{2}}\frac{\frac{1}{M}-\cos\theta}{\sin\theta},$$

$$R_0^2 = X_0^2+Y_0^2, R^2 = X^2+Y^2.$$

Taking inverse Fourier transform of (D28) by using Eq. (2.6) we get Eq. (4.11).

Appendix E

EVALUATION OF INTEGRAL IN Eq. (4.11)

In this appendix, we present the evaluation of one of the integrals appearing in Eq. (4.11). The other integrals can be evaluated similarly. We consider the following integral

$$\widetilde{I} = \int_{-\infty}^{\infty} \frac{e^{iK\left[\widetilde{Y}(Z-Z_0)+(R+R_0)\sqrt{1-\widetilde{Y}^2}\right]} F(|\widetilde{Q}|)}{\sqrt{K(1-\widetilde{Y}^2)^{\frac{1}{2}} Q_+(Ks\cos\theta) Q_-(-Ks\cos\theta_0)\sqrt{1-M^2}}} d\widetilde{Y}, \qquad \text{(E1)}$$

where

$$\begin{aligned}
\widetilde{Y} &= \mu(1-M^2)^{\frac{1}{2}}, s^2 = 1-\widetilde{Y}^2, \\
F(|\widetilde{Q}|) &= F\left[\mu_1\sqrt{K\sqrt{1-\widetilde{Y}^2}}\right], \mu_1 = \frac{\cos\theta+\cos\theta_0}{\sin\theta}\sqrt{\frac{R}{2}}.
\end{aligned}$$

Making use of the result

$$\int_q^\infty e^{i\lambda_2 t^2}dt = e^{i\lambda_2 q^2}\frac{F(\sqrt{\lambda_2}q)}{\sqrt{\lambda_2}}. \tag{E2}$$

Eq. $(E1)$ can be written as

$$\widetilde{I} = \int_{\mu_1}^\infty \int_{-\infty}^\infty \frac{e^{iK[\widetilde{Y}(Z-Z_0)+\left[(t^2-\mu_1^2+R+R_0)\sqrt{1-\widetilde{Y}^2}\right]}}{Q_+(Ks\cos\theta)Q_-(-Ks\cos\theta_0)\sqrt{1-M^2}}d\widetilde{Y}dt. \tag{E3}$$

Now, consider the integral

$$\widetilde{I}' = \int_{-\infty}^\infty \frac{e^{iK\left[\widetilde{Y}(Z-Z_0)+\sqrt{1-\widetilde{Y}^2}P_1\right]}}{Q_+(Ks\cos\theta)Q_-(-Ks\cos\theta_0)}d\widetilde{Y}. \tag{E4}$$

By the substitutions

$$\widetilde{Y} = \cos\xi,\ Z - Z_0 = R_{12}\cos\Theta_{12},\ P_1 = R_{12}\sin\Theta_{12}. \tag{E5}$$

$\widetilde{I}'$ takes the form

$$\widetilde{I}' = \int_{-\infty}^\infty f_1(\zeta)e^{iKR_{12}\cos(\zeta-\Theta_{12})}d\zeta, \tag{E6}$$

where

$$f_1(\zeta) = \frac{-\sin\zeta}{Q_+(K\sin\zeta\cos\theta)Q_-(-K\sin\zeta\cos\theta_0)}.$$

We apply the method of steepest descent to solve the integral $\widetilde{I}'$. For that, we deform the contour of integration to pass through the point of steepest descent $\zeta = \Theta_{12}$, such that the major part of the integrand is given by the integration over the part of the deformed contour near Θ_{12}, with $f_1(\zeta)$ slowly varying around it [82]. Hence, we can write

$$\widetilde{I}' \approx \pi f_1(\Theta_{12}) H_0^{(1)}(KR_{12}) \qquad \text{(E7)}$$

$$\approx -\frac{\pi H_0^{(1)}\left(K\sqrt{(Z-Z_0)^2+P_1^2}\right)}{Q_+(K\xi_1\cos\theta)Q_-(-K\xi_1\cos\theta_0)}\xi_1.$$

where

$$\xi_1 = \frac{P_1}{\sqrt{(Z-Z_0)^2+P_1^2}}.$$

Using (E7), we can write (E3) as

$$\widetilde{I} = \frac{-\pi}{\sqrt{1-M^2}}\int_{\mu_1}^{\infty}\frac{H_0^{(1)}\left(K\sqrt{(Z-Z_0)^2+(t^2-\mu_1^2+R+R_0)^2}\right)}{Q_+(K\xi_1\cos\theta)Q_-(-K\xi_1\cos\theta_0)} \qquad \text{(E8)}$$

$$\times\frac{(t^2-\mu_1^2+R+R_0)}{\sqrt{(Z-Z_0)^2+(t^2-\mu_1^2+R+R_0)^2}}dt.$$

If we make the substitutions

$$t^2 = -A_1+\left(A_1^2+R_{11}^2\sinh^2 u\right)^{\frac{1}{2}}, R_{11}^2 = (Z-Z_0)^2+A_1^2,$$

$$A_1 = R+R_0-\mu_1^2,$$

in $(E8)$, we obtain

$$\widetilde{I} = -\frac{\pi}{2}\int_{\zeta_1}^{\infty}\frac{H_0^{(1)}(KR_{11}\cosh u)\sqrt{A_1+\left(A_1^2+R_{11}^2\sinh^2 u\right)^{\frac{1}{2}}}}{Q_+(K\xi_2\cos\theta)Q_-(-K\xi_2\cos\theta_0)\sqrt{1-M^2}}du, \qquad \text{(E9)}$$

where

$$\xi_2 = \frac{\left(A_1^2 + R_{11}^2 \sinh^2 u\right)^{\frac{1}{2}}}{R_{11} \cosh u}, \zeta_1 = \sinh^{-1}\left[\frac{\sqrt{u^2 + 2A_1\mu_1}}{R_{11}}\right]. \qquad \text{(E10)}$$

The integral $(E9)$ can be solved asymptotically by taking $KR_{11} \cosh u >> 1$. Therefore, we can replace the Hankel function by the first term of this asymptotic expansion to give

$$\widetilde{I} = -\frac{\sqrt{\pi}e^{\frac{-i\pi}{4}}}{\sqrt{2KR_{11}(1-M^2)}} \int_{\varsigma}^{\infty} \frac{\sqrt{A_1 + \left(A_1^2 + R_{11}^2 \sinh^2 u\right)^{\frac{1}{2}}}\, e^{iKR_{11}\cosh u}}{Q_+(K\xi_2 \cos\theta)Q_-(K\xi_2 \cos\theta_0)\sqrt{\cosh u}} du. \qquad \text{(E11)}$$

If we let $\tau_1 = \sqrt{2KR_{11}} \sinh(u/2)$, then

$$\widetilde{I} = -\frac{\sqrt{2\pi}e^{i(KR_{11}-\frac{\pi}{4})}}{\sqrt{K(1-M^2)}} \int_{\tau_{R_{12}}}^{\infty} e^{i\tau_1^2} f_2(\tau_1) d\tau_1, \qquad \text{(E12)}$$

where

$$f_2(\tau_1) = \left[\frac{\sqrt{\tau_1^2\left(\tau_1^2 + 2KR_{11}\right) + A_1^2K^2} + A_1K}{\left(\tau_1^2 + KR_{11}\right)\left(\tau_1^2 + 2KR_{11}\right)}\right]^{\frac{1}{2}} \times \frac{1}{Q_+(K\xi_2 \cos\theta)Q_-(-K\xi_2 \cos\theta_0)},$$

and

$$\xi_2 = \frac{\sqrt{\tau_1^2\left(\tau_1^2 + 2KR_{11}\right) + A_1^2K^2}}{\left(\tau_1^2 + KR_{11}\right)}, \tau_{R_{12}} = \sqrt{K\left(R_{12} - R_{11}\right)},$$
$$R_{12}^2 = (Z - Z_0)^2 + (R + R_0)^2.$$

An asymptotic expansion of $\tilde{I}$ then follows by putting τ_1 equal to its lower limit value in the non-exponential factor of the integrand plus the contribution from $\tau_1 = 0$ depending if zero lies in the interval of integration. Hence

$$\begin{aligned}\tilde{I} &= -\sqrt{\frac{2\pi}{K(1-M^2)}}e^{iKR_{11}}I_0H(-\epsilon_1) - \epsilon_1\frac{e^{i(KR_{12}-\frac{\pi}{4})}}{\sqrt{K(1-M^2)}}F(\tau_{R_{12}}) \quad \text{(E13)}\\ &\times\frac{\sqrt{2\pi(A_1+R+R_0)}}{\sqrt{K(R_{12}+R_{11})R_{12}}Q_+(K\zeta_2\cos\theta)Q_-(-K\zeta_2\cos\theta_0)}.\end{aligned}$$

where

$$\begin{aligned}I_0 &= \frac{1}{2}\frac{\sqrt{\pi A_1}}{\sqrt{K}R_{11}}\frac{1}{Q_+(K\zeta_2\cos\theta)Q_-(-K\zeta_2\cos\theta_0)},\\ \epsilon_1 &= sgn\ \tau_{R_{12}},\end{aligned}$$

$H(\bullet)$ is the usual Heaviside function, and

$$\zeta_2 = \frac{R+R_0}{R_{12}}, \zeta_1 = \frac{A_1}{R_{11}}.$$

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PUBLISHED WORK FROM THE THESIS OF THE AUTHOR

Department of Mathematics
University of Manchester
Manchester M13 9PL, U.K.

THE UNIVERSITY
of MANCHESTER

Dr. P. A. Martin
Telephone: 0161 275 5883
Fax: 0161 275 5819
Email: pamartin@manchester.ac.uk

January 28, 1998

Prof. J.G. Harris
Dept. of Theoretical & Applied Mechanics
University of Illinois
Urbana, IL 61801
U.S.A.

Dear John,

"Diffraction by a slit in an infinite porous barrier"
by S. Asghar, T. Hayat and J.G. Harris

Thank you for submitting a revised version of the above paper. I have read it and your covering letter. Consequently, I am pleased to inform you that your paper has been accepted for publication in a special issue of the journal *Wave Motion*, dedicated to the memory of Gerry Wickham.

I apologise for my delay in responding. I think that the special issue will be a fitting tribute to Gerry's memory, and I am very glad that it will contain your contribution.

With best wishes,

Yours sincerely,

Paul

Paul A. Martin

Contribution to the issue of *Wave Motion* honouring Gerry Wickham

DIFFRACTION BY A SLIT IN AN INFINITE POROUS BARRIER

S. Asghar and Tasawar Hayat
Mathematics Department, Quaid-i-Azam University
Islamabad, Pakistan

John G. Harris
Theoretical and Applied Mechanics, UIUC
216 Talbot Laboratory, 104 S. Wright St.
Urbana, IL 61801

(Received)

Abstract

The diffraction of an acoustic wave by a slit in an infinite, plane, porous barrier is investigated. The barrier is modeled as a rigid material filled with narrow pores, normal to the plane of the barrier, that provide sound damping. However, the barrier is thin enough that sound transmission takes place. An approximate boundary condition is derived that models both these effects. The source point is assumed far from the slit so that the incident spherical wave is locally plane. The slit is wide and the barrier thin, both with respect to wavelength. The principal purpose of the barrier is to reduce the reflected and transmitted sound so that we assume that the flow resistance of the pores is large. The diffracted field is calculated using integral transforms, the Wiener-Hopf technique and asymptotic methods. While a formal solution to the complete problem is given, only the diffracted wavefield is studied, and that only in the farfield of the slit. The diffracted field is the sum of the wavefields produced by the two edges of the slit and an interaction wavefield. The dependence on the barrier parameters of the power removed from the reflected wavefield by the diffraction at the slit is exhibited.

1. Introduction

An effective method of noise reduction is to use sound absorbent barriers in heavily built up areas [1,2]. In most calculations with such a barrier, no sound is assumed to be transmitted through it. However, many barriers are not sufficiently thick to completely prevent sound transmission. The aim of this work is to calculate the scattered wavefield excited by a spherical wave incident to a slit in a barrier exhibiting both absorption and transmission. The source is assumed to be sufficiently far from the slit that its wavefront is locally plane. Throughout we assume that the field is harmonic in time. In this paper we give a formal solution to the complete problem and demonstrate that, in the limit of a rigid barrier, the solution reduces to that calculated by the geometrical theory of diffraction. The asymptotic analysis of the resulting integrals is only carried far enough to permit the calculation of the diffracted wavefields far from the slit as well as the power removed from the reflected wavefield by interference with the diffracted one. We anticipate extending the analysis of these integrals, so that expressions for the wavefield in the slit and close to the barrier can be obtained, and have therefore given more details of the solution than is necessary to calculate only the farfield results.

Scattering from a slit or strip is a well-studied problem in diffraction theory. Asvestas and Kleinman [3, pp. 181-239] summarize and review much of the work done on it. Jones [4, pp. 602-607] and Noble [5, pp. 196-207] discusses diffraction from a slit or strip using the Wiener-Hopf method. We follow their approach very closely. To calculate the diffracted wavefield from the interaction between the edges we assume that the slit is large, with respect to wavelength, and asymptotically approximate several integrals using this assumption. Karp and Keller [6] calculate this interaction term for diffraction from a slit in a perfectly rigid barrier using the geometrical theory of diffraction (this theory also assumes that the slit is large with respect to wavelength). Their work is a limiting case for ours and we show that, in this limit, the power removed from the reflected wavefield by interference with the diffracted one, that we calculate, agrees with theirs. Lastly, the same overall approach used here has been taken by Asghar [7] in his study of scattering from an absorbing strip in a moving fluid.

Rawlins [8], continuing his earlier work on diffraction from an absorbing barrier [9], presented a model of an acoustically penetrable but absorbing half plane barrier, and calculated the diffraction from its edge. He used a boundary condition, having two parameters, that mixes the pressure and its normal derivative at each side of the barrier. The boundary condition produces discontinuities across the barrier in both the pressure and its normal derivative. The magnitudes of the discontinuities are set by the two parameters. They are chosen to give approximately the same reflection and transmission coefficients as those found for the case of a plane wave incident to a thin layer, whose governing equation is a scalar wave equation. Adopting the same form of boundary condition here, we identify the parameters in a different way. Using a simple theory of porous materials described in Morse and Ingard [10, pp. 252-256], our model assumes the barrier is made from a rigid material that is riddled with small pores that are approximately normal to the plane of the barrier. No particle velocity in the barrier parallel to its plane is permitted (a kinematic constraint). We take limited account of the compressibility of the gas in the pores. However, the gas in each pore behaves primarily as an incompressible cylinder, driven back and forth by the harmonic wavefield, but opposed by the frictional force generated at the pore walls (the flow resistance). The barrier is thin enough (with respect to wavelength) that sound is communicated from one side to the other by the motion of the numerous incompressible cylinders. The model is accurate provided $h\Phi/\rho c = O(1)$, where h is half the thickness, Φ the flow resistance and ρc the specific acoustic impedance of the surrounding gas.

There have been other attempts do derive approximate boundary conditions that model thin layers, though, unlike the one discussed here, they have not involved a kinematic constraint. Bovik[11] derives approximate boundary conditions for thin fluid and elastic layers in a differential form, using Taylor expansions as the basis of the approximation procedure. Wickham[12] takes a different approach and reduces the approximate boundary condition to an integral formulation that avoids the need to approximate the boundary conditions pointwise, but imposes instead a condition averaged over the boundary. Though our approach lies somewhat mid-way between the two, we end with a differential form because the boundary conditions are locally reacting. The gas in each pore responds only to the wavefield in its immediate neighborhood.

The final results are presented in the form of the power removed from the reflected wavefield by interference with the diffracted one. To make this calculation we adopt an argument given by Newton[13, pp. 18-20]. Normalized with respect to the reflected intensity times twice the width of the slit, this gives a measure of the effectiveness of the barrier, with the slit, at reducing sound transmission. This term is a function of the slit width and the properties of the barrier.

2. Formulation

We consider the diffraction of an acoustic wave excited by a point source located at (x_0, y_0, z_0) or (r_0, θ_0, z_0) by a slit in the plane $y = 0$ of width $2a$, $-a \le x \le a$. We shall also ask that $0 < \theta_0 \le \pi/2$. The geometry is shown in the Fig. 1. Throughout, the time harmonic factor $e^{-i\omega t}$ is understood. We shall work with the velocity potential σ, where the particle velocity $\mathbf{u}$ is given by $\mathbf{u} = -\nabla\sigma$. The total velocity potential σ_t satisfies

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} + k^2\right)\sigma_t = \delta(x - x_0)\delta(y - y_0)\delta(z - z_0) \tag{1}$$

where

$$k = k_1 + ik_2 \tag{2}$$

is the wavenumber. The wavenumber k is assumed to have a small positive imaginary part whenever this is needed to ensure the convergence (regularity) of the Fourier transform integrals defined subsequently. The term k_2 is otherwise set to zero. The boundary conditions satisfied by σ_t on $(-\infty < x \le -a) \cup (a \le x < \infty)$, $y = 0^{\pm}$ are

$$\pm\frac{\partial}{\partial y}\sigma_t(x, 0^{\pm}, z) + ik\alpha\sigma_t(x, 0^{\pm}, z) + ik\beta\sigma_t(x, 0^{\mp}, z) = 0 \tag{3}$$

We shall refer to this as the Rawlins[8] boundary condition. The parameters α and β will be identified shortly. The $0^{\pm}$ means that the field term is to be evaluated as $y \to 0$

through positive or negative values of y. The boundary conditions on $-a < x < a$, $y = 0^{\pm}$ are

$$\sigma_t(x, 0^+, z) = \sigma_t(x, 0^-, z) \tag{4}$$

and

$$\frac{\partial}{\partial y}\sigma_t(x, 0^+, z) = \frac{\partial}{\partial y}\sigma_t(x, 0^-, z) \tag{5}$$

In addition, we insist that σ_t satisfy the edge condition as $x \to -a^+$, a^-,

$$\sigma_t(x, 0, z) = O(1) \tag{6}$$

and

$$\frac{\partial}{\partial y}\sigma_t(x, 0, z) = O\left(x^{-1/2}\right) \tag{7}$$

The plus sign indicates a limit taken from the left and the minus sign one taken from the right.

It is useful to split the total field σ_t in two ways. To discuss the boundary condition we write

$$\sigma_t = \sigma_i + \sigma_s \tag{8}$$

where σ_i is the incident wave and σ_s is the scattered wavefield. We insist that σ_s represent an outward radiating wavefield. However, to discuss the diffraction problem, it is more useful to write σ_t as

$$\sigma_t = \begin{cases} \sigma_i + \sigma_r + \sigma, & y \geq 0^+ \\ \sigma & , y \leq 0^- \end{cases} \tag{9}$$

where σ_i is again the incident wave, σ_r is the wave reflected from a *perfectly rigid* barrier and σ is the scattered wavefield. It is comprised of the diffracted wave, a correction to the reflected wave and a transmitted wave.

3. The Boundary Condition

Figure 2 shows a porous barrier of thickness $2h$ extending to infinity in the $\pm x$ directions. No slit is present. The space is divided into three regions. The regions V^+ and V^- are those above and below the barrier and are occupied by a gas having density ρ and sound speed c. The region V_0 is that occupied by the porous barrier. Following a formulation that is identical to that given in Section I.B of Harris *et al.* [14], the velocity potential σ_s scattered from this barrier is represented by

$$\sigma_s(\mathbf{x}) = -\int_S [\sigma_g(\mathbf{x}',\mathbf{x})\nabla\sigma_t(\mathbf{x}') - \sigma_t(\mathbf{x}')\nabla'\sigma_g(\mathbf{x}',\mathbf{x})]\cdot\hat{\mathbf{n}}\, dS(\mathbf{x}'), \quad \mathbf{x} \in V^+ \cup V^- \tag{10}$$

where σ_t is the total potential given by Eq. (8) and σ_g is the three-dimensional, free-space Green's function. The surface S is comprised of the upper and lower surfaces of the barrier, $\hat{\mathbf{n}}$ is a unit normal vector pointing out of the barrier and ∇' indicates that the gradient is taken with respect to the argument $\mathbf{x}'$. The vector $\mathbf{x}$ indicates the observation point and lies outside the barrier, while the vector $\mathbf{x}'$ indicates the source point and lies on the surface S.

Asking that the unit normal $\hat{\mathbf{n}}$ now point only in the positive y direction, we define the discontinuities

$$[\nabla\sigma_t \cdot \hat{\mathbf{n}}] = \nabla\sigma_t(x,h,z)\cdot\hat{\mathbf{n}} - \nabla\sigma_t(x,-h,z)\cdot\hat{\mathbf{n}} \tag{11}$$

and

$$[\sigma_t] = \sigma_t(x,h,z) - \sigma_t(x,-h,z) \tag{12}$$

These are the sources of the scattered sound as can be seen by noting that, provided the discontinuities in Eqs. (11) and (12) are no larger than $O(1)$, then the integral Eq. (10) can be approximated to $O(kh)$ by evaluating the Green's terms at $y' = 0$. This leaves us with

$$\sigma_s(\mathbf{x}) = -\iint_S \left\{\sigma_g(x',0,z',\mathbf{x})[\nabla\sigma_t\cdot\hat{\mathbf{n}}] - [\sigma_t]\nabla'\sigma_g\cdot\hat{\mathbf{n}}\right\}dx'dz' + O(kh) \tag{13}$$

where $\mathbf{x}$ lies outside the volume enclosed by S. Note that we have approximated a function that we know and whose length scale is set by the wavenumber k and not by the wavenumber of the porous material. It is therefore the discontinuities, Eqs. (11) and (12), that Eq. (3) must mimic.

Returning to the Rawlins boundary condition, we note that if we take the limit $kh \to 0^{\pm}$ of the following

$$[\nabla\sigma_t\cdot\hat{\mathbf{n}}] = -ik(\alpha+\beta)[\sigma_t(x,h,z)+\sigma_t(x,-h,z)] \tag{14}$$

and

$$[\sigma_t] = -[ik(\alpha-\beta)]^{-1}[\nabla\sigma_t(x,h,z)\cdot\hat{\mathbf{n}} + \nabla\sigma_t(x,-h,z)\cdot\hat{\mathbf{n}}] \tag{15}$$

then, by adding and subtracting Eqs. (14) and (15), we recover Eq. (3). Accordingly, by estimating the discontinuities, Eqs. (11) and (12), we may use Eqs. (14) and (15) to determine the parameters α and β.

