EFFECTS OF HALL CURRENT ON HYDROMAGNETIC STEADY FLOW OF A SECOND ORDER FLUID PAST A POROUS PLATE

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بسم الله الرحمن الرحيم

CERTIFICATE

We accept this dissertation as conforming to the required standard
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Dedicated to

My Parents, Brothers

and

Sisters

Whose prayers are always with me

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Preface

A great deal of mathematical literature describes the solution of various types of fluids. The Newtonian fluid is the simplest to be solved, not only numerically but also analytically. However, its application is limited, as a few types of fluids obey the law of Newtonian fluid. In practice, such as chemical, mechanical and nuclear industries, geophysics and bioengineering, the behavior of several fluids is greatly deviated from Newtonian fluids. In mathematical literature, the non-Newtonian fluids are principally classified on the basis of their behavior in shear. A fluid with a linear relationship between the shear stress and the shear rate, is always characterized to be Newtonian fluid. Based on the knowledge of the solutions to Newtonian fluids, the different fluids can be extended. In recent years, interest in boundary layer flows of non-Newtonian fluids has increased. Amongst many models that have been used to describe the non-Newtonian behavior exhibited by the certain fluids, the fluids of differential type (including second order). Several exact solutions have been presented for the flow of these fluids (Rajagopal [1], Rajagopal and Gupta [2,3], Hayat et al [3-6],Siddiqui and Benharbit [7] and Siddiqui et al [9]).

This dissertation consists of three chapters. Chapter 1 includes the basic definitions of fluid mechanics and magnetohydrodynamics, the equation of continuity and momentum equation up to second order fluid.

Chapter 2 describes the problem [9] of hydromagnetic viscous flow past a porous plate with Hall effects.

In chapter 3, an investigation is made of the flow of a second order conducting liquid past an infinite porous plate taking Hall effects into account. The plate is subjected to suction or blowing and this study is likely to have bearing on the problem of transposition cooling of a vehicle. It is shown that the asymptotic solution for velocity and magnetic field exists both for suction or blowing at the plate.

Contents

Chapter 1

Basics of Fluid Mechanics

1.1 Introduction

This chapter includes definitions and derivations of some basic equations like continuity equation, equation of momentum, equation of law of conservation of charge and Ohm's law. Finally, the equations of motion for magnetohydrodynamic viscous and second order fluids are derived.

1.2 Definitions

Fluid

Fluid is a substance which cannot sustain shearing forces when at rest or fluid is a substance that deforms continuously under the influence of external forces.

Fluid Mechanics

The branch of applied sciences analyzing the properties and behavior of fluids both in motion and at rest is called fluid mechanics.

Types of fluids

Fluids are usually classified as follows:

Ideal fluid

An incompressible fluid with zero viscosity is called ideal fluid.

Incompressible fluid

A fluid is said to be incompressible when its density is constant.

Viscous fluid

A fluid with non zero viscosity is called viscous fluid. All real fluids are viscous fluids.

Types of flows

There are many types of flows yet the following are important from the subject point of view.

Uniform flow

A flow in which the velocities of liquid particles at all sections of channel are equal is called uniform flow.

Non -uniform flow

A flow in which the velocities of liquid particles at all sections of the channel are not equal is called a non-uniform flow

Laminar flow

A flow in which each fluid particle has a definite path and paths of individual particles do not cross each other is called laminar flow.

Turbulent flow

A flow in which each fluid particle does not have a definite path and the paths of individual particles cross each other is called a turbulent or irregular flow.

Steady flow

A flow in which the quantity of fluid flowing per second is constant is called a steady flow. A steady flow may be uniform or non-uniform. Mathematically

$$\eta = \eta\,(x, y, z) \;\text{ or }\; \frac{\partial \eta}{\partial t} = 0, \tag{1.1}$$

where η is any fluid quantity.

Unsteady flow

A flow in which the quantity of fluid flowing per second is not constant, is called unsteady flow.

Compressible flow

A flow in which the volume and thus density of the fluid changes during the flow is called a compressible flow. All the gases are generally considered to have compressible flows.

Rotational flow

A flow in which the fluid particles also rotate (i.e. have angular velocity) about their own axes while flowing is called rotational flow.

Irrotational flow

A flow in which the fluid particles do not rotate about their axes and retain their original orientations is called an irrotational flow.

Types of forces present in a moving fluid

When a fluid is in motion, some forces are always involved in the phenomenon of flow but there are always one or two forces which dominate the other forces. They governed the flow of the fluid and keep it in motion. The following forces which are present in moving fluid are important from the subject point of view.

Inertia force

The inertia force F_i is the product of mass and acceleration of the flowing fluid and is always existing in the phenomenon of fluid flow.

Viscous force

The viscous force F_v is the product of shear stress due to viscosity and the cross sectional area of flow.

Gravity force

The gravity force F_g is the product of mass of the fluid and acceleration of the flowing fluid due to gravity.

Surface tension force

The surface tension force F_t is the product of tension per unit length and the length of surface of the flowing fluid.

Pressure force

The pressure force F_p is the product of intensity of pressure and area of the flowing fluid.

Elastic force

The elastic force is the product of the elastic stress and the area of flowing fluid.

Magnetohydrodynamics

The branch of science that deals with study of motion of an electrically conducting fluid in the presence of magnetic field. The fluid under consideration is termed as magnetohydrodynamic fluid or simply MHD fluid.

Permeability

The absolute permeability is the ratio of the magnitude of the magnetic induction to the magnitude of magnetic intensity i.e.

$$\mu_e = \frac{B}{H}, \tag{1.2}$$

where B is magnitude of magnetic induction and H is magnitude of magnetic intensity.

Electric field intensity

It is defined as the force experienced by unit positive charge placed in an electric field at specific point. Mathematically

$$\mathrm{E} = \frac{\mathbf{F}}{q}, \tag{1.3}$$

where $\mathbf{E}$ is an electric field intensity and has unit Newton per Column and $\mathbf{F}$ is a force acting on unit positive test charge q.

Magnetic field intensity

The force acting on a unit north pole in a magnetic field at a particular point is called magnetic field intensity at that point i.e.

$$\mathrm{H} = \frac{\mathbf{F}}{m_n}, \tag{1.4}$$

where $\mathbf{H}$ is magnetic field intensity, $\mathbf{F}$ is force acting on north pole and m_n is strength of north pole.

Magnetic induction

The magnetic lines of force falling on a unit area is called magnetic flux density or magnetic induction. It is found that magnetic induction is proportional to magnetic field intensity i.e.

$$\mathbf{B} = \mu_e \mathbf{H}, \tag{1.5}$$

where μ_e is magnetic permeability.

Current density

The time rate of flow of charges per unit area is called current density. It is usually denoted by $\mathbf{J}$ given by

$$\mathbf{J} = \frac{1}{A}\frac{dQ}{dt}, \tag{1.6}$$

where A is area, Q is amount of charges passing through conductor under consideration.

Charge density

The amount of charges passing through per unit volume is called charge density. It is denoted by ρ_c and is given by

$$\rho_c = \frac{Q}{\widetilde{\mathrm{V}}}, \tag{1.7}$$

where Q is amount of charges and $\widetilde{\mathrm{V}}$ is volume through which Q is passing. In differential form

$$\rho_c = \frac{dQ}{d\widetilde{\mathrm{V}}} \tag{1.8}$$

and integral form is

$$Q = \int_{\widetilde{\mathrm{V}}} \rho_c d\widetilde{\mathrm{V}}. \tag{1.9}$$

Reynold number

The ratio of the inertial force to viscous force is called Reynold number i.e.

$$R = \frac{\text{Inertial force}}{\text{Viscous force}},$$

or

$$R = \frac{\text{Mass} \ \times \text{Acceleration}}{\text{Shear stress} \times \text{Cross sectional area}},$$

or

$$R = \frac{\text{Volume} \times \text{Mass density} \times \frac{\text{velocity}}{\text{Time}}}{\text{Shear stress} \times \text{Cross sectional area}},$$

or

$$R = \frac{\text{velocity} \times \text{Mass density} \times \text{velocity}}{\text{Shear stress}},$$

or

$$R = \frac{\rho U^2}{\mu \frac{du}{dy}} = \frac{\rho U^2}{\mu \frac{U}{L}} = \frac{UL}{\nu}, \tag{1.10}$$

where L is dimension of system under consideration and U is uniform velocity at infinity.

When the Reynold number of the system is small, then viscous force is predominant and the effect of viscosity is important. When the Reynold number is large, the inertial force is predominant and effect of viscosity is important only in the narrow boundary layer region near the solid boundary.

1.3 Newton's law of Viscosity

The Newton's law of viscosity states that the shear stress that deforms the fluid elements is directly proportional to the velocity gradient. Mathematically

$$\tau_{yx} = \mu \frac{du}{dy}, \tag{1.11}$$

where τ_{yx} is shear stress acting on fluid element and μ is constant of proportionality called coefficient of viscosity or simply viscosity.

Newtonian fluids.

The fluid that obeys Newton's law of viscosity is called Newtonian fluid.

Non-Newtonian fluids

The fluid which does not obey Newton's law of viscosity is called non-Newtonian fluid.

1.4 Continuity Equation

The law of conservation of mass states that matter cannot be created or destroyed in any classical system i.e. in any fluid flow system, fluid mass is conserved. This fact leads to establish a relation between the fluid density and fluid velocity at any point. Mathematical form of this relation is called equation of continuity which can be derived as follows:

Let us consider a control volume $\widetilde{V}$ enclosed by control surface S. Let $d\widetilde{V}$ be small volume element enclosed by surface element dS.

The fluid entering through small surface element $d\mathsf{S}$ is

$$\rho\mathbf{q}\cdot\mathbf{n}d\mathsf{S}, \tag{1.12}$$

where $\mathbf{n}$ is outward unit normal at surface element $d\mathsf{S}$.

The fluid entering through whole surface S is

$$\int_{\mathsf{S}} \rho\mathbf{q}\cdot\mathbf{n}d\mathsf{S}. \tag{1.13}$$

By divergence theorem we have

$$\int_{\mathsf{S}} \rho\mathbf{q}\cdot\mathbf{n}d\mathsf{S} = \int_{\widetilde{\mathrm{V}}} \boldsymbol{\nabla}\cdot(\rho\mathbf{q})\, d\widetilde{\mathrm{V}}. \tag{1.14}$$

The rate at which the fluid is leaving the volume $\widetilde{\mathrm{V}}$ is

$$-\frac{\partial}{\partial t}\int_{\widetilde{\mathrm{V}}} \rho d\widetilde{\mathrm{V}}. \tag{1.15}$$

Law of conservation of mass implies that

$$\int_{\widetilde{\mathrm{V}}} \boldsymbol{\nabla}\cdot(\rho\mathbf{q})\, d\widetilde{\mathrm{V}} = -\frac{\partial}{\partial t}\int_{\widetilde{\mathrm{V}}} \rho d\widetilde{\mathrm{V}}, \tag{1.16}$$

or

$$\int_{\widetilde{\mathrm{V}}} \left(\boldsymbol{\nabla}\cdot(\rho\mathbf{q}) + \frac{\partial\rho}{\partial t}\right) d\widetilde{\mathrm{V}} = 0. \tag{1.17}$$

Since $\widetilde{\mathrm{V}}$ is arbitrary, therefore integrand must vanish i.e.

$$\boldsymbol{\nabla}\cdot(\rho\mathbf{q}) + \frac{\partial\rho}{\partial t} = 0 \tag{1.18}$$

and is called continuity equation.

For steady flows $\frac{\partial\rho}{\partial t} = 0$ and thus Eq. (1.18) becomes

$$\boldsymbol{\nabla}\cdot(\rho\mathbf{q}) = 0. \tag{1.19}$$

If fluid is incompressible ($\rho =$ constant) then Eq. (1.19) takes the form

$$\nabla \cdot \mathbf{q} = 0 \tag{1.20}$$

which is continuity equation for steady flow of incompressible fluids.

1.5 Equation of momentum

Every particle of fluid at rest or in steady or accelerated motion obeys Newton's second law of motion which states that "the time rate of change of linear momentum is equal to the net force". To derive the differential form of the momentum equation, we shall apply Newton's second law to an infinitesimal fluid particle of mass dm. Thus for an infinitesimal system of mass dm, the Newton's second law can be written as

$$d\mathbf{F} = dm\frac{d\mathbf{q}}{dt}, \tag{1.21}$$

or

$$d\mathbf{F} = dm\left[u\frac{\partial \mathbf{q}}{\partial x} + v\frac{\partial \mathbf{q}}{\partial y} + w\frac{\partial \mathbf{q}}{\partial z} + \frac{\partial \mathbf{q}}{\partial t}\right], \tag{1.22}$$

where $\frac{d}{dt}$ is called the total or substantial derivative and $d\mathbf{F}$ is the net force acting on the infinitesimal system.

Since the forces acting on a fluid element may be classified as body forces and surface forces. Surface forces include both normal forces and tangential (shear) forces. We shall consider the x-component of the force, acting on a differential element of mass dm and volume $d\widetilde{\mathrm{V}} = dxdydz$. Only those stresses that act in the x-direction will give rise to surface forces in the x-direction. If the stresses at the center of the differential element are taken to be σ_{xx}, τ_{yx} and τ_{zx}, where σ_{xx} is the normal stress

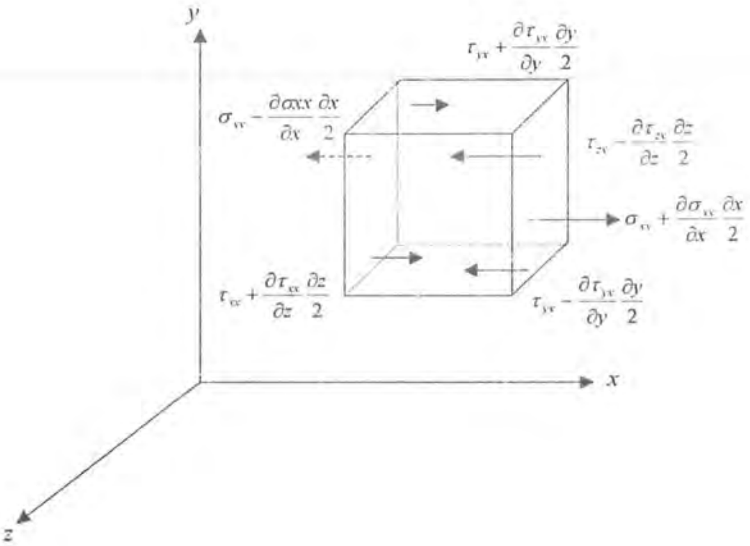

Figure 1.1:

and τ_{yx}, τ_{zx} are the shear or tangential stresses, then the stresses acting in the x-direction on all faces of the element (obtained by a Taylor series expansion about the centre of the element) are as shown in figure:

To obtain the net stress force in the x-direction, dF_{s_x}, we must sum the stress forces in the x-direction. Thus

$$\begin{aligned} dF_{s_x} &= \left(\sigma_{xx} + \frac{\partial \sigma_{xx}}{\partial x}\frac{dx}{2}\right) dydz - \left(\sigma_{xx} - \frac{\partial \sigma_{xx}}{\partial x}\frac{dx}{2}\right) dydz \\ &+ \left(\tau_{yx} + \frac{\partial \tau_{yx}}{\partial y}\frac{dy}{2}\right) dxdz - \left(\tau_{yx} - \frac{\partial \tau_{yx}}{\partial y}\frac{dy}{2}\right) dxdz \\ &+ \left(\tau_{zx} + \frac{\partial \tau_{zx}}{\partial z}\frac{dz}{2}\right) dxdy - \left(\tau_{zx} - \frac{\partial \tau_{zx}}{\partial z}\frac{dz}{2}\right) dxdy, \end{aligned} \tag{1.23}$$

or

$$dF_{s_x} = \left(\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z}\right) dxdydz. \tag{1.24}$$

Let ρB_x be the body force per unit volume in the x-direction. Then the net force in the x-direction dF_x is given by

$$dF_x = dF_{B_x} + dF_{S_x} = \left(\rho B_x + \frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z}\right) dxdydz. \tag{1.25}$$

Similarly the net forces in the y- and z-directions are

$$dF_y = dF_{B_y} + dF_{S_y} = \left(\rho B_y + \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \tau_{zy}}{\partial z}\right) dxdydz. \tag{1.26}$$

$$dF_z = dF_{B_z} + dF_{S_z} = \left(\rho B_z + \frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \sigma_{zz}}{\partial z}\right) dxdydz. \tag{1.27}$$

Using Eqs. (1.25) to (1.27) into the x-, y- and z-components of Eq. (1.22) and then dividing throughout by $dxdydz$, we obtain

$$\rho\frac{du}{dt} = \left(\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} + \rho B_x\right), \tag{1.28}$$

$$\rho\frac{dv}{dt} = \left(\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \tau_{zy}}{\partial z} + \rho B_y\right), \tag{1.29}$$

$$\rho\frac{dw}{dt} = \left(\frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \sigma_{zz}}{\partial z} + \rho B_z\right). \tag{1.30}$$

Equations (1.28) to (1.30) are the differential equations of motion for any fluid satisfying the continuum.

In vector form Eqs. (1.28) to (1.30) take the form

$$\rho\frac{d\mathbf{q}}{dt} = \rho\mathbf{B} + div\mathbf{T}, \tag{1.31}$$

where

$$\mathbf{T} = \begin{bmatrix} \sigma_{xx} & \tau_{yx} & \tau_{zx} \\ \tau_{xy} & \sigma_{yy} & \tau_{zy} \\ \tau_{xz} & \tau_{yz} & \sigma_{zz} \end{bmatrix}, \tag{1.32}$$

is the Cauchy stress tensor.

1.6 Law of conservation of charge

Let us consider a conducting medium of volume $\widetilde{\mathrm{V}}$ enclosed by closed surface S. The charge per unit time (current) leaving $\widetilde{\mathrm{V}}$ through surface S is

$$\frac{dQ}{dt} = \int_{\mathrm{S}} \mathbf{J} \cdot \mathbf{n} d\mathrm{S}, \tag{1.33}$$

where $\mathbf{n}$ is outward unit normal at small surface element $d\mathrm{S}$.

By divergence theorem we have

$$\frac{dQ}{dt} = \int_{\mathrm{S}} \mathbf{J} \cdot \mathbf{n} d\mathrm{S} = \int_{\widetilde{\mathrm{V}}} \boldsymbol{\nabla} \cdot \mathbf{J} d\widetilde{\mathrm{V}}. \tag{1.34}$$

Also

$$\frac{dQ}{dt} = -\frac{\partial}{\partial t} \int_{\widetilde{\mathrm{V}}} \rho_c d\widetilde{\mathrm{V}}. \tag{1.35}$$

From Eqs. (1.34) and (1.35) we have

$$\int_{\widetilde{\mathrm{V}}} \boldsymbol{\nabla} \cdot \mathbf{J} d\widetilde{\mathrm{V}} = -\int_{\widetilde{\mathrm{V}}} \frac{\partial \rho_c}{\partial t} d\widetilde{\mathrm{V}}, \tag{1.36}$$

or

$$\int_{\widetilde{\mathrm{V}}} \left(\boldsymbol{\nabla} \cdot \mathbf{J} + \frac{\partial \rho_c}{\partial t} \right) d\widetilde{\mathrm{V}} = 0. \tag{1.37}$$

Since $\widetilde{\mathrm{V}}$ is an arbitrary, therefore integrand must be identically zero i.e.

$$\boldsymbol{\nabla} \cdot \mathbf{J} + \frac{\partial \rho_c}{\partial t} = 0. \tag{1.38}$$

This is mathematical form of law of conservation of charge or continuity equation for flow of charges (currents).

For steady currents

$$\frac{\partial \rho_c}{\partial t} = 0 \tag{1.39}$$

and thus Eq. (1.38) takes the form

$$\nabla \cdot \mathbf{J} = 0 \tag{1.40}$$

which states that the net current entering or leaving any closed surface is zero.

1.7 Maxwell's equations

James Clerk Maxwell derived a set of equations known as Maxwell's equations and are given by

$$\nabla \times \mathbf{H} = \mathbf{J} + \frac{1}{\mu_e C^2}\frac{\partial \mathbf{E}}{\partial t}, \tag{1.41}$$

$$\nabla \cdot \mathbf{H} = 0, \tag{1.42}$$

$$\nabla \times \mathbf{E} = -\mu_e \frac{\partial \mathbf{H}}{\partial t}, \tag{1.43}$$

$$\nabla \cdot \mathbf{E} = \frac{\rho_c}{\epsilon}, \tag{1.44}$$

where **H** is the total magnetic field, C is the velocity of light, **E** is the electric field, ρ the charge density, **J** the current density, μ_e the magnetic permeability and ϵ is the permittivity.

1.8 Hall effect

In the presence of magnetic field the motion of electrons and holes give rise to a galvanometric effects. The most effective is the Hall effect discovered by H.E. Hall in 1879. When the current carrying conductor is placed in a magnetic field, then electric field is produced which is normal or perpendicular to both the direction of the current and the magnetic field. This phenomenon is known as Hall effect and the field produced is called Hall field.

Consider a conducting material of rectangular shaped bar. We first consider the situation when magnetic field is not introduced and there is an electric current flowing through the rectangular shaped bar and conduction electrons are drifted in the direction opposite to the direction of current. When transverse magnetic field is introduced then the moving charges experience a force called Lorentz force which is given by

$$\mathbf{F}_L = \mu_e e \left(\mathbf{V}_d \times \mathbf{H} \right), \tag{1.45}$$

where $\mathbf{V}_d$, e and $\mathbf{H}$ denote respectively drift velocity, charge on electron and imposed magnetic field. This force causes the electrons to bend downwards. The positive charge carriers are left on top of the specimen bar and the negative charge carriers are pushed down to the bottom of the bar. This effect is called Hall effect and produces electric field $\mathbf{E}_H$. The potential difference across the bar is called Hall potential difference (Hall voltage). The combination of positive charge carriers on the upper side and negative charge carriers on the lower side of the bar generates downward electric field called Hall field $\mathbf{E}_H$. It exerts an upward force on electrons partially cancelling the force of magnetic field. This force is called Hall force and equals to

$$\mathbf{F}_H = e\mathbf{E}_H. \tag{1.46}$$

The net force F on electrons is given by

$$F = \text{Lorentz force - Hall force} \tag{1.47}$$

or

$$\mathbf{F} = \mathbf{F}_L - \mathbf{F}_H. \tag{1.48}$$

In equilibrium both forces cancel out each other and hence

$$\mathbf{F}_L - \mathbf{F}_H = 0, \tag{1.49}$$

or

$$\mathbf{F}_L = \mathbf{F}_H. \tag{1.50}$$

From Eqs. (1.45) and (1.46) we have

$$e\mathbf{E}_H = \mu_e e\left(\mathbf{V}_d \times \mathbf{H}\right), \tag{1.51}$$

or

$$\mathbf{E}_H = \mu_e\left(\mathbf{V}_d \times \mathbf{H}\right). \tag{1.52}$$

The drift velocity $\mathbf{V}_d$ of n_e electrons in term of current density $\mathbf{J}$ is

$$\mathbf{V}_d = -\frac{1}{en_e}\mathbf{J}. \tag{1.53}$$

Using Eq. (1.53) in Eq. (1.52) we have

$$\mathbf{E}_H = -\frac{\mu_e}{en_e}\left(\mathbf{J} \times \mathbf{H}\right). \tag{1.54}$$

Above equation shows that $\mathbf{E}_H$ is proportional to $\mathbf{J} \times \mathbf{H}$ and $-\frac{\mu_e}{en_e}$ is constant of proportionality called Hall co-efficient and is denoted by R_H i.e.

$$R_H = -\frac{\mu_e}{en_e}. \tag{1.55}$$

A similar analysis can be carried out for P-type charge carriers. For P-type charge carriers Hall co-efficient is given by

$$R_H = -\frac{\mu_e}{ep}, \tag{1.56}$$

where p is equal to charge on electron.

Equations (1.55) and (1.56) are very useful for the measurement of carriers concentration in semi–conductor or metals.

1.9 Ohm's law

Let current $\mathbf{I}$ be passing through a conductor of length L and area of cross-section A. It is experimentally proved that the current passing through the conductor is proportional to the voltage $\mathbf{U}$ applied across the ends of conductor i.e.

$$\mathbf{U} = \mathbf{I}R, \tag{1.57}$$

where R is constant of proportionality and is called resistance of conductor.

The resistance of any conductor is defined as

$$R = \rho_r \frac{L}{A}, \tag{1.58}$$

where ρ_r is resistivity of the conductor and thus Eq. (1.57) becomes

$$\mathbf{U} = \rho_r \left(\frac{\mathbf{I}}{A} \right) L. \tag{1.59}$$

Since $\frac{\mathbf{I}}{A} = \mathbf{J}$ called current density, so Eq. (1.59) takes the form

$$\mathbf{J} = \sigma \frac{\mathbf{U}}{L}, \tag{1.60}$$

where σ is conductivity of conductor.

As

$$\mathbf{U} = \mathbf{E}L \tag{1.61}$$

so Eqs. (1.60) and (1.61) imply that

$$\mathbf{J} = \sigma\mathbf{E} \tag{1.62}$$

which is called Ohm's law and states that the current density **J** is directly proportional to electric field intensity **E**.

Now we consider the case of current carrying conductor moving in a uniform transverse magnetic field. There are three types of electric fields in this case.

1. Electric field **E** due to current **I**.
2. Electric field $\mathbf{E}_m$ due to induced magnetic field.
3. Electric field $\mathbf{E}_H$ due to Hall effect.

The electric field intensity $\mathbf{E}_m$ induced due to the motion of conductor across the transverse magnetic field is

$$\mathbf{E}_m = \mu_e \mathbf{q} \times \mathbf{H} \tag{1.63}$$

and electric field intensity $\mathbf{E}_H$ due to Hall effect is

$$\mathbf{E}_H = -\frac{\mu_e}{en_e}\mathbf{J} \times \mathbf{H}, \tag{1.64}$$

where n_e is number of electrons and e is charge on electron.

If **E** is electric field intensity due to current **I** passing through the conductor then net electric field intensity is

$$\mathbf{E}_t = \mathbf{E} + \mathbf{E}_m + \mathbf{E}_H, \tag{1.65}$$

or

$$\mathbf{E}_t = \mathbf{E} + \mu_e \mathbf{q} \times \mathbf{H} - \frac{\mu_e}{en_e}\mathbf{J} \times \mathbf{H}. \tag{1.66}$$

Equation (1.62) takes the form

$$\mathbf{J} = \boldsymbol{\sigma}\left[\mathbf{E}+\mu_e\mathbf{q}\times\mathbf{H}-\frac{\mu_e}{en_e}\mathbf{q}\times\mathbf{H}\right] \tag{1.67}$$

which is known as generalized Ohm's law.

