# ON ONE SIDED PRIME IDEALS OF SEMIGROUPS

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**A Dissertation**
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By the Grace of

# *ALLAH*

*THE MOST BENEFICIENT*
*THE MOST MERCIFUL*

# CERTIFICATE

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We accept this dissertation as conforming to the required standard.

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# Chapter 1

# Fundamental Concepts

In this introductory chapter we shall define basic concepts of semigroups and review some of the background material that will be of value for our later pursuits. For undefined terms and notations, we refer to [6] and [9].

## 1.1 Basic Concepts in Semigroups

### 1.1.1 Definition

A system $(S, *)$ consisting of a non-empty set $S$, together with an associative binary operation $*$ on $S$ is called a *semigroup*. Hence forth we shall refer to the binary operation as multiplication $\cdot$ on $S$ and write $x * y$ simply as $xy$. If $(S, \cdot)$ or more simply $S$ is a semigroup with the additional property that multiplication is commutative, then $S$ is called a *commutative semigroup*. $S$ is called a *monoid* if $S$ is a semigroup which contains an identity element. If $S$ has no identity element it is very easy to adjoin an identity element

1 to $S$ to form a monoid, by defining

$$1.s = s.1 = s \text{ for all } s \text{ in } S, \text{ and } 1.1 = 1.$$

We shall use the notation $S^1$ with the following meaning:

$$S^1 = \begin{cases} S, & \text{if } S \text{ has an identity element} \\ S \cup \{1\} & \text{otherwise} \end{cases}$$

and call $S^1$ the semigroup obtained from $S$ by adjoining an identity element.

If a semigroup $S$ with at least two elements contains a zero element 0 ($s.0 = 0.s = 0$ for all $s$ in $S$) then $S$ is called a *semigroup with zero*.

If $S$ has no zero element then it is easy to adjoin an extra element 0 to the set $S$, by defining

$$0.s = s.0 = 0 \text{ and } 0.0 = 0, \text{ for all } s \text{ in } S.$$

This makes the set $S \cup \{0\}$ a semigroup with zero element 0. We shall use the notation $S^0$ with the following meaning:

$$S^0 = \begin{cases} S, & \text{if S has a zero element} \\ S \cup \{0\} & \text{otherwise} \end{cases}$$

and call $S^0$ the semigroup obtained from $S$ by adjoining a zero (if necessary).

### 1.1.2 Example

$\mathbb{N}$, the set of natural numbers is a semigroup with the usual operation of addition and also with the usual operation of multiplication.

### 1.1.3 Example

Let $\mathbb{N}$ be the set of natural numbers and $M_{n\times n}$ be the set of all $n \times n$ matrices whose entries are from $\mathbb{N}$. Then $M_{n\times n}$ is a semigroup with respect to the usual addition of matrices. Also $M_{n\times n}$ is a semigroup with respect to the usual multiplication of matrices.

### 1.1.4 Example

Let $S = [0, 1]$, then $S$ is a semigroup with respect to the operations.

$$\begin{aligned} a * b &= \min\{a, b\} \text{ for all } a,\ b \text{ in } S. \\ a \circ b &= \max\{a, b\} \text{ for all } a,\ b \text{ in } S. \end{aligned}$$

### 1.1.5 Example

Let $X$ be a non-empty set. Define

$$\begin{aligned} a * b &= a \text{ for all } a,\ b \text{ in } X. \\ \text{and } a \circ b &= b \text{ for all } a,\ b \text{ in } X. \end{aligned}$$

Then $(X, *)$ and $(X, \circ)$ are semigroups.

### 1.1.6 Example

Let $X$ be a non-empty set. Denote $\tau(X)$, the set of all mappings from $X$ to $X$. Then $\tau(X)$ is a semigroup with respect to composition of mappings.

### 1.1.7 Definition

If $A$ and $B$ are subsets of a semigroup $S$, we write

$$AB = \{ab : a \in A \text{ and } b \in B\}.$$

### 1.1.8 Definition

A non-empty subset $T$ of a semigroup $S$ is called a subsemigroup of $S$ if for all $x$, $y \in T$, $xy \in T$. Thus $T$ is a subsemigroup of $S$ if $T^2 = T.T \subseteq T$.

A subsemigroup $T$ of a semigroup $S$ is called a subgroup of $S$ if $T$ is a group. Recall that a semigroup $S$ which has the property: $aS = S$ and $Sa = S$, for all $a$ in $S$, then $S$ is a group in the usual sense. Thus a non-empty subset $T$ of a semigroup $S$ is a subgroup of $S$ if and only if

$$aT = Ta = T \text{ for all } a \text{ in } T.$$

An element of a semigroup $S$ which commutes with every elements of $S$ is called a *central element* of $S$. The set of all central elements of $S$ is either empty or a subsemigroup of $S$, and in the latter case, is called the center of $S$.

Let $A$ be a non-empty subset of a semigroup $S$. The intersection of all subsemigroups of $S$ containing $A$ is a subsemigroup of $S$ and is denoted by $< A >$. Clearly $< A >$ contains $A$ and is contained in every other subsemigroup of $S$ which contains $A$, it is called the subsemigroup of $S$ generated by $A$.

### 1.1.9 Definition

A non-empty subset $A$ of a semigroup $S$ is called a left (right) ideal of $S$ if $SA \subseteq A$ ($AS \subseteq A$).

A non-empty subset $A$ of a semigroup $S$ is called a two-sided ideal, or simply an ideal of $S$ if it is both a left and a right ideal of $S$. Clearly $S$ is an ideal of $S$, and if $S$ has a zero element, then $\{0\}$ is an ideal of $S$. An ideal of $S$ different from these two ideals is called a proper ideal of $S$.

### 1.1.10 Definition

A relation $\rho$ on a semigroup $S$ is said to be right (left) compatible if for $a$, $b$ in $S$, $a\rho b$ implies that $as\rho bs$ ($sa\rho sb$) for all $s \in S$. A congruence on $S$ is an equivalence relation that is both right and left compatible. If $\rho$ is a congruence on $S$, then $S/\rho$ denotes the set of all equivalence classes of $S$ determined by $\rho$. If $a\rho$ denotes the equivalence class of $S$ containing the element $a$ ($a \in S$), then $S/\rho$ can be made into a semigroup by defining $(a\rho)(b\rho) = (ab)\rho$, $S/\rho$ is called the factor semigroup of $S$ modulo $\rho$. Let $I$ be an ideal of a semigroup $S$. Define a relation $\rho$ on $S$ by $a\rho b$ ($a$, $b \in S$) to mean that either $a = b$ or else both $a$, $b \in I$. Clearly $\rho$ is a congrouence on $S$, called the *Rees Congruence modulo I*. The equivalence classes of $S$ modulo $\rho$ are $I$ itself and every one element set $\{a\}$ with $a \in S \backslash I$. We shall write $S/I$ instead of $S/\rho$, and call $S/I$ the Rees factor semigroup of $S$ modulo $I$.