Adapting a simple theory of porous materials given in Morse and Ingard [10, pp. 252-256], the equations governing the acoustical behavior of the porous barrier are

$$i\omega\kappa_p\Omega p = du_2/dy \tag{16}$$

$$dp/dy = i\omega\rho_p\left[1+\left(i\Phi/\rho_p\omega\right)\right]u_2 \tag{17}$$

The particle velocity in the barrier u_2 is restricted to be in the normal direction *only*, the particle velocity in the tangential direction must be zero, and the acoustic pressure in the barrier is p. The parameters of the model are κ_p the compressibility of the gas in the pores, Ω the porosity or fraction of the volume occupied by the pores and hence by the gas, ρ_p the effective density of the gas in the pores and Φ the flow resistance. This last parameter determines the effective sound absorbing properties of the barrier. At the boundaries of the barrier the pressure and normal components of the particle velocity are continuous. No condition is placed on the tangential particle velocity components immediately outside the barrier. Integrating Eqs. (16) and (17), noting that p and u_2 are the total fields in the barrier and using the boundary conditions at the barrier walls gives

$$[\nabla\sigma_t\cdot\hat{\mathbf{n}}]=-\omega^2\rho\kappa_p\Omega(-i\omega\rho)^{-1}\int_{-h}^{h}p\,dy \tag{18}$$

and

$$[\sigma_t]=i\omega\rho_p\left[1+\left(i\Phi/\rho_p\omega\right)\right](-i\omega\rho)^{-1}\int_{-h}^{h}u_2\,dy \tag{19}$$

The barrier is both thin and absorbing. We wish to capture both these features. Defining $\kappa_e=\kappa_p\Omega$, $\rho_e=\rho_p\left[1+\left(i\Phi/\rho_p\omega\right)\right]$ and $c_e=(\rho_e\kappa_e)^{-1/2}$, the effective wavenumber in the barrier is $k_e=\omega/c_e$. We assume that p and u_2 vary slowly enough through the barrier to be approximated accurately by the first two terms of a Taylor series in the scaled thickness variable $k_eh(y/h)$. This assumes that the flow resistance is not so strong as to cause the wavefield in the barrier to very rapidly decay We are therefore able to relate Eqs. (14) and (15) to the porous barrier model by noting that

$$\frac{1}{(-i\omega\rho)2h}\int_{-h}^{h}p\,dy=\left[\sigma_t(x,h,z)+\sigma_t(x,-h,z)\right]/2+\mathrm{O}(k_eh)^2 \tag{20}$$

and

$$\frac{-1}{2h}\int_{-h}^{h}u_2\,dy=\left[\nabla\sigma_t(x,h,z)\cdot\hat{\mathbf{n}}+\nabla\sigma_t(x,-h,z)\cdot\hat{\mathbf{n}}\right]/2+\mathrm{O}(k_eh)^2 \tag{21}$$

Assuming that $(k_eh)^2$ is small, we find that

$$\alpha+\beta=-i\rho c^2\kappa_p\Omega kh \tag{22}$$

and

$$\alpha-\beta=i\rho\Big/kh\,\rho_p\left[1+\left(i\Phi/\rho_p\omega\right)\right] \tag{23}$$

Note that only $(\alpha-\beta)$ contains the flow resistance term.

To estimate the sizes of these terms assume that κ_p and ρ_p are equal to the compressibility κ and density ρ of the surrounding gas, so that $\kappa_p\rho_pc^2=1$. This is not quite the case because ρ_p can be larger than ρ, and κ_p can be the isothermal compressibility rather than the adiabatic compressibility κ. Nevertheless, if the barrier is to absorb the incident sound then $\Phi/\rho\omega$ must be moderately large. Morse and Ingard [10, pp. 252-256] suggest a value as high as 10 at 1000 Hz. We are therefore left with the following estimates

$$\alpha + \beta = -i\Omega kh \tag{24}$$

and

$$(\alpha - \beta)^{-1} = kh\,\Phi/\rho\omega \tag{25}$$

For kh small $(\alpha + \beta)$ is small because $\Omega < 1$, but $(\alpha - \beta)^{-1}$ need not be because, for effective sound absorption, $\Phi/\rho\omega > 1$. Moreover, $|k_e h| = kh(\Omega\Phi/\rho\omega)^{1/2}$. Examining the approximation in Eqs. (20) and (21), we note that, provided $kh\Phi/\rho\omega = O(1)$ or equivalently $h\Phi/\rho c = O(1)$, then the error leading to the approximate equivalence between Eqs. (14) and(15), and Eqs. (20) and (21) is $O(kh)$ throughout. As we continue with the calculation we shall find that some terms are proportional to $(\alpha + \beta)$ and can be dropped, while others contain $(\alpha - \beta)$ or $(\alpha - \beta)^{-1}$ and cannot. We could just set $(\alpha + \beta)$ to zero at this point, but, by carrying it through the calculation the different roles of the barrier thickness and absorption become clearer. Moreover, though we are assuming that $(\alpha - \beta)$ is not small, it can be set to zero to recover the case of a rigid barrier.

The reflection R and transmission T coefficients for the velocity potential using the boundary condition Eq. (3) are given in Rawlins [8] Eq. (38). Neglecting the $(\alpha + \beta)$, they are

$$R(\theta) = \frac{\sin\theta}{[\sin\theta + (\alpha - \beta)]} \tag{26}$$

and

$$T(\theta) = \frac{-2\beta}{[\sin\theta + (\alpha - \beta)]} \tag{27}$$

Note that $\alpha \approx -\beta$ and thus $-2\beta \approx (\alpha - \beta)$. The parameter β clearly controls transmission. For normal incidence, using the previous estimates $T(\pi/2)$ is approximately $-(\rho c/2h\Phi)$ so that the barrier allows weak transmission of sound. The coefficients have no poles on the real θ axis $(0 < \theta < \pi)$.

4. The Wiener-Hopf Problem

We now proceed with the calculation of the diffraction by the slit. The Fourier transform over z and its inverse are defined, respectively, as

$$\psi_t(x, y, \mu) = \frac{1}{(2\pi)^{1/2}} \int_{-\infty}^{\infty} \sigma_t(x, y, z) e^{-i\mu z} dz \tag{28}$$

and

$$\sigma_t(x, y, z) = \frac{1}{(2\pi)^{1/2}} \int_{-\infty}^{\infty} \psi_t(x, y, \mu) e^{i\mu z} d\mu \tag{29}$$

with identical definitions for the other potentials σ_i, σ_r and σ. The problem now becomes

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \gamma^2\right) \begin{matrix} \psi_i(x, y, \mu) \\ \psi_r(x, y, \mu) \end{matrix} = \frac{e^{-i\mu z_0}}{(2\pi)^{1/2}} \delta(x - x_0) \delta(y \mp y_0), \tag{30}$$

and

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \gamma^2\right) \psi(x, y, \mu) = 0, \tag{31}$$

where

$$\gamma = \left(k^2 - \mu^2\right)^{1/2}, \quad \operatorname{Im}\gamma > 0 \tag{32}$$

The boundary conditions at $y = 0$ are

$$\frac{\partial(\psi_i + \psi_r)}{\partial y} = 0 \tag{33}$$

for $(-\infty < x < \infty)$

$$\left(\frac{\partial}{\partial y} + ik\alpha\right) \psi(x, 0^+, \mu) + ik\alpha\left[\psi_i(x, 0, \mu) + \psi_r(x, 0, \mu)\right] + ik\beta\psi(x, 0^-, \mu) = 0 \tag{34}$$

$$\left(\frac{\partial}{\partial y} - ik\alpha\right) \psi(x, 0^-, \mu) - ik\beta\left[\psi_i(x, 0, \mu) + \psi_r(x, 0, \mu) + \psi(x, 0^+, \mu)\right] = 0 \tag{35}$$

for $(-\infty < x \le -a) \cup (a \le x < \infty)$ and

$$\psi(x, 0^+, \mu) - \psi(x, 0^-, \mu) = -\left[\psi_i(x, 0, \mu) + \psi_r(x, 0, \mu)\right] \tag{36}$$

$$\frac{\partial}{\partial y} \psi(x, 0^+, \mu) - \frac{\partial}{\partial y} \psi(x, 0^-, \mu) = 0 \tag{37}$$

for $(-a < x < a)$.

The solution to Eq. (30), giving the incident wave ψ_i, is

$$\psi_i(x, y, \mu) = -\frac{e^{-i\mu z_0}}{(2\pi)^{1/2} 4i} H_0^{(1)}\left(\gamma \left|\mathbf{r} - \mathbf{r}_0\right|\right) \tag{38}$$

where $\left|\mathbf{r} - \mathbf{r}_0\right| = \left[(x - x_0)^2 + (y - y_0)^2\right]^{1/2}$. The reflected wave ψ_r has the same form with the source point replaced by its reflected image source $(x_0, -y_0, z_0)$. As indicted in the introduction, we are interested in a situation where the source point is far from the slit. Accordingly, we may use the asymptotic approximation to the Hankel function, assuming that $|\gamma r_0| \to \infty$, to obtain

$$\psi_i = b(\mu) e^{-i\gamma(x\cos\theta_0 + y\sin\theta_0)} \tag{39}$$

and

$$\psi_r = b(\mu) e^{-i\gamma(x\cos\theta_0 - y\sin\theta_0)} \tag{40}$$

where $x_0 = r_0\cos\theta_0$ and $y_0 = r_0\sin\theta_0$ ($0 < \theta_0 \leq \pi/2$), and $x = r\cos\theta$ and $|y| = r\sin\theta$ ($0 < \theta < \pi$). The possibility that γ is near 0 can always be avoided. The term $b(\mu)$ is given by

$$b(\mu) = -\frac{e^{-i\mu z_0}}{(2\pi)^{1/2} 4i}\left(\frac{2}{\pi\gamma r_0}\right)^{1/2} e^{i(\gamma r_0 - \pi/4)}. \tag{41}$$

Note that by asking that $\operatorname{Im}\gamma > 0$, we have succeeded only in causing the incident and reflected disturbance to be damped in the negative x direction.

We next define the Fourier transform pair

$$\overline{\psi}(\nu, y, \mu) = \frac{1}{(2\pi)^{1/2}} \int_{-\infty}^{\infty} \psi(x, y, \mu) e^{i\nu x} dx \tag{42}$$

and

$$\psi(x, y, \mu) = \frac{1}{(2\pi)^{1/2}} \int_{-\infty}^{\infty} \overline{\psi}(\nu, y, \mu) e^{-i\nu x} d\nu \tag{43}$$

with identical definitions for the other wavefield terms. Note the different sign convention in the exponential terms from that used in Eqs. (28) and (29). We split $\overline{\psi}(\nu, y, \mu)$ as

$$\overline{\psi}(\nu, y, \mu) = \overline{\psi}_+(\nu, y, \mu) e^{i\nu a} + \overline{\psi}_-(\nu, y, \mu) e^{-i\nu a} + \overline{\psi}_1(\nu, y, \mu), \tag{44}$$

where

$$\overline{\psi}_\pm(\nu, y, \mu) = \frac{1}{(2\pi)^{1/2}} \int_{a, -\infty}^{\infty, -a} \psi(x, y, \mu) e^{i\nu(x \mp a)} dx \tag{45}$$

and

$$\overline{\psi}_1(\nu, y, \mu) = \frac{1}{(2\pi)^{1/2}} \int_{-a}^{a} \psi(x, y, \mu) e^{i\nu x} dx \tag{46}$$

In Eq. (45) the first (reading from left to right) set of limits accompany the plus sign and the second minus sign.

In calculating the partial transforms, Eq. (45), of ψ_i and ψ_r, Eqs. (39 and (40) care needs to be taken as $x \to \infty$. Accordingly, we shall assume that ψ_i and ψ_r are multiplied by $H(x-a)e^{-\varepsilon(x-a)}$ for $x > 0$ and by $H(x+a)e^{\varepsilon(x+a)}$ for $x < 0$. Later we shall let $\varepsilon \to 0$. This device allows us to sort out the regions of analyticity. Because ψ_r is that for a rigid rather than a porous barrier, the wavefield ψ will contain a transmitted term that behaves as $e^{-i\gamma(x\cos\theta_0)}e^{-\varepsilon(x-a)}$ for $x > a$, $e^{-i\gamma(x\cos\theta_0)}$ for $-a < x < a$ and $e^{-i\gamma(x\cos\theta_0)}e^{\varepsilon(x+a)}$ for $x < -a$. This fact will dominate the regions of analyticity. The term $\overline{\psi}_+(\nu, y, \mu)$ is regular for $\operatorname{Im}\nu > [\operatorname{Im}(\gamma\cos\theta_0) - i\varepsilon]$ and $\overline{\psi}_-(\nu, y, \mu)$ for $\operatorname{Im}\nu < [\operatorname{Im}(\gamma\cos\theta_0) + i\varepsilon]$. The function $\overline{\psi}_1(\nu, y, \mu)$ is an integral function. We shall end with two Wiener-Hopf problems one with the common region $\operatorname{Im}(\gamma\cos\theta_0 - i\varepsilon) < \operatorname{Im}\nu < \operatorname{Im}(\gamma\cos\theta_0)$ and one with the common region $\operatorname{Im}(\gamma\cos\theta_0) < \operatorname{Im}\nu < \operatorname{Im}(\gamma\cos\theta_0 + i\varepsilon)$.

Taking the Fourier transform over x of Eq. (31) and solving the resulting differential equation, so that the radiation condition is satisfied, gives

$$\overline{\psi}(\nu, y, \mu) = \begin{cases} A_1(\nu)e^{-\bar{\gamma}y} & y \ge 0^+ \\ A_2(\nu)e^{\bar{\gamma}y} & y \le 0^- \end{cases} \tag{47}$$

where

$$\bar{\gamma} = \left(\nu^2 - \gamma^2\right)^{1/2}, \quad \operatorname{Re}\bar{\gamma} > 0 \tag{48}$$

Transforming boundary conditions Eqs. (34) to (37), and using Eqs. (39) to (41) we get

$$\frac{d\overline{\psi}_-}{dy}(\nu, 0^\pm, \mu) \pm ik\left[\alpha\overline{\psi}_-(\nu, 0^\pm, \mu) + \beta\overline{\psi}_-(\nu, 0^\mp, \mu)\right] \pm \frac{2k\,{}^{\alpha}_{\beta}\, e^{i\gamma\cos\theta_0 a}\, b(\mu)}{(2\pi)^{1/2}\left[\nu - (\gamma\cos\theta_0 + i\varepsilon)\right]} = 0 \tag{49}$$

$$\frac{d\overline{\psi}_+}{dy}(\nu, 0^\pm, \mu) \pm ik\left[\alpha\overline{\psi}_+(\nu, 0^\pm, \mu) + \beta\overline{\psi}_+(\nu, 0^\mp, \mu)\right] \mp \frac{2k\,{}^{\alpha}_{\beta}\, e^{-i\gamma\cos\theta_0 a}\, b(\mu)}{(2\pi)^{1/2}\left[\nu - (\gamma\cos\theta_0 - i\varepsilon)\right]} = 0 \tag{50}$$

$$\overline{\psi}_1(\nu, 0^+, \mu) - \overline{\psi}_1(\nu, 0^-, \mu) = 2iG(\nu)b(\mu), \tag{51}$$

$$\frac{d\overline{\psi}_1}{dy}(\nu, 0^+, \mu) = \frac{d\overline{\psi}_1}{dy}(\nu, 0^-, \mu), \tag{52}$$

In Eqs. (49) and (50) the term α goes with the upper sign and β with the lower sign. The term $G(\nu)$ is given by

$$G(\nu)=\frac{e^{i(\nu-\gamma\cos\theta_0)a}-e^{-i(\nu-\gamma\cos\theta_0)a}}{(2\pi)^{1/2}(\nu-\gamma\cos\theta_0)} \tag{53}$$

From Eq. (47) and using the boundary conditions (49) to (52), we eliminate $d\overline{\psi}_+/dy$ and $d\overline{\psi}_-/dy$ to get

$$\begin{aligned}&e^{i\nu a}\,\overline{\eta}_+(\nu,0,\mu)[\overline{\gamma}-ik(\alpha-\beta)]+\frac{d\overline{\psi}_1}{dy}(\nu,0,\mu)+e^{-i\nu a}\,\overline{\eta}_-(\nu,0,\mu)[\overline{\gamma}-ik(\alpha-\beta)]\\&+i\overline{\gamma}\,G(\nu)b(\mu)+\frac{k\,b(\mu)(\alpha-\beta)e^{i(\nu-\gamma\cos\theta_0)a}}{(2\pi)^{1/2}[\nu-(\gamma\cos\theta_0-i\varepsilon)]}-\frac{k\,b(\mu)(\alpha-\beta)e^{-i(\nu-\gamma\cos\theta_0)a}}{(2\pi)^{1/2}[\nu-(\gamma\cos\theta_0+i\varepsilon)]}=0\end{aligned} \tag{54}$$

where

$$2\overline{\eta}_\pm(\nu,0,\mu)=\overline{\psi}_\pm(\nu,0^+,\mu)-\overline{\psi}_\pm(\nu,0^-,\mu) \tag{55}$$

Equation (54) is the Wiener-Hopf functional equation discussed by Noble [5, pp. 196-202]. Note how $(\alpha-\beta)$ enters this equation.

5. The Diffracted Wavefield

The unknown functions $\overline{\eta}_+(\nu,0,\mu)$ and $\overline{\eta}_-(\nu,0,\mu)$ have been determined by using the procedure discussed by Noble [5, pp. 196-202]. Several steps in the procedure are given in Appendix A. Terms multiplied by $(\alpha+\beta)$ are $O(kh)$ and are dropped, but terms containing $(\alpha-\beta)$ (that appear in $L(\nu)$ and $L_\pm(\nu)$) need not be small and are retained. Moreover, the procedure includes asymptotically evaluating the integrals appearing in Eqs. (A15) and (A17) for large ξa, where ξ scales with k. That is, we have taken ka to be large. With these approximations the functions are given by

$$\overline{\eta}_\pm(\nu,0,\mu)=-\frac{ib(\mu)}{(2\pi)^{1/2}S_\pm(\nu)}\left[G_{1,2}(\pm\nu)+T(\pm\nu)C_{1,2}(\gamma)\right], \tag{56}$$

where the subscript 1 accompanies the upper sign and the subscript 2 the lower. The $C_{1,2}(\gamma)$ are

$$C_{1,2}(\gamma)=\left[S_+(\gamma)G_{2,1}(\gamma)\;+\;T(\gamma)G_{1,2}(\gamma)\right]\left[S_+^2(\gamma)-T^2(\gamma)\right]^{-1} \tag{57}$$

The $G_{1,2}(\nu)$ are

$$G_{1,2}(\nu)=P_{1,2}(\nu)e^{\mp i\gamma\cos\theta_0 a}-R_{1,2}(\nu)e^{\pm i\gamma\cos\theta_0 a} \tag{58}$$

where

$$P_{1,2}(\nu)=\frac{S_+(\nu)-S_\pm(\gamma\cos\theta_0)}{(\nu\mp\gamma\cos\theta_0)}-\frac{ik(\alpha-\beta)}{S_\mp(\gamma\cos\theta_0\mp i\varepsilon)[\nu\mp(\gamma\cos\theta_0\mp i\varepsilon)]} \tag{59}$$

and

$$R_{1,2}(\nu) = \frac{E_0\gamma^{1/2}\left\{W_0\left[-i(\gamma \pm \gamma\cos\theta_0)2a\right] - W_0\left[-i(\gamma+\nu)2a\right]\right\}}{2\pi i\, L_+(\gamma)(\nu \mp \gamma\cos\theta_0)} \tag{60}$$

The first subscript accompanies the upper sign and the second the lower sign . The $T(\nu)$ is

$$T(\nu) = \frac{E_0\gamma^{1/2}W_0\left[-i(\gamma+\nu)2a\right]}{2\pi i\, L_+(\gamma)} \tag{61}$$

where

$$E_0 = 2e^{i\pi/2}\frac{e^{i2\gamma a}}{(2\gamma a)^{1/2}} \tag{62}$$

The definition of $W_0(z)$ needed in this paper is

$$W_0(-iy) = \pi^{1/2}\left\{1 + \pi^{1/2}e^{-iy}(-iy)^{1/2}\operatorname{erfc}\left[(-iy)^{1/2}\right]\right\} \tag{63}$$

where y is real and positive (for our work), and $\operatorname{erfc}(z)$ is the complimentary error function. It is closely related to the Fresnel integral.

6. Farfield Asymptotic Approximations to the Diffracted Wavefield

Substitution of Eqs. (56) and (57) in Eq. (A8) yields

$$\begin{aligned} A_{1,2}(\nu) = &-\operatorname{sgn}(y)\frac{ib(\mu)}{(2\pi)^{1/2}}\left\{\frac{e^{i\nu a}}{S_+(\nu)}\left[G_1(\nu) + C_1(\gamma)T(\nu)\right] + \frac{e^{-i\nu a}}{S_-(\nu)}\left[G_2(-\nu) + C_2(\gamma)T(-\nu)\right]\right\} \\ &+ i\operatorname{sgn}(y)G(\nu)b(\mu) \end{aligned} \tag{64}$$

where the first subscript corresponds to $y > 0$ and the second to $y < 0$, and therefore the wavefield $\psi(x, y, \mu)$. We divide ψ as $\psi = \psi_1(x, y, \mu) + \psi_2(x, y, \mu)$. Each part is given by

$$\begin{aligned} \psi_1(x, y, \mu) = \operatorname{sgn}(y)\frac{ib(\mu)}{2\pi}\int_{-\infty}^{\infty} d\nu\, e^{-i\nu x}e^{-\bar{\gamma}|y|}&\left\{\frac{S_+(\gamma\cos\theta_0)e^{i(\nu-\gamma\cos\theta_0)a}}{S_+(\nu)(\nu-\gamma\cos\theta_0)} - \frac{S_-(\gamma\cos\theta_0)e^{-i(\nu-\gamma\cos\theta_0)a}}{S_-(\nu)(\nu-\gamma\cos\theta_0)}\right. \\ &+ \frac{ik(\alpha-\beta)e^{i(\nu-\gamma\cos\theta_0)a}}{S_+(\nu)S_-(\gamma\cos\theta_0 - i\varepsilon)\left[\nu-(\gamma\cos\theta_0 - i\varepsilon)\right]} \\ &\left.- \frac{ik(\alpha-\beta)e^{-i(\nu-\gamma\cos\theta_0)a}}{S_-(\nu)S_+(\gamma\cos\theta_0 + i\varepsilon)\left[\nu-(\gamma\cos\theta_0 + i\varepsilon)\right]}\right\} \end{aligned} \tag{65}$$

and

$$\psi_2(x,y,\mu) = \mathrm{sgn}(y)\frac{ib(\mu)}{2\pi}\int_{-\infty}^{\infty} d\nu\, e^{-i\nu x} e^{-\bar{\gamma}|y|}\left\{\frac{e^{i\nu a}}{S_+(\nu)}\left[R_1(\nu)e^{i\gamma\cos\theta_0 a} - C_1(\gamma)T(\nu)\right]\right.$$
$$\left. + \frac{e^{-i\nu a}}{S_-(\nu)}\left[R_2(-\nu)e^{-i\gamma\cos\theta_0 a} - C_2(\gamma)T(-\nu)\right]\right\} \tag{66}$$

The first term $\psi_1(x,y,\mu)$ represents the field diffracted by the edges at $x = \pm a$, plus the geometrical wavefield, not included earlier. Note that there is one pole above the contour and a second below it. These terms are the transmitted wavefield. Once these pole contributions are captured we can let $\varepsilon \to 0$. The second term $\psi_2(x,y,\mu)$ gives the interaction of one edge with the other.

