Ricci Collineations of Cylindrically Symmetric Static Spacetimes

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by

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لطافت بے کثا فت جلوہ پیدا کر نہیں سکتی
چمن زنگار ہے آئینۂ بادِبہاری کا
غالب

The Ethereal cannot manifest itself without the Substantial

As the garden is like Verdigris for the mirror of Spring breeze

Ghalib (1797 – 1869)

(Translation from Urdu by Sajjad Shaikh)

(Riemannian Geometry, which is abstract, finds its manifestation in the study of the physical Universe, in General Relativity.)

To Ishfaq Ahmed

Prefatorial Note

Einstein's theory of general relativity is among the most active fields of research, advancing on theoretical, observational and experimental fronts. Like other fields of mathematical physics symmetries play an important role in this theory also. The Einstein field equations (EFE), which provide the foundations for this theory are highly nonlinear partial differential equations and it is very difficult to obtain their exact solutions. Spacetime symmetries not only make it possible to obtain exact solutions of the EFE but some of these also provide invariant bases for the classification of spacetimes. Isometries, homothetic motions, Ricci collineations (RCs) and matter collineations are some of these symmetries.

The spacetime metric, the Ricci and the curvature tensors play a significant role in understanding the geometric structure of spacetimes in relativity. While the isometries, which are the symmetries of the metric tensor, provide information of the symmetries inherent in the spacetime, the Ricci collineations, which are the symmetries of the Ricci tensor, are important both from the geometric as well as physical points of view. In this thesis cylindrically symmetric spacetimes have been classified according to their RCs. This has been done both for the degenerate as well as the non-degenerate Ricci tensor. The Lie algebras of the RCs have also been given. It is found that while the RC algebras for the non-degenerate tensor are always finite dimensional, those for the degenerate cases are mostly, *but not always*, infinite dimensional. The RCs have been compared with the isometries. It is well known that every isometry is an RC, but the converse is not true in general. Numerous cases of these non-isometric RCs have been found. In the course of finding the RCs, constraints on the components of the Ricci tensor are also obtained, so that the various cases of the RCs are characterized by these constraints. Solving these constraints give the metrics of the manifold. The EFE are used for the physical interpretation of the results.

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Abstract

Symmetries are used in general relativity not only to find the exact solutions of the Einstein Field Equations (EFE), but some of them provide invariant bases for the classification of spacetimes as well. Killing vectors (KVs), homothetic motions (HMs) and Ricci collineations (RCs) are some of these symmetries. In this thesis RCs are used to serve both of these purposes. A complete classification of cylindrically symmetric static spacetimes according to their RCs is provided. After the introductory chapter, where a survey of the related literature is given, we have obtained the RCs for the non-degenerate as well as the degenerate Ricci tensor. We have also provided their Lie algebras. It is found that the Lie algebras of the RCs of these spacetimes, for the non-degenerate Ricci tensor have dimension ranging from 3 to 10 excluding 8 and 9. For the degenerate case the Lie algebras are mostly infinite dimensional. However, cases of the algebras of dimensions 3, 4, 5 and10 have also been found. The comparison of the RCs with the KVs and HMs has given rise to numerous interesting cases of *proper* (or non-isometric) RCs. Corresponding to each Lie algebra there arise differential constraints (mostly non-linear) on the metric coefficients. We have solved these constraints to construct examples of metrics which include some exact solutions admitting proper RCs. Their physical interpretation is also given. The classification of plane symmetric static spacetimes emerges as a special case of this classification when the cylinder is unfolded. Some results are summarized in the form of theorems in the concluding chapter.

Contents

Chapter 1

Introduction

Symmetries play an important role not only in arts but in mathematics and theoretical physics also. We quote from Daniel Rockmore's review [1], on the book: *The Universe and the Teacup: The Mathematics of Truth and Beauty* by K. C. Cole.

> *"The connections between symmetry and beauty are a well-trodden area, with Hermann Weyl's Symmetry the classical reference. Ms. Cole sees invariance and symmetry as a way to get from truth to beauty, adding that deep truths can be defined as invariants - things that do not change no matter what; how invariants are defined by symmetries, which in turn define which properties of nature are conserved, no matter what. These are the selfsame symmetries that appeal to the senses in art and music and natural forms like snowflakes and galaxies. The fundamental truths are based on symmetry, and there's a deep kind of beauty in that."*

The Einstein Field Equations (EFE), which play a central role in Einstein's theory of general relativity, have symmetry consideration as one of the most important mathematical properties apart from their applications and implications for astrophysics and cosmology. This is why these equations have been a subject of extensive and intensive study both by mathematicians and physicists ever since they were put on paper in 1915. In the words of A. Trautman [2],

> *"One of the many unsolved problems connected with the general theory of relativity is whether the theory belongs to physics or rather to mathematics. One of my*

colleagues ... said that those who work in the theory of relativity do so because of its mathematical beauty rather than because they want to make predictions which could be checked against experiment. I think there is some truth in this statement, and probably I am no exception to it."

The situation today may not be the same as it was in the sixties (when this was written) because in the recent decades research in general relativity (GR) has seen a tremendous growth on observational and experimental fronts also. But, this does not mean that the theory has been deprived of its mathematical beauty. From a physical point of view, however, some of the most important solutions of the field equations, include the Schwarzschild, Reissner-Nordström and Kerr solutions for black holes, the Friedmann solution for cosmology and the solutions for gravitational waves. On the other hand if no particular matter distribution is assumed, then the solutions can be obtained by imposing symmetry conditions on or by restricting the algebraic structure of the metric, the Ricci tensor or the Riemannian tensor. Isometries, which are the symmetries of the metric, and Ricci collineations (RCs), which are the symmetries of the Ricci tensor, are two important examples of such symmetries. Apart from their significance in obtaining the exact solutions of the field equations, these symmetries provide various invariant bases for the classification of spacetimes also. The techniques used for this purpose include: application of groups of motions; algebraic classification of the Weyl tensor and spinor methods developed by Penrose [3] for what are called the Petrov types [4]. Since the Ricci tensor and the energy-momentum tensor are mathematically similar, the study of RCs is important from the point of view of the study of symmetries of matter (called matter collineations) also, apart from their geometrical significance. Other important symmetries in GR include homothetic motions (HMs), which are obtained when the Lie derivative of the metric is proportional to the metric, and curvature collineations — symmetries of the Riemann tensor which has an all important role in GR.

In this thesis we classify cylindrically symmetric static spacetimes according to their RCs. In the first chapter, we discuss different geometric symmetry properties, found in the literature on general relativity, and review different approaches to obtaining symmetries and classifying spacetimes. We briefly explain cylindrical symmetry and the relevance and significance of cylindrically symmetric solutions for astrophysics and cosmology. In Chapter 2 we construct and solve the RC equations for cylindrically symmetric static spacetimes for the non-degenerate as well as the degenerate Ricci tensor. In the next chapter we give the symmetry algebras for these spacetimes. Some physically interesting spacetimes are discussed in Chapter 4 where their RCs are also compared with the KVs and the spacetimes admitting non-isometric RCs are given. Some general observations on the results of the thesis and the theorems are given in the concluding chapter where the results are also summarized in the form of tables. There are three appendices also. Appendix A contains some definitions for the sake of completeness. Appendix B and C consist of the detailed calculations of which only the results are given in Chapter 2.

1.1 The Einstein Field Equations

We consider a four dimensional Lorentzian manifold M with signature $(+,-,-,-)$ and metric tensor g_{ab}, which is a function of the position given in coordinates by x^a, $(a, b, ...= 0, 1, 2, 3)$. If R_{ab} is the Ricci tensor and R is the Ricci scalar, the EFE (without cosmological constant) are [5]

$$R_{ab}-\frac{1}{2}g_{ab}R=\kappa T_{ab}\ , \tag{1.1}$$

where T_{ab} is the energy-momentum tensor of the matter and κ is called the Einstein gravitational constant. These are the basic equations of the general theory of relativity set up by Einstein in 1915. By virtue of these equations GR becomes the theory of the dependence of the metric of a Riemann manifold

$$ds^2=g_{ab}dx^a dx^b\ , \tag{1.2}$$

on the distribution and motion of matter.

Since R_{ab} is a non-linear function of g_{ab} and its first and second derivatives, the EFE are a system of 10 coupled highly non-linear second order partial differential equations for the 6 independent functions g_{ab} of four spacetime coordinates, x^a. Solving these equations amounts to determining the 10 components of the energy-momentum tensor, as well as the 10 components of g_{ab} of four variables; this makes the system undetermined. This problem of intractability is tackled by either making certain assumptions on the matter contents of the space (i.e. T_{ab}) or assuming some geometric symmetry properties of the metric tensor and/or the Ricci tensor.

1.2 The Lie Derivative

For each point p in a manifold M, a vector field $\mathbf{V}$ on M determines a unique curve $\alpha_p(t)$ such that $\alpha_p(o) = p$ and $\mathbf{V}$ is the tangent vector to the curve. Now, consider a mapping h_t dragging each point p, with coordinates x^i, along the curve $\alpha_p(t)$ through p into the image point $q = h_t(p)$, with coordinates $y^i(t)$. If t is very small the map h_t is a one-one map and induces a map $h_t^*\mathbf{T}$ of any tensor $\mathbf{T}$. The Lie derivative of $\mathbf{T}$ with respect to $\mathbf{V}$ is defined by [4, 6]

$$\pounds_{\mathbf{V}}\mathbf{T} = \lim_{t \to 0} \frac{1}{t}(h_t^*\mathbf{T} - \mathbf{T}) . \tag{1.3}$$

If $\mathbf{T}$ is of type (r, s), its Lie derivative is also a tensor of type (r, s). Using the coordinate bases $\{\partial_{x^i}\}$ and $\{\partial_{y^i}\}$, the Lie derivative of a vector $\mathbf{U}$ with respect to $\mathbf{V}$ in component form can be written as

$$\pounds_{\mathbf{V}}\mathbf{U}|_p = \left[\frac{\partial u^j}{\partial y^m}\frac{dy^m}{dt}\frac{\partial x^i}{\partial y^j} + u^j\frac{\partial}{\partial y^j}\left(\frac{dx^i}{dt}\right)\right]_{t=0}\frac{\partial}{\partial x^i} , \tag{1.4}$$

because we have

$$h_t^*\mathbf{U}|_p = u^j\left(y\left(x,t\right)\right)\frac{\partial x^i}{\partial y^j}\frac{\partial}{\partial x^i} . \tag{1.5}$$

Thus, we can write

$$\pounds_{\mathbf{V}}\mathbf{U} = v^m\frac{\partial}{\partial x^m}\left(u^i\frac{\partial}{\partial x^i}\right) - u^m\frac{\partial}{\partial x^m}\left(v^i\frac{\partial}{\partial x^i}\right) = [\mathbf{V}\,,\mathbf{U}] , \tag{1.6}$$

where the commutator $[\mathbf{V}\,,\mathbf{U}]$ is called the Lie bracket. It has the following properties:

(a) it is bilinear;

$$[a\mathbf{U}+b\mathbf{V}, \mathbf{W}] = a\,[\mathbf{U}, \mathbf{W}] + b\,[\mathbf{V}, \mathbf{W}] \text{ , } a \text{ and } b \text{ are scalars.}$$

(b) it is antisymmetric:

$$[\mathbf{V}, \mathbf{U}] = -[\mathbf{U}, \mathbf{V}], \tag{1.7}$$

(c) the Jacobi identity:

$$[\mathbf{U}, [\mathbf{V}, \mathbf{W}]] + [\mathbf{V}, [\mathbf{W}, \mathbf{U}]] + [\mathbf{W}, [\mathbf{U}, \mathbf{V}]] = 0. \tag{1.8}$$

A vector space together with a product [,] satisfying (a), (b) and (c) is called a Lie algebra. Let $\{\mathbf{X}_i, i = 1, ..., n\}$ be a basis for the Lie algebra, then we can always write

$$[\mathbf{X}_k, \mathbf{X}_l] = C^j_{kl}\mathbf{X}_j, \qquad C^j_{kl} = -C^j_{lk}, \tag{1.9}$$

where, we have assumed the summation convention. The Lie algebra is completely characterized by C^j_{kl}, called the structure constants. If all the structure constants vanish, the Lie algebra is said to be Abelian. Every Lie algebra defines a unique simply connected Lie group and vice versa. Naturally the basis $\{\mathbf{X}_i\}$ is not unique, and under a change of basis the numbers C^j_{kl} transform as components of a tensor. Every Lie group and algebra has a unique 'structure tensor' **C**.

Lie derivatives are used in theoretical physics to express the invariance of a tensor field under some transformation. We say that a tensor field **T** is invariant under a vector field **V** if the tensors $h^*_t\mathbf{T}$ and **T** coincide for t in some interval around 0, i.e., the Lie derivative vanishes

$$\mathcal{L}_{\mathbf{V}}\mathbf{T} = 0, \tag{1.10}$$

If **T** has physical importance — e.g., it might be the metric tensor, or a scalar field describing the potential energy of a particle, or a vector field of force — then those special vector fields under which **T** is invariant will also be important. For example, we know that angular momentum will be important in a physical problem only if the problem is invariant under rotations associated with at least one of the vector fields. For instance, if the system is invariant under rotations in some plane, it is said to be axially symmetric and the angular momentum associated with the vector generating those rotations is conserved. Suppose we have a set $F = \{\mathbf{T}_1, \mathbf{T}_2, ...\}$

of tensor fields whose invariance properties are being studied. Then the set of all vector fields $\mathbf{V}$ under which all fields in F are invariant is a Lie algebra.

The manifolds of interest in theoretical physics have metric tensors. It is of interest to know when the metric is invariant with respect to some vector field. The vector fields along which the metric remains invariant are called Killing vector (KV) fields or isometries. After the spacetime metric, the curvature and the Ricci tensors are other important candidates which play a significant role in understanding the geometric structure of spacetimes in GR. While the isometries provide information of the symmetries inherent in the spacetime, the Ricci Collineations (RCs), vector fields along which the Ricci tensor is invariant under Lie transport, are important from the physical point of view as well. These symmetry properties are described by continuous groups of motions or collineations and they lead to conservation laws.

It may be stated that besides using the symmetry groups admitted by the metric and the existence and structure of preferred vector fields as the bases for classifying exact solutions, there are other classification schemes which include algebraic classification of the conformal curvature (Petrov type), the algebraic classification of the Ricci tensor (Plebanski type) and the physical characterization of the energy-momentum tensor [4].

1.3 Symmetries in General Relativity

The EFE are the fundamental equations of the theory of gravitation as developed by Einstein, which relate the geometry of the space to its matter contents. The heart of the classification schemes for the solutions of these equations are the symmetry methods based on the Lie derivative. Here we formally define some of the symmetries used in general relativity [7, 8], particularly from the point of view of classifying the solutions.

1.3.1 Killing Vectors

A manifold M is said to admit a Killing vector (or motion) $\boldsymbol{\xi}$ if the Lie derivative of g with respect to $\boldsymbol{\xi}$ is conserved, i.e.

$$\pounds_{\xi}\mathbf{g} = 0\,. \tag{1.11}$$

This equation, called the Killing equation, in a torsion free space and in a coordinate basis, can also be written as

$$g_{ab,c}\xi^c + g_{ac}\xi^c_{,b} + g_{bc}\xi^c_{,a} = 0\,. \tag{1.12}$$

Its solutions are KVs. The set of all solutions of Eq. (1.12) forms a Lie algebra and generates a Lie group of transformations. Many explicit solutions of EFE have been found using Killing symmetries [4]. KVs can be used to derive the most general axially symmetric stationary metric [9]. These symmetries leave all the curvature quantities invariant and they help in describing the kinematic and dynamic properties of spaces.

Soon after Killing's discovery of the Killing equations at the end of the nineteenth century and Lie's classification over the complex numbers of all Lie algebras up to dimension 3, Bianchi [10] classified all 3 dimensional Riemannian manifolds according to their isometries. In this classification Bianchi derived the full Killing vector Lie algebra for each possible symmetry class of group actions and solved the Killing equations to derive the metric. He also gave a representative line element for a given symmetry type.

The attempts to classify all solution of the EFE on the basis of KVs faced problems initially because the energy-momentum tensor is arbitrary. A procedure was needed which could provide a list of all metrics according to a given isometry group and a complete list of all isometry groups. This is an alternate approach to solving the EFE for given T_{ab}. Using this approach Eisenhart succeeded [7] in classifying all 2 and 3 dimensional spaces. He also developed important general theorems concerning groups of motions. Petrov [11] gave an invariant classification of Riemannian spacetimes admitting groups of motions on the basis of their detailed group structure. But his extension to 4-dimensional spaces was incomplete as admitted by him. However, Bokhari, Qadir and Ziad [12] classified all static spherically symmetric spacetimes by solving the Killing equations for both g_{ab} and KVs. Qadir and Ziad [13] gave a complete classification of *all* spherically symmetric spacetimes according to their isometries and metrics. Using these methods, this work was extended to the plane symmetric [14] and then to the cylindrically symmetric static Lorentzian manifolds according to their KVs and metrics [15]. In these classifications metrics with their KVs and Lie algebras were given explicitly.

1.3.2 Homothetic Motions

The vector $\boldsymbol{\xi}$ is said to be an HM [16, 17] if the right hand side of Eq. (1.11) is replaced by ϕg, i.e.,

$$\mathcal{L}_{\xi}\mathrm{g} = \phi \mathrm{g}\,, \tag{1.13}$$

where ϕ is a nonzero constant.

As in the case of KVs, all the curvature quantities (except the scalar curvature which is preserved up to a constant factor) are invariant under a homothetic vector field.

Hall and Steele [16] investigated the Segré and Petrov types of spaces that admit proper homothety groups — HMs which are not KVs. They classified all such gravitational fields for homothety groups, H_m, $m \geq 6$, and gave some remarks for $m \leq 5$. Hall [17] has also discussed the relation between homotheties and singularities of spacetimes.

The spherically symmetric [18], the plane symmetric static [19] and the cylindrically symmetric static [20] Lorentzian manifolds have already been classified according to their homotheties and metrics. In these classifications, homothetic vector fields and the corresponding metrics were obtained explicitly. In the case of cylindrical symmetry, the global extension of some local homotheties was also considered. Later, Hall in a private communication [21], pointed out a few errors in the Segré and Petrov types in some of the cases of [20]. These errors have been rectified in [22].

1.3.3 Ricci Collineations

A vector field $\mathbf{B}$ is an RC if the Lie derivative of a Ricci tensor $\mathbf{R}$ with respect to $\mathbf{B}$ is conserved, i.e.

$$\mathcal{L}_{\mathbf{B}}\mathbf{R} = 0\,, \tag{1.14}$$

or in component form

$$B^c R_{ab,c} + R_{ac} B^c_{,b} + R_{bc} B^c_{,a} = 0\,. \tag{1.15}$$

Since the Ricci tensor is built from the metric tensor, it must inherit its symmetries. Thus if the Lie derivative of g vanishes, it must vanish for R also. Hence every KV is an RC but the converse may not be true. We call the RCs which are not KVs "proper RCs". For Einstein spaces, R $\propto$ g, therefore, in this case the RCs and isometries coincide.

As regards the physical significance of RCs, Davis, Green, Katzin and Norris [23] did the pioneering work on the important role of RCs and the related conservation laws that are admitted by particular types of matter fields. They showed that the existence of isometries and collineations leads to conservation laws in the form of integrals of a dynamical system. They also considered the application of these results to relativistic hydrodynamics and plasma physics. Oliver and Davis [24] obtained conservation expressions for perfect fluids using RCs. The properties of fluid spacetimes admitting RCs were studied by Tsamparlis and Mason [25]. They have studied perfect fluid spacetimes in detail and have also considered a variety of imperfect fluids with cosmological constant and with anisotropic pressure.

Núñez, Percoco and Villalba [26] gave the RCs of the Robertson-Walker spacetime. Melbo, Núñez, Percoco and Villalba [27] studied RC symmetry in Godel-type spacetimes and Hall, Roy and Vaz [28] studied RCs for various decomposable spacetimes and discussed the relationship between the RCs and the matter collineations. Bokhari and Qadir [29] and later Amir, Bokhari and Qadir [30] did some work on the RCs of static spherically symmetric spacetimes. Later Bertolotti et. al. [31] pointed out a few errors, and Qadir and Ziad [32] completed this work rectifying the earlier errors. Contreras, Núñez and Percoco [33] claimed to extend this to the non-static (and non-degenerate) case. Ziad [34] has recently completed the earlier attempts on classification of spherically symmetric Lorentzian manifolds according to their RCs. Plane symmetric static spacetimes were classified according to the RCs by Farid, Qadir and Ziad [35]. Plane symmetry may locally be thought of as a special case of the cylindrical symmetry [4], therefore, we will see that this classification can be obtained as a special case of the present work.

Apart from the symmetries described above, there are other symmetries discussed in the literature on GR. Conformal motions preserve angles between two directions at a point and map null geodesics into null geodesics. HMs scale all distances by the same constant factor, therefore, they lead to self-similar spacetimes. HMs also preserve the null geodesic affine parameters. Projective collineations map geodesics into geodesics and affine collineations preserve, in addition, the affine parameters on geodesics. We briefly define these spacetime symmetries in Appendix A and the relation between them is described in a diagram there. It is clear from the diagram, for example, that motions, affine collineations, and HMs are automatically curvature collineations which are in turn RCs, but the converse is not true in general. We see that Ricci collineations (and contracted Ricci collineations) are the most general of all the symmetries. In the next chapter we will set up the RC equations and solve them for the vector fields under which the Ricci tensor remains invariant.

1.4 Cylindrical Symmetry

Cylindrically symmetric fields are axisymmetric about an infinite axis (KV, ∂_θ) and translationally symmetric along that axis (KV, ∂_z) [4]. The two KVs ∂_θ and ∂_z are spacelike and generate an Abelian group G_2. The stationary cylindrically symmetric fields are hypersurface-homogeneous spacetimes and admit three Killing vectors, ∂_t, ∂_θ, ∂_z, as the minimal symmetry which has the algebra $\mathbb{R} \otimes SO(2) \otimes \mathbb{R}$. These admit an Abelian group G_3. We take $\left(x^0_, x^1_, x^2_, x^3\right) = (t, \rho, \theta, z)$, so that, the most general cylindrically symmetric static metric can be written as [4]

$$ds^2 = e^{\nu(\rho)}dt^2 - d\rho^2 - a^2 e^{\lambda(\rho)}d\theta^2 - e^{\mu(\rho)}dz^2. \tag{1.16}$$

It may be pointed out that Carot et. al. [36] have proposed another definition of cylindrical symmetry which may prove useful in some situations, but for our purpose we keep the above definition.

Ever since the first investigations of cylindrically symmetric spacetimes by Levi-Civita [37] and Weyl [38] and, later by Lewis [39], these spacetimes have been studied extensively for their mathematical and physical properties. Here we give some examples of well known cylindrically symmetric astrophysical and cosmological solutions discussed in the literature.

Levi-Civita gave a static vacuum solution of EFE. Some other vacuum solutions [40], gravitational field inside a rotating hollow cylinder [41] and static gravitational fields [42] have been discussed in literature. A number of Einstein-Maxwell fields [43] which consist of angular and longitudinal magnetic fields and radial electric fields are prominent examples. Some dust solutions [44] and a large number of perfect fluid solutions with and without rigid rotations are also cylindrically symmetric, see for example [45]. Islam [9] has discussed solutions with axial and cylindrical symmetry in detail. He has studied the cylindrically symmetric solutions for rigidly rotating charged as well as neutral dust and even given a global cylindrically symmetric solution of the field equations. Since the time of Einstein, cylindrical gravitational waves [46, 47] have been an active field of research in General Relativity. Interest in cylindrically symmetric spacetimes has seen a tremendous growth particularly in the context of black holes [48], gravitational waves, magnetic strings [49] and cosmic strings [47, 50, 51]. A renewed interest in Levi-Civita spacetimes has been seen quite recently.

Chapter 2

Ricci Collineations of Cylindrically Symmetric Static Spacetimes

In this chapter a complete solution of RC equations is provided for the cylindrically symmetric static Lorentzian manifolds. This gives RC vector fields explicitly along with the constraints on the components of the Ricci tensor. The solution is divided into two sections; first we solve the equations for the non-degenerate Ricci tensor, i.e. when $\det(R_{ab}) \neq 0$, and then for the degenerate case, when $\det(R_{ab}) = 0$. The results for the RCs of finite dimensional Lie algebras, obtained here, are also summarized in the form of tables at the end of Chapter 5.

2.1 The Ricci Collineation Equations

The RC equations will be written explicitly in this section. We consider the most general cylindrically symmetric static metric (Eq. (1.16)). As the metric is diagonal and the metric coefficients depend only on ρ, the only non-zero components of the Ricci tensor are

$$\begin{aligned}
R_{00} &= \tfrac{e^{\nu}}{4}\left(2\nu'' + \nu'^2 + \nu'\lambda' + \nu'\mu'\right), \\
R_{11} &= -\left(\tfrac{\nu''}{2} + \tfrac{\lambda''}{2} + \tfrac{\mu''}{2} + \tfrac{\nu'^2}{4} + \tfrac{\lambda'^2}{4} + \tfrac{\mu'^2}{4}\right), \\
R_{22} &= -\tfrac{a^2 e^{\lambda}}{4}\left(2\lambda'' + \nu'\lambda' + \lambda'^2 + \lambda'\mu'\right), \\
R_{33} &= -\tfrac{e^{\mu}}{4}\left(2\mu'' + \nu'\mu' + \lambda'\mu' + \mu'^2\right).
\end{aligned} \tag{2.1}$$

Here $'$ denotes differentiation with respect to ρ. The Ricci scalar is given by

$$\begin{aligned} R &= R_0^0 + R_1^1 + R_2^2 + R_3^3 \\ &= \nu'' + \lambda'' + \mu'' + \frac{1}{2}\left(\nu'^2 + \lambda'^2 + \mu'^2 + \nu'\lambda' + \nu'\mu' + \lambda'\mu'\right) . \end{aligned} \tag{2.2}$$

Using the EFE (Eq. (1.1)), the general form of the stress-energy tensor, T_b^a, is

$$\begin{aligned} T_0^0 &= -\tfrac{1}{4}\left(2\lambda'' + 2\mu'' + \lambda'^2 + \mu'^2 + \lambda'\mu'\right) , \\ T_1^1 &= -\tfrac{1}{4}\left(\nu'\lambda' + \nu'\mu' + \lambda'\mu'\right) , \\ T_2^2 &= -\tfrac{1}{4}\left(2\nu'' + 2\mu'' + \nu'^2 + \mu'^2 + \nu'\mu'\right) , \\ T_3^3 &= -\tfrac{1}{4}\left(2\nu'' + 2\lambda'' + \nu'^2 + \lambda'^2 + \nu'\lambda'\right) . \end{aligned} \tag{2.3}$$

For the sake of brevity writing $R_{ii} = R_i$, $\forall\, i = 0, 1, 2, 3$ the RC equations (Eq. (1.15)), by dropping the summation convention, can be written as

$$R_a' B^1 + 2R_a B^a_{,a} = 0 , \tag{2.4}$$

$$R_a B^a_{,b} + R_b B^b_{,a} = 0 , \ (a, b = 0, 1, 2, 3) . \tag{2.5}$$

Eqs. (2.4) give four equations and Eqs. (2.5) are six equations. These constitute together ten first order, non-linear coupled partial differential equations involving four components of the arbitrary RC vector $\mathbf{B} = (B^0, B^1, B^2, B^3)$, four components of the Ricci tensor R_0, R_1, R_2, R_3 and their partial derivatives. The B^i's depend on t, ρ, θ and z; and the R_i on ρ only.

2.2 Ricci Collineations for the Non-Degenerate Ricci Tensor

We will solve Eqs. (2.4), (2.5) in this section to obtain the components of the RC vector $\mathbf{B}$, for the non-degenerate Ricci tensor i.e., when $R_i \neq 0$, $i = 0, 1, 2, 3$. The procedure adopted will be as follows. We will first consider Eqs. (2.5), for $a = 0$, $b = 2, 3$ and for $a = 2$, $b = 3$ and solve them simultaneously to obtain the components of $\mathbf{B}$ in terms of arbitrary functions of the coordinates. Using this form of $\mathbf{B}$ in the remaining RC equations will give conditions on these arbitrary functions. We will go back and forth in this way until these functions are determined explicitly and we get the final form of B involving arbitrary constants. In the course of finding these solutions we will get constraints on the components of the Ricci tensor. Thus we will arrive at various cases of RCs corresponding to these constraints. Solving these constraints, which are often differential in nature, will give us the metrics of the spacetimes. Solution of these constraints and extraction of metrics from them will be the subject of discussion for Chapter 4. In the beginning we give the calculations in detail, in order to explain the procedure, and later on we only give the results. However, the detailed calculations are given in Appendix B.

We differentiate Eq. (2.5) for $a = 0$, $b = 3$ with respect to θ, and for $a = 0$, $b = 2$ with respect to z and subtract the latter from the former. Compare this equation with the derivative of Eq. (2.5) for $a = 2$, $b = 3$ relative to t, and the first of these equations to obtain $B^0_{,23} = B^2_{,03} = B^3_{,02} = 0$. Similarly, differentiating Eq. (2.4) for $a = 0$, with respect to θ and z, and using the above results we see that

$$R_0' B^1_{,23} = 0 \,. \tag{2.6}$$

This equation gives rise to two cases. Either

(A) $R_0' \neq 0$ (and $B^1_{,23} = 0$), or

(B) $R_0' = 0$.

We consider these cases one by one here.

2.2.1 Case A: $R_0' \neq 0$

Here, $B^1_{,23} = 0$, and we get from Eqs. (2.5) for $a = 1$, $b = 2, 3$, $B^2_{,13} = B^3_{,12} = 0$. Therefore, Eq. (2.5) for $a = 2$, $b = 3$ implies

$$\left(\frac{R_2}{R_3}\right)' B^2_{,3} = 0 \,. \tag{2.7}$$

Here we have two possibilities. Either

(I) $\left(\frac{R_2}{R_3}\right)' \neq 0$, or

(II) $\left(\frac{R_2}{R_3}\right)' = 0$.

Case A(I) $\left(\frac{R_2}{R_3}\right)' \neq 0$ (and $B^2_{,3} = 0$)

Eq. (2.5) for $a = 2$, $b = 3$ implies that $B^3_{,2} = 0$, and so from Eq. (2.4) for $a = 2,3$ we get

$$R'_2 B^1_{,3} = 0 = R'_3 B^1_{,2}. \tag{2.8}$$

Now, there are three possibilities.

(a) $R'_2 = 0$, $R'_3 \neq 0$,

(b) $R'_2 \neq 0$, $R'_3 = 0$,

(c) $R'_2 \neq 0$, $R'_3 \neq 0$.

We consider these one by one.

Case AI(a) $R'_2 = 0$, $R'_3 \neq 0$

In this case the RC equations imply $B^0_{,2} = B^1_{,2} = B^2_{,0} = B^2_{,1} = B^2_{,2} = B^2_{,3} = B^0_{,12} = 0$. Thus

$$B^2 = c_1, \tag{2.9}$$

where c_1 is a constant, and, therefore, integrating Eq. (2.4) for $a = 1$, with respect to ρ we get

$$B^1 = \frac{1}{\sqrt{R_1}} A_1(t, z). \tag{2.10}$$

where $A_1(t, z)$ is a function of integration to be determined. Using this value of B^1 in Eqs. (2.4), for $a = 0$, $b = 3$ and (2.5) for $a = 0$, $b = 1$, and $a = 1$, $b = 3$, respectively, gives

$$B^0_{,0} = -\frac{R'_0}{2R_0\sqrt{R_1}} A_1(t, z), \tag{2.11}$$

$$B^0_{,1} = -\frac{\sqrt{R_1}}{R_0} \dot{A}_1(t, z), \tag{2.12}$$

$$B^3_{,1} = -\frac{\sqrt{R_1}}{R_3} A_{1,3}(t, z), \tag{2.13}$$

$$B^3_{,3} = -\frac{R'_3}{2R_3\sqrt{R_1}} A_1(t, z), \tag{2.14}$$

where dot represents partial derivative with respect to t. Now, differentiating Eq. (2.11) with respect to ρ and Eq. (2.12) with respect to t and equating , we get

$$\dot{A}_1\,(t,z) - \frac{R_0}{\sqrt{R_1}}\left(\frac{R_0'}{2R_0\sqrt{R_1}}\right)' A_1\,(t,z) = 0\,. \tag{2.15}$$

Similarly differentiating Eq. (2.13) with respect to z and Eq. (2.14) with respect to ρ and equating gives

$$A_{1,33}\,(t,z) - \frac{R_3}{\sqrt{R_1}}\left(\frac{R_3'}{2R_3\sqrt{R_1}}\right)' A_1\,(t,z) = 0\,. \tag{2.16}$$

Here we write

$$\frac{R_0}{\sqrt{R_1}}\left(\frac{R_0'}{2R_0\sqrt{R_1}}\right)' = \alpha\;,\; \frac{R_3}{\sqrt{R_1}}\left(\frac{R_3'}{2R_3\sqrt{R_1}}\right)' = \beta, \tag{2.17}$$

where α and β are separation constants. The Eqs. (2.15) and (2.16) now become

$$\ddot{A}_1\,(t,z) - \alpha A_1\,(t,z) = 0\,, \tag{2.18}$$

$$A_{1,33}\,(t,z) - \beta A_1\,(t,z) = 0\,. \tag{2.19}$$

Here arise four cases depending upon whether one, both or none of the these constants is/are zero. We discuss them in turn.

Case AIa(1) $\qquad \alpha = 0,\, \beta = 0$

In this case, we let $\frac{R_0'}{2R_0\sqrt{R_1}} = -k_1$ and $\frac{R_3'}{2R_3\sqrt{R_1}} = -k_2$, where k_1 and k_2 are nonzero constants. From Eqs. (2.18) and (2.19) this means that

$$A_1\,(t,z) = c_8 tz + c_4 t + c_5 z + c_6\,, \tag{2.20}$$

where c_4, c_5, c_6 and c_8 are arbitrary constants. Putting this value of $A_1\,(t,z)$ in Eqs. (2.11) and (2.14), and integrating with respect to t and z respectively, we get

$$B^0 = k_1\left[c_8\frac{t^2}{2}z + c_4\frac{t^2}{2} + c_5 zt + c_6 t\right] + A_2\,(\rho,z)\,, \tag{2.21}$$

$$B^3 = k_2\left[c_8 t\frac{z^2}{2} + c_4 tz + c_5\frac{z^2}{2} + c_6 z\right] + A_3(\rho, z). \tag{2.22}$$

Eqs. (2.12) and (2.13) in view of Eqs. (2.20)-(2.22) on integration yield

$$A_2(\rho, z) = -(c_8 z + c_4)\int \frac{\sqrt{R_1}}{R_0}d\rho + B_3(z), \tag{2.23}$$

$$A_3(t, \rho) = -(c_8 t + c_5)\int \frac{\sqrt{R_1}}{R_3}d\rho + B_4(t). \tag{2.24}$$

Now, using Eqs. (2.21) and (2.22) with Eqs. (2.23) and (2.24)in Eq. (2.5) for $a = 0$, $b = 3$, we get

$$\begin{aligned}&\frac{R_0}{R_3}\left[-k_1\left(c_8\frac{t^2}{2} + c_5 t\right) + c_8\int \frac{\sqrt{R_1}}{R_0}d\rho - B_{3,3}(z)\right]\\ &+\left[-k_2\left(c_8\frac{z^2}{2} + c_4 z\right) + c_8\int \frac{\sqrt{R_1}}{R_3}d\rho - \dot{B}_4(t)\right] = 0.\end{aligned} \tag{2.25}$$

We note that

$$\int \frac{\sqrt{R_1}}{R_0}d\rho = \frac{1}{2k_1 R_0} \quad \text{and} \quad \int \frac{\sqrt{R_1}}{R_3}d\rho = \frac{1}{2k_2 R_3}. \tag{2.26}$$

From Eq. (2.25) we need to discuss two cases.

(i) $\left(\frac{R_0}{R_3}\right)' = 0$,

(ii) $\left(\frac{R_0}{R_3}\right)' \neq 0$.

Case AIa1(i) $\left(\frac{R_0}{R_3}\right)' = 0$

We put $R_0/R_3 = -k$ and note that in this $k_1 = k_2$, and therefore Eq. (2.25) yields

$$B_4(t) = \frac{kk_1c_5}{2}t^2 + kc_7 t + c_3, \tag{2.27}$$

$$B_3(z) = \frac{k_1 c_4}{k}\frac{z^2}{2} + c_7 z + c_1, \tag{2.28}$$

so that Eqs. (2.23) and (2.24) become

$$A_2(\rho,z) = -(c_8 z + c_4)\int \frac{\sqrt{R_1}}{R_0}d\rho + \frac{k_1 c_4}{k}\frac{z^2}{2} + c_7 z + c_1 \,, \tag{2.29}$$

$$A_3(t,\rho) = -(c_8 t + c_5)\int \frac{\sqrt{R_1}}{R_3}d\rho + \frac{k k_1 c_5}{2}t^2 + k c_7 t + c_3 \,. \tag{2.30}$$

Therefore, Eqs. (2.21), (2.10) , (2.9) and (2.22) give the final form of the RC vector as

$$\begin{aligned} &B^0 = c_4\left(\tfrac{k_1}{2}t^2 - \int \tfrac{\sqrt{R_1}}{R_0}d\rho + \tfrac{k_1}{2k}z^2\right) + k_1 c_5 t z + k_1 c_6 t + c_7 z + c_1 \,, \\ &B^1 = \tfrac{1}{\sqrt{R_1}}\left(c_4 t + c_5 z + c_6\right), \\ &B^2 = c_2 \,, \\ &B^3 = c_4 k_1 t z + c_5\left(\tfrac{k k_1}{2}t^2 - \int \tfrac{\sqrt{R_1}}{R_3}d\rho + \tfrac{k_1}{2}z^2\right) + k_1 c_6 z + k c_7 t + c_3 \,. \end{aligned} \tag{2.31}$$

Case AIa1(ii) $\quad \left(\frac{R_0}{R_3}\right)' \neq 0$

Here from Eq. (2.25) $c_4 = c_5 = c_8 = 0$, and therefore,

$$B_4(t) = c_3 \,, \quad B_3(z) = c_1 \,. \tag{2.32}$$

So, Eqs. (2.23) and (2.24) become

$$A_2(\rho,z) = c_1 \,, \tag{2.33}$$

$$A_3(t,\rho) = c_3 \,, \tag{2.34}$$

and, finally, we get from Eqs. (2.9), (2.10) and (2.20)-(2.22)

$$\begin{aligned} &B^0 = c_4 k_1 t + c_1 \,, \\ &B^1 = \tfrac{c_4}{\sqrt{R_1}} \,, \\ &B^2 = c_2 \,, \\ &B^3 = c_4 k_2 z + c_3 \,. \end{aligned} \tag{2.35}$$

Case AIa(2) $\qquad \alpha \neq 0, \quad \beta = 0$

Now, α can be greater or less than zero. We discuss both of these cases in turn.

Case AIa2(i) $\qquad \alpha > 0, \quad \beta = 0$

Here Eq. (2.18) gives

$$A_1\left(t_1 z\right) = B_1\left(z\right) \cosh \sqrt{\alpha} t + B_2\left(z\right) \sinh \sqrt{\alpha} t \,, \tag{2.36}$$

and the result is the minimal RCs

$$\begin{aligned} B^0 &= c_1 \,, \\ B^1 &= 0 \,, \\ B^2 &= c_2 \,, \\ B^3 &= c_3 \,. \end{aligned} \tag{2.37}$$

Case AIa2(ii) $\qquad \alpha < 0, \quad \beta = 0$

In this case Eq. (2.18) yields

$$A_1\left(t, z\right) = B_1\left(z\right) \cos \sqrt{\alpha} t + B_2\left(z\right) \sin \sqrt{\alpha} t \,. \tag{2.38}$$

So, Eq. (2.11) on integration with respect to t gives

$$B^0 = -\frac{R_0'}{2R_0\sqrt{R_1}\sqrt{\alpha}} \left[B_1\left(z\right) \sin \sqrt{\alpha} t - B_2\left(z\right) \cos \sqrt{\alpha} t\right] + A_2\left(\rho, z\right) , \tag{2.39}$$

where $A_2\left(\rho, z\right)$ is a function of integration which from Eq. (2.12) on integration is given by

$$A_2\left(\rho, z\right) = \left[\sqrt{\alpha} \int \frac{\sqrt{R_1}}{R_0} d\rho + \frac{R_0'}{2R_0\sqrt{R_1}\sqrt{\alpha}}\right] \left[B_1\left(z\right) \sin \sqrt{\alpha} t - B_2\left(z\right) \cos \sqrt{\alpha} t\right] + B_3\left(z\right) \,. \tag{2.40}$$

Eq. (2.39) becomes

$$B^0 = \sqrt{\alpha} \int \frac{\sqrt{R_1}}{R_0} d\rho \left[B_1\left(z\right) \sin \sqrt{\alpha} t - B_2\left(z\right) \cos \sqrt{\alpha} t\right] + B_3\left(z\right) \,. \tag{2.41}$$

Eq. (2.13) on integration gives

$$B^3 = -\int \frac{\sqrt{R_1}}{R_3} d\rho \left[B_{1,3}(z)\cos\sqrt{\alpha}t + B_{2,3}(z)\sin\sqrt{\alpha}t\right] + A_3(t,z) \,. \tag{2.42}$$

Using this in Eq. (2.14) and integrating with respect to z gives

$$\begin{aligned} A_3(t,z) &= \int \frac{\sqrt{R_1}}{R_3} d\rho \left[B_{1,3}(z)\cos\sqrt{\alpha}t + B_{2,3}(z)\sin\sqrt{\alpha}t\right] \\ &\quad + k\left[\int B_1(z)\,dz\cos\sqrt{\alpha}t + \int B_2(z)\,dz\sin\sqrt{\alpha}t\right] + A_4(t) \,. \end{aligned} \tag{2.43}$$

Therefore, Eq. (2.42) becomes

$$B^3 = k\left[\int B_1(z)\,dz\cos\sqrt{\alpha}t + \int B_2(z)\,dz\sin\sqrt{\alpha}t\right] + A_4(t) \,, \tag{2.44}$$

and from Eq. (2.10) we have

$$B^1 = \frac{1}{\sqrt{R_1}}\left[B_1(z)\cos\sqrt{\alpha}t + B_2(z)\sin\sqrt{\alpha}t\right] \,. \tag{2.45}$$

$$B^2 = c_2 \,. \tag{2.46}$$

Now, from Eqs. (2.44)and (2.5) for $a = 1$, $b = 3$, clearly $B^3_{,1} = 0$, $B^1_{,3} = 0$, and Eq. (2.45) yields $B_1(z) = c_4$, $B_2(z) = c_5$. So, Eqs. (2.41), (2.45) and (2.44) become

$$B^0 = \sqrt{\alpha}\int \frac{\sqrt{R_1}}{R_0} d\rho \left[c_4\sin\sqrt{\alpha}t - c_5\cos\sqrt{\alpha}t\right] + B_3(z) \,, \tag{2.47}$$

$$B^1 = \frac{1}{\sqrt{R_1}}\left(c_4\cos\sqrt{\alpha}t + c_5\sin\sqrt{\alpha}t\right) \,, \tag{2.48}$$

$$B^3 = k\left(c_4 z\cos\sqrt{\alpha}t + c_5 z\sin\sqrt{\alpha}t\right) + A_4(t) \,. \tag{2.49}$$

Eq. (2.5) for $a = 0$, $b = 3$, in this case gives

$$\frac{R_0}{R_3} B_{3,3}(z) = 0 \,. \tag{2.50}$$

This equation leads to two possibilities.

Case AIa2(ii)α $\qquad \left(\frac{R_0}{R_3}\right)' \neq 0$

Here $B_3(z) = c_1$, $A_4(t) = c_3$, and we have the minimal symmetry (from Eqs. (2.47)-(2.49)).

Case AIa2(ii)β $\qquad \left(\frac{R_0}{R_3}\right)' = 0$

In this case we put $\frac{R_0}{R_3}$ =constant $= -\gamma$ (say). Therefore, from Eq. (2.50)

$$B_3 = \frac{c_4 z}{\gamma} + c_1 \,, \tag{2.51}$$

$$A_4(t) = -c_4 t + c_3 \,. \tag{2.52}$$

Therefore, from Eqs. (2.47)-(2.49) we get

$$\begin{aligned} B^0 &= \tfrac{1}{\gamma} c_4 \, z + c_1 \,, \\ B^1 &= 0 \,, \\ B^2 &= c_2 \,, \\ B^3 &= c_4 t + c_3 \,. \end{aligned} \tag{2.53}$$

Case AIa(3) $\qquad \alpha = 0 \,, \quad \beta \neq 0$

We obtain this case just by interchanging the role of t and z in the previous case, Case AIa(2).

Case AIa(4) $\qquad \alpha \neq 0, \quad \beta \neq 0$

Now α and β can be positive or negative which gives rise to four further cases depending on whether one or both of these constants are greater than or less than zero. Now we discuss them.

Case AIa4(i) $\quad \alpha > 0 , \quad \beta > 0$

The solution for Eq. (2.18) in this case is

$$A_1 (t, z) = B_1 (z) \cosh \sqrt{\alpha} t + B_2 (z) \sinh \sqrt{\alpha} t \,. \tag{2.54}$$

Using this in Eq. (2.19) implies that

$$B_{1,33} (z) - \beta B_1 (z) = 0 \,, \tag{2.55}$$

$$B_{2,33} (z) - \beta B_2 (z) = 0 \,. \tag{2.56}$$

These can be solved to give

$$B_1 (z) = c_2 \cosh \sqrt{\beta} z + c_3 \sinh \sqrt{\beta} z \,, \tag{2.57}$$

$$B_2 (z) = c_4 \cosh \sqrt{\beta} z + c_5 \sinh \sqrt{\beta} z \,, \tag{2.58}$$

so that Eq. (2.54) becomes

$$\begin{aligned} A_1 (t, z) &= \left(c_2 \cosh \sqrt{\beta} z + c_3 \sinh \sqrt{\beta} z \right) \cosh \sqrt{\alpha} t \\ &\quad + \left(c_4 \cosh \sqrt{\beta} z + c_5 \sinh \sqrt{\beta} z \right) \sinh \sqrt{\alpha} t \,. \end{aligned} \tag{2.59}$$

Now, from Eqs. (2.11) and (2.12) we get

$$\begin{aligned} B^0 &= -\sqrt{\alpha} [\left(c_2 \cosh \sqrt{\beta} z + c_3 \sinh \sqrt{\beta} z \right) \sinh \sqrt{\alpha} t \\ &\quad + \left(c_4 \cosh \sqrt{\beta} z + c_5 \sinh \sqrt{\beta} z \right) \cosh \sqrt{\alpha} t] \int \frac{\sqrt{R_1}}{R_0} d\rho + B_3 (z) \,. \end{aligned} \tag{2.60}$$

Eqs. (2.13) and (2.14) yield

$$B^3 = -\frac{R_3'}{2R_3\sqrt{R_1}\sqrt{\beta}}[\left(c_2 \sinh\sqrt{\beta}z + c_3\cosh\sqrt{\beta}z\right)\cosh\sqrt{\alpha}t + \left(c_4\sinh\sqrt{\beta}z + c_5\cosh\sqrt{\beta}z\right)\sinh\sqrt{\alpha}t] + A_4(t)\,, \tag{2.61}$$

and from Eq. (2.10)

$$B^1 = \frac{1}{\sqrt{R_1}}[\left(c_2\cosh\sqrt{\beta}z + c_3\sinh\sqrt{\beta}z\right)\cosh\sqrt{\alpha}t + \left(c_4\cosh\sqrt{\beta}z + c_5\sinh\sqrt{\beta}z\right)\sinh\sqrt{\alpha}t]\,. \tag{2.62}$$

Now, satisfying Eq. (2.5) for $a=0$, $b=3$, gives

$$[\left(c_2\sinh\sqrt{\beta}z + c_3\cosh\sqrt{\beta}z\right)\sinh\sqrt{\alpha}t + \left(c_4\sinh\sqrt{\beta}z + c_5\cosh\sqrt{\beta}z\right)\cosh\sqrt{\alpha}t] \times \left[-R_0\sqrt{\alpha\beta}\int\frac{\sqrt{R_1}}{R_0}d\rho - \sqrt{\frac{\alpha}{\beta}}\frac{R_3'}{2\sqrt{R_1}}\right] + R_0B_{3,3}(z) + R_3\,\dot{A}_4(t) = 0\,. \tag{2.63}$$

In order that Eq. (2.63) be satisfied, there are two possibilities.