## 1.2 Regular Semigroups and inverse Semigroups

An element $x$ of a semigroup $S$ is said to be regular if there exists an element $x' \in S$ such that $xx'x = x$; $S$ is called a regular semigroup if every element of $S$ is regular. An element $x' \in S$ is said to be an inverse of $x \in S$ if and only if $xx'x = x$ and $x'xx' = x'$; $S$ is called an inverse semigroup if every element of $S$ has a unique inverse.

Let $a$ be a regular element of a semigroup $S$ and $a = axa$ for some $x \in S$, then

$$\begin{aligned} e &= ax = axax = ax.ax = e^2 \\ \text{and } f &= xa = x.axa = xa.xa = f^2. \end{aligned}$$

that is, $e = ax$ and $f = xa$ are idempotent elements in $S$ with the properties

$$ea = axa = a = af$$

Conversely, assume that for an element a there exist elements, $e$, $f$ and $b$ in $S$ such that

$$\begin{aligned} a &= ea = af \\ \text{and } ab &= e, \ ba = f \end{aligned}$$

$$\text{then } a = af = aba$$

Thus $a$ is a regular element. So we have the proposition.

### 1.2.1 Proposition

An element $a$ of a semigroup $S$ is regular if and only if there exist elements, $e$, $f$ and $b$ in $S$ such that $a = ea = af$ and $e = ab$, $f = ba$.

### 1.2.2 Remark

If $a$ is a regular element of a semigroup $S$, then the above property implies the principal left (principal right ideal) of $S$ generated by $a$ has the form $Sa\,(aS)$.

### 1.2.3 Proposition

The following conditions for a semigroup $S$ are equivalent:

(1) $S$ is regular;

(2) For every right ideal $R$ and left ideal $L$ of $S$

$$RL = R \cap L.$$

**Proof.** (1) $\Longrightarrow$ (2) :Let $R$ and $L$ be right and left ideals of $S$ respectively. Then $RL \subseteq R \cap L$. For the reverse inclusion, let $x \in R \cap L$. Since $S$ is regular, so there exists $y \in S$ such that

$$x = xyx = (xy)x \in RL \quad \because R \text{ is a right ideal.}$$

Therefore, we have $R \cap L \subseteq RL$. Hence $RL = R \cap L$.

(2) $\Longrightarrow$ (1) :Let $a \in S$. Let $R$ be the right ideal generated by $a$ and $L$ be the left ideal generated by $a$.

$$\text{Then } R = aS^1 \text{ and } L = S^1a.$$

By hypothesis,

$$aS^1 \cap S^1a = (aS^1)(S^1a).$$

As

$$\begin{aligned} a &\in aS^1 \cap S^1a, \text{ so } a \in (aS^1)(S^1a) = aS^1a \\ i.e.\ a &= axa \text{ for some } x \in S. \end{aligned}$$

Hence $a$ is a regular element. ■

### 1.2.4 Theorem

A semigroup $S$ is an inverse semigroup if and only if $S$ is a regular semigroup and idempotent elements of $S$ commute.

**Proof.** Let $S$ be an inverse semigroup, then clearly $S$ is a regular semigroup. Let $e$ and $f$ be idempotent elements of $S$. We show that $ef$ and $fe$ are also idempotent elements of $S$. Let $a$ be the (unique) inverse of $ef$, so that

$$(ef)a(ef) = ef, \quad a(ef)a = a.$$

Let $b = ae$. Then

$$\begin{aligned} (ef)b(ef) &= efae^2f = efaef = ef. \\ b(ef)b &= ae^2fae = aefae = ae = b. \end{aligned}$$

Hence $b$ is also an inverse of $ef$. Thus

$$ae = b = a.$$

Similarly we can show that $fa = a$.

$$\text{Hence} \quad a^2 = (ae)(fa) = a(ef)a = a.$$

But an idempotent is an inverse of itself. Thus $a = ef$. Hence $ef$ is an idempotent. Similarly $fe$ is an idempotent.

$$\begin{aligned} \text{As} \quad (ef)(fe)(ef) &= ef^2e^2f = efef = (ef)^2 = ef \\ \text{and } (fe)(ef)(fe) &= fefe = fefe = (fe)^2 = fe. \end{aligned}$$

So $ef$ and $fe$ are inverses of each other. Thus $ef$ and $fe$ are inverses of $ef$, and so

$$ef = fe$$

Conversely, suppose that $S$ is a regular semigroup in which idempotent elements commute. We show that $S$ is an inverse semigroup. Let $a \in S$ and suppose that $x \in S$ is such that $a = axa$.

$$\begin{aligned} \text{Notice that } (ax)^2 &= axax = ax \\ \text{and } (xa)^2 &= xaxa = xa. \end{aligned}$$

Let $a' = xax$.

$$\begin{aligned} \text{Then} \quad aa'a &= axaxa = (ax)^2a = axa = a \\ \text{and} \quad a'aa' &= xaxaxax = (xa)^3x = xax = a. \end{aligned}$$

Thus $a'$ is an inverse of $a$.

For uniqueness, let $y,\, z \in S$ are inverses of $a$.

$$\begin{aligned} \text{That is} \quad a &= aya,\ yay = y \\ \text{and} \quad a &= aza,\ zaz = z. \end{aligned}$$

Then $ay$, $ya$, $az$ and $za$ are idempotents.

$$\begin{aligned} y &= yay \\ &= y(aza)y \\ &= (ya)(za)y \\ &= (za)(ya)y \\ &= z(aya)y \\ &= zay \\ &= z(aza)y \\ &= z(az)(ay) \\ &= z(ay)(az) \\ &= z(aya)z \\ &= zaz \\ &= z. \end{aligned}$$

Hence $S$ is an inverse semigroup. ■

## 1.3 Weakly Regular Semigroups

In [5], Brown and McCoy considered the notion of weakly regular rings . These rings were latter studied by Ramamurthy [11], [12] and others . Adopting this notion of [1] weakly regular semigroups are defined as

### 1.3.1 Definition

A semigroup $S$ is right weakly regular if, for all $x \in S$, $x \in (xS)^2$.

Thus, if $S$ is commutative and weakly regular then, $S$ is regular.

### 1.3.2 Definition

A two-sided ideal $I$ of a semigroup $S$ is called *right (left)* pure if, for each $x \in I$, there exists $y \in I$, such that $x = xy$ ($x = yx$). In other words, $I$ is right pure if and only if, for every $a \in I$, the equation $a = ax$ has a solution in $I$.

The following results are from [1].

### 1.3.3 Lemma

Each two-sided ideal of a right weakly regular semigroup $S$ is right weakly regular as a semigroup.

**Proof.** Let I be a two-sided ideal of a right weakly regular semigroup S and let $x \in I$. Then, $x \in (xS)(xS)$. Hence, $x \in xSxSxSxS \subseteq x(SxS)x(SxS) \subseteq (xI)(xI)$. This means that $I$ is right weakly regular. ■

### 1.3.4 Proposition

For a monoid $S$ the following are equivalent:

(1) $S$ is right weakly regular ;

(2) $B^2 = B$ for all right ideals $B$ of $S$ ;

(3) $BA = B \cap A$ for all right ideals $B$ and all two-sided ideals $A$ of $S$ .

(4) Every two-sided ideal of $S$ is right pure.