The integrals appearing in Eqs. (65) and (66) can be evaluated asymptotically by using the method of steepest descents. Harris [15] shows that, beyond the Fresnel distance, $k(2a)^2/2\pi$, the exponential phase terms in the braces need not be considered and only the exponential with x and $|y|$ needs to be considered in making the steepest descents calculation. In other words we evaluate the diffracted wavefield at points sufficiently distant from the slit that it has evolved into a cylindrical wavefield (a spheroidal wavefield after the inversion in μ) with a radiation pattern. For that, we put $x = r\cos\theta$ and $|y| = r\sin\theta$, with $0 < \theta < \pi$, and deform the contour by the Sommerfeld transformation $\nu = -\gamma\cos(\tau)$. Hence, for large γr, the diffracted wavefields are

$$\psi_1(x,y,\mu) = \mathrm{sgn}(y)\frac{i\sin\theta\, b(\mu)}{(2\pi\gamma r)^{1/2}} F_1(-\gamma\cos\theta)e^{i(\gamma r - \pi/4)} \tag{67}$$

and

$$\psi_2(x,y,\mu) = \mathrm{sgn}(y)\frac{i\gamma\sin\theta\, b(\mu)}{(2\pi\gamma r)^{1/2}} F_2(-\gamma\cos\theta)e^{i(\gamma r - \pi/4)} \tag{68}$$

The radiation patterns are given by

$$F_1(-\gamma\cos\theta) = -\left\{\frac{S_+(\gamma\cos\theta_0)e^{-i\gamma(\cos\theta+\cos\theta_0)a}}{S_+(-\gamma\cos\theta)(\cos\theta+\cos\theta_0)} - \frac{S_-(\gamma\cos\theta_0)e^{i\gamma(\cos\theta+\cos\theta_0)a}}{S_-(-\gamma\cos\theta)(\cos\theta+\cos\theta_0)}\right.$$
$$\left. + \frac{ik(\alpha-\beta)e^{-i\gamma(\cos\theta+\cos\theta_0)a}}{S_+(-\gamma\cos\theta)S_-(\gamma\cos\theta_0)(\cos\theta+\cos\theta_0)} - \frac{ik(\alpha-\beta)e^{-i\gamma(\cos\theta+\cos\theta_0)a}}{S_-(-\gamma\cos\theta)S_+(\gamma\cos\theta_0)(\cos\theta+\cos\theta_0)}\right\} \tag{69}$$

and

$$F_2(-\gamma\cos\theta)=\left[R_1(-\gamma\cos\theta)e^{-i\gamma\cos\theta_0 a}-C_1(\gamma)T(-\gamma\cos\theta)\right]\left[\frac{e^{-i\gamma\cos\theta a}}{S_+(-\gamma\cos\theta)}\right]$$
$$+\left[R_2(\gamma\cos\theta)e^{i\gamma\cos\theta_0 a}-C_2(\gamma)T(\gamma\cos\theta)\right]\left[\frac{e^{i\gamma\cos\theta a}}{S_+(\gamma\cos\theta)}\right] \tag{70}$$

Next we take the inverse transform over μ using Eqs. (67) and (68) in Eq. (29).

$$\sigma_{d1}(x,y,z)=\mathrm{sgn}(y)\frac{i\sin\theta}{8\pi^2(rr_0)^{1/2}}\int_{-\infty}^{\infty}F_1(-\gamma\cos\theta)\frac{e^{i[\gamma(r+r_0)+\mu(z-z_0)]}}{\gamma}d\mu \tag{71}$$

$$\sigma_{d2}(x,y,z)=\mathrm{sgn}(y)\frac{i\sin\theta}{8\pi^2(rr_0)^{1/2}}\int_{-\infty}^{\infty}F_2(-\gamma\cos\theta)e^{i[\gamma(r+r_0)+\mu(z-z_0)]}d\mu \tag{72}$$

Introduce $r+r_0=r_{12}\sin\phi_{12}$ and $(z-z_0)=r_{12}\cos\phi_{12}$, with $0<\phi_{12}<\pi$. Using the transformation $\mu=k\cos(\tau)$, Eqs. (74) and (75) are approximated as

$$\sigma_{d1}(x,y,z)=\mathrm{sgn}(y)\frac{i\sin\theta}{4\pi(2\pi krr_0r_{12})^{1/2}}F_1(-k\cos\theta\sin\phi_{12})e^{i(kr_{12}-\pi/4)} \tag{73}$$

and

$$\sigma_{d2}(x,y,z)=\mathrm{sgn}(y)\frac{ik\sin\theta\sin\phi_{12}}{4\pi(2\pi krr_0r_{12})^{1/2}}F_2(-k\cos\theta\sin\phi_{12})e^{i(kr_{12}-\pi/4)} \tag{74}$$

where $F_{1,2}(-k\cos\theta\sin\phi_{12})$ are given by Eqs. (69) and (70), respectively.

7. Discussion

We are concerned with understanding how successfully the barrier reduces the sound transmission despite the presence of the slit. Moreover, we want to understand how the absorption of the barrier makes its presence felt. To do so we imagine that source lies on the positive y axis far from the slit and that the reflected sound is measured at a point on the y axis, also far from the slit. We take $b(\mu)=1$ and $\mu=0$, so that $\gamma=k$, in Eqs. (67) and (68). Moreover we set $\theta_0=\pi/2$. The power both carried by the reflected wavefield and by wavefield diffracted from the slit is then calculated in the farfield. The term resulting from their interference is then extracted. This term is the power removed from the reflected wavefield by that scattered by and transmitted through the slit, and by that absorbed by the barrier. It is then normalized by dividing by the reflected intensity times *twice* the width of the slit. This quantity is given by

$$\frac{\Gamma(ka,\alpha-\beta)}{4a}=\frac{1}{2ka}\mathrm{Im}\{[F_1(0)+kF_2(0)][1+(\alpha-\beta)]\} \tag{75}$$

The term $F_1(0)$ is given by

$$F_1(0) = \frac{2ika}{[1+(\alpha-\beta)]} - \left\{1 + \left[\frac{2(\alpha-\beta)}{1+(\alpha-\beta)}\right]^{1/2}\right\} \tag{76}$$

and $F_2(0)$ is given by

$$kF_2(0) = -\frac{e^{i(2ka-\pi/4)}}{(2\pi)^{1/2}(2ka)^{3/2}} \frac{1}{[L_+(k)]^2[1+(\alpha-\beta)]} \tag{77}$$

The interesting behavior is largely confined to the second term in Eq. (75). Setting

$$F(ka,\alpha-\beta) = \frac{1}{2ka}\mathrm{Im}\{kF_2(0)[1+(\alpha-\beta)]\} \tag{78}$$

Fig. 3 shows a plot of F against ka, for values of $\alpha-\beta$ from 0 to 1. The increasing effect of the absorption is apparent. The form of $kF_2(0)$ suggests that the interaction between the edges is affected more strongly by the properties of the barrier than are singly diffracted rays. However, because $2ka$ is large in our approximation, the interaction term is always small.

Note that the case $(\alpha-\beta)=0$ corresponds to a rigid barrier. In this case our expression for $\Gamma(ka,\alpha-\beta)/4a$ corresponds to the transmission cross-section given by Karp and Keller [6, Eq.(16)], namely,

$$\frac{\Gamma(ka,\alpha-\beta)}{4a} = 1 - \frac{\sin(2ka-\pi/4)}{(2\pi)^{1/2}(2ka)^{5/2}} \tag{79}$$

It is of interest to note how the parameters $(\alpha\pm\beta)$ enter the calculation. The parameter $(\alpha+\beta)$ represents essentially the thickness of the barrier and appears in the calculation separated from the other terms, while $(\alpha-\beta)$ represents the absorption of the barrier and is intimately included in the calculation through its role in the terms $L_\pm$ and L. We believe that the Rawlins boundary condition more adequately represents the mechanical response of a thin absorbing barrier than would a boundary condition with $\beta=0$.

While we have not explored our expressions in any completeness, we conjecture that they are more accurate than those calculated using the geometrical theory of diffraction and hence that they permit us to approximate the wavefields both near the slit and near the barrier itself.

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Appendix A The Solution to the Wiener-Hopf Problem

To solve Eq. (54), we make the following factorizations

$$\bar{\gamma} = K_+(\nu)K_-(\nu) = \left[e^{-i\pi/4}(\nu+\gamma)^{1/2}\right]\left[e^{-i\pi/4}(\gamma-\nu)^{1/2}\right], \tag{A1}$$

and

$$L(\nu) = \left[1 - ik(\alpha-\beta)/\bar{\gamma}\right] = L_+(\nu)L_-(\nu)\ , \tag{A2}$$

where $L_+(\nu)$ and $K_+(\nu)$ are regular for $\operatorname{Im}\nu > -\operatorname{Im}\gamma$, and $L_-(\nu)$ and $K_-(\nu)$ are regular for $\operatorname{Im}\nu < \operatorname{Im}\gamma$. Rawlins gives the exact factorization of Eq. (A2) in both [8] and [9]. The $L_\pm(0)$ are given by

$$L_+(0) = L_-(0) = \left[1 + (\alpha-\beta)k/\gamma\right]^{1/2} \tag{A3}$$

and

$$L_+(\gamma) = L_-(-\gamma) = \left(\frac{1+\cos\chi}{2}\right)^{1/2} \exp\left(\frac{-1}{2\pi}\int_{-\chi}^{\chi}\frac{u}{\sin u}du\right) \tag{A4}$$

where $\sin\chi = -(\alpha-\beta)(k/\gamma)$.

Using Eqs. (A1) and (A2), we rewrite Eq. (54) as

$$\begin{aligned} &e^{i\nu a}\bar{\eta}_+(\nu,0,\mu) + \left[S_+(\nu)S_-(\nu)\right]^{-1}\frac{d\bar{\psi}_1}{dy}(\nu,0,\mu) + e^{-i\nu a}\bar{\eta}_-(\nu,0,\mu) + iG(\nu)b(\mu) \\ &-\frac{kb(\mu)(\alpha-\beta)e^{ia(\nu-\gamma\cos\theta_0)}}{(2\pi)^{1/2}\left[S_+(\nu)S_-(\nu)\right]}\left[\frac{1}{(\nu-\gamma\cos\theta_0)} - \frac{1}{\left[\nu-(\cos\theta_0 - i\varepsilon)\right]}\right] \\ &+\frac{kb(\mu)(\alpha-\beta)e^{-ia(\nu-\gamma\cos\theta_0)}}{(2\pi)^{1/2}\left[S_+(\nu)S_-(\nu)\right]}\left[\frac{1}{(\nu-\gamma\cos\theta_0)} - \frac{1}{\left[\nu-(\cos\theta_0 + i\varepsilon)\right]}\right] = 0 \end{aligned} \tag{A5}$$

where.

$$S_\pm(\nu) = K_\pm(\nu)L_\pm(\nu) \tag{A6}$$

With the help of Eqs. (44), (47), and (49) to (52), the unknown functions $A_1(\nu)$ and $A_2(\nu)$ are given by

$$\begin{aligned} \pm 2A_{1,2}(\nu) &= e^{i\nu a}\left[\bar{\psi}_+(\nu,0^+,\mu) - \bar{\psi}_+(\nu,0^-,\mu)\right] + e^{-i\nu a}\left[\bar{\psi}_-(\nu,0^+,\mu) - \bar{\psi}_-(\nu,0^-,\mu)\right] + i2G(\nu)b(\mu) \\ &\pm\frac{ik(\alpha+\beta)}{\bar{\gamma}}\Big\{e^{i\nu a}\left[\bar{\psi}_+(\nu,0^+,\mu) + \bar{\psi}_+(\nu,0^-,\mu)\right] + e^{-i\nu a}\left[\bar{\psi}_-(\nu,0^+,\mu) + \bar{\psi}_-(\nu,0^-,\mu)\right] \\ &+\frac{i2b(\mu)e^{ia(\nu-\gamma\cos\theta_0)}}{\left[\nu-(\gamma\cos\theta_0 - i\varepsilon)\right]} - \frac{i2b(\mu)e^{-ia(\nu-\gamma\cos\theta_0)}}{\left[\nu-(\gamma\cos\theta_0 + i\varepsilon)\right]}\Big\} \end{aligned} \tag{A7}$$

The + sign is used with the subscript 1 and the - sign with the subscript 2. As we indicated in our discussion of the boundary conditions, terms multiplied by $(\alpha+\beta)$ are $O(kh)$(after the inverse transforms are taken) and are dropped, but terms containing $(\alpha-\beta)$, that appear in $L_{\pm}(\nu)$, need not be small and are retained. Thus, using this approximation, Eq. (A7) becomes

$$A_1(\nu)=-A_2(\nu)=e^{i\nu a}\,\overline{\eta}_+(\nu,0,\mu)+e^{-i\nu a}\,\overline{\eta}_-(\nu,0,\mu)+iG(\nu)b(\mu) \tag{A8}$$

By multiplying Eq. (A5) by $S_+(\nu)e^{-i\nu a}$ and using the general decomposition theorem, we obtain

$$\begin{aligned}
&S_+(\nu)\overline{\eta}_+(\nu,0,\mu)+\frac{ib(\mu)e^{-i\gamma\cos\theta_0 a}}{(2\pi)^{1/2}(\nu-\gamma\cos\theta_0)}\left[S_+(\nu)-S_+(\gamma\cos\theta_0)\right]+U_+(\nu)+V_+(\nu)\\
&+\frac{kb(\mu)(\alpha-\beta)e^{-i\gamma\cos\theta_0 a}}{(2\pi)^{1/2}S_-(\gamma\cos\theta_0-i\varepsilon)\left[\nu-(\gamma\cos\theta_0-i\varepsilon)\right]}=-\frac{e^{-i\nu a}}{S_-(\nu)}\frac{d\overline{\psi}_1}{dy}(\nu,0,\mu)\\
&-\frac{ib(\mu)e^{-i\gamma\cos\theta_0 a}S_+(\gamma\cos\theta_0)}{(2\pi)^{1/2}(\nu-\gamma\cos\theta_0)}-U_-(\nu)-V_-(\nu)+\frac{ib(\mu)e^{-i(2\nu-\gamma\cos\theta_0)a}S_+(\gamma\cos\theta_0)}{(2\pi)^{1/2}(\nu-\gamma\cos\theta_0)}\\
&+\frac{kb(\mu)(\alpha-\beta)e^{-i\gamma\cos\theta_0 a}}{(2\pi)^{1/2}S_-(\nu)(\nu-\gamma\cos\theta_0)}-\frac{kb(\mu)(\alpha-\beta)e^{-i\gamma\cos\theta_0 a}}{(2\pi)^{1/2}\left[\nu-(\gamma\cos\theta_0-i\varepsilon)\right]}\left[\frac{1}{S_-(\nu)}-\frac{1}{S_-(\gamma\cos\theta_0-i\varepsilon)}\right]\\
&-\frac{kb(\mu)(\alpha-\beta)e^{-i(2\nu-\gamma\cos\theta_0)a}}{(2\pi)^{1/2}S_-(\nu)}\left[\frac{1}{(\nu-\gamma\cos\theta_0)}-\frac{1}{\left[\nu-(\gamma\cos\theta_0+i\varepsilon)\right]}\right]
\end{aligned} \tag{A9}$$

The functions $U_{\pm}(\nu)$ and $V_{\pm}(\nu)$ are the decomposition [17] of

$$U(\nu)=S_+(\nu)\overline{\eta}_-(\nu,0,\mu)e^{-i2\nu a} \tag{A10}$$

and

$$V(\nu)=\frac{-ie^{-i(2\nu-\gamma\cos\theta_0)a}b(\mu)}{(2\pi)^{1/2}(\nu-\gamma\cos\theta_0)}\left[S_+(\nu)-S_+(\gamma\cos\theta_0)\right] \tag{A11}$$

Similarly, multiplying Eq. (A5) by $S_-(\nu)e^{i\nu a}$, we obtain

$$
\begin{aligned}
&S_-(\nu)\overline{\eta}_-(\nu,0,\mu)-\frac{ib(\mu)e^{i\gamma\cos\theta_0 a}}{(2\pi)^{1/2}(\nu-\gamma\cos\theta_0)}\left[S_-(\nu)-S_-(\gamma\cos\theta_0)\right]+P_-(\nu)-Q_-(\nu)\\
&-\frac{k\,b(\mu)(\alpha-\beta)e^{i\gamma\cos\theta_0 a}}{(2\pi)^{1/2}S_+(\gamma\cos\theta_0+i\varepsilon)\left[\nu-(\gamma\cos\theta_0+i\varepsilon)\right]}=-\frac{e^{i\nu a}}{S_+(\nu)}\frac{d\overline{\Psi}_1}{dy}(\nu,0,\mu)\\
&+\frac{ib(\mu)e^{i\gamma\cos\theta_0 a}S_-(\gamma\cos\theta_0)}{(2\pi)^{1/2}(\nu-\gamma\cos\theta_0)}-P_+(\nu)+Q_+(\nu)-\frac{ib(\mu)e^{i(2\nu-\gamma\cos\theta_0)a}S_-(\gamma\cos\theta_0)}{(2\pi)^{1/2}(\nu-\gamma\cos\theta_0)}\\
&-\frac{k\,b(\mu)(\alpha-\beta)e^{i\gamma\cos\theta_0 a}}{(2\pi)^{1/2}S_+(\nu)(\nu-\gamma\cos\theta_0)}+\frac{k\,b(\mu)(\alpha-\beta)e^{i\gamma\cos\theta_0 a}}{(2\pi)^{1/2}\left[\nu-(\gamma\cos\theta_0+i\varepsilon)\right]}\left[\frac{1}{S_+(\nu)}-\frac{1}{S_+(\gamma\cos\theta_0+i\varepsilon)}\right]\\
&+\frac{k\,b(\mu)(\alpha-\beta)e^{i(2\nu-\gamma\cos\theta_0)a}}{(2\pi)^{1/2}S_+(\nu)}\left[\frac{1}{(\nu-\gamma\cos\theta_0)}-\frac{1}{\left[\nu-(\gamma\cos\theta_0-i\varepsilon)\right]}\right]
\end{aligned}
\tag{A12}
$$

The functions $P_\pm(\nu)$ and $Q_\pm(\nu)$ are the decomposition of

$$P(\nu)=S_-(\nu)\overline{\eta}_+(\nu,0,\mu)e^{i2\nu a} \tag{A13}$$

and

$$Q(\nu)=\frac{-ib(\mu)e^{i(2\nu-\gamma\cos\theta_0)a}}{(2\pi)^{1/2}(\nu-\gamma\cos\theta_0)}\left[S_-(\nu)-S_-(\gamma\cos\theta_0)\right] \tag{A14}$$

Let $\bar{f}_1(\nu)$ define a function equal to both sides of Eq. (A9). The left hand side is regular for $\operatorname{Im}\nu>\operatorname{Im}(\gamma\cos\theta_0-i\varepsilon)$ and the right hand side is regular for $\operatorname{Im}\nu<\operatorname{Im}(\gamma\cos\theta_0)$. Therefore, by analytic continuation, the definition of $\bar{f}_1(\nu)$ can be extended throughout the complex ν plane. The form of $\bar{f}_1(\nu)$ is ascertained by examining the asymptotic behavior of the terms in Eq. (A9) as $|\nu|\to\infty$. We note from Rawlins [9] that $|L_\pm(\nu)|=O(1)$ as $|\nu|\to\infty$ and, with the help of the edge conditions Eqs. (6) and (7), we find that $d\overline{\Psi}_1/dy$ is of $O\left(|\nu|^{-1/2}\right)$ as $|\nu|\to\infty$. Using the extended form of Liouville's theorem, it can be seen that $\bar{f}_1(\nu)$ can only be a constant equal to zero. Hence, from Eq. (A9), we obtain

$$
\begin{aligned}
&S_+(\nu)\overline{\eta}_+^*(\nu,0,\mu)+\frac{1}{2\pi i}\int_{-\infty+ic}^{\infty+ic}\frac{K_+(\xi)\overline{\eta}_-^*(\xi,0,\mu)e^{-i2\xi a}}{L_-(\xi)(\xi-\nu)}d\xi\\
&-\frac{ib(\mu)e^{-i\gamma\cos\theta_0 a}S_+(\gamma\cos\theta_0)}{(2\pi)^{1/2}(\nu-\gamma\cos\theta_0)}+\frac{k\,b(\mu)(\alpha-\beta)e^{-i\gamma\cos\theta_0 a}}{(2\pi)^{1/2}S_-(\gamma\cos\theta_0-i\varepsilon)\left[\nu-(\gamma\cos\theta_0-i\varepsilon)\right]}=0
\end{aligned}
\tag{A15}
$$

where

$$\overline{\eta}_\pm^*(\nu,0,\mu)=\overline{\eta}_\pm(\nu,0,\mu)\pm\frac{ie^{\mp i\gamma\cos\theta_0 a}b(\mu)}{(2\pi)^{1/2}(\nu-\gamma\cos\theta_0)} \tag{A16}$$

Similarly, from the equality of both sides of Eq. (A12 in the strip $\mathrm{Im}(\gamma\cos\theta_0) < \nu < \mathrm{Im}(\gamma\cos\theta_0 + i\varepsilon)$, we have

$$S_-(\nu)\overline{\eta}_-^*(\nu,0,\mu) - \frac{1}{2\pi i}\int_{-\infty+id}^{\infty+id} \frac{K_-(\xi)\overline{\eta}_+^*(\xi,0,\mu)e^{i2\xi a}}{L_+(\xi)(\xi-\nu)}d\xi$$
$$+\frac{ib(\mu)e^{i\gamma\cos\theta_0 a}S_-(\gamma\cos\theta_0)}{(2\pi)^{1/2}(\nu-\gamma\cos\theta_0)} - \frac{k\,b(\mu)(\alpha-\beta)e^{i\gamma\cos\theta_0 a}}{(2\pi)^{1/2}S_+(\gamma\cos\theta_0+i\varepsilon)\left[\nu-(\gamma\cos\theta_0+i\varepsilon)\right]} = 0 \tag{A17}$$

The contours of the integrals in Eqs. (A15) and (A17) are such that $c < \mathrm{Im}(\gamma\cos\theta_0 - i\varepsilon)$ and $d > \mathrm{Im}(\gamma\cos\theta_0 + i\varepsilon)$. These integrals must be asymptotically approximated, as indicated in Noble [5, pp. 199-202]. Thus we arive at Eqs. (56) through (63).

