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Dedicated

To my **Abbu** and **Ammi**, who have always given me
love, care and cheer whose prayers have always
been a source of great inspiration for me
and whose sustained hope in me led
me to where I stand today

And

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the spirit to learn more and more.

Preface

It is now generally recognized that in practical applications non-Newtonian fluids are more significant and useful than Newtonian fluids. The flows of Newtonian fluids offer mathematicians, engineers and numerical specialists varied challenges in developing appropriate analytical and numerical solutions. This class of possible solution is still narrowed down for non-Newtonian fluids on account of highly non-linear nature of the governing equations. There is no single model, which completely explains all the properties of non-Newtonian fluids and numerous models have been developed for these fluids with their constitutive equations varying greatly in complexity. Since models are developed and derived on the basis of the first principle, therefore have great value from mathematical and practical points of view.

Some of the important models are of differential type such as viscoelastic second grade and third grade fluids. In general the problem of finding exact analytical solution for non-Newtonian fluids are extremely difficult. Nevertheless, it is possible to obtain exact solution in certain particular cases. These problems become even more complex and difficult in rotating system for non-Newtonian fluids since the equations are highly non-linear. Any analysis of non-Newtonian rotating fluids helps to understand the flow nature in the fluid regime. The oscillations and vibrations of plates bounded by Newtonian and non-Newtonian fluids is another aspect, which has great theoretical and industrial applications. The physical interpretation and the numerical computation of results obtained is essential to have a better understanding and for quantitative analysis of the results.

The aim of this thesis is to address these problems in a systematic way. Firstly, more realistic second and third grade fluids are considered. Then, rotation of the fluid and magnetic field are introduced and solutions are obtained thus extending the class of problems for which the exact analytical solutions are possible for non-Newtonian fluids. Numerical solutions are also presented for the quantitative analysis. Besides understanding the fluid flow for non-Newtonian fluids, these can be considered as model problems for real world situation in the following way. Rotating flows of electrically conducting fluids in the presence of magnetic fields, are encountered in cosmical and geophysical fluid dynamics. The effect of the Coriolis force due to the earth's rotation is found to be significant as compared to the inertial and viscous forces in the equations of motion. The Coriolis and electromagnetic forces are of comparable magnitude, the former having a strong effect on the hydromagnetic flow in the Earth's liquid core, which plays an important role in the mean geomagnetic field. The magnetohydrodynamic (MHD) rotating fluids are also useful in solar physics involved in sunspot development, solar cycle and the structure of rotating magnetic stars. It is hoped these problems will go a long way in extending the class of exact solutions for non-Newtonian fluids, modeling and answering some questions in cosmical and geophysical fluid dynamics, finding the numerical solution for rotating third grade fluid and advancing cause of applied mathematics.

1. An exact solution of hydromagnetic rotating flow of an electrically conducting fluid between two plates, one of which is at rest and the other oscillating in its own plane is obtained in chapter 2. The constitutive equation of second grade fluid is considered to give viscoelastic correction to the momentum equation. It is noted that the boundary layer thickness decreases with the increase in Hartmann number or the rotation parameter, however, with the increase in second grade fluid parameter the boundary layer thickness increases. Also, the present analysis exhibits a striking difference between the structure of hydrodynamic and hydromagnetic boundary layer.
2. In Chapter 3, a study is made of unsteady hydromagnetic flow engendered in a second grade rotating fluid by an infinite porous plate exhibiting non-torsional oscillations of a given frequency. The solution of the initial value problem is obtained using Laplace transform method. The effect of the material parameter on the flow is given explicitly. In second grade fluid, the difficulty involved in the hydrodynamic problem associated with the solution of blowing and resonant frequency has been resolved by the addition of the magnetic field.
3. Chapter 4 is devoted to the analytical and numerical solutions for the hydromagnetic rotating flow of a third grade fluid. The fluid is bounded by a porous plate performing elliptic harmonic oscillations in its own plane. The velocity fields are constructed after solving non-linear governing problem for the cases when twice of the angular velocity is greater than, smaller than or equal to the frequency of oscillations. Both perturbation and numerical solutions are given. The time required to attain steady flows for the cosine and sine oscillations is obtained. It is observed that the time required to attain steady flow for cosine oscillation is shorter than that for the sine oscillation.

4. An analysis of an incompressible, third grade fluid past an infinite porous plate is carried out in chapter 4 under the following assumptions: (i) suction velocity normal to the plate oscillates in magnitude but not in direction about a non-zero mean (ii) the free stream velocity oscillates in time about a constant mean. Approximate solution to the non-linear equation governing the flow is derived. It is noted that layer thickness decreases with the increase in material parameter of third grade fluid.

Contents

Chapter 0

Introduction

Fluid motion has fascinated many generations of engineers and scientists. Although many years of research have been devoted to the study of fluids of low molecular weight, which are well described by the Navier-Stokes equations, many challenging problems in both theory and applications remain. But even more challenging are polymeric liquids, whose motion cannot be described at all by the Navier-Stokes equations.

Understanding of non-Newtonian fluid dynamics is important in connection with plastics manufacture, performance of lubricants, applications of paints, processing of foodstuffs, and movement of biological fluids. There is no single model which clearly exhibits all the properties of non-Newtonian fluids. Therefore during the last several years, generalization of Navier-Stokes model to highly non-linear constitutive laws have been proposed and studied by Oldroyd [1], Truesdell and Noll [2] and Rajagopal [3] because of their interest in technological applications. In order to explain several non-standard features, such as normal stress effects, rod climbing, shear thinning and shear thickening, Rivlin-Ericksen fluids [4] of differential type were introduced. These fluids are rather complex form the point of view of partial differential equations theory. Nevertheless, several authors in fluid mechanics are now engaged with the equations of motion of non-Newtonian fluids of second and third grades. In particular, some authors are interested in studying $n - grade$ fluids as self consistent models and not as approximating models. Therefore, in studying dynamics they ask that all the flows meet the Clausius-Duhem inequality and that the specific Helmholtz free energy of the fluid is a minimum at equilibrium (see Dunn and Fosdick [5] and Fosdick and Rajagopal [6]). On the other hand, it is under the same hypothesis that the Navier-Stokes model is studied. That is, it is always assumed that some real fluids exist such that Navier-Stokes or $n - grade$ fluids are exact models, and not truncations of viscoelastic fluids. As noted by Rajagopal [7], different assumptions could heavily affect the rest state stability. Under these thermodynamical hypothesis Passerini and Videman [8] and Pilekas et al. [9] obtained several results concerning existence and stability.

Moreover, Rajagopal [10], Rajagopal and Kaloni [11] and Rajagopal and Gupta [12] studied the existence and uniqueness of solutions to the equations governing the flows of fluids of differential type. These equations are usually higher order partial differential equations than the Navier-Stokes equations. Hence the issue of whether the "no-slip" boundary condition is sufficient to have a well-posed problem is very important. This question cannot be answered in any generality in fluids of differential type of complexity n, for arbitrary n. However, if attention is confined to fluids of grade 2 or grade 3, then a study of Rajagopal [13] can provide some definite answers, while some partial answers are possible for fluids of grade n. With this fact in mind we consider second and third grade fluids in this thesis.

Using second grade fluid model, Rajagopal [14] examined some basic flows with and without pressure gradient. Mansutti et al. [15] discussed the non-similar flow of a non-Newtonian fluid past a wedge. Rajagopal and Gupta [16] also discussed the flow and stability of a second grade fluid between two parallel rotating plates. Hayat et al. [17,18] and Siddiqui et al. [19] investigated the periodic flows of a second grade fluid bounded by a plate (or between two plates). Benharbit and Siddiqui [20] examined certain solutions of the equations of the planar motion of a second grade fluid by assuming certain form of the stream function. The first and second problem of Stokes for third grade fluid have been discussed by Erdogan [21] and Rajagopal and Na [22], respectively. Ariel [23] investigated the steady laminar flow of a third grade fluid through a porous channel when the rate of injection (blowing) of the fluid at one boundary is equal to the rate of suction at the other boundary.

Rotation plays a significant role in several important phenomena in cosmical fluid dynamics. Similarly, a great deal of meteorology depends upon the dynamics of a revolving fluid. The large scale and the moderate motions of the atmosphere are greatly affected by the vorticity of the earth's rotation. In the case of infinite fluid rotating as a rigid body about an axis, the amount of energy possessed by the fluid is infinite and it is of great interest to know how small disturbances propagate in such a fluid.

Greenspan and Howard [24] have initiated the study of the dynamics of spin-up of an incompressible, homogeneous Newtonian rotating fluid. They have presented a detailed mathematical and physical analysis of the transient process by which the fluid adjusts to a small change in the rotation rate of its boundary. It has been demonstrated that the Ekman boundary layer is established on the horizontal boundary surfaces and it is primarily responsible for the adjustment process. Furthermore, the Ekman layer produces a secondary interior circulation throughout the fluid which transport angular momentum.

After the initiation by Greenspan and Howard [24], there have been many works on the dynamics of spin-up and spin-down of a homogeneous rotating fluid. Several investigations have considered rotating fluid in various geometries. Singh and Sathi [25] examined the rotating flow engendered in a Newtonian, incompressible fluid by an infinite plate suddenly set in motion. Mazumder [26] and Ganapthy [27] examined the oscillatory Couette flow in a non-inertial frame. The fluctuating and transient flows of elastico-viscous rotating fluid have been examined by Puri [28] and Puri and Kulshrestha [29], respectively. A study of nonlinear convective flows in rotating wavy channels has been made by Vajravelu and Debnath [30]. The axisymmetric flow of a thin Newtonian fluid layer subject to centrifugal and Coriolis forces, surface tension and gravity has been discussed by Myers and Charpin [31]. Govender [32] discussed the oscillatory convection induced by gravity and centrifugal forces in a rotating porous layer distant from the axis of rotation.

The analysis of the effects of rotation and magnetic field in fluid flows has been an active area of research because of its geophysical and technological importance. It is well known that a number of astronomical bodies (e.g. the Sun, Earth, Jupiter, Magnetic Stars, Pulsars) possesses fluid interiors and (at least surface) magnetic fields. Changes in the rotation rate of such objects suggest the possible importance of hydromagnetic spin-up. Hide and Roberts [33] have made a steady state investigations of the hydromagnetic rotating flow due to the oscillations of an infinite rigid wall. Chandrasekhar [34-36] has also made significant contributions to the theory of hydrodynamic and hydromagnetic flow phenomena. He pointed out the significant role of the Coriolis force on problems of thermal instability and on stability of a viscous hydromagnetic flow. In order to make some applications to solar physics, Lehnert [37,38] has presented a steady state analysis of the magnetohydrodynamic waves in a Newtonian incompressible rotating fluid with the same amplitude assumption. He predicted certain significant effects of the Coriolis force on the properties of the magnetohydrodynamic waves in the sun.

The rotating flow of a fluid bounded by a porous plate has also been a subject of much importance. It is apparent from physical considerations that suction and blowing have opposite effects. Gupta [39] discussed steady Newtonian flow past a porous plate in a rotating frame. He found, for uniform suction and blowing, the asymptotic profiles for the velocity distribution. Murthy and Ram [40] discussed the magnetohydrodynamic steady rotating flow on a porous plate. Debnath and Mukherjee [41] examined the multiple boundary layers on a porous plate with uniform suction or blowing in a rotating medium for Newtonian flow. The flow is induced in the fluid by oscillations of the plate in its own plane. In continuation, Debnath [42] discussed the magnetohydrodynamic rotating oscillatory flow of a Newtonian fluid on a porous and non-conducting plate. He showed that electromagnetic force is responsible to decrease the boundary layer thicknesses and solution is valid for all values of frequency.

In spite of the importance of the effects of rotation and electromagnetic force on the hydromagnetic Newtonian flow and their applications in cosmical fluid dynamics and solar physics, the simultaneous influence of these agencies appear to have been investigated in only a few specific cases. These even become rare if the non-Newtonian fluids are taken into account. Also, in fluid systems with rigid body rotation a resonance effect is found if an attempt is made to force oscillations with a frequency which is twice the angular velocity of rotation. In order to discover the above effects on hydromagnetic flow phenomena and to examine the structures of the associated boundary layers, the Couette flow of a rotating incompressible second grade fluid between two plates is considered. The fluid is electrically conducting in the presence of a transverse applied magnetic field. The lower plate is stationary and the upper one is oscillating in its own plane. Exact solution of the boundary value problem is constructed and several results of interest are discussed. The contents of this chapter has been **accepted** for publication in **Applied Mathematics Letters.**

In chapter 3, the unsteady motion of an electrically conducting, rotating, second grade fluid, initially at rest, occupying a half space and bounded by an infinite porous plate, also initially at rest is discussed. The non-conducting plate executes small amplitude non-torsional oscillations in its own plane. The influence of a magnetic field, material parameter of second grade fluid, suction and blowing is given. It is found that in second grade fluid, suction and magnetic field cause reduction in the boundary layer thickness, while increase in blowing and second grade parameters enhance the boundary layer thickness. These observations have been **accepted in Z. Angew. Math. Phys.(ZAMP: Zeitschrift für Angewandte Mathematik und Physik)**.

The problem of magnetohydrodynamic third grade fluid bounded by a porous and oscillating plate in a rotating frame of reference is investigated in chapter 4. The velocity fields are obtained for the cases when twice of the angular velocity is greater than, smaller than or equal to the frequency of oscillations. It is found that the material parameter of third grade fluid declerates the layer thickness for sine and cosine oscillations. Further, it is shown that asymptotic steady solution for blowing and resonance is possible. This work has been published in **Acta Mechanica 152, 177-190 (2001)**.

The present motivations in chapters 3 and 4 are important in the study of wind-generated ocean current on a rotating earth. These comes from a desire to understand, at least qualitatively, the effects of porosity and its role on the boundary layer likely to exist at the core-mantle interface of the Earth where rotation and magnetic field effects are simultaneously present in particular, to know how the geophysically important Ekman suction velocity is effected by magnetic field and rotation. Also, the magnetohydrodynamic analysis in these chapters provide an answer to the question of finding a meaningful solution for the case of blowing and resonant frequency in a non-Newtonian fluid.

The unsteady two-dimensional laminar flow was considered by Lighthill [43] for the velocity and thermal boundary layers. He has analyzed mathematically the equations of motion and energy when the velocity of the oncoming flow relative to the body oscillates in magnitude but not in direction.

After the initiation of Lighthill [43] the laminar Newtonian boundary layers which has a regular fluctuating flow superimposed on the mean flow have been studied extensively. Owing to mathematical difficulties, most of them include restrictions on an oscillation amplitude or a frequency in the course of their theoretical development. One of the solutions of the Navier-Stokes equation in which no restriction is placed on the amplitude and frequency was obtained by Stuart [44]. He examined the flow past a flat plate with free stream oscillations for constant suction and no heat transfer between the fluid and the plate. Messiha [45] examined the effect of variable suction velocity and heat transfer in Stuart's problem. Lal [46] examined the problem of Messiha [44] for free convection laminar flow when the dissipation term is neglected. Watson [47] and Kelly [48] discussed the solution to account for an arbitrary free stream velocity and suction velocity, respectively. The idea has also been extended to magnetohydrodynamic flows by Suryaprakasaro [49] and to the elastico-viscous flows by Kaloni [50] and Soundalgekar and Puri [51].

In technological fields, the boundary layer phenomenon of fluctuating flow of non-Newtonian fluids has recently become a fascinating problem, under a wide range of geometrical, dynamical and rheological conditions. Also due to the development of practical boundary layer control systems, the problem concerning suction has been important. With these facts in view, the effects of variable suction velocity on the flow fields of third-grade fluids past an infinite plate is investigated in chapter 5. Literature survey revealed no previous attempts on studying this problem, even in the constant suction velocity case. It is also pertinent to mention that flow of a second grade fluid with variable suction is still unattempted. The considered external flow velocity and suction velocity are of the periodic forms and analytic solution of the non-linear equation is given. The fluctuating parts of the solution are found to increase with the increase of material parameters of third grade fluid. Also, the velocity increases in case of third grade fluid when compared with Newtonian fluid and hence the layer thickness decreases. These observations have been published in **Archives of Mechanics 55, 305-324 (2003)**.

Chapter 1

Preliminaries

This chapter includes some basic concepts of non-Newtonian fluids, fluids of differential type, rotation, the equation of continuity and the methods of solution. The equation of motion for rotating third grade fluid is also modeled.

1.1 Non-Newtonian Fluids

Commercial fluid products comprise a wide variety of materials, with a wide range of consistencies. In general, polymer solutions and melts, emulsions, colloidal dispersions, and other suspensions of particulate solids at useful concentrations are non-Newtonian. However, there is a large category of fluids for which viscosity is not independent of strain as such fluids are referred to as non-Newtonian fluids. For liquids of complex molecular structure, particularly those with long-chain molecules, the expression for the deviatoric stress based on the Newtonian hypothesis may break down at quite moderate rates of strain and, for some rubber like liquids with "memory", the stress depends on the strain history as well as on the instantaneous value of the rate of strain.

Briefly, the non-Newtonian fluids are those fluids for which shear stress is directly proportional to the deformation rate in a non-linear manner i.e. for a unidirectional flow

$$\text{Shear stress} = k\left(\frac{du}{dy}\right)^n, \quad n \neq 1$$

in which k is consistency index and n is flow behavior index. For $n = 1$ and $k = \mu$ we obtain Newtonian fluids.

1.2 Differential Type Fluids

Geological materials, liquid foams, polymeric fluids, slurries, and food products are among the many substances which are capable of flowing but which exhibit flow characteristics that cannot be adequately described by the classical linearly viscous fluid model. In order to describe some of the departures from Newtonian behavior evinced by such materials, many idealized material models have been suggested. One of the earliest classes of such material models consists of what are commonly referred to as fluids of differential type or, informally, as Rivlin-Ericksen fluids. In such materials, only a very short (in fact, infinitesimal) part of the history of the deformation gradient has an influence on the stress. More specifically, in an incompressible fluid of differential type, apart from a constitutively indeterminate pressure, the stress is just a function of the velocity gradient and some number of its higher time derivatives. These materials thus lack a gradually "fading memory"; instead, their "memory" fades precipitously: with the instantaneous cessation of all local motion, the stress becomes a pure pressure. One consequence of this is that, while they can experience the phenomenon of creep, these materials do not exhibit the phenomenon of stress relaxation. Nevertheless, due to their mathematical simplicity, a great deal of interest has been evinced in recent years in the thermomechanics and stability of such fluids. Unfortunately, the results of some of these studies have sometimes been interpreted in ways that suggest logical paradoxes, physical impossibilities, or conflicts between "reality" and theory. A closer scrutiny reveals that all such "problems" with fluids of differential type are an illusion and stem from imprecise and inaccurate interpretations of both theoretical and experimental results.

Among fluids of differential type, one special subclass has received an especially large (perhaps even an inordinately large) amount attention. This is incompressible homogeneous second grade fluid, and much of confusion surrounding general fluids of differential type arose from and is rooted in the misunderstood consequences of the thermodynamics and stability of this model.

It is rate of growth of the first normal stress difference with the shear rate that is the key issue for many fluids of laboratory interest. While occasionally experimentalists have remarked that this growth rate is nonstandard for certain liquids, the point does not seem to have attracted the attention from theorists which we think it deserves.

In thermodynamic analysis of the general fluid of differential type we have constitutive restrictions on this material based on the hypothesis that a familiar dissipation principles holds and that the material possesses a stored energy function that is bounded either below or above. A noteworthy aspect of the analysis is that the restrictions established turn out to hold true even for the material's behavior on steady motions. As a consequence, the results not only complement those based on stability ideas, they also address questions of material response fundamentally inaccessible from a stability point of view.

We know that fluids of differential type which is thermodynamically consistent is necessarily forbidden from belonging to either the class studied by Joseph [52-53] or the (somewhat larger) class studied by Renardy [54], i.e. thermodynamically consistent fluids of differential type are to be found only among those materials not studied by Joseph and Renardy. From the vantage point of either stability theory or thermodynamics, the particular model materials examined by Joseph and Renardy are thus seen to be physically unnatural. Notwithstanding this, their results, like earlier, similar results of other authors, are sometimes interpreted in a way that incorrectly and unjustly impugns a far larger (sometimes the entire) class of fluids of differential type. There have been a consistent and long-standing pattern of misinterpretation of the retardation theorems of Coleman and Noll [55] and this has led to the adoption of certain flawed notions of model approximation.

It is remarked that, fluids of differential type cannot experience stress relaxation. One therefore hardly expect them to be useful in modeling those aspects, if any, of a particular real material's response that critically hinge on the material's ability to stress relax. To pick a concrete example, if all polymeric fluids are such that the capacity for stress relaxation is an ever present, though often times latent, aspect of their response, then differential type fluids would certainly seem to be inappropriate models for them. Rather than being used to model such "strongly viscoelastic" materials (if need that is what polymeric fluids are), fluids of differential type might well be better employed to model the usual materials involved in, say, slurry flows and food rheology where relaxation effects frequently seem to be rather insignificant. Work by Man and his co-workers [56-58] even suggests that polycrystalline ice might also be modeled. Examples of second grade fluids are dilute polymeric solutions (e.g. poly-iso-butylene, Methyl-Methacrylate in n-butyl acetate, polyethylene oxide in water, etc.).

1.2.1 Thermodynamic Fluids of Second and Third Grade

The Cauchy stress $\mathbf{T}$ in an incompressible homogeneous fluid of second grade is related to the fluid motion by the following constitutive equation

$$\mathbf{T} = -p_1\mathbf{I} + \mu\mathbf{A}_1 + \alpha_1\mathbf{A}_2 + \alpha_2\mathbf{A}_1^2, \tag{1.1}$$

where $-p_1\mathbf{I}$ is the spherical part of the stress due to constraint of incompressibility, p_1 is the pressure, $\mathbf{I}$ is the identity tensor, μ is the dynamic viscosity and α_1 and α_2 are normal stresses. The Rivlin-Ericksen kinematical tensors $\mathbf{A}_1, \mathbf{A}_2, ...$, are defined respectively through

$$\begin{aligned} \mathbf{A}_1 &= (grad\mathbf{V}) + (grad\mathbf{V})^T, \\ \mathbf{A}_n &= \frac{d\mathbf{A}_{n-1}}{dt} + \mathbf{A}_{n-1}(grad\mathbf{V}) + (grad\mathbf{V})^T\mathbf{A}_{n-1}, \quad n \geq 1, \end{aligned} \tag{1.2}$$

in which $\mathbf{V}$ is the velocity, t is the time and $\mathbf{A}_1^2 = \mathbf{A}_1\mathbf{A}_1$.

Dunn and Fosdick [5] assumed equation (1.1) to be an exact model and studied the thermodynamics and stability of such a fluid in general detail.

They showed that if the model is to be consistent with thermodynamics then it is necessary that

$$\mu \geq 0, \quad \alpha_1 \geq 0, \quad and \quad \alpha_1 + \alpha_2 = 0. \tag{1.3}$$

Dunn and Rajagopal [59] critically examined the status of the fluids of differential type. We shall not get involved in a lengthy discussion of the issues here and simply state that, if the material parameter α_1, is negative, it follows that the fluid exhibits undesirable stability properties (as in Fosdick and Rajagopal [6]). In the present thesis we shall restrict ourselves to the case:

$$\alpha_1 \geq 0. \tag{1.3a}$$

The study of Dunn and Fosdick [5] was followed by the work of Fosdick and Rajagopal [6] on the fluids of grade 3 where we get the first inkling that the conditions required by Renardy [54] might be at odds with thermodynamics.

The Cauchy stress $\mathbf{T}$ in an incompressible third grade fluid has the following constitutive structure:

$$\mathbf{T} = -p\mathbf{I} + \mu\mathbf{A}_1 + \alpha_1\mathbf{A}_2 + \alpha_2\mathbf{A}_1^2 + \beta_1\mathbf{A}_3 + \beta_2\left(\mathbf{A}_1\mathbf{A}_2 + \mathbf{A}_2\mathbf{A}_1\right) + \beta_3\left(tr\mathbf{A}_1^2\right)\mathbf{A}_1, \tag{1.4}$$

where β_1, β_2, and β_3 are additional material constants. Fosdick and Rajagopal [6] showed that if such a fluid is to be consistent with thermodynamics, it is necessary that

$$\begin{aligned} \mu &\geq 0, \quad \alpha_1 \geq 0, \quad |\alpha_1 + \alpha_2| \leq \sqrt{24\mu\beta_3}, \\ \beta_1 &= 0, \qquad \beta_2 = 0, \qquad \beta_3 \geq 0. \end{aligned} \tag{1.5}$$

We see at once that the ratio of the coefficient of $\mathbf{A}_2$ to that of $\mathbf{A}_3$ in the form of $\mathbf{T}$, i.e. the "ratio" $\alpha_1/0$, does not satisfy either the hypothesis of Joseph

[52,53] or the hypothesis of Renardy [54]. Thermodynamics is thus hinting here at conflict it has with Joseph's and Renardy's materials (although Renardy's instability result is for $n \geq 5$ and here $n = 3$). The study of Dundwoody [60] on the linearized stability of the rest state of fluids of differential type illustrates this point further. Dunwoody shows that if fluids of grade n are thermodynamically consistent for all retarded flows, then the rest state of the fluid is stable to infinitesimal disturbances. Assuming the response of fluids of complexity n, fluids are stable to infinitesimal disturbances. An interesting consequence of Dunwoody's analysis is that the criterion which Renardy [54] requires for instability can never be attained in any thermodynamically fit fluid of differential type. Equivalently, those fluids of grade n which Renardy shows have an unstable rest state are incompressible with thermodynamics. This around to conclude that, loosely speaking thermodynamic incompatibility implies stability. Far from rendering vacuous works on the thermodynamics of materials of differential type, the results of Joseph [52,53] and Renardy [54] both complement and emphasize the relevance of such investigations. The mathematical content of their linear stability analysis and the thermodynamic results here should remind once again of the frequent and intimate connection between thermodynamics and stability.

Some useful informations about the signs of α_1 that lead to the non-existence of solution have also been given in the studies by Fosdick and Straughan [61], Straughan [62-65] and Franchi and Straughan [66,67].

1.3 The Continuity Equation

Let us consider a fixed smooth surface $\mathbf{S}_0$ in the fluid contained in volume V_1. The rate of flow of mass into V_1 is

$$-\int_{S_0} \rho \mathbf{V}.d\mathbf{S}_0,$$

where the minus being because $d\mathbf{S}_0$ is out of $\mathbf{V}$, and density ρ being used to give a mass flux rather than a volume flux. The rate of increase of mass in V_1 is

$$\frac{d}{dt}\int_{V_1} \rho dV_1$$

and because V_1 is fixed region this is

$$\int_{V_1} \frac{\partial \rho}{\partial t} dV_1.$$

Because mass is conserved we must have

$$\int_{V_1} \frac{\partial \rho}{\partial t} dV_1 = -\int_{S_0} \rho \mathbf{V}.d\mathbf{S}_0. \tag{1.6}$$

Using the divergence theorem we obtain

$$\int_{V_1} \left\{ \frac{\partial \rho}{\partial t} + \nabla.(\rho \mathbf{V}) \right\} dV_1 = 0. \tag{1.7}$$

But this is for any volume V_1, hence, from the above expression

$$\frac{\partial \rho}{\partial t} + \nabla.(\rho \mathbf{V}) = 0 \tag{1.8}$$

at any point in the fluid. This is called the 'continuity equation' or the 'mass conservation equation'.

For incompressible fluid the density of any particle is invariable, so equation of continuity becomes

$$\nabla.\mathbf{V} = 0. \tag{1.9}$$

1.4 Rotation

One of the most compelling reasons for studying fluids which are rotating is that the earth is rotating and so any flow we observe here is actually a rotating one. As it turns out, the earth's rotation is still sometimes negligible, but for large scale motions like the flows in oceans and atmospheres rotation is dominant phenomena. Thinking further, we know that it is not just earth that rotates. All the planets of the solar system do so, most notably the Jovian planets because they rotate quickly and they are mostly fluid. The Sun also rotates and more massive stars rotate even faster; they are all fluids. As we go further out in the cosmos, rotation is still present: the great spiral galaxies are defined by their rotation.

1.5 Coriolis and Centripetal Forces

If we are stuck observing fluids in a rotating frame, we are not in an inertial reference frame. This means that Newton's second law does not apply; we can only apply it in an inertial frame. Let $\mathbf{r}_1$ and $\mathbf{r}$ be the position vector in inertial and rotating system respectively and $\mathbf{R}_1$ is the position vector of the origin of the rotating system in the inertial system then

$$\begin{aligned} \mathbf{r}_1 &= \mathbf{R}_1 + \mathbf{r}, \\ \frac{d\mathbf{r}_1}{dt} &= \frac{d\mathbf{R}_1}{dt} + \frac{d\mathbf{r}}{dt}. \end{aligned} \tag{1.10}$$

As

$$\mathbf{r} = x_i \mathbf{e}_i$$

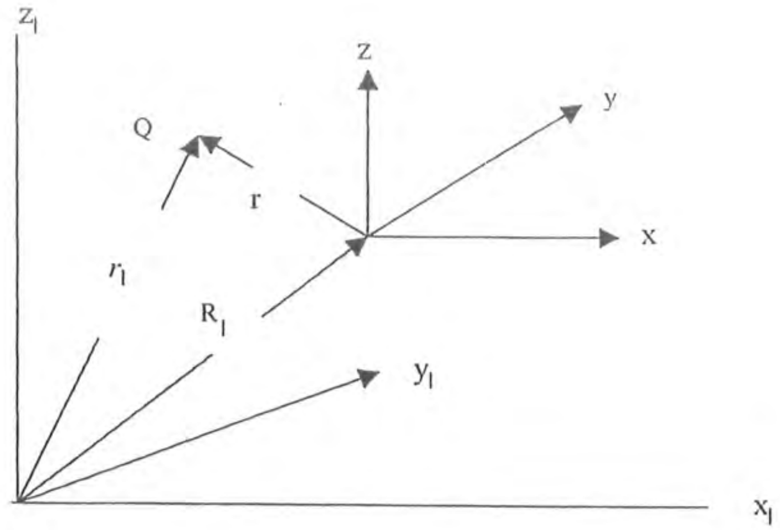

Figure 1.1: Inertial and rotating coordinate system.

$$\frac{d\mathbf{r}}{dt} = \mathbf{e}_i \frac{dx_i}{dt} + x_i \frac{d\mathbf{e}_i}{dt}, \tag{1.11}$$

where $\mathbf{e}_i$ is the unit vector.

