EFFECT OF HEAT TRANSFER ON THE PULSATILE FLOW OF A NON-NEWTONIAN FLUID

QUAID-I-AZAM UNIVERSITY
ISLAMABAD

By
Amer Bilal Mann

Supervised by

Dr. Saleem Asghar

Department of Mathematics
Quaid-I-Azam University
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بسم الله الرحمن الرحيم

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Dedicated Respectfully

to

Hazrat Pir Muhammad Naqib-ur-Rehman Sahib

CERTIFICATE

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Amer Bilal Mann

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We accept this dissertation as conforming to the required standard.

1. Saleem Asghar

Prof. Dr. Saleem Asghar
(Supervisor)

2. Qaiser Mushtaq

Prof. Dr. Qaiser Mushtaq
(Chairman)

3. ____________________

Brig. Dr. Muhammad Rafique
EME College, Peshawar Road,
NUST, Rawalpindi.

(External Examiner)

Department of Mathematics
Quaid-i-Azam University
Islamabad-Pakistan
2003

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Preface

Heat transfer plays an important role during the handling and processing of non-Newtonian fluids. The understanding of heat transfer in boundary layer flows of non-Newtonian fluids is of importance in many engineering applications such as transpiration cooling, drag reduction, thermal recovery of oil, etc. Several studies of boundary layer flows of non-Newtonian fluids have been carried out during the last decade. All these studies have been for non-Newtonian fluids of the differential or rate type [2] [4]-[8]. It is known that for the fluids of the differential type, except for the fluids of complexity $n = 1$, the equation of motion are an order higher than the Navier-Stokes equations, and the adherence boundary condition is insufficient to determine the solution completely. For such situations, the difficulty is removed by using perturbation expansion or by the argumentation of the boundary conditions. A systematic study of the boundary layer flow and heat transfer of non-Newtonian fluids has been provided by Shenoy and Mashelkar [7]. Recently Ghosh and Debnath [1] discussed the heat transfer to pulsatile flow of viscoelastic fluid. However, no attempt has been made to discuss the heat transfer analysis on the pulsatile flow of a magnetohydrodynamic viscoelastic fluid in a porous medium. Flow through a porous medium has been of considerable interest in recent years particularly among geophysical fluid dynamicts. Keeping this fact in view the arrangement of this dissertation is as follows:

In chapter one, we give the basic definitions of fluids and flows. The governing equations of continuity and momentum are included. The equation of motion for an Oldroyd-$\mathcal{B}$ fluid has been derived. Finally, the energy equation is presented.

Chapter two deals with the pulsatile flow of a viscoelastic fluid between two plates. The upper plate has higher temperature than the lower plate. The assumed pressure gradient is of the oscillating type.

Chapter three is devoted to the flow of a viscoelastic fluid between two infinite parallel plates. The motion is generated due to the application of unsteady pressure gradient. The lower plate has less temperature than the upper one. Expressions for the velocity and the rate of heat transfer at both the plates are given.

In chapter four, a study is made of a problem of heat transfer to pulsatile flow of a viscoelastic fluid between two parallel plates of which the upper one is at a temperature higher than the lower one. The fluid is electrically conducting by applying uniform magnetic field and the medium between the plates is porous. The solutions for the steady and fluctuating velocity and temperature distributions are obtained. The rate of heat transfer at the plates is also constructed.

Chapter 1

Some Basic Concepts and Governing Equations

1.1 Introduction

This chapter includes some definitions and basic concepts of fluid mechanics. Different kinds of flows and fluids are defined. The equation of continuity and energy equation for viscous fluid are given. The governing equation for unidirectional unsteady flow of an Oldroyd-$\mathcal{B}$ fluid is derived.

1.2 Definitions

Fluid

A fluid is a substance that deforms continuously under the application of shear stress no matter how small the shear stress may be.

Fluid Mechanics

The branch of engineering that examines the nature and properties of fluid, both at rest and motion is called *Fluid Mechanics*.

Viscosity

Viscosity is the physical property of fluid associated with the shearing deformation of the fluid particles subjected to the action of the applied force. In other words it is resistance of a fluid to its motion. It is ratio of shear stress to the rate of shear strain, i.e,

$$\text{Viscosity} = \mu = \frac{\text{shear stress}}{\text{rate of shear strain}}$$

The dimensions of μ is $ML^{-1}T^{-1}$. It is also called the coefficient of viscosity, absolute viscosity or dynamic viscosity. Note that for gases viscosity increases with the increase in temperature while for liquids viscosity decreases with increasing temperature. The SI unit of viscosity is Pascal-second (Pa.s).

Density

The mass per unit volume at a given temperature and pressure. Mathematically

$$\rho = \frac{m}{V_1}$$

where m is the mass and V_1 is the volume. Its dimension is ML^{-3}.

Kinematic Viscosity

It is the ratio of dynamic viscosity to mass density. It is denoted by ν, i.e,

$$\nu = \frac{\mu}{\rho}$$

The dimensions of ν is L^2T^{-1}. The SI units of kinematic viscosity are m^2/sec.

Eckert Number

The ratio between the square of viscosity and the temperature differences is called the Eckert number.

Prandtle Number

It is the ratio of the product of dynamic viscosity and specific heat with conductivity χ.

Specific Heat

Specific heat is the amount of heat required to change the temperature of a unit mass of a substance through one degree.

Thermal Conductivity

It is the conduction of heat through the medium due to a thermal gradient in the medium. Thermal conductivity χ of a substance depends upon the material of the substance.

1.3 Types of Flow

Steady Flow

A flow in which the quantity of fluid flowing per second is constant. In other words, if the properties at each point of fluid in the flow do not change with time, the flow is termed as steady flow. Mathematically the definition of steady flow is

$$\frac{\partial \xi}{\partial t} = 0$$

where ξ is any fluid property. For steady flow

$$\frac{\partial \rho}{\partial t} = 0, \qquad \rho = \rho(x, y, z),$$

where ρ is for density.

Unsteady Flow

A time dependent flow is known as unsteady flow.

Incompressible Flow

A flow in which the volume and thus the density of the fluid does not change during flow. All the liquid are generally considered to have the incompressible flow.

Coutte Flow

A flow between two parallel plates with one plate stationary and other plate moving parallel to it at a constant speed is known as Coutte flow. The pressure gradient for the Coutte flow is taken to be zero, i.e., p =constant and

$$\frac{dp}{dx} = 0.$$

Poiseuille Flow

When fluid is bounded between the two stationary plates and the flow is caused due to action of the constant pressure gradient then flow is known as Poiseuille flow.

1.4 Classification of Fluids

Real Fluids

All the fluids for which the coefficient of viscosity μ is not equal to zero are known as *real fluids.*

Ideal Fluids

The fluids for which the coefficient of viscosity μ is zero are called *ideal fluids.*

Newtonian Fluids

The viscous fluids are also known as *Newtonian fluids.* Alternatively the fluids which obey Newton's law of viscosity are known as Newtonian fluids. Mathematically the Newton's law of viscosity is

$$\tau_{yx} = \mu \frac{du}{dy},$$

where τ_{yx} is the shear stress, du/dy is the deformation rate, and μ is the constant of proportionality known as absolute or dynamic viscosity. Examples of Newtonian fluids are water, gasoline, etc.

Non-Newtonian Fluids

The fluids for which power law model holds, are called *non-Newtonian fluids*. Mathematically

$$\tau_{yx} \propto \left(\frac{du}{dy}\right)^n, \qquad n \neq 1$$

or

$$\tau_{yx} = k\left(\frac{du}{dy}\right)^n,$$

where n is the flow behavior index, and k is the consistency index. The examples of non-Newtonian fluids are blood, ketchup, toothpaste, lucite paints, biological fluids, etc. The non-Newtonian fluids are further divided into three broad groups, which are *time-independent*, *time-dependent* and *viscoelastic fluids*. The time-independent non-Newtonian fluids are further subdivided into three sub-classes, *pseudo plastic*, *dilatant* and *Bingham plastic*. These types are defined as:

Pseudo Plastic Fluids

Pseudo plastic fluids are those fluids in which the apparent viscosity decreases by increasing the deformation rate, i.e, $n < 1$. Pseudo plastic fluids are shear thinning fluids. Examples of such fluids are polymer solutions, paper pulp in water, etc.

Dilatant Fluids

The fluids in which the apparent viscosity increases by increasing the deformation rate, i.e, $n > 1$, are called *Dilatant*. Dilatant fluids are shear thickening fluids. Examples of this type are suspension of starch and sand.

Bingham Plastic Fluids

The fluid which behaves like a solid until a minimum yield stress τ_y is exceeded and subsequently possess a linear relation between stress and the deformation rate, called *Bingham plastic*. Mathematically it is given by

$$\tau_{yx} = \tau_y + \mu_p \frac{du}{dy}.$$

Examples of this kind are toothpaste, drilling mud and clay suspension.

The time-dependent non-Newtonian fluids are also subdivided into two groups, Thixotropic fluids and Rheopectic fluids, defined below as:

Thixotropic Fluids

The fluids in which η decreases with time under the constant applied shear stress are called *thixotropic fluids.* An example of this category is paint.

Rheopectic Fluids

The fluids which show increase in η with time are called *rheopectic fluids.*

Viscoelastic Fluids

The fluids in which the particles after deformation return to their original shape when applied shear stress is released are called *viscoelastic fluids.* The viscoelastic fluids are further subdivided into two main subgroups, namely, *linear viscoelastic* and *non-linear viscoelastic fluids.* The linear viscoelastic fluids are again further subdivided into three subgroups, *Maxwell*, *Kelvin Voigt* and *Jeffery's* models.

Similarly the non-linear viscoelastic fluids are further subdivided into *Walter's* $\mathcal{A}$ and $\mathcal{B}$ *fluid*, *Oldroyd's* $\mathcal{A}$ and $\mathcal{B}$ *fluid*, *Coleman* and *Noll* (second, third and forth grade fluids) and *Green correlational fluid.*

Since we are interested in the study of heat transfer to the pulsating flow of an Oldroyd-$\mathcal{B}$ fluid bounded by two infinitely parallel plates, so we will derive the governing equation for an Ordroyd-$\mathcal{B}$ fluid in this chapter later.

1.5 The Equation of Continuity

The equation of continuity is the mathematical expression of law of conservation of mass and is defined by

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{V}) = 0 \tag{1.5.1}$$

where V is the velocity.

For incompressible fluid the density of any particle is invariable, i.e, $\rho =$ constant, so equation of continuity (1.5.1) takes the form as

$$\nabla . \mathbf{V} = 0.$$

1.6 The Momentum Equation

The equation of motion in vector form can be written as

$$\rho \frac{d\mathbf{V}}{dt} = \rho \boldsymbol{\mathcal{B}} + div \mathcal{T}, \tag{1.6.1}$$

where the Cauchy stress tensor is

$$\mathcal{T} = \begin{bmatrix} \sigma_{xx} & \tau_{yx} & \tau_{zx} \\ \tau_{xy} & \sigma_{yy} & \tau_{zy} \\ \tau_{xz} & \tau_{yz} & \sigma_{zz} \end{bmatrix}, \tag{1.6.2}$$

and $\rho\mathcal{B}$ and d/dt are the body force per unit mass and material derivative respectively. For viscous fluid, Eq. (1.6.1) in components form can be written as

$$\rho \frac{du}{dt} = \left(\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} + \rho \mathcal{B}_x \right), \tag{1.6.3}$$

$$\rho\frac{dv}{dt} = \left(\frac{\partial\tau_{xy}}{\partial x} + \frac{\partial\sigma_{yy}}{\partial y} + \frac{\partial\tau_{zy}}{\partial z} + \rho\mathcal{B}_y\right), \tag{1.6.4}$$

$$\rho\frac{dw}{dt} = \left(\frac{\partial\tau_{xz}}{\partial x} + \frac{\partial\tau_{yz}}{\partial y} + \frac{\partial\sigma_{zz}}{\partial z} + \rho\mathcal{B}_z\right), \tag{1.6.5}$$

and are known as Navier-Strokes equations. Here u, v, w are the x, y and z components of velocity, σ_{xx}, σ_{yy} and σ_{zz} are used for normal stress and τ_{xy}, τ_{yz} and τ_{zx} are used for shear stress. The values of normal and shear stresses in Eqs. (1.6.3) to (1.6.5) for an incompressible fluid are given by

$$\sigma_{xx} = -p + 2\mu\frac{\partial u}{\partial x},$$

$$\sigma_{yy} = -p + 2\mu\frac{\partial v}{\partial y},$$

$$\sigma_{zz} = -p + 2\mu\frac{\partial w}{\partial z},$$

$$\tau_{xy} = \tau_{yx} = \mu\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right),$$

$$\tau_{yz} = \tau_{zy} = \mu\left(\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}\right),$$

$$\tau_{zx} = \tau_{xz} = \mu\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right),$$

where p is pressure.

1.7 Governing Equations for Unidirectional Flow of an Oldroyd-$\mathcal{B}$ Fluid

The constitutive relation for an incompressible Oldroyd-$\mathcal{B}$ fluid is

$$\mathcal{T} = -p\mathcal{I} + \mathcal{S}, \tag{1.7.1}$$

In the above equation p is the pressure, $\mathcal{I}$ the unit tensor and the extra stress $\mathcal{S}$ satisfies

$$\mathcal{S}+\lambda_1\frac{D\mathcal{S}}{Dt}=\mu\left(1+\lambda_2\frac{D}{Dt}\right)\mathcal{A}_1, \tag{1.7.2}$$

where λ_1 and λ_2 are the relaxation and retardation times, respectively, $\mathcal{A}_1$ is the first Rivlin-Ericksen tensor given by

$$\mathcal{A}_1=\mathcal{L}+\mathcal{L}^T=\begin{bmatrix} 0 & \frac{\partial u}{\partial y} \\ \frac{\partial u}{\partial y} & 0 \end{bmatrix}$$

and

$$\frac{D\mathcal{S}}{Dt}=\frac{\partial\mathcal{S}}{\partial t}+(\mathbf{V}\cdot\nabla)\mathcal{S}-\mathcal{L}\mathcal{S}-\mathcal{S}\mathcal{L}^T, \tag{1.7.4}$$

$$\frac{D\mathcal{A}_1}{Dt}=\frac{\partial\mathcal{A}_1}{\partial t}+(\mathbf{V}\cdot\nabla)\mathcal{A}_1-\mathcal{L}\mathcal{A}_1-\mathcal{A}_1\mathcal{L}^T. \tag{1.7.5}$$

Using Eqs. (1.7.4) and (1.7.5) in Eq. (1.7.2) we have

$$\mathcal{S}+\lambda_1\left[\frac{\partial\mathcal{S}}{\partial t}+(\mathbf{V}\cdot\nabla)\mathcal{S}-\mathcal{L}\mathcal{S}-\mathcal{S}\mathcal{L}^T\right]=$$
$$\mu\mathcal{A}_1+\mu\lambda_2\left[\frac{\partial\mathcal{A}_1}{\partial t}+(\mathbf{V}\cdot\nabla)\mathcal{A}_1-\mathcal{L}\mathcal{A}_1-\mathcal{A}_1\mathcal{L}^T\right]. \tag{1.7.6}$$

We seek a velocity field $\mathbf{V}$ for unidirectional flow of the form

$$\mathbf{V}=\left[\begin{array}{ccc} u(y,t), & 0, & 0 \end{array}\right]. \tag{1.7.7}$$

Using the above equation, continuity equation (1.5.1) is identically satisfied and equation of motion (1.6.1) in component form with no body force gives

$$\rho\frac{\partial u}{\partial t}=-\frac{\partial p}{\partial x}+(\nabla\cdot\mathcal{S})_x, \tag{1.7.8}$$

$$0=-\frac{\partial p}{\partial y}+(\nabla\cdot\mathcal{S})_y, \tag{1.7.9}$$

$$0=-\frac{\partial p}{\partial z}+(\nabla\cdot\mathcal{S})_z, \tag{1.7.10}$$

where the subscripts denote the components notation. Using Eq. (1.7.7) we get

$$\mathcal{L}=\begin{bmatrix} 0 & \frac{\partial u}{\partial y} \\ 0 & 0 \end{bmatrix}, \tag{1.7.11}$$

$$\mathcal{A}_1=\mathcal{L}+\mathcal{L}^T=\begin{bmatrix} 0 & \frac{\partial u}{\partial y} \\ \frac{\partial u}{\partial y} & 0 \end{bmatrix}, \tag{1.7.12}$$

$$\frac{\partial \mathcal{A}_1}{\partial t}=\begin{bmatrix} 0 & \frac{\partial^2 u}{\partial t\partial y} \\ \frac{\partial^2 u}{\partial t\partial y} & 0 \end{bmatrix}, \tag{1.7.13}$$

$$\left(\mathbf{V}\cdot\nabla\right)\mathcal{A}_1=0, \tag{1.7.14}$$

$$\mathcal{L}\mathcal{A}_1=\begin{bmatrix} \left(\frac{\partial u}{\partial y}\right)^2 & 0 \\ 0 & 0 \end{bmatrix}, \tag{1.7.15}$$

$$\mathcal{L}\mathcal{A}_1+\mathcal{A}_1\mathcal{L}^T=\begin{bmatrix} 2\left(\frac{\partial u}{\partial y}\right)^2 & 0 \\ 0 & 0 \end{bmatrix}. \tag{1.7.16}$$

Making use of above equations we obtain

$$\begin{aligned} &\mu\lambda_2\left[\frac{\partial \mathcal{A}_1}{\partial t}+\left(\mathbf{V}\cdot\nabla\right)\mathcal{A}_1-\mathcal{L}\mathcal{A}_1-\mathcal{A}_1\mathcal{L}^T\right]+\mu\mathcal{A}_1 \\ &\quad=\begin{bmatrix} -2\lambda_2\mu\left(\frac{\partial u}{\partial y}\right)^2 & \mu\frac{\partial u}{\partial y}+\mu\lambda_2\frac{\partial^2 u}{\partial t\partial y} \\ \mu\frac{\partial u}{\partial y}+\mu\lambda_2\frac{\partial^2 u}{\partial t\partial y} & 0 \end{bmatrix}. \end{aligned} \tag{1.7.17}$$

We also assume that

$$\mathcal{S}=\mathcal{S}(y,t)\widehat{i}, \tag{1.7.18}$$

where $\widehat{i}$ is the unit vector in the x-direction. From Eqs. (1.7.7) and (1.7.18) we obtain

$$\begin{aligned} \mathcal{S} \;+\; &\lambda_1\left[\frac{\partial \mathcal{S}}{\partial t}+\left(\mathbf{V}\cdot\nabla\right)\mathcal{S}-\left(\mathcal{L}\mathcal{S}+\mathcal{S}\mathcal{L}^T\right)\right] \\ =\; &\begin{bmatrix} S_{xx}+\lambda_1\left(\frac{\partial S_{xx}}{\partial t}-2S_{xy}\frac{\partial u}{\partial y}\right) & S_{xy}+\lambda_1\left(\frac{\partial S_{xy}}{\partial t}-S_{yy}\frac{\partial u}{\partial y}\right) \\ S_{yx}+\lambda_1\left(\frac{\partial S_{yx}}{\partial t}-S_{yy}\frac{\partial u}{\partial y}\right) & S_{yy}+\lambda_1\left(\frac{\partial S_{yy}}{\partial t}\right) \end{bmatrix}. \end{aligned} \tag{1.7.19}$$

From Eqs. (1.7.6), (1.7.17) and (1.7.19) we arrive at

$$S_{xx}+\lambda_1\left(\frac{\partial S_{xx}}{\partial t}-2S_{xy}\frac{\partial u}{\partial y}\right)=-2\lambda_2\mu\left(\frac{\partial u}{\partial y}\right)^2, \tag{1.7.20}$$

$$S_{xy}+\lambda_1\left(\frac{\partial S_{xy}}{\partial t}-S_{yy}\frac{\partial u}{\partial y}\right)=\mu\frac{\partial u}{\partial y}+\mu\lambda_2\frac{\partial^2 u}{\partial t\partial y}, \tag{1.7.21}$$

$$S_{yy}+\lambda_1\left(\frac{\partial S_{yy}}{\partial t}\right)=0. \tag{1.7.22}$$

With the help of Eq. (1.7.18), Eq. (1.7.10) is identically satisfied and Eqs. (1.7.8) and (1.7.9) give

$$\rho\frac{\partial u}{\partial t}=-\frac{\partial p}{\partial x}+\frac{\partial S_{xy}}{\partial y} \tag{1.7.23}$$

$$-\frac{\partial p}{\partial y}+\frac{\partial S_{yy}}{\partial y}=0. \tag{1.7.24}$$