**Proof.** (1) $\Longrightarrow$ (2) :Let $B$ be a right ideal of $S$. Clearly, $B^2 \subseteq B$. Let $x \in B$.

$$\text{Then} \quad x \in (xS)(xS) \subseteq B.B = B^2.$$

This proves that $\qquad B = B^2.$

(2) $\Longrightarrow$ (3) :Let $B$ be a right ideal and $A$ be a two-sided ideal of $S$. Clearly, $BA \subseteq B \cap A$. To prove the reverse inclusion, let $x \in B \cap A$. Since

$$x \in xS = (xS)(xS) = x(SxS) \subseteq xA \subseteq BA,$$

we have $B \cap A \subseteq BA$ and so $B \cap A = BA$.

(3) $\Longrightarrow$ (1) :Let $x \in S$ and let $A = SxS$ be the two-sided ideal generated by $x$. If $B$ is the right ideal $xS$ generated by $x$,

$$\text{then } x \in B \cap A = BA = (xS)(SxS) \subseteq xS^2xS \subseteq (xS)(xS).$$

This implies that $S$ is right weakly regular.

(1) $\Longrightarrow$ (4) : Suppose that $S$ is right weakly regular. Let $A$ be a two-sided ideal of $S$ and $a \in A$. Since $S$ is right weakly regular, $a \in (aS)(aS) = a(SaS) \subseteq aA$.

Hence, there exists an element $x \in A$ such that $a = ax$. Thus, $A$ is right pure.

(4) $\Longrightarrow$ (1) : Assume that each two-sided ideal of $S$ is right pure. In order to show that $S$ is right weakly regular, let $x \in S$ and $A = SxS$ be the two-sided ideal of $S$ generated by $x$. By the hypothesis, $x \in xA = x(SxS) = (xS)(xS)$. Hence, $S$ is right weakly regular.

■

## 1.4 Semisimple Semigroups

A semigroup $S$ is called semisimple if all its ideals are idempotent that is $I^2 = I$, for every two-sided ideal $I$ of $S$.

The following result is from [2].

### 1.4.1 Proposition

The following assertions for a semigroup $S$ are equivalent:

1) $S$ is semisimple;

2) for each pair of ideals $I$, $J$ of $S$, $I \cap J = IJ$;

3) for each right ideal $R$ and two-sided ideal $I$, $R \cap I \subseteq IR$;

4) for each left ideal $L$ and two-sided ideal $I$, $L \cap I \subseteq LI$.

**Proof.** (1) $\Longrightarrow$ (2) :Let $I$, $J$ be any ideals of $S$, then $IJ \subseteq I \cap J$. As $I \cap J$ is an ideal of $S$, so

$$I \cap J = (I \cap J)^2 \subseteq IJ.$$

Thus

$$IJ = I \cap J.$$

(2) $\Longrightarrow$ (3) : Let $R$ be a right ideal and $I$ be a an ideal of $S$, then $S^1R$ is a two-sided ideal of $S$. Now

$$R \cap I \subseteq S^1R \cap I = I(S^1R) = (IS^1)R = IR.$$

Thus

$$R \cap I \subseteq IR.$$

(3) $\Longrightarrow$ (4) : Let $L$ be a left ideal and $I$ be an ideal of $S$, then $LS^1$ is a two-sided ideal of $S$.

Hence $L \cap I \subseteq LS^1 \cap I \subseteq (LS^1)I = L(S^1I) = LI.$

(4) $\Longrightarrow$ (1) : Let $I$ be any ideal of $S$ then $I \cap I \subseteq I.I$ i.e. $I \subseteq I^2$. But $I^2 \subseteq I$ always. Hence

$$I = I^2.$$

■

# Chapter 2

# Fully Prime Semigroups

In this chapter we shall characterize semigroups whose every ideal is prime (fully prime semigroups). We prove that a semigroup $S$ is fully prime if and only if $S$ is semisimple and the set of ideals of $S$ is totally ordered under inclusion. We also prove that a commutative semigroup $S$ is fully prime if and only if it is an inverse semigroup and the set of ideals of $S$ is totally ordered under inclusion.

## 2.1 Prime ideals

We begin with the definition.

### 2.1.1 Definition

An ideal $I$ of a semigroup $S$ called a prime ideal of $S$ if $AB \subseteq I$ implies that either $A \subseteq I$ or $B \subseteq I$ for all ideals $A$, $B$ of $S$.

### 2.1.2 Proposition

The following conditions on an ideal $I$ of a semigroup $S$ are equivalent:

(1) $I$ is prime;

(2) $aS^1b \subseteq I$ if and only if $a \in I$ or $b \in I$;

(3) If $a$, $b$ are elements of a semigroup $S$ satisfying $< a >< b > \subseteq I$ then either $a \in I$ or $b \in I$. (where $< a >$ is the ideal generated by $a$).

**Proof.** (1) $\Longrightarrow$ (2) : Let $a, b \in S$ and set $I' = aS^1b$. If $a \in I$ or $b \in I$ then $I' \subseteq I$, since $I$ is an ideal.

Conversely, let $S^1aS^1$ and $S^1bS^1$ be the ideals of $S$ generated by $a$ and $b$ respectively.

$$\text{If } aS^1b \subseteq I \text{ then } S^1aS^1bS^1 \subseteq I$$

Hence $(S^1aS^1)(S^1bS^1) \subseteq I$. By (1) either $S^1aS^1 \subseteq I$ or $S^1bS^1 \subseteq I$, this implies that either $a \in I$ or $b \in I$.

(2) $\Longrightarrow$ (3) : Let $a$, $b$ be elements of $S$ such that $< a >< b > \subseteq I$. As $aS^1b \subseteq < a >< b >$, so $aS^1b \subseteq I$. Thus by (2) either $a \in I$ or $b \in I$.

(3) $\Longrightarrow$ (1) : Let $A$ and $B$ be ideals of $S$ such that $AB \subseteq I$. Suppose that $A \nsubseteq I$ and $a \in A$ such that $a \notin I$. Let $b$ be any arbitrary element of $B$ then $< a >< b > \subseteq AB \subseteq I$. By (3) either $a \in I$ or $b \in I$. As $a \notin I$ so $b \in I$, that is $B \subseteq I$.

■

### 2.1.3 Corollary

If $a$ and $b$ are elements of a semigroup $S$ then the following conditions on a prime ideal $I$ of $S$ are equivalent:

(1) If $ab \in I$ then $a \in I$ or $b \in I$;

(2) If $ab \in I$ then $ba \in I$.

**Proof.** (1) $\Longrightarrow$ (2) : Suppose that $ab \in I$ then $b(ab)a \in I$, since $I$ is an ideal of $S$. Thus $(ba)(ba) \in I$. By (1) $ba \in I$.

(2) $\Longrightarrow$ (1) : If $ab \in I$ then $abS^1 \subseteq I$. By (2) $bS^1a \subseteq I$. By above Proposition either $b \in I$ or $a \in I$. ■

### 2.1.4 Proposition

An ideal $I$ of a commutative semigroup $S$ is prime if and only if $ab \in I$ implies that $a \in I$ or $b \in I$ for all $a$ and $b$ of $S$.