In rotating system with the angular velocity $\boldsymbol{\Omega}$

$$\frac{d\mathbf{e}_i}{dt} = \boldsymbol{\Omega} \times \mathbf{e}_i,$$

and thus

$$\frac{d\mathbf{r}_1}{dt} = \frac{d\mathbf{R}_1}{dt} + \mathbf{e}_i \frac{dx_i}{dt} + \boldsymbol{\Omega} \times \mathbf{r}. \tag{1.12}$$

The first term on the right hand side represents the translation with respect to inertial system, the second term is the translation of the point Q with respect to the system, and the third term represents the rotation.

Differentiating equation (1.12), we obtain acceleration i.e.

$$\frac{d^2\mathbf{r}_1}{dt^2} = \frac{d^2\mathbf{R}_1}{dt^2} + \mathbf{e}_i \frac{d^2 x_i}{dt^2} + \frac{d\mathbf{e}_i}{dt}\frac{dx_i}{dt} + \frac{d\boldsymbol{\Omega}}{dt} \times \mathbf{r} + \boldsymbol{\Omega} \times \frac{d\mathbf{r}}{dt}, \tag{1.13}$$

Taking the acceleration of a particle as

$$\frac{d\mathbf{V}}{dt} = \mathbf{e}_i \frac{d^2 x_i}{dt^2}, \ \frac{d\mathbf{r}}{dt} = \mathbf{V} + \boldsymbol{\Omega} \times \mathbf{r}, \ \frac{d\mathbf{e}_i}{dt}\frac{dx_i}{dt} = \boldsymbol{\Omega} \times \mathbf{V}. \tag{1.13a}$$

Making use of equation (1.13a) in equation (1.13) we have

$$\frac{d^2\mathbf{r}_1}{dt^2} = \frac{d^2\mathbf{R}_1}{dt^2} + \frac{d\mathbf{V}}{dt} + 2\boldsymbol{\Omega} \times \mathbf{V} + \boldsymbol{\Omega}\times (\boldsymbol{\Omega} \times \mathbf{r}) + \frac{d\boldsymbol{\Omega}}{dt} \times \mathbf{r}. \tag{1.13b}$$

For equations to the rotating system, there is no translational acceleration, and the rotation is constant, so

$$\frac{d^2\mathbf{R}_1}{dt^2} = 0, \ \frac{d\boldsymbol{\Omega}}{dt} = 0.$$

With the help of above equation, equation (1.13b) can be written as

$$\frac{d^2\mathbf{r}_1}{dt^2} = \frac{d\mathbf{V}}{dt} + 2\boldsymbol{\Omega} \times \mathbf{V} + \boldsymbol{\Omega}\times (\boldsymbol{\Omega} \times \mathbf{r}).$$

There are two contributions to the difference between the way that acceleration is observed between two frames. The first $2\boldsymbol{\Omega} \times \mathbf{V}$ is known as the Coriolis acceleration,and it also appears in ordinary mechanics. The second term $\boldsymbol{\Omega}\times (\boldsymbol{\Omega} \times \mathbf{r})$ is known as centripetal acceleration. It is the same centripetal acceleration to observe we attach an object to a string and spin it above our head and you feel the centripetal acceleration; the tension in the string contracts it.

Notice that while the Coriolis acceleration depends on the flow through the velocity vector $\mathbf{V}$, the centripetal acceleration depends only on the frame rotation $\boldsymbol{\Omega}$ and the position vector $\mathbf{r}$. This is an important difference because it means we can simplify the appearance of the centripetal acceleration to some thing perhaps less threating than a double cross-product

1.6 Equation of Motion in Rotating System

We are principally concerned with the motion of an incompressible viscous liquid of constant material properties. The general problem of this type is formulated first and particular reductions of the theory are then discussed. The equations governing the motions of fluid are the conservation of mass and momentum:

$$\nabla.\mathbf{V} = 0,$$

$$\frac{d\mathbf{V}}{dt} + 2\boldsymbol{\Omega} \times \mathbf{V} + \boldsymbol{\Omega}\times(\boldsymbol{\Omega} \times \mathbf{r}) = \frac{1}{\rho} div\mathbf{T} + \pounds_1. \quad (1.14)$$

Here $\boldsymbol{\Omega}\left(= \Omega\widehat{\mathbf{k}}\right)$, $\widehat{\mathbf{k}}$ is a unit vector parallel to z-axis, ν and $\pounds_1$ represent respectively, the kinematic viscosity and body force per unit mass. The symbol $d\mathbf{V}/dt$ denotes here the substantial (material) acceleration which, like the substantive derivative of density, consists of the local contribution (in unsteady flow) $\partial\mathbf{V}/\partial t$, and the convective contribution (due to translation), $(\mathbf{V}.\nabla)\,\mathbf{V}$. Further $\rho\,(2\boldsymbol{\Omega} \times \mathbf{V})$ and $\rho\,[\boldsymbol{\Omega}\times(\boldsymbol{\Omega} \times \mathbf{r})]$ are the Coriolis and Centripetal acceleration.

1.7 The Equation of Motion for Conducting Third grade Fluid in a Rotating System

In this section, we are interested in constructing the governing equation for magnetohydrodynamic fluid of differential type in a rotating frame. The fluid is taken of the third grade type. Thus equation of motion (1.14) in a rotating frame for magnetohydrodynamic fluid is

$$\rho\left[\frac{d\mathbf{V}}{dt}+2\mathbf{\Omega}\times\mathbf{V}+\mathbf{\Omega}\times(\mathbf{\Omega}\times\mathbf{r})\right]=div\mathbf{T}+\mathbf{J}\times\mathbf{B}. \tag{1.15}$$

An incompressible simple fluid is defined as a material whose state of present stress is determined by the history of the deformation gradient without a preferred reference configuration [2]. Its constitutive equation can be written in the form of a functional

$$\mathbf{T}(t)=-p_1\mathbf{I}+\sum_{s=0}^{\infty}\left(F_t^t(s)\right), \tag{1.16}$$

where F is the deformation gradient.

Coleman and Noll [55] defined the incompressible fluid of differential type of grade n as the simple fluid obeying the constitutive equation

$$\mathbf{T}(t)=-p_1\mathbf{I}+\sum_{j=1}^{n}\widetilde{\mathbf{S}}_j, \tag{1.17}$$

obtained by asymptotic expansion of the functional in equation (1.17) through a retardation parameter. If $n=3$ the first three tensors $\widetilde{\mathbf{S}}_j$ are given by

$$\widetilde{\mathbf{S}}_1 = \mu\mathbf{A}_1, \tag{1.18}$$

$$\widetilde{\mathbf{S}}_2 = \alpha_1\mathbf{A}_2+\alpha_2\mathbf{A}_1^2, \tag{1.19}$$

$$\widetilde{\mathbf{S}}_3 = \beta_1\mathbf{A}_3+\beta_2(\mathbf{A}_1\mathbf{A}_2+\mathbf{A}_2\mathbf{A}_1)+\beta_3(tr\mathbf{A}_2)\mathbf{A}_1, \tag{1.20}$$

and constitutive equation is (1.4). For second grade fluid $\beta_i=0$ $(i=1,2,3)$ and constitutive equation is (1.1) . The thermodynamic constraints for second and third grade fluids are given in equations (1.3) and (1.5), respectively.

The last term on the right hand side of equation (1.15) represents the pondermotive force on the conducting fluid due to the interaction of **J** (current density) and **B** (magnetic induction), known as Lorentz force. Clearly, the Lorentz force depends not only on $\mathbf{B}$, but also on $\mathbf{E}$ (electric field) and $\mathbf{V}$. We assume that the induced magnetic field is negligible [68] in comparison to the applied magnetic field $(0, 0, B_0)$. This assumption is justified since the magnetic Reynold number is small, which is generally the case in normal aerodynamic applications. Since no external electric field is applied and the effect of polarization of the ionized fluid is negligible, we also can assume that the electric field $\mathbf{E} = 0$. This then corresponds to the case when no energy is added or extracted from the fluid by the electric field and thus

$$\mathbf{J} \times \mathbf{B} = -\sigma \mathbf{B}_0^2 \mathbf{V}. \tag{1.21}$$

We take the velocity field of the following form

$$\mathbf{V}(z,t) = [u(z,t), v(z,t), w(z,t)], \tag{1.22}$$

where u, v and w are x, y and z components of velocity.

In view of equation (1.2) we have

$$\mathbf{A}_1 = \begin{pmatrix} 0 & 0 & \frac{\partial u}{\partial z} \\ 0 & 0 & \frac{\partial v}{\partial z} \\ \frac{\partial u}{\partial z} & \frac{\partial v}{\partial z} & 2\frac{\partial w}{\partial z} \end{pmatrix}, \tag{1.23}$$

$$\mathbf{A}_1^2 = \begin{pmatrix} \left(\frac{\partial u}{\partial z}\right)^2 & \frac{\partial u}{\partial z}\frac{\partial v}{\partial z} & 2\frac{\partial u}{\partial z}\frac{\partial w}{\partial z} \\ \frac{\partial u}{\partial z}\frac{\partial v}{\partial z} & \left(\frac{\partial v}{\partial z}\right)^2 & 2\frac{\partial v}{\partial z}\frac{\partial w}{\partial z} \\ 2\frac{\partial u}{\partial z}\frac{\partial w}{\partial z} & 2\frac{\partial v}{\partial z}\frac{\partial w}{\partial z} & 4\left(\frac{\partial w}{\partial z}\right)^2 \end{pmatrix}, \tag{1.24}$$

$$\mathbf{A}_2 = \begin{bmatrix} 0 & 0 & \frac{\partial^2 u}{\partial z \partial t} + w\frac{\partial^2 u}{\partial z^2} + \frac{\partial u}{\partial z}\frac{\partial w}{\partial z} \\ 0 & 0 & \frac{\partial^2 v}{\partial z \partial t} + w\frac{\partial^2 v}{\partial z^2} + \frac{\partial v}{\partial z}\frac{\partial w}{\partial z} \\ \frac{\partial^2 u}{\partial z \partial t} + w\frac{\partial^2 u}{\partial z^2} + \frac{\partial u}{\partial z}\frac{\partial w}{\partial z} & \frac{\partial^2 v}{\partial z \partial t} + w\frac{\partial^2 v}{\partial z^2} + \frac{\partial v}{\partial z}\frac{\partial w}{\partial z} & \begin{matrix} 2\frac{\partial^2 w}{\partial z \partial t} + 2w\frac{\partial^2 w}{\partial z^2} \\ +2\left[\left(\frac{\partial u}{\partial z}\right)^2 + \left(\frac{\partial v}{\partial z}\right)^2\right] \end{matrix} \end{bmatrix}, \tag{1.25}$$

$$(trA_2)\,A_1 = \begin{bmatrix} \Phi_1 & \Phi_2 & \Phi_3 \\ \Psi_1 & \Psi_1 & \Psi_1 \\ F_1 & F_2 & F_3 \end{bmatrix}, \tag{1.26}$$

where

$$\Phi_1 = 0, \;\; \Phi_2 = 0, \quad \Phi_3 = \frac{\partial u}{\partial z}\begin{bmatrix} 2\frac{\partial^2 w}{\partial z\partial t} + 2w\frac{\partial^2 w}{\partial z^2} + 4(\frac{\partial w}{\partial z})^2 \\ +2\left[\left(\frac{\partial u}{\partial z}\right)^2 + \left(\frac{\partial v}{\partial z}\right)^2\right] \end{bmatrix}, \tag{1.27}$$

$$\Psi_1 = 0, \;\; \Psi_2 = 0, \quad \Psi_3 = \frac{\partial v}{\partial z}\begin{bmatrix} 2\frac{\partial^2 w}{\partial z\partial t} + 2w\frac{\partial^2 w}{\partial z^2} + 4(\frac{\partial w}{\partial z})^2 \\ +2\left[\left(\frac{\partial u}{\partial z}\right)^2 + \left(\frac{\partial v}{\partial z}\right)^2\right] \end{bmatrix}, \tag{1.28}$$

$$F_1 = \frac{\partial u}{\partial z}\begin{bmatrix} 2\frac{\partial^2 w}{\partial z\partial t} + 2w\frac{\partial^2 w}{\partial z^2} + 4(\frac{\partial w}{\partial z})^2 \\ +2\left[\left(\frac{\partial u}{\partial z}\right)^2 + \left(\frac{\partial v}{\partial z}\right)^2\right] \end{bmatrix}, \tag{1.29}$$

$$F_2 = \frac{\partial v}{\partial z}\begin{bmatrix} 2\frac{\partial^2 w}{\partial z\partial t} + 2w\frac{\partial^2 w}{\partial z^2} + 4(\frac{\partial w}{\partial z})^2 \\ +2\left[\left(\frac{\partial u}{\partial z}\right)^2 + \left(\frac{\partial v}{\partial z}\right)^2\right] \end{bmatrix}, \tag{1.30}$$

$$F_3 = 2\frac{\partial w}{\partial z}\begin{bmatrix} 2\frac{\partial^2 w}{\partial z\partial t} + 2w\frac{\partial^2 w}{\partial z^2} + 4(\frac{\partial w}{\partial z})^2 \\ +2\left[\left(\frac{\partial u}{\partial z}\right)^2 + \left(\frac{\partial v}{\partial z}\right)^2\right] \end{bmatrix}. \tag{1.31}$$

On substituting the values of equations (1.17) to (1.26) in equation (1.15) we get the following three scalar equations

$$\begin{aligned} & \rho\left[\frac{\partial u}{\partial t} + w\frac{\partial u}{\partial z} - 2v\Omega - x\Omega^2\right] \\ = \; & -\frac{\partial p_1}{\partial x} + \mu\frac{\partial^2 u}{\partial z^2} - \sigma B_0^2 u \\ & +\alpha_1\left(\frac{\partial^3 u}{\partial z^2\partial t} + w\frac{\partial^3 u}{\partial z^3} + 2\frac{\partial^2 u}{\partial z^2}\frac{\partial w}{\partial z} + \frac{\partial^2 w}{\partial z^2}\frac{\partial u}{\partial z}\right) \\ & +2\alpha_2\left(\frac{\partial^2 u}{\partial z^2}\frac{\partial w}{\partial z} + \frac{\partial^2 w}{\partial z^2}\frac{\partial u}{\partial z}\right) \\ & +\beta_3\frac{\partial}{\partial z}\left[\frac{\partial u}{\partial z}\begin{bmatrix} 2\frac{\partial^2 w}{\partial z\partial t} + 2w\frac{\partial^2 w}{\partial z^2} + 4(\frac{\partial w}{\partial z})^2 \\ +2\left[\left(\frac{\partial u}{\partial z}\right)^2 + \left(\frac{\partial v}{\partial z}\right)^2\right] \end{bmatrix}\right], \end{aligned} \tag{1.32}$$

$$
\begin{aligned}
& \rho\left[\frac{\partial v}{\partial t}+w\frac{\partial v}{\partial z}+2u\Omega-y\Omega^{2}\right] \\
= \quad & -\frac{\partial p_{1}}{\partial y}+\mu\frac{\partial^{2}v}{\partial z^{2}}-\sigma B_{0}^{2}v \\
& +\alpha_{1}\left(\frac{\partial^{3}v}{\partial z^{2}\partial t}+w\frac{\partial^{3}v}{\partial z^{3}}+\frac{\partial^{2}v}{\partial z^{2}}\frac{\partial w}{\partial z}+\frac{\partial^{2}w}{\partial z^{2}}\frac{\partial v}{\partial z}\right) \\
& +2\alpha_{2}\left(\frac{\partial^{2}v}{\partial z^{2}}\frac{\partial w}{\partial z}+\frac{\partial^{2}w}{\partial z^{2}}\frac{\partial v}{\partial z}\right) \\
& +\beta_{3}\frac{\partial}{\partial z}\left[\frac{\partial v}{\partial z}\left[\begin{array}{c} 2\frac{\partial^{2}w}{\partial z\partial t}+2w\frac{\partial^{2}w}{\partial z^{2}}+4(\frac{\partial w}{\partial z})^{2} \\ +2\left[\left(\frac{\partial u}{\partial z}\right)^{2}+\left(\frac{\partial v}{\partial z}\right)^{2}\right] \end{array}\right]\right], \qquad (1.33)
\end{aligned}
$$

$$
\begin{aligned}
& \rho\left[\frac{\partial w}{\partial t}+w\frac{\partial w}{\partial z}\right] \\
= \quad & -\frac{\partial p_{1}}{\partial z}+2\mu\frac{\partial^{2}w}{\partial z^{2}}-\sigma B_{0}^{2}w \\
& +2\alpha_{1}\left(\frac{\partial^{3}w}{\partial z^{2}\partial t}+w\frac{\partial^{3}w}{\partial z^{3}}+\frac{\partial^{2}v}{\partial z^{2}}\frac{\partial w}{\partial z}\right) \\
& +\left(2\alpha_{1}+\alpha_{2}\right)\frac{\partial}{\partial z}\left[\left(\frac{\partial u}{\partial z}\right)^{2}+\left(\frac{\partial v}{\partial z}\right)^{2}\right]+8\alpha_{2}\left(\frac{\partial w}{\partial z}\right)^{2}\frac{\partial w}{\partial z} \\
& +\beta_{3}\frac{\partial}{\partial z}\left[4\frac{\partial w}{\partial z}\left[\begin{array}{c} 2\frac{\partial^{2}w}{\partial z\partial t}+2w\frac{\partial^{2}w}{\partial z^{2}}+4(\frac{\partial w}{\partial z})^{2} \\ +2\left[\left(\frac{\partial u}{\partial z}\right)^{2}+\left(\frac{\partial v}{\partial z}\right)^{2}\right] \end{array}\right]\right]. \qquad (1.34)
\end{aligned}
$$

Considering the velocity field in equation (1.22) and continuity equation (1.9) one can obtain $w = constant$. For the case of uniform suction, equation of continuity gives $w = -W_0$ and hence equations (1.32) to (1.34) take the form as

$$
\begin{aligned}
& \rho\left[\frac{\partial u}{\partial t}-W_{0}\frac{\partial u}{\partial z}-2v\Omega\right] \\
= \quad & -\frac{\partial p^{*}}{\partial x}+\mu\frac{\partial^{2}u}{\partial z^{2}}-\sigma B_{0}^{2}u \\
& +\alpha_{1}\left(\frac{\partial^{3}u}{\partial z^{2}\partial t}-W_{0}\frac{\partial^{3}u}{\partial z^{3}}\right)
\end{aligned}
$$

$$+\beta_3\frac{\partial}{\partial z}\left[2\frac{\partial u}{\partial z}\left[\left(\frac{\partial u}{\partial z}\right)^2+\left(\frac{\partial v}{\partial z}\right)^2\right]\right], \tag{1.35}$$

$$\begin{aligned}
&\rho\left[\frac{\partial v}{\partial t}-W_0\frac{\partial v}{\partial z}+2u\Omega\right]\\
= &-\frac{\partial p^*}{\partial y}+\mu\frac{\partial^2 v}{\partial z^2}-\sigma B_0^2 v\\
&+\alpha_1\left(\frac{\partial^3 v}{\partial z^2\partial t}-W_0\frac{\partial^3 v}{\partial z^3}\right)\\
&+\beta_3\frac{\partial}{\partial z}\left[2\frac{\partial v}{\partial z}\left[\left(\frac{\partial u}{\partial z}\right)^2+\left(\frac{\partial v}{\partial z}\right)^2\right]\right],
\end{aligned} \tag{1.36}$$

$$\frac{\partial p^*}{\partial z}=\sigma B_0^2 W_0, \tag{1.37}$$

where the modified pressure

$$p^*=p_1-\frac{1}{2}\rho \mathrm{r}^2\Omega^2-(2\alpha_1+\alpha_2)\left[\left(\frac{\partial u}{\partial z}\right)^2+\left(\frac{\partial v}{\partial z}\right)^2\right]. \tag{1.38}$$

The equations for second grade are obtained by putting $\beta_3=0$ in equations (1.35) to (1.37).

1.8 Methods of Solution

- Laplace Transform Technique
- Perturbation Technique

1.8.1 Laplace Transform Technique

Laplace transform is essentially a mathematical tool which can be used to solve several problems in mathematics and engineering. This transform was first introduced by Laplace, a French mathematician, in the year 1970 in his work on probability theorem. This technique became popular when Heaviside applied to the solution of an ordinary differential equation referred hereafter as ordinary differential equation, representing a problem in electrical engineering. To the basic question as to why one should learn Laplace transform technique when other techniques are available, the answer is very simple. Transforms are used to accomplish the solution of certain problems with less effort and in a simple routine way.

Definition of Exponential Order

Suppose $f(t)$ is piecewise continuous function and if it has an additional property that there exist a real number ϵ_0 and a finite number Q_1 such that

$$\lim_{t\to\infty} |f(t)|\, e^{-\epsilon t} \le Q_1 \quad \text{for } \epsilon > \epsilon_0,$$

and the limit does not exist when $\epsilon > \epsilon_0$, then such a function is said to be of exponential order ϵ_0 also written as

$$|f(t)| = O\left(e^{-\epsilon_0 t}\right).$$

Variables such as velocity and current are always finite; which means that $f(t)$ is bounded. Thus for any bounded function $f(t)$, $|f(t)|\,(e^{-\epsilon t}) \to 0$ for all $\epsilon > 0$. The order of such a function is zero. However, variable such as electrical charge and mechanical displacement may increase without limit but of course proportional to t. Such functions are also of exponential order.

Definition Let $f(t)$ be continuous and single valued function of the real variable t defined for all t, $0 < t < \infty$, and is of exponential order. Then the Laplace transform of $f(t)$ is defined as a function $\overline{f}(p)$ denoted by the integral

$$\pounds[f(t)] = \overline{f}(p) = \int_0^\infty f(t)\, e^{-pt} dt, \; p > 0, \tag{1.39}$$

over that range of values of p for which the integral exist. Here p is a parameter, real or complex. Obviously, $\pounds[f(t)]$ is a function of p. Thus,

$$\begin{aligned} \pounds[f(t)] &= \overline{f}(p), \\ f(t) &= \pounds^{-1}\left[\overline{f}(p)\right] = \frac{1}{2\pi i}\int_{\lambda_1 - i\infty}^{\lambda_1 + i\infty} \overline{f}(p)\, e^{pt} dp, \end{aligned} \tag{1.40}$$

where $\pounds$ is the operator which transforms $f(t)$ into $\overline{f}(p)$, called Laplace transform operator, and $\pounds^{-1}$ is the inverse Laplace transform operator.

Theorem: If $f(t)$ is piecewise continuous in range $t \geq 0$ and is exponential order ϵ, then the Laplace transform $\overline{f}(p)$ of $f(t)$ exist for all $p > \epsilon$.

Proof: From the definition of Laplace transform,

$$\pounds[f(t)] = \overline{f}(p) = \int_0^\infty f(t)\, e^{-pt} dt = \int_0^{T_1} f(t)\, e^{-pt} dt + \int_{T_1}^\infty f(t)\, e^{-pt} dt = I_1 + I_2, \tag{1.41}$$

Since $f(t)$ is piecewise continuous on every finite interval $0 < T_1 < t$, I_1 exists, whereas

$$|I_2| \leq \int_{T_1}^\infty f(t)\, e^{-pt} dt. \tag{1.42}$$

But $f(t)$ is a function of exponential order; therefore,

$$|f(t)| \leq Q_1 e^{\epsilon t} \text{ for } \epsilon \text{ real.}$$

Hence

$$\left|e^{-pt} f(t)\right| < Q_1 e^{-(p-\epsilon)t}.$$

Thus,

$$|I_2| \leq \int_{T_1}^{\infty} Q_1 e^{-(p-\epsilon)t} dt = \frac{Q_1 e^{-(p-\epsilon)t}}{p-\epsilon}, \ p > \epsilon. \tag{1.43}$$

In other words, I_2 can be made as small as we like provided T_1 is large enough and, therefore, I_2 exists. Hence, $\pounds\left[f\left(t\right)\right]$ exists for $p > \epsilon$.

1.8.2 Perturbation Technique

The purpose of this section is to describe the application of perturbation expansion technique to the solution of differential equations and the approximation of integrals. Approximate expressions are generated in the form of asymptotic series. These may not and often do not converge but, in a truncated form only two or three terms provide a useful approximation to the original problem.

The techniques, being analytical rather than numerical, provide an alternative to a direct computer solution. An awareness of the perturbation approach is sometimes essential even when a direct numerical approach is adopted. An example of this occurs in boundary layer problems where there are regions of rapid change of quantities such as fluid velocity, temperature or concentration. Appropriate scaling of the boundary layer dimension is required before a numerical solution can be generated which will capture the behavior in the rapidly changing region.

Regular Perturbation

Applicable to:

Differential equations with small parameter

Yields:

A series of terms of decreasing magnitude that approximate the solution of the original differential equation.

Idea:

When an equation is changed by only a small amount, the solution will often only change by a small amount.

Procedure:

Expand the dependent variables in a power series depending on the small parameter in the problem. Substitute this series into the original equation(s), the boundary conditions(s), and the initial condition(s) . Expand every thing in a Taylor series, equate the terms corresponding to different powers of the small parameter, and solve the equations sequentially.

Chapter 2

Hydromagnetic Flow of a Second Grade Fluid in a Rotating System

In this chapter an exact solution is developed for an oscillatory boundary layer flow bounded by two horizontal flat plates, one of which is oscillating in its own plane and other at rest. The fluid and the plates are in a rotating frame with constant angular velocity about the $z - axis$ normal to the plates. The fluid considered is second grade, incompressible and electrically conducting. A uniform transverse magnetic field is applied. During the mathematical analysis, it is found that steady part of solution is identical to that of Newtonian fluid. The structure of the boundary layers is also discussed. Several known results of interest are found to follow as particular cases of the solution of the problem examined [26,27,70].

2.1 Basic Electromagnetic Concepts

It is known that an electrical conductor moving in a magnetic field generates an electromotive force (emf) that is proportional to its speed of motion and the magnetic field strengths. The coupling between the fluid flow equations and the electromagnetic field equations will take place. The fluid has been electrically conducting. The field of magnetohydrodynamics is complicate, since it involves the solution of both the momentum equations characterizing fluid flow and Maxwell equations for the magnetic field. In magneto fluid mechanics, Maxwell equations are presented as follows [69]:

$$\begin{aligned} \nabla \cdot \mathbf{B} &= 0, & (2.1)\\ \nabla \cdot \mathbf{E} &= 0, & (2.2)\\ \nabla \times \mathbf{B} &= \mu_e \mathbf{J}, & (2.3)\\ \nabla \times \mathbf{E} &= -\frac{\partial \mathbf{B}}{\partial \mathbf{t}}, & (2.4) \end{aligned}$$

where μ_e is the magnetic permeability. By Ohm's law the total current flow can be defined as

$$\mathbf{J} = \sigma (\mathbf{E} + \mathbf{V} \times \mathbf{B}), \qquad (2.5)$$

where σ is the electrical conductivity. In momentum equation we have to include the electromagnetic force $\mathbf{F}_m$. It is expressed as:

$$\mathbf{F}_m = \mathbf{J} \times \mathbf{B} = \sigma (\mathbf{V} \times \mathbf{B}) \times \mathbf{B}. \qquad (2.6)$$

2.2 Problem Formulation

The physical situation considered is that of the unsteady hydromagnetic flow of a second grade, incompressible and electrically conducting fluid bounded by an infinite parallel plates, distant d apart, when both the fluid and plates rotate with a constant angular velocity $\mathbf{\Omega}$ about the $z-axis$ taken normal to the plates. It is assumed that the plates are electrically non-conducting and an applied uniform magnetic field $\mathbf{B}_0$ is acting parallel to the $z-axis$. The lower plate is at rest and the upper plate oscillating in its own plane with a velocity $U_1(t) = U_0(1+ \in \cos\omega t)$ about a non-zero constant mean velocity U_0. Here ω is the frequency of oscillations. The origin is taken on the lower plate and the $x-axis$ parallel to the direction of motion of the upper plate. Since the plates are infinite in extent, all the physical quantities, except the pressure, depend on z and t only. In a coordinate system rotating with the fluid, the governing equations of continuity and motion are (1.9) and (1.15). The condition of incompressibility along with equation (1.22) yields $w = constant = 0$ since the plates are not porous and thus the velocity field is defined as

$$\mathbf{V} = (u(z,t), v(z,t), 0). \tag{2.7}$$

We are considering second grade fluid, so the Cauchy stress tensor here is given by equation (1.1). With the help of equations (1.1) – (1.22), equation (1.15) in component form can be written as

$$\frac{\partial u}{\partial t} = -\frac{1}{\rho}\frac{\partial p^*}{\partial x} + \left(\nu + \alpha\frac{\partial}{\partial t}\right)\frac{\partial^2 u}{\partial z^2} + 2\Omega v - \frac{\sigma B_0^2 u}{\rho}, \tag{2.8}$$

$$\frac{\partial v}{\partial t} = -\frac{1}{\rho}\frac{\partial p^*}{\partial y} + \left(\nu + \alpha\frac{\partial}{\partial t}\right)\frac{\partial^2 v}{\partial z^2} - 2\Omega u - \frac{\sigma B_0^2 v}{\rho}, \tag{2.9}$$

$$0 = -\frac{1}{\rho}\frac{\partial p^*}{\partial z}, \tag{2.10}$$

in which $\nu = (\mu/\rho)$ is the kinematic viscosity, $\alpha = \alpha_1/\rho$ and the modified pressure p^* is defined by equation (1.38). Equation (2.10) indicates that p^*

is not a function of z and hence p^* is at most a function of x, y and t. Considering

$$p^* = p^*(x, y, t).$$

The boundary conditions for the problem are

$$\begin{aligned} u &= v = 0, \quad at \;\; z = 0, \\ u &= U_1(t) = U_0(1 + \in \cos\omega t), \quad v = 0, \quad at \;\; z = d, \end{aligned} \tag{2.11}$$

where $\in$ is a constant.