The solution of Eq. (1.7.22) is given by

$$S_{yy}=f(y)e^{-\frac{t}{\lambda_1}}, \tag{1.7.25}$$

where $f(y)$ is an arbitrary function. Let $f(y)=0$, implies that $S_{yy}=0$ [3] and from Eq. (1.7.24) we conclude that p is independent of y or we can write $p=p(x,t)$. Eqs. (1.7.22) and (1.7.23) will now become

$$S_{xy}+\lambda_1\frac{\partial S_{xy}}{\partial t}=\mu\frac{\partial u}{\partial y}+\mu\lambda_2\frac{\partial^2 u}{\partial t\partial y}, \tag{1.7.26}$$

$$\rho\frac{\partial u}{\partial t}=-\frac{\partial p}{\partial x}+\frac{\partial S_{xy}}{\partial y}. \tag{1.7.27}$$

Multiplying Eq. (1.7.27) by λ_1 and differentiating the resulting equation with respect to t we get

$$\lambda_1\rho\frac{\partial^2 u}{\partial t^2}=-\lambda_1\frac{\partial^2 p}{\partial t\partial x}+\lambda_1\frac{\partial^2 S_{xy}}{\partial t\partial y}. \tag{1.7.28}$$

Adding Eqs. (1.7.27) and (1.7.28) we write

$$\rho\left(1+\lambda_1\frac{\partial}{\partial t}\right)\frac{\partial u}{\partial t}=\left(1+\lambda_1\frac{\partial}{\partial t}\right)\frac{\partial S_{xy}}{\partial y}-\left(1+\lambda_1\frac{\partial}{\partial t}\right)\frac{\partial p}{\partial x}. \tag{1.7.29}$$

Differentiating Eq. (1.7.26) with respect to y we have

$$\left(1+\lambda_1\frac{\partial}{\partial t}\right)\frac{\partial S_{xy}}{\partial y}=\mu\left(1+\lambda_2\frac{\partial}{\partial t}\right)\frac{\partial^2 u}{\partial y^2}. \tag{1.7.30}$$

Substitution of Eq. (1.7.30) into (1.7.29) gives

$$\rho\left(1+\lambda_1\frac{\partial}{\partial t}\right)\frac{\partial u}{\partial t}=-\left(1+\lambda_1\frac{\partial}{\partial t}\right)\frac{\partial p}{\partial x}+\mu\left(1+\lambda_2\frac{\partial}{\partial t}\right)\frac{\partial^2 u}{\partial y^2}. \tag{1.7.31}$$

1.8 Energy Equation for the Viscous Fluid

According to the law of conservation of energy we have

$$\rho\frac{d\mathcal{E}}{dt}=\mathcal{T}\cdot\mathcal{L}-\nabla\cdot\mathbf{q}+\rho\mathbf{r}, \tag{1.8.1}$$

where ρ is the density, $\mathcal{E}$ is the specific internal energy, $\mathbf{q}$ is the heat flux vector, $\mathbf{r}$ is the radiant heating and $\mathcal{L}$ is the gradient of the velocity vector. In the absence of the radiant heating, Eq. (1.8.1) will take the form

$$\rho\frac{d\mathcal{E}}{dt}=\mathcal{T}\cdot\mathcal{L}-\nabla\cdot\mathbf{q}, \tag{1.8.2}$$

where $\mathcal{E}=C_pT$ and $\mathbf{q}=-\chi\nabla T$, where C_p is the specific heat constant pressure and T is the temperature.

The Cauchy stress tenser $\mathcal{T}$ for the viscous fluid is given by

$$\mathcal{T}=-pI+\mu\mathcal{A}_1. \tag{1.8.3}$$

Choosing the velocity field of the form $V=(u(y,t),0,0)$, we obtain

$$\mathcal{L}=\begin{bmatrix}0 & \frac{\partial u}{\partial y}\\ 0 & 0\end{bmatrix}, \tag{1.8.4}$$

$$\mathcal{A}_1 = \mathcal{L} + \mathcal{L}^T = \begin{bmatrix} 0 & \frac{\partial u}{\partial y} \\ \frac{\partial u}{\partial y} & 0 \end{bmatrix}, \tag{1.8.5}$$

$$\text{tr}\mathcal{A}_1^2 = 2\left(\frac{\partial u}{\partial y}\right)^2, \tag{1.8.6}$$

$$\begin{aligned} \mathcal{A}_1 \cdot \mathcal{L} &= \begin{bmatrix} 0 & \frac{\partial u}{\partial y} \\ \frac{\partial u}{\partial y} & 0 \end{bmatrix} \cdot \begin{bmatrix} 0 & \frac{\partial u}{\partial y} \\ 0 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 0 & 0 \\ 0 & \left(\frac{\partial u}{\partial y}\right)^2 \end{bmatrix}, \end{aligned} \tag{1.8.7}$$

$$\text{tr}\mathcal{A}_1 \cdot \mathcal{L} = \left(\frac{\partial u}{\partial y}\right)^2. \tag{1.8.8}$$

Now

$$\begin{aligned} \mathcal{T} \cdot \mathcal{L} &= -p\mathcal{L} + \mu\mathcal{A}_1 \cdot \mathcal{L}, \\ &= -\text{tr}(p\mathcal{L}) + \mu\text{tr}(\mathcal{A}_1 \cdot \mathcal{L}), \\ &= \mu\left(\frac{\partial u}{\partial y}\right)^2. \end{aligned} \tag{1.8.9}$$

$$\text{div}\mathcal{T} = \begin{bmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \end{bmatrix} \begin{bmatrix} -p & 0 \\ 0 & -p \end{bmatrix} + \mu \begin{bmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \end{bmatrix} \begin{bmatrix} 0 & \frac{\partial u}{\partial y} \\ \frac{\partial u}{\partial y} & 0 \end{bmatrix}. \tag{1.8.10}$$

Now we consider the energy equation (1.8.2) neglecting radiant heat as

$$\rho\frac{d\mathcal{E}}{dt} = \mathcal{T} \cdot \mathcal{L} - \nabla \cdot \mathbf{q},$$

where $T = T(y,t)$ and $\mathbf{q} = -\chi\nabla T$, where χ is the thermal conductivity and T is the temperature. Then

$$\nabla T = \begin{bmatrix} 0 & \frac{\partial T}{\partial y} \\ 0 & 0 \end{bmatrix}, \tag{1.8.12}$$

and

$$\text{div}\mathbf{q} = -\chi\text{div}\nabla T, \tag{1.8.13}$$

or

$$\begin{aligned}(\mathrm{div}\mathbf{q})_x &= -\chi\frac{\partial^2 T}{\partial y^2}, &(1.8.14)\\(\mathrm{div}\mathbf{q})_y &= 0. &(1.8.15)\end{aligned}$$

Now as

$$\frac{d}{dt}=\frac{\partial}{\partial t}+(\mathbf{V}\cdot\nabla) \qquad (1.8.16)$$

thus the energy Eq. (1.8.2) can be written as

$$\begin{aligned}C_p\frac{dT}{dt} &= C_p\frac{\partial T}{\partial t}+C_p(\mathbf{V}\cdot\nabla)T\\ &= \frac{\partial T}{\partial t}+u\frac{\partial T}{\partial x}. &(1.8.17)\end{aligned}$$

Since $T=T(y,t)$ therefore we have

$$\frac{dT}{dt}=\frac{\partial T}{\partial t}. \qquad (1.8.18)$$

Also

$$\rho\frac{d\mathcal{E}}{dt}=\rho C_p\frac{dT}{dt}=\rho C_p\frac{\partial T}{\partial t}. \qquad (1.8.19)$$

Now from Eqs. (1.8.9), (1.8.13) and (1.8.19) we obtain

$$\rho C_p\frac{\partial T}{\partial t}=\chi\frac{\partial^2 T}{\partial y^2}+\mu\left(\frac{\partial u}{\partial y}\right)^2. \qquad (1.8.20)$$

which is the energy equation for the viscous fluid.

1.9 Maxwell's Equations

The electric and magnetic field vectors **E** and **H** in the electromagnetic theory are the solutions of the fundamental set of equations known as Maxwell's equations. Each of these equations is based on an experimentally observed property of the electric and magnetic fields. We do not present the mathematical derivation of these equations but writing them in *mks* units as follows:

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t},$$

$$\nabla \times \mathbf{H} = \frac{\partial \mathbf{D}}{\partial t} + \mathbf{J},$$

$$\nabla \cdot \mathbf{B} = 0,$$

$$\nabla \cdot \mathbf{D} = \rho_1.$$

In addition there are the constitutive relations that express **B**, **D** and **J** in terms of **E** and **H**,

$$\mathbf{D} = \epsilon_1 \mathbf{E}$$

$$\mathbf{B} = \mu_1 \mathbf{H}$$

$$\mathbf{J} = \sigma \mathbf{E}$$

where **D**, **B** and **J** are the electric displacement, the magnetic induction and the current density respectively. Further ρ_1 is the density of electric charge, σ is the electrical conductivity, ϵ_1 is the dielectric constant and μ_1 is the magnetic permeability.

Chapter 2

Effect of Heat Transfer to the Pulsatile Flow of a Viscoelastic Fluid

2.1 Introduction

In this chapter, a problem of heat transfer to flow of an Oldroyd-B fluid bounded by infinite parallel plates is examined. Both the plates are at rest. The lower plate is at lower temperature than the upper one. The flow is created due to the applied pressure gradient. Expressions for velocity and rate of heat transfer at the plates are given. The presented analysis is due to Ghosh and Debnath [1].

2.2 Heat Transfer to Pulsatile Flow Problem

We consider the flow of an Oldroyd-B fluid between two infinitely long parallel plates, a distance h apart, which is driven by the unsteady pressure gradient in the form

$$-\frac{1}{\rho}\frac{\partial p}{\partial x} = A\left\{1 + \epsilon \exp(\iota\omega t)\right\} \tag{2.2.1}$$

where A is a known constant, ϵ is a suitably chosen positive quantity and ω is the frequency. We suppose that the motion is slow and hence all the second order quantities can be neglected.

In the rectangular Cartesian coordinate system, the $x-axis$ is taken along the lower plate at $y=0$ and the $y-axis$ is normal to this plate. The lower plate at $y=0$ and the upper plate at $y=h$ are maintained at constant temperatures T_0 and T_1 $(T_1 > T_0)$ respectively. The governing equation of motion combined with constitutive relations of the viscoelastic fluid is given by

$$\left(1+\lambda_1\frac{\partial}{\partial t}\right)\frac{\partial u}{\partial t} = -\frac{1}{\rho}\left(1+\lambda_1\frac{\partial}{\partial t}\right)\frac{\partial p}{\partial x} + \nu\left(1+\lambda_2\frac{\partial}{\partial t}\right)\frac{\partial^2 u}{\partial y^2}, \tag{2.2.2}$$

where $\nu = \mu/\rho$ and above equation is modelled in chapter 1. Here λ_1 is the relaxation time, λ_2 is the retardation time, ρ is the density, p is the pressure, ν is kinematic viscosity and u is the fluid velocity in the $x-$direction. The energy equation is

$$C_p\frac{\partial T}{\partial t} = \frac{\chi}{\rho}\frac{\partial^2 T}{\partial y^2} + \nu\left(\frac{\partial u}{\partial y}\right)^2 \tag{2.2.3}$$

where C_p and χ are the specific heat and thermal conductivity , respectively.

The boundary conditions are

$$u = 0, \;\; T = T_0 \;\text{ at }\; y = 0, \tag{2.2.4}$$

$$u = 0, \;\; T = T_1 \;\text{ at }\; y = h. \tag{2.2.5}$$

Introducing the non-dimensional parameters

$$u^* = \frac{u}{Ah^2/\nu}, \qquad \tau = \omega t, \qquad \eta = \frac{y}{h}, \tag{2.2.6}$$

the boundary value problem takes the following form

$$\frac{\omega A h^2}{\nu}\left(1+\lambda_1\omega\frac{\partial}{\partial\tau}\right)\frac{\partial u^*}{\partial\tau} = -\frac{1}{\rho}\left(1+\lambda_1\omega\frac{\partial}{\partial\tau}\right)\frac{\partial p}{\partial x} + A\left(1+\lambda_2\omega\frac{\partial}{\partial\tau}\right)\frac{\partial^2 u^*}{\partial\eta^2} \tag{2.2.7}$$

with the boundary conditions as follow

$$u^* = 0 \qquad \text{at} \qquad \eta = 0, \tag{2.2.8}$$

$$u^* = 0 \qquad \text{at} \qquad \eta = 1. \tag{2.2.9}$$

We assume the solution of the following form

$$u^* = u_0(\eta) + \epsilon u_1(\eta)\exp(\iota\tau) \tag{2.2.10}$$

Substituting Eq. (2.2.10) and the boundary conditions (2.2.8) and (2.2.9) we get the following systems for velocities as:

System of order zero

$$\frac{d^2u_0}{d\eta^2} + 1 = 0, \tag{2.2.11}$$

$$u_0 = 0, \qquad \eta = 0, \tag{2.2.12}$$

$$u_0 = 0, \qquad \eta = 1. \tag{2.2.13}$$

System of order one

$$\frac{d^2u_1}{d\eta^2} - \iota R^2\beta^2 u_1 = -\beta^2, \tag{2.2.14}$$

$$u_1 = 0, \qquad \eta = 0, \tag{2.2.15}$$

$$u_1 = 0, \qquad \eta = 1, \tag{2.2.16}$$

where

$$R^2 = \frac{\omega h^2}{\nu}, \qquad \beta^2 = \frac{1+\iota F_1}{1+\iota F_1 F_2}, \qquad F_1 = \lambda_1\omega \quad \text{and} \quad F_2 = \frac{\lambda_2}{\lambda_1}(<1). \tag{2.2.17}$$

The solution of Eq. (2.2.11) is given by

$$u_0 = -\frac{\eta^2}{2} + A\eta + B. \tag{2.2.18}$$

Applying the boundary conditions (2.2.12) and (2.2.13) we get $B = 0$ and $A = \frac{1}{2}$ and thus the above solution becomes

$$u_0(\eta) = \frac{1}{2}\eta\,(1 - \eta)\,. \tag{2.2.19}$$

The Eq. (2.2.14) can be written as

$$\frac{d^2 u_1}{d\eta^2} - M^2 u_1 = -\beta^2, \tag{2.2.20}$$

where $M = (1 + \iota)\, m$ with $m = \frac{1}{\sqrt{2}} R\beta$. The general solution of the non-homogeneous Eq. (2.2.20) is

$$u_1 = Ce^{M\eta} + De^{-M\eta} + \frac{1}{\iota R^2}, \tag{2.2.21}$$

where C and D are constants. Making use of the boundary condition (2.2.15) we get

$$D = -\left(C + \frac{1}{\iota R^2}\right). \tag{2.2.22}$$

Substituting Eq. (2.2.22) in Eq. (2.2.21) we obtain

$$u_1 = C\left(e^{M\eta} - e^{-M\eta}\right) + \frac{1}{\iota R^2}\left(1 - e^{-M\eta}\right). \tag{2.2.23}$$

Using the boundary condition (2.2.16) in Eq. (2.2.23) we obtain

$$C = -\frac{1}{\iota R^2}\left(1 - e^{-M}\right)\frac{1}{e^{M} - e^{-M}} \tag{2.2.24}$$

and thus Eq. (2.2.23) gives

$$u_1 = \frac{1}{\iota R^2}\left[\left(e^{-M} - 1\right)\frac{\sinh M\eta}{\sinh M} + \left(1 - e^{-M\eta}\right)\right], \tag{2.2.25}$$

or

$$u_1 = \frac{1}{\iota R^2}\left[1 - \left\{\frac{e^{-M\eta}\sinh M - e^{M}\sinh M\eta + \sinh M\eta}{\sinh M}\right\}\right]. \tag{2.2.26}$$

Since

$$\begin{aligned}
\sinh(1+\iota)m(1-\eta) &= \sinh(M - M\eta) \\
&= \sinh M \cosh M\eta - \cosh M \sinh M\eta \\
&= \sinh M \left(\frac{e^{M\eta} + e^{-M\eta}}{2}\right) - \left(\frac{e^{M} + e^{-M}}{2}\right) \sinh M\eta \\
&= \sinh M \left(\sinh M\eta + e^{-M\eta}\right) - \left(\sinh M + e^{-M}\right) \sinh M\eta \\
&= e^{-M\eta} \sinh M - e^{-M} \sinh M\eta, \qquad (2.2.27)
\end{aligned}$$

so Eq. (2.2.26) can be written as

$$u_1 = \frac{1}{\iota R^2}\left[1 - \frac{\sinh M(1-\eta) + \sinh M}{\sinh M}\right], \qquad (2.2.28)$$

or

$$u_1 = -\frac{\iota}{R^2}\left[1 - \frac{\sinh(1+\iota)m(1-\eta) + \sinh(1+\iota)m\eta}{\sinh(1+\iota)m}\right]. \qquad (2.2.29)$$

From Eq. (2.2.10) the expression for u^* is

$$\begin{aligned}
u^* &= u_0 + \epsilon u_1 e^{\iota\tau} \qquad (2.2.30) \\
&= \frac{\eta}{2}(1-\eta) - \frac{\epsilon\iota}{R^2}\left[1 - \frac{\sinh(1+\iota)m(1-\eta) + \sinh(1+\iota)m\eta}{\sinh(1+\iota)m}\right] e^{\iota\tau}.
\end{aligned}$$

It is noted that the results for viscous fluid can be recovered as a special case by taking $\lambda_2 = \lambda_1 = 0$.

Now we come to the energy equation, i.e., Eq. (2.2.3) as follows

$$C_p \frac{\partial T}{\partial t} = \frac{\chi}{\rho}\frac{\partial^2 T}{\partial y^2} + \nu\left(\frac{\partial u}{\partial y}\right)^2.$$

Introducing

$$\begin{aligned}
\eta &= \frac{y}{h}, \qquad m = \frac{\beta R}{\sqrt{2}}, \qquad R^2 = \frac{\omega h^2}{\nu}, \qquad \beta^2 = \frac{1+\iota F_1}{1+\iota F_1 F_2}, \\
\tau &= \omega t, \qquad u^* = \frac{u\nu}{Ah^2}, \qquad F_1 = \lambda_1\omega, \qquad F_2 = \frac{\lambda_2}{\lambda_1}(<1),
\end{aligned}$$

and the non-dimensional temperature θ

$$\theta = \frac{T - T_0}{T_1 - T_0} \tag{2.2.32}$$

Eq. (2.2.3) takes the following form

$$\frac{\omega h^2}{\nu}\frac{\partial \theta}{\partial \tau} = \frac{\chi}{C_p \mu}\frac{\partial^2 \theta}{\partial \eta^2} + \frac{A^2 h^4}{\nu^2 C_p (T_1 - T_0)}\left(\frac{\partial u^*}{\partial \eta}\right)^2, \tag{2.2.33}$$

or

$$R^2 \frac{\partial \theta}{\partial \tau} = \frac{1}{P_r}\frac{\partial^2 \theta}{\partial \eta^2} + E_c\left(\frac{\partial u^*}{\partial \eta}\right)^2, \tag{2.2.34}$$

where

$$P_r = \frac{\mu C_p}{\chi}, \qquad E_c = \frac{A^2 h^4}{\nu^2 C_p (T_1 - T_0)}.$$

In the above equation P_r is the *Prandtl* number and E_c is the *Eckert* number.