Either

(α) $c_2 = c_3 = c_4 = c_5 = 0$ and $R_0B_{3,3}(z) = -R_3\,\dot{A}_4(t)$, or

(β) $\beta R_0\int\frac{\sqrt{R_1}}{R_0}d\rho + \frac{R_3'}{2\sqrt{R_1}} = 0$ and $R_0B_{3,3}(z) = -R_3\,\dot{A}_4(t)$.

We discuss these cases here.

Case AIa4(i)α

Here if $\left(\frac{R_0}{R_3}\right)' \neq 0$ we get minimal RCs. Otherwise we take $\frac{R_0}{R_3} = -\gamma$, a constant and the result is

$$\begin{aligned} B^o &= c_4z + c_1\,, \\ B^1 &= 0\,, \\ B^2 &= c_2\,, \\ B^3 &= \gamma c_4t + c_3\,. \end{aligned} \tag{2.64}$$

Case AIa4(i)β

Here

$$\beta \int \frac{\sqrt{R_1}}{R_0} d\rho + \frac{R_3'}{2R_0\sqrt{R_1}} = 0 \,. \qquad (2.65)$$

Also we can write $\int \frac{\sqrt{R_1}}{R_0} d\rho = \frac{1}{\alpha} \left(\frac{R_0'}{2R_0\sqrt{R_1}} \right)$.This case again gives minimal symmetry.

Case AIa4(ii) $\qquad \alpha > 0 \;, \quad \beta < 0$

The result will be similar to that obtained in the previous case, Case AIa4(i). The difference will be that in the solution of Eq. (2.19) the trigonometric functions involved will be circular instead of hyperbolic.

Case AIa4(iii) $\qquad \alpha < 0 \;, \quad \beta > 0$

The result will be similar to that obtained in Case AIa4(i). The difference will be that in the solution of Eq. (2.18) the trigonometric functions involved will be circular instead of hyperbolic.

Case AIa4(iv) $\qquad \alpha < 0 \;, \quad \beta < 0.$

The result will be similar to that obtained in Case AIa4(i). The difference will be that in the solution of Eq. (2.18) and Eq. (2.19) the trigonometric functions involved will be circular instead of hyperbolic.

Case AI(b) $\qquad R_2' \neq 0 \;, \; R_3' = 0$

Note that the RC equations, (2.4) and (2.5), are symmetric with respect to the interchange in θ and z (i.e. the indices 2 and 3). Thus this case is similar to the previous case, Case AI(a), except for the interchange of θ and z in all the equations.

Case AI(c) $\quad R_2' \neq 0\,, \quad R_3' \neq 0$

In this case Eq. (2.8) implies that $B^1_{,2} = B^1_{,3} = B^2_{,1} = B^3_{,1} = 0$. So, we have

$$\begin{aligned} B^0 &= B^0(t,\rho,\theta,z)\,, \\ B^1 &= B^1(t,\rho)\,, \\ B^2 &= B^2(t,\theta)\,, \\ B^3 &= B^3(t,z)\,. \end{aligned} \tag{2.66}$$

We further have the following constraints from the RC equations.

$$B^0_{,02} = B^0_{,12} = B^0_{,03} = B^0_{,23} = B^0_{,13} = B^2_{,22} = B^3_{,33} = 0\,. \tag{2.67}$$

Now, integrating Eq. (2.4) for $a = 1$ gives

$$B^1 = \frac{A_1(t)}{\sqrt{R_1}}\,. \tag{2.68}$$

Therefore, Eqs. (2.5), for $a = 0$, $b = 1$, and (2.4) for $a = 2, 3$ on integration give

$$B^0 = -\dot{A}_1(t)\int \frac{\sqrt{R_1}}{R_0}d\rho + A_3(t) + A_4(\theta) + A_5(z)\,. \tag{2.69}$$

$$B^2 = \left(-\frac{R_2'}{2R_2\sqrt{R_1}}\right)A_1(t)\,\theta + A_6(t)\,, \tag{2.70}$$

$$B^3 = \left(-\frac{R_3'}{2R_3\sqrt{R_1}}\right)A_1(t)\,z + A_7(t)\,. \tag{2.71}$$

Now, we note that either both the terms in the brackets in Eqs. (2.70) and (2.71) are constants or otherwise $A_1(t)$ will be zero. We discuss both of these possibilities here.

Case AIc(1) $\quad A_1(t) = 0$

Here, from Eq. (2.68) we get $B^1 = 0$, and therefore, from Eqs. (2.69) and (2.4) for $a = 0$, we get $A_3(t) = c_1$. Now, using Eqs. (2.69), (2.70) and (2.71) in Eqs. (2.5) for $a = 0$, $b = 2, 3$, gives

$$R_0 A_{4,2}(\theta) + R_2 \dot{A}_6(t) = 0\,, \tag{2.72}$$

$$R_0 A_{5,3}(z) + R_3 \dot{A}_7(t) = 0\,. \tag{2.73}$$

These imply that

$$A_6(t) = c_4 t + c_2\,, \tag{2.74}$$

$$A_7(t) = c_8 t + c_3\,. \tag{2.75}$$

Therefore, Eqs. (2.72) and (2.73) take the form

$$A_{4,2}(\theta) = -\frac{R_2}{R_0} c_4\,, \tag{2.76}$$

$$A_{5,3}(z) = -\frac{R_3}{R_0} c_8\,. \tag{2.77}$$

Here, we have three further cases depending upon whether one or none of $\frac{R_2}{R_0}$ and $\frac{R_3}{R_0}$ are constants. We note that both of these cannot be constants as in that case we will get $\frac{R_2}{R_3}$ as a constant, which will be a contradiction.

Case AIc1(i) $\quad \left(\frac{R_2}{R_0}\right)' = 0\,,\ \left(\frac{R_3}{R_0}\right)' \neq 0$

In this case $R_2/R_0 = -k_1$ and $c_8 = 0$, and we get from Eqs. (2.68), (2.69), (2.70) and (2.71)

$$\begin{aligned} B^0 &= k_1 c_4 \theta + c_1, \\ B^1 &= 0\,, \\ B^2 &= c_4 t + c_2\,, \\ B^3 &= c_3\,. \end{aligned} \tag{2.78}$$

Case AIc1(ii) $\quad \left(\frac{R_2}{R_0}\right)' \neq 0\,,\ \left(\frac{R_3}{R_0}\right)' = 0$

This is similar to the previous case; just the indices 2 and 3 are interchanged.

Case AIc1(iii) $\quad \left(\frac{R_2}{R_0}\right)' \neq 0\,,\ \left(\frac{R_3}{R_0}\right)' \neq 0$

Here, we have from Eqs. (2.76) and (2.77) $c_4 = c_8 = 0$, and Eqs. (2.68), (2.69), (2.70) and (2.71) give minimal symmetry.

Case AIc(2) $\left(\frac{R_2'}{2R_2\sqrt{R_1}}\right)' = 0$, $\left(\frac{R_3'}{2R_3\sqrt{R_1}}\right)' = 0$

We put $\frac{R_2'}{2R_2\sqrt{R_1}} = k_1$, and $\frac{R_3'}{2R_3\sqrt{R_1}} = k_2$, where k_1 and k_2 are nonzero constants and $k_1 \neq k_2$, because otherwise R_2 and R_3 will become proportional which will be a contradiction. Therefore, we have

$$B^2 = -k_1 A_1(t)\,\theta + A_6(t), \tag{2.79}$$

$$B^3 = -k_2 A_1(t)\,z + A_7(t). \tag{2.80}$$

Now, Eqs. (2.5) for $a = 0$, $b = 2, 3$, on integration with respect to θ and z respectively give

$$A_4(\theta) = \frac{R_2}{R_0}\left[k_1\,\dot{A}_1(t)\,\frac{\theta^2}{2} - \dot{A}_6(t)\,\theta\right] + c_6, \tag{2.81}$$

$$A_5(z) = \frac{R_3}{R_0}\left[k_2\,\dot{A}_1(t)\,\frac{z^2}{2} - \dot{A}_7(t)\,z\right] + c_7, \tag{2.82}$$

and so Eq. (2.69) becomes

$$\begin{aligned} B^0 = {} & -\int \frac{\sqrt{R_1}}{\sqrt{R_0}}d\rho\,\dot{A}_1(t) + A_3(t) + \frac{R_2}{R_0}\left[k_1\,\dot{A}_1(t)\,\frac{\theta^2}{2} - \dot{A}_6(t)\,\theta\right] \\ & + \frac{R_3}{R_0}\left[k_2\,\dot{A}_1(t)\,\frac{z^2}{2} - \dot{A}_7(t)\,z\right] + c_1, \end{aligned} \tag{2.83}$$

where $c_1 = c_6 + c_7$. Now, the value of B^i, $i = 0, 1, 2, 3$, as given by Eqs. (2.83), (2.68), (2.79) and (2.80) identically satisfy all the RC equations except Eq. (2.4) for $a = 0$, and Eq. (2.5) for $a = 0$, $b = 1$, which take the form

$$\begin{aligned} & \tfrac{R_0' A_1(t)}{2R_0\sqrt{R_1}} - \int \tfrac{\sqrt{R_1}}{R_0}d\rho\,\ddot{A}_1(t) + \dot{A}_3(t) \\ & + \tfrac{R_2}{R_0}\left[k_1\,\ddot{A}_1(t)\,\tfrac{\theta^2}{2} - \ddot{A}_6(t)\,\theta\right] + \tfrac{R_3}{R_0}\left[k_2\,\ddot{A}_1(t)\,\tfrac{z^2}{2} - \ddot{A}_7(t)\,z\right] = 0, \end{aligned} \tag{2.84}$$

$$\left(\frac{R_2}{R_0}\right)'\left[-k_1\,\dot{A}_1(t)\,\frac{\theta^2}{2}+\dot{A}_6(t)\,\theta\right]+\left(\frac{R_3}{R_0}\right)'\left[-k_2\,\dot{A}_1(t)\,\frac{z^2}{2}+\dot{A}_7(t)\,z\right]=0\,. \tag{2.85}$$

From Eq. (2.85) we see that we have to discuss three further cases depending upon whether one or none of $(R_2/R_0)'$ and $(R_3/R_0)'$ is/are equal to zero. We note that both of these cannot be equal to zero as it would contradict one of the conditions of this case, namely, $(R_2/R_3)' \neq 0$.

Case AIc2(i) $\left(\frac{R_2}{R_0}\right)'=0,\ \left(\frac{R_3}{R_0}\right)'\neq 0$

Let $R_2/R_0=-k_3$ which implies that $\frac{R_0'}{2R_0\sqrt{R_1}}=k_1$. In this case Eq. (2.85) yields

$$A_1(t)=c_4\,,\ A_7(t)=c_3\,. \tag{2.86}$$

Therefore, Eq. (2.84) takes the form

$$k_1c_4+\dot{A}_3(t)+k_3\ddot{A}_6(t)\,\theta=0\,. \tag{2.87}$$

which implies that

$$A_6(t)=c_5t+c_2\,, \tag{2.88}$$

$$A_3(t)=-k_1c_4t+c_6\,. \tag{2.89}$$

With values from Eqs. (2.86), (2.88) and (2.89), we finally get from Eqs. (2.83), (2.68), (2.79) and (2.80)

$$\begin{aligned}
B^0&=-k_1c_4t+k_3c_5\theta+c_1\,,\\
B^1&=\tfrac{c_4}{\sqrt{R_1}}\,,\\
B^2&=-k_1c_4\theta+c_5t+c_2\,,\\
B^3&=-k_2c_4z+c_3\,.
\end{aligned} \tag{2.90}$$

Case AIc2(ii) $\left(\frac{R_2}{R_0}\right)'\neq 0,\ \left(\frac{R_3}{R_0}\right)'=0$

This is similar to the previous case; only indices 2 and 3 (i.e. coordinates θ and z) are interchanged.

Case AIc2(iii) $\left(\frac{R_2}{R_0}\right)' \neq 0,\ \left(\frac{R_3}{R_0}\right)' \neq 0$

In this case Eq. (2.85) yields

$$A_1(t) = c_4\,,\ A_6(t) = c_2\,,\ A_7(t) = c_3\,, \tag{2.91}$$

and Eq. (2.84) takes the from

$$\frac{R_0'}{2R_0\sqrt{R_1}}c_4 + A_3(t) = 0\,. \tag{2.92}$$

Here again, as before, we have two cases.

Case AIc2(iii)α $\left(\frac{R_0'}{2R_0\sqrt{R_1}}\right)' = 0$

Putting $\frac{R_0'}{2R_0\sqrt{R_1}} = k_4$, and proceeding as previously, Eqs. (2.83), (2.68), (2.79) and (2.80) take the form

$$\begin{aligned} B^0 &= -k_4c_4t + c_1\,, \\ B^1 &= \tfrac{c_4}{\sqrt{R_1}}\,, \\ B^2 &= -k_1c_4\theta + c_2\,, \\ B^3 &= -k_2c_4z + c_3\,. \end{aligned} \tag{2.93}$$

Case AIc2(iii)β $\left(\frac{R_0'}{2R_0\sqrt{R_1}}\right)' \neq 0$

We have minimal RCs here.

Case A(II) $\left(\frac{R_2}{R_3}\right)' = 0$

This means that R_2/R_3 =constant= l (say), $l \neq 0$. We put $R_2 = lR_3$ in the RC equations and solve them as previously to get

$$B^0 = -\frac{R_2}{R_1}\left[\frac{1}{2}\left(\frac{1}{l}z^2 + \theta^2\right)\dot{B}_5(t,\rho) + \frac{1}{l}z\,\dot{B}_6(t,\rho) + \theta\,\dot{B}_7(t,\rho)\right] + B_8(t,\rho)\ , \tag{2.94}$$

$$B^1 = -\frac{R_2}{R_1}\left[\frac{1}{2}\left(\frac{1}{l}z^2 + \theta^2\right)B_5'(t,\rho) + \frac{1}{l}zB_6'(t,\rho) + \theta B_7'(t,\rho)\right] + B_9(t,\rho)\ , \tag{2.95}$$

$$B^2 = \frac{z^2}{2}(\theta c_9 + c_8) + z(c_7\theta + c_4) - \frac{l\theta^3}{6}c_9 - \frac{l\theta^2}{2}c_8 + \theta B_5(t,\rho) + B_7(t,\rho)\ , \tag{2.96}$$

$$B^3 = -\frac{l\theta^2}{2}\left(zc_9 + c_7\right) - l\theta\left(zc_8 + c_4\right) + \frac{z^3}{6}c_9 + \frac{z^2}{2}c_7 + zB_5\left(t,\rho\right) + B_6\left(t,\rho\right) \,. \tag{2.97}$$

In order to evaluate B_i, $i = 5,...9$, we substitute B^i from Eqs. (2.94)-(2.97) into Eqs. (2.4), for $a = 0,1,2$, (2.5) for $a = 0, b = 1$, and compare the coefficients of $\theta^2, \theta^1, \theta^0, z^2, z$ and z^0 in the equations thus obtained. This gives

$$\frac{R_0'}{R_1}B_j'\left(t,\rho\right) + 2\,\dot{B}_j\left(t,\rho\right) = 0, \quad j = 5,6,7\,; \tag{2.98}$$

$$R_0'B_9\left(t,\rho\right) + 2R_0\,\dot{B}_8\left(t,\rho\right) = 0\,, \tag{2.99}$$

$$2R_2\,\dot{B}_j'\left(t,\rho\right) + R_0\left(\frac{R_2}{R_0}\right)'\dot{B}_j\left(t,\rho\right) = 0\,, \quad j = 5,6,7\,; \tag{2.100}$$

$$R_0B_8'\left(t,\rho\right) + R_1\,\dot{B}_9\left(t,\rho\right) = 0\,, \tag{2.101}$$

$$\left(\frac{R_2}{\sqrt{R_1}}\right)'B_j'\left(t,\rho\right) + \frac{R_2}{\sqrt{R_1}}B_j''\left(t,\rho\right) = 0\,, j = 5,6,7\,; \tag{2.102}$$

$$R_1'B_9\left(t,\rho\right) + 2R_1B_9'\left(t,\rho\right) = 0\,, \tag{2.103}$$

$$\frac{R_2'}{lR_1}B_5'\left(t,\rho\right) + 2c_9 = 0\,, \tag{2.104}$$

$$\frac{R_2'}{lR_1}B_5'\left(t,\rho\right) - 2c_9 = 0\,, \tag{2.105}$$

$$\frac{R_2'}{lR_1}B_6'\left(t,\rho\right) - 2c_7 = 0\,, \tag{2.106}$$

$$\frac{R_2'}{lR_1}B_7'\left(t,\rho\right) + 2c_8 = 0\,, \tag{2.107}$$

$$R_2'B_9\left(t,\rho\right) + 2R_2B_5\left(t,\rho\right) = 0\,. \tag{2.108}$$

Solving these simultaneously yields

$$\frac{R_2'}{lR_2\sqrt{R_1}}f_6\left(t\right) - 2c_7 = 0\,, \tag{2.109}$$

$$\frac{R_2'}{lR_2\sqrt{R_1}}f_7\left(t\right) + 2c_8 = 0\,, \tag{2.110}$$

$$\frac{R_2'}{R_2\sqrt{R_1}}f_9\left(t\right) + 2B_5\left(t,\rho\right) = 0\,, \tag{2.111}$$

$$\frac{R_0'}{R_2\sqrt{R_1}} f_j(t) + 2\,\ddot{B}_j(t,\rho) = 0\,, \quad j = 5,6,7\,. \tag{2.112}$$

Now Eqs. (2.109)-(2.111) give rise to two possibilities.

(a) $\left(\frac{R_2'}{R_2\sqrt{R_1}}\right)' \neq 0\,,$

(b) $\left(\frac{R_2'}{R_2\sqrt{R_1}}\right)' = 0\,.$

We consider them now.

Case AII(a)

In this case $R_2' \neq 0$ and we arrive at the following two cases.

(1) $\left(\sqrt{R_0/R_2}\right)' = 0\,,$

(2) $\left(\sqrt{R_0/R_2}\right)' \neq 0\,.$

Case AIIa(1) $\left(\sqrt{\frac{R_0}{R_2}}\right)' = 0$

Let $\sqrt{R_0/R_2} = k$, so that Eqs. (2.94)-(2.97) become

$$\begin{aligned}
B^0 &= -\tfrac{1}{k}\left(c_6\tfrac{1}{l}z + c_5\theta\right) + c_1\,,\\
B^1 &= 0\,,\\
B^2 &= c_4 z + k c_5 t + c_2\,,\\
B^3 &= -l\theta c_4 + k c_6 t + c_3\,.
\end{aligned} \tag{2.113}$$

Case AIIa(2) $\left(\sqrt{\frac{R_0}{R_2}}\right)' \neq 0$

Eqs. (2.94)-(2.97) in this case give

$$\begin{aligned}
B^0 &= c_1\,,\\
B^1 &= 0\,,\\
B^2 &= c_4 z + c_2\,,\\
B^3 &= -c_4 l\theta + c_3\,.
\end{aligned} \tag{2.114}$$

Case AII(b) $\left(\frac{R_2'}{R_2\sqrt{R_1}}\right)' = 0$

Let $\frac{R_2'}{R_2\sqrt{R_1}} = \alpha$, a constant, which can be zero as well as nonzero.

(1) $R_2' \neq 0$,

(2) $R_2' = 0$.

Case AIIb(1) $\qquad R_2' \neq 0$

In this case from Eqs. (2.109) and (2.110) we have

$$f_6(t) = \frac{2lc_7}{\alpha}, \qquad f_7(t) = -\frac{2lc_8}{\alpha}, \tag{2.115}$$

and there arise further two cases.

(i) $(R_2/R_0)' \neq 0$,

(ii) $(R_2/R_0)' = 0$.

Case AIIb1(i) $\qquad \left(\frac{R_2}{R_0}\right)' \neq 0$

Now, we again have two possibilities.

(α) $\left(\frac{R_0'}{R_0\sqrt{R_1}}\right)' \neq 0$,

(β) $\left(\frac{R_0'}{R_0\sqrt{R_1}}\right)' = 0$.

Case AIIb1(i)α $\qquad \left(\frac{R_0'}{R_0\sqrt{R_1}}\right)' \neq 0$

Eqs. (2.94)-(2.97) in this case take the form

$$\begin{aligned} B^0 &= c_1 , \\ B^1 &= 0 , \\ B^2 &= c_4 z + c_2 , \\ B^3 &= -c_4 l\theta + c_3 . \end{aligned} \tag{2.116}$$

Case AIIb1(i)β $\qquad \left(\frac{R_0'}{R_0\sqrt{R_1}}\right)' = 0$

Let $\frac{R_0'}{R_0\sqrt{R_1}} = \beta \neq \alpha$, so that Eqs. (2.94)-(2.97) give the RCs as

$$\begin{aligned} B^0 &= \tfrac{\beta}{\alpha} c_5 t + c_1 , \\ B^1 &= -c_5 \tfrac{2}{\alpha\sqrt{R_1}} , \\ B^2 &= c_4 z + c_5 \theta + c_2 , \\ B^3 &= -c_4 l\theta + c_5 z + c_3 . \end{aligned} \tag{2.117}$$

Case AIIb1(ii) $\left(\frac{R_2}{R_0}\right)' = 0$

Here we write $R_2 = -\delta R_0$. The RCs from Eqs. (2.94)-(2.97), in this case are

$$\begin{aligned}
B^0 &= \tfrac{c_7}{2}\left(t^2 - \tfrac{4}{\alpha^2 R_0} + \delta\theta^2 + \tfrac{1}{l}\delta z^2\right) + c_6\tfrac{1}{l}\delta z - c_8 l\theta t + c_9 tz + c_5\delta\theta + c_{10}t + c_1\,,\\
B^1 &= -\tfrac{2}{\alpha\sqrt{R_1}}\left(c_7 t - c_8 l\theta + c_9 z + c_{10}\right)\,,\\
B^2 &= c_7\theta t + \tfrac{c_8}{2}\left(-\tfrac{lt^2}{\delta} + \tfrac{4l}{\alpha^2 R_2} - l\theta^2 + z^2\right) + c_5 t + c_9\theta z + c_{10}\theta + c_4 z + c_2\,,\\
B^3 &= c_7 zt - c_8 l\theta z - \tfrac{c_9}{2}\left(-\tfrac{lt^2}{\delta} + \tfrac{4l}{\alpha^2 R_2} + l\theta^2 - z^2\right) + c_6 t + c_{10} z - c_4 l\theta + c_3\,.
\end{aligned} \tag{2.118}$$

Case AIIb(2) $R_2' = 0$

This means that $R_2' = 0$ and this gives rise to two possibilities.

(i) $\left(\frac{(\sqrt{R_0})'}{\sqrt{R_1}}\right)' = 0\,,$

(ii) $\left(\frac{(\sqrt{R_0})'}{\sqrt{R_1}}\right)' \neq 0\,.$

We discuss each in turn.

Case AIIb2(i) $\left(\frac{(\sqrt{R_0})'}{\sqrt{R_1}}\right)' = 0$

Let $(\sqrt{R_0})'/\sqrt{R_1} = \gamma$, a constant, which cannot be zero, because otherwise R_0 becomes constant.

Eqs. (2.94)-(2.97) give the RCs as

$$\begin{aligned}
B^0 &= \tfrac{1}{\sqrt{R_0}}\left[\tfrac{1}{l}z\left(c_7\sin\gamma t - c_8\cos\gamma t\right) + \theta\left(c_5\sin\gamma t - c_6\cos\gamma t\right) - \left(c_9\sin\gamma t - c_{10}\cos\gamma t\right)\right] + c_1\,,\\
B^1 &= -\tfrac{1}{\sqrt{R_1}}\left[\tfrac{1}{l}z\left(c_7\cos\gamma t + c_8\sin\gamma t\right) + \theta\left(c_5\cos\gamma t + c_6\sin\gamma t\right) - \left(c_9\cos\gamma t + c_{10}\sin\gamma t\right)\right]\,,\\
B^2 &= \tfrac{\sqrt{R_0}}{\gamma R_2}\left(c_5\cos\gamma t + c_6\sin\gamma t\right) + c_4 z + c_2\,,\\
B^3 &= \tfrac{\sqrt{R_0}}{\gamma R_2}\left(c_7\cos\gamma t + c_8\sin\gamma t\right) - c_4 l\theta + c_3\,.
\end{aligned} \tag{2.119}$$

Case AIIb2(ii) $\left[\frac{(\sqrt{R_0})'}{\sqrt{R_1}}\right]' \neq 0$

In this case we have further two cases.

(α) $\frac{R_0}{2\sqrt{R_1}}\left(\frac{R_0'}{R_0\sqrt{R_1}}\right)' = \text{constant} = \eta$ (say),

(β) $\frac{R_0}{2\sqrt{R_1}}\left(\frac{R_0'}{R_0\sqrt{R_1}}\right)' \neq$ constant.

Case AIIb2(ii)α

Here we again discuss two cases depending on whether η is zero or nonzero.

Case AIIb2(ii)α_1 $\eta = 0$

This implies that $\frac{R_0'}{R_0\sqrt{R_1}}$ =constant $= \lambda \neq 0$. Here, Eqs. (2.94)-(2.97) yield

$$\begin{aligned}
B^0 &= c_5\left(\tfrac{1}{\lambda R_0} - \tfrac{\lambda}{4}t^2\right) - c_6\tfrac{\lambda}{2}t + c_1\,, \\
B^1 &= \tfrac{1}{\sqrt{R_1}}\left(c_5 t + c_6\right)\,, \\
B^2 &= c_4 z + c_2\,, \\
B^3 &= -c_4 l\theta + c_3\,.
\end{aligned} \tag{2.120}$$

Case AIIb2(ii)α_2 $\eta \neq 0$

In this case Eqs. (2.94)-(2.97) give the RCs as

$$\begin{aligned}
B^0 &= -\tfrac{1}{2\sqrt{\eta}}\tfrac{R_0'}{R_0\sqrt{R_1}}\left(c_5 e^{\sqrt{\eta}t} - c_6 e^{-\sqrt{\eta}t}\right) + c_1\,, \\
B^1 &= \tfrac{1}{\sqrt{R_1}}\left(c_5 e^{\sqrt{\eta}t} + c_6 e^{-\sqrt{\eta}t}\right)\,, \\
B^2 &= c_4 z + c_2\,, \\
B^3 &= -c_4 l\theta + c_3\,.
\end{aligned} \tag{2.121}$$

Case AIIb2(ii)β $\left[\frac{R_0}{\sqrt{R_1}}\left(\frac{R_0'}{R_0\sqrt{R_1}}\right)'\right]' \neq 0$

Eqs. (2.94)-(2.97) give the result as

$$\begin{aligned}
B^0 &= c_1\,, \\
B^1 &= 0\,, \\
B^2 &= c_4 z + c_2\,, \\
B^3 &= -c_4 l\theta + c_3\,.
\end{aligned} \tag{2.122}$$

2.2.2 Case B: $R_0' = 0$

Let $R_0 = -\alpha$, where α is a nonzero constant. In this case we observe from the RC equations that

$$B^0_{,0} = B^0_{,12} = B^0_{,13} = B^0_{,23} = B^1_{,00} = B^1_{,02} = B^1_{,03} = B^2_{,00} = B^2_{,03} = B^3_{,00} = B^3_{,02} = 0 \,. \quad (2.123)$$

Eq. (2.4) for $a = 1$ can thus be written as

$$B^1 = \frac{1}{\sqrt{R_1}} \left(c_{12} t + A_2 (\theta, z)\right) \,. \quad (2.124)$$

Using this in Eq. (2.5) for $a = 0, b = 1$ and integrating with respect to ρ gives

$$B^0 = \frac{c_{12}}{\alpha} \int \sqrt{R_1} d\rho + A_4 (\theta) + A_5 (z) \,. \quad (2.125)$$

Similarly, Eqs. (2.5) for $a = 0, b = 2, 3$ yield

$$B^2 = \frac{\alpha t}{R_2} A_{4,2} (\theta) + A_6 (\rho, \theta, z) \,, \quad (2.126)$$

$$B^3 = \frac{\alpha t}{R_3} A_{5,3} (z) + A_7 (\rho, \theta, z) \,, \quad (2.127)$$

Putting these values in Eqs. (2.5) for $a = 1, b = 2, 3$, we get

$$R_2' A_{4,2} (\theta) = 0 \,, \quad (2.128)$$

$$R_3' A_{5,3} (z) = 0 \,. \quad (2.129)$$

These equations suggest the following four cases.

(I) $R_2' = 0$, $R_3' = 0$,

(II) $R_2' = 0$, $R_3' \neq 0$,

(III) $R_2' \neq 0$, $R_3' = 0$,

(IV) $R_2' \neq 0$, $R_3' \neq 0$,

We discuss each in turn here.

Case B(I) $R_2' = 0$, $R_3' = 0$

We put $R_2 = \beta$ and $R_3 = \gamma$. Following the same procedure, the solution of the RC equations in this case is

$$\begin{aligned} B^0 &= -\tfrac{c_6}{\alpha} \int \sqrt{R_1} d\rho + c_7 \theta + c_8 z + c_1 \ , \\ B^1 &= \tfrac{1}{\sqrt{R_1}} \left(c_6 t + c_9 \theta + c_{10} z + c_5 \right) \ , \\ B^2 &= \tfrac{\alpha}{\beta} c_7 t - \tfrac{1}{\beta} c_9 \int \sqrt{R_1} d\rho + c_4 z + c_2 \ , \\ B^3 &= \tfrac{\alpha}{\gamma} c_8 t - \tfrac{1}{\gamma} c_{10} \int \sqrt{R_1} d\rho - \tfrac{\beta}{\gamma} c_4 \theta + c_3 \ . \end{aligned} \tag{2.130}$$

Case B(II) $R_2' = 0, \quad R_3' \neq 0,$

We call R_2 as β and Eq. (2.129) gives

$$A_5 (z) = c_0 \ . \tag{2.131}$$

Eq. (2.4) for $a = 2$ implies that

$$A_4 (\theta) = c_4 \theta \ + c_1 \ , \tag{2.132}$$

$$A_6 (\rho, \theta, z) = A_6 (\rho, z) \ , \tag{2.133}$$

Eq. (2.5) for $a = 1, b = 2$, becomes

$$A_2 (\theta, z) = \theta A_8 (z) + A_9 (z) \ , \tag{2.134}$$

so that

$$A_6 (\rho, z) = -\frac{1}{\beta} A_8 (z) \int \sqrt{R_1} d\rho + A_{10} (z) \ . \tag{2.135}$$

Now, Eq. (2.5) for $a = 1, b = 3$, yields

$$A_7(\rho,\theta,z) = -\left[\theta A_{8,3}(z) + A_{9,3}(z)\right]\int \frac{\sqrt{R_1}}{R_3}d\rho + A_{11}(\theta,z) , \tag{2.136}$$

and Eq. (2.5) for $a = 2, b = 3$ implies that

$$A_{11}(\theta,z) = \theta A_{12}(z) + A_{13}(z) . \tag{2.137}$$

Putting these values in Eq. (2.4) for $a = 3$, we get

$$A_{8,33}(z) - \left[\frac{R_3}{\sqrt{R_1}}\left(\frac{R_3'}{2R_3\sqrt{R_1}}\right)'\right]A_8(z) = 0 . \tag{2.138}$$

Depending upon whether the term in the square brackets in the last equation is constant or not, we need to discuss two further cases.

(a) $\left[\frac{R_3}{\sqrt{R_1}}\left(\frac{R_3'}{2R_3\sqrt{R_1}}\right)'\right]' \neq 0 ,$

(b) $\left[\frac{R_3}{\sqrt{R_1}}\left(\frac{R_3'}{2R_3\sqrt{R_1}}\right)'\right]' = 0 .$

Case BII(a) $\left[\frac{R_3}{\sqrt{R_1}}\left(\frac{R_3'}{2R_3\sqrt{R_1}}\right)'\right]' \neq 0$

In this case the RC vector becomes

$$\begin{aligned} B^0 &= c_4\theta + c_1 , \\ B^1 &= 0 , \\ B^2 &= \tfrac{\alpha}{\beta}c_4t + c_2 , \\ B^3 &= c_3 . \end{aligned} \tag{2.139}$$

Case BII(b) $\frac{R_3}{\sqrt{R_1}}\left(\frac{R_3'}{2R_3\sqrt{R_1}}\right)' = k_1$

Here there are three further possibilities: $k_1 \gtreqless 0$. We take them up one by one.

Case BIIb(1) $k_1 > 0$

In this case the solution of Eq. (2.138) can be written as

$$A_8(z) = c_7e^{\sqrt{k_1}z} + c_8e^{-\sqrt{k_1}z} , \tag{2.140}$$

and Eq. (2.4) for $a=3$ gives

$$A_9(z)=c_5e^{\sqrt{k_1}z}+c_6e^{-\sqrt{k_1}z}. \tag{2.141}$$

The result is

$$\begin{aligned}
B^0&=c_4\theta+c_1\ ,\\
B^1&=\tfrac{1}{\sqrt{R_1}}\left(c_5e^{\sqrt{k_1}z}+c_6e^{-\sqrt{k_1}z}\right)\ ,\\
B^2&=\tfrac{\alpha}{\beta}c_4t+c_2\ ,\\
B^3&=-\tfrac{1}{\sqrt{k_1}}\tfrac{R_3'}{2R_3\sqrt{R_1}}\left(c_5e^{\sqrt{k_1}z}-c_6e^{-\sqrt{k_1}z}\right)+c_3\ .
\end{aligned} \tag{2.142}$$

Case BIIb(2) $\qquad k_1<0$

This case is similar to the previous one, the difference being that in Eqs. (2.140) and (2.141) the arguments of the exponential functions will be complex instead of real and we get the similar RC vector with 6RCs.

Case BIIb(3) $\qquad k_1=0$

This means that $\frac{R_3'}{2R_3\sqrt{R_1}}=k_2$, where k_2 is a non-zero constant. From Eq. (2.138), we have $A_{8,33}(z)=0$, whose solution is

$$A_8(z)=c_7z+c_8\ , \tag{2.143}$$

yielding the final form of B^i as

$$\begin{aligned}
B^0&=c_4\theta+c_1\ ,\\
B^1&=\tfrac{1}{\sqrt{R_1}}\left(c_6z+c_5\right)\ ,\\
B^2&=\tfrac{\alpha}{\beta}c_4t+c_2\ ,\\
B^3&=-c_6\left(\int\tfrac{\sqrt{R_1}}{R_3}d\rho+k_2\tfrac{z^2}{2}\right)-k_2c_5z+c_3\ .
\end{aligned} \tag{2.144}$$

Case B(III) $\qquad R_2'\neq0\,,\quad R_3'=0$

As the RC equations remain unchanged if we interchange indices 2 and 3, the results for this case can be obtained by interchanging these indices (i.e. θ and z coordinates) in Case B(II).

Case B(IV) $\quad R_2' \neq 0\,, \quad R_3' \neq 0$

In this case Eqs. (2.128) and (2.129) yield

$$A_4(\theta) = c_0\,,\ A_5(z) = c_1\,, \tag{2.145}$$

and B^i from Eqs. (2.124)-(2.127) take the form

$$B^0 = c_0 + c_1\,, \tag{2.146}$$

$$B^1 = \frac{1}{\sqrt{R_1}} A_2(\theta, z)\,, \tag{2.147}$$

$$B^2 = A_6(\rho, \theta, z)\,, \tag{2.148}$$

$$B^3 = A_7(\rho, \theta, z)\,. \tag{2.149}$$

Therefore Eqs. (2.5) for $a = 1, b = 2$ and (2.4) for $a = 2$ yield

$$A_{2,22}(\theta, z) - \left[\frac{R_2}{\sqrt{R_1}}\left(\frac{R_2'}{2R_2\sqrt{R_1}}\right)'\right] A_2(\theta, z) = 0\,. \tag{2.150}$$

This equations gives rise to two possibilities (as before) depending upon whether the term in the square brackets is a function or a constant.

(a) $\quad \left[\frac{R_2}{\sqrt{R_1}}\left(\frac{R_2'}{2R_2\sqrt{R_1}}\right)'\right]' \neq 0\,,$

(b) $\quad \left[\frac{R_2}{\sqrt{R_1}}\left(\frac{R_2'}{2R_2\sqrt{R_1}}\right)'\right]' = 0\,.$

We discuss them here.

Case BIV(a) $\quad \left[\frac{R_2}{\sqrt{R_1}}\left(\frac{R_2'}{2R_2\sqrt{R_1}}\right)'\right]' \neq 0$

Here $A_2(\theta, z) = 0$, and the RC equations yield

$$A_6(z) = c_4 z + c_2\,, \tag{2.151}$$

$$A_7(\theta) = -\frac{R_2}{R_3} c_4 \theta + c_3\,. \tag{2.152}$$

Now, this is possible only if either $c_4 = 0$ or $\frac{R_2}{R_3}$ is constant.

Case BIVa(1) $\left(\frac{R_2}{R_3}\right)' \neq 0$

Here $c_4 = 0$ and we get minimal RCs.

Case BIVa(2) $\left(\frac{R_2}{R_3}\right)' = 0$.

Writing $R_2/R_3 = k$, the RC vector becomes

$$
\begin{aligned}
B^0 &= c_1 \, , \\
B^1 &= 0 \, , \\
B^2 &= c_4 z + c_2 \, , \\
B^3 &= -k c_4 \theta + c_3 \, .
\end{aligned}
\tag{2.153}
$$

Case BIV(b) $\left[\frac{R_2}{\sqrt{R_1}}\left(\frac{R_2'}{2R_2\sqrt{R_1}}\right)'\right]' = 0$

We put $\frac{R_2}{\sqrt{R_1}}\left(\frac{R_2'}{2R_2\sqrt{R_1}}\right)' = k_3$, and there are further three possibilities for the constant: $k_3 \gtreqless 0$. We discuss these in turn here.

Case BIVb(1) $k_3 > 0$

Eq. (2.150) becomes

$$A_{2,22}(\theta, z) - k_3 A_2(\theta, z) = 0 \, , \tag{2.154}$$

which can be solved to give

$$A_2(\theta, z) = A_8(z)\, e^{\sqrt{k_3}\theta} + A_9(z)\, e^{-\sqrt{k_3}\theta} \, . \tag{2.155}$$

where $A_8(z)$ and $A_9(z)$ satisfy

$$A_{8,33}(z) - \left[\frac{R_3}{\sqrt{R_1}}\left(\frac{R_3'}{2R_3\sqrt{R_1}}\right)'\right] A_8(z) = 0 \, , \tag{2.156}$$

$$A_{9,33}(z) - \left[\frac{R_3}{\sqrt{R_1}}\left(\frac{R_3'}{2R_3\sqrt{R_1}}\right)'\right] A_9(z) = 0 \, . \tag{2.157}$$

These equations again suggest two further possibilities depending upon whether the term in the square brackets is constant or not.

(i) $$\left[\frac{R_3}{\sqrt{R_1}}\left(\frac{R_3'}{2R_3\sqrt{R_1}}\right)'\right]' \neq 0 ,$$

(ii) $$\left[\frac{R_3}{\sqrt{R_1}}\left(\frac{R_3'}{2R_3\sqrt{R_1}}\right)'\right]' = 0 .$$

Case BIVb1(i) $\left[\frac{R_3}{\sqrt{R_1}}\left(\frac{R_3'}{2R_3\sqrt{R_1}}\right)'\right]' \neq 0$

Here the RC vector is given by

$$\begin{aligned}
B^0 &= c_1 ,\\
B^1 &= 0 ,\\
B^2 &= c_4 z + c_2 ,\\
B^3 &= -\frac{R_2}{R_3} c_4 \theta + c_3 .
\end{aligned} \tag{2.158}$$

We note that R_2/R_3 here, is a constant otherwise c_4 will be zero.

Case BIVb1(ii) $\left[\frac{R_3}{\sqrt{R_1}}\left(\frac{R_3'}{2R_3\sqrt{R_1}}\right)'\right]' = 0$

We put $\frac{R_3}{\sqrt{R_1}}\left(\frac{R_3'}{2R_3\sqrt{R_1}}\right)' = k_1$. Now, there are again three possibilities, $k_1 \gtreqless 0$. We first consider $k_1 > 0$. In this case the solution of Eqs. (2.156) and (2.157) can be written as

$$A_8(z) = c_5 e^{\sqrt{k_1} z} + c_6 e^{-\sqrt{k_1} z} , \tag{2.159}$$

$$A_9(z) = c_7 e^{\sqrt{k_1} z} + c_8 e^{-\sqrt{k_1} z} , \tag{2.160}$$

and the result in this case is

$$\begin{aligned}
B^0 &= c_1 ,\\
B^1 &= 0 ,\\
B^2 &= c_4 z + c_2 ,\\
B^3 &= -\frac{R_2}{R_3} c_4 \theta + c_3 .
\end{aligned} \tag{2.161}$$

Here again R_2/R_3 is a constant as before.

We see that for $k_1 < 0$ and $k_1 = 0$, we get the same result as in this case.

Case BIVb(2) $k_3 < 0$

In this case the solution (Eq. (2.155)) of Eq. (2.154) will involve complex arguments for the exponential functions and the results can be obtained similarly as in Case BIVb(1).

Case BIVb(3) $k_3 = 0$

In this case, we have $\frac{R_2'}{2R_2\sqrt{R_1}} = k_4$, where k_4 is a nonzero constant. From Eq. (2.150) therefore, we get the solution as

$$A_2(\theta, z) = A_8(z)\,\theta + A_9(z) \ . \tag{2.162}$$

where $A_8(z)$ and $A_9(z)$ again satisfy Eqs. (2.156) and (2.157). This again implies two possibilities, depending on whether the term in the square brackets of these equations is constant or not.

Case BIVb3(i) $\left[\frac{R_3}{\sqrt{R_1}}\left(\frac{R_3'}{2R_3\sqrt{R_1}}\right)'\right]' \neq 0$

Eqs. (2.156) and (2.157) imply that

$$A_8(z) = 0 = A_9(z) \ , \tag{2.163}$$

and the solution is similar to Case BIVb1(i).

Case BIVb3(ii) $\left[\frac{R_3}{\sqrt{R_1}}\left(\frac{R_3'}{2R_3\sqrt{R_1}}\right)'\right]' = 0$

We put $\frac{R_3}{\sqrt{R_1}}\left(\frac{R_3'}{2R_3\sqrt{R_1}}\right)' = k_1$, where k_1 is a constant which can be greater than, equal to or less than zero.

Case BIVb3(ii)α $k_1 > 0$

This solution here is similar to Case BIVb1(ii), where $k_1 = 0$, $k_3 > 0$.

Case BIVb3(ii)β $k_1 < 0$

This case is similar to the subcase of Case BIVb(2) where $k_1 = 0,\ k_3 < 0$.

Case BIVb3(ii)γ $k_1 = 0$

Here Eqs. (2.156) and (2.157) give

$$A_8(z) = c_{11}z + c_5 \ , \tag{2.164}$$

$$A_9(z) = c_6 z + c_4 \ , \tag{2.165}$$

and we get further two cases, depending upon, whether $\frac{R_3}{R_2}$ is a constant or not.

Case BIVb3(ii)γ_1 $\left(\frac{R_3}{R_2}\right)' \neq 0$

In this case the RC vector becomes

$$\begin{aligned} B^0 &= c_1 , \\ B^1 &= \frac{1}{\sqrt{R_1}} c_4 , \\ B^2 &= c_4\theta + c_2 , \\ B^3 &= -k_2 c_4 z + c_3 . \end{aligned} \tag{2.166}$$

Case BIVb3(ii)γ_2 $\left(\frac{R_3}{R_2}\right)' = 0$

Putting $R_3/R_2 = d$, the RCs become

$$\begin{aligned} B^0 &= c_1, \\ B^1 &= \frac{1}{\sqrt{R_1}} \left(c_6\theta + c_7 z + c_5\right) , \\ B^2 &= -c_6 \left(\int \frac{\sqrt{R_1}}{R_2} d\rho + k_2 \frac{\theta^2}{2} - d k_2 \frac{z^2}{2}\right) - k_2 c_7 \theta z - k_2 c_5 \theta - c_4 z + c_2 , \\ B^3 &= -k_2 c_6 \theta z - c_7 \left(\int \frac{\sqrt{R_1}}{R_2} d\rho - \frac{k_2}{2d}\theta^2 + k_2 \frac{z^2}{2}\right) - k_2 c_5 z + \frac{c_4}{d}\theta + c_3 . \end{aligned} \tag{2.167}$$

2.3 Ricci Collineations for the Degenerate Ricci Tensor

In this section we solve the RC equations (Eqs. (2.4)-and (2.5)) for the degenerate Ricci tensor i.e., when $\det(R_{ab}) = 0$. This can happen in any of the fifteen ways depending upon which one or more of the four components of Ricci tensor is/are zero. Here, we list these cases. In Cases I, ..., IV, one of the R_i $(i = 0,1,2,3)$ is zero (in order of increasing i), respectively; in Cases V, ..., X, two and in Cases XI, ..., XIV, three of the R_i are zero; and in Case XV all R_i are zero.

As the RC equations remain unchanged if any two of the three indices 0, 2 and 3 are interchanged, we note that the results for Cases I, III and IV would be 'similar'. In the same way Cases V, VIII and IX; Cases VI, VII and X; and Cases XI, XII and XIV are 'similar'. The results for the *similar* cases can be obtained by interchanging the role of any two of the coordinates t, θ and z. Cases II, XIII and XV are independent. We will see that only Case II ($R_1 = 0$, $R_i \neq 0$, $i = 0,2,3$) admits a finite dimensional algebra of RCs and for all other cases the algebra of RCs is infinite dimensional. Therefore, we discuss Case II first and in detail;

for other cases we merely give the results. For infinite dimensional cases also we solve the RC equations as far as possible and find the form of RCs to the extent the equations allow. Detailed calculations of Case I are given in Appendix C.

Case II $\qquad R_1 = 0\,,\ R_0 \neq 0\,,\ R_2 \neq 0\,,\ R_3 \neq 0$

Eqs. (2.5) for $a = 0, b = 1$ and for $a = 1, b = 2, 3$ yield $B^0_{,1} = B^2_{,1} = B^3_{,1} = 0$, so that, we have $B^i = B^i(t, \theta, z)\,,\ i = 0, 2, 3$. Differentiating Eq. (2.5) for $a = 0, b = 2$, with respect to z and for $a = 0, b = 3$, with respect to θ and subtracting gives

$$R_2 B^2_{,03} - R_3 B^3_{,02} = 0\,. \tag{2.168}$$

Differentiating Eq. (2.5) for $a = 2, b = 3$ with respect to t yields

$$R_2 B^2_{,03} + R_3 B^3_{,02} = 0\,. \tag{2.169}$$

Eqs. (2.168) and Eq. (2.169) imply

$$B^2_{,03} + B^3_{,02} = 0\,, \tag{2.170}$$

In view of Eq. (2.170) we note that $B^0_{,23} = 0$. Therefore, we can assume B^0 of the form

$$B^0 = A_1(t, \theta) + A_2(t, z)\,. \tag{2.171}$$

Substituting this in Eq. (2.4) for $a = 0$ we get

$$R_0' B^1 + 2R_0 \left[\dot{A}_1(t, \theta) + \dot{A}_2(t, z)\right] = 0\,. \tag{2.172}$$

We consider two possibilities depending upon whether R_0 is constant or not.