**Proof.** Note that, by commutativity, $ab \in I$ if and only if $aS^1b \subseteq I$. The result follows from above Proposition. ■

### 2.1.5 Definition

A non-empty subset $M$ of a semigroup $S$ is called an m-system if and only if $a$, $b \in M$ implies that there exists an element $x \in S^1$ such that $axb \in M$.

### 2.1.6 Corollary

An ideal $I$ of a semigroup $S$ is prime if and only if $S\backslash I$ is an m-system.

**Proof.** Suppose that $I$ is a prime ideal of $S$. Let $a$, $b \in S\backslash I$. Suppose that there does not exist $x \in S^1$ such that $axb \in S\backslash I$, this implies that $aS^1b \subseteq I$. By Proposition 2.1.2, either $a \in I$ or $b \in I$, which is a contradiction. Hence there exists an $x \in S^1$ such that $axb \in S\backslash I$.

Conversely, assume that $S\backslash I$ is an m-system . Let $a$, $b \in S$ such that $aS^1b \subseteq I$. If $a \notin I$ and $b \notin I$ then $a$, $b \in S\backslash I$ and as $S\backslash I$ is an m-system, so there exists an $x \in S^1$ such that $axb \in S\backslash I$ that is $aS^1b \nsubseteq I$, which is a contradiction. Hence either $a \in I$ or $b \in I$. ■

### 2.1.7 Proposition

Let $S$ be a monoid. Then every maximal ideal of $S$ is a prime ideal.

**Proof.** Let $P$ be a maximal ideal of $S$. Let $A$, $B$ be ideals of $S$ such that $AB \subseteq P$. Suppose that $A \nsubseteq P$ then $A \cup P = S$. As $1 \in S$, so $1 \in A \cup P$. Since $1 \notin P$, so $1 \in A$. Thus $A = S$. Now $B = SB = AB \subseteq P$. ■

### 2.1.8 Proposition

If $I$ is an ideal of a semigroup $S$ and $H$ is an ideal of $S$ minimal among those ideals of $S$ properly containing $I$ then $K = \{x \in S : xH \subseteq I\}$ is a prime ideal of $S$.

**Proof.** First we show that $K$ is an ideal of $S$. Let $x \in K$ and $s \in S$ then $xH \subseteq I$. Now $(sx)H \subseteq sI \subseteq I$, implies that $sx \in K$. Also $(xs)H = x(sH) \subseteq xH \subseteq I$, implies that $as \in K$. So $K$ is an ideal of $S$. Let $A$, $B$ be ideals of $S$ such that $AB \subseteq K$. Suppose that $B \nsubseteq K$. As $AB \subseteq K$ and $B \nsubseteq K$, we have $ABH \subseteq I$ and $BH \nsubseteq I$. .

Therefore $I \subset I \cup BH \subseteq H$ and by the minimality of $H$, we have $I \cup BH = H$. Therefore $AI \cup ABH = AH \subseteq H$ and so $A \subseteq K$. ■

## 2.2 Semiprime Ideals

### 2.2.1 Definition

An ideal $I$ of a semigroup $S$ is called a semiprime ideal if $A^2 \subseteq I$ implies that $A \subseteq I$ for all ideals $A$ of $S$.

### 2.2.2 Proposition

The following conditions on an ideal $I$ of a semigroup $S$ are equivalent:

(1) $I$ is semiprime;

(2) $aS^1a \subseteq I$ if and only if $a \in I$.

**Proof.** (1) $\Longrightarrow$ (2) : Let $a \in S$ and set $I' = aS^1a$. If $a \in I$ then $I' \subseteq I$, since $I$ is an ideal.

Conversely, let $S^1aS^1$ be the ideal of $S$ generated by $a$. If $aS^1a \subseteq I$ then $(S^1aS^1)(S^1aS^1) = S^1aS^1aS^1 \subseteq S^1IS^1 = I$. By (1) $S^1aS^1 \subseteq I \Longrightarrow a \in I$.

(2) $\Longrightarrow$ (1) : Let $A$ be an ideal of $S$ such that $A^2 \subseteq I$. Let $a \in A$ then $a \in S^1aS^1$ and $S^1aS^1 \subseteq A$. Also $aS^1a \subseteq (S^1aS^1)(S^1aS^1) \subseteq A^2 \subseteq I$. By (2) $a \in I$. Hence $A \subseteq I$. ■

### 2.2.3 Definition

A non-empty subset $A$ of a semigroup $S$ is called a p-system if and only if $a \in A$ implies that there exists an element $x \in S^1$ such that $axa \in A$.

### 2.2.4 Corollary

An ideal $I$ of a semigroup $S$ is semiprime if and only if $S\backslash I$ is a p-system.

**Proof.** Suppose that $I$ is a semiprime ideal of $S$ and $a \in S\backslash I$. If there does not exist $x \in S^1$ such that $axa \in S\backslash I$, then $aS^1a \subseteq I$. By above Proposition $a \in I$ which is a contradiction. Hence there exists $x \in S^1$ such that $axa \in S\backslash I$.

Conversely, assume that $S\backslash I$, is a p-system. Let $a \in S$ such that $aS^1a \subseteq I$. If $a \notin I$ then $a \in S\backslash I$, so there exist $x \in S^1$ such that $axa \in S\backslash I$, this implies that $aS^1a \nsubseteq I$, which is a contradiction. Hence $a \in I$. ■

### 2.2.5 Remark

(1) Clearly every prime ideal of a semigroup $S$ is semiprime ideal of $S$, but the converse is not necessarily true;

(2) Every m-system in a semigroup $S$ is a p-system but the converse may not be true;

(3) Intersection of prime ideals of a semigroup $S$ is a semiprime ideal.

### 2.2.6 Example

Consider the semigroup $S = \{0, 1, a, b\}$

| | 0 | 1 | a | b |
|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 |
| 1 | 0 | 1 | a | b |
| a | 0 | a | a | 0 |
| b | 0 | b | 0 | b |

Its two-sided ideals are

$\{0\}$, $\{0, a\}$, $\{0, b\}$, $S$

Each ideal is semiprime but $\{0\}$ is not a prime ideal.

$A = \{1, a, b\}$ is a p-system but not m-system.

Next we show that an ideal of a semigroup $S$ is prime if and only if it is semiprime and irreducible.

### 2.2.7 Definition

An ideal $I$ of a semigroup $S$ is called an irreducible (strongly irreducible ) if $A \cap B = I$ $(A \cap B \subseteq I)$ implies that either $A = I$ or $B = I$ (either $A \subseteq I$ or $B \subseteq I$) for every ideals $A$, $B$ of $S$.

A strongly irreducible ideal is an irreducible ideal.

### 2.2.8 Proposition

Let $a$ be an element of a semigroup $S$ and let $I$ be an ideal of $S$ not containing $a$. Then there exists an irreducible ideal $A$ of $S$ containing $I$ and not containing $a$.