The boundary conditions (2.2.4) and (2.2.5) in terms of θ are

$$\begin{aligned} \theta &= 0 \qquad \text{at} \qquad \eta = 0, \\ \theta &= 1 \qquad \text{at} \qquad \eta = 1. \end{aligned} \tag{2.2.35}$$

Now the temperature θ can be assumed in the form

$$\theta(\eta, \tau) = \theta_0(\eta) + \epsilon F(\eta) e^{\iota\tau} + \epsilon^2 G(\eta) e^{2\iota\tau} \tag{2.2.36}$$

From Eq. (2.2.30) we have

$$\frac{\partial u^*}{\partial \eta} = \left(\frac{1}{2} - \eta\right) + \frac{\epsilon\iota}{R^2}\left[\frac{M \cosh M\eta - M \cosh M(1-\eta)}{\sinh M}\right] e^{\iota\tau}, \tag{2.2.37}$$

or

$$\left(\frac{\partial u^*}{\partial \eta}\right)^2 = \left(\frac{1}{4} + \eta^2 - \eta\right) - \frac{\epsilon^2 e^{2\iota\tau}}{R^4 \sinh^2 M} g(\eta) + \frac{2\epsilon\iota e^{\iota\tau}}{R^2} f(\eta). \tag{2.2.38}$$

where

$$
\begin{aligned}
f(\eta) &= \left(\frac{1}{2}-\eta\right)\left[\frac{M\cosh M\eta - M\cosh M\,(1-\eta)}{\sinh M}\right], && (2.2.39)\\
g(\eta) &= M^2\cosh^2 M\eta + M^2\cosh^2 M\,(1-\eta)\\
&\quad -2M^2\cosh M\eta\cosh M(1-\eta). && (2.2.40)
\end{aligned}
$$

Now substituting Eq. (2.2.37) into Eq. (2.2.33), we obtain

$$R^2\frac{\partial\theta}{\partial\tau} = \frac{1}{P_r}\frac{\partial^2\theta}{\partial\eta^2} + E_c\left[\left(\frac{1}{4}+\eta^2-\eta\right) - \frac{\epsilon^2 e^{2\iota\tau}}{R^4\sinh^2 M}g(\eta) + \frac{2\epsilon\iota e^{\iota\tau}}{R^2}f(\eta)\right]. \quad (2.2.41)$$

Now differentiating Eq. (2.2.36) twice with respect to η we get

$$\frac{\partial^2\theta}{\partial\eta^2} = \frac{d^2\theta_0}{d\eta^2} + \epsilon\frac{d^2F}{d\eta^2}e^{\iota\tau} + \epsilon^2\frac{d^2G}{d\eta^2}e^{2\iota\tau}. \quad (2.2.42)$$

Also

$$\frac{\partial\theta}{\partial\tau} = \iota\epsilon F(\eta)e^{\iota\tau} + 2\iota\epsilon^2 e^{2\iota\tau}. \quad (2.2.43)$$

Using Eqs. (2.2.42) and (2.2.43) in Eq. (2.2.41) and equating the coefficients of $\epsilon^0 e^{0\iota\tau}$, $\epsilon e^{\iota\tau}$ and $\epsilon^2 e^{2\iota\tau}$ respectively, we have

$$
\begin{aligned}
\frac{d^2\theta_0}{d\eta^2} &= -P_r E_c\left(\frac{1}{4}+\eta^2-\eta\right), && (2.2.44)\\
\iota R^2 F(\eta) &= \frac{1}{P_r}\frac{d^2F}{d\eta^2} + \frac{2\iota E_c}{R^2}f(\eta), && (2.2.45)\\
2\iota R^2 G(\eta) &= \frac{1}{P_r}\frac{d^2G}{d\eta^2} - \frac{E_c}{R^4\sinh^2 M}g(\eta), && (2.2.46)
\end{aligned}
$$

with the following boundary conditions

$$\theta_0 = 0 \quad \text{at} \quad \eta = 0, \quad \text{and} \quad \theta_0 = 1 \quad \text{at} \quad \eta = 1, \quad (2.2.47)$$

$$F = 0 \quad \text{at} \quad \eta = 0, \quad \text{and} \quad F = 0 \quad \text{at} \quad \eta = 1, \quad (2.2.48)$$

$$G = 0 \quad \text{at} \quad \eta = 0, \quad \text{and} \quad G = 0 \quad \text{at} \quad \eta = 1, \quad (2.2.49)$$

where the functions $f(\eta)$ and $g(\eta)$ are defined in Eqs. (2.2.39) and (2.2.40), respectively.

Now integrating Eq. (2.2.44) with respect to η we have

$$\theta_0 = -P_r E_c \frac{3\eta^2 - 4\eta^3 + 2\eta^4}{24} + C_1\eta + C_2, \tag{2.2.50}$$

where C_1 and C_2 are the constants of integration. Now applying boundary conditions (2.2.47), we get

$$C_2 = 0, \qquad C_1 = 1 + \frac{P_r E_c}{24} \tag{2.2.51}$$

and thus Eq. (2.2.50) becomes

$$\theta_0 = \eta \left(1 + \frac{P_r E_c}{24} \left(1 - 3\eta + 4\eta^2 - 2\eta^3\right) \right). \tag{2.2.52}$$

Eq. (2.2.45) can also be written as

$$\frac{d^2F(\eta)}{d\eta^2} - \imath R^2 P_r F(\eta) + \frac{2\imath P_r E_c}{R^2} f(\eta) = 0. \tag{2.2.53}$$

Now making use of $N = (1 + \imath)\, n$ and $n = \left(\frac{P_r}{2}\right)^{\frac{1}{2}} R$ in Eq. (2.2.53) we have

$$\frac{d^2F(\eta)}{d\eta^2} - N^2F(\eta) + \frac{2\imath P_r E_c}{R^2} f(\eta) = 0. \tag{2.2.54}$$

The above equation after using Eq. (2.2.39) takes the following form

$$\frac{d^2F(\eta)}{d\eta^2} - N^2F(\eta) = A_1\eta e^{M\eta} + A_2\eta e^{-M\eta} + A_3 e^{M\eta} + A_4 e^{-M\eta}, \tag{2.2.55}$$

where

$$A_1 = \frac{2\imath M P_r E_c}{R^2 \sinh M} \frac{1 - e^{-M}}{2}, \quad A_2 = \frac{2\imath M P_r E_c}{R^2 \sinh M} \frac{1 - e^{M}}{2},$$

$$A_3 = -\frac{A_1}{2} \quad \text{and} \quad A_4 = -\frac{A_2}{2}. \tag{2.2.56}$$

The solution of above equation is the sum of complementary function and the particular integral. The complementary function is given by

$$F_c = c_1 e^{N\eta} + c_2 e^{-N\eta}. \tag{2.2.57}$$

For particular integral, we use the method of undetermined coefficients. Let

$$F_p = \overline{\alpha}\eta e^{M\eta} + \overline{\beta}\eta e^{-M\eta} + \overline{\gamma} e^{M\eta} + \overline{\delta} e^{-M\eta}, \tag{2.2.58}$$

where $\overline{\alpha}$, $\overline{\beta}$, $\overline{\gamma}$ and $\overline{\delta}$ are to be determined. Differentiating Eq. (2.2.58) with respect to η we arrive at

$$\begin{aligned}\frac{d^2F_p}{d\eta^2} &= \overline{\alpha}\left(2Me^{M\eta} + M^2\eta e^{M\eta}\right) + \overline{\beta}\left(-2Me^{-M\eta} + M^2\eta e^{-M\eta}\right) \\ &\quad +\overline{\gamma}M^2 e^{M\eta} + \overline{\delta}M^2 e^{-M\eta},\end{aligned}$$

or

$$\begin{aligned}\frac{d^2F_p(\eta)}{d\eta^2} - N^2F_p(\eta) &= \left(2\overline{\alpha}M + \overline{\gamma}M^2 - N^2\overline{\gamma}\right)e^{M\eta} + \left(-2\overline{\beta}M + \overline{\delta}M^2 - N^2\overline{\delta}\right)e^{-M\eta} \\ &\quad + \left(\overline{\alpha}M^2 - N^2\overline{\alpha}\right)\eta e^{M\eta} + \left(M^2\overline{\beta} - N^2\overline{\beta}\right)\eta e^{-M\eta}. \end{aligned} \tag{2.2.59}$$

Comparing Eqs. (2.2.55) and (2.2.59) we get

$$\begin{aligned} A_1 &= \left(M^2 - N^2\right)\overline{\alpha}, \\ A_2 &= \left(M^2 - N^2\right)\overline{\beta}, \\ A_3 &= 2\overline{\alpha}M + \left(M^2 - N^2\right)\overline{\gamma}, \\ A_4 &= -2\overline{\beta}M + \left(M^2 - N^2\right)\overline{\delta}, \end{aligned} \tag{2.2.60}$$

or we can write

$$\begin{aligned}\overline{\alpha} &= \frac{A_1}{(M^2-N^2)},\\ \overline{\beta} &= \frac{A_2}{(M^2-N^2)},\\ \overline{\gamma} &= \frac{A_3}{(M^2-N^2)} - \frac{2MA_1}{(M^2-N^2)^2},\\ \overline{\delta} &= \frac{A_4}{(M^2-N^2)} + \frac{2MA_2}{(M^2-N^2)^2}.\end{aligned} \tag{2.2.61}$$

Using Eq. (2.2.56) we can write $\overline{\alpha}$, $\overline{\beta}$, $\overline{\gamma}$ and $\overline{\delta}$ as

$$\overline{\alpha} = \frac{\iota M P_r E_c}{R^2 \sinh M}\left(\frac{1-e^{-M}}{M^2-N^2}\right), \tag{2.2.62}$$

$$\overline{\beta} = \frac{\iota M P_r E_c}{R^2 \sinh M}\left(\frac{1-e^{M}}{M^2-N^2}\right), \tag{2.2.63}$$

$$\overline{\gamma} = -\frac{\iota M P_r E_c}{R^2 \sinh M}\left(\frac{1-e^{-M}}{M^2-N^2}\right)\left(\frac{1}{2}+\frac{2M}{M^2-N^2}\right), \tag{2.2.64}$$

$$\overline{\delta} = -\frac{\iota M P_r E_c}{R^2 \sinh M}\left(\frac{1-e^{M}}{M^2-N^2}\right)\left(\frac{1}{2}-\frac{2M}{M^2-N^2}\right). \tag{2.2.65}$$

The complete solution of Eq. (2.2.54) is of the following form

$$\begin{aligned}F(\eta) &= F_c(\eta) + F_p(\eta),\\ &= c_1 e^{N\eta} + c_2 e^{-N\eta} + \overline{\alpha}\eta e^{M\eta} + \overline{\beta}\eta e^{-M\eta} + \overline{\gamma} e^{M\eta} + \overline{\delta} e^{-M\eta}.\end{aligned} \tag{2.2.66}$$

Using the boundary conditions (2.2.48) in Eq. (2.2.66) we get

$$\begin{aligned}0 &= c_1 + c_2 + \overline{\gamma} + \overline{\delta},\\ 0 &= c_1 e^{N} + c_2 e^{-N} + \overline{\alpha} e^{M} + \overline{\beta} e^{-M} + \overline{\gamma} e^{M} + \overline{\delta} e^{-M}.\end{aligned}$$

Solving above equations we have

$$\begin{bmatrix} 1 & 1 \\ e^{N} & e^{-N} \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} -F_p(0) \\ -F_p(1) \end{bmatrix},$$

or

$$\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ e^N & e^{-N} \end{bmatrix}^{-1} \begin{bmatrix} -F_p(0) \\ -F_p(1) \end{bmatrix}.$$

Because

$$\begin{bmatrix} 1 & 1 \\ e^N & e^{-N} \end{bmatrix}^{-1} = \frac{-1}{2\sinh N} \begin{bmatrix} e^{-N} & -1 \\ -e^N & 1 \end{bmatrix},$$

therefore

$$\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \frac{1}{2\sinh N} \begin{bmatrix} e^{-N}F_p(0) - F_p(1) \\ -e^N F_p(0) + F_p(1) \end{bmatrix},$$

or we can write

$$c_1 = \frac{1}{2\sinh N}\left[e^{-N}F_p(0) - F_p(1)\right], \tag{2.2.71}$$

$$c_2 = \frac{1}{2\sinh N}\left[-e^{N}F_p(0) + F_p(1)\right]. \tag{2.2.72}$$

Now from Eqs. (2.2.62) to (2.2.65) we obtain

$$\overline{\gamma} = -\overline{\alpha}\left(\frac{1}{2} + \frac{2M}{M^2 - N^2}\right), \tag{2.2.73}$$

$$\overline{\delta} = -\overline{\beta}\left(\frac{1}{2} - \frac{2M}{M^2 - N^2}\right), \tag{2.2.74}$$

$$\overline{\alpha} + \overline{\beta} = -\frac{2\iota M P_r E_c}{R^2 \sinh M}\left(\frac{\cosh M - 1}{M^2 - N^2}\right), \tag{2.2.75}$$

$$\overline{\alpha} - \overline{\beta} = \frac{2\iota M P_r E_c}{R^2\left(M^2 - N^2\right)}, \tag{2.2.76}$$

and

$$\overline{\alpha}e^M = -\overline{\beta}, \qquad \overline{\beta}e^{-M} = -\overline{\alpha}. \tag{2.2.77}$$

From above equations we can write

$$\overline{\gamma}e^M + \overline{\delta}e^{-M} = \frac{\overline{\alpha} + \overline{\beta}}{2} - \frac{2M}{M^2 - N^2}\left(\overline{\alpha} - \overline{\beta}\right). \tag{2.2.78}$$

From Eqs. (2.2.58) and (2.2.77) we have

$$\begin{aligned} F_p(1) &= \overline{\alpha}e^{M} + \overline{\beta}e^{-M} + \overline{\gamma}e^{M} + \overline{\delta}e^{-M}, \\ &= -\frac{\overline{\alpha}+\overline{\beta}}{2} - \frac{2M}{M^2-N^2}\left(\overline{\alpha}-\overline{\beta}\right), \\ &= \overline{\gamma}+\overline{\delta} = F_p(0). \end{aligned}$$

Let us define

$$F_p(0) = F_p(1) = L(0). \tag{2.2.79}$$

Now from Eqs. (2.2.75), (2.2.76) and (2.2.79) we have

$$L(0) = -\frac{\iota M P_r E_c}{R^2(M^2-N^2)\sinh M}\left(1 - \cosh M + \frac{4M}{M^2-N^2}\sinh M\right) \tag{2.2.80}$$

and thus from Eqs. (2.2.79) and (2.2.80) we can write

$$c_1 = \frac{1}{2\sinh N}\left(e^{-N}-1\right)L(0), \tag{2.2.81}$$

$$c_2 = \frac{1}{2\sinh N}\left(-e^{N}+1\right)L(0). \tag{2.2.82}$$

Now rewriting Eq. (2.2.66) as

$$F(\eta) = c_1 e^{N\eta} + c_2 e^{-N\eta} + L(\eta), \tag{2.2.83}$$

where

$$L(\eta) = \overline{\alpha}\eta e^{M\eta} + \overline{\beta}\eta e^{-M\eta} + \overline{\gamma}e^{M\eta} + \overline{\delta}e^{-M\eta}. \tag{2.2.84}$$

Now from Eqs. (2.2.81) and (2.2.82) we get

$$\begin{aligned} c_1 e^{N\eta} + c_2 e^{-N\eta} &= \frac{1}{2\sinh N}\left[(e^{-N}-1)e^{N\eta} + (1-e^{N})e^{-N\eta}\right]L(0), \\ &= \frac{-L(0)}{\sinh N}\left[\sinh N\eta + \sinh N(1-\eta)\right]. \end{aligned} \tag{2.2.85}$$

and thus from Eq. (2.2.83) we can write

$$F(\eta) = \frac{-L(0)}{\sinh N}\left[\sinh N\eta + \sinh N(1-\eta)\right] + L(\eta). \qquad (2.2.86)$$

With the help of Eqs. (2.2.73), (2.2.74) and (2.2.84) we have

$$\begin{aligned} L(\eta) &= \overline{\alpha}\eta e^{M\eta} + \overline{\beta}\eta e^{-M\eta} - \overline{\alpha}\left(\frac{1}{2} + \frac{2M}{M^2-N^2}\right)e^{M\eta} - \overline{\beta}\left(\frac{1}{2} - \frac{2M}{M^2-N^2}\right)e^{-M\eta}, \\ &= \left(\eta - \frac{1}{2}\right)\left(\overline{\alpha}e^{M\eta} + \overline{\beta}e^{-M\eta}\right) - \frac{2M}{M^2-N^2}\left(\overline{\alpha}e^{M\eta} - \overline{\beta}e^{-M\eta}\right). \qquad (2.2.87) \end{aligned}$$

Also from Eqs. (2.2.62) and (2.2.63) we obtain

$$\begin{aligned} \overline{\alpha}e^{M\eta} + \overline{\beta}e^{-M\eta} &= \frac{\iota M P_r E_c}{R^2 \sinh M}\frac{1}{M^2-N^2}\Big(\left(1-e^{-M}\right)e^{M\eta} + \left(1-e^{M}\right)e^{-M\eta}\Big), \\ &= \frac{2\iota M P_r E_c}{R^2 \sinh M}\left(\frac{\cosh M\eta - \cosh M(1-\eta)}{M^2-N^2}\right). \end{aligned}$$

Similarly

$$\overline{\alpha}e^{M\eta} - \overline{\beta}e^{-M\eta} = \frac{2\iota M P_r E_c}{R^2 \sinh M}\left(\frac{\sinh M\eta + \sinh M(1-\eta)}{M^2-N^2}\right).$$

Using above values in Eq. (2.2.87) we obtain

$$\begin{aligned} L(\eta) &= \frac{2\iota M P_r E_c}{R^2\left(M^2-N^2\right)\sinh M}\Bigg[\left(\eta - \frac{1}{2}\right)\left\{\cosh M\eta - \cosh M(1-\eta)\right\} \\ &\qquad - \frac{2M}{M^2-N^2}\left\{\sinh M\eta + \sinh M(1-\eta)\right\}\Bigg]. \qquad (2.2.89) \end{aligned}$$

Now from Eq. (2.2.46) we have

$$\frac{d^2G(\eta)}{d\eta^2} - 2\iota P_r R^2 G(\eta) = \frac{P_r E_c}{R^4 \sinh^2 M} g(\eta) \qquad (2.2.90)$$

where $g(\eta)$ is defined in Eq. (2.2.40) and boundary conditions are given in Eq.

(2.2.49). Eq. (2.2.40) can be written as

$$\begin{aligned}
g(\eta) &= M^2\left(\cosh^2 M\eta + \cosh^2 M(1-\eta) - 2\cosh M\eta \cosh M(1-\eta)\right) \\
&= \frac{M^2}{4}\Big(e^{2M\eta} + e^{-2M\eta} + 4 + e^{2M(1-\eta)} + e^{-2M(1-\eta)}, \\
&\qquad -2\left(e^{M\eta} + e^{-M\eta}\right)\left(e^{M(1-\eta)} + e^{-M(1-\eta)}\right)\Big), \\
&= \frac{M^2}{4}\Big[\left(1-e^{-M}\right)^2 e^{2M\eta} + \left(1-e^{M}\right)^2 e^{-2M\eta} + 4\left(1-\cosh M\right)\Big]. \quad (2.2.91)
\end{aligned}$$

Substituting Eq. (2.2.91) in Eq. (2.2.90) we obtain

$$\begin{aligned}
\frac{d^2G(\eta)}{d\eta^2} - 2\iota P_r R^2 G(\eta) &= \frac{P_r E_c M^2}{4R^4 \sinh^2 M}\Big[\left(1-e^{-M}\right)^2 e^{2M\eta} \\
&\qquad + \left(1-e^{M}\right)^2 e^{-2M\eta} + 4\left(1-\cosh M\right)\Big],
\end{aligned}$$

or

$$\frac{d^2G(\eta)}{d\eta^2} - 2\iota P_r R^2 G(\eta) = \widetilde{A}_1 e^{2M\eta} + \widetilde{A}_2 e^{-2M\eta} + \widetilde{A}_3, \quad (2.2.92)$$

where

$$\begin{aligned}
&\widetilde{A}_0 = \frac{P_r E_c M^2}{4R^4 \sinh^2 M}, \quad \widetilde{A}_1 = \widetilde{A}_0\left(1-e^{-M}\right)^2, \\
&\widetilde{A}_2 = \widetilde{A}_0\left(1-e^{M}\right)^2, \widetilde{A}_3 = 4\widetilde{A}_0\left(1-\cosh M\right). \quad (2.2.93)
\end{aligned}$$

We note that Eq. (2.2.92) is non-homogeneous second order ordinary differential equation. The complete solution is the sum of complementary function and particular integral. The complementary function is given by

$$G_c = a_1 e^{\sqrt{2\iota P_r} R\eta} + a_2 e^{-\sqrt{2\iota P_r} R\eta}, \quad (2.2.94)$$

where a_1 and a_2 are arbitrary constants. For particular integral, we use the method of undetermined coefficients. For that we write

$$G_p(\eta) = \widehat{\alpha} e^{2M\eta} + \widehat{\beta} e^{-2M\eta} + \widehat{\gamma}. \quad (2.2.95)$$

Differentiating the above equation twice with respect to η we get

$$\frac{d^2G_p}{d\eta^2} = 4M^2\widehat{\alpha}e^{2M\eta} + 4M^2\widehat{\beta}e^{-2M\eta}. \tag{2.2.96}$$

Substituting Eqs. (2.2.95) and (2.2.96) into Eq. (2.2.92) we get

$$\left(4M^2 - 2\iota P_rR^2\right)\left[\widehat{\alpha}e^{2M\eta} + \widehat{\beta}e^{-2M\eta}\right] - 2\iota P_rR^2\widehat{\gamma} = \widetilde{A}_1e^{2M\eta} + \widetilde{A}_2e^{-2M\eta} + \widetilde{A}_3. \tag{2.2.97}$$