Case II(A) $\qquad R_0' = 0$

We write $R_0 = \gamma$. In this case from the RC equations we get

$$B^0 = A_1(\theta) + A_2(z)\,, \tag{2.173}$$

$$B^2 = -\frac{\gamma}{R_2} t A_{1,2}(\theta) + A_3(\theta, z) \ , \tag{2.174}$$

$$B^3 = -\frac{\gamma}{R_3} t A_{2,3}(z) + A_4(\theta, z) \ . \tag{2.175}$$

These give rise to four possibilities depending upon whether both, one or none of R_2 and R_3 are/is constant. We discuss these one by one.

Case IIA(a) $\qquad R_2' = 0\,, R_3' = 0$

We put $R_2 = -\alpha, R_3 = -\beta$, and the RC equations give $B^2_{,2} = B^3_{,3} = 0$, so that Eqs. (2.174) and (2.175) take the form

$$B^2 = \frac{\gamma}{\alpha} t c_4 + A_3(z) \ , \tag{2.176}$$

$$B^3 = \frac{\gamma}{\beta} t c_5 + A_4(\theta) \ . \tag{2.177}$$

Substituting from Eqs. (2.173), (2.176)and (2.177) into the RC equations determines $A_1(\theta)$, $A_2(z)$, $A_3(z)$ and $A_4(\theta)$, explicitly yielding the result

$$\begin{aligned} B^0 &= c_4\theta + c_5 z + c_1 \ , \\ B^2 &= \tfrac{\gamma}{\alpha} c_4 t + c_6 z + c_2 \ , \\ B^3 &= \tfrac{\gamma}{\beta} c_5 t - \tfrac{\alpha}{\beta} c_6 \theta + c_3 \ , \end{aligned} \tag{2.178}$$

and $B^1(t, \rho, \theta, z)$ remains completely arbitrary.

Case IIA(b) $\qquad R_2' = 0\,, R_3' \neq 0$

We put $R_2 = -\alpha$. From Eqs. (2.173) - (2.175) we find that in this case $A_2(z)$ is constant and $A_1(\theta)$ is a linear function in θ. The RC equations yield the result

$$\begin{aligned} B^0 &= c_3\theta + c_1 \ , \\ B^1 &= -\tfrac{2R_3}{R_3'} A_{4,3}(z) \ , \\ B^2 &= \tfrac{\gamma}{\alpha} c_3 t + c_2 \ , \\ B^3 &= A_4(z) \ . \end{aligned} \tag{2.179}$$

Here $A_4(z)$ is arbitrary, therefore, the Lie algebra is again infinite dimensional.

Case IIA(c) $\qquad R_2' \neq 0\,, R_3' = 0$

This case is similar to the previous case.

Case IIA(d) $\quad R_2' \neq 0\,,\, R_3' \neq 0$

In this case $A_1(\theta)$ and $A_2(z)$ are constants and Eqs. (2.173) - (2.175) reduce to

$$B^0 = c_1 \;, \tag{2.180}$$

$$B^2 = A_3(\theta, z) \;, \tag{2.181}$$

$$B^3 = A_4(\theta, z) \;. \tag{2.182}$$

Substituting these in Eqs. (2.4) for $a = 2$ and 3, gives

$$B^1 = -\frac{2R_2}{R_2'} A_{3,2}(\theta, z) \;, \tag{2.183}$$

$$B^1 = -\frac{2R_3}{R_3'} A_{4,3}(\theta, z) \;, \tag{2.184}$$

and in Eq. (2.5) for $a = 2$ and $b = 3$, gives

$$A_{3,3}(\theta, z) + \frac{R_3}{R_2} A_{4,2}(\theta, z) = 0 \;. \tag{2.185}$$

Now, if R_3/R_2 is a constant, we again get infinite dimensional RCs. Otherwise, we find that

$$A_3(\theta, z) = A_3(\theta)\,, \;\; A_4(\theta, z) = A_4(z) \tag{2.186}$$

and equating Eqs. (2.183) and (2.184) gives

$$A_{3,2}(\theta) - \frac{R_2' R_3}{R_2 R_3'} A_{4,3}(z) = 0 \;. \tag{2.187}$$

We take $R_2' R_3 / R_2 R_3' = k$, a constant, otherwise we get the minimal symmetry. In this case from Eq. (2.187) we see that $A_3(\theta)$ and $A_4(z)$ are linear functions of θ and z respectively, yielding the final result as

$$\begin{aligned} B^0 &= c_1 \, , \\ B^1 &= -\tfrac{2R_2}{R_2'} k c_4 \, , \\ B^2 &= k c_4 \theta + c_2 \, , \\ B^3 &= c_4 z + c_3 \, . \end{aligned} \tag{2.188}$$

Case II(B) $\qquad R_0' \neq 0$

Putting B^0 from Eq. (2.171) into Eqs. (2.5) for $a = 0, b = 2, 3$, and integrating with respect to t, gives

$$B^2 = -\frac{R_0}{R_2} \int A_{1,2}(t, \theta)\, dt + A_3(\theta, z) \ , \tag{2.189}$$

$$B^3 = -\frac{R_0}{R_3} \int A_{2,3}(t, z)\, dt + A_4(\theta, z) \ . \tag{2.190}$$

Now, as $B^2_{,1} = B^3_{,1} = 0$ the above equations imply

$$\left(\frac{R_0}{R_2}\right)' \int A_{1,2}(t, \theta)\, dt = 0 \, , \tag{2.191}$$

$$\left(\frac{R_0}{R_3}\right)' \int A_{2,3}(t, z)\, dt = 0 \, . \tag{2.192}$$

Here we get four cases.

(a) $\left(\frac{R_0}{R_2}\right)' = 0 \, , \qquad \left(\frac{R_0}{R_3}\right)' = 0 \, ,$

(b) $\left(\frac{R_0}{R_2}\right)' = 0 \, , \qquad \left(\frac{R_0}{R_3}\right)' \neq 0 \, ,$

(c) $\left(\frac{R_0}{R_2}\right)' \neq 0 \, , \qquad \left(\frac{R_0}{R_3}\right)' = 0 \, ,$

(d) $\left(\frac{R_0}{R_2}\right)' \neq 0 \, , \qquad \left(\frac{R_0}{R_3}\right)' \neq 0 \, .$

Case IIB(a) $\qquad \left(\frac{R_0}{R_2}\right)' = 0 \, , \ \left(\frac{R_0}{R_3}\right)' = 0$

We put $R_0/R_2 = -\alpha$, $R_0/R_3 = -\beta$, where α and β are non-zero constants. In this case substituting from Eqs. (2.171)-(2.189) in Eqs. (2.4) for $a = 2, 3$ gives

$$\alpha \int A_{1,22}(t, \theta)\, dt - \dot{A}_1(t, \theta) - \dot{A}_2(t, z) + A_{3,2}(\theta, z) = 0 \ , \tag{2.193}$$

$$\beta \int A_{2,33}(t, z)\, dt - \dot{A}_1(t, \theta) - \dot{A}_2(t, z) + A_{4,3}(\theta, z) = 0 \ . \tag{2.194}$$

Subtracting these two equations yields

$$\alpha A_{1,22}\left(t,\theta\right)-\beta A_{2,33}\left(t,z\right)=0\ , \tag{2.195}$$

from where we can get the form of $A_1(t,\theta)$ and $A_2\left(t,z\right)$ as

$$A_1\left(t,\theta\right)=\frac{\theta^2}{2}A_5\left(t\right)+\theta A_6\left(t\right)+A_7\left(t\right)\ , \tag{2.196}$$

$$A_2\left(t,z\right)=\frac{\alpha}{\beta}\frac{z^2}{2}A_5\left(t\right)+zA_8\left(t\right)+A_9\left(t\right)\ . \tag{2.197}$$

Substituting these values in Eqs. (2.193) and (2.194) yields

$$A_5\left(t\right)=c_0t+c_7\ , \tag{2.198}$$

$$A_6\left(t\right)=c_8t+c_5\ , \tag{2.199}$$

$$A_8\left(t\right)=c_9t+c_6\ , \tag{2.200}$$

$$A_7\left(t\right)+A_9\left(t\right)=\alpha\left(c_0\frac{t^3}{6}+c_7\frac{t^2}{2}\right)+c_{10}t+c_1\ . \tag{2.201}$$

With these values in Eqs. (2.193) and (2.194), we get on integration with respect to θ and z respectively.

$$A_3\left(\theta,z\right)=c_0\frac{\theta^3}{6}+c_8\frac{\theta^2}{2}+\frac{\alpha}{\beta}c_0\frac{z^2}{2}+c_9\theta z+c_{10}\theta+A_{10}\left(z\right)\ , \tag{2.202}$$

$$A_4\left(\theta,z\right)=c_0\frac{\theta^2 z}{2}+c_8\theta z+\frac{\alpha}{\beta}c_0\frac{z^3}{6}+c_9\frac{z^2}{2}+c_{10}z+A_{11}\left(\theta\right)\ . \tag{2.203}$$

Substituting these values in Eqs. (2.189) and (2.190), we get from Eq. (2.5) for $a=2,b=3$

$$\beta\left[\frac{\alpha}{\beta}c_0\theta z+c_9\theta+A_{10,3}\left(z\right)\right]+\alpha\left[c_0\theta z+c_8 z+A_{11,2}\left(\theta\right)\right]=0\ , \tag{2.204}$$

which yields

$$c_0=0\ ,$$

$$A_{11}\left(\theta\right)=-\frac{\beta}{\alpha}c_9\frac{\theta^2}{2}+c_4\theta+c_3\ , \tag{2.205}$$

$$A_{10}(z) = -\frac{\alpha}{\beta}c_8\frac{z^2}{2} - \frac{\alpha}{\beta}c_4 z + c_2 \,. \tag{2.206}$$

Substituting from Eqs. (2.196)-(2.206), the final form of Eqs. (2.171)-(2.190) becomes

$$\begin{aligned}
B^0 &= c_7\left(\tfrac{\theta^2}{2} + \tfrac{\alpha}{\beta}\tfrac{z^2}{2} + \alpha\tfrac{t^2}{2}\right) + c_8 t\theta + c_5\theta + c_9 tz + c_6 z + c_{10}t + c_1 \,, \\
B^1 &= -\tfrac{2R_0}{R_0'}\left(\alpha c_7 t + c_8\theta + c_9 z + c_{10}\right) \,, \\
B^2 &= \alpha c_7 t\theta + c_8\left(\alpha\tfrac{t^2}{2} + \tfrac{\theta^2}{2} - \tfrac{\alpha}{\beta}\tfrac{z^2}{2}\right) + \alpha c_5 t + c_9\theta z + c_{10}\theta - \tfrac{\alpha}{\beta}c_4 z + c_2 \,, \\
B^3 &= \alpha c_7 tz + c_8\theta z + c_9\left(\beta\tfrac{t^2}{2} - \tfrac{\beta}{\alpha}\tfrac{\theta^2}{2} + \tfrac{z^2}{2}\right) + \beta c_6 t + c_{10}z + c_4\theta + c_3 \,.
\end{aligned} \tag{2.207}$$

This is a case of 10 RCs. This is the only case for degenerate Ricci tensor which admits 10 RCs.

Case IIB(b) $\quad \left(\frac{R_0}{R_2}\right)' = 0, \quad \left(\frac{R_0}{R_3}\right)' \neq 0$

In this case Eq. (2.192) implies

$$A_2(t,z) = A_2(t) \,. \tag{2.208}$$

With this in Eqs. (2.171)-(2.190), we get from Eqs. (2.4) for $a = 2, 3$

$$\dot{A}_1(t,\theta) + \dot{A}_2(t) - \alpha\int A_{1,22}(t,\theta)\,dt - A_{3,2}(\theta,z) = 0 \,, \tag{2.209}$$

$$\frac{R_3' R_0}{R_3 R_0'}\left[\dot{A}_1(t,\theta) + \dot{A}_2(t)\right] - A_{4,3}(\theta,z) = 0 \,. \tag{2.210}$$

The last equation implies

$$\left(\frac{R_3' R_0}{R_3 R_0'}\right)'\left[\dot{A}_1(t,\theta) + \dot{A}_2(t)\right] = 0 \,, \tag{2.211}$$

giving rise to further two cases depending upon whether $\frac{R_3' R_0}{R_3 R_0'}$ is a constant or not.

Case IIBb(1) $\quad \left(\frac{R_3' R_0}{R_3 R_0'}\right)' = 0$

Taking $\left(\frac{R_3' R_0}{R_3 R_0'}\right)' = k$, where k is nonzero constant, Eq. (2.210) gives

$$A_4(\theta,z) = zA_5(\theta) + A_6(\theta) \,. \tag{2.212}$$

Using this in Eq. (2.210) and subtracting from Eq. (2.209) yields

$$A_1(t,\theta) = \theta A_7(t) + A_8(t) \ ,$$

which on substitution again in Eq. (2.210) implies

$$A_7(t) = c_1 t + c_4 \ , \tag{2.213}$$

$$A_5(\theta) = k c_1 \theta + c_5 \ , \tag{2.214}$$

$$A_8(t) + A_2(t) = c_5 t + c_1 \ . \tag{2.215}$$

With these values in Eqs. (2.209) and (2.210), integrating with respect to θ and z respectively gives

$$A_3(\theta,z) = c_1\frac{\theta^2}{2} + c_5\theta + A_9(z) \ , \tag{2.216}$$

$$A_4(\theta,z) = k c_1 \theta z + k c_5 z + A_{10}(\theta) \ . \tag{2.217}$$

Now, Eq. (2.5) for $a = 2, b = 3$ takes the form

$$R_2 A_{9,3}(z) + R_3\left[k c_1 z + A_{10,2}(\theta)\right] = 0 \ , \tag{2.218}$$

which implies

$$c_1 = 0 \, , \ A_{10}(\theta) = c_3 \, , \ A_9(z) = c_2 \ . \tag{2.219}$$

The RC vector from Eqs. (2.171)-(2.190), therefore, is given by

$$\begin{aligned} B^0 &= c_4\theta + c_5 t + c_1 \ , \\ B^1 &= -\tfrac{2R_0}{R_0'} c_5 \ , \\ B^2 &= \alpha c_4 t + c_5\theta + c_2 \ , \\ B^3 &= k c_5 z + c_3 \ . \end{aligned} \tag{2.220}$$

Case IIBb(2) $\left(\frac{R_3'R_0}{R_3R_0'}\right)' \neq 0$

Here, Eq. (2.211) gives

$$\dot{A}_1(t,\theta) + \dot{A}_2(t) = 0\ , \tag{2.221}$$

which implies that

$$A_1(t,\theta) = A_5(t) + A_6(\theta)\ , \tag{2.222}$$

and Eq. (2.210) gives

$$A_4(\theta,z) = A_4(\theta)\ . \tag{2.223}$$

Now, Eq. (2.209) takes the form

$$\dot{A}_5(t) + \dot{A}_2(t) - \alpha t A_{6,22}(\theta) - A_{3,2}(\theta,z) = 0\ , \tag{2.224}$$

which yields

$$A_5(t) + A_2(t) = c_1\frac{t^2}{2} + c_2 t + c_3\ , \tag{2.225}$$

$$A_6(\theta) = \frac{1}{\alpha}c_1\frac{\theta^2}{2} + c_4\theta + c_5\ , \tag{2.226}$$

and therefore, Eq. (2.224) on integration with respect to θ gives

$$A_3(\theta,z) = c_2\theta + A_7(z)\ . \tag{2.227}$$

Eq. (2.5) for $a=2, b=3$ now becomes

$$R_2A_{7,3}(z) + R_3A_{4,2}(\theta) = 0\ , \tag{2.228}$$

which means that

$$A_4(\theta) = c_3\ ,\ A_7(z) = c_2\ . \tag{2.229}$$

Now, substituting from Eqs. (2.222)-(2.226) in Eq. (2.4) for $a=3$ gives

$$c_1 = c_2 = 0\ . \tag{2.230}$$

Finally, with values from Eqs. (2.222)-(2.230), the RC vector, from Eqs. (2.171)-(2.190), in this case becomes

$$\begin{aligned} B^0 &= c_4\theta + c_1 \, , \\ B^1 &= 0 \, , \\ B^2 &= \alpha c_4 t + c_2 \, , \\ B^3 &= c_3 \, , \end{aligned} \tag{2.231}$$

where the constant $c_3 + c_5$, has been replaced by c_1.

Case IIB(c) $\left(\frac{R_0}{R_2}\right)' \neq 0, \quad \left(\frac{R_0}{R_3}\right)' = 0$

As the RC equations (Eqs. (2.4) and (2.5)) are invariant under the interchange of indices 2 and 3, if we interchange these indices (i.e. coordinates θ and z) in Case II(b), we will get the results for this case.

Case IIB(d) $\left(\frac{R_0}{R_2}\right)' \neq 0, \quad \left(\frac{R_0}{R_3}\right)' \neq 0$

Now, Eqs. (2.191) and (2.192) imply

$$A_1(t,\theta) = A_1(t) \; , \; A_2(t,z) = A_2(t) \; . \tag{2.232}$$

In this case Eqs. (2.171)-(2.190) take the following form (writing $A_1(t) + A_2(t) = A(t)$)

$$B^0 = A(t) \; , \tag{2.233}$$

$$B^1 = -\frac{2R_0}{R_0'}\dot{A}(t) \; , \tag{2.234}$$

$$B^2 = A_3(\theta, z) \; , \tag{2.235}$$

$$B^3 = A_4(\theta, z) \; . \tag{2.236}$$

Now, Eqs. (2.4) for $a = 2, 3$ and (2.5) for $a = 2, b = 3$ become

$$\dot{A}(t) - \left(\frac{R_0' R_2}{R_0 R_2'}\right) A_{3,2}(\theta, z) = 0 \; , \tag{2.237}$$

$$\dot{A}(t) - \left(\frac{R_0' R_3}{R_0 R_3'}\right) A_{4,3}(\theta, z) = 0 \; . \tag{2.238}$$

This gives rise to four cases depending upon whether the terms in the brackets in the above two equations are constants or not.

Case IIBd(1) $\left(\frac{R_0' R_2}{R_0 R_2'}\right)' \neq 0\,, \quad \left(\frac{R_0' R_3}{R_0 R_3'}\right)' \neq 0$

Here, we have

$$A_3(\theta, z) = A_3(z) \,, \tag{2.239}$$

$$A_4(\theta, z) = A_4(\theta) \,, \tag{2.240}$$

$$\dot{A}(t) = c_1 \,, \tag{2.241}$$

so that Eq. (2.5) for $a = 2, b = 3$ becomes

$$R_2 A_{3,3}(z) + R_3 A_{4,2}(\theta) = 0 \,, \tag{2.242}$$

which implies that

$$A_3(z) = c_4 z + c_2 \,, \tag{2.243}$$

$$A_4(\theta) = -\frac{R_2}{R_3} c_4 \theta + c_3 \,, \tag{2.244}$$

and Eqs. (2.233)-(2.236) become

$$\begin{aligned} B^0 &= c_1 \,, \\ B^1 &= 0 \,, \\ B^2 &= c_4 z + c_2 \,, \\ B^3 &= -\tfrac{R_2}{R_3} c_4 \theta + c_3 \,. \end{aligned} \tag{2.245}$$

Here, we note that R_2/R_3 is a constant, otherwise c_4 will be zero to give the minimal symmetry.

Case IIBd(2) $\left(\frac{R_0' R_2}{R_0 R_2'}\right)' \neq 0\,, \quad \left(\frac{R_0' R_3}{R_0 R_3'}\right)' = 0$

In this case, from Eqs. (2.237) and (2.238), we see that

$$A_3(\theta, z) = A_3(z) \,, \tag{2.246}$$

$$\dot{A}(t) = 0 \,, \tag{2.247}$$

$$A_{4,3}(\theta, z) = 0 \,, \tag{2.248}$$

and so, it reduces to the previous case.

Case IIBd(3) $\left(\frac{R_0' R_2}{R_0 R_2'}\right)' = 0\,, \quad \left(\frac{R_0' R_3}{R_0 R_3'}\right)' \neq 0$

This is similar to Case IId(2).

Case IIBd(4) $\left(\frac{R_0' R_2}{R_0 R_2'}\right)' = 0\,, \quad \left(\frac{R_0' R_3}{R_0 R_3'}\right)' = 0$

We put $R_0' R_2 / R_0 R_2' = k_1$, $R_0' R_3 / R_0 R_3' = k_2$, and Eqs. (2.237) and (2.238) become

$$\dot{A}(t) - k_1 A_{3,2}(\theta, z) = 0 \,, \tag{2.249}$$

$$\dot{A}(t) - k_2 A_{4,3}(\theta, z) = 0 \,. \tag{2.250}$$

Subtracting these gives

$$k_1 A_{3,2}(\theta, z) - k_2 A_{4,3}(\theta, z) = 0 \,. \tag{2.251}$$

Now, from Eq. (2.249) we get

$$A(t) = c_5 t + c_1 \,, \tag{2.252}$$

$$A_3(\theta, z) = \frac{c_5}{k_1}\theta + A_5(z) \,, \tag{2.253}$$

which on substituting in Eq. (2.251) and integrating with respect to z yields

$$A_4(\theta, z) = \frac{c_5}{k_2} z + A_6(\theta) \,. \tag{2.254}$$

Using Eqs. (2.253) and (2.254) in Eqs. (2.235) and (2.236), Eq. (2.5) for $a = 2, b = 3$ takes the form

$$R_2 A_{5,3}(z) + R_3 A_{6,2}(\theta) = 0 \,, \tag{2.255}$$

which implies

$$A_5(z) = c_4 z + c_2 \,, \tag{2.256}$$

$$A_6(\theta) = -\frac{R_2}{R_3} c_4 \theta + c_3 \,. \tag{2.257}$$

Therefore, we finally have

$$\begin{aligned} B^0 &= c_5 t + c_1 \,, \\ B^1 &= -\tfrac{2R_0}{R_0'} c_5 \,, \\ B^2 &= \tfrac{c_5}{k_1}\theta + c_4 z + c_2 \,, \\ B^3 &= \tfrac{c_5}{k_2} z - \tfrac{R_2}{R_3} c_4 \theta + c_3 \,. \end{aligned} \tag{2.258}$$

Here R_2/R_3 is constant (which implies that $k_1 = k_2$), otherwise c_4 will be zero.

Case I $\quad R_0 = 0\,,\quad R_1 \neq 0\,,\quad R_2 \neq 0\,,\quad R_3 \neq 0$

In this case from the RC equations we see that B^0 is completely an arbitrary function of t, ρ and z.

From Eqs. (2.5) for $a = 0, b = 1, 2, 3$ we obtain $B^1_{,0} = B^2_{,0} = B^3_{,0} = 0$, and therefore, on integration, Eq. (2.4) for $a = 1$ gives

$$B^1 = \frac{A_1(\theta, z)}{\sqrt{R_1}} \,. \tag{2.259}$$

Similarly, from Eqs. (2.4) for $a = 2, 3$ we get

$$B^2 = -\frac{R_2'}{2R_2\sqrt{R_1}} \int A_1(\theta, z)\, d\theta + B_1(\rho, z) \,, \tag{2.260}$$

$$B^3 = -\frac{R_3'}{2R_3\sqrt{R_1}} \int A_1(\theta, z)\, dz + B_2(\rho, \theta) \,. \tag{2.261}$$

Now, differentiating Eq. (2.5) for $a = 1, b = 2$ with respect to θ and substituting from above yields

$$A_{1,22}(\theta, z) + \frac{R_2}{\sqrt{R_1}} \left(-\frac{R_2'}{2R_2\sqrt{R_1}} \right)' A_1(\theta, z) = 0 \,, \tag{2.262}$$

which suggests two possibilities.

(a) $\left[\frac{R_2}{\sqrt{R_1}} \left(-\frac{R_2'}{2R_2\sqrt{R_1}} \right)' \right]' \neq 0 \,,$

(b) $\left[\frac{R_2}{\sqrt{R_1}} \left(-\frac{R_2'}{2R_2\sqrt{R_1}} \right)' \right]' = 0 \,.$

We take these one by one.

Case I(a) $\left[\frac{R_2}{\sqrt{R_1}}\left(-\frac{R_2'}{2R_2\sqrt{R_1}}\right)'\right]' \neq 0$

In this case the solution is

$$\begin{aligned}
&B^0 \text{ is totally arbitrary,}\\
&B^1 = 0\,,\\
&B^2 = c_1 z + c_2\,,\\
&B^3 = c_3\theta + c_4\,.
\end{aligned} \tag{2.263}$$

Case I(b) $\left[\frac{R_2}{\sqrt{R_1}}\left(-\frac{R_2'}{2R_2\sqrt{R_1}}\right)'\right]' = 0$

We put $\frac{R_2}{\sqrt{R_1}}\left(-\frac{R_2'}{2R_2\sqrt{R_1}}\right)' = k_1$, a constant, and discuss three cases: $k_1 \gtreqless 0$.

Case Ib(1) $k_1 > 0$

The solution of Eq. (2.262) in this case can be written as

$$A_1(\theta, z) = B_3(z)\, e^{i\sqrt{k_1}\theta} + B_4(z)\, e^{-i\sqrt{k_1}\theta}\,. \tag{2.264}$$

Now differentiating Eq.(2.5) for $a = 1, b = 3$ with respect to z yields

$$e^{2i\sqrt{k_1}\theta}\left[B_{3,33}(z) + k_2 B_3(z)\right] + \left[B_{4,33}(z) + k_2 B_4(z)\right] = 0\,, \tag{2.265}$$

where $\frac{R_3}{\sqrt{R_1}}\left(-\frac{R_3'}{2R_3\sqrt{R_1}}\right)' = k_2$, is a constant and we get

$$B_{3,33}(z) + k_2 B_2(z) = 0\,, \tag{2.266}$$

$$B_{4,33}(z) + k_2 B_4(z) = 0\,. \tag{2.267}$$

Here, again we have three possibilities $k_2 \gtreqless 0$.

Case Ib1(i) $k_2 > 0$

In this case the solution of Eqs. (2.266) and (2.267) can be written as

$$B_3(z) = c_1 e^{i\sqrt{k_2}z} + c_2 e^{-i\sqrt{k_2}z}\,, \tag{2.268}$$

$$B_4(z) = c_3 e^{i\sqrt{k_2}z} + c_4 e^{-i\sqrt{k_2}z} . \tag{2.269}$$

Here, we have two subcases.

(α) $\sqrt{\frac{k_1}{k_2}} + \sqrt{\frac{k_2}{k_1}} \neq 0$,

(β) $\sqrt{\frac{k_1}{k_2}} + \sqrt{\frac{k_2}{k_1}} = 0$.

Case Ib1(i)α

$$\begin{aligned} B^1 &= 0 , \\ B^2 &= c_1 z + c_2 , \\ B^3 &= -c_1 k_3 \theta + c_3 , \end{aligned} \tag{2.270}$$

and B^0 is arbitrary. Here the constant $k_3 = R_2/R_3$.

Case Ib1(i)β

In this case we have

$$\begin{aligned} B^1 &= 0 , \\ B^2 &= c_1 z + c_2 , \\ B^3 &= c_1 \theta + c_3 , \end{aligned} \tag{2.271}$$

and B^0 is arbitrary.

Case Ib1(ii) $\qquad k_2 = 0 \quad \text{or} \quad \frac{R_3'}{2R_3\sqrt{R_1}} = k_3$, a constant.

In this case the solution of Eqs. (2.266) and (2.267) is

$$B_3(z) = c_1 z + c_2 , \tag{2.272}$$

$$B_4(z) = c_3 z + c_4 . \tag{2.273}$$

The final form of the RCs is

$$
\begin{aligned}
B^1 &= \tfrac{1}{\sqrt{R_1}}\left[e^{i\sqrt{k_1}\theta}\left(c_1 z + c_2\right) + e^{-i\sqrt{k_1}\theta}\left(c_3 z + c_4\right)\right] , \\
B^2 &= -\tfrac{iR_2'\sqrt{k_1}}{2R_2\sqrt{R_1}}\left[e^{i\sqrt{k_1}\theta}\left(c_1 z + c_2\right) - e^{-i\sqrt{k_1}\theta}\left(c_3 z + c_4\right)\right] + c_5 , \\
B^3 &= -k_3\left[e^{i\sqrt{k_1}\theta}\left(c_1 \tfrac{z}{2}^2 + c_2 z\right) + e^{-i\sqrt{k_1}\theta}\left(c_3 \tfrac{z}{2}^2 + c_4 z\right)\right] \\
&\quad - \left[e^{i\sqrt{k_1}\theta} c_1 + e^{-i\sqrt{k_1}\theta} c_3\right] \int \tfrac{\sqrt{R_1}}{R_3} d\rho + c_6 ,
\end{aligned}
\tag{2.274}
$$

with B^0 being arbitrary.

Case Ib1(iii) $\qquad k_2 < 0$

Here the solution of Eqs. (2.266) and (2.267)can be written as

$$
B_3(z) = c_1 e^{\sqrt{k_2} z} + c_2 e^{-\sqrt{k_2} z} , \tag{2.275}
$$

$$
B_4(z) = c_3 e^{\sqrt{k_2} z} + c_4 e^{-\sqrt{k_2} z} . \tag{2.276}
$$

If we compare these with Eqs. (2.268) and (2.269) of Case Ib1(i) we see that the only difference is that here the argument of exponential function is real whereas in the previous case it was complex. So, similar results (with this difference, of course) are obtained on parallel lines.

Case Ib(2) $\qquad k_1 = 0$

Here $-\frac{R_2'}{2R_2\sqrt{R_1}} = k_3$, a constant , and Eq. (2.262) has the solution

$$
A_1(\theta, z) = A_2(z)\,\theta + A_3(z) . \tag{2.277}
$$

Eqs. (2.259), (2.260) and (2.261), therefore, become

$$
B^1 = \frac{1}{\sqrt{R_1}}\left[A_2(z)\,\theta + A_3(z)\right] , \tag{2.278}
$$

$$
B^2 = k_3\left[A_2(z)\frac{\theta^2}{2} + A_3(z)\,\theta\right] + B_1(\rho, z) , \tag{2.279}
$$

$$B^3 = -\frac{R_3'}{2R_3\sqrt{R_1}}\left[\theta\int A_2(z)\,dz + \int A_3(z)\,dz\right] + B_2(\rho,\theta). \tag{2.280}$$

We discuss here two further cases.

(i) $\quad k_3 = 0 \quad$ i.e. $\quad R_2' = 0\,,$

(ii) $\quad k_3 \neq 0 \quad$ i.e. $\quad R_2' \neq 0\,.$

Case Ib2(i) $\qquad k_3 = 0 \quad (\text{or} \quad R_2' = 0)$

Here, if we put $\frac{R_3}{\sqrt{R_1}}\left(-\frac{R_3'}{2R_3\sqrt{R_1}}\right)' = k_4$, a constant , such that

$$A_{2,33}(z) + k_4 A_2(z) = 0\,, \tag{2.281}$$

$$A_{3,33}(z) + k_4 A_3(z) = 0\,, \tag{2.282}$$

and further discuss three cases: $k_4 \gtreqless 0$.

Case Ib2(i)α $\qquad k_4 > 0$

Here, the solution of Eqs. (2.281) and (2.282) can be written as

$$A_2(z) = c_1 e^{i\sqrt{k_4}z} + c_2 e^{-i\sqrt{k_4}z}\,, \tag{2.283}$$

$$A_3(z) = c_3 e^{i\sqrt{k_4}z} + c_4 e^{-i\sqrt{k_4}z}\,. \tag{2.284}$$

Therefore, Eqs. (2.278)-(2.280) reduce to

$$\begin{aligned}
B^1 &= \frac{1}{\sqrt{R_1}}\left[\left(c_1 e^{i\sqrt{k_4}z} + c_2 e^{-i\sqrt{k_4}z}\right)\right]\,,\\
B^2 &= 0\,,\\
B^3 &= -\frac{R_3'}{i2\sqrt{k_4}R_3\sqrt{R_1}}\left(c_1 e^{i\sqrt{k_4}z} - c_2 e^{-i\sqrt{k_4}z}\right)\,.
\end{aligned} \tag{2.285}$$

Case Ib2(i)β $\qquad k_4 = 0$

Here we can write $-\frac{R_3'}{2R_3\sqrt{R_1}} = k_5$, a constant, and get two further cases:

(1) $\quad k_5 \neq 0 \quad$ or $\quad R_3' \neq 0$,

(2) $\quad k_5 = 0 \quad$ or $\quad R_3' = 0$.

Case Ib2(i)β_1 $\qquad R_3' \neq 0 \quad$ or $\quad k_5 \neq 0$

The form of RCs here is

$$
\begin{aligned}
B^1 &= \frac{1}{\sqrt{R_1}}\left(c_1 z + c_2\right), \\
B^2 &= 0, \\
B^3 &= k_5\left(c_1\frac{z^2}{2} + c_2 z\right) - c_1\int\frac{\sqrt{R_1}}{R_3}d\rho .
\end{aligned}
\tag{2.286}
$$

Case Ib2(i)β_2 $\qquad R_3' = 0 \quad$ or $\quad k_5 = 0$

Here the result is

$$
\begin{aligned}
B^1 &= \frac{1}{\sqrt{R_1}}\left(c_1\theta + c_2 z + c_3\right), \\
B^2 &= -\frac{c_1}{R_2}\int\sqrt{R_1}d\rho , \\
B^3 &= -\frac{c_2}{R_3}\int\sqrt{R_1}d\rho .
\end{aligned}
\tag{2.287}
$$

Case Ib2(i)γ $\qquad k_4 < 0$

This case is similar to the Case Ib2(i)α except for the difference that now in Eqs. (2.283) and (2.284) the argument of the exponentials will be real and not complex.

Case Ib2(ii) $\qquad R_2' \neq 0 \quad$ or $\quad k_3 \neq 0$

Now, if the quantity $\frac{R_3}{\sqrt{R_1}}\left(-\frac{R_3'}{2R_3\sqrt{R_1}}\right)' = k_4$ is not a constant, then we will have the result as

$$
\begin{aligned}
B^1 &= 0\,, \\
B^2 &= c_1 z + c_2\,, \\
B^3 &= -\frac{R_2}{R_3} c_1 \theta + c_3\,.
\end{aligned}
\tag{2.288}
$$

On the other hand if k_4 is a constant, we have the cases: $k_4 \gtreqless 0$.

Case Ib2(ii)α $\qquad k_4 > 0$

Following the previous procedure, we get

$$
\begin{aligned}
B^1 &= 0\,, \\
B^2 &= c_1 z + c_2\,, \\
B^3 &= -\frac{R_2}{R_3} c_1 \theta + c_3\,.
\end{aligned}
\tag{2.289}
$$

Case Ib2(ii)β $\qquad k_4 = 0$

This means that $-\frac{R_3'}{2R_3\sqrt{R_1}} = k_5$, is a constant. In this case the solution reduces to

$$
\begin{aligned}
&B^1 = \tfrac{1}{\sqrt{R_1}} c_1\,, \\
&B^2 = k_3 c_1 \theta + c_2\,, \\
&B^3 = k_5 c_1 z + c_3\,.
\end{aligned}
\tag{2.290}
$$

Case Ib2(ii)γ $\qquad k_4 < 0$

The results will be similar to Case Ib2(ii)α.

Case III $\qquad R_2 = 0\,,\quad R_0 \neq 0\,,\quad R_1 \neq 0\,,\quad R_3 \neq 0$

As the RC equations (Eqs. (2.4) and (2.5)) are symmetric with respect to the interchange in indices 0 and 2, this case is similar to Case I. If we interchange the role of 0 and 2 in Case I, we get the results from this case.

Case IV $\quad R_3 = 0\,, \quad R_0 \neq 0\,, \quad R_1 \neq 0\,, \quad R_2 \neq 0$

As the RC equations (Eqs. (2.4) and (2.5)) remain unchanged if we interchange the indices 0 and 3, the results in this case can be obtained by interchanging these two indices in Case I.

Case V $\quad R_0 = 0\,, \quad R_1 = 0\,, \quad R_2 \neq 0\,, \quad R_3 \neq 0$

In this case Eqs. (2.4) and (2.5) give $B^2_{,0} = B^2_{,1} = B^3_{,0} = B^3_{,1} = 0$, and we are left with Eqs. (2.4) and (2.5) for $a = 2$, $b = 3$. This means $B^2 = B^2(\theta, z)$, $B^3 = B^3(\theta, z)$, and $B^0(t, \rho, \theta, z)$ is a completely arbitrary function. Now, differentiating Eq. (2.5) for $a = 2$, $b = 3$ with respect to ρ gives

$$\left(\frac{R_3}{R_2}\right)' B^3_{,2} = 0\,. \tag{2.291}$$

This equation gives rise to two cases.

(a) $\quad \left(\frac{R_3}{R_2}\right)' = 0\,,$

(b) $\quad \left(\frac{R_3}{R_2}\right)' \neq 0\,.$

Case V(a) $\quad \left(\frac{R_3}{R_2}\right)' = 0$

Let $R_3/R_2 = k \neq 0$. Now, subtracting Eq. (2.4) for $a = 3$ from the one for $a = 2$ and differentiating with respect to z and subtracting from the derivative of Eq. (2.5) for $a = 2$, $b = 3$ relative to θ gives

$$B^3_{,22} + \frac{1}{k} B^3_{,33} = 0\,. \tag{2.292}$$

Similarly, we see that

$$B^2_{,22} + \frac{1}{k} B^2_{,33} = 0\,. \tag{2.293}$$

For $k > 0$, the solution of Eqs. (2.292) and (2.293) can be written as

$$B^2 = f_+\left(\theta + \frac{iz}{\sqrt{k}}\right) + f_-\left(\theta - \frac{iz}{\sqrt{k}}\right)\,, \tag{2.294}$$

$$B^3 = g_+\left(\theta + \frac{iz}{\sqrt{k}}\right) + g_-\left(\theta - \frac{iz}{\sqrt{k}}\right)\,. \tag{2.295}$$

Therefore, from Eq. (2.4) for $a = 2$ we get

$$B^1 = -\frac{2R_2}{R_2'}\left[f_{+,2}\left(\theta + \frac{iz}{\sqrt{k}}\right) + f_{-,2}\left(\theta - \frac{iz}{\sqrt{k}}\right)\right] , \tag{2.296}$$

provided that $R_2' \neq 0$. B^0 is an arbitrary function of t, ρ, θ and z. If $R_2' = 0$, we see that from Eqs. (2.4) for $a = 2$ and 3, $B^2_{,3} = B^3_{,3} = 0$, and therefore, $B^2 = B^2(z)$, $B^3 = B^3(\theta)$. Hence from Eq. (2.5) for $a = 2$, $b = 3$ we get

$$\begin{aligned} B^2 &= c_1 z + c_2 , \\ B^3 &= c_1\theta + c_3 , \end{aligned} \tag{2.297}$$

and B^0 and B^1 become arbitrary.

Case V(b) $\left(\frac{R_3}{R_2}\right)' \neq 0$

In this case we get from Eqs. (2.291) and (2.5) for $a = 2$, $b = 3$ $B^2_{,3} = B^3_{,2} = 0$. Therefore, $B^2 = B^2(\theta)$, $B^3 = B^3(z)$ and we are only left with Eqs. (2.4) for $a = 2, 3$, for which we have the following three possibilities.

(1) $R_2' = 0$, $R_3' \neq 0$,

(2) $R_2' \neq 0$, $R_3' = 0$,

(3) $R_2' \neq 0$, $R_3' \neq 0$.

We discuss these now.

Case Vb(1) $R_2' = 0$, $R_3' \neq 0$

Here $B^2 = c_1$, and from Eq. (2.4) for $a = 3$, we see that $B^1_{,0} = B^1_{,2} = 0$, therefore, we get

$$\frac{R_3'}{2R_3}B^1(\rho, z) + B^3_{,3} = 0 , \tag{2.298}$$

which gives

$$\begin{aligned} B^1 &= \tfrac{R_3}{R_3'}A_4(z) , \\ B^2 &= c_1, \\ B^3 &= -\tfrac{1}{2}\int A_4(z)\,dz + c_2 . \end{aligned} \tag{2.299}$$

and $B^0(t, \rho, \theta, z)$ is arbitrary.

Case Vb(2) $\qquad R_2' \neq 0\,, \quad R_3' = 0$

Interchanging the role of indices 2 and 3 in the previous case gives the result for this case.

Case Vb(3) $\qquad R_2' \neq 0\,, \quad R_3' \neq 0$

From Eqs. (2.4) for $a = 2, 3$, we note that $B^1_{,0} = B^1_{,2} = B^1_{,3} = 0$. Therefore, $B^1 = B^1(\rho)$. So, Eqs. (2.4) for $a = 2, 3$ are

$$\frac{R_2'}{2R_2} B^1(\rho) + B^2_{,2}(\theta) = 0\,, \tag{2.300}$$

$$\frac{R_3'}{2R_3} B^1(\rho) + B^3_{,3}(z) = 0\,. \tag{2.301}$$

Eqs. (2.300) and (2.301) yield

$$\begin{aligned} B^1 &= \tfrac{R_2}{R_2'} c_1\,, \\ B^2 &= c_1\theta + c_2\,, \\ B^3 &= c_3 z + c_4\,, \end{aligned} \tag{2.302}$$

and $B^0(t, \rho, \theta, z)$ is arbitrary.

Case VI $\qquad R_0 = 0\,, \quad R_2 = 0\,, \quad R_1 \neq 0\,, \ R_3 \neq 0$

In this case we get from RC equations

$$B^1_{,0} = B^1_{,2} = B^2_{,0} = B^2_{,2} = B^3_{,0} = B^3_{,2} = 0\,, \tag{2.303}$$

and we are left with Eqs. (2.4) for $a = 1, 3$ and Eq. (2.5) $a = 1, b = 3$. Note that B^0 is an arbitrary function of t, ρ, θ and z; and B^2 is an arbitrary function of ρ and z. Also, $B^1 = B^1(\rho, z)$ and $B^3 = B^3(\rho, z)$. Eq. (2.4) for $a = 1$, therefore, yields

$$B^1 = \frac{A_1(z)}{\sqrt{R_1}}\,. \tag{2.304}$$

Putting this in Eqs.(2.5) $a = 1, b = 3$, and (2.4) for $a = 3$ and integrating these equations with respect to ρ and z respectively, we get

$$B^3 = -A_{1,3}(z) \int \frac{\sqrt{R_1}}{R_3} d\rho + A_2(z)\,, \tag{2.305}$$

$$B^3 = -\frac{R_3'}{2R_3\sqrt{R_1}} \int A_1(z)\, dz + A_3(\rho) \ . \tag{2.306}$$

Equating these two and differentiating with respect to ρ and z gives

$$A_{1,33}(z) - \frac{R_3}{\sqrt{R_1}} \left(\frac{R_3'}{2R_3\sqrt{R_1}} \right)' A_1(z) = 0 \ , \tag{2.307}$$

which implies

$$\left[\frac{R_3}{\sqrt{R_1}} \left(\frac{R_3'}{2R_3\sqrt{R_1}} \right)' \right]' A_1(z) = 0 \ . \tag{2.308}$$

The last equation gives rise to two cases.

(a) $\left[\frac{R_3}{\sqrt{R_1}} \left(\frac{R_3'}{2R_3\sqrt{R_1}} \right)' \right]' \neq 0 \,,$

(b) $\left[\frac{R_3}{\sqrt{R_1}} \left(\frac{R_3'}{2R_3\sqrt{R_1}} \right)' \right]' = 0 \,.$

Case VI(a)

In this case $A_1(z) = 0$, so that we have

$$\begin{aligned} B^1 &= 0 \ , \\ B^3 &= c_1 \ , \end{aligned} \tag{2.309}$$

with $B^0(t, \rho, \theta, z)$ and $B^2(\rho, z)$ arbitrary.

Case VI(b)

Here we write $\frac{R_3}{\sqrt{R_1}} \left(\frac{R_3'}{2R_3\sqrt{R_1}} \right) = k$, where k is a constant. Eq. (2.307), therefore, becomes

$$A_{1,33}(z) - kA_1(z) = 0 \ . \tag{2.310}$$

Here we discuss three cases: $k \gtreqless 0$.

Case VIb(1) $\qquad k > 0$

In this case the solution of Eq. (2.310) can be written as

$$A_1(z) = c_1 e^{\sqrt{k}z} + c_2 e^{-\sqrt{k}z} \ , \tag{2.311}$$

Eqs. (2.304) and (2.305) become

$$B^1 = \frac{1}{\sqrt{R_1}}\left(c_1 e^{\sqrt{k}z} + c_2 e^{-\sqrt{k}z}\right), \tag{2.312}$$

$$B^3 = -\sqrt{k}\left(c_1 e^{\sqrt{k}z} - c_2 e^{-\sqrt{k}z}\right)\int \frac{\sqrt{R_1}}{R_3} d\rho + A_2(z) . \tag{2.313}$$

Putting these values in Eq. (2.4) for $a = 3$ and integrating with respect to z yields

$$A_2(z) = \sqrt{k}\left(c_1 e^{\sqrt{k}z} - c_2 e^{-\sqrt{k}z}\right)\int \frac{\sqrt{R_1}}{R_3} d\rho - \frac{R_3'}{2R_3\sqrt{R_1}\sqrt{k}}\left(c_1 e^{\sqrt{k}z} - c_2 e^{-\sqrt{k}z}\right) + c_3 . \tag{2.314}$$

Using this in Eq. (2.313) gives B^3, with $B^0(t,\rho,\theta,z)$ and $B^2(\rho,z)$ arbitrary. (Note that $\int \frac{\sqrt{R_1}}{R_3} d\rho$ and $\frac{R_3'}{2R_3\sqrt{R_1}}$ are constants).

Case VIb(2) $\qquad k = 0$

Here the solution Eq. (2.310) can be written as

$$A_1(z) = c_1 z + c_2 . \tag{2.315}$$

Eqs. (2.304) and (2.305) become

$$B^1 = \frac{1}{\sqrt{R_1}}(c_1 z + c_2), \tag{2.316}$$

$$B^3 = -c_1 \int \frac{\sqrt{R_1}}{R_3} d\rho + A_2(z) . \tag{2.317}$$

Putting these in Eq. (2.4) for $a = 3$ and integrating with respect to z yields

$$A_2(z) = -\frac{R_3'}{2R_3\sqrt{R_1}}\left(c_1 \frac{z^2}{2} + c_2 z\right) + c_3 . \tag{2.318}$$

Using this in Eq. (2.317)gives B^3, with $B^0(t,\rho,\theta,z)$ and $B^2(\rho,z)$ arbitrary. (Note that $\frac{R_3'}{2R_3\sqrt{R_1}}$ is a constant).

Case VIb(3) $\qquad k < 0$

Here the solution from Eq. (2.310) can be written as

$$A_1(z) = c_1 e^{i\sqrt{k}z} + c_2 e^{-i\sqrt{k}z}, \tag{2.319}$$

and the results can be obtained similarly as in Case VIb(2).

Case VII $\qquad R_0 = 0\,,\quad R_3 = 0\,,\quad R_1 \neq 0\,,\quad R_2 \neq 0$

As the RC equations are symmetric with respect to the interchange in indices 2 and 3, we get the results for this case by interchanging these two indices in the previous case (Case VII).

Case VIII $\qquad R_1 = 0\,,\quad R_2 = 0\,,\quad R_0 \neq 0\,,\quad R_3 \neq 0$

Interchanging the role of indices 0 and 2 in Case V gives results from this case.