**Proof.** If $\{A_i : i \in \Omega\}$ is a chain of ideals in $S$ containing $I$ and not containing $a$ then $A = \cup_{i\in\Omega} Ai$ is an ideal of $S$ not containing $a$. Therefore by Zorn's Lemma the set of ideals of $S$ containing $I$ and not containing $a$ has a maximal element $A$. Suppose that $A = B \cap C$, where $B$ and $C$ are ideals of $S$ properly containing $A$. Then by choice of $A$, $a \in B$ and $a \in C$. Thus $a \in B \cap C = A$, which is a contradiction. Hence $A$ is irreducible. ■

### 2.2.9 Proposition

Any ideal $I$ of a semigroup $S$ is the intersection of all irreducible ideal containing it.

**Proof.** Let $I$ be an ideal of a semigroup $S$ and $\{A_i : i \in \Omega\}$ be the collection of irreducible ideals of $S$ containing $I$, then $I \subseteq \cap A_i$. For the reverse inclusion, let $x \in S$ such that $x \notin I$, then by above Proposition there exist an irreducible ideal $A$ of $S$ containing $I$ but $x \notin A$. Hence $x \in \cap A_i$. Thus $I = \cap A_i$. ■

### 2.2.10 Proposition

An ideal $I$ of a semigroup $S$ is prime if and only if it is semiprime and irreducible.

**Proof.** Suppose that $I$ is a prime ideal of $S$, then $I$ is a semiprime ideal of $S$. Moreover, if $A$ and $B$ are ideals of $S$ satisfying $A \cap B = I$. As $AB \subseteq A \cap B = I$ and $I$ is a prime ideal of $S$, so either $A \subseteq I$ or $B \subseteq I$. Since $I \subseteq A$ and $I \subseteq B$. Thus either $A = I$ or $B = I$. Hence $I$ is irreducible.

Conversely, assume that $I$ is an ideal of $S$ which is both semiprime and irreducible. If $A$ and $B$ are ideals of $S$ such that $AB \subseteq I$, then $(A \cap B)^2 \subseteq AB \subseteq I$ and so by semiprimeness, $A \cap B \subseteq I$. Also $(A \cap B) \cup I = I$.

This implies that $(A \cup I) \cap (B \cup I) = I$.

As $I$ is irreducible so either $A \cup I = I$ or $B \cup I = I$. That is either $A \subseteq I$ or $B \subseteq I$. Hence $I$ is a prime ideal of $S$. ■

## 2.3 Fully Prime Semigroups

In this section we give necessary and sufficient condition for a semigroup $S$ to be fully prime. We also show that a commutative semigroup $S$ is fully prime iff $S$ is an inverse semigroup and the set of ideals of $S$ is totally ordered under inclusion.

The following result is from [2].

### 2.3.1 Theorem

The following condition for a semigroup $S$ are equivalent :

(1) $S$ is semisimple;

(2) Each ideal of $S$ is semiprime;

(3) Each ideal of $S$ is the intersection of prime ideals which contain it;

**Proof.** (1) $\Rightarrow$ (2) : Let $I$ be an ideal of $S$. Let $A$ be any ideal of $S$ such that $A^2 \subseteq I$.

By (1) $A^2 = A$, so $A \subseteq I$. Thus $I$ is a semiprime ideal of $S$.

(2) $\Rightarrow$ (3) : By Proposition 2.2.9, any ideal $I$ of $S$ is the intersection of all irreducible

ideals containing $I$. By (2) each ideal of $S$ is semiprime. Thus every ideal of $S$ is the intersection of all irreducible semiprime ideals of $S$ which contain it. By Proposition 2.2.10 every irreducible semiprime ideal is prime ideal. Thus $I$ is the intersection of all prime ideals which contain it.

(3) $\Rightarrow$ (1) : Let $I$ be an ideal of $S$, by (3) it is the intersection of all prime ideals which contain it. Hence $I$ is a semiprime ideal. As $I^2 \subseteq I^2 \Rightarrow I \subseteq I^2$. But $I^2 \subseteq I$ always. Thus $I = I^2$. Hence $S$ is semisimple. ■

It is well known that for a ring $R$ we have the following result:

### 2.3.2 Proposition

A commutative ring $R$ is fully prime if and only if $R$ is a field.

Recently Blair and Tsutsui [4] proved that

### 2.3.3 Theorem

A ring $R$ is fully prime if and only if it is fully idempotent and the set of ideals of $R$ is totally ordered under inclusion.

Below we show that the same result is true for semigroups. However, a fully prime commutative monoid need not be a semifield. Instead of this we show that a commutative semigroup $S$ is fully prime if and only if $S$ is an inverse semigroup and the set of ideals of $S$ is totally ordered under inclusion. The proof below is essentially the same as the one given in [4].

### 2.3.4 Theorem

A semigroup $S$ is fully prime if and only if $S$ is semisimple and the set of ideals of $S$ is totally ordered under inclusion.

**Proof.** Suppose that $S$ is fully prime. Let $I$ be an ideal of $S$, then $I^2$ is also an ideal of $S$. As $I^2 \subseteq I^2 \Longrightarrow I \subseteq I^2$. But $I^2 \subseteq I$ always. Hence $I^2 = I$. Thus every ideal of $S$ is idempotent, so $S$ is semisimple. Let $P$, $Q$ be ideals of $S$. Since $PQ \subseteq P \cap Q$. As $P \cap Q$ is an ideal of $S$, so a prime ideal. Thus either $P \subseteq P \cap Q$ or $Q \subseteq P \cap Q$. That is either $P \subseteq Q$ or $Q \subseteq P$.

Conversely, assume that $S$ is a semisimple semigroup and the set of ideals of $S$ is totally ordered by inclusion. Let $I$, $J$, $P$ be ideals of $S$ with $IJ \subseteq P$. Since the set of ideals of S is totally ordered by inclusion, so either $I \subseteq J$ or $J \subseteq I$. Assume that $I \subseteq J$. Now $I = I^2 \subseteq IJ \subseteq P$. Hence $I \subseteq P$ and so $P$ is a prime ideal. ■

### 2.3.5 Corollary

A commutative semigroup $S$ is fully prime if and only if $S$ is an inverse semigroup and the set of ideals of $S$ is totally ordered under inclusion.

**Proof.** As commutative semisimple semigroup is regular and by Theorem 1.2.4 $S$ is an inverse semigroup. This proves the corollary. ■

### 2.3.6 Example

Let $S = \{0, 1, 2\}$

| $\cdot$ | 0 | 1 | 2 |
|---|---|---|---|
| 0 | 0 | 0 | 0 |
| 1 | 0 | 1 | 2 |
| 2 | 0 | 2 | 2 |

$S$ is a commutative semigroup

Ideals of $S$ are $\{0\}$, $\{0, 2\}$, $\{0, 1, 2\}$. As each ideal of $S$ is prime, so $S$ is fully prime but 2 does not have multiplicative inverse.

### 2.3.6 Example

Let $S = \{0, 1, 2\}$

| $\cdot$ | 0 | 1 | 2 |
|---|---|---|---|
| 0 | 0 | 0 | 0 |
| 1 | 0 | 1 | 2 |
| 2 | 0 | 2 | 2 |

$S$ is a commutative semigroup

Ideals of $S$ are $\{0\}$, $\{0, 2\}$, $\{0, 1, 2\}$. As each ideal of $S$ is prime, so $S$ is fully prime but 2 does not have multiplicative inverse.

