CONFORMAL EXTENSION OF ψN-FORMALISM

by

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A THESIS

SUBMITTED IN PARTIAL FULFILMENT OF

THE REQUIREMENTS FOR THE

DEGREE OF

DOCTOR OF PHILOSOPHY

IN THE

DEPARTMENT OF MATHEMATICS

QUAID-I-AZAM UNIVERSITY

ISLAMABAD

1985

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MY PARENTS

ACKNOWLEDGEMENTS

It is a pleasure to acknowledge my gratitude to my supervisor, Dr. Asghar Qadir, for his inspiring guidance and continuous encouragement during the completion of this thesis.

I deeply appreciate valuable discussions with Dr.Jawaid Quamar, particularly regarding the ψN-formalism. I am indebted, also, to Dr. M.Rafique for exchange of views on different matters during the work.

Thanks are also due to Mr. S.U.Khan for typing this thesis.

S.Ashfaque Hussain Shah Bokhari

ABSTRACT

An earlier idea of reintroducing the Newtonian concept of force in Relativity is extended to space-times admitting only a conformal time-like Killing vector. It is shown that space-times having metrics which are a conformal generalisation of Carter's circular metrics admit of a conformal analogue of ψN-forces and potentials. To be able to see the working of this formalism a toy cosmological model is constructed. It is shown that in this model a test particle can simultaneously experience a gravitational force and a cosmological expansion, unlike the models generally available.

CONTENTS

Chapter 1

PRELIMINARIES

§1.1 INTRODUCTION

The General Theory of Relativity, which is a field theory of gravitation, is described completely in terms of geometry[1]. It is this geometric description in favour of which the concept of force is normally avoided. The changes in the trajectories of test particles that are classically attributed to forces are formulated in terms of space-time curvature in Relativity. These trajectories, which test particles follow in a four-dimensional space-time, are assumed to be geodesics (optimal paths). The geodesics are specified by a geometry whose structure is determined (in terms of curvature) by the distribution of matter. However, our physical intution still relies on the concept of force. It was argued[2] that the re-introduction of the force concept in Relativity would provide new insights into its working and predictions.

The re-introduction of the concept of force in Relativity is based on the fact that a freely falling observer can detect the presence of a gravitating source towards which he is falling, by measuring the tidal forces (discussed in more detail in §1.2). The re-introduction of the force concept was extended further[3,4]. The extension of the force concept (called the ψN-force) derives from the fact that the tidal force is the gradient of the central gravitational force (§1.3). This formalism (called the ψN-formalism) was applied to a few goemetries. In the case of a Schwarzschild source a test particle experiences the classical force[2]. It turned out that, in the Reissner-Nordstrom geometry, even an uncharged particle would experience a (repulsive) force[2,5] due to the charge on the gravitational source. This fact indicates that naked singularities may be physically feasible[2,5]. It was suggested that these ideas might help to understand a demonstration[6] of the validity of Penrose's conjecture[7] about the amalgamation of the black hole singularity and the big crunch singularity in a closed cosmology.

The re-introduced concept of force was also applied to more general metrics such as the Kerr-Newmann metric. It was shown that these metrics admit of an analogue of Newtonian forces[3] and potentials[4,8] (§1.4). This analysis was used[9] to provide a relativistic explanation of the phenomenon of pulsar drift[10]. It was also applied to provide a physical interpretation of Carter's fourth invariant of motion[11]. It has also been used to provide a relativistic explanation for the inclination of planetary orbits[12].

Killing vectors play an important role in Relativity by defining the directions of symmetry (§2.1). Also it is easier to deal with space-times which admit of more Killing vectors than with those possessing fewer Killing vectors. To be able to find the number of independent Killing vectors, in a space-time, one needs to solve the Killing equations. Section 2.2 contains a brief description of the technique used in this thesis for finding Killing vectors in different, static spherically symmetric, space-time geometries. It is further shown (§2.3) how a reduction of symmetry in a space-time leads to the reduction of the number of Killing vectors. This reduction is considered in detail in Sections 2.4, 2.5 and 2.6 respectively. It would be interesting to see if a steady reduction of the number of Killing vectors could be achieved and local symmetries classified accordingly.

The ψN-formalism was applicable for metrics which admit of a time-like Killing vector. However, our aim is to deal with cosmological models in which there may often exist only a conformal time-like Killing vector. We extend the ψN-formalism, in Chapter 3, so as to be able to deal with such situations. ψN-forces and potentials were defined for a class of space-times possessing metrics which is a certain generalisation of Carter's "circular metrics"[13]. This generalisation is extended to the conformal case in §3.1. Section 3.2 deals with a conformal generalisation of tidal forces. The conformal generalisation[14] of ψN-forces and potentials is discussed in §3.3. This formalism is applied, in Section 3.4, to the three Friedmann cosmological models.

In Chapter 4, the problem addressed is that the standard cosmological models which evolve with time are homogeneous and therefore cannot contain gravitational sources. Admittedly, the Einstein-Straus model[1,15], which cuts a piece out of the Friedmann Universe and replaces it by a Schwarzschild geometry, is available. However, at any given point in this model the Universe either has a Friedmann geometry or a Schwarzschild geometry. To be able to deal with gravitational sources in a time-evolving model, with both detectable at the same place, a sort of cross between the Friedmann and Schwazschild metrics has been constructed[16] (§4.1). The stress-energy tensor for this model Universe is obtained in Section 4.2. Discussing the evolution of the Universe in Section 4.3 the stress-energy tensor is diagonalised in the next section. It is shown in Section 4.5 that the appropriately scaled (conformal) definition of the gravitational force is just the Newtonian gravitational force.

Chapter 5 contains a brief summary and discussion of the work.

§1.2 THE TIDAL FORCE

It is generally believed that a freely falling observer cannot detect the presence of a gravitating source towards which he is falling. However, the freely falling observer can detect the source by carrying along with him an accelerometer (an idealised version of which is shown in Fig. 1). This accelerometer has a spring of length L with two (unit) masses attached to its ends. One of the ends of the spring has a pointer which can move on the dial of the accelerometer. The zero of the accelerometer is where the gravitational attraction between the two masses is just balanced by the spring tension. The force exerted by the source pulls the mass near it more than the mass further away. Thus, the spring is stretched and the pointer moves in the positive direction. Now consider the force exerted by a large electric charge, if the two masses are given equal (small) charges. The zero of the accelerometer is now defined by the balance between the electrostatic repulsion, the gravitational attraction and the spring tension. Now, if the source has a charge opposite to that of the two ends of the accelerometer, the resultant

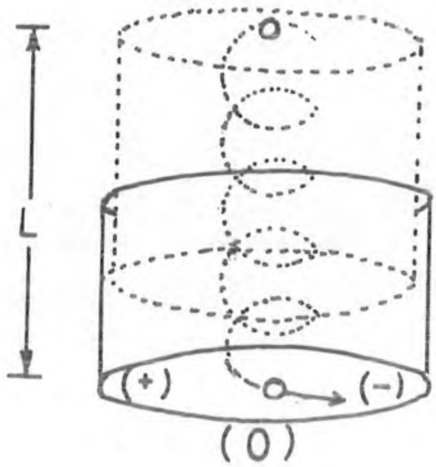

Fig. 1. The accelerometer has a spring of length L which connects two masses. One end of the spring has a needle that can rotate on a dial and represents attractive (positive) or repulsive (negative) forces. The zero on the dial represents no force being exerted.

attraction causes a stretching of the spring as before. However, if the source has a similar charge to that of the two ends of the accelerometer, the end closer to it would be repelled more than the end further away. Hence the spring would be compressed and the pointer would move in the negative direction. Thus, our accelerometer would show an attractive source if the pointer moved in the positive direction and a repulsive source if it moved in the negative direction. The strength of the source would be indicated by the extent that the pointer moved.

Geometrically this tidal force is measured by the geodesic deviation of two neighbouring geodesics having the same (time-like) tangent vector, $\underline{k}$. If the two geodesics are separated by a space-like vector, $\underline{\ell}$, the tidal force is

$$\underline{F}_T = \underline{k}[\underline{k}(\underline{\ell})] \quad . \tag{1.1}$$

Now, since $\underline{\ell}$ is to be Lie transported along $\underline{k}$, it is easily seen[1] that

$$\underline{F}_T^a = -R^a{}_{bcd}k^b\ell^c k^d \quad , \tag{1.2}$$

where $R^a{}_{bcd}$ is the Riemann-Christoffel curvature tensor. This formalism can be applied readily to yield tidal force in the Schwarzschild or Reissner-Nordstrom geometries[2]. To find the general expression for the tidal force in the Kerr-Newmann (K.N.) metric, we write it (the K.N. metric) in block diagonal form

$$g_{ab} = \begin{pmatrix} g_{ij}(x^k) & \vdots & 0 \\ \cdots\cdots & \vdots & \cdots\cdots \\ 0 & \vdots & g_{rs}(x^k) \end{pmatrix} \quad \begin{matrix} a,b = 0,1,2,3 \\ i,j,k = 1,2 \\ r,s = 0,3 \end{matrix} \quad , \qquad (1.3)$$

where the metric coefficients for the K.N. geometry are,

$$\left.\begin{matrix} g_{11} = -R^2/J \ , & g_{22} = -R^2, \\ g_{12} = g_{21} = 0 \ , & g_{33} = -\frac{A}{R^2} \sin^2\theta \ , \\ g_{00} = 1 - (2mr - Q^2)/R^2, & g_{03} = g_{30} = a(1 - g_{00}) \sin^2\theta \end{matrix}\right\} , \qquad (1.4)$$

and

$$\left.\begin{matrix} R^2 = r^2 + a^2\cos^2\theta \ , \\ J = r^2 - 2mr + a^2 + Q^2 \ , \\ A = (r^2 + a^2)^2 - Ja^2\sin^2\theta \end{matrix}\right\} . \qquad (1.5)$$

Using Riemann-normal coordinates[1] the expression for the tidal force given by Eq.(1.2) becomes

$$F_T{}^i = -\Gamma^i{}_{00,j}\ell^j = \frac{1}{2} g^{ij}g_{00,jk}\ell^k \ . \qquad (1.6)$$

If the accelerometer is turned about till a maximum reading is obtained on the dial the accelerometer will lie along the principal direction given by the eigen-value equation

$$\frac{1}{2} g^{ij}g_{00,jk}\ell^k = \lambda\ell^i \ . \qquad (1.7)$$

We will thus have operationally chosen the maximum magnitude of the eigen-value λ. Since in the K.N. geometry λ is always positive outside the horizon, we will have chosen the higher value of λ, λ_+. The eigen-values and eigen-vectors for the problem, determined from Eq.(1.7), are

$$\lambda_{\pm} = \frac{1}{4}\left[(g^{11}g_{00,11} + g^{22}g_{00,22}) \pm \left\{(g^{11}g_{00,11} - g^{22}g_{00,22})^2 + 4g^{11}g^{22}(g_{00,12})^2\right\}^{\frac{1}{2}}\right], \tag{1.8}$$

and

$$\frac{\ell^1_{\pm}}{\ell^2_{\pm}} = \frac{g^{11}g_{00,12}}{(2\lambda - g^{11}g_{00,11})}. \tag{1.9}$$

Here ℓ^i must satisfy the equation

$$g_{11}(\ell^1)^2 + g_{22}(\ell^2)^2 = -L^2. \tag{1.10}$$

Eqs.(1.9) and (1.10) together determine ℓ^1 and ℓ^2 given by

$$\left.\begin{aligned} \ell^1 &= Lg^{11}(-g^{22})^{\frac{1}{2}}(g_{00,12})\left\{(2\lambda - g^{11}g_{00,11})^2 + g^{11}g^{22}(g_{00,12})^2\right\}^{-\frac{1}{2}}, \\ \ell^2 &= L(-g^{22})^{\frac{1}{2}}(2\lambda - g^{11}g_{00,11})\left\{(2\lambda - g^{11}g_{00,11})^2 + g^{11}g^{22}(g_{00,12})^2\right\}^{\frac{1}{2}}. \end{aligned}\right\} \tag{1.11}$$

Since $F_T{}^i = \lambda_+\ell^i$, the tidal force is determined completely by Eqs.(1.8) and (1.11). The magnitude of the tidal force is λ_+L.

The general expression for the tidal force in the K.N.

metric is too complicated to provide any further understanding. Therefore the tidal force was calculated[3] in special cases only (i.e., at $\theta = \pi/2$ and $\theta = 0$ or π), to yield, respectively

$$\left.\begin{aligned} \lambda &= (1 - 2m/r + Q^2/r^2 + a^2/r^2)(2m/r - 3Q^2/r^2)1/r^2, \\ \ell^1 &= (1 - 2m/r + Q^2/r^2 + a^2/r^2)^{\frac{1}{2}}\ L, \\ \ell^2 &= 0\ , \end{aligned}\right\} \qquad (1.12)$$

and

$$\left.\begin{aligned} \lambda &= (1 - 2m/r + Q^2/r^2 + a^2/r^2)\left[\left(\frac{2m}{r^2} - \frac{3Q^2}{r^2}\right) - \right. \\ &\qquad \left. - \frac{a^2}{r^2}\left(\frac{6m}{r} - \frac{Q^2}{r^2}\right)\right]/r^2\left(1 + \frac{a^2}{r^2}\right)^4, \\ \ell^1 &= (1 - 2m/r + Q^2/r^2 + a^2/r^2)^{\frac{1}{2}}\ (1 + a^2/r^2)^{-\frac{1}{2}} L, \\ \ell^2 &= 0\ . \end{aligned}\right\} \qquad (1.13)$$

It may be noticed that Eqs.(1.12) are not in agreement with earlier calculations[2] for the tidal force in the K.N. geometry. This disagreement was a consequence of the naive restriction (of radial motion) imposed in the earlier calculations on the path followed by the freely falling observer. Infact, the resultant path was, in general, a geodesic.

In the case that $a = 0$ in Eqs.(1.4) and (1.5) the K.N. metric reduces to the Reissner-Nordstrom metric. The expression for the tidal force in the Reissner-Nordstrom metric has no contribution due to the θ term. Therefore the tidal force in this case becomes

$$\left.\begin{aligned} \lambda &= (1 - 2m/r + Q^2/r^2)(2m/r - 3Q^2/r^2)\cdot 1/r^2, \\ \ell^1 &= (1 - 2m/r + Q^2/r^2)^{\frac{1}{2}}\, L\ , \\ \ell^2 &= 0\ . \end{aligned}\right\} \qquad (1.14)$$

There is an extra local Lorentz factor in the expressions for the tidal force and the length of the accelerometer which can be removed by going into the local Lorentz rest frame [4]. In this rest frame the tidal force, which the observer experiences in the Reissner-Nordstrom geometry, becomes

$$\lambda = (2m/r - 3Q^2/r^2)\cdot 1/r^2, \qquad (1.15)$$

where the local Lorentz factor is given by $(1 - 2m/r + Q^2/r^2)$. In the case that Q is also zero we get the tidal force for the Schwarzschild metric

$$\lambda = 2m/r^3. \qquad (1.16)$$

§1.3 THE ψN-FORCE

Classically the tidal force is a gradient of the central force [1,4]. A generalisation of this result leads to a relativistic analogue of the Newtonian (central) force, which is called the pseudo-Newtonian (ψN) force [3,4]. In a space-time which admits of a time-like isometry, the geodesic equation is given by

$$x''^{i} + \Gamma^{i}_{ab}\, x'^{a} x'^{b} = 0\ , \tag{1.17}$$

where a prime denotes differentiation with respect to the proper time. Eq.(1.17) can be used in Eq.(1.4) to yield

$$F_{T}^{\ i} = x''^{i}_{\ ,j}\ \ell^{j}\ , \tag{1.18}$$

where x''^{i} is the analogue of the Newtonian force. It seems natural, here, to define the ψN-force as the second derivative of the relevant position vector. However, an integration constant is to be incorporated into the definition of the ψN-force. The value of this constant of integration is fixed by considering an analogous situation in classical physics while defining the gravitational potential. In that case the integration constant is fixed by setting the potential equal to zero at infinite distance from the gravitating source. Similarly, in the definition of the ψN-force the integration constant is chosen to be zero in the Minkowski space since there are not supposed to be any forces in it. Thus, the ψN-force is defined as

$$F^{i} = x''^{i} - x''^{\ i}_{M}\ , \tag{1.19}$$

where $x''^{\ i}_{M}$ are evaluated in the Minkowski space. The expression for the ψN-force, using Eqs.(1.17) and (1.19), becomes

$$F^i = (\Gamma_M{}^i{}_{ab} - \Gamma^i{}_{ab})\dot{x}^a\dot{x}^b \ , \tag{1.20}$$

where $\Gamma_M{}^i{}_{ab}$ are the Christoffel symbols to be evaluated in Minkowski space in the coordinates used for other calculations. In Cartesian coordinates $\Gamma_M{}^i{}_{ab}$ are zero but can be non-zero in Curvilinear coordinates. Eq.(1.20), which is a generalisation of the expression for force, gives the usual Newtonian force in the Schwarzschild metric. By using the polar type coordinates the magnitude [3,4] of the ψN-force is given by

$$F = [(F^1)^2 + r^2(F^2)^2]^{\frac{1}{2}}. \tag{1.21}$$

The angle made by $\underline{F}$ with the radial direction is

$$\chi = \tan^{-1}(r\ F^2/F^1) \ . \tag{1.22}$$

We consider the K.N. geometry to find the components of the ψN-force. The velocity components for the motion of a test particle moving in the rotational gravito-electric field outside the horizon of the K.N. blackhole are given by

$$\left.\begin{aligned} t' &= [(r^2 + a^2)U/J - a^2\varepsilon\ \sin^2\theta + aB]/R^2, \\ r' &= V/R^2 \ , \\ \theta' &= W/R^2 \ , \\ \phi' &= (aU/J - a\varepsilon + B/\sin^2\theta)/R^2 \ , \end{aligned}\right\} \tag{1.23}$$

where

$$\left.\begin{aligned} U &= \varepsilon(r^2+a^2) - aB, \\ V^2 &= U^2 - J(r^2 + na^2), \\ W^2 &= a^2[(n-\cos^2\theta) - (\varepsilon\sin^2\theta - B/a)^2/\sin^2\theta]. \end{aligned}\right\} \quad (1.24)$$

and ε, B and n are the integration constants of the geodesic equation of the particle. The n appearing here is related to Carter's fourth invariant of motion[11].

The ψN-force components, F^1 and F^2, obtained by using Eqs.(1.20), (1.23) and (1.24), are

$$\left.\begin{aligned} F^1(r\ \theta) = R^{-6}\Big\{ & a^2VW\sin2\theta - 2arU(a\varepsilon\sin^2\theta - B) \\ & + \{R^2(m-r)(r^2+na^2) + rJ(r^2+2na^2-a^2\cos^2\theta)\} \\ & - rR^2\{W^2+\sin^2\theta(aU/J - a\varepsilon + B/\sin^2\theta)^2\}\Big\}, \\ F^2(r\ \theta) = R^{-6}\Big\{ & -2rVW + a^2\sin2\theta\{a\varepsilon B - B^2/\sin^2\theta - \\ & - a^2\varepsilon^2\sin^2\theta + (r^2+na^2) - R^2(\varepsilon^2+1 - \\ & - B^2/a^2\sin^4\theta)/2\} + R^2\Big\{\frac{2VW}{r} - \sin2\theta(aU/J \\ & - a\varepsilon + B/\sin^2\theta)^2/2\Big\}\Big\}. \end{aligned}\right\} \quad (1.25)$$

Eqs.(1.25) show that the ψN-force has a radial and polar component.

Thus, unlike the classical force, the ψN-force is generally not directed radially inwards. Now if we put a = 0 in Eq.(1.25) we obtain

$$\left.\begin{aligned} F^1 &= -r^{-3}(mr + B^2/\sin^2\theta - Q^2) \ , \\ F^2 &= 0 \ . \end{aligned}\right\} \qquad (1.26)$$

§1.4 CIRCULAR METRICS AND ψN-POTENTIALS

As mentioned earlier the tidal force is the intrinsic derivative of the ψN-force along the separation vector. The question arises whether the ψN-force can be expressed as the gradient of some scalar quantity. This scalar quantity, conjectured already [17], which has been called the ψN-potential [8] is the relativistic analogue of the Newtonian potential. We require that the ψN-potential be zero in Minkowski space.