Case IX $\qquad R_1 = 0\,,\quad R_3 = 0\,,\quad R_0 \neq 0\,,\quad R_2 \neq 0$

The results in this case can be obtained by interchanging indices 0 and 3 in Case V.

Case X $\qquad R_2 = 0\,,\quad R_3 = 0\,,\quad R_0 \neq 0\,,\quad R_1 \neq 0$

The results in this case can be obtained by interchanging 0 and 3 in Case VI.

Case XI $\qquad R_0 = 0\,,\quad R_1 = 0\,,\quad R_2 = 0\,,\quad R_3 \neq 0$

From the RC equations we get $B^3_{,0} = B^3_{,1} = B^3_{,2} = 0$, and we are only left with Eq. (2.4) for $a = 0$. This means $B^3 = B^3(t)$ and from Eq. (2.4) for $a = 0$

$$B^1 = -\frac{2R_3}{R_3'}\dot{B}^3(t)\,, \tag{2.320}$$

provided that $R_3' \neq 0$, and B^0 and B^2 are completely arbitrary functions of t, ρ, θ and z.

Case XII $\qquad R_2 \neq 0\,,\quad R_0 = 0\,,\quad R_1 = 0\,,\quad R_3 = 0$

This case is similar to Case XI except for the interchange in indices 2 and 3.

Case XIII $\qquad R_1 \neq 0\,,\quad R_0 = 0\,,\quad R_2 = 0\,,\quad R_3 = 0$

In this case the RC equations yield $B^1_{,0} = B^1_{,2} = B^1_{,3} = 0$, and we are only left with Eq. (2.4) for $a = 1$ which yields

$$B^1 = \frac{c_1}{\sqrt{R_1}}\,, \tag{2.321}$$

and B^0, B^2 and B^3 are arbitrary function of t, ρ, θ and z.

Case XIV $\quad R_0 \neq 0\,, \quad R_1 = 0\,, \quad R_2 = 0\,, \quad R_3 = 0$

The result in this case can be obtained by interchanging indices 0 and 3 in Case XI.

Case XV $\quad R_0 = 0\,, \quad R_1 = 0\,, \quad R_2 = 0\,, \quad R_3 = 0$

In this case all B^i, $i = 0, 1, 2, 3$ are arbitrary functions of t, ρ, θ and z, which means that every direction is a Ricci collineation.

In this chapter we have solved the RC equations for both the non-degenerate as well as the degenerate Ricci tensor to obtain the explicit forms of the RC vectors. The dimensions of the Lie algebras admissible by the numerous sub-cases of the non-degenerate case are 3, 4, 5, 6, 7 and 10. These cases are characterized by the constraints on the components of Ricci tensor. Of the fifteen cases of the degenerate Ricci tensor, only one, Case II, admits finite dimensional Lie algebras; they are of dimensions 3, 4, 5 and 10. The forms of the Lie algebras generated by these vectors is the subject of the next chapter.

Chapter 3

Lie Algebras of the Ricci Collineations

As discussed in Chapter 1, if a set of vector fields on a manifold under the operation of Lie bracket (defined by the Lie derivative on a manifold) satisfies the conditions of bilinearity, anti-commutativity and the Jacobi identity, one gets a Lie algebra. In the last chapter, we obtained a set of vector fields along with constraints on the components of the Ricci tensor in each case. In this chapter we provide the Lie algebraic structure for these vector fields in each case and identify their nature. We also classify them into solvable and semisimple algebras and identify some of their sub-algebras. The constraints on the components of the Ricci tensor, whose solution will provide the corresponding gravitational fields, will be discussed in the subsequent chapter.

3.1 Lie Algebras of RCs for the Non-Degenerate Ricci Tensor

In this section the Lie algebras for the cases of non-degenerate Ricci tensor, i.e. the cases discussed in Section 2.2 are given explicitly in terms of their generators and commutation relations, and case numbers here refer to the case numbers of the previous chapter. In what follows $\langle \mathrm{X}_1, \mathrm{X}_2, \mathrm{X}_3 \rangle$ is the minimal symmetry representing translation in t, rotation in θ and translation in the z direction (some mathematicians regard $\mathrm{X}_2 = \partial_\theta$ also as translation). Though numerous cases give rise to minimal symmetries, we are listing it only once to avoid repetition.

We will see that in all the subcases of Case AII, where R_2 and R_3 are proportional, the minimal algebra gets 'extended' to include $\mathbf{X}_4 = z\partial_\theta - l\theta\partial_z$, the rotation in θ and z, as well, l being a constant. Also, $\langle \mathbf{X}_1, \mathbf{X}_2, \mathbf{X}_3, \mathbf{X}_4 \rangle$ is a sub-algebra of all the algebras in that case. This extra symmetry will be globally allowed only for the plane symmetric spacetimes and be disallowed for the cylindrically symmetric case.

3.1.1 Case A: $R_0' \neq 0$

These are the cases for which $R_0' \neq 0$.

Case AIa1(i)

Generators:

$$\begin{aligned}
\mathbf{X}_1 &= \partial_t\,,\\
\mathbf{X}_2 &= \partial_\theta\,,\\
\mathbf{X}_3 &= \partial_z\,,\\
\mathbf{X}_4 &= \left(\tfrac{k_1}{2}t^2 - \int \tfrac{\sqrt{R_1}}{R_0}d\rho + \tfrac{k_1}{2k}z^2\right)\partial_t + \tfrac{1}{\sqrt{R_1}}t\partial_\rho + k_1 tz\partial_z\,,\\
\mathbf{X}_5 &= k_1 tz\partial_t + \tfrac{1}{\sqrt{R_1}}z\partial_\rho + \left(\tfrac{kk_1}{2}t^2 - \int \tfrac{\sqrt{R_1}}{R_3}d\rho + \tfrac{k_1}{2}z^2\right)\partial_z\,,\\
\mathbf{X}_6 &= k_1 t\partial_t + \tfrac{1}{\sqrt{R_1}}\partial_\rho + k_1 z\partial_z\,,\\
\mathbf{X}_7 &= z\partial_t + kt\partial_z\,,.
\end{aligned}$$

Algebra:

$$\begin{aligned}
&[\mathbf{X}_1, \mathbf{X}_4] = k_1\mathbf{X}_6\,, && [\mathbf{X}_1, \mathbf{X}_5] = k_1\mathbf{X}_7\,, && [\mathbf{X}_1, \mathbf{X}_6] = k_1\mathbf{X}_1\,,\\
&[\mathbf{X}_1, \mathbf{X}_7] = \mathbf{X}_3\,, && [\mathbf{X}_3, \mathbf{X}_4] = k_1\mathbf{X}_7\,, && [\mathbf{X}_3, \mathbf{X}_5] = \mathbf{X}_6\,,\\
&[\mathbf{X}_3, \mathbf{X}_6] = k_1\mathbf{X}_3\,, && [\mathbf{X}_3, \mathbf{X}_7] = \mathbf{X}_1\,, && [\mathbf{X}_4, \mathbf{X}_6] = -k_1\mathbf{X}_4\,,\\
&[\mathbf{X}_4, \mathbf{X}_7] = -\mathbf{X}_5\,, && [\mathbf{X}_5, \mathbf{X}_6] = -k_1\mathbf{X}_5\,, && [\mathbf{X}_5, \mathbf{X}_7] = -\mathbf{X}_4\,,\\
&[\mathbf{X}_i, \mathbf{X}_j] = 0\,, \text{ otherwise.}
\end{aligned}$$

This is a solvable algebra. Here $\mathbf{X}_5$ is a scaling symmetry and $\mathbf{X}_6$ is what is called the *Lorentz boost* in the z-direction. Writing $G_4 = \langle \mathbf{X}_0, \mathbf{X}_1, \mathbf{X}_2, \mathbf{X}_6 \rangle$, $G_7 = \langle G_4, \mathbf{X}_3, \mathbf{X}_4, \mathbf{X}_5 \rangle$, where $G_4 = \left[SO(1,1) \times \mathbb{R}^2\right] \otimes SO(2)$ and '$\times$' represents the semi-direct and '$\otimes$' the direct product.

Case AIa1(ii)

Generators:

$$X_1 = \partial_t ,$$
$$X_2 = \partial_\theta ,$$
$$X_3 = \partial_z ,$$
$$X_4 = k_1 t\partial_t + \frac{1}{\sqrt{R_1}}\partial_\rho + k_2 z\partial_z .$$

Algebra:

$$[X_1, X_4] = k_1 X_1 , \quad [X_3, X_4] = k_2 X_3 , \quad [X_i, X_j] = 0 , \text{ otherwise.}$$

This is a solvable algebra. Here $\mathbf{X}_4$ is a scaling symmetry and $G_4 = \langle G_3, \mathbf{X}_4 \rangle$, where $G_3 = \mathbb{R} \otimes SO(2) \otimes \mathbb{R}$.

Case AIa2(i)

Generators:

$$X_1 = \partial_t ,$$
$$X_2 = \partial_\theta ,$$
$$X_3 = \partial_z .$$

Algebra:

$$[X_i, X_j] = 0 .$$

This is the minimal symmetry $G_3 = \mathbb{R} \otimes SO(2) \otimes \mathbb{R}$ and is clearly solvable.

Case AIa2(ii)β

Generators:

$$X_1 = \partial_t ,$$
$$X_2 = \partial_\theta ,$$
$$X_3 = \partial_z ,$$
$$X_4 = \frac{z}{\gamma}\partial_t + t\partial_z .$$

Algebra:

$$[X_1, X_4] = X_3 , \quad [X_3, X_4] = \frac{1}{\gamma} X_1 , \quad [X_i, X_j] = 0 , \text{ otherwise.}$$

This is $G_4 = SO(1,1) \times [SO(2) \otimes \mathbb{R}^2]$ which is a solvable algebra and $\mathbf{X}_4$ is the Lorentz boost in z direction. Notice that by rescaling z by $\sqrt{\gamma}$, or t by $1/\sqrt{\gamma}$, we can eliminate the γ. We have left it as such for ease of comparison with the earlier derivations. Similar remarks apply to all such generators occurring subsequently.

Case AIa4(i)α

Generators:

$$\begin{aligned}
\mathrm{X}_1 &= \partial_t\,, \\
\mathrm{X}_2 &= \partial_\theta\,, \\
\mathrm{X}_3 &= \partial_z\,, \\
\mathrm{X}_4 &= z\partial_t + \gamma t\partial_z\,.
\end{aligned}$$

Algebra:

$$[\mathrm{X}_1, \mathrm{X}_4] = \gamma \mathrm{X}_3\,, \quad [\mathrm{X}_3, \mathrm{X}_4] = \mathrm{X}_1\,, \quad [\mathrm{X}_i, \mathrm{X}_j] = 0\,, \text{ otherwise.}$$

This is similar to the above.

Case AIc1(i)

Generators:

$$\begin{aligned}
\mathrm{X}_1 &= \partial_t\,, \\
\mathrm{X}_2 &= \partial_\theta\,, \\
\mathrm{X}_3 &= \partial_z\,, \\
\mathrm{X}_4 &= k_1\theta\partial_t + t\partial_\theta\,.
\end{aligned}$$

Algebra:

$$[\mathrm{X}_1, \mathrm{X}_4] = \mathrm{X}_2\,, \quad [\mathrm{X}_2, \mathrm{X}_4] = k_1 \mathrm{X}_1\,, \quad [\mathrm{X}_i, \mathrm{X}_j] = 0\,, \text{ otherwise.}$$

This is the same as in the Case AIa2(ii)β above, except for the difference that X_4 here is the Lorentz boost in θ direction. Here the k_1 can only be absorbed into t and not θ for the cylindrically symmetric case as θ ranges from 0 to 2π.

Case AIc2(i)

Generators:

$$\begin{aligned}
\mathbf{X}_1 &= \partial_t\,, \\
\mathbf{X}_2 &= \partial_\theta\,, \\
\mathbf{X}_3 &= \partial_z\,, \\
\mathbf{X}_4 &= k_1 t\partial_t - \frac{1}{\sqrt{R_1}}\partial_\rho + k_1\theta\partial_\theta + k_2 z\partial_z\,, \\
\mathbf{X}_5 &= k_3\theta\partial_t + t\partial_\theta\,.
\end{aligned}$$

Algebra:

$$\begin{aligned}
&[\mathbf{X}_1,\mathbf{X}_4] = k_1\mathbf{X}_1\,, \quad [\mathbf{X}_1,\mathbf{X}_5] = \mathbf{X}_2\,, \quad [\mathbf{X}_2,\mathbf{X}_4] = k_1\mathbf{X}_2\,, \\
&[\mathbf{X}_2,\mathbf{X}_5] = k_3\mathbf{X}_1\,, \quad [\mathbf{X}_3,\mathbf{X}_4] = k_2\mathbf{X}_3\,, \quad [\mathbf{X}_i,\mathbf{X}_j] = 0\,, \text{ otherwise.}
\end{aligned}$$

Writing $G_4 = \langle \mathbf{X}_1, \mathbf{X}_2, \mathbf{X}_3, \mathbf{X}_5\rangle$, $G = \langle G_4, \mathbf{X}_4\rangle$, where $G_4 = SO(1,1) \times \left[SO(2) \otimes \mathbb{R}^2\right]$. This is a solvable algebra. Here $\mathbf{X}_4$ is a scaling symmetry and $\mathbf{X}_5$ is the Lorentz boost in θ direction.

Case AIc2(iii)α

Generators:

$$\begin{aligned}
\mathbf{X}_1 &= \partial_t\,, \\
\mathbf{X}_2 &= \partial_\theta\,, \\
\mathbf{X}_3 &= \partial_z\,, \\
\mathbf{X}_4 &= k_4 t\partial_t - \frac{1}{\sqrt{R_1}}\partial_\rho + k_1\theta\partial_\theta + k_2 z\partial_z\,.
\end{aligned}$$

Algebra:

$$\begin{aligned}
&[\mathbf{X}_1,\mathbf{X}_4] = k_4\mathbf{X}_1\,, \quad [\mathbf{X}_2,\mathbf{X}_4] = -k_1\mathbf{X}_2\,, \quad [\mathbf{X}_3,\mathbf{X}_4] = -k_2\mathbf{X}_3\,, \\
&[\mathbf{X}_i,\mathbf{X}_j] = 0\,, \text{ otherwise.}
\end{aligned}$$

This can be written as $G_4 = \langle G_3, \mathbf{X}_4\rangle$, where $G_3 = \mathbb{R} \otimes SO(2) \otimes \mathbb{R}$ and is solvable. $\mathbf{X}_4$ is a scaling symmetry.

Case AIIa(1)

Generators:

$$X_1 = \partial_t\,,$$
$$X_2 = \partial_\theta\,,$$
$$X_3 = \partial_z\,,$$
$$X_4 = z\partial_\theta - l\theta\partial_z\,,$$
$$X_5 = \theta\partial_t + k^2 t\partial_\theta\,,$$
$$X_6 = z\partial_t + lk^2 t\partial_z\,.$$

Algebra:

$$[X_1, X_5] = k^2 X_2\,, \quad [X_1, X_6] = lk^2 X_3\,, \quad [X_2, X_4] = -lX_3\,,$$
$$[X_2, X_5] = X_1\,, \quad [X_3, X_4] = X_2\,, \quad [X_3, X_6] = X_1\,,$$
$$[X_4, X_5] = X_6\,, \quad [X_4, X_6] = -lX_5\,, \quad [X_5, X_6] = -k^2 X_4\,,$$
$$[X_i, X_j] = 0\,, \text{ otherwise.}$$

This is $SO(1,2) \times \left[SO(2) \otimes \mathbb{R}^2\right]$ where $\mathbf{X}_5$ and $\mathbf{X}_6$ are the Lorentz boosts in θ and z directions. $\mathbf{X}_4$ which is a rotation in θ and z will appear in all the subcases of Case AII. This is a semisimple algebra having $\langle X_4, X_5, X_6\rangle$ as a subalgebra.

Case AIIa(2)

Generators:

$$X_1 = \partial_t\,,$$
$$X_2 = \partial_\theta\,,$$
$$X_3 = \partial_z\,,$$
$$X_4 = z\partial_\theta - l\theta\partial_z\,.$$

Algebra:

$$[X_2, X_4] = -lX_3\,, \quad [X_3, X_4] = X_2\,, \quad [X_i, X_j] = 0\,, \text{ otherwise.}$$

This can be written as $\{SO(2) \times [\mathbb{R} \otimes SO(2)]\} \otimes \mathbb{R}$ and is solvable, and can be regarded as the 'minimal' algebra of Case II.

Case AIIb1(i)α

Generators:

$$\begin{aligned} X_1 &= \partial_t\,, \\ X_2 &= \partial_\theta\,, \\ X_3 &= \partial_z\,, \\ X_4 &= z\partial_\theta - l\theta\partial_z\,. \end{aligned}$$

Algebra:

$$[X_2, X_4] = -lX_3\,, \quad [X_3, X_4] = X_2\,, \quad [X_i, X_j] = 0\,, \text{ otherwise.}$$

This is the same as the previous case.

Case AIIb1(i)β

Generators:

$$\begin{aligned} X_1 &= \partial_t\,, \\ X_2 &= \partial_\theta\,, \\ X_3 &= \partial_z\,, \\ X_4 &= z\partial_\theta - l\theta\partial_z\,, \\ X_5 &= \tfrac{\beta}{\alpha}t\partial_t - \tfrac{2}{\alpha\sqrt{R_1}}\partial_\rho + \theta\partial_\theta + z\partial_z\,. \end{aligned}$$

Algebra:

$$\begin{aligned} &[X_1, X_5] = \tfrac{\beta}{\alpha}X_1\,, \quad [X_2, X_4] = -lX_3\,, \quad [X_2, X_5] = X_2\,, \\ &[X_3, X_4] = X_2\,, \quad [X_3, X_5] = X_3\,, \quad [X_i, X_j] = 0\,, \text{ otherwise.} \end{aligned}$$

This is a solvable algebra which can be written as $G_5 = \langle G_4, X_5 \rangle$, where $G_4 = \{SO(2) \times [\mathbb{R} \otimes SO(2)]\} \otimes \mathbb{R}$.

Case AIIb1(ii)

Generators:

$$
\begin{aligned}
X_1 &= \partial_t\,, \\
X_2 &= \partial_\theta\,, \\
X_3 &= \partial_z\,, \\
X_4 &= z\partial_\theta - l\theta\partial_z\,, \\
X_5 &= \delta\theta\partial_t + t\partial_\theta\,, \\
X_6 &= \tfrac{1}{l}\delta z\partial_t + t\partial_z\,, \\
X_7 &= \tfrac{1}{2}\left(t^2 - \tfrac{4}{\alpha^2 R_0} + \delta\theta^2 + \tfrac{1}{l}\delta z^2\right)\partial_t - \tfrac{2}{\alpha\sqrt{R_1}}t\partial_\rho + \theta t\partial_\theta + zt\partial_z\,, \\
X_8 &= -l\theta t\partial_t + \tfrac{2l}{\alpha\sqrt{R_1}}\theta\partial_\rho + \tfrac{1}{2}\left(-\tfrac{lt^2}{\delta} + \tfrac{4l}{\alpha^2 R_2} - l\theta^2 + z^2\right)\partial_\theta - l\theta z\partial_z\,, \\
X_9 &= tz\partial_t - \tfrac{2}{\alpha\sqrt{R_1}}z\partial_\rho + \theta z\partial_\theta - \tfrac{1}{2}\left(-\tfrac{lt^2}{\delta} + \tfrac{4l}{\alpha^2 R_2} + l\theta^2 - z^2\right)\partial_z\,, \\
X_{10} &= t\partial_t - \tfrac{2}{\alpha\sqrt{R_1}}\partial_\rho + \theta\partial_\theta + z\partial_z\,.
\end{aligned}
$$

Algebra:

$$
\begin{array}{lll}
[X_1, X_5] = X_2\,, & [X_1, X_6] = X_3\,, & [X_1, X_7] = X_{10}\,, \\
[X_1, X_8] = \frac{l}{\delta}X_5\,, & [X_1, X_9] = \frac{l}{\delta}X_6\,, & [X_1, X_{10}] = X_1\,, \\
[X_2, X_4] = -lX_3, & [X_2, X_5] = \delta X_1\,, & [X_2, X_7] = X_5\,, \\
[X_2, X_8] = -lX_{10} & [X_2, X_9] = X_4\,, & [X_2, X_{10}] = X_2\,, \\
[X_3, X_4] = X_2\,, & [X_3, X_6] = \frac{\delta}{l}X_1\,, & [X_3, X_7] = X_6\,, \\
[X_3, X_8] = X_4\,, & [X_3, X_9] = X_{10}\,, & [X_3, X_{10}] = X_3\,, \\
[X_4, X_5] = lX_6\,, & [X_4, X_6] = -X_5\,, & [X_4, X_8] = -lX_9\,, \\
[X_4, X_9] = X_8\,, & [X_5, X_6] = -\frac{\delta}{l}X_4\,, & [X_5, X_7] = -\frac{\delta}{l}X_8\,, \\
[X_5, X_8] = -lX_7\,, & [X_6, X_7] = \frac{\delta}{l}X_9\,, & [X_6, X_9] = X_7\,, \\
[X_7, X_{10}] = -X_7\,, & [X_8, X_{10}] = -X_8\,, & [X_9, X_{10}] = -X_9\,, \\
[X_i, X_j] = 0\,,\ \text{otherwise.} & &
\end{array}
$$

This $SO(1,4)$ or $SO(2,3)$ is the maximal semisimple anti-deSitter algebra. It has 3 dimensional subalgebras $\{X_4, X_5, X_6\}$ of rotations and $\{X_8, X_9, X_{10}\}$; 4 dimensional subalgebras $\{X_1, X_2, X_3, X_4\}$ and $\{X_7, X_8, X_9, X_{10}\}$; and 6 dimensional subalgebras $\{X_1, X_2, X_3, X_4, X_5, X_6\}$ in it.

Case AIIb2(i)

Generators:

$$
\begin{aligned}
X_1 &= \partial_t\,,\\
X_2 &= \partial_\theta\,,\\
X_3 &= \partial_z\,,\\
X_4 &= z\partial_\theta - l\theta\partial_z\,,\\
X_5 &= \tfrac{1}{\sqrt{R_0}}\theta\sin\gamma t\partial_t - \tfrac{1}{\sqrt{R_1}}\theta\cos\gamma t\partial_\rho + \tfrac{\sqrt{R_0}}{\gamma R_2}\cos\gamma t\partial_\theta\,,\\
X_6 &= -\tfrac{1}{\sqrt{R_0}}\theta\cos\gamma t\partial_t - \tfrac{1}{\sqrt{R_1}}\theta\sin\gamma t\partial_\rho + \tfrac{\sqrt{R_0}}{\gamma R_2}\sin\gamma t\partial_\theta\,,\\
X_7 &= \tfrac{1}{l\sqrt{R_0}}z\sin\gamma t\partial_t - \tfrac{1}{l\sqrt{R_1}}z\cos\gamma t\partial_\rho + \tfrac{\sqrt{R_0}}{\gamma R_2}\cos\gamma t\partial_z\,,\\
X_8 &= -\tfrac{1}{l\sqrt{R_0}}z\cos\gamma t\partial_t - \tfrac{1}{l\sqrt{R_1}}z\sin\gamma t\partial_\rho + \tfrac{\sqrt{R_0}}{\gamma R_2}\sin\gamma t\partial_z\,,\\
X_9 &= -\tfrac{1}{\sqrt{R_0}}\sin\gamma t\partial_t + \tfrac{1}{\sqrt{R_1}}\cos\gamma t\partial_\rho\,,\\
X_{10} &= \tfrac{1}{\sqrt{R_0}}\cos\gamma t\partial_t + \tfrac{1}{\sqrt{R_1}}\sin\gamma t\partial_\rho\,.
\end{aligned}
$$

Algebra:

$$
\begin{array}{lll}
[X_1, X_5] = -X_6\,, & [X_1, X_6] = X_5\,, & [X_1, X_7] = -X_8\,,\\
[X_1, X_8] = X_7\,, & [X_1, X_9] = -X_{10}\,, & [X_1, X_{10}] = X_9\,,\\
[X_2, X_4] = -lX_3\,, & [X_2, X_5] = -X_9\,, & [X_2, X_6] = -X_{10}\,,\\
[X_3, X_4] = X_2\,, & [X_3, X_7] = -\frac{1}{l}X_9\,, & [X_3, X_8] = -\frac{1}{l}X_{10}\,,\\
[X_4, X_5] = lX_7\,, & [X_4, X_6] = lX_8\,, & [X_4, X_7] = -X_5\,,\\
[X_4, X_8] = -X_6\,, & [X_5, X_6] = -\frac{1}{\gamma R_2}X_1\,, & [X_5, X_7] = \frac{1}{lR_2}X_4\,,\\
[X_5, X_9] = -\frac{1}{R_2}X_2\,, & [X_6, X_8] = \frac{1}{lR_2}X_4\,, & [X_6, X_{10}] = -\frac{1}{R_2}X_2\,,\\
[X_7, X_8] = -\frac{1}{\gamma l R_2}X_1\,, & [X_7, X_9] = -\frac{1}{R_2}X_3\,, & [X_8, X_{10}] = -\frac{1}{R_2}X_3\,,\\
[X_i, X_j] = 0\,,\ \text{otherwise.} & &
\end{array}
$$

This is again a 10 dimensional semisimple algebra and has $\{X_1, X_2, X_3, X_4\}$ as a subalgebra.

Case AIIb2(ii)α_1

Generators:

$$
\begin{aligned}
X_1 &= \partial_t\,,\\
X_2 &= \partial_\theta\,,\\
X_3 &= \partial_z\,,\\
X_4 &= z\partial_\theta - l\theta\partial_z\,,\\
X_5 &= \left(\tfrac{1}{\lambda R_0} - \tfrac{\lambda}{4}t^2\right)\partial_t + \tfrac{1}{\sqrt{R_1}}t\partial_\rho\,,\\
X_6 &= -\tfrac{\lambda}{2}t\partial_t + \tfrac{1}{\sqrt{R_1}}\partial_\rho\,.
\end{aligned}
$$

Algebra:

$$
\begin{aligned}
&[X_1, X_5] = X_6\,, \quad [X_1, X_6] = -\tfrac{\lambda}{2}X_1\,, \quad [X_2, X_4] = -lX_3\,,\\
&[X_3, X_4] = X_2\,, \quad [X_5, X_6] = \tfrac{\lambda}{2}X_5\,, \quad [X_i, X_j] = 0\,, \text{ otherwise.}
\end{aligned}
$$

This is a semisimple algebra having $\{X_1, X_2, X_3, X_4\}$ and $\{X_1, X_5, X_6\}$ as subalgebras.

Case AIIb2(ii)α_2

Generators:

$$
\begin{aligned}
X_1 &= \partial_t\,,\\
X_2 &= \partial_\theta\,,\\
X_3 &= \partial_z\,,\\
X_4 &= z\partial_\theta - l\theta\partial_z\,,\\
X_5 &= -\tfrac{R_0'}{2\sqrt{\eta}R_0\sqrt{R_1}}e^{\sqrt{\eta}t}\partial_t + \tfrac{1}{\sqrt{R_1}}e^{\sqrt{\eta}t}\partial_\rho\,,\\
X_6 &= \tfrac{R_0'}{2\sqrt{\eta}R_0\sqrt{R_1}}e^{-\sqrt{\eta}t}\partial_t + \tfrac{1}{\sqrt{R_1}}e^{-\sqrt{\eta}t}\partial_\rho\,.
\end{aligned}
$$

Algebra:

$$
\begin{aligned}
&[X_1, X_5] = \sqrt{\eta}X_5\,, \quad [X_1, X_6] = -\sqrt{\eta}X_6\,, \quad [X_2, X_4] = -lX_3\,,\\
&[X_3, X_4] = X_2\,, \quad [X_5, X_6] = KX_1\,, \quad [X_i, X_j] = 0\,, \text{ otherwise.}
\end{aligned}
$$

Here $K = \frac{1}{\sqrt{\eta}}\left[\frac{1}{2}\left(\frac{R_0'}{R_0\sqrt{R_1}}\right)^2 + \frac{1}{\sqrt{R_1}}\left(\frac{R_0'}{R_0\sqrt{R_1}}\right)'\right]$ is a constant.

Case AIIb2(ii)β

Generators:

$$
\begin{aligned}
X_1 &= \partial_t \,, \\
X_2 &= \partial_\theta \,, \\
X_3 &= \partial_z \,, \\
X_4 &= z\partial_\theta - l\theta\partial_z \,.
\end{aligned}
$$

Algebra:

$$[X_2, X_4] = -lX_3 \,, \quad [X_3, X_4] = X_2 \,, \quad [X_i, X_j] = 0 \,, \text{ otherwise.}$$

This is the same as in Case AIIa(2).

3.1.2 Case B: $R_0' = 0$

Here we list the Lie algebras of those cases for which $R_0' = 0$.

Case B(I)

Generators:

$$
\begin{aligned}
X_1 &= \partial_t \,, \\
X_2 &= \partial_\theta \,, \\
X_3 &= \partial_z \,, \\
X_4 &= z\partial_\theta - \tfrac{\beta}{\gamma}\theta\partial_z \,, \\
X_5 &= \tfrac{1}{\sqrt{R_1}}\partial_\rho \,, \\
X_6 &= -\tfrac{1}{\alpha}\int\sqrt{R_1}d\rho\partial_t + \tfrac{1}{\sqrt{R_1}}t\partial_\rho \,, \\
X_7 &= \theta\partial_t + \tfrac{\alpha}{\beta}t\partial_\theta \,, \\
X_8 &= z\partial_t + \tfrac{\alpha}{\gamma}t\partial_z \,, \\
X_9 &= \tfrac{\theta}{\sqrt{R_1}}\partial_\rho - \tfrac{1}{\beta}\int\sqrt{R_1}d\rho\partial_\theta \,, \\
X_{10} &= \tfrac{z}{\sqrt{R_1}}\partial_\rho - \tfrac{1}{\gamma}\int\sqrt{R_1}d\rho\partial_z \,.
\end{aligned}
$$

Algebra:

$$
\begin{array}{lll}
[X_1, X_6] = X_5\,, & [X_1, X_7] = \frac{\alpha}{\beta} X_2\,, & [X_1, X_8] = \frac{\alpha}{\gamma} X_3\,, \\
[X_2, X_4] = -\frac{\beta}{\gamma} X_3\,, & [X_2, X_7] = X_1\,, & [X_2, X_9] = X_5\,, \\
[X_3, X_4] = X_2\,, & [X_3, X_8] = X_1\,, & [X_3, X_{10}] = X_5\,, \\
[X_4, X_7] = X_8\,, & [X_4, X_8] = -\frac{\beta}{\gamma} X_7\,, & [X_4, X_9] = X_{10}\,, \\
[X_4, X_{10}] = -\frac{\beta}{\gamma} X_9\,, & [X_5, X_6] = \frac{1}{\alpha} X_1\,, & [X_5, X_9] = -\frac{1}{\beta} X_2\,, \\
[X_5, X_{10}] = -\frac{1}{\gamma} X_3\,, & [X_6, X_7] = -X_9\,, & [X_6, X_8] = -X_{10}\,, \\
[X_6, X_9] = -\frac{1}{\alpha} X_7\,, & [X_6, X_{10}] = -\frac{1}{\alpha} X_8\,, & [X_7, X_8] = -\frac{\alpha}{\beta} X_4\,, \\
[X_7, X_9] = \frac{\alpha}{\beta} X_6\,, & [X_8, X_{10}] = \frac{\alpha}{\gamma} X_6\,, & [X_9, X_{10}] = \frac{1}{\beta} X_4\,, \\
[X_i, X_j] = 0\,, \text{ otherwise.} & &
\end{array}
$$

This algebra is similar to that of Case AIIb2(i)β.

Case BII(a)

Generators:

$$
\begin{aligned}
X_1 &= \partial_t\,, \\
X_2 &= \partial_\theta\,, \\
X_3 &= \partial_z\,, \\
X_4 &= \theta\partial_t + \tfrac{\alpha}{\beta} t\partial_\theta\,,
\end{aligned}
$$

Algebra:

$$
[X_1, X_4] = \tfrac{\alpha}{\beta} X_2\,, \quad [X_2, X_4] = X_1\,, \quad [X_i, X_j] = 0\,, \text{ otherwise.}
$$

This is similar to Case AIc1(i).

Case BIIb(1)

Generators:

$$\begin{aligned}
X_1 &= \partial_t\,, \\
X_2 &= \partial_\theta\,, \\
X_3 &= \partial_z\,, \\
X_4 &= \theta\partial_t + \tfrac{\alpha}{\beta}t\partial_\theta\,, \\
X_5 &= \tfrac{e^{\sqrt{k_1}z}}{\sqrt{R_1}}\partial_\rho - \tfrac{1}{\sqrt{k_1}}\tfrac{R_3' e^{\sqrt{k_1}z}}{2R_3\sqrt{R_1}}\partial_z\,, \\
X_6 &= \tfrac{e^{-\sqrt{k_1}z}}{\sqrt{R_1}}\partial_\rho + \tfrac{1}{\sqrt{k_1}}\tfrac{R_3' e^{-\sqrt{k_1}z}}{2R_3\sqrt{R_1}}\partial_z\,.
\end{aligned}$$

Algebra:

$$\begin{aligned}
&[X_1, X_4] = \tfrac{\alpha}{\beta}X_2\,, && [X_2, X_4] = X_1\,, && [X_3, X_5] = \sqrt{k_1}X_5\,, \\
&[X_3, X_6] = -\sqrt{k_1}X_6\,, && [X_5, X_6] = kX_3\,, && [X_i, X_j] = 0\,, \text{ otherwise,}
\end{aligned}$$

where $k = \frac{2}{\sqrt{k_1}}\left[\frac{1}{\sqrt{R_1}}\left(\frac{R_3'}{2R_3\sqrt{R_1}}\right)' + \left(\frac{R_3'}{2R_3\sqrt{R_1}}\right)^2\right]$ is a constant. This is a semisimple algebra having $\{X_3, X_5, X_6\}$ as a subalgebra.

Case BIIb(3)

Generators:

$$\begin{aligned}
X_1 &= \partial_t\,, \\
X_2 &= \partial_\theta\,, \\
X_3 &= \partial_z\,, \\
X_4 &= \theta\partial_t + \tfrac{\alpha}{\beta}t\partial_\theta\,, \\
X_5 &= \tfrac{1}{\sqrt{R_1}}\partial_\rho - k_2 z\partial_z\,, \\
X_6 &= \tfrac{z}{\sqrt{R_1}}\partial_\rho - \left(\int \tfrac{\sqrt{R_1}}{R_3}d\rho + k_2\tfrac{z^2}{2}\right)\partial_z\,.
\end{aligned}$$

Algebra:

$$\begin{aligned}
&[X_1, X_4] = \tfrac{\alpha}{\beta}X_2\,, && [X_2, X_4] = X_1\,, && [X_3, X_5] = -k_2X_3\,, \\
&[X_3, X_6] = X_5\,, && [X_5, X_6] = k_2X_6\,, && [X_i, X_j] = 0\,, \text{ otherwise.}
\end{aligned}$$

This is a semisimple algebra having $\{X_3, X_5, X_6\}$ as a subalgebra.

Case BIVa(2)

Generators:

$$\begin{aligned} X_1 &= \partial_t \,, \\ X_2 &= \partial_\theta \,, \\ X_3 &= \partial_z \,, \\ X_4 &= z\partial_\theta - k\theta\partial_z \;. \end{aligned}$$

Algebra:

$$[X_2, X_4] = -kX_3 \,, \quad [X_3, X_4] = X_2 \,, \quad [X_i, X_j] = 0 \,, \text{ otherwise.}$$

Its structure is similar to that of Case AIIa(2).

Case BIVb1(i)

Generators:

$$\begin{aligned} X_1 &= \partial_t \,, \\ X_2 &= \partial_\theta \,, \\ X_3 &= \partial_z \,, \\ X_4 &= z\partial_\theta - \tfrac{R_2}{R_3}\theta\partial_z \;. \end{aligned}$$

Algebra:

$$[X_2, X_4] = -\tfrac{R_2}{R_3}X_3 \,, \quad [X_3, X_4] = X_2 \,, \quad [X_i, X_j] = 0 \,, \text{ otherwise.}$$

This is similar to the previous case.

Case BIVb1(ii) Same as above

Case BIVb3(ii)γ_1

Generators:

$$\begin{aligned} X_1 &= \partial_t \,, \\ X_2 &= \partial_\theta \,, \\ X_3 &= \partial_z \,, \\ X_4 &= \tfrac{1}{\sqrt{R_1}}\partial_\rho + \theta\partial_\theta - k_2 z\partial_z \;. \end{aligned}$$

Algebra:

$$[X_2, X_4] = X_2\,, \quad [X_3, X_4] = -k_2X_3\,, \quad [X_i, X_j] = 0\,, \text{ otherwise.}$$

This is $G_4 = \{SO(2) \times [\mathbb{R} \otimes SO(2)]\} \otimes \mathbb{R}$ and is a solvable algebra.

Case BIVb3(ii)γ_2

Generators:

$$\begin{aligned}
X_1 &= \partial_t\,,\\
X_2 &= \partial_\theta\,,\\
X_3 &= \partial_z\,,\\
X_4 &= -z\partial_\theta + \tfrac{\theta}{d}\partial_z\,,\\
X_5 &= \tfrac{1}{\sqrt{R_1}}\partial_\rho - k_2\theta\partial_\theta - k_2z\partial_z\,,\\
X_6 &= \tfrac{\theta}{\sqrt{R_1}}\partial_\rho - \left(\int \tfrac{\sqrt{R_1}}{R_2}d\rho + k_2\tfrac{\theta^2}{2} - dk_2\tfrac{z^2}{2}\right)\partial_\theta - k_2\theta z\partial_z\,,\\
X_7 &= \tfrac{z}{\sqrt{R_1}}\partial_\rho - k_2\theta z\partial_\theta - \left(\int \tfrac{\sqrt{R_1}}{R_2}d\rho - \tfrac{k_2}{2d}\theta^2 + k_2\tfrac{z^2}{2}\right)\partial_z\,.
\end{aligned}$$

Algebra:

$$\begin{aligned}
&[X_2, X_4] = \tfrac{1}{d}X_3\,, && [X_2, X_5] = -k_2X_2\,, && [X_2, X_6] = X_5\,,\\
&[X_2, X_7] = k_2X_4\,, && [X_3, X_4] = -X_2\,, && [X_3, X_5] = -k_2X_3\,,\\
&[X_3, X_6] = -dk_2X_4\,, && [X_3, X_7] = X_5\,, && [X_4, X_6] = -X_7\,,\\
&[X_4, X_7] = -X_6\,, && [X_5, X_6] = -X_6\,, && [X_5, X_7] = -X_7\,,\\
&[X_6, X_7] = KX_4\,, && [X_i, X_j] = 0\,, \text{ otherwise.}
\end{aligned}$$

where $K = -2k_2\left(\int \frac{\sqrt{R_1}}{R_2}d\rho + \frac{1}{2k_2R_2}\right)$ is a constant. This is a semisimple algebra having $\{X_1, X_2, X_3, X_4\}$ and $\{X_4, X_5, X_6, X_7\}$ as 4 dimensional subalgebras and a 6 dimensional subalgebra $\{X_2, X_3, X_4, X_5, X_6, X_7\}$ in it. We write this as $G_7 = \langle G_4, X_5, X_6, X_7\rangle$ where $G_4 = \{SO(2) \times [\mathbb{R} \otimes SO(2)]\} \otimes \mathbb{R}$.

3.2 Lie Algebras of RCs for the Degenerate Ricci Tensor

We have found in our classification that the algebras for the degenerate Ricci tensor are infinite dimensional except in Case II, where $R_1 = 0$, and $R_i \neq 0$, $i = 0, 2, 3$. In this case we have algebras of dimensions 3, 4, 5, and 10.

Case IIA(d)

Generators:

$$\begin{aligned}
X_1 &= \partial_t\,,\\
X_2 &= \partial_\theta\,,\\
X_3 &= \partial_z\,,\\
X_4 &= -\tfrac{2R_2}{R_2'}k\partial_\rho + k\theta\partial_\theta + z\partial_z\,.
\end{aligned}$$

Algebra:

$$[X_2, X_4] = kX_2\,, \quad [X_3, X_4] = X_3\,, \quad [X_i, X_j] = 0\,, \text{ otherwise.}$$

This is $G_4 = \{SO(2) \times [\mathbb{R} \otimes SO(2)]\} \otimes \mathbb{R}$ and is a solvable algebra.

Case IIB(a)

Generators:

$$\begin{aligned}
X_1 &= \partial_t\,,\\
X_2 &= \partial_\theta\,,\\
X_3 &= \partial_z\,,\\
X_4 &= -\tfrac{\alpha}{\beta}z\partial_\theta + \theta\partial_z\,,\\
X_5 &= \theta\partial_t + \alpha t\partial_\theta\,,\\
X_6 &= z\partial_t + \beta t\partial_z\,,\\
X_7 &= \left(\tfrac{\theta^2}{2} + \tfrac{\alpha}{\beta}\tfrac{z^2}{2} + \alpha\tfrac{t^2}{2}\right)\partial_t - \tfrac{2\alpha R_0}{R_0'}t\partial_\rho + \alpha t\theta\partial_\theta + \alpha tz\partial_z\,,\\
X_8 &= t\theta\partial_t - \tfrac{2R_0}{R_0'}\theta\partial_\rho + \left(\alpha\tfrac{t^2}{2} + \tfrac{\theta^2}{2} - \tfrac{\alpha}{\beta}\tfrac{z^2}{2}\right)\partial_\theta + \theta z\,\partial_z,\\
X_9 &= tz\partial_t - \tfrac{2R_0}{R_0'}z\partial_\rho + \theta z\partial_\theta + \left(\beta\tfrac{t^2}{2} - \tfrac{\beta}{\alpha}\tfrac{\theta^2}{2} + \tfrac{z^2}{2}\right)\partial_z\,,\\
X_{10} &= t\partial_t - \tfrac{2R_0}{R_0'}\partial_\rho + \theta\partial_\theta + z\partial_z.
\end{aligned}$$

Algebra:

$$
\begin{array}{lll}
[X_1, X_5] = \alpha X_2 \,, & [X_1, X_6] = \beta X_3 \,, & [X_1, X_7] = \alpha X_{10} \,, \\
[X_1, X_8] = X_5 \,, & [X_1, X_9] = X_6 \,, & [X_1, X_{10}] = X_1 \,, \\
[X_2, X_4] = X_3 \,, & [X_2, X_5] = X_1 \,, & [X_2, X_7] = X_5 \,, \\
[X_2, X_8] = X_{10} \,, & [X_2, X_9] = -\frac{\beta}{\alpha} X_4 \,, & [X_2, X_{10}] = X_2 \,, \\
[X_3, X_4] = -\frac{\alpha}{\beta} X_2 \,, & [X_3, X_6] = X_1 \,, & [X_3, X_7] = \frac{\alpha}{\beta} X_6 \,, \\
[X_3, X_8] = X_4 \,, & [X_3, X_9] = X_{10} \,, & [X_3, X_{10}] = X_3 \,, \\
[X_4, X_5] = -\frac{\alpha}{\beta} X_6 \,, & [X_4, X_6] = X_5 \,, & [X_4, X_8] = -\frac{\alpha}{\beta} X_9 \,, \\
[X_4, X_9] = X_8 \,, & [X_5, X_6] = \beta X_4 \,, & [X_5, X_7] = \alpha X_8 \,, \\
[X_5, X_8] = X_7 \,, & [X_6, X_7] = -X_9 \,, & [X_6, X_9] = \frac{\beta}{\alpha} X_7 \,, \\
[X_7, X_{10}] = -X_7 \,, & [X_8, X_{10}] = -X_8 \,, & [X_9, X_{10}] = -X_9 \,, \\
[X_i, X_j] = 0 \,, \text{ otherwise.} & &
\end{array}
$$

It has 3 dimensional subalgebras $\{X_4, X_5, X_6\}$ of rotations and $\{X_8, X_9, X_{10}\}$; 4 dimensional subalgebras $\{X_1, X_2, X_3, X_4\}$ and $\{X_7, X_8, X_9, X_{10}\}$; and 6 dimensional subalgebras $\{X_1, X_2, X_3, X_4, X_5, X_6\}$ in it. This $SO(1,4)$ or $SO(2,3)$ anti-deSitter Lie algebra is the maximal semisimple algebra of the degenerate case.

Case IIBb(1)

Generators:

$$
\begin{aligned}
X_1 &= \partial_t \,, \\
X_2 &= \partial_\theta \,, \\
X_3 &= \partial_z \,, \\
X_4 &= \theta \partial_t + \alpha t \partial_\theta , \\
X_5 &= t\partial_t - \tfrac{2R_0}{R_0'} \partial_\rho + \theta \partial_\theta + kz\partial_z .
\end{aligned}
$$

Algebra:

$$
\begin{array}{lll}
[X_1, X_4] = \alpha X_2 \,, & [X_1, X_5] = X_1 \,, & [X_2, X_4] = X_1 \,, \\
[X_2, X_5] = X_2 \,, & [X_3, X_5] = kX_3 \,, & [X_i, X_j] = 0 \,, \text{ otherwise.}
\end{array}
$$

Writing $G_4 = \langle X_1, X_2, X_3, X_4 \rangle$, $G = \langle G_4, X_5 \rangle$, where $G_4 = SO(1,2) \times \left[SO(2) \otimes \mathbb{R}^2\right]$. X_4 is the Lorentz boost in the θ direction. This is a solvable algebra.

Case IIBb(2)

Generators:

$$\begin{aligned}
X_1 &= \partial_t\,, \\
X_2 &= \partial_\theta\,, \\
X_3 &= \partial_z\,, \\
X_4 &= \theta\partial_t + \alpha t\partial_\theta.
\end{aligned}$$

Algebra:

$$[X_1, X_4] = \alpha X_2\,, \quad [X_2, X_4] = X_1\,, \quad [X_i, X_j] = 0\,, \text{ otherwise.}$$

This is $G_4 = SO(1,1) \times \left[SO(2) \otimes \mathbb{R}^2\right]$ which is a solvable algebra and X_4 is the Lorentz boost in the θ direction.

Case IIBd(1)

Generators:

$$\begin{aligned}
X_1 &= \partial_t\,, \\
X_2 &= \partial_\theta\,, \\
X_3 &= \partial_z\,, \\
X_4 &= z\partial_\theta - \tfrac{R_2}{R_3}\theta\partial_z.
\end{aligned}$$

Algebra:

$$[X_2, X_4] = -\tfrac{R_2}{R_3}X_3\,, \quad [X_3, X_4] = X_2\,, \quad [X_i, X_j] = 0\,, \text{ otherwise.}$$

This is $\{SO(2) \times [\mathbb{R} \otimes SO(2)]\} \otimes \mathbb{R}$ and is solvable.

Case IIBd(4)

Generators:

$$\begin{aligned}
X_1 &= \partial_t\,, \\
X_2 &= \partial_\theta\,, \\
X_3 &= \partial_z\,, \\
X_4 &= z\partial_\theta - \tfrac{R_2}{R_3}\theta\partial_z, \\
X_5 &= t\partial_t - \tfrac{2R_0}{R_0'}\partial_\rho + \tfrac{1}{k_1}\theta\partial_\theta + \tfrac{1}{k_1}z\partial_z.
\end{aligned}$$

Algebra:

$$[X_1, X_5] = X_1\,, \quad [X_2, X_4] = -\frac{R_2}{R_3}X_3\,, \quad [X_2, X_5] = \frac{1}{k_1}X_2\,,$$
$$[X_3, X_4] = X_2\,, \quad [X_3, X_5] = \frac{1}{k_1}X_3\,, \quad [X_i, X_j] = 0\,, \text{ otherwise.}$$

This is a solvable algebra which can be written as $G = \langle G_4, X_5 \rangle$, where $G_4 = \{SO(2) \times [\mathbb{R} \otimes SO(2)]\} \otimes \mathbb{R}$.