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Figure 1 Asghar, *et al.*

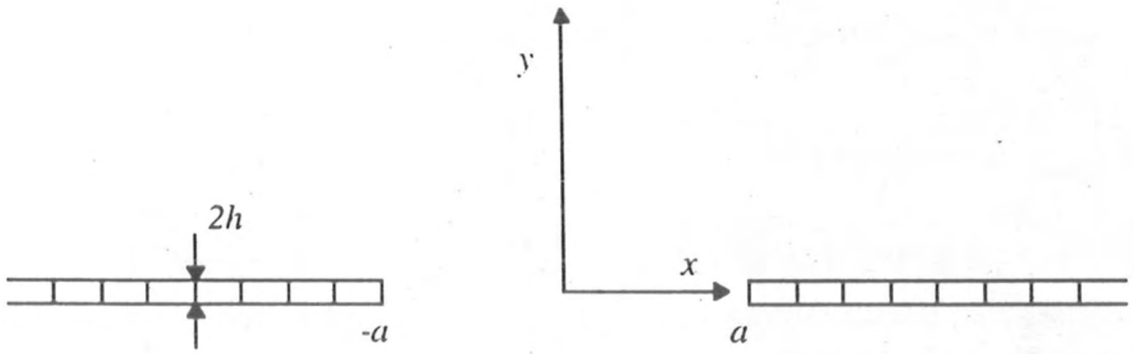
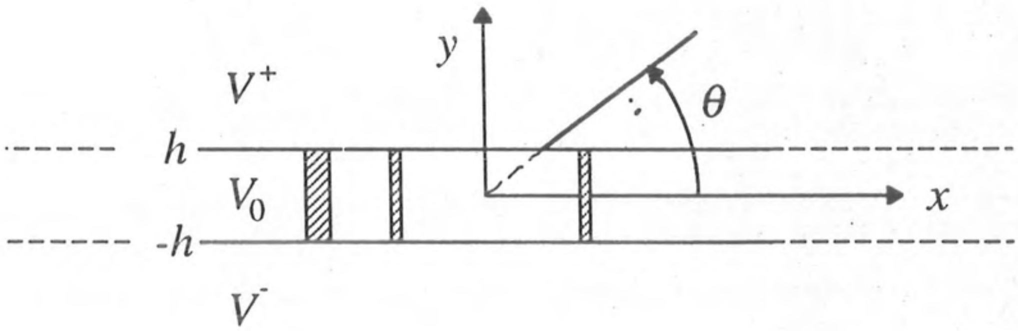
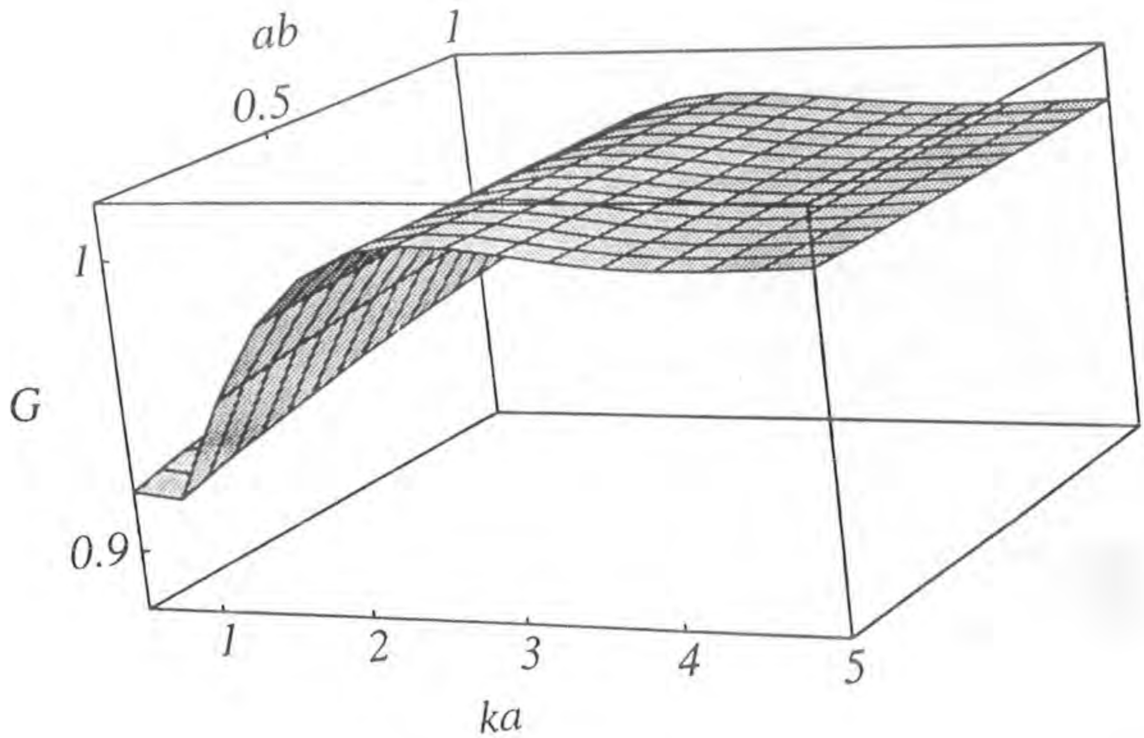


Figure 2 Asghar, *et al.*





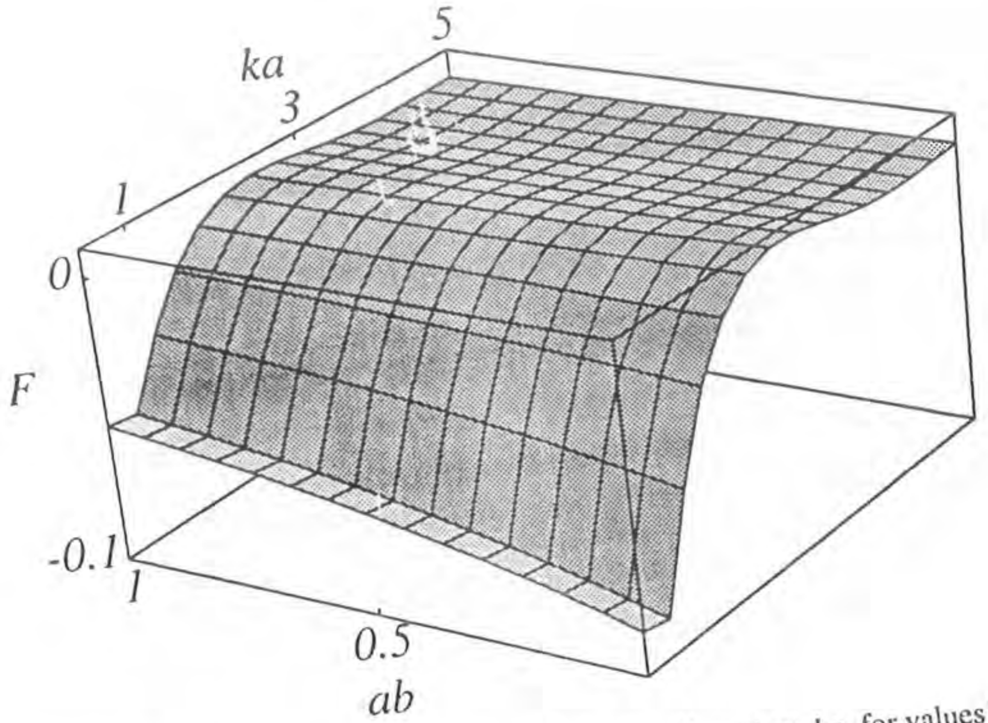


Figure 3 A three-dimensional graph of $F(ka, \alpha - \beta)$ against ka, for values between 0.5 and 10, and against $(\alpha - \beta)$ for values from 0 to 1.0. The $(\alpha - \beta)$ axis is labeled ab.

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SCATTERING OF A SPHERICAL SOUND WAVE BY A RIGID SCREEN WITH AN ABSORBENT EDGE IN A MOVING FLUID

S. ASGHAR AND TASAWAR HAYAT

ABSTRACT. The scattering of a spherical wave (emanating due to a point source) by a semi-infinite plane in the presence of a moving fluid is investigated. A finite region in the vicinity of the edge has an absorbing boundary condition; the remaining part of the half plane is rigid. The problem which is solved is a mathematical model for a rigid barrier with an absorbent edge in the presence of a moving fluid. It is found that the absorbing material that comprises the edge need only be of the order of a wavelength long to have approximately the same effect on the sound attenuation in the shadow region of the barrier as a semi-infinite absorbent barrier. Also the softer the absorbent lining the greater the attenuation in the shadow region of the barrier. In the illuminated region a reduction in the sound intensity level can be achieved by a suitable choice of the absorptive material of the strip and its length. This investigation is important in the sense that point source is regarded as fundamental radiating device. It is found that a diffracted field is the sum of fields produced by the two edges. Finally, physical interpretation of the result is discussed.

1. Introduction. Acoustic waves have a wide range of applications in modern science. These are used as diagnostic tools in determining the mechanical parameters of fluids and solids. In addition their application ranges from geophysics (seismic exploration techniques, bore-hole sounding) to quantitative nondestructive evaluation of mechanical structures, and acoustic tomography for medical purposes. Another important application is the problem of noise reduction. Noise from motorways, railways and airports especially can be shielded by a barrier which intercepts the line of hearing from the noise source to the receiver. The acoustic field in the shadow region of a barrier (when transmission through the barrier is negligible) is due to diffraction at the edge alone. For this reason Butler [6] suggested that the region in the immediate vicinity of the edge should be lined with absorbent material to reduce the sound level in the shadow region. This technique

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has potential applications in engine noise shielding by aircraft wings. In case of noise radiated from aero-engines and inside wind tunnels, analysis of the acoustic diffraction from absorbing planes in moving fluids is required. Rawlins [20] has presented theoretical work on this model by considering diffraction of an acoustic wave by an absorbing half-plane in the presence of a moving fluid. This analysis was further extended to a point source by Asghar, et al. [1]. Jones [11] considered a time harmonic line source parallel to a semi-infinite rigid plane in still air as well as in a moving inviscid fluid. It was further extended by Balasubramanyam [4], and to the diffraction of a cylindrical pulse by Rienstra [24].

If the wavelength of the sound is much smaller than the length scale associated with the barrier, the diffraction process is governed to all intents and purposes by the solution to the canonical problem of diffraction by a semi-infinite rigid plane with absorbent edge. The aim here is to solve this mixed boundary value problem in the presence of a moving fluid. The present analysis is also related to Wiener-Hopf solutions for structural elements composed of flat plates joined end to end [19, 2, 3, 7, 5, 22, 17, 8, 9, 10, 23, 24], although such configurations are quite distinct from the present one. The solution of the problem is obtained in terms of two Fredholm integral equations. The mathematical method used to obtain these Fredholm integral equations is Jones method and the Wiener-Hopf technique [16]. A key attribute of such a technique is that it is not fundamentally numerical in nature and thus allows additional insight into the mathematical and physical structure of the diffracted field. The difficulty that arises is the solution of the integrals occurring in solving the Wiener-Hopf functional equations. These integrals are normally difficult to handle because of the presence of branch points and are only amenable to solution using asymptotic approximations. The analytic solution of these integrals is thus obtained and finally the far field is presented.

2. Formulation of the problem. We consider a small amplitude sound wave on a main stream moving with a velocity U parallel to the x-axis. A semi-infinite plane is assumed to occupy $y = 0$, $x \leq 0$ as shown in Figure 1. The half plane is assumed to be infinitely thin, and over the interval $-l < x < 0$ there is an absorbing substance satisfying

$$p - u_n z_a = 0, \quad [14] \tag{1}$$

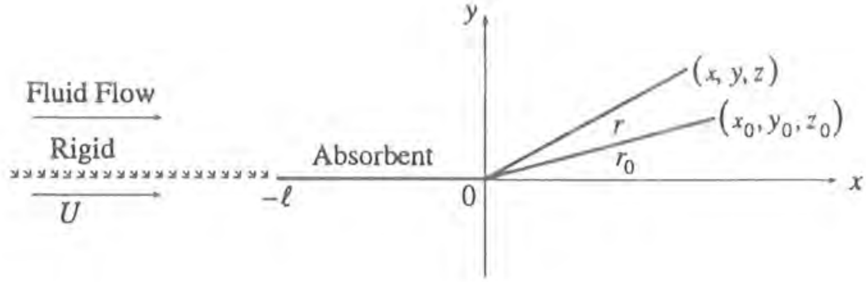

FIGURE 1.

on both sides of the surface and the remainder $-\infty < x < -l$, of the half plane is rigid. In Equation (1) p is the pressure, u_n the normal component of the perturbation velocity, z_a is acoustic impedance of the plane, and n is a normal vector pointing from the fluid into the surface.

The perturbation velocity u of the irrotational sound wave can be expressed in terms of a total velocity potential $\chi_t(x, y, z)$ by $u = \operatorname{grad} \chi_t(x, y, z)$. The resulting pressure in the sound field is given by

$$p = -\rho_0(\partial/\partial t + U\,\partial/\partial x)\chi_t, \tag{2}$$

where ρ_0 is the density of the undisturbed stream. We shall restrict our consideration to the time harmonic variation $e^{-i\omega t}$ (ω is the angular frequency) and suppose there is a point source at (x_0, y_0, z_0), $y_0 > 0$. Then our problem becomes one of solving the convective wave equation

$$\left((1 - M^2)\frac{\partial^2}{\partial x^2} + 2ikM\frac{\partial}{\partial x} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} + k^2\right)\chi_t(x, y, z) = \delta(x - x_0)\delta(y - y_0)\delta(z - z_0), \tag{3}$$

subject to the following boundary conditions:

$$\partial\chi_t(x, 0^\pm, z)/\partial y = 0, \quad x < -l, \tag{4}$$

$$\left(\frac{\partial}{\partial y} \mp \beta M\frac{\partial}{\partial x} \pm ik\beta\right)\chi_t(x, 0^\pm, z) = 0, \quad -l < x < 0, \tag{5}$$

and

$$\chi_t(x,0^+,z) = \chi_t(x,0^-,z),$$
$$(6) \qquad \frac{\partial \chi_t(x,0^+,z)}{\partial y} = \frac{\partial \chi_t(x,0^-,z)}{\partial y},$$
$$x > 0,$$

where k $(= \omega/c)$ is the free space wave number, c is the velocity of sound, β $(= \rho_0 c/z_a)$ is the specific admittance of the absorbent surface, and $M = U/c$ is the Mach number. For subsonic flow $|M| < 1$ and for acoustic absorption $\mathrm{Re}\,(z_a) > 0$.

It is assumed that a solution can be written in the form

$$(7) \qquad \chi_t(x,y,z) = \chi_0(x,y,z) + \chi(x,y,z),$$

where χ_0 is the incident wave which accounts for the inhomogeneous source term and $\chi(x,y,z)$ is the solution of homogeneous wave equation (3) that corresponds to the diffracted field.

In addition, for a unique solution of the boundary value problem Equations (3)–(7) [**18**], we insist that χ represents an outward travelling wave as $R = (x^2+y^2+z^2)^{1/2} \rightarrow \infty$ and also satisfies the 'edge condition' [**12**]

$$(8) \qquad \begin{aligned} &\chi_t(x,0,z) = O(1) \quad \text{and} \quad \frac{\partial \chi_t(x,0,z)}{\partial y} = O(x^{-1/2}) \\ &\qquad \text{as } x \longrightarrow 0^+, \\ &\chi_t(x,0,z) = O(1) \quad \text{and} \quad \frac{\partial \chi_t(x,0,z)}{\partial y} = O((x+l)^{-1/2}) \\ &\qquad \text{as } x \longrightarrow -l. \end{aligned}$$

3. Solution of the problem. We define the Fourier transform pair by

$$(9) \qquad \begin{aligned} \eta_t(x,y,s) &= \int_{-\infty}^{\infty} \chi_t(x,y,z) e^{-iksz}\, dz, \\ \chi_t(x,y,z) &= (k/2\pi) \int_{-\infty}^{\infty} \eta_t(x,y,s) e^{iksz}\, ds. \end{aligned}$$

In (9) the transform parameter is taken conveniently to be ks, s is nondimensional. The decomposition (9) is common in other field theories as well, for example, Fourier optics [**15, 13**]. For analytic convenience one can write $k = k_r + ik_i$, $k_r, k_i > 0$. Transforming equations (3), (7) and the boundary conditions (4)–(6) with respect to z by using (9), we obtain

$$\left((1-M^2)\frac{\partial^2}{\partial x^2} + 2ikM\frac{\partial}{\partial x} + \frac{\partial^2}{\partial y^2} - k^2s^2 + k^2\right)\eta_t(x,y,s) = e^{-iksz_0}\delta(x-x_0)\delta(y-y_0). \tag{10}$$

$$\frac{\partial \eta_t(x,0^\pm,s)}{\partial y} = 0, \qquad x < -l, \tag{11}$$

$$\left(\frac{\partial}{\partial y} \mp \beta M\frac{\partial}{\partial x} \pm ik\beta\right)\eta_t(x,0^\pm,s) = 0, \qquad -l < x < 0, \tag{12}$$

$$\begin{gathered} \eta_t(x,0^+,s) = \eta_t(x,0^-,s), \\ \frac{\partial \eta_t(x,0^+,s)}{\partial y} = \frac{\partial \eta_t(x,0^-,s)}{\partial y} \\ x > 0, \end{gathered} \tag{13}$$

and

$$\eta_t(x,y,s) = \chi_0(x,y,s) + \chi(x,y,s). \tag{14}$$

Since we are dealing with subsonic flow, we can introduce the following substitutions:

$$\begin{gathered} x = (1-M^2)^{1/2}X, \qquad x_0 = (1-M^2)^{1/2}X_0, \\ y = Y, \qquad y_0 = Y_0, \qquad z = Z, \qquad z_0 = Z_0 \\ k = (1-M^2)^{1/2}K, \qquad \beta = (1-M^2)^{1/2}\mathcal{B}, \qquad l = (1-M^2)^{1/2}L, \end{gathered} \tag{15}$$

which, together with the substitution

$$\eta_t(x,y,s) = \phi_t(X,Y,s)e^{-iKMX}, \tag{16}$$

reduce the boundary value problem to

$$\left(\frac{\partial^2}{\partial X^2} + \frac{\partial^2}{\partial Y^2} + K^2\lambda^2\right)\phi_0(X,Y,s) = \mathring{a}\delta(X - X_0)\delta(Y - Y_0), \tag{17}$$

$$\left(\frac{\partial^2}{\partial X^2} + \frac{\partial^2}{\partial Y^2} + K^2\lambda^2\right)\phi(X,Y,s) = 0, \tag{18}$$

$$\frac{\partial\phi(X,0^\pm,s)}{\partial Y} = -\frac{\partial\phi_0}{\partial Y}, \qquad X < -L, \tag{19}$$

$$\begin{gathered}\left(\frac{\partial}{\partial Y} \mp \mathcal{B}M\frac{\partial}{\partial X} \pm iK\mathcal{B}\right)[\phi_0 + \phi(X,0^\pm,s)] = 0,\\ -L < X < 0,\end{gathered} \tag{20}$$

$$\begin{gathered}\phi(X,0^+,s) = \phi(X,0^-,s),\\ \frac{\partial\phi(X,0^+,s)}{\partial Y} = \frac{\partial\phi(X,0^-,s)}{\partial Y},\\ X > 0,\end{gathered} \tag{21}$$

where

$$\begin{gathered}\mathring{a} = \frac{e^{iKMX_0 - iK(1-M^2)^{1/2}sZ_0}}{(1-M^2)^{1/2}},\\ \lambda^2 = [1 - s^2(1-M^2)],\end{gathered} \tag{22}$$

and we have used ϕ_t $(= \phi_0 + \phi)$ in writing Equations (17)–(21).

Now we define the Fourier transform $\bar{\phi}(\alpha, Y, s)$ of $\phi(X,Y,s)$ as

$$\bar{\phi}(\alpha,Y,s) = \int_{-\infty}^{\infty} \phi(X,Y,s)e^{i\alpha X}\,dX, \tag{23}$$

and inverse transform as

$$\phi(X,Y,s) = \frac{1}{(2\pi)^{1/2}}\int_{-\infty+i\tau}^{\infty+i\tau} \bar{\phi}(\alpha,Y,s)e^{-i\alpha X}\,d\alpha, \tag{24}$$

where $\alpha = \sigma + i\tau$. The transform (23) and its inverse (24) will exist provided $-K_i\lambda_i < \tau < K_i\lambda_i$. The solution of Equation (17) can be written as

$$\begin{aligned}\phi_0(X,Y,s) &= -\frac{a^*}{4i}H_0^{(1)}[K\lambda((X-X_0)^2+(Y-Y_0)^2)^{1/2}] \\ &= \frac{a^*}{4\pi i}\int_{-\infty+i\tau}^{\infty+i\tau}\frac{e^{-i\alpha(X-X_0)+i(K^2\lambda^2-\alpha^2)^{1/2}|Y-Y_0|}}{(K^2\lambda^2-\alpha^2)^{1/2}}\,d\alpha.\end{aligned} \tag{25}$$

Introducing the transformations $X_0 = R_0\cos\vartheta_0$, $Y_0 = R_0\sin\vartheta_0$, $0 < \vartheta_0 < \pi$, in (25) and letting $R_0 \to \infty$, we obtain, using the asymptotic form for the Hankel function

$$\phi_0(X,Y,s) = b(s)e^{-iK\lambda(X\cos\vartheta_0+Y\sin\vartheta_0)}, \tag{26}$$

where

$$b(s) = -\frac{a^*}{4i}\left(\frac{2}{\pi K\lambda R_0}\right)^{1/2}e^{i(K\lambda R_0-\pi/4)}, \tag{27}$$

and ϑ_0 is the angle of incidence measured from the x-axis.