In developing the "no hair" theorem Carter defined circular metrics (CM_s) [13]. These metrics can be partitioned in the block diagonal form such as given by Eq.(1.3). To be able to compute the ψN-potential, which gives the corresponding ψN-force, it is convenient to use the local Lorentz rest frame of a freely falling observer. In the local frame, in which $\ell^0 = 0$ and ℓ^i are non-zero, the eigen-value problem for the tidal force is well-determined if

$$\left.\begin{aligned} g^{0i} g_{00,ij} \ell^j &= 0 \quad , \\ g_{0i} \ell^i &= 0 \quad . \end{aligned}\right\} \qquad (1.27)$$

The circular metrics given by Eq.(1.3) satisfy Eqs.(1.27). A natural extension of the CMs can be obtained by allowing i,j,k to 1 and r,s to 2,3,0 or i,j,k to 1,2,3 and r,s to 0 only. These generalised circular metrics (GCMs) are called GCM1 and GCM2 respectively[4,8]. Now the CM allows two Killing vector fields, one time-like and one space-like. Similarly GCM1 admits of one time-like and two space-like Killing vector fields and GCM2 admits of only a time-like Killing vector field. For GCM1 the eigen-value problem for the tidal force is

$$\lambda = \frac{1}{2} g^{11} g_{00,11} \quad , \qquad (1.28)$$

and for GCM2

$$\begin{vmatrix} g^{1i} g_{00,1i} - 2\lambda & g^{1i} g_{00,2i} & g^{1i} g_{00,3i} \\ g^{2j} g_{00,1j} & g^{2j} g_{00,2j} - 2\lambda & g^{2j} g_{00,3j} \\ g^{3k} g_{00,1k} & g^{3k} g_{00,2k} & g^{3k} g_{00,3k} - 2\lambda \end{vmatrix} = 0 \quad . \qquad (1.29)$$

Notice that GCM1 corresponds to a trivial one-dimensional eigen-value problem whereas GCM2 gives rise to a three dimensional eigen-value problem giving a cubic equation in λ. This cubic equation can be solved for the real maximal roots outside the horizon.

In the local Lorentz rest frame of the observer the four velocity of the observer is given by

$$\acute{x}^a = (1,0,0,0). \tag{1.30}$$

On lowering the indices in Eq.(1.20) and remembering that $\underset{M}{g}_{00} = +1$ in every coordinate system, the ψN-potential becomes

$$\phi = (\underline{k}\cdot\underline{k} - 1)/2 \; . \tag{1.31}$$

The ψN-force corresponding to this potential is given by

$$F_i = -\phi_{,i} \; . \tag{1.32}$$

Using these values in the tidal force formula, we get

$$F_i^{\,T} = -\phi_{,ij}\ell^j = [(1 - \underline{k}\cdot\underline{k})/2]_{,ij}\ell^j \; . \tag{1.33}$$

Eq.(1.33) shows that the tidal force can be expressed as the second derivative of the square of the magnitude of the Killing vector $\underline{k}$. Those space-times which admit of ψN-force and potential have been called ψN-space-times[3,4].

Applying the ψN-formalism in the K.N. geometry, the ψN-potential becomes

$$\phi = -(2mr - Q^2)/2R^2 \; . \tag{1.34}$$

The corresponding expression for the ψN-force in the K.N. geometry becomes

$$\left. \begin{aligned} F_1 &= \frac{-mr^2 + Q^2 r + a^2 m\cos^2\theta}{(r^2 + a^2\cos^2\theta)^2} , \\ F_2 &= \frac{a^2(2mr - Q^2)\sin 2\theta}{2(r^2 + a^2\cos^2\theta)^2} . \end{aligned} \right\} \qquad (1.35)$$

These components of the ψN-force correspond to the local Lorentz rest frame in terms of the coordinates relevant for an observer at infinity. The tidal force in the same rest frame of the observer in the K.N. geometry is determined from Eqs.(1.8) and (1.11). Setting $a = 0$ in Eqs.(1.34) and (1.35) we obtain the expressions for the ψN-potential and ψN-force in the Reissner-Nordstrom geometry

$$\left. \begin{aligned} \phi &= -(2mr - Q^2)/2r^2 , \\ F_1 &= -(mr - Q^2)/r^3 . \end{aligned} \right\} \qquad (1.36)$$

The tidal force in this case is given by Eq.(1.15). Setting $Q = 0$ in Eqs.(1.36) we obtain the expressions for the ψN-force and potential in the Schwarzschild geometry

$$\left. \begin{aligned} \phi &= -m/r , \\ F_1 &= -m/r^2 . \end{aligned} \right\} \qquad (1.37)$$

and the corresponding tidal force is given by Eq.(1.16).

Notice that in the Reissner-Nordstrom geometry the ψN-formalism gives a "$-Q^2/a^3$" correction to the force where Q is the charge of the gravitating source. As such even an uncharged particle experiences a radially directed repulsive tidal force proportional to $3Q^2/a^4$ (Eq.1.15). This force is completely repulsive if $r < 3Q^2/2m$.

Chapter 2

KILLING VECTORS AND THE REDUCTION OF SYMMETRY

Killing vectors are extensively used in General Relativity in the study of the symmetry structure of different space-times[1,18]. They are the infinitesimal generators of the symmetry groups. The motivation of this chapter is not to study the groups generated by the Killing vectors but to understand how a reduction of symmetry reduces the number of independent Killing vectors for different spherically symmetric, static space-times.

§2.1 KILLING VECTORS

By Noether's theorem[1,19] the symmetries of a Lagrangian imply the existance of conserved quantities. These symmetries have been used[1] to obtain the constants of motion for the trajectories of freely falling particles in the field of a gravitating source, e.g. in the Schwarzschild, Reissner-Nordstrom and K.N. geometeries (where, infact, a Killing tensor appears [20,21]).

A Killing vector is a vector, $\underline{k}$, relative to which the Lie derivative of the metric tensor, g, is zero, i.e.,

$$\underset{\underline{k}}{\pounds}\, g = 0 \ , \tag{2.1}$$

where the Lie derivative of a tensor $\underline{T}$ relative to a vector $\underline{v}$ is given by

$$(\underset{\underline{v}}{\pounds}\, T)^{i\cdots k}{}_{p\cdots r} = v^m T^{i\cdots j}{}_{p\cdots r;m} - T^{m\cdots k}{}_{p\cdots q}\, V^i{}_{;m} - \cdots - T^{i\cdots m}{}_{p\cdots q}\, V^j{}_{;m} + T^{i\cdots k}{}_{m\cdots r}\, V^m{}_{;p} + \cdots + T^{i\cdots j}{}_{p\cdots m} \times V^m{}_{;q} \ . \tag{2.2}$$

These Killing vectors characterise the symmetry properties of pseudo-Riemanian spaces in an invariant way[18]. Eqs.(2.1) and (2.2) give

$$k^c g_{ab;c} + g_{ac} k^c{}_{;b} + g_{bc} k^c{}_{;a} = 0 \ . \tag{2.3}$$

Using the fact that the covariant derivative of the metric tensor is zero and rearranging terms in Eq.(2.3), we get

$$k_{a;b} + k_{b;a} = 0 \ , \tag{2.4}$$

which are known as the Killing equations. In a torsion-free space, in a coordinate basis, Eqs.(2.3) can be written in the form

$$g_{ab,c}k^c + g_{ac}k^c{}_{,b} + g_{bc}k^c{}_{,a} = 0 , \qquad (2.5)$$

since the Christoffel symbols from the last two terms of Eq.(2.3) combine to give the first term in Eq.(2.5). The Killing equations, Eqs.(2.5), constitute a system of first order linear partial differential equations.

§2.2 THE PROCEDURE FOR FINDING KILLING VECTORS

Let us consider an n-dimensional Euclidean (or Minkowski) space-time in Cartesian coordinates. In this case the derivatives of the metric coefficients are zero. Therefore Eq.(2.4) gives

$$k_{a,b} + k_{b,a} = 0 , \qquad (2.6)$$

and hence (dropping the summation convention)

$$k_{a,a} = 0 . \qquad (2.7)$$

Differentiating Eq.(2.6) with respect to x^a, we obtain

$$k_{a,ba} + k_{b,aa} = 0 . \qquad (2.8)$$

Eqs.(2.7) and (2.8) give

$$k_{(a,b)b} = 0 . \qquad (2.9)$$

Here k_a is a linear function of x^b. Thus, Eqs.(2.9) yield

$$k_a = C_{ab}\, x^b + D_a . \qquad (2.10)$$

For consistency it is easily seen that Eqs.(2.6) and (2.10) imply

$$C_{ab} = -C_{ba} . \tag{2.11}$$

There are n constants in D_a and $n(n-1)/2$ in C_{ab}. Therefore the n-dimensional Euclidean (or Minkowski) space is characterised by $n(n+1)/2$ independent Killing vectors. The Killing vectors generate the corresponding continuous symmetry group (E^n) in which $C_{ab}\, x^b$ generates the rotation group So(n) and D_a the translation group. In Minkowski space-time there are $(4 \times 5/2)$ Killing vectors of which $C_{ab}\, x^b$ gives the generators of the proper Lorentz group So(1,3) and D_a the generators of the group of the translation T_4. Thus, the Killing vectors, k_a, in the Minkowski space-time generate the Poincarre group P(4).

§2.3 THE REDUCTION OF SYMMETRY

We now discuss how a reduction of symmetry reduces the number of Killing vectors. Consider a static and spherically symmetric space-time whose metric is written in the form

$$ds^2 = e^{\nu(r)}\, dt^2 - e^{-\nu(r)}\, dr^2 - r^2 d\theta^2 - r^2 \sin^2\theta\, d\phi^2 . \tag{2.12}$$

Eqs. (2.5) for this metric, become

$$\nu'(r) k^1 + 2k^0{}_{,0} = 0 , \tag{2.13}$$

$$e^{\nu(r)}k^0{}_{,1} - e^{-\nu(r)}k^1{}_{,0} = 0 \quad , \tag{2.14}$$

$$e^{\nu(r)}k^0{}_{,2} - r^2k^2{}_{,0} = 0 \quad , \tag{2.15}$$

$$e^{\nu(r)}k^0{}_{,3} - r^2\sin^2\theta k^3{}_{,0} = 0 \quad , \tag{2.16}$$

$$\nu'(r)k^1 - 2k^1{}_{,1} = 0 \quad , \tag{2.17}$$

$$e^{-\nu(r)}k^1{}_{,2} + r^2k^2{}_{,1} = 0 \quad , \tag{2.18}$$

$$e^{-\nu(r)}k^1{}_{,3} + r^2\sin^2\theta k^3{}_{,1} = 0 \quad , \tag{2.19}$$

$$k^1 + rk^2{}_{,2} = 0 \quad , \tag{2.20}$$

$$k^2{}_{,3} + \sin^2\theta k^3{}_{,2} = 0 \quad , \tag{2.21}$$

$$k^1 + r\cot\theta k^2 + rk^3{}_{,3} = 0 \quad , \tag{2.22}$$

where prime denotes differentiation with respect to r. Solving Eq.(2.17) gives

$$k^1 = g(t,\theta,\phi)e^{\nu/2} \ . \tag{2.23}$$

Differentiating Eqs.(2.18) and (2.20) with respect to θ and r respectively and comparing, give

$$\frac{g(t,\theta,\phi)_{,22}}{g(t,\theta,\phi)} = -\left\{1 - r\nu'(r)/2\right\} e^{\nu(r)} \ . \tag{2.24}$$

The left hand side of Eq.(2.24) is a function of t, θ and ϕ only whereas the right hand side is a function of r only. Thus, both sides equal the same constant $-\alpha$, whence

$$\{1 - r\nu'(r)/2\}e^{\nu(r)} = \alpha , \tag{2.25}$$

and

$$g(t,\theta,\phi)_{,22} + \alpha g(t,\theta,\phi) = 0 . \tag{2.26}$$

Eq.(2.25) can be solved to give

$$e^{\nu(r)} = \alpha + \beta r^2 , \tag{2.27}$$

where β is a constant of integration. Also, Eq.(2.26) gives

$$g(t,\theta,\phi) = \gamma(t,\phi)\cos\sqrt{\alpha}\theta + \delta(t,\phi)\sin\sqrt{\alpha}\theta . \tag{2.28}$$

Clearly $\alpha \neq 0$, since that would yield a negative value of $e^{\nu(r)}$ at $r = 0$, which is not permissible. We can now consider the following cases.

§2.4 THE MINKOWSKI METRIC

If $\alpha > 0$, without loss of generality, we can choose $\alpha = 1$. Thus, Eqs.(2.27) and (2.28) become

$$e^{\nu(r)} = 1 + \beta r^2 , \tag{2.29}$$

$$g(t,\theta,\phi) = \gamma(t,\phi)\cos\theta + \delta(t,\phi)\sin\theta . \tag{2.30}$$

Choosing $\underline{\beta = 0}$ in Eqs.(2.29) and (2.12), gives the metric for Minkowski space-time in spherical polar coordinates

$$ds^2 = dt^2 - dr^2 - r^2d\theta^2 - r^2\sin^2\theta d\phi^2 . \tag{2.31}$$

The Killing vectors could be directly obtained from Eq.(2.10) by a coordinate transformation. However, it is instructive to follow the procedure through as an example to be followed later. The Killing equations for this metric are

$$k^0{}_{,0} = 0 \ , \tag{2.32}$$

$$k^0{}_{,1} - k^1{}_{,0} = 0 \ , \tag{2.33}$$

$$k^0{}_{,2} - r^2 k^2{}_{,0} = 0 \ , \tag{2.34}$$

$$k^0{}_{,3} - r^2 \sin^2\theta k^3{}_{,0} = 0 \ , \tag{2.35}$$

$$k^1{}_{,1} = 0 \ , \tag{2.36}$$

$$k^1{}_{,2} + r^2 k^2{}_{,1} = 0 \ , \tag{2.37}$$

$$k^1{}_{,3} + r^2 \sin^2\theta k^3{}_{,1} = 0 \ . \tag{2.38}$$

The other Killing equations for this metric are as given by Eqs.(2.20), (2.21) and (2.22). Already Eq.(2.23) gives the form of k^1 which satisfies Eq.(2.36). Eq.(2.32) gives

$$k^0 = f(r,\theta,\phi) \ . \tag{2.39}$$

Using the values of k^0 and k^1 in Eq.(2.33), we obtain

$$\partial f(r,\theta,\phi)/\partial r = \partial g(t,\theta,\phi)/\partial t \ .$$

Since the left hand side is independent of t and the right hand side of r, we get

$$k^0 = rf_1(\theta,\phi) + f_2(\theta,\phi) \ , \tag{2.40}$$

$$k^1 = tf_1(\theta,\phi) + g_1(\theta,\phi) \ . \tag{2.41}$$

Eqs.(2.20) and (2.41) imply that

$$k^2_{\ ,2} = -\frac{t}{r} f_1(\theta,\phi) - \frac{1}{r} g_1(\theta,\phi) \ . \tag{2.42}$$

Differentiating this equation with respect to r yields

$$k^2_{\ ,12} = \frac{t}{r^2} f_1(\theta,\phi) + \frac{1}{r^2} g_1(\theta,\phi) \ . \tag{2.43}$$

Now differentiating Eq.(2.37) with respect to θ and using Eq.(2.41), we get

$$k^2_{\ ,12} = -\frac{1}{r^2} k^1_{\ ,22} = -\frac{1}{r^2}\left(t \frac{\partial^2 f_1(\theta,\phi)}{\partial\theta^2} + \frac{\partial^2 g_1(\theta,\phi)}{\partial\theta^2}\right). \tag{2.44}$$

Comparing Eqs.(2.43) and (2.44) gives

$$\frac{\partial^2 f_1(\theta,\phi)}{\partial\theta^2} + f_1(\theta,\phi) = 0 \ , \tag{2.45}$$

$$\frac{\partial^2 g_1(\phi,\theta)}{\partial\theta^2} + g_1(\theta,\phi) = 0 \ , \tag{2.46}$$

whose solutions are

$$f_1(\theta,\phi) = h_1(\phi)\cos\theta + h_2(\phi)\sin\theta \ , \tag{2.47}$$

$$g_1(\theta,\phi) = h_3(\phi)\cos\theta + h_4(\phi)\sin\theta \ . \tag{2.48}$$

Integrating θ in Eq.(2.42) and using Eqs.(2.47) and (2.48) gives

$$k^2 = -\frac{t}{r}\left[h_1(\phi)\sin\theta - h_2(\phi)\cos\theta\right] - \frac{1}{r}\left[h_3(\phi)\sin\theta - h_4(\phi)\cos\theta\right] + H(t,r,\phi) \ . \tag{2.49}$$

Differentiate Eqs.(2.41) and (2.49) with respect to θ and r, respectively, to obtain

$$k^1{}_{,2} = t\left[-h_1(\phi)\sin\theta + h_2(\phi)\cos\theta\right] + \left[-h_3(\phi)\sin\theta + h_4(\phi)\cos\theta\right], \qquad (2.50)$$

$$k^2{}_{,1} = \frac{t}{r^2}\left[h_1(\phi)\sin\theta - h_2(\phi)\cos\theta\right] + \frac{1}{r^2}\left[h_3(\phi)\sin\theta - h_4(\phi)\cos\theta\right] + H'(t,r,\phi). \qquad (2.51)$$

Using Eqs.(2.50) and (2.51) in Eq.(2.37) imply that the last term in Eq.(2.51) is zero, i.e., $H = H(t,\phi)$. Differentiating Eqs.(2.40) and (2.49) with respect to θ and t respectively, we obtain

$$k^0{}_{,2} = r\left[-h_1(\phi)\sin\theta + h_2(\phi)\cos\theta\right] + \frac{\partial f_2(\theta,\phi)}{\partial\theta}, \qquad (2.52)$$

$$k^2{}_{,0} = -\frac{1}{r}\left[h_1(\phi)\sin\theta - h_2(\phi)\cos\theta\right] + \frac{\partial H(t,\phi)}{\partial t}. \qquad (2.53)$$

Using these results in Eq.(2.34) it can be easily seen that

$$\frac{\partial f_2(\theta,\phi)}{\partial\theta} = 0 = \frac{\partial H(t,\phi)}{\partial t}, \qquad (2.54)$$

which implies that f_2 and H are functions of ϕ only. Let us write k^0, k^1 and k^2

$$k^0 = r\left[h_1(\phi)\cos\theta + h_2(\phi)\sin\theta\right] + f_2(\phi), \qquad (2.55)$$

$$k^1 = t\left[h_1(\phi)\cos\theta + h_2(\phi)\sin\theta\right] + \left[h_3(\phi)\cos\theta + h_4(\phi)\sin\theta\right], \qquad (2.56)$$

$$k^2 = -\frac{t}{r}\left[h_1(\phi)\sin\theta - h_2(\phi)\cos\theta\right] - \frac{1}{r}\left[h_3(\phi)\sin\theta - h_4(\phi)\cos\theta\right] + H(\phi). \qquad (2.57)$$

Inserting Eqs.(2.56) and (2.57) in Eq.(2.22) and simplifying yields

$$k^3{}_{,3} = -\frac{t}{r\sin\theta} h_2(\phi) - \frac{1}{r\sin\theta} h_4(\phi) - \text{Cot}\theta\, H(\phi). \quad (2.58)$$

Differentiating Eqs.(2.55) and (2.58) with respect to ϕ and t gives

$$k^0{}_{,3} = r\left[\frac{\partial h_1(\phi)}{\partial\phi}\cos\theta + \frac{\partial h_2(\phi)}{\partial\phi}\sin\theta\right] + \frac{\partial f_2(\phi)}{\partial\phi}, \quad (2.59)$$

$$k^3{}_{,03} = -\frac{1}{r\sin\theta} h_2(\phi) . \quad (2.60)$$

Now Eq.(2.35) gives

$$k^3{}_{,03} = \frac{1}{r^2\sin^2\theta} k^0{}_{,33} \quad (2.61)$$

Inserting the values from Eqs.(2.59) and (2.60) in Eq.(2.61) and comparing we obtain

$$\left.\begin{aligned} &\partial^2 h_1(\phi)/\partial\phi^2 = 0 , \\ &\partial^2 h_2(\phi)/\partial\phi^2 + h_2(\phi) = 0 , \\ &\partial^2 f_2(\phi)/\partial\phi^2 = 0 . \end{aligned}\right\} \quad (2.62)$$