We have classified the RCs of cylindrically symmetric static spacetimes according to their Lie algebras here. It is found that for the non-degenerate case these algebras are of dimensions 3, 4, 5, 6, 7 and 10 only. For the degenerate case, however, the algebras are infinite dimensional except in Case II, where $R_1 = 0$, and $R_i \neq 0$, $i = 0, 2, 3$, in which case we have 3, 4, 5 and 10 dimensional algebras. The minimal symmetry has translations in t, and z, and rotation in θ. All the higher dimensional algebras include rotations in θ and z, Lorentz rotations along θ and z directions, some simple and some complicated scaling symmetries, and occasionally translations in the ρ-direction as well. Similarly, both the solvable and semisimple algebras have been found. Examples of spacetimes admitting these algebras are discussed in the next chapter.

Chapter 4

Metrics with their Isometries and Ricci Collineations

The classification of cylindrically symmetric static manifolds according to their RCs, presented in the previous chapters, has given rise to numerous cases where each case is characterized by constraints on the components of the Ricci tensor, R_i. Now, solving these constraints gives the explicit form of metrics corresponding to these cases. In some cases we find particular spacetimes while in others a class or family of metrics is obtained. The physical nature of some of these spacetimes is also discussed. These metrics for the non-degenerate and the degenerate cases are listed here, and the RCs of these metrics are compared with their KVs [15] and HMs [20]. It is well known that the Lie algebra of KVs is a subalgebra of HMs and every HM is an RC. The converse is not true in general [8]. This gives rise to the interesting cases where one obtains RCs which are not KVs. When we compare our results with the classification of cylindrically symmetric spacetimes according to KVs [15] and HMs [20], we find numerous cases of the proper (i.e. non-isometric) RCs.

4.1 Metrics with the Non-Degenerate Ricci Tensor

It is well known that the spaces of non-zero constant curvature have RCs coinciding with their KVs, for example, the anti-de Sitter universe with 10 KVs and 10 RCs (Case AIIb1(ii)). These are not listed below. However, there are some non-trivial cases of coinciding RCs and KVs which do not have constant curvature, and they are included in the list. The solution of constraints on the Ricci tensor to obtain the metrics is explained in the first example; for other examples, only the metrics are given.

N1. The constraints on the components of the Ricci tenser, R_i, for Case BIVb3(ii)γ_2 are

$$(R_3/R_2)' = 0, R_0' = 0, R_2' \neq 0, \text{and } \left(\frac{R_2'}{R_2\sqrt{R_1}}\right)' = 0\,.$$

These suggest taking $R_3 \propto R_2$. Taking the constant of proportionality as -1, one gets $\lambda = \mu$ (in Eq. (1.16)), so that R_0 becomes

$$R_0 = \frac{e^{\upsilon}\upsilon''}{2} + \frac{e^{\upsilon}\upsilon'}{4}\left(\upsilon' + 2\lambda'\right)\,.$$

Now, $R_0' = 0$ suggests a possibility that

$$e^{\upsilon}\upsilon'' = 2k^2\,,$$

where k is a constant, and

$$\upsilon' + 2\lambda' = 0\,.$$

The solution of these equations gives

$$\upsilon = \ln \cos h^2 k\rho\,,$$

$$\lambda = \mu = \ln\left(\cos hk\rho\right)^{-1}\,.$$

Thus all the constraints for this case are satisfied and the line element becomes

$$ds^2 = \cos h^2 k\rho dt^2 - d\rho^2 - a^2\left(\cos hk\rho\right)^{-1} d\theta^2 - \left(\cos hk\rho\right)^{-1} dz^2\,.$$

It admits 7 RCs and 4 KVs and, therefore, is a case of proper RCs. Now, solving the EFE for the energy momentum tensor gives

$$\begin{aligned} T_0^0 &= \tfrac{k^2}{4}\left(4 - 7\tanh^2 k\rho\right), \\ T_1^1 &= \tfrac{3k^2}{4}\tanh^2 k\rho, \\ T_2^2 &= -\tfrac{k^2}{4}\left(2 + \tanh^2 k\rho\right) = T_3^3. \end{aligned} \tag{4.1}$$

$T_2^2 = T_3^3$ is the pressure on a cylinder of radius ρ and T_1^1 is the pressure transverse to it. Therefore, we have an anisotropic fluid with energy density positive for $0 \leq \rho < \frac{1}{k}\tanh^{-1}\frac{2}{\sqrt{7}}$ and negative for $\rho \geq \frac{1}{k}\tanh^{-1}\frac{2}{\sqrt{7}}$. However, with a cosmological constant greater than $\frac{3}{4}k^2$, the energy density becomes positive definite.

N2.

$$ds^2 = \sqrt{A\rho + B}\left(dt^2 - a^2 d\theta^2 - dz^2\right) - d\rho^2,$$

A and B are constants. We must have $A\rho + B > 0$, because it becomes singular at $A\rho + B = 0$. It admits 10 RCs (Case AIIb1(ii)) and 6 KVs. This is a case of non-trivial RCs. For this the energy momentum tensor (Eqs. 2.3) becomes

$$\begin{aligned} T_0^0 &= \tfrac{5A^2}{4(A\rho+B)^2} = T_2^2 = T_3^3, \\ T_1^1 &= -\tfrac{3A^2}{16(A\rho+B)^2}. \end{aligned} \tag{4.2}$$

It is a spacetime with non-zero cosmological constant and represents an anisotropic inhomogeneous perfect fluid.

N3.

$$ds^2 = (\rho/\rho_0)^{2a}\, dt^2 - d\rho^2 - (\rho/\rho_0)^{2b}\, \alpha^2 d\theta^2 - (\rho/\rho_0)^{2c}\, dz^2,$$

a, b, c, α and ρ_0 are constants. For this metric R_{ab} from Eq. (2.1) are given by

$$
\begin{aligned}
R_0 &= a\rho_0^{-2a}\left(-1+a+b+c\right)\rho^{2a-2},\\
R_1 &= -\left(-a-b-c+a^2+b^2+c^2\right)\rho^{-2},\\
R_2 &= -\alpha^2 b\rho_0^{-2b}\left(-1+a+b+c\right)\rho^{2b-2},\\
R_3 &= -c\rho_0^{-2c}\left(-1+a+b+c\right)\rho^{2c-2}.
\end{aligned}
\tag{4.3}
$$

Now, for the metrics of this form in this section a, b and c must not be such that $a+b+c=1$ or $a+b+c=a^2+b^2+c^2$, as these will make the Ricci tensor degenerate (which is the subject of next section). The energy momentum tensor (Eqs. 2.3) for this metric can be written as

$$
\begin{aligned}
T_0^0 &= \left(b+c-b^2-c^2-bc\right)/\rho^2,\\
T_1^1 &= -\left(ab+bc+ca\right)/\rho^2,\\
T_2^2 &= \left(a+c-a^2-c^2-ac\right)/\rho^2,\\
T_3^3 &= \left(a+b-a^2-b^2-ab\right)/\rho^2.
\end{aligned}
\tag{4.4}
$$

If a, b, $c \neq 0$, 1, it admits 4 RCs (Case AIc2(iii)α). If one takes $b=c\neq a$ with a, $b\neq 0$, 1 one gets 3 KVs, 4 HMs and 5 RCs (Case AIIb1(i)β_1). In this case we have

$$
\begin{aligned}
T_0^0 &= \left(2b-3b^2\right)/\rho^2,\\
T_1^1 &= -\left(2ab+b^2\right)/\rho^2,\\
T_2^2 &= \left(a+b-a^2-b^2-ab\right)/\rho^2 = T_3^3.
\end{aligned}
\tag{4.5}
$$

For the energy density to be positive we must have $0<b<2/3$. It is a perfect fluid spacetime when

$$b = a(a-1)/(a+1),$$

and a non-null electromagnetic field for

$$b = a+1.$$

N4. Taking $a = b = c \neq 0, 1$, in metric (N3) gives a metric admitting 6 KVs, 7 HMs and 10 RCs (Case AIIb1(ii)). Here the energy momentum tensor is

$$\begin{aligned} T_0^0 &= \left(2a - 3a^2\right)/\rho^2 = T_2^2 = T_3^3, \\ T_1^1 &= -3a^2/\rho^2 . \end{aligned} \tag{4.6}$$

Here $0 < a < 2/3$ otherwise it becomes tachyonic. It is singular at $\rho = 0$.

N5. Setting $\mu = \nu$ (in Eq. (1.16)), and solving the constraints on R_i, for Case AIa1(i) gives the metric

$$ds^2 = (coshk\rho)^{-1} dt^2 - d\rho^2 - a^2 cosh^2 k\rho d\theta^2 - (coshk\rho)^{-1} dz^2 ,$$

where k is a constant. The energy momentum tensor for this metric is

$$\begin{aligned} T_0^0 &= -\tfrac{k^2}{4}\left(2 + \tanh^2 k\rho\right) = T_3^3 , \\ T_1^1 &= \tfrac{3k^2}{4} \tanh^2 k\rho , \\ T_2^2 &= \tfrac{k^2}{4}\left(4 - 7\tanh^2 k\rho\right) . \end{aligned} \tag{4.7}$$

This is anisotropic tachyonic fluid. It admits 7 RCs and minimal isometry group and hence is a case of proper RCs.

N6. Taking $a = 1$, $b = c \neq 1$ in metric (N3) gives a metric with 7 RCs (Case BIVb3(ii)γ_2) and 4 isometries. Hence, this is a case of proper RCs. Here

$$\begin{aligned} T_0^0 &= \left(2b - 3b^2\right)/\rho^2, \\ T_1^1 &= -\left(2b + b^2\right)/\rho^2 , \\ T_2^2 &= -b^2/\rho^2 = T_3^3 . \end{aligned} \tag{4.8}$$

So, the energy density is positive for $0 < a < 2/3$. The trace, $T = 0$.

N7. Taking $b = c = 1$, $a \neq 0, 1$, in metric (N3) gives a metric with 4 KVs and 6 RCs (Case AIIb2(ii)α). For this T^a_b is

$$\begin{aligned} T^0_0 &= -1/\rho^2\,, \\ T^1_1 &= -\left(2a+1\right)/\rho^2\,, \\ T^2_2 &= -a^2/\rho^2 = T^3_3\,, \end{aligned} \tag{4.9}$$

and

$$T = -2\left(a^2+a+1\right)/\rho^2\,.$$

The energy density is negative and cannot be made positive by introducing a cosmological constant, therefore, it is unphysical.

N8.

$$ds^2 = e^{\mu}\left(dt^2 - dz^2\right) - d\rho^2 - e^{\lambda}a^2 d\theta^2,\ (\lambda'' \neq 0,\ \mu'' \neq 0)\,.$$

It admits 4 KVs and 4 RCs given by Case AIc1(ii).

N9.

$$ds^2 = e^{\nu}\left(dt^2 - a^2 d\theta^2\right) - d\rho^2 - e^{\mu}dz^2,\ (\nu'' \neq 0,\ \mu'' \neq 0)\,.$$

It admits 4 KVs and 4 RCs given by Case AIc1(i).

N10. Taking $a = b \neq c$; $a,\ c \neq 0, 1$ in metric (N3) gives a metric with 4 KVs and 5 RCs (Case AIc2(i)).

N11. Taking $a = c \neq b$; $a,\ b \neq 0, 1$ in metric (N3) gives a metric with 4 KVs and 5 RCs (Case AIc2(ii)).

N12.

$$ds^2 = e^{\nu}\left(dt^2 - a^2 d\theta^2 - dz^2\right) - d\rho^2,\ (\nu'' \neq 0)\,.$$

It has 6 KVs and 6 RCs given in the Case AIIa(1).

N13.

$$ds^2 = e^{A\rho}\left(dt^2 - a^2 d\theta^2\right) - e^{B\rho}dz^2 - d\rho^2\,,$$

A and B are constants ($A \neq B$). It has 5 KVs and 5 RCs (Case AIc2(i)). It does not seem to have physical significance.

N14.

$$ds^2 = e^{A\rho}dt^2 - d\rho^2 - e^{B\rho}\left(a^2 d\theta^2 + dz^2\right)\,,$$

A and B are constants. It has 5 KVs and 5 RCs (Case AIIb1(i)β). It does not seem to have physical significance.

N15.

$$ds^2 = e^{A\rho}\left(dt^2 - dz^2\right) - d\rho^2 - e^{B\rho}a^2 d\theta^2\,,$$

A and B are constants. It has 5 KVs and 5 RCs (Case AIc2(ii)). It does not seem to have physical significance.

N16.

$$ds^2 = e^{A\rho}dt^2 - d\rho^2 - e^{B\rho}a^2 d\theta^2 - e^{C\rho}dz^2\,,$$

A, B and C are constants. It has 4 KVs and 4 RCs (Case AIc2(iii)α). It appears to be non-physical unless a cosmological constant is introduced.

N17.

$$ds^2 = e^{\nu}dt^2 - d\rho^2 - e^{\mu}\left(a^2 d\theta^2 + dz^2\right)\,,\ (\nu'' \neq 0,\, \mu'' \neq 0)\,.$$

It admits 4 KVs and 4 RCs given by Case AIIa(2).

4.2 Metrics with the Degenerate Ricci tensor

As we have seen that for a degenerate Ricci tensor, the RCs are mostly arbitrary functions of coordinates t, ρ, θ and z. However, in Case II where $R_1 = 0$ and R_i, $i = 0, 2, 3$, are non-zero we obtain RCs of dimensions 10, 5, 4 and 3. Here, we give metrics which have a degenerate Ricci tensor.

D1. If $a = (1 \pm \sqrt{3})/2$, $b = c = 1/2$, in metric (N3), one gets $R_1 = 0$ and R_i are non-zero for $i = 0, 2, 3$. It admits 5 RCs which are given in Case IIBd4(i) and 4 KVs, and therefore, is a case of proper RCs. Case II is the only case of degenerate Ricci tensor which admits RC vectors with finite dimensional Lie algebra. For $a = (1 + \sqrt{3})/2$, we have

$$\begin{aligned} T_0^0 = 1/4\rho^2, T_1^1 &= -\left(3 + 2\sqrt{3}\right)/4\rho^2, \\ T_2^2 = T_3^3 &= -\left(2 + \sqrt{3}\right)/4\rho^2. \end{aligned} \tag{4.10}$$

It is an exact solution of the EFE and represents an anisotropic inhomogeneous fluid.

D2. If $a = b = c = 1$, in metric (N3), one gets $R_1 = 0$ and R_i are non-zero constants for $i = 0, 2, 3$. It admits infinitely many RCs which are given in Case II(A) and 6 KVs, and therefore, is a case of proper RCs. Here

$$\begin{aligned} T_0^0 &= -1/\rho^2 = T_2^2 = T_3^3, \\ T_1^1 &= -3a^2/\rho^2. \end{aligned} \tag{4.11}$$

It is an exact solution of the EFE and represents an anisotropic inhomogeneous perfect fluid.

D3. The conditions of Case IV (where $R_3 = 0$ and R_i are non-zero, otherwise) of the degenerate Ricci tensor are satisfied if we take the Ricci tensor to be of the form

$$\begin{aligned} R_0 &= 2ck^2 \cosh^2 k\rho, \\ R_1 &= -2k^2, \\ R_2 &= -2c \sinh^2 k\rho, \\ R_3 &= 0, \end{aligned} \tag{4.12}$$

c and k being non-zero constants. Now, setting

$$\mu = 0\,, \tag{4.13}$$

and choosing ν, λ and the constants appropriately gives the form of the metric as

$$ds^2 = \left(c + c\sinh^2 k\rho\right) dt^2 - d\rho^2 - \frac{c}{k^2}\sinh^2 k\rho d\theta^2 - dz^2\,,$$

which admits minimal KVs and infinitely many RCs. Taking $c > 0$, $c = b^2$, if the coordinate ρ is transformed according to the relation

$$r = \frac{b}{k}\sinh k\rho\,, \tag{4.14}$$

where $0 \leq r < \infty$, the above metric corresponds to the string solution

$$ds^2 = \left(c + k^2 r^2\right) dt^2 - \left(c + k^2 r^2\right)^{-1} dr^2 - r^2 d\theta^2 - dz^2\,.$$

The string is naked for $c > 0$. If $c < 0$, we put $c = -b^2$ and set

$$r = \frac{b}{k}\cosh k\rho\,, \tag{4.15}$$

in which case r has the range $b/k \leq r < \infty$. The transformation

$$\rho = \frac{1}{k}\ln r\,, \tag{4.16}$$

corresponds to the solution for $c = 0$.

D4.

$$ds^2 = dt^2 - d\rho^2 - \frac{\rho^2}{A^2}a^2 d\theta^2 - dz^2\,.$$

It has 10 KVs and infinitely many RCs (Case XV). This is wrapped Minkowski spacetime.

D5.

$$ds^2 = dt^2 - d\rho^2 - a^2 d\theta^2 - \frac{\rho^2}{A^2} dz^2 \, .$$

It has 10 KVs and the RCs have infinite dimensional Lie algebra (Case XV). It is Bertotti-Robinson like metric.

D6.

$$ds^2 = dt^2 - d\rho^2 - \alpha^2 d\theta^2 - dz^2 \, .$$

It admits 11 HMs and infinitely many RCs (Case XV). It is wrapped Minkowski with zero curvature.

D7.

$$ds^2 = (\rho/\rho_0)^2 \, dt^2 - d\rho^2 - \alpha^2 d\theta^2 - dz^2 \, .$$

It has 11 HMs and the RCs have infinite dimensional Lie algebra (Case XV). It is wrapped Minkowski.

D8.

$$ds^2 = e^{A\rho} \left(dt^2 - dz^2 \right) - d\rho^2 - a^2 d\theta^2 \, ,$$

A is a non-zero constant. It has 7 KVs and infinitely many RCs. Their form is given in Case III. It is anti-Einstein and anisotropic with negative energy.

D9.

$$ds^2 = e^{A\rho} \left(dt^2 - a^2 d\theta^2 \right) - d\rho^2 - dz^2 \, ,$$

A is a non-zero constant. It has 7 KVs and the RCs have infinite dimensional Lie algebra. The form of the RCs is as given in Case IV. It is anti-Einstein and anisotropic with negative energy.

D10.

$$ds^2 = dt^2 - d\rho^2 - e^{A\rho}a^2d\theta^2 - e^{B\rho}dz^2\,,$$

A and B are non-zero constants. It admits infinitely many RCs whose form is given in Case Ib(ii). It admits 7 KVs if $A = B$, and it is anti-Einstein. When $A \neq B$, it admits 4 KVs and is non-physical unless a cosmological constant is introduced.

D11.

$$ds^2 = dt^2 - d\rho^2 - a^2d\theta^2 - e^{A\rho}dz^2\,,$$

$A \neq 0$. It has 6 KVs and the RCs have infinite dimensional Lie algebra of the form given in Case VI. It is anisotropic with negative energy.

D12.

$$ds^2 = dt^2 - d\rho^2 - e^{A\rho}a^2d\theta^2 - dz^2\,,$$

$A \neq 0$. It has 6 KVs and infinitely many RCs (Case VII). It is anisotropic with negative energy.

D13.

$$ds^2 = e^{A\rho}dt^2 - d\rho^2 - a^2d\theta^2 - dz^2\,,$$

$A \neq 0$. It has 6 KVs and infinite dimensional RC algebra (Case X). It has zero energy, zero radial pressure and is isotropic along the cylindrical direction.

D14.

$$ds^2 = cosh^2(A + B\rho)dt^2 - d\rho^2 - \alpha^2d\theta^2 - dz^2\,,$$

A, B and α are constants. It has 6 KVs and the RCs have infinite dimensional Lie algebra (Case X). It is a Bertotti-Robinson-like metric.

D15.

$$ds^2 = dt^2 - d\rho^2 - \alpha^2 d\theta^2 - cosh^2(A + B\rho)dz^2 ,$$

A, B and α are constants. It has 6 KVs and infinitely many RCs of the form given in Case VI. It is a Bertotti-Robinson-like metric.

D16.

$$ds^2 = dt^2 - d\rho^2 - cosh^2(A + B\rho)\alpha^2 d\theta^2 - dz^2 ,$$

A, B and α are constants. It has 6 KVs and infinite dimensional RCs (Case VII). It is Bertotti-Robinson with traceless stress-energy tensor.

D17.

$$ds^2 = (\rho/\rho_0)^2 dt^2 - d\rho^2 - \rho^2 d\theta^2 - dz^2 .$$

It admits 5 HMs and infinitely many RCs (Case IX).

D18.

$$ds^2 = (\rho/\rho_0)^2 dt^2 - d\rho^2 - \alpha^2 d\theta^2 - (\rho/\rho_0)^2 dz^2 .$$

It has 5 HMs and the RC algebras is infinite dimensional (Case VIII).

D19.

$$ds^2 = dt^2 - d\rho^2 - \alpha^2 (\rho/\rho_0)^2 d\theta^2 - (\rho/\rho_0)^2 dz^2 ,$$

α is a constant. It has 5 HMs and infinitely many RCs (Case V). It represents a non-null electromagnetic field.

D20. If $R_1 = 0$ is taken for metric (N3), one gets

$$a + b + c = a^2 + b^2 + c^2 . \qquad (4.17)$$

Now, if the Kasner conditions,

$$a + b + c = a^2 + b^2 + c^2 = 1\,, \tag{4.18}$$

are applied, one gets

$$R_i = 0,\ i = 0, 2, 3\,, \tag{4.19}$$

corresponding to Case XV for infinite dimensional RC algebra. These conditions are satisfied by metrics representing different physical situations [52]. For example, the standard metric for a conical spacetime with a deficit angle

$$ds^2 = dt^2 - d\rho^2 - (1-k)^2 \rho^2 d\theta^2 - dz^2\,,$$

which is a flat metric, or the solution

$$\begin{aligned} ds^2 &= [1 + A\ln(\rho/\rho_0)]\left(dt^2 - dz^2\right) - \\ &\quad [1 + B\ln(\rho/\rho_0)]^{-1/2} (\rho_0/\rho)^2 \left(d\rho^2 + \rho^2 d\theta^2\right)\,, \end{aligned}$$

where A and B are non-zero constant, which may represent a gravitationally collapsed cylindrical matter distribution totally disconnected from the external space. Another solution is

$$ds^2 = [1 + A\ln(\rho/\rho_0)]^2 dt^2 - (\rho_0/\rho)^2 \left(d\rho^2 + \rho^2 d\theta^2\right) - dz^2\,,$$

which is again a flat spacetime with cylindrical topology.

If one sets

$$a = \frac{2m(1+2m)}{4m^2 + 2m + 1}, b = \frac{2m+1}{4m^2 + 2m + 1}\,,\ c = \frac{-2m}{4m^2 + 2m + 1}\,, \tag{4.20}$$

the Kasner conditions are satisfied and with an appropriately chosen ρ_0, one gets the cosmic string solution

$$\begin{aligned} ds^2 &= (1-8m-8\varepsilon)^{\frac{-2}{2(4m^2+2m+1)}}\left(4m^2+2m+1\right)^{2a}\rho^{2a}dt^2 \\ &\quad -d\rho^2-(1-8m-8\varepsilon)^{\frac{4m+2}{2(4m^2+2m+1)}}\left(4m^2+2m+1\right)^{2b}\rho^{2b}d\theta^2 \\ &\quad -(1-8m-8\varepsilon)^{\frac{-4m}{2(4m^2+2m+1)}}\left(4m^2+2m+1\right)^{2c}\rho^{2c}dz^2\,, \end{aligned}$$

which, on redefining the coordinate ρ as

$$r=(1-8m-8\varepsilon)^{\frac{1}{2(4m^2+2m+1)}}\left(4m^2+2m+1\right)^{\frac{1}{4m^2+2m+1}}\rho^{\frac{1}{4m^2+2m+1}}$$

can be cast into the following form given in [50].

$$ds^2=(1-8m-8\varepsilon)^{-1}r^{4m(1+2m)}\left(dt^2-dr^2\right)-r^{4m+2}d\varphi^2-r^{-4m}dz^2.$$

Here m is the mass per specific length and ε is the energy per specific length. If m is replaced by $-\sigma$ in the above definition of the so-called Kasner parameters a, b, c and, the constants are redefined, this metric takes the form

$$\begin{aligned} ds^2 &= a\left(4\sigma^2-2\sigma+1\right)^{2c}\rho^{2c}dt^2-d\rho^2-\left(4\sigma^2-2\sigma+1\right)^{2b}\rho^{2b}d\theta^2 \\ &\quad -\left(4\sigma^2-2\sigma+1\right)^{2a}\rho^{2a}dz^2\,, \end{aligned}$$

which is the Levi-Civita spacetime discussed in [53], written in different coordinates. When $\sigma=0$, the spacetime is flat. When $\sigma=1/2$, it is again flat but for $\sigma=-1/2$, it becomes

$$ds^2=a\,(3)^{-2/3}\rho^{-2/3}dt^2-d\rho^2-(3)^{4/3}\rho^{4/3}d\theta^2-(3)^{4/3}\rho^{4/3}dz^2\,,$$

which is not flat and admits an extra KV. This type of metric can also be obtained by putting $a=-1/3$, $b=c=2/3$, in metric (N3), which corresponds to Einstein-Maxwell field of Petrov type D. It admits 4 HMs and infinitely many RCs.

D21.

$$ds^2 = (\rho/\rho_0)^{2a}\, dt^2 - d\rho^2 - \rho^2 d\theta^2 - dz^2\,,$$

$a \neq 0, 1$ and ρ_0 are constants. It admits 4 HMs and RCs with infinite dimensional Lie algebra (Case IX).

D22.

$$ds^2 = (\rho/\rho_0)^{2a}\, dt^2 - d\rho^2 - \alpha^2 d\theta^2 - (\rho/\rho_0)^2\, dz^2\,,$$

a, α and ρ_0 are constants. It admits 4 HMs and the RC algebra is infinite dimensional (Case VIII).

D23.

$$ds^2 = e^{A\rho} dt^2 - d\rho^2 - a^2 d\theta^2 - e^{B\rho} dz^2\,,$$

$A \neq B$. It has 4 KVs and infinitely many RCs (Case III). It appears to be non-physical unless a cosmological constant is introduced.

D24.

$$ds^2 = e^{A\rho} dt^2 - d\rho^2 - e^{B\rho} a^2 d\theta^2 - dz^2\,,$$

$A \neq B$. It has 4 KVs and infinite dimensional RC algebra (Case IV). It appears to be non-physical unless a cosmological constant is introduced.

In this chapter we have given metrics for the non-degenerate as well as degenerate Ricci tensor with their KVs and RCs, and discussed their physical significance. We have found many spacetimes admitting non-isometric RCs.

Chapter 5

Summary and Conclusion

We have classified cylindrically symmetric static spacetimes according to their RCs. The RC equations have been solved for the non-degenerate as well as the degenerate Ricci tensor. The procedure adopted has given the explicit form of the RC vectors along with constraints on the components of the Ricci tensor. Solving these constraints explicitly has given metrics or in some cases a family of metrics. Using EFE to obtain the stress energy tensor we got the physical interpretation.

For the non-degenerate Ricci tensor, we have found RCs of 10, 7, 6, 5, 4 and 3 dimensions. 10 dimensional RCs are given in Cases AIIb1(ii), AIIb2(i)α, AIIb2(i)β and B(I); 7 RCs in Case BIVb3(ii)γ_2; 6 RCs in Cases AIIa(1), AIIb2(ii)α_1, AIIb2(ii)α_2, BIIb(1) and BIIb(3); 5 RCs in Cases AIa1(i), AIc2(i), AIIb1(i)β and BIVb3(ii)γ_1; 4 RCs in Cases AIa1(ii), AIa2(ii)β, AIa4(i)α_1, AIc1(i), AIc2(iii), AIIa(2), AIIb1(i)α, AIIb2(ii)β, BII(a), BIVa(2), BIVb1(i), BIVb1(ii), BIVb3(i) and BIVb3(ii)α; and minimal RCs in Cases AIa2(i), AIa2(ii)α, AIa4(i)α_2 AIa4(i)β, AIc1(iii), AIc2(iii)β and BIVa(1). It is worth noting that, as the RC equations are invariant under the interchange of any two of the three indices 0, 2, 3, the cases mentioned above are not the only cases, but there are numerous *similar* cases; similar in the sense that the results for those cases can be obtained by interchanging any two of these indices (or coordinates t, θ and z).

For the non-degenerate Ricci tensor, we have 15 cases, but again by using the above symmetry, they reduce to 7. We note that the Cases I, III and IV are *similar*. In the same way Cases V, VIII and IX; Cases VI, VII and X; and Cases XI, XII and XIV are similar. We have found that while the algebras of the RCs for the non-degenerate Ricci tensor are always finite, those for the degenerate cases are mostly, *but not always*, infinite; Case II of the degenerate case has RCs of dimensions 10 (Case II(a)), 5 (Cases IIb(1) and IId4(i)), 4 (Cases IIb(2), IId1(i) and IId(2)) and 3 (Cases IId1(ii) and IId4(ii)). All these results have been summarized in the tables at the end of this conclusion. The Lie algebra structure of all these cases has also been given. For the infinite dimensional cases also, we have solved the RC equations as far as the equations allow, to obtain their form as arbitrary functions of the coordinates.

The comparison of the present classification with the classification of cylindrically symmetric spacetimes according to KVs has given rise to the interesting cases of proper RCs given in Chapter 4. We have found the non-isometric RCs of 10, 7, 6, 5, 4 and 3 dimensions. We have also solved the constraints for the different cases of degenerate and non-degenerate RCs given in the previous chapters, to obtain the metrics. The physical significance has also been given where possible. Taking plane symmetry (locally) as a special case of cylindrical symmetry, the classification of plane symmetric static metrics [35] can be obtained as a special case from this classification.

Some of the general observations on the work done are listed below:

1. The RC equations are invariant under the interchange of any two of the three indices 0, 2 and 3 (or coordinates t, θ and z).

2. $< \partial_t, \partial_\theta, \partial_z >$, translations in t and z, and rotation in θ, is the minimal symmetry for cylindrically symmetric static spacetimes which has the algebra $\mathbb{R} \otimes SO(2) \otimes \mathbb{R}$.

3. The classification of plane symmetric static metrics according to RCs is a special case of the present classification.

4. We saw that in all the subcases of Case AII, where the R_{22} and R_{33} components of Ricci tensor are proportional the minimal symmetry gets *extended* to include $z\partial_\theta - l\theta\partial_z$, rotation in θ and z, as well. Also, $\langle \mathbf{X}_0, \mathbf{X}_1, \mathbf{X}_2, \mathbf{X}_3 \rangle$ is a sub-algebra of all the algebras in that case. This extra symmetry is globally allowed for the plane and not for cylinders.

In the light of the results obtained, we state the following theorems.

Theorem 1 *Cylindrically symmetric static spacetimes with non-degenerate Ricci tensor admit RCs with Lie algebras of dimensions 10, 7, 6, 5, 4 and 3. There are no 8- or 9-dimensional Lie algebras.*

Theorem 2 *For the degenerate Ricci tensor the RCs have infinite dimensional Lie algebras except when $R_{11} = 0$ and R_{ii} $(i = 0, 2, 3)$ are non-zero (Case II in this thesis), the Lie algebras are 10-, 5-, 4- and 3-dimensional.*

Theorem 3 *For the non-degenerate Ricci tensor, if any of the R_{00}, R_{22} or R_{33} components are nonzero constants, the space admits non-isometric RCs.*

Theorem 3 holds for the plane symmetric spacetimes also (in the light of our third observation, above). It would be interesting to see if something similar can be said about spherically symmetric spacetimes as well. The results of this thesis have been summarized in [54].

Now, we provide tables on the following pages for the RCs of finite dimensional Lie algebras, obtained in this thesis .

The Table of Tables for the Ricci Collineations of Cylindrically Symmetric Static Spacetimes				
The Non-Degenerate Ricci Tensor				The Degenerate Ricci Tensor
Case A: $R_0' \neq 0$			Case B: $R_0' = 0$	Case II: $R_1 = 0\,,\, R_0 \neq 0\,,\, R_2 \neq 0\,,\, R_3 \neq 0$
Case A(I): $\left(\frac{R_2}{R_3}\right)' \neq 0$		Case A(II): $\left(\frac{R_2}{R_3}\right)' = 0$		
(a) $R_2' = 0\,,\, R_3' \neq 0$	(c) $R_2' \neq 0\,,\, R_3' \neq 0$			
(b) $R_2' \neq 0\,,\, R_3' = 0$				
Table 1	Table 2	Table 3	Table 4	Table 5

Table 1: The Non-Degenerate Case **A(I)** $R_0' \neq 0$

(I) $\left(\frac{R_2}{R_3}\right)' \neq 0$	(a) $R_2' = 0$, $R_3' \neq 0$	(1) $\alpha = 0,\ \beta = 0$	(i) $\left(\frac{R_0}{R_3}\right)' = 0$			7 RCs (Eqs. 2.31)
			(ii) $\left(\frac{R_0}{R_3}\right)' \neq 0$			4 RCs (Eqs. 2.35)
		(2) $\alpha \neq 0,\ \beta = 0$	(i) $\alpha > 0$			3 RCs (Eqs. 2.37)
			(ii) $\alpha < 0$	(α) $\left(\frac{R_0}{R_3}\right)' \neq 0$		3 RCs
				(β) $\left(\frac{R_0}{R_3}\right)' = 0$		4 RCs (Eqs. 2.53)
		(3) $\alpha = 0,\ \beta \neq 0$				Similar to (2)
		(4) $\alpha \neq 0,\ \beta \neq 0$	(i) $\alpha > 0,\ \beta > 0$	(α) $\beta \int \frac{\sqrt{R_1}}{R_0} d\rho + \frac{R_3'}{2\sqrt{R_1}} \neq 0$	(α_1) $\left(\frac{R_0}{R_3}\right)' = 0$	4 RCs (Eqs. 2.64)
					(α_2) $\left(\frac{R_0}{R_3}\right)' \neq 0$	3 RCs
				(β) $\beta \int \frac{\sqrt{R_1}}{R_0} d\rho + \frac{R_3'}{2\sqrt{R_1}} = 0$		3 RCs
			(ii) $\alpha > 0,\ \beta < 0$			Similar to (i)
			(iii) $\alpha < 0,\ \beta > 0$			Similar to (i)
			(iv) $\alpha < 0,\ \beta < 0$			Similar to (i)
	(b) $R_2' \neq 0$, $R_3' = 0$					Similar to (a)

Definitions

$$\alpha = \frac{R_0}{\sqrt{R_1}} \left(\frac{R_0'}{2R_0\sqrt{R_1}}\right)' \qquad k_1 = -\frac{R_0'}{2R_0\sqrt{R_1}}$$

$$\beta = \frac{R_3}{\sqrt{R_1}} \left(\frac{R_3'}{2R_3\sqrt{R_1}}\right)' \qquad k_2 = -\frac{R_3'}{2R_3\sqrt{R_1}}$$

Table 2: The Non-Degenerate Case A(I) $\quad R_0' \neq 0 \quad$ (continued)

(I) $\left(\frac{R_2}{R_3}\right)' \neq 0$	(c) $R_2' \neq 0$, $R_3' \neq 0$	(1) $\left(\frac{R_2'}{2R_2\sqrt{R_1}}\right)' \neq 0$, $\left(\frac{R_3'}{2R_3\sqrt{R_1}}\right)' \neq 0$	(i) $\left(\frac{R_2}{R_0}\right)' = 0$, $\left(\frac{R_3}{R_0}\right)' \neq 0$		4 RCs (Eqs. 2.78)
			(ii) $\left(\frac{R_2}{R_0}\right)' \neq 0$, $\left(\frac{R_3}{R_0}\right)' = 0$		Similar to (i)
			(iii) $\left(\frac{R_2}{R_0}\right)' \neq 0$, $\left(\frac{R_3}{R_0}\right)' \neq 0$		3 RCS
		(2) $\left(\frac{R_2'}{2R_2\sqrt{R_1}}\right)' = 0$, $\left(\frac{R_3'}{2R_3\sqrt{R_1}}\right)' = 0$	(i) $\left(\frac{R_2}{R_0}\right)' = 0$, $\left(\frac{R_3}{R_0}\right)' \neq 0$		5 RCs (Eqs. 2.90)
			(ii) $\left(\frac{R_2}{R_0}\right)' \neq 0$, $\left(\frac{R_3}{R_0}\right)' = 0$		Similar to (i)
			(iii) $\left(\frac{R_2}{R_0}\right)' \neq 0$, $\left(\frac{R_3}{R_0}\right)' \neq 0$	(α) $\left(\frac{R_0'}{2R_0\sqrt{R_1}}\right)' = 0$	4 RCs (Eqs. 2.93)
				(β) $\left(\frac{R_0'}{2R_0\sqrt{R_1}}\right)' \neq 0$	3 RCs

Table 3: The Non-Degenerate Case A(II) $\quad R_0' \neq 0$

(II) $\left(\frac{R_2}{R_3}\right)' = 0$	(a) $\left(\frac{R_2'}{2R_2\sqrt{R_1}}\right)' \neq 0$	(1) $\left(\sqrt{\frac{R_0}{R_2}}\right)' = 0$				6 RCs (Eqs. 2.113)
		(2) $\left(\sqrt{\frac{R_0}{R_2}}\right)' \neq 0$				4 RCs (Eqs. 2.114)
	(b) $\left(\frac{R_2'}{2R_2\sqrt{R_1}}\right)' = 0$	(1) $R_2' \neq 0$	(i) $\left(\frac{R_2}{R_0}\right)' \neq 0$	(α) $\left(\frac{R_0'}{2R_0\sqrt{R_1}}\right)' \neq 0$		4 RCs (Eqs. 2.116)
				(β) $\left(\frac{R_0'}{2R_0\sqrt{R_1}}\right)' = 0$		5 RCs (Eqs. 2.117)
			(ii) $\left(\frac{R_2}{R_0}\right)' = 0$			10 RCs (Eqs. 2.118)
		(2) $R_2' = 0$,	(i) $\left(\frac{(\sqrt{R_0})'}{\sqrt{R_1}}\right)' = 0$			10 RCs (Eqs. 2.119)
			(ii) $\left(\frac{(\sqrt{R_0})'}{\sqrt{R_1}}\right)' \neq 0$	(α) $\left[\frac{R_0}{2\sqrt{R_1}}\left(\frac{R_0'}{R_0\sqrt{R_1}}\right)'\right]' = 0$	(α_1) $\eta = 0$	6 RCs (Eqs. 2.120)
					(α_2) $\eta \neq 0$	6 RCs (Eqs. 2.121)
				(β) $\left[\frac{R_0}{2\sqrt{R_1}}\left(\frac{R_0'}{R_0\sqrt{R_1}}\right)'\right]' \neq 0$		4 RCs (Eqs. 2.122)

Definitions $\quad \eta = \frac{R_0}{2\sqrt{R_1}}\left(\frac{R_0'}{R_0\sqrt{R_1}}\right)'$

Table 4: The Non-Degenerate Case B $\quad R_0' = 0$

(I) $R_2' = 0\,,\ R_3' = 0$						10 RCs (Eqs. 2.130)
(II) $R_2' = 0\,,\ R_3' \neq 0$	(a) $\left[\frac{R_3}{\sqrt{R_1}}\left(\frac{R_3'}{2R_3\sqrt{R_1}}\right)'\right]' \neq 0$					4 RCs (Eqs. 2.139)
	(b) $\left[\frac{R_3}{\sqrt{R_1}}\left(\frac{R_3'}{2R_3\sqrt{R_1}}\right)'\right]' = 0$	(1) $k_1 > 0$				6 RCs (Eqs. 2.142)
		(2) $k_1 < 0$				Similar to (1)
		(3) $k_1 = 0$				6 RCs (Eqs. 2.144)
(III) $R_2' \neq 0\,,\ R_3' = 0$						Similar to (II)
(IV) $R_2' \neq 0\,,\ R_3' \neq 0$	(a) $\left[\frac{R_2}{\sqrt{R_1}}\left(\frac{R_2'}{2R_2\sqrt{R_1}}\right)'\right]' \neq 0$	(1) $\left(\frac{R_2}{R_3}\right)' \neq 0$				3 RCs
		(2) $\left(\frac{R_2}{R_3}\right)' = 0$				4 RCs (Eqs. 2.153)
	(b) $\left[\frac{R_2}{\sqrt{R_1}}\left(\frac{R_2'}{2R_2\sqrt{R_1}}\right)'\right]' = 0$	(1) $k_3 > 0$	(i) $\left[\frac{R_3}{\sqrt{R_1}}\left(\frac{R_3'}{2R_3\sqrt{R_1}}\right)'\right]' \neq 0$			4 RCs (Eqs. 2.158)
			(ii) $\left[\frac{R_3}{\sqrt{R_1}}\left(\frac{R_3'}{2R_3\sqrt{R_1}}\right)'\right]' = 0$			4 RCs (Eqs. 2.161)
		(2) $k_3 < 0$				Similar to (1)
		(3) $k_3 = 0$	(i) $\left[\frac{R_3}{\sqrt{R_1}}\left(\frac{R_3'}{2R_3\sqrt{R_1}}\right)'\right]' \neq 0$			4 RCs (Eqs. 2.158)
			(ii) $\left[\frac{R_3}{\sqrt{R_1}}\left(\frac{R_3'}{2R_3\sqrt{R_1}}\right)'\right]' = 0$	(α) $k_1 > 0$		4 RCs (Eqs. 2.161)
				(β) $k_1 < 0$		Similar to (2)
				(γ) $k_1 = 0$	(γ_1) $\left(\frac{R_3}{R_2}\right)' \neq 0$	4 RCs (Eqs. 2.166)
					(γ_2) $\left(\frac{R_3}{R_2}\right)' = 0$	7 RCs (Eqs. 2.167)

Definitions	$k_1 = \frac{R_3}{\sqrt{R_1}}\left(\frac{R_3'}{2R_3\sqrt{R_1}}\right)'$ $k_3 = \frac{R_2}{\sqrt{R_1}}\left(\frac{R_2'}{2R_2\sqrt{R_1}}\right)'$

Table 5: The Degenerate Case II $\qquad R_1 = 0\,,\, R_0 \neq 0\,,\, R_2 \neq 0\,,\, R_3 \neq 0$

(A) $R_0' = 0$	$R_2' \neq 0\,,\, R_3' \neq 0$	(1) $\left(\frac{R_2' R_3}{R_2 R_3'}\right)' = 0$		4 RCs (Eqs. 2.188)
		(2) $\left(\frac{R_2' R_3}{R_2 R_3'}\right)' \neq 0$		3 RCs
	Otherwise			Infinitely many RCs
(B) $R_0' \neq 0$	(a) $\left(\frac{R_0}{R_2}\right)' = 0, \left(\frac{R_0}{R_3}\right)' = 0$			10 RCs (Eqs. 2.207)
	(b) $\left(\frac{R_0}{R_2}\right)' = 0, \left(\frac{R_0}{R_3}\right)' \neq 0$	(1) $\left(\frac{R_3' R_0}{R_3 R_0'}\right)' = 0$		5 RCs (Eqs. 2.220)
		(2) $\left(\frac{R_3' R_0}{R_3 R_0'}\right)' \neq 0$		4 RCs (Eqs. 2.231)
	(c) $\left(\frac{R_0}{R_2}\right)' \neq 0, \left(\frac{R_0}{R_3}\right)' = 0$			Similar to (b)
	(d) $\left(\frac{R_0}{R_2}\right)' \neq 0, \left(\frac{R_0}{R_3}\right)' \neq 0$	(1) $\left(\frac{R_0' R_2}{R_0 R_2'}\right)' \neq 0\,,\, \left(\frac{R_0' R_3}{R_0 R_3'}\right)' \neq 0$	(i) $\left(\frac{R_2}{R_3}\right)' = 0$	4 RCs (Eqs. 2.245)
			(ii) $\left(\frac{R_2}{R_3}\right)' \neq 0$	3 RCs
		(2) $\left(\frac{R_0' R_2}{R_0 R_2'}\right)' \neq 0\,,\, \left(\frac{R_0' R_3}{R_0 R_3'}\right)' = 0$		4 RCs (Eqs. 2.245)
		(3) $\left(\frac{R_0' R_2}{R_0 R_2'}\right)' = 0\,,\, \left(\frac{R_0' R_3}{R_0 R_3'}\right)' \neq 0$		Similar to (2)
		(4) $\left(\frac{R_0' R_2}{R_0 R_2'}\right)' = 0\,,\, \left(\frac{R_0' R_3}{R_0 R_3'}\right)' = 0$	(i) $\left(\frac{R_2}{R_3}\right)' = 0$	5 RCs (Eqs. 2.258)
			(ii) $\left(\frac{R_2}{R_3}\right)' \neq 0$	4 RCs

Appendix A

Symmetries in General Relativity

We briefly define here spacetime symmetries discussed in the literature on GR and the relation between these symmetries is described in the inclusion diagram, Figure 1 (taken from [8]).

1. Weyl Projective Collineation (WPC)

$$\pounds_\xi W^a_{bcd} = 0 ,$$

where W^a_{bcd} is the Weyl projective curvature tensor given by

$$W^a_{bcd} = R^a_{bcd} - (n-1)^{-1} \left(\delta^a_d R_{bc} - \delta^a_c R_{bd}\right) ,$$

n being the dimension of the space.

2. Projective Collineation (PC)

$$\pounds_\xi \Pi^a_{bc} = 0 ,$$

where, the projective connection

$$\Pi^a_{bc} = \Gamma^a_{bc} - \frac{1}{n+1}\left(\delta^a_b \Gamma^d_{dc} + \delta^a_c \Gamma^d_{db}\right) .$$

3. Special Projective Collineation (SPC)

$$\mathcal{L}_\xi \Gamma^a_{bc} = \delta^a_b \phi_{;c} + \delta^a_c \phi_{;b} \, , \, \phi_{;bc} = 0 \, .$$

4. Ricci Collineation (RC)

$$\mathcal{L}_\xi R_{ab} = 0 \, .$$

5. Curvature Collineation (CC)

$$\mathcal{L}_\xi R^a_{bcd} = 0 \, .$$

6. Special Curvature Collineation (SCC)

$$(\mathcal{L}_\xi \Gamma^a_{bc})_{;d} = 0 \, .$$

7. Affine Collineation (AC)

$$\mathcal{L}_\xi \Gamma^a_{bc} = 0 \, .$$

8. Homothetic Motion (HM)

$$\mathcal{L}_\xi g_{ab} = 2\phi g_{ab} \, , \, \phi \text{ is a constant.}$$

9. Killing Vector (KV)

$$\mathcal{L}_\xi g_{ab} = 0 \, .$$

10. Special Conformal Collineation (S Conf C)

$$\mathcal{L}_\xi \Gamma^a_{bc} = \delta^a_b \sigma_{;c} + \delta^a_c \sigma_{;b} - g_{bc} g^{ad} \sigma_{;d} \, , \, \sigma_{;bc} = 0 \, .$$

11. Special Conformal Motion (S Conf M)

$$\pounds_\xi g_{ab} = 2\sigma g_{ab}\,,\ \sigma_{;bc} = 0\,.$$

12. Weyl Conformal Motion (W Conf C)

$$\pounds_\xi C^a_{bcd} = 0\,,$$

where C^a_{bcd} is the conformal curvature tensor.