Elimination of p^* from equations (2.8) – (2.10) by cross differentiation gives

$$\frac{\partial^2 u}{\partial z \partial t} = \left(\nu + \alpha \frac{\partial}{\partial t}\right) \frac{\partial^3 u}{\partial z^3} + 2\Omega \frac{\partial v}{\partial z} - \frac{\sigma B_0^2}{\rho} \frac{\partial u}{\partial z}, \tag{2.12}$$

$$\frac{\partial^2 v}{\partial z \partial t} = \left(\nu + \alpha \frac{\partial}{\partial t}\right) \frac{\partial^3 v}{\partial z^3} - 2\Omega \frac{\partial u}{\partial z} - \frac{\sigma B_0^2}{\rho} \frac{\partial v}{\partial z}. \tag{2.13}$$

Integration of above equations gives the following equations

$$\frac{\partial u}{\partial t} = \left(\nu + \alpha \frac{\partial}{\partial t}\right) \frac{\partial^2 u}{\partial z^2} + 2\Omega v - \frac{\sigma B_0^2 u}{\rho} + \widetilde{A}(t), \tag{2.14}$$

$$\frac{\partial v}{\partial t} = \left(\nu + \alpha \frac{\partial}{\partial t}\right) \frac{\partial^2 v}{\partial z^2} - 2\Omega u - \frac{\sigma B_0^2 v}{\rho} + \widetilde{B}(t), \tag{2.15}$$

where $\widetilde{A}$ and $\widetilde{B}$ are functions of integration. The resulting boundary layer equations, of equations (2.14) and (2.15) can be combined into following partial differential equation

$$\frac{\partial q}{\partial t} = \left(\nu + \alpha \frac{\partial}{\partial t}\right) \frac{\partial^2 q}{\partial z^2} + \frac{\partial U_1}{\partial t} - 2i\Omega (q - U_1) - \frac{\sigma B_0^2}{\rho} (q - U_1) \tag{2.16}$$

and the corresponding boundary conditions (2.11) are

$$\begin{aligned} q &= 0, \quad at \;\; z = 0, \\ q &= U_1(t) \;\; at \;\; z = d, \end{aligned} \tag{2.17}$$

where

$$q = u + iv \tag{2.18}$$

is the fluid velocity in the complex form. It should be noted that equation (2.16) includes the Newtonian fluid as a special case for $\alpha = 0$. If $\Omega = 0$, the equation reduces to that of second grade fluid in an inertial frame. Moreover if $B_0 = 0$ the equation governing the flow of a non-conducting second grade fluid is obtained.

2.2.1 Solution of the Problem

In order to solve equation (2.16) subject to the boundary conditions (2.17), we look for the solution of the form [27]

$$q(\eta, t) = U_0 \left[q_0(\eta) + \frac{\in}{2} \left\{ q_1(\eta) e^{i\omega t} + q_2(\eta) e^{-i\omega t} \right\} \right], \tag{2.19}$$

where

$$\eta = \frac{z}{d}, \; q_0(\eta) = u_0(\eta) + iv_0(\eta) \text{ and } q_1(\eta) e^{i\omega t} + q_2(\eta) e^{-i\omega t} = u_1(\eta, t) + iv_1(\eta, t). \tag{2.20}$$

Using equation (2.19) into equation (2.17) and boundary conditions (2.18) and then collecting harmonic and nonharmonic terms, we obtain

$$\frac{d^2 q_0}{d\eta^2} - \left(2iK + M^2\right) q_0 = -\left(2iK + M^2\right), \tag{2.21}$$

$$\frac{d^2 q_1}{d\eta^2} - (1 + \lambda\beta)^{-1} \left[2iK + M^2 + i\lambda\right] q_1 = -(1 + \lambda\beta)^{-1} \left[2iK + M^2 + i\lambda\right], \tag{2.22}$$

$$\frac{d^2 q_2}{d\eta^2} - (1 + \lambda\beta)^{-1} \left[2iK + M^2 - i\lambda\right] q_2 = -(1 + \lambda\beta)^{-1} \left[2iK + M^2 - i\lambda\right], \tag{2.23}$$

$$
\begin{aligned}
q_0 &= q_1 = q_2 = 0, \quad at \quad \eta = 0, \\
q_0 &= q_1 = q_2 = 1, \quad at \quad \eta = 1.
\end{aligned} \tag{2.24}
$$

In above equations, $K = \Omega d^2/\nu$ is the dimensionless rotation parameter, $\lambda = \omega d^2/\nu$ is the dimensionless oscillating parameter, $\beta = \alpha/d^2$ is the dimensionless second grade parameter and $M = B_0 d\,(\sigma/\mu)^{\frac{1}{2}}$ is the Hartmann number. The solution of equation (2.21) under the boundary conditions (2.24), is of the form

$$
q_0(\eta) = 1 + Ae^{l\eta} + Be^{-l\eta}, \tag{2.25}
$$

where A, B are constants and

$$
l = \sqrt{2iK + M^2}. \tag{2.26}
$$

Using boundary conditions (2.24) in equation (2.25), we obtain

$$
A = \frac{1}{2\sinh l}, \; B = -\frac{1}{2\sinh l}.
$$

Making use of A and B in equation (2.25) we finally get

$$
q_0(\eta) = 1 - \frac{\sinh l\,(1-\eta)}{\sinh l}. \tag{2.27}
$$

Employing the same procedure as for q_0, the solutions of equations (2.22) and (2.23) subject to boundary conditions (2.24) are given by

$$
q_1(\eta) = 1 - \frac{\sinh m\,(1-\eta)}{\sinh m}, \tag{2.28}
$$

$$
q_2(\eta) = 1 - \frac{\sinh n\,(1-\eta)}{\sinh n}, \tag{2.29}
$$

where

$$
m = \left[\frac{2iK + M^2 + i\lambda}{(1+\lambda\beta)}\right]^{\frac{1}{2}}, \; n = \left[\frac{2iK + M^2 - i\lambda}{(1+\lambda\beta)}\right]^{\frac{1}{2}}. \tag{2.30}
$$

We note that the result of Singh [70] can be recovered when the dimensionless material parameter of the second grade fluid is zero. The solution (2.27) corresponds to the steady part which gives u_0 and v_0 as the primary and secondary velocity components, respectively. From equations (2.27) we have for large K

$$u_0 \approx 1 - e^{-l_R \eta} \cos l_I \eta, \tag{2.31}$$

$$v_0 \approx e^{-l_R \eta} \sin l_I \eta, \tag{2.32}$$

where

$$l_R = \frac{1}{\sqrt{2}} \left[M^2 + \sqrt{M^4 + 4K^2} \right]^{\frac{1}{2}}, \; l_I = \frac{1}{\sqrt{2}} \left[-M^2 + \sqrt{M^4 + 4K^2} \right]^{\frac{1}{2}}$$

and R and I in the subscripts indicate the real and imaginary parts.

The solutions (2.28) and (2.29) together give the unsteady part of the flow. These solutions depend on β. For large K, the primary and secondary velocity components u_1 and v_1, respectively, for the fluctuating flow are given by

$$u_1(\eta, t) \approx 2\cos\omega t - e^{-m_R \eta} \cos(m_I \eta - \omega t) - e^{-n_R \eta} \cos(n_I \eta + \omega t), \tag{2.33}$$

$$v_1(\eta, t) \approx e^{-m_R \eta} \sin(m_I \eta - \omega t) + e^{-n_R \eta} \sin(n_I \eta + \omega t), \tag{2.34}$$

in which

$$m_R = (C_1)^{-1} \left[\sqrt{\sqrt{A_1^2 + B_1^2} + A_1} \right],$$

$$m_I = (C_1)^{-1} \left[\sqrt{\sqrt{A_1^2 + B_1^2} - A_1} \right],$$

$$
\begin{aligned}
n_R &= (C_1)^{-1}\left[\sqrt{\sqrt{A_2^2+B_2^2}+A_2}\right], \\
n_I &= (C_1)^{-1}\left[\sqrt{\sqrt{A_2^2+B_2^2}-A_2}\right], \\
A_1 &= M^2+(2K+\lambda)\,\lambda\beta, \\
B_1 &= (2K+\lambda)-M^2\lambda\beta, \\
A_2 &= M^2+(2K-\lambda)\,\lambda\beta,
\end{aligned}
$$

$$
\begin{aligned}
B_2 &= (2K-\lambda)-M^2\lambda\beta, \\
C_1 &= \sqrt{2\left(1+\lambda^2\beta^2\right)}.
\end{aligned}
$$

We note that steady solution (2.27) is independent on β. It means that primary and secondary velocity components u_0 and v_0 respectively for present steady flow do not depend upon nature of the fluid. The amplitudes and phase differences in terms of u_0 and v_0 are given by

$$
R_0=\sqrt{u_0^2+v_0^2},\ \theta_0=\tan^{-1}\frac{v_0}{u_0}, \tag{2.35}
$$

and for unsteady flow

$$
R_1=\sqrt{u_1^2+v_1^2},\ \theta_1=\tan^{-1}\frac{v_1}{u_1}. \tag{2.36}
$$

2.3 Results and Discussion

The investigations of the velocity, magnetic field, frequency and non-Newtonian effects on the flow of an incompressible conducting fluid bounded between two rigid non-conducting parallel plates have been carried out in the preceding paragraphs. The solutions are obtained for steady and unsteady velocity field from equations (2.27) to (2.29). Numerical computations are presented graphically and discussed in the following points:

1. The primary velocity u_0, secondary velocity v_0, resultant velocity R_0 and phase angle θ_0 are shown graphically in Figs. 2.1-2.4 for various values of K and M. It is observed from Fig. 2.1 that u_0 increases with increase of M for small K, however, for large rotation parameter K, u_0 decreases with the increases of M and is approximately one for large K in the upper half of the channel width. Fig. 2.2 shows that v_0 increases in the lower half of the channel for small K and becomes approximately zero in the upper half of the channel width. These observations can also be expected from equations (2.31) and (2.32). These equations show the existence of a thin boundary layer of order $O(l_R^{-1})$ in the vicinity of the lower plate which decreases with the increase in Hartmann number M or the rotation parameter K. The behavior of R_0 in Fig. 2.3 is almost the same as that of u_0. Fig. 4 shows that θ_0 decreases with increasing M for any value of rotation large or small. It is also evident that θ_0 increases with small rotation whereas it decreases with large rotation and is approximately zero in the upper half of the channel.

2. In Figs. 2.5,2.6 and 2.9,2.10, u_1 and v_1 are plotted versus η for $M = 2$, $\lambda = 5$, $\beta = 0, 0.25, 0.5, 0.75, 1$ and $\omega t = \pi/4$ respectively. We see from these figures that for $K = 10$ and $K = 2$ an increase in β decreases the velocity. The resultant velocity R_1 and θ_1 for these cases are plotted in Figs. 2.7 and 2.8. With the increase in β the amplitude as well as phase angle decreases.

3. The expressions (2.33) and (2.34) represent the shear oscillations spreading out from the oscillating plate. The velocities of these oscillations are ω/m_I and ω/n_I and the amplitude of these oscillations decay exponentially with η. These expressions also show the emergence of a boundary layer of thickness of order $O(m_R^{-1})$ superimposed with a boundary layer of thickness of order $O(n_R^{-1})$. These boundary layers which are a direct consequence of the cyclonic and anticyclonic components of the imposed harmonic oscillations decrease with increase in M, β and K. It may be noted that in second grade fluid the boundary layer thickness increases. Also, the present analysis exhibits a striking difference between the structure of hydrodynamic and the hydromagnetic boundary layers.

In case of resonance $(2\Omega - \omega = 0 \ or \ 2K - \lambda = 0)$, the solution of equation (2.30) is

$$q_2(\eta) = 1 - \frac{\sinh \widetilde{M}\,(1-\eta)}{\sinh \widetilde{M}} \tag{2.37}$$

where

$$\widetilde{M}^2 = M^2\,(1 + i\lambda\beta)^{-1}. \tag{2.38}$$

We note that when $M = 0$ then $q_2(\eta)$ for Newtonian and second grade fluids is the same and is given by

$$q_2(\eta) = \eta.$$

The above solution for Newtonian fluid is the same as in the paper by Singh [70]. Even in the case of resonance, differential equation (5) in Ganapathy [27] yields $q_2(\eta) = \eta \neq 0$. This is a contradiction to the claim of Ganapathy that $q_2(\eta) = 0$ and the solution

$$q(\eta, t) = U_0 \left[q_0(\eta) + \frac{\in}{2} q_1(\eta) e^{i\omega t} \right], \tag{2.39}$$

of Mazumder [26] is valid for the special case.

Chapter 3

Unsteady Rotating Flow of a Conducting Second Grade Fluid on Oscillating Plate

In this chapter, an initial value investigation is made of the oscillatory hydromagnetic flow in a semi-infinite expanse of an electrically conducting, incompressible, homogeneous, second grade fluid bounded by a porous plate with uniform suction or blowing. The plate and the fluid are in a state of rigid body rotation about an axis normal to the flow. The plate is non-conducting and the magnetic field is applied transversely to the direction of the flow. Analytic solution of the governing equation is obtained by using the Laplace transform treatment. Asymptotic analysis is carried out to determine the unsteady flow field for small and large times. The structure of the velocity distribution and the associated boundary layers for hydromagnetic and hydrodynamic flows are examined for various cases related to resonant and non-resonant frequencies and uniform suction or blowing. The analysis of the obtained results shows that the flow field is appreciably influenced by the material parameter of the second grade fluid, the applied magnetic field, the imposed frequency, rotation and suction and blowing parameters. It is observed in a second grade fluid that a steady asymptotic hydromagnetic solution exists for blowing and resonance which is different from the hydrodynamic situation for a viscous fluid [41].

3.1 Governing Equations

We introduce a Cartesian coordinate system (x, y, z) rotating uniformly with an angular velocity Ω about the $z - axis$, taken positive in the vertically upward direction, with the plate coinciding with the plane $z = 0$ and being uniformly porous. The fluid is incompressible second grade and is permeated by a uniform magnetic field B_0 in the $z - direction$. At time $t \leq 0$, the fluid and the plate are assumed to be at rest. The hydromagnetic flow is generated in the uniformly rotating fluid system by non-torsional oscillations of frequency ω of the plate in its own plane at $t = 0^+$.

Referred to the rotating frame of reference the unsteady motion of electrically conducting, incompressible second grade fluid is governed by equations (1.36) – (1.38) when $\beta_3 = 0$ and equation of continuity (1.9).

The condition of incompressibility yields $w = -W_0$ ($W_0 > 0$ for the suction and $W_0 < 0$ for blowing). With the help of equations (1.35) – (1.36) and (2.7) we can write

$$\frac{\partial u}{\partial t} - W_0\frac{\partial u}{\partial z} - 2\Omega v + \frac{\sigma B_0^2 u}{\rho} = -\frac{1}{\rho}\frac{\partial p^*}{\partial x} + \nu\frac{\partial^2 u}{\partial z^2} + \alpha\left(\frac{\partial^3 u}{\partial z^2 \partial t} - W_0\frac{\partial^3 u}{\partial z^3}\right), \tag{3.1}$$

$$\frac{\partial v}{\partial t} - W_0\frac{\partial v}{\partial z} + 2\Omega u + \frac{\sigma B_0^2 v}{\rho} = -\frac{1}{\rho}\frac{\partial p^*}{\partial y} + \nu\frac{\partial^2 v}{\partial z^2} + \alpha\left(\frac{\partial^3 v}{\partial z^2 \partial t} - W_0\frac{\partial^3 v}{\partial z^3}\right), \tag{3.2}$$

$$\sigma B_0^2 W_0 = \frac{\partial p^*}{\partial z}. \tag{3.3}$$

The proposed boundary and initial conditions for the problem are

$$\begin{aligned} u &= 0,\ v = 0,\ \ \textit{for all}\ \ z > 0\ \ \textit{and}\ \ t \le 0, \\ u &= -U + U_0(ae^{i\omega t} + be^{-i\omega t}),\ v = 0,\ \ \textit{at}\ \ z = 0,\ t > 0, \\ u &\to 0,\ \ v \to 0,\ \ \ \textit{as}\ \ z \to \infty,\ t > 0, \end{aligned} \tag{3.4}$$

where U and U_0 are the real constants with the dimension of velocity and a, b, are complex constants.

From equation (3.3) we note that p^* is a function of x, y, z and t. Taking p^* of the following form

$$p^* = \sigma B_0^2 W_0 z + f_1(x, y, t). \tag{3.5}$$

With the help of equation (3.5) we get the following boundary equations from equations (3.1) and (3.2)

$$\frac{\partial u}{\partial t} - W_0\frac{\partial u}{\partial z} - 2\Omega v + \frac{\sigma B_0^2 u}{\rho} = \nu\frac{\partial^2 u}{\partial z^2} + \alpha\left(\frac{\partial^3 u}{\partial z^2 \partial t} - W_0\frac{\partial^3 u}{\partial z^3}\right), \tag{3.6}$$

$$\frac{\partial v}{\partial t} - W_0\frac{\partial v}{\partial z} + 2\Omega u + \frac{\sigma B_0^2 v}{\rho} = \nu\frac{\partial^2 v}{\partial z^2} + \alpha\left(\frac{\partial^3 v}{\partial z^2 \partial t} - W_0\frac{\partial^3 v}{\partial z^3}\right), \tag{3.7}$$

From equations (3.6) and (3.7) we arrive at

$$\frac{\partial q}{\partial t} - W_0\frac{\partial q}{\partial z} + (2i\Omega + n^*)\, q = \nu\frac{\partial^2 q}{\partial z^2} + \alpha\left(\frac{\partial^3 q}{\partial z^2 \partial t} - W_0\frac{\partial^3 q}{\partial z^3}\right), \tag{3.8}$$

where

$$n^* = \frac{\sigma B_0^2}{\rho}. \tag{3.9}$$

The initial and boundary conditions are

$$q = 0, \quad \textit{for all} \quad z > 0 \quad \textit{and} \quad t \leq 0, \tag{3.10}$$

$$q = -U + U_0(ae^{i\omega t} + be^{-i\omega t}), \quad at \quad z = 0,\ t > 0, \tag{3.11}$$

$$q \to 0, \quad as \quad z \to \infty,\ t > 0.$$

Using the dimensionless variables

$$\widehat{z} = zU_0/\nu, \quad \widehat{t} = \Omega t, \quad \widehat{q} = q/U_0,$$

equations (3.8) , (3.10) and (3.11) , after dropping the hats, become

$$\frac{\partial^2 q}{\partial z^2} + \alpha\left(\nu_1 \frac{\partial^3 q}{\partial z^2 \partial t} - \nu_2 S \frac{\partial^3 q}{\partial z^3}\right) + S\frac{\partial q}{\partial z} = \frac{E}{2}\frac{\partial q}{\partial t} + (iE + N)\, q, \tag{3.12}$$

$$q = 0, \quad \textit{for all} \ z > 0 \quad \textit{and} \quad t \leq 0, \tag{3.13}$$

$$q = -U/U_0 + (ae^{i\sigma_1 t} + be^{-i\sigma_1 t}), \quad at \quad z = 0,\ t > 0, \tag{3.14}$$

$$q \to 0, \qquad as \quad z \to \infty, \qquad t > 0,$$

where

$$S = W_0/U_0,\ E = 2\Omega\nu/U_0^2,\ \sigma_1 = \omega/\Omega,\ \nu_1 = \Omega/\nu,\ \nu_2 = U_0^2/\nu^2,\ N = \frac{n^*\nu}{U_0^2}.$$

Equations (3.12) – (3.14) comprise the linear initial boundary value problem we are now going to solve.

3.2 Solution of the Problem for Suction

Let

$$\overline{q}(z,p) = \int_0^\infty q(z,t)\; e^{-pt} dt, \tag{3.15}$$

be the Laplace transform of q. Then multiplying equation (3.12) and conditions (3.14), respectively by e^{-pt} and integrating between the limits 0 to ∞ and using equation (3.13) yields the following transformed boundary value problem

$$\frac{d^2\overline{q}}{dz^2} + S\frac{d\overline{q}}{dz} - \left(\frac{E}{2}p + iE + N\right)\overline{q} + \alpha\left(p\nu_1\frac{d^2\overline{q}}{dz^2} - \nu_2 S\frac{d^3\overline{q}}{dz^3}\right) = 0, \tag{3.16}$$

$$\begin{aligned} \overline{q} &= -\frac{U}{U_0 p} + \frac{a}{p - i\sigma_1} + \frac{b}{p + i\sigma_1},\; at\; z = 0, \\ \overline{q} &\rightarrow 0,\; as\; z \rightarrow \infty. \end{aligned} \tag{3.17}$$

Equation (3.16) is a third order ordinary differential equation when $\alpha \neq 0$, which for $\alpha = 0$ reduces to an equation governing the Newtonian fluid. Hence, the analysis of the flow of the second grade fluids, in particular, and the viscoelastic fluids, in general is more challenging mathematically and computationally, because of a peculiarity in equations governing the fluid motion, namely, the order of differential equation (s) characterizing the flow of these fluids is larger than the number of available boundary conditions. The difficulty is further accentuated by the fact that a non-Newtonian parameter of the fluid (for example α, for a second grade fluid) usually occur in the coefficient of the highest derivative. The usual attempt to resolve this difficulty centered around seeking a perturbation solution assuming the non-Newtonian parameter to be small, the classical papers being by Kaloni

[50], Beard and Walters [71] and Siddiqui et al. [72]. Here we also write the solution in the following form

$$\overline{q} = \overline{q}_{01} + \alpha\overline{q}_{11} + O(\alpha^2). \tag{3.18}$$

Substituting equation (3.18) into equations (3.16) and (3.17), and equating the coefficient of various powers of α, we obtain the following systems of equations, along with the boundary conditions:

3.2.1 Zeroth-Order System

$$\frac{d^2\overline{q}_{01}}{dz^2} + S\frac{d\overline{q}_{01}}{dz} - \left(\frac{E}{2}p + iE + N\right)\overline{q}_{01} = 0, \tag{3.19}$$

$$\begin{aligned}\overline{q}_{01} &= -\frac{U}{U_0 p} + \frac{a}{p - i\sigma_1} + \frac{b}{p + i\sigma_1}, \, at \ z = 0, \\ \overline{q}_{01} &\to 0, \, as \ z \to \infty.\end{aligned} \tag{3.20}$$

3.2.2 First-Order System

$$\frac{d^2\overline{q}_{11}}{dz^2} + S\frac{d\overline{q}_{11}}{dz} - \left(\frac{E}{2}p + iE + N\right)\overline{q}_{11} = -p\nu_1\frac{d^2\overline{q}_{01}}{dz^2} + \nu_2 S\frac{d^3\overline{q}_{01}}{dz^3}, \tag{3.21}$$

$$\begin{aligned}\overline{q}_{11} &= 0, \, at \ z = 0, \\ \overline{q}_{11} &\to 0, \, as \ z \to \infty.\end{aligned} \tag{3.22}$$

3.2.3 Zeroth-Order Solution

The solution of equation (3.19) is

$$\overline{q}_{01} = A_3 e^{-m_1 z} + B_3 e^{-n_1 z}, \tag{3.23}$$

where A_3 and B_3 are constants and

$$m_1 = \frac{1}{2}\left(S + \sqrt{S^2 + 4iE + 2Ep + 4N}\right), \tag{3.23a}$$

$$n_1 = \frac{1}{2}\left(S - \sqrt{S^2 + 4iE + 2Ep + 4N}\right). \tag{3.23b}$$

Making use of boundary conditions (3.20), we can write equation (3.23) as

$$\overline{q}_{01} = \left(\frac{a}{p - i\sigma_1} + \frac{b}{p + i\sigma_1} - \frac{U}{U_0 p}\right) e^{-m_1 z}. \tag{3.24}$$

3.2.4 First-Order Solution

With the help of equation (3.24), equation (3.21) takes the form

$$\begin{aligned} &\frac{d^2\overline{q}_{11}}{dz^2} + S\frac{d\overline{q}_{11}}{dz} - \left(\frac{E}{2}p + iE + N\right)\overline{q}_{11} = \\ &-p\nu_1 m_1^2\left(\frac{a}{p - i\sigma_1} + \frac{b}{p + i\sigma_1} - \frac{U}{U_0 p}\right) e^{-m_1 z} \\ &-\nu_2 S m_1^3\left(\frac{a}{p - i\sigma_1} + \frac{b}{p + i\sigma_1} - \frac{U}{U_0 p}\right) e^{-m_1 z}. \end{aligned} \tag{3.25}$$

The solution of equation (3.25) satisfying the boundary conditions (3.22) after using equation (3.23a) is given by

$$\begin{aligned} \overline{q}_{11} = \; & \frac{\nu_1}{2}\left[\frac{pze^{-\frac{1}{2}\left(S+\sqrt{S^2+4iE+2Ep+4N}\right)z}\left(\frac{a}{p-i\sigma_1} + \frac{b}{p+i\sigma_1} - \frac{U}{U_0 p}\right)}{\left(S + \sqrt{S^2 + 4iE + 2Ep + 4N} - \frac{4iE+2Ep+4N}{2\sqrt{S^2+4iE+2Ep+4N}}\right)}\right] \\ & + \frac{\nu_2 S}{4}\left[\frac{ze^{-\frac{1}{2}\left(S+\sqrt{S^2+4iE+2Ep+4n}\right)z}\left(\frac{a}{p-i\sigma_1} + \frac{b}{p+i\sigma_1} - \frac{U}{U_0 p}\right)}{\left(\begin{array}{c} 2S\sqrt{S^2 + 4iE + 2Ep + 4N} \\ -\frac{4iE+2Ep+4N}{2\sqrt{S^2+4iE+2Ep+4N}} + 2S^2 + 2iE + Ep + 2N \end{array}\right)}\right]. \end{aligned} \tag{3.26}$$

Substitution of equations (3.24) and (3.26) in equation (3.18) yields

$$\overline{q}\,(z,p) = \left(\frac{a}{p-i\sigma_1} + \frac{b}{p+i\sigma_1} - \frac{U}{U_0 p}\right) e^{-\frac{1}{2}\left(S+\sqrt{S^2+4iE+2Ep+4N}\right)z} + \alpha z e^{-Sz/2}$$

$$\times \left[\begin{array}{c} \nu_3 p + \nu_4 + \nu_{10}\, p\sqrt{p+\frac{S^2+4iE+4N}{2E}} \\ -\nu_{11}\, p\left(\sqrt{p+\frac{S^2+4iE+4N}{2E}}\right)^{-1} - \nu_{12}\, p^2\left(\sqrt{p+\frac{S^2+4iE+4N}{2E}}\right)^{-1} \\ +\nu_{13}\left(\sqrt{p+\frac{S^2+4iE+4N}{2E}}\right) - \nu_{14}\left(\sqrt{p+\frac{S^2+4iE+4N}{2E}}\right)^{-1} \end{array}\right] F, \tag{3.27}$$

where

$$F = \left(\frac{a}{p-i\sigma_1} + \frac{b}{p+i\sigma_1} - \frac{U}{U_0 p}\right) e^{-z\sqrt{E/2}\left(p+\frac{S^2+4iE+4N}{2E}\right)^{1/2}},$$

$$\begin{aligned}
\nu_3 &= \frac{S}{2}\left(\nu_1 + \frac{E\nu_2}{2}\right),\ \nu_4 = \frac{S\nu_2}{2}\left(S^2+iE+N\right),\ \nu_5 = \frac{\nu_1}{2},\\
\nu_6 &= (iE+N)\,\nu_1 + \frac{ES^2\nu_2}{4},\ \nu_7 = \frac{E\nu_1}{2},\ \nu_8 = \frac{S^2\nu_2}{2},\ \nu_9 = \frac{(iE+N)\,S^2\nu_2}{2},\\
\nu_{10} &= \nu_5\sqrt{2E},\ \nu_{11} = \nu_6\left(\sqrt{2E}\right)^{-1},\ \nu_{12} = \nu_7\left(\sqrt{2E}\right)^{-1},\\
\nu_{13} &= \nu_8\sqrt{2E},\ \nu_{14} = \nu_9\left(\sqrt{2E}\right)^{-1}.
\end{aligned}$$