Eqs.(2.62) can be solved to give

$$\left.\begin{aligned} &h_1(\phi) = C_1 + C_2\phi , \\ &h_2(\phi) = C_3\cos\phi + C_4\sin\phi , \\ &f_2(\phi) = C_5 + C_6\phi . \end{aligned}\right\} \quad (2.63)$$

Thus, Eqs.(2.55), (2.56),(2,57) and (2.58) become

$$k^0 = r\left[(C_1 + C_2\phi)\cos\theta + (C_3\cos\phi + C_4\sin\phi)\sin\theta\right] + (C_5 + C_6\phi), \quad (2.64)$$

$$k^1 = t\left[(C_1 + C_2\phi)\cos\theta + (C_3\cos\phi + C_4\sin\phi)\sin\theta\right] + \left[h_3(\phi)\cos\theta + h_4(\phi)\sin\theta\right], \quad (2.65)$$

$$k^2 = -\frac{t}{r}\left[(C_1 + C_2\phi)\sin\theta - (C_3\cos\phi + C_4\sin\phi)\cos\theta\right] - \frac{1}{r}\left[h_3(\phi)\sin\theta - h_4(\phi)\cos\theta\right] + H(\phi), \quad (2.66)$$

$$k^3{}_{,3} = -\frac{t}{r\sin\theta}(C_3\cos\phi + C_4\sin\phi) - \frac{1}{r\sin\theta}h_4(\phi) - \cot\theta\, H(\phi). \quad (2.67)$$

Now Eqs.(2.35) and (2.64) give

$$k^3{}_{,0} = \frac{1}{r^2\sin^2\theta}\left[r\left\{(C_2\cos\theta) + (-C_3\sin\phi + C_4\cos\phi)\sin\theta\right\} + C_6\right]$$

Integrating this equation over t we obtain

$$k^3 = \frac{t}{r^2\sin^2\theta}\left[r\left\{(C_2\cos\theta) + (-C_3\sin\phi + C_4\cos\phi)\sin\theta\right\} + C_6\right] + F(r,\theta,\phi). \quad (2.68)$$

Differentiating Eq.(2.68) with respect to ϕ and comparing with Eq.(2.67) yields

$$\frac{\partial F(r,\theta,\phi)}{\partial\phi} = -\frac{1}{r\sin\theta}h_4(\phi) - \cot\theta\, H(\phi). \quad (2.69)$$

Now differentiate Eqs.(2.65) and (2.68) with respect to ϕ and r respectively to get

$$k^1{}_{,3} = t\left[C_2\cos\theta + (-C_3\sin\phi + C_4\cos\phi)\sin\theta\right]$$

$$+ \left[\frac{\partial h_3(\phi)}{\partial\phi}\cos\theta + \frac{\partial h_4(\phi)}{\partial\phi}\sin\theta\right] , \qquad (2.70)$$

$$k^3{}_{,1} = - \frac{-t}{r^2\sin^2\theta}\left[C_2\cos\theta + (-C_3\sin\phi + C_4\cos\phi)\sin\theta\right]$$

$$- \frac{2t}{r^3\sin^2\theta} C_6 + \frac{\partial F(r,\theta,\phi)}{\partial r} . \qquad (2.71)$$

These equations together with Eq.(2.39) imply that

$$2tC_6 + r\left[\frac{\partial h_3(\phi)}{\partial\phi}\cos\theta + \frac{\partial h_4(\phi)}{\partial\phi}\sin\theta + r^2\sin^2\theta\,\frac{\partial F(r,\theta,\phi)}{\partial r}\right] = 0 ,$$

which gives us

$$C_6 = 0 , \qquad (2.72)$$

$$\frac{\partial F(r,\theta,\phi)}{\partial r} + \left[\frac{\partial h_3(\phi)}{\partial\phi}\cos\theta + \frac{\partial h_4(\phi)}{\partial\phi}\sin\theta\right]/r^2\sin^2\theta = 0 . \qquad (2.73)$$

Now differentiate Eqs.(2.69) and (2.73) with respect to r and ϕ respectively and compare to obtain

$$\left.\begin{aligned} &\frac{\partial^2 h_4(\phi)}{\partial\phi^2} + h_4(\phi) = 0 , \\ &\frac{\partial^2 h_3(\phi)}{\partial\phi^2} = 0 . \end{aligned}\right\} \qquad (2.74)$$

Therefore Eqs.(2.74) yield

$$\left.\begin{aligned} h_4 &= C_7\cos\phi + C_8\sin\phi , \\ h_3 &= C_9 + C_{10}\phi . \end{aligned}\right\} \qquad (2.75)$$

Thus, Eqs. (2.64) to (2.66) and (2.68) become

$$k^0 = r\left[(C_1 + C_2\phi)\cos\theta + (C_3\cos\phi + C_4\sin\phi)\sin\theta\right] + C_5, \qquad (2.76)$$

$$k^1 = t\left[(C_1 + C_2\phi)\cos\theta + (C_3\cos\phi + C_4\sin\phi)\sin\theta\right]$$

$$+ (C_9 + C_{10}\phi)\cos\theta + (C_7\cos\phi + C_8\sin\phi)\sin\theta\ , \qquad (2.77)$$

$$k^2 = -\frac{t}{r}\left[(C_1 + C_2\phi)\sin\theta - (C_3\cos\phi + C_4\sin\phi)\cos\theta\right] - \frac{1}{r}\left[(C_9 + C_{10}\phi)\sin\theta\right.$$

$$\left. - (C_7\cos\phi + C_8\sin\phi)\cos\theta\right] + H(\phi)\ , \qquad (2.78)$$

$$k^3 = \frac{C_2 t\cos\theta}{r\sin^2\theta} - \frac{t}{r\sin\theta}(C_3\sin\phi - C_4\cos\phi) + F(r,\theta,\phi), \qquad (2.79)$$

where

$$\frac{\partial F(r,\theta,\phi)}{\partial r} = -\frac{C_{10}\cos\theta}{r^2\sin^2\theta} + \frac{C_7\sin\phi - C_8\cos\phi}{r^2\sin\theta}\ , \qquad (2.80)$$

$$\frac{\partial F(r,\theta,\phi)}{\partial\phi} = -\frac{(C_7\cos\phi + C_8\sin\phi)}{r\sin\theta} - \operatorname{Cot}\theta\ H(\phi)\ . \qquad (2.81)$$

Differentiate Eqs.(2.78) and (2.79) with respect to ϕ and θ to give

$$k^2{}_{,3} = -\frac{t}{r}\left[C_2\sin\theta + (C_3\sin\phi - C_4\cos\phi)\cos\theta\right] - \frac{1}{r}\left[C_{10}\sin\theta\right.$$

$$\left. + (C_7\sin\phi - C_8\cos\phi)\cos\theta\right] + \frac{\partial H(\phi)}{\partial\phi}\ , \qquad (2.82)$$

$$k^3{}_{,2} = -\frac{C_2 t}{r\sin\theta} - 2\frac{C_2 t\cos^2\theta}{r\sin^3\theta} + \frac{t\cos\theta}{r\sin^2\theta}(C_3\sin\phi - C_4\cos\phi)$$

$$+ \frac{\partial F(r,\theta,\phi)}{\partial\theta}\ . \qquad (2.83)$$

Inserting these values in Eq.(2.21) gives

$$- 2C_2t + r\sin\theta \left(- \frac{1}{r}\left\{C_{10}\sin\theta + (C_7\sin\phi - C_8\cos\phi)\cos\theta\right.\right.$$

$$\left.\left. + \frac{\partial H(\phi)}{\partial\phi} + \sin^2\theta \frac{\partial F(r,\theta,\phi)}{\partial\theta}\right\}\right) = 0 \; , \tag{2.84}$$

which is satisfied if

$$C_2 = 0 \; , \tag{2.85}$$

and

$$\frac{\partial F(r,\theta,\phi)}{\partial\theta} = \frac{C_{10}}{r\sin\theta} + \frac{(C_7\sin\phi - C_8\cos\phi)\cos\theta}{r\sin^2\theta} - \frac{1}{\sin^2\theta}\frac{\partial H(\phi)}{\partial\phi} \; . \tag{2.86}$$

Using these results in Eqs.(2.76) to (2.79), we obtain

$$k^0 = r\left(C_1\cos\theta + (C_3\cos\phi + C_4\sin\phi)\sin\theta\right) + C_5 \; , \tag{2.87}$$

$$k^1 = t\left(C_1\cos\theta + (C_3\cos\phi + C_4\sin\phi)\sin\theta\right) + (C_9 + C_{10}\phi)\cos\theta$$

$$+ (C_7\cos\phi + C_8\sin\phi)\sin\theta \; , \tag{2.88}$$

$$k^2 = - \frac{t}{r}\left(C_1\sin\theta - (C_3\cos\phi + C_4\sin\phi)\cos\theta\right) - \frac{1}{r}\left((C_9 + C_{10}\phi)\sin\theta\right.$$

$$\left. - (C_7\cos\phi + C_8\sin\phi)\cos\theta\right) + H(\phi) \; , \tag{2.89}$$

$$k^3 = - \frac{t}{r\sin\theta}(C_3\sin\phi - C_4\cos\phi) + F(r,\theta,\phi) \; . \tag{2.90}$$

Now differentiating Eqs.(2.81) and (2.86) with respect to θ and ϕ respectively and comparing, we have

$$\frac{\partial^2 H(\phi)}{\partial\phi^2} + H(\phi) = 0 \; , \tag{2.91}$$

which yields

$$H = C_{11}\cos\phi + C_{12}\sin\phi \; . \tag{2.92}$$

It can also be observed that differentiating Eqs.(2.80) and (2.86) with respect to θ and r, give

$$C_{10} = 0 . \tag{2.93}$$

Inserting the values from Eq.(2.92) into Eq.(2.81) and integrating ϕ, we obtain

$$F = - \frac{(C_7 \sin\phi - C_8 \cos\phi)}{r \sin\theta} - \text{Cot}\theta \, (C_4 \sin\phi - C_{12} \cos\phi) + C_{13} . \tag{2.94}$$

Using these results in Eqs.(2.87) to (2.90) and relabeling the parameters, we obtain the Killing vectors for the Minkowski space-time

$$\left.\begin{aligned}
k^0 &= r\left(C_1 \cos\theta + C_2 \cos(\phi + C_3)\right) + C_4 , \\
k^1 &= t\left(C_1 \cos\theta + C_2 \cos(\phi + C_3) \sin\theta\right) + C_5 \cos\theta \\
&\quad + C_6 \cos(\phi + C_7) \sin\theta , \\
k^2 &= - \frac{t}{r}\left(C_1 \sin\theta - C_2 \cos(\phi + C_3) \cos\theta\right) - \frac{1}{r}\left(C_5 \sin\theta \right. \\
&\quad \left. - C_6 \cos(\phi + C_7) \cos\theta\right) + C_8 \cos(\phi + C_9) , \\
k^3 &= - \frac{t}{r \sin}\left(C_2 \sin(\phi + C_3) - \frac{C_6 \sin(\phi + C_7)}{r \sin\theta}\right. \\
&\quad - C_8 \sin(\phi + C_9) \text{Cot}\theta + C_{10} .
\end{aligned}\right\} \tag{2.91}$$

§2.5 THE DE-SITTER METRIC

Consider $e^{\nu} \to 0$ at some R. This corresponds to the choice $\beta = -1/R^2$ in Eq.(2.27). The metric given by Eq.(2.12) becomes

$$ds^2 = (1 - r^2/R^2)dt^2 - \frac{dr^2}{1 - r^2/R^2} - r^2 d\theta^2 - r^2 \sin^2\theta d\phi^2 \quad .(2.92)$$

In this case Eqs.(2.23) and (2.31) yield

$$k^1 = \sqrt{1 - r^2/R^2}\left[\gamma(t,\phi)\cos\theta + \delta(t,\phi)\sin\theta\right]. \qquad (2.93)$$

The remaining Killing equations for the same space-time become

$$k^0{}_{,0} = \frac{r/R^2}{\sqrt{1 - r^2/R^2}}\left[\gamma(t,\phi)\cos\theta + \delta(t,\phi)\sin\theta\right], \qquad (2.94)$$

$$k^0{}_{,1} = \frac{1}{(1 - r^2/R^2)^{3/2}}\left[\gamma(t,\phi)_{,0}\cos\theta + \delta(t,\phi)_{,0}\sin\theta\right], \qquad (2.95)$$

$$k^0{}_{,2} = \frac{r^2}{1 - r^2/R^2}\, k^2{}_{,0} \quad , \qquad (2.96)$$

$$k^0{}_{,3} = \frac{r^2 \sin^2\theta}{1 - r^2/R^2}\, k^3{}_{,0} \quad , \qquad (2.97)$$

$$k^2{}_{,1} = \frac{1}{r^2\sqrt{1 - r^2/R^2}}\left[\gamma(t,\phi)\sin\theta - \delta(t,\phi)\cos\theta\right], \qquad (2.98)$$

$$k^3{}_{,1} = \frac{-1}{r^2\sqrt{1 - r^2/R^2}\,\sin^2\theta}\left[\gamma(t,\phi)_{,3}\cos\theta + \delta(t,\phi)_{,3}\sin\theta\right], \qquad (2.99)$$

$$k^2{}_{,2} = -\frac{\sqrt{1 - r^2/R^2}}{r}\left[\gamma(t,\phi)\cos\theta + \delta(t,\phi)\sin\theta\right], \qquad (2.100)$$

$$k^3{}_{,2} = -k^2{}_{,3}/\sin^2\theta \quad , \qquad (2.101)$$

$$k^3{}_{,3} = -\frac{\sqrt{1 - r^2/R^2}}{r}\left[\gamma(t,\phi)\cos\theta + \delta(t,\phi)\sin\theta\right] - \mathrm{Cot}\,\theta k^2. \quad (2.102)$$

Differentiating Eqs.(2.95) and (2.96) with respect to r and t respectively yield

$$k^0{}_{,01} = \frac{1/R^2}{(1 - r^2/R^2)^{3/2}} \left[\gamma(t,\phi)\cos\theta + \delta(t,\phi)\sin\theta\right] , \qquad (2.103)$$

$$k^0{}_{,01} = \frac{1}{(1 - r^2/R^2)^{3/2}} \left[\gamma(t,\phi)_{,00}\cos\theta + \delta(t,\phi)_{,00}\sin\theta\right] . \qquad (2.104)$$

Comparing these two equations, we obtain

$$\left[\gamma(t,\phi)_{,00} - \frac{1}{R^2}\,\gamma(t,\phi)\right]\cos\theta + \left[\delta(t,\phi)_{,00} - \frac{1}{R^2}\,\delta(t,\phi)\right]\sin\theta = 0,$$

which is satisfied if the coefficients of $\cos\theta$ and $\sin\theta$ are separately zero, i.e.,

$$\gamma(t,\phi)_{,00} - \frac{1}{R^2}\,\gamma(t,\phi) = 0 \quad , \qquad (2.105)$$

$$\delta(t,\phi)_{,00} - \frac{1}{R^2}\,\delta(t,\phi) = 0 \quad . \qquad (2.106)$$

These equations can be solved to yield

$$\gamma(t,\phi) = \alpha(\phi)\cosh t/R + \beta(\phi)\sinh t/R \; , \qquad (2.107)$$

$$\delta(t,\phi) = \eta(\phi)\cosh t/R + \lambda(\phi)\sinh t/R \; . \qquad (2.108)$$

Rewriting equations involving γ and δ, keeping in mind Eqs.(2.107) and (2.108), we obtain

$$k^1 = \sqrt{1 - r^2/R^2}\,\left[\left\{\alpha(\phi)\cosh\frac{t}{R} + \beta(\phi)\sinh\frac{t}{R}\right\}\cos\theta + \left\{\eta(\phi)\cosh\frac{t}{R} + \lambda(\phi)\sinh\frac{t}{R}\right\}\sin\theta\right] , \qquad (2.109)$$

$$k^0{}_{,0} = \frac{r/R^2}{\sqrt{1 - r^2/R^2}}\,\left[\left\{\alpha(\phi)\cosh\frac{t}{R} + \beta(\phi)\sinh\frac{t}{R}\right\}\cos\theta + \left\{\eta(\phi)\cosh\frac{t}{R} + \lambda(\phi)\sinh\frac{t}{R}\right\}\sin\theta\right] , \qquad (2.110)$$

$$k^0{}_{,1} = \frac{1/R}{(1-r^2/R^2)^{3/2}} \left[\left\{\alpha(\phi)\sinh\frac{t}{R} + \beta(\phi)\cosh\frac{t}{R}\right\}\cos\theta + \left\{\eta(\phi)\sinh\frac{t}{R} + \lambda(\phi)\cosh\frac{t}{R}\right\}\sin\theta\right], \tag{2.111}$$

$$k^2{}_{,1} = \frac{1}{r^2\sqrt{1-r^2/R^2}} \left[\left\{\alpha(\phi)\cosh\frac{t}{R} + \beta(\phi)\sinh\frac{t}{R}\right\}\sin\theta - \left\{\eta(\phi)\cosh\frac{t}{R} + \lambda(\phi)\sinh\frac{t}{R}\right\}\cos\theta\right], \tag{2.112}$$

$$k^2{}_{,2} = -\frac{\sqrt{1-r^2/R^2}}{r} \left[\left\{\alpha(\phi)\cosh\frac{t}{R} + \beta(\phi)\sinh\frac{t}{R}\right\}\cos\theta + \left\{\eta(\phi)\cosh\frac{t}{R} + \lambda(\phi)\sinh\frac{t}{R}\right\}\sin\theta\right], \tag{2.113}$$

$$k^3{}_{,1} = \frac{-1}{r^2\sqrt{1-r^2/R^2}\sin^2\theta} \left[\left\{\alpha(\phi)_{,3}\cosh\frac{t}{R} + \beta(\phi)_{,3}\sinh\frac{t}{R}\right\}\cos\theta + \left\{\eta(\phi)_{,3}\cosh\frac{t}{R} + \lambda(\phi)_{,3}\sinh\frac{t}{R}\right\}\sin\theta\right], \tag{2.114}$$

$$k^3{}_{,3} = -\frac{\sqrt{1-r^2/R^2}}{r} \left[\left\{\alpha(\phi)\cosh\frac{t}{R} + \beta(\phi)\sinh\frac{t}{R}\right\}\cos\theta + \left\{\eta(\phi)\cosh\frac{t}{R} + \lambda(\phi)\sinh\frac{t}{R}\right\}\sin\theta\right] - \mathrm{Cot}\,\theta k^2 . \tag{2.115}$$

Integrate θ in Eq.(2.113) to obtain

$$k^2 = -\frac{\sqrt{1-r^2/R^2}}{r} \left[\left\{\alpha(\phi)\cosh\frac{t}{R} + \beta(\phi)\sinh\frac{t}{R}\right\}\sin\theta - \left\{\eta(\phi)\cosh\frac{t}{R} + \lambda(\phi)\sinh\frac{t}{R}\right\}\cos\theta\right] + D(t,r,\phi) . \tag{2.116}$$

Differentiating this equation with respect to r and comparing with Eq.(2.112) it is seen that D cannot depend on r. Thus

$$D = E(t,\phi) . \tag{2.117}$$

Incorporating this value of D, Eq.(2.116) yields

$$k^2 = -\frac{\sqrt{1-r^2/R^2}}{r}\left[\left\{\alpha(\phi)\cosh\frac{t}{R}+\beta(\phi)\sinh\frac{t}{R}\right\}\sin\theta - \left\{\eta(\phi)\cosh\frac{t}{R}+\lambda(\phi)\sinh\frac{t}{R}\right\}\cos\theta\right] + E(t,\phi) \ . \qquad (2.118)$$

Integrating θ in Eq.(2.96) implies

$$k^0 = \frac{r/R}{\sqrt{1-r^2/R^2}}\left[\left\{\alpha(\phi)\sinh\frac{t}{R}+\beta(\phi)\cosh\frac{t}{R}\right\}\cos\theta + \left\{\eta(\phi)\sinh\frac{t}{R}+\lambda(\phi)\cosh\frac{t}{R}\right\}\sin\theta\right] + \frac{r^2\theta}{1-r^2/R^2}E(t,\phi)_{,0} + F(t,r,\phi) \ . \qquad (2.119)$$