1.10 Equation of motion for magnetohydrodynamic viscous fluid

All fluids in motion obey law of conservation of momentum which is given by

$$\rho\frac{d\mathbf{q}}{dt} = div\mathbf{T} + \mu_e\mathbf{J}\times\mathbf{H}, \tag{1.68}$$

where ρ mass density of the fluid, μ_e magnetic permeability of the fluid and $\mathbf{T}$ is Cauchy stress tensor defined by

$$\mathbf{T} = -p\mathbf{I} + \mu\mathbf{A}_1, \tag{1.69}$$

where μ is coefficient of viscosity, p is pressure and $\mathbf{A}_1$ is kinematical tensor which is defined by

$$\mathbf{A}_1 = grad\mathbf{q} + (grad\mathbf{q})^T. \tag{1.70}$$

We assume the velocity field of the form

$$\mathbf{q} = [u(x,y,z,t), v(x,y,z,t), w(x,y,z,t)]. \tag{1.71}$$

From above equation we have

$$grad\mathbf{q} = \mathbf{L} = \begin{bmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{bmatrix}, \tag{1.72}$$

$$(grad\mathbf{q})^T = \mathbf{L}^T = \begin{bmatrix} \frac{\partial u}{\partial x} & \frac{\partial v}{\partial x} & \frac{\partial w}{\partial x} \\ \frac{\partial u}{\partial y} & \frac{\partial v}{\partial y} & \frac{\partial w}{\partial y} \\ \frac{\partial u}{\partial z} & \frac{\partial v}{\partial z} & \frac{\partial w}{\partial z} \end{bmatrix}. \tag{1.73}$$

Using Eqs. (1.72) and (1.73) in Eq. (1.70) we obtain

$$\mathbf{A}_1 = \mathbf{L} + \mathbf{L}^T = \begin{bmatrix} 2\frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} & \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} \\ \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} & 2\frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} \\ \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} & \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} & 2\frac{\partial w}{\partial z} \end{bmatrix}. \tag{1.74}$$

Taking divergence of (1.69) we obtain

$$div\mathbf{T} = div\left(-p\mathbf{I}\right) + \mu div\mathbf{A}_1. \tag{1.75}$$

Writing $div\mathbf{A}_1$ and $div\left(-p\mathbf{I}\right)$ in component form we have

$$\left(div\left(-p\mathbf{I}\right)\right)_x = -\frac{\partial p}{\partial x}, \tag{1.76}$$

$$\left(div\left(-p\mathbf{I}\right)\right)_y = -\frac{\partial p}{\partial y}, \tag{1.77}$$

$$\left(div\left(-p\mathbf{I}\right)\right)_z = -\frac{\partial p}{\partial z}, \tag{1.78}$$

$$div\left(\mathbf{A}_1\right)_x = 2\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 v}{\partial x \partial y} + \frac{\partial^2 u}{\partial z^2} + \frac{\partial^2 w}{\partial x \partial z}, \tag{1.79}$$

$$div\left(\mathbf{A}_1\right)_y = \frac{\partial^2 u}{\partial x \partial y} + \frac{\partial^2 v}{\partial x^2} + 2\frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2} + \frac{\partial^2 w}{\partial y \partial z}, \tag{1.80}$$

$$div\left(\mathbf{A}_1\right)_z = \frac{\partial^2 u}{\partial x \partial z} + \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 v}{\partial y \partial z} + \frac{\partial^2 w}{\partial y^2} + 2\frac{\partial^2 w}{\partial z^2}. \tag{1.81}$$

Taking x, y, z-components of Eq. (1.65) we get

$$\left(div\mathbf{T}\right)_x = -\frac{\partial p}{\partial x} + \mu\left[2\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 v}{\partial x \partial y} + \frac{\partial^2 u}{\partial z^2} + \frac{\partial^2 w}{\partial x \partial z}\right], \tag{1.82}$$

$$\left(div\mathbf{T}\right)_y = -\frac{\partial p}{\partial y} + \mu\left[\frac{\partial^2 u}{\partial x \partial y} + \frac{\partial^2 v}{\partial x^2} + 2\frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2} + \frac{\partial^2 w}{\partial y \partial z}\right], \tag{1.83}$$

$$(div\mathbf{T})_z = -\frac{\partial p}{\partial z} + \mu\left[\frac{\partial^2 u}{\partial x \partial z} + \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 v}{\partial y \partial z} + \frac{\partial^2 w}{\partial y^2} + 2\frac{\partial^2 w}{\partial z^2}\right]. \quad (1.84)$$

Writing $\mathbf{J} \times \mathbf{H}$ in component form we have

$$(\mathbf{J} \times \mathbf{H})_x = J_y H_z - J_z H_y, \quad (1.85)$$

$$(\mathbf{J} \times \mathbf{H})_y = J_z H_x - J_x H_z, \quad (1.86)$$

$$(\mathbf{J} \times \mathbf{H})_x = J_x H_y - J_y H_x. \quad (1.87)$$

In Eq. (1.68) $\frac{d}{dt} = \frac{\partial}{\partial t} + \mathbf{q} \cdot \boldsymbol{\nabla}$ is called material derivative and so Eq. (1.68) can be written as

$$\rho\left[\frac{\partial \mathbf{q}}{\partial t} + (\mathbf{q} \cdot \boldsymbol{\nabla})\,\mathbf{q}\right] = div\mathbf{T} + \mu_e \mathbf{J} \times \mathbf{H}. \quad (1.88)$$

Now making use of the Eqs. (1.82) to (1.87) in component form of above equation we get

$$\begin{aligned}\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} &= -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\left[2\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 v}{\partial x \partial y} + \frac{\partial^2 u}{\partial z^2} + \frac{\partial^2 w}{\partial x \partial z}\right] \\ &\quad + \frac{\mu_e}{\rho}\left[J_y H_z - J_z H_y\right], \quad (1.89)\end{aligned}$$

$$\begin{aligned}\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} &= -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\left[2\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 v}{\partial x \partial y} + \frac{\partial^2 u}{\partial z^2} + \frac{\partial^2 w}{\partial x \partial z}\right] \\ &\quad + \frac{\mu_e}{\rho}\left[J_z H_x - J_x H_z\right], \quad (1.90)\end{aligned}$$

$$\begin{aligned}\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} &= -\frac{1}{\rho}\frac{\partial p}{\partial z} + \nu\left[\frac{\partial^2 u}{\partial x \partial z} + \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 v}{\partial y \partial z} + \frac{\partial^2 w}{\partial y^2} + 2\frac{\partial^2 w}{\partial z^2}\right] \\ &\quad + \frac{\mu_e}{\rho}\left[J_x H_y - J_y H_x\right], \quad (1.91)\end{aligned}$$

where $\nu = \frac{\mu}{\rho}$ is called kinematic viscosity.

1.11 Equation of motion for magnetohydrodynamic second order fluid

The Cauchy stress tensor for an incompressible second order fluid is given by

$$\mathbf{T} = -p\mathbf{I} + \mu\mathbf{A}_1 + \alpha_1\mathbf{A}_2 + \alpha_2\mathbf{A}_1^2, \tag{1.92}$$

where μ is coefficient of viscosity, α_1 and α_2 are material moduli which are usually referred to as the normal stress moduli, p is pressure and $\mathbf{A}_1$ and $\mathbf{A}_2$ are kinematical tensors defined by

$$\mathbf{A}_1 = grad\mathbf{q} + (grad\mathbf{q})^T, \tag{1.93}$$

$$\mathbf{A}_2 = \frac{d\mathbf{A}_1}{dt} + \mathbf{A}_1 grad\mathbf{q} + (grad\mathbf{q})^T \mathbf{A}_1. \tag{1.94}$$

If an incompressible fluid of second order is to have motions which are incompatible with thermodynamics in the sense of Clausius-Duhem inequality and the condition that the Helmholtz free energy be minimum when the fluid is at rest, then following conditions must be satisfied

$$\mu \geq 0,\ \alpha_1 > 0,\ \alpha_1 + \alpha_2 = 0. \tag{1.95}$$

We assume the velocity field of the form

$$\mathbf{q} = [u(x, y, z, t), v(x, y, z, t), w(x, y, z, t)] \tag{1.96}$$

and hence we have

$$(grad\mathbf{q})^T = \mathbf{L}^T = \begin{bmatrix} \frac{\partial u}{\partial x} & \frac{\partial v}{\partial x} & \frac{\partial w}{\partial x} \\ \frac{\partial u}{\partial y} & \frac{\partial v}{\partial y} & \frac{\partial w}{\partial y} \\ \frac{\partial u}{\partial z} & \frac{\partial v}{\partial z} & \frac{\partial w}{\partial z} \end{bmatrix}, \tag{1.97}$$

$$\mathbf{A}_1 = \mathbf{L} + \mathbf{L}^T = \begin{bmatrix} 2\frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} & \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} \\ \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} & 2\frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} \\ \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} & \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} & 2\frac{\partial w}{\partial z} \end{bmatrix}, \tag{1.98}$$

$$\mathbf{A}_1^2 = \begin{bmatrix} \begin{bmatrix} 4\left(\frac{\partial u}{\partial x}\right)^2 \\ +\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)^2 \\ +\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right)^2 \end{bmatrix} & \begin{bmatrix} 2\frac{\partial u}{\partial x}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) \\ +2\frac{\partial v}{\partial y}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) \\ +\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right) \\ +\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right) \end{bmatrix} & \begin{bmatrix} 2\frac{\partial u}{\partial x}\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right) \\ +\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) \\ +\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right) \\ +2\frac{\partial w}{\partial z}\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right) \end{bmatrix} \\ \begin{bmatrix} 2\frac{\partial u}{\partial x}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) \\ +2\frac{\partial v}{\partial y}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) \\ +\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right) \\ +\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right) \end{bmatrix} & \begin{bmatrix} \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)^2 \\ +4\left(\frac{\partial v}{\partial y}\right)^2 + \\ \left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right)^2 \end{bmatrix} & \begin{bmatrix} \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) \\ +\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right) \\ +2\frac{\partial v}{\partial y}\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right) \\ +2\frac{\partial w}{\partial z}\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right) \end{bmatrix} \\ \begin{bmatrix} 2\frac{\partial u}{\partial x}\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right) \\ +\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right) \\ +\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) \\ +2\frac{\partial w}{\partial z}\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right) \end{bmatrix} & \begin{bmatrix} \left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right) \\ +\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) \\ +2\frac{\partial v}{\partial y}\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right) \\ +2\frac{\partial w}{\partial z}\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right) \end{bmatrix} & \begin{bmatrix} \left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right)^2 \\ +\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right)^2 \\ +4\left(\frac{\partial w}{\partial z}\right)^2 \end{bmatrix} \end{bmatrix}, \tag{1.99}$$

$$\frac{\partial A_1}{\partial t} = \begin{bmatrix} 2\frac{\partial^2 u}{\partial x \partial t} & \frac{\partial^2 u}{\partial y \partial t} + \frac{\partial^2 v}{\partial x \partial t} & \frac{\partial^2 u}{\partial z \partial t} + \frac{\partial^2 w}{\partial x \partial t} \\ \frac{\partial^2 u}{\partial y \partial t} + \frac{\partial^2 v}{\partial x \partial t} & 2\frac{\partial^2 v}{\partial y \partial t} & \frac{\partial^2 v}{\partial z \partial t} + \frac{\partial^2 w}{\partial y \partial t} \\ \frac{\partial^2 u}{\partial z \partial t} + \frac{\partial^2 w}{\partial x \partial t} & \frac{\partial^2 v}{\partial z \partial t} + \frac{\partial^2 w}{\partial y \partial t} & 2\frac{\partial^2 w}{\partial z \partial t} \end{bmatrix}, \tag{1.100}$$

$$
\mathbf{A}_1 grad\mathbf{q} = \begin{bmatrix}
\begin{bmatrix} 2\left(\frac{\partial u}{\partial x}\right)^2 \\ +\frac{\partial v}{\partial x}\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right) \\ +\frac{\partial w}{\partial x}\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}\right) \end{bmatrix} &
\begin{bmatrix} 2\frac{\partial u}{\partial x}\frac{\partial u}{\partial y} \\ +\frac{\partial v}{\partial y}\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right) \\ +\frac{\partial w}{\partial y}\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}\right) \end{bmatrix} &
\begin{bmatrix} 2\frac{\partial u}{\partial x}\frac{\partial u}{\partial z} \\ +\frac{\partial v}{\partial z}\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right) \\ +\frac{\partial w}{\partial z}\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}\right) \end{bmatrix} \\
\begin{bmatrix} \frac{\partial u}{\partial x}\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right) \\ +2\frac{\partial v}{\partial x}\frac{\partial v}{\partial y} \\ +\frac{\partial w}{\partial x}\left(\frac{\partial v}{\partial z}+\frac{\partial w}{\partial y}\right) \end{bmatrix} &
\begin{bmatrix} \frac{\partial u}{\partial y}\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right) \\ +2\left(\frac{\partial v}{\partial y}\right)^2 \\ +\frac{\partial w}{\partial y}\left(\frac{\partial v}{\partial z}+\frac{\partial w}{\partial y}\right) \end{bmatrix} &
\begin{bmatrix} \frac{\partial u}{\partial z}\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right) \\ +2\frac{\partial v}{\partial y}\frac{\partial v}{\partial z} \\ +\frac{\partial w}{\partial z}\left(\frac{\partial v}{\partial z}+\frac{\partial w}{\partial y}\right) \end{bmatrix} \\
\begin{bmatrix} \frac{\partial u}{\partial x}\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}\right) \\ +\frac{\partial v}{\partial x}\left(\frac{\partial v}{\partial z}+\frac{\partial w}{\partial y}\right) \\ +2\frac{\partial w}{\partial x}\frac{\partial w}{\partial z} \end{bmatrix} &
\begin{bmatrix} \frac{\partial u}{\partial y}\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}\right) \\ +\frac{\partial v}{\partial y}\left(\frac{\partial v}{\partial z}+\frac{\partial w}{\partial y}\right) \\ +2\frac{\partial w}{\partial y}\frac{\partial w}{\partial z} \end{bmatrix} &
\begin{bmatrix} \frac{\partial u}{\partial z}\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}\right) \\ +\frac{\partial v}{\partial z}\left(\frac{\partial v}{\partial z}+\frac{\partial w}{\partial y}\right) \\ +2\left(\frac{\partial w}{\partial x}\right)^2 \end{bmatrix}
\end{bmatrix}, \tag{1.101}
$$

$$
(grad\mathbf{q})^T \mathbf{A}_1 = \begin{bmatrix}
\begin{bmatrix} 2\left(\frac{\partial u}{\partial x}\right)^2 \\ +\frac{\partial v}{\partial x}\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right) \\ +\frac{\partial w}{\partial x}\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}\right) \end{bmatrix} &
\begin{bmatrix} \frac{\partial u}{\partial x}\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right) \\ +2\frac{\partial v}{\partial x}\frac{\partial v}{\partial y} \\ +\frac{\partial w}{\partial x}\left(\frac{\partial v}{\partial z}+\frac{\partial w}{\partial y}\right) \end{bmatrix} &
\begin{bmatrix} \frac{\partial u}{\partial x}\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}\right) \\ +\frac{\partial v}{\partial x}\left(\frac{\partial v}{\partial z}+\frac{\partial w}{\partial y}\right) \\ +2\frac{\partial w}{\partial x}\frac{\partial w}{\partial z} \end{bmatrix} \\
\begin{bmatrix} 2\frac{\partial u}{\partial x}\frac{\partial u}{\partial y} \\ +\frac{\partial v}{\partial y}\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right) \\ +\frac{\partial w}{\partial y}\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}\right) \end{bmatrix} &
\begin{bmatrix} \frac{\partial u}{\partial y}\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right) \\ +2\left(\frac{\partial v}{\partial y}\right)^2 \\ +\frac{\partial w}{\partial y}\left(\frac{\partial v}{\partial z}+\frac{\partial w}{\partial y}\right) \end{bmatrix} &
\begin{bmatrix} \frac{\partial u}{\partial y}\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}\right) \\ +\frac{\partial v}{\partial y}\left(\frac{\partial v}{\partial z}+\frac{\partial w}{\partial y}\right) \\ +2\frac{\partial w}{\partial y}\frac{\partial w}{\partial z} \end{bmatrix} \\
\begin{bmatrix} 2\frac{\partial u}{\partial x}\frac{\partial u}{\partial z} \\ +\frac{\partial v}{\partial z}\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right) \\ +\frac{\partial w}{\partial z}\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}\right) \end{bmatrix} &
\begin{bmatrix} \frac{\partial u}{\partial z}\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right) \\ 2\frac{\partial v}{\partial z}\frac{\partial v}{\partial y} \\ +\frac{\partial w}{\partial z}\left(\frac{\partial v}{\partial z}+\frac{\partial w}{\partial y}\right) \end{bmatrix} &
\begin{bmatrix} \frac{\partial u}{\partial z}\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}\right) \\ +\frac{\partial v}{\partial z}\left(\frac{\partial v}{\partial z}+\frac{\partial w}{\partial y}\right) \\ +2\left(\frac{\partial w}{\partial x}\right)^2 \end{bmatrix}
\end{bmatrix}, \tag{1.102}
$$

$$
(\mathbf{q}\cdot\nabla)\,\mathbf{A}_1 = \begin{bmatrix}
2\begin{bmatrix} u\frac{\partial^2 u}{\partial x^2} \\ +v\frac{\partial^2 u}{\partial x\partial y} \\ +w\frac{\partial^2 u}{\partial x\partial z}\end{bmatrix} & \begin{bmatrix} u\frac{\partial^2 u}{\partial x\partial y}+v\frac{\partial^2 u}{\partial y^2} \\ +w\frac{\partial^2 u}{\partial y\partial z}+u\frac{\partial^2 v}{\partial x^2} \\ +v\frac{\partial^2 v}{\partial x\partial y}+w\frac{\partial^2 v}{\partial x\partial z}\end{bmatrix} & \begin{bmatrix} u\frac{\partial^2 u}{\partial x\partial z}+v\frac{\partial^2 u}{\partial y\partial z} \\ +w\frac{\partial^2 u}{\partial z^2}+u\frac{\partial^2 w}{\partial x^2} \\ +v\frac{\partial^2 w}{\partial x\partial y}+w\frac{\partial^2 w}{\partial x\partial z}\end{bmatrix} \\
\begin{bmatrix} u\frac{\partial^{2u}}{\partial x\partial y}+v\frac{\partial^2 u}{\partial y^2} \\ +w\frac{\partial^2 u}{\partial y\partial z}+u\frac{\partial^2 v}{\partial x^2} \\ +v\frac{\partial^2 v}{\partial x\partial y}+w\frac{\partial^2 v}{\partial x\partial z}\end{bmatrix} & 2\begin{bmatrix} u\frac{\partial^2 v}{\partial x\partial y} \\ +v\frac{\partial^2 v}{\partial y^2} \\ +w\frac{\partial^2 v}{\partial y\partial z}\end{bmatrix} & \begin{bmatrix} u\frac{\partial^2 v}{\partial x\partial z}+v\frac{\partial^2 v}{\partial y\partial z} \\ +w\frac{\partial^2 v}{\partial z^2}+u\frac{\partial^2 w}{\partial x\partial y} \\ +v\frac{\partial^2 w}{\partial y^2}+w\frac{\partial^2 w}{\partial y\partial z}\end{bmatrix} \\
\begin{bmatrix} u\frac{\partial^2 u}{\partial x\partial z}+v\frac{\partial^2 u}{\partial y\partial z} \\ +w\frac{\partial^2 u}{\partial z^2}+u\frac{\partial^2 w}{\partial x^2} \\ +v\frac{\partial^2 w}{\partial x\partial y}+w\frac{\partial^2 w}{\partial x\partial z}\end{bmatrix} & \begin{bmatrix} u\frac{\partial^2 v}{\partial x\partial z}+v\frac{\partial^2 v}{\partial y\partial z} \\ +w\frac{\partial^2 v}{\partial z\partial z}+u\frac{\partial^2 w}{\partial x\partial y} \\ +v\frac{\partial^2 w}{\partial y\partial y}+w\frac{\partial^2 w}{\partial z\partial y}\end{bmatrix} & 2\begin{bmatrix} u\frac{\partial^2 w}{\partial x\partial z} \\ +v\frac{\partial^2 w}{\partial y\partial z} \\ +w\frac{\partial^2 w}{\partial z^2}\end{bmatrix}
\end{bmatrix}. \tag{1.103}
$$

In component form the divergence of Eqs. (1.98) to (1.103) yields

$$(div\mathbf{A}_1)_x = 2\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 v}{\partial x\partial y}+\frac{\partial^2 u}{\partial z^2}+\frac{\partial^2 w}{\partial x\partial z}, \tag{1.104}$$

$$(div\mathbf{A}_1)_y = \frac{\partial^2 u}{\partial x\partial y}+\frac{\partial^2 v}{\partial x^2}+2\frac{\partial^2 v}{\partial y^2}+\frac{\partial^2 v}{\partial z^2}+\frac{\partial^2 w}{\partial y\partial z}, \tag{1.105}$$

$$(div\mathbf{A}_1)_z = \frac{\partial^2 u}{\partial x\partial z}+\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 v}{\partial y\partial z}+\frac{\partial^2 w}{\partial y^2}+2\frac{\partial^2 w}{\partial z^2}, \tag{1.106}$$

$$\left(div\frac{\partial \mathbf{A}_1}{\partial t}\right)_x = 2\frac{\partial^3 u}{\partial x^2\partial t}+\frac{\partial^3 u}{\partial y^2\partial t}+\frac{\partial^3 v}{\partial x\partial y\partial t}+\frac{\partial^3 u}{\partial z^2\partial t}+\frac{\partial^3 w}{\partial x\partial z\partial t}, \tag{1.107}$$

$$\left(div\frac{\partial \mathbf{A}_1}{\partial t}\right)_y = \frac{\partial^3 u}{\partial x\partial y\partial t}+\frac{\partial^3 v}{\partial x^2\partial t}+2\frac{\partial^3 v}{\partial y^2\partial t}+\frac{\partial^3 v}{\partial z^2\partial t}+\frac{\partial^3 w}{\partial y\partial z\partial t}, \tag{1.108}$$

$$\left(div\frac{\partial \mathbf{A}_1}{\partial t}\right)_z = \frac{\partial^3 u}{\partial x\partial z\partial t}+\frac{\partial^3 w}{\partial x^2\partial t}+\frac{\partial^3 v}{\partial y\partial z\partial t}+\frac{\partial^3 w}{\partial y^2\partial t}+2\frac{\partial^3 w}{\partial z^2\partial t}, \tag{1.109}$$

$$
\begin{aligned}
(div\,((\mathbf{q}\cdot\nabla)\,\mathbf{A}_1))_x \;=\;& 2u\frac{\partial^3 u}{\partial x^3}+2\frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial x^2}+2v\frac{\partial^3 u}{\partial x^2\partial y}+2\frac{\partial v}{\partial x}\frac{\partial^2 u}{\partial x\partial y}+2w\frac{\partial^3 u}{\partial x^2\partial z} \\
&+2\frac{\partial w}{\partial x}\frac{\partial^2 u}{\partial x\partial z}+u\frac{\partial^3 u}{\partial x\partial y^2}+\frac{\partial u}{\partial y}\frac{\partial^2 u}{\partial x\partial y}+v\frac{\partial^3 u}{\partial y^3}+\frac{\partial v}{\partial y}\frac{\partial^2 u}{\partial y^2}
\end{aligned}
$$

$$
\begin{aligned}
&+w\frac{\partial^3 u}{\partial y^2\partial z}+\frac{\partial w}{\partial y}\frac{\partial^2 u}{\partial y\partial z}+u\frac{\partial^3 w}{\partial x^2\partial y}+\frac{\partial u}{\partial y}\frac{\partial^2 w}{\partial x^2}+v\frac{\partial^3 w}{\partial x\partial y^2}\\
&+\frac{\partial v}{\partial y}\frac{\partial^2 w}{\partial x\partial y}+w\frac{\partial^3 w}{\partial x\partial y\partial z}+\frac{\partial w}{\partial y}\frac{\partial^2 w}{\partial x\partial z}+\frac{\partial}{\partial z}u\frac{\partial^3 u}{\partial x\partial z^2}+\frac{\partial u}{\partial z}\frac{\partial^2 u}{\partial x\partial z}\\
&+v\frac{\partial^3 u}{\partial y\partial z^2}+\frac{\partial v}{\partial z}\frac{\partial^2 u}{\partial y\partial z}+w\frac{\partial^3 u}{\partial z^3}+\frac{\partial w}{\partial z}\frac{\partial^2 u}{\partial z^2}+u\frac{\partial^3 w}{\partial x^2\partial z}\\
&+\frac{\partial u}{\partial z}\frac{\partial^2 w}{\partial x^2}+v\frac{\partial^3 w}{\partial x\partial y\partial z}+\frac{\partial v}{\partial z}\frac{\partial^2 w}{\partial x\partial y}+w\frac{\partial^3 w}{\partial x\partial z^2}\\
&+\frac{\partial w}{\partial z}\frac{\partial^2 w}{\partial x\partial z},
\end{aligned}
\tag{1.110}
$$

$$
\begin{aligned}
(div\left((\mathbf{q}\cdot\nabla)\,\mathbf{A}_1\right))_y \;=\; & u\frac{\partial^3 u}{\partial x^2\partial y}+\frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial x\partial y}+v\frac{\partial^3 u}{\partial x\partial y^2}+\frac{\partial v}{\partial x}\frac{\partial^2 u}{\partial y^2}+w\frac{\partial^3 u}{\partial x\partial y\partial z}\\
&+\frac{\partial w}{\partial x}\frac{\partial^2 u}{\partial y\partial z}+u\frac{\partial^3 v}{\partial x^3}+\frac{\partial u}{\partial x}\frac{\partial^2 v}{\partial x^2}+v\frac{\partial^3 v}{\partial x^2\partial y}+\frac{\partial v}{\partial x}\frac{\partial^2 v}{\partial x\partial y}\\
&+w\frac{\partial^3 v}{\partial x^2\partial z}+\frac{\partial w}{\partial x}\frac{\partial^2 v}{\partial x\partial z}+2u\frac{\partial^3 v}{\partial x\partial y^2}+2\frac{\partial u}{\partial y}\frac{\partial^2 v}{\partial x\partial y}+2v\frac{\partial^3 v}{\partial y^3}\\
&+2\frac{\partial v}{\partial y}\frac{\partial^2 v}{\partial y^2}+2w\frac{\partial^3 v}{\partial y^2\partial z}+2\frac{\partial w}{\partial y}\frac{\partial^2 v}{\partial y\partial z}+u\frac{\partial^3 v}{\partial x\partial z^2}+\frac{\partial u}{\partial z}\frac{\partial^2 v}{\partial x\partial z}\\
&+v\frac{\partial^3 v}{\partial y\partial z^2}+\frac{\partial v}{\partial z}\frac{\partial^2 v}{\partial y\partial z}+w\frac{\partial^3 v}{\partial z^3}+\frac{\partial w}{\partial z}\frac{\partial^2 v}{\partial z^2}+u\frac{\partial^3 w}{\partial x\partial y\partial z}\\
&+\frac{\partial u}{\partial z}\frac{\partial^2 w}{\partial x\partial y}+v\frac{\partial^3 w}{\partial y^2\partial z}+\frac{\partial v}{\partial z}\frac{\partial^2 w}{\partial y^2}+w\frac{\partial^2 w}{\partial y\partial z^2}\\
&+\frac{\partial w}{\partial z}\frac{\partial^2 w}{\partial y\partial z},
\end{aligned}
\tag{1.111}
$$