Note that the first term in equation (3.27) corresponds to zeroth-order solution while all the other terms correspond to the first order solution. The inverse Laplace transform of equation (3.27) is given by

$$q(z,t) = \frac{1}{2\pi i}\int_{\lambda_1-i\infty}^{\lambda_1+i\infty} \overline{q}(z,p)\, e^{pt}dp,\ (\lambda_1 > 0)$$

$$= \frac{1}{2\pi i}\int_{\lambda_1-i\infty}^{\lambda_1+i\infty} \left(\frac{a}{p-i\sigma_1} + \frac{b}{p+i\sigma_1} - \frac{U}{U_0 p}\right) e^{-\frac{1}{2}\left(S+\sqrt{S^2+4iE+2Ep+4N}\right)z+pt}dp$$

$$+\frac{\alpha}{2\pi i}\int_{\lambda_1-i\infty}^{\lambda_1+i\infty}\nu_3 p\left(\frac{a}{p-i\sigma_1}+\frac{b}{p+i\sigma_1}-\frac{U}{U_0 p}\right)ze^{-\frac{Sz}{2}-\sqrt{\frac{E}{2}}z\sqrt{p+\frac{S^2+4iE+4N}{2E}}+pt}dp$$

$$+\frac{\alpha}{2\pi i}\int_{\lambda_1-i\infty}^{\lambda_1+i\infty}\nu_4\left(\frac{a}{p-i\sigma_1}+\frac{b}{p+i\sigma_1}-\frac{U}{U_0 p}\right)ze^{-\frac{Sz}{2}-\sqrt{\frac{E}{2}}z\sqrt{p+\frac{S^2+4iE+4N}{2E}}+pt}dp$$

$$+\frac{\alpha}{2\pi i}\int_{\lambda_1-i\infty}^{\lambda_1+i\infty}\left[\begin{array}{c}\nu_{10}p\sqrt{p+\frac{S^2+4iE+4N}{2E}}\\ \times\left(\frac{a}{p-i\sigma_1}+\frac{b}{p+i\sigma_1}-\frac{U}{U_0 p}\right)\end{array}\right]ze^{-\frac{Sz}{2}-\sqrt{\frac{E}{2}}z\sqrt{p+\frac{S^2+4iE+4N}{2E}}+pt}dp$$

$$-\frac{\alpha}{2\pi i}\int_{\lambda_1-i\infty}^{\lambda_1+i\infty}\nu_{11}p\left(\frac{a}{p-i\sigma_1}+\frac{b}{p+i\sigma_1}-\frac{U}{U_0 p}\right)\frac{ze^{-\frac{Sz}{2}-\sqrt{\frac{E}{2}}z\sqrt{p+\frac{S^2+4iE+4N}{2E}}+pt}}{\sqrt{p+\frac{S^2+4iE+4N}{2E}}}dp$$

$$-\frac{\alpha}{2\pi i}\int_{\lambda_1-i\infty}^{\lambda_1+i\infty}\nu_{11}p^2\left(\frac{a}{p-i\sigma_1}+\frac{b}{p+i\sigma_1}-\frac{U}{U_0 p}\right)\frac{ze^{-\frac{Sz}{2}-\sqrt{\frac{E}{2}}z\sqrt{p+\frac{S^2+4iE+4N}{2E}}+pt}}{\sqrt{p+\frac{S^2+4iE+4N}{2E}}}dp$$

$$+\frac{\alpha}{2\pi i}\int_{\lambda_1-i\infty}^{\lambda_1+i\infty}\left[\begin{array}{c}\nu_{13}p\sqrt{p+\frac{S^2+4iE+4N}{2E}}\\ \times\left(\frac{a}{p-i\sigma_1}+\frac{b}{p+i\sigma_1}-\frac{U}{U_0 p}\right)\end{array}\right]ze^{-\frac{Sz}{2}-\sqrt{\frac{E}{2}}z\sqrt{p+\frac{S^2+4iE+4N}{2E}}+pt}dp$$

$$-\frac{\alpha}{2\pi i}\int_{\lambda_1-i\infty}^{\lambda_1+i\infty}\nu_{14}\left(\frac{a}{p-i\sigma_1}+\frac{b}{p+i\sigma_1}-\frac{U}{U_0 p}\right)\frac{ze^{-\frac{Sz}{2}-\sqrt{\frac{E}{2}}z\sqrt{p+\frac{S^2+4iE+4N}{2E}}+pt}}{\sqrt{p+\frac{S^2+4iE+4N}{2E}}}dp. \tag{3.28}$$

In order to evaluate the integrals appearing in equation (3.28) we use the substitution method and after lengthy calculations we have

$$\frac{1}{2\pi i}\int_{\lambda_1-i\infty}^{\lambda_1+i\infty}\frac{pa\nu_3}{p-i\sigma_1}e^{-\sqrt{\frac{E}{2}}z\sqrt{p+\frac{S^2+4iE+4N}{2E}}+pt}dp=\frac{\nu_3 a}{2}\frac{z}{t}\sqrt{\frac{E}{2\pi t}}e^{-\left(\frac{S^2+4iE+4N}{2E}\right)t-Ez^2/8t}$$
$$+\frac{\nu_3 a}{2}i\sigma_1 e^{i\sigma_1 t}\left[\begin{array}{c}e^{\sqrt{\frac{E}{2}}z\sqrt{\frac{S^2+4iE+4N}{2E}+i\sigma_1}}\operatorname{erf}c\left(\frac{z}{2}\sqrt{\frac{E}{2t}}+\sqrt{\frac{S^2+4iE+4N}{2E}+i\sigma_1}\sqrt{t}\right)\\ +e^{-\sqrt{\frac{E}{2}}z\sqrt{\frac{S^2+4iE+4N}{2E}+i\sigma_1}}\operatorname{erf}c\left(\frac{z}{2}\sqrt{\frac{E}{2t}}-\sqrt{\frac{S^2+4iE+4N}{2E}+i\sigma_1}\sqrt{t}\right)\end{array}\right], \tag{3.29}$$

$$\frac{1}{2\pi i}\int_{\lambda_1-i\infty}^{\lambda_1+i\infty}\frac{pa\nu_3}{p+i\sigma_1}e^{-\sqrt{\frac{E}{2}}z\sqrt{p+\frac{S^2+4iE+4N}{2E}}+pt}dp=\frac{\nu_3 b}{2}\frac{z}{t}\sqrt{\frac{E}{2\pi t}}e^{-\left(\frac{S^2+4iE+4N}{2E}\right)t-Ez^2/8t}$$

$$-\frac{\nu_3 b}{2} i\sigma_1 e^{i\sigma_1 t}\left[\begin{array}{c} e^{\sqrt{\frac{E}{2}}z\sqrt{\frac{S^2+4iE+4N}{2E}-i\sigma_1}}\operatorname{erf}c\left(\frac{z}{2}\sqrt{\frac{E}{2t}}+\sqrt{\frac{S^2+4iE+4N}{2E}-i\sigma_1\sqrt{t}}\right) \\ +e^{-\sqrt{\frac{E}{2}}z\sqrt{\frac{S^2+4iE+4N}{2E}-i\sigma_1}}\operatorname{erf}c\left(\frac{z}{2}\sqrt{\frac{E}{2t}}-\sqrt{\frac{S^2+4iE+4N}{2E}-i\sigma_1\sqrt{t}}\right)\end{array}\right], \tag{3.30}$$

$$-\frac{\nu_3 U}{U_0 2\pi i}\int_{\lambda_1-i\infty}^{\lambda_1+i\infty} e^{-\sqrt{\frac{E}{2}}z\sqrt{p+\frac{S^2+4iE+4N}{2E}}+pt}dp = -\frac{\nu_3 U}{U_0\sqrt{\pi t}}e^{-\left(\frac{S^2+4iE+4N}{2E}\right)t-Ez^2/8t} \tag{3.31}$$

$$\begin{aligned} & \frac{1}{2\pi i}\int_{\lambda_1-i\infty}^{\lambda_1+i\infty}\frac{a\nu_4}{p-i\sigma_1}e^{-\frac{Sz}{2}-\sqrt{\frac{E}{2}}z\sqrt{p+\frac{S^2+4iE+4N}{2E}}+pt}dp \\ = & \frac{\nu_4 a}{2}e^{i\sigma_1 t}\left[\begin{array}{c} e^{\sqrt{\frac{E}{2}}z\sqrt{\frac{S^2+4iE+4N}{2E}+i\sigma_1}}\operatorname{erf}c\left(\frac{z}{2}\sqrt{\frac{E}{2t}}+\sqrt{\frac{S^2+4iE+4N}{2E}+i\sigma_1\sqrt{t}}\right) \\ +e^{-\sqrt{\frac{E}{2}}z\sqrt{\frac{S^2+4iE+4N}{2E}+i\sigma_1}}\operatorname{erf}c\left(\frac{z}{2}\sqrt{\frac{E}{2t}}-\sqrt{\frac{S^2+4iE+4N}{2E}+i\sigma_1\sqrt{t}}\right)\end{array}\right], \end{aligned} \tag{3.32}$$

$$\begin{aligned} & \frac{1}{2\pi i}\int_{\lambda_1-i\infty}^{\lambda_1+i\infty}\frac{a\nu_4}{p+i\sigma_1}e^{-\frac{Sz}{2}-\sqrt{\frac{E}{2}}z\sqrt{p+\frac{S^2+4iE+4N}{2E}}+pt}dp \\ = & \frac{\nu_4 b}{2}e^{-i\sigma_1 t}\left[\begin{array}{c} e^{\sqrt{\frac{E}{2}}z\sqrt{\frac{S^2+4iE+4N}{2E}-i\sigma}}\operatorname{erf}c\left(\frac{z}{2}\sqrt{\frac{E}{2t}}+\sqrt{\frac{S^2+4iE+4N}{2E}-i\sigma_1\sqrt{t}}\right) \\ +e^{-\sqrt{\frac{E}{2}}z\sqrt{\frac{S^2+4iE+4N}{2E}-i\sigma_1}}\operatorname{erf}c\left(\frac{z}{2}\sqrt{\frac{E}{2t}}-\sqrt{\frac{S^2+4iE+4N}{2E}-i\sigma_1\sqrt{t}}\right)\end{array}\right], \end{aligned} \tag{3.33}$$

$$\begin{aligned} & -\frac{1}{2\pi i}\int_{\lambda_1-i\infty}^{\lambda_1+i\infty}\frac{U\nu_4}{U_0 p}e^{-\frac{Sz}{2}-\sqrt{\frac{E}{2}}z\sqrt{p+\frac{S^2+4iE+4N}{2E}}+pt}dp \\ = & -\frac{\nu_4}{2}\frac{U\nu_4}{U_0}\left[\begin{array}{c} e^{\sqrt{\frac{E}{2}}z\sqrt{\frac{S^2+4iE+4N}{2E}}}\operatorname{erf}c\left(\frac{z}{2}\sqrt{\frac{E}{2t}}+\sqrt{\frac{S^2+4iE+4N}{2E}}\sqrt{t}\right) \\ +e^{-\sqrt{\frac{E}{2}}z\sqrt{\frac{S^2+4iE+4N}{2E}}}\operatorname{erf}c\left(\frac{z}{2}\sqrt{\frac{E}{2t}}-\sqrt{\frac{S^2+4iE+4N}{2E}}\sqrt{t}\right)\end{array}\right], \end{aligned} \tag{3.34}$$

$$\frac{1}{2\pi i}\int_{\lambda_1-i\infty}^{\lambda_1+i\infty}\left[\nu_{10}p\sqrt{p+\frac{S^2+4iE+4N}{2E}}\frac{a}{p-i\sigma_1}\right]e^{-\sqrt{\frac{E}{2}}z\sqrt{p+\frac{S^2+4iE+4N}{2E}}+pt}dp$$

$$
\begin{aligned}
= \quad & \frac{i\sigma_1 a\nu_{10}}{4}\frac{e^{-\left(\frac{S^2+4iE+4N}{2E}\right)t-Ez^2/8t}}{\sqrt{\pi t}}+\frac{i\sigma_1 a\nu_{10}}{8}\sqrt{\frac{S^2+4iE+4N}{2E}+i\sigma_1} \\
& \times e^{i\sigma_1 t}\left[\begin{array}{l} e^{-\sqrt{\frac{E}{2}}z\sqrt{\frac{S^2+4iE+4N}{2E}+i\sigma_1}}\operatorname{erf}c\left(\frac{z}{2}\sqrt{\frac{E}{2t}}-\sqrt{\frac{S^2+4iE+4N}{2E}+i\sigma_1}\sqrt{t}\right) \\ -e^{\sqrt{\frac{E}{2}}z\sqrt{\frac{S^2+4iE+4N}{2E}+i\sigma_1}}\operatorname{erf}c\left(\frac{z}{2}\sqrt{\frac{E}{2t}}+\sqrt{\frac{S^2+4iE+4N}{2E}+i\sigma_1}\sqrt{t}\right)\end{array}\right],
\end{aligned} \tag{3.35}
$$

$$
\begin{aligned}
& \frac{1}{2\pi i}\int_{\lambda_1-i\infty}^{\lambda_1+i\infty}\left[\nu_{10}p\sqrt{p+\frac{S^2+4iE+4N}{2E}}\frac{b}{p+i\sigma_1}\right]e^{-\sqrt{\frac{E}{2}}z\sqrt{p+\frac{S^2+4iE+4N}{2E}}+pt}dp \\
= \quad & -\frac{i\sigma_1 b\nu_{10}}{4}\frac{e^{-\left(\frac{S^2+4iE+4N}{2E}\right)t-Ez^2/8t}}{\sqrt{\pi t}}-\frac{i\sigma_1 b\nu_{10}}{8}\sqrt{\frac{S^2+4iE+4N}{2E}-i\sigma_1} \\
& \times e^{-i\sigma_1 t}\left[\begin{array}{l} e^{-\sqrt{\frac{E}{2}}z\sqrt{\frac{S^2+4iE+4N}{2E}-i\sigma_1}}\operatorname{erf}c\left(\frac{z}{2}\sqrt{\frac{E}{2t}}-\sqrt{\frac{S^2+4iE+4N}{2E}-i\sigma_1}\sqrt{t}\right) \\ -e^{\sqrt{\frac{E}{2}}z\sqrt{\frac{S^2+4iE+4N}{2E}-i\sigma_1}}\operatorname{erf}c\left(\frac{z}{2}\sqrt{\frac{E}{2t}}+\sqrt{\frac{S^2+4iE+4N}{2E}-i\sigma_1}\sqrt{t}\right)\end{array}\right],
\end{aligned} \tag{3.36}
$$

$$
\begin{aligned}
& \frac{1}{2\pi i}\int_{\lambda_1-i\infty}^{\lambda_1+i\infty}\left[\nu_{11}p\frac{a}{(p-i\sigma_1)\sqrt{p+\frac{S^2+4iE+4N}{2E}}}\right]e^{-\sqrt{\frac{E}{2}}z\sqrt{p+\frac{S^2+4iE+4N}{2E}}+pt}dp \\
= \quad & \frac{-a\nu_{11}e^{-\left(\frac{S^2+4iE+4N}{2E}\right)t-Ez^2/8t}}{\sqrt{\pi t}}-\frac{i\sigma_1 a\nu_{11}e^{i\sigma_1 t}}{2\sqrt{\frac{S^2+4iE+4N}{2E}+i\sigma_1}} \\
& \times\left[\begin{array}{l} e^{-\sqrt{\frac{E}{2}}z\sqrt{\frac{S^2+4iE+4N}{2E}+i\sigma_1}}\operatorname{erf}c\left(\frac{z}{2}\sqrt{\frac{E}{2t}}-\sqrt{\frac{S^2+4iE+4N}{2E}+i\sigma_1}\sqrt{t}\right) \\ -e^{\sqrt{\frac{E}{2}}z\sqrt{\frac{S^2+4iE+4N}{2E}+i\sigma_1}}\operatorname{erf}c\left(\frac{z}{2}\sqrt{\frac{E}{2t}}+\sqrt{\frac{S^2+4iE+4N}{2E}+i\sigma_1}\sqrt{t}\right)\end{array}\right],
\end{aligned} \tag{3.37}
$$

$$
\frac{1}{2\pi i}\int_{\lambda_1-i\infty}^{\lambda_1+i\infty}\left[\nu_{11}p\frac{b}{(p+i\sigma_1)\sqrt{p+\frac{S^2+4iE+4N}{2E}}}\right]e^{-\sqrt{\frac{E}{2}}z\sqrt{p+\frac{S^2+4iE+4N}{2E}}+pt}dp
$$

$$= \frac{-b\nu_{11}e^{-\left(\frac{S^2+4iE+4N}{2E}\right)t-Ez^2/8t}}{\sqrt{\pi t}} - \frac{i\sigma_1 b\nu_{11}e^{-i\sigma_1 t}}{2\sqrt{\frac{S^2+4iE+4N}{2E}-i\sigma_1}}$$

$$\times \left[\begin{array}{l} e^{-\sqrt{\frac{E}{2}}z\sqrt{\frac{S^2+4iE+4N}{2E}-i\sigma_1}} \operatorname{erf} c\left(\frac{z}{2}\sqrt{\frac{E}{2t}} - \sqrt{\frac{S^2+4iE+4N}{2E}-i\sigma_1}\sqrt{t}\right) \\ -e^{\sqrt{\frac{E}{2}}z\sqrt{\frac{S^2+4iE+4N}{2E}-i\sigma_1}} \operatorname{erf} c\left(\frac{z}{2}\sqrt{\frac{E}{2t}} + \sqrt{\frac{S^2+4iE+4N}{2E}-i\sigma_1}\sqrt{t}\right) \end{array} \right], \tag{3.38}$$

$$\frac{1}{2\pi i}\int_{\lambda_1-i\infty}^{\lambda_1+i\infty}\left[\frac{\nu_{11}U}{U_0\sqrt{p+\frac{S^2+4iE+4N}{2E}}}\right]e^{-\sqrt{\frac{E}{2}}z\sqrt{p+\frac{S^2+4iE+4N}{2E}}+pt}dp$$

$$= \frac{U\nu_{11}e^{-\left(\frac{S^2+4iE+4N}{2E}\right)t-Ez^2/8t}}{U_0\sqrt{\pi t}}, \tag{3.39}$$

$$\frac{1}{2\pi i}\int_{\lambda_1-i\infty}^{\lambda_1+i\infty}\left[\frac{-a\nu_{12}p^2}{(p-i\sigma_1)\sqrt{p+\frac{S^2+4iE+4N}{2E}}}\right]e^{-\sqrt{\frac{E}{2}}z\sqrt{p+\frac{S^2+4iE+4N}{2E}}+pt}dp$$

$$= \frac{-a\nu_{12}\left(i\sigma_1 - \frac{S^2+4iE+4N}{2E}\right)e^{-\left(\frac{S^2+4iE+4N}{2E}\right)t-Ez^2/8t}}{\sqrt{\pi t}} + \frac{\sigma_1^2 a\nu_{12}e^{i\sigma_1 t}}{2\sqrt{\frac{S^2+4iE+4N}{2E}+i\sigma_1}}$$

$$\times \left[\begin{array}{l} e^{-\sqrt{\frac{E}{2}}z\sqrt{\frac{S^2+4iE+4N}{2E}+i\sigma_1}} \operatorname{erf} c\left(\frac{z}{2}\sqrt{\frac{E}{2t}} - \sqrt{\frac{S^2+4iE+4N}{2E}+i\sigma_1}\sqrt{t}\right) \\ -e^{\sqrt{\frac{E}{2}}z\sqrt{\frac{S^2+4iE+4N}{2E}+i\sigma_1}} \operatorname{erf} c\left(\frac{z}{2}\sqrt{\frac{E}{2t}} + \sqrt{\frac{S^2+4iE+4N}{2E}+i\sigma_1}\sqrt{t}\right) \end{array} \right], \tag{3.40}$$

$$\frac{1}{2\pi i}\int_{\lambda_1-i\infty}^{\lambda_1+i\infty}\left[\frac{-b\nu_{12}p^2}{(p+i\sigma_1)\sqrt{p+\frac{S^2+4iE+4N}{2E}}}\right]e^{-\sqrt{\frac{E}{2}}z\sqrt{p+\frac{S^2+4iE+4N}{2E}}+pt}dp$$

$$= \frac{b\nu_{12}\left(i\sigma_1 + \frac{S^2+4iE+4N}{2E}\right)e^{-\left(\frac{S^2+4iE+4N}{2E}\right)t-Ez^2/8t}}{\sqrt{\pi t}} + \frac{\sigma_1^2 b\nu_{12}e^{-i\sigma_1 t}}{2\sqrt{\frac{S^2+4iE+4N}{2E}-i\sigma_1}}$$

$$\times \left[\begin{array}{l} e^{-\sqrt{\frac{E}{2}}z\sqrt{\frac{S^2+4iE+4N}{2E}-i\sigma_1}} \operatorname{erf} c\left(\frac{z}{2}\sqrt{\frac{E}{2t}} - \sqrt{\frac{S^2+4iE+4N}{2E} - i\sigma_1}\sqrt{t}\right) \\ -e^{\sqrt{\frac{E}{2}}z\sqrt{\frac{S^2+4iE+4N}{2E}-i\sigma_1}} \operatorname{erf} c\left(\frac{z}{2}\sqrt{\frac{E}{2t}} + \sqrt{\frac{S^2+4iE+4N}{2E} - i\sigma_1}\sqrt{t}\right) \end{array} \right], \tag{3.41}$$

$$\frac{1}{2\pi i}\int_{\lambda_1-i\infty}^{\lambda_1+i\infty}\left[\frac{U\nu_{12}p}{U_0\sqrt{p+\frac{S^2+4iE+4N}{2E}}}\right]e^{-\sqrt{\frac{E}{2}}z\sqrt{p+\frac{S^2+4iE+4N}{2E}}+pt}dp$$

$$= \frac{U\nu_{12}\left(\frac{S^2+4iE+4n}{2E}\right)e^{-\left(\frac{S^2+4iE+4N}{2E}\right)t-Ez^2/8t}}{U_0\sqrt{\pi t}}, \tag{3.42}$$

$$\frac{1}{2\pi i}\int_{\lambda_1-i\infty}^{\lambda_1+i\infty}\left[\frac{a\nu_{13}\sqrt{p+\frac{S^2+4iE+4N}{2E}}}{(p-i\sigma_1)}\right]e^{-\sqrt{\frac{E}{2}}z\sqrt{p+\frac{S^2+4iE+4N}{2E}}+pt}dp$$

$$= \frac{a\nu_{13}e^{-\left(\frac{S^2+4iE+4N}{2E}\right)t-Ez^2/8t}}{\sqrt{\pi t}} + \frac{a\nu_{13}\sqrt{\frac{S^2+4iE+4N}{2E}+i\sigma_1}e^{i\sigma_1 t}}{2}$$

$$\times \left[\begin{array}{l} e^{-\sqrt{\frac{E}{2}}z\sqrt{\frac{S^2+4iE+4N}{2E}+i\sigma_1}} \operatorname{erf} c\left(\frac{z}{2}\sqrt{\frac{E}{2t}} - \sqrt{\frac{S^2+4iE+4N}{2E} + i\sigma_1}\sqrt{t}\right) \\ -e^{\sqrt{\frac{E}{2}}z\sqrt{\frac{S^2+4iE+4N}{2E}+i\sigma_1}} \operatorname{erf} c\left(\frac{z}{2}\sqrt{\frac{E}{2t}} + \sqrt{\frac{S^2+4iE+4N}{2E} + i\sigma_1}\sqrt{t}\right) \end{array} \right], \tag{3.43}$$

$$\frac{1}{2\pi i}\int_{\lambda_1-i\infty}^{\lambda_1+i\infty}\left[\frac{b\nu_{14}\sqrt{p+\frac{S^2+4iE+4N}{2E}}}{(p+i\sigma_1)}\right]e^{-\sqrt{\frac{E}{2}}z\sqrt{p+\frac{S^2+4iE+4N}{2E}}+pt}dp$$

$$= \frac{b\nu_{13}e^{-\left(\frac{S^2+4iE+4N}{2E}\right)t-Ez^2/8t}}{\sqrt{\pi t}} + \frac{b\nu_{13}\sqrt{\frac{S^2+4iE+4N}{2E}-i\sigma_1}e^{-i\sigma_1 t}}{2}$$

$$\times \left[\begin{array}{l} e^{-\sqrt{\frac{E}{2}}z\sqrt{\frac{S^2+4iE+4N}{2E}-i\sigma_1}} \operatorname{erf} c\left(\frac{z}{2}\sqrt{\frac{E}{2t}} - \sqrt{\frac{S^2+4iE+4N}{2E} - i\sigma_1}\sqrt{t}\right) \\ -e^{\sqrt{\frac{E}{2}}z\sqrt{\frac{S^2+4iE+4N}{2E}-i\sigma_1}} \operatorname{erf} c\left(\frac{z}{2}\sqrt{\frac{E}{2t}} + \sqrt{\frac{S^2+4iE+4N}{2E} - i\sigma_1}\sqrt{t}\right) \end{array} \right], \tag{3.44}$$

$$\frac{1}{2\pi i}\int_{\lambda_1-i\infty}^{\lambda_1+i\infty}\left[\frac{-U\nu_{14}\sqrt{p+\frac{S^2+4iE+4N}{2E}}}{U_0}\right]e^{-\sqrt{\frac{E}{2}}z\sqrt{p+\frac{S^2+4iE+4N}{2E}}+pt}dp$$

$$= -\frac{U}{U_0}\frac{\nu_{13}e^{-\left(\frac{S^2+4iE+4N}{2E}\right)t-Ez^2/8t}}{\sqrt{\pi t}}$$

$$-\frac{U}{U_0}\frac{\nu_{13}\sqrt{\frac{S^2+4iE+4N}{2E}}}{2}\left[\begin{array}{l} e^{-\sqrt{\frac{E}{2}}z\sqrt{\frac{S^2+4iE+4N}{2E}}}\operatorname{erf}c\left(\frac{z}{2}\sqrt{\frac{E}{2t}}-\sqrt{\frac{S^2+4iE+4N}{2E}}\sqrt{t}\right) \\ -e^{\sqrt{\frac{E}{2}}z\sqrt{\frac{S^2+4iE+4N}{2E}}}\operatorname{erf}c\left(\frac{z}{2}\sqrt{\frac{E}{2t}}+\sqrt{\frac{S^2+4iE+4N}{2E}}\sqrt{t}\right)\end{array}\right], \tag{3.45}$$

$$\frac{1}{2\pi i}\int_{\lambda_1-i\infty}^{\lambda_1+i\infty}\left[\frac{-a\nu_{14}}{(p-i\sigma_1)\sqrt{p+\frac{S^2+4iE+4N}{2E}}}\right]e^{-\sqrt{\frac{E}{2}}z\sqrt{p+\frac{S^2+4iE+4N}{2E}}+pt}dp$$

$$= \frac{-a\nu_{14}e^{i\sigma_1 t}}{2\sqrt{\frac{S^2+4iE+4N}{2E}+i\sigma_1}}$$

$$\times\left[\begin{array}{l} e^{-\sqrt{\frac{E}{2}}z\sqrt{\frac{S^2+4iE+4N}{2E}+i\sigma_1}}\operatorname{erf}c\left(\frac{z}{2}\sqrt{\frac{E}{2t}}-\sqrt{\frac{S^2+4iE+4N}{2E}+i\sigma_1}\sqrt{t}\right) \\ -e^{\sqrt{\frac{E}{2}}z\sqrt{\frac{S^2+4iE+4N}{2E}+i\sigma_1}}\operatorname{erf}c\left(\frac{z}{2}\sqrt{\frac{E}{2t}}+\sqrt{\frac{S^2+4iE+4N}{2E}+i\sigma_1}\sqrt{t}\right)\end{array}\right], \tag{3.46}$$

$$\frac{1}{2\pi i}\int_{\lambda_1-i\infty}^{\lambda_1+i\infty}\left[\frac{-b\nu_{14}}{(p+i\sigma_1)\sqrt{p+\frac{S^2+4iE+4N}{2E}}}\right]e^{-\sqrt{\frac{E}{2}}z\sqrt{p+\frac{S^2+4iE+4N}{2E}}+pt}dp$$

$$= \frac{-b\nu_{14}e^{-i\sigma_1 t}}{2\sqrt{\frac{S^2+4iE+4N}{2E}-i\sigma_1}}$$

$$\times\left[\begin{array}{l} e^{-\sqrt{\frac{E}{2}}z\sqrt{\frac{S^2+4iE+4N}{2E}-i\sigma_1}}\operatorname{erf}c\left(\frac{z}{2}\sqrt{\frac{E}{2t}}-\sqrt{\frac{S^2+4iE+4N}{2E}-i\sigma_1}\sqrt{t}\right) \\ -e^{\sqrt{\frac{E}{2}}z\sqrt{\frac{S^2+4iE+4N}{2E}-i\sigma_1}}\operatorname{erf}c\left(\frac{z}{2}\sqrt{\frac{E}{2t}}+\sqrt{\frac{S^2+4iE+4N}{2E}-i\sigma_1}\sqrt{t}\right)\end{array}\right], \tag{3.47}$$

$$\frac{1}{2\pi i}\int_{\lambda_1-i\infty}^{\lambda_1+i\infty}\left[\frac{U\nu_{14}}{U_0\sqrt{p+\frac{S^2+4iE+4N}{2E}}}\right]e^{-\sqrt{\frac{E}{2}}z\sqrt{p+\frac{S^2+4iE+4N}{2E}}+pt}dp$$

$$= \frac{U\nu_{14}e^{-i\sigma_1 t}}{2U_0\sqrt{\frac{S^2+4iE+4N}{2E}}}\left[\begin{array}{l} e^{-\sqrt{\frac{E}{2}}z\sqrt{\frac{S^2+4iE+4N}{2E}}}\operatorname{erf}c\left(\frac{z}{2}\sqrt{\frac{E}{2t}}-\sqrt{\frac{S^2+4iE+4N}{2E}}\sqrt{t}\right) \\ -e^{\sqrt{\frac{E}{2}}z\sqrt{\frac{S^2+4iE+4N}{2E}}}\operatorname{erf}c\left(\frac{z}{2}\sqrt{\frac{E}{2t}}+\sqrt{\frac{S^2+4iE+4N}{2E}}\sqrt{t}\right)\end{array}\right], \tag{3.48}$$

where $\operatorname{erf}c(x+iy)$ is the complementary error function of the complex argument which can be calculated in terms of tabulated functions [73]. The tables given in [73] do not give $\operatorname{erf}c\,(x+iy)$ directly but an auxiliary function $W_1\,(x+iy)$, which is defined as

$$\operatorname{erf}c\,(x+iy) = W_1\,(-y+ix)\,e^{-(x+iy)^2}.$$

It is easily shown that some properties of $W_1\,(x+iy)$ are

$$\begin{aligned} W_1\,(-x+iy) &= W_2\,(x+iy)\,, \\ W_1\,(x-iy) &= 2e^{-(x+iy)^2} - W_2\,(x+iy)\,, \end{aligned}$$

where $W_2\,(x+iy)$ is complex conjugate of $W_1\,(x+iy)$.