# Chapter 3

# Prime Right Ideals in Semigroups

In this chapter we shall define prime right ideals and semiprime right ideals of a semigroup. We shall also characterize semigroups where every right ideal is semiprime.

## 3.1 Prime right ideals

In [9] K. Koh has defined that a right ideal $I$ in a ring $R$ is of a prime type if $AB \subseteq I$, $A$, $B$ are right ideals, implies that either $A \subseteq I$ or $B \subseteq I$. In [8] F. Hansen call these ideals prime right ideals. Adopting this notion we have the following definition.

### 3.1.1 Definition

A right ideal $I$ of a semigroup $S$ is called a prime right ideal of $S$ if $AB \subseteq I$ implies that either $A \subseteq I$ or $B \subseteq I$ for all right ideals $A$, $B$ of $S$.

### 3.1.2 Proposition

The following conditions on a right ideal $I$ of a semigroup $S$ are equivalent:

(1) $I$ is prime right ideal ;

(2) $aS^1b \subseteq I \Longrightarrow a \in I$ or $b \in I$;

(3) If $a$, $b$ are elements of $S$ satisfying $(a)_r(b)_r \subseteq I$ then either $a \in I$ or $b \in I$.

**Proof.** (1) $\Longrightarrow$ (2) : Let $a,\ b \in S$ such that $aS^1b \subseteq I$, then $aS^1bS^1 \subseteq IS^1 = I$. By (1) either $aS^1 \subseteq I$ or $bS^1 \subseteq I$. This implies that either $a \in I$ or $b \in I$.

(2) $\Longrightarrow$ (3) : Let $a$, $b$ be elements of $S$ such that $(a)_r(b)_r \subseteq I$. As $aS^1b \subseteq (a)_r(b)_r \subseteq I$, so by (2) either $a \in I$ or $b \in I$.

(3) $\Longrightarrow$ (1) : Let $A$ and $B$ be right ideals of $S$ such that $AB \subseteq I$. Suppose that $A \nsubseteq I$, then there exists $a \in A$ such that $a \notin I$. Let $b$ be any arbitrary element of $B$ then $(a)_r(b)_r \subseteq AB \subseteq I$. By (3) either $a \in I$ or $b \in I$. As $a \notin I$ so $b \in I$, that is $B \subseteq I$. ■

### 3.1.3 Proposition

A right ideal $I$ of a semigroup $S$ is prime right ideal if and only if $S \backslash I$ is an m-system.

**Proof.** Let $I$ be a right ideal of $S$. Suppose that $I$ is a prime right ideal of $S$. We show that $S \backslash I$ is an m-system. Let $a,\ b \in S \backslash I$. Suppose that there does not exist $x \in S^1$ such that $axb \in S \backslash I$. Then $aS^1b \subseteq I$. By Proposition 3.1.2 either $a \in I$ or $b \in I$, which is a contradiction. Hence there exists $x \in S^1$ such that $axb \in S \backslash I$.

Conversely, assume that $S \backslash I$ is an m-system. Let $aS^1b \subseteq I$, such that $a \notin I$ and $b \notin I$. Then $a,\ b \in S \backslash I$. As $S \backslash I$ is an m-system, so there exists $x \in S^1$ such that $axb \in S \backslash I$, which is a contradiction. Hence either $a \in I$ or $b \in I$. ■

### 3.1.4 Proposition

If $S$ is a monoid, then every maximal right ideal of $S$ is a prime right ideal.

**Proof.** Let $P$ be a maximal right ideal of $S$. Let $A$, $B$ be right ideals of $S$ such that $AB \subseteq P$. Suppose that $A \nsubseteq P$ then $A \cup P = S$. As $1 \in S$ so $1 \in A \cup P$. But $1 \notin P$ (since $P$ is maximal right ideal of $S$), thus $1 \in A$, so $A = S$.

$$\text{Hence } B \subseteq SB = AB \subseteq P.$$

■

### 3.1.5 Proposition

Let $P$ be a prime right ideal of $S$. Then $s(P) = \{s \in S : S^1 s \subseteq P\}$ is the largest two sided ideal of $S$ contained in $P$. Also $s(P)$ is a prime right ideal of $S$.

**Proof.** First we show that $s(P) = \{s \in S : S^1 s \subseteq P\}$ is the largest two-sided ideal of $S$ contained in $P$. Let $s \in s(P)$ and $x \in S$,

$$\begin{aligned} \text{then } S^1(sx) &= (S^1 s)x \subseteq Px \subseteq P \Longrightarrow sx \in s(P). \\ \text{and } S^1(xs) &= (S^1 x)s \subseteq S^1 s \subseteq P \Longrightarrow xs \in s(P). \end{aligned}$$

So $s(P)$ is a two-sided ideal of $S$. Clearly $s(P) \subseteq P$.

Let $J$ be a two-sided ideal of $S$ such that $J \subseteq P$. Let $x \in J$, then $Sx \subseteq J \subseteq P \Longrightarrow x \in s(P)$. Hence $J \subseteq s(P)$. Thus $s(P)$ is the largest two-sided ideal of $S$ contained in $P$.

Let $A$, $B$ be right ideals of $S$ such that $AB \subseteq s(P) \subseteq P$. Now

$$S^1(AB) \subseteq S^1(s(P)) \subseteq s(P) \subseteq P.$$

$$(S^1A)(S^1B) = S^1(AS^1)B \subseteq S^1AB \subseteq s(P) \subseteq P.$$

As $P$ is a prime right ideal, so either $S^1A \subseteq P$ or $S^1B \subseteq P$.

Since $s(P)$ is the largest two-sided ideal of $S$ contained in $P$, so either $S^1A \subseteq s(P)$ or $S^1B \subseteq s(P)$. Thus either $A \subseteq s(P)$ or $B \subseteq s(P)$. Hence $s(P)$ is a prime right ideal of $S$. ■

### 3.1.6 Proposition

Let $S$ be a monoid. If every prime right ideal of $S$ is a maximal right ideal of $S$. Then every maximal right ideal is a two-sided ideal.

**Proof.** Let $P$ be a maximal right ideal of $S$. Then by Proposition 3.1.4 $P$ is a prime right ideal of $S$. By Proposition 3.1.5 $s(P)$ is a prime right ideal of $S$. Hence $s(P)$ is a maximal right ideal. But $s(P) \subseteq P$, thus $P = s(P)$. Since $s(P)$ is a two-sided ideal, so $P$ is a two-sided ideal of $S$. ■

### 3.1.7 Proposition

If $I$ is a prime right ideal of a semigroup $S$ with zero, then $(I : x) = \{s \in S : xs \in I\}$ is also a prime right ideal of $S$ for any $x \in S$.

**Proof.** Clearly $(I : x) \neq \Phi$ because $0 \in (I : x)$. Also if $s \in (I : x)$ and $t \in S$ then $st \in (I : x)$. Hence $(I : x)$ is a right ideal of $S$. Let $A$, $B$ be any right ideals of a semigroup

$S$ such that $AB \subseteq (I : x)$ then $xAB \subseteq I \Longrightarrow x(AS)B \subseteq I \Longrightarrow (xA)(SB) \subseteq I$.

$\Longrightarrow$ $(xA)(xB) \subseteq I$, for fixed $x \in S$.