Differentiating this equation with respect to r and comparing with (2.111) yields

$$\theta\left(\frac{r^2}{1-r^2/R^2}\right)' E(t,\phi)_{,0} + F'(t,r,\phi) = 0 \quad ,$$

which is satisfied only if

$$E(t,\phi)_{,0} = 0 \ , \ F'(t,r,\phi) = 0 \ . \qquad (2.120)$$

Thus, Eqs.(2.120) imply that

$$E = G(\phi) \quad , \qquad (2.121)$$

$$F = H(t,\phi) \quad . \qquad (2.122)$$

Incorporating these values of E and F, Eqs.(2.118) and (2.119) become

$$k^0 = \frac{r/R}{\sqrt{1-r^2/R^2}}\left[\left\{\alpha(\phi)\sinh\frac{t}{R}+\beta(\phi)\cosh\frac{t}{R}\right\}\cos\theta + \left\{\eta(\phi)\sinh\frac{t}{R}+\lambda(\phi)\cosh\frac{t}{R}\right\}\sin\theta\right] + H(t,\phi) \ , \qquad (2.123)$$

$$k^2 = -\frac{\sqrt{1-r^2/R^2}}{r}\left[\left\{\alpha(\phi)\cosh\frac{t}{R}+\beta(\phi)\sinh\frac{t}{R}\right\}\sin\theta - \left\{\eta(\phi)\cosh\frac{t}{R}+\lambda(\phi)\sinh\frac{t}{R}\right\}\cos\theta\right] + G(\phi) \quad . \tag{2.124}$$

Integrating Eq.(2.114) in r, we obtain

$$k^3 = \frac{\sqrt{1-r^2/R^2}}{r\sin^2\theta}\left[\left\{\alpha(\phi)_{,3}\cosh\frac{t}{R}+\beta(\phi)_{,3}\sinh\frac{t}{R}\right\}\cos\theta + \left\{\eta(\phi)_{,3}\cosh\frac{t}{R}+\lambda(\phi)_{,3}\sinh\frac{t}{R}\right\}\sin\theta\right] + L(t,\theta,\phi). \tag{2.125}$$

Using Eqs.(2.123) and (2.125) in Eq.(2.97) yield

$$\sin^2\theta L(t,\theta,\phi)_{,0} = \frac{1-r^2/R^2}{r^2} H(t,\phi)_{,3} \quad ,$$

which is satisfied only if

$$L(t,\theta,\phi)_{,0} = 0 = H(t,\phi)_{,3} \; . \tag{2.126}$$

This equation implies that

$$L = M(\theta,\phi) \quad , \tag{2.127}$$

$$H = N(t) \quad . \tag{2.128}$$

Inserting these values Eqs.(2.123) and (2.125) become

$$k^0 = \frac{r/R}{\sqrt{1-r^2/R^2}}\left[\left\{\alpha(\phi)\sinh\frac{t}{R}+\beta(\phi)\cosh\frac{t}{R}\right\}\cos\theta + \left\{\eta(\phi)\sinh\frac{t}{R}+\lambda(\phi)\cosh\frac{t}{R}\right\}\sin\theta\right] + N(t) \quad . \tag{2.129}$$

$$k^3 = \frac{\sqrt{1-r^2/R^2}}{r\sin^2\theta}\left[\left\{\alpha(\phi)_{,3}\cosh\frac{t}{R}+\beta(\phi)_{,3}\sinh\frac{t}{R}\right\}\cos\theta + \left\{\eta(\phi)_{,3}\cosh\frac{t}{R}+\lambda(\phi)_{,3}\sinh\frac{t}{R}\right\}\sin\theta\right] + M(\theta,\phi). \tag{2.130}$$

Using Eqs.(2.124) and (2.130), Eq.(2.101) imply

$$\left.\begin{aligned} \alpha(\phi)_{,3} &= 0 = \beta(\phi)_{,3} \\ M(\theta,\phi)_{,2} &= -G(\phi)_{,3}/\sin^2\theta . \end{aligned}\right\} \qquad (2.131)$$

Thus,

$$\left.\begin{aligned} &\alpha = C_1, \quad \beta = C_2, \\ &M(\theta,\phi) = G(\phi)_{,3}\cot\theta + R(\phi). \end{aligned}\right\} \qquad (2.132)$$

We rewrite k^0, k^1, k^2 and k^3 using Eqs.(2.132) to get

$$k^0 = \frac{r/R}{\sqrt{1-r^2/R^2}}\left[\left\{C_1\sinh\frac{t}{R} + C_2\cosh\frac{t}{R}\right\}\cos\theta + \left\{\eta(\phi)\sinh\frac{t}{R} + \lambda(\phi)\cosh\frac{t}{R}\right\}\sin\theta\right] + N(t) , \qquad (2.133)$$

$$k^1 = \sqrt{1-r^2/R^2}\left[\left\{C_1\cosh\frac{t}{R} + C_2\sinh\frac{t}{R}\right\}\cos\theta + \left\{\eta(\phi)\cosh\frac{t}{R} + \lambda(\phi)\sinh\frac{t}{R}\right\}\sin\theta\right] , \qquad (2.134)$$

$$k^2 = -\frac{\sqrt{1-r^2/R^2}}{r}\left[\left\{C_1\cosh\frac{t}{R} + C_2\sinh\frac{t}{R}\right\}\sin\theta - \left\{\eta(\phi)\cosh\frac{t}{R} + \lambda(\phi)\sinh\frac{t}{R}\right\}\cos\theta\right] + G(\phi), \qquad (2.135)$$

$$k^3 = \frac{\sqrt{1-r^2/R^2}}{r\sin\theta}\left[\left\{\eta(\phi)_{,3}\cosh\frac{t}{R} + \lambda(\phi)_{,3}\sinh\frac{t}{R}\right\}\right] + R(\phi) + G(\phi)_{,3}\cot\theta . \qquad (2.136)$$

Differentiating Eq.(2.136) and comparing it with Eq.(2.102), we obtain

Using Eqs.(2.124) and (2.130), Eq.(2.101) imply

$$\left.\begin{aligned} &\alpha(\phi)_{,3} = 0 = \beta(\phi)_{,3} = G(\phi)_{,3} \\ &M(\theta,\phi)_{,2} = 0 \ . \end{aligned}\right\} \qquad (2.131)$$

Thus,

$$\left.\begin{aligned} &\alpha = C_1, \ \beta = C_2, \ G = C_3 \\ &M = R\ (\phi) \ . \end{aligned}\right\} \qquad (2.132)$$

We rewrite k^0, k^1, k^2 and k^3 using Eqs.(2.132) to get

$$k^0 = \frac{r/R}{\sqrt{1 - r^2/R^2}} \left[\left\{C_1 \sinh\frac{t}{R} + C_2 \cosh\frac{t}{R}\right\}\cos\theta + \left\{\eta(\phi)\sinh\frac{t}{R} + \lambda(\phi)\cosh\frac{t}{R}\right\}\sin\theta\right] + N(t) \ , \qquad (2.133)$$

$$k^1 = \sqrt{1 - r^2/R^2} \left[\left\{C_1 \cosh\frac{t}{R} + C_2 \sinh\frac{t}{R}\right\}\cos\theta + \left\{\eta(\phi)\cosh\frac{t}{R} + \lambda(\phi)\sinh\frac{t}{R}\right\}\sin\theta\right] \ , \qquad (2.134)$$

$$k^2 = -\frac{\sqrt{1 - r^2/R^2}}{r} \left[\left\{C_1 \cosh\frac{t}{R} + C_2 \sinh\frac{t}{R}\right\}\sin\theta - \left\{\eta(\phi)\cosh\frac{t}{R} + \lambda(\phi)\sinh\frac{t}{R}\right\}\cos\theta\right] + C_3 \ , \qquad (2.135)$$

$$k^3 = \frac{\sqrt{1 - r^2/R^2}}{r\sin\theta} \left[\left\{\eta(\phi)_{,3}\cosh\frac{t}{R} + \lambda(\phi)_{,3}\sinh\frac{t}{R}\right\}\right] + R(\phi) . \qquad (2.136)$$

Differentiating Eq.(2.136) and comparing it with Eq.(2.102), we obtain

$$\frac{\sqrt{1-r^2/R^2}}{r}\left[\left\{\eta(\phi)_{,33}+\eta(\phi)\right\}\cosh\frac{t}{R}+\left\{\lambda(\phi)_{,33}+\lambda(\phi)\right\}\sinh\frac{t}{R}\right]$$
$$+\left[G(\phi)_{,33}+G(\phi)\right]\cos\theta+R(\phi)_{,3}\sin\theta=0,$$

which is satisfied if

$$\left.\begin{aligned}&\eta(\phi)_{,33}+\eta(\phi)=0\quad,\\&\lambda(\phi)_{,33}+\lambda(\phi)=0\quad,\\&G(\phi)_{,33}+G(\phi)=0=R(\phi)_{,3}\end{aligned}\right\}\tag{2.137}$$

These equations give

$$\left.\begin{aligned}&\eta(\phi)=C_3\cos\phi+C_4\sin\phi\quad,\\&\lambda(\phi)=C_5\cos\phi+C_6\sin\phi\quad,\\&G(\phi)=C_7\cos\phi+C_8\sin\phi\quad,\\&R(\phi)=C_9\;.\end{aligned}\right\}\tag{2.138}$$

Differentiating Eq.(2.133) with respect to t and comparing with Eq.(2.94) (using other results) gives that N is a constant C_{10}. Thus, relabeling the parameters the Killing vectors are

$$k^0=C_1+\frac{r/R}{\sqrt{1-r^2/R^2}}\left[\left[C_2\sinh\frac{t}{R}+C_3\cosh\frac{t}{R}\right]\cos\theta\right.$$
$$+\left\{\left[C_4\cos\phi+C_5\sin\phi\right]\sinh\frac{t}{R}+\left[C_6\cos\phi+C_7\sin\phi\right]\cosh\frac{t}{R}\right\}$$
$$\left.\times\sin\theta\right]\;,\tag{2.139a}$$

$$k^1=\sqrt{1-r^2/R^2}\left[\left[C_2\cosh\frac{t}{R}+C_3\sinh\frac{t}{R}\right]\cos\theta+\left\{(C_4\cos\phi+C_5\sin\phi)\cosh\frac{t}{R}\right.\right.$$
$$\left.\left.+(C_6\cos\phi+C_7\sin\phi)\sinh\frac{t}{R}\right\}\sin\theta\right]\;,\tag{2.139b}$$

$$\frac{\sqrt{1 - r^2/R^2}}{r} \left[\left\{\eta(\phi)_{,33} + \eta(\phi)\right\}\cosh \frac{t}{R} + \left\{\lambda(\phi)_{,33} + \lambda(\phi)\right\}\sinh \frac{t}{R}\right]$$

$$+ C_3\cos\theta + R(\phi)_{,3} = 0 ,$$

which is satisfied if

$$\left.\begin{aligned} &\eta(\phi)_{,33} + \eta(\phi) = 0 , \\ &\lambda(\phi)_{,33} + \lambda(\phi) = 0 , \\ &C_3 = 0 = R(\phi)_{,3} . \end{aligned}\right\} \qquad (2.137)$$

These equations give

$$\left.\begin{aligned} &\eta(\phi) = C_4\cos\phi + C_5\sin\phi , \\ &\lambda(\phi) = C_6\cos\phi + C_7\sin\phi , \\ &R = C_8 . \end{aligned}\right\} \qquad (2.138)$$

Differentiating Eq.(2.133) with respect to t and comparing with Eq.(2.94) (using other results) gives that N is a constant C_9. Thus, relabeling the parameters the Killing vectors are

$$k^0 = C_1 + \frac{r/R}{\sqrt{1 - r^2/R^2}} \left[\left(C_2\sinh \frac{t}{R} + C_3\cosh \frac{t}{R}\right)\cos\theta + \left\{\left(C_4\cos\phi + C_5\sin\phi\right)\sinh \frac{t}{R} + \left(C_6\cos\phi + C_7\sin\phi\right)\cosh \frac{t}{R}\right\} \times \sin\theta\right] , \qquad (2.139a)$$

$$k^1 = \sqrt{1 - r^2/R^2} \left[\left(C_2\cosh \frac{t}{R} + C_3\sinh \frac{t}{R}\right)\cos\theta + \left\{(C_4\cos\phi + C_5\sin\phi)\cosh \frac{t}{R} + (C_6\cos\phi + C_7\sin\phi)\sinh \frac{t}{R}\right\}\sin\theta\right] , \qquad (2.139b)$$

$$k^2 = -\frac{\sqrt{1-r^2/R^2}}{r}\left(\left[C_2\cosh\frac{t}{R} + C_3\sinh\frac{t}{R}\right]\sin\theta - \left\{(C_4\cos\phi + C_5\sin\phi)\cosh\frac{t}{R} + (C_6\cos\phi + C_7\sin\phi)\sinh\frac{t}{R}\right\}\cos\theta\right) + (C_8\cos\phi + C_9\sin\phi) \; . \quad (2.139c)$$

$$k^3 = C_{10} + \frac{\sqrt{1-r^2/R^2}}{r\sin\theta}\left\{(-C_4\sin\phi + C_5\cos\phi)\cosh\frac{t}{R} + (-C_6\sin\phi + C_7\cos\phi)\sinh\frac{t}{R}\right\} + \cot\theta(-C_8\sin\phi + C_9\cos\phi) . \quad (2.139d)$$

§2.6 A METRIC WITH SCHWARZSCHILD SYMMETRY

Choosing $\alpha = 0$ we must have $\beta > 0$ so that $e^{\nu(r)} > 0$. In this case Eqs.(2.26) and (2.27) give

$$e^{\nu(r)} = \beta r^2 \quad , \qquad (2.140)$$

$$g(t,\theta,\phi)_{,22} = 0 \quad . \qquad (2.141)$$

Here the space-time metric becomes

$$ds^2 = \beta r^2 dt^2 - \frac{1}{\beta r^2} dr^2 - r^2 d\theta^2 - r^2 \sin^2\theta d\phi^2 \; . \qquad (2.142)$$

Now Eq.(2.141) gives

$$g(t,\theta,\phi) = A(t,\phi) + \theta D(t,\phi) \; . \qquad (2.143)$$

Using this equation and Eq.(2.140) we can write the Killing equations for this metric as follows

$$k^0{}_{,0} = -k^1/r \quad , \qquad (2.144)$$

$$k^0{}_{,1} = \frac{1}{\beta^2 r^4} k^1{}_{,0} \quad , \qquad (2.145)$$

$$k^2 = -\frac{\sqrt{1-r^2/R^2}}{r}\left[\left\{C_2\cosh\frac{t}{R} + C_3\sinh\frac{t}{R}\right\}\sin\theta - \left\{(C_4\cos\phi + C_5\sin\phi)\cosh\frac{t}{R} + (C_6\cos\phi + C_7\sin\phi)\sinh\frac{t}{R}\right\}\cos\theta\right]. \quad (2.139c)$$

$$k^3 = C_8 + \frac{\sqrt{1-r^2/R^2}}{r\sin\theta}\left[(-C_4\sin\phi + C_5\cos\phi)\cosh\frac{t}{R} + (-C_6\sin\phi + C_7\cos\phi)\sinh\frac{t}{R}\right] . \quad (2.139d)$$

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$$k^0{}_{,2} = \frac{1}{\beta} k^2{}_{,0} \quad , \tag{2.146}$$

$$k^0{}_{,3} = \frac{\sin^2\theta}{\beta} k^3{}_{,0} \quad , \tag{2.147}$$

$$k^1 = \sqrt{\beta}\, r \left[A(t,\phi) + \theta D(t,\phi)\right] \quad , \tag{2.148}$$

$$k^2{}_{,1} = -\frac{D(t,\phi)}{\sqrt{\beta}\, r^3} \quad , \tag{2.149}$$

$$k^3{}_{,1} = -\frac{1}{\sqrt{\beta}\, r^3 \sin^2\theta}\left[A(t,\phi)_{,3} + \theta D(t,\phi)_{,3}\right] , \tag{2.150}$$

$$k^2{}_{,2} = -\sqrt{\beta}\left[A(t,\phi) + \theta D(t,\phi)\right] \quad , \tag{2.151}$$

$$k^2{}_{,3} = -\sin^2\theta k^3{}_{,2} \quad , \tag{2.152}$$

$$k^3{}_{,3} = -\sqrt{\beta}\left[A(t,\phi) + \theta D(t,\phi)\right] - \mathrm{Cot}\,\theta k^2 \; . \tag{2.153}$$

Integrating r in Eq.(2.149) gives

$$k^2 = \frac{D(t,\phi)}{2\sqrt{\beta}\, r^2} + E(t,\theta,\phi) \quad . \tag{2.154}$$

Differentiating this equation with respect to θ and comparing with Eq.(2.151) yields

$$E(t,\theta,\phi)_{,2} = -\sqrt{\beta}\left[A(t,\phi) + \theta D(t,\phi)\right] . \tag{2.155}$$

Integrating θ in this equation gives

$$E(t,\theta,\phi) = -\sqrt{\beta}\left[\theta A(t,\phi) + \frac{\theta^2}{2} D(t,\phi)\right] + F(t,\phi) \; . \tag{2.156}$$

Thus

$$k^2 = \frac{D(t,\phi)}{2\sqrt{\beta}\, r^2} - \sqrt{\beta}\left[\theta A(t,\phi) + \frac{\theta^2}{2} D(t,\phi)\right] + F(t,\phi). \tag{2.157}$$

We now integrate r in Eq.(2.150) which implies

$$k^3 = \frac{1}{2\sqrt{\beta}\ r^2 \sin^2\theta}\left[A(t,\phi)_{,3} + \theta D(t,\phi)_{,3}\right] + G(t,\theta,\phi) \ . \qquad (2.158)$$

Inserting Eqs.(2.157) and (2.158) in Eq.(2.152), it is clear that

$$A(t,\phi)_{,3} = 0 = D(t,\phi)_{,3} \ , \qquad (2.159)$$

$$G(t,\theta,\phi)_{,3} = -\frac{1}{\sin^2\theta} F(t,\phi)_{,3} \ . \qquad (2.160)$$

These equations imply that

$$\left.\begin{aligned} &A = H(t) \ , \ D = N(t), \\ &G(t,\theta,\phi) = \operatorname{Cot}\theta\ F(t,\phi)_{,3} + P(t,\phi) \ . \end{aligned}\right\} \qquad (2.161)$$

Thus, k^1, k^2, k^3 can be rewritten as

$$k^1 = \sqrt{\beta}\ r\left[H(t) + \theta N(t)\right] \ , \qquad (2.162)$$

$$k^2 = \frac{N(t)}{2\sqrt{\beta}r^2} - \sqrt{\beta}\left[\theta H(t) + \frac{\theta^2}{2} N(t)\right] + F(t,\phi) \ , \qquad (2.163)$$

$$k^3 = \operatorname{Cot}\theta F(t,\phi)_{,3} + P(t,\phi) \ . \qquad (2.164)$$

Using these results in Eq.(2.153) and comparing, imply

$$F(t,\phi) = \ell(t)\cos\phi + m(t)\sin\phi \ , \qquad (2.165)$$

$$P(t,\phi) = Q(t) \ , \qquad (2.166)$$

$$H(t) = 0 = N(t) \ . \qquad (2.167)$$

Thus, using these results in Eqs.(2.162) to (2.164), we obtain

$$k^1 = 0 \quad , \tag{2.168}$$

$$k^2 = \ell(t)\cos\phi + m(t)\sin\phi \quad , \tag{2.169}$$

$$k^3 = \left[-\ell(t)\sin\phi + m(t)\cos\phi\right]\mathrm{Cot}\,\theta + Q(t) \quad . \tag{2.170}$$

We now use Eq.(2.168) in Eqs.(2.144) and (2.145) to get

$$k^0 = R(\theta,\phi) \ . \tag{2.171}$$

Differentiating Eqs.(2.169) and (2.171) with respect to t and θ respectively and using Eq.(2.146) gives

$$R(\theta,\phi)_{,2} = \frac{1}{\beta}\left[\ell(t)_{,0}\cos\phi + m(t)_{,0}\sin\phi\right] \ . \tag{2.172}$$

Integrating θ in this equation yields

$$R(\theta,\phi) = \frac{\theta}{\beta}\left[\ell(t)_{,0}\cos\phi + m(t)_{,0}\sin\phi\right] + u(\phi) \ .$$

Thus, Eq.(2.171) becomes

$$k^0 = \frac{\theta}{\beta}\left[\ell(t)_{,0}\cos\phi + m(t)_{,0}\sin\phi\right] + u(\phi) \ . \tag{2.173}$$

It is easily checked that Eqs.(2.170) and (2.173) along with Eq.(2.147) imply

$$\left.\begin{array}{l} u = C_1 \ , \ \ell = C_2 \\ m = C_3 \ , \ Q = C_4 \ . \end{array}\right\} \tag{2.174}$$

Using these results, the Killing vectors for the metric given by

Eq.(2.142) become

$$\left.\begin{aligned} k^0 &= C_1 \\ k^1 &= 0 \\ k^2 &= C_2\cos\phi + C_3\sin\phi \\ k^3 &= C_4 + (-C_2\sin\phi + C_3\cos\phi)\mathrm{Cot}\,\theta \end{aligned}\right\} \quad (2.175)$$

Notice that the Schwarzschild and Reissner-Nordstrom geometries possess the same four Killing vectors.