Substitution of equations (3.29) – (3.48) into equation (3.28) finally yields the following result for suction case

$$\begin{aligned} q(z,t) &= \frac{a}{2}e^{i\sigma_1 t-Sz/2} \\ &\times\left\{\begin{array}{l} e^{(x_1+iy_1)z}\operatorname{erf}c\left(\frac{\xi}{2\sqrt{t}}+(x_1+iy_1)\sqrt{\frac{2t}{E}}\right) \\ +e^{-(x_1+iy_1)z}\operatorname{erf}c\left(\frac{\xi}{2\sqrt{t}}-(x_1+iy_1)\sqrt{\frac{2t}{E}}\right)\end{array}\right\} \\ &+\frac{b}{2}e^{-i\sigma_1 t-Sz/2}\times \end{aligned}$$

$$\times\left\{\begin{array}{c} e^{(x_2+iy_2)z}\,\mathrm{erf}\,c\left(\frac{\xi}{2\sqrt{t}}+(x_2+iy_2)\sqrt{\frac{2t}{E}}\right) \\ +e^{-(x_2+iy_2)z}\,\mathrm{erf}\,c\left(\frac{\xi}{2\sqrt{t}}-(x_2+iy_2)\sqrt{\frac{2t}{E}}\right) \end{array}\right\}$$

$$-\frac{U}{2U_0}e^{-Sz/2}\left\{\begin{array}{c} e^{(x_3+iy_3)z}\,\mathrm{erf}\,c\left(\frac{\xi}{2\sqrt{t}}+(x_3+iy_3)\sqrt{\frac{2t}{E}}\right) \\ +e^{-(x_3+iy_3)z}\,\mathrm{erf}\,c\left(\frac{\xi}{2\sqrt{t}}-(x_3+iy_3)\sqrt{\frac{2t}{E}}\right) \end{array}\right\}$$

$$+\alpha z e^{-Sz/2}\left\{\frac{\nu_3(a+b)z}{2t}\sqrt{\frac{E}{2}}+H_1\right\}\frac{e^{-Ez^2/8t-(x_3+iy_3)\sqrt{\frac{2}{E}}}}{\sqrt{\pi t}}$$

$$+\alpha z\ e^{i\sigma_1 t-Sz/2}\left\{\begin{array}{c} (H_2+H_3)\ e^{-z(x_1+iy_1)}\times \\ \mathrm{erf}\,c\left(\frac{\xi}{2\sqrt{t}}-(x_1+iy_1)\sqrt{\frac{2t}{E}}\right) \\ +\ (H_2-H_3)\ e^{z(x_1+iy_1)}\times \\ \mathrm{erf}\,c\left(\frac{\xi}{2\sqrt{t}}+(x_1+iy_1)\sqrt{\frac{2t}{E}}\right) \end{array}\right\}$$

$$+\alpha z\ e^{-i\sigma_1 t-Sz/2}\left\{\begin{array}{c} (H_4+H_5)\ e^{-z(x_3+iy_3)}\times \\ \mathrm{erf}\,c\left(\frac{\xi}{2\sqrt{t}}-(x_2+iy_2)\sqrt{\frac{2t}{E}}\right) \\ +\ (H_4-H_5)\ e^{z(x_2+iy_2)}\times \\ \mathrm{erf}\,c\left(\frac{\xi}{2\sqrt{t}}+(x_2+iy_2)\sqrt{\frac{2t}{E}}\right) \end{array}\right\}$$

$$-\alpha\frac{Uz}{U_0}e^{-Sz/2}\left\{\begin{array}{c} (H_6+H_7)\ e^{-z(x_3+iy_3)}\times \\ \mathrm{erf}\,c\left(\frac{\xi}{2\sqrt{t}}-(x_3+iy_3)\sqrt{\frac{2t}{E}}\right) \\ +\ (H_6-H_7)\ e^{(x_3+iy_3)}\times \\ \mathrm{erf}\,c\left(\frac{\xi}{2\sqrt{t}}+(x_3+iy_3)\sqrt{\frac{2t}{E}}\right) \end{array}\right\}, \tag{3.49}$$

where

$$\xi = z\sqrt{\frac{E}{2}},$$

$$x_1 = \frac{1}{2\sqrt{2}}\sqrt{(S^2+4N)+\sqrt{(S^2+4N)^2+4E^2(2+\sigma_1)^2}},$$

$$x_2 = \frac{1}{2\sqrt{2}}\sqrt{(S^2+4N)+\sqrt{(S^2+4N)^2+4E^2(2-\sigma_1)^2}},$$

$$
\begin{aligned}
x_3 &= \frac{1}{2\sqrt{2}}\sqrt{(S^2+4N)+\sqrt{(S^2+4N)^2+16E^2}}, \\
y_1 &= \frac{1}{2\sqrt{2}}\sqrt{-(S^2+4N)+\sqrt{(S^2+4N)^2+4E^2(2+\sigma_1)^2}}, \\
y_2 &= \frac{1}{2\sqrt{2}}\sqrt{-(S^2+4N)+\sqrt{(S^2+4N)^2+4E^2(2-\sigma_1)^2}}, \\
y_3 &= \frac{1}{2\sqrt{2}}\sqrt{-(S^2+4N)+\sqrt{(S^2+4N)^2+16E^2}}
\end{aligned}
$$

$$
\begin{aligned}
H_1 &= -\frac{\nu_3 U}{U_0}+\frac{i\sigma_1\nu_{10}(a-b)}{4}-(\nu_{11}-\nu_{13})\left(a+b-\frac{U}{U_0}\right) \\
&\quad -\nu_{12}\left[\begin{array}{c} a\left\{i\sigma_1-\left(\frac{S^2+4iE+4N}{2E}\right)\right\}-b\left\{i\sigma_1+\left(\frac{S^2+4iE+4N}{2E}\right)\right\} \\ +\frac{U}{U_0}\left(\frac{S^2+4iE+4N}{2E}\right)\end{array}\right],
\end{aligned}
$$

$$
H_2 = \frac{a}{2}\left(i\sigma_1\nu_3+\nu_4\right),
$$

$$
H_3 = \frac{a}{2}\left[\begin{array}{c} \left(\frac{i\sigma_1\nu_{10}}{4}+\nu_{13}\right)\sqrt{i\sigma_1+\frac{S^2+4iE+4N}{2E}} \\ +\ (\sigma_1^2\nu_{12}-i\sigma_1\nu_{11}-\nu_{14})\left(\sqrt{i\sigma_1+\frac{S^2+4iE+4N}{2E}}\right)^{-1}\end{array}\right],
$$

$$
H_4 = \frac{b}{2}\left(\nu_4-i\sigma_1\nu_3\right),
$$

$$
H_5 = \frac{b}{2}\left[\begin{array}{c} \left(\nu_{13}-\frac{i\sigma_1\nu_{10}}{4}\right)\sqrt{\frac{S^2+4iE+4N}{2E}-i\sigma_1} \\ +\ (\sigma_1^2\nu_{12}+i\sigma_1\nu_{11}-\nu_{14})\left(\sqrt{\frac{S^2+4iE+4N}{2E}-i\sigma_1}\right)^{-1}\end{array}\right],
$$

$$
H_6 = \frac{\nu_4}{2},\ H_7 = \frac{1}{2}\left[\nu_{13}\sqrt{\frac{S^2+4iE+4N}{2E}}-\nu_{14}\left(\sqrt{\frac{S^2+4iE+4N}{2E}}\right)^{-1}\right].
$$

3.3 Blowing Solution

Here $S<0$, and we take $-S=\widetilde{S}>0$. The solution in this case is given by

$$
\widetilde{q}(z,t)=\widetilde{q}_0(z,t)+\alpha\widetilde{q}_1(z,t) \tag{3.50}
$$

with

$$
\begin{aligned}
\widetilde{q}(z,t) &= \frac{a}{2}e^{i\sigma_1 t+\widetilde{S}z/2}\times \\
&\times\left\{\begin{array}{c} e^{\left(\widetilde{a}_1+i\widetilde{b}_1\right)z}\operatorname{erf}c\left(\frac{\xi}{2\sqrt{t}}+\left(\widetilde{a}_1+i\widetilde{b}_1\right)\sqrt{\frac{2t}{E}}\right) \\ +e^{-\left(\widetilde{a}_1+i\widetilde{b}_1\right)z}\operatorname{erf}c\left(\frac{\xi}{2\sqrt{t}}-\left(\widetilde{a}_1+i\widetilde{b}_1\right)\sqrt{\frac{2t}{E}}\right)\end{array}\right\} \\
&+\frac{b}{2}e^{-i\sigma_1 t+\widetilde{S}z/2} \\
&\times\left\{\begin{array}{c} e^{\left(\widetilde{a}_2+i\widetilde{b}_2\right)z}\operatorname{erf}c\left(\frac{\xi}{2\sqrt{t}}+\left(\widetilde{a}_2+i\widetilde{b}_2\right)\sqrt{\frac{2t}{E}}\right) \\ +e^{-\left(\widetilde{a}_2+i\widetilde{b}_2\right)z}\operatorname{erf}c\left(\frac{\xi}{2\sqrt{t}}-\left(\widetilde{a}_2+i\widetilde{b}_2\right)\sqrt{\frac{2t}{E}}\right)\end{array}\right\} \\
&-\frac{U}{2U_0}e^{\widetilde{S}z/2}\left\{\begin{array}{c} e^{\left(\widetilde{a}_3+i\widetilde{b}_3\right)z}\operatorname{erf}c\left(\frac{\xi}{2\sqrt{t}}+\left(\widetilde{a}_3+i\widetilde{b}_3\right)\sqrt{\frac{2t}{E}}\right) \\ +e^{-\left(\widetilde{a}_3+i\widetilde{b}_3\right)z}\operatorname{erf}c\left(\frac{\xi}{2\sqrt{t}}-\left(\widetilde{a}_3+i\widetilde{b}_3\right)\sqrt{\frac{2t}{E}}\right)\end{array}\right\} \\
&+\alpha z e^{\widetilde{S}z/2}\left\{\widetilde{H}_1-\frac{S_1(\nu_1+\frac{E\nu_2}{2})(a+b)z}{4t}\sqrt{\frac{E}{2}}\right\}\frac{e^{-Ez^2/8t-\left(\widetilde{a}_3+i\widetilde{b}_3\right)\sqrt{\frac{2}{E}}}}{\sqrt{\pi t}} \\
&+\alpha z\ e^{i\sigma_1 t+\widetilde{S}z/2}\left\{\begin{array}{c} (\widetilde{H}_2+\widetilde{H}_3)\ e^{-z\left(\widetilde{a}_1+i\widetilde{b}_1\right)}\times \\ \operatorname{erf}c\left(\frac{\xi}{2\sqrt{t}}-\left(\widetilde{a}_1+i\widetilde{b}_1\right)\sqrt{\frac{2t}{E}}\right) \\ +\ (\widetilde{H}_2-\widetilde{H}_3)\ e^{z\left(\widetilde{a}_2+i\widetilde{b}_2\right)}\times \\ \operatorname{erf}c\left(\frac{\xi}{2\sqrt{t}}+\left(\widetilde{a}_1+i\widetilde{b}_1\right)\sqrt{\frac{2t}{E}}\right)\end{array}\right\} \\
&+\alpha z\ e^{-i\sigma_1 t+\widetilde{S}z/2}\left\{\begin{array}{c} (\widetilde{H}_4+\widetilde{H}_5)\ e^{-z\left(\widetilde{a}_3+i\widetilde{b}_3\right)}\times \\ \operatorname{erf}c\left(\frac{\xi}{2\sqrt{t}}-\left(\widetilde{a}_2+i\widetilde{b}_2\right)\sqrt{\frac{2t}{E}}\right) \\ +\ (\widetilde{H}_4-\widetilde{H}_5)\ e^{z\left(\widetilde{a}_2+i\widetilde{b}_2\right)}\times \\ \operatorname{erf}c\left(\frac{\xi}{2\sqrt{t}}+\left(\widetilde{a}_2+i\widetilde{b}_2\right)\sqrt{\frac{2t}{E}}\right)\end{array}\right\} \\
&-\alpha\frac{Uz}{U_0}e^{\widetilde{S}z/2}\left\{\begin{array}{c} (\widetilde{H}_6+\widetilde{H}_7)\ e^{-z\left(\widetilde{a}_3+i\widetilde{b}_3\right)}\times \\ \operatorname{erf}c\left(\frac{\xi}{2\sqrt{t}}-\left(\widetilde{a}_2+i\widetilde{b}_2\right)\sqrt{\frac{2t}{E}}\right) \\ +\ (\widetilde{H}_6-\widetilde{H}_7)\ e^{\left(\widetilde{a}_3+i\widetilde{b}_3\right)}\times \\ \operatorname{erf}c\left(\frac{\xi}{2\sqrt{t}}+\left(\widetilde{a}_3+i\widetilde{b}_3\right)\sqrt{\frac{2t}{E}}\right)\end{array}\right\}, \qquad (3.51)
\end{aligned}
$$

where $\widetilde{H}_r$ $(r = 1$ to $7)$, and $\widetilde{a}_j$ and $\widetilde{b}_j$ $(j = 1$ to $3)$ are given by H_r, x_j and y_j with S replaced by $-\widetilde{S}$.

3.4 Graphical Results

In order to study the effects of suction, magnetic field, rotation and second grade parameter on the velocity distribution, we have plotted u and v via z.

1. Fig. 3.1(a,b) are plotted for various values of suction and blowing when $N = 2, a = b = 1, \sigma_1 = 1, E = 0.5, \alpha = 0, t = 12$. In this case, we see that with the increase in suction parameter, the boundary layer thickness decreases and with the increase in blowing parameter the boundary layer thickness increases as compared to the case of suction.

2. Figs. $3.2(a,b)$, $3.3(a,b)$ and $3.4(a,b)$ depict the effects of magnetic field and rotation when $\sigma_1 = 2, a = b = 1, \alpha = 0, t = 12$. We observe that the increase in magnetic field declerates the boundary layer thickness for Newtonian fluid. Similar results hold in the case of rotation. We also note from Figs. $4(a,b)$ that the steady blowing solution exists for resonant frequency.

3. The variation of second grade parameter α on u and v are shown in Figs. $3.5(a,b)$. Here, we found that for small values of α the results are identical to that of Newtonian fluid. However, α starts influencing the flow field for its value near 0.01. Also the boundary layer thickness increases with the increase of α.

4. *Figs.* 3.6(a, b) and 3.7 (a, b) shows the effect of S and N on u and v when $\alpha \neq 0$. The boundary layer thickness is found to decrease with the increase of suction and magnetic field for the case of second grade fluid.

3.5 Discussion

The solutions (3.49) and (3.51) represent the general features of the unsteady hydromagnetic boundary layer flows of a second grade fluid in the rotating system including the effects of uniform suction and blowing respectively.

In particular, when $\alpha = 0$, solution (3.49) is identical with that of Debnath [18]. When $\alpha = 0$ and $N = 0$, solution (3.49) reduces to that of Debnath and Mukherjee [41]. Further when $\alpha = 0, U = 0, S = 0$ and $N = 0$, solution (3.49) recovers to that of Thornley [74].

Finally, it can be seen that unless $S = 0$ or $N = 0$

$$\begin{aligned} Lim_{t\to\infty}\, Lim_{\sigma_1 \to 2}\, q(z,t) &= Lim_{\sigma_1 \to 2}\, Lim_{t\to\infty}\, q(z,t) \\ &\sim [a + 2\alpha z\,(H_2^* + H_3^*)]\, e^{2it - \left(\frac{S}{2} + a_1^* + ib_1^*\right)z} \\ &\quad + [b + 2\alpha z\,(H_4^* + H_5^*)]\, e^{-2it - \left(\frac{S}{2} + a_2^* + ib_2^*\right)z} \\ &\quad - [1 + 2\alpha z\,(H_6^* + H_7^*)]\, \frac{U}{U_0} e^{-\left(\frac{S}{2} + a_3^* + ib_3^*\right)z}, \end{aligned} \tag{3.52}$$

where H_r^* $(r = 2$ to $7)$ and a_j^*, $b_j^*\,(j = 1$ to $3)$ are the limiting values of H_r and x_j, y_j respectively as $\sigma_1 \to 2$.

In order to determine the steady state structure of the solution (3.49) we use the asymptotic formula for the complimentary error function

$$\operatorname{erf}c\left(\frac{\xi}{2\sqrt{t}}\pm(x_j+iy_j)\sqrt{\frac{2t}{E}}\right)\rightarrow(0,2)\ \text{as}\ t\rightarrow\infty. \tag{3.53}$$

Evidently, in the limit $t\rightarrow\infty$, solution (3.49) , yields

$$\begin{aligned} q_s(z,t) \sim\ & [a+2\alpha z\ (H_2+H_3)]\, e^{i\sigma_1 t-\frac{S}{2}z-(x_1+\ iy_1)z} \\ & +[b+2\alpha z\ (H_4+H_5)]\, e^{-i\sigma_1 t-\frac{S}{2}z-(x_2+\ iy_2)z} \\ & -[1+2\alpha z\ (H_6+H_7)]\,\frac{U}{U^*}e^{-\frac{S}{2}z-(x_3+\ iy_3)z}. \end{aligned} \tag{3.54}$$

This result describes physically meaningful hydromagnetic boundary layer flow of a second grade fluid for both resonant ($\sigma_1=2$) and non-resonant ($\sigma_1\neq2$) frequency. It may be noted that the effects of the suction and material parameters, Coriolis and electromagnetic forces are reflected in the unsteady and the ultimate steady velocity fields. Solution (3.54) indicates the existence of three distinct boundary layers of thicknesses of order $f_j=\nu/U_0\left(x_j+\frac{S}{2}\right)$ with $f_1<f_3<f_2$ for suction. Clearly, the solution (3.54) and the associated boundary layers are modified due to the presence of S,α,N and σ_1. It is noted that the thicknesses of the boundary layers decrease with an increase of the external magnetic field or suction parameter and remain bounded for all values of the imposed oscillations, α,N and S.

In the presence of an external magnetic field ($N\neq0$) and $S=0$, the ultimate steady state solution does not depend on the order of the double limit operation $t\rightarrow\infty\ \sigma_1\rightarrow2$. In fact, the double limiting procedure with $N\neq0$ and $S=0$ yields

$$Lim_{t\rightarrow\infty}\ Lim_{\sigma_1\rightarrow2}\ q(z,t)\ =\ Lim_{\sigma_1\rightarrow2}\ Lim_{t\rightarrow\infty}\ q(z,t)$$

$$
\begin{aligned}
&= [a + 2\alpha z L_1]\, e^{2it-(h_1+ig_1)z} \\
&\quad + [b + 2\alpha z L_2]\, e^{-2it-h_2 z} - \frac{U}{U_0} e^{-(h_3+ig_3)z},
\end{aligned}
\tag{3.55}
$$

where

$$
\begin{aligned}
L_1 &= \frac{a}{2}\sqrt{\frac{E}{2}}\left[\frac{i\nu_1}{2}\sqrt{4i} + \frac{4\nu_1}{\sqrt{4i}}\right]\sqrt{\frac{2N}{4iE} + 1}, \\
L_2 &= \frac{i\nu_1}{4} b\sqrt{N}, \\
h_1 &= \frac{1}{\sqrt{2}}\left[N + \sqrt{N^2 + 16E^2}\right]^{1/2},\ h_2 = \sqrt{N}, \\
h_3 &= \frac{1}{\sqrt{2}}\left[N + \sqrt{N^2 + E^2}\right]^{1/2}, \\
g_1 &= \frac{1}{\sqrt{2}}\left[-N + \sqrt{N^2 + 16E^2}\right]^{1/2},\ g_2 = 0, \\
g_3 &= \frac{1}{\sqrt{2}}\left[-N + \sqrt{N^2 + E^2}\right]^{1/2}
\end{aligned}
$$

and solution is identical to that of Newtonian fluid for fixed viscosity and $\Omega = 0$.

It should be pointed out that the mathematical nature and physical content of the solution obtained by the three limiting procedures $\sigma_1 \to 2$ $t \to \infty$ $N \to 0$ in this case or in the reverse order $t \to \infty$ $\sigma_1 \to 2$ $N \to 0$ are radically different. In fact from equation (3.55)

$$
\begin{aligned}
&Lim_{\sigma_1 \to 2}\ Lim_{t \to \infty}\ Lim_{N \to 0}\ q(z,t) \\
&= a\,(1 + 2\alpha z M^*)\, e^{2it - z\sqrt{2E}(1+i)} + be^{-2it} - \frac{U}{U_0} e^{-z\sqrt{2E}(1+i)},
\end{aligned}
\tag{3.56}
$$

and

$$
Lim_{t \to \infty}\ Lim_{\sigma_1 \to 2}\ Lim_{N \to 0}\ q(z,t)
$$

$$= a\left(1+2\alpha z M^*\right)e^{2it-z\sqrt{2iE}} + be^{-2it}\operatorname{erf}c\left(\frac{z}{2}\sqrt{\frac{E}{2t}}\right) - \frac{U}{U_0}e^{-z\sqrt{iE}}, \tag{3.57}$$

where

$$M^* = \frac{\sqrt{E/2}}{2}\left[\frac{i\nu_1}{2}\sqrt{4i} + \frac{4\nu_1}{\sqrt{4i}}\right].$$

Clearly, the above two results are different. Result (3.57) does not even qualify for solution because it does not satisfy the boundary condition at infinity unless $b = 0$. Result (3.56) satisfies all the conditions and is a correct solution. In case of blowing ultimate steady-state solution is obtained from (3.57) by taking the limit $t \to \infty$ and has the form

$$\begin{aligned} \widetilde{q}_s(z,t) \sim & \left[a + 2\alpha z\left(\widetilde{H}_2 + \widetilde{H}_3\right)\right]e^{i\sigma_1 t + \frac{\widetilde{S}}{2}z - (\widetilde{a}_1 + i\widetilde{b}_1)z} \\ & + \left[b + 2\alpha z\left(\widetilde{H}_4 + \widetilde{H}_5\right)\right]e^{-i\sigma t + \frac{\widetilde{S}}{2}z - (\widetilde{a}_2 + i\widetilde{b}_2)z} \\ & - \left[1 + 2\alpha z\left(\widetilde{H}_6 + \widetilde{H}_7\right)\right]\frac{U}{U_0}e^{\frac{\widetilde{S}}{2}z - (\widetilde{a}_3 + i\widetilde{b}_3)z}. \end{aligned} \tag{3.58}$$

The above solution describes the hydromagnetic oscillations which decay exponentially within the three boundary layers of thicknesses of the order $\widetilde{f}_j = \nu/U_0\left(\widetilde{a}_j + \frac{\widetilde{S}}{2}\right)$ with $\widetilde{f}_1 < \widetilde{f}_3 < \widetilde{f}_2$. It is worth noting that the thicknesses of these boundary layers are significantly modified by the magnetic field. The most important feature of (3.58) is that unlike the hydrodynamic situation for the resonant case, (3.58) satisfies the boundary condition at infinity for all values of σ_1 including the resonant frequency. Consequently, the associated boundary layers remain bounded for the resonant case. In contrast to the hydrodynamic solution for the case of blowing and resonance where the blowing promotes the spreading of the shear oscillations far away from the plate, the hydromagnetic solution (3.58) represents the oscillatory boundary layer flow confined to the ultimate boundary layers for the frequencies including the resonant frequency. The physical implication of this conclusion is that for the case of resonance, the unbounded spreading of oscillations away from the plate is controlled by the magnetic field.

Chapter 4

Oscillatory Rotating Flow of a Magnetohydrodynamic Third Grade Fluid Bounded by a Porous Plate

In this chapter, the analytical and numerical solutions of the hydromagnetic boundary layer equation for an incompressible, third grade, electrically conducting and rotating fluid with constant properties are obtained to examine the effects of the Coriolis and electromagnetic forces. The fluid is bounded by an oscillating and porous plate. It is found that an external magnetic field has the same effect on the flow as the material parameters of the third grade fluid.

4.1 Mathematical Formulation

Let us consider hydromagnetic boundary layer flow generated in a semi-infinite expanse of an electrically conducting third grade fluid. The fluid is bounded by an infinite non-conducting porous plate at $z = 0$ subject to uniform suction and blowing. Here $z - axis$ is taken in the vertical upward direction. The system is uniformly rotating with an angular velocity $\mathbf{\Omega}$ about the $z - axis$ in a rigid body rotation. A uniform magnetic field $\mathbf{B}_0$ acts in the vertical direction. The flow is generated due to elliptic harmonic oscillations of the plate in its own plane at $t > 0$.

The condition of incompressibility yields $w = -W_0$. Clearly $W_0 > 0$ is the suction velocity and $W_0 < 0$ is the blowing velocity. For the convenience of the readers we write equations (1.35) – (1.36) as

$$\begin{aligned} \frac{\partial u}{\partial t} - W_0\frac{\partial u}{\partial z} - 2\Omega v &= -\frac{1}{\rho}\frac{\partial p^*}{\partial x} + \nu\frac{\partial^2 u}{\partial z^2} + \alpha\left(\frac{\partial^3 u}{\partial z^2 \partial t} - W_0\frac{\partial^3 u}{\partial z^3}\right) \\ &\quad -\frac{\sigma B_0^2}{\rho}u + 2\frac{\beta_3}{\rho}\frac{\partial}{\partial z}\left(\frac{\partial u}{\partial z}\left(\left(\frac{\partial u}{\partial z}\right)^2 + \left(\frac{\partial v}{\partial z}\right)^2\right)\right), \end{aligned} \tag{4.1}$$

$$\begin{aligned} \frac{\partial v}{\partial t} - W_0\frac{\partial v}{\partial z} + 2\Omega u &= -\frac{1}{\rho}\frac{\partial p^*}{\partial y} + \nu\frac{\partial^2 v}{\partial z^2} + \alpha\left(\frac{\partial^3 v}{\partial z^2 \partial t} - W_0\frac{\partial^3 v}{\partial z^3}\right) \\ &\quad -\frac{\sigma B_0^2}{\rho}v + 2\frac{\beta_3}{\rho}\frac{\partial}{\partial z}\left(\frac{\partial v}{\partial z}\left(\left(\frac{\partial u}{\partial z}\right)^2 + \left(\frac{\partial v}{\partial z}\right)^2\right)\right), \end{aligned} \tag{4.2}$$

$$\sigma B_0^2 W_0 = \frac{\partial p^*}{\partial z}, \tag{4.3}$$

where p^* is defined in equation (1.38).

The flow is set up in rotating system by the elliptic harmonic oscillations of the porous plate so that relevant boundary conditions of the problem are

$$\begin{aligned} u &= U_0(ae^{i\omega t} + be^{-i\omega t}),\ v = 0 \ \ at\ z = 0, \quad t > 0, \\ u &\rightarrow 0,\ v \rightarrow 0, \quad as \quad z \rightarrow \infty, \quad t > 0, \end{aligned} \tag{4.4}$$

where a and b are complex.

