On Soft Topological Rings

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A THESIS SUBMITTED IN THE PARTIAL FULFILLMENT OF THE REQUIREMENTS FOR THE DEGREE OF

MASTER OF PHILOSOPHY

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CERTIFICATE

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We accept this thesis as conforming to the required standard.

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Salma Shaheen

PREFACE

Change is the sole constant in the universe. Most of our traditional tools for modeling, reasoning, and computing are crisp, deterministic, and precise in character. However, there are many convoluted problems in economics, engineering, medical sciences, environment, social sciences, etc., that involve data which are not always crisp. We cannot use classical methods because of various types of uncertainties present in these problems. There are several well-known theories to describe uncertainty. For instance fuzzy set theory [25], vague set theory, rough set theory, interval mathematics theory [6] and other mathematical tools. But all of these theories have their own difficulties as pointed out by Molodtsov in [16].

To triumph over these difficulties, Molodtsov introduced the concept of soft set as a new mathematical tool for dealing with uncertainties that is free from difficulties effecting existing methods. Soft systems supply a very universal outline with the participation of parameters. Work on soft set theory and its applications in various fields are moving ahead fast in recent years.

Maji [15] offered an application of soft sets in decision making problems that is based on the drops of parameters to maintain the finest choice objects. Appliance of soft set theory in algebraic structures was introduced by Aktaş and Çağman [2]. They discussed the opinion of soft groups and consequently obtained some fundamental properties. Feng et. al. [9] worked on soft semirings, soft ideals and idealistic soft semirings. On the other hand, in [1], Acar et al. introduce the basic notion of a soft ring, which is in fact a parameterized family of sub rings\ideals of a ring, over a ring. In [7] soft subrings and soft ideals over a ring are introduced, moreover in [7] soft subfield over a field and soft sub-module over a left R-module has been introduced. Celik, et. al [6] defined a new binary relation and some new operations on soft sets, also they introduced the notion of a soft ring and soft ideal over a ring. However, in [17] Sk. Nazmul and Sk Samnta introduce the basic idea of a soft topological group, its subsystem and morphism over a topological group.

In this dissertation we introduce the notion of a soft topological group over a group, and examine some of its properties. Furthermore, we extend the study of soft topological algebraic structures by defining soft topological ring and idealistic soft topological ring, which we define over a ring as an initial universe with a fixed set of parameters and investigate some of their basic properties . We also provide the condition under which every soft ring becomes a soft topological ring.

Furthermore, we also deal with some of the algebraic properties of soft topological groups and soft topological rings. Also we introduce the concept of soft topological ring homomorphism, and examine some of its properties. In the end we give the idea of closure of a soft topological ring, which helps in defining a soft topological divisor of zero. Whereas, in general, a soft topological divisor of zero is different from a soft zero divisor of [22].

Contents

Chapter 1

Topological Algebraic Structures

1.1 Introduction

This chapter consists of basic concepts of topological groups, topological rings and topological divisors of zero. We have partitioned it into three sections. In the first section we discuss basic definitions, examples and results of topological groups. Second section is devoted to the basic definitions, examples and results of topological rings. It also includes discussion on subrings and ideals of topological rings. In the last section we discuss topological homomorphism of groups and rings.

1.2 Topological Groups

Definition 1 *[12] An Abelian group $\mathcal{A}$ is called a topological group if a topology is defined on the set $\mathcal{A}$ and the following conditions are satisfied:*

- Addition Continuity Condition (AC). This condition implies that the mapping $(a, b) \to a + b$ of the topological space $\mathcal{A} \times \mathcal{A}$ onto the topological space $\mathcal{A}$ is continuous.

- Additive Inversion Continuity Condition (AIC). This condition implies that the mapping $a \to -a$ of the topological space $\mathcal{A}$ onto itself is continuous.

Here $\mathcal{A} \times \mathcal{A}$ is viewed as a topological space by using the product topology.

Remark 2 *[4] In the terms of neighborhoods, conditions (AC) and (AIC) exhibit respectively the following:*

For any two elements $a, b \in \mathcal{A}$ and arbitrary neighborhood U of the element $a+b$ there exist such neighborhoods V and W of elements a and b, respectively, that $V + W \subseteq U$.

For any element $a \in \mathcal{A}$ and any arbitrary neighborhood U of the element $-a$ there exist a neighborhood V of the element such that $-V \subseteq U$.

Theorem 3 *[12] An Abelian group $\mathcal{A}$ is a topological group with a topology τ, if and only if, the mapping $(a, b) \to a - b$ of $\mathcal{A} \times \mathcal{A}$ onto $\mathcal{A}$ is continuous.*

This condition is named as Subtraction continuity condition (SC). For any $a, b \in \mathcal{A}$ and arbitrary neighborhood U of the element $a - b$ there exist neighborhoods V and W of elements a and b respectively such that $V + W = U$.

It is advantageous to use the topological structure to uncover facts about the algebraic structure, and vice versa.

Example 4 *The real numbers $\mathbb{R}$ form a topological group, together with its interval topology and the addition as operation . More generally, Euclidean n-space $\mathbb{R}^n$ with standard topology is a topological group.*

Example 5 *With the Euclidean topology $(\mathbb{R}^{\times}, \cdot)$ is a topological group. Note that $\mathbb{R}^{\times} = \mathbb{R} - \{0\}$ and the multiplicative function defined by $(x, y) \to x \cdot y$.*

Example 6 *The groups $(\mathbb{C}, +)$ and $(\mathbb{C}^{\times}, \cdot)$ with the complex norm topology are topological groups.*

Example 7 *$\mathbb{R}^n$ is an additive Abelian Lie group (topological group which is also manifold). Group of all invertible $n \times n$ matrices $GL(n, \mathbb{R})$ with real entries is topological group with the topology as viewing $GL(n, \mathbb{R})$ as a subset of $\mathbb{R}^{n \times n}$.*

Theorem 8 *[4] Let B and C be the subsets of $\mathcal{A}$ (topological group) and $a \in \mathcal{A}$. Then the followings are true:*

(1) $\psi_a : \mathcal{A} \to \mathcal{A}$ and $\psi : \mathcal{A} \to \mathcal{A}$, where $\psi_a(x) = x + a$ and $\psi(x) = -x$, are homeomorphic mappings of the topological space $\mathcal{A}$ onto itself.

(2) The followings are equivalent:

(a) Subset B is open (closed).

(b) Subset $-B$ is open (closed).

(c) Subset $B+a$ is open (closed). (Among other things a subset $B \subseteq \mathcal{A}$ is a neighborhood of the element a if and only if $B-a$ is a neighborhood of 0.)

(3) If B is open, then $B+C$ is also an open subset.

Proposition 9 *[4] Let B and C are subsets of a topological group $\mathcal{A}$. Then the followings are true:*

(1) $[B+C]_{\mathcal{A}} \supseteq [B]_{\mathcal{A}} + [C]_{\mathcal{A}}$.

(2) $[-B]_{\mathcal{A}} = -[B]_{\mathcal{A}}$.

(3) $[B-C]_{\mathcal{A}} \supseteq [B]_{\mathcal{A}} - [C]_{\mathcal{A}}$.

1.2.1 Hausdorff Groups

Theorem 10 *[4] Consider a topological group $\mathcal{A}$. The following are equivalent:*

(1) $\mathcal{A}$ is a totally regular space.

(2) $\mathcal{A}$ is a regular space.

(3) $\mathcal{A}$ is a Hausdorff space.

(4) $\{0\}$ is a closed subset in $\mathcal{A}$.

(5) $\mathcal{A}$ is a T_0-space.

(6) $\mathcal{A}$ is a T_1-space.

If $\mathcal{A}$ (topological group) satisfies one of the conditions $(1)-(6)$ of Theorem 10 then $\mathcal{A}$ is called a Hausdorff group.

1.2.2 Subgroups of Topological Groups

Let $(\mathcal{A}, \tau)$ be a topological group. A set B of elements of $\mathcal{A}$ is called a subgroup of topological group $(\mathcal{A}, \tau)$ if B is a subgroup of $\mathcal{A}$ and B is a topological space with the topology $\tau|_B$, induced by the topology τ.

Remark 11 *[4] An algebraic subgroup of a topological group is a topological group itself.*

Indeed, let $b_1, b_2 \in B$ and U be a neighborhood of the element $b_1 - b_2$ in the topological space $(B, \tau|_B)$. Then $U = U_1 \cap B$, where U_1 is a neighborhood of the element $b_1 - b_2$ in the topological space $(\mathcal{A}, \tau)$. Let V_1' and V_1'' be neighborhood of elements b_1 and b_2 correspondingly in the topological space $(\mathcal{A}, \tau)$ such that $V_1' - V_1'' \subseteq U_1$. Then $V' = V_1' \cap B$ and $V'' = V_1'' \cap B$ are neighborhoods of elements b_1 and b_2 respectively in the topological space B. Besides

$$V' - V'' \subseteq (V_1' - V_1'') \cap B \subseteq U_1 \cap B = U$$

i.e. *B with the induced topology satisfies condition (SC) (see 1.2). Hence, $(B, \tau|_B)$ is a topological group. Clearly, if $\mathcal{A}$ is a Hausdorff topological group so is B.*

Proposition 12 *[12] If B is a subgroup of a topological group $\mathcal{A}$. Then $[B]_{\mathcal{A}}$ is a subgroup of the topological group $\mathcal{A}$, too. If B is invariant (normal or distinguished) subgroup of $\mathcal{A}$, so is $[B]_{\mathcal{A}}$.*

Proposition 13 *[12] Every open subgroup B of a topological group $\mathcal{A}$ is closed.*

Proposition 14 *[12] The center of a Hausdorff topological group $\mathcal{A}$ is a closed invariant subgroup.*

Proposition 15 *[4] If a topological group $\mathcal{A}$ contains at least one Hausdorff closed subgroup, then $\mathcal{A}$ is Hausdorff group, too.*

Proposition 16 *[4] Any discrete subgroup B of a Hausdorff topological group $\mathcal{A}$ is closed.*

1.3 Topological Rings

Definition 17 *[4] A topological ring is a ring R with a topology τ (ring topology) making R into an additive abelian topological group, such that the following condition is valid:*

- Multiplication Continuity Condition (MC). This condition implies that the mapping $(r_1, r_2) \to r_1 \cdot r_2$ of the topological space $\mathcal{R} \times \mathcal{R}$ to the topological space $\mathcal{R}$ is continuous.

Remark 18 *[4] Condition (MC) exhibits the following:*

For any two elements $r_1, r_2 \in \mathcal{R}$ *and arbitrary neighborhood* U *of the element* $r_1 \cdot r_2$*. Then there exist neighborhoods* V *and* W *of elements* r_1 *and* r_2*, respectively, such that* $V \cdot W \subseteq U$

In the following we are given some examples of topological rings.

Example 19 *[4] The real numbers* $\mathbb{R}$*, together with canonical operations of addition and multiplication and its ordinary topology (interval topology), form a topological ring.*

Example 20 *[4] Consider a commutative ring* $\mathcal{R}$ *and non-zero ideal* I *of ring* $\mathcal{R}$*. The system* $\mathcal{B} = \{I^n \mid n = 1, 2, ...\}$ *of ideals of the ring* $\mathcal{R}$ *defines a topology on* $\mathcal{R}$*. This topology is called* I*-adic topology.* I*-adic topology is the discrete topology if and only if ideal* I *is nilpotent. The anti-discrete topology is unique* I*-adic topology on* $\mathcal{R}$ *if ring is simple non-nilpotent ring. Consider* $I = r\mathbb{Z}$ *being the ideal of the ring of integers* $\mathbb{Z}$*, subsequently* I*-adic topology is known as* r*-adic topology on* $\mathbb{Z}$*.*

1.3.1 Normed (Pseudonormed) Rings

A ring $\mathcal{R}$ is called a normed (pseudonormed) ring if a non-negative real function ψ is specified onto $\mathcal{R}$, and this function satisfies the following conditions:

(NR1): $\psi(r) = 0$ if and only if $r = 0$.

(NR2): $\psi(r_1 - r_2) \leq \psi(r_1) + \psi(r_2)$, for any $r_1, r_2 \in \mathcal{R}$.

(NR3): $\psi(r_1 \cdot r_2) = \psi(r_1) \cdot \psi(r_2)$ (respectively, $\psi(r_1 \cdot r_2) \leq \psi(r_1) \cdot \psi(r_2)$), for every $r_1, r_2 \in \mathcal{R}$.

The number $\psi(r)$ is called a norm (pseudonorm) of the element $r \in \mathcal{R}$, and the function ψ is called a norm (pseudonorm) on the ring $\mathcal{R}$. It is well understood that any norm on the ring $\mathcal{R}$ is a pseudonorm on this ring.

Remark 21 *[4] Let* ψ *be a pseudonorm on the ring* R*, then the following are valid:*

(1) $\psi(-r) = -\psi(r)$ *for any* $r \in \mathcal{R}$*.*

(2) $\psi(r_1) - \psi(r_2) \leq \psi(r_1 - r_2)$ *for any* $r_1, r_2 \in \mathcal{R}$*.*

Theorem 22 *[23] Let* ψ *be a norm on a ring* $\mathcal{R}$*. The topology given by the metric* d *defined by* ψ *is a ring topology.*

Example 23 *[4] Any pseudonorm ring is a topological ring.*

Example 24 *[4] The function corresponding complex number $z = a + ib \in \mathbb{C}$ to its absolute value $|z| = \sqrt{a^2 + b^2}$ satisfies conditions (NR1) to (NR3), and, hence, $\mathbb{C}$ is a normed ring,* `i.e.` *it is a topological ring.*

Example 25 *[4] Let some prime number p be fixed. For each non-zero rational number $r \in \mathbb{Q}$ there exist a unique integer k such that $r = \frac{r_1}{r_2} \cdot p^k$, where $r_1, r_2 \in \mathbb{Z}$, and r_1, r_2 are not divisible by p. Define $\psi_p(r) = \frac{1}{2^k}$ and $\psi_p(0) = 0$. Then the non-negative real valued function ψ_p is defined on $\mathbb{Q}$. Clearly, the function ψ_p satisfies conditions (NR1) to (NR3). Hence the ring $\mathbb{Q}$ is a topological ring in topology τ_p defined by the norm ψ_p.*

Theorem 26 *[4]Let a topological ring $\mathcal{R}$, $r \in \mathcal{R}$, and let P and Q be subsets in $\mathcal{R}$. Then the followings are valid:*

(1) $\phi_r : \mathcal{R} \to \mathcal{R}$ and $\phi'_r : \mathcal{R} \to \mathcal{R}$, where $\phi_r(x) = x \cdot r$ and $\phi'_r(x) = r \cdot x$, are continuous mappings of the topological space $\mathcal{R}$ into itself.

(2) $[P \cdot Q]_{\mathcal{R}} \supseteq [P]_{\mathcal{R}} \cdot [Q]_{\mathcal{R}}$

Corollary 27 *[4] Consider a topological ring $\mathcal{R}$ and $r \in \mathcal{R}$ be an invertible element and let $Q \subseteq \mathcal{R}$. Then following are equivalent:*

(1) Q is open (closed).

(b) $r \cdot Q$ is open (closed).

(c) $Q \cdot r$ is open (closed).

For any topological ring R the statements (1)-(6) of 10 are equivalent.

Hausdorff ring

If a topological ring satisfies one of the conditions $(1) - (6)$ of Theorem 10 then ring is known as Hausdorff ring.

1.3.2 Subrings and Ideals of Topological Rings

Let $\mathcal{R}$ be a topological ring. A subset Q of $\mathcal{R}$ is called a subring of topological ring $\mathcal{R}$ if Q is a subring of $\mathcal{R}$ and ring Q is topological space with the topology induced by the topology of the ring $\mathcal{R}$.