13. Conformal Collineation (Conf C)

$$\pounds_\xi \Gamma^a_{bc} = \delta^a_b \sigma_{;c} + \delta^a_c \sigma_{;b} - g_{bc} g^{ad} \sigma_{;d}\,.$$

14. Conformal Motion (Conf M)

$$\pounds_\xi g_{ab} = 2\sigma g_{ab}\,.$$

15. Null Geodesic Collineation (NC)

$$\pounds_\xi \Gamma^a_{bc} = g_{bc} g^{ad} \psi_{;d}\,.$$

16. Special Null Geodesic Collineation (SNC)

$$\pounds_\xi \Gamma^a_{bc} = g_{bc} g^{ad} \psi_{;d}\,,\ \psi_{;bc} = 0\,.$$

17. Contracted Ricci Collineation (CRC)

$$g^{ab} \pounds_\xi R g_{ab} = 0\,.$$

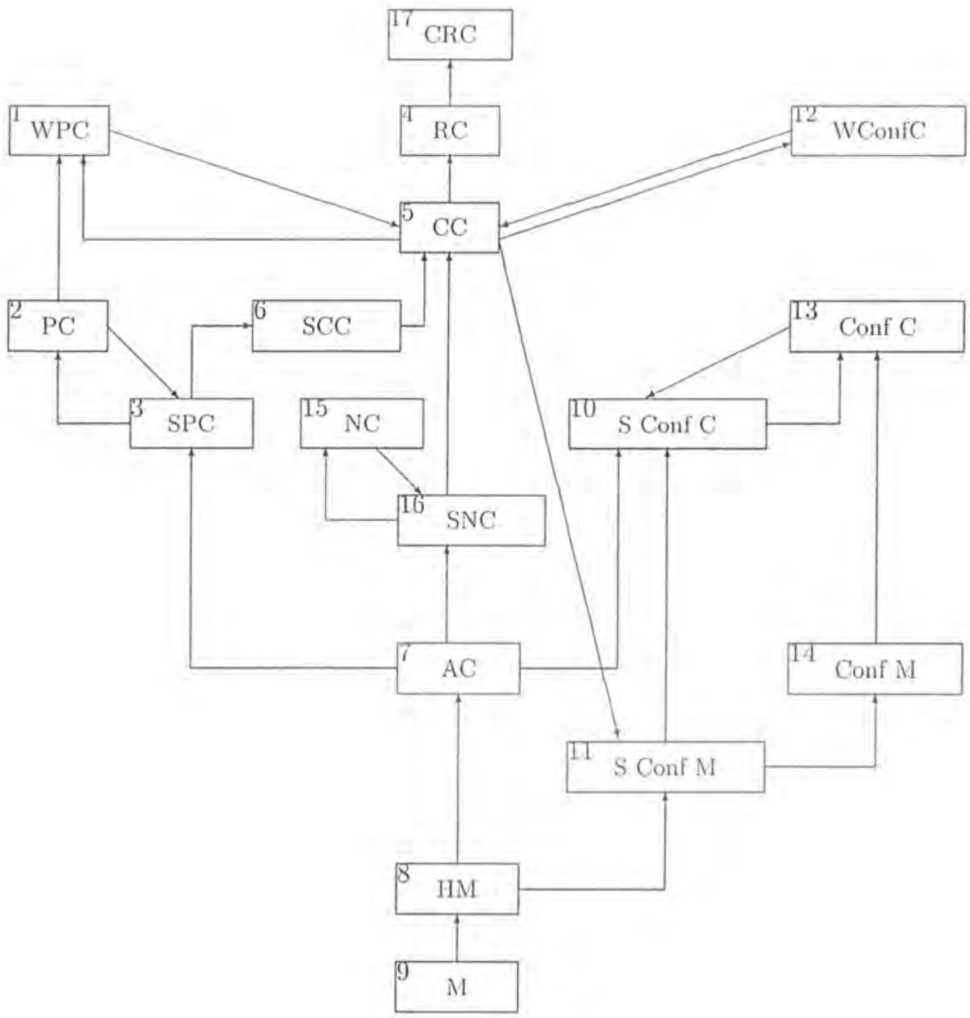

Fig.1. Relations between symmetries.
The leant arrows relate symmetries for which: $R_{ij} = 0$.

Appendix B

Ricci Collineations for the Non-Degenerate Ricci Tensor

In this appendix, we give the details of the calculations done to get the RCs given in Chapter 2.

B.1 Case A: $R_0' \neq 0$

Here we give the calculations in detail for the following cases of non-degenerate Ricci tensor for which only the results are given in Section 2.2 of the main text.

Case AIa2(i) $\alpha > 0, \quad \beta = 0$

Here $\frac{R_3'}{2R_3\sqrt{R_1}}$ =(constant) $= k$ (say) and Eq. (2.18) gives

$$A_1\left(t_1 z\right) = B_1\left(z\right)\cosh\sqrt{\alpha}t + B_2\left(z\right)\sinh\sqrt{\alpha}t\,, \tag{B.1}$$

Therefore, Eq. (2.11) on integration yields

$$B^0 = -\frac{R_0'}{2R_0\sqrt{R_1}\sqrt{\alpha}}\left[B_1\left(z\right)\sinh\sqrt{\alpha}t + B_2\left(z\right)\cosh\sqrt{\alpha}t\right] + A_2\left(\rho, z\right)\,. \tag{B.2}$$

Eq. (2.12) on integration with respect to ρ gives

$$A_2(\rho,z) = \left[\frac{1}{\sqrt{\alpha}}\frac{R_0'}{2R_0\sqrt{R_1}} - \sqrt{\alpha}\int\frac{\sqrt{R_1}}{R_0}d\rho\right]\left[B_1(z)\sinh\sqrt{\alpha}t + B_2(z)\cosh\sqrt{\alpha}t\right] + B_3(z)\ , \tag{B.3}$$

so that Eq. (B.2) becomes

$$B^0 = -\sqrt{\alpha}\left[B_1(z)\sinh\sqrt{\alpha}t + B_2(z)\cosh\sqrt{\alpha}t\right]\int\frac{\sqrt{R_1}}{R_0}d\rho + B_3(z)\ . \tag{B.4}$$

Now, Eq. (2.13) on integration with respect to ρ gives

$$B^3 = -\left[B_{1,3}(z)\cosh\sqrt{\alpha}t + B_{2,3}(z)\sinh\sqrt{\alpha}t\right]\int\frac{\sqrt{R_1}}{R3}d\rho + A_3(t,z)\ . \tag{B.5}$$

Therefore, Eq. (2.14) on integration with respect to z yields

$$\begin{aligned} A_3(t,z) &= \left[B_{1,3}(z)\cosh\sqrt{\alpha}t + B_{2,3}(z)\sinh\sqrt{\alpha}t\right]\int\frac{\sqrt{R_1}}{R_3}d\rho \\ &\quad -k\left[\cosh\sqrt{\alpha}t\int B_1(z)\,dz + \sinh\sqrt{\alpha}t\int B_2(z)\,dz\right] + A_4(t)\ , \end{aligned} \tag{B.6}$$

so that from Eq. (B.5) we have

$$B^3 = -k_1\left[\cosh\sqrt{\alpha}t\int B_1(z)\,dz + \sinh\sqrt{\alpha}t\int B_2(z)\,dz\right] + A_4(t)\ , \tag{B.7}$$

and from Eqs. (2.9) and (2.10) we see that

$$B^2 = c_2\ , \tag{B.8}$$

$$B^1 = \frac{1}{\sqrt{R_1}}\left[B_1(z)\cosh\sqrt{\alpha}t + B_2(z)\sinh\sqrt{\alpha}t\right]\ . \tag{B.9}$$

Now, from Eq. (2.5) for $a=1$, $b=3$, we see that $B^1_{,3}=0$ (because $B^3_{,1}=0$). This means that

$$B_{1,3}(z) = 0 = B_{2,3}(z)\ , \tag{B.10}$$

which implies

$$B_1(z) = c_4 \quad \text{and} \quad B_2(z) = c_5 \,. \tag{B.11}$$

Therefore, Eqs. (B.2), (B.9) and (B.7) can be written as

$$B^0 = -\sqrt{\alpha}\left(c_4 \sinh\sqrt{\alpha}t + c_5 \cosh\sqrt{\alpha}t\right)\int \frac{\sqrt{R_1}}{R_0}d\rho + B_3(z) \,, \tag{B.12}$$

$$B^1 = \frac{1}{\sqrt{R_1}}\left[c_4 \cosh\sqrt{\alpha}t + c_5 \sinh\sqrt{\alpha}t\right] , \tag{B.13}$$

$$B^3 = -k\left[c_4 z \cosh\sqrt{\alpha}t + c_5 z \sinh\sqrt{\alpha}t\right] + A_4(t) \,. \tag{B.14}$$

Putting these values in Eq. (2.5) for $a = 0$, $b = 3$, we get

$$\frac{R_0}{R_3}B_{3,3}(z) - \sqrt{\alpha}k\left(c_4 z \sinh\sqrt{\alpha}t + c_5 z \cosh\sqrt{\alpha}t\right) + \dot{A}_4(t) = 0 \,. \tag{B.15}$$

This implies

$$B_{3,3}(z) = 0 \,, \tag{B.16}$$

$$\dot{A}_4(t) = 0 \,, \tag{B.17}$$

$$c_4 = c_5 = 0 \,, \tag{B.18}$$

or

$$B_3(z) = c_1, \quad A_4(t) = c_3 \,. \tag{B.19}$$

So, we get the solution as given in Eqs. (2.37).

Case AIa2(ii) $\qquad \alpha < 0 \quad , \quad \beta = 0$

In this case Eq. (2.18) yields

$$A_1(t,z) = B_1(z)\cos\sqrt{\alpha}t + B_2(z)\sin\sqrt{\alpha}t\,. \tag{B.20}$$

So, Eq. (2.11) on integration with respect to t gives

$$B^0 = -\frac{R_0'}{2R_0\sqrt{R_1}\sqrt{\alpha}}\left[B_1(z)\sin\sqrt{\alpha}t - B_2(z)\cos\sqrt{\alpha}t\right] + A_2(\rho,z)\,, \tag{B.21}$$

where $A_2(\rho,z)$ is a function of integration. But integrating Eq. (2.12) with respect to ρ yields

$$A_2(\rho,z) = \left[\sqrt{\alpha}\int\frac{\sqrt{R_1}}{R_0}d\rho + \frac{R_0'}{2R_0\sqrt{R_1}\sqrt{\alpha}}\right]\left[B_1(z)\sin\sqrt{\alpha}t - B_2(z)\cos\sqrt{\alpha}t\right] + B_3(z)\,. \tag{B.22}$$

Eq. (2.39) becomes

$$B^0 = \sqrt{\alpha}\left[B_1(z)\sin\sqrt{\alpha}t - B_2(z)\cos\sqrt{\alpha}t\right]\int\frac{\sqrt{R_1}}{R_0}d\rho + B_3(z)\,. \tag{B.23}$$

Now, from Eq. (2.13) we have on integration

$$B^3 = -\left[B_{1,3}(z)\cos\sqrt{\alpha}t + B_{2,3}(z)\sin\sqrt{\alpha}t\right] + \int\frac{\sqrt{R_1}}{R_3}d\rho + A_3(t,z)\,. \tag{B.24}$$

Using this in Eq. (2.14) we get on integration with respect to z

$$A_3(t,z) = \left[B_{1,3}(z)\cos\sqrt{\alpha}t + B_{2,3}(z)\sin\sqrt{\alpha}t\right]\int\frac{\sqrt{R_1}}{R_3}d\rho \tag{B.25}$$

$$+k\left[\cos\sqrt{\alpha}t\int B_1(z)\,dz + \sin\sqrt{\alpha}t\int B_2(z)\,dz\right] + A_4(t)\,. \tag{B.26}$$

Therefore, Eq. (B.24) becomes

$$B^3 = k\left[\cos\sqrt{\alpha}t\int B_1(z)\,dz + \sin\sqrt{\alpha}t\int B_2(z)\,dz\right] + A_4(t)\,, \tag{B.27}$$

and from Eq. (2.10) we have

$$B^1 = \frac{1}{\sqrt{R_1}}\left[B_1(z)\cos\sqrt{\alpha}t + B_2(z)\sin\sqrt{\alpha}t\right]\,, \tag{B.28}$$

$$B^2 = c_2 \,. \tag{B.29}$$

Now, from Eq. (B.27), clearly $B^3_{,1} = 0$. Therefore, Eq. (2.5) for $a = 1$, $b = 3$,implies $B^1_{,3} = 0$. So, Eq. (B.28) yields

$$B_1(z) = c_4 \,,\ B_2(z) = c_5 \,. \tag{B.30}$$

So, Eqs. (B.23), (B.28) and (B.27) become

$$B^0 = \sqrt{\alpha}\left[c_4 \sin\sqrt{\alpha}t - c_5 \cos\sqrt{\alpha}t\right]\int \frac{\sqrt{R_1}}{R_0}d\rho + B_3(z) \,, \tag{B.31}$$

$$B^1 = \frac{1}{\sqrt{R_1}}\left(c_4 \cos\sqrt{\alpha}t + c_5 \sin\sqrt{\alpha}t\right) \,, \tag{B.32}$$

$$B^2 = c_2 \,, \tag{B.33}$$

$$B^3 = k\left(c_4 z \cos\sqrt{\alpha}t + c_5 z \sin\sqrt{\alpha}t\right) + A_4(t) \,. \tag{B.34}$$

Now, we evaluate $A_4(t)$ and $B_3(z)$. Eq. (2.5) for $a = 0$, $b = 3$, in this case gives

$$\frac{R_0}{R_3}B_{3,3}(z) + k\sqrt{\alpha}\left(c_4 z \sin\sqrt{\alpha}t - c_5 z \cos\sqrt{\alpha}t\right) + \dot{A}_4(t) = 0 \,. \tag{B.35}$$

This implies that $c_4 = 0 = c_5$. This again gives rise to two cases.

Case AIa2(ii)α $\quad \left(\frac{R_0}{R_3}\right)' \neq 0$

Here we have

$$B_{3,3}(z) = 0 = \dot{A}_4(t) \,, \tag{B.36}$$

or

$$B_3(z) = c_1 \,, \quad A_4(t) = c_3 \,, \tag{B.37}$$

and we have the result (from Eqs. (2.47)-(2.49)) as the minimal RCs.

Case AIa2(ii)β $\quad\left(\frac{R_0}{R_3}\right)'=0$

In this case we put $\frac{R_0}{R_3}$ =constant $=-\gamma$ (say) . Therefore, from Eq. (B.35)

$$\gamma B_{3,3}(z)=-\dot{A}_4(t)=c_4\,(\text{say})\,, \tag{B.38}$$

or

$$B_3=\frac{c_4 z}{\gamma}+c_1\,, \tag{B.39}$$

$$A_4(t)=-c_4 t+c_3\,. \tag{B.40}$$

Therefore, Eqs. (2.47)-(2.49) give the RCs given in Eqs. (2.53).

Case AIa(3) $\quad\alpha=0\,,\quad\beta\neq 0$

We obtain this case just by interchanging the role of t and z in the previous case, Case AIa(2).

Case AIa(4) $\quad\alpha\neq 0,\quad\beta\neq 0$

Now α and β can be positive or negative which gives rise to four further cases depending on whether one or both of these constants are greater than or less than zero. Let us discuss them.

Case AIa4(i) $\quad\alpha>0\,,\quad\beta>0$

The solution for Eq. (2.18) in this case is

$$A_1(t,z)=B_1(z)\cosh\sqrt{\alpha}t+B_2(z)\sinh\sqrt{\alpha}t\,. \tag{B.41}$$

Using this in Eq. (2.19) implies that

$$B_{1,33}(z)-\beta B_1(z)=0\,, \tag{B.42}$$

$$B_{2,33}(z)-\beta B_2(z)=0\,, \tag{B.43}$$

which can be solved to give

$$B_1(z) = c_2 \cosh\sqrt{\beta}z + c_3 \sinh\sqrt{\beta}z\,, \tag{B.44}$$

$$B_2(z) = c_4 \cosh\sqrt{\beta}z + c_5 \sinh\sqrt{\beta}z\,, \tag{B.45}$$

so that Eq. (B.41) becomes

$$\begin{aligned} A_1(t,z) &= \left(c_2 \cosh\sqrt{\beta}z + c_3 \sinh\sqrt{\beta}z\right)\cosh\sqrt{\alpha}t \\ &\quad + \left(c_4 \cosh\sqrt{\beta}z + c_5 \sinh\sqrt{\beta}z\right)\sinh\sqrt{\alpha}t\,. \end{aligned} \tag{B.46}$$

Now, Eq. (2.11) can be written as

$$\begin{aligned} B^0 &= -\frac{R_0'}{2R_0\sqrt{R_1}\sqrt{\alpha}}\left[\left(c_2 \cosh\sqrt{\beta}z + c_3 \sinh\sqrt{\beta}z\right)\sinh\sqrt{\alpha}t \right. \\ &\quad \left. + \left(c_4 \cosh\sqrt{\beta}z + c_5 \sinh\sqrt{\beta}z\right)\cosh\sqrt{\alpha}t\right] + A_2(\rho,z)\,, \end{aligned} \tag{B.47}$$

and from Eq. (2.12) on integration with respect to ρ we get

$$\begin{aligned} A_2(\rho,z) &= \frac{R_0'}{2R_0\sqrt{R_1}\sqrt{\alpha}}\left[\left(c_2 \cosh\sqrt{\beta}z + c_3 \sinh\sqrt{\beta}z\right)\sinh\sqrt{\alpha}t \right. \\ &\quad \left. + \left(c_4 \cosh\sqrt{\beta}z + c_5 \sinh\sqrt{\beta}z\right)\cosh\sqrt{\alpha}t\right] \\ &\quad -\sqrt{\alpha}\int\frac{\sqrt{R_1}}{R_0}d\rho\left[\left(c_2 \cosh\sqrt{\beta}z + c_3 \sinh\sqrt{\beta}z\right)\sinh\sqrt{\alpha}t \right. \\ &\quad \left. + \left(c_4 \cosh\sqrt{\beta}z + c_5 \sinh\sqrt{\beta}z\right)\cosh\sqrt{\alpha}t\right] + B_3(z)\,. \end{aligned} \tag{B.48}$$

So, Eq. (B.47) becomes

$$B^0 = -\sqrt{\alpha}\left[\left(c_2 \cosh\sqrt{\beta}z + c_3 \sinh\sqrt{\beta}z\right)\sinh\sqrt{\alpha}t\right.$$

$$+ \left(c_4 \cosh\sqrt{\beta}z + c_5 \sinh\sqrt{\beta}z\right) \cosh\sqrt{\alpha}t\Big] \int \frac{\sqrt{R_1}}{R_0} d\rho + B_3(z) \,. \tag{B.49}$$

Now, Eq. (2.13) can be written as

$$\begin{aligned} B^3 &= -\sqrt{\beta}\Big[\left(c_2 \sinh\sqrt{\beta}z + c_3 \cosh\sqrt{\beta}z\right) \cosh\sqrt{\alpha}t \\ &+ \left(c_4 \sinh\sqrt{\beta}z + c_5 \cosh\sqrt{\beta}z\right) \sinh\sqrt{\alpha}t\Big] \int \frac{\sqrt{R_1}}{R_3} d\rho + A_3(t,z) \,, \end{aligned} \tag{B.50}$$

so that Eq. (2.14) becomes

$$\begin{aligned} A_3(t,z) &= -\frac{R_3'}{2R_3\sqrt{R_1}\sqrt{\beta}}\Big[\left(c_2 \sinh\sqrt{\beta}z + c_3 \cosh\sqrt{\beta}z\right) \cosh\sqrt{\alpha}t \\ &+ \left(c_4 \sinh\sqrt{\beta}z + c_5 \cosh\sqrt{\beta}z\right) \sinh\sqrt{\alpha}t\Big] \\ &+ \Big[\left(c_2 \sinh\sqrt{\beta}z + c_3 \cosh\sqrt{\beta}z\right) \cosh\sqrt{\alpha}t \\ &+ \left(c_4 \sinh\sqrt{\beta}z + c_5 \cosh\sqrt{\beta}z\right) \sinh\sqrt{\alpha}t\Big] \int \frac{\sqrt{R_1}}{R_3} d\rho + A_4(t) \,. \end{aligned} \tag{B.51}$$

Substituting from Eq. (B.51) into Eq. (B.50) yields

$$\begin{aligned} B^3 &= -\frac{R_3'}{2R_3\sqrt{R_1}\sqrt{\beta}}\Big[\left(c_2 \sinh\sqrt{\beta}z + c_3 \cosh\sqrt{\beta}z\right) \cosh\sqrt{\alpha}t \\ &+ \left(c_4 \sinh\sqrt{\beta}z + c_5 \cosh\sqrt{\beta}z\right) \sinh\sqrt{\alpha}t\Big] + A_4(t) \,, \end{aligned} \tag{B.52}$$

and from Eq. (2.10)

$$\begin{aligned} B^1 &= \frac{1}{\sqrt{R_1}}\Big[\left(c_2 \cosh\sqrt{\beta}z + c_3 \sinh\sqrt{\beta}z\right) \cosh\sqrt{\alpha}t \\ &+ \left(c_4 \cosh\sqrt{\beta}z + c_5 \sinh\sqrt{\beta}z\right) \sinh\sqrt{\alpha}t\Big] \,. \end{aligned} \tag{B.53}$$

Now, satisfying Eq. (2.5) for $a = 0$, $b = 3$, yields

$$\left[\left(c_2 \sinh\sqrt{\beta}z + c_3 \cosh\sqrt{\beta}z\right)\sinh\sqrt{\alpha}t \right.$$
$$\left. + \left(c_4 \sinh\sqrt{\beta}z + c_5 \cosh\sqrt{\beta}z\right)\cosh\sqrt{\alpha}t\right]$$
$$\left[-R_0\sqrt{\alpha\beta}\int\frac{\sqrt{R_1}}{R_0}d\rho - \sqrt{\frac{\alpha}{\beta}}\frac{R_3^{'}}{2\sqrt{R_1}}\right] + R_0 B_{3,3}(z) + R_3\, \dot{A}_4(t) = 0\,. \tag{B.54}$$

In order that Eq. (B.54) holds, there are two possibilities.

Either

(α) $c_2 = c_3 = c_4 = c_5 = 0$ and $R_0 B_{3,3}(z) = -R_3\, \dot{A}_4(t)\,,$ or

(β) $R_0\beta\int\frac{\sqrt{R_1}}{R_0}d\rho - \frac{R_3^{'}}{2\sqrt{R_1}} = 0$ and $R_0 B_{3,3}(z) = -R_3\, \dot{A}_4(t)\,.$

We discuss each in turn.

Case AIa4(i)α

Here again we have two possibilities.

If $\left(\frac{R_0}{R_3}\right)' \neq 0$ then

$$B_{3,3}(z) = 0\,, \quad \dot{A}_4(t) = 0\,, \tag{B.55}$$

or

$$B_3(z) = c_6\,, \quad A_4(t) = c_7\,, \tag{B.56}$$

and we get the minimal symmetry. In the second case $\left(\frac{R_0}{R_3}\right)' = 0$; put $\frac{R_0}{R_3} = -\gamma$, then

$$B_{3,3}(z) = -\dot{A}_4(t) = c_6\,, \tag{B.57}$$

$$B_3(z) = c_6 z + c_7\,, \tag{B.58}$$

$$A_4(t) = \gamma c_6 t + c_8\,. \tag{B.59}$$

So, we have the result from Eqs. (2.64) by renaming c_7, c_8, and c_6 as c_1, c_3 and c_4.

Case AIa4(i)β

Here

$$\beta \int \frac{\sqrt{R_1}}{R_0} d\rho + \frac{R_3'}{2R_0\sqrt{R_1}} = 0 . \tag{B.60}$$

Now, from Eq. (2.17) we can write

$$\int \frac{\sqrt{R_1}}{R_0} d\rho = \frac{1}{\alpha} \left(\frac{R_0'}{2R_0\sqrt{R_1}} \right) . \tag{B.61}$$

Substituting this in Eq. (B.60) we get

$$R_0' = -\frac{\alpha}{\beta} R_3' , \tag{B.62}$$

which on integration yields

$$R_0 = -\frac{\alpha}{\beta} R_3 + \gamma . \tag{B.63}$$

We note that here $R_0 \neq R_3$, because otherwise from Eq. (B.62) we would have $R_0' = 0$ (as $\alpha \neq 0, \quad \beta \neq 0$) which is not possible. Therefore, as above,

$$B_3(z) = c_6 , \qquad A_4(t) = c_7 , \tag{B.64}$$

and the result is the minimal symmetry.

Case AIa4(ii) $\qquad \alpha > 0 , \quad \beta < 0$

The result will be similar to that obtained in the previous case, Case AIa4(i). The difference will be that in the solution of Eq. (2.19) the trigonometric functions involved will be circular instead of hyperbolic.

Case AIa4(iii) $\qquad \alpha < 0 , \quad \beta > 0$

The result will be similar to that obtained in Case AIa4(i). The difference will be that in the solution of Eq. (2.18) the trigonometric functions involved will circular instead of hyperbolic.

Case AIa4(iv) $\qquad \alpha < 0 , \quad \beta < 0.$

The result will be similar to that obtained in Case AIa4(i). The difference will be that in the solution of Eq. (2.18) and Eq. (2.19) the trigonometric functions involved will be circular instead of hyperbolic.

Case AI(b) $\quad R_2' \neq 0\ ,\ \ R_3' = 0$

Note that the RC Equations, (2.4) and (2.5), are symmetric with respect to the interchange in θ and z (i.e. the indices 2 and 3). Thus this case is similar to the previous case, Case AI(a), except for the interchange of θ and z in all the equations.

Case AI(c) $\quad R_2' \neq 0\ ,\ \ R_3' \neq 0$

In this case Eq. (2.8) implies that $B^1_{,2} = B^1_{,3} = B^2_{,1} = B^3_{,1} = 0$. So, we have

$$\begin{aligned} B^0 &= B^0(t,\rho,\theta,z)\ , \\ B^1 &= B^1(t,\rho)\ , \\ B^2 &= B^2(t,\theta)\ , \\ B^3 &= B^3(t,z)\ . \end{aligned} \tag{B.65}$$

We further have the following constraints from the RC equations.

$$B^0_{,02} = B^0_{,12} = B^0_{,03} = B^0_{,23} = B^0_{,13} = B^2_{,22} = B^3_{,33} = 0\,. \tag{B.66}$$

Now, integrating Eq. (2.4) for $a = 1$, gives

$$B^1 = \frac{A_1(t)}{\sqrt{R_1}}\,. \tag{B.67}$$

Therefore, Eq. (2.5) for $a = 0$, $b = 1$, on integration with respect to ρ gives

$$B^0 = -\dot{A}_1(t)\int \frac{\sqrt{R_1}}{R_0}d\rho + A_2(t,\theta,z)\,. \tag{B.68}$$

But Eq. (B.66) implies that

$$B^0 = -\dot{A}_1(t)\int \frac{\sqrt{R_1}}{R_0}d\rho + A_3(t) + A_4(\theta) + A_5(z)\,. \tag{B.69}$$

Similarly, Eqs. (2.4) for $a = 2$ and 3, on integrating with respect to θ and z respectively yield

$$B^2 = \left(-\frac{R_2'}{2R_2\sqrt{R_1}}\right) A_1(t)\,\theta + A_6(t) \ , \quad \text{(B.70)}$$

$$B^3 = \left(-\frac{R_3'}{2R_3\sqrt{R_1}}\right) A_1(t)\,z + A_7(t) \ . \quad \text{(B.71)}$$

Now, as $B^2_{,1} = B^3_{,1} = 0$, we note that either both the terms in the brackets in Eqs. (2.70) and (2.71) are constants or otherwise $A_1(t)$ will be zero. We discuss these two cases.

Case AIc(1) $\qquad A_1(t) = 0$

Here from Eq. (B.67) we get

$$B^1 = 0 \, , \quad \text{(B.72)}$$

and therefore, from Eqs. (B.69) and (2.4) for $a = 0$, we get

$$A_3(t) = c_1 \, . \quad \text{(B.73)}$$

Now, using Eqs. (B.69), (B.70) and (B.71) in Eqs. (2.5) for $a = 0$, $b = 2$ and 3, gives

$$R_0 A_{4,2}(\theta) + R_2 \dot{A}_6(t) = 0 \, , \quad \text{(B.74)}$$

$$R_0 A_{5,3}(z) + R_3 \dot{A}_7(t) = 0 \, . \quad \text{(B.75)}$$

These imply that

$$A_6(t) = c_4 t + c_2 \, , \quad \text{(B.76)}$$

$$A_7(t) = c_8 t + c_3 \, . \quad \text{(B.77)}$$

Therefore, Eqs. (B.74) and (B.75) take the form

$$A_{4,2}(\theta) = -\frac{R_2}{R_0} c_4 \, , \quad \text{(B.78)}$$

$$A_{5,3}(z) = -\frac{R_3}{R_0} c_8 \, . \quad \text{(B.79)}$$

Here, we have three further cases depending upon whether one or none of $\frac{R_2}{R_0}$ and $\frac{R_3}{R_0}$ are constants. We note that both of these cannot be constants as in that case we will get $\frac{R_2}{R_3}$ as a constant, which will be a contradiction.

Case AIc1(i) $\quad \left(\frac{R_2}{R_0}\right)' = 0\,,\ \left(\frac{R_3}{R_0}\right)' \neq 0$

In this case $\frac{R_2}{R_0} = k_1$ and $c_4 = 0$, therefore, Eqs. (B.78) and (B.79) give

$$\begin{aligned} A_4(\theta) &= k_1 c_4 \theta + c_6\,, && \text{(B.80)} \\ A_5(z) &= c_7\,, && \text{(B.81)} \end{aligned}$$

and we finally get from Eqs. (B.67), (B.69), (B.70) and (B.71) the RCs as given in Eqs. (2.78), where we have written $c_1 + c_6 + c_7$ as c_1.

Case AIc1(ii) $\quad \left(\frac{R_2}{R_0}\right)' \neq 0\,,\ \left(\frac{R_3}{R_0}\right)' = 0$

This is similar to the previous case; just the indices 2 and 3 are interchanged.

Case AIc1(iii) $\quad \left(\frac{R_2}{R_0}\right)' \neq 0\,,\ \left(\frac{R_3}{R_0}\right)' \neq 0$

Here, we have from Eqs. (B.78) and (B.79) $c_4 = c_8 = 0$, therefore,

$$A_4(\theta) = c_6\,,\ A_5(z) = c_7\,, \qquad \text{(B.82)}$$

and Eqs. (B.67), (B.69), (B.70) and (B.71) give the minimal symmetry.

Case AIc(2) $\quad \left(\frac{R_2'}{2R_2\sqrt{R_1}}\right)' = 0\,,\ \left(\frac{R_3'}{2R_3\sqrt{R_1}}\right)' = 0$

We put $\frac{R_2'}{2R_2\sqrt{R_1}} = k_1$, $\frac{R_3'}{2R_3\sqrt{R_1}} = k_2$, where $k_1$and k_2 are nonzero constants and $k_1 \neq k_2$, because otherwise R_2 and R_3 will become proportional which will be a contradiction. Therefore, we have

$$\begin{aligned} B^2 &= -k_1 A_1(t)\,\theta + A_6(t)\,, && \text{(B.83)} \\ B^3 &= -k_2 A_1(t)\,z + A_7(t)\,. && \text{(B.84)} \end{aligned}$$

Now, Eq. (2.5) for $a = 0$, $b = 2$, on integration with respect to θ gives

$$A_4(\theta) = \frac{R_2}{R_0}\left[k_1\, \dot{A}_1(t)\,\frac{\theta^2}{2} - \dot{A}_6(t)\,\theta\right] + c_6\,. \tag{B.85}$$

Similarly, Eq. (2.5) for $a = 0$, $b = 3$, on integration with respect to z gives

$$A_5(z) = \frac{R_3}{R_0}\left[k_2\, \dot{A}_1(t)\,\frac{z^2}{2} - \dot{A}_7(t)\,z\right] + c_7\,, \tag{B.86}$$

and so Eq. (B.68) becomes

$$\begin{aligned} B^0 &= -\dot{A}_1(t)\int \frac{\sqrt{R_1}}{\sqrt{R_0}}d\rho + A_3(t) + \frac{R_2}{R_0}\left[k_1\, \dot{A}_1(t)\,\frac{\theta^2}{2} - \dot{A}_6(t)\,\theta\right] \\ &\quad + \frac{R_3}{R_0}\left[k_2\, \dot{A}_1(t)\,\frac{z^2}{2} - \dot{A}_7(t)\,z\right] + c_3\,, \end{aligned} \tag{B.87}$$

where $c_1 = c_6 + c_7$. Now, the value of B^i, $i = 0, 1, 2, 3$, as given by Eqs. (B.87), (B.67), (B.83) and (B.84) identically satisfy all the RC equations except Eq. (2.4) for $a = 0$, and Eq. (2.5) for $a = 0$, $b = 1$, which take the form

$$\begin{aligned} &\tfrac{R_0' A_1(t)}{2R_0\sqrt{R_1}} - \int \tfrac{\sqrt{R_1}}{R_0}d\rho\, \dot{A}_1(t) + \dot{A}_3(t) \\ &+\tfrac{R_2}{R_0}\left[k_1\, \ddot{A}_1(t)\,\tfrac{\theta^2}{2} - \ddot{A}_6(t)\,\theta\right] \\ &-\tfrac{R_3}{R_0}\left[k_2\, \ddot{A}_1(t)\,\tfrac{z^2}{2} - \ddot{A}_7(t)\,z\right] = 0\,, \end{aligned} \tag{B.88}$$

$$\begin{aligned} &\left(\frac{R_2}{R_0}\right)'\left[k_1\, \dot{A}_1(t)\,\frac{\theta^2}{2} - \dot{A}_6(t)\,\theta\right] \\ &+\left(\frac{R_3}{R_0}\right)'\left[k_2\, \ddot{A}_1(t)\,\frac{z^2}{2} - \ddot{A}_7(t)\,z\right] = 0\,. \end{aligned} \tag{B.89}$$

From Eq. (B.89) we see that we have to discuss three further cases depending upon whether one or none of $\left(\frac{R_2}{R_0}\right)'$ and $\left(\frac{R_3}{R_0}\right)'$ is/are equal to zero. We note that both of these cannot be equal to zero as it would contradict one of the conditions of this case, namely, $\left(\frac{R_2}{R_3}\right)' \neq 0$.

Case AIc2(i) $\left(\frac{R_2}{R_0}\right)' = 0$, $\left(\frac{R_3}{R_0}\right)' \neq 0$

Let $\frac{R_2}{R_0} = -k_3$ which implies that $\frac{R_0'}{2R_0\sqrt{R_1}} = k_1$. In this case Eq. (B.89) yields

$$A_1(t) = \dot{A}_7(t) = 0, \tag{B.90}$$

or

$$A_1(t) = c_4\,,\; A_7(t) = c_3 \tag{B.91}$$

Therefore, Eq. (B.88) takes the form

$$k_1 c_4 + \dot{A}_3(t) + k_3 \ddot{A}_6(t)\,\theta = 0\,. \tag{B.92}$$

which implies that

$$A_6(t) = c_5 t + c_2\,. \tag{B.93}$$

$$A_3(t) = k_1\; c_4 t + c_6\,. \tag{B.94}$$

With values from Eqs. (B.91), (B.93) and (B.94), we finally get from Eqs. (B.87), (B.67), (B.83) and (B.84) the RCs as given in Eqs. (2.90).

Case AIc2(ii) $\left(\frac{R_2}{R_0}\right)' \neq 0,\ \left(\frac{R_3}{R_0}\right)' = 0$

This is similar to the previous case; only indices 2 and 3 (i.e. coordinates θ and z) are interchanged.

Case AIc2(iii) $\left(\frac{R_2}{R_0}\right)' \neq 0,\ \left(\frac{R_3}{R_0}\right)' \neq 0$

In this case Eq. (B.89) implies

$$A_1(t) = c_4\,,\; A_6(t) = c_2\,,\; A_7(t) = c_3\,, \tag{B.95}$$

and Eq. (B.88) takes the from

$$\frac{R_0'}{2R_0\sqrt{R_1}} c_4 + \dot{A}_3(t) = 0\,. \tag{B.96}$$

Here again, as before, we have two cases.

Case AIc2(iii)α $\left(\frac{R_0'}{2R_0\sqrt{R_1}}\right)' = 0$

Putting $\frac{R_0'}{2R_0\sqrt{R_1}} = k_4$, Eq. (B.92) implies

$$A_3(t) = k_4\ c_4 t + c_5\,. \tag{B.97}$$

So, Eqs. (B.87), (B.67), (B.83) and (B.84) take the form as given in Eq. (2.93).

Case AIc2(iii)β $\qquad \left(\frac{R_0'}{2R_0\sqrt{R_1}}\right)' \neq 0$

Eq. (B.92) in this case gives $c_4 = 0$ and $A_3(t) = c_5$. Therefore, Eqs. (B.87), (B.67), (2.79) and (B.84) yield the minimal symmetry.

Case A(II) $\qquad \left(\frac{R_2}{R_3}\right)' = 0$

This means that $\frac{R_2}{R_3}$ =constant= l (say), $l \neq 0$. We put $R_3 = R_2/l$ in the RC equations and eliminate R_3 from Eqs. (2.5), for $a =$ 0,1,2 and $b = 3$; and Eq. (2.4) for $a = 3$, which become

$$lR_0B^0_{,3} + R_2B^3_{,0} = 0\,, \tag{B.98}$$

$$lR_1B^1_{,3} + R_2B^3_{,1} = 0\,, \tag{B.99}$$

$$lB^2_{,3} + B^3_{,2} = 0\,, \tag{B.100}$$

$$R_2'B^1 + 2R_2B^3_{,3} = 0\,. \tag{B.101}$$

Differentiating Eq. (2.5) for $a = 1$, $b = 2$, with respect to z and Eq. (B.99) with respect to θ, we get

$$R_1B^1_{,23} + R_2B^2_{,13} = 0\,, \tag{B.102}$$

$$lR_1B^1_{,23} + R_2B^3_{,12} = 0\,. \tag{B.103}$$

Using Eq. (B.100) in Eq. (B.103) we get

$$R_1 B^1_{,23} - R_2 B^2_{,13} = 0 \, . \tag{B.104}$$

Now adding Eqs. (B.102) and (B.104) gives $B^1_{,23} = 0$. Therefore, Eq. (2.4) for $a = 2$, gives $B^2_{,223} = 0$, which on integrating twice with respect to θ using arbitrary functions of integration gives

$$B^2_{,3} = \theta A_1 (t, \rho, z) + A_2 (t, \rho, z) \ , \tag{B.105}$$

and, therefore, Eq. (B.100) on integration with respect to θ gives

$$B^3 = -\frac{l\theta^2}{2} A_1 (t, \rho, z) - l\theta A_2 (t, \rho, z) + A_3 (t, \rho, z) \, . \tag{B.106}$$

Also, Eqs. (2.4) for $a = 2,3$ and (2.45) mean that $B^2_{,23} = B^2_{,33}$ which on using Eqs. (B.105) and (B.106) becomes

$$A_1 (t, \rho, z) = -\frac{l\theta^2}{2} A_{1,33} (t, \rho, z) - l\theta A_{2,33} (t, \rho, z) + A_{3,33} (t, \rho, z) \, . \tag{B.107}$$

Comparing the coefficients of θ^2, θ^1 and θ^0 in this gives

$$A_{1,33} (t, \rho, z) = 0 \, , \tag{B.108}$$

$$A_{2,33} (t, \rho, z) = 0 \, , \tag{B.109}$$

$$A_{3,33} (t, \rho, z) = A_1 (t, \rho, z) \ , \tag{B.110}$$

which on integrating twice with respect to z yield

$$A_1 (t, \rho, z) = z B_1 (t, \rho) + B_2 (t, \rho) \ , \tag{B.111}$$

$$A_2 (t, \rho, z) = z B_3 (t, \rho) + B_4 (t, \rho) \, . \tag{B.112}$$

Putting from Eq. (B.111) in Eq. (B.110) and integrating twice with respect to z gives

$$A_3(t,\rho,z)=\frac{z^3}{6}B_1(t,\rho)+\frac{z^2}{2}B_2(t,\rho)+zB_5(t,\rho)+B_6(t,\rho)\,. \tag{B.113}$$

Using Eqs. (B.111), (B.112) and (B.113), B^3 from Eq. (B.106) becomes

$$\begin{aligned} B^3 &= -\frac{l\theta^2}{2}\left[zB_1(t,\rho)+B_2(t,\rho)\right]-l\theta\left[zB_3(t,\rho)+B_4(t,\rho)\right] \\ &\quad +\frac{z^3}{6}B_1(t,\rho)+\frac{z^2}{2}B_2(t,\rho)+zB_5(t,\rho)+B_6(t,\rho)\,. \end{aligned} \tag{B.114}$$

And, Eq. (B.105) on integration with respect to z gives

$$B^2=\theta\left[\frac{z^2}{2}B_1(t,\rho)+zB_2(t,\rho)\right]+\frac{z^2}{2}B_3(t,\rho)+zB_4(t,\rho)+A_4(t,\rho,\theta)\,. \tag{B.115}$$

Now, as $B^2_{,2}=B^3_{,3}=0$, substituting from the above two equations and integrating with respect to θ yields

$$A_4(t,\rho,\theta)=-\frac{\theta^3}{6}B_1(t,\rho)-\frac{l\theta^2}{2}B_3(t,\rho)+\theta B_5(t,\rho)+B_7(t,\rho)\,, \tag{B.116}$$

so that Eq. (B.115) becomes

$$\begin{aligned} B^2 &= \theta\left[\frac{z^2}{2}B_1(t,\rho)+zB_2(t,\rho)\right]+\frac{z^2}{2}B_3(t,\rho)+zB_4(t,\rho) \\ &\quad -\frac{l\theta^3}{6}B_1(t,\rho)-\frac{l\theta^2}{2}B_3(t,\rho)+\theta B_5(t,\rho)+B_7(t,\rho)\,. \end{aligned} \tag{B.117}$$

Now using Eq. (B.114) in Eq. (B.98) and integrating with respect to z gives

$$\begin{aligned} B^0 &= \frac{R_2}{lR_0}\left\{\frac{l\theta^2}{2}\left[\frac{z^2}{2}\dot{B}_1(t,\rho)+z\dot{B}_2(t,\rho)\right]+l\theta\left[\frac{z^2}{2}\dot{B}_3(t,\rho)+z\dot{B}_4(t,\rho)\right]\right. \\ &\quad \left.-\frac{z^4}{24}\dot{B}_1(t,\rho)-\frac{z^3}{6}\dot{B}_2(t,\rho)-\frac{z^2}{2}\dot{B}_5(t,\rho)-z\dot{B}_6(t,\rho)\right\}+A_5(t,\rho)\,. \end{aligned} \tag{B.118}$$

Similarly, using Eq. (B.114) in Eq. (B.99) and integrating with respect to z gives

$$\begin{aligned} B^1 &= \frac{R_2}{R_1}\left\{\frac{\theta^2}{2}\left[\frac{z^2}{2}B_1'(t,\rho)+zB_2'(t,\rho)\right]+\theta\left[\frac{z^2}{2}B_3'(t,\rho)+zB_4'(t,\rho)\right]\right. \\ &\left.-\frac{z^4}{24l}B_1'(t,\rho)-\frac{z^3}{6l}B_2'(t,\rho)-\frac{z^2}{2l}B_5'(t,\rho)-\frac{1}{l}zB_6'(t,\rho)\right\}+A_6(t,\rho,\theta)\ . \end{aligned} \quad (B.119)$$

Now Eq. (2.5) for $a=0$, $b=2$, can be put in the form

$$\theta\left[z^2\,\dot{B}_1(t,\rho)+2z\,\dot{B}_2(t,\rho)\right]+\left[z^2\,\dot{B}_3(t,\rho)+2z\,\dot{B}_4(t,\rho)\right]$$
$$+\frac{R_0}{R_2}A_{5,2}(t,\rho,\theta)-\frac{l\theta^3}{6}\,\dot{B}_1(t,\rho)-\frac{l\theta^2}{2}\,\dot{B}_3(t,\rho)+\theta\,\dot{B}_5(t,\rho)+\dot{B}_7(t.\rho)=0\ .$$

Comparing the coefficients of z^2, z and z^0 gives

$$\theta\,\dot{B}_1(t,\rho)+\dot{B}_3(t,\rho)=0\,, \quad (B.120)$$

$$\theta\,\dot{B}_2(t\rho)+\dot{B}_4(t,\rho)=0\,, \quad (B.121)$$

$$A_{5,2}(t,\rho,\theta)=\frac{R_2}{R_0}\left[\frac{l\theta^3}{6}\,\dot{B}_1(t,\rho)+\frac{l\theta^2}{2}\,\dot{B}_3(t,\rho)-\theta\,\dot{B}_5(t,\rho)-\dot{B}_7(t,\rho)\right]. \quad (B.122)$$

From Eqs. (B.120) and (B.121) it is clear that

$$\dot{B}_1(t,\rho)=0=\dot{B}_2(t,\rho)=\dot{B}_3(t,\rho)=\dot{B}_4(t,\rho)\,. \quad (B.123)$$

Therefore, Eq. (B.122) on integration with respect to θ gives

$$A_5(t,\rho,\theta)=-\frac{R_2}{R_0}\left[\frac{\theta^2}{2}\,\dot{B}_5(t,\rho)+\theta\,\dot{B}_7(t,\rho)\right]+B_8(t,\rho)\,. \quad (B.124)$$

Now Eq. (2.5) for $a = 1$, $b = 2$, can be written as

$$\begin{aligned} &z^2\left[\theta B_1'(t,\rho) + B_3'(t,\rho)\right] + 2z\left[\theta B_2'(t,\rho) + B_4'(t,\rho)\right] \\ &-\tfrac{\theta^3}{6}B_1'(t,\rho) - \tfrac{l\theta^2}{2}B_3'(t,\rho) + \theta B_5'(t,\rho) + B_7'(t,\rho) + \tfrac{R_1}{R_2}A_{6,2}(t,\rho,\theta) = 0\,. \end{aligned} \tag{B.125}$$

Comparing the coefficients of z^2, z and z^0 gives

$$\theta B_1'(t,\rho) + B_3'(t,\rho) = 0\,, \tag{B.126}$$

$$\theta B_2'(t,\rho) + B_4'(t,\rho) = 0\,, \tag{B.127}$$

$$A_{6,2}(t,\rho,\theta) = \frac{R_2}{R_1}\left[\frac{\theta^3}{6}B_1'(t,\rho) + \frac{l\theta^2}{2}B_3'(t,\rho) - \theta B_5'(t,\rho) - B_7'(t,\rho)\right]. \tag{B.128}$$

It is clear from Eqs. (B.127) and (B.128) that

$$B_1'(t,\rho) = 0 = B_2'(t,\rho) = B_3'(t,\rho) = B_4'(t,\rho)\,. \tag{B.129}$$

Therefore, Eq. (B.128) on integration with respect to θ gives

$$A_6(t,\rho,\theta) = -\frac{R_2}{R_1}\left[\frac{\theta^2}{2}B_5'(t,\rho) + \theta B_7'(t,\rho)\right] + B_9(t,\rho)\,. \tag{B.130}$$