$$
\begin{aligned}
(div\left((\mathbf{q}\cdot\nabla)\,\mathbf{A}_1\right))_z \;=\; & u\frac{\partial^3 u}{\partial x^2\partial z}+\frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial x\partial z}+v\frac{\partial^3 u}{\partial x\partial y\partial z}+\frac{\partial v}{\partial x}\frac{\partial^2 u}{\partial y\partial z}+w\frac{\partial^3 u}{\partial x\partial z^2}\\
&+\frac{\partial w}{\partial x}\frac{\partial^2 u}{\partial z^2}+u\frac{\partial^3 w}{\partial x^3}+\frac{\partial u}{\partial x}\frac{\partial^2 w}{\partial x^2}+v\frac{\partial^3 w}{\partial x^2\partial y}+\frac{\partial v}{\partial x}\frac{\partial^2 w}{\partial x\partial y}\\
&+w\frac{\partial^3 w}{\partial x^2\partial z}+\frac{\partial w}{\partial x}\frac{\partial^2 w}{\partial x\partial z}+u\frac{\partial^3 v}{\partial x\partial y\partial z}+\frac{\partial u}{\partial y}\frac{\partial^2 v}{\partial x\partial z}+v\frac{\partial^3 v}{\partial y^2\partial z}\\
&+\frac{\partial v}{\partial y}\frac{\partial^2 v}{\partial y\partial z}+w\frac{\partial^3 v}{\partial y\partial z^2}+\frac{\partial w}{\partial y}\frac{\partial^2 v}{\partial z^2}+u\frac{\partial^3 w}{\partial x\partial y^2}+\frac{\partial u}{\partial y}\frac{\partial^2 w}{\partial x\partial y}\\
&+v\frac{\partial^3 w}{\partial y^3}+\frac{\partial v}{\partial y}\frac{\partial^2 w}{\partial y^2}+w\frac{\partial^3 w}{\partial y^2\partial z}+\frac{\partial w}{\partial y}\frac{\partial^2 w}{\partial y\partial z}+2u\frac{\partial^3 w}{\partial x\partial z^2}\\
&+2\frac{\partial u}{\partial z}\frac{\partial^2 w}{\partial x\partial z}+2v\frac{\partial^3 w}{\partial y\partial z^2}+2\frac{\partial v}{\partial z}\frac{\partial^2 w}{\partial y\partial z}+2w\frac{\partial^3 w}{\partial z^3}\\
&+2\frac{\partial w}{\partial z}\frac{\partial^2 w}{\partial z^2},
\end{aligned}
\tag{1.112}
$$

$$
\begin{aligned}
(div\,(\mathbf{A_1} grad\mathbf{q}))_x &= 4\frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 v}{\partial x^2}\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\frac{\partial^2 u}{\partial x \partial y} + 2\frac{\partial v}{\partial x}\frac{\partial^2 v}{\partial x^2} + \frac{\partial w}{\partial x}\frac{\partial^2 u}{\partial x \partial z} \\
&+ \frac{\partial^2 w}{\partial x^2}\frac{\partial u}{\partial z} + 2\frac{\partial w}{\partial x}\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 u}{\partial x \partial y}\frac{\partial u}{\partial y} + \frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial x \partial y}\frac{\partial v}{\partial x} \\
&+ \frac{\partial u}{\partial x}\frac{\partial^2 v}{\partial x \partial y} + 2\frac{\partial^2 v}{\partial x \partial y}\frac{\partial v}{\partial y} + 2\frac{\partial v}{\partial x}\frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 w}{\partial x \partial y}\frac{\partial v}{\partial z} + \frac{\partial w}{\partial x}\frac{\partial^2 v}{\partial y \partial z} \\
&+ \frac{\partial^2 w}{\partial x \partial y}\frac{\partial w}{\partial y} + \frac{\partial w}{\partial x}\frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 u}{\partial x \partial z}\frac{\partial u}{\partial z} + \frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial z^2} + \frac{\partial^2 u}{\partial x \partial z}\frac{\partial w}{\partial x} \\
&+ \frac{\partial u}{\partial x}\frac{\partial^2 w}{\partial x \partial z} + \frac{\partial^2 v}{\partial x \partial z}\frac{\partial v}{\partial z} + \frac{\partial v}{\partial x}\frac{\partial^2 v}{\partial z^2} + \frac{\partial^2 v}{\partial x \partial z}\frac{\partial w}{\partial y} + \frac{\partial v}{\partial x}\frac{\partial^2 w}{\partial y \partial z} \\
&+ 2\frac{\partial^2 w}{\partial x \partial z}\frac{\partial w}{\partial z} + 2\frac{\partial w}{\partial x}\frac{\partial^2 w}{\partial z^2},
\end{aligned} \tag{1.113}
$$

$$
\begin{aligned}
(div\,(\mathbf{A_1} grad\mathbf{q}))_y &= 2\frac{\partial^2 u}{\partial x^2}\frac{\partial u}{\partial y} + 2\frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial x \partial y} + \frac{\partial^2 v}{\partial x \partial y}\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}\frac{\partial^2 u}{\partial x \partial y} + \frac{\partial^2 v}{\partial x \partial y}\frac{\partial v}{\partial x} \\
&+ \frac{\partial v}{\partial y}\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 w}{\partial x \partial y}\frac{\partial u}{\partial z} + \frac{\partial w}{\partial y}\frac{\partial^2 u}{\partial x \partial z} + \frac{\partial^2 w}{\partial x \partial y}\frac{\partial w}{\partial x} + \frac{\partial w}{\partial y}\frac{\partial^2 w}{\partial x^2} \\
&+ 2\frac{\partial u}{\partial y}\frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial y^2}\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\frac{\partial^2 v}{\partial x \partial y} + 4\frac{\partial v}{\partial y}\frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 w}{\partial y^2}\frac{\partial v}{\partial z} \\
&+ \frac{\partial w}{\partial y}\frac{\partial^2 v}{\partial y \partial z} + 2\frac{\partial w}{\partial y}\frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 u}{\partial y \partial z}\frac{\partial u}{\partial z} + \frac{\partial u}{\partial y}\frac{\partial^2 u}{\partial z^2} + \frac{\partial^2 u}{\partial y \partial z}\frac{\partial w}{\partial x} \\
&+ \frac{\partial u}{\partial y}\frac{\partial^2 w}{\partial x \partial z} + \frac{\partial^2 v}{\partial y \partial z}\frac{\partial v}{\partial z} + \frac{\partial v}{\partial y}\frac{\partial^2 v}{\partial z^2} + \frac{\partial^2 v}{\partial y \partial z}\frac{\partial w}{\partial y} + \frac{\partial v}{\partial y}\frac{\partial^2 w}{\partial y \partial z} \\
&+ 2\frac{\partial^2 w}{\partial y \partial z}\frac{\partial w}{\partial z} + 2\frac{\partial w}{\partial y}\frac{\partial^2 w}{\partial z 2},
\end{aligned} \tag{1.114}
$$

$$
\begin{aligned}
(div\,(\mathbf{A_1} grad\mathbf{q}))_z &= 2\frac{\partial^2 u}{\partial x^2}\frac{\partial u}{\partial z} + 2\frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial x \partial z} + \frac{\partial^2 v}{\partial x \partial z}\frac{\partial u}{\partial y} + \frac{\partial v}{\partial z}\frac{\partial^2 u}{\partial x \partial y} + \frac{\partial^2 v}{\partial x \partial z}\frac{\partial v}{\partial x} \\
&+ \frac{\partial v}{\partial z}\frac{\partial^2 v}{\partial x^2} + \frac{\partial u}{\partial z}\frac{\partial^2 w}{\partial x \partial z} + \frac{\partial^2 u}{\partial x \partial z}\frac{\partial w}{\partial z} + \frac{\partial^2 w}{\partial x \partial z}\frac{\partial w}{\partial x} + \frac{\partial w}{\partial z}\frac{\partial^2 w}{\partial x^2} \\
&+ \frac{\partial^2 u}{\partial y \partial z}\frac{\partial u}{\partial y} + \frac{\partial u}{\partial z}\frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial y \partial z}\frac{\partial v}{\partial x} + \frac{\partial u}{\partial z}\frac{\partial^2 v}{\partial x \partial y} + 2\frac{\partial^2 v}{\partial y^2}\frac{\partial v}{\partial z} \\
&+ 2\frac{\partial v}{\partial y}\frac{\partial^2 v}{\partial y \partial z} + \frac{\partial^2 w}{\partial y \partial z}\frac{\partial v}{\partial z} + \frac{\partial w}{\partial z}\frac{\partial^2 v}{\partial y \partial z} + \frac{\partial^2 w}{\partial y \partial z}\frac{\partial w}{\partial y} + \frac{\partial w}{\partial z}\frac{\partial^2 w}{\partial y^2} \\
&+ 2\frac{\partial u}{\partial z}\frac{\partial^2 u}{\partial z^2} + \frac{\partial^2 u}{\partial z^2}\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\frac{\partial^2 w}{\partial z \partial x} + 2\frac{\partial v}{\partial z}\frac{\partial^2 v}{\partial z^2} + \frac{\partial^2 v}{\partial z^2}\frac{\partial w}{\partial y} \\
&+ \frac{\partial v}{\partial z}\frac{\partial^2 w}{\partial z \partial y} + 4\frac{\partial w}{\partial x}\frac{\partial^2 w}{\partial x \partial z},
\end{aligned} \tag{1.115}
$$

$$
\left(div\left((grad\mathbf{q})^T \mathbf{A}_1\right)\right)_x = 4\frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 v}{\partial x^2}\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\frac{\partial^2 u}{\partial x \partial y} + 2\frac{\partial v}{\partial x}\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 w}{\partial x^2}\frac{\partial u}{\partial z}
$$

$$
\begin{aligned}
&+\frac{\partial w}{\partial x}\frac{\partial^2 u}{\partial x\partial z}+2\frac{\partial w}{\partial x}\frac{\partial^2 w}{\partial x^2}+2\frac{\partial^2 u}{\partial x\partial y}\frac{\partial u}{\partial y}+2\frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 v}{\partial y^2}\frac{\partial u}{\partial y}\\
&+\frac{\partial v}{\partial y}\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 v}{\partial y^2}\frac{\partial v}{\partial x}+\frac{\partial v}{\partial y}\frac{\partial^2 v}{\partial x\partial y}+\frac{\partial^2 w}{\partial y^2}\frac{\partial u}{\partial z}+\frac{\partial w}{\partial y}\frac{\partial^2 u}{\partial z\partial y}\\
&+\frac{\partial^2 w}{\partial y^2}\frac{\partial w}{\partial x}+\frac{\partial w}{\partial y}\frac{\partial^2 w}{\partial x\partial y}+2\frac{\partial^2 u}{\partial x\partial z}\frac{\partial u}{\partial z}+2\frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial z^2}+\frac{\partial^2 v}{\partial z^2}\frac{\partial u}{\partial y}\\
&+\frac{\partial v}{\partial z}\frac{\partial^2 u}{\partial y\partial z}+\frac{\partial^2 v}{\partial z^2}\frac{\partial v}{\partial x}+\frac{\partial v}{\partial z}\frac{\partial^2 v}{\partial x\partial z}+\frac{\partial^2 w}{\partial z^2}\frac{\partial u}{\partial z}+\frac{\partial w}{\partial z}\frac{\partial^2 u}{\partial z^2}\\
&+\frac{\partial^2 w}{\partial x\partial z}\frac{\partial w}{\partial z}+\frac{\partial w}{\partial x}\frac{\partial^2 w}{\partial z^2}, \qquad (1.116)
\end{aligned}
$$

$$
\begin{aligned}
\left(div\left((grad\mathbf{q})^T\mathbf{A}_1\right)\right)_y \;=\;& \frac{\partial^2 u}{\partial x^2}\frac{\partial u}{\partial y}+\frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial x\partial y}+\frac{\partial^2 u}{\partial x^2}\frac{\partial v}{\partial x}+\frac{\partial u}{\partial x}\frac{\partial^2 v}{\partial x^2}+2\frac{\partial^2 v}{\partial x^2}\frac{\partial v}{\partial y}\\
&+2\frac{\partial v}{\partial x}\frac{\partial^2 v}{\partial x\partial y}+\frac{\partial^2 w}{\partial x^2}\frac{\partial v}{\partial z}+\frac{\partial w}{\partial x}\frac{\partial^2 v}{\partial x\partial z}+\frac{\partial^2 w}{\partial x^2}\frac{\partial w}{\partial y}+\frac{\partial w}{\partial x}\frac{\partial^2 w}{\partial x\partial y}\\
&+2\frac{\partial u}{\partial y}\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial y^2}\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}\frac{\partial^2 v}{\partial x\partial y}+4\frac{\partial v}{\partial y}\frac{\partial^2 v}{\partial y^2}+\frac{\partial^2 w}{\partial y^2}\frac{\partial v}{\partial z}\\
&+\frac{\partial w}{\partial y}\frac{\partial^2 v}{\partial z\partial y}+2\frac{\partial w}{\partial y}\frac{\partial^2 w}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}\frac{\partial v}{\partial x}+\frac{\partial u}{\partial z}\frac{\partial^2 v}{\partial x\partial z}+\frac{\partial^2 u}{\partial z^2}\frac{\partial u}{\partial y}\\
&+\frac{\partial u}{\partial z}\frac{\partial^2 u}{\partial y\partial z}+2\frac{\partial^2 v}{\partial z^2}\frac{\partial v}{\partial y}+2\frac{\partial v}{\partial z}\frac{\partial^2 v}{\partial y\partial z}+\frac{\partial^2 w}{\partial z^2}\frac{\partial v}{\partial z}+\frac{\partial w}{\partial z}\frac{\partial^2 v}{\partial z^2}\\
&+\frac{\partial^2 w}{\partial z^2}\frac{\partial w}{\partial y}+\frac{\partial w}{\partial z}\frac{\partial^2 w}{\partial y\partial z}, \qquad (1.117)
\end{aligned}
$$

$$
\begin{aligned}
div\left((grad\mathbf{q})^T\mathbf{A}_1\right)_z \;=\;& \frac{\partial^2 u}{\partial x^2}\frac{\partial u}{\partial z}+\frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial x\partial z}+\frac{\partial^2 u}{\partial x^2}\frac{\partial w}{\partial x}+\frac{\partial u}{\partial x}\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 v}{\partial x^2}\frac{\partial v}{\partial z}\\
&+\frac{\partial v}{\partial x}\frac{\partial^2 v}{\partial x\partial z}+\frac{\partial^2 v}{\partial x^2}\frac{\partial w}{\partial y}+\frac{\partial v}{\partial x}\frac{\partial^2 w}{\partial x\partial y}+2\frac{\partial w}{\partial z}\frac{\partial^2 w}{\partial x^2}+2\frac{\partial w}{\partial x}\frac{\partial^2 w}{\partial x\partial z}\\
&+\frac{\partial^2 u}{\partial y^2}\frac{\partial u}{\partial z}+\frac{\partial u}{\partial y}\frac{\partial^2 u}{\partial y\partial z}+\frac{\partial^2 u}{\partial y^2}\frac{\partial w}{\partial x}+\frac{\partial u}{\partial y}\frac{\partial^2 w}{\partial x\partial y}+\frac{\partial^2 v}{\partial y^2}\frac{\partial v}{\partial z}\\
&+\frac{\partial v}{\partial y}\frac{\partial^2 v}{\partial y\partial z}+\frac{\partial^2 v}{\partial y^2}\frac{\partial w}{\partial y}+\frac{\partial v}{\partial y}\frac{\partial^2 w}{\partial y^2}+2\frac{\partial^2 w}{\partial y^2}\frac{\partial w}{\partial z}+2\frac{\partial w}{\partial y}\frac{\partial^2 w}{\partial y\partial z}\\
&+2\frac{\partial u}{\partial z}\frac{\partial^2 u}{\partial z^2}+\frac{\partial^2 u}{\partial z^2}\frac{\partial w}{\partial x}+\frac{\partial u}{\partial z}\frac{\partial^2 w}{\partial x\partial z}+2\frac{\partial v}{\partial z}\frac{\partial^2 v}{\partial z^2}+\frac{\partial^2 v}{\partial z^2}\frac{\partial w}{\partial y}\\
&+\frac{\partial v}{\partial z}\frac{\partial^2 w}{\partial y\partial z}+4\frac{\partial w}{\partial x}\frac{\partial^2 w}{\partial x\partial z}, \qquad (1.118)
\end{aligned}
$$

$$
div\left(\mathbf{A}_1^2\right)_x \;=\; 8\frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial x^2}+2\frac{\partial u}{\partial y}\frac{\partial^2 u}{\partial x\partial y}+2\frac{\partial u}{\partial y}\frac{\partial^2 v}{\partial x^2}+2\frac{\partial^2 u}{\partial x\partial y}\frac{\partial v}{\partial x}+2\frac{\partial^2 v}{\partial x^2}\frac{\partial v}{\partial x}
$$

$$
\begin{aligned}
&+2\frac{\partial^2 u}{\partial x\partial y}\frac{\partial u}{\partial z}+2\frac{\partial u}{\partial y}\frac{\partial^2 w}{\partial x^2}+2\frac{\partial w}{\partial x}\frac{\partial^2 u}{\partial x\partial z}+2\frac{\partial w}{\partial x}\frac{\partial^2 w}{\partial x^2}\\
&+2\frac{\partial^2 u}{\partial x\partial y}\frac{\partial u}{\partial y}+2\frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial y^2}+2\frac{\partial^2 u}{\partial x\partial y}\frac{\partial v}{\partial x}+2\frac{\partial u}{\partial x}\frac{\partial^2 v}{\partial x\partial y}\\
&+2\frac{\partial^2 v}{y^2}\frac{\partial u}{\partial y}+2\frac{\partial v}{\partial x}\frac{\partial^2 v}{\partial y^2}+2\frac{\partial^2 u}{\partial y^2}\frac{\partial v}{\partial y}+2\frac{\partial v}{\partial y}\frac{\partial^2 v}{\partial x\partial y}+\frac{\partial v}{\partial z}\frac{\partial^2 u}{\partial y\partial z}\\
&+\frac{\partial w}{\partial y}\frac{\partial^2 u}{\partial y\partial z}+\frac{\partial^2 w}{\partial x\partial y}\frac{\partial v}{\partial z}+\frac{\partial w}{\partial y}\frac{\partial^2 w}{\partial x\partial y}+\frac{\partial^2 v}{\partial z\partial y}\frac{\partial u}{\partial z}+\frac{\partial u}{\partial z}\frac{\partial^2 w}{\partial y^2}\\
&+\frac{\partial^2 v}{\partial y\partial z}\frac{\partial w}{\partial x}+\frac{\partial w}{\partial x}\frac{\partial^2 w}{\partial y^2}+2\frac{\partial^2 u}{\partial x\partial z}\frac{\partial u}{\partial z}+2\frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial z^2}+2\frac{\partial^2 u}{\partial x\partial z}\frac{\partial w}{\partial x}\\
&+2\frac{\partial u}{\partial x}\frac{\partial^2 w}{\partial x\partial z}+\frac{\partial u}{\partial y}\frac{\partial^2 v}{\partial z^2}+\frac{\partial v}{\partial x}\frac{\partial^2 v}{\partial z^2}+\frac{\partial^2 w}{\partial y\partial z}\frac{\partial u}{\partial y}+\frac{\partial w}{\partial y\partial z}\frac{\partial v}{\partial x}\\
&+\frac{\partial^2 u}{\partial y\partial z}\frac{\partial v}{\partial z}+\frac{\partial v}{\partial z}\frac{\partial^2 v}{\partial x\partial z}+\frac{\partial^2 u}{\partial x\partial y}\frac{\partial w}{\partial y}+\frac{\partial w}{\partial y}\frac{\partial^2 v}{\partial x\partial z}+2\frac{\partial^2 w}{\partial z^2}\frac{\partial u}{\partial z}\\
&+2\frac{\partial w}{\partial z}\frac{\partial^2 u}{\partial z^2}+2\frac{\partial^2 w}{\partial z^2}\frac{\partial w}{\partial x}+2\frac{\partial w}{\partial z}\frac{\partial^2 w}{\partial x\partial z},
\end{aligned}
\tag{1.119}
$$

$$
\begin{aligned}
\left(div\mathbf{A}_1^2\right)_y = {}& 2\frac{\partial^2 u}{\partial x^2}\frac{\partial u}{\partial y}+2\frac{\partial^2 u}{\partial x^2}\frac{\partial v}{\partial x}+2\frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial x\partial y}+2\frac{\partial u}{\partial x}\frac{\partial^2 v}{\partial x^2}+2\frac{\partial^2 v}{\partial x\partial y}\frac{\partial u}{\partial y}\\
&+2\frac{\partial^2 v}{\partial x\partial y}\frac{\partial v}{\partial x}+2\frac{\partial v}{\partial y}\frac{\partial^2 u}{\partial x\partial y}+2\frac{\partial v}{\partial y}\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 u}{\partial x\partial z}\frac{\partial v}{\partial z}+\frac{\partial^2 w}{\partial x^2}\frac{\partial v}{\partial z}\\
&+\frac{\partial^2 u}{\partial x\partial z}\frac{\partial w}{\partial y}+\frac{\partial^2 w}{\partial x^2}\frac{\partial w}{\partial y}+\frac{\partial u}{\partial z}\frac{\partial^2 v}{\partial x\partial z}+\frac{\partial u}{\partial z}\frac{\partial^2 w}{\partial x\partial y}+\frac{\partial w}{\partial x}\frac{\partial^2 v}{\partial x\partial z}\\
&+\frac{\partial w}{\partial x}\frac{\partial^2 w}{\partial x\partial y}+2\frac{\partial u}{\partial y}\frac{\partial^2 u}{\partial y^2}+2\frac{\partial u}{\partial y}\frac{\partial^2 v}{\partial x\partial y}+2\frac{\partial^2 u}{\partial y^2}\frac{\partial v}{\partial x}+2\frac{\partial v}{\partial x}\frac{\partial^2 v}{\partial x\partial y}\\
&+8\frac{\partial v}{\partial y}\frac{\partial^2 v}{\partial y^2}+2\frac{\partial v}{\partial z}\frac{\partial^2 v}{\partial y\partial z}+2\frac{\partial v}{\partial z}\frac{\partial^2 w}{\partial y^2}+2\frac{\partial^2 v}{\partial y\partial z}\frac{\partial w}{\partial y}+2\frac{\partial w}{\partial y}\frac{\partial^2 w}{\partial y^2}\\
&+\frac{\partial^2 u}{\partial z^2}\frac{\partial u}{\partial y}+\frac{\partial^2 w}{\partial x\partial z}\frac{\partial u}{\partial y}+\frac{\partial^2 u}{\partial z^2}\frac{\partial v}{\partial x}+\frac{\partial v}{\partial x}\frac{\partial^2 w}{\partial x\partial z}+\frac{\partial^2 u}{\partial y\partial z}\frac{\partial u}{\partial z}\\
&+\frac{\partial^2 v}{\partial x\partial z}\frac{\partial u}{\partial z}+\frac{\partial^2 u}{\partial y\partial z}\frac{\partial w}{\partial x}+\frac{\partial^2 v}{\partial x\partial z}\frac{\partial w}{\partial x}+2\frac{\partial^2 v}{\partial y\partial z}\frac{\partial v}{\partial z}+2\frac{\partial^2 v}{\partial y\partial z}\frac{\partial w}{\partial y}\\
&+2\frac{\partial v}{\partial y}\frac{\partial^2 v}{\partial z^2}+2\frac{\partial v}{\partial y}\frac{\partial^2 w}{\partial y\partial z}+2\frac{\partial^2 w}{\partial z^2}\frac{\partial v}{\partial z}+2\frac{\partial^2 w}{\partial z^2}\frac{\partial w}{\partial y}+2\frac{\partial w}{\partial z}\frac{\partial^2 v}{\partial z^2}\\
&+2\frac{\partial w}{\partial z}\frac{\partial^2 w}{\partial y\partial z},
\end{aligned}
\tag{1.120}
$$

$$
\begin{aligned}
\left(div\mathbf{A}_1^2\right)_z = {}& 2\frac{\partial^2 u}{\partial x^2}\frac{\partial u}{\partial z}+2\frac{\partial^2 u}{\partial x^2}\frac{\partial w}{\partial x}+2\frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial x\partial z}+2\frac{\partial u}{\partial x}\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 u}{\partial x\partial y}\frac{\partial v}{\partial z}\\
&+\frac{\partial^2 v}{\partial x^2}\frac{\partial v}{\partial z}+\frac{\partial^2 u}{\partial x\partial y}\frac{\partial w}{\partial y}+\frac{\partial^2 v}{\partial x^2}\frac{\partial w}{\partial y}+\frac{\partial u}{\partial y}\frac{\partial^2 v}{\partial x\partial z}+\frac{\partial u}{\partial y}\frac{\partial^2 w}{\partial x\partial y}\\
&+\frac{\partial v}{\partial x}\frac{\partial^2 v}{\partial x\partial z}+\frac{\partial v}{\partial x}\frac{\partial^2 w}{\partial x\partial y}+2\frac{\partial^2 w}{\partial x\partial z}\frac{\partial u}{\partial z}+2\frac{\partial^2 w}{\partial x\partial z}\frac{\partial w}{\partial x}+2\frac{\partial w}{\partial z}\frac{\partial^2 u}{\partial x\partial z}
\end{aligned}
$$

$$
\begin{aligned}
&+2\frac{\partial w}{\partial z}\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\frac{\partial u}{\partial z}+\frac{\partial^2 v}{\partial x\partial y^2}\frac{\partial u}{\partial z}+\frac{\partial^2 u}{\partial y^2}\frac{\partial w}{\partial x}+\frac{\partial^2 v}{\partial x\partial y^2}\frac{\partial w}{\partial x}\\
&+\frac{\partial^2 u}{\partial y\partial z}\frac{\partial u}{\partial y}+\frac{\partial u}{\partial y}\frac{\partial^2 w}{\partial x\partial y}+\frac{\partial^2 u}{\partial y\partial z}\frac{\partial v}{\partial x}+\frac{\partial^2 w}{\partial x\partial y}\frac{\partial v}{\partial x}+2\frac{\partial^2 v}{\partial y^2}\frac{\partial v}{\partial z}\\
&+2\frac{\partial^2 v}{\partial y^2}\frac{\partial w}{\partial y}+2\frac{\partial v}{\partial y}\frac{\partial^2 v}{\partial y\partial z}+2\frac{\partial v}{\partial y}\frac{\partial^2 w}{\partial y^2}+2\frac{\partial^2 w}{\partial y\partial z}\frac{\partial v}{\partial z}+2\frac{\partial^2 w}{\partial y\partial z}\frac{\partial w}{\partial y}\\
&+2\frac{\partial w}{\partial z}\frac{\partial^2 v}{\partial y\partial z}+2\frac{\partial w}{\partial z}\frac{\partial^2 w}{\partial y^2}+2\frac{\partial u}{\partial z}\frac{\partial^2 u}{\partial z^2}+2\frac{\partial u}{\partial z}\frac{\partial^2 w}{\partial x\partial z}+2\frac{\partial w}{\partial x}\frac{\partial^2 u}{\partial z^2}\\
&+2\frac{\partial w}{\partial x}\frac{\partial^2 w}{\partial x\partial z}+2\frac{\partial v}{\partial z}\frac{\partial^2 v}{\partial z^2}+2\frac{\partial v}{\partial z}\frac{\partial^2 w}{\partial y\partial z}+2\frac{\partial^2 v}{\partial z^2}\frac{\partial w}{\partial y}+2\frac{\partial^2 w}{\partial y\partial z}\frac{\partial w}{\partial y}\\
&+8\frac{\partial w}{\partial z}\frac{\partial^2 w}{\partial z^2}. \qquad (1.121)
\end{aligned}
$$