This analysis of symmetry reduction shows the reduction of the number of independent Killing vectors from Minkowski to a metric with symmetries corresponding to the Schwarzschild (or Reissner-Nordstrom) metric. In the De-Sitter metric the number of Killing vectors reduce to eight. The metric which results from the choice $\alpha = 0$ admits of four Killing vectors.

The K.N. space-time is more general than the Schwarzchild (or Reissner-Nordstrom) space-times. Here the spherical symmetry reduces to axial symmetry. This reduction of symmetry leads to a further reduction of Killing vectors in the K.N. metric down to only two.

§2.7 CONCLUSION

It is observed that by reducing symmetry we can obtain the Killing vectors for different geometries. For this reduction in symmetry (for the particular form of the metric considered here) from Minkowski to the Schwarzschild metric the number of Killing vectors goes down by the sequence $10 \longrightarrow 4$. For a general spherically symmetric static metric

$$ds^2 = e^{\nu(r)}dt^2 - e^{\lambda(r)}dr^2 - r^2d\theta^2 - r^2\sin^2\theta d\phi^2 \ , \qquad (2.176)$$

the symmetry reduction has still to be worked out. Clearly it will contain the Friedmann case of 6 Killing vectors and cannot go lower than 4 (for time-like and spherical symmetry to hold).

The metric with local Schwarzschild symmetry satisfies the Einstein field equation with cosmological constant (with $\Lambda = 3$) and stress-energy tensor

$$T_{\mu\nu} = \frac{1}{8\pi}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -r^{-4} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad (2.177)$$

We shall be using the procedure detailed above to workout the conformal Killing vectors in various relevant situations.

§2.7 CONCLUSION

It is observed that by reducing symmetry we can obtain the Killing vectors for different geometries. For this reduction in symmetry (for the particular form of the metric considered here) from Minkowski to the Schwarzschild metric the number of Killing vectors goes down by the sequence $10 \to 8 \to 4$. For a general spherically symmetric static metric

$$ds^2 = e^{\nu(r)}dt^2 - e^{\lambda(r)}dr^2 - r^2d\theta^2 - r^2\sin^2\theta d\phi^2 , \qquad (2.176)$$

the symmetry reduction has still to be worked out. Clearly it will contain the Friedmann case of 6 Killing vectors and cannot go lower than 4 (for time-like and spherical symmetry to hold).

The metric with local Schwarzschild symmetry satisfies the Einstein field equation with cosmological constant (with $\Lambda = 3$) and stress-energy tensor

$$T_{\mu\nu} = \frac{1}{8\pi}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -r^{-4} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad (2.177)$$

We shall be using the procedure detailed above to workout the conformal Killing vectors in various relevant situations.

Chapter 3

A CONFORMAL EXTENSION OF PSEUDO-NEWTONIAN FORCES AND POTENTIALS

Classically, energy is defined in terms of a time translational invariance of the Hamiltonian[1,22]. Relativistically, in a space-time which admits a time-like Killing vector, the energy of a test particle is defined by

$$E = \underline{k}\cdot\underline{p} \ , \tag{3.1}$$

where $\underline{k}$ is a time-like Killing vector and $\underline{p}$ the four-momentum of the test particle. Generally, we do not expect a time-like Killing vector to exist in a cosmological context. However, there may exist a conformal time-like Killing vector. Therefore, in these space-times (admitting conformal time-like Killing vectors) only a conformal analogue of the usual energy can be defined

$$\tilde{E} = \underline{\tilde{k}}\cdot\underline{\tilde{p}} \ , \tag{3.2}$$

where, $\underline{\tilde{k}}$, is the conformal time-like Killing vector.

The earlier programme[2,3,4,8] of re-introducing the concept of force in Relativity was implemented in space-times whose metric is a particular generalisation of Carter's circular metrics[13](still retaining the requirement of a time-like Killing vector). This chapter deals with the conformal extension of the earlier programme[3,4,8]. It is shown that a space-time whose metric is the corresponding conformal generalisation of Carter's circular metrics (requiring a conformal time-like Killing vector) can admit of a conformal analogue[14] of ψN-force[3] and ψN-potential[4,8]. These space-times have been called conformal pseudo-Newtonian (cψN) space-times. The conformal analogue of ψN-force and potential has been called the cψN-force and cψN-potential.

§3.1 CONFORMAL CIRCULAR MATRICS

To be able to deal with space-times which admit of a conformal time-like isometry, we need to extend the idea of Carter's circular metrics[13]. These are the metrics, as mentioned earlier, having at least one time-like and one space-like isometry and can be expressed in the block diagonal form such as given by Eq.(1.3).

A conformal Killing vector, $\underline{\tilde{k}}$, satisfies the conformal Killing equation

$$\tilde{k}_{(a;b)} = \sigma g_{ab} , \qquad (3.3)$$

which can be written in the contravariant form

$$g_{ab,c}\,\tilde{k}^{c} + g_{ac}\,\tilde{k}^{c}{}_{,b} + g_{bc}\,\tilde{k}^{c}{}_{,a} = 2\sigma\, g_{ab} \ . \qquad (3.4)$$

The conformal transformation $\tilde{g} = \Omega^2 g$ reduces the above conformal Killing equation to give the Killing equation

$$\tilde{g}_{ab,c}\,\tilde{k}^{c} + \tilde{g}_{ac}\,\tilde{k}^{c}{}_{,b} + \tilde{g}_{bc}\,\tilde{k}^{c}{}_{,a} = 0 \ . \qquad (3.5)$$

CMs can be extended to space-times admitting only conformal time-like Killing vectors by requiring that the conformally transformed metric, $\tilde{g}$, be circular (or a GCM). Such a metric will be called a conformal circular metric (CCM) or conformal GCM (CGCM). Such a metric has the following interesting properties.

Theorem

In a space-time whose metric is a CGCM, there exists a conformal time-like Killing vector which is a function of time only. The spatial components of the conformally transformed metric are time independent.

Proof

A time-like solution of Eq.(3.5) can be chosen to have only the time component

$$\tilde{k}^{a} = f(t,x^{i})\,\delta^{a}_{0} \ , \qquad (3.6)$$

where t is time in these coordinates and i = 1,2,3. Eqs.(3.5) and (3.6) give

$$\left.\begin{aligned} &\tilde{g}_{00,0}f + 2\tilde{g}_{00}f_{,0} = 0 ,\\ &\tilde{g}_{0i,0}f + 2\tilde{g}_{0(i}f_{,0)} = 0 ,\\ &\tilde{g}_{ij,0}f + 2\tilde{g}_{0(i}f_{;j)} = 0 .\end{aligned}\right\} \qquad (3.7)$$

On solving Eqs.(3.7), we get

$$\tilde{g}_{00} = A/f^2 , \qquad (3.8)$$

$$\tilde{g}_{0i} = \{Aa_i + B_i\}/f , \qquad (3.9)$$

$$\tilde{g}_{ij} = 2\left\{Aa_{(i}a_{j)} - \int a_{(i}b_{j)}\,dt - B_{(i}a_{j)}\right\} + C_{ij} , \qquad (3.10)$$

where A, B_i and C_{ij} can depend on space coordinates only and

$$a_i = -\int (1/f)_{,i}\,dt,\ b_i = a_{i,0} . \qquad (3.11)$$

Since the metric is a CGCM the right hand side of Eq.(3.9) is zero. Also, since a_i is a function of time and A and B_i are not, a_i and B_i must be separately zero. Now from Eq.(3.11), since $a_i = 0$, $f_{,i} = 0$. Hence f is a function of time only. This completes the proof of the first part of the theorem. Using the fact that $B_i = 0 = a_i$ in Eq.(3.10), we see that

$$\tilde{g}_{ij} = C_{ij} . \qquad (3.12)$$

Now $\tilde{g}_{00}$ can be recast in CGCM form by rescalling the time coordinate by

$$\tau = \int dt/f \ , \tag{3.13}$$

to give the zero-zero metric component equal to A, which is a function of spatial coordinates only.

§3.2 THE CONFORMAL EXTENSION OF TIDAL FORCES

In a space-time admitting a conformal time-like Killing vector the accelerometer will register the effect of expansion as seen by a deflection of the needle. However, we can treat this deflection as a time-dependent shift of the zero-point of the accelerometer. Thus, even in the expanding Universe we can have a well-defined zero of the tidal force.

The tidal force given by Eq.(1.2) has the conformal analogue

$$\tilde{F}_T{}^a = -\tilde{R}^a{}_{bcd} \ \tilde{k}^b \ell^c \tilde{k}^d \ , \tag{3.14}$$

where $\tilde{k}^a$ is the conformal time-like Killing vector and $\tilde{R}^a{}_{bcd}$ is the Riemann Christoffel tensor defined by using the conformally transformed metric. The conformal factor to be used must not disturb the spatial symmetries. Thus, it must be purely time dependent. It is clear from Eq.(3.9) that it must, therefore, be simply f^{-2}, i.e., the conformal time-like Killing vector discussed earlier has as its only component the inverse square root of the conformal factor. To be able to deal with general space-times, we still use the Riemann-normal coordinates. In these coordinates the expression for the conformally transformed Riemann Christoffel curvature tensor reduces to

$$\tilde{R}^a{}_{bcd} = \tilde{\Gamma}^a{}_{bd,c} - \tilde{\Gamma}^a{}_{bc,d} , \tag{3.15}$$

where $\tilde{\Gamma}^a{}_{bc}$ are the Christoffel symbols defined by the conformally transformed metric. Taking $\tilde{k}^a$ to define the time direction, Eq.(3.14) becomes

$$\tilde{F}_T{}^a = -\tilde{R}^a{}_{0b0}\, \tilde{k}^0 \ell^b \tilde{k}^0 , \tag{3.16}$$

where ℓ^b is the separation vector. Eqs.(3.6) and (3.16) give

$$\tilde{F}_T{}^a = -f^2 \tilde{R}^a{}_{0b0}\, \ell^b . \tag{3.17}$$

Consider an observer (equipped with the accelerometer shown in Figure 1) falling freely towards a gravitating source. He can detect the presence of the gravitating source by the dial effect of the accelerometer. The maximal effect of the tidal force is obtained by adjusting the direction of the accelerometer. Then the conformal analogue of the tidal force due to the source is obtained by solving the eigen-value problem

$$(\tilde{R}^a{}_{0b0} + \delta^a{}_b\, \lambda/f^2)\ell^b = 0 . \tag{3.18}$$

In a local Lorentz frame $\ell^r = 0$ and $\ell^i \neq 0$ in general. The eigen-value problem given by Eq.(3.18) reduces to a three-dimensional eigen-value problem

$$(\tilde{R}^i{}_{0j0} + \delta^i{}_j \ \lambda/f^2)\ell^j = 0 \ , \tag{3.19}$$

which is well determined only if

$$\frac{1}{2}\, \tilde{g}^{0i}(\tilde{g}_{00,ij} + \tilde{g}_{ij,00})\ell^j = 0 \ . \tag{3.20}$$

In a CGCM Eq.(3.20) is satisfied automatically. It may also be satisfied for non-CGCM's if $(\tilde{g}_{00,ij} + \tilde{g}_{ij,00})$ is orthogonal to the separation vector. Thus, we get an additional constraint. A space-time for which Eq.(3.20) is satisfied is what we have called a cψN-space-time.

In CGCM1, the eigen-value determined by Eq.(3.18) is given by

$$\lambda = (f^2/2)\tilde{g}^{11}\tilde{g}_{00,11} \ . \tag{3.21}$$

In CGCM2 one has to deal with a cubic equation in λ. In principle this problem can be solved as discussed in chapter one. However, we will not present the solution here as it does not provide any further insights. If the CGCM2 is diagonal, as in the case of Friedmann cosmologies, the problem becomes much simpler, as will be seen later. In the case of a CCM, the conformal analogue of tidal force components become

$$\left.\begin{aligned}\tilde{F}^1_T &= (f^2/2)\tilde{g}^{11}\tilde{g}_{00,1i}\ell^i \ , \\ \tilde{F}^2_T &= (f^2/2)\tilde{g}^{22}\tilde{g}_{00,2i}\ell^i \ .\end{aligned}\right\} \qquad (3.22)$$

From Eq.(3.19) maximum and minimum tidal forces are given by

$$\lambda_\pm = (f^2/4)\left[(\tilde{g}^{11}\tilde{g}_{00,11} + \tilde{g}^{22}\tilde{g}_{00,22}) \pm \left\{(\tilde{g}^{11}\tilde{g}_{00,11} - \tilde{g}^{22}\tilde{g}_{00,22})^2 + 4\tilde{g}^{11}\tilde{g}^{22}(\tilde{g}_{00,12})^2\right\}^{\frac{1}{2}}\right]. \qquad (3.23)$$

The corresponding eigen-vectors are determined by

$$\ell^1_\pm/\ell^2_\pm = f^2\tilde{g}^{11}\tilde{g}_{00,12}/(2\lambda_\pm - \tilde{g}^{11}\tilde{g}_{00,11}) \ . \qquad (3.24)$$

The discriminant in Eq.(3.23) is always positive outside the event-horizon of the gravitating source. In the case that there is no event-horizon the discriminant is always positive. Thus, $\lambda_+ > \lambda_-$ always. λ_+ and λ_- are equal if and only if the discriminant is zero. If the length of the accelerometer is L, then

$$\tilde{g}_{11}(\ell^1)^2 + \tilde{g}_{22}(\ell^2)^2 = -L^2 \ . \qquad (3.25)$$

Eqs.(3.24) and (3.25) together determine ℓ^1 and ℓ^2 :

$$\left.\begin{aligned}\ell^1 &= f^2L(-\tilde{g}^{22})^{\frac{1}{2}}\ \tilde{g}^{11}\tilde{g}_{00,12}\left\{(2\lambda - f^2\tilde{g}^{11}\tilde{g}_{00,11})^2 + f^4\tilde{g}^{11}\tilde{g}^{22}(\tilde{g}_{00,12})^2\right\}^{-\frac{1}{2}} \ , \\ \ell^2 &= L(-\tilde{g}^{22})^{\frac{1}{2}}\ (2\lambda - f^2\tilde{g}^{11}\tilde{g}_{00,11})\left\{(2\lambda - f^2\tilde{g}^{11}\tilde{g}_{00,11})^2 + f^4\tilde{g}^{11}\tilde{g}^{22}(\tilde{g}_{00,12})^2\right\}^{-\frac{1}{2}} \ .\end{aligned}\right\} \qquad (3.26)$$

Thus, we see that in a CCM, the conformal analogue of the tidal force is given completely by Eqs.(3.23) and (3.26) since

$$\tilde{F}_T{}^i = \lambda \ell^i \ . \tag{3.27}$$

Thus, λL is the magnitude of the conformal analogue of the tidal force.

§3.3 THE cψN-FORCE AND POTENTIAL

It is easy to see, from Eq.(3.17), that in a CCM or CGCM the conformal analogue of the tidal force is given by

$$\tilde{F}_T{}^i = -f^2 \tilde{\Gamma}^i{}_{00,j} \ell^j \ . \tag{3.28}$$

Using the geodesic equation and removing the local Lorentz factor, the expression for the conformal analogue of tidal force becomes,

$$\tilde{F}_T{}^i = f^2 \acute{x}^i{}_{,j} \ell^j \ , \tag{3.29}$$

where $\acute{x}^i = d^2x^i/ds^2$. Thus, it turns out that, as in the case of ψN-forces, the conformal analogue of tidal force in a cψN-space-time is proportional to the gradient along the separation vector of the second derivative of the position vector. However, here it has to be multiplied by the inverse of the conformal factor. The quantity of which the conformal analogue of the tidal force is the gradient is the desired cψN-force

$$\tilde{F}^i = f^2(\acute{\acute{x}}^i - \acute{\acute{x}}^i{}_M) \ , \tag{3.30}$$

where $\acute{\acute{x}}^i{}_M$ is evaluated in the Minkowski space-time.

The $\acute{\acute{x}}^i{}_M$ was introduced earlier[3,4] so that there should be no force in Minkowski space. Here we require that the cψN-force be zero when the space-time is conformally flat. Using the geodesic equation we then obtain, from Eq.(3.30)

$$\tilde{F}^i = f^2(\Gamma_M{}^i{}_{ab} - \tilde{\Gamma}^i{}_{ab})\acute{x}^a\acute{x}^b \ , \tag{3.31}$$

where $\Gamma_M{}^i{}_{ab}$ are the Minkowski space Christoffel symbols evaluated in the relevant coordinates. Thus, the expression for the cψN-force turns out to be the same as the expression for the ψN-force with Γ replaced by $\tilde{\Gamma}$ and the total expression multiplied by the inverse of the conformal factor.

Classically the Newtonian gravitational force is the gradient of a potential. It was found that [3,4] (reviewed in first chapter) in a GCM the ψN-force is the gradient of a potential. This idea is extended to the space-times admitting only a conformal time-like Killing vector. Eq.(3.31) gives

$$\tilde{F}_i = (f^2/2)\ \tilde{g}_{00,i} \ . \tag{3.32}$$

Since the conformal factor is given by f^{-2}, Eq.(3.32) gives

$$\tilde{F}_i = g_{00,i}/2 \ . \tag{3.33}$$

Thus, we can define the quantity of which the cψN-force is the gradient, the cψN-potential, $\tilde{\phi}$, by

$$\tilde{\phi} = (\tilde{k}\cdot\tilde{k} - 1)/2 \ , \tag{3.34}$$

since $g_{00} = \tilde{k}\cdot\tilde{k}$. Notice that $\tilde{\phi}$ is defined to be zero in a flat space-time, as usual. Further it should be borne in mind that we have been able to construct the cψN-force and potential only in CCM's or CGCM's. The cψN-force can be defined for a non-CCM (or non-CGCM) if Eq.(3.20) holds. However, this force will not be the gradient of a potential, in general, unless the space-time is a CCM (or CGCM) as there is an extra contribution from the space part of the metric tensor. The relativistic potential is then, the zero-zero component of the metric tensor.

§3.4 APPLICATION TO THE FRIEDMANN GEOMETRY

As an example we consider the three Friedmann cosmological models. The closed Friedmann model of the Universe (K = -1) has the metric,

$$ds^2 = dt^2 - a^2(t)\left\{d\chi^2 + \sin^2\chi(d\theta^2 + \sin^2\theta d\phi^2)\right\} \ . \tag{3.35}$$

A time-like Killing vector exists if there is a non-zero solution of the Killing equations:

$$k^0{}_{,0} = 0 \ . \tag{3.36}$$

$$k^0{}_{,1} - a^2(t)k^1{}_{,0} = 0 \ . \tag{3.37}$$

$$a(t)_{,0}\, k^0 + a(t)\, k^1{}_{,1} = 0 \ . \tag{3.38}$$

Differentiating Eqs.(3.37) and (3.38) with respect to χ and t respectively and comparing, gives

$$k^0{}_{,11} = \left\{ a^2(t)_{,0} - a(t)a(t)_{,00} \right\} k^0 \ , \tag{3.39}$$

which gives

$$k^0 = A(\theta,\phi)e^{\alpha\chi} + B(\theta,\phi)e^{-\alpha\chi} \ , \tag{3.40}$$

where

$$\alpha^2 = a^2(t)_{,0} - a(t)a(t)_{,00} > 0 \ , \tag{3.41}$$

Now differentiating Eq.(3.40) with respect to t, bearing in mind Eq.(3.41), Eq.(3.36) requires that $k^0 = 0$. Thus, the closed Friedmann model does not possess any time-like Killing vector. This fact could have been anticipated by noticing that the density of the Friedmann Universe changes with time. Hence no time-like isometry should have been expected. The full set of Killing vectors for the closed Friedmann cosmological model is given in Appendix I.