Remark 28 *[4] A subring Q of a topological ring $\mathcal{R}$ is a topological ring.*

Indeed, let $p_1, p_2 \in P$ and U be a neighborhood of the element $p_1 \cdot p_2$ in the topological space P. Then $U = U_1 \cap Q$, where U_1 is a neighborhood of the element $p_1 \cdot p_2$ in the topological space $\mathcal{R}$. Let V_1 and W_1 be neighborhood of elements p_1 and p_2 correspondingly in the topological space $\mathcal{R}$ such that $V_1 \cdot W_1 \subseteq U_1$. Then $V = V_1 \cap Q$ and $W = W_1 \cap Q$ are neighborhoods of elements p_1 and p_2 correspondingly in the topological space Q. Besides,

$$V \cdot W \subseteq (V_1 \cap Q) \cdot (W_1 \cap Q) \subseteq (V_1 \cdot W_1) \cap Q \subseteq U_1 \cap Q = U$$

i.e. *Q with the induced topology satisfies condition MC (see 1.3 and 18). Hence, Q is a topological ring.*

Proposition 29 *[4] Let Q be a subring of a topological ring $\mathcal{R}$. Then $[Q]_{\mathcal{R}}$ is a subring of the topological ring $\mathcal{R}$.*

Corollary 30 *[4] Let Q be a dense ($[Q]_{\mathcal{R}} = \mathcal{R}$) subring of a topological ring $\mathcal{R}$, and $\mathcal{I}$ be a left ideal of the ring Q. Then $[\mathcal{I}]_{\mathcal{R}}$ is a left ideal of the ring $\mathcal{R}$. In particular the closure of a left ideal of the ring $\mathcal{R}$ is also left ideal of the ring $\mathcal{R}$. The result is also valid in case of right and two sided ideal.*

Corollary 31 *[4] Let Q be a dense left ideal of a topological ring $\mathcal{R}$, and $\mathcal{I}$ be a left ideal of the ring Q. The $[\mathcal{I}]_Q$ is a left ideal of the ring $\mathcal{R}$. The result is also valid in case of right and two sided ideal.*

Proposition 32 *[4] In a Hausdorff ring $\mathcal{R}$ the closure $[Q]_{\mathcal{R}}$ of a commutative subring Q is a commutative subring, too.*

Proposition 33 *[4] The center of a Hausdorff topological ring $\mathcal{R}$ is a closed commutative subring.*

1.4 Topological Homomorphism

Definition 34 *[23] Let $\mathcal{A}_1$ and $\mathcal{A}_2$ be topological groups. A mapping $\psi : \mathcal{A}_1 \to \mathcal{A}_2$ is called a topological homomorphism if*

(1) ψ is a group homomorphic mapping from $\mathcal{A}_1$ into $\mathcal{A}_2$.

(2) ψ is the continuous and open mapping of the topological space $\mathcal{A}_1$ into the topological space $\mathcal{A}_2$.

A homomorphic mapping ψ of a topological group $\mathcal{A}_1$ into a topological group $\mathcal{A}_2$ is called open (continuous) if ψ is an open (continuous) mapping of the topological space $\mathcal{A}_1$ into the topological space $\mathcal{A}_2$.

Definition 35 *[23] A mapping ψ of topological group $\mathcal{A}_1$ on a topological group $\mathcal{A}_2$ is called topological isomorphism if*

(1) ψ is an group isomorphic mapping of the abstract group $\mathcal{A}_1$ on the abstract group $\mathcal{A}_2$.

(2) ψ the homeomorphic mapping of the topological space $\mathcal{A}_1$ on the topological space $\mathcal{A}_2$.

Two topological groups are called isomorphic if there exist an isomorphic mapping of one group on the other. Two topological groups may be isomorphic as abstract groups, but not isomorphic as topological groups.

The subgroup $\psi(\mathcal{A}_1)$ of the topological group $\mathcal{A}_2$ is called a continuous, open and topologically homomorphic image of the topological group $\mathcal{A}_1$, correspondingly. An isomorphism of topological group $\mathcal{A}$ into $\mathcal{A}$ itself is called an automorphism of the group $\mathcal{A}$.

Two topological groups may be isomorphic as abstract groups, but not isomorphic as topological groups.

Remark 36 *[4] Let $\mathcal{A}_1$ and $\mathcal{A}_2$ be topological groups. A mapping $\psi : \mathcal{A}_1 \to \mathcal{A}_2$ be an isomorphism of these groups. Then the following are equivalent:*

(1) ψ is open mapping.

(2) $\psi^{-1} : \mathcal{A}_2 \to \mathcal{A}_1$ is continuous mapping.

Consequently, a topological isomorphism of topological groups is an isomorphism of these groups, being a homeomorphism of the corresponding spaces.

Definition 37 *[4] Let $\mathcal{R}_1$ and $\mathcal{R}_2$ be topological rings, $\phi : \mathcal{R}_1 \to \mathcal{R}_2$ be a homomorphic (isomorphic) mapping of the ring $\mathcal{R}_1$ and $\mathcal{R}_2$. We call ϕ a continuous, open or topological homomorphism (isomorphism) of the topological ring $\mathcal{R}_1$ to (onto) the topological rings $\mathcal{R}_2$ if ϕ is correspondingly continuous, open or topological homomorphism (isomorphism) of the additive topological group of $\mathcal{R}_1$ to (onto) the additive topological group of $\mathcal{R}_2$. Meantime, the subring $\phi(\mathcal{R}_1)$ of $\mathcal{R}_2$ is called a continuous, open or topological homomorphic (isomorphic) image of $\mathcal{R}_1$.*

Topological Divisors of Zero

Definition 38 *An element r of a ring $\mathcal{R}$ is called a left (right) topological divisor of zero if there exist a subset $L \subset \mathcal{R}$ such that:*

(1) $0 \notin [L]_{\mathcal{R}}$.

(2) $0 \in [r \cdot L]_{\mathcal{R}}$ *(Correspondingly,* $0 \in [L \cdot r]_{\mathcal{R}}$*).*

If an element r is a left and a right topological divisor of zero then r is known as topological divisor of zero. i.e. there exist $L_1 \subset \mathcal{R}$ and $L_2 \subset \mathcal{R}$ such that $0 \notin [L_1]_{\mathcal{R}}$ and $0 \notin [L_2]_{\mathcal{R}}$, but as well $0 \in [r \cdot L_1]_{\mathcal{R}}$ and $0 \in [r \cdot L_2]_{\mathcal{R}}$.

Sometimes the topological divisors of zero are called generalized divisors of zero.

Example 39 *[4] Let p_1 and p_2 be the different prime numbers. Consider $\mathcal{B}_0(\mathbb{Z}) = \{Q_k \,|k = 1, 2, ...\}$ of the subsets of the ring $\mathbb{Z}$ of integers, where Q_k is the ideal in $\mathbb{Z}$ generated by $p_1 \cdot p_2^k$,* i.e. *$Q_k = p_1 p_2^k \cdot \mathbb{Z}$ for $k = 1, 2, 3....$ The system $\mathcal{B}_0(\mathbb{Z})$ defines on $\mathbb{Z}$ the Hausdorff ring topology $\tau_{p_1 \cdot p_2}$ and $(\mathbb{Z}, \tau_{p_1 \cdot p_2})$ is a Hausdorff topological ring. The subset $L = \{p_2^k \,|k = 1, 2, ...\}$ does not contain the numbers, divisible by p_1, hence $L \cap Q_k n = \Phi$,* i.e.*$0 \notin [L]_{(\mathbb{Z}, \tau_{p_1 \cdot p_2})}$. Since $p_1 \cdot p_2^k \in A_k$, then $(p_1 \cdot L) \cap Q_k \neq \Phi$ for any $k = 1, 2, ...,$* i.e. *$0 \in [p_1 \cdot L]_{(\mathbb{Z}, \tau_{p_1 \cdot p_2})}$. Therefore p_1 is a topological divisor of zero in the ring $(\mathbb{Z}, \tau_{p_1 \cdot p_2})$.*

Remark 40 *[4] In a Hausdorff topological ring any left (right) zero divisor is a left (right) topological divisor of zero.*

Example 41 *[4] A topological ring with anti-discrete topology has neither left nor right topological divisors of zero, though it can be a ring with zero multiplication.*

Remark 42 *[4] Let* $\mathcal{R}$ *be a topological ring,* Q *be a subring of* $\mathcal{R}$ *and an element* $r \in \mathcal{R}$ *be a left (right) in the topological divisor of zero in* $\mathcal{R}$*. Then* r *is a left (right) topological divisor of zero in* Q*.*

Proposition 43 *[4] Let* Q *be a subring of a topological ring* $\mathcal{R}$ *with the unitary element. If an element* $r \in \mathcal{R}$ *is invertible from left (right) in* Q*, then* r *is not a left (right) topological divisor of zero in the topological ring* $\mathcal{R}$*.*

Proposition 44 *[4] In a topological ring* $\mathcal{R}$ *let* r_1 *be a left topological divisor of zero . Then for any* $r_2 \in \mathcal{R}$ *the element* $r_2 \cdot r_1$ *is a left topological divisor of zero in* $\mathcal{R}$*.*

Proposition 45 *[4] Let* $\mathcal{R}$ *be a Hausdorff compact topological ring and an element* $r \in \mathcal{R}$ *be not a left (right) divisor of zero in* $\mathcal{R}$*. Then* r *is not a left (right) topological divisor of zero in* $\mathcal{R}$*.*

Proposition 46 *[4] Let* $\mathcal{R}$ *be a topological ring and* $r_1, r_2 \in \mathcal{R}$*. If* $r_1 \cdot r_2$ *is a left (right) topological divisor of zero in* $\mathcal{R}$*, then either* r_1 *or* r_2 *is a left (right) topological divisor of zero.*

Corollary 47 *[4] Let* $l(\mathcal{R})$ *be the system of all left, and* $r(\mathcal{R})$ *be the system of all right topological divisor of zero in the topological ring* $\mathcal{R}$*. Then the subsets* $\mathcal{R} \diagdown l(\mathcal{R})$ *and* $\mathcal{R} \diagdown r(\mathcal{R})$ *are the sub-semigroup of the multiplicative semigroup of the ring* $\mathcal{R}$*.*

Chapter 2

Operations on Soft Sets

2.1 Introduction

This chapter is of introductory nature. In the first section we give basic definition of soft set and review some of the background material that will be value for our later chapter. In the second section we deal the soft algebraic structures along with few examples to illustrate these definitions. Particularly, we discuss soft groups, soft rings, idealistic soft rings, soft ideals and soft zero divisors.

2.2 Soft Sets

Definition 48 *[16] Let $\mathcal{X}$ be the initial universe and the set of parameters be $\mathcal{Y}$. Let $2^{\mathcal{X}}$ denotes the power set of $\mathcal{X}$ and Υ be a non-empty subset of $\mathcal{Y}$. A pair (ξ, Υ) is called a soft set over $\mathcal{X}$, where ξ is a mapping given by $\xi : \Upsilon \rightarrow 2^{\mathcal{X}}$.*

Soft set is a parametrized family of subsets of the universe X. A soft set (ξ, Υ) is a collection of fairly accurate description of an object. Obviously soft set is not a set in the crisp sense.

Example 49 *[15] Consider a soft set (ξ, Υ) which depicts the "attractiveness of mobile phones" that Mr.Z (say) wants to buy for his family and friends.*

Suppose that there are six types of mobile phones that are available in the universe i.e. $\mathcal{X} =$

$\{m_1, m_2, m_3, m_4, m_5, m_6\}$, *and to select mobile phones the set of parameters is*

$$\mathcal{Y} = \{y_1, y_2, y_3, y_4, y_5, y_6, y_7, y_8\}, \textit{where}$$

y_1 *is for 'beautiful outlook',*	y_2 *is for 'modern'*
y_3 *is for'larger coverage' ,*	y_4 *is for 'cheap'*
y_5 *is for'in good repair',*	y_6 *is for 'expensive'*
y_7 *is for 'in bad repair',*	y_8 *is for 'more functions'*

Take $\Upsilon = \{$*modern; larger coverage; cheap; in bad repair; expensive;*$\}$ *and suppose that*

$$\xi(y_1) = \{m_1, m_3\}$$
$$\xi(y_3) = \{m_4, m_6\}$$
$$\xi(y_4) = \{m_2, m_4, m_5\}$$
$$\xi(y_6) = \{m_3, m_4, m_6\}$$
$$\xi(y_7) = \{m_5\}$$

The soft set (ξ, Υ) *is a parametrized family* $\{\xi(y_1), \xi(y_3), \xi(y_4), \xi(y_6), \xi(y_7)\}$ *of subsets of the set* $\mathcal{X}$ *and gives us a description of an object's collection. To define a soft set we means to point out modern mobile phones and so on. Therefore,* $\xi(y_1)$ *means "mobile phones (modern)" whose value-set is* $\{m_1, m_3\}$.

Thus, the soft set (ξ, Υ) *can be viewed as a collection of approximations as:* $(\xi, \Upsilon) = \{$*modern mobile phones* $= \{m_1, m_3\}$, *larger coverage mobile phones* $= \{m_4, m_6\}$, *cheap mobile phones* $= \{m_2, m_4, m_5\}$, *expensive mobile phones* $= \{m_3, m_4, m_6\}$, *in bad repair mobile phones* $= \{m_5\}\}$, *where each has two parts:*

(i) a predicate.

(ii) a value-set.

For example, for "expensive mobile phones $= \{m_3, m_4, m_6\}$*", we have the following:*

(i) the predicate name is expensive mobile phones and

(ii) the value set is $\{m_3, m_4, m_6\}$.

Tabular Representation

Tabular representation of soft sets were given by Lin [11] and Yao [24]. Here an almost similar representation in the form of a binary table is given. For the purpose of storing a soft set in a computer memory, we could represent a soft set in the form of Table 1, (corresponding to the soft set in the Example 49).

$\mathcal{X}$	'modern'	'larger coverage'	'cheap'	'expensive'	'in bad repair'
m_1	1	0	0	0	0
m_2	0	0	1	0	0
m_3	1	0	0	1	0
m_4	0	1	1	1	0
m_5	0	0	1	0	1
m_6	0	1	0	1	0

Table 1

If $m_i \in \xi(y)$ then $m_{ij} = 1$, otherwise $m_{ij} = 0$, where m_{ij} are the entries in Table 1.

Definition 50 *Let (ξ, Υ) be a soft set. Then the collection of all of its value sets is said to be value class and is denoted by $C_{(\xi,\Upsilon)}$.*

Clearly, $C_{(\xi,\Upsilon)} \subseteq 2^{\mathcal{X}}$.

Definition 51 *[15] For two soft sets (ξ_1, Υ_1) and (ξ_2, Υ_2) over $\mathcal{X}$, we say that (ξ_1, Υ_1) is a soft subset of (ξ_2, Υ_2), denoted by $(\xi_1, \Upsilon_1)\widetilde{\subset}(\xi_2, \Upsilon_2)$, if*

(a) $\Upsilon_1 \subset \Upsilon_2$ and

(b) $\xi_1(y) \subseteq \xi_2(y)$, for each $y \in \Upsilon_1$.

Similarly, (ξ_1, Υ_1) is said to be a soft super set of (ξ_2, Υ_2), if (ξ_2, Υ_2) is a soft subset of (ξ_1, Υ_1) and it is denoted by $(\xi_1, \Upsilon_1)\widetilde{\supset}(\xi_2, \Upsilon_2)$.

Definition 52 *[15] Two soft sets (ξ_1, Υ_1) and (ξ_2, Υ_2) over a universe $\mathcal{X}$ are said to be soft equal if (ξ_1, Υ_1) is a soft subset of (ξ_2, Υ_2) and (ξ_2, Υ_2) is a soft subset of (ξ_1, Υ_1).*

Definition 53 *Let (ξ, Υ) be a soft set over $\mathcal{X}$ and $x \in \mathcal{X}$. We say that $x \in (\xi, \Upsilon)$ read as x belongs to the soft set (ξ, Υ) whenever $x \in \xi(y)$, $\forall\ y \in \Upsilon$.*

Note that for any $x \in \mathcal{X}$, $x \notin (\xi, \Upsilon)$, if $x \notin \xi(y)$, for some $y \in \Upsilon$.