Also, from Eqs. (B.123) and (B.129) it is clear that

$$B_i(t,\rho) = c_i \quad , \qquad i = 4,7,8,9\,. \tag{B.131}$$

Therefore, in view of Eqs. (B.124), (B.130) and (B.131), Eqs. (B.118), (B.119), (B.117) and (B.114) become

$$B^0 = -\frac{R_2}{R_1}\left[\frac{1}{2}\left(\frac{1}{l}z^2 + \theta^2\right)\dot{B}_5(t,\rho) + \frac{1}{l}z\,\dot{B}_6(t,\rho) + \theta\,\dot{B}_7(t,\rho)\right] + B_8(t,\rho)\,, \tag{B.132}$$

$$B^1 = -\frac{R_2}{R_1}\left[\frac{1}{2}\left(\frac{1}{l}z^2+\theta^2\right)B_5'(t,\rho)+\frac{1}{l}zB_6'(t,\rho)+\theta B_7'(t,\rho)\right]+B_9(t,\rho)\ , \tag{B.133}$$

$$B^2 = \frac{z^2}{2}(\theta c_9+c_8)+z(c_7\theta+c_4)-\frac{l\theta^3}{6}c_9-\frac{l\theta^2}{2}c_8+\theta B_5(t,\rho)+B_7(t,\rho)\ , \tag{B.134}$$

$$\begin{aligned} B^3 &= -\frac{l\theta^2}{2}(zc_9+c_7)-l\theta(zc_8+c_4)+ \\ &\quad \frac{z^3}{6}c_9+\frac{z^2}{2}c_7+zB_5(t,\rho)+B_6(t,\rho)\ . \end{aligned} \tag{B.135}$$

Now substituting B^i from Eqs. (B.132)-(B.135) into Eqs. (2.4), for $a=0$,1,2 and Eq. (2.5) for $a=0$, $b=1$, gives

$$\begin{aligned} &\frac{R_0'R_2}{R_1}\left[\frac{1}{2}\left(\frac{1}{l}z^2+\theta^2\right)B_5'(t,\rho)+\frac{1}{l}zB_6'(t,\rho)+\theta B_7'(t,\rho)\right]-R_0'B_9(t,\rho) \\ &+2R_2\left[\frac{1}{2}\left(\frac{1}{l}z^2+\theta^2\right)\ddot{B}_5(t,\rho)+\frac{1}{l}z\,\ddot{B}_6(t,\rho)+\theta\,\ddot{B}_7(t,\rho)\right]-2R_0\,\dot{B}_8(t,\rho)=0 \end{aligned} \tag{B.136}$$

$$\begin{aligned} &R_0\left\{\tfrac{R_2}{R_0}\left[\tfrac{1}{2}\left(\tfrac{1}{l}z^2+\theta^2\right)\dot{B}_5'(t,\rho)+\tfrac{1}{l}z\,\dot{B}_6'(t,\rho)+\theta\,\dot{B}_7'(t,\rho)\right]-B_8'(t,\rho)\right. \\ &\left.+\left(\tfrac{R_2}{R_0}\right)'\left[\tfrac{1}{2}\left(\tfrac{1}{l}z^2+\theta^2\right)\dot{B}_5(t,\rho)+\tfrac{1}{l}z\,\dot{B}_6(t,\rho)+\theta\,\dot{B}_7(t,\rho)\right]\right\} \\ &+R_1\left\{\tfrac{R_2}{R_1}\left[\tfrac{1}{2}\left(\tfrac{1}{l}z^2+\theta^2\right)\dot{B}_5'(t,\rho)+\tfrac{1}{l}z\,\dot{B}_6'(t,\rho)+\theta\,\dot{B}_7'(t,\rho)\right]-\dot{B}_9(t,\rho)\right\}=0\,, \end{aligned} \tag{B.137}$$

$$\begin{aligned} &R_1'\left\{\tfrac{R_2}{R_1}\left[\tfrac{1}{2}\left(\tfrac{1}{l}z^2+\theta^2\right)B_5'(t,\rho)+\tfrac{1}{l}zB_6'(t,\rho)+\theta B_7'(t,\rho)\right]-B_9(t,\rho)\right\} \\ &+2R_1\left(\tfrac{R_2}{R_1}\right)'\left[\tfrac{1}{2}\left(\tfrac{1}{l}z^2+\theta^2\right)B_5'(t,\rho)+\tfrac{1}{l}zB_6'(t,\rho)+\theta B_7'(t,\rho)\right] \\ &+2R_1\left\{\tfrac{R_2}{R_1}\left[\tfrac{1}{2}\left(\tfrac{1}{l}z^2+\theta^2\right)B_5''(t,\rho)+\tfrac{1}{l}zB_6''(t,\rho)+\theta B_7''(t,\rho)\right]-B_9'(t,\rho)\right\}=0, \end{aligned} \tag{B.138}$$

$$R_2'\left\{\frac{R_2}{R_1}\left[\frac{1}{2}\left(\frac{1}{l}z^2+\theta^2\right)B_5'(t,\rho)+\frac{1}{l}zB_6'(t,\rho)+\theta B_7'(t,\rho)\right]-B_9(t,\rho)\right\}$$
$$-2R_1\left[\frac{1}{2}\left(z^2+l\theta^2\right)c_9+c_7z-l\theta c_8+B_5(t,\rho)\right]=0\,. \tag{B.139}$$

Comparing the coefficients of $\theta^2, \theta^1, \theta^0$, z^2, z^1 and z^0 in Eqs. (B.136)-(B.139) gives

$$\frac{R_0'}{R_1}B_j'(t,\rho)+2\,\ddot{B}_j(t,\rho)=0,\quad j=5,6,7\,; \tag{B.140}$$

$$R_0'B_9(t,\rho)+2R_0\,\dot{B}_8(t,\rho)=0\,, \tag{B.141}$$

$$2R_2\,\dot{B}_j'(t,\rho)+R_0\left(\frac{R_2}{R_0}\right)'\dot{B}_j(t,\rho)=0\,,\quad j=5,6,7\,; \tag{B.142}$$

$$R_0B_8'(t,\rho)+R_1\,\dot{B}_9(t,\rho)=0\,, \tag{B.143}$$

$$\left(\frac{R_2}{\sqrt{R_1}}\right)'B_j'(t,\rho)+\frac{R_2}{\sqrt{R_1}}B_j''(t,\rho)=0\,,\,j=5,6,7\,; \tag{B.144}$$

$$R_1'B_9(t,\rho)+2R_1B_9'(t,\rho)=0\,, \tag{B.145}$$

$$\frac{R_2'}{lR_1}B_5'(t,\rho)+2c_9=0\,, \tag{B.146}$$

$$\frac{R_2'}{lR_1}B_5'(t,\rho)-2c_9=0\,, \tag{B.147}$$

$$\frac{R_2'}{lR_1}B_6'(t,\rho)-2c_7=0\,, \tag{B.148}$$

$$\frac{R_2'}{lR_1}B_7'(t,\rho)+2c_8=0\,, \tag{B.149}$$

$$R_2' B_9(t,\rho) + 2R_2 B_5(t,\rho) = 0\,. \qquad \text{(B.150)}$$

Eqs. (B.144), (B.142) and (B.145) on integration with respect to ρ give

$$\dot{B}_j(t,\rho) = \sqrt{\frac{R_0}{R_2}}h_j(t)\,, \quad j = 5,6,7\,; \qquad \text{(B.151)}$$

$$B_j'(t,\rho) = \sqrt{\frac{R_1}{R_2}}f_j(t)\,, \quad j = 5,6,7\,; \qquad \text{(B.152)}$$

$$B_9(t,\rho) = \frac{1}{\sqrt{R_1}}f_9(t)\,, \qquad \text{(B.153)}$$

where $h_j(t)$ and $f_j(t)$, $j = 5,6,7,9$ are functions of integration. Also, from Eqs. (B.146) and (B.147) we see that $c_9 = 0$, and therefore (as $l \neq 0$)

$$R_2' B_5'(t,\rho) = 0 \qquad \text{(B.154)}$$

Using Eq. (B.153) in Eqs. (B.141) and (B.143) gives

$$\dot{B}_8(t,\rho) = -\frac{R_0'}{2R_0\sqrt{R_1}}f_9(t)\,, \qquad \text{(B.155)}$$

$$B_8'(t,\rho) = -\frac{\sqrt{R_1}}{R_0}\dot{f}_9(t)\,. \qquad \text{(B.156)}$$

Now, if Eq. (B.155) is differentiated with respect to ρ and Eq. (B.156) with respect to t, $f_9(t)$ can be seen to satisfy the equation

$$\frac{\sqrt{R_1}}{R_0}\ddot{f}_9(t) - \left(\frac{R_0'}{2R_0\sqrt{R_1}}\right)' f_9(t) = 0\,. \qquad \text{(B.157)}$$

Now substituting from Eqs. (B.152) and (B.153) into Eqs. (B.148)-(B.150), we get

$$\frac{R_2'}{lR_2\sqrt{R_1}}f_6(t) - 2c_7 = 0\,, \qquad \text{(B.158)}$$

$$\frac{R_2'}{lR_2\sqrt{R_1}}f_7(t)+2c_8=0\,, \tag{B.159}$$

$$\frac{R_2'}{R_2\sqrt{R_1}}f_9(t)+2B_5(t,\rho)=0\,, \tag{B.160}$$

and using Eq. (B.152) in Eq. (2.98) gives

$$\frac{R_0'}{R_2\sqrt{R_1}}f_j(t)+2\,\ddot{B}_j(t,\rho)=0\,,\quad j=5,6,7\,. \tag{B.161}$$

Now Eqs. (B.158)-(B.160) give rise two possibilities.

(a) $\left(\frac{R_2'}{\sqrt{R_1R_2}}\right)'\neq 0\,,$

(b) $\left(\frac{R_2'}{\sqrt{R_1R_2}}\right)'=0\,.$

Now, we discuss these.

Case AII(a)

In this case $R_2'\neq 0$ and, therefore, from Eq. (B.154)

$$B_5'(t,\rho)=0\,. \tag{B.162}$$

Differentiating Eqs. (B.158)-(B.160) with respect to ρ and using Eq. (B.156), we get

$$\left(\frac{R_2'}{R_2\sqrt{R_1}}\right)'\frac{1}{l}f_6(t)=0\,, \tag{B.163}$$

$$\left(\frac{R_2'}{R_2\sqrt{R_1}}\right)'\frac{1}{l}f_7(t)=0\,, \tag{B.164}$$

$$\left(\frac{R_2'}{R_2\sqrt{R_1}}\right)'f_9(t)=0\,. \tag{B.165}$$

These imply that

$$f_6(t)=0=f_7(t)=f_9(t)\,, \tag{B.166}$$

which in turn imply that

$$c_7 = 0 = c_8 = B_5(t,\rho) \ . \tag{B.167}$$

Also from Eqs. (B.155), (B.156) and (B.153) we see that

$$B_8(t,\rho) = \text{ Constant} = c_1 \text{ (say)} \ , \qquad B_9(t,\rho) = 0 \ . \tag{B.168}$$

Using these results in Eqs. (B.140), (B.151) and (B.152), we get

$$B_6'(t,\rho) = 0 = B_7'(t,\rho) \ , \tag{B.169}$$

$$\ddot{B}_6(t,\rho) = 0 = \ddot{B}_7(t,\rho) \ , \tag{B.170}$$

$$h_5(t) = 0 = f_5(t) = \dot{h}_6(t) = \dot{h}_7(t) \ . \tag{B.171}$$

From Eq. (B.164) we have

$$h_6(t) = c_6 \ , \quad h_7(t) = c_5 \ . \tag{B.172}$$

So, Eq. (B.151) on integration with respect to t gives

$$B_6(t,\rho) = \sqrt{\frac{R_0}{R_2}}c_6 t + c_3 \ , \tag{B.173}$$

$$B_7(t,\rho) = \sqrt{\frac{R_0}{R_2}}c_5 t + c_2 \ . \tag{B.174}$$

Differentiating Eqs. (B.173) and (B.174) with respect to ρ and using Eq. (m) we get

$$\left(\sqrt{\frac{R_0}{R_2}}\right)' c_6 = 0 = \left(\sqrt{\frac{R_0}{R_2}}\right)' c_5 \ . \tag{B.175}$$

We have further two possibilities here.

(1) $\left(\sqrt{\frac{R_0}{R_2}}\right)' = 0 \ ,$

(2) $\left(\sqrt{\frac{R_0}{R_2}}\right)' \neq 0$, (and $c_6 = 0 = c_5$).

Case AIIa(1) $\left(\sqrt{\frac{R_0}{R_2}}\right)' = 0$

Let $\sqrt{\frac{R_0}{R_2}} = k$, so that Eqs. (B.132)-(B.135) give the RCs (Eqs. (2.113).

Case AIIa(2) $c_6 = c_7 = 0$

Eqs. (B.132)-(B.135) in this case give the result as given in Eq. (2.114).

Case AII(b) $\left(\frac{R_2'}{R_2\sqrt{R_1}}\right)' = 0$

Let $\frac{R_2'}{R_2\sqrt{R_1}} = \alpha$ (a constant) for which there are two possibilities.

(1) $\alpha \neq 0$,

(2) $\alpha = 0$.

Case AIIb(1) $\alpha \neq 0$

So, from Eqs. (B.158) and (B.159)we have

$$f_6(t) = \frac{2lc_7}{\alpha}, \qquad f_7(t) = -\frac{2lc_8}{\alpha}, \tag{B.176}$$

and from Eq. (B.154)

$$B_5'(t,\rho) = 0. \tag{B.177}$$

In view of Eqs. (B.173) and (B.174) , we have from Eq. (B.152)

$$\dot{B}_6'(t,\rho) = 0 = \dot{B}_7'(t,\rho) , \tag{B.178}$$

so that Eq. (B.142) becomes

$$\left(\frac{R_2}{R_0}\right)' \dot{B}_j(t,\rho) = 0, \quad j = 5,6,7; \tag{B.179}$$

which further gives rise to two cases.

(i) $\left(\frac{R_2}{R_0}\right)' \neq 0$,

(ii) $\left(\frac{R_2}{R_0}\right)' = 0$.

Case AIIb1(i) $\left(\frac{R_2}{R_0}\right)' \neq 0$

In this case from Eq. (B.179) we have

$$\dot{B}_5(t,\rho) = 0 = \dot{B}_6(t,\rho) = \dot{B}_7(t,\rho)\,, \tag{B.180}$$

and therefore from Eq. (B.152) along with Eq. (B.176) we get

$$B_6'(t,\rho) = \frac{lc_7}{\alpha}\frac{2\sqrt{R_1}}{R_2}\,, \tag{B.181}$$

which on integration with respect to ρ gives

$$B_6(t,\rho) = \frac{2lc_7}{\alpha}\int \frac{\sqrt{R_1}}{R_2}d\rho + c_3\,. \tag{B.182}$$

Similarly

$$B_7(t,\rho) = -\frac{2lc_8}{\alpha}\int \frac{\sqrt{R_1}}{R_2}d\rho + c_2\,, \tag{B.183}$$

and from Eq. (B.177)

$$B_5(t,\rho) = c_5\,. \tag{B.184}$$

Eq. (B.160) now can be written as

$$f_9(t) = -\frac{2c_5}{\alpha}\,. \tag{B.185}$$

So, Eq. (B.153) becomes

$$B_9(t,\rho) = -\frac{2c_5}{\sqrt{R_1}\alpha}\,, \tag{B.186}$$

and Eqs. (B.140) and (B.157) reduce to

$$R_0'B_j'(t,\rho) = 0\,, \qquad j = 5,6,7\,; \tag{B.187}$$

$$\left(\frac{R_0'}{R_0\sqrt{R_1}}\right)' c_5 = 0 \; . \tag{B.188}$$

Here again there are further two cases.

(α) $\left(\frac{R_0'}{R_0\sqrt{R_1}}\right)' \neq 0$,

(β) $\left(\frac{R_0'}{R_0\sqrt{R_1}}\right)' = 0$.

Case AIIb1(i)α $\left(\frac{R_0'}{R_0\sqrt{R_1}}\right)' \neq 0$

This implies that $R_0' \neq 0$. Therefore, from Eqs. (B.187) and (B.188) we have

$$c_5 = 0 = B_j'(t,\rho) \; , \qquad j = 5,6,7 \; ; \tag{B.189}$$

which, when used in Eqs. (B.182)-(B.184), mean

$$c_7 = 0 = c_8 = B_5(t,\rho) \; . \tag{B.190}$$

Now, from Eq. (B.160) we see that $f_9(t) = 0$, so that Eqs. (B.155), (B.156) and (B.153) imply that

$$B_8(t,\rho) = c_1 \; , \quad B_9(t,\rho) = 0 \; . \tag{B.191}$$

So, we finally get Eqs. (2.116).

Case AIIb1(i)β $\left(\frac{R_0'}{R_0\sqrt{R_1}}\right)' = 0$

Let $\frac{R_0'}{R_0\sqrt{R_1}} = \beta \neq \alpha$. Now, as $R_0' \neq 0$, therefore, from Eq. (B.187) we have $B_j'(t,\rho) = 0$, $j = 5,6,7$. So, Eqs. (B.152) and (B.176) give $c_7 = c_8 = 0$, so that Eqs. (B.182) and (B.183) become $B_6 = c_3$, $B_7 = c_2$. Now, putting from Eq. (B.185) in (B.155) and integrating with respect to t gives

$$B_8 = \frac{\beta}{\alpha} c_5 t + c_1 \; . \tag{B.192}$$

Therefore, Eqs. (2.94)-(2.97) RCs as given in Eqs. (2.117).

Case AIIb1(ii) $\left(\frac{R_2}{R_0}\right)' = 0$

Here we write $R_2 = -\delta R_0$, where δ is a constant. So, we can write $R_2' = \delta R_0'$, which implies that $\frac{R_0'}{R_0\sqrt{R_1}} = \alpha$. Now, integrating Eq. (B.152) with respect to ρ gives

$$B_j\left(t,\rho\right) = f_j\left(t\right)\int \frac{\sqrt{R_1}}{R_2}d\rho + g_j\left(t\right)\ , \quad j = 5, 6, 7\ . \tag{B.193}$$

This, in view of Eqs. (B.161), (B.176) and (B.177) (and the condition $B^3_{,1} = 0$) gives

$$\ddot{g}_5\left(t\right) = 0\,, \tag{B.194}$$

$$\ddot{g}_6\left(t\right) = \frac{lc_9}{\delta}\,, \tag{B.195}$$

$$\ddot{g}_7\left(t\right) = -\frac{lc_8}{\delta}\,. \tag{B.196}$$

Integrating Eqs. (B.194)-(B.196) gives

$$g_5\left(t\right) = c_7 t + c_{10}\,, \tag{B.197}$$

$$g_6\left(t\right) = \frac{lc_9}{2\delta}t^2 + c_6 t + c_3\ , \tag{B.198}$$

$$g_7\left(t\right) = -\frac{lc_8}{2\delta}t^2 + c_5 t + c_2\,. \tag{B.199}$$

Putting these in Eq. (B.193), we get

$$B_5\left(t,\rho\right) = c_7 t + c_{10}\,, \tag{B.200}$$

$$B_6\left(t,\rho\right) = -\left(\frac{2}{\alpha^2 R_2} - \frac{t^2}{2\delta}\right) lc_9 + c_6 t + c_3\,, \tag{B.201}$$

$$B_7\left(t,\rho\right) = \left(\frac{2}{\alpha^2 R_2} - \frac{t^2}{2\delta}\right) lc_8 + c_5 t + c_2\,, \tag{B.202}$$

where, we have also used the fact that $\int \frac{\sqrt{R_1}}{R_2} d\rho = -\frac{1}{\alpha R_2}$ .So, from Eqs. (B.153) and (B.160) we get

$$B_9\left(t,\rho\right) = -\frac{2}{\alpha\sqrt{R_0}}\left(c_7 t + c_{10}\right) . \tag{B.203}$$

Therefore, from Eqs. (B.155) and (B.156) we have

$$B_8'\left(t,\rho\right) = \frac{2\sqrt{R_1}}{\alpha R_0} c_7 , \tag{B.204}$$

$$\dot{B}_8\left(t,\rho\right) = c_7 t + c_{10} . \tag{B.205}$$

Now, integrating Eq. (B.204) with respect to ρ and substituting in Eq. (B.205) and integrating again with respect to t gives

$$B_8\left(t,\rho\right) = \left(-\frac{2}{\alpha^2 R_0} + \frac{t^2}{2}\right) c_7 + t c_{10} + c_1 . \tag{B.206}$$

Therefore, Eqs. (B.132)-(B.135) , give the RCs as given in Eqs. (2.118).

Case AIIb(2) $\qquad \alpha = 0$

This means that $R_2' = 0$, i.e., R_2 is a constant. So that from Eqs. (B.148)-(B.150) we have $c_2 = c_3 = 0$ and $B_5\left(t,\rho\right) = 0$. Now, differentiating Eqs. (B.151) and (B.152) with respect to ρ and t respectively and equating gives

$$\dot{f}_j\left(t\right) - \frac{\left(\sqrt{R_0}\right)'\sqrt{R_2}}{\sqrt{R_1}} h_j\left(t\right) = 0 , \qquad j = 6,7 ; \tag{B.207}$$

which on differentiating again with respect to t gives

$$\ddot{f}_j\left(t\right) - \frac{\left(\sqrt{R_0}\right)'\sqrt{R_2}}{\sqrt{R_1}} \dot{h}_j\left(t\right) = 0 , \qquad j = 6,7 . \tag{B.208}$$

Now, substituting Eqs. (B.151) and (B.152) in Eq. (B.140) gives

$$\frac{\left(\sqrt{R_0}\right)'}{\sqrt{R_1}\sqrt{R_2}} f_j\left(t\right) + \dot{h}_j\left(t\right) = 0 , \quad j = 6,7 . \tag{B.209}$$

Eliminating $h_j(t)$ from Eqs. (B.208) and (B.209) yields

$$\ddot{f}_j(t) + \left(\frac{(\sqrt{R_0})'}{\sqrt{R_1}}\right)^2 f_j(t) = 0\,, \quad j = 6,7\,. \tag{B.210}$$

This gives rise to two possibilities.

(i) $\left(\frac{(\sqrt{R_0})'}{\sqrt{R_1}}\right)' = 0\,,$

(ii) $\left(\frac{(\sqrt{R_0})'}{\sqrt{R_1}}\right)' \neq 0\,.$

We discuss them one by one.

Case AIIb2(i) $\left(\frac{(\sqrt{R_0})'}{\sqrt{R_1}}\right)' = 0$

Let $\frac{(\sqrt{R_0})'}{\sqrt{R_1}} = \gamma$, a constant which cannot be zero because otherwise R_0 becomes costant. Now, solving Eq. (B.210) gives

$$f_j(t) = c_{1j}\cos\gamma t + c_{2j}\sin\gamma t\,, \qquad j = 6,7\,. \tag{B.211}$$

Substituting this in Eq. (B.207) gives

$$h_j(t) = \frac{1}{\sqrt{R_2}}\left[-c_{1j}\sin\gamma t + c_{2j}\cos\gamma t\right]\,, \qquad j = 6,7\,. \tag{B.212}$$

Therefore, Eqs. (B.151) and (B.152) become

$$\dot{B}_j(t,\rho) = \frac{\sqrt{R_0}}{R_2}\left(-c_{1j}\sin\gamma t + c_{2j}\cos\gamma t\right)\,, \qquad j = 6,7\,; \tag{B.213}$$

$$B_j'(t,\rho) = \frac{\sqrt{R_1}}{R_2}\left(c_{1j}\cos\gamma t + c_{2j}\sin\gamma t\right)\,, \quad j = 6,7\,. \tag{B.214}$$

Solving these two simultaneously yields

$$B_j(t,\rho) = \frac{\sqrt{R_0}}{\gamma R_2}\left(c_{1j}\cos\gamma t + c_{2j}\sin\gamma t\right) + c_{3j}\,, \quad j = 6,7\,. \tag{B.215}$$

Now, noting that

$$\left(\frac{R_0'}{\sqrt{R_1}R_0}\right)' = -\frac{\gamma R_0'}{R_0^{\frac{3}{2}}}\,, \tag{B.216}$$

Eq. (B.157) becomes

$$\ddot{f}_9\,(t) + \gamma^2 f_9\,(t) = 0\,, \tag{B.217}$$

and its solution can be written as

$$f_9\,(t) = c_9 \cos\gamma t + c_{10}\sin\gamma t\,. \tag{B.218}$$

Therefore, Eq. (B.153) takes the form

$$B_9\,(t,\rho) = \frac{1}{\sqrt{R_1}}\left(c_9\cos\gamma t + c_{10}\sin\gamma t\right)\,, \tag{B.219}$$

and Eq. (B.155) and (B.156) become

$$\dot{B}_8\,(t,\rho) = -\frac{\gamma}{\sqrt{R_0}}\left(c_9\cos\gamma t + c_{10}\sin\gamma t\right)\,, \tag{B.220}$$

$$B_8'\,(t,\rho) = \frac{\gamma\sqrt{R_1}}{R_0}\left(c_9\sin\gamma t - c_{10}\cos\gamma t\right)\,. \tag{B.221}$$

Integrating Eq. (B.220) with respect to t and using Eq. (B.221) to evaluate the function of integration yields,

$$B_8\,(t,\rho) = -\frac{1}{\sqrt{R_0}}\left(c_9\sin\gamma t - c_{10}\cos\gamma t\right) + c_1\,. \tag{B.222}$$

Therefore, Eqs. (2.94)-(2.97) finally give the result as Eqs. (2.119) where we have called c_{37}, c_{36}, c_{17}, c_{27}, c_{16} and c_{26} as c_2, c_3, c_5, c_6, c_7 and c_8 respectively.

Case AIIb2(ii) $\left[\frac{(\sqrt{R_0})'}{\sqrt{R_1}}\right]' \neq 0$

In this case Eq. (B.207) and (B.209) give

$$f_j\,(t) = h_j\,(t) = 0\,, \qquad j = 6, 7\,. \tag{B.223}$$

And, from Eq. (B.151) and (B.152) $B_j = c_{2j}$, $j = 6,7$. Now, Eq. (B.157) gives rise to two cases.

(α) $\frac{R_0}{2\sqrt{R_1}}\left(\frac{R_0'}{R_0\sqrt{R_1}}\right)' = \text{constant} = \eta$ (say),

(β) $\frac{R_0}{2\sqrt{R_1}}\left(\frac{R_0'}{R_0\sqrt{R_1}}\right)' \neq \text{constant}.$

Case AIIb2(ii)α

Here we again discuss two cases depending on whether η is zero or nonzero.

Case AIIb2(ii)α_1 $\qquad \eta = 0$

This implies that $\frac{R_0'}{R_0\sqrt{R_1}}$ =constant $= \lambda \neq 0$. So, Eq. (B.157) yields

$$f_9(t) = c_5 t + c_6 \,, \tag{B.224}$$

so that Eq. (B.178) becomes

$$B_9(t,\rho) = \frac{1}{\sqrt{R_1}}(c_5 t + c_6) \,, \tag{B.225}$$

and Eqs. (B.155) and (B.156) give

$$\dot{B}_8(t,\rho) = -\frac{\lambda}{2}(c_5 t + c_6) \,, \tag{B.226}$$

$$B_8'(t,\rho) = -\frac{\sqrt{R_1}}{R_0}c_5 \,. \tag{B.227}$$

These give

$$B_8(t,\rho) = \left(\frac{1}{\lambda R_0} - \frac{\lambda t^2}{4}\right)c_5 - \frac{\lambda t}{2}c_6 + c_1 \,, \tag{B.228}$$

where we have used the relation $\frac{1}{\lambda}\left(\frac{1}{R_0}\right)' = -\frac{\sqrt{R_1}}{R_0}$. So, finally Eqs. (B.132)-(B.135) yield the RCs as given in Eqs. (2.120). (We have renamed c_{27} and c_{26} as c_2 and c_3.)

Case AIIb2(ii)α_2 $\qquad \eta \neq 0$

In this case Eq. (B.157) has the solution

$$f_9(t) = c_5 e^{\sqrt{\eta}t} + c_6 e^{-\sqrt{\eta}t}, \quad \text{(B.229)}$$

and therefore Eq. (B.153) gives

$$B_9(t,\rho) = \frac{1}{\sqrt{R_1}}\left(c_5 e^{\sqrt{\eta}t} + c_6 e^{-\sqrt{\eta}t}\right). \quad \text{(B.230)}$$

Eqs. (B.155) and (B.156) become

$$\dot{B}_8(t,\rho) = -\frac{R_0'}{2R_0\sqrt{R_1}}\left(c_5 e^{\sqrt{\eta}t} + c_6 e^{-\sqrt{\eta}t}\right), \quad \text{(B.231)}$$

$$B_8'(t,\rho) = -\frac{\sqrt{R_1}\sqrt{\eta}}{R_0}\left(c_5 e^{\sqrt{\eta}t} - c_6 e^{-\sqrt{\eta}t}\right), \quad \text{(B.232)}$$

and we get

$$B_8 = -\frac{1}{2\sqrt{\eta}}\frac{R_0'}{R_0\sqrt{R_1}}\left(c_5 e^{\sqrt{\eta}t} - c_6 e^{-\sqrt{\eta}t}\right) + c_1. \quad \text{(B.233)}$$

So, finally in this case Eqs. (2.94)-(2.97) yield RCs given in Eqs. (2.121).

Case AIIb2(ii)β $\quad \left[\frac{R_0}{\sqrt{R_1}}\left(\frac{R_0'}{R_0\sqrt{R_1}}\right)'\right]' \neq 0$

In this case Eq. (B.157) yields $f_9(t) = 0$, so that, Eqs. (B.153), (B.155) and (B.156) give $B_8 = c_1$, $B_9 = 0$, and finally Eqs. (B.132)-(B.135) yield the result as given in Eqs. (2.122).

B.2 Case B: $\quad R_0' = 0$

Let $R_0 = -\alpha$, where α is a nonzero constant. The RC Eqs. give further conditions as

$$B^0_{,0} = B^0_{,12} = B^0_{,13} = B^0_{,23} = B^1_{,00} = B^1_{,02} = B^1_{,03} = B^2_{,00} = B^2_{,03} = B^3_{,00} = B^3_{,02} = 0\ . \quad \text{(B.234)}$$

Eq. (2.4) for $a = 1$ can thus be written as

$$B^1 = \frac{1}{\sqrt{R_1}} \left(c_{12} t + A_2 (\theta, z) \right) . \tag{B.235}$$

Using this in Eq. (2.5) for $a = 0$, $b = 1$, and integrating with respect to ρ keeping in mind Eq. (B.234) gives

$$B^0 = \frac{c_{12}}{\alpha} \int \sqrt{R_1} d\rho + A_4 (\theta) + A_5 (z) . \tag{B.236}$$

Similarly, Eqs. (2.5) for $a = 0$, $b = 2$ and 3, yield

$$B^2 = \frac{\alpha t}{R_2} A_{4,2} (\theta) + A_6 (\rho, \theta, z) , \tag{B.237}$$

$$B^3 = \frac{\alpha t}{R_3} A_{5,3} (z) + A_7 (\rho, \theta, z) , \tag{B.238}$$

Putting these values in Eqs. (2.5) for $a = 1$, $b = 2$ and 3, we get

$$R_2' A_{4,2} (\theta) = 0 , \tag{B.239}$$

$$R_3' A_{5,3} (z) = 0 . \tag{B.240}$$

These give rise to four cases.

I	$R_2' = 0$,	$R_3' = 0$,
II	$R_2' = 0$,	$R_3' \neq 0$,
III	$R_2' \neq 0$,	$R_3' = 0$,
IV	$R_2' \neq 0$,	$R_3' \neq 0$.

We discuss each in turn.

Case B(I) $\quad R_2' = 0 , \quad R_3' = 0$

We put $R_2 = \beta$ and $R_3 = \gamma$. In this case Eqs. (2.4) for $a = 2$ and 3, take the form

$$\frac{\alpha t}{\beta} A_{4,22} (\theta) + A_{6,2} (\rho, \theta, z) = 0 , \tag{B.241}$$

$$\frac{\alpha t}{\gamma} A_{5,33}(z) + A_{7,3}(\rho, \theta, z) = 0 \,. \tag{B.242}$$

These imply that

$$A_{4,22}(\theta) = A_{5,33}(z) = 0 \,, \tag{B.243}$$

or

$$A_4(\theta) = c_7 \theta + c_1 \,, \tag{B.244}$$

$$A_5(z) = c_8 z + c_0 \,. \tag{B.245}$$

Also, from Eqs. (B.241) and (B.242) we deduce that

$$A_6(\rho, \theta, z) = A_6(\rho, z) \,, \tag{B.246}$$

$$A_7(\rho, \theta, z) = A_7(\rho, \theta) \tag{B.247}$$

Therefore, from Eqs. (2.5) for $a = 1$, $b = 2$ and 3, we get

$$A_{2,22}(\theta, z) = 0 \,, \tag{B.248}$$

$$A_{2,33}(\theta, z) = 0 \,, \tag{B.249}$$

which can be solved to give

$$A_2(\theta, z) = \theta (c_{11} z + c_9) + (c_{10} z + c_5) \,. \tag{B.250}$$

Substituting this value in Eqs. (2.5) for $a = 1$, $b = 2$ and 3, and integrating with respect to ρ yields.

$$A_6(\rho, z) = -\frac{1}{\beta}(c_{11}z + c_9)\int \sqrt{R_1}d\rho + A_8(z) \ , \tag{B.251}$$

$$A_7(\rho, z) = -\frac{1}{\gamma}(c_{11}\theta + c_{10})\int \sqrt{R_1}d\rho + A_9(\theta) \ . \tag{B.252}$$

On substituting these values, Eqs. (2.5) for $a = 2$, $b = 3$, become

$$c_{11}\int \sqrt{R_1}d\rho + \beta A_{8,3}(z) = c_{11}\int \sqrt{R_1}d\rho + \alpha A_{9,2}(\theta) \ , \tag{B.253}$$

which implies that

$$c_{11} = 0 \ , \tag{B.254}$$

$$A_8(z) = c_4 z + c_2 \ , \tag{B.255}$$

$$A_9(\theta) = \frac{\beta}{\alpha}c_4\theta + c_3 \ . \tag{B.256}$$

So, finally the RCs in this case are as given in Eqs. (2.130). (We have replaced $c_0 + c_1$ by c_1)

Case B(II) $\qquad R_2' = 0, \quad R_3' \neq 0,$

We call R_2 as β and Eq. (B.240) gives $A_5(z) = c_0$. Eq. (2.4) for $a = 2$ implies that

$$A_4(\theta) = c_4\theta + c_1 \ , \tag{B.257}$$

$$A_6(\rho, \theta, z) = A_6(\rho, z) \ . \tag{B.258}$$

Eq. (2.4) for $a = 3$ takes the form

$$\frac{R_3'}{2R_3\sqrt{R_1}}[c_5 t + A_2(\theta, z)] + A_{7,3}(\rho, \theta, z) = 0 \, , \tag{B.259}$$

which implies that $c_5 = 0$. Therefore, Eq. (2.5) for $a = 1$, $b = 2$ becomes

$$\sqrt{R_1}A_{2,2}(\theta,z)+\beta A_6'(\rho,z)=0\,, \tag{B.260}$$

which gives

$$A_2(\theta,z)=\theta A_8(z)+A_9(z)\ , \tag{B.261}$$

and therefore

$$A_6(\rho,z)=-\frac{1}{\beta}A_8(z)\int\sqrt{R_1}d\rho+A_{10}(z)\ . \tag{B.262}$$

Now, Eq. (2.5) for $a=1$, $b=3$ on integrating takes the form

$$A_7(\rho,\theta,z)=-\left[\theta A_{8,3}(z)+A_{9,3}(z)\right]\int\frac{\sqrt{R_1}}{R_3}d\rho+A_{11}(\theta,z)\ . \tag{B.263}$$

Eq. (2.5) for $a=2$, $b=3$, implies that

$$A_{11}(\theta,z)=\theta A_{12}(z)+A_{13}(z)\ . \tag{B.264}$$

Putting these values in Eq. (2.4) for $a=3$, we get

$$\frac{R_3'}{2R_3\sqrt{R_1}}\left[\theta A_8(z)+A_9(z)\right]-\left[\theta A_{8,33}(z)+A_{9,33}(z)\right]\int\frac{\sqrt{R_1}}{R_3}d\rho+\theta A_{12,3}(z)+A_{13,3}(z)=0\,. \tag{B.265}$$

This means that

$$A_{8,33}(z)-\left[\frac{R_3}{\sqrt{R_1}}\left(\frac{R_3'}{2R_3\sqrt{R_1}}\right)'\right]A_8(z)=0\,. \tag{B.266}$$

Depending upon whether the term in the square brackets in the last equation is constant or not, we need to discuss two further cases.

(a) $\left[\frac{R_3}{\sqrt{R_1}}\left(\frac{R_3'}{2R_3\sqrt{R_1}}\right)'\right]'\neq 0\,,$

(b) $\left[\frac{R_3}{\sqrt{R_1}}\left(\frac{R_3'}{2R_3\sqrt{R_1}}\right)'\right]'=0\,.$

Case BII(a) $\left[\frac{R_3}{\sqrt{R_1}}\left(\frac{R_3'}{2R_3\sqrt{R_1}}\right)'\right]' \neq 0$

From Eq. (B.266) we have $A_8(z) = 0$. Therefore, from Eq. (B.265) we get $A_{12}(z) = c_6$, and Eq. (2.4) for $a = 3$, gives

$$A_{9,33}(z) - \left[\frac{R_3}{\sqrt{R_1}}\left(\frac{R_3'}{2R_3\sqrt{R_1}}\right)'\right] A_9(z) = 0 . \tag{B.267}$$

Now, as the quantity in the square brackets is not constant, therefore,

$$A_9(z) = 0 . \tag{B.268}$$

Further from Eq. (2.5) for $a = 2$, $b = 3$, we get

$$\beta A_{10,3}(z) + R_3 c_6 = 0 , \tag{B.269}$$

which implies that

$$c_6 = 0 , \tag{B.270}$$

$$A_{10}(z) = c_3 . \tag{B.271}$$

Eq. (B.265) gives

$$A_{13}(z) = c_3 . \tag{B.272}$$

Finally, we get the RC vector as in Eqs. (2.139).

Case BII(b) $\frac{R_3}{\sqrt{R_1}}\left(\frac{R_3'}{2R_3\sqrt{R_1}}\right)' = k_1$

Here there are three further possibilities: $k_1 \gtreqless 0$. We take them up one by one.

Case BIIb(1) $k_1 > 0$

In this case the solution of Eq. (B.266) can be written as

$$A_8(z) = c_7 e^{\sqrt{k_1} z} + c_8 e^{-\sqrt{k_1} z} . \tag{B.273}$$

Substituting this value in Eq. (2.134), Eq. (2.124) becomes (keeping in mind that $c_{12} = 0$)

$$B^1 = \frac{1}{\sqrt{R_1}} \left[\theta \left(c_7 e^{\sqrt{k_1} z} + c_8 e^{-\sqrt{k_1} z} \right) + A_9 (z) \right] . \tag{B.274}$$

Similarly, on using Eqs. (B.273), (2.134), (B.264) and (2.136) in Eq. (2.127), we get

$$B^3 = - \left[\theta \sqrt{k_1} \left(c_7 e^{\sqrt{k_1} z} - c_8 e^{-\sqrt{k_1} z} \right) + A_{9,3} (z) \right] \int \frac{\sqrt{R_1}}{R_3} d\rho + \theta A_{12} (z) + A_{13} (z) . \tag{B.275}$$

Therefore, Eq. (2.4) for $a = 3$, takes the form

$$\begin{aligned} &\tfrac{R_3'}{2R_3\sqrt{R_1}} \left[\theta \left(c_7 e^{\sqrt{k_1} z} + c_8 e^{-\sqrt{k_1} z} \right) + A_9 (z) \right] \\ &- \left[\theta k_1 \left(c_7 e^{\sqrt{k_1} z} + c_8 e^{-\sqrt{k_1} z} \right) + A_{9,33} (z) \right] \int \tfrac{\sqrt{R_1}}{R_3} d\rho \\ &+ \theta A_{12,3} (z) + A_{13,3} (z) = 0 . \end{aligned} \tag{B.276}$$

From here, we see that

$$A_{9,33} (z) - k_1 A_9 (z) = 0 , \tag{B.277}$$

where the solution can be written as

$$A_9 (z) = c_5 e^{\sqrt{k_1} z} + c_6 e^{-\sqrt{k_1} z} . \tag{B.278}$$

Now, Eq. (B.237) on using Eqs. (B.257), (B.262) and (B.274) takes the form

$$B^2 = -\frac{\alpha}{\beta} t c_3 - \frac{1}{\beta} \left(c_7 e^{\sqrt{k_1} z} + c_8 e^{-\sqrt{k_1} z} \right) \int \sqrt{R_1} d\rho + A_{10} (z) . \tag{B.279}$$

Putting from Eqs. (B.275) and (B.279) in Eq. (B.239) we get

$$-\sqrt{R_1} \left(c_7 e^{\sqrt{k_1} z} - c_8 e^{-\sqrt{k_1} z} \right) \left[\int \sqrt{R_1} d\rho + R_3 \int \frac{\sqrt{R_1}}{R_3} d\rho \right] + A_{10,3} (z) + R_3 A_{12} (z) = 0 . \tag{B.280}$$

On using Eq. (B.278), Eq. (B.276) becomes

$$\left[\theta\left(c_7e^{\sqrt{k_1}z}+c_8e^{-\sqrt{k_1}z}\right)+\left(c_5e^{\sqrt{k_1}z}+c_6e^{-\sqrt{k_1}z}\right)\right]\left[\frac{R_3'}{2R_3\sqrt{R_1}}-k_1\int\frac{\sqrt{R_1}}{R_3}d\rho\right] \tag{B.281}$$
$$+\theta A_{12,3}(z)+A_{13,3}(z)=0\,,$$

which yields the values

$$A_{12}(z)=-\frac{1}{\sqrt{k_1}}\left(c_7e^{\sqrt{k_1}z}-c_8e^{-\sqrt{k_1}z}\right)\left[\frac{R_3'}{2R_3\sqrt{R_1}}-k_1\int\frac{\sqrt{R_1}}{R_3}d\rho\right], \tag{B.282}$$

$$A_{13}(z)=-\frac{1}{\sqrt{k_1}}\left(c_5e^{\sqrt{k_1}z}-c_6e^{-\sqrt{k_1}z}\right)\left[\frac{R_3'}{2R_3\sqrt{R_1}}-k_1\int\frac{\sqrt{R_1}}{R_3}d\rho\right]+c_3\,. \tag{B.283}$$

Using Eq. (B.282) we get from Eq. (B.280)

$$\begin{aligned} A_{10,3}(z) &= \sqrt{k_1}\left(c_7e^{\sqrt{k_1}z}-c_8e^{-\sqrt{k_1}z}\right)\left[\int\sqrt{R_1}d\rho+R_3\int\frac{\sqrt{R_1}}{R_3}d\rho\right]\\ &-\frac{R_3}{\sqrt{k_1}}\left(c_7e^{\sqrt{k_1}z}-c_8e^{-\sqrt{k_1}z}\right)\left[\frac{R_3'}{2R_3\sqrt{R_1}}-k_1\int\frac{\sqrt{R_1}}{R_3}d\rho\right]+R_3c_9\,. \end{aligned} \tag{B.284}$$

Now, this holds only if

$$c_7=c_8=c_9=0\,, \tag{B.285}$$

$$A_{10}(z)=c_2\,. \tag{B.286}$$

So, finally Eqs. (B.236), (B.274), (B.279) and (B.275) take the form of Eqs. (2.142).

Case BIIb(2) $\quad k_1<0$

This case is similar to the previous one, the difference being that in Eqs. (2.140) and (2.141) the arguments of the exponential functions will be complex instead of real and we get the similar RC vector with 6RCs.

Case BIIb(3) $\quad k_1=0$

This means that $\frac{R_3'}{2R_3\sqrt{R_1}} = k_2$, where k is a non-zero constant. From Eq. (2.138), we have $A_{8,33}(z) = 0$, whose solution is

$$A_8(z) = c_7 z + c_8 \,. \tag{B.287}$$

Using this value in Eqs. (B.261), (B.262), and (B.263), we get corresponding values of B^1, B^2 and B^3 from Eqs. (B.235), (B.237) and (B.238). Substituting these in RC equations yields

$$A_{9,33}(z) = 0 \,, \tag{B.288}$$

$$A_{12,3}(z) = -k_2 (c_7 z + c_8) \,, \tag{B.289}$$

$$A_{13,3}(z) = -k_2 (c_6 z + c_5) \,. \tag{B.290}$$

On integrating, these give

$$A_9(z) = c_6 z + c_5 \,, \tag{B.291}$$

$$A_{12}(z) = -k_2 \left(c_7 \frac{z^2}{2} + c_8 z \right) + c_9 \,, \tag{B.292}$$

$$A_{13}(z) = -k_2 \left(c_6 \frac{z^2}{2} + c_5 z \right) + c_3 \,. \tag{B.293}$$

Now, substituting these values in Eq. (2.5) for $a = 2$, $b = 3$, gives

$$-c_7 \int \sqrt{R_1} d\rho - c_7 R_3 \int \frac{\sqrt{R_1}}{R_3} d\rho + R_3 c_9 - R_3 k_2 \left(c_7 \frac{z^2}{2} + c_8 z \right) + A_{10,3}(z) = 0 \,. \tag{B.294}$$

This implies

$$c_7 = c_8 = c_9 = 0 \,, \tag{B.295}$$

$$A_{10}(z) = c_2 \,. \tag{B.296}$$

So B^i, finally is given by Eqs. (2.144).

Case B(III) $\qquad R_2' \neq 0 \,, \quad R_3' = 0$

As the RC equations remain unchanged if we interchange indices 2 and 3, the results for this case can be obtained by interchanging these indices (i.e. θ and z coordinates) in Case B(II).

Case B(IV) $\qquad R_2' \neq 0\,, \quad R_3' \neq 0$

In this case Eqs. (B.239) and (B.240) yield

$$A_4(\theta) = c_0\,,\ A_5(z) = c_1\,, \tag{B.297}$$

If we substitute Eqs. (B.235)-(B.238) in RC equations, the only surviving equations are

$$\sqrt{R_1}A_{2,2}(\theta,z) + R_2A_6'(\rho,\theta,z) = 0\,, \tag{B.298}$$

$$\sqrt{R_1}A_{2,3}(\theta,z) + R_3A_7'(\rho,\theta,z) = 0\,, \tag{B.299}$$

$$\frac{R_2'}{2R_2\sqrt{R_1}}\left[c_1t + A_2(\theta,z)\right] + A_{6,2}(\rho,\theta,z) = 0\,, \tag{B.300}$$

$$R_2A_{6,3}(\rho,\theta,z) + R_3A_{7,2}(\rho,\theta,z) = 0\,, \tag{B.301}$$

$$\frac{R_3'}{2R_3\sqrt{R_1}}\left[c_1t + A_2(\theta,z)\right] + A_{7,3}(\rho,\theta,z) = 0\,. \tag{B.302}$$

It is clear from Eqs. (B.300) and (B.302) $c_1 = 0$. So, the B^i from Eqs. (B.235)-(B.238) have the form

$$B^0 = c_0 + c_1\,, \tag{B.303}$$

$$B^1 = \frac{1}{\sqrt{R_1}}A_2(\theta,z)\,, \tag{B.304}$$

$$B^2 = A_6(\rho,\theta,z)\,, \tag{B.305}$$

$$B^3 = A_7(\rho,\theta,z)\,. \tag{B.306}$$

Differentiating Eq. (B.298) with respect to θ and Eq. (B.300) with respect to ρ and subtracting yields

$$A_{2,22}(\theta,z) - \left[\frac{R_2}{\sqrt{R_1}}\left(\frac{R_2'}{2R_2\sqrt{R_1}}\right)'\right]A_2(\theta,z) = 0\,. \tag{B.307}$$

This gives rise to two possibilities (as before) depending upon whether the term in the square brackets is a function or a constant.

(a) $\qquad \left[\frac{R_2}{\sqrt{R_1}}\left(\frac{R_2'}{2R_2\sqrt{R_1}}\right)'\right]' \neq 0\,,$

(b) $\qquad \left[\frac{R_2}{\sqrt{R_1}}\left(\frac{R_2'}{2R_2\sqrt{R_1}}\right)'\right]' = 0\,.$

Here we will discuss them.