Applying the transform (23) to (18) gives

$$\bar{\phi}(\alpha,Y,s) = A_1(\alpha)e^{i\kappa Y}, \qquad Y > 0, \tag{28}$$

$$= A_2(\alpha)e^{-i\kappa Y}, \qquad Y < 0, \tag{29}$$

where $\kappa = (K^2\lambda^2-\alpha^2)^{1/2}$ is defined on the cut sheet for which $\mathrm{Im}(\kappa) > 0$.

From Equations (28) and (29),

$$e^{-i\alpha L}\bar{\phi}_-(\alpha,0^+,s) + \bar{\phi}_1(\alpha,0^+,s) + \bar{\phi}_+(\alpha,0,s) = A_1(\alpha), \tag{30a}$$

$$e^{-i\alpha L}\bar{\phi}'_-(\alpha,0,s) + \bar{\phi}'_1(\alpha,0^+,s) + \bar{\phi}'_+(\alpha,0,s) = i\kappa A_1(\alpha), \tag{30b}$$

$$e^{-i\alpha L}\bar{\phi}_-(\alpha,0^-,s) + \bar{\phi}_1(\alpha,0^-,s) + \bar{\phi}_+(\alpha,0,s) = A_2(\alpha), \tag{31a}$$

$$e^{-i\alpha L}\bar{\phi}'_-(\alpha,0,s) + \bar{\phi}'_1(\alpha,0^-,s) + \bar{\phi}'_+(\alpha,0,s) = -i\kappa A_2(\alpha), \tag{31b}$$

where

$$\bar{\phi}_+(\alpha, Y, s) = \int_0^\infty \phi(X, Y, s) e^{i\alpha X}\, dX,$$
(32)
$$\bar{\phi}_-(\alpha, Y, s) = \int_{-\infty}^{-L} \phi(X, Y, s) e^{i\alpha(X+L)}\, dX,$$
$$\bar{\phi}_1(\alpha, Y, s) = \int_{-L}^{0} \phi(X, Y, s) e^{i\alpha X}\, dX.$$

The primes denote differentiation with respect to Y, and the $\pm$ subscripts denote functions which are regular and analytic in the upper ($\mathrm{Im}(\alpha) > -K_i\lambda_i$) and lower ($\mathrm{Im}(\alpha) < K_i\lambda_i$) α-plane. Eliminating $A_1(\alpha)$ from (30a), (30b), and $A_2(\alpha)$ from (31a), (31b), and using the boundary condition (19) to obtain an expression for $\bar{\phi}'_-(\alpha, 0, s)$ gives the following two equations

(33a)
$$-e^{-i\alpha L}\xi'_-(\alpha,0,s)+\bar{\phi}'_1(\alpha,0^\pm,s)+\bar{\phi}'_+(\alpha,0,s)$$
(33b)
$$= \pm i\kappa(e^{-i\alpha L}\bar{\phi}_-(\alpha,0^\pm,s)+\bar{\phi}_1(\alpha,0^\pm,s)+\bar{\phi}_+(\alpha,0,s)),$$

where

(34)
$$\xi'_-(\alpha, Y, s) = \int_{-\infty}^{-L} \frac{\partial\phi_0(X,Y,s)}{\partial Y} e^{i\alpha(X+L)}\, dX.$$

On taking the Fourier transform of the boundary condition (20), we get

(35a)
$$\bar{\phi}'_1(\alpha, 0^\pm, s) \pm i(K + M\alpha)\mathcal{B}\bar{\phi}_1(\alpha, 0^\pm, s)$$
(35b)
$$= -[\xi'_1(\alpha, 0, s) \pm i(K + M\alpha)\mathcal{B}\xi_1(\alpha, 0, s)],$$

where

(36a)
$$\xi_1(\alpha, Y, s) = \int_{-L}^{0} \phi_0(X, Y, s) e^{i\alpha X}\, dX,$$

(36b)
$$\xi'_1(\alpha, Y, s) = \frac{\partial}{\partial Y}\xi_1(\alpha, Y, s).$$

After eliminating $\bar{\phi}_1(\alpha, 0^+, s)$ from (33a) and (35a) and $\bar{\phi}_1(\alpha, 0^-, s)$ from (33b) and (35b), we arrive at

(37a)

$$\pm i\kappa\Big(e^{-i\alpha L}\bar{\phi}_-(\alpha, 0^\pm, s) \mp \frac{1}{i\mathcal{B}(K+M\alpha)}\{\bar{\phi}_1'(\alpha, 0^\pm, s) + [\xi_1'(\alpha, 0, s) \pm i(K+M\alpha)\mathcal{B}\xi_1(\alpha, 0, s)]\} + \bar{\phi}_+(\alpha, 0, s)\Big)$$

(37b)

$$= -e^{-i\alpha L}\xi_-'(\alpha, 0, s) + \bar{\phi}_1'(\alpha, 0^\pm, s) + \bar{\phi}_+'(\alpha, 0, s).$$

In a similar way, by eliminating $\bar{\phi}_1'(\alpha, 0^+, s)$ from (33a) and (35a) and $\bar{\phi}_1'(\alpha, 0^-, s)$ from (33b) and (35b), we get

(38a)

$$-e^{-i\alpha L}\xi_-'(\alpha, 0, s) \mp i\mathcal{B}(K+M\alpha)\bar{\phi}_1(\alpha, 0^\pm, s) - [\xi_1'(\alpha, 0, s) \pm i\mathcal{B}(K+M\alpha)\xi_1(\alpha, 0, s)] + \bar{\phi}_+'(\alpha, 0, s)$$

(38b)

$$= \pm(e^{-i\alpha L}\bar{\phi}_-(\alpha, 0^\pm, s) + \bar{\phi}_1(\alpha, 0^\pm, s) + \bar{\phi}_+(\alpha, 0, s)).$$

Subtracting (37a) from (37b) and adding (38a) to (38b) gives, on a slight rearrangement of the resulting expressions, the following two equations:

$$e^{-i\alpha L}\psi_-(\alpha) + Q(\alpha)\psi_1(\alpha) + \psi_+(\alpha) = S(\alpha), \tag{39}$$

$$e^{-i\alpha L}\Lambda_-(\alpha) + \sqrt{K\lambda - \alpha}\,Q(\alpha)\Lambda_1(\alpha) + \Lambda_+(\alpha) = N(\alpha), \tag{40}$$

where

$$\begin{aligned}
\psi_-(\alpha) &= \frac{1}{2}(\bar{\phi}_-(\alpha, 0^+, s) + \bar{\phi}_-(\alpha, 0^-, s)),\\
\psi_1(\alpha) &= \frac{i}{2}(\bar{\phi}_1'(\alpha, 0^+, s) - \bar{\phi}_1'(\alpha, 0^-, s)),\\
\psi_+(\alpha) &= \bar{\phi}_+(\alpha, 0, s),\\
S(\alpha) &= \xi_1(\alpha, 0, s),\\
\Lambda_-(\alpha) &= \frac{-i\sqrt{K\lambda - \alpha}}{2}(\bar{\phi}_-(\alpha, 0^+, s) - \bar{\phi}_-(\alpha, 0^-, s)),\\
\Lambda_+(\alpha) &= \frac{\bar{\phi}_+'(\alpha, 0, s)}{\sqrt{K\lambda + \alpha}},
\end{aligned}$$

$$\Lambda_1(\alpha) = -\frac{i\mathcal{B}(K+M\alpha)}{2}(\bar{\phi}_1(\alpha,0^+,s) - \bar{\phi}_1(\alpha,0^-,s)),$$

$$N(\alpha) = \frac{[\xi_1'(\alpha,0,s) + e^{-i\alpha L}\xi_-'(\alpha,0,s)]}{\sqrt{K\lambda+\alpha}}$$

$$Q(\alpha) = \left(\frac{1}{\kappa} + \frac{1}{\mathcal{B}(M\alpha+K)}\right) = Q_+(\alpha)Q_-(\alpha). \tag{41}$$

Explicit expressions for the functions $Q_\pm(\alpha)$ are given in [**1**]. Now for proceeding further with Equations (39) and (40), it is necessary to know that how the various quantities in (41) grow as $|\alpha| \to \infty$. Equation (8) means that the transformed functions satisfy the following growth estimates as $|\alpha| \to \infty$:

$$\begin{aligned}
&\psi_-(\alpha) \approx O(|\alpha|^{-1}), \qquad \Lambda_-(\alpha) \approx O(|\alpha|^{-1/2}),\\
&\psi_1(\alpha) \approx O(|\alpha|^{-1/2}), \qquad \Lambda_1(\alpha) \approx O(|\alpha|^{-1}),\\
&Q_-(\alpha) \approx O((|K+M\alpha|\mathcal{B})^{-1}), \quad \text{in } \operatorname{Im}(\alpha) < K_i\lambda_i;\\
&\psi_+(\alpha) \approx O(|\alpha|^{-1}), \qquad \Lambda_+(\alpha) \approx O(|\alpha|^{-1}),\\
&\psi_1(\alpha) \approx O(|\alpha|^{-1}e^{-i\alpha L}),\\
&\Lambda_1(\alpha) \approx O(|\alpha|^{-1}e^{-i\alpha L}),\\
&Q_+(\alpha) \approx O((|K+M\alpha|\mathcal{B})^{-1}), \quad \text{in } \operatorname{Im}(\alpha) > -K_i\lambda_i.
\end{aligned}$$

Now we see that Equations (39) and (40) cannot be split by the standard Wiener-Hopf argument because of the second term on the lefthand side of the equations. However, after using the above estimates and the procedure used in [**16**, pp. 196–199], we get Fredholm integral equations of the second kind. Although in (39) and (40) the righthand sides of the equality sign are of a more general form than that considered by Noble [**16**] the basic technique used follows through, mutatis mutandis, and so the exact details may be omitted here. Thus, for (39) one obtains, noting that the coefficient of $\psi_1(\alpha)$ is an even function of α and M (see [**1, 16**]),

$$\begin{aligned}
\frac{\Gamma_+(\alpha)}{Q_+(\alpha)} &= \frac{1}{2\pi i}\int_{-\infty+ia}^{\infty+ia} \frac{e^{i\varepsilon L}\Gamma_+(\varepsilon)}{Q_-(\varepsilon)(\varepsilon+\alpha)}\,d\varepsilon\\
&\quad - \frac{1}{2\pi i}\int_{-\infty+ia}^{\infty+ia} \frac{[e^{i\varepsilon L}S(\varepsilon) + S(-\varepsilon)]}{Q_-(\varepsilon)(\varepsilon+\alpha)}\,d\varepsilon,
\end{aligned} \tag{42}$$

$$\frac{\gamma_+(\alpha)}{Q_+(\alpha)} = -\frac{1}{2\pi i}\int_{-\infty+ia}^{\infty+ia} \frac{e^{i\varepsilon L}\gamma_+(\varepsilon)}{Q_-(\varepsilon)(\varepsilon+\alpha)}\,d\varepsilon - \frac{1}{2\pi i}\int_{-\infty+ia}^{\infty+ia} \frac{[S(-\varepsilon) - e^{i\varepsilon L}S(\varepsilon)]}{Q_-(\varepsilon)(\varepsilon+\alpha)}\,d\varepsilon, \tag{43}$$

$\mathrm{Im}\,(\alpha) > -a > -K_i\lambda_i$, where

$$\begin{aligned}\Gamma_+(\alpha) &= \psi_+(\alpha) + \psi_-(-\alpha),\\ \gamma_+(\alpha) &= \psi_+(\alpha) - \psi_-(-\alpha).\end{aligned} \tag{44}$$

It is clear from (40) that the coefficient of $\Lambda_1(\alpha)$ is not an even function of α and M and therefore, using the technique of Noble [**16**] and the above estimates, we get the slightly different expressions

$$\begin{aligned}\frac{\Lambda_+(\alpha)}{Q_+(\alpha)} &= \frac{1}{2\pi i}\int_{-\infty+ic}^{\infty+ic} \frac{e^{-i\varepsilon L}\Lambda_-(\varepsilon)}{Q_+(\varepsilon)(\varepsilon-\alpha)}\,d\varepsilon\\ &\quad - \frac{1}{2\pi i}\int_{-\infty+ic}^{\infty+ic} \frac{N(\varepsilon)}{Q_+(\varepsilon)(\alpha-\varepsilon)}\,d\varepsilon,\\ &\quad \mathrm{Im}\,(\alpha) > c > -K_i\lambda_i,\end{aligned} \tag{45}$$

$$\begin{aligned}\frac{\Lambda_-(\alpha)}{Q_-(\alpha)\sqrt{K\lambda-\alpha}} &= -\frac{1}{2\pi i}\int_{-\infty+id}^{\infty+id} \frac{e^{i\varepsilon L}\Lambda_+(\varepsilon)}{Q_-(\varepsilon)(\alpha-\varepsilon)\sqrt{K\lambda-\varepsilon}}\,d\varepsilon\\ &\quad - \frac{1}{2\pi i}\int_{-\infty+id}^{\infty+id} \frac{e^{i\varepsilon L}N(\varepsilon)}{Q_-(\varepsilon)(\varepsilon-\alpha)\sqrt{K\lambda-\varepsilon}}\,d\varepsilon,\\ &\quad \mathrm{Im}\,(\alpha) < d < K_i\lambda_i.\end{aligned} \tag{46}$$

From Equations (26), (34) and (36), we have

$$\begin{aligned}\xi'_-(\alpha,0,s) &= \frac{-K\lambda\sin\vartheta_0 b(s)}{(\alpha - K\lambda\cos\vartheta_0)}e^{iK\lambda L\cos\vartheta_0},\\ \xi_1(\alpha,0,s) &= \frac{ib(s)}{(\alpha - K\lambda\cos\vartheta_0)}\{e^{-i(\alpha-K\lambda\cos\vartheta_0)L} - 1\},\\ \xi'_1(\alpha,0,s) &= \frac{K\lambda\sin\vartheta_0 b(s)}{(\alpha - K\lambda\cos\vartheta_0)}\{e^{-i(\alpha-K\lambda\cos\vartheta_0)L} - 1\},\end{aligned} \tag{47}$$

so that

$$S(\alpha) = \frac{ib(s)}{(\alpha - K\lambda\cos\vartheta_0)}\{e^{-i(\alpha - K\lambda\cos\vartheta_0)L} - 1\}, \tag{48}$$

$$N(\alpha) = -\frac{b(s)K\lambda\sin\vartheta_0}{\sqrt{K\lambda+\alpha}(\alpha - K\lambda\cos\vartheta_0)}. \tag{49}$$

4. Approximate solution of equations (42) and (43) for $K\lambda L \geq 1$. Restricting the path of integration in expression (42) to the band $K_i\lambda_i\cos\vartheta_0 < a < K_i\lambda_i$ and then using (48), into equation (42) and making the further substitution

$$\Gamma_+(\alpha) = G_+(\alpha) - \frac{ibe^{iK\lambda L\cos\vartheta_0}}{(\alpha + K\lambda\cos\vartheta_0)} - \frac{ib}{(\alpha - K\lambda\cos\vartheta_0)}, \tag{50}$$

gives

$$\begin{aligned} \frac{G_+(\alpha)}{Q_+(\alpha)} &= \frac{ibe^{iK\lambda L\cos\vartheta_0}}{(\alpha + K\lambda\cos\vartheta_0)Q_+(\alpha)} + \frac{ib}{(\alpha - K\lambda\cos\vartheta_0)Q_+(\alpha)} \\ &\quad - \frac{b}{2\pi}\int_{-\infty+ia}^{\infty+ia}\frac{1}{Q_-(\varepsilon)(\varepsilon+\alpha)} \\ &\qquad \cdot\left\{\frac{1}{(\varepsilon + K\lambda\cos\vartheta_0)} + \frac{e^{iK\lambda L\cos\vartheta_0}}{(\varepsilon - K\lambda\cos\vartheta_0)}\right\} \\ &\quad + \frac{1}{2\pi i}\int_{-\infty+ia}^{\infty+ia}\frac{e^{i\varepsilon L}G_+(\varepsilon)}{Q_-(\varepsilon)(\varepsilon+\alpha)}\,d\varepsilon, \\ &\quad K_i\lambda_i\cos\vartheta_0 < a < K_i\lambda_i. \end{aligned} \tag{51}$$

The first two integrals appearing in (51) can be evaluated by distorting the path of integration into the lower half of the ε-plane. The only poles captured will be $\varepsilon = -\alpha$ and $\varepsilon = -K\lambda\cos\vartheta_0$. Thus

$$\begin{aligned} \frac{G_+(\alpha)}{Q_+(\alpha)} &= \frac{ibe^{iK\lambda L\cos\vartheta_0}}{(\alpha + K\lambda\cos\vartheta_0)Q_-(K\lambda\cos\vartheta_0)} \\ &\quad + \frac{ib}{(\alpha - K\lambda\cos\vartheta_0)Q_+(K\lambda\cos\vartheta_0)} \\ &\quad + \frac{1}{2\pi i}\int_{-\infty+ia}^{\infty+ia}\frac{e^{i\varepsilon L}G_+(\varepsilon)Q_+(\varepsilon)}{Q(\varepsilon)(\varepsilon+\alpha)}\,d\varepsilon, \\ &\quad K_i\lambda_i\cos\vartheta_0 < a < K_i\lambda_i. \end{aligned} \tag{52}$$

In a similar manner by using Equation (48) along with (43), where $K_i\lambda_i \cos\vartheta_0 < a < K_i\lambda_i$, and making the substitution

$$\gamma_+(\alpha) = g_+(\alpha) + \frac{ibe^{iK\lambda L\cos\vartheta_0}}{(\alpha + K\lambda\cos\vartheta_0)} - \frac{ib}{(\alpha - K\lambda\cos\vartheta_0)}. \tag{53}$$

Thus one obtains eventually

$$\begin{aligned} \frac{g_+(\alpha)}{Q_+(\alpha)} = &- \frac{ibe^{iK\lambda L\cos\vartheta_0}}{(\alpha + K\lambda\cos\vartheta_0)Q_-(K\lambda\cos\vartheta_0)} \\ &+ \frac{ib}{(\alpha - K\lambda\cos\vartheta_0)Q_+(K\lambda\cos\vartheta_0)} \\ &- \frac{1}{2\pi i}\int_{-\infty+ia}^{\infty+ia} \frac{e^{i\varepsilon L}g_+(\varepsilon)Q_+(\varepsilon)}{Q(\varepsilon)(\varepsilon+\alpha)}\,d\varepsilon, \\ &K_i\lambda_i\cos\vartheta_0 < a < K_i\lambda_i. \end{aligned} \tag{54}$$

For the solutions of Equations (52) and (54), we use the technique given by Jones [**12**], and the approximate expressions are respectively given by

$$G_+(\alpha) = (bS_1(\alpha) + \mathcal{G}_+(\alpha))Q_+(\alpha), \tag{55}$$

$$g_+(\alpha) = (bS_2(\alpha) - \mathfrak{g}_+(\alpha))Q_+(\alpha). \tag{56}$$

In Equations (55) and (56),

$$\begin{aligned} S_1(\alpha) = &\frac{ie^{iK\lambda L\cos\vartheta_0}}{(\alpha + K\lambda\cos\vartheta_0)Q_-(K\lambda\cos\vartheta_0)} \\ &+ \frac{i}{(\alpha - K\lambda\cos\vartheta_0)Q_+(K\lambda\cos\vartheta_0)}, \end{aligned} \tag{57}$$

$$\begin{aligned} S_2(\alpha) = &\frac{i}{(\alpha - K\lambda\cos\vartheta_0)Q_+(K\lambda\cos\vartheta_0)} \\ &- \frac{ie^{iK\lambda L\cos\vartheta_0}}{(\alpha + K\lambda\cos\vartheta_0)Q_-(K\lambda\cos\vartheta_0)}, \end{aligned} \tag{58}$$

$$\mathcal{G}_+(\alpha) = \frac{1}{2\pi i}\int_{-\infty+ia}^{\infty+ia} \frac{e^{i\varepsilon L}G_+(\varepsilon)Q_+(\varepsilon)}{Q(\varepsilon)(\varepsilon+\alpha)}\,d\varepsilon, \tag{59}$$

$$\mathfrak{g}_+(\alpha) = \frac{1}{2\pi i}\int_{-\infty+ia}^{\infty+ia} \frac{e^{i\varepsilon L}g_+(\varepsilon)Q_+(\varepsilon)}{Q(\varepsilon)(\varepsilon+\alpha)}\,d\varepsilon. \tag{60}$$

Putting the values of $G_+(\varepsilon)$ from Equation (55) into (59) and $g_+(\varepsilon)$ from (56) into Equation (60), we have

(61)

$$\mathcal{G}_+(\alpha) = \frac{1}{2\pi i}\int_{-\infty+ia}^{\infty+ia} \frac{e^{i\varepsilon L}Q_+^2(\varepsilon)}{Q(\varepsilon)(\varepsilon+\alpha)}\{bS_1(\varepsilon)+\mathcal{G}_+(\varepsilon)\}\,d\varepsilon,$$

(62)

$$\mathfrak{g}_+(\alpha) = \frac{1}{2\pi i}\int_{-\infty+ia}^{\infty+ia} \frac{e^{i\varepsilon L}Q_+^2(\varepsilon)}{Q(\varepsilon)(\varepsilon+\alpha)}\{bS_2(\varepsilon)-\mathfrak{g}_+(\varepsilon)\}\,d\varepsilon.$$

If the contour of integration in expressions (61) and (62) is distorted into the region Im $(\alpha) > a$, then the integrals can be asymptotically approximated, for $K\lambda L \geq 1$, by the integrals with its path of integration wrapped around the branch cut $\varepsilon = K\lambda$. The part of the integrands of expressions (61) and (62) within the curly brackets are regular and analytic in this region and, provided $\vartheta_0 \neq 0, \pi$, this term will vary slowly in the vicinity of $\varepsilon = K\lambda$. Thus, since the dominant part of the integrands comes from the region $\varepsilon = K\lambda$, the terms in the curly brackets can be removed under the integral signs and ε can be replaced by $K\lambda$. The remaining integrals can be replaced by the asymptotic approximation (A2) of the Appendix. Hence,

$$\mathcal{G}_+(\alpha) \approx Q_+^2(K\lambda)\{bS_1(K\lambda)+\mathcal{G}_+(K\lambda)\}W(\alpha), \tag{63}$$

$$\mathfrak{g}_+(\alpha) \approx Q_+^2(K\lambda)\{bS_2(K\lambda)-\mathfrak{g}_+(K\lambda)\}W(\alpha), \tag{64}$$

where

$$W(\alpha) = \frac{(2K\lambda)^{1/2}}{(1-\lambda_1^2(\alpha+K\lambda))} \cdot \{W_0[(L(\alpha+K\lambda))^{1/2}] - W_0[\sqrt{L}/\lambda_1]\},$$

$$\lambda_1 = \frac{(2K\lambda)^{1/2}}{\mathcal{B}(K+M\alpha)},$$

and $\mathcal{G}_+(K\lambda)$ and $\mathfrak{g}_+(K\lambda)$ are obtained by putting $\alpha = K\lambda$ in Equations (63) and (64), respectively, and solving the resulting equations for $\mathcal{G}_+(K\lambda)$ and $\mathfrak{g}_+(K\lambda)$.