Definition 54 *[?] Let $x \in \mathcal{X}$, then $(x, \mathcal{Y})$ denotes the soft set over $\mathcal{X}$ for which $x(y) = \{x\}$, for all $y \in \mathcal{Y}$.*

Definition 55 *[15] Let $\mathcal{Y} = \{y_1, y_2, ..., y_n\}$ be a set of parameters. The NOT set of $\mathcal{Y}$ denoted by $\rceil\mathcal{Y}$ is defined as $\rceil\mathcal{Y} = \{_\rceil y_1, _\rceil y_2, ..., _\rceil y_n\}$ where $_\rceil y_i = not\ y_i$, for all i.*

Proposition 56 *[15] Let $\mathcal{Y} = \{y_1, y_2, ..., y_n\}$ be a set of parameters and Υ, Υ_1 and Υ_2 be the non-empty subsets of Y. Then*

(1) $\rceil(\rceil\Upsilon) = \Upsilon$.

(2) $\rceil(\Upsilon_1 \cup \Upsilon_2) = \rceil\Upsilon_1 \cup \rceil\Upsilon_2$.

(3) $\rceil(\Upsilon_1 \cap \Upsilon_2) = \rceil\Upsilon_1 \cap \rceil\Upsilon_2$.

Definition 57 *[15] Let (ξ, Υ) be the soft set. Then the complement of (ξ, Υ) is represented by $(\xi, \Upsilon)^c$ and is defined by $(\xi, \Upsilon)^c = (\xi^c, \rceil\Upsilon')$, where $\xi^c : \rceil\Upsilon \to 2^{\mathcal{X}}$ is a mapping defined by $\xi^c(y) = \mathcal{X} \setminus \xi(_\rceil y)$, for all $y \in \rceil\Upsilon$.*

The soft complement function of ξ is ξ^c. Obviously $(\xi^c)^c$ is the same as ξ and $((\xi, \Upsilon)^c)^c = (\xi, \Upsilon)$.

Definition 58 *[15] A soft set (ξ, Υ) over $\mathcal{X}$ is called a NULL soft set, if for all $y \in \Upsilon$, $\xi(y) = \emptyset$ (null set) and denoted by Φ.*

Definition 59 *[15] A soft set (ξ, Υ) over $\mathcal{X}$ is to be Absoulte soft set by denoted by $\widetilde{\mathcal{Y}}$, if for all $y \in \Upsilon$, $\xi(y) = \mathcal{X}$. Obviously $\widetilde{\mathcal{Y}}^c = \emptyset$ and $\emptyset^c = \widetilde{\mathcal{Y}}$.*

Definition 60 *[15] If (ξ_1, Υ_1) and (ξ_2, Υ_2) are two soft sets over $\mathcal{X}$, then "(ξ_1, Υ_1) AND (ξ_2, Υ_2)" denoted by $(\xi_1, \Upsilon_1)\tilde{\wedge}(\xi_2, \Upsilon_2)$ is defined by $(\xi_1, \Upsilon_1)\tilde{\wedge}(\xi_2, \Upsilon_2) = (\xi', \Upsilon_1 \times \Upsilon_2)$, where $\xi'((y_1, y_2)) = \xi_1(y_1) \cap \xi_2(y_2)$, for all $(y_1, y_2) \in \Upsilon_1 \times \Upsilon_2$.*

Definition 61 *[15] If (ξ_1, Υ_1) and (ξ_2, Υ_2) are two soft sets over $\mathcal{X}$, then "(ξ_1, Υ_1) OR (ξ_2, Υ_2)" denoted by $(\xi_1, \Upsilon_1)\tilde{\vee}(\xi_2, \Upsilon_2)$ is defined by $(\xi_1, \Upsilon_1)\tilde{\vee}(\xi_2, \Upsilon_2) = (\xi', \Upsilon_1 \times \Upsilon_2)$ where, $\xi'((y_1, y_2)) = \xi_1(y_1) \cup \xi_2(y_2)$, for all $(y_1, y_2) \in \Upsilon_1 \times \Upsilon_2$.*

Proposition 62 *[3] For two soft set* (ξ_1, Υ_1) *and* (ξ_2, Υ_2) *over* $\mathcal{X}$ *we have the following results.*

(1) $((\xi_1, \Upsilon_1)\tilde{\vee}(\xi_2, \Upsilon_2))^c = (\xi_1, \Upsilon_1)^c\tilde{\wedge}(\xi_2, \Upsilon_2)^c$.

(2) $((\xi_1, \Upsilon_1)\tilde{\wedge}(\xi_2, \Upsilon_2))^c = (\xi_1, \Upsilon_1)^c\tilde{\vee}(\xi_2, \Upsilon_2)^c$.

Definition 63 *[15] Union of two soft sets* (ξ_1, Υ_1) *and* (ξ_2, Υ_2) *over a common universe* $\mathcal{X}$ *is the soft set* (ξ', Υ'), *where* $\Upsilon' = \Upsilon_1 \cup \Upsilon_2$, *and for all* $y \in \Upsilon'$,

$$\xi'(y) = \begin{cases} \xi_1(y) & if\ y \in \Upsilon_1 - \Upsilon_2 \\ \xi_2(y) & if\ y \in \Upsilon_2 - \Upsilon_1 \\ \xi_1(y) \cup \xi_2(y) & if\ y \in \Upsilon_1 \cap \Upsilon_2 \end{cases}$$

We write $(\xi_1, \Upsilon_1)\tilde{\cup}(\xi_2, \Upsilon_2) = (\xi', \Upsilon')$.

Definition 64 *[9] Bi-intersection of two soft sets* (ξ_1, Υ_1) *and* (ξ_2, Υ_2) *over the common universe* $\mathcal{X}$ *is the soft set* (ξ', Υ'), *denoted by* $(\xi_1, \Upsilon_1)\widetilde{\sqcap}(\xi_2, \Upsilon_2)$, *is defined as* $(\xi_1, \Upsilon_1)\widetilde{\sqcap}(\xi_2, \Upsilon_2) = (\xi', \Upsilon')$, *where* $\Upsilon = \Upsilon_1 \cap \Upsilon_2$, *and* $\xi'(y) = \xi_1(y) \cap \xi_2(y)$, *for all* $y \in \Upsilon$.

Definition 65 *[9] Let* (ξ_1, Υ_1) *and* (ξ_2, Υ_2) *be two soft sets over* $\mathcal{X}$. *The intersection* (ξ_1, Υ_1) *and* (ξ_2, Υ_2) *is defined as the soft set* (ξ', Υ') *satisfying the following conditions:*

(1) $\Upsilon' = \Upsilon_1 \cap \Upsilon_2$.

(2) For all $y \in \Upsilon'$, $\xi'(y) = \xi_1(y)$ *or* $\xi_2(y)$.

In this case, we write $(\xi_1, \Upsilon_1)\widetilde{\cap}(\xi_2, \Upsilon_2) = (\xi', \Upsilon')$.

Definition 66 *[3] Extended intersection of two soft sets* (ξ_1, Υ_1) *and* (ξ_2, Υ_2) *over* $\mathcal{X}$ *is the soft set* (ξ', Υ'), *where* $\Upsilon' = \Upsilon_1 \cup \Upsilon_2$, *and for all* $y \in \Upsilon'$,

$$\xi'(y) = \begin{cases} \xi_1(y) & if\ y \in \Upsilon_1 - \Upsilon_2 \\ \xi_2(y) & if\ y \in \Upsilon_2 - \Upsilon_1 \\ \xi_1(y) \cap \xi_2(y) & if\ y \in \Upsilon_1 \cap \Upsilon_2 \end{cases}$$

We write $(\xi_1, \Upsilon_1) \cap_{\Upsilon} (\xi_2, \Upsilon_2) = (\xi', \Upsilon')$.

Definition 67 *The restricted intersection of two soft sets* (ξ_1, Υ_1) *and* (ξ_2, Υ_2) *over a common universe* $\mathcal{X}$ *is denoted by* $(\xi_1, \Upsilon_1) \cap (\xi_2, \Upsilon_2)$, *defined as* $(\xi_1, \Upsilon_1) \Cap (\xi_2, \Upsilon_2) = (\xi', \Upsilon')$, *where* $\Upsilon' = \Upsilon_1 \cap \Upsilon_2$ *and* $\xi'(y) = \xi_1(y) \cap \xi_2(y)$, *for all* $y \in \Upsilon'$.

Note that restricted intersection was also known as bi-intersection ($\widetilde{\sqcap}$) in Feng et al. , and extended union was first introduce and called union by Maji et al.

Definition 68 *[8] The extended sum of two soft sets* (ξ_1, Υ_1) *and* (ξ_2, Υ_2) *over a common universe* $\mathcal{X}$ *is denoted by* $(\xi_1, \Upsilon_1) \oplus_{\cup} (\xi_2, \Upsilon_2)$*, defined as* $(\xi_1, \Upsilon_1) \oplus_{\cup} (\xi_2, \Upsilon_2) = (\xi', \Upsilon')$*, where* $\Upsilon' = \Upsilon_1 \cup \Upsilon_2$*, and for all* $y \in \Upsilon$*.*

$$\xi'(y) = \begin{cases} \xi_1(y) & if\ y \in \Upsilon_1 - \Upsilon_2 \\ \xi_2(y) & if\ y \in \Upsilon_2 - \Upsilon_1 \\ \xi_1(y) + \xi_2(y) & if\ y \in \Upsilon_1 \cap \Upsilon_2 \end{cases}$$

Definition 69 *[8] The restricted sum of two soft sets* (ξ_1, Υ_1) *and* (ξ_2, Υ_2) *over* $\mathcal{X}$ *is denoted by* $(\xi_1, \Upsilon_1) \oplus_{\cap} (\xi_2, \Upsilon_2)$*, is defined as* $(\xi_1, \Upsilon_1) \oplus_{\cap} (\xi_2, \Upsilon_2) = (\xi', \Upsilon')$*, where* $\Upsilon' = \Upsilon_1 \cap \Upsilon_2$ *and* $\xi'(y) = \xi_1(y) + \xi_2(y)$*, for all* $y \in \Upsilon'$*.*

Definition 70 *[8] Extended product of two soft sets* (ξ_1, Υ_1) *and* (ξ_2, Υ_2) *over a common universe* $\mathcal{X}$ *is denoted by* $(\xi_1, \Upsilon_1) \odot_{\cup} (\xi_2, \Upsilon_2)$*, is defined as* $(\xi_1, \Upsilon_1) \odot_{\cup} (\xi_2, \Upsilon_2) = (\xi', \Upsilon')$*, where* $\Upsilon' = \Upsilon_1 \cup \Upsilon_2$*, and for all* $y \in \Upsilon'$*.*

$$\xi'(y) = \begin{cases} \xi_1(y) & if\ y \in \Upsilon_1 - \Upsilon_2 \\ \xi_2(y) & if\ y \in \Upsilon_2 - \Upsilon_1 \\ \xi_1(y) \cdot \xi_2(y) & if\ y \in \Upsilon_1 \cap \Upsilon_2 \end{cases}$$

Definition 71 *[8] The restricted product of two soft sets* (ξ_1, Υ_1) *and* (ξ_2, Υ_2) *over a common universe* $\mathcal{X}$ *is denoted by* $(\xi_1, \Upsilon_1) \odot_{\cap} (\xi_2, \Upsilon_2)$*, is defined as* $(\xi_1, \Upsilon_1) \odot_{\cap} (\xi_2, \Upsilon_2) = (\xi', \Upsilon')$*, where* $\Upsilon' = \Upsilon_1 \cap \Upsilon_2$ *and* $\xi'(y) = \xi_1(y) \cdot \xi_2(y)$*, for all* $y \in \Upsilon'$*.*

Definition 72 *[1] Let* (ξ, Υ) *be a soft set. The set Supp*$(\xi, \Upsilon) = \{y \in \Upsilon : \xi(y) \neq \varnothing\}$ *is called the support of the soft set* (ξ, Υ)*. If the support of a soft set is non-empty then the soft set is said to be non-null.*

2.3 Some Algebraic Notions on Soft Sets

Soft Groups

Definition 73 *[2] Let (ξ, Υ) be a non-null soft set over a ring A. Then (ξ, Υ) is called a soft group over A if $\xi(y)$ is a subgroup of A, for all $y \in \Upsilon$.*

Example 74 *Consider the group $\mathcal{A} = D_4 = \{e, a, a^2, a^3, b, ab, a^2b, a^3b\}$ which is generated by two elements a and b satisfying the relations $a^4 = e$, $b^2 = e$ and $ba = a^3b$. Consider $\Upsilon = \{y_1, y_2, y_3\}$ and define mapping $\xi : \Upsilon \to 2^{\mathcal{A}}$ such that $\xi(y_1) = \{e, a, a^2, a^3\}$, $\xi(y_2) = \{e, a^2, b, a^2b\}$ and $\xi(a_3) = \{e, ab, a^2, a^3b\}$. For each parameter, $\xi(y)$ is a subgroup of D_4. This implies that the pair (ξ, Υ) is a soft group over $\mathcal{A}$. Any soft group can be represented in the form of table such as*

$\mathcal{A}$	y_1	y_2	y_3
e	1	1	1
a	1	0	0
a^2	1	1	1
a^3	1	0	0
b	0	1	0
ab	0	0	1
a^2b	0	1	0
a^3b	0	0	1

Table 2

For every soft set one can not produce a soft group. For example if $\mathcal{A} = \xi_3$ and $\xi(y) = \{y' \in \mathcal{A} : o(y) = o(y')\}$, where $o(y)$ is order of y in $\mathcal{A}$, then (ξ, Υ) is not a soft group over $\mathcal{A}$.

Definition 75 *[2] Let (ξ_1, Υ_1) and (ξ_2, Υ_2) be two soft sets over $\mathcal{A}$. Then (ξ_1, Υ_1) is a soft subgroup of (ξ_2, Υ_2), written $(\xi_1, \Upsilon_1)\tilde{\prec}(\xi_2, \Upsilon_2)$, if*

(1) $\Upsilon_1 \subset \Upsilon_2$.

(2) $\xi_1(y) < \xi_2(y)$, for all $y \in \Upsilon_1$.

Example 76 *[2] Let $\mathcal{A} = \Upsilon = \xi_3$, $\Upsilon_1 = S_3$ and $\Upsilon_2 = A_3 \prec \xi_3$ and $\Upsilon_1 = \xi_3$. Define the functions $\xi_1(y) = \{y' \in \xi_3 : y' = y^n, n \in N\}$ and $\xi_2(y) = \{y' \in \Upsilon_3 : y' \in < y >\}$. Clearly, $(\xi_2, \Upsilon_2)\tilde{\prec}(\xi_1, \Upsilon_1)$ because $A_3 \subset \xi_3$ and $\xi_2(y) < \xi_1(y)$, for all $y \in \Upsilon_3$.*

Theorem 77 *[2] (ξ, Υ) is a soft group over $\mathcal{A}$ and $\{(\xi_i, \Upsilon_i) : i \in I\}$ be a non-empty family of soft subgroups of (ξ, Υ). Then*

(1) $\cap_{i \in I}(\xi_i, \Upsilon_i)$ is a soft subgroup of (ξ, Υ).

(2) $\tilde{\wedge}_{i \in I}(\xi_i, \Upsilon_i)$ is a soft subgroup of (ξ, Υ).

(3) $\tilde{\vee}_{i \in I}(\xi_i, \Upsilon_i)$ is a soft subgroup of (ξ, Υ) if $\Upsilon_i \cap \Upsilon_j = \emptyset$ for all $i, j \in \wedge$, $i \neq j$.

Soft Rings

Definition 78 *[1] Let (ξ, Υ) be a non-null soft set over a ring $\mathcal{R}$. Then (ξ, Υ) is called a soft ring over $\mathcal{R}$ if $\xi(y)$ is a subring of $\mathcal{R}$, for all $y \in \Upsilon$.*

Definition 79 *[1] Let (ξ_1, Υ_1) and (ξ_2, Υ_2) be soft rings over $\mathcal{R}$. Then (ξ_1, Υ_1) is called a soft subring of (ξ_2, Υ_2) if*

(a) $\Upsilon_1 \subset \Upsilon_2$.