Comparing the coefficients of $e^{0\eta}$, $e^{2M\eta}$, and $e^{-2M\eta}$ we obtain

$$\widehat{\alpha} = \frac{\widetilde{A}_1}{(4M^2 - 2\iota P_rR^2)}, \tag{2.2.98}$$

$$\widehat{\beta} = \frac{\widetilde{A}_2}{(4M^2 - 2\iota P_rR^2)}, \tag{2.2.99}$$

$$\widehat{\gamma} = -\frac{\widetilde{A}_3}{2\iota P_rR^2}. \tag{2.2.100}$$

With the help of Eq. (2.2.93) we can write $\widehat{\alpha}$, $\widehat{\beta}$, and $\widehat{\gamma}$ as

$$\widehat{\alpha} = \left(\frac{P_rE_cM^2}{4R^4\sinh^2 M}\right)\frac{\left(1 - e^{-M}\right)^2}{(4M^2 - 2\iota P_rR^2)}, \tag{2.2.101}$$

$$\widehat{\beta} = \left(\frac{P_rE_cM^2}{4R^4\sinh^2 M}\right)\frac{\left(1 - e^{M}\right)^2}{(4M^2 - 2\iota P_rR^2)}, \tag{2.2.102}$$

$$\widehat{\gamma} = -\left(\frac{P_rE_cM^2}{4R^4\sinh^2 M}\right)\frac{4\,(1 - \cosh M)}{2\iota P_rR^2}. \tag{2.2.103}$$

The complete solution of Eq. (2.2.90) can thus be written as

$$\begin{aligned} G(\eta) &= G_c(\eta) + G_p(\eta), \\ &= a_1e^{\sqrt{2\iota P_r}R\eta} + a_2e^{-\sqrt{2\iota P_r}R\eta} + \widehat{\alpha}e^{2M\eta} + \widehat{\beta}e^{-2M\eta} + \widehat{\gamma}. \end{aligned} \tag{2.2.104}$$

To find a_1 and a_2 we apply boundary conditions (2.2.49) in Eq. (2.2.104) and get

$$\begin{aligned} 0 &= a_1 + a_2 + \widehat{\alpha} + \widehat{\beta} + \widehat{\gamma}, \\ 0 &= a_1e^{\sqrt{2\iota P_r}R} + a_2e^{-\sqrt{2\iota P_r}R} + \widehat{\alpha}e^{2M\eta} + \widehat{\beta}e^{-2M\eta} + \widehat{\gamma}. \end{aligned}$$

Since $N = (1+\imath)n$ and $n = R\left(\frac{P_r}{2}\right)^{\frac{1}{2}}$, we can write $\sqrt{2\imath P_r}R = \sqrt{2}N$ and thus from above equations we have

$$\begin{bmatrix} 1 & 1 \\ e^{\sqrt{2}N} & e^{-\sqrt{2}N} \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} -G_p(0) \\ -G_p(1) \end{bmatrix},$$

or

$$a_1 = \frac{1}{2\sinh\sqrt{2}N}\left[e^{-\sqrt{2}N}G_p(0) - G_p(1)\right], \tag{2.2.106}$$

$$a_2 = \frac{1}{2\sinh\sqrt{2}N}\left[-e^{\sqrt{2}N}G_p(0) + G_p(1)\right]. \tag{2.2.107}$$

Now from Eqs. (2.2.101) and (2.2.102) we have

$$\widehat{\beta}e^{-2M} = \widehat{\alpha}. \tag{2.2.108}$$

Making use of Eq. (2.2.108) in Eq. (2.2.95) we have

$$G_p(0) = G_p(1) = -Q(0) \qquad \text{(say)}, \tag{2.2.109}$$

and thus from Eqs. (2.2.101), (2.2.102) and (2.2.103) we have

$$\begin{aligned} Q(0) &= -\left(\widehat{\alpha} + \widehat{\beta} + \widehat{\gamma}\right), \\ &= -\left(\frac{P_r E_c M^2}{4R^4 \sinh^2 M}\right)\left[\frac{\left(1-e^{-M}\right)^2 + \left(1-e^{-M}\right)^2}{(4M^2 - 2\imath P_r R^2)} - \frac{4\,(1-\cosh M)}{2\imath P_r R^2}\right], \end{aligned}$$

or

$$Q(0) = \frac{M^2 P_r E_c}{2R^4 \sinh^2 M}\left[\frac{1-\cosh M}{N^2} - \frac{1+\cosh 2M - 2\cosh M}{2\,(2M^2 - N^2)}\right]. \tag{2.2.110}$$

From Eqs. (2.2.107), (2.2.108) and (2.2.110) we obtain

$$a_1 = \frac{Q(0)}{2\sinh\sqrt{2}N}\left(1 - e^{-\sqrt{2}N}\right),$$

$$a_2 = \frac{Q(0)}{2\sinh\sqrt{2}N}\left(e^{\sqrt{2}N} - 1\right).$$

Using above equations we get

$$a_1e^{\sqrt{2}N\eta} + a_2e^{-\sqrt{2}N\eta} = \frac{Q(0)}{2\sinh\sqrt{2}N}\left[\left(1-e^{-\sqrt{2}N}\right)e^{\sqrt{2}N\eta} + \left(e^{\sqrt{2}N}-1\right)e^{-\sqrt{2}N\eta}\right],$$
$$= \frac{Q(0)}{\sinh\sqrt{2}N}\left[\sinh M\eta + \sinh M(1-\eta)\right] \qquad (2.2.113)$$

and so from Eq. (2.2.104) we can write

$$G(\eta) = \frac{Q(0)}{\sinh\sqrt{2}N}\left[\sinh\sqrt{2}N\eta + \sinh\sqrt{2}N(1-\eta)\right] - Q(\eta), \qquad (2.2.114)$$

where

$$Q(\eta) = -G_p(\eta) = -\left(\widehat{\alpha}e^{2M\eta} + \widehat{\beta}e^{-2M\eta} + \widehat{\gamma}\right). \qquad (2.2.115)$$

Now from Eqs. (2.2.101), (2.2.102) and (2.2.103) we obtain

$$Q(\eta) = -\left(\frac{P_rE_cM^2}{4R^4\sinh^2 M}\right)\left[\frac{\left(1-e^{-M}\right)^2e^{2M\eta} + \left(1-e^{M}\right)^2e^{-2M\eta}}{(4M^2-2\iota P_rR^2)} - \frac{4\left(1-\cosh M\right)}{2\iota P_rR^2}\right],$$
$$= \frac{P_rE_cM^2}{2R^4\sinh^2 M}\left[\frac{(1-\cosh M)}{N^2} - \frac{\left(1-e^{-M}\right)^2+\left(1-e^{M}\right)^2}{4\left(2M^2-N^2\right)}\cosh 2M\eta\right.$$
$$\left. + \frac{\left(1-e^{M}\right)^2-\left(1-e^{-M}\right)^2}{4\left(2M^2-N^2\right)}\sinh 2M\eta\right]. \qquad (2.2.116)$$

As

$$\left(1-e^{-M}\right)^2 + \left(1-e^{M}\right)^2 = 2\left(1+\cosh 2M - 2\cosh M\right),$$

or

$$\left(1-e^{M}\right)^2 - \left(1-e^{-M}\right)^2 = 2\left(\sinh 2M - 2\sinh M\right),$$

Equation (2.2.116) becomes

$$Q(\eta) = \frac{M^2P_rE_c}{2R^4\sinh^2 M}\left[\frac{(1-\cosh M)}{N^2} - \frac{1+\cosh 2M - 2\cosh M}{2\left(2M^2-N^2\right)}\cosh 2M\eta\right.$$
$$\left. + \frac{\sinh 2M - 2\sinh M}{2\left(2M^2-N^2\right)}\sinh 2M\eta\right] \qquad (2.2.119)$$

2.3 Rate of Heat Transfer

The rate of heat transfer per unit area at the plate $\eta = 0$ and $\eta = 1$ are respectively given by

$$\left(\frac{\partial\theta}{\partial\eta}\right)_{\eta=0} = \left(\frac{d\theta_0}{d\eta}\right)_{\eta=0} + \epsilon e^{\iota\omega t}\left(\frac{dF}{d\eta}\right)_{\eta=0} + \epsilon^2 e^{2\iota\omega t}\left(\frac{dG}{d\eta}\right)_{\eta=0}, \tag{2.3.1}$$

and

$$\left(\frac{\partial\theta}{\partial\eta}\right)_{\eta=1} = \left(\frac{d\theta_0}{d\eta}\right)_{\eta=1} + \epsilon e^{\iota\omega t}\left(\frac{dF}{d\eta}\right)_{\eta=1} + \epsilon^2 e^{2\iota\omega t}\left(\frac{dG}{d\eta}\right)_{\eta=1}. \tag{2.3.2}$$

Differentiating Eqs. (2.2.52), (2.2.86) ,(2.2.89), (2.2.114) and (2.2.119) with respect to η we have

$$\frac{d\theta_0}{d\eta} = \left(1 + \frac{P_r E_c}{24}\left(1 - 6\eta + 12\eta^2 - 8\eta^3\right)\right), \tag{2.3.3}$$

$$\frac{dF}{d\eta} = \frac{-NL(0)}{\sinh N}\left[\cosh N\eta - \cosh N(1-\eta)\right] + \frac{dL}{d\eta}, \tag{2.3.4}$$

$$\frac{dG}{d\eta} = \frac{\sqrt{2}NQ(0)}{\sinh\sqrt{2}N}\left(\cosh\sqrt{2}N\eta - \cosh\sqrt{2}N(1-\eta)\right) - \frac{dQ}{d\eta}, \tag{2.3.5}$$

$$\frac{dL}{d\eta} = \frac{2\iota M P_r E_c}{R^2\left(M^2 - N^2\right)\sinh M}\left[\left(1 - \frac{2M^2}{M^2 - N^2}\right)\left\{\cosh M\eta - \cosh M(1-\eta)\right\}\right.$$
$$\left. + \left(\eta - \frac{1}{2}\right)\left\{M\sinh M\eta - M\sinh M(1-\eta)\right\}\right], \tag{2.3.6}$$

$$\frac{dQ}{d\eta} = \frac{M^2 P_r E_c}{2R^4\sinh^2 M}\left[2M\frac{\sinh 2M - 2\sinh M}{2\left(2M^2 - N^2\right)}\cosh 2M\eta\right.$$
$$\left. - 2M\frac{1 + \cosh 2M - 2\cosh M}{2\left(2M^2 - N^2\right)}\sinh 2M\eta\right]. \tag{2.3.7}$$

Using above equations in Eq. (2.3.1) we get

$$\left(\frac{\partial\theta}{\partial\eta}\right)_{\eta=0} = 1 + \frac{P_r E_c}{24} - \epsilon e^{\iota\omega t}\left\{\frac{2\iota M P_r E_c}{R^2\left(M^2 - N^2\right)\sinh M}\left[\frac{M^2 + N^2}{M^2 - N^2}\left(1 - \cosh M\right)\right.\right.$$
$$\left.\left. + \frac{M}{2}\sinh M\right] + \frac{NL(0)}{\sinh N}\left(1 - \cosh N\right)\right\}$$

$$+\epsilon^2 e^{2\iota\omega t}\left[-\frac{M^3 P_r E_c}{2R^4 \sinh^2 M\,(2M^2-N^2)}\Big(\sinh 2M - 2\sinh M\Big)\right.$$
$$\left.+\frac{\sqrt{2}NQ(0)}{\sinh\sqrt{2}N}\Big(1-\cosh\sqrt{2}N\Big)\right]. \qquad (2.3.8)$$

Similarly the rate of heat transfer per unit area at the plate $\eta = 1$ is given by

$$\left(\frac{\partial\theta}{\partial\eta}\right)_{\eta=1} = 1 - \frac{P_r E_c}{24} + \epsilon e^{\iota\omega t}\left[\frac{NL(0)}{\sinh N}(1-\cosh N)\right.$$
$$\left.-\frac{2\iota M P_r E_c}{R^2(M^2-N^2)\sinh M}\left\{\frac{M^2+N^2}{M^2-N^2}(1-\cosh M) - \frac{M}{2}\sinh M\right\}\right]$$

$$+\epsilon^2 e^{2\iota\omega t}\left[\sqrt{2}NQ(0) - \frac{M^3 P_r E_c}{R^4\sinh^2 M}\times\right.$$
$$\left.\left\{\frac{-\sinh 2M - 2\sinh M\cosh 2M + 2\cosh M\sinh 2M}{2(2M^2-N^2)}\right\}\right]. \qquad (2.3.9)$$

2.4 Numerical Results and Discussions

For the problem under investigation θ_0 represents the steady temperature in the fluid containing one linear term corresponding to the fluid at rest and added to it a biquadratic term which arises due to viscous friction. The expression for θ_0 given by (2.2.52) remains the same for both a viscous and a visco-elastic fluid of Oldroyed type under similar conditions. The temperature profiles corresponding to θ_0 are shown in Figure-2.1 for various values of $P_r E_c$.

Regarding the rate of heat transfer in the steady-state condition the reversal of heat flux from the fluid to the hotter plate take place when $P_r E_c > 24$ which, in term, makes the hotter plate more hot. In fact, the value of $P_r E_c$ provides a measure of the amount of heat generated due to friction which, in the present case, increases with the

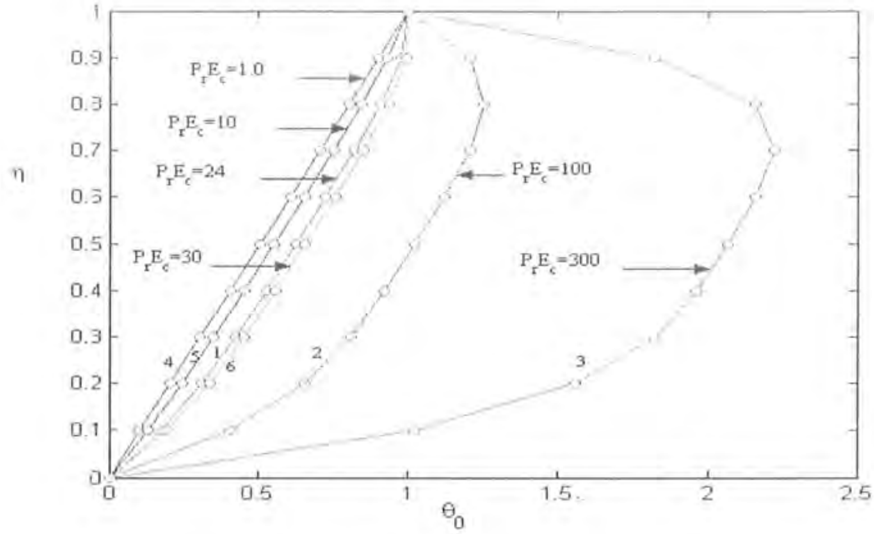

Figure 2.1: Steady temperature profiles.

increase of the pressure gradient. As a result, if the temperature difference between the plates is fixed, heat flows from the hotter plate to the fluid as long as the pressure gradient does not exceed a certain value i.e., for $P_rE_c \not> 24$. This phenomenon is important for cooling at high pressure gradients.

Fixing P_r to 100,R to 1.0 and E_c to 1.0 or 3.0, the instantaneous temperature profiles are plotted in Figure (2.2)-(2.5) to find the effect of changing values of the elastic parameters F_1 and F_2. Figure (2.2) represents the instantaneous temperature profiles for a viscous fluid where F_2 is taken as unity and $E_c = 1.0$. It has already been pointed out that the results for $F_2 = 1$ always represent the case of a viscous fluid irrespective of the values of F_1.The non-steady temperature profiles for $E_c = 1$, $F_1 = 0.1$ $F_2 = 0.01$ are shown in Figure (2.3). Comparison of Figures (2.2) and (2.3) shows that the presence of the elasticity of the fluid increases the temperature in a region near the plate and diminishes the same at the central part of the channel.

Further, from Figure (2.4) we see that the temperature in a viscoelastic fluid increases rapidly with the increase of E_c which in similar to that in a viscous fluid. Figure (2.5) shows that the increase of temperature near the plate occurs mainly due to the increase of relaxation time of the fluid while the increase in retardation time of the fluid produces a slight decrease of temperature at the central part of the channel.

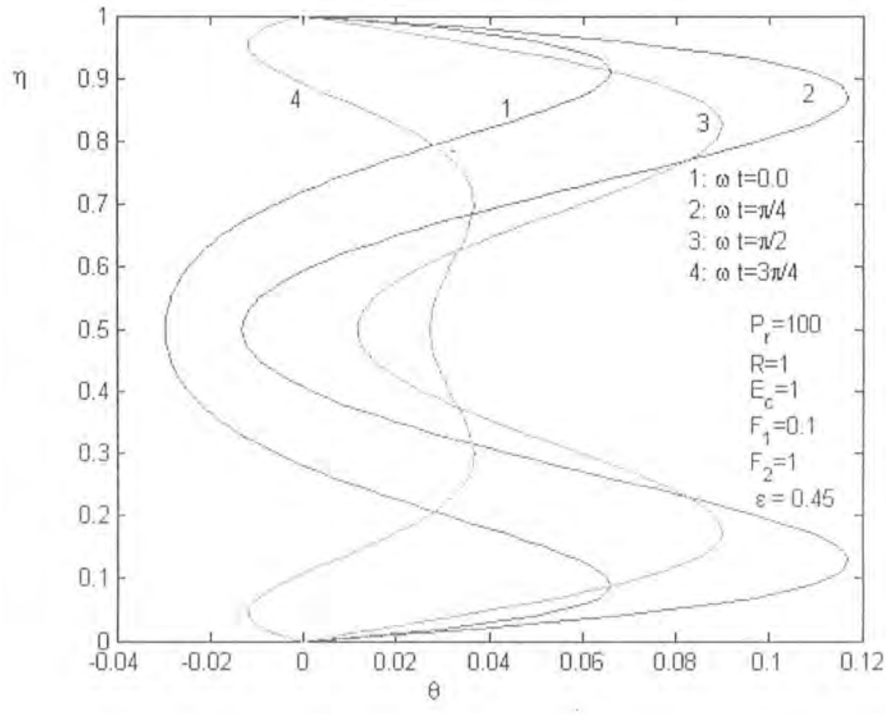

Figure 2.2: Unsteady temperature profiles in viscous fluid.

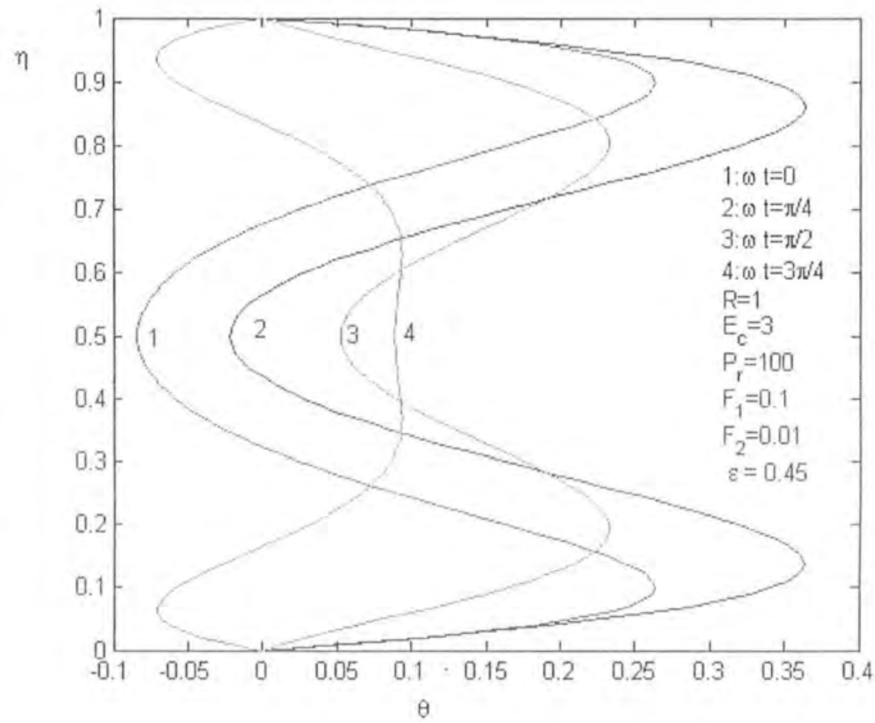

Figure 2.3: Unsteady temperature profiles in viscoelastic fluid.