From the expressions (44), (50), (53), (55), (56), (63) and (64), we have

$$(65)\qquad \begin{aligned} \psi_+(\alpha) = &-\frac{ib}{(\alpha - K\lambda\cos\vartheta_0)}\left(1 - \frac{Q_+(\alpha)}{Q_+(K\lambda\cos\vartheta_0)}\right) \\ &+ \frac{bQ_+^2(K\lambda)W(\alpha)Q_+(\alpha)}{2} \\ &\cdot\left(S_1(K\lambda) - S_2(K\lambda) + \frac{\mathcal{G}_+(K\lambda)}{b} + \frac{\mathfrak{g}_+(K\lambda)}{b}\right), \end{aligned}$$

$$(66)\qquad \begin{aligned} \psi_-(\alpha) = &\frac{ibe^{iK\lambda L\cos\vartheta_0}}{(\alpha - K\lambda\cos\vartheta_0)}\left(1 - \frac{Q_-(\alpha)}{Q_-(K\lambda\cos\vartheta_0)}\right) \\ &+ \frac{bQ_+^2(K\lambda)W(-\alpha)Q_-(\alpha)}{2} \\ &\cdot\left(S_1(K\lambda) + S_2(K\lambda) + \frac{\mathcal{G}_+(K\lambda)}{b} - \frac{\mathfrak{g}_+(K\lambda)}{b}\right). \end{aligned}$$

5. Approximate solution of equations (45) and (46) for $K\lambda L > 1$. From Equations (45), (46) and (49), we have

$$(67)\qquad \begin{aligned} \frac{\Lambda_+(\alpha)}{Q_+(\alpha)} = &\frac{K\lambda b\sin\vartheta_0}{2\pi i} \\ &\cdot\int_{-\infty+ic}^{\infty+ic}\frac{d\varepsilon}{Q_+(\varepsilon)(\alpha-\varepsilon)(\varepsilon-K\lambda\cos\vartheta_0)(\varepsilon+K\lambda)^{1/2}} \\ &-\frac{1}{2\pi i}\int_{-\infty+ic}^{\infty+ic}\frac{e^{-i\varepsilon L}\Lambda_-(\varepsilon)}{Q_+(\varepsilon)(\varepsilon-\alpha)}\,d\varepsilon, \quad \operatorname{Im}(\alpha) > c > -K_i\lambda_i, \end{aligned}$$

$$(68)\qquad \begin{aligned} \frac{\Lambda_-(\alpha)}{Q_-(\alpha)\sqrt{K\lambda-\alpha}} = &-\frac{1}{2\pi i} \\ &\cdot\int_{-\infty+id}^{\infty+id}\frac{e^{i\varepsilon L}\Lambda_+(\varepsilon)}{Q_-(\varepsilon)(\alpha-\varepsilon)\sqrt{K\lambda-\varepsilon}}\,d\varepsilon + \frac{K\lambda b\sin\vartheta_0}{2\pi i} \\ &\cdot\int_{-\infty+id}^{\infty+id}\frac{e^{i\varepsilon L}d\varepsilon}{Q_-(\varepsilon)(\varepsilon-\alpha)(\varepsilon-K\lambda\cos\vartheta_0)(K^2\lambda^2-\varepsilon^2)^{1/2}}, \\ &\operatorname{Im}(\alpha) < d < K_i\lambda_i\cos\vartheta_0. \end{aligned}$$

For the evaluation of the first integral in Equation (67), we distorted the path of integration into the upper half ε-plane. Since $\mathrm{Im}\,(\alpha) > c > -K_i\lambda$, then the poles at $\varepsilon = \alpha$ and $\varepsilon = K\lambda\cos\vartheta_0$ will give rise to residue contributions. Hence,

$$
\begin{aligned}
\frac{\Lambda_+(\alpha)}{Q_+(\alpha)} = & -\frac{K\lambda b\sin\vartheta_0}{Q_+(\alpha)(\alpha - K\lambda\cos\vartheta_0)(\alpha + K\lambda)^{1/2}} \\
& + \frac{b(K\lambda(1-\cos\vartheta_0))^{1/2}}{Q_+(K\lambda\cos\vartheta_0)(\alpha - K\lambda\cos\vartheta_0)} \\
& - \frac{1}{2\pi i}\int_{-\infty+ic}^{\infty+ic} \frac{e^{-i\varepsilon L}\{\Lambda_-(\varepsilon)(K\lambda-\varepsilon)^{1/2}Q_-(\varepsilon)\}}{Q(\varepsilon)(\varepsilon-\alpha)(K\lambda-\varepsilon)^{1/2}}\,d\varepsilon \\
& \mathrm{Im}\,(\alpha) > c > -K_i\lambda_i.
\end{aligned} \tag{69}
$$

Now the evaluation of the second integral in (68) is best achieved by distorting the path of integration into the upper half of the ε-plane. However, this requires a knowledge of the singularities of $Q_-(\varepsilon)$ in $\mathrm{Im}\,(\varepsilon) > -K_i\lambda_i$. It can be shown, by a method used in [**1**], that the only singularities of $Q(\varepsilon)$ are the branch points at $\varepsilon = \pm K\lambda$; no poles occur in the cut plane. Hence, moving the path of integration vertically until it crosses the pole $\varepsilon = K\lambda\cos\vartheta_0$, but not the branch point $\varepsilon = K\lambda$, gives

$$
\begin{aligned}
& \frac{\Lambda_-(\alpha)}{Q_-(\alpha)\sqrt{K\lambda-\alpha}} \\
& = -\frac{be^{iK\lambda L\cos\vartheta_0}}{(\alpha - K\lambda\cos\vartheta_0)Q_-(K\lambda\cos\vartheta_0)} + \frac{1}{2\pi 1} \\
& \cdot \int_{-\infty+id}^{\infty+id}\left\{\frac{K\lambda b\sin\vartheta_0 Q_+(\varepsilon)}{\sqrt{K\lambda+\varepsilon}(\varepsilon - K\lambda\cos\vartheta_0)} + \Lambda_+(\varepsilon)Q_+(\varepsilon)\right\} \\
& \cdot \frac{e^{i\varepsilon L}}{Q(\varepsilon)\sqrt{K\lambda-\varepsilon}(\varepsilon-\alpha)}, \quad K_i\lambda_i\cos\vartheta_0 < d < K_i\lambda_i.
\end{aligned} \tag{70}
$$

For $K\lambda L > 1$, the dominant contribution of the integral in (69) comes from the region $\varepsilon = -K\lambda$; and of the integral in (70) from the region $\varepsilon = K\lambda$. Provided $\vartheta_0 \neq 0$, the term in the curly bracket of the integrands in Equations (69) and (70) are slowly varying in the vicinity of $\varepsilon = \pm K\lambda$. One can therefore replace ε by $-K\lambda$ in this part of the integrand in (69) and remove it from under the integral sign. Similarly one can replace ε by $K\lambda$ in the curly part of the integrand in (70) and remove it from under the integral sign. The integrals remaining can be replaced by the asymptotic approximation (A6) and (A8) of the Appendix. Thus,

$$\begin{aligned}\frac{\Lambda_+(\alpha)}{Q_+(\alpha)} &= -\frac{K\lambda b\sin\vartheta_0}{Q_+(\alpha)(\alpha-K\lambda\cos\vartheta_0)(\alpha+K\lambda)^{1/2}} \\ &\quad+\frac{b(K\lambda(1-\cos\vartheta_0))^{1/2}}{Q_+(K\lambda\cos\vartheta_0)(\alpha-K\lambda\cos\vartheta_0)} \\ &\quad+W(\alpha)Q_+(K\lambda)\Lambda_-(-K\lambda),\end{aligned} \tag{71}$$

$$\begin{aligned}\frac{\Lambda_-(\alpha)}{Q_-(\alpha)\sqrt{K\lambda-\alpha}} &= -\frac{be^{iK\lambda L\cos\vartheta_0}}{(\alpha-K\lambda\cos\vartheta_0)Q_-(K\lambda\cos\vartheta_0)} \\ &\quad-\frac{W(-\alpha)Q_+(K\lambda)}{\mathcal{B}(K+M\alpha)}\left\{\frac{\sin\vartheta_0}{(1-\cos\vartheta_0)}\right. \\ &\quad\left.+\sqrt{2K\lambda}\Lambda_+(K\lambda)\right\},\end{aligned} \tag{72}$$

where the constants $\Lambda_\pm(\pm K\lambda)$ are obtained by putting $\alpha = K\lambda$ in Equation (71) and $\alpha = -K\lambda$ in (72) and solving the resulting two equations for the two unknowns $\Lambda_\pm(\pm K\lambda)$.

The Equations (41) in conjunction with Equations (65), (66), (71) and (72) will now give explicit expressions for $\bar\phi_+(\alpha,0,s)$, $\bar\phi'_+(\alpha,0,s)$, $\bar\phi_-(\alpha,0^+,s)$ and $\bar\phi_-(\alpha,0^-,s)$. $A_1(\alpha)$ and $A_2(\alpha)$ can now be given in terms of these known quantities. For example, multiplying (30a) throughout by $i(K+M\alpha)\mathcal{B}$, adding the resulting expression to (30b) and using (34) and (36a) gives an explicit expression for $A_1(\alpha)$. Similarly, an explicit expression for $A_2(\alpha)$ is obtained by multiplying (31a) throughout by $i(K+M\alpha)\mathcal{B}$, subtracting the resulting expression from (31b) and using the expressions (34) and (36b). Thus, in carrying out the above, we have

$$A_1(\alpha) = bA_3(\alpha), \tag{73}$$

$$A_2(\alpha) = bA_4(\alpha), \tag{74}$$

where

$$
\begin{aligned}
A_3(\alpha) = {} & \frac{1}{\kappa Q(\alpha)}\Bigg(\frac{i}{(\alpha - K\lambda\cos\vartheta_0)} \\
& \cdot\left\{\frac{Q_+(\alpha)}{Q_+(K\lambda\cos\vartheta_0)} - \frac{2Q_-(\alpha)e^{-i(\alpha-K\lambda\cos\vartheta_0)L}}{Q_-(K\lambda\cos\vartheta_0)}\right. \\
& \left. - \frac{(K\lambda(1-\cos\vartheta_0))^{1/2}(\alpha+K\lambda)^{1/2}Q_+(\alpha)}{\mathcal{B}(K+M\alpha)Q_+(K\lambda\cos\vartheta_0)}\right\} \\
& + e^{-i\alpha L}Q_+(K\lambda)Q_-(\alpha) \\
& \cdot\left\{\frac{Q_+(K\lambda)}{2}\left(S_1(K\lambda)+S_2(K\lambda)+\frac{\mathcal{G}_+(K\lambda)}{b}-\frac{\mathfrak{g}_+(K\lambda)}{b}\right)\right. \\
& \left. - \frac{i}{\mathcal{B}(K+M\alpha)}\left(\frac{\sin\vartheta_0}{(1-\cos\vartheta_0)}+\frac{\sqrt{2K\lambda}}{b}\Lambda_+(K\lambda)\right)\right\}W(-\alpha) \\
& + \frac{Q_+^2(K\lambda)W(\alpha)Q_+(\alpha)}{2} \\
& \cdot\left\{S_1(K\lambda)-S_2(K\lambda)+\frac{\mathcal{G}_+(K\lambda)}{b}+\frac{\mathfrak{g}_+(K\lambda)}{b}\right\} \\
& + \frac{W(\alpha)Q_+(K\lambda)Q_+(\alpha)\sqrt{K\lambda+\alpha}\,\Lambda_-(-K\lambda)}{i\mathcal{B}b(K+M\alpha)}\Bigg),
\end{aligned}
\tag{75}
$$

$$
\begin{aligned}
A_4(\alpha) = {} & \frac{1}{\kappa Q(\alpha)}\Bigg(\frac{i}{(\alpha - K\lambda\cos\vartheta_0)} \\
& \cdot\left\{\frac{Q_+(\alpha)}{Q_+(K\lambda\cos\vartheta_0)} + \frac{(K\lambda(1-\cos\vartheta_0))^{1/2}(\alpha+K\lambda)^{1/2}Q_+(\alpha)}{\mathcal{B}(K+M\alpha)Q_+(K\lambda\cos\vartheta_0)}\right\} \\
& + e^{-i\alpha L}Q_+(K\lambda)Q_-(\alpha) \\
& \cdot\left\{\frac{Q_+(K\lambda)}{2}\left(S_1(K\lambda)+S_2(K\lambda)+\frac{\mathcal{G}_+(K\lambda)}{b}-\frac{\mathfrak{g}_+(K\lambda)}{b}\right)\right. \\
& \left. + \frac{i}{\mathcal{B}(K+M\alpha)}\left(\frac{\sin\vartheta_0}{(1-\cos\vartheta_0)}+\frac{\sqrt{2K\lambda}}{b}\Lambda_+(K\lambda)\right)\right\}W(-\alpha) \\
& + \frac{Q_+^2(K\lambda)W(\alpha)Q_+(\alpha)}{2} \\
& \cdot\left\{S_1(K\lambda)-S_2(K\lambda)+\frac{\mathcal{G}_+(K\lambda)}{b}+\frac{\mathfrak{g}_+(K\lambda)}{b}\right\}
\end{aligned}
$$

$$- \frac{W(\alpha)Q_+(K\lambda)Q_+(\alpha)\sqrt{K\lambda+\alpha}\Lambda_-(-K\lambda)}{i\mathcal{B}b(K+M\alpha)}\Bigg). \tag{76}$$

From Equations (24), (28), (29), (73) and (74), we obtain

$$\phi(X,Y,s) = \frac{b(s)}{(2\pi)^{1/2}} \int_{-\infty+i\tau}^{\infty+i\tau} A_3(\alpha) e^{-i\alpha X + i\kappa Y}\, d\alpha, \quad Y > 0, \tag{77}$$

$$= \frac{b(s)}{(2\pi)^{1/2}} \int_{-\infty+i\tau}^{\infty+i\tau} A_4(\alpha) e^{-i\alpha X - i\kappa Y}\, d\alpha, \quad Y < 0, \tag{78}$$

$$-K_i\lambda_i < \tau < K_i\lambda_i \cos\vartheta_0.$$

Now we find that, for rigid part and the absorbing strip of the half-plane, the reflected waves (RF_1 and RF_2, respectively) as given by

$$RF_{1,2} = b(s) R_{1,2} e^{-iK\lambda R\cos(\vartheta+\vartheta_0)}, \tag{78a}$$

can be calculated from (77) and (78) by deforming the contour in the upper half-plane, when the pole $\alpha = K\lambda\cos\vartheta_0$ is captured and we obtain the reflection coefficients $R_{1,2}$ as given by

$$R_1 = 1, \qquad R_2 = \frac{|\lambda\sin\vartheta_0| - \mathcal{B}(1+M\lambda\cos\vartheta_0)}{|\lambda\sin\vartheta_0| + \mathcal{B}(1+M\lambda\cos\vartheta_0)}, \tag{78b}$$

where $X = R\cos\vartheta$, $Y = R\sin\vartheta$.

In order to solve the integrals appearing in Equations (77) and (78) for the diffracted field, we put $X = R\cos\vartheta$, $Y = R\sin\vartheta$ and deform the contour by the transformation $\alpha = -K\lambda\cos(\vartheta + iq_1)$, $-\infty < q_1 < \infty$. Hence, for large $K\lambda R$,

$$\begin{aligned} \phi(X,Y,s) &= -b(s)\sin\vartheta A_3(-K\lambda\cos\vartheta)(K\lambda/R)^{1/2} e^{i(K\lambda-\pi/4)}, \\ &\qquad Y > 0, \\ &= -b(s)\sin\vartheta A_4(-K\lambda\cos\vartheta)(K\lambda/R)^{1/2} e^{i(K\lambda-\pi/4)}, \\ &\qquad Y < 0. \end{aligned} \tag{79}$$

Now using Equations (16), (22), (27) and (79), we arrive at

$$\chi(x,y,z) = \frac{-K\sin\vartheta e^{-iKM(X-X_0)}}{4\pi(2\pi RR_0)^{1/2}} \tag{80}$$
$$\cdot \int_{-\infty}^{\infty} A_3(-K\lambda\cos\vartheta)e^{iK\mathcal{F}(Z-Z_0)+iK\lambda(R+R_0)}\,ds, \quad Y>0,$$

$$= \frac{-K\sin\vartheta e^{-iKM(X-X_0)}}{4\pi(2\pi RR_0)^{1/2}} \tag{81}$$
$$\cdot \int_{-\infty}^{\infty} A_4(-K\lambda\cos\vartheta)e^{iK\mathcal{F}(Z-Z_0)+iK\lambda(R+R_0)}\,ds, \quad Y<0,$$

where

$$\mathcal{F} = s(1-M^2)^{1/2}.$$

The integrals appearing in (80) and (81) can be evaluated asymptotically. For that we introduce the transformations $Z - Z_0 = R_{12}\cos\theta$, $R + R_0 = R_{12}\sin\theta$, $s = \cos(\theta + iq_2)(1-M^2)^{-1/2}$. Hence, for large KR_{12},

$$\chi(x,y,z) = \frac{K\sin\vartheta\sin\theta A_3(-K\cos\vartheta\sin\theta)e^{-iKM(X-X_0)}}{4\pi(KRR_0R_{12})^{1/2}(1-M^2)^{1/2}} \cdot e^{i(KR_{12}+3\pi/4)}, \quad Y>0, \tag{82}$$

$$\chi(x,y,z) = \frac{K\sin\vartheta\sin\theta A_4(-K\cos\vartheta\sin\theta)e^{-iKM(X-X_0)}}{4\pi(KRR_0R_{12})^{1/2}(1-M^2)^{1/2}} \cdot e^{i(KR_{12}+3\pi/4)}, \quad Y<0, \tag{83}$$

where $A_3(-K\cos\vartheta\sin\theta)$ and $A_4(-K\cos\vartheta\sin\theta)$ are given by (75) and (76), respectively.

6. Conclusions. We have solved a new canonical diffraction problem of a spherical wave in the presence of a moving fluid. From Equations (82) and (83) we observe that, as a result of fluid motion, the field is increased by the factor $(1-M^2)^{-1/2}$ in comparison to still fluid. Also, the field is independent of the direction of flow since the fluid velocity U appears as $|U|^2$ in the factor $(1-M^2)$. The results for the still air case can be obtained by putting $M=0$.

It is also interesting to note that Equations (82) and (83) represent fields diffracted from the edges $x = 0$ and $x = -l$. The radiated sound intensity in the illuminated region $0 < \vartheta < \pi$ is due to constructive/destructive interference between the incident wave; the diffracted fields from the edges (0,0) and the joint $(0, -l)$ between the absorptive strip and rigid region of the screen. For a given value of ϑ_0, a value of the absorptive parameter given by

$$\mathcal{B} = \frac{|\lambda \sin \vartheta_0|}{(1 + M\lambda \cos \vartheta_0)}$$

can make the reflected field RF_2 vanish. This will reduce the maximum intensity of sound. The criterion that the reflected wave should vanish means physically that the strip absorbs all the energy incident upon it and does not reflect any.

In Rawlins' paper [**21**] the situation where the edge region of a rigid barrier was connected to a soft ($|\beta| \to \infty$) strip was analyzed. The analysis showed that the strip need only be the order of a wavelength long to have the same effect on the sound attenuation in the shadow region as a soft half plane. By using the concept of a perfectly absorbing strip, it was shown in a qualitative sense that the same was true for an absorbing strip. The analysis presented here is concerned with the more general and practical case where β is finite. A major use of the presented analysis is to design a barrier which would reduce noise both in the illuminated and shadow region. The lengths and absorptive properties of the strip would need to be different for the lining on the illuminated and shadow side of the barrier.

Appendix

Consider the integral

$$I_1 = \frac{1}{2\pi i} \int_{-\infty+ia}^{\infty+ia} \frac{e^{i\varepsilon L}}{Q(\varepsilon)(\varepsilon + \alpha)}\, d\varepsilon, \tag{A1}$$

$$\operatorname{Im}(\alpha) > -a, \quad K_i\lambda_i \cos\vartheta_0 < a < K_i\lambda_i.$$

From the way κ has been defined, it has been shown [**1**] that for $\operatorname{Re}(\mathcal{B}) > 0$, $Q(\varepsilon)$ has no poles or zeros in the cut plane. Thus, in the region $\operatorname{Im}(\alpha) > a$ the only singularity is a branch cut at $\varepsilon = K\lambda$.