(b) $\xi_1(y)$ is a subring of $\xi_2(y)$, for all $y \in Supp(\xi_2, \Upsilon_2)$.

Theorem 80 *[1] Let $(\xi_i, \Upsilon_i)_{i \in I}$ be a non-empty family of soft rings over $\mathcal{R}$. Then*

(1) $\tilde{\wedge}_{i \in I}(\xi_i, \Upsilon_i)$ is a soft ring over $\mathcal{R}$ if it is non-null.

(2) $\widetilde{\sqcap}_{i \in I}(\xi_i, \Upsilon_i)$ is a soft ring over $\mathcal{R}$ if it is non-null.

(3) $\widetilde{\cup}_{i \in I}(\xi_i, \Upsilon_i)$ is a soft ring over $\mathcal{R}$ if $\Upsilon_i \cap \Upsilon_j = \emptyset$, for all $i, j \in \wedge$, $i \neq j$.

Definition 81 *[1] Let (ξ, Υ) be a soft set over $\mathcal{R}$. Then (ξ, Υ) is called an idealistic soft ring over $\mathcal{R}$ if $\xi(y)$ is an ideal of $\mathcal{R}$, for all $y \in \sup p(\xi, \Upsilon)$.*

Theorem 82 *[8] Let $\{(\xi_i, \Upsilon_i) | i \in \wedge\}$ be a non-empty family of idealistic soft rings over $\mathcal{R}$. Then*

(1) The restricted intersection of the family $\{(\xi_i, \Upsilon_i) | i \in \wedge\}$ is an idealistic soft ring over $\mathcal{R}$ if it is non-null.

(2) If $\Upsilon_i \cap \Upsilon_j = \emptyset$ for all $i, j \in \wedge$, $i \neq j$, then the extended union of the family$\{(\xi_i, \Upsilon_i) | i \in \wedge\}$ is an idealistic soft ring over $\mathcal{R}$.

Soft Ideals

Definition 83 *[1] Let* (ξ_1, Υ_1) *and* (ξ_2, Υ_2) *be soft rings over* $\mathcal{R}$*. Then* (ξ_1, Υ_1) *is called a soft ideal of* (ξ_2, Υ_2)*, which will be denoted by* $(\xi_1, \Upsilon_1)\tilde{\lhd}(\xi_2, \Upsilon_2)$*, if it satisfies the following conditions:*

(a) $\Upsilon_1 \subset \Upsilon_2$*.*

(b) $\xi_1(y)$ *is an ideal of* $\xi_2(y)$*, for all* $y \in Supp(\xi_1, \Upsilon_1)$*.*

Theorem 84 *[1] Let* (ξ_1', Υ_1') *and* (ξ_2', Υ_2') *be soft ideals of soft rings* (ξ_1, Υ_1) *and* (ξ_2, Υ_2) *over* $\mathcal{R}$ *respectively. Then* $(\xi_1', \Upsilon_1')\widetilde{\sqcap}(\xi_2', \Upsilon_2')$ *is a soft ideal of* $(\xi_1, \Upsilon_1)\widetilde{\sqcap}(\xi_2, \Upsilon_2)$ *if it is non-null.*

Definition 85 *[22] (a) Let* (ξ, Υ) *be a soft ring over the ring* $\mathcal{R}$*. Let* $\{0\} \neq \xi(y_1) \in (\xi, \Upsilon)$*, then* $\xi(y_1)$ *is said to be a soft left (resp. soft right) zero divisor in* (ξ, Υ) *if there exist some elements* $\{0\} \neq \xi(y_2) \in (\xi, \Upsilon)$ *such that* $\xi(y_1) \cdot \xi(y_2) = \{0\}$ *(resp.* $\xi(y_2) \cdot \xi(y_1) = \{0\}$*).*

(b) A soft zero divisor in (ξ, Υ) *is either a soft left zero divisor or a soft right zero divisor.*

Chapter 3

Soft Topological Groups and Rings

3.1 Introduction

In this chapter we are initiating the study of soft topological structures. Nazmul and Samanta (2010) initiated a soft set over a topological group and call it a soft topological group. In the first section we generalize the concept of a soft topological group which was defined by Nazamul and Samanta in [18]. Furthermore, we also dealt with some of the algebraic properties of soft topological groups. In the second section we deal with the basic definition of the soft topological ring, examples of soft topological rings and theorems of the soft topological ring. It is also provided in this section that under what condition every soft ring becomes a soft topological ring. In the third section the idealistic approach of soft topological rings is discussed. In the last section we define closure of a soft topological ring which helps in defining a soft topological divisor of zero which, in general, are different from a soft zero divisors of (Tariq Shah and Zoya Abdullah).

3.2 Soft Topological Groups

In this section, $\mathcal{A}$ denotes a group and $(\mathcal{A}, \tau)$ denotes a topological group. We start with the following definition.

Definition 86 *Let τ be a topology and (ξ, Υ) be a non-null soft set defined over $\mathcal{A}$. Then the triplet (ξ, Υ, τ) is called the soft topological group over $\mathcal{A}$ if* *(a) For all $y \in \Upsilon$, $\xi(y)$ is a subgroup of $\mathcal{A}$.*

(b) For all $y \in \Upsilon$, the mapping $(a, b) \mapsto a - b$ of the topological space $\xi(y) \times \xi(y)$ onto $\xi(y)$ is continuous.

If $\mathcal{A}$ is a topological group, then Definition 86 coincide with [18, Defination 3.1].

In terms of neighborhoods, conditions (b) of Definition 86 imply that for any $a, b \in \xi(y)$ and arbitrary neighborhood W of $a - b$ there exist neighborhoods W_1 and W_2 of elements a and b, respectively, such that $W_1 - W_2 \subset W$.

Example 87 *Take $\mathcal{A} = S_3 = \{e, (12), (13), (23), (123), (132)\}$, $\Upsilon = \{y_1, y_2, y_3\}$ and base for the topology τ is $B = \{\varnothing, \{e\}, \{(12)\}, \{(123)\}, \{(132)\}, S_3\}$. The set valued function ξ is defined by*

$$\xi(y_1) = \{e\},\ \xi(y_2) = \{e, (12)\} \text{ and } \xi(y_3) = \{e, (123), (132)\}$$

Clearly, $\xi(y)$ is a subgroup of $\mathcal{A}$, for all $y \in \Upsilon$. Also condition (b) of Definition 86 is satisfied. Hence (ξ, Υ, τ) is a soft topological group.

Remark 88 *Let $\mathcal{A}$ be a group. Then every soft group can be transformed into a soft topological group over $\mathcal{A}$ by endowing $\mathcal{A}$ with discrete or anti-discrete topology. It is easy to verify that any soft group satisfies the condition (b) of Definition 86 in both topologies. In this manner any soft group can be considered as a soft topological group in the discrete or anti-discrete topology.*

Theorem 89 *Every soft group over a topological group (non-discrete) is a soft topological group.*

Proof. Let $(\mathcal{A}, \tau)$ be topological group and (ξ, Υ) be a soft group over $\mathcal{A}$. So for all $y \in \Upsilon$, $\xi(y)$ is a subgroup of $\mathcal{A}$. Since $\mathcal{A}$ is a topological group and the mapping $(a, b) \to a - b$ of the topological space $\mathcal{A} \times \mathcal{A}$ onto $\mathcal{A}$ is continuous, so its restriction from $\xi(y) \times \xi(y)$ onto $\xi(y)$ is also continuous. Hence (ξ, Υ, τ) is a soft topological group over $(\mathcal{A}, \tau)$. ■

Remark 90 *Every Soft group over a group need not to be a soft topological group. Since in Example 87, (ξ, Υ, τ) is a soft topological group with defined topology. Take*

$$\tau = \{\varnothing, \{e\}, \{e, (12)\}, \{e, (123), (132)\}, S_3\}$$

Then $\xi(y_3) = \{e, (123), (132)\}$ does not satisfy the condition (b) of Definition 86. Hence (ξ, Υ, τ) is not a soft topological group. Thus, this example demonstrates that the condition of being a topological group for the group $\mathcal{A}$ in Theorem 89 is important.

Theorem 91 *If $(\xi_1, \Upsilon_1, \tau)$ and $(\xi_2, \Upsilon_2, \tau)$ are two soft topological groups over $\mathcal{A}$. Then*

(1) The bi-intersection $(\xi_1, \Upsilon_1, \tau)\widetilde{\sqcap}(\xi_2, \Upsilon_2, \tau)$ is a topological group over $\mathcal{A}$ if it is non-null.

(2) Extended intersection $(\xi_1, \Upsilon_1, \tau) \cap_\varepsilon (\xi_2, \Upsilon_2, \tau)$ is a soft topological group over $\mathcal{A}$.

Proof. (1) Since $(\xi_1, \Upsilon_1, \tau)$ and $(\xi_2, \Upsilon_2, \tau)$ are soft topological groups over $\mathcal{A}$. Therefore by Definition 65 their bi-intersection over $\mathcal{A}$ is the soft topological set (ξ', Υ', τ), where $\Upsilon' = \Upsilon_1 \cap \Upsilon_2$, and for all $y \in \Upsilon'$, it is defined as $\xi'(y) = \xi_1(y) \cap \xi_2(y)$. Since both $\xi_1(y)$ and $\xi_2(y)$ are subgroups, therefore $\xi'(y)$ is a subgroup of $\mathcal{A}$, for all $y \in \Upsilon_1 \cap \Upsilon_2$. Also $\xi'(y) \subset \xi_1(y)$ and $\xi_2(y)$ and condition (b) of Definition 86 holds for $\xi_1(y)$. So its also holds for $\xi'(y)$, for all $y \in \Upsilon'$. Hence $(\xi_1, \Upsilon_1, \tau)\widetilde{\sqcap}(\xi_2, \Upsilon_2, \tau)$ is a topological group over $\mathcal{A}$.

(2) Obvious. ■

Theorem 92 *If $(\xi_1, \Upsilon_1, \tau)$ and $(\xi_2, \Upsilon_2, \tau)$ are two soft topological groups over $\mathcal{A}$, where τ is a topology defined over $\mathcal{A}$. Then $(\xi_1, \Upsilon_1, \tau)\tilde{\wedge}(\xi_2, \Upsilon_2, \tau)$ is a soft topological group over $\mathcal{A}$ if it is non-null.*

Proof. The proof is similar to the proof of 91. ■

If $\mathcal{A}$ is a topological group, then Theorem 91 coincide with [18, Theorem 3.4].

Theorem 93 *If $(\xi_1, \Upsilon_1, \tau)$ and $(\xi_2, \Upsilon_2, \tau)$ are two soft topological groups over $\mathcal{A}$, where τ is a topology defined over $\mathcal{A}$. If Υ_1 and Υ_2 are disjoint, then $(\xi_1, \Upsilon_1, \tau)\tilde{\cup}(\xi_2, \Upsilon_2, \tau)$ is a topological group over $\mathcal{A}$.*

Proof. The proof is similar to the proof of Theorem 91. ■

Definition 94 *A soft topological group (ξ, Υ, τ) is said to be soft trivial if $\xi(y) = \{0\}$, for all $y \in \Upsilon$ and whole if $\xi(y) = \mathcal{A}$, for all $y \in \Upsilon$.*

If $\mathcal{A}$ is a topological group, then Definition 94 coincides with [18, Definatin 3.5].

Definition 95 *Let* $(\xi_1, \Upsilon_1, \tau)$ *be a soft topological group over* $\mathcal{A}$*. Then* $(\xi_2, \Upsilon_2, \tau)$ *is said to be a soft topological subgroup (resp. normal subgroup) of* $(\xi_1, \Upsilon_1, \tau)$ *if*

(a) $\Upsilon_2 \subset \Upsilon_1$ *and* $\xi_2(y)$ *is a subgroup (resp. normal subgroup) of* $\xi_1(y)$*, for all* $y \in \sup(\xi_2, \Upsilon_2)$*.*

(b) For all $y \in \Upsilon_2$*, the mapping* $(a, b) \to a - b$ *of the topological space* $\xi_2(y) \times \xi_2(y)$ *onto* $\xi_2(y)$ *is continuous.*

If $\mathcal{A}$ is a topological group and $\Upsilon_1 = \Upsilon_2$, then Definition 95 coincides with [18, Definatin 3.7].

Definition 96 *Let* $(\xi_1, \Upsilon_1, \tau_1)$ *and* $(\xi_2, \Upsilon_2, \tau_2)$ *be the soft topological groups over* $\mathcal{A}_1$ *and* $\mathcal{A}_2$*, where* τ_1 *and* τ_2 *are topologies defined over* $\mathcal{A}_1$ *and* $\mathcal{A}_2$ *respectively. Let* $\psi : \mathcal{A}_1 \to \mathcal{A}_2$ *and* $\phi : \Upsilon_1 \to \Upsilon_2$ *be two mappings. Then the pair* (ψ, ϕ) *is called a soft topological group homomorphism if the following conditions are satisfied:*

(a) ψ *is group epimorphism and* ϕ *is surjection.*

(b) $\psi(\xi_1(y)) = \xi_2(\phi(y))$*.*

(c) $\psi_y : (\xi_1(y), \tau_{1_{\xi_1(y)}}) \to (\xi_2(\phi(y)), \tau_{2_{\xi_2(\phi(y))}})$ *is continuous.*

Then $(\xi_1, \Upsilon_1, \tau_1)$ is said to be soft topologically homomorphic to $(\xi_2, \Upsilon_2, \tau_2)$ and denoted by $(\xi_1, \Upsilon_1, \tau_1) \sim (\xi_2, \Upsilon_2, \tau_2)$.

If ψ is a group isomorphism, ϕ is bijective and ψ_y is continuous as well as open, then (ψ, ϕ) is called a soft topological group isomorphism. In this case $(\xi_1, \Upsilon_1, \tau_1)$ is soft topological isomorphic to $(\xi_2, \Upsilon_2, \tau_2)$, which is denoted by $(\xi_1, \Upsilon_1, \tau_1) \simeq (\xi_2, \Upsilon_2, \tau_2)$.

Example 97 *Let* (ξ_1, Υ_1) *and* (ξ_2, Υ_2) *be the two soft homomorphic groups defined over* $\mathcal{A}_1$*and* $\mathcal{A}_2$ *respectively. Then* $(\xi_1, \Upsilon_1, \tau_1)$ *is soft topologically homomorphic to* $(\xi_2, \Upsilon_2, \tau_2)$ *with discrete or anti-discrete topology. In this manner any soft homomorphic groups can be considered as soft topological homomorphic groups in the discrete or anti-discrete topology.*

Definition 98 *Let* (ξ, Υ, τ) *be a soft topological group over* $\mathcal{A}$*. Then we can associate with* (ξ, Υ, τ) *a soft set over* $\mathcal{A}$ *denoted by* $([\xi]_{\mathcal{A}}, \Upsilon, \tau)$*, named closure of* (ξ, Υ, τ)*, and defined as:*

$$[\xi]_{\mathcal{A}}(y) = [\xi(y)]_{\mathcal{A}}$$

where $[\xi(y)]_{\mathcal{A}}$ *is the closure of* $\xi(y)$ *in topology defined on* $\mathcal{A}$.

Theorem 99 *Let* (ξ, Υ, τ) *be a soft topological group over a topological group* $(\mathcal{A}, \tau)$. *Then*

(1) $([\xi]_{\mathcal{A}}, \Upsilon, \tau)$ *is also a soft topological group over* $(\mathcal{A}, \tau)$,

(2) $(\xi, \Upsilon, \tau)\tilde{\subset}([\xi]_{\mathcal{A}}, \Upsilon, \tau)$,

(3) If $(\xi_1, \Upsilon_1, \tau)$ *and* $(\xi_2, \Upsilon_2, \tau)$ *be the soft topological sets over* $(\mathcal{A}, \tau)$, *then* $([\xi_1]_{\mathcal{A}}, \Upsilon_1, \tau) \oplus_{\cup} ([\xi_2]_{\mathcal{A}}, \Upsilon_2, \tau)\tilde{\subseteq}[(\xi_1, \Upsilon_1, \tau) \oplus_{\cup} (\xi_2, \Upsilon_2, \tau)]_{\mathcal{A}}$.