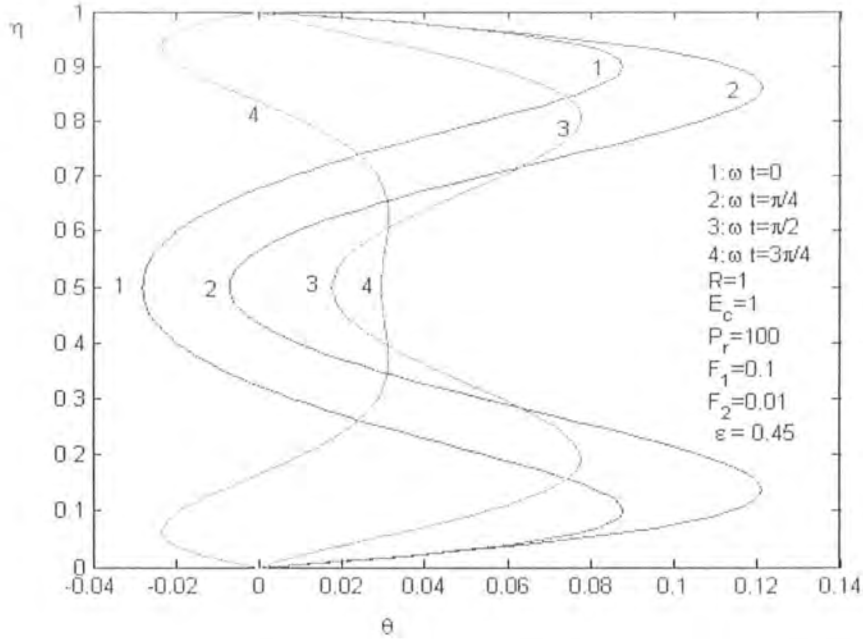

Figure 2.4: Unsteady temperature profiles in viscoelastic fluid.

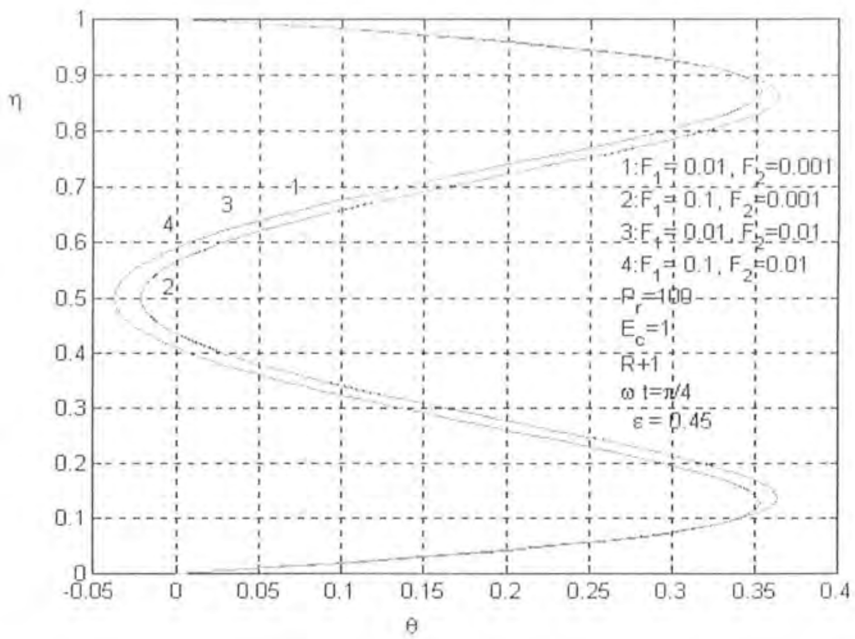

Figure 2.5: Unsteady temperature profiles in viscoelastic fluid.

Chapter 3

Effect of Heat Transfer on the Viscoelastic Flow Due to Unsteady Pressure Gradient

3.1 Introduction

An attempt is made in this chapter to investigate the effect of heat transfer on flow of an Oldroyd-$\mathcal{B}$ between the two rigid plates. The flow is generated by applying an unsteady pressure gradient when both the plates are at rest. The upper plate has the higher temperature than the lower one. The resulting partial differential equations have been solved analytically and the results for steady and unsteady velocity profiles are constructed. Finally the rate of heat transfer at the plates are given.

3.2 Formulation of the Problem

Let us consider the flow of fluid between two infinitely parallel plates. The plates are at a distance h apart and the fluid is taken an Oldroyd-$\mathcal{B}$ fluid. The flow is due to unsteady pressure gradient which is of the following form

$$-\frac{1}{\rho}\frac{\partial p}{\partial x} = A\left\{1+\epsilon\exp(\alpha t)\right\}. \tag{3.2.1}$$

In above equation A is a known constant, ϵ is a suitably chosen positive quantity and $\alpha = (-\delta + \iota\omega,\ \delta > 0)$ is a complex.

We consider a rectangular Cartesian coordinate system with x-axis along the lower plate and y-axis normal to it. The lower and upper plates are at $y = 0$ and $y = h$ and have constant temperatures T_0 and T_1, respectively. It is assumed that $T_1 > T_0$. The governing equation is of the following form

$$\frac{\partial u}{\partial t} + \lambda_1\frac{\partial^2 u}{\partial t^2} = -\frac{1}{\rho}\left(1+\lambda_1\frac{\partial}{\partial t}\right)\frac{\partial p}{\partial x} + \nu\left(1+\lambda_2\frac{\partial}{\partial t}\right)\frac{\partial^2 u}{\partial y^2}, \tag{3.2.2}$$

where λ_1 and λ_2 are the relaxation and retardation times, ρ is the density, p is the pressure, ν is the kinematic viscosity and u is the velocity in the x-direction. The energy equation is given by

$$\rho\, C_p\frac{\partial T}{\partial t} - \chi\frac{\partial^2 T}{\partial y^2} = \mu\left(\frac{\partial u}{\partial y}\right)^2. \tag{3.2.3}$$

In above equation C_p , χ and μ are the specific heat, thermal conductivity and coefficient of dynamic viscosity, respectively.

For the problem under consideration, the boundary conditions are

$$u = 0,\ \ T = T_0 \ \text{ at } \ y = 0$$

$$u = 0,\ \ T = T_1 \ \text{ at } \ y = h$$

Defining

$$u^{**} = \frac{u}{Ah^2/\nu}, \qquad \tau_1 = -\iota\alpha t, \qquad \eta = \frac{y}{h}, \tag{3.2.6}$$

the boundary value problem takes the following form

$$\frac{-\iota\alpha\Lambda h^2}{\nu}\left(1-\iota\lambda_1\alpha\frac{\partial}{\partial\tau_1}\right)\frac{\partial u^{**}}{\partial\tau_1}=-\frac{1}{\rho}\left(1-\iota\lambda_1\alpha\frac{\partial}{\partial\tau_1}\right)\frac{\partial p}{\partial x}+A\left(1-\iota\lambda_2\alpha\frac{\partial}{\partial\tau_1}\right)\frac{\partial^2 u^{**}}{\partial\eta^2} \tag{3.2.7}$$

with the boundary conditions as follow

$$u^{**}=0 \qquad \text{at} \qquad \eta=0, \tag{3.2.8}$$

$$u^{**}=0 \qquad \text{at} \qquad \eta=1. \tag{3.2.9}$$

We take the solution of the form as

$$u^{**}=u_0^{**}(\eta)+\epsilon u_1^{**}(\eta)\exp\left(\iota\tau_1\right). \tag{3.2.10}$$

Making use of Eq. (3.2.10) into Eqs. (3.2.7) to (3.2.9) we have

$$\frac{d^2u_0^{**}}{d\eta^2}+1=0, \tag{3.2.11}$$

$$u_0^{**}=0, \qquad \eta=0,$$

$$u_0^{**}=0, \qquad \eta=1. \tag{3.2.12}$$

System of order one

$$\frac{d^2u_1^{**}}{d\eta^2}-\iota R_1^2\beta_1^2u_1^{**}=-\beta_1^2, \tag{3.2.13}$$

$$u_1^{**}=0, \qquad \eta=0,$$

$$u_1^{**}=0, \qquad \eta=1. \tag{3.2.14}$$

In above systems

$$R_1^2=-\iota\frac{\alpha h^2}{\nu}, \qquad \beta_1^2=\frac{1+\widetilde{F}_1}{1+\widetilde{F}_1\widetilde{F}_2}, \qquad \widetilde{F}_1=\lambda_1\alpha, \qquad \widetilde{F}_2=\frac{\lambda_2}{\lambda_1}(<1). \tag{3.2.15}$$

Following the same method of solution as in chapter 2, the solutions of the systems are given by

$$u_0^{**}(\eta) = \frac{1}{2}\eta(1-\eta), \tag{3.2.16}$$

$$u_1^{**} = \frac{1}{\iota R_1^2}\left[1 - \left\{\frac{e^{-M_1\eta}\sinh M_1 - e^{M_1}\sinh M_1\eta + \sinh M_1\eta}{\sinh M_1}\right\}\right], \tag{3.2.17}$$

where $M_1 = (1+\iota)\,m_1$ with $m_1 = \frac{1}{\sqrt{2}}R_1\beta_1$. We can write

$$\begin{aligned}
\sinh(1+\iota)\,m_1(1-\eta) &= \sinh(M_1 - M_1\eta) \\
&= \sinh M_1 \cosh M_1\eta - \cosh M_1 \sinh M_1\eta \\
&= \sinh M_1\left(\frac{e^{M_1\eta} + e^{-M_1\eta}}{2}\right) - \left(\frac{e^{M_1} + e^{-M_1}}{2}\right)\sinh M_1\eta \\
&= \sinh M_1\left(\sinh M_1\eta + e^{-M_1\eta}\right) - \left(\sinh M_1 + e^{-M_1}\right)\sinh M_1\eta \\
&= e^{-M_1\eta}\sinh M_1 - e^{-M_1}\sinh M_1\eta.
\end{aligned} \tag{3.2.18}$$

After using Eq. (3.2.18) in Eq. (3.2.17) we have

$$u_1^{**} = \frac{1}{\iota R_1^2}\left[1 - \frac{\sinh M_1(1-\eta) + \sinh M_1}{\sinh M_1}\right], \tag{3.2.19}$$

or

$$u_1^{**} = -\frac{\iota}{R_1^2}\left[1 - \frac{\sinh(1+\iota)\,m_1(1-\eta) + \sinh(1+\iota)\,m_1\eta}{\sinh(1+\iota)\,m_1}\right]. \tag{3.2.20}$$

From Eqs. (3.2.16) and (3.2.20) we obtain the expression for u^{**} as

$$\begin{aligned}
u^{**} &= u_0^{**} + \epsilon u_1^{**} e^{\iota\tau_1} \qquad (3.2.21)\\
&= \frac{\eta}{2}(1-\eta) - \frac{\epsilon\iota}{R_1^2}\left[1 - \frac{\sinh(1+\iota)\,m_1(1-\eta) + \sinh(1+\iota)\,m_1\eta}{\sinh(1+\iota)\,m_1}\right]e^{\iota\tau_1}.
\end{aligned}$$

Now let us consider the energy equation (3.2.3). Now defining

$$\eta = \frac{y}{h}, \qquad m_1 = \frac{\beta_1 R_1}{\sqrt{2}}, \qquad R_1^2 = -\iota\frac{\alpha h^2}{\nu}, \qquad \beta_1^2 = \frac{1+\widetilde{F}_1}{1+\widetilde{F}_1\widetilde{F}_2}, \tau_1 = -\iota\alpha t,$$
$$u^{**} = \frac{u\nu}{Ah^2}, \qquad \widetilde{F}_1 = \lambda_1\alpha, \qquad \widetilde{F}_2 = \frac{\lambda_2}{\lambda_1}(<1), \quad \overline{\theta} = \frac{T-T_0}{T_1-T_0},$$

Eq. (3.2.3) becomes

$$R_1^2 \frac{\partial \overline{\theta}}{\partial \tau_1} = \frac{1}{P_r} \frac{\partial^2 \overline{\theta}}{\partial \eta^2} + E_c \left(\frac{\partial u^{**}}{\partial \eta} \right)^2, \tag{3.2.22}$$

where the Prandle number P_r and Eckert number E_c are given by

$$P_r = \frac{\mu C_p}{\chi}, \qquad E_c = \frac{A^2 h^4}{\nu^2 C_p (T_1 - T_0)},$$

and $\overline{\theta}$ is the non-dimensional temperature .

From Eqs. (2.2.4) and (2.2.5) the boundary conditions in terms of $\overline{\theta}$ are given by

$$\overline{\theta} = 0 \qquad \text{at} \qquad \eta = 0,$$

$$\overline{\theta} = 1 \qquad \text{at} \qquad \eta = 1. \tag{3.2.23}$$

Assume $\overline{\theta}$ to be of the following form

$$\overline{\theta}(\eta, \tau_1) = \overline{\theta}_0(\eta) + \epsilon \overline{F}(\eta) e^{\iota \tau_1} + \epsilon^2 \overline{G}(\eta) e^{2\iota \tau_1}. \tag{3.2.24}$$

From Eqs. (3.2.21) and (3.2.24) we have

$$\left(\frac{\partial u^{**}}{\partial \eta} \right)^2 = \left(\frac{1}{4} + \eta^2 - \eta \right) - \frac{\epsilon^2 e^{2\iota \tau_1}}{R_1^4 \sinh^2 M_1} \overline{g}(\eta) + \frac{2\epsilon \iota e^{\iota \tau_1}}{R_1^2} \overline{f}(\eta), \tag{3.2.25}$$

$$\frac{\partial^2 \overline{\theta}}{\partial \eta^2} = \frac{d^2 \overline{\theta}_0}{d\eta^2} + \epsilon \frac{d^2 \overline{F}}{d\eta^2} e^{\iota \tau_1} + \epsilon^2 \frac{d^2 \overline{G}}{d\eta^2} e^{2\iota \tau_1}, \tag{3.2.26}$$

$$\frac{\partial \overline{\theta}}{\partial \tau_1} = \iota \epsilon \overline{F}(\eta) e^{\iota \tau_1} + 2\iota \epsilon^2 e^{2\iota \tau_1}, \tag{3.2.27}$$

where

$$\overline{f}(\eta) = \left(\frac{1}{2} - \eta \right) \left[\frac{M_1 \cosh M_1 \eta - M_1 \cosh M_1 (1 - \eta)}{\sinh M_1} \right], \tag{3.2.28}$$

$$\overline{g}(\eta) = M_1^2 \cosh^2 M_1 \eta + M_1^2 \cosh^2 M_1 (1 - \eta) - 2M_1^2 \cosh M_1 \eta \cosh M_1 (1 - \eta). \tag{3.2.29}$$

Using Eqs. (3.2.25) to (3.2.27) into Eq. (3.2.22) and equating the coefficients of $\epsilon^0 e^{0\iota\tau_1}$, $\epsilon e^{\iota\tau_1}$ and $\epsilon^2 e^{2\iota\tau_1}$, we arrive at

$$\frac{d^2\overline{\theta}_0}{d\eta^2} = -P_r E_c\left(\frac{1}{4}+\eta^2-\eta\right), \tag{3.2.30}$$

$$\iota R_1^2\overline{F}(\eta) = \frac{1}{P_r}\frac{d^2\overline{F}}{d\eta^2}+\frac{2\iota E_c}{R_1^2}\overline{f}(\eta), \tag{3.2.31}$$

$$2\iota R_1^2\overline{G}(\eta) = \frac{1}{P_r}\frac{d^2\overline{G}}{d\eta^2}-\frac{E_c}{R_1^4\sinh^2 M_1}\overline{g}(\eta). \tag{3.2.32}$$

From Eqs. (3.2.23) and (3.2.24) we have

$$\overline{\theta}_0 = 0 \quad \text{at} \quad \eta = 0, \quad \text{and} \quad \overline{\theta}_0 = 1 \quad \text{at} \quad \eta = 1, \tag{3.2.33}$$

$$\overline{F} = 0 \quad \text{at} \quad \eta = 0, \quad \text{and} \quad \overline{F} = 0 \quad \text{at} \quad \eta = 1, \tag{3.2.34}$$

$$\overline{G} = 0 \quad \text{at} \quad \eta = 0, \quad \text{and} \quad \overline{G} = 0 \quad \text{at} \quad \eta = 1, \tag{3.2.35}$$

The solution for $\overline{\theta}_0$ is of the following form

$$\overline{\theta}_0 = \eta\left(1+\frac{P_r E_c}{24}\left(1-3\eta+4\eta^2-2\eta^3\right)\right). \tag{3.2.36}$$

From Eq. (3.2.31)

$$\frac{d^2\overline{F}(\eta)}{d\eta^2}-\iota R_1^2 P_r\overline{F}(\eta)+\frac{2\iota P_r E_c}{R_1^2}\overline{f}(\eta) = 0. \tag{3.2.37}$$

Using $N_1 = (1+\iota)\,n$ and $n_1 = \left(\frac{P_r}{2}\right)^{\frac{1}{2}} R_1$ and Eq. (3.2.31) in above equation we get

$$\frac{d^2\overline{F}(\eta)}{d\eta^2}-N_1^2\overline{F}(\eta) = B_1\eta e^{M_1\eta}+B_2\eta e^{-M_1\eta}+B_3 e^{M_1\eta}+B_4 e^{-M_1\eta}, \tag{3.2.38}$$

where

$$B_1 = \frac{2\iota M_1 P_r E_c}{R_1^2\sinh M_1}\frac{1-e^{-M_1}}{2}, \quad B_2 = \frac{2\iota M_1 P_r E_c}{R_1^2\sinh M_1}\frac{1-e^{M_1}}{2}, \quad B_3 = -\frac{B_1}{2}, \quad B_4 = -\frac{B_2}{2}. \tag{3.2.39}$$

We note that Eq. (3.2.38) is non-homogeneous ordinary differential equation and the solution is the sum of complementary function and the particular integral. The complementary function is given by

$$\overline{F}_c = \overline{c}_1 e^{N_1\eta} + \overline{c}_2 e^{-N_1\eta}, \tag{3.2.40}$$

where $\overline{c}_1$ and $\overline{c}_2$ are constants. Using the method of undetermined coefficients the particular integral is given by

$$\overline{F}_p = \overline{\alpha}_1 \eta e^{M_1\eta} + \overline{\beta}_1 \eta e^{-M_1\eta} + \overline{\gamma}_1 e^{M_1\eta} + \overline{\delta}_1 e^{-M_1\eta}, \tag{3.2.41}$$

where

$$\overline{\alpha}_1 = \frac{\iota M_1 P_r E_c}{R_1^2 \sinh M_1}\left(\frac{1-e^{-M_1}}{M_1^2-N_1^2}\right), \tag{3.2.42}$$

$$\overline{\beta}_1 = \frac{\iota M_1 P_r E_c}{R_1^2 \sinh M_1}\left(\frac{1-e^{M_1}}{M_1^2-N_1^2}\right), \tag{3.2.43}$$

$$\overline{\gamma}_1 = -\frac{\iota M_1 P_r E_c}{R_1^2 \sinh M_1}\left(\frac{1-e^{-M_1}}{M_1^2-N_1^2}\right)\left(\frac{1}{2}+\frac{2M_1}{M_1^2-N_1^2}\right), \tag{3.2.44}$$

$$\overline{\delta}_1 = -\frac{\iota M_1 P_r E_c}{R_1^2 \sinh M_1}\left(\frac{1-e^{M_1}}{M_1^2-N_1^2}\right)\left(\frac{1}{2}-\frac{2M_1}{M_1^2-N_1^2}\right). \tag{3.2.45}$$

The complete solution of Eq. (3.2.38) is thus given by

$$\begin{aligned} \overline{F}(\eta) &= \overline{F}_c(\eta) + \overline{F}_p(\eta), \\ &= \overline{c}_1 e^{N_1\eta} + \overline{c}_2 e^{-N_1\eta} + \overline{\alpha}_1 \eta e^{M_1\eta} + \overline{\beta}_1 \eta e^{-M_1\eta} + \overline{\gamma}_1 e^{M_1\eta} + \overline{\delta}_1 e^{-M_1\eta}. \end{aligned} \tag{3.2.46}$$

Using the boundary conditions (3.2.34) in above equation we get

$$0 = \overline{c}_1 + \overline{c}_2 + \overline{\beta}_1 + \overline{\delta}_1, \tag{3.2.47}$$

$$0 = \overline{c}_1 e_1^N + \overline{c}_2 e^{-N_1} + \overline{\alpha}_1 e^{M_1} + \overline{\beta}_1 e^{-M_1} + \overline{\gamma}_1 e^{M_1} + \overline{\delta}_1 e^{-M_1}. \tag{3.2.48}$$