Using equation (4.3) ,we can express

$$p^* = \sigma B_0^2 W_0 z + f(x, y, t),$$

which in term gives

$$\frac{\partial^2 p^*}{\partial z \partial x} = 0, \quad \frac{\partial^2 p^*}{\partial z \partial y} = 0.$$

Thus in order to eliminate p^* we take partial derivatives of (4.1) and (4.2). Whereas, this helps us to eliminate p^* it increases the order of differential equation by one. However, the condition at infinity helps us to determine three constants at a time, thus yielding a unique solution of the differential equation

$$\begin{aligned} \frac{\partial^2 u}{\partial z \partial t} - W_0 \frac{\partial^2 u}{\partial z^2} - 2\Omega \frac{\partial v}{\partial z} &= \nu \frac{\partial^3 u}{\partial z^3} + \alpha\left(\frac{\partial^4 u}{\partial z^3 \partial t} - W_0 \frac{\partial^4 u}{\partial z^4}\right) \\ &\quad - \frac{\sigma B_0^2}{\rho}\frac{\partial u}{\partial z} + 2\frac{\beta_3}{\rho}\frac{\partial^2}{\partial z^2}\left(\frac{\partial u}{\partial z}\left(\left(\frac{\partial u}{\partial z}\right)^2 + \left(\frac{\partial v}{\partial z}\right)^2\right)\right), \end{aligned} \tag{4.5}$$

$$\begin{aligned} \frac{\partial^2 v}{\partial z \partial t} - W_0 \frac{\partial^2 v}{\partial z^2} + 2\Omega \frac{\partial u}{\partial z} &= \nu \frac{\partial^3 v}{\partial z^3} + \alpha\left(\frac{\partial^4 v}{\partial z^3 \partial t} - W_0 \frac{\partial^4 v}{\partial z^4}\right) \\ &\quad - \frac{\sigma B_0^2}{\rho}\frac{\partial v}{\partial z} + 2\frac{\beta_3}{\rho}\frac{\partial^2}{\partial z^2}\left(\frac{\partial v}{\partial z}\left(\left(\frac{\partial u}{\partial z}\right)^2 + \left(\frac{\partial v}{\partial z}\right)^2\right)\right). \end{aligned} \tag{4.6}$$

The equations (4.5) and (4.6) can be combined as

$$\frac{\partial^2 q}{\partial z \partial t} - W_0 \frac{\partial^2 q}{\partial z^2} + 2i\Omega \frac{\partial q}{\partial z} = \nu \frac{\partial^3 q}{\partial z^3} + \alpha \left(\frac{\partial^4 q}{\partial z^3 \partial t} - W_0 \frac{\partial^4 q}{\partial z^4} \right) - \frac{\sigma B_0^2}{\rho} \frac{\partial q}{\partial z} + 2\frac{\beta_3}{\rho} \frac{\partial^2}{\partial z^2} \left(\left(\frac{\partial q}{\partial z} \right)^2 \frac{\partial q^*}{\partial z} \right), \tag{4.7}$$

where

$$q^* = u - iv. \tag{4.8}$$

The boundary conditions now takes the following form

$$q = U_0(ae^{i\omega t} + be^{-i\omega t}), \quad at\ z = 0, \quad t > 0, \tag{4.9}$$

$$q \to 0, \quad as \quad z \to \infty, \quad t > 0. \tag{4.10}$$

4.2 The Solution of the Problem for Suction

Introducing the dimensionless variables

$$\eta_1 = \frac{zW_0}{\nu}, \quad \tau = \frac{W_0^2 t}{\nu}, \quad q = U_0 \widehat{q}(\eta_1, \tau) \tag{4.11}$$

the boundary value problem consisting of equation (4.7) and conditions (4.9) and (4.10) after dropping the hats become

$$\frac{\partial^2 q}{\partial \eta_1 \partial \tau} - \frac{\partial^2 q}{\partial \eta_1^2} + \frac{2i\Omega\nu}{W_0^2} \frac{\partial q}{\partial \eta_1} = \frac{\partial^3 q}{\partial \eta_1^3} + \frac{\alpha W_0^2}{\nu^2} \left(\frac{\partial^4 q}{\partial \eta_1^3 \partial \tau} - \frac{\partial^4 q}{\partial \eta_1^4} \right) - \frac{\sigma B_0^2}{\rho W_0^2} \frac{\partial q}{\partial \eta_1} + 2\frac{\beta_3 W_0^2 U_0^2}{\rho \nu^3} \frac{\partial^2}{\partial \eta_1^2} \left(\left(\frac{\partial q}{\partial \eta_1} \right)^2 \frac{\partial q^*}{\partial \eta_1} \right), \tag{4.12}$$

$$q = (ae^{i\delta\tau} + be^{-i\delta\tau}), \quad at\ \eta_1 = 0, \quad \tau > 0,$$

$$q \to 0, \quad as \quad \eta_1 \to \infty, \quad \tau > 0, \tag{4.13}$$

where

$$\delta = \frac{\omega\nu}{W_0^2}.$$

Here it is worth emphasizing that the equation for a fluid of second and third grades are in general of higher order than the Navier-Stokes equation. Thus, in general, one needs conditions in addition to the usual no slip condition to solve the flow problem of these fluids. Furthermore the equations for second grade and Newtonian flow are linear and equation (4.12) is non-linear. As a result, it seems to be impossible to obtain the general solution in closed form for arbitrary values of all parameters arising in equation (4.12). One possible way to overcoming this difficulty is to employ the solution of the problem as the power series expansion in small parameter

$$\in^* = \frac{2\beta_3}{\rho}\frac{W_0^2 U_0^2}{\nu^3}. \tag{4.14}$$

Accordingly we assume that q can be expanded in powers of $\in^*$ as follows:

$$q = q_{02} + \in^* q_{12} + \ldots \tag{4.15}$$

Substituting equation (4.15) into equation (4.12) and conditions (4.13) and then equating terms of like powers of $\in^*$, one obtains the following systems of partial differential equations along with boundary conditions.

4.2.1 Zeroth-Order System

$$\begin{aligned}
&\frac{\partial^2 q_{02}}{\partial\eta_1\partial\tau} - \frac{\partial^2 q_{02}}{\partial\eta_1^2} + \frac{2i\Omega\nu}{W_0^2}\frac{\partial q_{02}}{\partial\eta_1} \\
= \;& \frac{\partial^3 q_{02}}{\partial\eta_1^3} - \frac{\sigma B_0^2\nu}{\rho W_0^2}\frac{\partial q_{02}}{\partial\eta_1} + \frac{\alpha W_0^2}{\nu^2}\left(\frac{\partial^4 q_{02}}{\partial\eta_1^3\partial\tau} - \frac{\partial^4 q_{02}}{\partial\eta_1^4}\right), \qquad (4.16)\\
q_{02} \;=\;& (ae^{i\delta\tau} + be^{-i\delta\tau}), \quad at\; \eta_1 = 0, \quad \tau > 0, \\
q_{02} \;\rightarrow\;& 0, \quad as \quad \eta_1 \rightarrow \infty, \quad \tau > 0. \qquad (4.17)
\end{aligned}$$

4.2.2 First-Order System

$$\begin{aligned} & \frac{\partial^2 q_{12}}{\partial\eta_1\partial\tau} - W_0\frac{\partial^2 q_{12}}{\partial\eta_1^2} + \frac{2i\Omega\nu}{W_0^2}\frac{\partial q_{12}}{\partial\eta_1} \\ = & \frac{\partial^3 q_{12}}{\partial\eta_1^3} + \frac{\alpha W_0^2}{\nu^2}\left(\frac{\partial^4 q_{12}}{\partial\eta_1^3\partial\tau} - W_0\frac{\partial^4 q_{12}}{\partial\eta_1^4}\right) \\ & -\frac{\sigma B_0^2}{\rho W_0^2}\frac{\partial q_{12}}{\partial\eta_1} + \frac{\partial^2}{\partial\eta_1^2}\left(\left(\frac{\partial q_{12}}{\partial\eta_1}\right)^2\frac{\partial q_{12}^*}{\partial\eta_1}\right), \end{aligned} \tag{4.18}$$

$$\begin{aligned} q_{12} &= 0, \quad at\ \eta_1 = 0, \quad \tau > 0, \\ q_{12} &\to 0, \quad as \quad \eta_1 \to \infty, \quad \tau > 0. \end{aligned} \tag{4.19}$$

The boundary conditions suggest, for an oscillatory flow, a solution of the form

$$q_{02} = \phi_{01}(\eta_1) + a\phi_{02}(\eta_1)e^{i\delta\tau} + b\phi_{03}(\eta_1)e^{-i\delta\tau}, \delta > 0, \tag{4.20}$$

$$q_{12} = \phi_{11}(\eta_1) + a\phi_{12}(\eta_1)e^{i\delta\tau} + b\phi_{13}(\eta_1)e^{-i\delta\tau}. \tag{4.21}$$

Substituting equation (4.20) in equations (4.16) and (4.17) and equation (4.21) in equations (4.18) and (4.19) we get the following six ordinary differential conditions:

$$\lambda_2\frac{d^4\varphi_{01}}{d\eta_1^4} - \frac{d^3\varphi_{01}}{d\eta_1^3} - \frac{d^2\varphi_{01}}{d\eta_1^2} + (\Omega_2 + i\Omega_1)\frac{d\varphi_{01}}{d\eta_1} = 0, \tag{4.22}$$

$$\lambda_2\frac{d^4\varphi_{02}}{d\eta_1^4} - (1 + i\delta\lambda_2)\frac{d^3\varphi_{02}}{d\eta_1^3} - \frac{d^2\varphi_{02}}{d\eta_1^2} + (\Omega_2 + i(\delta + \Omega_1))\frac{d\varphi_{02}}{d\eta_1} = 0, \tag{4.23}$$

$$\lambda_2\frac{d^4\varphi_{03}}{d\eta_1^4} + (i\delta\lambda_2 - 1)\frac{d^3\varphi_{03}}{d\eta_1^3} - \frac{d^2\varphi_{03}}{d\eta_1^2} + (\Omega_2 - i(\delta - \Omega_1))\frac{d\varphi_{03}}{d\eta_1} = 0, \tag{4.24}$$

$$\lambda_2\frac{d^4\varphi_{11}}{d\eta_1^4} - \frac{d^3\varphi_{11}}{d\eta_1^3} - \frac{d^2\varphi_{11}}{d\eta_1^2} + (\Omega_2 + i\Omega_1)\frac{d\varphi_{11}}{d\eta_1}$$

$$= \frac{d^2}{d\eta_1^2}\left[\begin{array}{c} \left(\frac{d\varphi_{01}}{d\eta_1}\right)^3 + 2a^2\left(\frac{d\varphi_{01}}{d\eta_1}\right)\left(\frac{d\varphi_{02}}{d\eta_1}\right)^2 \\ +2ab\left(\frac{d\varphi_{01}}{d\eta_1}\right)\left(\frac{d\varphi_{02}}{d\eta_1}\right)\left(\frac{d\varphi_{03}}{d\eta_1}\right) + 2b^2\left(\frac{d\varphi_{01}}{d\eta_1}\right)\left(\frac{d\varphi_{03}}{d\eta_1}\right)^2 \end{array}\right], \tag{4.25}$$

$$\lambda_2\frac{d^4\varphi_{12}}{d\eta_1^4} - (1+i\delta\lambda_2)\frac{d^3\varphi_{12}}{d\eta_1^3} - \frac{d^2\varphi_{12}}{d\eta_1^2} + (\Omega_2 + i(\delta+\Omega_1))\frac{d\varphi_{12}}{d\eta_1}$$
$$= \frac{d^2}{d\eta_1^2}\left[\begin{array}{c} \frac{b}{a}\left(\frac{d\varphi_{01}}{d\eta_1}\right)^2\left(\frac{d\varphi_{03}}{d\eta_1}\right) + a^2\left(\frac{d\varphi_{02}}{d\eta_1}\right)^3 \\ +2\left(\frac{d\varphi_{01}}{d\eta_1}\right)^2\left(\frac{d\varphi_{02}}{d\eta_1}\right) + 2b^2\left(\frac{d\varphi_{02}}{d\eta_1}\right)\left(\frac{d\varphi_{03}}{d\eta_1}\right)^2 \end{array}\right], \tag{4.26}$$

$$\lambda_2\frac{d^4\varphi_{13}}{d\eta_1^4} + (i\delta\lambda_2 - 1)\frac{d^3\varphi_{13}}{d\eta_1^3} - \frac{d^2\varphi_{13}}{d\eta_1^2} + (\Omega_2 - i(\delta-\Omega_1))\frac{d\varphi_{13}}{d\eta_1}$$
$$= \frac{d^2}{d\eta_1^2}\left[\begin{array}{c} \frac{a}{b}\left(\frac{d\varphi_{01}}{d\eta_1}\right)^2\left(\frac{d\varphi_{02}}{d\eta_1}\right) + b^2\left(\frac{d\varphi_{03}}{d\eta_1}\right)^3 \\ +2a^2\left(\frac{d\varphi_{02}}{d\eta_1}\right)^2\left(\frac{d\varphi_{03}}{d\eta_1}\right) + 2\left(\frac{d\varphi_{01}}{d\eta_1}\right)^2\left(\frac{d\varphi_{03}}{d\eta_1}\right) \end{array}\right], \tag{4.27}$$

with the boundary conditions

$$\varphi_{01} = \varphi_{11} = \varphi_{12} = \varphi_{13} = 0,\ \varphi_{02} = \varphi_{03} = 1,\ on\ \eta_1 = 0, \tag{4.28}$$

$$\varphi_{01} = \varphi_{02} = \varphi_{03} = \varphi_{11} = \varphi_{12} = \varphi_{13} \to 0\ \ as\ \eta_1 \to \infty, \tag{4.29}$$

and a physical condition that solutions of systems (4.22)–(4.29) reduce to Newtonian case when $\lambda_2 \to 0$.

In above equations

$$\lambda_2 = \frac{\alpha W_0^2}{\nu^2},\ \Omega_1 = \frac{2\Omega\nu}{W_0^2},\ \ \Omega_2 = \frac{n^*\nu}{W_0^2}. \tag{4.30}$$

The characteristic polynomial of equation (4.22) is

$$\lambda_2 m_2^4 - m_2^3 - m_2^2 + (\Omega_2 - i\Omega_1)\,m_2 = 0. \tag{4.31}$$

From equation (4.31) either

$$m_2 = 0, \tag{4.32}$$

or

$$\lambda_2 m_2^3 - m_2^2 - m_2 + (\Omega_2 + i\Omega_1) = 0. \tag{4.33}$$

For small values of λ_2, the roots of equation (4.33) can be obtained by perturbation expansion method [75]. For that m_2 can be expressed as

$$m_2 = \frac{\in_{-1}}{\lambda_2} + \in_0 + \in_1 \lambda_2 + \in_2 \lambda_2^2 + ... \tag{4.34}$$

Now using equation (4.34) in equation (4.33) we get $\in_{-1} = 0, 0, 1$. The corresponding roots for three values of $\in_{-1}$ are given by

$$m_{11} \approx \in_0 + \in_1 \lambda_2 + \in_2 \lambda_2^2, \tag{4.35}$$

$$m_{12} \approx \widetilde{\in}_0 + \widetilde{\in}_1 \lambda_2 + \widetilde{\in}_2 \lambda_2^2, \tag{4.36}$$

$$m_{13} \approx \frac{1}{\lambda} + \in_0^* + \in_1^* \lambda + \in_2^* \lambda^2, \tag{4.37}$$

where

$$\in_0 = -\left(\frac{1 + \sqrt{1 + 4(\Omega_2 + i\Omega_1)}}{2}\right), \; \in_1 = \frac{\in_0^3}{1 + 2 \in_0}, \; \in_2 = \frac{3 \in_0^2 \in_1 - \in_1^2}{1 + 2 \in_0}, \tag{4.38}$$

$$\widetilde{\in}_0 = \left(\frac{-1 + \sqrt{1 + 4(\Omega_2 + i\Omega_1)}}{2}\right), \; \widetilde{\in}_1 = \frac{\widetilde{\in}_0^3}{1 + 2\widetilde{\in}_0}, \; \widetilde{\in}_2 = \frac{3\widetilde{\in}_0^2 \widetilde{\in}_1 - \widetilde{\in}_1^2}{1 + 2\widetilde{\in}_0}, \tag{4.39}$$

$$\in_0^* = 1, \quad \in_1^* = -1 - \Omega_2 + i\Omega_1, \quad \in_2^* = \frac{\Omega_2 + i\Omega_1}{5}. \tag{4.40}$$

The solution of equation (4.22) satisfying the boundary conditions (4.28) and (4.29) along with the physical condition is given by

$$\varphi_{01} = 0. \tag{4.41}$$

Similarly the solutions of equations (4.23)–(4.27) satisfying conditions (4.28) and (4.29) are given by

$$\begin{aligned}
\varphi_{02} &= e^{-(a_1+ib_1)\eta_1}, \quad \delta > \Omega_1, & (4.42)\\
&= e^{-(a_2+ib_2)\eta_1}, \quad \delta < \Omega_1, & (4.43)\\
&= e^{-(a_3+ib_3)\eta_1}, \quad \delta = \Omega_1, & (4.44)
\end{aligned}$$

$$\begin{aligned}
\varphi_{03} &= e^{-(a_4+ib_4)\eta_1}, \ \delta > \Omega_1, & (4.45)\\
&= e^{-(a_4+ib_4)\eta_1}, \ \delta < \Omega_1,\\
&= e^{-(a_4+ib_4)\eta_1}, \ \delta = \Omega_1,
\end{aligned}$$

$$\begin{aligned}
\varphi_{11} &= 0, \ \ \delta > \Omega_1, & (4.46)\\
&= 0, \ \ \delta < \Omega_1,\\
&= 0, \ \ \delta = \Omega_1,
\end{aligned}$$

$$\begin{aligned}
\varphi_{12} &= (A_{4R} + iA_{4I})\left[e^{-3(a_1+ib_1)\eta_1} - e^{-(a_1+ib_1)\eta_1}\right]\\
&\quad + (A_{5R} + iA_{5I})\left[e^{-(c_1+ic_2)\eta_1} - e^{-(a_1+ib_1)\eta_1}\right], \ \delta > \Omega_1\ , & (4.47)\\
\varphi_{12} &= (A_{10R} + iA_{10I})\left[e^{-3(a_2+ib_2)\eta_1} - e^{-(a_2+ib_2)\eta_1}\right]\\
&\quad + (A_{11R} + iA_{11I})\left[e^{-(c_3+ic_4)\eta_1} - e^{-(a_2+ib_2)\eta_1}\right], \ \delta < \Omega_1\ , & (4.48)\\
\varphi_{12} &= (A_{16R} + iA_{16I})\left[e^{-3(a_3+ib_3)\eta_1} - e^{-(a_3+ib_3)\eta_1}\right]\\
&\quad + (A_{17R} + iA_{17I})\left[e^{-(c_1+ic_2)\eta_1} - e^{-(a_1+ib_1)\eta_1}\right], \ \delta = \Omega_1\ , & (4.49)\\
\varphi_{13} &= (A_{22R} + iA_{22I})\left[e^{-3(a_4+ib_4)\eta_1} - e^{-(a_4+ib_4)\eta_1}\right]\\
&\quad + (A_{23R} + iA_{23I})\left[e^{-(c_7+ic_8)\eta_1} - e^{-(a_4+ib_4)\eta_1}\right], \ \delta > \Omega_1\ , & (4.50)
\end{aligned}$$

$$\varphi_{13} = (A_{22R} + iA_{22I})\left[e^{-3(a_4+ib_4)\eta_1} - e^{-(a_4+ib_4)\eta_1}\right]$$
$$+ (A_{26R} + iA_{26I})\left[e^{-(c_9+ic_{10})\eta_1} - e^{-(a_4+ib_4)\eta_1}\right], \ \delta < \Omega_1, \quad (4.51)$$
$$\varphi_{13} = (A_{22R} + iA_{22I})\left[e^{-3(a_4+ib_4)\eta_1} - e^{-(a_4+ib_4)\eta_1}\right]$$
$$+ (A_{29R} + iA_{29I})\left[e^{-(c_{11}+ic_{12})\eta_1} - e^{-(a_4+ib_4)\eta_1}\right], \ \delta = \Omega_1, \quad (4.52)$$

where

$$A_{4R} = \frac{A_R A_{2R} + A_I A_{2I}}{A_{2R}^2 + A_{2I}^2}, \quad A_{4I} = \frac{A_I A_{2R} - A_R A_{2I}}{A_{2R}^2 + A_{2I}^2},$$
$$A_{5R} = \frac{A_{1R} A_{3R} + A_{1I} A_{3I}}{A_{3R}^2 + A_{3I}^2}, \quad A_{5I} = \frac{A_{1I} A_{3R} - A_{1R} A_{3I}}{A_{3R}^2 + A_{3I}^2},$$
$$A_{10R} = \frac{A_{6R} A_{8R} + A_{6I} A_{8I}}{A_{8R}^2 + A_{8I}^2}, \quad A_{10I} = \frac{A_{6I} A_{8R} - A_{6R} A_{8I}}{A_{8R}^2 + A_{8I}^2},$$
$$A_{11R} = \frac{A_{7R} A_{9R} + A_{7I} A_{9I}}{A_{9R}^2 + A_{9I}^2}, \quad A_{11I} = \frac{A_{7I} A_{9R} - A_{7R} A_{9I}}{A_{9R}^2 + A_{9I}^2},$$
$$A_{16R} = \frac{A_{12R} A_{14R} + A_{12I} A_{14I}}{A_{14R}^2 + A_{14I}^2}, \quad A_{16I} = \frac{A_{12I} A_{14R} - A_{12R} A_{14I}}{A_{14R}^2 + A_{14I}^2},$$
$$A_{17R} = \frac{A_{13R} A_{15R} + A_{13I} A_{15I}}{A_{15R}^2 + A_{15I}^2}, \quad A_{17I} = \frac{A_{13I} A_{15R} - A_{13R} A_{15I}}{A_{15R}^2 + A_{15I}^2},$$
$$A_{22R} = \frac{A_{18R} A_{20R} + A_{18I} A_{20I}}{A_{20R}^2 + A_{20I}^2}, \quad A_{22I} = \frac{A_{18I} A_{20R} - A_{18R} A_{20I}}{A_{20R}^2 + A_{20I}^2},$$
$$A_{23R} = \frac{A_{19R} A_{21R} + A_{19I} A_{21I}}{A_{21R}^2 + A_{21I}^2}, \quad A_{23I} = \frac{A_{19I} A_{21R} - A_{19R} A_{21I}}{A_{21R}^2 + A_{21I}^2},$$
$$A_{26R} = \frac{A_{24R} A_{25R} + A_{24I} A_{25I}}{A_{25R}^2 + A_{25I}^2}, \quad A_{26I} = \frac{A_{24I} A_{25R} - A_{24R} A_{25I}}{A_{25R}^2 + A_{25I}^2},$$
$$A_{29R} = \frac{A_{27R} A_{28R} + A_{27I} A_{28I}}{A_{28R}^2 + A_{28I}^2}, \quad A_{29I} = \frac{A_{27I} A_{28R} - A_{27R} A_{28I}}{A_{28R}^2 + A_{28I}^2},$$
$$a_1 = c_{0R} - \lambda_2 c_{1R} - \lambda_2^2 c_{2R}, \quad a_2 = n_{0R} - \lambda_2 n_{1R} - \lambda_2^2 n_{2R},$$
$$a_3 = h_0 - \lambda_2 h_{1R} - \lambda_2^2 h_{2R}, \quad a_4 = k_{0R} - \lambda_2 k_{1R} - \lambda_2^2 k_{2R},$$
$$b_1 = c_{0I} - \lambda_2 c_{1I} - \lambda_2^2 c_{2I}, \quad b_2 = n_{0I} - \lambda_2 n_{1I} - \lambda_2^2 n_{2I},$$

$$b_3 = \lambda_2 h_{1I} - \lambda_2^2 h_{2I}, \quad b_4 = k_{0I} - \lambda_2 k_{1I} - \lambda_2^2 k_{2I},$$

$$\begin{aligned} A_{1R} &= 2\left(b_R^2 - b_I^2\right)\left[\begin{array}{c} \{a_1\left(b_4^2 - a_4^2\right) + 2b_1a_4b_4\}\left(c_1^2 - c_2^2\right) \\ +2\{b_1\left(a_4^2 - b_4^2\right) - 2a_1a_4b_4\}\left(c_1c_2\right) \end{array}\right] \\ &\quad -4b_Rb_I\left[\begin{array}{c} \{b_1\left(a_4^2 - b_4^2\right) - 2a_1a_4b_4\}\left(c_1^2 - c_2^2\right) \\ +2\{a_1\left(b_4^2 - a_4^2\right) + 2b_1a_4b_4\}\left(c_1c_2\right) \end{array}\right], \end{aligned}$$

$$\begin{aligned} A_{1I} &= 2\left(b_R^2 - b_I^2\right)\left[\begin{array}{c} \{b_1\left(a_4^2 - b_4^2\right) - 2a_1a_4b_4\}\left(c_1^2 - c_2^2\right) \\ +2\{a_1\left(b_4^2 - a_4^2\right) + 2b_1a_4b_4\}\left(c_1c_2\right) \end{array}\right] \\ &\quad +4b_Rb_I\left[\begin{array}{c} \{a_1\left(b_4^2 - a_4^2\right) + 2b_1a_4b_4\}\left(c_1^2 - c_2^2\right) \\ +2\{b_1\left(a_4^2 - b_4^2\right) - 2a_1a_4b_4\}\left(c_1c_2\right) \end{array}\right], \end{aligned}$$

$$\begin{aligned} A_{2R} &= 3b_1\left[27\lambda_2 b_1^3 - 81a_1^2b_1\lambda_2 - 9\left(a_1^2 - b_1^2\right) + 18a_1b_1\delta\lambda_2 + 3a_1 + \Omega_2\right] \\ &\quad -3a_1\left[-27\lambda_2 a_1^3 + 81a_1b_1^2\lambda_2 - 9\delta\lambda_2\left(a_1^2 - b_1^2\right) - 18a_1b_1 + 3b_1 + \delta - \Omega_1\right], \end{aligned}$$

$$\begin{aligned} A_{2I} &= -3a_1\left[27\lambda_2 b_1^3 - 81a_1^2b_1\lambda_2 - 9\delta\lambda_2\left(a_1^2 - b_1^2\right) - 18a_1b_1 + 3b_1 + \delta - \Omega_1\right] \\ &\quad -3b_1\left[-27\lambda_2 a_1^3 + 81a_1b_1^2\lambda_2 - 9\left(a_1^2 - b_1^2\right) + 18a_1b_1\delta\lambda_2 + 3a_1 + \Omega_2\right], \end{aligned}$$

$$\begin{aligned} A_{3R} &= -c_1\left[-\lambda_2 c_1^3 + 3\lambda_2 c_1c_2^2 + 2c_1c_2\delta\lambda_2 + c_1 + \Omega_2 - \left(c_1^2 - c_2^2\right)\right] \\ &\quad +c_2\left[\lambda_2 c_2^3 - 3c_1^2c_2\lambda_2 - \delta\lambda_2\left(c_1^2 - c_2^2\right) - 2c_1c_2 + c_2 + \delta - \Omega_1\right], \end{aligned}$$

$$\begin{aligned} A_{3I} &= -c_1\left[\lambda_2 c_2^3 - 3c_1^2c_2\lambda_2 - \delta\lambda_2\left(c_1^2 - c_2^2\right) - 2c_1c_2 + c_2 + \delta - \Omega_1\right] \\ &\quad -c_2\left[-\lambda_2 c_1^3 + 3\lambda_2 c_1c_2^2 + 2c_1c_2\delta\lambda_2 + c_1 + \Omega_2 - \left(c_1^2 - c_2^2\right)\right], \end{aligned}$$

$$\begin{aligned} A_{6R} &= \left(a_R^2 - a_I^2\right)\left[9\left(a_2^2 - b_2^2\right)\left(3a_2b_2^2 - a_2^3\right) + 18a_2b_2\left(3a_2^2b_2 - b_2^3\right)\right] \\ &\quad -4a_Ra_I\left[9\left(a_2^2 - b_2^2\right)\left(b_2^3 - 3a_2^2b_2\right) - 18a_2b_2\left(a_2^3 - 3a_2b_2^2\right)\right], \end{aligned}$$

$$
\begin{aligned}
A_{6I} &= \left(a_R^2 - a_I^2\right)\left[9\left(a_2^2 - b_2^2\right)\left(b_2^3 - 3a_2^2 b_2\right) - 18a_2 b_2\left(a_2^3 - 3a_2 b_2^2\right)\right] \\
&\quad + 4a_R a_I\left[9\left(a_2^2 - b_2^2\right)\left(3a_2 b_2^2 - a_2^3\right) + 18a_2 b_2\left(3a_2^2 b_2 - b_2^3\right)\right],
\end{aligned}
$$

$$
\begin{aligned}
A_{7R} &= 2\left(b_R^2 - b_I^2\right)\left[\begin{array}{l} \{a_2\left(b_4^2 - a_4^2\right) + 2b_2 a_4 b_4\}\left(c_3^2 - c_4^2\right) \\ +2\{b_2\left(a_4^2 - b_4^2\right) - 2a_2 a_4 b_4\}\left(c_3 c_4\right) \end{array}\right] \\
&\quad - 4b_R b_I\left[\begin{array}{l} \{b_2\left(a_4^2 - b_4^2\right) - 2a_2 a_4 b_4\}\left(c_3^2 - c_4^2\right) \\ +2\{a_2\left(b_4^2 - a_4^2\right) + 2b_2 a_4 b_4\}\left(c_3 c_4\right) \end{array}\right],
\end{aligned}
$$