Distorting the path of integration in Equation (A1) into the upper ε-plane until it runs around the branch cut $\varepsilon = K\lambda$, gives

$$I_1 = \frac{\sqrt{2K\lambda}}{2\pi i}\Bigg(- i\int_{\infty K\lambda}^{K\lambda} \frac{e^{i\varepsilon L}\sqrt{\varepsilon - K\lambda}}{(\varepsilon+\alpha)(1 - i\lambda_1\sqrt{\varepsilon - K\lambda})}\,d\varepsilon + i\int_{K\lambda}^{\infty K\lambda} \frac{e^{i\varepsilon L}\sqrt{\varepsilon - K\lambda}}{(\varepsilon+\alpha)(1 + i\lambda_1\sqrt{\varepsilon - K\lambda})}\,d\varepsilon\Bigg),$$

where $\lambda_1 = \sqrt{2K\lambda}/(K + M\alpha)\mathcal{B}$ is obtained by replacing the smoothly varying function $\sqrt{K\lambda+\alpha}$ by $\sqrt{2K\lambda}$. Making an obvious change of variables, one obtains

$$\begin{aligned} I_1 &= \frac{\sqrt{2K\lambda}e^{iK\lambda L}}{\pi(1 - \lambda_1^2(\alpha + K\lambda))}\int_0^\infty e^{iuL}u^{1/2}\left\{\frac{1}{u+K\lambda+\alpha} - \frac{1}{u+\lambda_1^{-2}}\right\}du \\ &= \frac{\sqrt{2K\lambda}}{(1 - \lambda_1^2(\alpha + K\lambda))}\{W_0[(L(\alpha+K\lambda))^{1/2}] - W_0[\sqrt{L}/\lambda_1]\} \\ &= W(\alpha). \end{aligned} \tag{A2}$$

W_0 can be expressed in terms of the Fresnel integral $F(z_1)$ by

$$W_0[\sqrt{z_1 L}] = \frac{e^{i(K\lambda L + \pi/4)}}{\sqrt{L\pi}}\{1 + 2i\sqrt{z_1 L}F(\sqrt{z_1 L})\}, \quad L > 0, \quad |\arg(z_1)| < \pi. \tag{A3}$$

Also note the asymptotic expansion

$$W_0[\sqrt{z_1 L}] \approx -\frac{e^{i(K\lambda L - \pi/4)}}{2\pi^{1/2}L^{3/2}z_1}. \tag{A4}$$

Consider the integral

$$I_2 = \frac{1}{2\pi i}\int_{-\infty+id}^{\infty+id} \frac{e^{i\varepsilon L}}{Q(\varepsilon)(\varepsilon-\alpha)(K\lambda-\varepsilon)^{1/2}}\,d\varepsilon, \quad \mathrm{Im}\,(\alpha) < d < K_i\lambda_i. \tag{A5}$$

Anticipating that the major contribution from the smooth functions under the integral sign occurs at $\varepsilon = K\lambda$, one obtains, on distorting the path of integration round the branch cut $\varepsilon = K\lambda$,

$$I_2 = \frac{\sqrt{2K\lambda}}{2\pi i}\left(\int_{\infty K\lambda}^{K\lambda} \frac{e^{i\varepsilon L}}{(\varepsilon-\alpha)(1-i\lambda_1\sqrt{\varepsilon - K\lambda})}\,d\varepsilon + \int_{K\lambda}^{\infty K\lambda} \frac{e^{i\varepsilon L}}{(\varepsilon-\alpha)(1+i\lambda_1\sqrt{\varepsilon - K\lambda})}\,d\varepsilon\right),$$

(A6)

$$\begin{aligned} I_2 &= \frac{-\sqrt{2K\lambda}e^{iK\lambda L\lambda_1}}{(1+\lambda_1^2(\alpha - K\lambda))}\int_0^\infty e^{iuL}u^{1/2}\left\{\frac{1}{u+K\lambda-\alpha} - \frac{1}{u+\lambda_1^{-2}}\right\}du \\ &= \frac{-\lambda_1\sqrt{2K\lambda}}{(1+\lambda_1^2(\alpha - K\lambda))}\{W_0[(L(K\lambda-\alpha))^{1/2}] - W_0\sqrt{L}/\lambda_1]\} \\ &= -\lambda_1 W(-\alpha), \\ &\qquad L>0, \quad |\arg(K\lambda-\alpha)|<\pi, \quad |\arg(\lambda_1^{-1})|<\pi. \end{aligned}$$

Consider the integral

$$\text{(A7)} \qquad I_3 = \frac{1}{2\pi i}\int_{-\infty+ic}^{\infty+ic}\frac{e^{-i\varepsilon L}}{Q(\varepsilon)(\varepsilon-\alpha)(K\lambda-\varepsilon)^{1/2}}, \quad \operatorname{Im}(\alpha) > c > K_i\lambda_i.$$

Letting ε be replaced by $(-\varepsilon)$, c by $-a$), and using the fact that $Q(-\varepsilon) = Q(\varepsilon)$, $(K\lambda+\varepsilon)^{1/2} \approx (2K\lambda)^{1/2}$, gives

$$\text{(A8)} \qquad \begin{aligned} I_3 &= -\frac{1}{2\pi i(2K\lambda)^{1/2}}\int_{-\infty+ia}^{\infty+ia}\frac{e^{i\varepsilon L}}{Q(\varepsilon)(\varepsilon+\alpha)}\,d\varepsilon \\ &= -I_1/(2K\lambda)^{1/2}. \end{aligned}$$

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MATHEMATICS DEPARTMENT, QUAID-I-AZAM UNIVERSITY, ISLAMABAD, PAKISTAN

MATHEMATICS DEPARTMENT, QUAID-I-AZAM UNIVERSITY, ISLAMABAD, PAKISTAN

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Scattering of a Spherical Gaussian Pulse Near an Absorbing Half Plane

S. Asghar,* T. Hayat, M. Ayub and B. Ahmad

Department of Mathematics, Quaid-Azam University, Islamabad, Pakistan

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ABSTRACT

The space–time acoustic wave diffraction due to a spherical Gaussian pulse near an absorbing half plane introducing the Kutta–Joukowski condition (wake condition) is considered. The temporal Fourier transform is used to calculate the diffracted field. It is found that the field produced by the Kutta–Joukowski condition will be substantially in excess of that in its absence when the source is near the edge. © 1998 Elsevier Science Ltd. All rights reserved

Keywords: Scattering theory, Kutta–Joukowski condition, Gaussian pulse, moving fluid, absorbing half plane.

INTRODUCTION

It was shown by Ffowcs Williams and Hall[1] that the aerodynamic sound scattered by a sharp edge is proportional in intensity to the fifth power of the flow velocity and inversely to the cube of the distance of the source from the edge. Thus, the edge is likely to be the dominant sound source, especially when the source is very close to the edge. Their findings were however, based upon the fact that there is a potential flow near the sharp edge with velocity becoming infinite there. In the case of a non-viscous flow the boundary condition for the flow about a body is simply that the normal velocity component of the surface vanishes. The proper boundary condition in a viscous fluid is that the fluid adheres to the bounding surface. Thus both the normal and tangential velocities relative to the body must vanish. At a small distance

*To whom correspondence should be addressed.

from the surface, the velocity reaches a value of the order of the free stream value and the influence of viscosity is restricted to a small boundary layer with strong vorticity near the surface. However for thin wings the vortex layer is also thin. The vortices are carried along with the flow and for a thin vortex wake behind the wing. The strength of this wake can be determined approximately by what is called the Kutta–Joukowski condition. Jones[2] observed that in order to satisfy the Kutta–Joukowski condition the velocity cannot be taken as infinite at the edge. To take care of this situation he introduced the wake condition to examine the effect of the Kutta–Joukowski condition at the edge of a half plane. It was observed by him that near the edge the field produced in the presence of the wake was considerably larger than if no wake was present. Furthermore, in still air the Kutta–Joukowski condition altered the M^5 (M is the Mach number) prediction of Ffowcs Williams and Hall to M^3. Moreover, he showed that when field was convected the dependence of the intensity on Mach number was M^5 whether these conditions were applied or not. Thus, the observation of the sound intensity at low Mach number in a moving fluid failed to predict whether or not the Kutta–Joukowski conditions had been imposed. It was further established that near the wake there was an acoustic surface wave which was much stronger than the distant field which did not decay down stream. This problem was further extended to the point source excitation by Balasubramanyam[3] and to the diffraction by a cylindrical impulse by Rienstra.[4] Later on Rawlins[5] addressed the diffraction of a cylindrical acoustic wave by an absorbing half plane in a moving fluid in the presence of a wake. This analysis was further extended to a spherical wave emanating from the point source by Asghar *et al.*,[6] Jones,[2] Balasubramanyam,[3] and Rawlins[5] have assumed the wave harmonic in time. Although harmonic waves are of great importance, there are significant fields whose time variation is not harmonic. The interpretation of these results nay not be easy for all values of time but it is usually possible to estimate the integrals asymptotically for large values of time.

In this paper, investigations are presented for the diffraction of spherical Gaussian pulse from an absorbing half plane in a moving fluid introducing the wake condition to examine the effect of the Kutta–Joukowski condition. The time dependence of the field requires a temporal Fourier transform in addition to spatial Fourier transforms. The spatial integrals appearing in the solution for the diffracted field are solved asymptotically in the far field approximation. It is found that the field due to a Gaussian pulse with the Kutta–Joukowski condition will be substantially in excess of that in its absence when the source is near the edge. In other words, there is greater attenuation of the sound level in the trailing edge (wake producing situation) than the leading edge situation. We observe that the diffraction of a spherical

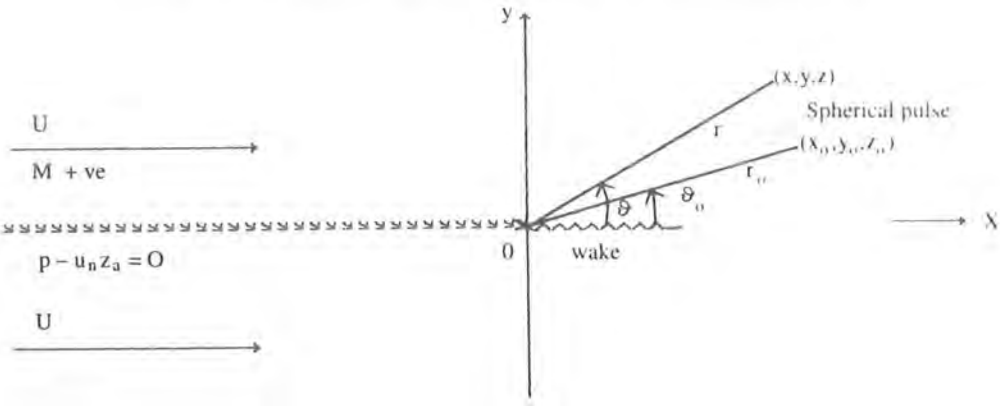

Fig. 1. Geometry of the scattering problem.

Gaussian pulse from a rigid barrier can be obtained as a special case of this problem by taking the absorption parameter equal to zero. The results for still air can be deduced easily by putting $M = 0$. This investigation is also important since the wave field due to any transient source can be expressed as a linear combination of the Gaussian pulses. The wave field for the impulsive source can be easily obtained by using the representation of the Dirac delta function in terms of Gaussian pulse.

FORMULATION OF THE PROBLEM

Consider a small amplitude sound wave on a main stream moving with subsonic velocity U parallel to the x-axis. A semi-infinite absorbing plane is assumed to occupy $y = 0$, $x \leq 0$ as shown in Fig. 1. The half plane is assumed to be infinitely thin and satisfying an absorbing boundary condition,[7] i.e. $p - u_n Z_a = 0$ on both sides of the surface where p is the pressure, u_n the normal component of the perturbation velocity at a point on the surface of the semi-infinite plane and Z_a is the acoustic impedance of the material which makes up the half plane. The perturbation velocity u of the irrotational sound wave can be expressed in terms of velocity potential ψ by $u = \text{grad}\psi$. The resulting pressure in the sound field is given by

$$p = -\rho_o(\partial/\partial t + U\partial/\partial x)\psi \qquad (*)$$

where ρ_o is the density of the undisturbed stream. We consider a spherical Gaussian pulse from a source located at the position (x_o, y_o, z_o). The convective wave equation satisfied by ψ in the presence of the point source pulse is

$$\nabla^2\psi - \left\{\frac{1}{c}\frac{\partial}{\partial t} + M\frac{\partial}{\partial x}\right\}^2 \psi = \pi^{-1/2} s e^{-s^2 t^2} \delta(x - x_o)\delta(y - y_o)\delta(z - z_o) \quad (1)$$

subject to the boundary conditions

$$\left\{\frac{\partial}{\partial y} \mp \beta M \frac{\partial}{\partial x} \mp \frac{\beta}{c}\frac{\partial}{\partial t}\right\}\psi(x, 0^{\pm}, z; t) = 0, \; x < 0 \quad (2)$$

where ∇^2 is the usual Laplacian and $\beta = \rho_o c/Z_a$, c is the velocity of sound, $M = U/c$ is the Mach number. We choose the co-efficients of the Gaussian pulse to be "$s/\pi^{1/2}$" so that the strength of the pulse $_{-\infty}\int^{\infty} \pi^{-1/2} s e^{-s^2 t^2} dt$, is unity. We shall assume that the flow is subsonic, $-1 < M < 1$ (for a leading edge situation $-1 < M \leq 0$ and for a trailing edge situation $0 < M < 1$) and $Re(\beta) > 0$. The trailing edge problem adds the complication of a trailing vortex sheet or wake attached to the absorbing half plane. The usual edge conditions give rise to a field which is singular at the origin for the trailing edge situation. Therefore, the Kutta–Joukowski condition is imposed to obtain a unique solution of the problem. In order to satisfy the Kutta–Joukowski condition at the edge, Jones[2] introduced a discontinuity in the field at $0 < x < \infty$ and postulated the existence of a wake condition. According to him, ψ is discontinuous while $\frac{\partial \psi}{\partial y}$ and the pressure remain continuous or $y = 0, x > 0$. The boundary conditions on $y = 0, x > 0$ can thus be expressed as

$$\left.\begin{aligned} \tfrac{\partial}{\partial y}\psi(x, 0^+, z; t) &= \tfrac{\partial}{\partial y}\psi(x, 0^-, z; t) \\ \left(\tfrac{\partial}{\partial t} + U\tfrac{\partial}{\partial x}\right)\psi(x, 0^+, z; t) &= \left(\tfrac{\partial}{\partial t} + U\tfrac{\partial}{\partial x}\right)\psi(x, 0^-, z; t), \end{aligned}\right\} x > 0 \quad (3a, b)$$

SOLUTION OF THE PROBLEM

We define the temporal Fourier transform and its inverse by

$$\hat{\psi}(x, y, z; \omega) = \int_{-\infty}^{\infty} \psi(x, y, z; t) e^{i\omega t} dt \quad (4)$$

$$\psi(x, y, z; t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{\psi}(x, y, z; \omega) e^{-i\omega t} d\omega \quad (5)$$

By analogy to the time harmonic problem, we use ω as the variable of the Fourier transform. Transforming eqns (1)–(3), we obtain

$$(1 - M^2)\hat{\psi}_{xx} + 2ikM\hat{\psi}_x + \hat{\psi}_{yy} + \hat{\psi}_{zz} + k^2\hat{\psi} = \delta(x - x_o)\delta(y - y_o)\delta(z - z_o)e^{-w^2/4s^2} \tag{6}$$

$$\left\{\frac{\partial}{\partial y} \mp \beta M \frac{\partial}{\partial x} \pm ik\beta\right\}\hat{\psi}(x, 0^{\pm}, z; \omega) = 0, x < 0 \tag{7}$$

$$\left.\begin{array}{r}\frac{\partial}{\partial y}\hat{\psi}(x, 0^+, z; \omega) = \frac{\partial}{\partial y}\hat{\psi}(x, 0^-, z; \omega)\\ (-ik + M\frac{\partial}{\partial x})\hat{\psi}(x, 0^+, z; \omega) = (-ik + M\frac{\partial}{\partial x})\hat{\psi}(x, 0^-, z; \omega),\end{array}\right\} x > 0 \tag{8a, b}$$

where $k = \omega/c$. The boundary condition (8b) can be written in the alternative form as

$$\hat{\psi}(x, 0^+, z; \omega) - \hat{\psi}(x, 0^-, z; \omega) = \alpha(z)e^{ikx/M} \tag{8c}$$

In eqn (8c), $\alpha(z)$ can be determined by means of the Kutta–Joukowski condition. We note that $\alpha(z) = 0$ corresponds to the no wake situation.

The solution of the boundary value problem consisting of eqns (6)–(8) has been obtained employing the procedure used by the authors in Ref. 6. In order to avoid repetition, the computational details are omitted and the diffracted field $\hat{\psi}_d$ for the spherical Gaussian pulse is directly given by

$$\begin{aligned}\hat{\psi}_d(x, y, z; \omega) = \tilde{C}_{11} \int_{-\infty}^{\infty} &\frac{e^{iK\gamma(R+R_o)+iK\sqrt{(1-M^2)}\eta(z-z_o)}F(|\tilde{Q}|)}{\sqrt{1-(1-M^2)\eta^2}\left(K\sqrt{\{1-(1-M^2)y^2\}}\right)^{1/2}N(\gamma)}d\eta\\ &+ \tilde{C}_{22} \int_{-\infty}^{\infty} \frac{e^{iK\gamma(R+R_o)+iK\sqrt{(1-M^2)}\eta(z-z_o)}}{\sqrt{K\sqrt{\{1-(1-M^2)y^2\}}N(\gamma)}} F(|\tilde{Q}|)d\eta\\ &- \tilde{C}_{33} \int_{-\infty}^{\infty} \frac{e^{iK\gamma(R+R_o)+iK\sqrt{(1-M^2)}\eta(z-z_o)}}{\sqrt{K\sqrt{\{1-(1-M^2)\eta^2\}}N(\gamma)}} F(|\tilde{Q}'|)d\eta\end{aligned} \tag{9}$$

where

$$\tilde{C}_{11} = \frac{\mathcal{B}K\sqrt{(1-M^2)}}{2\pi}\tilde{C}_1 e^{-\omega^2/4s^2}, \gamma = \left\{1-(1-M^2)\eta^2\right\}^{1/2}$$

$$\tilde{C}_{22} = \frac{K\sqrt{(1-M^2)}}{2\pi}[\mathcal{B}M\cos\Theta_o - 2\sin(\Theta/2)\sin(\Theta_o/2)]\tilde{C}_1 e^{-\omega^2/4s^2}$$

$$\tilde{C}_{33} = \frac{K\sqrt{(1-M^2)}}{\pi}\sin(\Theta/2)\sin(\Theta_o/2)\tilde{C}_1 e^{-\omega^2/4s^2}$$

$$N(\gamma) = \tilde{L}_+(K\gamma\cos\Theta)\tilde{L}(-K\gamma\cos\Theta_o)$$

$$\tilde{C}_1 = \frac{-e^{-iKM(X-X_o)}}{2\pi\sin\Theta\sqrt{2R_o(1-M^2)}}$$

$$|\tilde{Q}| = \left(\frac{KR}{2}\sqrt{1-(1-M^2)\eta^2}\right)^{1/2}\frac{\cos\Theta-\cos\Theta_o}{\sin\Theta}$$

$$|\tilde{Q}'| = \left(\frac{KR}{2}\sqrt{1-(1-M^2)\eta^2}\right)^{1/2}\frac{1/M-\cos\Theta}{\sin\Theta}$$

$$x = \sqrt{1-M^2}X, x_o = \sqrt{1-M^2}X_o, \beta = \sqrt{1-M^2}\mathcal{B}, k = \sqrt{1-M^2}K$$

$$y = Y, z = Z, y_o = Y_o, z_o = Z_o, F(\tilde{Q}) = e^{-i\tilde{Q}^2}\int_{\tilde{Q}}^{\infty} e^{i\lambda^2}d\lambda$$

$$X = R\cos\Theta, Y = R\sin\Theta, X_o = R_o\cos\Theta_o, Y_o = R_o\sin\Theta_o$$

$$x = r\cos\Theta, y = r\sin\Theta, x_o = r_o\cos\Theta_o, y_o = r_o\sin\Theta_o$$

$$R = r\left(\frac{1 - M^2\{\sin\Theta\}^2}{(1 - M^2)}\right)^{1/2}, \cos\Theta = \frac{\cos\Theta}{(1 - M^2\{\sin\Theta\}^2)^{1/2}}$$

$$\tilde{L}(\nu) = 1 + \mathcal{B}(K - M\nu)/\sqrt{K^2\gamma^2 - \nu^2} = \tilde{L}_+(\nu)\tilde{L}_-(\nu)$$

$$\tilde{L}_+(\nu) = \tilde{L}_+(0)\exp\int_0^\nu \mathcal{G}_+(\nu)\mathrm{d}\nu, \tilde{L}_+(0) = \tilde{L}_-(0) = (1 + \mathcal{B}/\gamma)^{1/2}$$

$$\mathcal{G}_+(\nu) = -\frac{1}{2(\nu + K\gamma)} + \frac{\mathcal{B}K(\gamma^2 M - \nu_1)}{\pi(1 + \mathcal{B}^2 M^2)(\nu_1 - \nu_2)}\mathcal{F}(\nu, K\nu_1) - \frac{\mathcal{B}K(\gamma^2 M - \nu_2)}{\pi(1 + \mathcal{B}^2 M^2)(\nu_1 - \nu_2)}\mathcal{F}(\nu, K\nu_2)$$

$$\mathcal{G}_-(-\nu)|_{M=-M} = -\mathcal{G}_+(\nu)$$

$$\mathcal{F}(\nu, \nu_o) = \frac{1}{(\nu - \nu_o)}[f(\nu) - f(\nu_o)]$$

$$\nu_1 = \frac{1}{(1 + \mathcal{B}^2 M^2)}\left\{M\mathcal{B}^2 + \sqrt{(\gamma^2 - \mathcal{B}^2 + M^2\mathcal{B}^2\gamma^2)}\right\}$$

$$\nu_2 = \frac{1}{(1 + \mathcal{B}^2 M^2)}\left\{M\mathcal{B}^2 - \sqrt{(\gamma^2 - \mathcal{B}^2 + M^2\mathcal{B}^2\gamma^2)}\right\}$$

$$f(p) = \int_{K\gamma}^{\infty K\gamma} \frac{\mathrm{d}\zeta}{\sqrt{(\zeta^2 - K^2\gamma^2)}(\zeta + p)} = \frac{\cos^{-1}(p/K\gamma)}{\sqrt{(K^2\gamma^2 - p^2)}}, \mathcal{R}e(p + K\gamma) > 0$$