Case BIV(a) $\left[\frac{R_2}{\sqrt{R_1}}\left(\frac{R_2'}{2R_2\sqrt{R_1}}\right)'\right]' \neq 0$

We have $A_2(\theta, z) = 0$. In this case Eqs. (B.298)-(B.302) yield

$$A_6(\rho, \theta, z) = A_6(\theta, z) \ , \tag{B.308}$$

$$A_7(\rho, \theta, z) = A_7(\theta, z) \ , \tag{B.309}$$

$$A_6(\theta, z) = A_6(z) \ , \tag{B.310}$$

$$A_7(\theta, z) = A_7(\theta) = 0 \ , \tag{B.311}$$

$$R_2 A_{6,3}(z) + R_3 A_{7,2}(\theta) = 0 \ . \tag{B.312}$$

This gives

$$A_6(z) = c_4 z + c_2 \ , \tag{B.313}$$

$$A_7(\theta) = -\frac{R_2}{R_3} c_4 \theta + c_3 \ . \tag{B.314}$$

Now, this is possible only if either $c_4 = 0$ or $\frac{R_2}{R_3}$ is constant.

Case BIVa(1) $\left(\frac{R_2}{R_3}\right)' \neq 0$

So, $c_4 = 0$ and the result is the minimal symmetry.

Case BIVa(2) $\left(\frac{R_2}{R_3}\right)' = 0$

Writing $R_2/R_3 = k$, and $c_0 + c_1$ as c_1 the RC vector in this case is given by Eqs. (2.153).

Case BIV(b) $\left[\frac{R_2}{\sqrt{R_1}}\left(\frac{R_2'}{2R_2\sqrt{R_1}}\right)'\right]' = 0$

We put $\frac{R_2}{\sqrt{R_1}}\left(\frac{R_2'}{2R_2\sqrt{R_1}}\right)' = k_3$, where there are further three possibilities for the constant: $k_3 \gtreqless 0$.

Case BIVb(1) $k_3 > 0$

Eq. (B.307) becomes

$$A_{2,22}(\theta, z) - k_3 A_2(\theta, z) = 0 \ , \tag{B.315}$$

which can be solved to give

$$A_2(\theta, z) = A_8(z)\, e^{\sqrt{k_3}\theta} + A_9(z)\, e^{-\sqrt{k_3}\theta} . \tag{B.316}$$

Substituting this value in Eq. (B.300) and (B.302) and integrating with respect to θ and z respectively, yields

$$A_6(\rho, \theta, z) = -\frac{1}{\sqrt{k_3}}\left(\frac{R_2'}{2R_2\sqrt{R_1}}\right)\left[A_8(z)\, e^{\sqrt{k_3}\theta} - A_9(z)\, e^{-\sqrt{k_3}\theta}\right] + A_{10}(\rho, z) , \tag{B.317}$$

$$A_7(\rho, \theta, z) = -\left(\frac{R_3'}{2R_3\sqrt{R_1}}\right)\left[e^{\sqrt{k_3}\theta}\int A_8(z)\, dz + e^{-\sqrt{k_3}\theta}\int A_9(z)\, dz\right] + A_{11}(\rho, z) . \tag{B.318}$$

Now, Eq. (B.298) implies

$$A_{10}(\rho, z) = A_{10}(z) . \tag{B.319}$$

Eq. (B.299) yields two equations

$$A_{8,33}(z) - \left[\frac{R_3}{\sqrt{R_1}}\left(\frac{R_3'}{2R_3\sqrt{R_1}}\right)'\right] A_8(z) = 0 , \tag{B.320}$$

$$A_{9,33}(z) - \left[\frac{R_3}{\sqrt{R_1}}\left(\frac{R_3'}{2R_3\sqrt{R_1}}\right)'\right] A_9(z) = 0 . \tag{B.321}$$

These equations again suggest two further possibilities depending upon whether the term in the square brackets is constant or not.

(i) $\left[\frac{R_3}{\sqrt{R_1}}\left(\frac{R_3'}{2R_3\sqrt{R_1}}\right)'\right]' \neq 0$,

(ii) $\left[\frac{R_3}{\sqrt{R_1}}\left(\frac{R_3'}{2R_3\sqrt{R_1}}\right)'\right]' = 0$.

Case BIVb1(i) $\left[\frac{R_3}{\sqrt{R_1}}\left(\frac{R_3'}{2R_3\sqrt{R_1}}\right)'\right]' \neq 0$

Here, from Eqs. (B.320) and (B.321) we have $A_8(z) = 0$, $A_9(z) = 0$, i.e., $A_2(\theta, z) = 0$. Therefore, from Eqs. (B.298)-(B.300) and (B.302), we note that

$$A_6(\rho, \theta, z) = A_6(z) \ , \ A_7(\rho, \theta, z) = A_7(\theta) \ , \tag{B.322}$$

and from Eq. (B.301) we find that

$$A_6(z) = c_4 z + c_2 , \tag{B.323}$$

$$A_7(\theta) = -\frac{R_2}{R_3} c_4 \theta + c_3 . \tag{B.324}$$

The RC vector finally is as given in Eqs. (2.158).

Case BIVb1(ii) $\qquad \left[\frac{R_3}{\sqrt{R_1}}\left(\frac{R_3'}{2R_3\sqrt{R_1}}\right)'\right]' = 0$

We put $\frac{R_3}{\sqrt{R_1}}\left(\frac{R_3'}{2R_3\sqrt{R_1}}\right)' = k_1$. Now, there are again three possibilities, $k_1 \gtreqless 0$. We first consider $k_1 > 0$. In this case the solution of Eqs. (B.320)-(B.321) can be written as

$$A_8(z) = c_5 e^{\sqrt{k_1} z} + c_6 e^{-\sqrt{k_1} z} , \tag{B.325}$$

$$A_9(z) = c_7 e^{\sqrt{k_1} z} + c_8 e^{-\sqrt{k_1} z} . \tag{B.326}$$

Substituting these values in Eqs. (B.316)-(B.319) and (B.299), we get

$$A_{11}(\rho,\theta) = A_{11}(\theta) , \tag{B.327}$$

$$\begin{aligned} A_6(\rho,\theta,z) &= -\frac{1}{\sqrt{R_3}}\left(\frac{R_2'}{2R_2\sqrt{R_1}}\right)\left\{\left[c_5 e^{\sqrt{k_1}z} + c_6 e^{-\sqrt{k_1}z}\right] e^{\sqrt{k_3}\theta}\right. \\ &\quad \left. - \left[c_7 e^{\sqrt{k_1}z} + c_8 e^{-\sqrt{k_1}z}\right] e^{-\sqrt{k_3}\theta}\right\} + A_{10}(z) , \end{aligned} \tag{B.328}$$

$$\begin{aligned} A_7(\rho,\theta,z) &= -\frac{1}{\sqrt{k_1}}\left(\frac{R_3'}{2R_3\sqrt{R_1}}\right)\left\{\left[c_5 e^{\sqrt{k_1}z} - c_6 e^{-\sqrt{k_1}z}\right] e^{\sqrt{k_3}\theta}\right. \\ &\quad \left. + \left[c_7 e^{\sqrt{k_1}z} - c_8 e^{-\sqrt{k_1}z}\right] e^{-\sqrt{k_3}\theta}\right\} + A_{11}(\theta) . \end{aligned} \tag{B.329}$$

Now, with these values all the RC equations are satisfied except Eq. (2.5) for $a = 2$, $b = 3$, which takes the following form.

$$R_2 A_{10,3}(z) + R_3 A_{11,2}(\theta) - \left\{\left[c_5 e^{\sqrt{k_1}z} - c_6 e^{-\sqrt{k_1}z}\right] e^{\sqrt{k_3}\theta} - \left[c_7 e^{\sqrt{k_1}z} - c_8 e^{-\sqrt{k_1}z}\right] e^{-\sqrt{k_3}\theta}\right\}$$
$$\times \left\{R_2\left(\frac{R_2'}{2R_2\sqrt{R_1}}\right)\frac{\sqrt{k_1}}{\sqrt{k_3}} + R_3\left(\frac{R_3'}{2R_3\sqrt{R_1}}\right)\frac{\sqrt{k_3}}{\sqrt{k_1}}\right\} = 0 \,. \tag{B.330}$$

This implies that

$$c_5 = c_6 = c_7 = c_8 = 0 \,, \tag{B.331}$$

$$A_{10}(z) = c_4 z + c_2 \,, \tag{B.332}$$

$$A_{11}(\theta) = c_4\theta + c_3 \,. \tag{B.333}$$

So, the result in this case is given by Eqs. (2.161).

We see that for $k_1 < 0$ and $k_1 = 0$, we get the same result as in this case.

Case BIVb(2) $\qquad k_3 < 0$

In this case the solution (Eq. (B.316)) of Eq. (B.315) will involve complex arguments for exponential functions and the results can be obtained similarly as in Case BIVb(1).

Case BIVb(3) $\qquad k_3 = 0$

In this case, we have $\frac{R_2'}{2R_2\sqrt{R_1}} = k_4$, where k_4 is a nonzero constant. From Eq. (B.307) therefore, we get $A_{2,22}(\theta, z) = 0$, which has the solution

$$A_2(\theta, z) = \theta A_8(z) + A_9(z) \,. \tag{B.334}$$

Eqs. (B.298)-(B.299), on integration with respect to θ and z respectively yield

$$A_6(\rho, \theta, z) = -A_8(z)\int \frac{\sqrt{R_1}}{R_2}d\rho + A_{10}(\theta, z) \,, \tag{B.335}$$

$$A_7(\rho, \theta, z) = -\left[\theta A_{8,3}(z) + A_{9,3}(z)\right]\int \frac{\sqrt{R_1}}{R_3}d\rho + A_{11}(\theta, z) \,. \tag{B.336}$$

Using these in Eqs. (B.300) and (B.302) we get

$$A_{10}(\theta, z) = -k_4\left[\frac{\theta^2}{2}A_8(z) + \theta A_9(z)\right] + A_{12}(z) \,, \tag{B.337}$$

$$\frac{R_3}{\sqrt{R_1}}\left(\frac{R_3'}{2R_3\sqrt{R_1}}\right)'\left[\theta A_8\left(z\right)+A_9\left(z\right)\right]-\left[\theta A_{8,33}\left(z\right)+A_{9,33}\left(z\right)\right] . \quad \text{(B.338)}$$

which implies that

$$A_{8,33}\left(z\right)-\left[\frac{R_3}{\sqrt{R_1}}\left(\frac{R_3'}{2R_3\sqrt{R_1}}\right)'\right]A_8\left(z\right)=0 , \quad \text{(B.339)}$$

$$A_{9,33}\left(z\right)-\left[\frac{R_3}{\sqrt{R_1}}\left(\frac{R_3'}{2R_3\sqrt{R_1}}\right)'\right]A_9\left(z\right)=0 . \quad \text{(B.340)}$$

This again implies two possibilities, depending on whether the term in the square brackets is constant or not.

Case BIVb3(i) $\quad \left[\frac{R_3}{\sqrt{R_1}}\left(\frac{R_3'}{2R_3\sqrt{R_1}}\right)'\right]\neq 0$

Eqs. (B.339) and (B.340) imply that $A_8\left(z\right)=0=A_9\left(z\right)$, and the solution is similar to Case BIVb1(i).

Case BIVb3(ii) $\quad \left[\frac{R_3}{\sqrt{R_1}}\left(\frac{R_3'}{2R_3\sqrt{R_1}}\right)'\right]=0$

We put $\frac{R_3}{\sqrt{R_1}}\left(\frac{R_3'}{2R_3\sqrt{R_1}}\right)'=k_1$, where k_1 is a constant which can be greater than, equal to or less than zero.

Case BIVb3(ii)α $\quad k_1>0$

This solution here is similar to Case BIVb1(ii)γ, where $k_1=0$, $k_3>0$.

Case BIVb3(ii)β $\quad k_1<0$

This case is similar to the subcase of Case BIVb(2), where $k_1=0,\ k_3<0$.

Case BIVb3(ii)γ $\quad k_1=0$

Here Eqs. (B.339) and (B.340) give

$$A_8\left(z\right)=c_{11}z+c_5 , \quad \text{(B.341)}$$

$$A_9\left(z\right)=c_6z+c_4 . \quad \text{(B.342)}$$

In this case Eqs. (B.335)-(B.337) take the form

$$A_6\left(\rho,\theta,z\right)=-\left(c_{11}z+c_5\right)\int\frac{\sqrt{R_1}}{R_2}d\rho-k_4\left[\frac{\theta^2}{2}\left(c_{11}z+c_5\right)+\theta\left(c_6z+c_4\right)\right]+A_{12}\left(z\right) , \quad \text{(B.343)}$$

$$A_7(\rho,\theta,z) = -(\theta c_{11} z + c_6)\int \frac{\sqrt{R_1}}{R_3} + A_{11}(\theta,z) \ . \tag{B.344}$$

Substituting these values in Eq. (B.302) and integrating with respect to z yields

$$A_{11}(\theta,z) = -k_2\left[\theta\left(c_{11}\frac{z^2}{2} + c_5 z\right) + c_6\frac{z^2}{2} + c_7 z\right] + A_{13}(\theta) \ . \tag{B.345}$$

Eq. (B.301) now, takes the form

$$\begin{aligned} &R_2\left[-c_{11}\int\frac{\sqrt{R_1}}{R_2}d\rho - k_4\left(c_{11}\frac{\theta^2}{2} + c_6\theta\right) + A_{12,3}(z)\right] \\ &+R_3\left[-c_{11}\int\frac{\sqrt{R_1}}{R_3}d\rho - k_2\left(c_{11}\frac{z^2}{2} + c_5 z\right) + A_{13,2}(\theta)\right] = 0 \, , \end{aligned} \tag{B.346}$$

which implies that

$$-k_4(\theta c_{11} + c_6) + \frac{R_3}{R_2}A_{13,22}(\theta) = 0 \tag{B.347}$$

or

$$\left(\frac{R_3}{R_2}\right)' A_{13,22}(\theta) = 0 \ . \tag{B.348}$$

The last equation gives rise to further two cases, depending upon, whether $\frac{R_3}{R_2}$ is a constant or not.

Case BIVb3(ii)γ_1 $\quad\quad\left(\frac{R_3}{R_2}\right)' \neq 0$

In this case, we get from Eqs. (B.347) and (B.348), $c_{11} = c_6 = 0$,

$$A_{13}(\theta) = c_8\theta + c_3 \ , \tag{B.349}$$

and Eq. (B.346) further implies $c_5 = c_8 = 0$ and $A_{12}(z) = c_2$ .So, the RC vector (Eqs. (B.303)-(B.306)) in this case is given by Eqs. (2.166).

Case BIVb3(ii)γ_2 $\quad\quad\left(\frac{R_3}{R_2}\right)' = 0$

We put $\frac{R_3}{R_2} = -d$. In this case k_2 becomes equal to k_4. Eq. (B.346) in this case gives $c_{11} = 0$,

$$A_{12}(z) = -dk_2c_6\frac{z^2}{2} - c_4 z + c_2 \ . \tag{B.350}$$

Eq. (B.347) gives

$$A_{13}(\theta) = -k_2 c_7 \frac{\theta^2}{2d} - \frac{c_4}{d}\theta + c_3 \,. \tag{B.351}$$

Using those values in Eqs. (B.343)-(B.344) and (B.345), we finally get from Eqs. (B.303)-(B.306), the RCs as given in Eqs. (2.167).

Appendix C

Ricci Collineations for the Degenerate Ricci Tensor

In this appendix we give the detailed calculations of Case I of Section 2.3.

Case I $\qquad R_0 = 0\,, \quad R_1 \neq 0\,, \quad R_2 \neq 0\,, \quad R_3 \neq 0$

In this case from the RC equations we see that B^0 is completely an arbitrary function of t, ρ and z. Further, from Eqs. (2.5) for $a = 0$ $b = 1, 2, 3$, we obtain $B^1_{,0} = B^2_{,0} = B^3_{,0} = 0$, and therefore, on integration Eq. (2.4) for $a = 1$, gives

$$B^1 = \frac{A_1(\theta, z)}{\sqrt{R_1}}\,. \tag{C.1}$$

Putting this in Eq. (2.4) for $a = 2$, and integrating with respect to θ gives

$$B^2 = -\frac{R_2'}{2R_2\sqrt{R_1}} \int A_1(\theta, z)\, d\theta + B_1(\rho, z)\,. \tag{C.2}$$

Similarly, from Eq. (2.4) for $a = 3$, we get

$$B^3 = -\frac{R_3'}{2R_3\sqrt{R_1}} \int A_1(\theta, z)\, dz + B_2(\rho, \theta)\,. \tag{C.3}$$

Now, differentiating Eq. (2.5) for $a = 1$, $b = 2$, with respect to θ and substituting from above yields

$$A_{1,22}(\theta,z)+\frac{R_2}{\sqrt{R_1}}\left(-\frac{R_2'}{2R_2\sqrt{R_1}}\right)'A_1(\theta,z)=0\,, \tag{C.4}$$

which suggests two possibilities.

(a) $\left[\frac{R_2}{\sqrt{R_1}}\left(-\frac{R_2'}{2R_2\sqrt{R_1}}\right)'\right]' \neq 0\,,$

(b) $\left[\frac{R_2}{\sqrt{R_1}}\left(-\frac{R_2'}{2R_2\sqrt{R_1}}\right)'\right]' = 0\,.$

We take these one by one.

Case I(a)

In this case from Eqs. (C.1), (C.2) and (C.3) we have $B^1=0$, $B^2=B_1(\rho,z)$, $B^3=B_2(\rho,\theta)$, and Eqs. (2.5) for $a=1$, $b=2$ and 3, give $B^2=B_1(z)$, $B^2=B_2(\theta)$. From Eq. (2.5) for $a=2$, $b=3$ we have

$$R_2B_{1,3}(z)+R_3B_{2,2}(\theta)=0\,, \tag{C.5}$$

which yields

$$B_1(z)=c_1z+c_2\,, \tag{C.6}$$

$$B_2(\theta)=c_3\theta+c_4\,. \tag{C.7}$$

So, the complete solution for this case is as given in Eqs. (2.263).

Case I(b)

Here, we put $\frac{R_2}{\sqrt{R_1}}\left(-\frac{R_2'}{2R_2\sqrt{R_1}}\right)'=k_1$, a constant, and discuss three cases: $k_1 \gtreqless 0$.

Case Ib(1) $\quad k_1>0$

The solution of Eq. (C.4) in this case can be written as

$$A_1(\theta,z)=B_3(z)\,e^{i\sqrt{k_1}\theta}+B_4(z)\,e^{-i\sqrt{k_1}\theta}\,. \tag{C.8}$$

Now differentiating Eq. (2.5) for $a=1$, $b=3$, with respect to z yields

$$\begin{aligned}&\left[B_{3,33}(z)\, e^{i\sqrt{k_1}\theta} + B_{4,33}(z)\, e^{-i\sqrt{k_1}\theta}\right]\\&+\tfrac{R_3}{\sqrt{R_1}}\left(-\tfrac{R_3'}{2R_3\sqrt{R_1}}\right)'\left[B_3(z)\, e^{i\sqrt{k_1}\theta} + B_4(z)\, e^{-i\sqrt{k_1}\theta}\right] = 0\ .\end{aligned} \tag{C.9}$$

Note that $\frac{R_3}{\sqrt{R_1}}\left(-\frac{R_3'}{2R_3\sqrt{R_1}}\right)' = k_2$, is a constant. So, we have

$$e^{2i\sqrt{k_1}\theta}\left[B_{3,33}(z) + k_2 B_3(z)\right] + \left[B_{4,33}(z) + k_2 B_4(z)\right] = 0\ , \tag{C.10}$$

which yields

$$B_{3,33}(z) + k_2 B_3(z) = 0\ , \tag{C.11}$$

$$B_{4,33}(z) + k_2 B_4(z) = 0\ . \tag{C.12}$$

Here, again we have the following possibilities: $k_2 \gtreqless 0$.

Case Ib1(i) $\qquad k_2 > 0$

In this case the solution of Eqs. (C.11) and (C.12) can be written as

$$B_3(z) = c_1 e^{i\sqrt{k_2}z} + c_2 e^{-i\sqrt{k_2}z}\ , \tag{C.13}$$

$$B_4(z) = c_3 e^{i\sqrt{k_2}z} + c_4 e^{-i\sqrt{k_2}z}\ . \tag{C.14}$$

Substituting from Eqs. (C.8), (C.13) and (C.14), we get from Eqs. (C.1), (C.2) and (C.3)

$$B^1 = \frac{1}{\sqrt{R_1}}\left[e^{i\sqrt{k_1}\theta}\left(c_1 e^{i\sqrt{k_2}z} + c_2 e^{-i\sqrt{k_2}z}\right) + e^{-i\sqrt{k_1}\theta}\left(c_3 e^{i\sqrt{k_2}z} + c_4 e^{-i\sqrt{k_2}z}\right)\right]\ , \tag{C.15}$$

$$\begin{aligned}B^2 = &\tfrac{iR_2'}{2R_2\sqrt{R_1}\sqrt{k_1}}\left[e^{i\sqrt{k_1}\theta}\left(c_1 e^{i\sqrt{k_2}z} + c_2 e^{-i\sqrt{k_2}z}\right)\right.\\&\left.- e^{-i\sqrt{k_1}\theta}\left(c_3 e^{i\sqrt{k_2}z} + c_4 e^{-i\sqrt{k_2}z}\right) + B_1(\rho, z)\right]\ ,\end{aligned} \tag{C.16}$$

$$B^3 = \frac{iR_3'}{2R_3\sqrt{R_1}\sqrt{k_2}}\left[e^{i\sqrt{k_1}\theta}\left(c_1 e^{i\sqrt{k_2}z} - c_2 e^{-i\sqrt{k_2}z}\right)\right.$$
$$\left. + e^{-i\sqrt{k_1}\theta}\left(c_3 e^{i\sqrt{k_2}z} - c_4 e^{-i\sqrt{k_2}z}\right)\right] + B_2(\rho, z)\,. \tag{C.17}$$

Putting these in Eq. (2.5) for $a = 1$, $b = 2$ and integrating with respect to ρ gives

$$B_1(\rho, z) = -i\sqrt{k_1}\left[e^{i\sqrt{k_1}\theta}\left(c_1 e^{i\sqrt{k_2}z} + c_2 e^{-i\sqrt{k_2}z}\right)\right.$$
$$\left. - e^{-i\sqrt{k_1}\theta}\left(c_3 e^{i\sqrt{k_2}z} + c_4 e^{-i\sqrt{k_2}z}\right)\right]\int\frac{\sqrt{R_1}}{R_2}d\rho - \frac{iR_2'}{2R_2\sqrt{R_1}\sqrt{k_1}}\left[e^{i\sqrt{k_1}\theta}\left(c_1 e^{i\sqrt{k_2}z} + c_2 e^{-i\sqrt{k_2}z}\right)\right.$$
$$\left. - e^{-i\sqrt{k_1}\theta}\left(c_3 e^{i\sqrt{k_2}z} + c_4 e^{-i\sqrt{k_2}z}\right)\right] + B_3(z)\,. \tag{C.18}$$

But $B_1(\rho, z)$ is a function of ρ and z only and we must have $B_{1,2}(\rho, z) = 0$. This makes all the terms on the right hand side of Eq. (C.18) vanish except the last one , i.e., $B_1(\rho, z) = B_3(z)$, and similarly from Eq. (2.5) for $a = 1$, $b = 3$ we find that $B_2(\rho, \theta) = B_4(\theta)$. Now, after putting these values in Eqs. (C.16) and (C.17), we get from Eq. (2.5) for $a = 2$, $b = 3$

$$\frac{1}{2}\left(\sqrt{\frac{k_1}{k_2}} + \sqrt{\frac{k_2}{k1}}\right)\left[e^{i\sqrt{k_1}\theta}\left(c_1 e^{i\sqrt{k_2}z} - c_2 e^{-i\sqrt{k_2}z}\right)\right.$$
$$\left. - e^{-i\sqrt{k_1}\theta}\left(c_3 e^{i\sqrt{k_2}z} - c_4 e^{-i\sqrt{k_2}z}\right)\right]$$
$$-R_2\sqrt{R_1}B_{3,3}(z) - R_3\sqrt{R_1}B_{4,2}(\theta) = 0\,. \tag{C.19}$$

Differentiating this, first with respect to θ and then with respect to z yields

$$\left(\sqrt{\frac{k_1}{k_2}} + \sqrt{\frac{k_2}{k_1}}\right)\left[e^{i\sqrt{k_1}\theta}\left(c_1 e^{i\sqrt{k_2}z} + c_2 e^{-i\sqrt{k_2}z}\right) + e^{-i\sqrt{k_1}\theta}\left(c_3 e^{i\sqrt{k_2}z} + c_4 e^{-i\sqrt{k_2}z}\right)\right] = 0\,. \tag{C.20}$$

This gives rise to two cases depending upon which of the two factors is zero.

Case Ib1(i)α

First, we consider

$$e^{i\sqrt{k_1}\theta}\left(c_1 e^{i\sqrt{k_2}z} + c_2 e^{-i\sqrt{k_2}z}\right) + e^{-i\sqrt{k_1}\theta}\left(c_3 e^{i\sqrt{k_2}z} + c_4 e^{-i\sqrt{k_2}z}\right) = 0\,. \tag{C.21}$$

This expression and its derivatives with respect to θ and z make the terms in the square brackets of Eqs. (C.15), (C.16) and (C.17) vanish and we are left with

$$B^1 = 0\ ,\ B^2 = B_3(z)\ ,\ B^3 = B_4(\theta)\ , \tag{C.22}$$

and Eq. (C.19) implies that

$$B_3(z) = c_5 z + c_6\ , \tag{C.23}$$

$$B_{4,2}(\theta) = -\frac{R_2}{R_3} c_5\ , \tag{C.24}$$

which implies that $\frac{R_2}{R_3}$ =constant= k_3(say). Therefore,

$$B_4(\theta) = -c_5 k_3 \theta + c_7\ . \tag{C.25}$$

So, we finally get the result as given in Eqs. (2.270).

Case Ib1(i)β

Here $\sqrt{\frac{k_1}{k_2}} + \sqrt{\frac{k_2}{k_1}} = 0$, which implies that $k_1 = -k_2$. In this case again we get Eqs. (C.23) and (C.25). Therefore, we write Eqs. (C.16), (C.17) and (C.15) as

$$B^1 = \frac{1}{\sqrt{R_1}}\left[e^{i\sqrt{k_1}\theta}\left(c_1 e^{i\sqrt{k_2}z} + c_2 e^{-i\sqrt{k_2}z}\right) + e^{-i\sqrt{k_1}\theta}\left(c_3 e^{i\sqrt{k_2}z} + c_4 e^{-i\sqrt{k_2}z}\right)\right]\ , \tag{C.26}$$

$$\begin{aligned} B^2 = {} & \tfrac{iR_2'}{2R_2\sqrt{R_1}\sqrt{k_1}}\left[e^{i\sqrt{k_1}\theta}\left(c_1 e^{i\sqrt{k_2}z} + c_2 e^{-i\sqrt{k_2}z}\right) - e^{-i\sqrt{k_1}\theta}\left(c_3 e^{i\sqrt{k_2}z} + c_4 e^{-i\sqrt{k_2}z}\right)\right] \\ & + c_5 z + c_6\ , \end{aligned} \tag{C.27}$$

$$\begin{aligned} B^3 = {} & \tfrac{iR_3'}{2R_3\sqrt{R_1}\sqrt{k_2}}\left[e^{i\sqrt{k_1}\theta}\left(c_1 e^{i\sqrt{k_2}z} - c_2 e^{-i\sqrt{k_2}z}\right) + e^{-i\sqrt{k_1}\theta}\left(c_3 e^{i\sqrt{k_2}z} - c_4 e^{-i\sqrt{k_2}z}\right)\right] \\ & + c_5 \tfrac{R_2}{R_3}\theta + c_7\ , \end{aligned} \tag{C.28}$$

Now, putting these in Eq. (2.5) for $a = 2$, $b = 3$, gives

$$\frac{\sqrt{k_2}}{2\sqrt{R_1}\sqrt{k_1}}\left[e^{i\sqrt{k_1}\theta}\left(c_1e^{i\sqrt{k_2}z}-c_2e^{-i\sqrt{k_2}z}\right)-e^{-i\sqrt{k_1}\theta}\left(c_3e^{i\sqrt{k_2}z}-c_4e^{-i\sqrt{k_2}z}\right)\right]\left(R_2'-R_3'\right)=0\,. \tag{C.29}$$

Now, $R_2' \neq R_3'$, because if $R_2' = R_3'$ we get $k_1 = k_2$, which contradicts that $k_1 = -k_2$. So, the factor in the square bracket in Eq. (C.29) must be zero. This gives the same solution as in the previous case in Eq. (2.270) with the only difference that here $k_3 = -1$. So, here we have the RCs from Eqs. (2.271).

Case Ib1(ii) $\qquad k_2 = 0 \quad$ (or $\frac{R_3'}{2R_3\sqrt{R_1}} = k_3$, a constant.)

In this case the solution of Eqs. (2.266) and (2.267) is

$$B_3(z) = c_1 z + c_2\,, \tag{C.30}$$

$$B_4(z) = c_3 z + c_4\,. \tag{C.31}$$

Therefore, Eq. (2.264) becomes

$$A_1(\theta, z) = e^{i\sqrt{k_1}\theta}(c_1 z + c_2) + e^{-i\sqrt{k_1}\theta}(c_3 z + c_4)\;, \tag{C.32}$$

so that Eqs. (C.1), (C.2) and (C.3) take he form

$$B^1 = \frac{1}{\sqrt{R_1}}\left[e^{i\sqrt{k_1}\theta}(c_1 z + c_2) + e^{-i\sqrt{k_1}\theta}(c_3 z + c_4)\right]\;, \tag{C.33}$$

$$B^2 = -\frac{iR_2'\sqrt{k_1}}{2R_2\sqrt{R_1}}\left[e^{i\sqrt{k_1}\theta}(c_1 z + c_2) - e^{-i\sqrt{k_1}\theta}(c_3 z + c_4)\right] + B_1(\rho, z)\;, \tag{C.34}$$

$$B^3 = -k_3\left[e^{i\sqrt{k_1}\theta}\left(c_1\frac{z^2}{2} + c_2 z\right) + e^{-i\sqrt{k_1}\theta}\left(c_3\frac{z^2}{2} + c_4 z\right)\right] + B_2(\rho, z)\,. \tag{C.35}$$

Now, putting these in Eq. (2.5) for $a = 1$, $b = 2$, gives

$$\left[e^{i\sqrt{k_1}\theta}\left(c_1z+c_2\right)-e^{-i\sqrt{k_1}\theta}\left(c_3z+c_4\right)\right]\left(i\sqrt{k_1}-i\left(k_1\right)^{\frac{3}{2}}\right)+\frac{R_2}{\sqrt{R_1}}B_1'\left(\rho,z\right)=0\,, \qquad (C.36)$$

which implies

$$B_1\left(\rho,z\right)=B_1\left(z\right)\,. \qquad (C.37)$$

Similarly, from Eq. (2.5) for $a=1$, $b=3$, on integrating this with respect to ρ we get

$$B_{1,3}\left(z\right)=0=B_{5,2}\left(\theta\right)\,, \qquad (C.38)$$

or

$$B_2\left(\rho,\theta\right)=-\left[c_1e^{i\sqrt{k_1}\theta}+c_3e^{-i\sqrt{k_1}\theta}\right]\int\frac{\sqrt{R_1}}{R_3}d\rho+B_5\left(\theta\right)\,. \qquad (C.39)$$

Eq. (2.5) for $a=2$, $b=3$, therefore, becomes

$$\begin{array}{l}\frac{iR_2'\sqrt{k_1}}{\sqrt{R_1}}\left[c_1e^{i\sqrt{k_1}\theta}-c_3e^{-i\sqrt{k_1}\theta}\right]+B_{1,3}\left(z\right)\\ -iR_3k_3\sqrt{k_1}\left[e^{i\sqrt{k_1}\theta}\left(c_1\frac{z}{2}^2+c_2z\right)-e^{-i\sqrt{k_1}\theta}\left(c_3\frac{z}{2}^2+c_4z\right)\right]\\ -iR_3\sqrt{k_1}\left[c_1e^{i\sqrt{k_1}\theta}-c_3e^{-i\sqrt{k_1}\theta}\right]\int\frac{\sqrt{R_1}}{R_3}d\rho+B_{5,2}\left(\theta\right)=0\,,\end{array} \qquad (C.40)$$

which shows that

$$B_1=c_5, \qquad B_5=c_6\,. \qquad (C.41)$$

So, finally the RCs from Eqs. (C.33)-(C.35) take the form as given in Eqs. (2.274).

Case Ib1(iii) $\qquad k_2<0$

Here the solution of Eqs. (C.11) and (C.12) can be written as

$$B_3\left(z\right)=c_1e^{\sqrt{k_2}z}+c_2e^{-\sqrt{k_2}z}\,, \qquad (C.42)$$

$$B_4\left(z\right)=c_3e^{\sqrt{k_2}z}+c_4e^{-\sqrt{k_2}z}\,. \qquad (C.43)$$

If we compare these with Eqs. (C.13) and (C.14) of Case Ib1(i) we see that the only difference is that here the argument of exponential functions is real whereas in the previous case it was complex. So, similar results (with this difference, of course) are obtained on parallel lines.

Case Ib(2) $\quad k_1 = 0$

Here we get $-\frac{R_2'}{2R_2\sqrt{R_1}} = k_3$, a constant, and Eq. (C.4) has the solution

$$A_1(\theta, z) = A_2(z)\,\theta + A_3(z)\,. \tag{C.44}$$

Eqs. (C.1), (C.2) and (C.3), therefore, become

$$B^1 = \frac{1}{\sqrt{R_1}}\left[A_2(z)\,\theta + A_3(z)\right]\,, \tag{C.45}$$

$$B^2 = k_3\left[A_2(z)\frac{\theta^2}{2} + A_3(z)\,\theta\right] + B_1(\rho, z)\,, \tag{C.46}$$

$$B^3 = -\frac{R_3'}{2R_3\sqrt{R_1}}\left[\theta\int A_2(z)\,dz + \int A_3(z)\,dz\right] + B_2(\rho, \theta)\,. \tag{C.47}$$

We discuss here two further cases.

(i) $\quad k_3 = 0 \quad$ i.e. $\quad R_2' = 0\,,$

(ii) $\quad k_3 \neq 0 \quad$ i.e. $\quad R_2' \neq 0\,.$

Case Ib2(i) $\quad k_3 = 0 \quad (\text{or} \quad R_2' = 0)$

Note that in this case Eq. (2.5) for $a = 2$, $b = 3$, gives $B_{2,22}(\rho, \theta) = 0$, or

$$B_2(\rho, \theta) = B_3(\rho)\,\theta + B_4(\rho)\,. \tag{C.48}$$

From Eq. (2.5) for $a = 1$, $b = 3$, we have

$$\begin{aligned} &A_{2,3}(z)\,\theta + A_{3,3}(z) + \tfrac{R_3}{\sqrt{R_1}}\left(-\tfrac{R_3'}{2R_3\sqrt{R_1}}\right)'\left[\theta\textstyle\int A_2(z)\,dz + \int A_3(z)\,dz\right] \\ &+ \tfrac{R_3}{\sqrt{R_1}}B_3'(\rho)\,\theta + \tfrac{R_3}{\sqrt{R_1}}B_4'(\rho) = 0\,. \end{aligned} \tag{C.49}$$

Here, if we put $\frac{R_3}{\sqrt{R_1}}\left(-\frac{R_3'}{2R_3\sqrt{R_1}}\right)' = k_4$, a constant, we get

$$A_{2,33}(z) + k_4 A_2(z) = 0 , \tag{C.50}$$

$$A_{3,33}(z) + k_4 A_3(z) = 0 . \tag{C.51}$$

Otherwise we will have

$$A_2(z) = A_3(z) = B_3'(\rho) = B_4'(\rho) = 0 , \tag{C.52}$$

and Eqs. (C.45)-(C.47) will give the result.

Now, from Eqs. (C.50) and (C.51) we further discuss three cases: $k_4 \gtreqless 0$.

Case Ib2(i)α $\qquad k_4 > 0$

Here, the solution of Eqs. (C.50) and (C.51) can be written as

$$A_2(z) = c_1 e^{i\sqrt{k_4}z} + c_2 e^{-i\sqrt{k_4}z} , \tag{C.53}$$

$$A_3(z) = c_3 e^{i\sqrt{k_4}z} + c_4 e^{-i\sqrt{k_4}z} . \tag{C.54}$$

And from Eq. (2.5) for $a = 1$, $b = 2$ we get after integrating with respect to ρ,

$$B_1(\rho, \theta) = -\left(c_1 e^{i\sqrt{k_4}z} + c_2 e^{-i\sqrt{k_4}z}\right) \int \frac{\sqrt{R_1}}{R_2} d\rho . \tag{C.55}$$

Therefore, Eqs. (C.45)-(C.47) become

$$B^1 = \frac{1}{\sqrt{R_1}} \left[\left(c_1 e^{i\sqrt{k_4}z} + c_2 e^{-i\sqrt{k_4}z}\right)\theta + c_3 e^{i\sqrt{k_4}z} + c_4 e^{-i\sqrt{k_4}z}\right] , \tag{C.56}$$

$$B^2 = -\frac{1}{R_2}\left(c_1 e^{i\sqrt{k_4}z} + c_2 e^{-i\sqrt{k_4}z}\right) \int \sqrt{R_1} d\rho , \tag{C.57}$$

$$B^3 = -\frac{R_3'}{i2\sqrt{k_4}R_3\sqrt{R_1}} \left[\theta\left(c_1 e^{i\sqrt{k_4}z} - c_2 e^{-i\sqrt{k_4}z}\right) + \left(c_3 e^{i\sqrt{k_4}z} - c_4 e^{-i\sqrt{k_4}z}\right)\right]$$

$$+B_3(\rho)\,\theta + B_4(\rho)\ , \tag{C.58}$$

Now from Eq. (2.5) for $a=1$, $b=3$, we get

$$B_3'(\rho) = 0\ , \qquad B_3(\rho) = c_5\ , \quad B_4(\rho) = 0\,. \tag{C.59}$$

Finally, from Eq. (2.5) for $a=2$, $b=3$ we see that

$$\begin{aligned} & i\sqrt{k_4}\left(c_1 e^{i\sqrt{k_4}z} - c_2 e^{-i\sqrt{k_4}z}\right)\int\sqrt{R_1}d\rho \\ & -\tfrac{R_3}{i\sqrt{k_4}}\left(-\tfrac{R_3'}{2R_3\sqrt{R_1}}\right)\left(c_1 e^{i\sqrt{k_4}z} - c_2 e^{-i\sqrt{k_4}z}\right) - R_3 c_5 = 0\ , \end{aligned} \tag{C.60}$$

which implies that $c_1 = c_2 = c_5 = 0$. Hence, the result from Eqs. (C.56)-(C.58) reduce to Eqs. (2.285).

Case Ib2(i)β $\qquad k_4 = 0$

Here we can write $-\frac{R_3'}{2R_3\sqrt{R_1}} = k_5$, a constant. In this case Eqs. (C.50) and (C.51) can be solved to give

$$A_2(z) = c_1 z + c_2\ , \tag{C.61}$$

$$A_3(z) = c_3 z + c_4\ . \tag{C.62}$$

So, Eqs. (C.45)-(C.47) yield

$$B^1 = \frac{1}{\sqrt{R_1}}\left[(c_1 z + c_2)\,\theta + c_3 z + c_4\right]\ , \tag{C.63}$$

$$B^2 = B_1(\rho, z)\ , \tag{C.64}$$

$$B^3 = k_5\left[\theta\left(c_1\frac{z^2}{2} + c_2 z\right) + c_3\frac{z^2}{2} + c_4 z\right] + B_2(\rho,\theta)\ . \tag{C.65}$$

Now, Eqs. (2.5) for $a=1$, $b=2$ and 3, give

$$B_1(\rho, z) = -\ (c_1 z + c_2)\,\frac{1}{R_2}\int\sqrt{R_1}d\rho\ , \tag{C.66}$$

$$B_2(\rho,\theta) = -(c_1\theta + c_3)\int \frac{\sqrt{R_1}}{R_3}d\rho\,, \tag{C.67}$$

so that Eq. (2.5) for $a = 2$, $b = 3$, becomes

$$-\frac{c_1}{R_3}\int \sqrt{R_1}d\rho + k_5\left(c_1\frac{z^2}{2} + c_2 z\right) - c_1\int \frac{\sqrt{R_1}}{R_3}d\rho = 0\,. \tag{C.68}$$

This equation gives rise to further two possibilities.

(1) $R_3' \neq 0$ or $k_5 \neq 0$,

(2) $R_3' = 0$ or $k_5 = 0$.

Case Ib2(i)β_1 $\quad R_3' \neq 0$ or $k_5 \neq 0$

From Eq. (C.68) we see that $c_1 = c_2 = 0$, and Eqs. (C.63)-(C.65) give the RCs as given in Eqs. (2.286).

Case Ib2(i)β_2 $\quad R_3' = 0$ or $k_5 = 0$

In this case Eq. (2.5) for $a = 2$, $b = 3$, yields $c_1 = 0$, so that Eqs. (C.63)-(C.65) in view of Eqs. (C.66) and (C.67) take the form of Eqs. (2.287).

Case Ib2(i)γ $\quad k_4 < 0$

This case is similar to the Case Ib2(i)α except for the difference that now in Eqs. (C.53) and (C.54) the argument of the exponentials will be real and not complex.

Case Ib2(ii) $\quad R_2' \neq 0$ or $k_3 \neq 0$

Now, using Eqs. (C.45) and (C.47) in Eq. (2.5) for $a = 1$, $b = 3$, and differentiating with respect to z yields

$$A_{2,33}(z)\,\theta + A_{3,33}(z) + \frac{R_3}{\sqrt{R_1}}\left(-\frac{R_3'}{2R_3\sqrt{R_1}}\right)'\left[\theta A_2(z) + A_3(z)\right] = 0\,. \tag{C.69}$$

Now, if the quantity $\frac{R_3}{\sqrt{R_1}}\left(-\frac{R_3'}{2R_3\sqrt{R_1}}\right)' = k_4$, is not a constant, then we will have $A_2(z) = A_3(z) = 0$, in which case Eqs. (2.5) for $a = 1$, $b = 2, 3$, yield $B^2_{,1} = B^3_{,1} = 0$, and Eqs. (C.45) and (C.47) become

$$B^1 = 0\,,\; B^2 = B_1(z)\;,\; B^3 = B_2(\theta)\;. \tag{C.70}$$

Eq. (2.5) for $a = 2$, $b = 3$, therefore, implies

$$B_1(z) = c_1 z + c_2 , \tag{C.71}$$

$$B_2(\theta) = -\frac{R_2}{R_3} c_1 \theta + c_3 , \tag{C.72}$$

where $\frac{R_2}{R_3}$ is constant. So, the final result is as given in Eqs. (2.288).

On the other hand if k_4 is a constant, we have the following cases: $k_4 \gtreqless 0$.

We discuss these one by one.

Case Ib2(ii)α $\qquad k_4 > 0$

Now, Eq. (C.69) yields

$$A_{2,33}(z) + k_4 A_2(z) = 0 , \tag{C.73}$$

$$A_{3,33}(z) + k_4 A_3(z) = 0 . \tag{C.74}$$

The solution in this case is

$$A_2(z) = c_1 e^{i\sqrt{k_4}z} + c_2 e^{-i\sqrt{k_4}z} , \tag{C.75}$$

$$A_3(z) = c_3 e^{i\sqrt{k_4}z} + c_4 e^{-i\sqrt{k_4}z} . \tag{C.76}$$

From Eq. (2.5) for $a = 1$, $b = 2$, we get on integration with respect to ρ

$$B_1(\rho, z) = -\left(c_1 e^{i\sqrt{k_4}z} + c_2 e^{-i\sqrt{k_4}z}\right) \int \frac{\sqrt{R_1}}{R_2} d\rho + A_4(z) . \tag{C.77}$$

Now, differentiating Eq. (2.5) for $a = 1$, $b = 2$, with respect to z and putting values from Eqs. (C.45)-(C.47) and (C.75)-(C.77), we get

$$\begin{aligned}&R_2k_3k_4\left[\left(c_1e^{i\sqrt{k_4}z}+c_2e^{-i\sqrt{k_4}z}\right)\tfrac{\theta^2}{2}+\left(c_3e^{i\sqrt{k_4}z}+c_4e^{-i\sqrt{k_4}z}\right)\theta\right]\\&+R_2k_4\left(c_1e^{i\sqrt{k_4}z}+c_2e^{-i\sqrt{k_4}z}\right)\int\tfrac{\sqrt{R_1}}{R_2}d\rho+R_2A_{4,3}(z)\\&+R_3\left(-\tfrac{R_3'}{2R_3\sqrt{R_1}}\right)\left(c_1e^{i\sqrt{k_4}z}+c_2e^{-i\sqrt{k_4}z}\right)=0\,,\end{aligned}\tag{C.78}$$

which is satisfied if and only if $c_1 = c_2 = c_3 = c_4 = 0$. And Eqs. (C.45)-(C.47) reduce to Eqs. (2.289) (as done in the previous case).

Case Ib2(ii)β $\qquad k_4 = 0$

This means that $-\frac{R_3'}{2R_3\sqrt{R_1}} = k_5$, is a constant. In this case the solution of Eqs. (C.73) and (C.74) is

$$A_2(z) = c_1 z + c_2\,, \tag{C.79}$$

$$A_3(z) = c_3 z + c_4\,. \tag{C.80}$$

Therefore, Eqs. (C.45) and (C.47) take the following form:

$$B^1 = \frac{1}{\sqrt{R_1}}\left[(c_1 z + c_2)\,\theta + (c_3 z + c_4)\right]\,, \tag{C.81}$$

$$B^2 = k_3\left[(c_1 z + c_2)\frac{\theta^2}{2} + (c_3 z + c_4)\,\theta\right] + B_1(\rho, z)\,, \tag{C.82}$$

$$B^3 = k_5\left[\left(c_1\frac{z^2}{2} + c_2 z\right)\theta + c_3\frac{z^2}{2} + c_4 z\right] + B_2(\rho,\theta)\,. \tag{C.83}$$

Using these in Eqs. (2.5) for $a = 1$, $b = 2$, 3, and integrating with respect to ρ yields

$$B_1(\rho, z) = -(c_1 z + c_2)\int\frac{\sqrt{R_1}}{R_2}d\rho + A_4(z)\,, \tag{C.84}$$

$$B_2(\rho,\theta) = -(c_1\theta + c_3)\int\frac{\sqrt{R_1}}{R_3}d\rho + A_5(\theta)\,. \tag{C.85}$$

Now, Eq. (2.5) for $a = 2$, $b = 3$, becomes

$$\begin{aligned}&k_3R_2\left(c_1\frac{\theta^2}{2}+c_3\theta\right)-R_2c_1\int\frac{\sqrt{R_1}}{R_2}d\rho+R_2A_{4,3}\left(z\right)\\&+k_5R_3\left(c_1\frac{z^2}{2}+c_2z\right)-R_3c_1\int\frac{\sqrt{R_1}}{R_2}d\rho+R_3A_{5,2}\left(\theta\right)=0\ ,\end{aligned}\tag{C.86}$$

which implies that $c_1 = c_2 = c_3 = 0$, $A_4(z) = c_5$ and $A_5(\theta) = c_6$. Therefore, Eqs. (C.81)-(C.83) give the RCs as given in Eqs. (2.290).

Case Ib2(ii)γ $\quad k_4 < 0$

In this case the arguments of the exponential functions in the solutions, Eqs. (C.75) and (C.76) of the Eqs. (C.73) and (C.74) will be real instead of complex and the results can be obtained exactly similarly as in the Case Ib2(ii)α.

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