$$
\begin{aligned}
A_{7I} &= 2\left(b_R^2 - b_I^2\right)\left[\begin{array}{l} \{b_2\left(a_4^2 - b_4^2\right) - 2a_2 a_4 b_4\}\left(c_3^2 - c_4^2\right) \\ +2\{a_2\left(b_4^2 - a_4^2\right) + 2b_2 a_4 b_4\}\left(c_3 c_4\right) \end{array}\right] \\
&\quad + 4b_R b_I\left[\begin{array}{l} \{a_2\left(b_4^2 - a_4^2\right) + 2b_2 a_4 b_4\}\left(c_3^2 - c_4^2\right) \\ +2\{b_2\left(a_4^2 - b_4^2\right) - 2a_2 a_4 b_4\}\left(c_3 c_4\right) \end{array}\right],
\end{aligned}
$$

$$
\begin{aligned}
A_{8R} &= 3b_2\left[27\lambda_2 b_2^3 - 81a_2^2 b_2 \lambda_2 - 9\delta\lambda_2\left(a_2^2 - b_2^2\right) - 18a_2 b_2 + 3b_2 + \delta - \Omega_1\right] \\
&\quad - 3a_2\left[-27\lambda_2 a_2^3 + 81a_2 b_2^2 \lambda_2 - 9\left(a_2^2 - b_2^2\right) + 18a_2 b_2 \delta\lambda_2 + 3a_2 + \Omega_2\right],
\end{aligned}
$$

$$
\begin{aligned}
A_{8I} &= -3a_2\left[27\lambda_2 b_2^3 - 81a_2^2 b_2 \lambda_2 - 9\delta\lambda_2\left(a_2^2 - b_2^2\right) - 18a_2 b_2 + 3b_2 + \delta - \Omega_1\right] \\
&\quad - 3b_2\left[-27\lambda_2 a_2^3 + 81a_2 b_2^2 \lambda_2 - 9\left(a_2^2 - b_2^2\right) + 18a_2 b_2 \delta\lambda_2 + 3a_2 + \Omega_2\right],
\end{aligned}
$$

$$
\begin{aligned}
A_{9R} &= -c_3\left[-\lambda_2 c_3^3 + 3\lambda_2 c_3 c_4^2 + 2c_3 c_4 \delta\lambda_2 + c_3 + \Omega_2 - \left(c_3^2 - c_4^2\right)\right] \\
&\quad + c_4\left[\lambda_2 c_4^3 - 3c_3^2 c_3 \lambda_2 - \delta\lambda_2\left(c_3^2 - c_4^2\right) - 2c_3 c_4 + c_4 + \delta - \Omega_1\right],
\end{aligned}
$$

$$
\begin{aligned}
A_{9I} &= -c_3\left[\lambda_2 c_4^3 - 3c_3^2 c_4 \lambda_2 - \delta\lambda_2\left(c_3^2 - c_4^2\right) - 2c_3 c_4 + c_4 + \delta - \Omega_1\right] \\
&\quad - c_4\left[-\lambda_2 c_3^3 + 3\lambda_2 c_3 c_4^2 + 2c_3 c_4 \delta\lambda_2 + c_3 + \Omega_2 - \left(c_3^2 - c_4^2\right)\right],
\end{aligned}
$$

$$
\begin{aligned}
A_{12R} &= \left(a_R^2 - a_I^2\right)\left[9\left(a_3^2 - b_3^2\right)\left(3a_3b_3^2 - a_3^3\right) + 18a_3b_3\left(3a_3^2b_3 - b_3^3\right)\right] \\
&\quad -4a_Ra_I\left[9\left(a_3^2 - b_3^2\right)\left(b_3^3 - 3a_3^2b_3\right) - 18a_3b_3\left(a_3^3 - 3a_3b_3^2\right)\right],
\end{aligned}
$$

$$
\begin{aligned}
A_{12I} &= \left(a_R^2 - a_I^2\right)\ \left[9\left(a_3^2 - b_3^2\right)\left(b_3^3 - 3a_3^2b_3\right) - 18a_3b_3\left(a_3^3 - 3a_3b_3^2\right)\right] \\
&\quad +4a_Ra_I\ \left[9\left(a_3^2 - b_3^2\right)\left(3a_3b_3^2 - a_3^3\right) + 18a_3b_3\left(3a_3^2b_3 - b_3^3\right)\right],
\end{aligned}
$$

$$
\begin{aligned}
A_{13R} &= 2\left(b_R^2 - b_I^2\right)\left[\begin{array}{c} \left\{a_3\left(b_4^2 - a_4^2\right) + 2b_3a_4b_4\right\}\left(c_5^2 - c_6^2\right) \\ +2\left\{b_3\left(a_4^2 - b_4^2\right) - 2a_3a_4b_4\right\}\left(c_5c_6\right) \end{array}\right] \\
&\quad -4b_Rb_I\left[\begin{array}{c} \left\{b_3\left(a_4^2 - b_4^2\right) - 2a_3a_4b_4\right\}\left(c_5^2 - c_6^2\right) \\ +2\left\{a_3\left(b_4^2 - a_4^2\right) + 2b_3a_4b_4\right\}\left(c_5c_6\right) \end{array}\right],
\end{aligned}
$$

$$
\begin{aligned}
A_{13I} &= 2\left(b_R^2 - b_I^2\right)\left[\begin{array}{c} \left\{b_3\left(a_4^2 - b_4^2\right) - 2a_3a_4b_4\right\}\left(c_5^2 - c_6^2\right) \\ +2\left\{a_3\left(b_4^2 - a_4^2\right) + 2b_3a_4b_4\right\}\left(c_5c_6\right) \end{array}\right] \\
&\quad +4b_Rb_I\left[\begin{array}{c} \left\{a_3\left(b_4^2 - a_4^2\right) + 2b_3a_4b_4\right\}\left(c_5^2 - c_6^2\right) \\ +2\left\{b_3\left(a_4^2 - b_4^2\right) - 2a_3a_4b_4\right\}\left(c_5c_6\right) \end{array}\right],
\end{aligned}
$$

$$
\begin{aligned}
A_{14R} &= 3b_3\left[27\lambda_2b_3^3 - 81a_3^2b_3\lambda_2 - 9\delta\lambda_2\left(a_3^2 - b_3^2\right) - 18a_3b_3 + 3b_3\right] \\
&\quad -3a_3\left[-27\lambda_2a_3^3 + 81a_3b_3^2\lambda_2 - 9\left(a_3^2 - b_3^2\right) + 18a_3b_3\delta\lambda_2 + 3a_3 + \Omega_2\right],
\end{aligned}
$$

$$
\begin{aligned}
A_{14I} &= -3a_3\left[27\lambda_2b_3^3 - 81a_3^2b_3\lambda_2 - 9\delta\lambda_2\left(a_3^2 - b_3^2\right) - 18a_3b_3 + 3b_3\right] \\
&\quad -3b_3\left[-27\lambda_2a_3^3 + 81a_3b_3^2\lambda_2 - 9\left(a_3^2 - b_3^2\right) + 18a_3b_3\delta\lambda_2 + 3a_3 + \Omega_2\right],
\end{aligned}
$$

$$
\begin{aligned}
A_{15R} &= -c_5\left[-\lambda_2c_5^3 + 3\lambda_2c_5c_6^2 + 2c_5c_6\delta\lambda_2 + c_5 + \Omega_2 - \left(c_5^2 - c_6^2\right)\right] \\
&\quad +c_6\left[\lambda_2c_6^3 - 3c_5^2c_6\lambda_2 - \delta\lambda_2\left(c_5^2 - c_6^2\right) - 2c_5c_6 + c_6\right],
\end{aligned}
$$

$$A_{15I} = -c_5\left[\lambda_2 c_6^3 - 3c_5^2 c_6 \lambda_2 - \delta\lambda_2\left(c_5^2 - c_6^2\right) - 2c_5 c_6 + c_6\right]$$
$$-c_6\left[-\lambda_2 c_5^3 + 3\lambda_2 c_5 c_6^2 + 2c_5 c_6 \delta\lambda_2 + c_5 + \Omega_2 - \left(c_5^2 - c_6^2\right)\right],$$

$$A_{18R} = \left(b_R^2 - b_I^2\right)\left[9\left(a_4^2 - b_4^2\right)\left(3a_4 b_4^2 - a_4^3\right) - 18a_4 b_4\left(3a_4^2 b_4 - b_4^3\right)\right]$$
$$-4a_R a_I\left[9\left(a_4^2 - b_4^2\right)\left(b_4^3 - 3a_4^2 b_4\right) + 18a_4 b_4\left(a_4^3 - 3a_4 b_4^2\right)\right],$$

$$A_{18I} = \left(b_R^2 - b_I^2\right)\quad\left[9\left(a_4^2 - b_4^2\right)\left(b_4^3 - 3a_4^2 b_4\right) + 18a_4 b_4\left(a_4^3 - 3a_4 b_4^2\right)\right]$$
$$+4a_R a_I\quad\left[9\left(a_4^2 - b_4^2\right)\left(3a_4 b_4^2 - a_4^3\right) - 18a_4 b_4\left(3a_4^2 b_4 - b_4^3\right)\right],$$

$$A_{19R} = 2\left(a_R^2 - a_I^2\right)\begin{bmatrix}\left\{b_4\left(b_1^2 - a_1^2\right) + 2b_1 a_1 b_4\right\}\left(c_7^2 - c_8^2\right)\\ +2\left\{b_4\left(a_1^2 - b_1^2\right) + 2a_1 a_4 b_1\right\}\left(c_7 c_8\right)\end{bmatrix}$$
$$-4a_R a_I\begin{bmatrix}\left\{b_4\left(b_1^2 - a_1^2\right) + 2b_1 a_1 b_4\right\}\left(c_7 c_8\right)\\ -\left\{b_4\left(a_1^2 - b_1^2\right) + 2a_1 a_4 b_1\right\}\left(c_7^2 - c_8^2\right)\end{bmatrix},$$

$$A_{19I} = 2\left(a_R^2 - a_I^2\right)\begin{bmatrix}\left\{b_4\left(b_1^2 - a_1^2\right) + 2b_1 a_1 b_4\right\}\left(c_7 c_8\right)\\ -\left\{b_4\left(a_1^2 - b_1^2\right) + 2a_1 a_4 b_1\right\}\left(c_7^2 - c_8^2\right)\end{bmatrix}$$
$$+4b_R b_I\begin{bmatrix}\left\{b_4\left(b_1^2 - a_1^2\right) + 2b_1 a_1 b_4\right\}\left(c_7^2 - c_8^2\right)\\ +2\left\{b_4\left(a_1^2 - b_1^2\right) + 2a_1 a_4 b_1\right\}\left(c_7 c_8\right)\end{bmatrix},$$

$$A_{20R} = 3b_4\left[27\lambda_2 b_4^3 - 81a_4^2 b_4\lambda_2 + 9\delta\lambda_2\left(a_4^2 - b_4^2\right) - 18a_4 b_4 + 3b_4 - \left(\delta + \Omega_1\right)\right]$$
$$-3a_4\left[-27\lambda_2 a_4^3 + 81a_4 b_4^2\lambda_2 - 9\left(a_4^2 - b_4^2\right) - 18a_4 b_4\delta\lambda_2 + 3a_4 + \Omega_2\right],$$

$$A_{20I} = 3b_4\left[27\lambda_2 b_4^3 - 81a_4 b_4^2\lambda_2 + 9\left(a_4^2 - b_4^2\right) + 18a_4 b_4\delta\lambda_2 - 3a_4 - \Omega_2\right]$$
$$-3a_4\left[27\lambda_2 b_4^3 - 81a_4^2 b_4\lambda_2 + 9\delta\lambda_2\left(a_4^2 - b_4^2\right) - 18a_4 b_4 + 3b_4 - \left(\delta + \Omega_1\right)\right],$$

$$
\begin{aligned}
A_{21R} &= -c_7\left[-\lambda_2 c_7^3 + 3\lambda_2 c_7 c_8^2 + 2c_7c_8\delta\lambda_2 + c_7 + \Omega_2 - \left(c_7^2 - c_8^2\right)\right] \\
&\quad + c_8\left[\lambda_2 c_8^3 - 3c_7^2 c_8\lambda_2 + \delta\lambda_2\left(c_7^2 - c_8^2\right) - 2c_7c_8 + c_8 - (\delta + \Omega_1)\right],
\end{aligned}
$$

$$
\begin{aligned}
A_{21I} &= -c_7\left[\lambda_2 c_8^3 - 3c_7^2 c_8\lambda_2 + \delta\lambda_2\left(c_7^2 - c_8^2\right) - 2c_7c_8 + c_8 + (\delta + \Omega_1)\right] \\
&\quad - c_8\left[-\lambda_2 c_7^3 + 3\lambda_2 c_7 c_8^2 + 2c_7c_8\delta\lambda_2 + c_7 + \Omega_2 - \left(c_7^2 - c_8^2\right)\right],
\end{aligned}
$$

$$
\begin{aligned}
A_{24R} &= 2\left(a_R^2 - a_I^2\right)\begin{bmatrix} \{b_4(b_2^2 - a_2^2) + 2b_2a_2b_4\}(c_9^2 - c_{10}^2) \\ +2\{b_4(a_2^2 - b_2^2) + 2a_2a_4b_2\}(c_9c_{10}) \end{bmatrix} \\
&\quad -4a_Ra_I\begin{bmatrix} \{b_4(b_2^2 - a_2^2) + 2b_2a_2b_4\}(c_9c_{10}) \\ -\{b_4(a_2^2 - b_2^2) + 2a_2a_4b_2\}(c_9^2 - c_{10}^2) \end{bmatrix},
\end{aligned}
$$

$$
\begin{aligned}
A_{24I} &= 2\left(a_R^2 - a_I^2\right)\begin{bmatrix} \{b_4(b_2^2 - a_2^2) + 2b_2a_2b_4\}(c_9c_{10}) \\ -\{b_4(a_2^2 - b_2^2) + 2a_2a_4b_2\}(c_9^2 - c_{10}^2) \end{bmatrix} \\
&\quad +4b_Rb_I\begin{bmatrix} \{b_4(b_2^2 - a_2^2) + 2b_2a_2b_4\}(c_9^2 - c_{10}^2) \\ +2\{b_4(a_2^2 - b_2^2) + 2a_2a_4b_2\}(c_9c_{10}) \end{bmatrix},
\end{aligned}
$$

$$
\begin{aligned}
A_{25R} &= -c_9\left[-\lambda_2 c_9^3 + 3\lambda_2 c_9 c_{10}^2 - 2c_9c_{10}\delta\lambda_2 + c_9 + \Omega_2 - \left(c_9^2 - c_{10}^2\right)\right] \\
&\quad + c_{10}\left[\lambda_2 c_{10}^3 - 3c_9^2 c_{10}\lambda_2 + \delta\lambda_2\left(c_9^2 - c_{10}^2\right) + 2c_9c_{10} + c_{10} - (\delta + \Omega_1)\right],
\end{aligned}
$$

$$
\begin{aligned}
A_{25I} &= c_{10}\left[\lambda_2 c_9^3 - 3\lambda_2 c_9 c_{10}^2 + 2c_9c_{10}\delta\lambda_2 - c_9 - \Omega_2 + \left(c_9^2 - c_{10}^2\right)\right] \\
&\quad - c_9\left[\lambda_2 c_{10}^3 - 3c_9^2 c_{10}\lambda_2 + \delta\lambda_2\left(c_9^2 - c_{10}^2\right) + 2c_9c_{10} + c_{10} - (\delta + \Omega_1)\right],
\end{aligned}
$$

$$
\begin{aligned}
A_{27R} &= 2\left(a_R^2 - a_I^2\right)\begin{bmatrix} \{b_4(b_3^2 - a_3^2) + 2b_3a_3b_4\}(c_{11}^2 - c_{12}^2) \\ +2\{b_4(a_3^2 - b_3^2) + 2a_3a_4b_3\}(c_{11}c_{12}) \end{bmatrix} \\
&\quad -4a_Ra_I\begin{bmatrix} \{b_4(b_3^2 - a_3^2) + 2b_3a_3b_4\}(c_{11}c_{12}) \\ -\{b_4(a_3^2 - b_3^2) + 2a_3a_4b_3\}(c_{11}^2 - c_{12}^2) \end{bmatrix},
\end{aligned}
$$

$$A_{27I} = 2\left(a_R^2 - a_I^2\right)\left[\begin{array}{c} \left\{b_4\left(b_3^2 - a_3^2\right) + 2b_3a_3b_4\right\}\left(c_{11}c_{12}\right) \\ -\left\{b_4\left(a_3^2 - b_3^2\right) + 2a_3a_4b_3\right\}\left(c_{11}^2 - c_{12}^2\right) \end{array}\right]$$
$$+4b_Rb_I\left[\begin{array}{c} \left\{b_4\left(b_3^2 - a_3^2\right) + 2b_3a_3b_4\right\}\left(c_{11}^2 - c_{12}^2\right) \\ +2\left\{b_4\left(a_3^2 - b_3^2\right) + 2a_3a_4b_3\right\}\left(c_{11}c_{12}\right) \end{array}\right],$$

$$A_{28R} = c_{11}\left[\lambda_2 c_{11}^3 - 3\lambda_2 c_{11}c_{12}^2 + 2c_{11}c_{12}\delta\lambda_2 - c_{11} - \Omega_2 + \left(c_{11}^2 - c_{12}^2\right)\right]$$
$$+c_{12}\left[\lambda_2 c_{12}^3 - 3c_{11}^2 c_{12}\lambda_2 + \delta\lambda_2\left(c_{11}^2 - c_{12}^2\right) + 2c_{11}c_{12} + c_{12} - 2\Omega_1\right],$$

$$A_{28I} = c_{12}\left[\lambda_2 c_{11}^3 - 3\lambda_2 c_{11}c_{12}^2 + 2c_{11}c_{12}\delta\lambda_2 - c_{11} - \Omega_2 + \left(c_{11}^2 - c_{12}^2\right)\right]$$
$$-c_9\left[\lambda_2 c_{12}^3 - 3c_{11}^2 c_{12}\lambda_2 + \delta\lambda_2\left(c_{11}^2 - c_{12}^2\right) - 2c_{11}c_{12} + c_{12} - 2\Omega_1\right],$$

$$\sigma_{11} = [\frac{1}{2}\{\sqrt{(1+4\Omega_2)^2 + 16(\delta - \Omega_1)^2} + (1+4\Omega_2)\}]^{\frac{1}{2}},$$
$$\sigma_{22} = [\frac{1}{2}\{\sqrt{(1+4\Omega_2)^2 + 16(\Omega_1 - \delta)^2} + (1+4\Omega_2)\}]^{\frac{1}{2}},$$
$$\sigma_{33} = [\frac{1}{2}\{\sqrt{(1+4\Omega_2)^2 + 16(\Omega_1 + \delta)^2} + (1+4\Omega_2)\}]^{\frac{1}{2}},$$
$$\xi_1 = [\frac{1}{2}\{\sqrt{(1+4\Omega_2)^2 + 16(\delta - \Omega_1)^2} - (1+4\Omega_2)\}]^{\frac{1}{2}},$$
$$\xi_2 = [\frac{1}{2}\{\sqrt{(1+4\Omega_2)^2 + 16(\Omega_1 - \delta)^2} - (1+4\Omega_2)\}]^{\frac{1}{2}},$$
$$\xi_3 = [\frac{1}{2}\{\sqrt{1+4\Omega_2 + 16(\Omega_1 + \delta)^2} - (1+4\Omega_2)\}]^{\frac{1}{2}},$$
$$c_1 = a_1 + 2a_4,\ \ c_2 = b_1 + 2b_4,\ \ c_3 = a_2 + 2a_4,\ \ c_4 = b_2 + 2b_4,$$
$$c_5 = a_3 + 2a_4,\ \ c_6 = b_3 + 2b_4,\ \ c_7 = 2a_1 + a_4,\ \ c_8 = 2b_1 + b_4,$$
$$c_9 = 2a_2 + a_4,\ \ c_{10} = 2b_2 + b_4,\ \ c_{11} = 2a_3 + a_4,\ \ c_{12} = 2b_3 + b_4,$$
$$c_{0R} = \frac{1+\sigma_{11}}{2},\ \ n_{0R} = \frac{1+\sigma_{22}}{2},\ \ k_{0R} = \frac{1+\sigma_{33}}{2},$$
$$c_{0I} = \frac{\xi_1}{2},\ \ n_{0I} = \frac{\xi_2}{2},\ \ k_{0I} = \frac{\xi_3}{2},$$

$$c_{1R} = \frac{\left[\begin{array}{c} (1-2c_{0R})(3c_{0R}c_{0I}^2 - c_{0R}^3 + 2\delta c_{0R}c_{0I}) \\ -2c_{0I}\left((c_{0I}^3 - 3c_{0R}^2 c_{0I}) - \delta\left(c_{0R}^2 - c_{0I}^2\right)\right) \end{array}\right]}{[(1-2c_{0R})^2 + 4c_{0I}^2]},$$

$$c_{1I} = \frac{\left[\begin{array}{c} 2c_{0I}\left((3c_{0R}c_{0I}^2 - c_{0R}^3) + 2\delta c_{0R}c_{0I}\right) \\ +\left(c_{0I}^3 - 3c_{0R}^2 c_{0I} - \delta\left(c_{0R}^2 - c_{0I}^2\right)\right)(1-2c_{0R}) \end{array}\right]}{[(1-2c_{0R})^2 + 4c_{0I}^2]},$$

$$c_{2R} = \frac{\left[\begin{array}{c} \left\{\begin{array}{c} 3(c_{0R}^2 - c_{0I}^2)c_{1R} - 6c_{0R}c_{0I}c_{1I} \\ -2\delta c_{0I}c_{1R} - 2\delta c_{0R}c_{1I} - (c_{1R}^2 - c_{1I}^2) \end{array}\right\}(1-2c_{0R}) \\ -\left\{\begin{array}{c} 6c_{0R}c_{0I}c_{1R} + 3c_{1I}(c_{0R}^2 - c_{0I}^2) \\ -2c_{1R}c_{1I} + 2\delta c_{0I}c_{0R} - 2\delta c_{0I}c_{1I} \end{array}\right\}2c_{0I} \end{array}\right]}{[(1-2c_{0R})^2 + 4c_{0I}^2]},$$

$$c_{2I} = \frac{\left[\begin{array}{c} 2c_{0I}\left\{\begin{array}{c} 3(c_{0R}^2 - c_{0I}^2)c_{1R} - 6c_{0R}c_{0I}c_{1I} \\ -(c_{1R}^2 - c_{1I}^2)\} - 2\delta c_{0I}c_{1R} - 2\delta c_{0R}c_{1I} \end{array}\right\} \\ +(1-2c_{0R})\left\{\begin{array}{c} 6c_{0R}c_{0I}c_{1R} + 3c_{1I}(c_{0R}^2 - c_{0I}^2) \\ -2c_{1R}c_{1I} + 2\delta c_{0I}c_{0R} - 2\delta c_{0I}c_{1I} \end{array}\right\} \end{array}\right]}{[(1-2c_{0R})^2 + 4c_{0I}^2]},$$

$$n_{1R} = \frac{\left[\begin{array}{c} (1-2n_{0R})(3n_{0R}n_{0I}^2 - n_{0R}^3 + 2\delta n_{0R}n_{0I}) \\ -2n_{0I}\left((n_{0I}^3 - 3n_{0R}^2 n_{0I}) - \delta\left(n_{0R}^2 - n_{0I}^2\right)\right) \end{array}\right]}{[(1-2n_{0R})^2 + 4n_{0I}^2]},$$

$$n_{1I} = \frac{\left[\begin{array}{c} 2n_{0I}\left((3n_{0R}n_{0I}^2 - n_{0R}^3) + 2\delta n_{0R}n_{0I}\right) \\ +\left(n_{0I}^3 - 3n_{0R}^2 n_{0I} - \delta\left(n_{0R}^2 - n_{0I}^2\right)\right)(1-2n_{0R}) \end{array}\right]}{[(1-2n_{0R})^2 + 4n_{0I}^2]},$$

$$n_{2R} = \frac{\left[\begin{array}{c} \left\{\begin{array}{c} 3(n_{0R}^2 - n_{0I}^2)n_{1R} - 6n_{0R}n_{0I}n_{1I} \\ -2\delta n_{0I}n_{1R} - 2\delta n_{0R}n_{1I} - (n_{1R}^2 - n_{1I}^2) \end{array}\right\}(1 - 2n_{0R}) \\ -\left\{\begin{array}{c} 6n_{0R}n_{0I}n_{1R} + 3n_{1I}(n_{0R}^2 - n_{0I}^2) \\ -2n_{1R}n_{1I} + 2\delta n_{0I}n_{0R} - 2\delta n_{0I}n_{1I} \end{array}\right\}2n_{0I} \end{array}\right]}{[(1 - 2n_{0R})^2 + 4n_{0I}^2]},$$

$$n_{2I} = \frac{\left[\begin{array}{c} 2n_{0I}\left\{\begin{array}{c} 3(n_{0R}^2 - n_{0I}^2)n_{1R} - 6n_{0R}n_{0I}n_{1I} \\ -(n_{1R}^2 - n_{1I}^2))\} - 2\delta n_{0I}n_{1R} - 2\delta n_{0R}n_{1I} \end{array}\right\} \\ +(1 - 2n_{0R})\left\{\begin{array}{c} 6n_{0R}n_{0I}n_{1R} + 3n_{1I}(n_{0R}^2 - n_{0I}^2) \\ -2n_{1R}n_{1I} + 2\delta n_{0I}n_{0R} - 2\delta n_{0I}n_{1I} \end{array}\right\} \end{array}\right]}{[(1 - 2n_{0R})^2 + 4n_{0I}^2]},$$

$$h_0 = \frac{[1 + \sqrt{1 + 4\Omega_2}]}{2},$$

$$h_{1R} = \frac{h_0^3}{1 + 2h_0}, \quad h_{1I} = \frac{-\delta h_0^2}{1 + 2h_0},$$

$$h_{2R} = \frac{3h_0^2 h_{1R} + 2\delta h_0 h_{1R} + (h_{1R}^2 - h_{1I}^2)}{1 + 2h_0},$$

$$h_{2I} = \frac{3h_0^2 h_{1I} - 2\delta h_0 h_{1R} - 2h_{1R}h_{1I}}{1 + 2h_0},$$

$$k_{1R} = \frac{\left[\begin{array}{c} (1 - 2k_{0R})(3k_{0R}k_{0I}^2 - k_{0R}^3 - 2\delta k_{0R}k_{0I}) \\ -2k_{0I}\left((k_{0I}^3 - 3k_{0R}^2 k_{0I}) + \delta\left(k_{0R}^2 - k_{0I}^2\right)\right) \end{array}\right]}{[(1 - 2k_{0R})^2 + 4k_{0I}^2]},$$

$$k_{1I} = \frac{\left[\begin{array}{c} 2k_{0I}\left((3k_{0R}k_{0I}^2 - k_{0R}^3) - 2\delta k_{0R}k_{0I}\right) \\ +\left(k_{0I}^3 - 3k_{0R}^2 k_{0I} + \delta\left(k_{0R}^2 - k_{0I}^2\right)\right)(1 - 2k_{0R}) \end{array}\right]}{[(1 - 2k_{0R})^2 + 4k_{0I}^2]},$$

$$k_{2R} = \frac{\left[\begin{array}{c} \left\{\begin{array}{c} 3(k_{0R}^2 - k_{0I}^2)k_{1R} - 6k_{0R}k_{0I}k_{1I} \\ +2\delta k_{0I}k_{1R} + 2\delta k_{0R}k_{1I} \end{array}\right\}(1-2k_{0R}) \\ -\left\{\begin{array}{c} 6k_{0R}k_{0I}k_{1R} + 3k_{1I}(k_{0R}^2 - k_{0I}^2) \\ -2k_{1R}k_{1I} + 2\delta k_{0I}k_{0R} - 2\delta k_{0I}k_{1I} \end{array}\right\}2k_{0I} \end{array}\right]}{[(1-2k_{0R})^2 + 4k_{0I}^2]},$$

$$k_{2I} = \frac{\left[\begin{array}{c} 2k_{0I}\left\{\begin{array}{c} 3(k_{0R}^2 - k_{0I}^2)k_{1R} - 6k_{0R}k_{0I}k_{1I} \\ +2\delta k_{0I}k_{1R} + 2\delta k_{0R}k_{1I} \end{array}\right\} \\ +(1-2k_{0R})\left\{\begin{array}{c} 6k_{0R}k_{0I}k_{1R} + 3k_{1I}(k_{0R}^2 - k_{0I}^2) \\ -2\delta k_{1R}k_{0R} + 2\delta k_{0I}k_{1I} \end{array}\right\} \end{array}\right]}{[(1-2k_{0R})^2 + 4k_{0I}^2]}.$$