Taking divergence of Eq. (1.94) we have

$$
div\left(\mathbf{A}_2\right)=div\left(\frac{\partial \mathbf{A}_1}{\partial t}\right)+div\left(\left(\mathbf{q}\cdot\boldsymbol{\nabla}\right)\mathbf{A}_1\right)+div\left(\mathbf{A}_1 grad\mathbf{q}\right)+div\left(\left(grad\mathbf{q}\right)^T\mathbf{A}_1\right). \qquad (1.122)
$$

Now using Eqs. (1.107) to (1.121) in Eq. (1.122) we have

$$
\begin{aligned}
\left(div\mathbf{A}_2\right)_x = &\frac{\partial^3 u}{\partial x\partial y\partial t}+\frac{\partial^3 v}{\partial x^2\partial t}+2\frac{\partial^3 v}{\partial y^2\partial t}+\frac{\partial^3 v}{\partial z^2\partial t}+\frac{\partial^3 w}{\partial y\partial z\partial t}+2u\frac{\partial^3 u}{\partial x^3}+2v\frac{\partial^3 u}{\partial x^2\partial y}\\
&+2w\frac{\partial^3 u}{\partial x^2\partial z}+u\frac{\partial^3 u}{\partial x\partial y^2}+v\frac{\partial^3 u}{\partial y^3}+w\frac{\partial^3 u}{\partial y^2\partial z}+u\frac{\partial^3 w}{\partial x^2\partial y}+v\frac{\partial^3 w}{\partial x\partial y^2}\\
&+w\frac{\partial^3 w}{\partial x\partial y\partial z}+u\frac{\partial^3 u}{\partial x\partial z^2}+v\frac{\partial^3 u}{\partial y\partial z^2}+w\frac{\partial^3 u}{\partial z^3}+u\frac{\partial^3 w}{\partial x^2\partial z}+v\frac{\partial^3 w}{\partial x\partial y\partial z}\\
&+w\frac{\partial^3 w}{\partial x\partial z^2}+10\frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial x^2}+3\frac{\partial^2 v}{\partial x^2}\frac{\partial u}{\partial y}+5\frac{\partial v}{\partial x}\frac{\partial^2 u}{\partial x\partial y}+4\frac{\partial v}{\partial x}\frac{\partial^2 v}{\partial x^2}\\
&+3\frac{\partial^2 w}{\partial x^2}\frac{\partial u}{\partial z}+5\frac{\partial w}{\partial x}\frac{\partial^2 u}{\partial x\partial z}+4\frac{\partial w}{\partial x}\frac{\partial^2 w}{\partial x^2}+4\frac{\partial^2 u}{\partial x\partial y}\frac{\partial u}{\partial y}+3\frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial y^2}\\
&+\frac{\partial u}{\partial x}\frac{\partial^2 v}{\partial x\partial y}+4\frac{\partial v}{\partial y}\frac{\partial^2 v}{\partial x\partial y}+3\frac{\partial^2 v}{\partial y^2}\frac{\partial v}{\partial x}+2\frac{\partial v}{\partial z}\frac{\partial^2 w}{\partial x\partial y}+\frac{\partial w}{\partial x}\frac{\partial^2 u}{\partial y\partial z}\\
&+2\frac{\partial w}{\partial y}\frac{\partial^2 w}{\partial z\partial y}+2\frac{\partial^2 w}{\partial y^2}\frac{\partial w}{\partial x}+4\frac{\partial^2 u}{\partial x\partial z}\frac{\partial u}{\partial z}+3\frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial z^2}+\frac{\partial u}{\partial x}\frac{\partial^2 w}{\partial x\partial z}\\
&+2\frac{\partial v}{\partial z}\frac{\partial^2 v}{\partial x\partial z}+2\frac{\partial^2 v}{\partial z^2}\frac{\partial v}{\partial x}+\frac{\partial w}{\partial y}\frac{\partial^2 v}{\partial x\partial z}+\frac{\partial^2 w}{\partial y\partial z}\frac{\partial v}{\partial x}+4\frac{\partial^2 w}{\partial x\partial z}\frac{\partial w}{\partial z}\\
&+3\frac{\partial w}{\partial x}\frac{\partial^2 w}{\partial z^2}+\frac{\partial u}{\partial y}\frac{\partial^2 v}{\partial y^2}+2\frac{\partial^2 u}{\partial y^2}\frac{\partial v}{\partial y}+\frac{\partial u}{\partial z}\frac{\partial^2 w}{\partial y^2}+\frac{\partial u}{\partial y}\frac{\partial^2 v}{\partial z^2}\\
&+2\frac{\partial v}{\partial z}\frac{\partial^2 u}{\partial y\partial z}+\frac{\partial u}{\partial z}\frac{\partial^2 w}{\partial z^2}+2\frac{\partial w}{\partial z}\frac{\partial^2 u}{\partial z^2}+\frac{\partial w}{\partial y}\frac{\partial^2 w}{\partial x\partial z}, \qquad (1.123)
\end{aligned}
$$

$$
\begin{aligned}
(div\,(\mathbf{A}_2))_y \;=\;& \frac{\partial^3 u}{\partial x\partial y\partial t}+\frac{\partial^3 v}{\partial x^2\partial t}+2\frac{\partial^3 v}{\partial y^2\partial t}+\frac{\partial^3 v}{\partial z^2\partial t}+\frac{\partial^3 w}{\partial y\partial z\partial t}+u\frac{\partial^3 u}{\partial x^2\partial y}\\
&+v\frac{\partial^3 u}{\partial x\partial y^2}+w\frac{\partial^3 u}{\partial x\partial y\partial z}+u\frac{\partial^3 v}{\partial x^3}+v\frac{\partial^3 u}{\partial y^3}+w\frac{\partial^3 v}{\partial x^2\partial z}\\
&+2u\frac{\partial^3 v}{\partial x\partial y^2}+2v\frac{\partial^3 v}{\partial y^3}+2w\frac{\partial^3 v}{\partial y^2\partial z}+u\frac{\partial^3 v}{\partial x\partial z^2}+v\frac{\partial^3 v}{\partial y\partial z^2}\\
&+w\frac{\partial^3 v}{\partial z^3}+u\frac{\partial^3 w}{\partial x\partial y\partial z}+v\frac{\partial^3 w}{\partial y^2\partial z}+w\frac{\partial^2 w}{\partial y\partial z^2}+2u\frac{\partial^3 w}{\partial x\partial z^2}\\
&+2v\frac{\partial^3 w}{\partial y\partial z^2}+w\frac{\partial^2 w}{\partial y\partial z^2}+3\frac{\partial^2 u}{\partial x^2}\frac{\partial u}{\partial y}+4\frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial x\partial y}+\frac{\partial^2 u}{\partial x^2}\frac{\partial v}{\partial x}\\
&+2\frac{\partial u}{\partial x}\frac{\partial^2 v}{\partial x^2}+3\frac{\partial^2 v}{\partial x^2}\frac{\partial v}{\partial y}+4\frac{\partial v}{\partial x}\frac{\partial^2 v}{\partial x\partial y}+\frac{\partial^2 w}{\partial x^2}\frac{\partial v}{\partial z}+2\frac{\partial w}{\partial x}\frac{\partial^2 v}{\partial x\partial z}\\
&+2\frac{\partial^2 w}{\partial x^2}\frac{\partial w}{\partial y}+2\frac{\partial w}{\partial x}\frac{\partial^2 w}{\partial x\partial y}+4\frac{\partial u}{\partial y}\frac{\partial^2 u}{\partial y^2}+3\frac{\partial^2 u}{\partial y^2}\frac{\partial v}{\partial x}+4\frac{\partial u}{\partial y}\frac{\partial^2 v}{\partial x\partial y}\\
&+10\frac{\partial v}{\partial y}\frac{\partial^2 v}{\partial y^2}+3\frac{\partial^2 w}{\partial y^2}\frac{\partial v}{\partial z}+5\frac{\partial w}{\partial y}\frac{\partial^2 v}{\partial z\partial y}+4\frac{\partial w}{\partial y}\frac{\partial^2 w}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}\frac{\partial v}{\partial x}\\
&+2\frac{\partial u}{\partial z}\frac{\partial^2 v}{\partial x\partial z}+2\frac{\partial^2 u}{\partial z^2}\frac{\partial u}{\partial y}+2\frac{\partial u}{\partial z}\frac{\partial^2 u}{\partial y\partial z}+3\frac{\partial^2 v}{\partial z^2}\frac{\partial v}{\partial y}+4\frac{\partial v}{\partial z}\frac{\partial^2 v}{\partial y\partial z}\\
&+\frac{\partial^2 w}{\partial z^2}\frac{\partial v}{\partial z}+2\frac{\partial w}{\partial z}\frac{\partial^2 v}{\partial z^2}+3\frac{\partial^2 w}{\partial z^2}\frac{\partial w}{\partial y}+4\frac{\partial w}{\partial z}\frac{\partial^2 w}{\partial y\partial z}+\frac{\partial^2 u}{\partial x\partial y}\frac{\partial v}{\partial y}\\
&+2\frac{\partial u}{\partial z}\frac{\partial^2 w}{\partial y\partial z}+\frac{\partial^2 u}{\partial x\partial z}\frac{\partial w}{\partial y}+2\frac{\partial w}{\partial x}\frac{\partial^2 u}{\partial y\partial z}+\frac{\partial u}{\partial y}\frac{\partial^2 w}{\partial x\partial z}\\
&+\frac{\partial^2 w}{\partial y\partial z}\frac{\partial v}{\partial y},
\end{aligned}
\tag{1.124}
$$

$$
\begin{aligned}
(div\mathbf{A}_2)_z \;=\;& \frac{\partial^3 u}{\partial x\partial z\partial t}+\frac{\partial^3 w}{\partial x^2\partial t}+\frac{\partial^3 v}{\partial y\partial z\partial t}+\frac{\partial^3 w}{\partial y^2\partial t}+2\frac{\partial^3 w}{\partial z^2\partial t}\\
&+u\frac{\partial^3 u}{\partial x^2\partial z}+v\frac{\partial^3 u}{\partial x\partial y\partial z}+w\frac{\partial^3 u}{\partial x\partial z^2}+u\frac{\partial^3 w}{\partial x^3}+v\frac{\partial^3 w}{\partial x^2\partial y}\\
&+w\frac{\partial^3 w}{\partial x^2\partial z}+u\frac{\partial^3 v}{\partial x\partial y\partial z}+v\frac{\partial^3 v}{\partial y^2\partial z}+w\frac{\partial^3 v}{\partial y\partial z^2}+u\frac{\partial^3 w}{\partial x\partial y^2}\\
&+v\frac{\partial^3 w}{\partial y^3}+w\frac{\partial^3 w}{\partial y^2\partial z}+2u\frac{\partial^3 w}{\partial x\partial z^2}+2v\frac{\partial^3 w}{\partial y\partial z^2}+2w\frac{\partial^3 w}{\partial z^3}+\\
&+3\frac{\partial u}{\partial z}\frac{\partial^2 u}{\partial x^2}+4\frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial x\partial x}+2\frac{\partial u}{\partial y}\frac{\partial^2 v}{\partial x\partial z}+\frac{\partial v}{\partial z}\frac{\partial^2 u}{\partial x\partial y}+2\frac{\partial v}{\partial x}\frac{\partial^2 v}{\partial x\partial z}\\
&+2\frac{\partial v}{\partial z}\frac{\partial^2 v}{\partial x^2}+\frac{\partial u}{\partial z}\frac{\partial^2 w}{\partial x\partial z}+\frac{\partial^2 u}{\partial x\partial z}\frac{\partial w}{\partial z}+3\frac{\partial w}{\partial z}\frac{\partial^2 w}{\partial x^2}+2\frac{\partial^2 u}{\partial y\partial z}\frac{\partial u}{\partial y}\\
&+2\frac{\partial u}{\partial z}\frac{\partial^2 u}{\partial y^2}+2\frac{\partial v}{\partial x}\frac{\partial^2 u}{\partial y\partial z}+\frac{\partial u}{\partial z}\frac{\partial^2 v}{\partial x\partial y}+3\frac{\partial v}{\partial z}\frac{\partial^2 v}{\partial y^2}+2\frac{\partial w}{\partial z}\frac{\partial^2 w}{\partial z^2}\\
&+4\frac{\partial v}{\partial y}\frac{\partial^2 v}{\partial y\partial z}+5\frac{\partial^2 w}{\partial y\partial z}\frac{\partial v}{\partial z}+\frac{\partial^2 v}{\partial y\partial z}\frac{\partial w}{\partial z}+4\frac{\partial w}{\partial y}\frac{\partial^2 w}{\partial y\partial z}+3\frac{\partial w}{\partial z}\frac{\partial^2 w}{\partial y^2}
\end{aligned}
$$

$$+4\frac{\partial u}{\partial z}\frac{\partial^2 u}{\partial z^2}+3\frac{\partial w}{\partial x}\frac{\partial^2 u}{\partial z^2}+4\frac{\partial u}{\partial z}\frac{\partial^2 w}{\partial x\partial z}+4\frac{\partial v}{\partial z}\frac{\partial^2 v}{\partial z^2}+3\frac{\partial^2 v}{\partial z^2}\frac{\partial w}{\partial y}$$
$$+10\frac{\partial w}{\partial x}\frac{\partial^2 w}{\partial x\partial z}+\frac{\partial w}{\partial x}\frac{\partial^2 u}{\partial x^2}+2\frac{\partial u}{\partial x}\frac{\partial^2 w}{\partial x^2}+\frac{\partial w}{\partial y}\frac{\partial^2 v}{\partial x^2}+2\frac{\partial^2 w}{\partial x\partial y}\frac{\partial v}{\partial x}$$
$$+\frac{\partial w}{\partial x}\frac{\partial^2 u}{\partial y^2}+2\frac{\partial^2 w}{\partial x\partial y}\frac{\partial u}{\partial y}+\frac{\partial^2 v}{\partial y^2}\frac{\partial w}{\partial y}+2\frac{\partial^2 w}{\partial y^2}\frac{\partial v}{\partial y}. \tag{1.125}$$

Taking divergence of Eq. (1.92) we have

$$div\mathbf{T}=div\left(-p\mathbf{I}\right)+\mu div\left(\mathbf{A}_1\right)+\alpha_1 div\mathbf{A}_2+\alpha_2 div\mathbf{A}_1^2 \tag{1.126}$$

and in component form we have

$$\begin{aligned}
(div\mathbf{T})_x = & -\frac{\partial p}{\partial x}+\mu\left[2\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 v}{\partial x\partial y}+\frac{\partial^2 u}{\partial z^2}+\frac{\partial^2 w}{\partial x\partial z}\right]+\frac{\partial^3 u}{\partial x\partial y\partial t} \\
& +\frac{\partial^3 v}{\partial x^2\partial t}+2\frac{\partial^3 v}{\partial y^2\partial t}+\frac{\partial^3 v}{\partial z^2\partial t}+\frac{\partial^3 w}{\partial y\partial z\partial t}+2u\frac{\partial^3 u}{\partial x^3}+2v\frac{\partial^3 u}{\partial x^2\partial y} \\
& +2w\frac{\partial^3 u}{\partial x^2\partial z}+u\frac{\partial^3 u}{\partial x\partial y^2}+v\frac{\partial^3 u}{\partial y^3}+w\frac{\partial^3 u}{\partial y^2\partial z}+u\frac{\partial^3 w}{\partial x^2\partial y}+v\frac{\partial^3 w}{\partial x\partial y^2} \\
& +w\frac{\partial^3 w}{\partial x\partial y\partial z}+u\frac{\partial^3 u}{\partial x\partial z^2}+v\frac{\partial^3 u}{\partial y\partial z^2}+w\frac{\partial^3 u}{\partial z^3}+u\frac{\partial^3 w}{\partial x^2\partial z}+v\frac{\partial^3 w}{\partial x\partial y\partial z} \\
& +w\frac{\partial^3 w}{\partial x\partial z^2}+\alpha_1\left[3\frac{\partial^2 w}{\partial x^2}\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}\frac{\partial^2 u}{\partial y\partial z}+2\frac{\partial w}{\partial y}\frac{\partial^2 w}{\partial z\partial y}\right] \\
& +\alpha_1\left[\frac{\partial w}{\partial y}\frac{\partial^2 w}{\partial x\partial z}\right]+\alpha_2\left[2\frac{\partial^2 u}{\partial x\partial y}\frac{\partial u}{\partial z}+2\frac{\partial u}{\partial y}\frac{\partial^2 w}{\partial x^2}+\frac{\partial w}{\partial y}\frac{\partial^2 w}{\partial x\partial y}\right] \\
& +\alpha_2\left[+\frac{\partial^2 v}{\partial z\partial y}\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}\frac{\partial^2 v}{\partial y\partial z}+\frac{\partial^2 w}{\partial y\partial z}\frac{\partial u}{\partial y}+\frac{\partial w}{\partial y\partial z}\frac{\partial v}{\partial x}\right] \\
& +a_3\left[\frac{\partial w}{\partial x}\frac{\partial^2 u}{\partial x\partial z}+\frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial z^2}+2\frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial x^2}+\frac{\partial v}{\partial x}\frac{\partial^2 u}{\partial x\partial y}\right] \\
& +a_4\left[\frac{\partial^2 v}{\partial x^2}\frac{\partial u}{\partial y}+\frac{\partial^2 v}{\partial y^2}\frac{\partial v}{\partial x}+\frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial y^2}+3\frac{\partial w}{\partial x}\frac{\partial^2 w}{\partial z^2}\right] \\
& +a_5\left[\frac{\partial w}{\partial y}\frac{\partial^2 v}{\partial x\partial z}+\frac{\partial^2 w}{\partial y\partial z}\frac{\partial v}{\partial x}+2\frac{\partial^2 u}{\partial y^2}\frac{\partial v}{\partial y}+\frac{\partial u}{\partial z}\frac{\partial^2 w}{\partial y^2}\right] \\
& +a_5\left[2\frac{\partial v}{\partial z}\frac{\partial^2 u}{\partial y\partial z}+2\frac{\partial w}{\partial z}\frac{\partial^2 u}{\partial z^2}+4\frac{\partial^2 u}{\partial x\partial y}\frac{\partial u}{\partial y}+\frac{\partial u}{\partial y}\frac{\partial^2 v}{\partial z^2}\right] \\
& +a_6\left[\frac{\partial u}{\partial x}\frac{\partial^2 v}{\partial x\partial y}+\frac{\partial u}{\partial x}\frac{\partial^2 w}{\partial x\partial z}+\frac{\partial u}{\partial y}\frac{\partial^2 v}{\partial y^2}+\frac{\partial u}{\partial z}\frac{\partial^2 w}{\partial z^2}\right] \\
& +a_7\left[2\frac{\partial v}{\partial y}\frac{\partial^2 v}{\partial x\partial y}+\frac{\partial v}{\partial z}\frac{\partial^2 w}{\partial x\partial y}+\frac{\partial^2 w}{\partial y^2}\frac{\partial w}{\partial x}+2\frac{\partial^2 u}{\partial x\partial z}\frac{\partial u}{\partial z}\right]
\end{aligned}$$

$$+a_7\left[\frac{\partial^2 v}{\partial z^2}\frac{\partial v}{\partial x}+2\frac{\partial^2 w}{\partial x\partial z}\frac{\partial w}{\partial z}+2\frac{\partial v}{\partial x}\frac{\partial^2 v}{\partial x^2}+2\frac{\partial w}{\partial x}\frac{\partial^2 w}{\partial x^2}\right]$$
$$+a_7\left[\frac{\partial v}{\partial z}\frac{\partial^2 v}{\partial x\partial z}\right] \tag{1.127}$$

$$\begin{aligned}
(div\mathbf{T})_y \;=\; & -\frac{\partial p}{\partial y}+\mu\left[\frac{\partial^2 u}{\partial x\partial y}+\frac{\partial^2 v}{\partial x^2}+2\frac{\partial^2 v}{\partial y^2}+\frac{\partial^2 v}{\partial z^2}+\frac{\partial^2 w}{\partial y\partial z}\right]+\frac{\partial^3 u}{\partial x\partial y\partial t}+\frac{\partial^3 v}{\partial x^2\partial t}\\
&+2\frac{\partial^3 v}{\partial y^2\partial t}+v\frac{\partial^3 u}{\partial y^3}+w\frac{\partial^3 v}{\partial x^2\partial z}+u\frac{\partial^3 w}{\partial x\partial y\partial z}\\
&+\frac{\partial^3 v}{\partial z^2\partial t}+\frac{\partial^3 w}{\partial y\partial z\partial t}+u\frac{\partial^3 u}{\partial x^2\partial y}+v\frac{\partial^3 u}{\partial x\partial y^2}+w\frac{\partial^3 u}{\partial x\partial y\partial z}+u\frac{\partial^3 v}{\partial x^3}\\
&+2u\frac{\partial^3 v}{\partial x\partial y^2}+2v\frac{\partial^3 v}{\partial y^3}+2w\frac{\partial^3 v}{\partial y^2\partial z}+u\frac{\partial^3 v}{\partial x\partial z^2}+v\frac{\partial^3 v}{\partial y\partial z^2}+w\frac{\partial^3 v}{\partial z^3}\\
&+v\frac{\partial^3 w}{\partial y^2\partial z}+w\frac{\partial^2 w}{\partial y\partial z^2}+2u\frac{\partial^3 w}{\partial x\partial z^2}+2v\frac{\partial^3 w}{\partial y\partial z^2}+w\frac{\partial^2 w}{\partial y\partial z^2}\\
&+\alpha_1\left[\frac{\partial^2 u}{\partial y^2}\frac{\partial v}{\partial x}+\frac{\partial^2 u}{\partial x\partial y}\frac{\partial v}{\partial y}+2\frac{\partial u}{\partial z}\frac{\partial^2 w}{\partial y\partial z}\right]+a_3\left[2a_3\frac{\partial v}{\partial y}\frac{\partial^2 v}{\partial y^2}\right]\\
&+a_3\left[\frac{\partial w}{\partial y}\frac{\partial^2 v}{\partial z\partial y}\right]+\alpha_2\left[\frac{\partial v}{\partial x}\frac{\partial^2 w}{\partial x\partial z}+2\frac{\partial^2 u}{\partial y^2}\frac{\partial v}{\partial x}\right]\\
&+\alpha_2\left[2\frac{\partial v}{\partial y}\frac{\partial^2 u}{\partial x\partial y}+\frac{\partial^2 w}{\partial x^2}\frac{\partial v}{\partial z}+\frac{\partial u}{\partial z}\frac{\partial^2 w}{\partial x\partial y}+\frac{\partial w}{\partial x}\frac{\partial^2 v}{\partial x\partial z}\right]\\
&+a_4\left[\frac{\partial^2 u}{\partial x^2}\frac{\partial u}{\partial y}+\frac{\partial^2 v}{\partial x^2}\frac{\partial v}{\partial y}+\frac{\partial^2 w}{\partial y^2}\frac{\partial v}{\partial z}+\frac{\partial^2 v}{\partial z^2}\frac{\partial v}{\partial y}+\frac{\partial^2 w}{\partial z^2}\frac{\partial w}{\partial y}\right]\\
&+a_5\left[2\frac{\partial u}{\partial x}\frac{\partial^2 v}{\partial x^2}+4\frac{\partial v}{\partial x}\frac{\partial^2 v}{\partial x\partial y}+2\frac{\partial w}{\partial z}\frac{\partial^2 v}{\partial z^2}+\frac{\partial^2 w}{\partial x^2}\frac{\partial v}{\partial z}+\frac{\partial^2 u}{\partial x\partial z}\frac{\partial w}{\partial y}\right]\\
&+a_5\left[\frac{\partial^2 u}{\partial z^2}\frac{\partial v}{\partial x}+2\frac{\partial u}{\partial z}\frac{\partial^2 v}{\partial x\partial z}+4\frac{\partial v}{\partial z}\frac{\partial^2 v}{\partial y\partial z}+4\frac{\partial u}{\partial y}\frac{\partial^2 v}{\partial x\partial y}\right]\\
&+a_6\left[\frac{\partial^2 u}{\partial x^2}\frac{\partial v}{\partial x}+\frac{\partial^2 w}{\partial z^2}\frac{\partial v}{\partial z}+\frac{\partial^2 w}{\partial y\partial z}\frac{\partial v}{\partial y}\right]+a_7\left[2\frac{\partial u}{\partial y}\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}\frac{\partial u}{\partial y}\right]\\
&+a_7\left[2\frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial x\partial y}+\frac{\partial w}{\partial x}\frac{\partial^2 v}{\partial x\partial z}+\frac{\partial^2 w}{\partial x^2}\frac{\partial w}{\partial y}+\frac{\partial w}{\partial x}\frac{\partial^2 w}{\partial x\partial y}\right]\\
&+a_7\left[\frac{\partial u}{\partial z}\frac{\partial^2 u}{\partial y\partial z}+2\frac{\partial w}{\partial z}\frac{\partial^2 w}{\partial y\partial z}+\frac{\partial w}{\partial x}\frac{\partial^2 u}{\partial y\partial z}+\frac{\partial u}{\partial y}\frac{\partial^2 w}{\partial x\partial z}\right],
\end{aligned} \tag{1.128}$$