As no time-like Killing vector exists, we try to obtain a conformal time-like Killing vector satisfying Eqs.(3.5) for the metric given by Eq.(3.35). Now from the theorem in section 3.1

$$\Omega^2(t) = a^{-2}(t) \ , \tag{3.42}$$

and

$$\tilde{k}^a = a(t)\ \delta^a{}_0 , \tag{3.43}$$

as shown in Appendix II.

The Friedmann models are of the type CGCM2. The cψN-force given by Eq.(3.31), in the case K = -1, has three components:

$$\left.\begin{aligned} \tilde{F}^1 &= f^2\left\{(\Gamma_M{}^1{}_{22} - \tilde{\Gamma}^1{}_{22})(\theta')^2 + (\Gamma_M{}^1{}_{33} - \tilde{\Gamma}^1{}_{33})(\phi')^2\right\}, \\ \tilde{F}^2 &= f^2\left\{2(\Gamma_M{}^2{}_{12} - \tilde{\Gamma}^2{}_{12})(\chi'\theta') + (\Gamma_M{}^2{}_{33} - \tilde{\Gamma}^2{}_{33})(\phi)^2\right\} , \\ \tilde{F}^3 &= 2f^2\left\{(\Gamma_M{}^3{}_{13} - \tilde{\Gamma}^3{}_{13})(\chi'\phi') + (\Gamma_M{}^3{}_{23} - \tilde{\Gamma}^3{}_{23})(\theta'\phi')\right\}. \end{aligned}\right\} \tag{3.44}$$

Now the geodesic equations

$$\chi^{a''} + \tilde{\Gamma}^a{}_{bc}\chi^{b'}\chi^{c'} = 0 , \tag{3.45}$$

become

$$\left.\begin{aligned} & t'' + a(t)_{,0}/a(t)\ (t')^2 = 0 , \\ & \chi'' - \sin\chi\cos\chi(\theta')^2 - \sin\chi\cos\chi\sin^2\theta(\phi')^2 = 0 , \\ & \theta'' + 2\mathrm{Cot}\,\chi(\chi'\theta') - \sin\theta\cos\theta(\phi')^2 = 0 , \\ & \phi'' + 2\mathrm{Cot}\,\chi(\chi'\phi') + 2\mathrm{Cot}\,\theta(\theta'\phi') = 0 . \end{aligned}\right\} \tag{3.46}$$

where

$$\chi^{b'} = d\chi^b/ds \text{ and } a(t)_{,0} = da(t)/dt = a(t)'/t' . \tag{3.47}$$

Taking $\theta' = \phi' = 0$, gives

$$\left.\begin{aligned} t' &= Ca(t) \ , \\ \chi' &= D \ . \end{aligned}\right\} \qquad (3.48)$$

where C and D are arbitrary constants. Eqs.(3.44) and (3.48) yield the cψN-force equal to zero as would be the natural requirement. In other words there is no cψN-force acting on a particle at rest in the closed Friedmann Universe. The cψN-potential given by Eq.(3.34), for the closed Friedmann model, is also zero (taking $\alpha_0 = 1$) as required.

In the open and flat Friedmann models (K = 1 and 0) we replace $\sin\chi$ in Eq.(3.35) by $\sinh\chi$ and χ, respectively. Again there donot exist any time-like Killing vectors. The conformally transformed metrics for these cosmological models, with the conformal factor given by Eq.(3.42), possess conformal time-like Killing vectors given by Eq.(3.43). The full set of Killing vectors are given in Appendix III. It can be easily verified that here again the cψN-force and potential are zero.

Chapter 4

A GRAVITATING SOURCE IN AN EVOLVING UNIVERSE

The cψN-formalism, applied to the Friedmann cosmological models, gives the trivial result that, modulo the rescaling, particles in a Friedmann Universe experience no force. In this chapter a cosmological model is constructed which admits of a conformal time-like Killing vector. It is shown that a test particle in this cosmological model simultaneously experiences a gravitational force and a cosmological expansion. Such a situation does not exist in the models available in the literature, like the Einstein-straus model[1,15].

§4.1 THE METRIC ANSATZ

Standard cosmological models evolve with time but have no gravitational sources present. On the other hand, the more manageable metrics which represent gravitational sources are static. A "marriage" of the two types of metrics has been managed by cutting a region out of the Friedmann Universe and replacing it by the Schwarzschild exterior metric[1]. At the boundary between the two regions a matching condition has to be met. Since the Schwarzschild metric does not change with time the matching condition is trivial in the coordinates used in that region. The metric of the boundary, in the coordinates relevant for the Friedmann region, is

$$ds^2 = a^2(\eta)\left[d\eta^2 - \sin^2\chi_0(d\theta^2 + \sin^2\theta d\phi^2)\right] , \qquad (4.1)$$

where η is the cosmological time parameter and χ_0 the hyper-spherical angle locating the boundary.

In this cosmological model each region has its own geometry and neither really influences the other. The boundary between the regions does appear to expand according to the Friedmann based observer, but this expansion is no different from the expansion of any other hypersurface. A test particle in one region experiences only the geometry of that region and is not effected by the other region. Thus, the "marriage" does not appear to be very fruitful. Admittedly a test particle could go from one region to the other, but even then it would only experience one of the regions at any instant. There would be no way to consider a test particle experiencing a gravitational force and an expansion of the Universe simultaneously. This is an unsatisfactory model in that it does not permit an interplay of the effects in any significant way. Here we present an alternative way of constructing a cosmological model satisfactory for this purpose — though it has weaknesses of its own, of course.

To be able to deal with the effects of both, time-evolution and a gravitational source, we construct ("by hand" as it were) a new metric. The spatial part of this metric is the same as the Schwarzschild metric, but multiplied by a time dependent scale factor. The time component is the usual Schwarzschild metric time component

$$ds^2 = e^{\nu(r)}dt^2 - a^2(t)\left[e^{-\nu(r)}dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)\right]. \tag{4.2}$$

This metric admits of a conformal time-like Killing vector. The interesting feature (to be seen later) of this metric is that a test particle in this model Universe experiences the conformal analogue of the ψN-force.

§4.2 THE STRESS-ENERGY TENSOR

The method adopted here is to use the metric given by Eq.(4.2) with

$$e^{\nu} = 1 - 2m/r , \tag{4.3}$$

and use the Einstein field equations

$$R_{ab} - (\tfrac{1}{2})Rg_{ab} = 8\pi T_{ab} , \tag{4.4}$$

to define the stress-energy tensor. It is realised that this procedure is not generally adopted because any metric would automatically yield some stress-energy tensor. Thus, the procedure is arbitrary and will, generally, yield a non-physical stress-energy tensor. The justification for it is that we can mimic the features of the Friedmann cosmological model in a situation where we have an inhomogeniety and, in fact, a gravitational source.

The surviving Christoffel symbols for the metric given by Eq.(4.2) are listed in Apprndix IV. Using these Christoffel symbols the expression for the Ricci tensor yields

$$R_{00} = \frac{1}{2a^2}\left\{\nu''(r) + \nu'^2(r) + 2\nu'(r)/r\right\}e^{2\nu(r)} - 3a_{,00}/a, \quad (4.5)$$

$$R_{11} = \left\{aa_{,00} + 2(a_{,0})^2\right\}e^{-2\nu(r)} - \left\{\nu''(r) + \nu'^2(r) + \frac{2\nu'(r)}{r}\right\}/2 , \quad (4.6)$$

$$R_{22} = r^2\left\{aa_{,00} + 2(a_{,0})^2\right\}e^{-\nu(r)} + \left[-\left\{1 + r\nu'(r)\right\}e^{\nu(r)} + 1\right], \quad (4.7)$$

$$R_{33} = R_{22}\sin^2\theta , \quad (4.8)$$

$$R_{01} = \frac{a_{,0}}{a}\,\nu'(r) . \quad (4.9)$$

All other R_{ij} are zero. Incorporating the value of $e^{\nu(r)}$ from Eq.(4.3), the above equations reduce to

$$R_{00} = -3a_{,00}/a , \quad (4.10)$$

$$R_{11} = \frac{1}{a}\left\{a^2a_{,0}\right\}_{,0} e^{-2\nu} , \quad (4.11)$$

$$R_{22} = \frac{r^2}{a}\left\{a^2 a_{,0}\right\}_{,0} e^{-\nu(r)} \,, \tag{4.12}$$

$$R_{33} = R_{22}\sin^2\theta \,, \tag{4.13}$$

$$R_{01} = \frac{a_{,0}}{a}\,\nu'(r) \,. \tag{4.14}$$

In this case the Ricci scalar is

$$R = -6\,\frac{\left[aa_{,00} + (a_{,0})^2\right] e^{-\nu(r)}}{a^2} \,. \tag{4.15}$$

In view of Eqs.(4.4), (4.10) to (4.15) the non-vanishing components of the stress-energy tensor are

$$\left.\begin{aligned}
T^{00} &= \frac{(3/8\pi)(a_{,0}/a)^2}{(1-2m/r)^2} = A \text{ (say)}, \\
T^{11} &= -\left\{\sqrt{a}\left[\sqrt{a}\; a_{,0}\right]_{,0}\right\}/4\pi a^4 \,, \\
T^{22} &= T^{11}/(r^2 - 2mr) \,, \\
T^{33} &= T^{22}/\sin^2\theta \,, \\
T^{01} &= \frac{(-m/8\pi)(a_{,0}/a^3)}{(r^2-2mr)} = B \text{ (say)} \,.
\end{aligned}\right\} \tag{4.16}$$

§4.3 EVOLUTION OF THE UNIVERSE

Since the Einstein field equations have been used to define the stress-energy tensor, they cannot be used to derive the time evolution of the cosmological model. Given the freedom of choice of the time evolution we could choose to get rid of some of the components of the stress-energy tensor. The natural choice would have been to eliminate T^{01}, but this would also remove the other components. A convenient choice is to set $T^{11} = 0$ (whence $T^{22} = T^{33} = 0$ also) by taking

$$\left(\sqrt{a}\; a_{,0}\right)_{,0} = 0 \; . \tag{4.17}$$

Integrating Eq.(4.17), we obtain

$$a(t) = C(t + b)^{2/3} \; , \tag{4.18}$$

where b and C are the constants of integration. By appropriate choice of the origin of time we can set b = 0 to obtain the time evolution of the flat Friedmann model

$$a(t) = C\; t^{2/3} \; . \tag{4.19}$$

We can also consider the choice that Eq.(4.18) is replaced by

$$\left(\sqrt{a}\; a_{,0}\right)_{,0} = \xi \quad , \tag{4.20}$$

where ξ is a positive or negative constant. Then, with the appropriate choice of the origin of time, integrating Eq.(4.20) yields

$$a(t) = C\; t^{2/3}(1 + \zeta t)^{2/3} \; , \tag{4.21}$$

where ζ has the same sign as ξ. Clearly, if we have $\zeta < 0$, we get a Universe which expands faster than the flat Friedmann. Thus, the choice of ζ here corresponds to the choice of K in the Friedmann models, giving a Universe of finite or infinite duration and a time evolution sufficiently similar to the Friedmann models. It is apparent from this discussion that generally we will have a time evolution analogous to the Friedmann models, which would be exactly the "flat" case if the diagonal stresses are chosen to be zero. For convenience we will restrict our attention mainly to the "flat" case.

§4.4 DIAGONALISATION OF THE STRESS-ENERGY TENSOR

Let us consider the stress-energy tensor we are left with in Eq.(4.16). It has the zero-zero component A, the one-one component zero and the zero-one component B. Thus, there is an energy and momentum with no stresses. In the appropriate frame there should be no momentum. We can convert to a different set of coordinates, "comoving" in some sense, in which there would be no momentum, i.e., the off-diagonal terms would be zero.

To find a matrix which diagonalises the stress-energy tensor consider the characteristic equation

$$\begin{vmatrix} A-\lambda & B \\ B & -\lambda \end{vmatrix} = 0 \ , \qquad (4.22)$$

which gives

$$\lambda^2 - A\lambda - B^2 = 0 \ . \tag{4.23}$$

The eigen-values corresponding to the above equation are

$$\lambda_\pm = A(1 \pm \sqrt{1 + 4B^2/A^2})/2 \ . \tag{4.24}$$

The eigen-vectors corresponding to λ_+ and λ_- can be obtained by solving

$$\begin{pmatrix} A-\lambda & B \\ B & -\lambda \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = 0 \ . \tag{4.25}$$

This equation readily yields the eigen-vectors given by

$$\underline{X} = \begin{pmatrix} 1 \\ B/\lambda_+ \end{pmatrix} \quad \text{and} \quad \underline{Y} = \begin{pmatrix} 1 \\ B/\lambda_- \end{pmatrix} \ . \tag{4.26}$$

Normalizing these vectors gives the matrix which diagonalises the stress-energy tensor

$$\begin{pmatrix} \dfrac{1}{\sqrt{1 + B^2/\lambda_+^2}} & -\dfrac{1}{\sqrt{1 + B^2/\lambda_-^2}} \\ \dfrac{1}{\sqrt{1 + \lambda_+^2/B^2}} & \dfrac{1}{\sqrt{1 + \lambda_-^2/B^2}} \end{pmatrix} , \tag{4.27}$$

where $\lambda_\pm$ is given by Eq.(4.24).

The above coordinate transformation leads to a new set of coordinates (τ, R) instead of (t, r) given by the transformation matrix (4.27). Inverting it we obtain

$$\begin{pmatrix} dt \\ \\ dr \end{pmatrix} = \begin{pmatrix} \dfrac{1}{\sqrt{1 + B^2/\lambda_+^2}} & \dfrac{1}{\sqrt{1 + \lambda_+^2/B^2}} \\ \\ -\dfrac{1}{\sqrt{1 + B^2/\lambda_-^2}} & \dfrac{1}{\sqrt{1 + \lambda_-^2/B^2}} \end{pmatrix} \begin{pmatrix} d\tau \\ \\ dR \end{pmatrix} . \qquad (4.28)$$

The new coordinates are therefore related to the old coordinates by the transformation equation

$$\left. \begin{aligned} dt &= \frac{d\tau}{\sqrt{1 + B^2/\lambda_+^2}} + \frac{dR}{\sqrt{1 + \lambda_+^2/B^2}} , \\ dr &= \frac{-d\tau}{\sqrt{1 + B^2/\lambda_-^2}} + \frac{dR}{\sqrt{1 + \lambda_-^2/B^2}} . \end{aligned} \right\} \qquad (4.29)$$

In view of Eq.(4.19) and the above transformation equations, the metric, Eq.(4.2), in the new coordinates becomes

$$d\hat{s}^2 = \alpha d\tau^2 - \beta dR^2 + 2\gamma d\tau dR - a^2 r^2 (d\theta^2 + \sin^2\theta d\phi^2) , \qquad (4.30)$$

where

$$\left. \begin{aligned} \alpha &= \frac{(1 - 2m/r)}{(1 + B^2/\lambda_+^2)} - \frac{C^2 t^{4/3}/(1 - 2m/r)}{(1 + B^2/\lambda_-^2)} , \\ \beta &= \frac{C^2 t^{4/3}/(1 - 2m/r)}{(1 + \lambda_-^2/B^2)} - \frac{(1 - 2m/r)}{(1 + \lambda_+^2/B^2)} , \\ \gamma &= \frac{C^2 t^{4/3}/(1 - 2m/r)}{\sqrt{2 + B^2/\lambda_-^2 + \lambda_-^2/B^2}} + \frac{(1 - 2m/r)}{\sqrt{2 + B^2/\lambda_+^2 + \lambda_+^2/B^2}} . \end{aligned} \right\} \qquad (4.31)$$

In these messy coordinates the diagonalised stress-energy tensor takes the simple form

$$\left.\begin{aligned} T^{\hat{0}\hat{0}} &= A(1 + \sqrt{1 + 4B^2/A^2})/2 \ , \\ T^{\hat{1}\hat{1}} &= A(1 - \sqrt{1 + 4B^2/A^2})/2 \ . \end{aligned}\right\} \tag{4.32}$$

Here the zero-zero component gives the energy-density as a positive quantity and the one-one component gives a negative radial pressure. Notice that

$$\left.\begin{aligned} A &= \left\{6\pi t^2(1 - 2m/r)\right\}^{-1} \ , \\ B &= -m\left\{12\pi C^2 r^2 t^{7/3}(1 - 2m/r)\right\}^{-1} \ . \end{aligned}\right\} \tag{4.33}$$

Clearly, for large values of t, B/A is small. Thus asymptotically, we have

$$T^{\hat{0}\hat{0}} \sim t^{-2}, \quad T^{\hat{1}\hat{1}} \sim t^{-8/3} \ . \tag{4.34}$$

Writing the stress-energy tensor in the form

$$T^{\hat{a}\hat{b}} = \ u^{\hat{a}} \ u^{\hat{b}} + S^{\hat{a}\hat{b}} \ , \tag{4.35}$$

where ρ is the energy density, $u^{\hat{a}}$ the four-velocity and $S^{\hat{a}\hat{b}}$ the stress-tensor, we have

$$u^{\hat{a}} = S\,{}^{\hat{a}}_{\hat{0}} \ / \sqrt{g_{\hat{0}\hat{0}}} \ . \tag{4.36}$$

Thus, the energy-density becomes

$$\rho = \alpha A(1 + \sqrt{1 + 4B^2/A^2})/2 \ . \tag{4.37}$$

Notice that the energy-density is a decreasing function of time and position in this model, as is the pressure. If the diagonal components of Eq.(4.16) had not been removed we could have had an anisotropic pressure with non-radial components as well. In that case the pressure would have "caused" the Universe to collapse if $\zeta < 0$ or expand faster if $\zeta > 0$. It must be admitted that the dependence of density on the radial parameter makes the model unrealistic.

§4.5 THE GRAVITATIONAL FORCE AND POTENTIAL

To be able to discuss the gravitational force experienced by a test particle in the space-time given by Eq.(4.2), we look for a time-like Killing vector. Since the above metric, Eq.(4.2), possesses only a conformal time-like Killing vector (Appendix V) we must turn to the cψN-formalism[14] developed in the last chapter.

The complete set of conformal Killing vectors is, (Appendix V)

$$\tilde{k}^a = \begin{pmatrix} \alpha_0 a(t) \\ 0 \\ \alpha_1 \cos\phi + \alpha_2 \sin\phi \\ (-\alpha_1 \sin\phi + \alpha_2 \cos\phi)\operatorname{Cot}\theta + \alpha_3 \end{pmatrix} , \qquad (4.38)$$

for the conformal factor $a^{-2}(t)$. Here we can take $\alpha_1 = \alpha_2 = \alpha_3 = 0$

and $\alpha_0 = 1$ for the definition of the required time-like Killing vector. The cψN-potential given by Eq.(3.33) yields the value

$$\tilde{\phi} = -m/r \quad , \qquad (4.39)$$

giving the cψN-force

$$\tilde{F}_i = -m/r^2 \delta^1_i \quad , \qquad (4.40)$$

which is the usual gravitational force.

The cψN-force is the conformal generalisation of the relativistic analogue of the Newtonian force of gravitation. The fact that we get the usual Newtonian force, as happens for the Schwarzschild metric, shows that our metric does, infact, give the effect of a gravitating particle of mass m. However, to be able to define "force" we need to have a well-defined "energy". Since energy can only be well-defined if there is time-translational invariance, the usual definition of energy does not apply. Instead the energy of a test particle is defined in terms of a scale factor. In the sense in which the energy is defined with scaling, the gravitational potential, and force, are defined without that scaling.

The choice of the Schwarzschild factor, taken in Eq.(4.2), was not forced on us. We could have taken for example, the

Reissner-Nordstrom factor

$$e^{\nu(r)} = 1 - 2m/r + Q^2/r^2 \; . \qquad (4.41)$$

This would change the values of A and B and also of the stress-energy tensor components. However, there would be no difference in the basic features of the cosmological model. As such the undesirable feature of too much freedom of choice in the model extends to the r-dependence. Nevertheless, the model is useful to see how forces can be dealt with in genuinely time-evolving situations. The force corresponding to the choice given by Eq.(4.41) is

$$F_i = -m/r^2 + Q^2/r^3 \; . \qquad (4.42)$$

as was found for the Reissner-Nordstrom geometry[3,4]. Of course, no such model can be put forward seriously due to the global requirement of charge neutrality.