Proof. (1) Since (ξ, Υ, τ) is a soft topological group over $(\mathcal{A}, \tau)$. Therefore $(\xi(y), \tau_{\xi(y)})$ is a topological subgroup of $(\mathcal{A}, \tau)$, for all $y \in \Upsilon$. So $\xi(y)$ is a subgroup of $\mathcal{A}$ and from [4, Proposition 1.4.5] closure of any subgroup of a topological group is also a subgroup of $\mathcal{A}$. Therefore $[\xi(y)]_{\mathcal{A}}$ is a subgroup of topological group $\mathcal{A}$ together with the topology defined on $\mathcal{A}$. So $([\xi(y)]_{\mathcal{A}}, \tau_{\xi(y)})$ is a topological subgroup of $(\mathcal{A}, \tau)$. Hence $([\xi]_{\mathcal{A}}, \Upsilon, \tau)$ is also a soft topological group over $(\mathcal{A}, \tau)$.

(2) Obvious.

(3) For $y \in \Upsilon_1 - \Upsilon_2$

$$\begin{aligned} [([\xi_1]_{\mathcal{A}}, \Upsilon_1, \tau) \oplus_{\cup} ([\xi_2]_{\mathcal{A}}, \Upsilon_2, \tau)](y) &= ([\xi_1]_{\mathcal{A}}, \Upsilon_1, \tau)(y) = [\xi_1(y)]_{\mathcal{A}} \\ &= [(\xi_1, A_1, \tau) \oplus_{\cup} (\xi_2, B, \tau)]_{\mathcal{A}}(y) \end{aligned}$$

For $y \in \Upsilon_2 - \Upsilon_1$

$$\begin{aligned} [([\xi_1]_{\mathcal{A}}, \Upsilon_1, \tau) \oplus_{\cup} ([\xi_2]_{\mathcal{A}}, \Upsilon_2, \tau)](y) &= ([\xi_2]_{\mathcal{A}}, \Upsilon_2, \tau)(y) = [\xi_2(y)]_{\mathcal{A}} \\ &= [(\xi_1, \Upsilon_1, \tau) \oplus_{\cup} (\xi_2, \Upsilon_2, \tau)]_{\mathcal{A}}(y) \end{aligned}$$

For $y \in \Upsilon_1 \cap \Upsilon_2$

$$\begin{aligned} [([\xi_1]_{\mathcal{A}}, \Upsilon_1, \tau) \oplus_{\cup} ([\xi_2]_{\mathcal{A}}, \Upsilon_2, \tau)](y) &= [\xi_1]_{\mathcal{A}}(y) + [\xi_2]_{\mathcal{A}}(y) \\ &= [\xi_1(y)]_{\mathcal{A}} + [\xi_2(y)]_{\mathcal{A}} \\ &\subseteq [\xi_1(y) + \xi_2(y)]_{\mathcal{A}} \\ &= [(\xi_1, \Upsilon_1, \tau) \oplus_{\cup} (\xi_2, B, \tau)]_{\mathcal{A}}(y) \end{aligned}$$

Hence $([\xi_1]_{\mathcal{A}}, \Upsilon_1, \tau) \oplus_{\cup} ([\xi_2]_{\mathcal{A}}, \Upsilon_2, \tau) \tilde{\subseteq} [(\xi_1, \Upsilon_1, \tau) \oplus_{\cup} (\xi_2, B, \tau)]_{\mathcal{A}}$ ■

Definition 100 *A soft topological group (ξ, Υ, τ) is said to be closed if $[\xi]_{\mathcal{A}}(y) = \xi(y)$, where $([\xi]_{\mathcal{A}}, \Upsilon, \tau)$ being the corresponding soft topological set.*

Definition 101 *A soft topological group (ξ, Υ, τ) is said to be dense if $[\xi]_{\mathcal{A}}(y) = \mathcal{A}$, where $([\xi]_{\mathcal{A}}, \Upsilon, \tau)$ being the corresponding soft topological set.*

3.3 Soft Topological Rings

From now on, $\mathcal{R}$ denotes a (unitary) commutative ring and $(\mathcal{R}, \tau)$ denotes a topological ring.

Definition 102 *Let τ be a topology and a non-null soft set (ξ, Υ) be defined over $\mathcal{R}$. Then the triplet (ξ, Υ, τ) is called the soft topological ring over $\mathcal{R}$ if*

(a) For all $y \in \Upsilon$, $\xi(y)$ is a subring of $\mathcal{R}$.

(b) For all $y \in \Upsilon$, the mapping $(a, b) \to a - b$ of the topological space $\xi(y) \times \xi(y)$ onto $\xi(y)$ is continuous.

(c) For all $y \in \Upsilon$, the mapping $(a, b) \to a \cdot b$ of the topological space $\xi(y) \times \xi(y)$ to $\xi(y)$ is continuous.

In terms of neighborhoods, conditions (b) and (c) implies that for any $a, b \in \xi(y)$ and arbitrary neighborhoods W of $a + b$ (resp. $a \cdot b$) there exist neighborhoods W_1 and W_2 of elements a and b, respectively, such that $W_1 + W_2 \subset W$ (resp. $W_1 \cdot W_2 \subset W$).

Example 103 *Take $\mathcal{R} = \mathbb{Z}_4$, $\Upsilon = \{2, 3\}$ and $\tau = \{\Phi, \{0\}, \{0, 2\}, \{0, 1\}, \{0, 1, 2\}, \mathbb{Z}_4\}$. Consider a function $\xi : \Upsilon \to 2^{\mathcal{R}}$ defined by*

$$\xi(y) = \{y' \in \mathcal{R} : y \cdot y' = 0\}$$

Then $\xi(2) = \{0, 2\}$ and $\xi(3) = \{0\}$ which are subrings of $\mathcal{R}$ and conditions (b) and (c) of Definition 102 are also satisfied. Hence (ξ, Υ, τ) is soft topological ring over $\mathcal{R}$.

Remark 104 *Every soft ring can be transformed into a soft topological ring over a ring $\mathcal{R}$ by endowing $\mathcal{R}$ with discrete or anti-discrete topology. It is easy to verify that any soft ring* *satisfies the conditions (b) and (c) of Definition 102 in both topologies. In this manner any soft ring can be considered as a soft topological ring in the discrete or anti-discrete topology.*

Theorem 105 *Every soft ring over a topological ring (non-discrete) is a soft topological ring.*

Proof. Let $(\mathcal{R}, \tau)$ be topological ring. Let (ξ, Υ) be a soft ring over $\mathcal{R}$. So for all $y \in \Upsilon$, $\xi(y)$ is a subring of $\mathcal{R}$. Since $\mathcal{R}$ is a topological ring and from [4, Remark 1.4.4] subring of a topological ring is itself a topological ring. Hence (ξ, Υ, τ) is a soft topological ring over $(\mathcal{R}, \tau)$. ■

Example 106 *Take* $\Upsilon = \mathbb{Z}^+$, $\mathcal{R} = \mathbb{R}$ *and* $\tau =$*The interval topology defined on* $\mathbb{R}$ *and* $\xi : \Upsilon \to 2^{\mathcal{R}}$ *be soft set defined by*

$$\xi(y) = \begin{cases} \mathbb{Q}[\sqrt{y}] & \textit{if } y \textit{ not a perfect square integer} \\ \mathbb{Q} & \textit{otherwise} \end{cases}$$

For each $y \in \mathbb{Z}^+$, $\xi(y)$ *is a subring of* $\mathbb{R}$. *Therefore for each* $y \in \mathbb{Z}^+$, $(\xi(y), \tau_{\xi(y)})$ *is topological subring of* $(\mathbb{R}, \tau)$. *Hence* (ξ, Υ, τ) *is a soft topological ring over* $(\mathbb{R}, \tau)$.

Remark 107 *Every soft ring over a ring need not to be a topological ring. For instance, consider* $\mathcal{R} = \mathbb{Z}_8$, $\Upsilon = \{0, 1, 2\}$ *and* $\tau = \{\Phi, \{0\}, \{0, 2\}, \{0, 4\}, \{0, 2, 4, 6\}, \mathbb{Z}_8\}$.*The set-valued function* $\xi : \Upsilon \to 2^{\mathcal{R}}$ *defined by*

$$\xi(y) = \{y \in \mathcal{R} : y \cdot y' = \{0, 4\}\}$$

Then $\xi(0) = \mathcal{R}$ *and* $\xi(1) = \{0, 4\}$ *and* $\xi(2) = \{0, 2, 4, 6\}$ *are all subrings of* $\mathcal{R}$. *Condition (2) and (3) of Definition 102 are satisfied for* $\xi(1)$ *and* $\xi(2)$ *but not for* $\xi(0)$. *Hence* (ξ, Υ, τ) *is not soft topological ring.*

Theorem 108 *Let* $(\xi_1, \Upsilon_1, \tau)$ *and* $(\xi_2, \Upsilon_2, \tau)$ *be the two soft topological rings over* $\mathcal{R}$. *Then*

(1) The bi-intersection $(\xi_1, \Upsilon_1, \tau) \tilde{\sqcap} (\xi_2, \Upsilon_2, \tau)$ *is a topological ring over* $\mathcal{R}$ *if it is non-null.*

(2) Extended intersection $(\xi_1, \Upsilon_1, \tau) \cap_{\varepsilon} (\xi_2, \Upsilon_2, \tau)$ *is a soft topological ring over* $\mathcal{R}$.

Proof. (1) Since $(\xi_1, \Upsilon_1, \tau)$ and $(\xi_2, \Upsilon_2, \tau)$ are soft topological rings over $\mathcal{R}$. Therefore by Definition 65 their bi-intersection over $\mathcal{R}$ is the soft topological set (ξ', Υ', τ) , where $\Upsilon' =$

$\Upsilon_1 \cap \Upsilon_2$, and for all $y \in \Upsilon$ and defined as $\xi'(y) = \xi_1(y) \cap \xi_2(y)$. Since both $\xi_1(y)$ and $\xi_2(y)$ are subrings of $\mathcal{R}$, therefore $\xi'(y)$ is a subring of $\mathcal{R}$, for all $y \in \Upsilon_1 \cap \Upsilon_2$. Also $\xi'(y) \subset \xi_1(y)$, $\xi'(y) \subset \xi_2(y)$. Hence $(\xi_1, \Upsilon_1, \tau)\widetilde{\sqcap}(\xi_2, \Upsilon_2, \tau)$ is a topological ring over $\mathcal{R}$.

(2) Obvious result. ■

Theorem 109 *For two soft topological rings $(\xi_1, \Upsilon_1, \tau)$ and $(\xi_2, \Upsilon_2, \tau)$ over $\mathcal{R}$, $(\xi_1, \Upsilon_1, \tau)$ $\tilde{\wedge}$ $(\xi_2, \Upsilon_2, \tau)$ is a soft topological ring if it is non-null.*

Proof. The proof is easily seen by Definitions 60 and 102. ■

Theorem 110 *If $(\xi_1, \Upsilon_1, \tau)$ and $(\xi_2, \Upsilon_2, \tau)$ be the two soft topological rings over $\mathcal{R}$. If $\Upsilon_1 \cap \Upsilon_2 = \varnothing$, then $(\xi_1, \Upsilon_1, \tau)\tilde{\cup}(\xi_2, \Upsilon_2, \tau)$ is a topological ring over $\mathcal{R}$.*

Definition 111 *A soft topological ring (ξ, Υ, τ) is said to be soft trivial if $\xi(y) = \{0\}$ for all $y \in \Upsilon$ and soft whole if $\xi(y) = \mathcal{R}$, for all $y \in \Upsilon$.*

Definition 112 *Let $(\xi_1, \Upsilon_1, \tau)$ be a soft topological ring over $\mathcal{R}$. Then $(\xi_2, \Upsilon_2, \tau)$ is said to be a soft topological subring (resp. ideal) of $(\xi_1, \Upsilon_1, \tau)$ if*

(a) $\Upsilon_2 \subset \Upsilon_1$ and $\xi_2(y)$ is a subring (resp. ideal) of $\xi_1(y)$, for all $y \in \sup(\xi_2, \Upsilon_2)$.

(b) For all $y \in \Upsilon$, the mapping $(a, b) \to a - b$ of the topological space $\xi_2(y) \times \xi_2(y)$ onto $\xi_2(y)$ is continuous.

(c) For all $y \in \Upsilon$, the mapping $(a, b) \to a \cdot b$ of the topological space $\xi_2(y) \times \xi_2(y)$ to $\xi_2(y)$ is continuous.

Example 113 *In Example (106), take $\Upsilon_2 = 2\mathbb{Z}^+$ and $\xi_2 : \Upsilon_2 \to 2^{\mathcal{R}}$ defined by $\xi_2(y) = y\mathbb{Z}$. Clearly, $(\xi_2, \Upsilon_2, \tau)$ is a soft topological subring of (ξ, Υ, τ) .*

Example 114 *Take $\Upsilon_1 = \mathbb{Z}^+$, $\mathcal{R} = \mathbb{R}$ and $(\mathbb{R}, \tau)$ is a topological ring, where τ is the interval topology defined on $\mathbb{R}$. Let $\xi_1 : \Upsilon_1 \to 2^{\mathcal{R}}$ be defined by*

$$\xi_1(y) \begin{cases} \mathbb{Q}[\pi] & \text{if } 7 \mid y \\ \mathbb{Q} & \text{otherwise} \end{cases}$$

Here $\mathbb{Q}[\pi]$ is the subring of $\mathbb{R}$ generated by the subset $\mathbb{Q} \cup \{\pi\}$. Clearly, $(\xi_1, \Upsilon_1, \tau)$ is a soft topological ring over $(\mathbb{R}, \tau)$. Now take $\Upsilon_2 = 7\mathbb{Z}^+$ and define a soft set (ξ_2, Υ_2) by

$$\xi_2(y) = \pi^y \cdot \mathbb{Q}[\pi], \textit{ for all } y \in \Upsilon_2$$

Then $(\xi_1', \Upsilon_1', \tau)$ is a soft topological ideal of (ξ, Υ, τ).

Theorem 115 *Let $(\xi_1, \Upsilon_1, \tau)$ and $(\xi_2, \Upsilon_2, \tau)$ be two soft topological rings over $\mathcal{R}$.*

(1) If $\xi_2(y) \subset \xi_1(y)$, for all $y \in \Upsilon_2 \subset \Upsilon_1$. Then $(\xi_2, \Upsilon_2, \tau)$ is a soft topological subring of $(\xi_1, \Upsilon_1, \tau)$.

(2) $(\xi_1, \Upsilon_1, \tau)\widetilde{\sqcap}(\xi_2, \Upsilon_2, \tau)$ is a soft topological subring (ideal) of both $(\xi_1, \Upsilon_1, \tau)$ and $(\xi_2, \Upsilon_2, \tau)$.

Proof. The proof is straightforward. ■

Theorem 116 *Every soft subring (resp. ideal) of a soft topological ring is a soft topological subring (resp. ideal).*

Proof. Let $(\xi_1, \Upsilon_1, \tau)$ be a soft topological ring over $\mathcal{R}$ and (ξ_2, Υ_2) be a soft subring (resp. ideal) of (ξ_1, Υ_1). Then for each $y \in \sup(\xi_2, \Upsilon_2)$, $\xi_2(y)$ is a subring (resp. ideal) of $\xi_1(y)$. And conditions (b) and (c) of Definition 102 hold for every subset of $\xi_1(y)$. So for $\xi_2(y)$, being a subring (resp. ideal) of $\xi_1(y)$. Hence $(\xi_2, \Upsilon_2, \tau)$ is a soft topological subring (resp. ideal) of $(\xi_1, \Upsilon_1, \tau)$. ■

3.4 Idealistic Soft Topological Rings

Definition 117 *Let τ be a topology defined on a ring $\mathcal{R}$. Let (ξ, Υ) be a non-null soft set defined over $\mathcal{R}$. Then the triplet (ξ, Υ, τ) is called idealistic soft topological ring over $\mathcal{R}$ if*

(a) For every $y \in \Upsilon$, $\xi(y)$ is an ideal of $\mathcal{R}$.