$$\begin{aligned}
(div\mathbf{T})_z \;=\; & -\frac{\partial p}{\partial z}+\mu\left[\frac{\partial^2 u}{\partial x\partial z}+\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 v}{\partial y\partial z}+\frac{\partial^2 w}{\partial y^2}+2\frac{\partial^2 w}{\partial z^2}\right]+\frac{\partial^3 u}{\partial x\partial z\partial t}+\frac{\partial^3 w}{\partial x^2\partial t}\\
&+\frac{\partial^3 v}{\partial y\partial z\partial t}+\frac{\partial^3 w}{\partial y^2\partial t}+2\frac{\partial^3 w}{\partial z^2\partial t}+u\frac{\partial^3 u}{\partial x^2\partial z}+v\frac{\partial^3 u}{\partial x\partial y\partial z}+w\frac{\partial^3 u}{\partial x\partial z^2}+u\frac{\partial^3 w}{\partial x^3}
\end{aligned}$$

$$
\begin{aligned}
&+v\frac{\partial^3 w}{\partial x^2\partial y}+w\frac{\partial^3 w}{\partial x^2\partial z}+u\frac{\partial^3 v}{\partial x\partial y\partial z}+v\frac{\partial^3 v}{\partial y^2\partial z}+w\frac{\partial^3 v}{\partial y\partial z^2}+u\frac{\partial^3 w}{\partial x\partial y^2}+v\frac{\partial^3 w}{\partial y^3}\\
&+w\frac{\partial^3 w}{\partial y^2\partial z}+2u\frac{\partial^3 w}{\partial x\partial z^2}+2v\frac{\partial^3 w}{\partial y\partial z^2}+2w\frac{\partial^3 w}{\partial z^3}\\
&+\alpha_1\left[4\frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial x^2}+\frac{\partial u}{\partial z}\frac{\partial^2 v}{\partial x\partial y}+\frac{\partial w}{\partial x}\frac{\partial^2 w}{\partial x\partial z}\right]+a_3\left[2\frac{\partial w}{\partial z}\frac{\partial^2 w}{\partial z^2}+\frac{\partial u}{\partial z}\frac{\partial^2 w}{\partial x\partial z}\right]\\
&+a_3\left[\frac{\partial^2 w}{\partial y\partial z}\frac{\partial v}{\partial z}\right]+\alpha_2\left[\frac{\partial^2 u}{\partial x\partial z}\frac{\partial u}{\partial x}+\frac{\partial u}{\partial z}\frac{\partial^2 v}{\partial y^2}+\frac{\partial^2 w}{\partial x\partial y}\frac{\partial v}{\partial x}+\frac{\partial^2 v}{\partial x\partial y}\frac{\partial w}{\partial x}\right]\\
&+a_4\left[\frac{\partial w}{\partial z}\frac{\partial^2 w}{\partial x^2}+\frac{\partial v}{\partial z}\frac{\partial^2 v}{\partial y^2}+\frac{\partial w}{\partial z}\frac{\partial^2 w}{\partial y^2}+\frac{\partial w}{\partial x}\frac{\partial^2 u}{\partial z^2}+\frac{\partial^2 v}{\partial z^2}\frac{\partial w}{\partial y}+\frac{\partial u}{\partial z}\frac{\partial^2 u}{\partial x^2}\right]\\
&+a_5\left[4\frac{\partial w}{\partial y}\frac{\partial^2 w}{\partial y\partial z}+2\frac{\partial u}{\partial x}\frac{\partial^2 w}{\partial x^2}+\frac{\partial w}{\partial y}\frac{\partial^2 v}{\partial x^2}+\frac{\partial w}{\partial x}\frac{\partial^2 u}{\partial y^2}\right]\\
&+a_5\left[2\frac{\partial^2 w}{\partial x\partial y}\frac{\partial u}{\partial y}+2\frac{\partial^2 w}{\partial y^2}\frac{\partial v}{\partial y}\right]+a_6\left[\frac{\partial v}{\partial z}\frac{\partial^2 u}{\partial x\partial y}+\frac{\partial^2 u}{\partial x\partial z}\frac{\partial w}{\partial z}\right]\\
&+a_6\left[\frac{\partial^2 v}{\partial y\partial z}\frac{\partial w}{\partial z}+\frac{\partial w}{\partial x}+\frac{\partial^2 v}{\partial y^2}\frac{\partial w}{\partial y}\right]+a_7\left[2\frac{\partial v}{\partial z}\frac{\partial^2 v}{\partial z^2}+\frac{\partial^2 w}{\partial x\partial y}\frac{\partial v}{\partial x}\right]\\
&+a_7\left[2\frac{\partial u}{\partial z}\frac{\partial^2 u}{\partial z^2}+\frac{\partial v}{\partial x}\frac{\partial^2 v}{\partial x\partial z}+\frac{\partial v}{\partial z}\frac{\partial^2 v}{\partial x^2}+\frac{\partial u}{\partial y}\frac{\partial^2 v}{\partial x\partial z}\right]\\
&+a_7\left[\frac{\partial^2 u}{\partial y\partial z}\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}\frac{\partial^2 u}{\partial y^2}+\frac{\partial v}{\partial x}\frac{\partial^2 u}{\partial y\partial z}+2\frac{\partial v}{\partial y}\frac{\partial^2 v}{\partial y\partial z}\right]. \qquad (1.129)
\end{aligned}
$$

Now using Eqs. (1.127) to (1.129) in Eq. (1.88) we obtain

$$
\begin{aligned}
\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z} \;=\; & -\frac{1}{\rho}\frac{\partial p}{\partial x}+\nu\left[2\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 v}{\partial x\partial y}+\frac{\partial^2 u}{\partial z^2}+\frac{\partial^2 w}{\partial x\partial z}\right]\\
&+\frac{\partial^3 u}{\partial x\partial y\partial t}+\frac{\partial^3 v}{\partial x^2\partial t}+2\frac{\partial^3 v}{\partial y^2\partial t}+\frac{\partial^3 v}{\partial z^2\partial t}+\frac{\partial^3 w}{\partial y\partial z\partial t}\\
&+2u\frac{\partial^3 u}{\partial x^3}+2w\frac{\partial^3 u}{\partial x^2\partial z}+u\frac{\partial^3 u}{\partial x\partial y^2}+v\frac{\partial^3 u}{\partial y^3}+w\frac{\partial^3 u}{\partial y^2\partial z}\\
&+w\frac{\partial^3 w}{\partial x\partial y\partial z}+u\frac{\partial^3 u}{\partial x\partial z^2}+v\frac{\partial^3 u}{\partial y\partial z^2}+w\frac{\partial^3 u}{\partial z^3}+u\frac{\partial^3 w}{\partial x^2\partial z}\\
&+v\frac{\partial^3 w}{\partial x\partial y^2}+v\frac{\partial^3 w}{\partial x\partial y\partial z}+2v\frac{\partial^3 u}{\partial x^2\partial y}+u\frac{\partial^3 w}{\partial x^2\partial y}\\
&+w\frac{\partial^3 w}{\partial x\partial z^2}+\alpha_1\left[3\frac{\partial^2 w}{\partial x^2}\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}\frac{\partial^2 u}{\partial y\partial z}+2\frac{\partial w}{\partial y}\frac{\partial^2 w}{\partial z\partial y}\right]\\
&+\alpha_1\left[\frac{\partial w}{\partial y}\frac{\partial^2 w}{\partial x\partial z}\right]+\alpha_2\left[\frac{\partial w}{\partial x}\frac{\partial^2 v}{\partial y\partial z}\right]\\
&+\alpha_2\left[2\frac{\partial^2 u}{\partial x\partial y}\frac{\partial u}{\partial z}+2\frac{\partial u}{\partial y}\frac{\partial^2 w}{\partial x^2}+\frac{\partial w}{\partial y}\frac{\partial^2 w}{\partial x\partial y}+\frac{\partial^2 v}{\partial z\partial y}\frac{\partial u}{\partial z}\right]\\
&+\alpha_2\left[\frac{\partial^2 w}{\partial y\partial z}\frac{\partial u}{\partial y}+\frac{\partial w}{\partial y\partial z}\frac{\partial v}{\partial x}\right]+a_3\left[2\frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial x^2}\right]
\end{aligned}
$$

$$
\begin{aligned}
&+a_3\left[\frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial z^2}+\frac{\partial w}{\partial x}\frac{\partial^2 u}{\partial x\partial z}+\frac{\partial v}{\partial x}\frac{\partial^2 u}{\partial x\partial y}\right]+a_4\left[\frac{\partial^2 v}{\partial x^2}\frac{\partial u}{\partial y}\right]\\
&+a_4\left[3\frac{\partial w}{\partial x}\frac{\partial^2 w}{\partial z^2}+\frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 v}{\partial y^2}\frac{\partial v}{\partial x}\right]+a_5\left[\frac{\partial u}{\partial z}\frac{\partial^2 w}{\partial y^2}\right]\\
&+a_5\left[\frac{\partial w}{\partial y}\frac{\partial^2 v}{\partial x\partial z}+\frac{\partial^2 w}{\partial y\partial z}\frac{\partial v}{\partial x}+2\frac{\partial^2 u}{\partial y^2}\frac{\partial v}{\partial y}+\frac{\partial u}{\partial y}\frac{\partial^2 v}{\partial z^2}\right]\\
&+a_5\left[2\frac{\partial v}{\partial z}\frac{\partial^2 u}{\partial y\partial z}+2\frac{\partial w}{\partial z}\frac{\partial^2 u}{\partial z^2}+4\frac{\partial^2 u}{\partial x\partial y}\frac{\partial u}{\partial y}\right]\\
&+a_6\left[\frac{\partial u}{\partial x}\frac{\partial^2 v}{\partial x\partial y}+\frac{\partial u}{\partial x}\frac{\partial^2 w}{\partial x\partial z}\right]+a_6\left[\frac{\partial u}{\partial y}\frac{\partial^2 v}{\partial y^2}+\frac{\partial u}{\partial z}\frac{\partial^2 w}{\partial z^2}\right]\\
&+a_7\left[2\frac{\partial v}{\partial y}\frac{\partial^2 v}{\partial x\partial y}+\frac{\partial v}{\partial z}\frac{\partial^2 w}{\partial x\partial y}+\frac{\partial^2 w}{\partial y^2}\frac{\partial w}{\partial x}+2\frac{\partial^2 u}{\partial x\partial z}\frac{\partial u}{\partial z}\right]\\
&+a_7\left[2\frac{\partial w}{\partial x}\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 v}{\partial z^2}\frac{\partial v}{\partial x}+2\frac{\partial^2 w}{\partial x\partial z}\frac{\partial w}{\partial z}+2\frac{\partial v}{\partial x}\frac{\partial^2 v}{\partial x^2}\right]\\
&+a_7\left[\frac{\partial v}{\partial z}\frac{\partial^2 v}{\partial x\partial z}\right]+\frac{\mu_e}{\rho}\left[J_yH_z-J_zH_y\right], \qquad (1.130)
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+w\frac{\partial v}{\partial z} \;=\;& -\frac{1}{\rho}\frac{\partial p}{\partial y}+\nu\left[\frac{\partial^2 u}{\partial x\partial y}+\frac{\partial^2 v}{\partial x^2}+2\frac{\partial^2 v}{\partial y^2}+\frac{\partial^2 v}{\partial z^2}+\frac{\partial^2 w}{\partial y\partial z}\right]\\
&+\frac{\partial^3 u}{\partial x\partial y\partial t}+\frac{\partial^3 v}{\partial x^2\partial t}+2\frac{\partial^3 v}{\partial y^2\partial t}+\frac{\partial^3 v}{\partial z^2\partial t}+\frac{\partial^3 w}{\partial y\partial z\partial t}\\
&+u\frac{\partial^3 u}{\partial x^2\partial y}+v\frac{\partial^3 u}{\partial x\partial y^2}+w\frac{\partial^3 u}{\partial x\partial y\partial z}+u\frac{\partial^3 v}{\partial x^3}+v\frac{\partial^3 u}{\partial y^3}\\
&+w\frac{\partial^3 v}{\partial x^2\partial z}+2u\frac{\partial^3 v}{\partial x\partial y^2}+2v\frac{\partial^3 v}{\partial y^3}+2w\frac{\partial^3 v}{\partial y^2\partial z}\\
&+v\frac{\partial^3 v}{\partial y\partial z^2}+w\frac{\partial^3 v}{\partial z^3}+u\frac{\partial^3 w}{\partial x\partial y\partial z}+v\frac{\partial^3 w}{\partial y^2\partial z}+u\frac{\partial^3 v}{\partial x\partial z^2}\\
&+w\frac{\partial^2 w}{\partial y\partial z^2}+2u\frac{\partial^3 w}{\partial x\partial z^2}+2v\frac{\partial^3 w}{\partial y\partial z^2}+w\frac{\partial^2 w}{\partial y\partial z^2}\\
&+\alpha_1\left[\frac{\partial^2 u}{\partial x\partial y}\frac{\partial v}{\partial y}+2\frac{\partial u}{\partial z}\frac{\partial^2 w}{\partial y\partial z}\right]+\alpha_2\left[2\frac{\partial v}{\partial y}\frac{\partial^2 u}{\partial x\partial y}\right]\\
&+\alpha_2\left[\frac{\partial u}{\partial z}\frac{\partial^2 w}{\partial x\partial y}+\frac{\partial w}{\partial x}\frac{\partial^2 v}{\partial x\partial z}+\frac{\partial v}{\partial x}\frac{\partial^2 w}{\partial x\partial z}\right]\\
&+\alpha_2\left[\frac{\partial^2 w}{\partial x^2}\frac{\partial v}{\partial z}+2\frac{\partial^2 u}{\partial y^2}\frac{\partial v}{\partial x}\right]+\alpha_1\left[\frac{\partial^2 u}{\partial y^2}\frac{\partial v}{\partial x}\right]\\
&+a_3\left[2a_3\frac{\partial v}{\partial y}\frac{\partial^2 v}{\partial y^2}+\frac{\partial w}{\partial y}\frac{\partial^2 v}{\partial z\partial y}\right]+a_4\left[\frac{\partial^2 u}{\partial x^2}\frac{\partial u}{\partial y}\right]\\
&+a_4\left[+\frac{\partial^2 w}{\partial y^2}\frac{\partial v}{\partial z}+\frac{\partial^2 v}{\partial z^2}\frac{\partial v}{\partial y}+\frac{\partial^2 w}{\partial z^2}\frac{\partial w}{\partial y}+\frac{\partial^2 v}{\partial x^2}\frac{\partial v}{\partial y}\right]
\end{aligned}
$$

$$
\begin{aligned}
&+a_5\left[2\frac{\partial u}{\partial x}\frac{\partial^2 v}{\partial x^2}+4\frac{\partial u}{\partial y}\frac{\partial^2 v}{\partial x\partial y}\right]\\
&+a_5\left[+4\frac{\partial v}{\partial x}\frac{\partial^2 v}{\partial x\partial y}+2\frac{\partial w}{\partial z}\frac{\partial^2 v}{\partial z^2}+\frac{\partial^2 w}{\partial x^2}\frac{\partial v}{\partial z}\right]\\
&+a_5\left[\frac{\partial^2 u}{\partial x\partial z}\frac{\partial w}{\partial y}+\frac{\partial^2 u}{\partial z^2}\frac{\partial v}{\partial x}+2\frac{\partial u}{\partial z}\frac{\partial^2 v}{\partial x\partial z}+4\frac{\partial v}{\partial z}\frac{\partial^2 v}{\partial y\partial z}\right]\\
&+a_6\left[\frac{\partial^2 u}{\partial x^2}\frac{\partial v}{\partial x}+\frac{\partial^2 w}{\partial z^2}\frac{\partial v}{\partial z}+\frac{\partial^2 w}{\partial y\partial z}\frac{\partial v}{\partial y}\right]\\
&+a_7\left[2\frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial x\partial y}+2\frac{\partial u}{\partial y}\frac{\partial^2 u}{\partial y^2}\right]\\
&+a_7\left[+\frac{\partial w}{\partial x}\frac{\partial^2 v}{\partial x\partial z}+\frac{\partial^2 w}{\partial x^2}\frac{\partial w}{\partial y}+\frac{\partial w}{\partial x}\frac{\partial^2 w}{\partial x\partial y}+2\frac{\partial u}{\partial y}\frac{\partial^2 u}{\partial y^2}\right]\\
&+a_7\left[\frac{\partial^2 u}{\partial z^2}\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}\frac{\partial^2 u}{\partial y\partial z}+2\frac{\partial w}{\partial z}\frac{\partial^2 w}{\partial y\partial z}+\frac{\partial w}{\partial x}\frac{\partial^2 u}{\partial y\partial z}\right]\\
&+a_7\left[\frac{\partial u}{\partial y}\frac{\partial^2 w}{\partial x\partial z}\right]+\frac{\mu_e}{\rho}\left[J_zH_x-J_xH_z\right], \qquad (1.131)
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial w}{\partial t}+u\frac{\partial w}{\partial x}+v\frac{\partial w}{\partial y}+w\frac{\partial w}{\partial z} \;=\;& -\frac{1}{\rho}\frac{\partial p}{\partial z}+\nu\left[\frac{\partial^2 u}{\partial x\partial z}+\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 v}{\partial y\partial z}+\frac{\partial^2 w}{\partial y^2}+2\frac{\partial^2 w}{\partial z^2}\right]\\
&+\frac{\partial^3 w}{\partial x^2\partial t}+\frac{\partial^3 v}{\partial y\partial z\partial t}+\frac{\partial^3 w}{\partial y^2\partial t}+2\frac{\partial^3 w}{\partial z^2\partial t}+w\frac{\partial^3 u}{\partial x\partial z^2}\\
&+u\frac{\partial^3 u}{\partial x^2\partial z}+v\frac{\partial^3 u}{\partial x\partial y\partial z}+\frac{\partial^3 u}{\partial x\partial z\partial t}+u\frac{\partial^3 w}{\partial x^3}\\
&+v\frac{\partial^3 w}{\partial x^2\partial y}+w\frac{\partial^3 w}{\partial x^2\partial z}+u\frac{\partial^3 v}{\partial x\partial y\partial z}+w\frac{\partial^3 v}{\partial y\partial z^2}\\
&+v\frac{\partial^3 w}{\partial y^3}+w\frac{\partial^3 w}{\partial y^2\partial z}+2u\frac{\partial^3 w}{\partial x\partial z^2}+u\frac{\partial^3 w}{\partial x\partial y^2}\\
&+2v\frac{\partial^3 w}{\partial y\partial z^2}+v\frac{\partial^3 v}{\partial y^2\partial z}+2w\frac{\partial^3 w}{\partial z^3}+\alpha_1\left[4\frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial x^2}\right]\\
&+\alpha_1\left[\frac{\partial u}{\partial z}\frac{\partial^2 v}{\partial x\partial y}+\frac{\partial w}{\partial x}\frac{\partial^2 w}{\partial x\partial z}\right]+\alpha_2\left[\frac{\partial^2 u}{\partial x\partial z}\frac{\partial u}{\partial x}\right]\\
&+\alpha_2\left[\frac{\partial^2 w}{\partial x\partial y}\frac{\partial v}{\partial x}+\frac{\partial^2 v}{\partial x\partial y}\frac{\partial w}{\partial x}+\frac{\partial u}{\partial z}\frac{\partial^2 v}{\partial y^2}\right]\\
&+a_3\left[\frac{\partial^2 w}{\partial y\partial z}\frac{\partial v}{\partial z}+\frac{\partial u}{\partial z}\frac{\partial^2 w}{\partial x\partial z}\right]+a_4\left[\frac{\partial u}{\partial z}\frac{\partial^2 u}{\partial x^2}\right]\\
&+a_4\left[\frac{\partial w}{\partial x}\frac{\partial^2 u}{\partial z^2}+\frac{\partial^2 v}{\partial z^2}\frac{\partial w}{\partial y}+\frac{\partial w}{\partial z}\frac{\partial^2 w}{\partial y^2}+\frac{\partial v}{\partial z}\frac{\partial^2 v}{\partial y^2}\right]\\
&+a_4\left[\frac{\partial w}{\partial z}\frac{\partial^2 w}{\partial x^2}\right]+a_3\left[2\frac{\partial w}{\partial z}\frac{\partial^2 w}{\partial z^2}\right]
\end{aligned}
$$

$$+a_5\left[2\frac{\partial^2 w}{\partial y^2}\frac{\partial v}{\partial y}+4\frac{\partial w}{\partial y}\frac{\partial^2 w}{\partial y\partial z}+2\frac{\partial u}{\partial x}\frac{\partial^2 w}{\partial x^2}+\frac{\partial w}{\partial y}\frac{\partial^2 v}{\partial x^2}\right]$$
$$+a_5\left[\frac{\partial w}{\partial x}\frac{\partial^2 u}{\partial y^2}+2\frac{\partial^2 w}{\partial x\partial y}\frac{\partial u}{\partial y}\right]+a_6\left[\frac{\partial v}{\partial z}\frac{\partial^2 u}{\partial x\partial y}\right]$$
$$+a_6\left[\frac{\partial^2 v}{\partial y\partial z}\frac{\partial w}{\partial z}+\frac{\partial w}{\partial x}\frac{\partial^2 u}{\partial z^2}+\frac{\partial^2 v}{\partial y^2}\frac{\partial w}{\partial y}+\frac{\partial^2 u}{\partial x\partial z}\frac{\partial w}{\partial z}\right]$$
$$+a_7\left[2\frac{\partial v}{\partial z}\frac{\partial^2 v}{\partial z^2}+\frac{\partial v}{\partial z}\frac{\partial^2 v}{\partial x^2}+\frac{\partial v}{\partial x}\frac{\partial^2 u}{\partial y\partial z}\right]$$
$$+a_7\left[\frac{\partial^2 w}{\partial x\partial y}\frac{\partial v}{\partial x}+2\frac{\partial u}{\partial z}\frac{\partial^2 u}{\partial z^2}+\frac{\partial v}{\partial x}\frac{\partial^2 v}{\partial x\partial z}\right]$$
$$+a_7\left[\frac{\partial u}{\partial y}\frac{\partial^2 v}{\partial x\partial z}+\frac{\partial^2 u}{\partial y\partial z}\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}\frac{\partial^2 u}{\partial y^2}\right]$$
$$+a_7\left[2\frac{\partial v}{\partial y}\frac{\partial^2 v}{\partial y\partial z}\right]+\frac{\mu_e}{\rho}\left[J_z H_x - J_x H_z\right], \tag{1.132}$$

where

$$a_3 = 4\alpha_2 + 5\alpha_1, \tag{1.133}$$

$$a_4 = 2\alpha_2 + 3\alpha_1, \tag{1.134}$$

$$a_5 = \alpha_2 + \alpha_1, \tag{1.135}$$

$$a_6 = 2\alpha_2 + \alpha_1, \tag{1.136}$$

$$a_7 = \alpha_2 + 2\alpha_1. \tag{1.137}$$

Chapter 2

Hydromagnetic Flow of a Viscous Fluid Past a Porous Infinite Plate

2.1 Introduction

In this chapter an investigation is made of the steady flow of an electrically conducting incompressible fluid past an infinite porous non-conducting plate taking Hall effects into account. The fluid is being permeated by a uniform transverse magnetic field. It is shown that the asymptotic solutions for the velocity and the magnetic field exist both for suction and blowing at the plate. Finally, the case when magnetic Reynold number is very small is also discussed. This problem is due to A. S. Gupta [9].

2.2 Mathematical formulation

Consider the steady flow of an electrically conducting incompressible fluid past an infinite non-conducting porous plate coinciding with the plane $y = 0$ such that x-axis is along the plate and parallel to the flow with y-axis normal to the plate. A uniform transverse magnetic field $\mathbf{H}_o$ is imposed along the y-axis and the plate is taken as electrically non-conducting. Since the plate is infinite, all physical variables depend on y only in the steady state. Taking z-axis normal to xy-plane and assuming that the velocity $\mathbf{q}$ and the induced magnetic field $\mathbf{H}$ have components (u, v, w) and (H_x, H_y, H_z) respectively.

The continuity equation for incompressible fluid is given by

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0. \tag{2.1}$$

We assume the velocity field of the form

$$\mathbf{q} = [u(y), v(y), w(y)]. \tag{2.2}$$

From Eqs. (2.1) and (2.2) we have

$$\frac{dv}{dy} = 0. \tag{2.3}$$

The above equation shows that $v \neq v(y)$. Hence

$$v = -v_o, \tag{2.4}$$

where the constant $v_o > 0$ for suction and $v_o < 0$ for blowing.

By the solenoidal relation one yields

$$\frac{\partial H_x}{\partial x} + \frac{\partial H_y}{\partial y} + \frac{\partial H_z}{\partial z} = 0 \tag{2.5}$$

which implies that

$$\frac{dH_y}{dy} = 0. \tag{2.6}$$

Integration of above equation yields

$$H_y = H_{o,} \tag{2.7}$$

were H_o is imposed uniform magnetic field.

If $[J_x, J_y, J_z]$ are the components of electric current density $\mathbf{J}$, then equation of conservation of charge is

$$\nabla \cdot [J_x(y), J_y(y), J_z(y)] = 0 \tag{2.8}$$

or

$$J_y = cons\tan t. \tag{2.9}$$

For simplicity, we take the value of constant to be zero and thus

$$J_y = 0 \tag{2.10}$$

at the plate because it is electrically non-conducting and thus $J_y = 0$ everywhere in the flow.

The equation of motion for magnetohydrodynamic fluid is

$$\rho \frac{d\mathbf{q}}{dt} = div\mathbf{T} + \mu_e \mathbf{J} \times \mathbf{H}, \tag{2.11}$$

where ρ is mass density of the fluid, $\mathbf{T}$ is Cauchy stress tensor and μ_e is the magnetic permeability of the fluid.