Solving Eqs. (3.2.47) and (3.2.48) we have

$$\begin{bmatrix} 1 & 1 \\ e_1^N & e^{-N_1} \end{bmatrix} \begin{bmatrix} \overline{c}_1 \\ \overline{c}_2 \end{bmatrix} = \begin{bmatrix} -\overline{F}_p(0) \\ -\overline{F}_p(1) \end{bmatrix}, \tag{3.2.49}$$

or

$$\begin{bmatrix} \overline{c}_1 \\ \overline{c}_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ e_1^N & e^{-N_1} \end{bmatrix}^{-1} \begin{bmatrix} -\overline{F}_p(0) \\ -\overline{F}_p(1) \end{bmatrix}. \tag{3.2.50}$$

Since

$$\begin{bmatrix} 1 & 1 \\ e_1^N & e^{-N_1} \end{bmatrix}^{-1} = \frac{-1}{2\sinh N_1} \begin{bmatrix} e^{-N_1} & -1 \\ -e_1^N & 1 \end{bmatrix}$$

so Eq. (3.2.50) gives

$$\begin{bmatrix} \overline{c}_1 \\ \overline{c}_2 \end{bmatrix} = \frac{1}{2\sinh N_1} \begin{bmatrix} e^{-N_1}\overline{F}_p(0) - \overline{F}_p(1) \\ -e_1^N\overline{F}_p(0) + \overline{F}_p(1) \end{bmatrix},$$

or

$$\overline{c}_1 = \frac{1}{2\sinh N_1} \left(e^{-N_1}\overline{F}_p(0) - \overline{F}_p(1)\right), \tag{3.2.53}$$

$$\overline{c}_2 = \frac{1}{2\sinh N_1} \left(-e^{N_1}\overline{F}_p(0) + \overline{F}_p(1)\right). \tag{3.2.54}$$

From Eqs. (3.2.42) to (3.2.45) we obtain

$$\overline{\gamma}_1 = -\overline{\alpha}_1 \left(\frac{1}{2} + \frac{2M_1}{M_1^2 - N_1^2}\right), \tag{3.2.55}$$

$$\overline{\delta}_1 = -\overline{\beta}_1 \left(\frac{1}{2} - \frac{2M_1}{M_1^2 - N_1^2}\right), \tag{3.2.56}$$

$$\overline{\alpha}_1 + \overline{\beta}_1 = -\frac{2\iota M_1 P_r E_c}{R_1^2 \sinh M_1} \left(\frac{\cosh M_1 - 1}{M_1^2 - N_1^2}\right), \tag{3.2.57}$$

$$\overline{\alpha}_1 - \overline{\beta}_1 = \frac{2\iota M_1 P_r E_c}{R_1^2\,(M_1^2 - N_1^2)}, \tag{3.2.58}$$

where

$$\overline{\beta}_1 e^{-M_1} = -\overline{\alpha}_1. \tag{3.2.59}$$

From Eqs. (3.2.55) and (3.2.56) we can write

$$\overline{\gamma}_1 e_1^{M} + \overline{\delta}_1 e^{-M_1} = \frac{\overline{\alpha}_1 + \overline{\beta}_1}{2} - \frac{2M_1}{M_1^2 - N_1^2}\left(\overline{\alpha}_1 - \overline{\beta}_1\right). \tag{3.2.60}$$

With the help of Eqs. (3.2.41), (3.2.59) and (3.2.60) we have

$$\begin{aligned} \overline{F}_p(1) &= \overline{\alpha}_1 e^{M_1} + \overline{\beta}_1 e^{-M_1} + \overline{\gamma}_1 e^{M_1} + \overline{\delta}_1 e^{-M_1}, \\ &= -\frac{\overline{\alpha}_1 + \overline{\beta}_1}{2} - \frac{2M_1}{M_1^2 - N_1^2}\left(\overline{\alpha}_1 - \overline{\beta}_1\right), \\ &= \overline{\gamma}_1 + \overline{\delta}_1 = \overline{F}_p(0). \end{aligned}$$

Let us define

$$\overline{F}_p(0) = \overline{F}_p(1) = \overline{L}(0). \tag{3.2.61}$$

Now from Eqs. (3.2.57), (3.2.58) and (3.2.61) we have

$$\overline{L}(0) = -\frac{\iota M_1 P_r E_c}{R_1^2(M_1^2 - N_1^2)\sinh M_1}\left(1 - \cosh M_1 + \frac{4M_1}{M_1^2 - N_1^2}\sinh M_1\right), \tag{3.2.62}$$

and thus from Eqs. (3.2.61) and (3.2.62) we get

$$\overline{c}_1 = \frac{1}{2\sinh N_1}\left(e^{-N_1} - 1\right)\overline{L}(0), \tag{3.2.63}$$

$$\overline{c}_2 = \frac{1}{2\sinh N_1}\left(-e^{N_1} + 1\right)\overline{L}(0). \tag{3.2.64}$$

Now rewriting Eq. (3.2.46) as

$$\overline{F}(\eta) = \overline{c}_1 e^{N_1\eta} + \overline{c}_2 e^{-N_1\eta} + \overline{L}(\eta), \tag{3.2.65}$$

where

$$\overline{L}(\eta) = \overline{\alpha}_1 \eta e^{M_1\eta} + \overline{\beta}_1 \eta e^{-M_1\eta} + \overline{\gamma}_1 e^{M_1\eta} + \overline{\delta}_1 e^{-M_1\eta}. \tag{3.2.66}$$

From Eqs. (3.2.63) and (3.2.64) we have

$$\begin{aligned} \overline{c}_1 e^{N_1\eta} + \overline{c}_2 e^{-N_1\eta} &= \frac{1}{2\sinh N_1}\left[(e^{-N_1} - 1)e^{N_1\eta} + (1 - e^{N_1})e^{-N_1\eta}\right]\overline{L}(0), \\ &= \frac{-\overline{L}(0)}{\sinh N_1}\left[\sinh N_1\eta + \sinh N_1(1-\eta)\right], \end{aligned} \tag{3.2.67}$$

and thus from Eq. (3.2.65) we have

$$\overline{F}(\eta) = \frac{-\overline{L}(0)}{\sinh N_1}\left[\sinh N_1\eta + \sinh N_1(1-\eta)\right] + \overline{L}(\eta). \tag{3.2.68}$$

With the help of Eqs. (3.2.55), (3.2.56) and (3.2.66) we obtain

$$\begin{aligned}\overline{L}(\eta) &= \overline{\alpha}_1\eta e^{M_1\eta} + \overline{\beta}_1\eta e^{-M_1\eta} - \overline{\alpha}_1\left(\frac{1}{2} + \frac{2M_1}{M_1^2 - N_1^2}\right)e^{M_1\eta} - \overline{\beta}_1\left(\frac{1}{2} - \frac{2M_1}{M_1^2 - N_1^2}\right)e^{-M_1\eta},\\ &= \left(\eta - \frac{1}{2}\right)\left(\overline{\alpha}_1 e^{M_1\eta} + \overline{\beta}_1 e^{-M_1\eta}\right) - \frac{2M_1}{M_1^2 - N_1^2}\left(\overline{\alpha}_1 e^{M_1\eta} - \overline{\beta}_1 e^{-M_1\eta}\right). \end{aligned} \tag{3.2.69}$$

From Eqs. (3.2.42) and (3.2.43) we have

$$\begin{aligned}\overline{\alpha}_1 e^{M_1\eta} + \overline{\beta}_1 e^{-M_1\eta} &= \frac{\iota M_1 P_r E_c}{R_1^2\sinh M_1}\frac{1}{M_1^2 - N_1^2}\Big(\left(1 - e^{-M_1}\right)e^{M_1\eta} + \left(1 - e^{M_1}\right)e^{-M_1\eta}\Big),\\ &= \frac{2\iota M_1 P_r E_c}{R_1^2\sinh M_1}\left(\frac{\cosh M_1\eta - \cosh M_1(1-\eta)}{M_1^2 - N_1^2}\right), \end{aligned} \tag{3.2.70}$$

and

$$\overline{\alpha}_1 e^{M_1\eta} - \overline{\beta}_1 e^{-M_1\eta} = \frac{2\iota M_1 P_r E_c}{R_1^2\sinh M_1}\left(\frac{\sinh M_1\eta + \sinh M_1(1-\eta)}{M_1^2 - N_1^2}\right). \tag{3.2.71}$$

Using above equations, Eq. (3.2.69) becomes

$$\begin{aligned}\overline{L}(\eta) = \frac{2\iota M_1 P_r E_c}{R_1^2\left(M_1^2 - N_1^2\right)\sinh M_1}\Bigg[&\left(\eta - \frac{1}{2}\right)\left\{\cosh M_1\eta - \cosh M_1(1-\eta)\right\}\\ &-\frac{2M_1}{M_1^2 - N_1^2}\left\{\sinh M_1\eta + \sinh M_1(1-\eta)\right\}\Bigg]. \end{aligned} \tag{3.2.72}$$

From Eq. (3.2.32) we get

$$\frac{d^2\overline{G}(\eta)}{d\eta^2} - 2\iota P_r R_1^2\overline{G}(\eta) = \frac{P_r E_c}{R_1^4\sinh^2 M_1}\overline{g}(\eta). \tag{3.2.73}$$

Making use of Eq. (3.2.29) and following the same procedure as in chapter 2 we finally write $\overline{G}(\eta)$ in the following form

$$\overline{G}(\eta) = \frac{\overline{Q}(0)}{\sinh\sqrt{2}N_1}\left(\sinh\sqrt{2}N_1\eta + \sinh\sqrt{2}N_1(1-\eta)\right) - \overline{Q}(\eta), \tag{3.2.74}$$

where

$$\overline{Q}(\eta) = \frac{M_1^2 P_r E_c}{2R_1^4 \sinh^2 M_1}\left[\frac{(1-\cosh M_1)}{N_1^2} - \frac{1+\cosh 2M_1 - 2\cosh M_1}{2\left(2M_1^2 - N_1^2\right)}\cosh 2M_1\eta \right.$$
$$\left. + \frac{\sinh 2M_1 - 2\sinh M_1}{2\left(2M_1^2 - N_1^2\right)}\sinh 2M_1\eta\right]. \quad (3.2.75)$$

3.3 Rate of Heat Transfer

The rate of heat transfer per unit area at the plate are given by

$$\left(\frac{\partial\overline{\theta}}{\partial\eta}\right)_{\eta=0} = \left(\frac{d\overline{\theta}_0}{d\eta}\right)_{\eta=0} + \epsilon e^{\iota\omega t}\left(\frac{d\overline{F}}{d\eta}\right)_{\eta=0} + \epsilon^2 e^{2\iota\omega t}\left(\frac{d\overline{G}}{d\eta}\right)_{\eta=0}, \quad (3.3.1)$$

$$\left(\frac{\partial\overline{\theta}}{\partial\eta}\right)_{\eta=1} = \left(\frac{d\overline{\theta}_0}{d\eta}\right)_{\eta=1} + \epsilon e^{\iota\omega t}\left(\frac{d\overline{F}}{d\eta}\right)_{\eta=1} + \epsilon^2 e^{2\iota\omega t}\left(\frac{d\overline{G}}{d\eta}\right)_{\eta=1}. \quad (3.3.2)$$

Differentiating Eqs. (3.2.36), (3.2.68), (3.2.72), (3.2.74) and (3.2.75) we obtain

$$\frac{d\overline{\theta}_0}{d\eta} = \left(1 + \frac{P_r E_c}{24}\left(1 - 6\eta + 12\eta^2 - 8\eta^3\right)\right), \quad (3.3.3)$$

$$\frac{d\overline{F}}{d\eta} = \frac{-N_1\overline{L}(0)}{\sinh N_1}\left[\cosh N_1\eta - \cosh N_1(1-\eta)\right] + \frac{d\overline{L}}{d\eta}, \quad (3.3.4)$$

$$\frac{d\overline{G}}{d\eta} = \frac{\sqrt{2}N_1\overline{Q}(0)}{\sinh\sqrt{2}N_1}\left(\cosh\sqrt{2}N_1\eta - \cosh\sqrt{2}N_1(1-\eta)\right) - \frac{d\overline{Q}}{d\eta}, \quad (3.3.5)$$

$$\frac{d\overline{L}}{d\eta} = \frac{2\iota M_1 P_r E_c}{R_1^2\left(M_1^2 - N_1^2\right)\sinh M_1}\left[\left(1 - \frac{2M_1^2}{M_1^2 - N_1^2}\right)\left\{\cosh M_1\eta - \cosh M_1(1-\eta)\right\}\right.$$
$$\left. + \left(\eta - \frac{1}{2}\right)\left\{M_1\sinh M_1\eta - M_1\sinh M_1(1-\eta)\right\}\right] \quad (3.3.6)$$

$$\frac{d\overline{Q}}{d\eta} = \frac{M_1^2 P_r E_c}{2R_1^4\sinh^2 M_1}\left[2M_1\frac{\sinh 2M_1 - 2\sinh M_1}{2\left(2M_1^2 - N_1^2\right)}\cosh 2M_1\eta\right.$$
$$\left. - 2M_1\frac{1+\cosh 2M_1 - 2\cosh M_1}{2\left(2M_1^2 - N_1^2\right)}\sinh 2M_1\eta\right]. \quad (3.3.7)$$

Using above equations in Eq. (3.3.1) we get

$$\left(\frac{\partial\overline{\theta}}{\partial\eta}\right)_{\eta=0} = 1+\frac{P_rE_c}{24}-\epsilon e^{\iota\tau_1}\left\{\frac{2\iota M_1P_rE_c}{R_1^2\,(M_1^2-N_1^2)\sinh M_1}\left[\frac{M_1^2+N_1^2}{M_1^2-N_1^2}\Big(1-\cosh M_1\Big)\right.\right.$$
$$\left.\left.+\frac{M_1}{2}\sinh M_1\right]+\frac{N_1\overline{L}(0)}{\sinh N_1}\left(1-\cosh N_1\right)\right\}$$
$$+\epsilon^2e^{2\iota\tau_1}\left[-\frac{M_1^3P_rE_c}{2R_1^4\sinh^2 M_1\,(2M_1^2-N_1^2)}\Big(\sinh 2M_1-2\sinh M_1\Big)\right.$$
$$\left.+\frac{\sqrt{2}N_1\overline{Q}(0)}{\sinh\sqrt{2}N_1}\Big(1-\cosh\sqrt{2}N_1\Big)\right] \qquad (3.3.8)$$

Similarly the rate of heat transfer per unit area at the plate $\eta=1$ is given by

$$\left(\frac{\partial\overline{\theta}}{\partial\eta}\right)_{\eta=1} = 1-\frac{P_rE_c}{24}+\epsilon e^{\iota\tau_1}\left[\frac{N_1\overline{L}(0)}{\sinh N_1}\left(1-\cosh N_1\right)\right.$$
$$\left.-\frac{2\iota M_1P_rE_c}{R_1^2\,(M_1^2-N_1^2)\sinh M_1}\left\{\frac{M_1^2+N_1^2}{M_1^2-N_1^2}\left(1-\cosh M_1\right)-\frac{M_1}{2}\sinh M_1\right\}\right]$$
$$+\epsilon^2e^{2\iota\tau_1}\left[\sqrt{2}N_1\overline{Q}(0)-\frac{M_1^3P_rE_c}{R_1^4\sinh^2 M_1}\times\right.$$
$$\left.\left\{\frac{-\sinh 2M_1-2\sinh M_1\cosh 2M_1+2\cosh M_1\sinh 2M_1}{2\,(2M_1^2-N_1^2)}\right\}\right]. \qquad (3.3.9)$$

3.4 Conclusions and Discussions

Analytical solutions of the momentum transfer and the energy equations has been developed for the flow between two parallel infinite plates. The pulsatile flow of incompressible fluid has been considered to be laminar. The flow arises due to the application of unsteady pressure gradient. Calculations for velocity distribution and rate of the heat transfer at the plates are given. Numerical graphs are plotted for velocity and temperature. From the graphs we conclude the following points:

1. Figure (3.1) is depicted to see the change in δ on the velocity distribution. From the graph, one finds that velocity decreases with the increase of δ.

2. The fluctuating part of the temperature distribution is sketched against η in Figure (3.2). Here, we find that fluctuating temperature increases for large values of time.

3. Figure (3.3) shows that the increase of temperature near the plate occurs only due to the increase of relaxation time. Further, it is noted that increase in retardation time produces a slight decrease of temperature.

4. From Eqs. (3.2.16) and (3.2.36) we note that steady parts of velocity and temperature are independent of the time dependent amplitude of the pressure gradient.

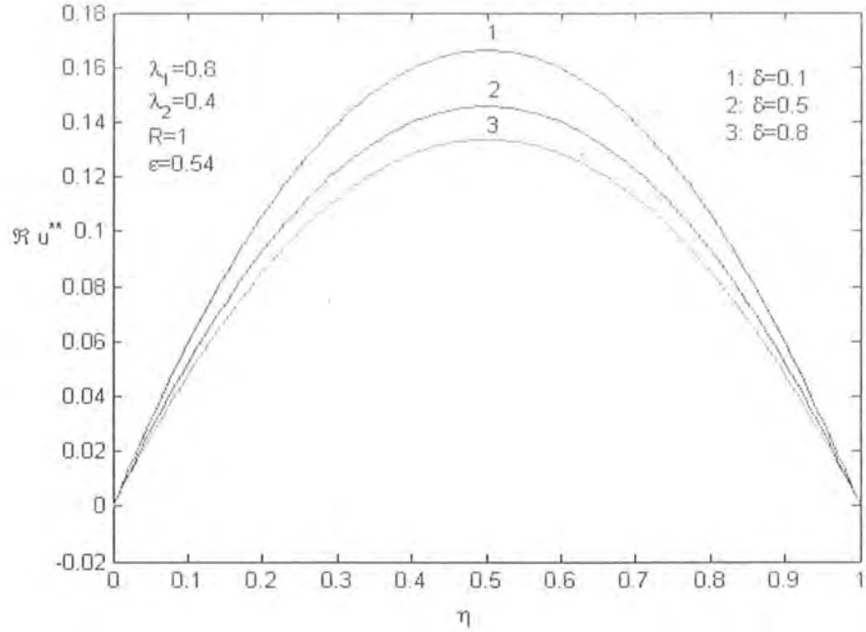

Figure 3.1: Effect of δ on the velocity.

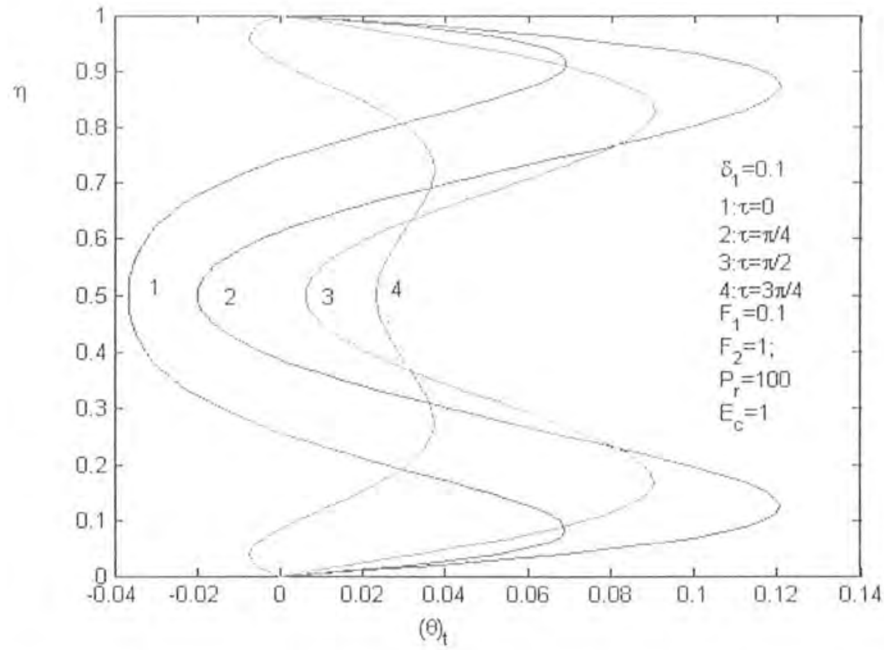

Figure 3.2: Fluctuating temperature distribution in a viscous fluid θ_t.