Chapter 5

SUMMARY AND DISCUSSION

In an attempt to understand Relativity in terms of forces and potentials the ψN-formalism was developed[3,4]. This formalism provided an operational definition of the relativistic analogue of the gravitational force as would be seen by a freely falling observer (Chapter 1). It required the existance of a time-like isometry. However, there are space-times which do not have any time-like Killing vectors but do possess conformal time-like Killing vectors. Therefore, the energy of a test particle is not well-defined. However, a conformal analogue of energy was defined in these space-times (Chapter 3). It provides the usual definition of energy with, however, a time dependent scale factor.

This (cψN) formalism has been developed for those space-times whose metrics are a conformal generalisation of Carter's circular metrics[13]. This generalisation led to an interesting result (Chapter 3) regarding these metrics. The conformally generalised circular metrics admit of a conformal time-like Killing vector <u>which could be chosen to depend only on time</u>. It turned out that these space-times admit of a conformal analogue of ψN-forces and potentials[14].

These results were applied to the three Friedmann cosmological models which possess only a conformal time-like isometry. It was found (Chapter 3) that the cψN-force experienced by a stationary particle is zero, exactly as would be required.

To obtain a non-zero cψN-force a new cosmological model needed to be constructed. For this purpose we considered a gravitational source in a time-evolving situation. The metric of this cosmological model was obtained by constructing a cross between a Friedmann metric and a Schwarzschild metric[16] (Chapter 4). The result was a metric which could be chosen to evolve like a flat Friedmann Universe. The distribution of matter was unsatisfactory for any realistic cosmological model. In appropriate, comoving, coordinates we found that there was a radial pressure acting inwards. The appropriately scaled (conformal) definition of the gravitational force is just the Newtonian gravitational force.

Here the model Universe constructed has a single gravitational source with a distribution of dust throughout the rest of the Universe. If a more realistic model were sought it would have to allow many gravitational sources with a distribution of dust between them. Our purpose, here, was not to construct a realistic cosmological model, but merely a toy model showing some essential features. As such we have not computed, for example, the volume of the model Universe in any of the cases, etc.

It would be of interest to extend our analysis to the Reissner-Nordstrom metric crossed with the Friedmann metric, as indicated earlier. The problem here is the requirement that the Universe be charge neutral. Thus, two sources would have to be taken, one positively and the other negatively charged. Of still greater interest would be an attempt to use the Kerr metric instead. New features of the cψN-force could be expected to emerge there and the pressure structure should be particularly interesting. It is also necessary to look at the global aspects of such models.

A question which still remains open is to consider a more realistic model which has many discretely distributed gravitational sources in it. One of the possible ways to try this problem is to consider two source patched together. The gravitational effects of each of the sources can then be looked at.

A hope has been expressed[4] that a theory of gravitation depending on the variation of a scalar field could be developed. Such a theory would have to incorporate time variation of the scalar field. As such the cψN-formalism would ultimately have to be used in such attempts.

Also of interest is the connection between the reduction of symmetry and the number of independent Killing vectors (Chapter 2). Can one reduce symmetry steadily from 10 to 9 etc. down to 1? The approach to answer this question would be to restrict our attention to the case of spherically symmetric and static metrics and see if we can proceed steadily down to 4. If not we can then remove the restrictions and then see if the gaps can be filled in. It is hoped that such a procedure would lead to fresh insights into the structure of space-times.

APPENDIX I

In the Friedmann cosmological models $k^0 = 0$ (section 3.4). Therefore, for the metric tensor given by Eq.(3.35), the Killing equations give

$$k^i{}_{,0} = 0 \qquad (i = 1,2,3.); \tag{I.1}$$

$$k^1{}_{,1} = 0 ; \tag{I.2}$$

$$k^1{}_{,2} + \sin^2\chi\, k^2{}_{,1} = 0 ; \tag{I.3}$$

$$k^1{}_{,3} + \sin^2\chi\, \sin^2\theta\, k^3{}_{,1} = 0 ; \tag{I.4}$$

$$k^2{}_{,2} + \mathrm{Cot}\chi\, k^1 = 0 ; \tag{I.5}$$

$$k^2{}_{,3} + \sin^2\theta\, k^3{}_{,2} = 0 ; \tag{I.6}$$

$$k^3{}_{,3} + \mathrm{Cot}\theta\, k^2 + \mathrm{Cot}\chi\, k^1 = 0 ; \tag{I.7}$$

Eqs.(I.1) and (I.2) give

$$k^1 = A(\theta,\phi) . \tag{I.8}$$

Differentiating Eqs.(I.3) and (I.4) with respect to χ and using Eq.(I.2) yields

$$k^2 = -b(\theta,\phi)\, \mathrm{Cot}\chi + C(\theta,\phi) . \tag{I.9}$$

$$k^3 = -d(\theta,\phi)\, \mathrm{Cot}\chi + f(\theta,\phi) . \tag{I.10}$$

Now (to check consistancy) Eqs.(I.3) and (I.4) give

$$b(\theta,\phi) = -\frac{\partial A(\theta,\phi)}{\partial\theta} . \tag{I.11}$$

$$d(\theta,\phi) = -\frac{\partial A(\theta,\phi)}{\sin^2\theta\partial\phi} . \tag{I.12}$$

Eqs.(I.11) and (I.12) with Eqs.(I.9) and (I.10) together give

$$k^2 = \frac{\partial A(\theta,\phi)}{\partial\theta} \operatorname{Cot}\chi + C(\theta,\phi) . \tag{I.13}$$

$$k^3 = \frac{A(\theta,\phi)}{\partial\phi} \operatorname{Cot}\chi \operatorname{cosec}^2\theta + f(\theta,\phi) . \tag{I.14}$$

Now differentiating Eq.(I.15) with respect to χ and using Eqs.(I.12) and (I.13), we get

$$A(\theta,\phi) = g(\phi)\cos\theta + h(\phi)\sin\theta , \tag{I.15}$$

and C is a function of ϕ only. Using Eq.(I.15), Eqs.(I.8), (I.13) and (I.14) yield

$$\left.\begin{aligned} k^1 &= g(\phi)\cos\theta + h(\phi)\sin\theta , \\ k^2 &= \left\{-g(\phi)\sin\theta + h(\phi)\cos\theta\right\}\operatorname{Cot}\chi + C(\phi), \\ k^3 &= \left\{\frac{\partial g(\phi)}{\partial\phi}\cos\theta + \frac{\partial h(\phi)}{\partial\phi}\sin\theta\right\}\operatorname{Cot}\chi\operatorname{cosec}^2\theta \\ &\qquad + f(\theta,\phi) . \end{aligned}\right\} \tag{I.16}$$

Then Eq.(I.6) gives

$$f(\theta,\phi) = \frac{\partial C(\phi)}{\partial\phi} \operatorname{Cot}\theta + j(\phi) , \tag{I.17}$$

and g is a constant, α_1. Now Eq.(I.7) gives

$$\left.\begin{aligned} h(\phi) &= \alpha_2\cos\phi + \alpha_3\sin\phi \ , \\ C(\phi) &= \alpha_4\cos\phi + \alpha_5\sin \ , \end{aligned}\right\} \qquad (I.18)$$

and j is a constant, α_6. Using Eqs.(I.18) and (I.16) yield the six Killing vectors

$$\left.\begin{aligned} k^0 &= 0 \\ k^1 &= \left\{\alpha_1\cos\theta + (\alpha_2\cos\phi + \alpha_3\sin\phi)\sin\theta\right\} \ , \\ k^2 &= \left\{-\alpha_1\sin\theta + (\alpha_2\cos\phi + \alpha_3\sin\phi)\cos\theta\right\}\mathrm{Cot}\,\chi \\ &\quad + (\alpha_4\cos\phi + \alpha_5\sin\phi) \ , \\ k^3 &= \left\{-\alpha_2\sin\phi + \alpha_3\cos\phi\right\}\mathrm{Cot}\,\chi\,\mathrm{cosec}\,\theta \\ &\quad + (-\alpha_4\sin\phi + \alpha_5\cos\phi)\mathrm{Cot}\,\theta + \alpha_6 . \end{aligned}\right\} \qquad (I.19)$$

APPENDIX II

Consider the conformally transformed (with the conformal factor given by Eq.(3.42)) closed Friedmann metric. The conformal Killing equations are

$$k^0{}_{,0}/k^0 = a(t)_{,0}/a(t) \; ; \qquad (II.1)$$

$$k^0{}_{,1} - a^2(t)k^1{}_{,0} = 0 \; ; \qquad (II.2)$$

$$k^0{}_{,2} - a^2(t)\sin^2\chi \; k^2{}_{,0} = 0 \; ; \qquad (II.3)$$

$$k^0{}_{,3} - a^2(t)\sin^2\chi\sin^2\theta \; k^3{}_{,0} = 0 \; ; \qquad (II.4)$$

$$k^1{}_{,1} = 0 \; ; \qquad (II.5)$$

$$k^1{}_{,2} + \sin^2\chi \; k^2{}_{,1} = 0 \; ; \qquad (II.6)$$

$$k^1{}_{,3} + \sin^2\chi\sin^2\theta \; k^3{}_{,1} = 0 \; ; \qquad (II.7)$$

$$k^2{}_{,2} + \mathrm{Cot}\chi \; k^1 = 0 \; ; \qquad (II.8)$$

$$k^2{}_{,3} + \sin^2\theta \; k^3{}_{,2} = 0 \; ; \qquad (II.9)$$

$$k^3{}_{,3} + \mathrm{Cot}\theta \; k^2 + \mathrm{Cot}\chi \; k^1 = 0 \; . \qquad (II.10)$$

Eqs.(II.1) and (II.5) give

$$k^0 = a(t)A(\chi,\theta,\phi) \; . \qquad (II.11)$$

$$k^1 = B(t,\theta,\phi) \; . \qquad (II.12)$$

Differentiating Eq.(II.6) with respect to χ and using Eq.(II.5) gives

$$k^2 = -\cot\chi \; C(t,\theta,\phi) + D(t,\theta,\phi) \; . \qquad (II.13)$$

Eq.(II.7), when differentiated with respect to χ, and compared with Eq.(II.5) gives

$$k^3 = -\cot\chi \; E(t,\theta,\phi) + F(t,\theta,\phi) \; . \qquad (II.14)$$

To check consistancy use these results in Eqs.(II.6) and (II.7) to get

$$\left.\begin{aligned} C(t,\theta,\phi) &= -\frac{\partial B(t,\theta,\phi)}{\partial\theta} \; , \\ E(t,\theta,\phi) &= -\frac{B(t,\theta,\phi)}{\partial\phi} \; . \end{aligned}\right\} \qquad (II.15)$$

Using Eqs.(II.15) in Eqs.(II.13) and (II.14) give

$$\left.\begin{aligned} k^2 &= \frac{\partial B(t,\theta,\phi)}{\partial\theta} \cot\chi + D(t,\theta,\phi) \; , \\ k^3 &= \frac{\partial B(t,\theta,\phi)}{\partial\phi} \cot\chi \operatorname{cosec}^2\theta + F(t,\theta,\phi) \; . \end{aligned}\right\} \qquad (II.16)$$

Eq.(II.8) together with Eqs.(II.12) and (II.16) give

$$\left.\begin{aligned} B(t,\theta,\phi) &= G(t,\phi)\cos\theta + H(t,\phi)\sin\theta \; , \\ D(t,\theta,\phi) &= j(t,\phi) \; . \end{aligned}\right\} \qquad (II.17)$$

Thus, Eqs.(II.16) and (II.17) give

$$\left.\begin{aligned} k^2 &= \left[-G(t,\phi)\sin\theta + H(t,\phi)\cos\theta\right]\cot\chi + j(t,\phi) \; , \\ k^3 &= \left[\frac{\partial G(t,\phi)}{\partial\phi}\cos\theta + \frac{\partial H(t,\phi)}{\partial\phi}\sin\theta\right]\cot\chi \operatorname{cosec}^2\theta + F(t,\theta,\phi) \end{aligned}\right\} (II.18)$$

Eqs.(II.9) and (II.18) imply that

$$\left.\begin{aligned} G(t,\phi) &= i(t) \ , \\ F(t,\theta,\phi) &= \mathrm{Cot}\,\theta\,\frac{\partial j(t,\phi)}{\partial\phi} + \ell(t,\phi) \ . \end{aligned}\right\} \qquad (II.19)$$

Thus, using Eqs.(II.17) and (II.19) in Eqs.(II.12) and (II.18) give

$$\left.\begin{aligned} k^1 &= i(t)\cos\theta + H(t,\phi)\sin\theta \ , \\ k^2 &= \left[-i(t)\sin\theta + H(t,\phi)\cos\theta\right]\mathrm{Cot}\,\chi + j(t,\phi) \ , \\ k^3 &= \frac{\partial H(t,\phi)}{\partial\phi}\,\mathrm{Cot}\,\chi\,\mathrm{cosec}\,\theta + \frac{\partial j(t,\phi)}{\partial\phi}\,\mathrm{Cot}\,\theta + \ell(t,\phi) . \end{aligned}\right\} \qquad (II.20)$$

Taking Eq.(II.10) and using Eq.(II.20), we get

$$\left.\begin{aligned} H(t,\phi) &= m(t)\cos\phi + n(t)\sin\phi \ , \\ j(t,\phi) &= p(t)\cos\phi + q(t)\sin\phi \ , \\ \ell(t,\phi) &= r(t). \end{aligned}\right\} \qquad (II.21)$$

Using Eqs.(II.21) in Eqs.(II.20) give

$$\left.\begin{aligned} k^1 &= i(t)\cos\theta + \left[m(t)\cos\phi + n(t)\sin\phi\right]\sin\theta \ , \\ k^2 &= \left[-i(t)\sin\theta + \left\{m(t)\cos\phi + n(t)\sin\phi\right\}\cos\theta\right]\mathrm{Cot}\,\chi \\ &\quad + \left\{p(t)\cos\phi + q(t)\sin\phi\right\} \ , \\ k^3 &= \left\{-m(t)\sin\phi + n(t)\cos\phi\right\}\mathrm{Cot}\,\chi\,\mathrm{cosec}\,\theta \\ &\quad + \left\{-p(t)\sin\phi + q(t)\cos\phi\right\}\mathrm{Cot}\,\chi + r(t) \ . \end{aligned}\right\} \qquad (II.22)$$

Consider Eqs.(II.2) which gives

$$A(\chi,\theta,\phi) = \chi a(t)\left[\frac{\partial i(t)}{\partial t}\cos\theta + \left\{\frac{\partial m(t)}{\partial t}\cos\phi + \frac{\partial n(t)}{\partial t} \times \sin\phi\right\}\sin\theta\right] + u(\theta,\phi) . \quad \text{(II.23)}$$

Eq.(II.23) implies that

$$\left.\begin{aligned} i(t) &= \int \frac{\alpha}{a(t)}\, dt + \alpha_1 , \\ m(t) &= \int \frac{\beta}{a(t)}\, dt + \alpha_2 , \\ n(t) &= \int \frac{\gamma}{a(t)}\, dt + \alpha_3 . \end{aligned}\right\} \quad \text{(II.24)}$$

Using the values of k^0 and k^2 [keeping in view Eqs.(II.23) and (II.24)] into Eq.(II.4) we obtain

$$\left[\left\{-\alpha\sin\theta + (\beta\cos\phi + \gamma\sin\phi)\cos\theta\right\}(\chi - \text{xin}\chi\cos\chi) + \frac{\partial u(\theta,\phi)}{\partial\theta}\right]\text{cosec}^2\chi - \left[\frac{\partial p(t)}{\partial t}\cos\phi + \frac{\partial q(t)}{\partial t}\sin\phi\right]a(t) = 0 .$$

This equation is satisfied if

$$\left.\begin{aligned} &\alpha = \beta = \gamma = 0 , \\ &\frac{\partial p(t)}{\partial t} = \frac{\partial q(t)}{\partial t} = \frac{\partial u(\theta,\phi)}{\partial\theta} = 0 . \end{aligned}\right\} \quad \text{(II.25)}$$

Thus, $p = \alpha_4$ and $q = \alpha_5$. (II.26)

and u is a function of ϕ only. Eq.(II.4) gives the constraint that r(t) and $u(\phi)$ are constants. Choosing $r(t) = \alpha_6$ and $u = \alpha_0$, the Killing equations give

$$\left.\begin{aligned}
k^0 &= \alpha_0 a(t) \ , \\
k^1 &= \alpha_1 \cos\theta + (\alpha_2 \cos\phi + \alpha_3 \sin\phi)\sin\theta \ , \\
k^2 &= \left\{-\alpha_1 \sin\theta + (\alpha_2 \cos\phi + \alpha_3 \sin\phi)\cos\theta\right\}\mathrm{Cot}\,\chi \\
&\quad + (\alpha_4 \cos\phi + \alpha_5 \sin\phi) \ , \\
k^3 &= (-\alpha_2 \sin\phi + \alpha_3 \cos\phi)\mathrm{Cot}\,\chi \,\mathrm{cosec}\,\theta \\
&\quad + (-\alpha_4 \sin\phi + \alpha_5 \cos\phi)\mathrm{Cot}\,\theta + \alpha_6 .
\end{aligned}\right\} \qquad (\mathrm{II}.27)$$

APPENDIX III

The complete set of Killing vectors for the flat Friedmann (k = 0) model are

$$\left.\begin{aligned}
k^0 &= 0 \;, \\
k^1 &= \alpha_1\cos\theta + (\alpha_2\cos\phi + \alpha_3\sin\phi)\sin\theta \;, \\
k^2 &= \left\{-\alpha_1\sin\theta + (\alpha_2\cos\phi + \alpha_3\sin\phi)\cos\theta\right\}\frac{1}{\chi} \\
&\quad + (\alpha_4\cos\phi + \alpha_5\sin\phi) \;, \\
k^3 &= (-\alpha_2\sin\phi + \alpha_3\cos\phi)\text{cosec}\,\theta\cdot\frac{1}{\chi} \\
&\quad + (-\alpha_4\sin\phi + \alpha_5\cos\phi)\text{Cot}\,\theta + \alpha_6 .
\end{aligned}\right\} \quad (\text{III}.1)$$

The complete set of Killing vectors for the open Friedmann (k = 1) model are

$$\left.\begin{aligned}
k^0 &= 0 \;, \\
k^1 &= \alpha_1\cos\theta + (\alpha_2\cos\phi + \alpha_3\sin\phi)\sin\theta \;, \\
k^2 &= \left\{-\alpha_1\sin\theta + (\alpha_2\cos\phi + \alpha_3\sin\phi)\cos\theta\right\}\text{Coth}\,\chi \\
&\quad + (\alpha_4\cos\phi + \alpha_5\sin\phi) \;, \\
k^3 &= (-\alpha_2\sin\phi + \alpha_3\cos\phi)\text{Coth}\,\chi\,\text{cosec}\,\theta \\
&\quad + (-\alpha_4\sin\phi + \alpha_5\cos\phi)\text{Cot}\,\theta + \alpha_6 .
\end{aligned}\right\} \quad (\text{III}.2)$$

The complete set of conformal Killing vectors for the flat Friedmann model, with the conformal factor given by Eq.(3.42), are

$$
\left.\begin{aligned}
k^0 &= a(t)\left[\alpha_0 + \left\{\alpha_1\cos\theta + (\alpha_2\cos\phi + \alpha_3\sin\phi)\sin\theta\right\}\chi\right], \\
k^1 &= 3\sqrt{a(t)}\left[\alpha_1\cos\theta + (\alpha_2\cos\phi + \alpha_3\sin\phi)\sin\theta\right] \\
&\quad + \left[\alpha_4\cos\theta + (\alpha_5\cos\phi + \alpha_6\sin\phi)\sin\theta\right], \\
k^2 &= 3\sqrt{a(t)}\left[-\alpha_1\sin\theta + (\alpha_2\cos\phi + \alpha_3\sin\phi)\cos\theta\right]\frac{1}{\chi} \\
&\quad + \left[-\alpha_4\sin\theta + (\alpha_5\cos\phi + \alpha_6\sin\phi)\cos\theta\right]\frac{1}{\chi} \\
&\quad + (\alpha_7\cos\phi + \alpha_8\sin\phi), \\
k^3 &= 3\sqrt{a(t)}(-\alpha_2\sin\phi + \alpha_3\cos\phi)\frac{\operatorname{cosec}\theta}{\chi} + (-\alpha_5\sin\phi \\
&\quad + \alpha_6\cos\phi)\frac{\operatorname{cosec}\theta}{\chi} + (-\alpha_7\sin\phi + \alpha_8\cos\phi)\operatorname{Cot}\theta + \alpha_9.
\end{aligned}\right\} \quad \text{(III.3)}
$$