The integrals appearing on the right side of eqn (9) can be evaluated asyptotieally using the method of steepest descent (see Appendix) and the diffracted field ψ_d is written as

$$\hat{\psi}_{\mathrm{d}} = \hat{\psi}_{\mathrm{dA}} + \hat{\psi}_{\mathrm{dW}} \tag{10}$$

where $\hat{\psi}_{\mathrm{dA}}$ denotes that part of $\hat{\psi}$ which arises when there is no wake and $\hat{\psi}_{\mathrm{dW}}$ when there is a wake and are explicitly given by

$$\begin{aligned} \hat{\psi}_{\mathrm{dA}} = {} & \left(\frac{\mathcal{B}R_{12}}{R+R_{\mathrm{o}}} + \mathcal{B}M\cos\Theta_{\mathrm{o}} - 2\sin(\Theta/2)\sin(\Theta_{\mathrm{o}}/2)\right)\mathrm{e}^{-\omega^2/4s^2} \\ & x\frac{\mathrm{e}^{-iKM(X-X_{\mathrm{o}})}\mathrm{e}^{i(KR_{12}-\pi/4)}\sqrt{R+R_{\mathrm{o}}+A_1}F(\tau_{R_{12}})}{4\pi\sqrt{\pi R_{\mathrm{o}}}\sin\Theta\sqrt{(1-M^2)}\sqrt{R_{12}(R_{12}+R_{11})}N(\zeta_2)} \end{aligned} \tag{11}$$

and

$$\begin{aligned} \hat{\psi}_{\mathrm{dW}} = {} & -\frac{\sin(\Theta_{\mathrm{o}}/2)\mathrm{e}^{-iKM(X-X_{\mathrm{o}})-\omega^2/4s^2}}{4\pi\sqrt{\pi R_{\mathrm{o}}}\sqrt{(1-M^2)}\cos(\Theta/2)}\left\{\frac{\sqrt{\pi A_1'}\mathrm{e}^{iKR_{11}'}}{2R_{11}'N(\zeta_1')}H(-\varepsilon_1')\right. \\ & \left. + \varepsilon_1'\mathrm{e}^{i(KR_{12}-\pi/4)}\left(\frac{\sqrt{R+R_{\mathrm{o}}+A_1'}}{\sqrt{R_{12}(R_{12}+R_{11})}}\right)\frac{F(\tau_{R_{12}}')}{N(\zeta_2)}\right\} \end{aligned} \tag{12}$$

where

$$R_{12}^2 = (z-z_{\mathrm{o}})^2 + (R+R_{\mathrm{o}})^2$$

$$R_{11}^2 = (z-z_{\mathrm{o}})^2 + A_1^2 \qquad R_{11}'^2 = (z-z_{\mathrm{o}})^2 + A_1'^2$$

$$A_1 = R + R_{\mathrm{o}} - \mu^2 \qquad A_1' = R + R_{\mathrm{o}} - \mu'^2$$

$$\mu = \frac{\cos\Theta + \cos\Theta_{\mathrm{o}}}{\sin\Theta}\sqrt{R/2} \qquad \mu' = \frac{1/M - \cos\Theta}{\sin\Theta}\sqrt{R/2}$$

$$\tau_{R_{12}} = \frac{\sqrt{K(R+R_{\mathrm{o}}+A_1)}}{\sqrt{R_{12}+R_{11}}}\mu \qquad \tau_{R_{12}}' = \frac{\sqrt{K(R+R_{\mathrm{o}}+A_1')}}{\sqrt{R_{12}'+R_{11}'}}\mu'$$

$$\zeta_2 = \frac{R+R_{\mathrm{o}}}{R_{12}} \qquad \zeta_1' = \frac{A_1'}{R_{11}'},\ \varepsilon_1' = \mathrm{sgn}\,\tau_{R_{12}}'$$

When the souree is very lose to the edge ($KR_o \ll 1$) and the point of observation is at a large distance from the source but not near the wake, the dominant part of $\hat{\psi}_d$ denoted by $\hat{\psi}_{d1}$ is given by

$$\hat{\psi}_{d1} = \hat{\psi}_{dA1} + \hat{\psi}_{dW1} \tag{13}$$

where

$$\hat{\psi}_{dA1} = -\left(\frac{\mathcal{B}\tilde{R}_{12}}{R+R_o} + \mathcal{B}M\cos\Theta_o - 2\sin(\Theta/2)\sin(\Theta_o/2)\right)e^{-\omega^2/4s^2}$$
$$x\left(\frac{KRR_o}{2\pi\tilde{R}_{12}(1-M^2)}\right)^{1/2}\frac{e^{-iKM(X-X_o)+iK\tilde{R}_{12}-i\pi/4}}{2\pi\tilde{R}_{12}N(\zeta_2)} \tag{14}$$

$$\hat{\psi}_{dW1} = -\frac{i\exp[-iKM(X-X_o)+iK\tilde{R}_{12}-i\pi/4]\sin(\Theta/2)\sin(\Theta_o/2)}{2\pi\sqrt{2\pi KRR_o\tilde{R}_{12}(1-M^2)}N(\zeta_2)(\cos\Theta-1/M)}e^{-\omega^2/4s^2} \tag{15}$$

$$\tilde{R}_{12} = \sqrt{(z-z_o)^2+R^2}$$

Now using $k=\omega/c$, $F(\nu)\approx -\frac{i}{2\nu}$ (when $\nu\to\infty$) and $N(\zeta_2)$ in eqn (13), we obtain

$$\hat{\psi}_d(x,y,z;\omega) \approx (b+b_1)\frac{e^{-i\omega[M(X-X_o)-R_{12}]/Q}e^{-\omega^2/4s^2}}{\sqrt{\omega}} \tag{16}$$

where

$$b = \left\{\frac{\mathcal{B}R_{12}}{R+R_o} + \mathcal{B}M\cos\Theta_o - 2\sin(\Theta/2)\sin\{\Theta_o/2)\right\}\{Q/(1-M^2)\}^{1/2}$$
$$\times\frac{1}{4\pi\sqrt{2\pi RR_oR_{12}}(\cos\Theta+\cos\Theta_o)\tilde{L}_+(K\zeta_2\cos\Theta)\tilde{L}_-(-K\zeta_2\cos\Theta_o)} \tag{17}$$

$$b_1 = \{Q/(1-M^2)\}^{1/2}\frac{\sin(\Theta/2)\sin(\Theta_o/2)(1/M-\cos\Theta)^{-1}}{2\pi\sqrt{2\pi RR_oR_{12}}(\cos\Theta+\cos\Theta_o)}$$
$$\times\frac{1}{\tilde{L}_+(K\zeta_2\cos\Theta)\tilde{L}_-(-K\zeta_2\cos\Theta_o)} \tag{18}$$

$$K=\omega/Q,\ Q=c\sqrt{1-M^2}$$

It is important to note that $\tilde{L}_+(K\zeta_2 \cos\Theta)$ and $\tilde{L}_-(-K\zeta_2 \cos\Theta_o)$, appearing in eqns (17) and (18) are independent of ω (see Asghar *et al.*[6]).

In order to calculate the field $\psi_d(x, y, z; t)$, we need to find the inverse temporal Fourier transform of eqn (16). This gives, on using eqn (5):

$$\psi(x, y, z; t) \approx \frac{b + b_1}{2\pi} \int_{-\infty}^{\infty} \frac{1}{\sqrt{\omega}} e^{-i\omega(MR' \cos\Theta'/Q - R_{12}/Q)} \times e^{-\omega^2/4s^2} e^{-i\omega t} d\omega \tag{19}$$

where

$$X - X_o = R' \cos\Theta', \; Y - Y_o = R' \sin\Theta'$$

The integral appearing in eqn (19) may be evaluated with the help of mathematica.[8] Thus

$$\psi(x, y, z; t) \approx \frac{(b + b_1)}{2}(1 - i)^{3/4} e^{i\pi/8} s \, e^{-\frac{s^2(b-t)^2}{2}} \times (b - t)^{-5/8} \left\{ (1 - i)^{1/4}(b - t)^{3/4} \text{Bessel } I\left(\frac{-1}{4}, \frac{-s^2(b - t)^2}{2}\right) + 2^{3/4} e^{i\pi/4} \pi^{-1/4} \text{Bessel } I\left(\frac{1}{4}, \frac{-s^2(b - t)^2}{2}\right) \right\} \tag{20}$$

where $b = [R_{12} - MR' \cos\Theta']/Q$ and b, b_1 are given by eqns (17) and (18), respectively. In eqn (20) Bessel I (n,t) gives the modified Bessel function of the first kind of order n in t.

CONCLUSIONS

A complete analytical description has been provided for the scattering of a spherical Gaussian pulse for trailing edge (wake present) situation. Of particular significance are the following points:

(1) From eqn (20), we observe that as a result of fluid motion the field is increased by the factor $(1 - M^2)^{-1/2}$ in comparison to still fluid. Also, the field is independent of the direction of flow since the fluid velocity U appears as $|U|^2$ in the factor $(1 - M^2)$.

(2) It is perhaps well to note that the wave profile at $y = y_o$, $z = z_o$ moves along the direction of x-axis at the velocity of $c + U$ due to the fact that the fluid is moving in the x-direction.

(3) The ratio of $\hat{\psi}_{dA1}$ to $\hat{\psi}_{dW1}$ is found to be

$$\frac{\hat{\psi}_{dA1}}{\hat{\psi}_{dW1}} \simeq i\frac{[\mathcal{B}(1 + M\cos\Theta_o) - 2\sin(\Theta/2)\sin(\Theta_o/2)]}{\sin(\Theta/2)\sin(\Theta_o/2)} \times KR_o(\cos\Theta - 1/M) \tag{21}$$

Equation (21) gives the ratio of the diffracted wave when the wake is absent to that due to the wake for the point source. If we calculate this ratio for the line source situation (see Rawlins,[5] who has not explicitly shown it), we find that both the ratios are exactly the same. Thus the ratio of no wake to wake situation is independent of the type of acoustic sources.

(4) For the rigid half plane if we put $\beta = 0$ in eqn (21), this ratio becomes

$$\frac{\hat{\psi}_{dA1}}{\hat{\psi}_{dW1}} = -2iKR_o(\cos\Theta - 1/M) \tag{22}$$

We note that this ratio is the same as calculated by Balasubramanyam.[3] For small Mach number, the ratio (22) is effectively $2iKR_o/M$ and is independent of angle. If KR_o is of the order of πM this ratio is of the order of 2π. Consequently, the dependence of the intensity on Mach number would be M^5 whether the Kutta–Joukowski condition were imposed or not. At any rate, observations of the sound intensity at low Mach number in a moving medium would fail to detect whether or not a Kutta–Joukowski condition has been imposed, if the observations are not near the wake and are limited to the dependence on angle and Mach number. This conclusion remains unmodified for quadruples since the ratio of the two terms is not essentially altered by derivatives with respect to either R_o or Θ_o.

(5) We also conclude from eqn (21) that for point sources near the edge of absorbing half plane ($R_o \to 0$), the field caused by the Kutta–Joukowski condition will be substantially in excess of that in its absence. Also, the imposition of the Kutta–Joukowski condition and the associated wake has the effect of producing a stronger scattered field away from the wake than that in the neighbourhood of the wake an intense sound is created; it is much stronger than the scattered field away from the wake and does not decay downstream. This is true whether or not the sound be near the edge.

(6) If the analysis of Ffowcs Williams and Hall be regarded as relevant to a leading edge and the present analysis to a trailing edge, it would seem that the sound caused by a source near the trailing edge of an aerofoil will be more substantial than that due to a source of the same magnitude near the leading edge.

(7) Near the wake Θ is small and an additional term is required in eqn (10). This extra term is given by

$$\frac{\sin(\Theta_o/2)_e^{-iKM(X-X_o)+iKR_2}\sqrt{2M}\operatorname{sgn}Y}{2\pi R_2\sqrt{(1-M^2)}\sqrt{(1+M)}\tilde{L}_-(-K\bar{l}\cos\Theta_o/R_2)\tilde{L}_+(K\bar{l}/MR_2)} \times\left(1+\frac{X+\sqrt{M^2-1}|Y|}{MR_o}\right)^{1/2}$$

where

$$R_2^2=(z-z_o)^2+R_o^2, \qquad \bar{l}=R_o+X/M+\sqrt{M^2-1}|Y|/M$$

It is imperative to note the smaller that M becomes the more closly is the surface wave confined to the wake. It is the pressure of this wave which is the main distinguishing feature in the radiated-sound between the absence or otherwise of the Kutta–Joukowski condition. It is good to note that the surface wave disappears from the pressure, as given by eqn (*), but remains in velocity. Therefore, measurements of the pressure fluctuations alone will not indicate the existence of the surface wave. However, if the product of pressure and velocity is taken as a measure of energy, differences in the energy due to the surface wave will be manifest.

(8) As regards an aerofoil at zero incidence in a moving stream let us assume that the leading and trailing edges are sufficiently far apart for their acoustic interaction to be of secondary importance. Then, the intensity of sound produced by a fixed source near either edge ($KR_o \approx \pi M$) will be about the same away from the wake. Near the wake the sound will be more intense than in other parts of the distant field.

(9) It should be emphasized that this theory has dealt with the radiation from fixed given sources. Where the sources are created by the flow itself, as in turbulence, the production of the sources themselves may be profoundly affected by the Kutta–Joukowski condition and conclusions we have drawn would not necessarily be applicable.

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APPENDIX

In this appendix, we present the evaluation of one of the integrals appearing in eqn (9). The other integrals can be evaluated similarly. We consider the following integral:

$$I = \int_{-\infty}^{\infty} \frac{e^{iK[\tilde{\eta}(z-z_o)+(1-\tilde{\eta}^2)^{1/2}(R+R_o)]F(|\tilde{Q}|)d\tilde{\eta}}}{[K(1-\tilde{\eta}^2)^{1/2}]^{1/2}\tilde{L}_+(K\tilde{\gamma}\cos\Theta)\tilde{L}_-(-K\tilde{\gamma}\cos\Theta_o)(1-M^2)^{1/2}} \tag{A1}$$

where

$$\tilde{\eta} = \eta(1-M^2)^{1/2}, \tilde{\eta} = (1-\tilde{\eta}^2)^{1/2}$$

$$F(|\tilde{Q}|) = F(\mu[K(1-\tilde{\eta}^2)^{1/2}]^{1/2}), \mu = \tfrac{\cos\Theta+\cos\Theta_o}{\sin\Theta}\sqrt{(R/2)}$$

Making use of the result

$$\int_{v}^{\infty} e^{i\lambda t^2}dt = e^{i\lambda v^2}\frac{F(\sqrt{\lambda}v)}{\sqrt{\lambda}} \tag{A2}$$

Equation (A1) can be written as

$$I = \int_{\mu}^{\infty} \int_{-\infty}^{\infty} \frac{e^{iK[\tilde{\eta}(z-z_o)+(1-\tilde{\eta}^2)^{1/2}(t^2-\mu^2+R+R_o)]}}{\tilde{L}_+(K\tilde{\gamma}\cos\Theta)\tilde{L}_-(-K\tilde{\gamma}\cos\Theta_o)(1-M^2)^{1/2}} \mathrm{d}\tilde{\eta}\mathrm{d}t \qquad (A3)$$

Now, consider the integral

$$I' = \int_{-\infty}^{\infty} \frac{e^{iK[\tilde{\eta}(z-z_o)+(1-\tilde{\eta}^2)^{1/2}P]}}{\tilde{L}_+(K\tilde{\gamma}\cos\Theta)\tilde{L}_-(-K\tilde{\gamma}\cos\Theta_o)} \mathrm{d}\tilde{\eta} \qquad (A4)$$

By the substitutions

$$\tilde{\eta} = \cos\zeta, z - z_o = R_1\cos\Theta, P = R_1\sin\Theta, (1-\tilde{\eta}^2)^{1/2} = \sin\zeta \qquad (A5)$$

I' takes the form

$$I' = \int_{-\infty}^{\infty} f(\zeta)e^{iKR_1\cos(\zeta-\Theta)}\mathrm{d}\zeta \qquad (A6)$$

where

$$f(\zeta) = \frac{-\sin\zeta}{\tilde{L}_+(K\sin\zeta\cos\Theta)\tilde{L}_-(-K\sin\zeta\cos\Theta_o)}$$

We apply the method of steepest descent to solve the integral I'. For that, we deform the contour of integration to pass through the point of steepest descent $\zeta = \Theta$, such that the major part of the integrand is given by the integration over the part of the deformed contour near Θ, with $f(\zeta)$ slowly varying around it.[9] Hence, we can write

$$\begin{aligned} I' &\cong \pi f(\Theta)H_o^{(1)}(KR_1) \\ &\cong -\frac{\pi H_o^{(1)}(K[(z-z_o)^2+P^2]^{1/2})}{\tilde{L}_+(K\zeta\cos\Theta)\tilde{L}_-(-K\zeta\cos\Theta_o)}\zeta \end{aligned} \qquad (A7)$$

where

$$\zeta = \frac{P}{\left[(z-z_o)^2 + P^2\right]^{1/2}}$$

Using eqn (A7), we can write eqn (A3) as

$$I = \frac{-\pi}{(1-M^2)^{1/2}} \int_{\mu}^{\infty} \frac{H_o^{(1)}(K\left[(z-z_o)^2 + (t^2+R+R_o-\mu^2)^2\right]^{1/2})}{\tilde{L}_+(K\zeta\cos\Theta)\tilde{L}_-(-K\zeta\cos\Theta_o)} \times \frac{\left[t^2+R+R_o-\mu^2\right]}{[(z-z_o^2)+(t^2+R+R_o-\mu^2)^2]^{1/2}} \mathrm{d}t \quad \text{(A8)}$$

If we make the substitutions

$$t^2 = -A_1 + \left[A_1^2 + R_{11}^2 \sin h^2 u\right]^{1/2}, R_{11}^2 = (z-z_o)^2 + A_1^2$$
$$A_1 = R + R_o - \mu^2$$

In eqn (A8), we obtain

$$I = -\frac{\pi}{2} \int_{\eta_{11}}^{\infty} \frac{H_o^{(1)}(KR_{11}\cosh u)\left[(R_{11}^2\sinh^2 u + A_1^2)^{1/2} + A_1\right]^{1/2}}{\tilde{L}_+(K\tilde{\zeta}\cos\Theta)\tilde{L}_-(-K\tilde{\zeta}\cos\Theta_o)(1-M^2)^{1/2}} \mathrm{d}u \quad \text{(A9)}$$

where

$$\zeta = \frac{\left[R_{11}^2\sinh^2 u + A_1^2\right]^{1/2}}{R_{11}\cosh u}, \eta_{11} = \sinh^{-1}\left[(\mu^2+2A_1)^{1/2}\mu/R_{11}\right] \quad \text{(A10)}$$

The integral eqn (A9) can be solved asymptotically by taking $KR_{11}\cosh u \gg 1$. Therefore, we can replace the Hankel function by the first term of its asymptotic expansion to give

$$I = -\frac{\sqrt{\pi}\mathrm{e}^{-i\pi/4}}{\left[2KR_{11}(1-M^2)\right]^{1/2}} \int_{\eta_{11}}^{\infty} \frac{\left[(R_{11}^2\sinh^2 u + A_1^2)^{1/2} + A_1\right]^{1/2}}{\tilde{L}_+(K\tilde{\zeta}\cos\Theta)\tilde{L}_-(K\tilde{\zeta}\cos\Theta_o)\sqrt{\cosh u}} \times \exp\{iKR_{11}\cosh u\}\mathrm{d}u \quad \text{(A11)}$$

If we let $\tau = \sqrt{(2KR_{11})}\sinh(u/2)$, then

$$I = -\frac{\sqrt{2\pi}e^{iKR_{11}-i\pi/4}}{[K(1-M^2)]^{1/2}}\int_{\tau_{R_{12}}}^{\infty} e^{i\tau^2}\tilde{f}_2(\tau)d\tau \tag{A12}$$

where

$$\tilde{f}_2(\tau) = \left[\frac{\left[\tau^2(\tau^2+2KR_{11})+A_1^2K^2\right]^{1/2}+A_1K}{(\tau^2+KR_{11})(\tau^2+2KR_{11})}\right]^{1/2} \times \frac{1}{\tilde{L}_+(K\tilde{\zeta}\cos\Theta)L_-(-K\tilde{\zeta}\cos\Theta_o)}$$

and

$$\tilde{\zeta} = \frac{\left[(\tau^2+2KR_{11})\tau^2+A_1^2K^2\right]^{1/2}}{\tau^2+KR_{11}}, \quad \tau_{R_{12}} = \sqrt{\left[K(R_{12}-R_{11})\right]}$$

$$R_{12} = \left[(z-z_o)^2+(R+R_o)^2\right]^{1/2}$$

An asymptotic expansion of I_2 then follows by putting τ equal to its lower limit value in the non-exponential factor of the integrand plus the contribution from $\tau = 0$ depending if zero lies in the interval of integration. Hence

$$I = -\frac{\sqrt{2\pi}}{\sqrt{K(1-M^2)}}e^{iKR_{11}}I_oH(-\varepsilon_1) - \varepsilon_1\frac{e^{iKR_{12}-i\pi/4}}{\sqrt{K(1-M^2)}}F(\tau R_{12}) \times \frac{\sqrt{2\pi(A_1+R+R_o)}}{\sqrt{K(R_{12}+R_{11}R_{12}}\tilde{L}_+(K\zeta_2\cos\Theta)\tilde{L}_-(-K\zeta_2\cos\Theta_o)} \tag{A13}$$

where

$$I_o = \frac{1}{2}\frac{\sqrt{\pi A_1}}{\sqrt{KR_{11}}}\frac{1}{\tilde{L}_+(K\zeta_1\cos\Theta)\tilde{L}_-(-K\zeta_1\cos\Theta_o)}, \varepsilon_1 = \operatorname{sgn}\tau_{R_{12}}$$

H (.) is the usual Heaviside function, and

$$\zeta_2 = \frac{R+R_o}{R_{12}}, \zeta_1 = \frac{A_1}{R_{11}}.$$