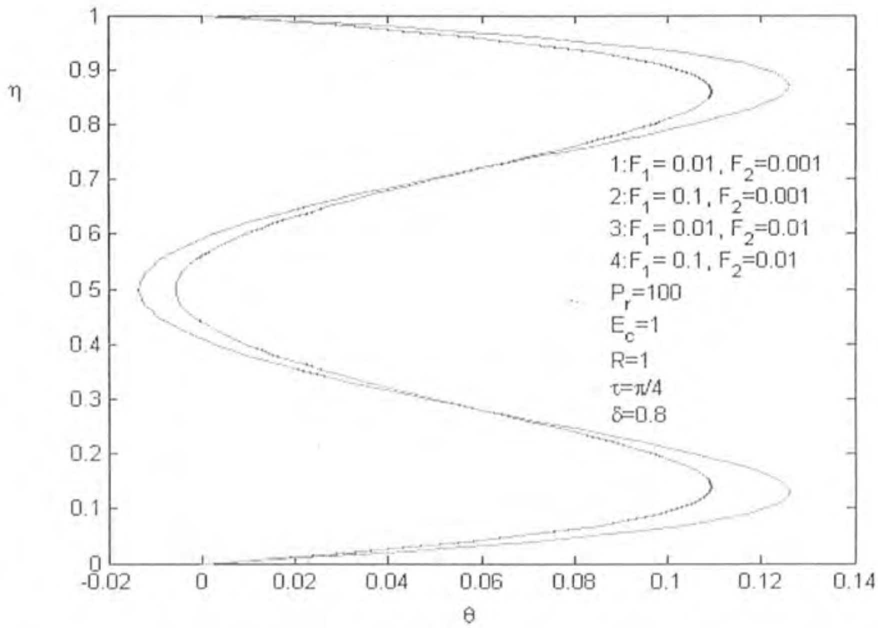

Figure 3.3: Fluctuating temperature distribution in a viscoelastic fluid.

Chapter 4

Magnetohydrodynamic Flow and Heat Transfer of Viscoelastic Fluid Between Parallel Plates in a Porous Medium

4.1 Introduction

This chapter studies the magnetohydrodynamic flow and heat transfer of an Oldroyd-$\mathcal{B}$ fluid between two plates in a porous medium. The adopted mathematical model leads to a problem, in which the channel width, porosity and permeability combine into a shape parameter. The fluid is driven by an oscillating pressure gradient and an external uniform magnetic field is applied perpendicular to the plates. The analysis is valid for small magnetic Reynold number and the governing partial differential equations are solved analytically. Exact solutions are derived for the velocity and the rate of heat transfer per unit area at the plates.

4.2 Problem Formulation

We consider the flow of an Oldroyd-B fluid between two infinitely long parallel plates, a distance h apart, which is driven by the unsteady pressure gradient in the form

$$-\frac{1}{\rho}\frac{\partial p}{\partial x} = A\left\{1 + \epsilon \exp(\iota\omega t)\right\} \tag{4.2.1}$$

where ω is the frequency and the medium between the two plates is porous. The Oldroyd-$\mathcal{B}$ fluid is assumed incompressible and electrically conducting with constant properties and the solid matrix is treated as homogeneous with respect to its porosity characteristics. The magnetic Reynold's number is assumed to be small so that the magnetic field is neglected. A uniform magnetic field $\mathcal{B}_0$ is applied parallel to y-axis. The x-axis is taken along the lower plate at $y = 0$ and y-axis is normal to it. The lower and upper plates, at $y = 0$ and $y = h$, have constant temperatures T_0 and T_1, respectively. The volume-averaged equation governing the unsteady transport of stream-wise momentum can be expressed as

$$\begin{aligned}\left(1 + \lambda_1\frac{\partial}{\partial t}\right)\frac{\partial u}{\partial t} &= -\frac{1}{\rho}\left(1 + \lambda_1\frac{\partial}{\partial t}\right)\frac{\partial p}{\partial x} + \nu\left(1 + \lambda_2\frac{\partial}{\partial t}\right)\frac{\partial^2 u}{\partial y^2} \\ &\quad - \left(\frac{\sigma\mathcal{B}_0^2}{\rho} + \frac{\mu\epsilon_1}{\rho k}\right)\left(1 + \lambda_1\frac{\partial}{\partial t}\right)u. \end{aligned} \tag{4.2.2}$$

In Eq. (4.2.2) σ is the electrical conductivity of the fluid, ρ is the density, ϵ_1 and k are the porosity and permeability of the porous medium. The energy equation is given by Eq. (1.8.20). The boundary conditions are given by

$$u = 0, \;\; T = T_0 \;\text{ at }\; y = 0,$$

$$u = 0, \;\; T = T_1 \;\text{ at }\; y = h.$$

Introducing the non-dimensional parameters (2.2.6) we obtain the boundary value problem of the following form

$$A\frac{\omega h^2}{\nu}\left(1+\lambda_1\omega\frac{\partial}{\partial\tau}\right)\frac{\partial u^*}{\partial\tau} = -\frac{1}{\rho}\left(1+\lambda_1\omega\frac{\partial}{\partial\tau}\right)\frac{\partial p}{\partial x} + A\left(1+\lambda_2\omega\frac{\partial}{\partial\tau}\right)\frac{\partial^2 u^*}{\partial\eta^2}$$
$$-(\Lambda+\Psi)\frac{\omega h^2}{\nu}\left(1+\lambda_1\omega\frac{\partial}{\partial\tau}\right)u^*, \quad (4.2.5)$$

$$u^* = 0 \quad \text{at} \quad \eta = 0,$$
$$u^* = 0 \quad \text{at} \quad \eta = 1. \quad (4.2.6)$$

where

$$\Lambda = \frac{\sigma\mathcal{B}_0^2}{\rho\omega}, \quad (4.2.7)$$

and

$$\Psi = \frac{\epsilon_1\nu}{k\omega} \quad (4.2.8)$$

is the shape parameter. Using the assumed form of the solution (2.2.12) in Eq. (4.2.5) and the boundary conditions and then separating the harmonic and non-harmonic parts we obtain the following systems:

System of order zero

$$\frac{d^2u_0}{d\eta^2} - R^2B^2u_0 + 1 = 0,$$

$$u_0 = 0, \quad \eta = 0, \quad (4.2.10)$$
$$u_0 = 0, \quad \eta = 1.$$

System of order one

$$\frac{d^2u_1}{d\eta^2} - C^{*2}u_1 = -\beta^2,$$

$$u_1 = 0, \quad \eta = 0, \tag{4.2.12}$$
$$u_1 = 0, \quad \eta = 1,$$

where

$$R^2 = \frac{\omega h^2}{\nu}, \qquad B^2 = (\Lambda + \Psi),$$
$$C^* = \beta R\sqrt{B^2 + \iota}, \qquad \beta^2 = \frac{1 + \iota F_1}{1 + \iota F_1 F_2}. \tag{4.2.13}$$

The general solution of system of order zero is

$$u_0 = \overline{\xi}_1 e^{RB\eta} + \overline{\xi}_2 e^{-RB\eta} + \frac{1}{R^2 B^2}, \tag{4.2.14}$$

where $\overline{\xi}_1$ and $\overline{\xi}_2$ are constants. Making use of the boundary conditions we get

$$\overline{\xi}_1 = -\frac{1}{R^2B^2}\left(1 - e^{-RB}\right)\frac{1}{e^{RB} - e^{-RB}}, \tag{4.2.15}$$

$$\overline{\xi}_2 = -\left(\overline{\xi}_1 + \frac{1}{R^2B^2}\right), \tag{4.2.16}$$

and thus Eq. (4.2.14) gives

$$u_0 = \frac{1}{R^2B^2}\left[\left(e^{-RB} - 1\right)\frac{\sinh RB\eta}{\sinh RB} + \left(1 - e^{-RB\eta}\right)\right],$$

or

$$u_0 = \frac{1}{R^2B^2}\left[1 - \frac{\sinh RB\eta - \sinh RB(\eta - 1)}{\sinh RB}\right]. \tag{4.2.18}$$

Similarly the solution of system of order one is given by

$$u_1 = \frac{\beta^2}{C^{*2}}\left[1 - \frac{\sinh C^*\eta - \sinh C^*(\eta - 1)}{\sinh C^*}\right]. \tag{4.2.19}$$

From Eqs. (4.2.14), (4.2.18) and (4.2.19) the expression for u^* is

$$\begin{aligned} u^* = \quad & \frac{1}{R^2B^2}\left[1 - \frac{\sinh RB\eta - \sinh RB(\eta - 1)}{\sinh RB}\right] \\ & -\frac{\beta^2}{C^{*2}}\left[1 - \frac{\sinh C^*\eta - \sinh C^*(\eta - 1)}{\sinh C^*}\right]\epsilon e^{\iota T}. \end{aligned} \tag{4.2.20}$$

Now the energy equation in non-dimensional form is given in Eq. (2.2.35) and for the convenience of the readers the boundary value problem in terms θ is given by (Eqs. (2.2.36)) and (2.2.37)), i.e.,

$$R^2\frac{\partial\theta}{\partial\tau}=\frac{1}{P_r}\frac{\partial^2\theta}{\partial\eta^2}+E_c\left(\frac{\partial u^*}{\partial\eta}\right)^2,$$

$$\theta=0 \qquad \text{at} \qquad \eta=0, \tag{4.2.22}$$

$$\theta=1 \qquad \text{at} \qquad \eta=1,$$

where

$$P_r=\frac{\mu C_p}{\chi}, \qquad E_c=\frac{A^2h^4}{\nu^2C_p(T_0-T_1)}.$$

Assuming θ of the following form

$$\theta(\eta,\tau)=\theta_0^*(\eta)+\epsilon F^*(\eta)e^{\iota\tau}+\epsilon^2G^*(\eta)e^{2\iota\tau} \tag{4.2.23}$$

and using in the system (4.2.22) and equating the coefficients of $\epsilon^0e^{0\iota\tau}$, $\epsilon e^{\iota\tau}$ and $\epsilon^2e^{2\iota\tau}$ we have

$$\frac{d^2\theta_0^*}{d\eta^2}+P_rE_c\left(\frac{du_0}{d\eta}\right)^2=0, \tag{4.2.24}$$

$$\frac{d^2F^*}{d\eta^2}-\iota R^2P_rF^*(\eta)=-2E_cP_r\left(\frac{du_0}{d\eta}\right)\left(\frac{du_1}{d\eta}\right), \tag{4.2.25}$$

$$\frac{d^2G^*}{d\eta^2}-2\iota R^2P_rG^*(\eta)=-E_cP_r\left(\frac{du_1}{d\eta}\right)^2, \tag{4.2.26}$$

with the boundary conditions

$$\theta_0^*(0) = 0, \qquad \theta_0^*(1)=1, \tag{4.2.27}$$

$$F^*(0) = 0, \qquad F^*(1)=0, \tag{4.2.28}$$

$$G^*(0) = 0, \qquad G^*(1)=0. \tag{4.2.29}$$

From Eqs. (4.2.18) and (4.2.19) we have

$$\left(\frac{du_0}{d\eta}\right)^2 = \frac{1}{R^2B^2\sinh^2 RB}\Big[\cosh^2 RB\eta + \cosh^2 RB(\eta-1) \\ -2\cosh RB\eta\cosh RB(\eta-1)\Big],$$

or

$$\left(\frac{du_0}{d\eta}\right)^2 = \frac{1}{4R^2B^2\sinh^2 RB}\Big[(1-e^{-RB})^2 e^{2RB\eta} + (e^{RB}-1)^2 e^{-2RB\eta} \\ +4(1-\cosh RB)\Big], \qquad (4.2.30)$$

and

$$\left(\frac{du_1}{d\eta}\right)^2 = \frac{\beta^4}{C^{*2}\sinh^2 C^*}\Big[\cosh^2 C^*\eta + \cosh^2 C^*(\eta-1) - 2\cosh C^*\eta\cosh C^*(\eta-1)\Big],$$

or

$$\left(\frac{du_1}{d\eta}\right)^2 = \frac{\beta^4}{4C^{*2}\sinh^2 C^*}\Big[(1-e^{-C^*})^2 e^{2C^*\eta} + (e^{C^*}-1)^2 e^{-2C^*\eta} + 4(1-\cosh C^*)\Big]. \qquad (4.2.32)$$

Also

$$\left(\frac{du_0}{d\eta}\right)\left(\frac{du_1}{d\eta}\right) = \frac{\beta^2}{C^*BR\sinh C^*\sinh RB}\Big[\cosh RB\eta\cosh C^*\eta \\ -\cosh C^*\eta\cosh RB(\eta-1) + \cosh RB(\eta-1)\cosh C^*(\eta-1) \\ -\cosh BR\eta\cosh C^*(\eta-1)\Big],$$

or

$$\left(\frac{du_0}{d\eta}\right)\left(\frac{du_1}{d\eta}\right) = \Big[A_1^* e^{M_1^*\eta} + A_2^* e^{M_2^*\eta} + A_3^* e^{-M_2^*\eta} + A_4^* e^{-M_1^*\eta}\Big], \qquad (4.2.33)$$

where

$$A_1^* = \frac{\beta^2(1-e^{-C^*})(1-e^{-BR})}{C^*BR\sinh C^*\sinh RB}, \qquad A_2^* = \frac{\beta^2(1-e^{C^*})(1-e^{-BR})}{C^*BR\sinh C^*\sinh RB},$$

$$A_3^* = \frac{\beta^2(1-e^{-C^*})(1-e^{BR})}{C^*BR\sinh C^*\sinh RB}, \quad A_4^* = \frac{\beta^2(1-e^{C^*})(1-e^{BR})}{C^*BR\sinh C^*\sinh RB} \quad \text{and}$$

$$M_1^* = BR + C^*, \qquad M_2^* = BR - C^*.$$

From Eqs. (4.2.24) and (4.2.30) we have

$$\frac{d^2\theta_0^*}{d\eta^2} = -\frac{P_r E_c}{4R^2B^2\sinh^2 RB}\Big[(1-e^{-RB})^2 e^{2RB\eta} + (e^{RB}-1)^2 e^{-2RB\eta} + 4(1-\cosh RB)\Big]. \quad (4.2.34)$$

Integrating Eq. (4.2.34) with respect to η we have

$$\theta_0^* = -\frac{P_r E_c}{4R^2B^2\sinh^2 RB}\Big[\frac{(1-e^{-RB})^2}{4R^2B^2} e^{2RB\eta} + \frac{(e^{RB}-1)^2}{4R^2B^2} e^{-2RB\eta} + 2(1-\cosh RB)\eta^2\Big] + \xi_3\eta + \xi_4, \quad (4.2.35)$$

where ξ_3 and ξ_4 are the constants of integration. Now applying the boundary conditions (4.2.27) we get

$$\xi_3 = 1 + \frac{2P_r E_c}{(2RB\sinh RB)^2}\Big[1-\cosh BR\Big], \quad (4.2.36)$$

$$\xi_4 = \frac{P_r E_c}{2(2R^2B^2\sinh RB)^2}\Big[1+\cosh 2BR - 2\cosh BR\Big], \quad (4.2.37)$$

and thus Eq. (4.2.35) will take the form as

$$\begin{aligned}\theta_0^* &= \eta - \frac{P_r E_c}{(2RB\sinh RB)^2}\Big[\frac{1}{2B^2R^2}\big(\cosh 2BR\eta + \cosh 2BR(\eta-1) \\ &\quad -2\cosh 2BR(\eta - \frac{1}{2})\big) + 2(1-\cosh RB)(\eta^2-\eta)\Big] \\ &\quad + \frac{P_r E_c}{2(2R^2B^2\sinh RB)^2}\Big[1+\cosh 2BR - 2\cosh BR\Big]. \quad (4.2.38)\end{aligned}$$

From Eqs. (4.2.25) and (4.2.33) and using $N = (1+\iota)n$, $n = R\sqrt{P_r/2}$, we obtain

$$\frac{d^2F^*}{d\eta^2} - N^2F^*(\eta) = -2E_cP_r\Big[A_1^* e^{M_1^*\eta} + A_2^* e^{M_2^*\eta} + A_3^* e^{-M_2^*\eta} + A_4^* e^{-M_1^*\eta}\Big], \quad (4.2.39)$$

where A_1^*, A_2^*, A_3^*, A_4^*, M_1^* and M_2^* are given in Eq. (4.2.33). The solution of Eq. (4.2.39) is sum of complementary function and the particular integral. The complementary function is given by

$$F_c^* = c_1^* e^{N\eta} + c_2^* e^{-N\eta}. \quad (4.2.40)$$

Using the method of undetermined coefficients, the particular integral can be written as

$$F_p^* = \overline{\alpha} e^{M_1^* \eta} + \overline{\beta}^* e^{M_2^* \eta} + \overline{\gamma}^* e^{-M_2^* \eta} + \overline{\delta}^* e^{-M_1^* \eta}, \tag{4.2.41}$$

where $\overline{\alpha}^*$, $\overline{\beta}^*$, $\overline{\gamma}^*$ and $\overline{\delta}^*$ are to be determined. Differentiating Eq. (4.2.41) we arrive at

$$\frac{d^2 F_p^*}{d\eta^2} = \overline{\alpha}^* M_1^{*2} e^{M_1^* \eta} + \overline{\beta}^* M_2^2 e^{M_2^* \eta} + \overline{\gamma}^* M_2^2 e^{-M_2^* \eta} + \overline{\delta}^* M_1^{*2} e^{-M_1^* \eta},$$

or

$$\begin{aligned} \frac{d^2 F_p^*(\eta)}{d\eta^2} - N^2 F_p^*(\eta) &= \overline{\alpha}^* \left(M_1^{*2} - N^2\right) e^{M_1^* \eta} + \overline{\beta}^* \left(M_2^2 - N^2\right) e^{M_2^* \eta} \\ &+ \overline{\gamma}^* \left(M^2 - N^2\right) e^{-M_2^* \eta} + \overline{\delta}^* \left(M^2 - N^2\right) e^{-M_1^* \eta} \end{aligned} \tag{4.2.42}$$

Comparing Eqs. (4.2.39) and (4.2.42) we get

$$\overline{\alpha}^* = \frac{-E_c P_r A_1^*}{M_1^{*2} - N^2}, \tag{4.2.43}$$

$$\overline{\beta}^* = \frac{-E_c P_r A_2^*}{M_2^{*2} - N^2}, \tag{4.2.44}$$

$$\overline{\gamma}^* = \frac{-E_c P_r A_3^*}{M_2^{*2} - N^2}, \tag{4.2.45}$$

$$\overline{\delta}^* = \frac{-E_c P_r A_4^*}{M_1^{*2} - N^2}. \tag{4.2.46}$$

Let us define $F_p^*(\eta) = L^*(\eta)$, i.e.,

$$L^*(\eta) = \overline{\alpha}^* e^{M_1^* \eta} + \overline{\beta}^* e^{M_2^* \eta} + \overline{\gamma}^* e^{-M_2^* \eta} + \overline{\delta}^* e^{-M_1^* \eta}. \tag{4.2.47}$$

The complete solution of Eq. (4.2.39) is of the following form

$$\begin{aligned} F^*(\eta) &= F_c^*(\eta) + F_p^*(\eta), \\ &= c_1^* e^{N\eta} + c_2^* e^{-N\eta} + L^*(\eta). \end{aligned} \tag{4.2.48}$$

Using the boundary conditions (4.2.28) in Eq. (4.2.48) we get

$$0 = c_1^* + c_2^* + L^*(0), \tag{4.2.49}$$

$$0 = c_1^* e^N + c_2^* e^{-N} + L^*(1), \tag{4.2.50}$$

where

$$L^*(0) = \overline{\alpha}^* + \overline{\beta}^* + \overline{\gamma}^* + \overline{\delta}^*, \tag{4.2.51}$$

$$L^*(1) = \overline{\alpha}^* e^{M_1^*} + \overline{\beta}^* e^{M_2^*} + \overline{\gamma}^* e^{-M_2^*} + \overline{\delta}^* e^{-M_1^*}. \tag{4.2.52}$$

From Eqs. (4.2.33) and (4.2.36) to (4.2.46) we have

$$\overline{\alpha}^* e^{M_1^*} = \overline{\delta}^* \quad \text{and} \quad \overline{\beta}^* e^{M_2^*} = \overline{\gamma}^*. \tag{4.2.53}$$

Using Eq. (4.2.53) in Eq. (4.2.52) we obtain

$$L^*(1) = L^*(0). \tag{4.2.54}$$

Solving Eqs. (4.2.49) and (4.2.50) we have

$$\begin{bmatrix} 1 & 1 \\ e^N & e^{-N} \end{bmatrix} \begin{bmatrix} c_1^* \\ c_2^* \end{bmatrix} = \begin{bmatrix} -L^*(0) \\ -L^*(1) \end{bmatrix},$$

or

$$\begin{bmatrix} c_1^* \\ c_2^* \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ e^N & e^{-N} \end{bmatrix}^{-1} \begin{bmatrix} -L^*(0) \\ -L^*(1) \end{bmatrix}.$$