(b) For every $y \in \Upsilon$, the mapping $(a, b) \to a - b$ of the topological space $\xi(y) \times \xi(y)$ onto $\xi(y)$ is continuous.

(c) For every $y \in \Upsilon$, the mapping $(r, a) \to r \cdot a$ of the topological space $\mathcal{R} \times \xi(y)$ to $\xi(y)$ is continuous.

Example 118 *Let $\mathcal{R}$ be a ring. Then every idealistic soft ring can be transformed into a soft topological ring over $\mathcal{R}$ by endowing $\mathcal{R}$ with discrete or anti-discrete topology. It is easy to verify that any idealistic soft ring satisfies the condition (b) and (c) of Definition 117 in both topologies. In this manner any ring can be considered as a soft topological ring in the discrete or anti-discrete topology.*

Theorem 119 *Every idealistic soft ring over a topological ring (non-discrete) is an idealistic soft topological ring.*

Proof. Let $(\mathcal{R}, \tau)$ be a topological ring. Let (ξ, Υ) be an idealistic soft ring over $\mathcal{R}$. So $\xi(y)$ is an ideal of $\mathcal{R}$ for every $y \in \Upsilon$. Since $\mathcal{R}$ is a topological ring and ideal of a topological ring is itself a topological ideal and satisfies the conditions (b) and (c) of Definition 117. Hence (ξ, Υ, τ) is a soft topological ring over $(\mathcal{R}, \tau)$. ∎

Theorem 120 *Let $(\xi_1, \Upsilon_1, \tau)$ and $(\xi_2, \Upsilon_2, \tau)$ be the two idealistic soft topological rings over $\mathcal{R}$. Then*

(1) $(\xi_1, \Upsilon_1, \tau)\tilde{\wedge}(\xi_2, \Upsilon_2, \tau)$ is an idealistic soft topological ring over $\mathcal{R}$ if it is non-null.

(2) The bi-intersection $(\xi_1, \Upsilon_1, \tau)\tilde{\sqcap}(\xi_2, \Upsilon_2, \tau)$ is an idealistic soft topological ring over $\mathcal{R}$ if it is non-null.

(3) If $\Upsilon_1 \cap \Upsilon_2 = \varnothing$, then $(\xi_1, \Upsilon_1, \tau)\tilde{\cup}(\xi_2, \Upsilon_2, \tau)$ is an idealistic soft topological ring over $\mathcal{R}$.

Theorem 121 *Let $(\xi_1, \Upsilon_1, \tau)$ and $(\xi_2, \Upsilon_2, \tau)$ be the two idealistic soft topological rings over $(\mathcal{R}, \tau)$. Then $(\xi_1, \Upsilon_1, \tau) \oplus_{\cup} (\xi_2, \Upsilon_2, \tau)$ is an idealistic soft topological ring over $(\mathcal{R}, \tau)$.*

Proof. Since $(\xi_1, \Upsilon_1, \tau)$ and $(\xi_2, \Upsilon_2, \tau)$ are idealistic soft topological rings over $(\mathcal{R}, \tau)$. Therefore by Definition 68 their extended sum over $\mathcal{R}$ is a soft topological set (ξ', Υ', τ), where $\Upsilon' = \Upsilon_1 \cup \Upsilon_2$, and for all $y \in \Upsilon'$, it is defined as

$$\xi'(y) = \begin{cases} \xi_1(y) & if\ y \in \Upsilon_1 - \Upsilon_2 \\ \xi_2(y) & if\ y \in \Upsilon_2 - \Upsilon_1 \\ \xi_1(y) + \xi_2(y) & if\ y \in \Upsilon_1 \cap \Upsilon_2 \end{cases}$$

In first two cases either $\xi'(y) = \xi_1(y)$ or $\xi'(y) = \xi_2(y)$. If $y \in \Upsilon_1 \cap \Upsilon_2$, then $\xi'(y) = \xi_1(y) + \xi_2(y)$. Since both $\xi_1(y)$ and $\xi_2(y)$ are ideals, therefore $\xi'(y)$ is an ideal of $\mathcal{R}$, for all $y \in \Upsilon$. Hence

$\xi'(y)$ is an ideal of $\mathcal{R}$, for all $y \in \Upsilon$. Hence by Theorem 119 $(\xi', \Upsilon') = (\xi_1, \Upsilon_1) \oplus_\cup (\xi_2, \Upsilon_2)$ is an idealistic soft topological ring over $(\mathcal{R}, \tau)$. ■

Remark 122 *If $(\xi_1, \Upsilon_1, \tau)$ and $(\xi_2, \Upsilon_2, \tau)$ are idealistic soft topological rings over $\mathcal{R}$. Then $(\xi_1, \Upsilon_1, \tau) \oplus_\cup (\xi_2, \Upsilon_2, \tau)$ need not to be an idealistic soft topological ring over $\mathcal{R}$.For instance, consider the ring $\mathcal{R} = \mathbb{Z}_{10} = \mathcal{Y}$, base for topology τ is $\mathcal{B} = \{\{0\},\{2\},\{4\},\{5\},\{6\},\{8\}\}$, $\Upsilon_1 = \{1, 3, 5\}$ and $\Upsilon_2 = \{1, 2, 3\}$. Let us consider the set valued function $\xi_1 : \Upsilon_1 \to 2^{\mathcal{R}}$ given by*

$$\xi_1(y) = \{y' \in \Upsilon_1 : y \cdot y' = \{0, 2, 4, 6, 8\}\}$$

Then $\xi_1(1) = \xi_1(3) = \xi_1(5) = \{0, 2, 4, 6, 8\}$. Now soft set $\xi_2 : \Upsilon_2 \to 2^{\mathcal{R}}$ is defined as

$$\xi_2(y) = \{y' \in \Upsilon_2 : y \cdot y' = \{0, 5\}\}$$

Since $(\xi_1, \Upsilon_1, \tau) \oplus_\cup (\xi_2, \Upsilon_2, \tau) = (\xi', \Upsilon', \tau)$, where $\Upsilon' = \Upsilon_1 \cup \Upsilon_2$ and

$$\xi'(y) = \begin{cases} \xi_1(y) & if\ y \in \Upsilon_1 - \Upsilon_2 \\ \xi_2(y) & if\ y \in \Upsilon_2 - \Upsilon_1 \\ \xi_1(y) + \xi_2(y) & if\ y \in \Upsilon_1 \cap \Upsilon_2 \end{cases}$$

$\xi'(1) = \xi_1(1) + \xi_2(1) = \mathbb{Z}_{10}$, $\xi'(2) = \xi_2(2)$, $\xi'(3) = \xi_1(3) + \xi_2(3) = \mathbb{Z}_{10}$ and $\xi'(5) = \xi_1(5)$. Here $\xi'(1)$ and $\xi'(3)$ does not satisfy the condition (b) and (c) of Definition 117. Hence (ξ', Υ', τ) is not an idealistic soft topological ring over $\mathcal{R}$.

Theorem 123 *Let $(\xi_1, \Upsilon_1, \tau)$ and $(\xi_2, \Upsilon_2, \tau)$ be the two idealistic soft topological rings over $(\mathcal{R}, \tau)$. Then $(\xi_1, \Upsilon_1, \tau) \odot_\cup (\xi_2, \Upsilon_2, \tau)$ is an idealistic soft topological ring over $(\mathcal{R}, \tau)$.*

Proof. Since $(\xi_1, \Upsilon_1, \tau)$ and $(\xi_2, \Upsilon_2, \tau)$ are idealistic soft topological rings over $(\mathcal{R}, \tau)$, therefore by Definition 70 their extended product over $\mathcal{R}$ is the soft topological set (ξ', Υ', τ), where $\Upsilon' = \Upsilon_1 \cup \Upsilon_2$, and for all $y \in \Upsilon'$ and it is defined as

$$\xi'(y) = \begin{cases} \xi_1(y) & if\ y \in \Upsilon_1 - \Upsilon_2 \\ \xi_2(y) & if\ y \in \Upsilon_2 - \Upsilon_1 \\ \xi_1(y) \cdot \xi_2(y) & if\ y \in \Upsilon_1 \cap \Upsilon_2 \end{cases}$$

In first two cases either $\xi'(y) = \xi_1(y)$ or $\xi'(y) = \xi_2(y)$. If $y \in \Upsilon_1 \cap \Upsilon_2$, then $\xi'(y) = \xi_1(y) \cdot \xi_2(y)$. Since both $\xi_1(y)$ and $\xi_2(y)$ are ideals, therefore $\xi'(y)$ is an ideal of $\mathcal{R}$, for all $y \in \Upsilon_1 \cap \Upsilon_2$. Hence by Theorem 119 $(\xi', \Upsilon', \tau) = (\xi_1, \Upsilon_1, \tau) \oplus_\cup (\xi_2, \Upsilon_2, \tau)$ is an idealistic soft topological ring over $(\mathcal{R}, \tau)$. ■

Remark 124 *If $(\xi_1, \Upsilon_1, \tau)$ and $(\xi_2, \Upsilon_2, \tau)$ are idealistic soft topological rings over the ring $\mathcal{R}$. Then $(\xi_1, \Upsilon_1, \tau) \odot_\cup (\xi_2, \Upsilon_2, \tau)$ need not to be an idealistic soft topological ring over $\mathcal{R}$.*

Theorem 125 *Let $(\xi_1, \Upsilon_1, \tau)$ and $(\xi_2, \Upsilon_2, \tau)$ be the two idealistic soft topological rings over $(\mathcal{R}, \tau)$. Then*

(1) $(\xi_1, \Upsilon_1, \tau) \oplus_\cap (\xi_2, \Upsilon_2, \tau)$ is an idealistic soft topological ring over $(\mathcal{R}, \tau)$.

(2) $(\xi_1, \Upsilon_1, \tau) \odot_\cap (\xi_2, \Upsilon_2, \tau)$ is an idealistic soft topological ring over $(\mathcal{R}, \tau)$.

The following Remark is obvious.

Remark 126 *If $(\xi_1, \Upsilon_1, \tau)$ and $(\xi_2, \Upsilon_2, \tau)$ are idealistic soft topological rings over the ring $\mathcal{R}$. Then*

(1) $(\xi_1, \Upsilon_1, \tau) \oplus_\cap (\xi_2, \Upsilon_2, \tau)$ need not to be an idealistic soft topological ring over $\mathcal{R}$.

(2) $(\xi_1, \Upsilon_1, \tau) \odot_\cap (\xi_2, \Upsilon_2, \tau)$ need not to be an idealistic soft topological ring over $\mathcal{R}$.

Definition 127 *Let (ξ, Υ, τ) be a soft topological ring over $\mathcal{R}$ and $f : \mathcal{R}_1 \to \mathcal{R}_2$ be a ring homomorphism. Define the set $K_f(y)$ by*

$K_f(y) = [K(f)]_{\xi(y)} = Ker(f) \cap \xi(y) = \{r \in \xi(y) : f(r) = 1_{\mathcal{R}_2}\}$, *for all* $y \in \Upsilon$.

It is clear that (K_f, Υ, τ) is a soft topological ring over $\mathcal{R}$.

Proposition 128 *(K_f, Υ, τ) is an ideal of soft topological ring (ξ, Υ, τ).*

Proof. Since $Ker(f)$ is an ideal of $\mathcal{R}$ and $\xi(y)$ is subring of $\mathcal{R}$. Therefore $K_f(y) = Ker(f) \cap \xi(y)$ is an ideal of $\xi(y)$, for all $y \in \Upsilon$. Hence by Theorem 116 (K_f, Υ, τ) is an ideal of of soft topological ring (ξ, Υ, τ). ■

Topological Ring Homomorphism

Definition 129 *Let $(\xi_1, \Upsilon_1, \tau_1)$ and $(\xi_2, \Upsilon_2, \tau_2)$ are soft topological rings over $\mathcal{R}_1$ and $\mathcal{R}_2$, where τ_1 and τ_2 are topologies defined over $\mathcal{R}_1$ and $\mathcal{R}_2$, respectively. Let $\psi : \mathcal{R}_1 \to \mathcal{R}_2$ and $\phi : \Upsilon_1 \to \Upsilon_2$ be two mappings. Then the pair (ψ, ϕ) is called a soft topological ring homomorphism if the following conditions are satisfied:*

(a) ψ is ring epimorphism and ϕ is surjection.

(b) $\psi(\xi_1(y)) = \xi_2(\phi(y))$.

(c) $\psi_y : (\xi_1(y), \tau_{1_{\xi_1(y)}}) \to (\xi_2(\phi(y)), \tau'_{2_{(\xi_2\phi(y))}})$ is continuous.

Then $(\xi_1, \Upsilon_1, \tau_1)$ is said to be soft topologically homomorphic to $(\xi_2, \Upsilon_2, \tau_2)$ and denoted by $(\xi, \Upsilon, \tau) \sim (\mathcal{A}, B, \tau')$.

If ψ is a ring isomorphism, ϕ is bijective and ψ_y is continuous as well as open, then (ψ, ϕ) is called a soft topological ring isomorphism. In this case $(\xi_1, \Upsilon_1, \tau_1)$ is soft topologically isomorphic to $(\xi_2, \Upsilon_2, \tau_2)$, which is denoted by $(\xi_1, \Upsilon_1, \tau_1) \simeq (\xi_2, \Upsilon_2, \tau_2)$.

Definition 130 *Let $(\xi_1, \Upsilon_1, \tau_1)$ and $(\xi_2, \Upsilon_2, \tau_2)$ be the soft topological rings over $\mathcal{R}_1$ and $\mathcal{R}_2$ respectively. Let $(\xi_1, \Upsilon_1, \tau_1)$ is topological soft homomorphic to $(\xi_2, \Upsilon_2, \tau_2)$. Then*

(a) Define $\psi\xi_1 : \Upsilon_2 \to 2^{\mathcal{R}_2}$ by $\psi\xi_1(y_2) = \psi(\xi_1(y_1))$ where $y_2 = \phi(y_1)$, for some $y_1 \in \Upsilon_1$.

(b) Define $\psi^{-1}\xi_2 : \Upsilon_1 \to 2^{\mathcal{R}_1}$ by $\psi^{-1}\xi_2(y_1) = \psi^{-1}(\xi_2(\phi(y_1)))$, for all $y_1 \in \Upsilon_1$.

Theorem 131 *Let $(\xi_1, \Upsilon_1, \tau_1)$ and $(\xi_2, \Upsilon_2, \tau_2)$ be the soft topological rings over $(\mathcal{R}_1, \tau_1)$ and $(\mathcal{R}_2, \tau_2)$ respectively. If $(\xi_1, \Upsilon_1, \tau_1)$ is topological soft homomorphic to $(\xi_2, \Upsilon_2, \tau_2)$ and (ψ, ϕ) be the corresponding soft topological homomorphism. Then*

(1) $(\psi\xi_1, \Upsilon_2, \tau_2)$ is soft topological ring over $(\mathcal{R}_2, \tau_2)$.

(2) $(\psi^{-1}\xi_2, \Upsilon_1, \tau_1)$ is a soft topological ring over $(\mathcal{R}_1, \tau_1)$.

Proof. (1) Since ϕ is surjective, so there exists $y_1 \in \Upsilon_1$ such that $y_2 = \phi(y_1)$, for all $y_2 \in \Upsilon_2$. Since ψ is the corresponding algebraic homomorphism from $\mathcal{R}_1$ to $\mathcal{R}_2$. Also for every $y' \in \Upsilon_1$, $\psi_{y'} : (\xi(y'), \tau_{1_{\xi(y)}}) \to (\xi_2(\phi(y')), \tau_{2_{\xi_2(\phi(y'))}})$ is continuous. So $\psi\xi_1(y_2) = \psi(\xi(y_1))$ is a subring of $\xi_2(y_2)$ and hence R$_2$. Thus $(\psi\xi_1(y_2), \tau_{2_{\psi\xi_1(y_2)}})$ is topological subring of $(\xi_2(y_2), \tau_{2_{\xi_2(y_2)}})$ and $(\xi_2(y_2), \tau_{2_{\xi_2(y_2)}})$ is topological subring of (R_2, τ_2). Consequently $(\psi\xi_1(y_2), \tau_{2_{\psi\xi_1(y_2)}})$ is topological subring of $(\mathcal{R}_2, \tau_2)$. Hence, by Theorem 116 $(\psi\xi_1, \Upsilon_2, \tau_2)$ is soft topological ring over $(\mathcal{R}_2, \tau_2)$.