The Cauchy stress tensor $\mathbf{T}$ for viscous fluid is given by

$$\mathbf{T} = -p\mathbf{I} + \mu \mathbf{A}_1, \tag{2.12}$$

where p is pressure, $\mathbf{I}$ is unit tensor, μ is coefficient of viscosity and $\mathbf{A}_1$ is kinematical tensor defined by

$$\mathbf{A}_1 = grad\mathbf{q} + (grad\mathbf{q})^T. \tag{2.13}$$

Equation (2.11) can also be written as

$$\rho\left[\frac{\partial \mathbf{q}}{\partial t} + (\mathbf{q}\cdot\nabla)\mathbf{q}\right] = div\mathbf{T} + \mu_e \mathbf{J}\times\mathbf{H} \tag{2.14}$$

which for steady state situation gives that

$$\rho\,(\mathbf{q}\cdot\nabla)\mathbf{q} = \mathbf{divT} + \mu_e(\mathbf{J}\times\mathbf{H})\,. \tag{2.15}$$

From Eqs. (2.2) and (2.4) we obtain

$$grad\mathbf{q} = \begin{bmatrix} 0 & \frac{du}{dy} & 0 \\ 0 & 0 & 0 \\ 0 & \frac{dw}{dy} & 0 \end{bmatrix}, \tag{2.16}$$

$$(grad\mathbf{q})^T = \begin{bmatrix} 0 & 0 & 0 \\ \frac{du}{dy} & 0 & \frac{dw}{dy} \\ 0 & 0 & 0 \end{bmatrix}, \tag{2.17}$$

$$\mathbf{A}_1 = \begin{bmatrix} 0 & \frac{du}{dy} & 0 \\ \frac{du}{dy} & 0 & \frac{dw}{dy} \\ 0 & \frac{dw}{dy} & 0 \end{bmatrix}, \tag{2.18}$$

$$(div\mathbf{A}_1)_x = \frac{d^2u}{dy^2}, \tag{2.19}$$

$$(div\mathbf{A}_1)_y = 0, \tag{2.20}$$

$$(div\mathbf{A}_1)_z = \frac{d^2w}{dy^2}, \tag{2.21}$$

$$(div\,(-p\mathbf{I}))_x = -\frac{\partial p}{\partial x}, \quad (2.22)$$

$$(div\,(-p\mathbf{I}))_y = -\frac{\partial p}{\partial y}, \quad (2.23)$$

$$(div\,(-p\mathbf{I}))_z = -\frac{\partial p}{\partial z}. \quad (2.24)$$

Taking divergence of Eq. (2.12) one obtains

$$div\mathbf{T} = div\,(-p\mathbf{I}) + \mu div\mathbf{A}_1. \quad (2.25)$$

Making use of Eqs. (2.19) to (2.24) in Eq. (2.25) we have

$$(div\mathbf{T})_x = \mu\frac{d^2u}{dy^2} - \frac{\partial p}{\partial x}, \quad (2.26)$$

$$(div\mathbf{T})_y = -\frac{\partial p}{\partial y}, \quad (2.27)$$

$$(div\mathbf{T})_z = \mu\frac{d^2w}{dz^2} - \frac{\partial p}{\partial z}. \quad (2.28)$$

Now writing $\mathbf{J}\times\mathbf{H}$ in component form as follows

$$(\mathbf{J}\times\mathbf{H})_x = -H_oJ_z, \quad (2.29)$$

$$(\mathbf{J}\times\mathbf{H})_y = H_xJ_z - H_zJ_x, \quad (2.30)$$

$$(\mathbf{J}\times\mathbf{H})_z = H_oJ_x. \quad (2.31)$$

Now using Eqs. (2.26) to (2.31) in Eq. (2.15) we arrive at

$$-v_o\frac{du}{dy} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\frac{d^2u}{dy^2} - \frac{\mu_e}{\rho}H_oJ_z, \quad (2.32)$$

$$\frac{\partial p}{\partial y} = \frac{\mu_e}{\rho}\left(H_xJ_z - H_zJ_x\right), \quad (2.33)$$

$$-v_o\frac{dw}{dy} = -\frac{1}{\rho}\frac{\partial p}{\partial z} + \nu\frac{d^2w}{dy^2} + \frac{\mu_e}{\rho}H_oJ_x, \quad (2.34)$$

where μ_e is the magnetic permeability.

Elimination of p from Eqs. (2.32) and (2.34) gives

$$-v_o\frac{d^2u}{dy^2}=\nu\frac{d^3u}{dy^3}-\frac{\mu_e H_o}{\rho}\frac{dJ_z}{dy}, \tag{2.35}$$

$$-v_o\frac{d^2w}{dy^2}=\nu\frac{d^3w}{dy^3}+\frac{\mu_e H_o}{\rho}\frac{dJ_x}{dy}. \tag{2.36}$$

Integrating Eqs. (2.35) and (2.36) with respect to y we obtain

$$-v_o\frac{du}{dy}=\nu\frac{d^2u}{dy^2}-\frac{\mu_e H_o J_z}{\rho}+B_{1,} \tag{2.37}$$

$$-v_o\frac{dw}{dy}=\nu\frac{d^2w}{dy^2}+\frac{\mu_e H_o J_x}{\rho}+B_{2,} \tag{2.38}$$

where B_1 and B_2 are constants of integration.

In free stream, the magnetic field is uniform so that there is no current and thus

$$J_x\to 0,\ J_y\to 0 \text{ as } y\to\infty. \tag{2.39}$$

Further

$$u\to U,\ w\to 0,\ H_x\to 0,\ H_z\to 0,\ \text{as } y\to\infty, \tag{2.40}$$

where U is the uniform free stream velocity.

The use of the boundary conditions (2.39) and (2.40) in Eqs. (2.37) and (2.38) gives that

$$B_1=0,\ B_2=0 \tag{2.41}$$

and hence Eqs. (2.37) and (2.38) become

$$-v_o\frac{du}{dy}=\nu\frac{d^2u}{dy^2}-\frac{\mu_e H_o}{\rho}J_z, \tag{2.42}$$

$$-v_o\frac{dw}{dy}=\nu\frac{d^2w}{dy^2}+\frac{\mu_e H_o}{\rho}J_x. \tag{2.43}$$

In M.K.S. system the Maxwell's equations are

$$\nabla \times \mathbf{H} = \mathbf{J}, \tag{2.44}$$

$$\nabla \times \mathbf{E} = 0, \tag{2.45}$$

where $\mathbf{E}$ is electric field.

In component form, Eqs. (2.44) and (2.45) are

$$J_y = 0,\ J_x = \frac{dH_z}{dy},\ J_z = -\frac{dH_x}{dy},\ E_x = \text{cons tan t},\ E_z = \text{cons tan t} \tag{2.46}$$

and the generalized Ohm's law is [10]

$$\mathbf{J} = \sigma\left[\mathbf{E} + \mu_e \mathbf{q} \times \mathbf{H} - \frac{\mu_e}{en_e}\mathbf{J} \times \mathbf{H} + \frac{1}{en_e}\nabla p_e\right], \tag{2.47}$$

where σ, e, n_e and p_e denote respectively the electrical conductivity in absence of magnetic field, electric charge, the number density of electrons and the electron pressure. The third and last terms on the right hand side of Eq. (2.47) represent the effects due to Hall current and electron pressure gradient. In writing Eq. (2.47), when ion slip and thermoelectric effects are neglected $(\nabla p_e = 0)$. Further it is assumed that $\omega_e \tau_e \sim O(1)$ and $\omega_i \tau_i \leqslant 1$, where ω_e, ω_i are cyclotron frequencies of electrons and ions, and τ_e and τ_i are the collision times of electrons and ions. Thus from Eq. (2.47) one obtains

$$\mathbf{J} = \sigma\left[\mathbf{E} + \mu_e \mathbf{q} \times \mathbf{H} - \frac{\mu_e}{en_e}\mathbf{J} \times \mathbf{H}\right]. \tag{2.48}$$

The above expression in component form can be written as

$$J_x = \sigma\left[E_x - \mu_e\left(v_o H_z + H_o w\right) + \frac{\mu_e H_o}{en_e} J_z\right], \tag{2.49}$$

$$J_z = \sigma\left[E_z + \mu_e\left(uH_o + v_oH_x\right) - \frac{\mu_e H_o}{en_e}J_x\right]. \tag{2.50}$$

Solving for J_x and J_z we get

$$J_x = \frac{\sigma}{(1+m^2)}\left[E_x - \mu_e\left(v_oH_z + H_ow\right) + m\left(E_z + \mu_e uH_o + \mu_e v_oH_x\right)\right], \tag{2.51}$$

$$J_z = \frac{\sigma}{(1+m^2)}\left[E_z + \mu_e uH_o + \mu_e v_oH_x - m\left(E_x - \mu_e v_oH_z - \mu_e H_ow\right)\right]. \tag{2.52}$$

Here m stands for the Hall parameter [10] and is given by

$$m = \omega_e\tau_e \tag{2.53}$$

with

$$\sigma = \frac{e^2 n_e\tau_e}{m_e}, \tag{2.54}$$

$$\omega_e = \frac{e\mu_e H_o}{m_e}. \tag{2.55}$$

Multiplying Eq. (2.43) by i and then adding into Eq. (2.42) we get

$$-v_o\frac{d}{dy}(u+iw) = \nu\frac{d^2}{dy^2}(u+iw) - \frac{\mu_e H_o}{\rho}\left(J_z - iJ_x\right). \tag{2.56}$$

Introducing

$$u + iw = V, H_x + iH_z = H \tag{2.57}$$

equation (2.56) becomes

$$-v_o\frac{dV}{dy} = \nu\frac{d^2V}{dy^2} + \frac{\mu_e H_o}{\rho}\frac{dH}{dy}. \tag{2.58}$$

Mutiplying Eq. (2.52) by i and then adding the resulting equation into Eq. (2.51) one obtains

$$-i\frac{dH}{dy} = \frac{\sigma(m+i)}{(1+m^2)}\left[-iE + \mu_e\left(H_oV + v_oH\right)\right], \tag{2.59}$$

where

$$E = E_x + iE_z. \tag{2.60}$$

Now using the boundary conditions given by Eqs. (2.39) and (2.40) in Eqs. (2.51) and (2.52) we get

$$E_x = 0, E_z = -\mu_e U H_o. \tag{2.61}$$

Equations (2.61) show that E_x and E_z are constants because H_o, U and μ_e are constants that confirms Eq. (2.46).

From Eqs. (2.59), (2.60) and (2.61) we can write

$$\frac{dH}{dy} = \frac{\sigma(im-1)}{(1+m^2)}\left[\mu_e H_o (V-U) + \mu_e v_o H\right]. \tag{2.62}$$

Integration of Eq. (2.58) with respect to y yields

$$-v_o V = \nu \frac{dV}{dy} + \frac{\mu H_o H}{\rho} + B_3, \tag{2.63}$$

where B_3 is constant of integration.

The boundary conditions given by Eqs. (2.39) and (2.40) imply that

$$B_3 = -v_o U \tag{2.64}$$

and thus from Eq. (2.63) we have

$$H = \frac{\rho v_o U}{\mu_e H_o} - \frac{\rho v_o}{\mu_e H_o} V - \frac{\rho v}{\mu_e H_o}\frac{dV}{dy} \tag{2.65}$$

which gives induced magnetic field.

Introducing the following dimensionless quantities

$$S = \frac{v_o}{U},\ M = \frac{\sigma \mu_e^2 H_o^2 \nu}{\rho U^2},\ \eta = \frac{Uy}{\nu},\ \alpha = \frac{v_o}{V_A},\ V_A^2 = \frac{\mu_e H_o^2}{\rho},\ \overline{V} = \frac{V}{U} \tag{2.66}$$

and using Eqs. (2.65) and (2.66) in Eq.(2.62) we have

$$\frac{d^2\overline{V}}{d^2\eta}+\left[S+\frac{M\alpha^2\left(1-im\right)}{S\left(1+m^2\right)}\right]\frac{d\overline{V}}{d\eta}-\frac{M\left(1-\alpha^2\right)\left(1-im\right)}{\left(1+m^2\right)}\left(\overline{V}-1\right)=0, \tag{2.67}$$

where $M^{\frac{1}{2}}$ and V_A represent the Hartmann number and Alfven velocity respectively.

The boundary conditions in term of $\overline{V}$ reflecting no-slip at the wall and the uniform free stream at infinity are

$$\overline{V}=0 \text{ at } \eta=0,\ \overline{V}\longrightarrow 1 \text{ as } \eta\longrightarrow\infty. \tag{2.68}$$

Since the plate is electrically non-conducting, the magnetic boundary conditions are given by

$$H\left(0\right)=0,\ H\left(\infty\right)=0. \tag{2.69}$$

Equation (2.67) is non-homogeneous which can be made homogeneous by making the following substitution

$$1-\overline{V}=T\left(\eta\right) \tag{2.70}$$

which by differentiating gives

$$\frac{d\overline{V}}{d\eta}=-\frac{dT}{d\eta} \text{ and } \frac{d^2\overline{V}}{d\eta^2}=-\frac{d^2T}{d\eta^2} \tag{2.71}$$

and hence Eq. (2.67) becomes

$$\frac{d^2T}{d\eta^2}+\left[S+\frac{M\alpha^2\left(1-im\right)}{S\left(1+m^2\right)}\right]\frac{dT}{d\eta}-\frac{M\left(1-\alpha^2\right)\left(1-im\right)}{\left(1+m^2\right)}T\left(\eta\right)=0 \tag{2.72}$$

and the boundary conditions are

$$T\left(0\right)=1 \text{ and } T\left(\infty\right)=0. \tag{2.73}$$

The characteristic equation corresponding to differential Eq. (2.72) is

$$D^2 + \left[S + \frac{M\alpha^2 (1 - im)}{S (1 + m^2)}\right] D - \frac{M (1 - \alpha^2) (1 - im)}{(1 + m^2)} = 0. \quad (2.74)$$

.The roots of above equation are

$$D_1 = \frac{-\left(S + \frac{M\alpha^2(1-im)}{S(1+m^2)}\right) - \sqrt{\left(S + \frac{M\alpha^2(1-im)}{S(1+m^2)}\right)^2 + \frac{4M(1-\alpha^2)(1-im)}{(1+m^2)}}}{2}, \quad (2.75)$$

$$D_2 = \frac{-\left(S + \frac{M\alpha^2(1-im)}{S(1+m^2)}\right) + \sqrt{\left(S + \frac{M\alpha^2(1-im)}{S(1+m^2)}\right)^2 + \frac{4M(1-\alpha^2)(1-im)}{(1+m^2)}}}{2}. \quad (2.76)$$

The solution of Eq. (2.72) is

$$T(\eta) = B_4 \exp(D_1\eta) + B_5 \exp(D_2\eta), \quad (2.77)$$

where conditions (2.73) give that

$$B_4 = 1, \; B_5 = 0 \quad (2.78)$$

and thus Eq. (2.77) becomes

$$T(\eta) = \exp(D_1\eta) \quad (2.79)$$

From Eqs. (2.75) and (2.79) we have

$$\frac{u}{U} = 1 - \exp\left[-\frac{S^2(1+m^2) + M\alpha^2 + B_7 S(1+m^2)}{S(1+m^2)}\frac{\eta}{2}\right] \times \cos\left(\frac{M\alpha^2 m + B_8 S(1+m^2)}{S(1+m^2)}\right)\frac{\eta}{2}, \quad (2.80)$$

$$\frac{w}{U} = -\exp\left[-\left(\frac{S^2(1+m^2) + M\alpha^2 + B_7 S(1+m^2)}{S(1+m^2)}\right)\frac{\eta}{2}\right] \times \sin\left(\frac{M\alpha^2 m + B_8 S(1+m^2)}{S(1+m^2)}\right)\frac{\eta}{2}, \quad (2.81)$$

where

$$-B_7 + iB_8 = \sqrt{\left(S + \frac{M\alpha^2(1-im)}{S(1+m^2)}\right)^2 + \frac{4M(1-\alpha^2)(1-im)}{(1+m^2)}}. \qquad (2.82)$$

Separating real and imaginary parts we have

$$B_7 = \left(\frac{\lambda}{2}\right)^{\frac{1}{2}} \left[(t^2+1)^{\frac{1}{2}} + 1\right]^{\frac{1}{2}}, \qquad (2.83)$$

$$B_8 = \left(\frac{\lambda}{2}\right)^{\frac{1}{2}} \left[(t^2+1)^{\frac{1}{2}} - 1\right]^{\frac{1}{2}}, \qquad (2.84)$$

$$\lambda = \frac{M(1-m^2)\alpha^4 - 2MS^2(1+m^2)\alpha^2 + S^2(1+m^2)(4M+S^2+S^2m)}{S^2(1+m^2)^2}, \qquad (2.85)$$

$$t = \frac{2\left[M^2m\alpha^4 - S^2Mm(1+m^2)\alpha^2 + 2S^2Mm(1+m^2)\right]}{M(1-m^2)\alpha^4 - 2MS^2(1+m^2)\alpha^2 + S^2(1+m^2)(4M+S^2+S^2m)}. \qquad (2.86)$$

When Hall effects are absent i.e. $m = 0$, there will not be cross flow so that $w = 0$ and Eq. (2.67) reduces to the equation derived by Kakutani [11]. It can be shown from Eq. (2.67) that when $m = 0$, an asymptotic solution for velocity and magnetic field satisfying the boundary conditions (2.68) and (2.69) will exist only when there is suction at the wall ($v_o > 0$) and ($v_o > V_A$) as found by Kakutani [11]. On the other hand when $\omega_e\tau_e \neq 0$, a little analysis will show that asymptotic solution for velocity and magnetic field satisfying Eqs. (2.68) and (2.69) exists both for suction and blowing at the plate.

2.3 The solution of the problem in case of blowing

Since, for blowing $S < 0$ so we put $S = -S_o$ for $S_o > 0$ in Eq. (2.72) i.e.

$$\frac{d^2T}{d^2\eta} - \left[S_o + \frac{M\alpha^2(1-im)}{S_o(1+m^2)}\right]\frac{dT}{d\eta} - \frac{M(1-\alpha^2)(1-im)}{(1+m^2)}T(\eta) = 0 \qquad (2.87)$$

with

$$T(0) = 1, \text{ and } T(\infty) = 0. \tag{2.88}$$

Following the same method of solution as for suction we have the solution of the following form

$$\begin{aligned}\frac{u}{U} &= 1 - \exp\left[-\left(\frac{B_9 S_o (1+m^2) - S_o^2 (1+m^2) - M\alpha^2}{S_o (1+m^2)}\right)\frac{\eta}{2}\right] \times \\ &\quad \cos\left(\frac{M\alpha^2 m - B_{10} S_o (1+m^2)}{S_o (1+m^2)}\right)\frac{\eta}{2},\end{aligned} \tag{2.89}$$

$$\begin{aligned}\frac{w}{U} &= \exp\left[-\left(\frac{B_9 S_o (1+m^2) - S_o^2 (1+m^2) - M\alpha^2}{S_o (1+m^2)}\right)\frac{\eta}{2}\right] \times \\ &\quad \sin\left(\frac{M\alpha^2 m - B_{10} S_o (1+m^2)}{S_o (1+m^2)}\right)\frac{\eta}{2},\end{aligned} \tag{2.90}$$

where

$$B_9 = \left(\frac{\lambda_1}{2}\right)^{\frac{1}{2}} \left[\left(q_1^2 + 1\right)^{\frac{1}{2}} + 1\right]^{\frac{1}{2}}, \tag{2.91}$$

$$B_{10} = \left(\frac{\lambda_1}{2}\right)^{\frac{1}{2}} \left[\left(q_1^2 + 1\right)^{\frac{1}{2}} - 1\right]^{\frac{1}{2}}, \tag{2.92}$$

$$\lambda_1 = \frac{M(1-m^2)\alpha^4 - 2MS_o^2(1+m^2)\alpha^2 + S_o^2(1+m^2)(4M + S_o^2 + S_o^2 m)}{S_o^2(1+m^2)^2}, \tag{2.93}$$

$$q_1 = \frac{2\left[M^2 m\alpha^4 - S_o^2 M m (1+m^2)\alpha^2 - 2S_o^2 M m (1+m^2)\right]}{M(1-m^2)\alpha^4 - 2MS_o^2(1+m^2)\alpha^2 + S_o^2(1+m^2)(4M + S_o^2 + S_o^2 m)}. \tag{2.94}$$

2.4 The solution of the problem for suction when magnetic Reynold number is small

Now we discuss the solution of the problem when the magnetic Reynold number is small. This number is defined as

$$M = R_\sigma R_p, \tag{2.95}$$

where R_p is called the magnetic pressure number and equals to

$$R_p = \frac{\mu_e H_o^2}{\rho U^2}. \tag{2.96}$$

It is well known that R_ρ is a measure of the extent to which the magnetic lines of force are distorted by the flow [10]. When $R_\sigma \ll 1$, the induced magnetic field can be neglected in comparison with imposed magnetic field H_o i.e. $0 = H = H_x + iH_z$, so that the components of the current density $\mathbf{J}$ given by Eqs. (2.51) and (2.52) reduce to

$$J_x = \frac{\sigma \mu_e H_o}{(1+m^2)} \left[-w + m(u - U)\right], \tag{2.97}$$

$$J_z = \frac{\sigma \mu_e H_o}{(1+m^2)} \left[u - U + mw\right]. \tag{2.98}$$

Combining Eqs. (2.42) and (2.43) and using Eqs. (2.65), (2.97) and (2.98) we arrive at

$$\frac{d^2\overline{V}}{d\eta^2} + S\frac{d\overline{V}}{d\eta} - \frac{(1-im)}{(1+m^2)} M\left[\overline{V} - 1\right] = 0. \tag{2.99}$$

The boundary conditions in term of $\overline{V}$ are

$$\overline{V} = 0 \text{ at } \eta = 0,\ \overline{V} \to 1 \text{ as } \eta \to \infty. \tag{2.100}$$

Defining

$$W(\eta) = 1 - \overline{V} \tag{2.101}$$

Eqs. (2.100) and (2.101) become

$$\frac{d^2W}{d\eta^2} + S\frac{dW}{d\eta} - \frac{(1-im)M}{(1+m^2)} W(\eta) = 0. \tag{2.102}$$

$$W(0) = 1,\ W(\infty) = 0. \tag{2.103}$$

The roots of Eq. (2.102) are given by

$$D_1 = \frac{1}{2}\left[-S - \sqrt{S^2 + \frac{4M(1-im)}{1+m^2}}\right], \tag{2.104}$$

$$D_2 = \frac{1}{2}\left[-S + \sqrt{S^2 + \frac{4M(1-im)}{1+m^2}}\right]. \tag{2.105}$$

The solution of Eq. (2.102) can be written as

$$W(\eta) = C_4 \exp(D_1\eta) + C_5 \exp(D_2\eta). \tag{2.106}$$

The boundary conditions given by Eqs. (2.103) imply that

$$C_4 = 1,\ C_5 = 0 \tag{2.107}$$

which gives

$$W(\eta) = \exp(D_1\eta) \tag{2.108}$$

and hence

$$\frac{u}{U} = 1 - \exp\left[-\left(\frac{S+A_2}{2}\right)\eta\right]\cos\frac{A_3\eta}{2}, \tag{2.109}$$

$$\frac{w}{U} = -\exp\left[-\left(\frac{S+A_2}{2}\right)\eta\right]\sin\frac{A_3\eta}{2}, \tag{2.110}$$

where

$$-A_2 + iA_3 = \sqrt{S^2 + \frac{4M(1-im)}{1+m^2}}. \tag{2.111}$$

Separating real and imaginary parts we have

$$A_2 = \left(\frac{\lambda_2}{2}\right)^{\frac{1}{2}}\left[\sqrt{(q_2^2+1)} + 1\right]^{\frac{1}{2}}, \tag{2.112}$$

$$A_3 = \left(\frac{\lambda_2}{2}\right)^{\frac{1}{2}}\left[\sqrt{(q_2^2+1)} - 1\right]^{\frac{1}{2}}, \tag{2.113}$$

$$\lambda_2 = S^2 + \frac{4M}{1+m^2}, \tag{2.114}$$

$$q_2 = \frac{4mM}{(1+m^2)\,S^2 + 4M}. \tag{2.115}$$

It is interesting to note that this velocity distribution is in the form of a logarithmic spiral and similar to Ekman velocity spiral for flow past a flat plate in a rotating fluid Batchelor [12]. Thus we may conclude that for a small magnetic Reynold number, Hall effects, which introduce a cross-flow, are similar to that of rotation . It is also clear from Eqs. (2.109) and (2.110) that the flow exhibits a boundary layer behavior with boundary layer thickness of $O(\frac{2}{S+A_2})$. This shows that increase in suction parameter S causes thinning of the boundary layer.

Now from Eqs. (2.112) to (2.115) we have

$$2A_2^2 = \left[S^2 + \frac{4M\,(S^2+4M)}{(1+m^2)\,S^2+4M}\right]^{\frac{1}{2}} + \left[S^2 + \frac{4M}{1+m^2}\right]^{\frac{1}{2}}. \tag{2.116}$$

Since for fixed S and M, each term on right hand side of Eq. (2.109) decreases with increase in the Hall parameter m, it follows that A_2 decreases with increase in m. Thus for fixed S and M, the boundary layer thickness, being of O $(\frac{2}{S+A_2})$ and increases with increase in m.

Again from Eqs. (2.109) and (2.110) we have

$$\frac{1}{U}\left(\frac{du}{d\eta}\right)_{\eta=0} = \frac{S+A_2}{2},\; \frac{1}{U}\left(\frac{dw}{d\eta}\right)_{\eta=0} = -\frac{A_3}{2}. \tag{2.117}$$

Since S, A_2 and A_3 are all positive, Eq. (2.117) shows that the boundary layer can never separate. Since A_2 increases with increase in m, we conclude from Eq. (2.117) that the skin friction due to the primary flow decreases with increase in Hall parameter for fixed S and M.