$\Longrightarrow$ $xA \subseteq I$ or $xB \subseteq I$. $\because$ $xA$ and $xB$ are right ideals of $S$ and $I$ is a prime right ideal of $S$.

$\Longrightarrow$ $A \subseteq (I : x)$ or $B \subseteq (I : x)$

■

### 3.1.8 Proposition

Let $\{P_\alpha : \alpha \in \Lambda\}$ be a chain of prime right ideals of a semigroup $S$, then $\cap_{\alpha\in\Lambda} P_\alpha$ is empty or a prime right ideal of $S$.

**Proof.** Assume that $\{P_\alpha : \alpha \in \Lambda\}$ is a chain of right ideals of $S$. Let $P = \cap_{\alpha\in\Lambda} P_\alpha$. Since the intersection of right ideals is empty or a right ideal, so $P$ is empty or a right ideal of $S$. If $P$ is a non-empty, let $I$, $J$ be right ideals of $S$ such that $IJ \subseteq P = \cap_{\alpha\in\Lambda} P$, for all $\alpha \in \Lambda$.

If there exists $\alpha \in \Lambda$ such that $I \nsubseteq P_\alpha$, then $J \subseteq P_\alpha$, because $P_\alpha$ is a prime right ideal. Now for $\alpha \leq \beta \Longrightarrow J \subseteq P_\beta$, as $P_\alpha \subseteq P_\beta$. If there exist $\gamma \leq \alpha$ such that $J \nsubseteq P_\gamma$, then $I \subseteq P_\gamma$ so $I \subseteq P_\alpha$, a contradiction. Hence for all $\alpha \in \Lambda \Longrightarrow J \subseteq P_\alpha \Longrightarrow J \subseteq \cap_{\alpha\in\Lambda} P_\alpha = P$.

■

## 3.2 Semiprime right ideals

### 3.2.1 Definition

A right ideal I of a semigroup $S$ is called a semiprime right ideal of $S$ if $A^2 \subseteq I$ implies that $A \subseteq I$ for all right ideals $A$ of $S$.

Clearly every prime right ideal of $S$ is a semiprime right ideal of $S$.

### 3.2.2 Proposition

The following conditions on a right ideal $I$ of a semigroup $S$ are equivalent:

(1) $I$ is a semiprime right ideal;

(2) $aS^1a \subseteq I \Longrightarrow a \in I$;

(3) $(< a >_r)^2 \subseteq I \Longrightarrow a \in I$.

**Proof.** (1) $\Longrightarrow$ (2) : Let $a \in S$ such that $aS^1a \subseteq I$. Then $aS^1aS^1 \subseteq IS^1 = I$. By (1) $aS^1 \subseteq I \Longrightarrow a \in I$.

(2) $\Longrightarrow$ (3) : Let $a \in S$ such that $(< a >_r)^2 \subseteq I$.

$$\text{As} \quad aS^1a \subseteq (< a >_r)^2 \subseteq I,$$

so by (2) $a \in I$.

(3) $\Longrightarrow$ (1) : Let $A$ be a right ideal of $S$, such that $A^2 \subseteq I$. If $A \nsubseteq I$ then there exist $a \in A$ such that $a \notin I$.

$$\text{But} \quad (< a >_r)^2 \subseteq A^2 \subseteq I.$$

By (3) $a \in I$, a contradiction. Hence $A \subseteq I$. ■

### 3.2.3 Corollary

A right ideal $I$ of a semigroup $S$ is semiprime if and only if $S \backslash I$ is a p-system.

**Proof.** Suppose that $I$ is a semiprime right ideal of $S$ and $a \in S \backslash I$. If there does not exist $x \in S^1$ such that $axa \in S \backslash I$, then $aS^1a \subseteq I$. By above Proposition $a \in I$, which is a contradiction. Hence there exist $x \in S^1$ such that $axa \in S \backslash I$.

Conversely, assume that $S \backslash I$ is a p-system. Let $a \in S$ such that $aSa \subseteq I$. If $a \notin I$, then $a \in S \backslash I$, so there exist an $x \in S^1$ such that $axa \in S \backslash I$, this implies that $aS^1a \nsubseteq I$, which is a contradiction. Hence $a \in I$. ■



### 3.2.4 Definition

A right ideal $I$ of a semigroup $S$ is called an irreducible right ideal of $S$ if $A \cap B = I$ implies that either $A = I$ or $B = I$, for every right ideals $A$ and $B$ of $S$.

### 3.2.5 Proposition

Let $a$ be an element of a semigroup $S$ and $I$ be a right ideal of $S$ not containing $a$. Then there exist an irreducible right ideal $A$ of $S$ containing $I$ and not containing $a$.

**Proof.** If $\{A_i : i \in \Omega\}$ is a chain of right ideals of $S$ containing $I$ and not containing $a$ then $\cup A_i$ is a right ideal of $S$ containing $I$ and not containing $a$. Therefore by Zorn's Lemma, the set of all ideals of $S$ containing $I$ and not containing $a$ has a maximal element $A$. Suppose that $A = B \cap C$, where $B$ and $C$ are both right ideals of $S$, properly containing $A$. Then by choice of $A$, $a \in B$ and $a \in C$. Thus $a \in B \cap C = A$, which is a contradiction. Hence $A$ is an irreducible right ideal of $S$. ■

### 3.2.6 Proposition

Any right ideal $I$ of a semigroup $S$ is the intersection of all irreducible right ideals of $S$ containing $I$.

**Proof.** Let $I$ be a right ideal of $S$ and $\{A_i : i \in \Omega\}$ be the collection of irreducible right ideals of $S$ containing $I$, then $I \subseteq \cap A_i$. For the reverse inclusion, let $x \in S$ such that $x \notin I$, then by Proposition 3.2.5. there exist an irreducible right ideal $A$ of $S$ containing $I$ but not containing $x$. Hence $x \notin \cap A_i$. Thus $I = \cap A_i$. ∎

### 3.2.7 Lemma

Let $S$ be a semigroup. Let $I$ be an irreducible semiprime right ideal of $S$, then $I$ is a prime right ideal of $S$.

**Proof.** Let $A$, $B$ be any right ideals of $S$ such that $AB \subseteq I$. Then $S^1B$ is a two sided ideal of $S$ generated by $B$. $A \cap S^1B$ is a right ideal of $S$. ($A \cap S^1B$ is non-empty, because $AS^1B \subseteq A$ and $AS^1B \subseteq SB$). As $(A \cap S^1B)^2 \subseteq A(S^1B) = (AS^1)B \subseteq AB \subseteq I$ and $I$ is a semiprime right ideal, so

$$A \cap S^1B \subseteq I.$$

$$\text{Thus} \quad (A \cap S^1B) \cup I = I.$$

$$\text{But} \quad (A \cap S^1B) \cup I = (A \cup I) \cap (S^1B \cup I).$$

$$\text{So} \quad (A \cup I) \cap (S^1B \cup I) = I.$$

As $I$ is an irreducible right ideal, so either

$$A \cup I = I \text{ or } S^1 B \cup I = I$$

$$\Longrightarrow \text{either } A \subseteq I \text{ or } S^1 B \cup I = I$$

$$\Longrightarrow \text{either } A \subseteq I \text{ or } B \subseteq S^1 B \subseteq I$$

$\Longrightarrow I$ is a prime right ideal. ■

### 3.2.8 Lemma

Intersection of prime right ideals of a semigroup $S$ is a semiprime right ideal.