Because

$$\begin{bmatrix} 1 & 1 \\ e^N & e^{-N} \end{bmatrix}^{-1} = \frac{-1}{2\sinh N} \begin{bmatrix} e^{-N} & -1 \\ -e^N & 1 \end{bmatrix},$$

therefore

$$\begin{bmatrix} c_1^* \\ c_2^* \end{bmatrix} = \frac{1}{2\sinh N} \begin{bmatrix} e^{-N} L^*(0) - L^*(1) \\ -e^N L^*(0) + L^*(1) \end{bmatrix},$$

or we can write

$$c_1^* = \frac{L^*(0)}{2\sinh N}\left(e^{-N} - 1\right), \qquad (4.2.59)$$

$$c_2^* = \frac{L^*(0)}{2\sinh N}\left(-e^{N} + 1\right). \qquad (4.2.60)$$

Thus from Eq. (4.2.48) we can write

$$F^*(\eta) = \frac{L^*(0)}{\sinh N}\left[\sinh N(\eta - 1) - \sinh N\eta\right] + L^*(\eta). \qquad (4.2.61)$$

From Eqs. (4.2.33) and (4.2.43) to (4.2.46) we have

$$L^*(0) = \frac{-2\beta^2 E_c P_r(2 - \cosh C^* - \cosh BR)}{C^* BR \sinh C^* \sinh BR}\left\{\frac{1}{M_1^{*2} - N^2} + \frac{1}{M_2^{*2} - N^2}\right\} \qquad (4.2.62)$$

and

$$\begin{aligned} L^*(\eta) \;=\; & \frac{-2\beta^2 E_c P_r}{C^* BR \sinh C^* \sinh BR}\left\{\frac{1 + \cosh M_1^*\eta}{M_1^{*2} - N^2} + \frac{1 + \cosh M_2^*\eta}{M_2^{*2} - N^2}\right. \\ & - \frac{\cosh\left(M_1^*\eta - C^*\right) + \cosh\left(M_1^*\eta - BR\right)}{M_1^{*2} - N^2} \\ & \left. - \frac{\cosh\left(M_2^*\eta + C^*\right) + \cosh\left(M_2^*\eta - BR\right)}{M_2^{*2} - N^2}\right\}. \end{aligned} \qquad (4.2.63)$$

With the help of Eqs. (4.2.26) and (4.2.32) we get

$$\frac{d^2 G^*(\eta)}{d\eta^2} - 2\iota P_r R^2 G^*(\eta) = a_1^* e^{2C^*\eta} + a_2^* e^{-2C^*\eta} + a_3^*, \qquad (4.2.64)$$

where

$$a_1^* = \frac{-P_r E_c \beta^4 (1 - e^{-C^*})^2}{4C^{*2}\sinh^2 C^*},\; a_2^* = \frac{-P_r E_c \beta^4 (e^{C^*} - 1)^2}{4C^{*2}\sinh^2 C^*},\; a_3^* = \frac{-P_r E_c \beta^4 (1 - \cosh C^*)}{C^{*2}\sinh^2 C^*}.$$

We note that Eq. (4.2.64) is non-homogeneous second order ordinary differential equation. The complete solution is the sum of complementary function and particular integral. Employing the same method of solution as for $F^*(\eta)$ we obtain

$$G^*(\eta) = \frac{Q^*(0)}{\sinh\sqrt{2}N}\left(\sinh\sqrt{2}N\eta + \sinh\sqrt{2}N(1 - \eta)\right) + Q^*(\eta), \qquad (4.2.66)$$

where

$$Q^*(\eta) = \frac{-P_r E_c \beta^4}{C^{*2}\sinh^2 C^*}\Bigg[\frac{(1-\cosh C^*)}{N^2} - \frac{1+\cosh 2C^* - 2\cosh C^*}{2\,(2C^{*2}-N^2)}\cosh 2C^*\eta + \frac{\sinh 2C^* - 2\sinh C^*}{2\,(2C^{*2}-N^2)}\sinh 2C^*\eta\Bigg] \quad (4.2.67)$$

$$Q^*(0) = \frac{-P_r E_c \beta^4}{C^{*2}\sinh^2 C^*}\left[\frac{1-\cosh C^*}{N^2} - \frac{1+\cosh 2C^* - 2\cosh C^*}{2\,(2C^{*2}-N^2)}\right]. \quad (4.2.68)$$

4.3 Rate of Heat Transfer

The rate of heat transfer per unit area at the plate $\eta = 0$ is defined by

$$\left(\frac{\partial\theta}{\partial\eta}\right)_{\eta=0} = \left(\frac{d\theta_0^*}{d\eta}\right)_{\eta=0} + \epsilon e^{\iota\omega t}\left(\frac{dF^*}{d\eta}\right)_{\eta=0} + \epsilon^2 e^{2\iota\omega t}\left(\frac{dG^*}{d\eta}\right)_{\eta=0}. \quad (4.3.1)$$

Now from Eqs. (4.2.38), (4.2.61), (4.2.63), (4.2.64) and (4.2.68) we obtain

$$\frac{d\theta_0^*}{d\eta} = 1 - \frac{P_r E_c}{(2RB\sinh RB)^2}\Bigg[\frac{1}{BR}\Big(\sinh 2BR\eta + \sinh 2BR(\eta-1) - 2\sinh 2BR(\eta - \frac{1}{2})\Big) + 2(1-\cosh RB)(2\eta-1)\Bigg], \quad (4.3.2)$$

$$\frac{dF^*}{d\eta} = \frac{-NL^*(0)}{\sinh N}\left[\cosh N\eta - \cosh N(1-\eta)\right] + \frac{dL^*}{d\eta}, \quad (4.3.3)$$

$$\frac{dG^*}{d\eta} = \frac{\sqrt{2}NQ^*(0)}{\sinh\sqrt{2}N}\Big(\cosh\sqrt{2}N\eta - \cosh\sqrt{2}N(1-\eta)\Big) - \frac{dQ^*}{d\eta}, \quad (4.3.4)$$

$$\frac{dL^*}{d\eta} = \frac{-2\beta^2 E_c P_r}{C^* BR\sinh C^*\sinh BR}\Bigg\{\frac{M_1^*\sinh M_1^*\eta}{M_1^{*2}-N^2} + \frac{M_2^*\sinh M_2^*\eta}{M_2^{*2}-N^2} - \frac{M_1^*\left[\sinh(M_1^*\eta - C^*) + \sinh(M_1^*\eta - BR)\right]}{M_1^{*2}-N^2} - \frac{M_2^*\left[\sinh(M_2^*\eta + C^*) + \sinh(M_2^*\eta - BR)\right]}{M_2^{*2}-N^2}\Bigg\}, \quad (4.3.5)$$

$$\frac{dQ^*}{d\eta} = \frac{-2P_rE_c\beta^4}{C^*\sinh^2 C^*}\left[\frac{\sinh 2C^* - 2\sinh C^*}{2\left(2C^{*2}-N^2\right)}\cosh 2C^*\eta - \frac{1+\cosh 2C^* - 2\cosh C^*}{2\left(2C^{*2}-N^2\right)}\sinh 2C^*\eta\right]. \tag{4.3.6}$$

Using above equations in Eq. (4.3.1) we get

$$\begin{aligned}\left(\frac{\partial\theta}{\partial\eta}\right)_{\eta=0} &= 1+\frac{P_rE_c}{(2RB\sinh RB)^2}\left[\frac{1}{BR}\Big(\sinh 2BR - 2\sinh BR\Big)+2(1-\cosh RB)\right]\\ &+\epsilon e^{\imath\omega t}\left[\frac{-2\beta^2E_cP_r}{C^*BR\sinh C^*\sinh BR}\left\{\Big(\frac{M_1^*}{M_1^{*2}-N^2}-\frac{M_2^*}{M_2^{*2}-N^2}\Big)\sinh C^* +\right.\right.\\ &\left.\left.\Big(\frac{M_1^*}{M_1^{*2}-N^2}+\frac{M_2^*}{M_2^{*2}-N^2}\Big)\sinh BR\right\} - \frac{NL^*(0)}{\sinh N}(1-\cosh N)\right] \\ &+\epsilon^2e^{2\imath\omega t}\left[\frac{\sqrt{2}NQ^*(0)}{\sinh\sqrt{2}N}\Big(1-\cosh\sqrt{2}N\Big)-\frac{2P_rE_c\beta^4}{C^*\sinh^2 C^*}\left[\frac{\sinh 2C^*-2\sinh C^*}{2\left(2C^{*2}-N^2\right)}\right]\right].\end{aligned} \tag{4.3.7}$$

The rate of heat transfer per unit area at the plate $\eta = 1$ is given by

$$\begin{aligned}\left(\frac{\partial\theta}{\partial\eta}\right)_{\eta=1} &= 1-\frac{P_rE_c}{(2RB\sinh RB)^2}\left[\frac{1}{BR}\Big(\sinh 2BR - 2\sinh BR\Big)+2(1-\cosh RB)\right]\\ &+\epsilon e^{\imath\omega t}\left[\frac{NL^*(0)}{\sinh N}(1-\cosh N)-\frac{2\beta^2E_cP_r}{C^*BR\sinh C^*\sinh BR}\left\{\frac{M_1^*\sinh M_1^*}{M_1^{*2}-N^2}\right.\right.\\ &+\frac{M_2^*\sinh M_2^*}{M_2^{*2}-N^2}-\Big(\frac{M_1^*}{M_1^{*2}-N^2}-\frac{M_2^*}{M_2^{*2}-N^2}\Big)\sinh C^*\\ &\left.\left.-\Big(\frac{M_1^*}{M_1^{*2}-N^2}+\frac{M_2^*}{M_2^{*2}-N^2}\Big)\sinh BR\right\}\right]\\ &+\epsilon^2e^{2\imath\omega t}\left[-\frac{\sqrt{2}NQ^*(0)}{\sinh\sqrt{2}N}\Big(1-\cosh\sqrt{2}N\Big)\right.\\ &\left.-\frac{2P_rE_c\beta^4}{C^*\sinh^2 C^*}\left[\frac{-\sinh 2C^*+2\sinh C^*}{2(2C^{*2}-N^2)}\right]\right].\end{aligned} \tag{4.3.8}$$

4.4 Concluding Remarks

Calculations have been carried out to study the flow between two infinite parallel plates. The effect of an external uniform magnetic field as well as the action of a shape parameter together with the influence of relaxation and retardation times on the velocity distribution and the rate of heat transfer at the plates are reported.

From the analytical solutions and graphs it is found that the velocity distribution is noticeably influenced by the presence of applied magnetic field, shape parameter, relaxation and retardation times and oscillating frequency. More precisely:

1. The magnetic filed accelerates the fluid motion when the relaxation time is greater than the retardation time (see Figure (4.1)).

2. From Figure (4.2) it is noted that the velocity increases with the increase of oscillating frequency which results in decrease of boundary layer thickness. Further, it is also concluded that velocity increases for the large values of time.

3. The change in relaxation time (λ_1) on u is depicted in Figure (4.3). Here one finds that u increases for greater values of λ_1.

4. Figure (4.4) is prepared to see the effect of retardation time on the velocity distribution. From the graphs, it is clearly seen that velocity decreases with the increase in retardation time.

5. Figure (4.5) represents the influence of shape parameter Ψ on the velocity. It is indicated from the graph that increasing the shape parameter Ψ decreases the velocity.

6. From Eqs. (4.2.18), (4.2.19), (4.2.23), (4.2.38), (4.2.61) and (4.2.66), it is concluded that steady parts of the velocity and temperature do not depend on the relaxation and retardation times where as the fluctuating parts depend. highly on the relaxation and retardation times. Further, it is also seen that steady and fluctuating parts of velocity and temperature are dependent on magnetic and shape parameters.

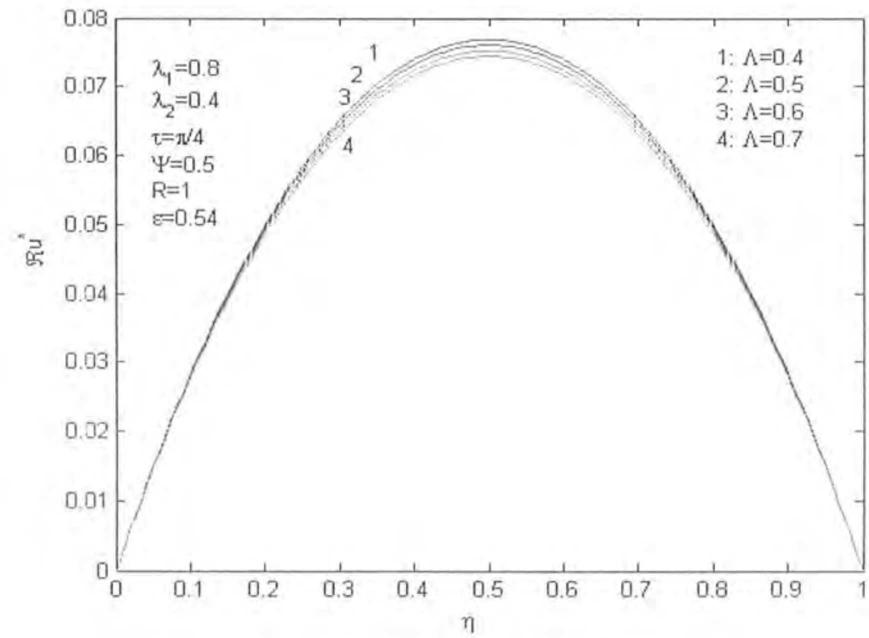

Figure 4.1: Variation of magnetic field parameter Λ.

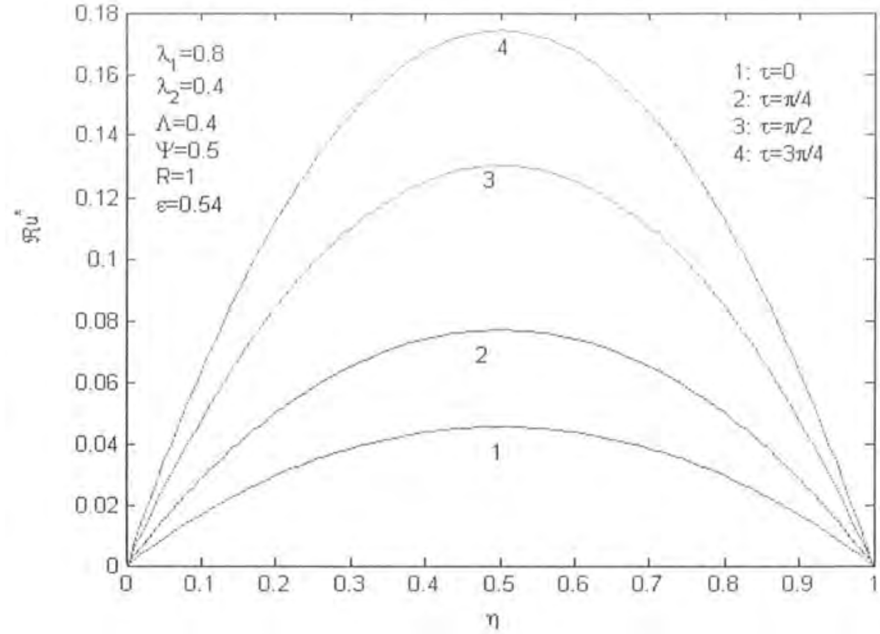

Figure 4.2: Variation of oscillating frequency $\omega = \tau/t$.

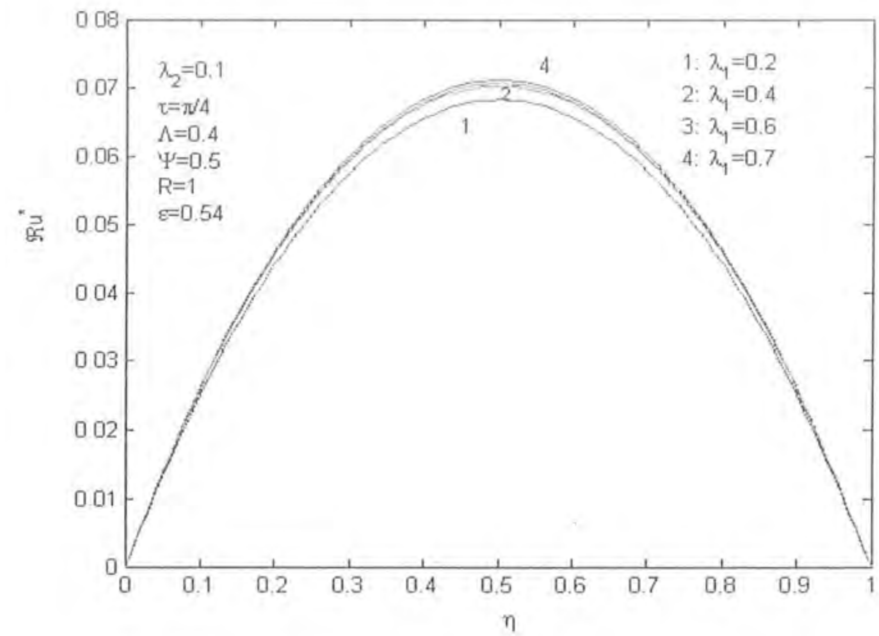

Figure 4.3: Variation of relaxation time (λ_1).

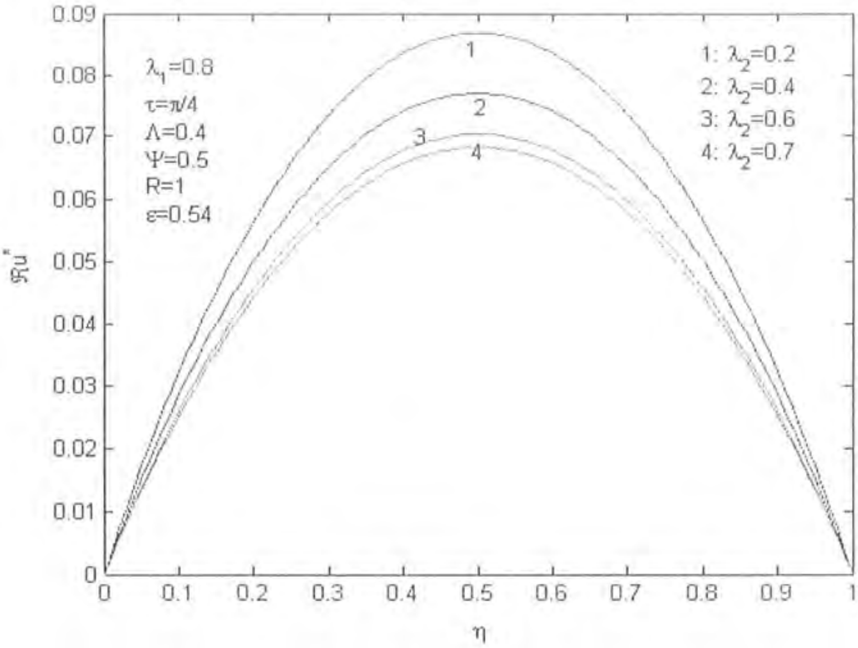

Figure 4.4: Variation of retardation time (λ_2).

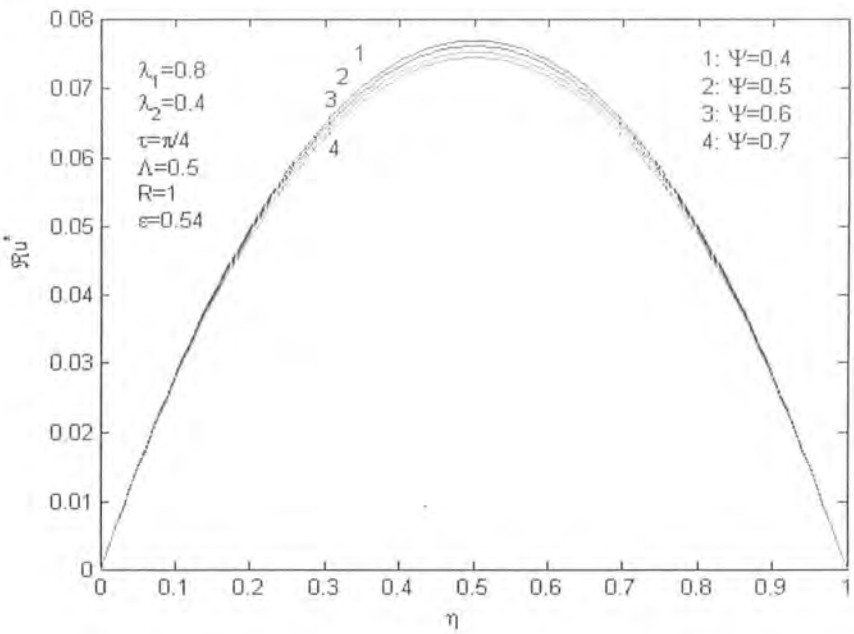

Figure 4.5: Variation of the shape parameter (Ψ).

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