2.5 The solution of the problem in case of blowing ($S < 0$) when magnetic Reynold number is small

In this section, we discuss the existence of solution for the blowing case at the plate .Since, for blowing $S < 0$ so we put $S = -S_o$ for $S_o > 0$. In this case, the solution of Eq. (2.102) satisfying Eq. (2.103) is

$$\frac{u}{U} = 1 - \exp\left[\left(\frac{S_o - A_4}{2}\right)\eta\right]\cos\frac{A_5\eta}{2}, \tag{2.118}$$

$$\frac{w}{U} = -\exp\left[\left(\frac{S_o - A_4}{2}\right)\eta\right]\sin\frac{A_5\eta}{2}, \tag{2.119}$$

where

$$A_4 = \left(\frac{\lambda_3}{2}\right)^{\frac{1}{2}}\left[\sqrt{(q_3^2 + 1)} + 1\right]^{\frac{1}{2}}, \tag{2.120}$$

$$A_5 = \left(\frac{\lambda_3}{2}\right)^{\frac{1}{2}}\left[\sqrt{(q_3^2 + 1)} - 1\right]^{\frac{1}{2}}, \tag{2.121}$$

$$\lambda_3 = S_o^2 + \frac{4M}{1 + m^2}, \tag{2.122}$$

$$q_3 = \frac{4mM}{(1 + m^2)\,S_o^2 + 4M}. \tag{2.123}$$

Since from Eqs .(2.120) and (2.122), $A_4 > \sqrt{\lambda_3} > S_o$, $W(\eta)$ clearly tends to zero as $\eta \to \infty$. Equations (2.118) and (2.119) also represent velocity field similar to Ekman spiral. The solution given by Eqs. (2.118) and (2.119) confirms our previous assertion that asymptotic solution for velocity exists even for the case of blowing at plate when Hall effects are considered.

2.6 Remarks

The presented analysis brings out two results of physical interest.

1. In the presence of Hall effect the asymptotic solutions for the velocity and the magnetic field exist both for suction and blowing at the plate.

2. When magnetic Reynold number is very small, the flow pattern with Hall effects is remarkably similar to that of non-conducting flow past a flat plate in a rotating frame of course, the assumption of very small magnetic Reynold number will be valid for flow of liquid metals or slightly ionized gas. For slightly ionized gas, the last term in Eq. (2.47) is significant with $p_e = \frac{p}{2}$, while ion slip term can be neglected.

Chapter 3

Hydromagnetic Flow of a Second Order Fluid Past a Porous Infinite Plate

3.1 Introduction

In this chapter the influence of Hall current on generalized Hartmann flow of a second order fluid is investigated. Effects of uniform suction (or blowing) and Hall parameter on the flow phenomena are analyzed. The solutions for small magnetic Reynold's number are also constructed. Several known results of interest are found as particular cases of the problem considered.

3.2 Problem formulation

We consider the steady hydromagnetic flow induced in a semi-infinite expanse of an electrically conducting second order fluid bounded by an infinite porous plate at $y = 0$. A uniform transverse magnetic field $\mathbf{H}_o$ is applied along the $y - axis$ normal to the plate with uniform suction or blowing. The steady hydromagnetic flow is governed by following the equations of motion, continuity and the Maxwell's equation in the form

$$\rho\left(\mathbf{q} \cdot \nabla\right)\mathbf{q} = \mathbf{divT} + \mu_e(\mathbf{J} \times \mathbf{H}), \tag{3.1}$$

$$\nabla \cdot \mathbf{q} = 0, \tag{3.2}$$

$$\nabla \cdot \mathbf{H} = 0, \tag{3.3}$$

$$\nabla \cdot \mathbf{J} = 0, \tag{3.4}$$

$$\nabla \times \mathbf{H} = \mathbf{J}, \tag{3.5}$$

$$\nabla \times \mathbf{E} = \mathbf{0}, \tag{3.6}$$

where $\mathbf{q}$, ρ, $\mathbf{T}$, $\mathbf{J}$, $\mathbf{H}$ and $\mathbf{E}$ denote respectively the velocity field, mass density of fluid, Cauchy stress tensor, current density, magnetic field and electric field.

Since the plate is infinite, all physical variables depend on y in steady flow and if (H_x, H_y, H_z) and (J_x, J_y, J_z) are the components of magnetic field $\mathbf{H}$ and electric current density $\mathbf{J}$ then Eqs. (3.3) and (3.4) give

$$H_y = H_o, \tag{3.7}$$

$$J_y = 0 \tag{3.8}$$

at the plate because it is electrically non-conducting and thus $J_y = 0$ everywhere in the flow.

In component form Eqs. (3.5) and (3.6) are given by

$$J_y = 0,\ J_x = \frac{dH_z}{dy},\ J_z = -\frac{dH_x}{dy},\ E_x = \text{cons tan t},\ E_z = \text{cons tan t}. \tag{3.9}$$

It is assumed that there is no applied or polarization voltage so that $\mathbf{E} = \mathbf{0}$ and the induced magnetic field is negligible so that the total magnetic field $\mathbf{H} = (0, 0, H_o)$, where H_o is applied magnetic field parallel to $y-axis$ normal to the plate. The latter assumption is justified in flow of liquid metals. When the strength of the magnetic field is very large, the generalized Ohm's law is modified to induce Hall current so that

$$\mathbf{J}+\frac{\omega_e\tau_e}{H_o}\left(\mathbf{J}\times\mathbf{H}\right)=\boldsymbol{\sigma}\left[\mathbf{E}+\mu_e\mathbf{q}\times\mathbf{H}-\frac{1}{en_e}\boldsymbol{\nabla}p_e\right], \tag{3.10}$$

where ω_e is a cyclotron frequency, τ_e is the electron collision time, e is charge on electron and p_e is electron pressure. The ion-slip and thermoelectric effects are not included in Eq. (3.10) i.e. $\boldsymbol{\nabla}p_e = \mathbf{0}$. Further, it is assumed that $\omega_e\tau_e \sim O(1)$ and $\omega_i\tau_i \leq 1$, where ω_i and τ_i are cyclotron frequency and collision time for ions respectively. Thus

$$\mathbf{J}+\frac{\omega_e\tau_e}{H_o}\left(\mathbf{J}\times\mathbf{H}\right)=\boldsymbol{\sigma}\left[\mathbf{E}+\mu_e\mathbf{q}\times\mathbf{H}\right]. \tag{3.11}$$

Above equation in component form can be written as

$$J_x = \sigma\left[E_x - \mu_e\left(v_oH_z + H_ow\right) + \frac{\mu_eH_o}{en_e}J_z\right], \tag{3.12}$$

$$J_z = \sigma\left[E_z + \mu_e\left(uH_o + v_oH_z\right) - \frac{\mu_eH_o}{en_e}J_x\right]. \tag{3.13}$$

Simplifying Eqs. (3.12) and (3.13) one obtains

$$J_x = \frac{\sigma}{(1+m^2)}\left[E_x - \mu_e\left(v_o H_z + H_o w\right) + m\left(E_z + \mu_e u H_o + \mu_e v_o H_x\right)\right], \tag{3.14}$$

$$J_z = \frac{\sigma}{(1+m^2)}\left[E_z + \mu_e u H_o + \mu_e v_o H_x - m\left(E_x - \mu_e v_o H_z - \mu_e H_o w\right)\right], \tag{3.15}$$

where

$$m = \omega_e \tau_e,\ \omega_e = \frac{e\mu_e H_o}{m_e},\ \sigma = \frac{e^2 n_e \tau_e}{m_e}. \tag{3.16}$$

The Cauchy stress tensor for an incompressible second order fluid is given by

$$\mathbf{T} = -p\mathbf{I} + \mu\mathbf{A}_1 + \alpha_1\mathbf{A}_2 + \alpha_2\mathbf{A}_1^2, \tag{3.17}$$

where μ is coefficient of viscosity, α_1 and α_2 are material moduli which are usually referred to as the normal stress moduli, p is pressure and $\mathbf{A}_1$ and $\mathbf{A}_2$ are kinematical tensors defined by

$$\mathbf{A}_1 = grad\mathbf{q} + (grad\mathbf{q})^T, \tag{3.18}$$

$$\mathbf{A}_2 = \frac{d\mathbf{A}_1}{dt} + \mathbf{A}_1 grad\mathbf{q} + (grad\mathbf{q})^T\mathbf{A}_1. \tag{3.19}$$

If an incompressible fluid of second order is to have motions which are compatible with thermodynamics in the sense of Clausius-Duhem inequality and the condition that the Helmholtz free energy be minimum when the fluid is at rest, then following conditions must be satisfied

$$\mu \geq 0,\ \alpha_1 > 0,\ \alpha_1 + \alpha_2 = 0. \tag{3.20}$$

We assume the velocity field of the form

$$\mathbf{q} = \left(u(y), v(y), w(y)\right). \tag{3.21}$$

Equations (3.2) and (3.21) yield

$$v = -v_o, \tag{3.22}$$

where $v_o > 0$ for suction and $v_o < 0$ for blowing. Hence Eq. (3.21) becomes

$$\mathbf{q} = \left(u\left(y\right), -v_o, w\left(y\right)\right). \tag{3.23}$$

From Eqs. (3.18), (3.19) and (3.23) we have

$$grad\mathbf{q} = \begin{bmatrix} 0 & \frac{du}{dy} & 0 \\ 0 & 0 & 0 \\ 0 & \frac{dw}{dy} & 0 \end{bmatrix}, \tag{3.24}$$

$$\left(grad\mathbf{q}\right)^T = \begin{bmatrix} 0 & 0 & 0 \\ \frac{du}{dy} & 0 & \frac{dw}{dy} \\ 0 & 0 & 0 \end{bmatrix}, \tag{3.25}$$

$$\mathbf{A}_1 = \begin{bmatrix} 0 & \frac{du}{dy} & 0 \\ \frac{du}{dy} & 0 & \frac{dw}{dy} \\ 0 & \frac{dw}{dy} & 0 \end{bmatrix}, \tag{3.26}$$

$$\mathbf{A}_1^2 = \begin{bmatrix} \left(\frac{du}{dy}\right)^2 & 0 & \frac{du}{dy}\frac{dw}{dy} \\ 0 & \left(\frac{du}{dy}\right)^2 + \left(\frac{dw}{dy}\right)^2 & 0 \\ \frac{du}{dy}\frac{dw}{dy} & 0 & \left(\frac{dw}{dy}\right)^2 \end{bmatrix}, \tag{3.27}$$

$$\mathbf{A}_2 = \begin{bmatrix} 0 & -v_o\frac{d^2u}{d^2y} & 0 \\ -v_o\frac{d^2u}{d^2y} & 2\left(\left(\frac{du}{dy}\right)^2 + \left(\frac{dw}{dy}\right)^2\right) & -v_o\frac{d^2w}{d^2y} \\ 0 & -v_o\frac{d^2w}{d^2y} & 0 \end{bmatrix}. \tag{3.28}$$

Taking divergence of Eq. (3.17) we have

$$divT = div[-pI] + \mu div A_1 + \alpha_1 div A_2 + \alpha_2 div A_1^2. \tag{3.29}$$

Writing $divA_1$, $divA_1^2$, $divA_2$ and $div(-pI)$ in component form we obtain

$$(divA_1)_x = \frac{d^2u}{dy^2}, \tag{3.30}$$

$$(divA_1)_y = 0, \tag{3.31}$$

$$(divA_1)_z = \frac{d^2w}{dy^2}, \tag{3.32}$$

$$(divA_2)_x = -v_o\frac{d^3u}{dy^3}, \tag{3.33}$$

$$(divA_2)_y = 4\left(\frac{du}{dy}\frac{d^2u}{dy^2} + \frac{dw}{dy}\frac{d^2w}{dy^2}\right), \tag{3.34}$$

$$(divA_2)_z = -v_o\frac{d^3w}{dy^3}, \tag{3.35}$$

$$\left(divA_1^2\right)_x = 0, \tag{3.36}$$

$$\left(divA_1^2\right)_y = 2\left(\frac{du}{dy}\frac{d^2u}{dy^2} + \frac{dw}{dy}\frac{d^2w}{dy^2}\right), \tag{3.37}$$

$$\left(divA_1^2\right)_z = 0, \tag{3.38}$$

$$(div(-pI))_x = -\frac{\partial p}{\partial x}, \tag{3.39}$$

$$(div(-pI))_y = -\frac{\partial p}{\partial y}, \tag{3.40}$$

$$(div(-pI))_z = -\frac{\partial p}{\partial z}. \tag{3.41}$$

Taking the x, y, $z-$components of Eq. (3.29) we have

$$(divT)_x = \mu\frac{d^2u}{dy^2} - \alpha_1 v_o\frac{d^3u}{dy^3}, \tag{3.42}$$

$$(divT)_y = -\frac{\partial p}{\partial y} + 2(2\alpha_1 + \alpha_2)\left(\frac{du}{dy}\frac{d^2u}{dy^2} + \frac{dw}{dy}\frac{d^2w}{dy^2}\right), \tag{3.43}$$

$$(div\mathbf{T})_z = -\frac{\partial p}{\partial z} + \mu\frac{d^2w}{dz^2} - \alpha_1 v_o \frac{d^3w}{dy^3}. \tag{3.44}$$

Now $\mathbf{J}\times\mathbf{H}$ in component form can be written as

$$(\mathbf{J}\times\mathbf{H})_x = -H_o J_z, \tag{3.45}$$

$$(\mathbf{J}\times\mathbf{H})_y = H_x J_z - H_z J_x, \tag{3.46}$$

$$(\mathbf{J}\times\mathbf{H})_z = H_o J_x \tag{3.47}$$

and also

$$\rho\left(\mathbf{q}\cdot\boldsymbol{\nabla}\right)\mathbf{q} = \left(-v_o\frac{du}{dy}, 0, -v_o\frac{dw}{dy}\right). \tag{3.48}$$

Using Eqs. (3.42) to (3.48) in Eq. (3.1) we have

$$-v_o\frac{du}{dy} = -\frac{1}{\rho}\frac{\partial p}{\partial x} - \frac{\alpha_1 v_o}{\rho}\frac{d^3u}{dy^3} + \nu\frac{d^2u}{dy^2} - \frac{\mu_e}{\rho}H_o J_z, \tag{3.49}$$

$$\frac{\partial p}{\partial y} = 2\left(2\alpha_1+\alpha_2\right)\left(\frac{du}{dy}\frac{d^2u}{dy^2} + \frac{dw}{dy}\frac{d^2w}{dy^2}\right) + \frac{\mu_e}{\rho}\left(H_x J_z - H_z J_x\right), \tag{3.50}$$

$$-v_o\frac{dw}{dy} = -\frac{1}{\rho}\frac{\partial p}{\partial z} - \frac{\alpha_1 v_o}{\rho}\frac{d^3w}{dy^3} + \nu\frac{dw}{dy} + \frac{\mu_e}{\rho}H_o J_x, \tag{3.51}$$

where $\nu = \frac{\mu}{\rho}$ is called kinematic viscosity and μ_e is the magnetic permeability.

From Eq. (3.50) we have

$$\frac{\partial\,\widetilde{p}}{\partial y} = \frac{\mu_e}{\rho}\left(H_x J_z - H_z J_x\right), \tag{3.52}$$

where

$$\widetilde{p} = p - \left(2\alpha_1+\alpha_2\right)\left[\left(\frac{du}{dy}\right)^2 + \left(\frac{dw}{dy}\right)^2\right]. \tag{3.53}$$

Taking partial derivative with respect to x and z of Eq. (3.53) we get

$$\frac{\partial\,\widetilde{p}}{\partial x} = \frac{\partial p}{\partial x} \text{ and } \frac{\partial\,\widetilde{p}}{\partial y} = \frac{\partial p}{\partial y} \tag{3.54}$$

and hence Eqs. (3.49) and (3.51) becomes

$$-v_o\frac{du}{dy}=-\frac{1}{\rho}\frac{\partial\,\widetilde{p}}{\partial x}-\frac{\alpha_1 v_o}{\rho}\frac{d^3u}{dy^3}+\nu\frac{d^2u}{dy^2}+\frac{\mu_e}{\rho}H_oJ_z, \tag{3.55}$$

$$-v_o\frac{dw}{dy}=-\frac{1}{\rho}\frac{\partial\,\widetilde{p}}{\partial z}-\frac{\alpha_1 v_o}{\rho}\frac{d^3w}{dy^3}+\nu\frac{d^2w}{dy^2}-\frac{\mu_e}{\rho}H_oJ_x. \tag{3.56}$$

On eliminating the pressure gradient from Eqs. (3.55) and (3.56) we have

$$-v_o\frac{d^2u}{dy^2}=-\frac{\alpha_1 v_o}{\rho}\frac{d^4u}{dy^4}+\nu\frac{d^3u}{dy^3}-\frac{\mu_e}{\rho}H_o\frac{dJ_z}{dy}, \tag{3.57}$$

$$-v_o\frac{d^2w}{dy^2}=-\frac{\alpha_1 \nu_o}{\rho}\frac{d^4w}{dy^4}+\nu\frac{d^3w}{dy^3}+\frac{\mu_e}{\rho}H_o\frac{dJ_x}{dy}. \tag{3.58}$$

Integration of equations yield

$$-v_o\frac{du}{dy}=-\frac{\alpha_1 v_o}{\rho}\frac{d^3u}{dy^3}+\nu\frac{d^2u}{dy^2}-\frac{\mu_e}{\rho}H_oJ_z+C_6, \tag{3.59}$$

$$-v_o\frac{dw}{dy}=-\frac{\alpha_1 v_o}{\rho}\frac{d^3w}{dy^3}+\nu\frac{d^2w}{dy^2}+\frac{\mu_e}{\rho}H_oJ_x+C_7, \tag{3.60}$$

where C_6 and C_7 are constants of integration.

The boundary conditions are of the form

$$J_x\to 0, J_y\to 0 \text{ as } y\to\infty, \tag{3.61}$$

$$u\to U,\ w\to 0,\ H_x\to 0,\ H_z\to 0,\ \text{as } y\to\infty, \tag{3.62}$$

where U is the uniform free stream velocity.

Making use of the boundary conditions (3.61) and (3.62) in Eqs. (3.59) and (3.60) gives that $C_6=0=C_7$ and thus from Eqs.(3.61) and (3.62)we have

$$-v_o\frac{du}{dy}=-\frac{\alpha_1 v_o}{\rho}\frac{d^3u}{dy^3}+\nu\frac{d^2u}{dy^2}-\frac{\mu_eH_o}{\rho}J_z, \tag{3.63}$$

$$-v_o\frac{dw}{dy}=-\frac{\alpha_1 v_o}{\rho}\frac{d^3w}{dy^3}+\nu\frac{dw}{dy}+\frac{\mu_eH_o}{\rho}J_x. \tag{3.64}$$

Multiplying Eq. (3.64) by i and then adding into Eq. (3.63) we get

$$-v_o\frac{dV}{dy}=-\frac{\alpha_1 v_o}{\rho}\frac{d^3V}{dy^3}+\nu\frac{d^2V}{dy^2}+\frac{\mu_e H_o}{\rho}\frac{dH}{dy}, \tag{3.65}$$

where

$$V=u+iw,\ H=H_x+iH_z. \tag{3.66}$$

Using the boundary conditions (3.61) and (3.62) in Eqs. (3.12) and (3.13) we obtain

$$E_x=0,\ E_z=-\mu_e UH_o \tag{3.67}$$

which shows that E_x and E_z are constants that confirms the Eq. (3.9). Mutiplying Eq. (3.15) by i and then adding the resulting equation into Eq. (3.14) we have

$$-i\frac{dH}{dy}=\frac{\sigma(m+i)}{(1+m^2)}\left[-iE+\mu_e\left(H_oV+v_oH\right)\right], \tag{3.68}$$

where

$$E=E_x+iE_z. \tag{3.69}$$

From Eqs. (3.67) and (3.68) we have

$$\frac{dH}{dy}=\frac{\sigma(im-1)}{(1+m^2)}\left[\mu_e H_o(V-U)+\mu_e v_o H\right]. \tag{3.70}$$

Integration of Eq. (3.65) with respect to y gives

$$-v_oV=-\frac{\alpha_1 v_o}{\rho}\frac{d^2V}{dy^2}+\nu\frac{dV}{dy}+\frac{\mu_e H_o}{\rho}H+C_8, \tag{3.71}$$

where C_8 is constant of integration.

From Eqs. (3.62) and (3.71) we have

$$C_8=-v_oU \tag{3.72}$$

and thus Eq. (3.71) becomes

$$H = \frac{\alpha_1 v_o}{\mu_e H_o}\frac{d^2V}{dy^2} - \frac{\rho\nu}{\mu_e H_o}\frac{dV}{dy} - \frac{\rho v_o}{\mu_e H_o}V + \frac{\rho v_o}{\mu_e H_o}U. \tag{3.73}$$

Now making use of Eq. (3.73) in Eq. (3.70) we obtain

$$\frac{\alpha_1 v_o}{\mu_e}\frac{d^3V}{dy^3} - \left[\frac{\rho\nu}{\mu_e} + \sigma\alpha_1 v_o^2\right]\frac{d^2V}{dy^2} - \left[\frac{\rho v_o}{\mu_e} - \frac{\sigma\rho\nu v_o\,(im-1)}{1+m^2}\right]\frac{dV}{y}$$
$$-\frac{\sigma\,(im-1)\,(\mu_e H_o^2 - \rho v_o^2)}{1+m^2}\,(V-U) = 0. \tag{3.74}$$

Defining

$$S = \frac{v_o}{U},\ M = \frac{\sigma\mu_e^2 H_o^2\nu}{\rho U^2},\ \eta = \frac{Uy}{v},\ \alpha = \frac{v_o}{V_A},\ V_A = \frac{\mu_e H_o}{\rho},\ \overline{V} = \frac{V}{U} \tag{3.75}$$

equation (3.74) yields

$$\frac{\alpha_1 v_o U}{v^2\rho}\frac{d^3\overline{V}}{d\eta^3} - \gamma\frac{d^2\overline{V}}{d\eta^2} - \left[S + \frac{M\alpha^2\,(1-im)}{S\,(1+m^2)}\right]\frac{d\overline{V}}{d\eta} + \frac{M\,(1-\alpha^2)\,(1-im)}{1+m^2}\,(\overline{V}-1) = 0, \tag{3.76}$$

where $M^{\frac{1}{2}}$ and V_A represent the Hartmann number and Alfven velocity respectively and

$$\frac{M\alpha^2}{S} = \frac{\sigma\mu_e\nu v_o}{U},\ \gamma = 1 + \frac{\sigma\alpha_1 v_o^2}{\rho\nu}, \tag{3.77}$$

$$M\left(1-\alpha^2\right) = \frac{\sigma\mu_e^2 H_o^2 v}{\rho U^2}\left(1 - \frac{\rho v_o^2}{\mu_e H_o^2}\right). \tag{3.78}$$

The boundary conditions are

$$\overline{V} \to 0 \text{ at } \eta = 0,\ \overline{V} \to 1 \text{ as } \eta \to \infty. \tag{3.79}$$

Using

$$F(\eta) = 1 - \overline{V} \tag{3.80}$$

in Eq. (3.76) we have

$$\beta\frac{d^3F}{d\eta^3}-\gamma\frac{d^2F}{d\eta^2}-\left[S+\frac{M\alpha^2\left(1-im\right)}{S\left(1+m^2\right)}\right]\frac{dF}{d\eta}+\frac{M\left(1-\alpha^2\right)\left(1-im\right)}{1+m^2}F\left(\eta\right)=0, \quad (3.81)$$

where

$$F\left(0\right)=1,\ F\left(\infty\right)=0, \quad (3.82)$$

$$\beta=\frac{\alpha_1 v_o U}{\rho v^2}. \quad (3.83)$$

The characteristic equation corresponding to Eq. (3.81) is

$$\beta D^3-\gamma D^2-\left[S+\frac{M\alpha^2\left(1-im\right)}{S\left(1+m^2\right)}\right]D+\frac{M\left(1-\alpha^2\right)\left(1-im\right)}{1+m^2}=0. \quad (3.84)$$

For small value of β, one can find the roots of Eq. (3.84) by using perturbation expansion method i.e.

$$D=\frac{C_{-1}}{\beta}+C_o+C_1\beta+C_2\beta^2+..... \quad (3.85)$$

Using Eq. (3.85) in Eq. (3.84) and comparing like powers of β we have

$$\beta^{-2}:C_{-1}^3-\gamma C_{-1}^2=0, \quad (3.86)$$

$$\beta^{-1}:3C_{-1}^2C_o-2\gamma C_{-1}C_o-\left[S+\frac{M\alpha^2\left(1-im\right)}{S\left(1+m^2\right)}\right]C_{-1}=0, \quad (3.87)$$

$$\beta^o:3C_{-1}C_o^2-\gamma C_o^2-2\gamma C_1C_{-1}-\left[S+\frac{M\alpha^2\left(1-im\right)}{S\left(1+m^2\right)}\right]C_o$$

$$+\frac{M\left(1-\alpha^2\right)\left(1-im\right)}{1+m^2}=0, \quad (3.88)$$

$$\beta:C_o^3-2\gamma C_oC_1-\left[S+\frac{M\alpha^2\left(1-im\right)}{1+m^2}\right]C_1=0, \quad (3.89)$$

$$\beta^2:\gamma C_1^2+\left[S+\frac{M\alpha^2\left(1-im\right)}{1+m^2}\right]C_2=0. \quad (3.90)$$

Solving Eqs. (3.86) to (3.90) we get

$$C_{-1} = (0, 0, \gamma), \tag{3.91}$$

$$C_o = -\frac{1}{2\gamma}\left[S + \frac{M\alpha^2}{1+m^2} + A_6 - i\left(\frac{Mm\alpha^2}{1+m^2} + A_7\right)\right], \tag{3.92}$$

$$C_1 = \frac{S(1+m^2)C_o^3}{2\gamma S(1+m^2)C_o + M\alpha^2 - iMm\alpha^2}, \tag{3.93}$$

$$C_2 = -\frac{\gamma S(1+m^2)C_1^2}{S^2(1+m^2) + M\alpha^2(1-im)}, \tag{3.94}$$

$$A_6 = \left(\frac{\lambda_4}{2}\right)^{\frac{1}{2}}\left[(q_4^2+1)^{\frac{1}{2}} + 1\right]^{\frac{1}{2}}, \tag{3.95}$$

$$A_7 = \left(\frac{\lambda_4}{2}\right)^{\frac{1}{2}}\left[(q_4^2+1)^{\frac{1}{2}} - 1\right]^{\frac{1}{2}}, \tag{3.96}$$

$$\lambda_4 = \frac{\left[\begin{array}{c} M^2(1-m^2)\alpha^4 + 2S^2M^2(1+m^2)(1-2\gamma)\alpha^2 + \\ S^2(1+m^2)(4M\gamma + S^2 + S^2m^2) \end{array}\right]}{S^2(1+m^2)^2}, \tag{3.97}$$

$$q_4 = \frac{2\left[M^2\alpha^4 + S^2M^2m(1+m^2)(1-4\gamma)\alpha^2 + 4MS^2m\gamma(1+m^2)\right]}{\left[\begin{array}{c} M^2(1-m^2)\alpha^4 + 2S^2M^2(1+m^2)(1-2\gamma)\alpha^2 \\ +S^2(1+m^2)(4M\gamma + S^2 + S^2m^2) \end{array}\right]}, \tag{3.98}$$

$$\widetilde{C}_o = -\frac{1}{2\gamma}\left[S + \frac{M\alpha^2}{1+m^2} + A_6 - i\left(\frac{Mm\alpha^2}{1+m^2} - A_7\right)\right], \tag{3.99}$$

$$\widetilde{C}_1 = \frac{S(1+m^2)\widetilde{C}_o^3}{2\gamma S(1+m^2)\widetilde{C}_o + S^2(1+m^2) + M\alpha^2 - iMm\alpha^2}, \tag{3.100}$$

$$\widetilde{C}_2 = -\frac{\gamma S(1+m^2)\widetilde{C}_1^2}{S^2(1+m^2) + M\alpha^2(1-im)}, \tag{3.101}$$

$$\widehat{C}_o = \frac{S^2 + S^2m^2 + M\alpha^2 - iMm\alpha^2}{\gamma S^2(1+m^2)}, \tag{3.102}$$

$$\widehat{C}_1 = \frac{S(1+m^2)\widehat{C}_o^3}{2\gamma(1+m^2)C_o + S^2(1+m^2) + M\alpha^2(1-im)}, \tag{3.103}$$

$$\widehat{C}_2 = -\frac{\gamma S(1+m^2)\widehat{C}_1^2}{S^2(1+m^2) + M\alpha^2(1-im)}. \tag{3.104}$$