## 3.3 Fully Prime Right Semigroups

A semigroup $S$ is called a fully prime right semigroup if all its right ideals are prime right ideals.

### 3.3.1 Theorem

Let $S$ be a semigroup with multiplicative identity 1. Then the following are equivalent:

(1) $S$ is right weakly regular;

(2) Every right ideal of $S$ is semiprime right ideal of $S$;

(3) Every right ideal of $S$ is the intersection of prime right ideals of $S$ which contain it.

**Proof.** (1) $\Longrightarrow$ (2): Let $I$ be a right ideal of $S$ and $A^2 \subseteq I$ where $A$ is a right ideal of $S$. By Proposition 1.3.4, $A^2 = A$, so $A \subseteq I$. Thus $I$ is a semiprime right ideal of $S$.

(2) $\Longrightarrow$ (3): First we show that each proper right ideal of $S$ is contained in an irreducible right ideal of $S$. Let $I$ be a proper right ideal of $S$ and $x \in S \backslash I$. Let $P_x$ be any right ideal maximal with respect to $I \subseteq P_x$ but $x \notin P_x$. Suppose that $P_x = A \cap B$, where $A$ and $B$ are right ideals of $S$ with $P_x \neq A$ and $B \neq P_x$.The maximality of $P_x$ requires that $x \in A$ and $x \in B$. But then $x \in A \cap B = P_x$, which is a contradiction. Hence $P_x$ is irreducible. Let $\{P_x : x \in S \backslash I\}$ be the family of proper irreducible right ideals containing $I$, then $I \subseteq \cap P_x$. For the reverse inclusion, let $y \in S$ such that $y \notin I$. Then as argued above, there exists an irreducible right ideal $P_y$ containing $I$ and $y \notin P_y$. This implies that $y \notin \cap_{x \in S \backslash I} P_x$. Hence by contrapositivity $\cap_{x \in S \backslash I} P_x \subseteq I$. Thus $I = \cap_{x \in S \backslash I} P_x$. Hence every right ideal is the intersection of irreducible semiprime right ideals. By Proposition 3.2.7, every right ideal of $S$ is the intersection of prime right ideals of $S$ which contain it.

(3) $\Longrightarrow$ (1): By Lemma 3.2.8, the intersection of prime right ideals is semiprime. Let $I$ be a right ideal of $S$, then $I^2$ is a semiprime right ideal of $S$. $I^2 \subseteq I^2 \Longrightarrow I \subseteq I^2$. Hence $I = I^2$ that is $S$ is right weakly regular. ■

### 3.3.2 Definition

A semigroup without zero is called simple if it has no proper ideal. A semigroup $S$ with zero is called 0-simple if

(1) $\{0\}$ and $S$ are its only ideals;

(2) $S^2 \neq \{0\}$.

The following Theorem is due to Koh [9].

### 3.3.3 Theorem

Let $R$ be a ring. Then every right ideal of $R$ is prime if and only if $R$ is simple and $a \in aR$ for all $a \in R$.

Example 1.3.7 shows that the above Theorem is not true in case of semigroups. However,

### 3.3.4 Proposition

Let $S$ be a semigroup. If $S$ is simple and $a \in aS$ for all $a \in S$, then every right ideal of $S$ is prime.

**Proof.** Let $S$ be a simple semigroup and $a \in aS$ for all $a \in S$. Let $I$ be a right ideal of $S$ and suppose that $A$ and $B$ are right ideals of $S$ such that $AB \subseteq I$. As $SB$ is an ideal of $S$ and so $SB = S$. Hence $A = AS = A(SB) = (AS)B = AB \subseteq I$. Thus $I$ is a prime right ideal of $S$. ■

### 3.3.5 Proposition

Let $S$ be a semigroup with multiplicative identity. If every right ideal of $S$ is prime then $S$ is right weakly regular and the set of ideals of $S$ is totally ordered under inclusion.

**Proof.** Let $S$ be a semigroup with multiplicative identity and every right ideal of $S$ is prime. Let $I$ be a right ideal of $S$ then

$$I^2 \subseteq I^2 \Longrightarrow I \subseteq I^2.$$

Hence $I = I^2$. By Proposition 1.3.4, $S$ is right weakly regular. Let $A$, $B$ be ideals of $S$ then

$$AB \subseteq A \cap B.$$

As every right ideal of $S$ is prime,

so either $A \subseteq A \cap B$ or $B \subseteq A \cap B$ that is either $A \subseteq B$ or $B \subseteq A$.

■

The converse of above result is not true.

### 3.3.6 Example

Consider the semigroup $S = \{1, a, b, c, d\}$

| | 1 | a | b | c | d |
|---|---|---|---|---|---|
| 1 | 1 | a | b | c | d |
| a | a | a | a | a | a |
| b | b | b | a | b | b |
| c | c | c | a | c | c |
| d | d | d | d | d | d |

$S$ is a regular semigroup, so right weakly regular.

Its right ideals are

$$\{a\}, \{d\}, \{a,b\}, \{a,d\}, \{a,c\}, \{a,b,c\}, \{a,b,c,d\}, \{1,a,b,c,d\}$$

$$\begin{aligned} \{a,b\}\{d\} &= \{a\} \\ \text{but neither } \{a,b\} &\subseteq \{a\} \text{ nor } \{d\} \subseteq \{a\}. \end{aligned}$$



So $\{a\}$ is not prime right ideal.

### 3.3.7 Proposition

If $S$ is a right weakly regular semigroup with multiplicative identity and the set of right ideals of $S$ is totally ordered by inclusion then every right ideal of $S$ is prime.

**Proof.** Let $I$, $J$, $K$ be right ideals of $S$ such that $IJ \subseteq K$. As the set of right ideals of $S$ is totally ordered, therefore either $I \subseteq J$ or $J \subseteq I$.

$$\begin{aligned} \text{If } I &\subseteq J \text{ then } I = I^2 \subseteq IJ \subseteq K. \\ \text{If } J &\subseteq I \text{ then } J = J^2 \subseteq IJ \subseteq K. \end{aligned}$$

So $K$ is a prime right ideal. ■

It is not necessary that if every right ideal of $S$ is prime then the set of right ideals of $S$ is totally ordered.

### 3.3.8 Example

Let $X$ be any non-empty set, define the binary operation on $X$ as

$$ab = a \text{ for all } a, b \in X$$

Then every non-empty subset of $X$ is a right ideal of $X$. Also if $A$, $B$ are two right ideals of $X$ then

$$A = AB$$

so every right ideal of $X$ is prime, but the set of right ideals of $X$ is not totally ordered.

### 3.3.9 Theorem

If $S$ is a semigroup with multiplicative identity and the set of right ideals of $S$ is totally ordered then the following are equivalent:

(1) $S$ is right weakly regular;

(2) every right ideal of $S$ is prime.

**Proof.** By Proposition 3.3.5 and 3.3.7. ■

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