(2) ψ is the corresponding algebraic homomorphism from $\mathcal{R}_1$ to $\mathcal{R}_2$. Also for each $y \in \Upsilon$, $\psi_{y'} : (\xi(y'), \tau_{1_{\xi(y)}}) \to (\xi_2(\phi(y')), \tau_{2_{\xi_2(\phi(y'))}})$ is continuous. Therefore $\psi^{-1}\xi_2(y_1) = \psi^{-1}(\xi_2(\phi(y_1)))$ is a subring of $\xi_1(y_1)$ and hence a subring of $\mathcal{R}_1$. Thus by Theorem 116 $(\psi^{-1}\xi_2, \Upsilon_1, \tau_1)$ is a soft topological ring over $(\mathcal{R}_1, \tau_1)$. ■

Theorem 132 *Let* $(\xi_1, \Upsilon_1, \tau_1)$ *and* $(\xi_2, \Upsilon_2, \tau_2)$ *be the idealistic soft topological rings over* $(\mathcal{R}_1, \tau_1)$ *and* $(\mathcal{R}_2, \tau_2)$ *respectively. If* $(\xi_1, \Upsilon_1, \tau_1)$ *is soft topological homomorphic to* $(\xi_2, \Upsilon_2, \tau_2)$ *and* (ψ, ϕ) *be the corresponding soft topological homomorphism. Then*

(1) $(\psi\xi_1, \Upsilon_2, \tau_2)$ *is an idealistic soft topological ring over* $(\mathcal{R}_1, \tau_1)$.

(2) $(\psi^{-1}\xi_2, \Upsilon_1, \tau_1)$ *is an idealistic soft topological ring over* $(\mathcal{R}_2, \tau_2)$.

Proof. Proof is obvious. ■

Theorem 133 *Let* $(\xi_1, \Upsilon_1, \tau_1)$ *and* $(\xi_2, \Upsilon_2, \tau_2)$ *be the soft topological rings over* $(\mathcal{R}_1, \tau_1)$ *and* $(\mathcal{R}_2, \tau_2)$ *respectively. Further assume that* $(\xi_1, \Upsilon_1, \tau_1)$ *is soft topological homomorphic to* $(\xi_2, \Upsilon_2, \tau_2)$ *and* (ψ, ϕ) *be the corresponding soft topological homomorphisms. If* $(\xi_1', \Upsilon_1', \tau_1)$ *be a soft topological Ideal (resp. subring) of* $(\xi_1, \Upsilon_1, \tau_1)$. *Then*

(1) $(\psi\xi_1', \Upsilon_2, \tau_2)$ *is a soft topological ideal(resp. subring) of* $(\psi\xi_1, \Upsilon_2, \tau_2)$.

(2) $(\psi^{-1}\xi_1, \Upsilon_1', \tau_1)$ *is a soft topological ideal(resp. subring) of* $(\psi^{-1}\xi_2, \Upsilon_1, \tau_1)$.

Proof. (1) Since ϕ is surjective, so there exists $y_1 \in \Upsilon_1$ such that $y_2 = \phi(y_1)$, for all $y_2 \in \Upsilon_2$. Also ψ_{y_1} is the corresponding algebraic homomorphism from $\xi_1(y_1)$ to $\xi_2(\phi(y_1))$. As $\psi\xi_1(y_1)$ and $\psi\xi_1'(y_1)$ are the subrings of $\xi_2(\phi(y_1))$ and $\xi_1'(y_1)$ is an ideal (resp. subring) of $\xi_1(y_1)$. So $\psi\xi_1'(y_1))$ is an ideal (resp. subring) of $\psi\xi_1(y_1))$. Consequently $(\psi\xi_1'(y_1)), \tau_{2_{\psi\xi_1'(y_1)}})$ is a topological ideal of $(\psi\xi_1(y_1), \tau_{2_{\psi\xi_1(y_1)}})$. Thus by Theorem 119 $(\psi\xi_1', \Upsilon_2, \tau_2)$ is a soft topological ideal (resp. subring) of $(\psi\xi_1, \Upsilon_2, \tau_2)$.

(2) $\psi^{-1}\xi_1'(y_1) = \psi_{y_1}^{-1}(\xi_1(y_1))$ for each $y_1 \in \Upsilon_1$. And ψ_{y_1} is the corresponding algebraic homomorphism from $\xi_1(y_1)$ to $\xi_1(\phi(y_1))$. So $\psi_{y_1}^{-1}(\xi_1(y_1))$ is an ideal (resp. subring) of $\psi_{y_1}^{-1}(\xi_2(\phi(y_1))$. Hence by Theorem 116 $(\psi^{-1}\xi_1, \Upsilon_1', \tau_1)$ is a soft topological ideal (resp. subring) of $(\psi^{-1}\xi_2, \Upsilon_1, \tau_1)$. ■

3.5 Soft Topological Divisors of Zero

We define the closure of a soft topological ring as follows:

Definition 134 *Let (ξ, Υ, τ) be a soft topological ring over $\mathcal{R}$. Then we can associate with (ξ, Υ, τ) a soft set over $\mathcal{R}$, denoted by $([\xi]_{\mathcal{R}}, \Upsilon, \tau)$, named closure of (ξ, Υ, τ) and defined as:*

$$[\xi]_{\mathcal{R}}(y) = [\xi(y)]_{\mathcal{R}}, \text{ for each } y \in \Upsilon$$

where $[\xi(y)]_{\mathcal{R}}$ is the closure of $\xi(y)$ in topology defined on $\mathcal{R}$.

Theorem 135 *Let (ξ, Υ, τ) be a soft topological ring over $(\mathcal{R}, \tau)$. Then*

(1) $([\xi]_{\mathcal{R}}, \Upsilon, \tau)$ is a soft topological ring over $(\mathcal{R}, \tau)$.

(2) $(\xi, \Upsilon, \tau)\tilde{\subset}([\xi]_{\mathcal{R}}, \Upsilon, \tau)$.

Proof. (1) Since (ξ, Υ, τ) is a soft topological ring over $(\mathcal{R}, \tau)$. Therefore every $y \in \Upsilon$, $(\xi(y), \tau_{\xi(y)})$ is a topological subring of $(\mathcal{R}, \tau)$. Hence $\xi(y)$ is a subring of $\mathcal{R}$ and from [4, Proposition 1.4.7] closure of any subring of a topological ring is also a subring of $\mathcal{R}$. Therefore $[\xi(y)]_{\mathcal{R}}$ is a subring of topological ring $\mathcal{R}$ together with the topology defined on $\mathcal{R}$. Hence $([\xi]_{\mathcal{R}}, \Upsilon, \tau)$ is a soft topological ring over $(\mathcal{R}, \tau)$.

(2) Obvious. ■

Remark 136 *If (ξ, Υ, τ) be a soft topological ring over $\mathcal{R}$, then $([\xi]_{\mathcal{R}}, \Upsilon, \tau)$ need not to be a soft topological ring. For instance, consider $\mathcal{R} = \mathbb{Z}_6$, $\Upsilon = \{y_1, y_2, y_3\}$ and*

$$\tau = \{\varnothing, \{0\}, \{1\}, \{2\}, \{3\}, \{0, 1, 2\}, \{0, 2, 3\}, \{0, 1, 2, 3, \}, \mathbb{Z}_6\}$$

Let us consider the set valued function $\xi : \Upsilon \to 2^{\mathcal{R}}$ defined by

$$\xi(y_1) = \{0\}, \ \xi(y_2) = \{0, 1\} \text{ and } \xi(y_3) = \{0, 3\}$$

Clearly, (ξ, Υ, τ) to be a soft topological ring over $\mathbb{Z}_6$. But $([\xi]_{\mathcal{R}}, \Upsilon, \tau)$ is not a soft topological ring over $\mathcal{R}$ because $[\xi]_{\mathcal{R}}(y_1) = \{0, 1, 2\}$ which is not a subring of $\mathcal{R}$.

Theorem 137 *Let* $(\xi_1, \Upsilon_1, \tau)$ *and* $(\xi_2, \Upsilon_2, \tau)$ *be the soft topological sets over* $(\mathcal{R}, \tau)$*. Then we have the following:*

(1) $([\xi_1]_{\mathcal{R}}, \Upsilon_1, \tau) \odot_{\cup} ([\xi_2]_{\mathcal{R}}, \Upsilon_2, \tau) \tilde{\subseteq} [(\xi_1, \Upsilon_1, \tau) \odot_{\cup} (\xi_2, \Upsilon_2, \tau)]_{\mathcal{R}}$.

(2) $[(\xi_1, \Upsilon_1, \tau) \cap_{\varepsilon} (\xi_2, \Upsilon_2, \tau)]_{\mathcal{R}} \tilde{\subseteq} ([\xi_1]_{\mathcal{R}}, \Upsilon_1, \tau) \cap_{\varepsilon} ([\xi_2]_{\mathcal{R}}, \Upsilon_2, \tau)$.

(3) $[(\xi_1, \Upsilon_1, \tau) \tilde{\cup} (\xi_2, \Upsilon_2, \tau)]_{\mathcal{R}} = ([\xi_1]_{\mathcal{R}}, \Upsilon_1, \tau) \tilde{\cup} ([\xi_2]_{\mathcal{R}}, \Upsilon_2, \tau)$.

(4) If $(\xi_1, \Upsilon_1, \tau) \tilde{\subset} (\xi_2, \Upsilon_2, \tau)$*. Then* $([\xi_1]_{\mathcal{R}}, \Upsilon_1, \tau) \tilde{\subset} ([\xi_2]_{\mathcal{R}}, \Upsilon_2, \tau)$.

Proof. (1) For $y \in \Upsilon_1 - \Upsilon_2$

$$\begin{aligned} [([\xi_1]_{\mathcal{R}}, \Upsilon_1, \tau) \odot_{\cup} ([\xi_2]_{\mathcal{R}}, \Upsilon_2, \tau)](y) &= ([\xi_2]_{\mathcal{R}}, \Upsilon_2, \tau)(y) = [\xi_2(y)]_{\mathcal{R}} \\ &= [(\xi_1, \Upsilon_1, \tau) \odot_{\cup} (\xi_2, \Upsilon_2, \tau)]_{\mathcal{R}}(y) \end{aligned}$$

For $y \in \Upsilon_2 - \Upsilon_1$

$$\begin{aligned} [([\xi_1]_{\mathcal{R}}, \Upsilon_1, \tau) \odot_{\cup} ([\xi_2]_{\mathcal{R}}, \Upsilon_2, \tau)](y) &= ([\xi_2]_{\mathcal{R}}, \Upsilon_2, \tau)(y) = [\xi_2(y)]_{\mathcal{R}} \\ &= [(\xi_1, \Upsilon_1, \tau) \odot_{\cup} (\xi_2, \Upsilon_2, \tau)]_{\mathcal{R}}(y) \end{aligned}$$

For $y \in \Upsilon_1 \cap \Upsilon_2$

$$\begin{aligned} [([\xi_1]_{\mathcal{R}}, y_1, \tau) \odot_{\cup} ([\xi_2]_{\mathcal{R}}, \Upsilon_2, \tau)](y) &= [\xi_1]_{\mathcal{R}}(y) \cdot [\xi_2]_{\mathcal{R}}(y) \\ &= [\xi_1(y)]_{\mathcal{R}} \cdot [\xi_2(y)]_{\mathcal{R}} \\ &\subseteq [\xi_1(y) \cdot \xi_2(y)]_{\mathcal{R}} \\ &= [(\xi_1, \Upsilon_1, \tau) \odot_{\cup} (\xi_2, \Upsilon_2, \tau)]_{\mathcal{R}}(y). \end{aligned}$$

Hence $([\xi_1]_{\mathcal{R}}, \Upsilon_1, \tau) \odot_{\cup} ([\xi_2]_{\mathcal{R}}, \Upsilon_2, \tau) \tilde{\subseteq} [(\xi_1, \Upsilon_1, \tau) \odot_{\cup} (\xi_2, \Upsilon_2, \tau)]_{\mathcal{R}}$.

(2) For $y \in \Upsilon_1 - \Upsilon_2$

$$\begin{aligned} [([\xi_1]_{\mathcal{R}}, \Upsilon_1, \tau) \cap_{\varepsilon} ([\xi_2]_{\mathcal{R}}, \Upsilon_2, \tau)](y) &= ([\xi_1]_{\mathcal{R}}, \Upsilon_1, \tau)(y) = [\xi_1(y)]_{\mathcal{R}} \\ &= [(\xi_1, \Upsilon_1, \tau) \cap_{\varepsilon} (\xi_2, \Upsilon_2, \tau)]_{\mathcal{R}}(y) \end{aligned}$$

For $y \in \Upsilon_2 - \Upsilon_1$

$$\begin{aligned} [([\xi_1]_{\mathcal{R}}, \Upsilon_1, \tau) \cap_\varepsilon ([\xi_2]_{\mathcal{R}}, \Upsilon_2, \tau)](y) &= ([\xi_2]_{\mathcal{R}}, \Upsilon_2, \tau)(y) = [\xi_2(y)]_{\mathcal{R}} \\ &= [(\xi_1, \Upsilon_1, \tau) \cap_\varepsilon (\xi_2, \Upsilon_2, \tau)]_{\mathcal{R}}(y) \end{aligned}$$

For $y \in \Upsilon_2 \cap \Upsilon_1$

$$\begin{aligned} [([\xi_1]_{\mathcal{R}}, \Upsilon_1, \tau) \cap_\varepsilon ([\xi_2]_{\mathcal{R}}, \Upsilon_2, \tau)](y) &= [\xi_1(y)]_{\mathcal{R}} \cap [\xi_2(y)]_{\mathcal{R}} \\ &\supseteq [\xi_1(y) \cap \xi_2(y)]_{\mathcal{R}} \\ &= [(\xi_1, \Upsilon_1, \tau) \cap_\varepsilon (\xi_2, \Upsilon_2, \tau)]_{\mathcal{R}}(y) \end{aligned}$$

Consequently, $[(\xi_1, \Upsilon_1, \tau) \cap_\varepsilon (\xi_2, \Upsilon_2, \tau)]_{\mathcal{R}} \tilde{\subseteq} ([\xi_1]_{\mathcal{R}}, \Upsilon_1, \tau) \cap_\varepsilon ([\xi_2]_{\mathcal{R}}, \Upsilon_2, \tau)$.

(3) Similar.