The complete set of conformal Killing vectors for the open Friedmann model, with the conformal factor given by Eq.(3.42), are

$$
\left.\begin{aligned}
k^0 &= \alpha_0 a(t), \\
k^1 &= \left[\alpha_1\cos\theta + (\alpha_2\cos\phi + \alpha_3\sin\phi)\sin\theta\right], \\
k^2 &= \left[-\alpha_1\sin\theta + (\alpha_2\cos\phi + \alpha_3\sin\phi)\cos\theta\right]\operatorname{Coth}\chi \\
&\quad + (\alpha_4\cos\phi + \alpha_5\sin\phi), \\
k^3 &= (-\alpha_2\sin\phi + \alpha_3\cos\phi)\operatorname{Coth}\chi\operatorname{cosec}\theta \\
&\quad + (-\alpha_4\sin\phi + \alpha_5\cos\phi)\operatorname{Cot}\theta + \alpha_6.
\end{aligned}\right\} \quad \text{(III.4)}
$$

APPENDIX IV

The non-vanishing christoffel symbols for the metric given in Eq.(4.2) are:

$$\left.\begin{aligned}
\Gamma^0{}_{11} &= \nu'(r)/2 \ , \\
\Gamma^0{}_{22} &= r^2\, a(t)a(t)_{,0}\ e^{-\nu(r)} \\
\Gamma^0{}_{33} &= \Gamma^0{}_{22}\sin^2\theta \ , \\
\Gamma^1{}_{00} &= \nu'(r)e^{2\nu(r)}/2a^2(t) \ , \\
\Gamma^1{}_{10} &= \Gamma^2{}_{20} = \Gamma^3{}_{30} = a(t)_{,0}/a(t) \ , \\
\Gamma^1{}_{11} &= -\nu'(r)/2 \ , \\
\Gamma^1{}_{22} &= -re^{\nu(r)} \ , \\
\Gamma^1{}_{33} &= \Gamma^1{}_{22}\sin^2\theta \ , \\
\Gamma^2{}_{21} &= \Gamma^3{}_{31} = 1/r \ , \\
\Gamma^2{}_{33} &= -\sin\theta\cos\theta \ , \\
\Gamma^3{}_{32} &= \mathrm{Cot}\,\theta \ .
\end{aligned}\right\}$$

APPENDIX V

The conformal Killing equations for the zero component of the conformal Killing vector of the metric conformally related, by the conformal factor $a^{-2}(t)$, to the metric given by Eq.(4.2) (suppressing the tilde) are

$$a(t)_{,0}\,\frac{k^0}{a^3(t)} + \nu'(r)e^{\nu(r)}\,\frac{k^1}{2} + \frac{k^0{}_{,0}}{a^2(t)} = 0\ , \tag{V.1}$$

$$k^1{}_{,0} = a^{-2}(t)e^{2\nu(r)}\,k^0{}_{,1}\ , \tag{V.2}$$

$$k^2{}_{,0} = a^{-2}(t)r^{-2}e^{\nu(r)}\,k^0{}_{,2}\ , \tag{V.3}$$

$$k^3{}_{,0} = a^{-2}(t)r^{-2}e^{\nu(r)}\mathrm{cosec}^2\theta\,k^0{}_{,3}\ . \tag{V.4}$$

The Killing equations for spatial components are:

$$\left.\begin{aligned}
&\nu'(r)k^1 - 2k^1{}_{,1} = 0\ ,\\
&e^{-\nu(r)}k^1{}_{,2} + r^2k^2{}_{,1} = 0\ ,\\
&e^{-\nu(r)}k^1{}_{,3} + r^2\sin^2\theta k^3{}_{,1} = 0\ ,\\
&k^1 + rk^2{}_{,2} = 0\ ,\\
&k^2{}_{,3} + \sin^2\theta k^3{}_{,2} = 0\ ,\\
&k^1 + r\mathrm{Cot}\,\theta k^2 + rk^3{}_{,3} = 0\ .
\end{aligned}\right\} \tag{V.5}$$

These Killing equations are exactly the same as the Killing equations of the spatial components of the Schwarzschild metric.

Thus, the Killing vectors of the spatial components for the conformally transformed metric

$$d\tilde{s}^2 = a^{-2}(t)dt^2 - dr^2 - r^2 d\theta^2 - r^2 \sin^2\theta d\phi^2 \ , \tag{V.6}$$

are

$$k^i = \begin{pmatrix} 0 \\ \alpha_1 \cos\phi + \alpha_2 \sin\phi \\ (-\alpha_1 \sin\phi + \alpha_2 \cos\phi)\mathrm{Cot}\,\theta + \alpha_3 \end{pmatrix} . \tag{V.7}$$

In view of Eq.(V.7), Eqs.(V.1) to (V.4) give

$$a(t)_{,0} \frac{k^0}{a(t)} + k^0_{\ ,0} = 0 \ , \tag{V.8}$$

$$k^0_{\ ,1} = 0 = k^0_{\ ,2} = k^0_{\ ,3} \ . \tag{V.9}$$

Integrating t in Eq.(V.8) readily yields

$$k^0 = f(r,\theta,\phi)a(t) \ . \tag{V.10}$$

Eqs.(V.9) and (V.10) imply that f is a constant α_0. Thus, the conformal time-like Killing vector, for the conformally transformed metric, Eq.(V.6), is

$$k^0 = \alpha_0 a(t) \tag{V.11}$$

Now the conformal transformation, since it only involves a time factor, does not change the spatial Killing vectors[14].

Using this fact in the one-one Killing equation for the metric given by Eq.(4.2), we see that in the untransformed case, we get

$$k^0 = 0 \ . \tag{V.12}$$

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Reprint from

ZAMP

Zeitschrift für angewandte Mathematik und Physik
Journal of Applied Mathematics and Physics
Journal de Mathématiques et de Physique appliquées

A prescription for *n*-dimensional Vierbeins

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Vol. 36
Nr. 1
Januar 1985
pages 184–188

Birkhäuser Verlag
Basel · Boston · Stuttgart

Journal of Applied Mathematics and Physics (ZAMP)
Vol. 36, January 1985
0044-2275/85/001184-05 $ 2.50/0

A prescription for n-dimensional Vierbeins

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With the advent of supergravity theories [1] the Vierbein formalism [2] has come into extensive use. Basically a Vierbein, e_a^i, converts a flat metric in Cartesian co-ordinates, $\bar{g}_{ij}$, into a general metric, g_{ab}, according to the equation

$$e_a^i e_b^j \bar{g}_{ij} = g_{ab}. \tag{1}$$

In some sense, then, the Vierbein may be regarded as a "four-dimensional square root" of the general metric tensor. In fact this applies in more or less the same sense that the Dirac spinor is regarded as "the square root of the Minkowski 4-vector". The curved space-time generalization of the flat space-time Dirac matrices, $\bar{\gamma}_i$, given by the commutation relations,

$$[\gamma_a, \gamma_b] = 2 g_{ab} \tag{2}$$

transform according to the rule

$$\gamma_a = e_a^i \bar{\gamma}_i. \tag{3}$$

These generalized γ-matrices are frequency used when dealing with spinor fields in general relativity [3]. It would be useful to be able to write down Vierbeins in an arbitrary space-time. Further, higher dimensional Vierbeins would be useful when dealing, for example, with the eleven-dimensional supergravity which reduces to $SU(8)$ supergravity [4], or in higher dimensional theories of the Kaluza-Klein variety such as those of Chodos and Detweiler or Halpern [5]. In this paper we provide a prescription to be able to write down a Vierbein given a metric in any number of dimensions.

Before going on we need to introduce certain notation. To make that notation clear we shall first discuss the Vierbein prescription in 2-dimensions in full detail, in 3-dimensions in somewhat less detail, and in 4-dimensions in much less detail. First we notice that for a diagonal metric tensor in an arbitrary number of dimensions the Vierbein can always be written as

$$e_a^i = (g_{aa})^{1/2} \delta_a^i, \tag{4}$$

where we have dropped the summation convention for the present. Notice that we *could* have other solutions in which the Vierbein need not become the identity when we revert to flat space in Cartesian co-ordinates. We shall not be much concerned with such prescriptions, but shall require that our general prescription reduces to Eq. (4) in the case that the metric tensor tends to the diagonal form.

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For a two-dimensional Riemannian metric there are two prescriptions for the Vierbein which satisfy our criteria

$$e_a^i = \begin{pmatrix} \pm\sqrt{g_{11}} & \pm g_{12}/\sqrt{g_{11}} \\ 0 & \pm\sqrt{(g_{11}g_{22} - g_{12}^2)/g_{11}} \end{pmatrix} (i, a = 1, 2) \tag{5}$$

and the equivalent expression obtained by interchanging g_{11} and g_{22}. An example of the type of Vierbein which we are not bothering about here, is

$$e_1^1 = e_2^2 = 0; \quad e_2^1 = \pm\sqrt{g_{11}}, \quad e_1^2 = \pm\sqrt{g_{22}}. \tag{5'}$$

Notice that the diagonal elements of the metric tensor cannot be zero here. Notice, also, that the signs of each row must go together (positive with positive and negative with negative) but for separate rows are independent.

If the metric is non-Riemannian (writing the co-ordinates as 0 and 1 co-ordinates), the Vierbein will now be written as

$$e_a^i = \begin{pmatrix} \pm\sqrt{g_{00}} & \pm g_{01}/\sqrt{g_{00}} \\ 0 & \pm\sqrt{(g_{01}^2 - g_{00}g_{11})/g_{00}} \end{pmatrix} (i, a = 0, 1). \tag{5''}$$

In this case the diagonal elements of the metric tensor *can* be zero. If $g_{11} = 0$ no problem arises, but if $g_{00} = 0$ we have to interchange g_{00} and $-g_{11}$ in Eq. (5''). (Here we have taken the convention that the metric is positive for a time-like vector). In the case that both the diagonal components are zero, the Vierbein components must satisfy the equations

$$e_0^0 = \pm e_0^1; \quad e_1^0 = \mp e_1^1; \quad e_0^0 e_1^0 = g_{01}/2 \tag{5'''}$$

none of whose infinitely many solutions can satisfy our earlier requirements.

We now present some notation which will be used shortly. Consider the entire (symmetric) matrix $g_{ab}(a, b = 1, \ldots, n)$. We shall denote the determinant of the submatrix composed of the first m rows and m columns by $D(m)$. The determinant of the submatrix obtained by replacing the p^{th} row ($p \leq m$) by the corresponding part of the j^{th} row of the entire matrix will be denoted by $\underset{p\to j}{D}(m)$, and the determinant of the submatrix obtained by replacing the q^{th} column ($q \leq m$) by the corresponding part of the k^{th} column will be denoted by $\underset{q\to k}{D}(m)$. We are now in a position to generalize the two-dimensional prescription to higher dimensions.

The required three-dimensional Vierbeins in a Riemannian space are

$$e_a^i = \begin{pmatrix} \dfrac{\pm \underset{1\to1}{D}(1)}{\sqrt{D(1)}} & \dfrac{\pm \underset{1\to2}{D}(1)}{\sqrt{D(1)}} & \dfrac{\pm \underset{1\to3}{D}(1)}{\sqrt{D(1)}} \\ 0 & \dfrac{\pm \underset{2\to2}{D}(2)}{\sqrt{D(1)\cdot D(2)}} & \dfrac{\pm \underset{2\to3}{D}(2)}{\sqrt{D(1)\cdot D(2)}} \\ 0 & 0 & \dfrac{\pm \underset{3\to3}{D}(3)}{\sqrt{D(2)\cdot D(3)}} \end{pmatrix} \tag{6}$$

provided that $D(1), D(2), D(3) \neq 0$. Clearly $D(3) \neq 0$ if the metric is nonsingular. We also have prescriptions which interchange the 1, 2, 3 co-ordinates. Even if one of the prescriptions goes singular, one of the others must be non-singular in the Riemannian space. For a pseudo-Riemannian space this need not be true. Nevertheless, there will exist the solutions which we are discarding on account of our criterion. For example

$$\left.\begin{aligned} e_0^0 &= 2\,g_{01}\,g_{02}/g_{03} = \pm\, e_0^1; \quad e_0^2 = e_1^1 = e_2^1 = 0; \\ e_1^0 &= g_{03}/2\,g_{02} = \pm\, e_1^2; \quad e_2^0 = g_{03}/2\,g_{01} = \mp\, e_2^2 \end{aligned}\right\}, \tag{6'}$$

where $g_{00} = g_{11} = g_{22} = 0$ and the Cartesian metric signature is $(+, -, -)$. We will not discuss such solutions further.

For the four-dimensional Vierbeins in a Riemannian space the required prescription is

$$e_a^i = \begin{pmatrix} \dfrac{\underset{1\to 1}{D(1)}}{\sqrt{D(1)}} & \dfrac{\underset{1\to 2}{D(1)}}{\sqrt{D(1)}} & \dfrac{\underset{1\to 3}{D(1)}}{\sqrt{D(1)}} & \dfrac{\underset{1\to 4}{D(1)}}{\sqrt{D(1)}} \\ 0 & \dfrac{\underset{2\to 2}{D(2)}}{\sqrt{D(1)\cdot D(2)}} & \dfrac{\underset{2\to 3}{D(2)}}{\sqrt{D(1)\cdot D(2)}} & \dfrac{\underset{2\to 4}{D(2)}}{\sqrt{D(1)\cdot D(2)}} \\ 0 & 0 & \dfrac{\underset{3\to 3}{D(3)}}{\sqrt{D(2)\cdot D(3)}} & \dfrac{\underset{3\to 4}{D(3)}}{\sqrt{D(2)\cdot D(3)}} \\ 0 & 0 & 0 & \dfrac{\underset{4\to 4}{D(4)}}{\sqrt{D(3)\cdot D(4)}} \end{pmatrix}, \tag{7}$$

where the sign ambiguity is left implicit for convenience. Again, the required prescriptions will be given by Eq. (7) and all possible relabellings of the co-ordinates. We assume that there will exist one prescription, at least, which is non-singular, i.e. we are not dealing with the cases where this assumption does not hold.

To be able to write the formulae for the Vierbeins more compactly we define $D(0) = +1$ in a Riemannian space and -1 in a space-time. More general cases can be similarly dealt with. The pattern emerging leads us to expect that, given an n-dimensional metric tensor, g_{ab}, such that for some suitable numbering of co-ordinates $D(m) \neq 0$ for all $m \leq n$, our Vierbeins will be given by

$$\left.\begin{aligned} e_a^i &= \underset{i\to a}{D(i)}/\sqrt{D(i-1)\,D(i)} \qquad i \leq a \\ &= 0 \qquad\qquad\qquad\qquad\quad i > a \end{aligned}\right\}. \tag{8}$$

To verify this conjecture consider the expression

$$G_{ab} = e_a^i\, e_b^j\, \tilde{g}_{ij}, \tag{9}$$

where $\tilde{g}_{ij} = 1$ *if* $i = j$ and 0 otherwise (or with -1 instead of 1 for the space co-ordinates in a space-time metric). We want to verify that $G_{ab} = g_{ab}$. This would be done by expanding Eq. (9) using Eq. (8) and verifying that the coefficient of g_{ab} is unity and of all g_{cd}, $c \neq a$, $d \neq b$ is zero. Now, from Eq. (8), the summation is over all $i \leq a$ and $j \leq b$. Thus

$$G_{ab} = \sum_{i=1}^{\inf(a,b)} \frac{\underset{i\to a}{D(i)} \cdot \underset{i\to b}{D(i)}}{D(i-1)\cdot D(i)}. \tag{10}$$

Let $a = \inf.(a, b)$. Then, expanding out for $i = a$ we have

$$G_{ab} = \frac{\underset{a\to b}{D(a)}}{D(a-1)} + \sum_{i=1}^{a-1} \frac{\underset{i\to a}{D(i)} \cdot \underset{i\to b}{D(i)}}{D(i-1)\cdot D(i)}. \tag{11}$$

Now the first term on the right-hand side of Eq. (11) can be expanded to give

$$\underset{a\to b}{D(a)} = g_{ab}\cdot D(a-1) + \sum_{i=1}^{a-1} g_{ib}\cdot \overset{i\to a}{D(a-1)}\cdot(-1)^{a-i}. \tag{12}$$

Inserting Eq. (12) into Eq. (11) we see that

$$G_{ab} = g_{ab} + \sum_{i=1}^{a-1}\left|(-1)^{a-i}\cdot g_{ib}\cdot\frac{\underset{i\to a}{D(a-1)}}{D(a-1)} + \frac{\underset{i\to a}{D(i)}\cdot\underset{i\to b}{D(i)}}{D(i-1)\cdot D(i)}\right|. \tag{13}$$

We now have to demonstrate that the term in the brackets, call it H_{ab}, is zero. Again expanding for $i = a-1$

$$H_{ab} = -g_{a-1,b}\frac{\underset{1\to a}{D(a-1)}}{D(a-1)} + \frac{\underset{a-1\to a}{D(a-1)}\cdot\underset{a-1\to b}{D(a-1)}}{D(a-2)\cdot D(a-1)}$$

$$+\sum_{i=1}^{a-2}\left[g_{ib}\cdot(-1)^{a-i}\frac{\underset{i\to a}{D(a-1)}}{D(a-1)} + \frac{\underset{i\to a}{D(i)}\cdot\underset{i\to b}{D(i)}}{D(i-1)\cdot D(i)}\right]. \tag{14}$$

Using Eq. (12) with a replaced by $a-1$ for the second term in Eq. (14), we see that the coefficients of $g_{a-1,b}$ cancel and we get

$$H_{ab} = \sum_{i=1}^{a-2}\left|\frac{\underset{i\to a}{D(i)}\cdot\underset{i\to b}{D(i)}}{D(i-1)\cdot D(i)} + (-1)^{a-i}\cdot g_{ib}\cdot\frac{\underset{i\to a}{D(a-1)}}{D(a-1)}\right.$$

$$\left. + (-1)^{a-i}\cdot g_{ib}\cdot\frac{\underset{a-1\to a}{D(a-1)}\cdot\underset{i\to a-1}{D(a-2)}}{D(a-2)\cdot D(a-1)}\right|. \tag{15}$$

This procedure can be repeated for the coefficients of $g_{a-2,b}$, $g_{a-3,b}$ and so on by expanding out $i = a-2$, $i = a-3$, and so on, successively. In the first case we get a determinant which is zero, in the next two determinants which cancel, in the next a determinant of determinants which is zero, and so on. We can continue this procedure till we reach g_{3b}, for which we already know the formula works, from Eq. (7). Thus we see that $G_{ab} = g_{ab}$ and so the Vierbeins are given by Eq. (8).

Apart from the main theme of this paper there are two points of mathematical interest in a prescription for writing down Vierbeins. The first point is that we have provided a procedure for evaluating the "square roots" of a symmetric matrix. It is immediately apparent that there are many linearly independent square roots of symmetric matrices. It would be interesting to find out the number of linearly independent Hermitian square roots of unity for an $n\times n$ identity matrix. For the 2×2 case there are the four Pauli spin matrices. What do we have for higher dimensions?

The second point of interest is the fact that the Vierbein may be regarded as a co-ordinate transformation. Thus we can write

$$e_a^i = \partial u^i/\partial x^a. \tag{16}$$

Reading Eq. (16) with Eq. (1), we would have a set of $n(n+1)/2$ non-linear, partial differential equations of the n functions u^i, of the n variables x^a, in terms of the $n(n+1)/2$ functions $g_{ab}(x^c)(b \geqq a)$. In the case of a Riemannian metric we get elliptical equations

$$\sum_{i=1}^{n}(\partial u^i/\partial x^a)(\partial u^i/\partial x^b) = g_{ab}(x^c) \tag{17}$$