Thus roots of Eq. (3.84) are

$$D_1 \simeq C_o + C_1\beta + C_2\beta^2, \tag{3.105}$$

$$D_2 \simeq \widetilde{C}_o + \widetilde{C}_1\beta + \widetilde{C}_2\beta^2, \tag{3.106}$$

$$D_3 \simeq \frac{\gamma}{\beta} + \widehat{C}_o + \widehat{C}_1\beta + \widehat{C}_2\beta^2 \tag{3.107}$$

and the solution of Eq. (3.81) is given by

$$F(\eta) = C_9 e^{D_1\eta} + C_{10} e^{D_2\eta} + C_{11} e^{D_3\eta}. \tag{3.108}$$

Now using physical condition that velocity reduces to the Newtonian (viscous) case as $\beta \to 0$, therefore we neglect the solution corresponding to the root D_3. So we have

$$F(\eta) = C_9 e^{D_1\eta} + C_{10} e^{D_2\eta}. \tag{3.109}$$

The boundary conditions (3.82) implies that

$$C_9 = 1,\ C_{10} = 0 \tag{3.110}$$

and thus Eq. (3.109) becomes

$$F(\eta) = e^{D_1\eta}. \tag{3.111}$$

Let

$$D_1 = \xi_1 + i\xi_2, \tag{3.112}$$

where

$$\xi_1 = C_{oR} + C_{1R}\beta + C_{2R}\beta^2, \tag{3.113}$$

$$\xi_2 = C_{oI} + C_{1I}\beta + C_{2I}\beta^2. \tag{3.114}$$

Separating real and imaginary parts of C_o, C_1 and C_2 we have

$$C_{oR} = -\frac{1}{2\gamma}\left[S + \frac{M\alpha^2}{1+m^2} + A_6\right], \tag{3.115}$$

$$C_{oI} = \frac{1}{2\gamma}\left[\left(\frac{Mm\alpha^2}{1+m^2} + A_7\right)\right], \tag{3.116}$$

$$C_{1R} = \frac{S\left(1+m^2\right)\left[\begin{array}{c} 2\gamma S\left(1+m^2\right)C_{oR}^4 + \left(S + Sm^2 + M\alpha^2\right)C_{oR}^3 \\ -Mm\left(Mm\alpha^2 - 2\gamma S - 2\gamma Sm^2\right)C_{oI}C_{oR}^2 \\ +\left(Mm\alpha^2 - 2\gamma S - 2\gamma Sm^2\right)C_{oI}^3 \end{array}\right]}{\left[\begin{array}{c} \left[2S\gamma\left(1+m^2\right)C_{oR} + S^2\left(1+m^2\right) + M^2\alpha^2\right]^2 \\ +\left[2S\gamma\left(1+m^2\right)C_{oI} - Mm\alpha^2\right]^2 \end{array}\right]}, \tag{3.117}$$

$$C_{1I} = \frac{S(1+m^2)\left[\begin{array}{c} \left(4\gamma SC_{oI} + 4\gamma Sm^2C_{oI} + Mm\alpha^2\right)C_{oR}^3 + \\ 3\left(S^2 + S^2m^2 + M\alpha^2\right)C_{oI}C_{oR}^2 \\ +4\gamma S\left(1+m^2\right)C_{oI}^3C_{oR} \\ -\left(S^2 + S^2m^2 + M\alpha^2\right)C_{oI}^3 \end{array}\right]}{\left[\begin{array}{c} \left[2S\gamma\left(1+m^2\right)C_{oR} + S^2\left(1+m^2\right) + M^2\alpha^2\right]^2 \\ +\left[2S\gamma\left(1+m^2\right)C_{oI} - Mm\alpha^2\right]^2 \end{array}\right]}, \tag{3.118}$$

$$C_{2R} = -\frac{\left[\begin{array}{c} MS\gamma\left(1+m^2\right)\left(C_{1R}^2 - C_{1I}^2 - 2mC_{1I}C_{1R}\right)\alpha^2 \\ +S^3\gamma\left(1+m^2\right)^2\left(C_{1R}^2 - C_{1I}^2\right) \end{array}\right]}{M\left(1+m^2\right)\left(M+2S^2\right)\alpha^4 + S^4\left(1+m^2\right)^2}, \tag{3.119}$$

$$C_{2I} = -\frac{\left[\begin{array}{c} MS\gamma\left(1+m^2\right)\left(mC_{1R}^2 - mC_{1I}^2 + 2C_{1I}C_{1R}\right)\alpha^2 \\ +2S^3\gamma\left(1+m^2\right)^2C_{1I}C_{1R} \end{array}\right]}{M\left(1+m^2\right)\left(M+2S^2\right)\alpha^4 + S^4\left(1+m^2\right)^2}. \tag{3.120}$$

With the help of Eqs. (3.111) and (3.112) the solution of the problem is

$$u = U\left[1 - exp\left(\frac{\xi_1 U}{\nu}y\right)cos\left(\frac{\xi_2 U}{\nu}\right)y\right], \tag{3.121}$$

$$w = -U\left[exp\left(\frac{\xi_1 U}{\nu}y\right) sin\left(\frac{\xi_2 U}{\nu}\right) y\right]. \tag{3.122}$$

Skin friction

The skin friction is defined as the shear stress at the surface of the plate over which fluid is flowing i.e.

$$\left(\mu\frac{du}{dy} - \alpha_1 v_o \frac{d^2u}{dy^2}\right)_{y=0}, \left(\mu\frac{dw}{dy} - \alpha_1 v_o \frac{d^2w}{dy^2}\right)_{y=0}. \tag{3.123}$$

From (3.121), (3.122) and (3.123) we have

$$\left(\mu\frac{du}{dy} - \alpha_1 v_o \frac{d^2u}{dy^2}\right)_{y=0} = \frac{\alpha_1 v_o U^3}{\nu^2}\left(\xi_{1-}^2 \xi_2^2\right) - \rho U^2 \xi_1, \tag{3.124}$$

$$\left(\mu\frac{dw}{dy} - \alpha_1 v_o \frac{d^2w}{dy^2}\right)_{y=0}. = U^2 \xi_2 \left(\frac{2\alpha_1 v_o \xi_2 \xi_1 U - \rho\nu^2}{\nu^2}\right). \tag{3.125}$$

Since ρ, U, ξ_1 and ξ_2 are constant so boundary layer separation does not occur.

Induced magnetic field H

Using Eqs. (3.121) and (3.122) in Eq. (3.73) we get

$$H_x = \frac{1}{\mu_e \nu^2 H_o^2} exp\left(\frac{\xi_1 U}{\nu}y\right)\left[a_1 cos\frac{\xi_2 U}{\nu}y - a_2 sin\left(\frac{\xi_2 U}{\nu}\right) y\right], \tag{3.126}$$

$$H_y = H_o, \tag{3.127}$$

$$H_z = \frac{1}{\mu_e \nu^2 H_o^2} exp\left(\frac{\xi_1 U}{\nu}y\right)\left[a_1 cos\frac{\xi_2 U}{\nu}y + a_2 sin\left(\frac{\xi_2 U}{\nu}\right) y\right]. \tag{3.128}$$

Electric field E

From Eqs. (3.67) we have

$$E_x = 0, \tag{3.129}$$

$$E_y = 0, \tag{3.130}$$

$$E_z = -\mu_e U H_o. \tag{3.131}$$

Current density J

Using Eqs. (3.126) and (3.127) in Eqs. (3.9) we obtain

$$J_x = \frac{U}{\mu_e \nu^3 H_o^2} exp\left(\frac{\xi_1 U}{\nu} y\right) \begin{bmatrix} (a_1\xi_2 + a_2\xi_1)\, cos\frac{\xi_2 U}{\nu} y \\ +(a_1\xi_1 - a_2\xi_2)\, sin\left(\frac{\xi_2 U}{\nu}\right) y \end{bmatrix}, \tag{3.132}$$

$$J_y = 0, \tag{3.133}$$

$$J_z = \frac{-U}{\mu_e \nu^3 H_o^2} exp\left(\frac{\xi_1 U}{\nu} y\right) \begin{bmatrix} (a_1\xi_1 - a_2\xi_2)\, cos\left(\frac{\xi_2 U}{\nu} y\right) \\ +(a_1\xi_2 + a_2\xi_1)\, sin\left(\frac{\xi_2 U}{\nu}\right) y \end{bmatrix}, \tag{3.134}$$

$$a_1 = \alpha_1 v_o U^3 \left(\xi_2^2 - \xi_1^2\right) + \rho\nu^2 U^2 \xi_1 + \rho v_o \nu^2, \tag{3.135}$$

$$a_2 = \rho\nu^2 U^2 \xi_2 - 2\alpha_1 v_o U^3 \xi_1 \xi_2. \tag{3.136}$$

Pressure p

Now using Eqs. (3.121) to (3.134) in Eqs. (3.50) we have

$$\frac{\partial p}{\partial y} = \frac{U\xi_1}{H_o^2 \mu_e \nu^5}\left[2H_o^2 U^4 \mu_e \nu^2 (2\alpha_1 + \alpha_2)\left(\xi_1^2 + \xi_2^2\right) - a_1^2 - a_2^2\right] exp\left(\frac{2\xi_1 U}{\nu} y\right). \tag{3.137}$$

Integtrating above equation with respect to y we have

$$p = \frac{\left[2H_o^2 U^4 \mu_e \nu^2 (2\alpha_1 + \alpha_2)\left(\xi_1^2 + \xi_2^2\right) - a_1^2 - a_2^2\right]}{2H_o^2 \mu_e \nu^4} exp\left(\frac{2\xi_1 U}{\nu} y\right) + C, \tag{3.138}$$

where C is constant of integration.

3.3 The solution of the problem in case of blowing

In this case the suction parameter $S < 0$ and $v_o < 0$, so we put $S = -S_o$ $(S_o > 0)$ and $v_o = -v_1\,(v_1 > 0)$ in Eq. (3.81) to obtain

$$\beta_1 \frac{d^3F}{d\eta^3} + \gamma\frac{d^2F}{d\eta^2} - \left[S_o + \frac{M\alpha^2(1-im)}{S_o(1+m^2)}\right]\frac{dF}{d\eta} - \frac{M(1-\alpha^2)(1-im)}{1+m^2}F(\eta) = 0, \tag{3.139}$$

$$F(0) = 1, \; F(\infty) = 0, \tag{3.140}$$

where

$$\beta_1 = \frac{\alpha_1 v_1 U}{\rho v^2}. \tag{3.141}$$

The solution of the boundary value problem is of the following form

$$u = U\left(1 - \exp\left(\frac{\overline{\xi}_1 U}{\nu} y\right) \cos \frac{\overline{\xi}_2 U}{\nu} y\right), \tag{3.142}$$

$$w = -U\left(\exp\left(\frac{\overline{\xi}_1 U}{\nu} y\right) \sin \frac{\overline{\xi}_2 U}{\nu} y\right), \tag{3.143}$$

where

$$\overline{\xi}_1 \simeq \overline{C}_{oR} + \overline{C}_{1R}\beta_1 + \overline{C}_{2R}\beta_1^2, \tag{3.144}$$

$$\overline{\xi}_2 \simeq \overline{C}_{oI} + \overline{C}_{1I}\beta_1 + \overline{C}_{2I}\beta_1^2, \tag{3.145}$$

$$\overline{C}_{oR} = -\frac{1}{2\gamma}\left[-S_0 + \frac{M\alpha^2}{1+m^2} + A_8\right], \tag{3.146}$$

$$\overline{C}_{oI} = -\frac{1}{2\gamma}\left[\left(\frac{Mm\alpha^2}{1+m^2} - A_9\right)\right], \tag{3.147}$$

$$\overline{C}_{1R} = \frac{S_o(1+m^2)\begin{bmatrix} 2\gamma S_o(1+m^2)\overline{C}_{oR}^4 + (S_o + S_o m^2 - M\alpha^2)\overline{C}_{oR}^3 \\ +mM(Mm\alpha^2 + 2S_o\gamma + 2\gamma S_o m^2)\overline{C}_{oI}\overline{C}_{oR}^2 \\ -(Mm\alpha^2 + 2\gamma S_o + 2\gamma S_o m^2)\overline{C}_{oI}^3 \end{bmatrix}}{\begin{array}{c}\left[2S_o\gamma(1+m^2)\overline{C}_{oR} - S_o^2(1+m^2) - M^2\alpha^2\right]^2 \\ + \left[2S_o\gamma(1+m^2)\overline{C}_{oI} + Mm\alpha^2\right]^2\end{array}}, \tag{3.148}$$

$$\overline{C}_{1I} = \frac{S_o(1+m^2)\begin{bmatrix} \left(4\gamma S_o C_{oI} + 4\gamma S_o m^2 C_{oI} - Mm\alpha^2\right) C_{oR}^3 \\ -3\left(S_o^2 + S_o^2 m^2 + M\alpha^2\right) C_{oI} C_{oR}^2 \\ +4\gamma S_o\left(1+m^2\right) C_{oI}^3 C_{oR} \\ +\left(S_o^2 + S_o^2 m^2 + M\alpha^2\right) C_{oI}^3 \end{bmatrix}}{\begin{matrix} \left[2S_o\gamma\left(1+m^2\right)\overline{C}_{oR} - S_o^2\left(1+m^2\right) - M^2\alpha^2\right]^2 \\ + \left[2S_o\gamma\left(1+m^2\right)\overline{C}_{oI} + Mm\alpha^2\right]^2 \end{matrix}}, \tag{3.149}$$

$$\overline{C}_{2R} = \frac{\begin{bmatrix} MS_o\gamma\left(1+m^2\right)\left(\overline{C}_{1R}^2 - \overline{C}_{1I}^2 - 2m\overline{C}_{1I}\overline{C}_{1R}\right)\alpha^2 \\ +S_o^3\gamma\left(1+m^2\right)^2\left(\overline{C}_{1R}^2 - \overline{C}_{1I}^2\right) \end{bmatrix}}{M\left(1+m^2\right)\left(M+2S_o^2\right)\alpha^4 + S_o^4\left(1+m^2\right)^2}, \tag{3.150}$$

$$\overline{C}_{2I} = \frac{\begin{bmatrix} MS_o\gamma\left(1+m^2\right)\left(m\overline{C}_{1R}^2 - m\overline{C}_{1I}^2 + 2\overline{C}_{1I}\overline{C}_{1R}\right)\alpha^2 \\ +2S_o^3\gamma\left(1+m^2\right)^2\overline{C}_{1I}\overline{C}_{1R} \end{bmatrix}}{M\left(1+m^2\right)\left(M+2S_o^2\right)\alpha^4 + S_o^4\left(1+m^2\right)^2}. \tag{3.151}$$

$$A_8 = \left(\frac{\lambda_5}{2}\right)^{\frac{1}{2}}\left[\left(q_5^2+1\right)^{\frac{1}{2}} + 1\right]^{\frac{1}{2}}, \tag{3.152}$$

$$A_9 = \left(\frac{\lambda_5}{2}\right)^{\frac{1}{2}}\left[\left(q_5^2+1\right)^{\frac{1}{2}} - 1\right]^{\frac{1}{2}}, \tag{3.153}$$

$$\lambda_5 = \frac{\begin{bmatrix} M^2\left(1-m^2\right)\alpha^4 + 2S_o^2M^2\left(1+m^2\right)\left(1-2\gamma\right)\alpha^2 + \\ S_o^2\left(1+m^2\right)\left(4M\gamma + S_o^2 + S_o^2m^2\right) \end{bmatrix}}{S_o^2\left(1+m^2\right)^2}, \tag{3.154}$$

$$q_5 = \frac{2\left[M^2\alpha^4 + S_o^2M^2m\left(1+m^2\right)\left(1-4\gamma\right)\alpha^2 + 4MS_o^2m\gamma\left(1+m^2\right)\right]}{\begin{bmatrix} M^2\left(1-m^2\right)\alpha^4 + 2S_o^2M^2\left(1+m^2\right)\left(1-2\gamma\right)\alpha^2 \\ +S_o^2\left(1+m^2\right)\left(4M\gamma + S_o^2 + S_o^2m^2\right) \end{bmatrix}}. \tag{3.155}$$

3.4 The solution of the problem for suction case when magnetic Reynold number is small

The small magnetic Reynold number is defined by

$$M = R_\sigma R_p, \tag{3.156}$$

where

$$R_p = \frac{\mu_e H_o^2}{\rho U^2}. \tag{3.157}$$

When $R_\sigma << 1$, the induced magnetic field $\mathbf{H}$ can be neglected in comparison with imposed magnetic field H_o i.e. $0 = H = H_x + iH_z$, so that the components of the current density $\mathbf{J}$ given by Eqs.(3.14) and (3.15) together with Eq. (3.67), reduce to

$$J_x = \frac{\sigma\mu_e H_o}{(1+m^2)}\left[-w + m\left(u - U\right)\right], \tag{3.158}$$

$$J_z = \frac{\sigma\mu_e H_o}{(1+m^2)}\left[u - U + mw\right]. \tag{3.159}$$

Multiplying Eq. (3.159) by i and then adding into Eq. (3.158) we get

$$J_z + iJ_x = \frac{\sigma\mu_e H_o\left(m+i\right)}{\rho\left(1+m^2\right)}\left[V - U\right]. \tag{3.160}$$

From Eqs. (3.9), (3.58) and (3.160) we have

$$\frac{dH}{dy} = \frac{\sigma\mu_e H_o\left(1-im\right)}{\rho\left(1+m^2\right)}\left[V - U\right]. \tag{3.161}$$

Substitution of Eq. (3.161) in Eq. (3.65) yields

$$\frac{\alpha_1\nu_o}{\rho}\frac{d^3V}{dy^3} - \nu\frac{d^2V}{dy^2} - v_o\frac{dV}{dy} + \frac{\sigma\mu_e^2 H_o^2\left(1-im\right)}{\rho\left(1+m^2\right)}\left[V-U\right] = 0. \tag{3.162}$$

Now using Eqs. (3.75) in Eq. (3.162) we obtain

$$\beta \frac{d^3\overline{V}}{d\eta^3} - \frac{d^2\overline{V}}{d\eta^2} - S\frac{d\overline{V}}{d\eta} + \frac{M(1-im)}{(1+m^2)}\left(\overline{V}-1\right) = 0, \tag{3.163}$$

where

$$\overline{V} = 0 \text{ at } \eta = 0, \overline{V} \to \infty \text{ as } \eta \to \infty. \tag{3.164}$$

Substituting $W(\eta) = 1 - \overline{V}$ in Eqs. (3.163) and (3.164) we obtain

$$\beta \frac{d^3W}{d\eta^3} - \frac{d^2W}{d\eta^2} - S\frac{dW}{d\eta} + \frac{M(1-im)}{(1+m^2)}W(\eta) = 0 \tag{3.165}$$

with

$$W(0) = 1, \; W(\infty) = 0. \tag{3.166}$$

Following the same procedure as in suction case, the solution is given by

$$u = U\left[1 - \exp\left(\frac{\tilde{\xi}_1 U}{\nu}y\right)\cos\frac{\tilde{\xi}_2 U}{\nu}y\right], \tag{3.167}$$

$$w = -U\left[\exp\left(\frac{\tilde{\xi}_1 U}{\nu}y\right)\sin\frac{\tilde{\xi}_2 U}{\nu}y\right], \tag{3.168}$$

where

$$\tilde{\xi}_1 \simeq G_{oR} + G_{1R}\beta_1 + G_{2R}\beta_1^2, \tag{3.169}$$

$$\tilde{\xi}_2 \simeq G_{oI} + G_{1I}\beta_1 + G_{2I}\beta_1^2, \tag{3.170}$$

$$G_{oR} = -\left[\frac{S}{2} + \left(\frac{\lambda_6}{2}\right)\left[\left(q_6^2+1\right)^{\frac{1}{2}} + 1\right]^{\frac{1}{2}}\right], \tag{3.171}$$

$$G_{oI} = -\left(\frac{\lambda_6}{2}\right)\left[\left(q_6^2+1\right)^{\frac{1}{2}} - 1\right]^{\frac{1}{2}}, \tag{3.172}$$

$$G_{1R} = \frac{2G_{oR}^4 + SG_{oR}^3 - 3SG_{oR}G_{oI}^2 - 2G_{oI}^4}{\left(2G_{oR}+S\right)^2 + 4G_{oI}^2}, \tag{3.173}$$

$$G_{1I} = -\frac{4G_{oR}^3G_{oI} + 3SG_{oR}^2G_{oI} + 4G_{oR}G_{oI}^3 - SG_{oI}^3}{\left(2G_{oR}+S\right)^2 + 4G_{oI}^2}, \tag{3.174}$$

$$G_{2R} = \frac{G_{1I}^2 - G_{1R}^2}{S}, \tag{3.175}$$

$$G_{2I} = -\frac{2G_{1R}G_{1I}}{S}, \tag{3.176}$$

$$\lambda_6 = S^2 + \frac{4M}{1+m^2}, \tag{3.177}$$

$$q_6 = \frac{4mM}{(1+m^2)\,S^2 + 4M}. \tag{3.178}$$

3.5 The solution of the problem in case of blowing when magnetic Reynold number is small

Since in case of blowing at the plate, the suction parameter $S < 0$ and $v_o < 0$, so we put $S = -S_o$ and $v_o = -v_1$ in Eq. (3.165) and get

$$\beta_1 \frac{d^3W}{d\eta^3} + \frac{d^2W}{d\eta^2} - S_o\frac{dW}{d\eta} - \frac{M\,(1-im)}{(1+m^2)}W(\eta) = 0 \tag{3.179}$$

with

$$W(0) = 1,\ W(\infty) = 0, \tag{3.180}$$

$$\beta_1 = \frac{\alpha_1 v_1 U}{\rho v^2}. \tag{3.181}$$

The solution of the above problem can be written as

$$u = U\left[1 - \exp\left(\frac{\widehat{\xi}_1 U}{\nu}y\right)\cos\frac{\widehat{\xi}_2 U}{\nu}y\right] \tag{3.182}$$

$$w = -U\left[\exp\left(\frac{\widehat{\xi}_1 U}{\nu}y\right)\sin\frac{\widehat{\xi}_2 U}{\nu}y\right] \tag{3.183}$$

where

$$\widehat{\xi}_1 \simeq \widehat{G}_{oR} + G_{1R}\beta_1 + \widehat{G}_{2R}\beta_1^2, \tag{3.184}$$

$$\widehat{\xi}_2 \simeq \widehat{G}_{oI} + \widehat{G}_{1I}\beta_1 + \widehat{G}_{2I}\beta_1^2, \tag{3.185}$$

$$\widehat{G}_{oR} = -\left[\frac{S_o}{2} - \left(\frac{\lambda_7}{2}\right)\left[\left(q_7^2+1\right)^{\frac{1}{2}}+1\right]^{\frac{1}{2}}\right], \tag{3.186}$$

$$\widehat{G}_{oR} = \left(\frac{\lambda_7}{2}\right)\left[\left(q_7^2+1\right)^{\frac{1}{2}}-1\right]^{\frac{1}{2}}, \tag{3.187}$$

$$\widehat{G}_{1R} = \frac{2\widehat{G}_{oR}^4 - S_o\widehat{G}_{oR}^3 + 3S_o\widehat{G}_{oR}\widehat{G}_{oI}^2 - 2\widehat{G}_{oI}^4}{\left(2\widehat{G}_{oR} - S_o\right)^2 + 4\widehat{G}_{oI}^2}, \tag{3.188}$$

$$\overline{G}_{1I} = -\frac{4\widehat{G}_{oR}^3\widehat{G}_{oI} - 3S_o\widehat{G}_{oR}^2\widehat{G}_{oI} + 4\widehat{G}_{oR}\widehat{G}_{oI}^3 + S_o\widehat{G}_{oI}^3}{\left(2\widehat{G}_{oR} - S_o\right)^2 + 4\widehat{G}_{oI}^2}, \tag{3.189}$$

$$\widehat{G}_{2R} = -\frac{\widehat{G}_{1I}^2 - \widehat{G}_{1R}^2}{S_o}, \tag{3.190}$$

$$\widehat{G}_{2I} = \frac{2\widehat{G}_{1R}\widehat{G}_{1I}}{S_o}, \tag{3.191}$$

$$\lambda_7 = S_o^2 + \frac{4M}{1+m^2}, \tag{3.192}$$

$$q_7 = \frac{4mM}{\left(1+m^2\right)S_o^2 + 4M}. \tag{3.193}$$

3.6 Conclusions

In this work, we have studied the flow of a hydrodynamically compatible incompressible fluid of order two past an infinite porous plate. The fluid is conducting . The governing equations are made dimensionless and the resulting ordinary differential equation is solved analytically. It is demonstrated that the perturbation method yields physically acceptable results which are valid through the region. From the results (3.121), (3.122), (3.142), (3.143), (3.167), (3.168), (3.182) and (3.183), it is concluded that the flow field is noticeable influenced by the presence of the applied magnetic field, Hall, suction and blowing parameters. More precisely:

1. The asymptotic solutions for suction and blowing exist.

2. As the value of the suction parameter is increased, the stream lines are pushing towards the plate, indicating the boundary layer thickness decreasing, which is as expected.

3. The magnetic field decelerates the fluid motion.

4. The boundary layer separation does not occur as the skin friction at the surface of the plate is constant.

5. It is noted that the boundary layer thickness increases with the increase of material modulus (α_1) of the second order fluid.