(4) Since, $(\xi_1, \Upsilon_1, \tau) \tilde{\subset} (\xi_2, \Upsilon_2, \tau)$. So $\xi_1(y) \subset \xi_2(y)$, for all $y \in \Upsilon_1$. So $[\xi_1(y)]_{\mathcal{R}} \subset [\xi_2(y)]_{\mathcal{R}}$ and so $[\xi_1]_{\mathcal{R}}(y) \subset [\xi_2]_{\mathcal{R}}(y)$, for all $y \in \Upsilon_1$. Hence $([\xi_1]_{\mathcal{R}}, \Upsilon_1, \tau) \tilde{\subset} ([\xi_2]_{\mathcal{R}}, \Upsilon_2, \tau)$. ■

Definition 138 *Let (ξ, Υ, τ) be a soft topological ring over a ring $\mathcal{R}$. Then (ξ, Υ, τ) is said to be closed if $[\xi]_{\mathcal{R}}(y) = \xi(y)$, for all $y \in \Upsilon$, where $([\xi]_{\mathcal{R}}, \Upsilon, \tau)$ being the corresponding soft topological set.*

Definition 139 *Let (ξ, Υ, τ) be a soft topological ring over a ring $\mathcal{R}$. Then (ξ, Υ, τ) is said to be dense if $[\xi]_{\mathcal{R}}(y) = \mathcal{R}$, for all $y \in \Upsilon$, where $([\xi]_{\mathcal{R}}, \Upsilon, \tau)$ being the corresponding soft topological set.*

Example 140 *Take $\mathcal{R} = \mathbb{R} = \mathcal{Y}$, $\Upsilon = \mathbb{Z}^+$, $\tau =$The interval topology defined on $\mathbb{R}$ and define $\xi : \Upsilon \to 2^{\mathcal{R}}$ by*

$$\xi(y) = y\mathbb{Z}, \text{ for all } y \in \Upsilon$$

Clearly, for every $y \in \Upsilon$, $(\xi(y), \tau_{\xi(y)})$ is a topological subring of topological ring $(\mathbb{R}, \tau)$. So (ξ, Υ, τ) is a soft topological ring over $(\mathcal{R}, \tau)$. Also $[\xi(y)]_{\mathcal{R}} = [y\mathbb{Z}]_{\mathcal{R}} = y\mathbb{Z}$, for all $y \in \Upsilon$. So (ξ, Υ, τ) is closed soft topological ring over $(\mathcal{R}, \tau)$ because $([\xi]_{\mathcal{R}}, \Upsilon, \tau) = (\xi, \Upsilon, \tau)$.

Now define $\xi : \Upsilon \to 2^{\mathcal{R}}$ *by*

$$\xi(y) = \mathbb{Z}[\sqrt{y}], \textit{ for all } y \in \Upsilon$$

(ξ, Υ, τ) *is a soft topological ring over* $(\mathcal{R}, \tau)$*. Also* $[\xi(y)]_{\mathcal{R}} = \mathbb{R}$*, for all* $y \in \Upsilon$*. So* (ξ, Υ, τ) *is a dense soft topological ring over* $(\mathcal{R}, \tau)$*.*

Also if we take $\Upsilon = \mathbb{Q}^c$ *and define* $\xi : \Upsilon \to 2^{\mathcal{R}}$ *by*

$$\xi(y) = \mathbb{Z} + y\mathbb{Z}, \textit{ for all } y \in \Upsilon$$

Then (ξ, Υ, τ) *is a dense soft topological ring over* $(\mathcal{R}, \tau)$ *because for each* $y \in \Upsilon$*,* $\xi(y)$ *is a discrete subgroup of* $\mathbb{R}$*.*

Remark 141 *Every Soft topological ring with discrete topology is closed and every soft topological ring with anti-discrete topology is dense.*

Theorem 142 *A trivial soft topological ring over a Hausdorff ring is closed.*

Proof. Let $(\mathcal{R}, \tau)$ be a Hausdorff ring and (ξ, Υ, τ) be a soft topological ring over $(\mathcal{R}, \tau)$ such that $\xi(y) = \{0\}$, for all $y \in \Upsilon$. Since $\mathcal{R}$ is a Hausdorff space so $\{0\}$ is closed in $\mathcal{R}$. Therefore

$$[\xi(y)]_{\mathcal{R}} = \{0\}, \text{ for all } y \in \Upsilon$$

Hence (ξ, Υ, τ) is closed. ■

Example 143 *Take* $\mathcal{R} = \mathbb{Z}_6 = \mathcal{Y}$*,* $\Upsilon = \{0, 1, 2\}$ *and* $\tau = \{\varnothing, \mathbb{Z}_6\}$*. Define* $\xi : \Upsilon \to 2^{\mathcal{R}}$ *by*

$$\xi(y) = \{0\}, \textit{ for all } y \in \Upsilon$$

Then $[\xi(y)]_{\mathcal{R}} = \mathbb{Z}_6$*, for all* $y \in \Upsilon$*. Hence* (ξ, Υ, τ) *is not closed. This example demonstrates that the condition of being Hausdorff for the ring* $\mathcal{R}$ *in Theorem (142) is important.*

Theorem 144 *Let* $(\xi_1, \Upsilon_1, \tau)$ *and* $(\xi_2, \Upsilon_2, \tau)$ *be soft topological ring over the ring* $\mathcal{R}$*. Then*

(1) If $(\xi_1, \Upsilon_1, \tau)$ *and* $(\xi_2, \Upsilon_2, \tau)$ *are dense soft topological rings, then* $(\xi_1, \Upsilon_1, \tau)\tilde{\wedge}(\xi_2, \Upsilon_2, \tau)$ *is a dense soft topological ring over* $\mathcal{R}$*.*

(2) If $(\xi_1, \Upsilon_1, \tau)$ and $(\xi_2, \Upsilon_2, \tau)$ are closed soft topological rings, then $(\xi_1, \Upsilon_1, \tau)\tilde{\vee}(\xi_2, \Upsilon_2, \tau)$ is a closed soft topological ring over $\mathcal{R}$.

(3) If $(\xi_1, \Upsilon_1, \tau)$ and $(\xi_2, \Upsilon_2, \tau)$ are dense soft topological rings, then $(\xi_1, \Upsilon_1, \tau)\tilde{\vee}(\xi_2, \Upsilon_2, \tau)$ is a dense soft topological ring $\mathcal{R}$.

(4) If $(\xi_1, \Upsilon_1, \tau)$ and $(\xi_2, \Upsilon_2, \tau)$ are dense (closed) soft topological rings, then $(\xi_1, \Upsilon_1, \tau)\tilde{\cup}(\xi_2, \Upsilon_2, \tau)$ is a dense soft topological ring over $\mathcal{R}$.

(5) If $(\xi_1, \Upsilon_1, \tau)$ and $(\xi_2, \Upsilon_2, \tau)$ are dense soft topological rings, then $(\xi_1, \Upsilon_1, \tau) \cap_\varepsilon (\xi_2, \Upsilon_2, \tau)$ is a dense soft topological ring over $\mathcal{R}$.

(6) If $(\xi_1, \Upsilon_1, \tau)$ and $(\xi_2, \Upsilon_2, \tau)$ are dense soft topological rings, then $(\xi_1, \Upsilon_1, \tau) \Cap (\xi_2, \Upsilon_2, \tau)$ is a dense soft topological ring over $\mathcal{R}$.

(7) If $(\xi_1, \Upsilon_1, \tau)$ and $(\xi_2, \Upsilon_2, \tau)$ are closed soft topological rings, then $(\xi_1, \Upsilon_1, \tau)\tilde{\cup}(\xi_2, \Upsilon_2, \tau)$ is a closed soft topological ring over $\mathcal{R}$.

(8) If $(\xi_1, \Upsilon_1, \tau)$ and $(\xi_2, \Upsilon_2, \tau)$ are dense soft topological rings, then $(\xi_1, \Upsilon_1, \tau)\oplus_\cup(\xi_2, \Upsilon_2, \tau)$ is a dense soft topological ring over $\mathcal{R}$.

(9) If $(\xi_1, \Upsilon_1, \tau)$ and $(\xi_2, \Upsilon_2, \tau)$ are dense soft topological rings, then $(\xi_1, \Upsilon_1, \tau)\oplus_\cap(\xi_2, \Upsilon_2, \tau)$ is a dense soft topological ring over $\mathcal{R}$.

(10) If $(\xi_1, \Upsilon_1, \tau)$ and $(\xi_2, \Upsilon_2, \tau)$ are dense soft topological rings, then $(\xi_1, \Upsilon_1, \tau)\odot_\cup(\xi_2, \Upsilon_2, \tau)$ is a dense soft topological ring over $\mathcal{R}$.

(11) If $(\xi_1, \Upsilon_1, \tau)$ and $(\xi_2, \Upsilon_2, \tau)$ are dense soft topological rings, then $(\xi_1, \Upsilon_1, \tau)\odot_\cap(\xi_2, \Upsilon_2, \tau)$ is a dense soft topological ring over $\mathcal{R}$.

(12) If $(\xi_1, \Upsilon_1, \tau)$ is a closed soft topological ring and $(\xi_2, \Upsilon_2, \tau)$ is a dense soft topological ring over $\mathcal{R}$, then $(\xi_1, \Upsilon_1, \tau) \cap_\varepsilon (\xi_2, \Upsilon_2, \tau)$ is a closed soft topological ring over $\mathcal{R}$.

(13)If $(\xi_1, \Upsilon_1, \tau)$ is a closed soft topological ring and $(\xi_2, \Upsilon_2, \tau)$ is a dense soft topological ring over $\mathcal{R}$, then $(\xi_1, \Upsilon_1, \tau)\tilde{\cup}(\xi_2, \Upsilon_2, \tau)$ is a dense soft topological ring over $\mathcal{R}$.

Proof. *Obvious.* ■

In topological ring theory, the concept of topological divisors of zero is well known. Now we use the concept of the idealistic soft topological ring because the motivations are due to the product of two ideals in a unitary commutative ring is again an ideal.

Definition 145 *(a) Let (ξ, Υ, τ) be a soft topological ring over $\mathcal{R}$. Let $\{0\} \neq \xi(y) \in (\xi, \Upsilon, \tau)$,* *then $\xi(y)$ is said to be soft topological left (respectively soft topological right) divisor of zero in (ξ, Υ, τ) if there exist a soft subset (ξ', Υ') of soft set (ξ, Υ) such that*

(i) $\{0\} \notin ([\xi']_{\mathcal{R}}, \Upsilon', \tau)$.

(ii) $\{0\} \in \{[\xi(y) \cdot \xi'(y')]_{\mathcal{R}} : \xi'(y') \in (\xi', \Upsilon')\}$ (respectively $\{0\} \in \{[\xi'(y') \cdot \xi(y)\ ]_{\mathcal{R}} : \xi'(y') \in (\xi', \Upsilon')\}$).

(b) An element of (ξ, Υ, τ) is called a soft topological divisor of zero if it is soft left and right topological divisor of zero.

Example 146 *Take $\mathcal{R} = \mathbb{Z}_4$, $\Upsilon = \{1, 2\}$ and $\tau = \{\Phi, \{0\}, \{2\}, \{0, 2\}, \mathbb{Z}_4\}$. Define $\xi : \Upsilon \to 2^{\mathcal{R}}$ by*

$$\xi(1) = \{0\} \text{ and } \xi(2) = \{0, 2\}$$

Clearly, (ξ, Υ, τ) is a soft topological ring over $\mathcal{R}$. Here $\xi(2)$ is a soft topological divisor of zero in (ξ, Υ, τ).

Remark 147 *A soft topological ring (ξ, Υ, τ) with anti-discrete topology τ has neither soft left nor soft right topological divisor of zero. For instance, let us consider $\xi(y)$ be a soft topological divisor of zero. So there exist a soft subset (ξ', Υ') of (ξ, Υ) such that $\{0\} \notin ([\xi']_{\mathcal{R}}, \Upsilon', \tau)$. Since the topology is anti-discrete. So $[\xi']_{\mathcal{R}}(y') = \Phi$, for all $y' \in \Upsilon'$ and also*

$$\{[\xi(y) \cdot \xi'(y')]_{\mathcal{R}} : \xi'(y') \in (\xi', \Upsilon')\} = \Phi$$

Hence $\xi'(y')$ cannot be a soft topological divisor of zero. Hence (ξ, Υ, τ) has no soft topological divisor of zero.

Proposition 148 *In a soft topological ring with discrete topology soft left (resp. right) zero divisors are soft left (resp. right) topological divisors of zero.*

Proof. Let (ξ, Υ, τ) be a soft topological ring, where τ is the discrete topology. Let $\{0\} \neq \xi(y_1)$ be a soft zero divisor. So there exist $\{0\} \neq \xi(y_2) \in (\xi, \Upsilon, \tau)$ such that $\xi(y_1) \cdot \xi(y_2)$ $= \{0\}$. Consider a soft subset $(\xi', \Upsilon') = \{(\xi(y_2), y_2)\}$ of (ξ, Υ). Then clearly $\{0\} \notin ([\xi']_{\mathcal{R}}, \Upsilon', \tau)$. But

$$\{0\} \in \{[\xi(y_1) \cdot \xi'(y_2)]_{\mathcal{R}} : \xi'(y_2) \in (\xi', \Upsilon')\} = \{[\xi(y_1) \cdot \xi(y_2)]_{\mathcal{R}}\}$$

Hence the proof. ■

The concept of the soft zero divisor and soft topological divisor of zero, in general, are different.

Proposition 149 *Let (ξ, Υ, τ) be a soft topological ring over $\mathcal{R}$. Let $\xi(y_1)$ be a soft left (right) topological divisors of zero in (ξ, Υ, τ). Then for every $\xi(y_2) \in (\xi, \Upsilon, \tau)$ the element $\xi(y_2) \cdot \xi(y_1)$ is a soft left topological divisor of zero (correspondingly, the element $\xi(y_1) \cdot \xi(y_2)$ is a soft right topological divisor of zero).*

Proof. Let $\xi(y_1)$ be a soft left topological divisor of zero in (ξ, Υ, τ) and (ξ', Υ') be a soft subset of (ξ, Υ) such that

$$\{0\} \notin ([\xi']_{\mathcal{R}}, \Upsilon', \tau), \text{ and } \{0\} \in \{[\xi(y_1) \cdot \xi'(y')]_{\mathcal{R}} : \xi'(y') \in (\xi', \Upsilon')\}$$

Then

$$\begin{aligned} \{0\} &\in \{\xi(y_2) \cdot [\xi(y_1)\xi'(y')]_{\mathcal{R}} : \xi'(y') \in (\xi', \Upsilon')\} \\ &\subseteq \{[(\xi(y_2)\xi(y_1)) \cdot \xi'(y')]_{\mathcal{R}} : \xi'(y') \in (\xi', \Upsilon')\} \end{aligned}$$

Hence $\xi(y_2) \cdot \xi(y_1)$ is a soft left topological divisor of zero. ■

Analogously the case when $\xi(y_1)$ is a soft right topological divisor of zero in (ξ, Υ, τ).

Proposition 150 *Let (ξ, Υ, τ) be a soft topological ring over $\mathcal{R}$ and $\xi(y_1) \cdot \xi(y_2) \in (\xi, \Upsilon, \tau)$. If $\xi(y_1) \cdot \xi(y_2)$ is a soft left (resp. right) topological divisor of zero in (ξ, Υ, τ), then either $\xi(y_1)$ or $\xi(y_2)$ is a soft left (resp. right) topological divisor of zero.*

Proof. Let $\xi(y_1) \cdot \xi(y_2)$ is a soft left topological divisor of zero in (ξ, Υ, τ) and (ξ', Υ') be a soft subset of (ξ, Υ) such that

$$\{0\} \notin ([\xi']_{\mathcal{R}}, \Upsilon', \tau) \text{ and } \{0\} \in \{[(\xi(y_1)\xi(y_2)) \cdot \xi'(y')]_{\mathcal{R}} : \xi'(y') \in (\xi', \Upsilon')\}$$

If

$$\{0\} \in \{[\xi(y_2) \cdot \xi'(y')]_{\mathcal{R}} : \xi'(y') \in (\xi', \Upsilon')\}$$

Proof. Then $\xi(y_2)$ is a soft left topological divisor of zero.

If

$$\{0\} \notin \{[\xi(y_2) \cdot \xi'(y')]_{\mathcal{R}} : \xi'(y') \in (\xi', \Upsilon')\}$$

■

Then by using the fact that

$$\begin{aligned} \{0\} &\in \{[(\xi(y_1)\xi(y_2)) \cdot \xi'(y')]_{\mathcal{R}} : \xi'(y') \in (\xi', \Upsilon')\} \\ &= \{\{[\xi(y_1) \cdot (\xi(y_2)\xi'(y'))]_{\mathcal{R}} : \xi'(y') \in (\xi', \Upsilon')\} \end{aligned}$$

$\xi(y_1)$ is a soft left topological divisor of zero.

Similar is the case when $\xi(y_1) \cdot \xi(y_2)$ is a soft right topological divisor of zero in (ξ, Υ, τ). ■

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