Making use of equations (4.41)–(4.52) in equations (4.20) and (4.21) we have for $\delta > \Omega_1$

$$q_{02} = ae^{-a_1\eta_1 + i(\delta\tau - b_1\eta_1)} + be^{-a_4\eta_1 - i(\delta\tau + b_4\eta_1)}, \tag{4.53}$$

$$\begin{aligned} q_{12} &= ae^{i\delta\tau}\left[\begin{array}{c} (A_{4R} + iA_{4I})\left\{e^{-3(a_1+ib_1)\eta_1} - e^{-(a_1+ib_1)\eta_1}\right\} \\ (A_{5R} + iA_{5I})\left\{e^{-(c_1+ic_2)\eta_1} - e^{-(a_1+ib_1)\eta_1}\right\} \end{array}\right] \\ &\quad + be^{-i\delta\tau}\left[\begin{array}{c} (A_{22R} + iA_{22I})\left\{e^{-3(a_4+ib_4)\eta_1} - e^{-(a_4+ib_4)\eta_1}\right\} \\ (A_{23R} + iA_{23I})\left\{e^{-(c_7+ic_8)\eta_1} - e^{-(a_4+ib_4)\eta_1}\right\} \end{array}\right], \end{aligned} \tag{4.54}$$

for $\delta < \Omega_1$

$$q_{02} = ae^{-a_2\eta_1 + i(\delta\tau - b_2\eta_1)} + be^{-a_4\eta_1 - i(\delta\tau + b_4\eta_1)}, \tag{4.55}$$

$$\begin{aligned} q_{12} &= ae^{i\delta\tau}\left[\begin{array}{c} (A_{10R} + iA_{10I})\left\{e^{-3(a_2+ib_2)\eta_1} - e^{-(a_2+ib_2)\eta_1}\right\} \\ +(A_{11R} + iA_{11I})\left\{e^{-(c_3+ic_4)\eta_1} - e^{-(a_2+ib_2)\eta_1}\right\} \end{array}\right] \\ &\quad + be^{-i\delta\tau}\left[\begin{array}{c} (A_{22R} + iA_{22I})\left\{e^{-3(a_4+ib_4)\eta_1} - e^{-(a_4+ib_4)\eta_1}\right\} \\ +(A_{26R} + iA_{26I})\left\{e^{-(c_9+ic_{10})\eta_1} - e^{-(a_4+ib_4)\eta_1}\right\} \end{array}\right], \end{aligned} \tag{4.56}$$

for $\delta = \Omega_1$

$$q_{02} = ae^{-a_3\eta_1 + i(\delta\tau - b_3\eta_1)} + be^{-a_4\eta_1 - i(\delta\tau + b_4\eta_1)}, \tag{4.57}$$

$$\begin{aligned} q_{12} &= ae^{i\delta\tau}\left[\begin{array}{c} (A_{16R} + iA_{16I})\left\{e^{-3(a_3+ib_3)\eta_1} - e^{-(a_3+ib_3)\eta_1}\right\} \\ +(A_{17R} + iA_{17I})\left\{e^{-(c_5+ic_6)\eta_1} - e^{-(a_3+ib_3)\eta_1}\right\} \end{array}\right] \\ &+be^{-i\delta\tau}\left[\begin{array}{c} (A_{22R} + iA_{22I})\left\{e^{-3(a_4+ib_4)\eta_1} - e^{-(a_4+ib_4)\eta_1}\right\} \\ +(A_{29R} + iA_{29I})\left\{e^{-(c_{11}+ic_{12})\eta_1} - e^{-(a_4+ib_4)\eta_1}\right\} \end{array}\right]. \end{aligned} \tag{4.58}$$

Using equations (4.11), (4.15) and (4.53)–(4.58) we have for $\delta > \Omega_1$

$$q = U_0\left[\begin{array}{c} ae^{-a_1\eta_1 + i(\delta\tau - b_1\eta_1)} + be^{-a_4\eta_1 - i(\delta\tau + b_4\eta_1)} \\ + \in^* ae^{i\delta\tau}\left[\begin{array}{c} (A_{4R} + iA_{4I})\left\{e^{-3(a_1+ib_1)\eta_1} - e^{-(a_1+ib_1)\eta_1}\right\} \\ +(A_{5R} + iA_{5I})\left\{e^{-(c_1+ic_2)\eta_1} - e^{-(a_1+ib_1)\eta_1}\right\} \end{array}\right] \\ + \in^* be^{-i\delta\tau}\left[\begin{array}{c} (A_{22R} + iA_{22I})\left\{e^{-3(a_4+ib_4)\eta_1} - e^{-(a_4+ib_4)\eta_1}\right\} \\ +(A_{23R} + iA_{23I})\left\{e^{-(c_7+ic_8)\eta_1} - e^{-(a_4+ib_4)\eta_1}\right\} \end{array}\right] \end{array}\right], \tag{4.59}$$

for $\delta < \Omega_1$

$$q = U_0\left[\begin{array}{c} ae^{-a_2\eta_1 + i(\delta\tau - b_2\eta_1)} + be^{-a_4\eta_1 - i(\delta\tau + b_4\eta_1)} \\ + \in^* ae^{i\delta\tau}\left[\begin{array}{c} (A_{10R} + iA_{10I})\left\{e^{-3(a_2+ib_2)\eta_1} - e^{-(a_2+ib_2)\eta_1}\right\} \\ +(A_{11R} + iA_{11I})\left\{e^{-(c_3+ic_4)\eta_1} - e^{-(a_2+ib_2)\eta_1}\right\} \end{array}\right] \\ + \in^* be^{-i\delta\tau}\left[\begin{array}{c} (A_{22R} + iA_{22I})\left\{e^{-3(a_4+ib_4)\eta_1} - e^{-(a_4+ib_4)\eta_1}\right\} \\ +(A_{26R} + iA_{26I})\left\{e^{-(c_9+ic_{10})\eta_1} - e^{-(a_4+ib_4)\eta_1}\right\} \end{array}\right] \end{array}\right], \tag{4.60}$$

for $\delta = \Omega_1$

$$q = U_0 \left[\begin{array}{c} ae^{-a_3\eta_1 + i(\delta t - b_3\eta_1)} + be^{-a_4\eta_1 - i(\delta t + b_4\eta_1)} \\ + \in^* ae^{i\delta t} \left[\begin{array}{c} (A_{16R} + iA_{16I}) \left\{ e^{-3(a_3+ib_3)\eta_1} - e^{-(a_3+ib_3)\eta_1} \right\} \\ + (A_{17R} + iA_{17I}) \left\{ e^{-(c_5+ic_6)\eta_1} - e^{-(a_3+ib_3)\eta_1} \right\} \end{array} \right] \\ + \in^* be^{-i\delta t} \left[\begin{array}{c} (A_{22R} + iA_{22I}) \left\{ e^{-3(a_4+ib_4)\eta_1} - e^{-(a_4+ib_4)\eta_1} \right\} \\ + (A_{29R} + iA_{29I}) \left\{ e^{-(c_{11}+ic_{12})\eta_1} - e^{-(a_4+ib_4)\eta_1} \right\} \end{array} \right] \end{array} \right]. \tag{4.61}$$

Equating the real and imaginary parts of equations (4.59)−(4.61) by taking $a = a_R + ia_I$, $b = b_R + ib_I$, we respectively get for $\delta > \Omega_1$

$$u = U_0 \left[\begin{array}{c} e^{-a_1\eta_1} \left\{ B_1^* \cos(\delta\tau - b_1\eta_1) + B_2^* \sin(\delta\tau - b_1\eta_1) \right\} \\ +e^{-3a_1\eta_1} \left\{ B_3^* \cos(\delta\tau - 3b_1\eta_1) - B_4 \sin(\delta\tau - 3b_1\eta_1) \right\} \\ +e^{-c_1\eta_1} \left\{ B_5 \cos(\delta\tau - c_2\eta_1) - B_6 \sin(\delta\tau - c_2\eta_1) \right\} \\ +e^{-a_4\eta_1} \left\{ B_7 \cos(\delta\tau + b_4\eta_1) - B_8 \sin(\delta\tau + b_4\eta_1) \right\} \\ +e^{-3a_4\eta_1} \left\{ B_9 \cos(\delta\tau + 3b_4\eta_1) + B_{10} \sin(\delta\tau + 3b_4\eta_1) \right\} \\ +e^{-c_7\eta_1} \left\{ B_{11} \cos(\delta\tau + c_8\eta_1) + B_{12} \sin(\delta\tau + c_8\eta_1) \right\} \end{array} \right], \tag{4.62}$$

$$v = U_0 \left[\begin{array}{c} e^{-a_1\eta_1} \left\{ B_1^* \sin(\delta\tau - b_1\eta_1) - B_2^* \cos(\delta\tau - b_1\eta_1) \right\} \\ +e^{-3a_1\eta_1} \left\{ B_3^* \sin(\delta\tau - 3b_1\eta_1) + B_4 \cos(\delta\tau - 3b_1\eta_1) \right\} \\ +e^{-c_1\eta_1} \left\{ B_5 \sin(\delta\tau - c_2\eta_1) + B_6 \cos(\delta\tau - c_2\eta_1) \right\} \\ -e^{-a_4\eta_1} \left\{ B_7 \sin(\delta\tau + b_4\eta_1) + B_8 \cos(\delta\tau + b_4\eta_1) \right\} \\ -e^{-3a_4\eta_1} \left\{ B_9 \sin(\delta\tau + 3b_4\eta_1) - B_{10} \cos(\delta\tau + 3b_4\eta_1) \right\} \\ -e^{-c_7\eta_1} \left\{ B_{11} \sin(\delta\tau + c_8\eta_1) - B_{12} \cos(\delta\tau + c_8\eta_1) \right\} \end{array} \right], \tag{4.63}$$

for $\delta < \Omega_1$

$$u = U_0 \begin{bmatrix} e^{-a_2\eta_1}\left\{B_{13}\cos\left(\delta\tau - b_2\eta_1\right) + B_{14}\sin\left(\delta\tau - b_2\eta_1\right)\right\} \\ +e^{-3a_2\eta_1}\left\{B_{15}\cos\left(\delta\tau - 3b_2\eta_1\right) - B_{16}\sin\left(\delta\tau - 3b_2\eta_1\right)\right\} \\ +e^{-c_3\eta_1}\left\{B_{17}\cos\left(\delta\tau - c_4\eta_1\right) - B_{18}\sin\left(\delta\tau - c_4\eta_1\right)\right\} \\ +e^{-a_4\eta_1}\left\{B_{19}\cos\left(\delta\tau + b_4\eta_1\right) - B_{20}\sin\left(\delta\tau + b_4\eta_1\right)\right\} \\ +e^{-3a_4\eta_1}\left\{B_{9}\cos\left(\delta\tau + 3b_4\eta_1\right) + B_{10}\sin\left(\delta\tau + 3b_4\eta_1\right)\right\} \\ +e^{-c_9\eta_1}\left\{B_{21}\cos\left(\delta\tau + c_{10}\eta_1\right) + B_{22}\sin\left(\delta\tau + c_{10}\eta_1\right)\right\} \end{bmatrix}, \quad (4.64)$$

$$v = U_0 \begin{bmatrix} e^{-a_2\eta_1}\left\{B_{13}\sin\left(\delta\tau - b_2\eta_1\right) - B_{14}\cos\left(\delta\tau - b_2\eta_1\right)\right\} \\ +e^{-3a_2\eta_1}\left\{B_{15}\sin\left(\delta\tau - 3b_2\eta_1\right) + B_{16}\cos\left(\delta\tau - 3b_2\eta_1\right)\right\} \\ +e^{-c_3\eta_1}\left\{B_{17}\sin\left(\delta\tau - c_4\eta_1\right) + B_{18}\cos\left(\delta\tau - c_4\eta_1\right)\right\} \\ -e^{-a_4\eta_1}\left\{B_{19}\sin\left(\delta\tau + b_4\eta_1\right) + B_{20}\cos\left(\delta\tau + b_4\eta_1\right)\right\} \\ -e^{-3a_4\eta_1}\left\{B_{9}\sin\left(\delta\tau + 3b_4\eta_1\right) - B_{10}\cos\left(\delta\tau + 3b_4\eta_1\right)\right\} \\ -e^{-c_9\eta_1}\left\{B_{21}\sin\left(\delta\tau + c_{10}\eta_1\right) - B_{22}\cos\left(\delta\tau + c_{10}\eta_1\right)\right\} \end{bmatrix}, \quad (4.65)$$

for $\delta = \Omega_1$

$$u = U_0 \begin{bmatrix} e^{-a_3\eta_1}\left\{B_{23}\cos\left(\delta\tau - b_3\eta_1\right) + B_{24}\sin\left(\delta\tau - b_3\eta_1\right)\right\} \\ +e^{-3a_3\eta_1}\left\{B_{25}\cos\left(\delta\tau - 3b_3\eta_1\right) - B_{26}\sin\left(\delta\tau - 3b_3\eta_1\right)\right\} \\ +e^{-c_5\eta_1}\left\{B_{27}\cos\left(\delta\tau - c_6\eta_1\right) - B_{28}\sin\left(\delta\tau - c_6\eta_1\right)\right\} \\ +e^{-a_4\eta_1}\left\{B_{29}\cos\left(\delta\tau + b_4\eta_1\right) - B_{30}\sin\left(\delta\tau + b_4\eta_1\right)\right\} \\ +e^{-3a_4\eta_1}\left\{B_{9}\cos\left(\delta\tau + 3b_4\eta_1\right) + B_{10}\sin\left(\delta\tau + 3b_4\eta_1\right)\right\} \\ +e^{-c_{11}\eta_1}\left\{B_{31}\cos\left(\delta\tau + c_{12}\eta_1\right) + B_{32}\sin\left(\delta\tau + c_{12}\eta_1\right)\right\} \end{bmatrix}, \quad (4.66)$$

$$v = U_0 \begin{bmatrix} e^{-a_3\eta_1}\left\{B_{23}\sin\left(\delta\tau - b_3\eta_1\right) - B_{24}\cos\left(\delta\tau - b_3\eta_1\right)\right\} \\ +e^{-3a_3\eta_1}\left\{B_{25}\sin\left(\delta\tau - 3b_3\eta_1\right) + B_{26}\cos\left(\delta\tau - 3b_3\eta_1\right)\right\} \\ +e^{-c_5\eta_1}\left\{B_{27}\sin\left(\delta\tau - c_6\eta_1\right) + B_{28}\cos\left(\delta\tau - c_6\eta_1\right)\right\} \\ -e^{-a_4\eta_1}\left\{B_{29}\sin\left(\delta\tau + b_4\eta_1\right) + B_{30}\cos\left(\delta\tau + b_4\eta_1\right)\right\} \\ -e^{-3a_4\eta_1}\left\{B_9\sin\left(\delta\tau + 3b_4\eta_1\right) - B_{10}\cos\left(\delta\tau + 3b_4\eta_1\right)\right\} \\ -e^{-c_{11}\eta_1}\left\{B_{31}\sin\left(\delta\tau + c_{12}\eta_1\right) - B_{32}\cos\left(\delta\tau + c_{12}\eta_1\right)\right\} \end{bmatrix}. \quad (4.67)$$

In equations (4.62)–(4.67)

$$B_1^* = a_R + \in^* \left[-a_R\left(A_{4R} + A_{5R}\right) + a_I\left(A_{4I} + A_{5I}\right)\right],$$

$$B_2^* = -a_I + \in^* \left[a_I\left(A_{4R} + A_{5R}\right) + a_R\left(A_{4I} + A_{5I}\right)\right],$$

$$B_3^* = \in^* \left[a_R A_{4R} - a_I A_{4I}\right], \quad B_4 = \in^* \left[a_R A_{4I} + a_I A_{4R}\right],$$

$$B_5 = \in^* \left[a_R A_{5R} - a_I A_{5I}\right], \quad B_6 = \in^* \left[a_R A_{5I} + a_I A_{5R}\right],$$

$$B_7 = b_R + \in^* \left[b_R\left(-A_{22R} - A_{23R}\right) + b_I\left(A_{22I} + A_{23I}\right)\right],$$

$$B_8 = -b_I + \in^* \left[b_I\left(A_{22R} + A_{23R}\right) + b_R\left(A_{22I} + A_{23I}\right)\right],$$

$$B_9 = \in^* \left[b_R A_{22R} - b_I A_{22I}\right], \quad B_{10} = \in^* \left[b_R A_{22I} + b_I A_{22R}\right],$$

$$B_{11} = \in^* \left[b_R A_{23R} - b_I A_{23I}\right], \quad B_{12} = \in^* \left[b_R A_{23I} + b_I A_{23R}\right],$$

$$B_{13} = a_R + \in^* \left[-a_R\left(A_{10R} + A_{11R}\right) + a_I\left(A_{10I} + A_{11I}\right)\right],$$

$$B_{14} = -a_I + \in^* \left[a_I\left(A_{10R} + A_{11R}\right) + a_R\left(A_{10I} + A_{11I}\right)\right],$$

$$B_{15} = \in^* \left[a_R A_{10R} - a_I A_{10I}\right], \quad B_{16} = \in^* \left[a_R A_{10I} + a_I A_{10R}\right],$$

$$B_{17} = \in^* \left[a_R A_{11R} - a_I A_{11I}\right], \quad B_{18} = \in^* \left[a_R A_{11I} + a_I A_{11R}\right],$$

$$B_{19} = b_R + \in^* \left[b_R\left(-A_{22R} - A_{26R}\right) + b_I\left(A_{22I} + A_{26I}\right)\right],$$

$$B_{20} = -b_I + \in^* \left[b_I\left(A_{22R} + A_{26R}\right) + b_R\left(A_{22I} + A_{26I}\right)\right],$$

$$B_{21} = \in^* [b_R A_{26R} - b_I A_{26I}], \quad B_{22} = \in^* [b_R A_{26I} + b_I A_{26R}],$$

$$B_{23} = a_R + \in^* [-a_R (A_{16R} + A_{17R}) + a_I (A_{16I} + A_{17I})],$$

$$B_{24} = -a_I + \in^* [a_I (A_{16R} + A_{17R}) + a_R (A_{16I} + A_{17I})],$$

$$B_{25} = \in^* [a_R A_{16R} - a_I A_{16I}], \quad B_{26} = \in^* [a_R A_{16I} + a_I A_{16R}],$$

$$B_{27} = \in^* [a_R A_{17R} - a_I A_{17I}], \quad B_{28} = \in^* [a_R A_{17I} + a_I A_{17R}],$$

$$B_{29} = b_R + \in^* [b_I (A_{22I} + A_{29I}) - b_R (A_{22R} + A_{29R})],$$

$$B_{30} = -b_I + \in^* [b_I (A_{22R} + A_{29R}) + b_R (A_{22I} + A_{29I})],$$

$$B_{31} = \in^* [b_R A_{29R} - b_I A_{29I}], \quad B_{32} = \in^* [b_R A_{29I} - b_I A_{29R}].$$

The most interesting part of solution is, however, the prediction of drag τ_{xz} and the lateral stress τ_{yz} at the plate and are given by

$$\tau_{xz} = \mu \frac{\partial u}{\partial z} + \alpha_1 \left(\frac{\partial^2 u}{\partial z \partial t} - W_0 \frac{\partial^2 u}{\partial z^2} \right) + 2\beta_3 \frac{\partial u}{\partial z} \left(\left(\frac{\partial u}{\partial z} \right)^2 + \left(\frac{\partial v}{\partial z} \right)^2 \right), \quad (4.68)$$

$$\tau_{yz} = \mu \frac{\partial v}{\partial z} + \alpha_1 \left(\frac{\partial^2 v}{\partial z \partial t} - W_0 \frac{\partial^2 v}{\partial z^2} \right) + 2\beta_3 \frac{\partial v}{\partial z} \left(\left(\frac{\partial u}{\partial z} \right)^2 + \left(\frac{\partial v}{\partial z} \right)^2 \right), \quad (4.69)$$

which in terms of the variables η_1 and τ become

$$\tau_{xz} = \mu \frac{W_0}{\nu} \frac{\partial u}{\partial \eta_1} + \alpha_1 \frac{W_0^3}{\nu^2} \left(\frac{\partial^2 u}{\partial \eta_1 \partial \tau} - \frac{\partial^2 u}{\partial \eta_1^2} \right) + 2\beta_3 \frac{W_0^3}{\nu^3} \frac{\partial u}{\partial \eta_1} \left(\left(\frac{\partial u}{\partial \eta_1} \right)^2 + \left(\frac{\partial v}{\partial \eta_1} \right)^2 \right), \quad (4.70)$$

$$\tau_{yz} = \mu \frac{W_0}{\nu} \frac{\partial v}{\partial \eta_1} + \alpha_1 \frac{W_0^3}{\nu^2} \left(\frac{\partial^2 v}{\partial \eta_1 \partial \tau} - \frac{\partial^2 v}{\partial \eta_1^2} \right) + 2\beta_3 \frac{W_0^3}{\nu^3} \frac{\partial v}{\partial \eta_1} \left(\left(\frac{\partial u}{\partial \eta_1} \right)^2 + \left(\frac{\partial v}{\partial \eta_1} \right)^2 \right). \quad (4.71)$$

Substituting the expressions for u and v, given by (4.62)–(4.67) in equations (4.70) and (4.71) and neglecting the coefficient of $\in^{*2}$ we get for $\delta > \Omega_1$

$$\begin{aligned}\tau_{xz} \mid_{\eta_1=0} &= \frac{\mu W_0}{\nu}\left\{K_1\cos\delta\tau + K_2\sin\delta\tau\right\} \\ &+\frac{\alpha_1 W_0^3}{\nu^2}\left\{\begin{array}{c}(K_5-K_3)\cos\delta\tau \\ +(K_6-K_4)\sin\delta\tau\end{array}\right\} \\ &+2\beta_3\frac{W_0^3}{\nu^3}\left\{\begin{array}{c}K_1\cos\delta\tau \\ +K_2\sin\delta\tau\end{array}\right\}\left[\begin{array}{c}\left\{K_1\cos\delta\tau + K_2\sin\delta\tau\right\}^2 \\ +\left\{K_7\cos\delta\tau + K_8\sin\delta\tau\right\}^2\end{array}\right],\end{aligned} \tag{4.72}$$

$$\begin{aligned}\tau_{yz} \mid_{\eta_1=0} &= \frac{\mu W_0}{\nu}\left\{K_7\cos\delta\tau + K_8\sin\delta\tau\right\} \\ &+\frac{\alpha_1 W_0^3}{\nu^2}\left\{(K_{11}-K_9)\cos\delta\tau + (K_{12}-K_{10})\sin\delta\tau\right\} \\ &+2\beta_3\frac{W_0^3}{\nu^3}\left\{K_7\cos\delta\tau + K_8\sin\delta\tau\right\}\left[\begin{array}{c}\left\{K_1\cos\delta\tau + K_2\sin\delta\tau\right\}^2 \\ +\left\{K_7\cos\delta\tau + K_8\sin\delta\tau\right\}^2\end{array}\right],\end{aligned} \tag{4.73}$$

for $\delta < \Omega_1$

$$\begin{aligned}\tau_{xz} \mid_{\eta_1=0} &= \frac{\mu W_0}{\nu}\left\{K_{13}\cos\delta\tau + K_{14}\sin\delta\tau\right\} \\ &+\frac{\alpha_1 W_0^3}{\nu^2}\left\{(K_{17}-K_{15})\cos\delta\tau + (K_{18}-K_{16})\sin\delta\tau\right\} \\ &+2\beta_3\frac{W_0^3}{\nu^3}\left\{K_{13}\cos\delta\tau + K_{14}\sin\delta\tau\right\}\left[\begin{array}{c}\left\{K_{13}\cos\delta\tau + K_{14}\sin\delta\tau\right\}^2 \\ +\left\{K_{19}\cos\delta\tau + K_{20}\sin\delta\tau\right\}^2\end{array}\right],\end{aligned} \tag{4.74}$$

$$\tau_{yz} \mid_{\eta_1=0} = \frac{\mu W_0}{\nu}\left\{K_{19}\cos\delta\tau + K_{20}\sin\delta\tau\right\}$$

$$+\frac{\alpha_1 W_0^3}{\nu^2}\left\{(K_{23}-K_{21})\cos\delta\tau+(K_{24}-K_{21})\sin\delta\tau\right\}$$
$$+2\beta_3\frac{W_0^3}{\nu^3}\left\{K_{19}\cos\delta\tau+K_{20}\sin\delta\tau\right\}\left[\begin{array}{c}\left\{K_{13}\cos\delta\tau+K_{14}\sin\delta\tau\right\}^2\\+\left\{K_{19}\cos\delta\tau+K_{20}\sin\delta\tau\right\}^2\end{array}\right], \tag{4.75}$$

for $\delta=\Omega_1$

$$\tau_{xz}\mid_{\eta_1=0}=\frac{\mu W_0}{\nu}\left\{K_{25}\cos\delta\tau+K_{26}\sin\delta\tau\right\}$$
$$+\frac{\alpha_1 W_0^3}{\nu^2}\left\{(K_{29}-K_{27})\cos\delta\tau+(K_{30}-K_{28})\sin\delta\tau\right\}$$
$$+2\beta_3\frac{W_0^3}{\nu^3}\left\{K_{25}\cos\delta\tau+K_{26}\sin\delta\tau\right\}\left[\begin{array}{c}\left\{K_{25}\cos\delta\tau+K_{26}\sin\delta\tau\right\}^2\\+\left\{K_{31}\cos\delta\tau+K_{32}\sin\delta\tau\right\}^2\end{array}\right], \tag{4.76}$$

$$\tau_{yz}\mid_{\eta_1=0}=\frac{\mu W_0}{\nu}\left\{K_{31}\cos\delta\tau+K_{32}\sin\delta\tau\right\}$$
$$+\frac{\alpha_1 W_0^3}{\nu^2}\left\{(K_{35}-K_{33})\cos\delta\tau+(K_{36}-K_{34})\sin\delta\tau\right\}$$
$$+2\beta_3\frac{W_0^3}{\nu^3}\left\{K_{31}\cos\delta\tau+K_{32}\sin\delta\tau\right\}\left[\begin{array}{c}\left\{K_{25}\cos\delta\tau+K_{26}\sin\delta\tau\right\}^2\\+\left\{K_{31}\cos\delta\tau+K_{32}\sin\delta\tau\right\}^2\end{array}\right], \tag{4.77}$$

where

$$\begin{aligned}K_1 &= U_0\left[-a_1B_1^*-b_1B_2^*+3a_1B_3^*+3b_1B_4-c_1B_5+c_2B_6-a_4B_7\right.\\&\quad\left.-b_4B_8-3a_4B_9+3b_4B_{10}-c_7B_{11}+c_8B_{12}\right],\end{aligned}$$

$$K_2=U_0\left[\begin{array}{c}-a_1B_2^*+b_1B_1^*+3a_1B_4+3b_1B_3^*+c_1B_6+c_2B_5+a_4B_8\\-b_4B_7-3a_4B_{10}+3b_4B_9-c_7B_{12}-c_8B_{11}\end{array}\right],$$

$$K_3 = U_0\left[a_1^2B_1^* + 2a_1b_1B_2^* - b_1^2B_1^* + 9a_1^2B_3^* - 18a_1b_1B_4 - 9b_1^2B_3^* + c_1^2B_5 - 2c_1c_2B_6 - c_2^2B_5 + a_4^2B_7 + 2a_4b_4B_8 - b_4^2B_7 + 9a_4^2B_9 - 18a_4b_4B_{10} - 9b_4^2B_9 + c_7^2B_{11} - 2c_7c_8B_{12} - c_8^2B_{11}\right],$$

$$K_4 = U_0\left[a_1^2B_2^* - 2a_1b_1B_1^* - b_1^2B_2^* - 9a_1^2B_4 - 18a_1b_1B_3^* + 9b_1^2B_4 - c_1^2B_6 - 2c_1c_2B_5 + c_2^2B_6 - a_4^2B_8 + 2a_4b_4B_7 + b_4^2B_8 + 9a_4^2B_{10} + 18a_4b_4B_9 - 9b_4^2B_{10} + c_7^2B_{12} + 2c_7c_8B_{11} - c_8^2B_{12}\right],$$

$$K_5 = U_0\left[\delta\left\{\begin{array}{c} -a_1B_2^* + b_1B_1^* + 3a_1B_4 + 3b_1B_3^* + c_1B_6 + c_2B_5 \\ +a_4B_8 - b_4B_7 - 3a_4B_{10} - 3b_4B_9 - c_7B_{12} - c_8B_{11} \end{array}\right\}\right],$$

$$K_6 = U_0\left[\delta\left\{\begin{array}{c} a_1B_1^* + b_1B_2^* + 3a_1B_3^* - 3b_1B_4 + c_1B_5 - c_2B_6 \\ +a_4B_7 + b_4B_8 + 3a_4B_9 - 3b_4B_{10} + c_7B_{11} - c_8B_{12} \end{array}\right\}\right],$$

$$K_7 = U_0\left[\begin{array}{c} a_1B_2^* - b_1B_1^* - 3a_1B_4 - 3b_1B_3^* - c_1B_6 - c_2B_5 \\ +a_4B_8 - b_4B_7 - 3a_4B_{10} - 3b_4B_9 - c_7B_{12} - c_8B_{11} \end{array}\right],$$

$$K_8 = U_0\left[\begin{array}{c} -a_1B_1^* - b_1B_2^* - 3a_1B_3^* + 3b_1B_4 - c_1B_5 + c_2B_6 \\ +a_4B_7 + b_4B_8 + 3a_4B_9 - 3b_4B_{10} + c_7B_{11} - c_8B_{12} \end{array}\right],$$

$$K_9 = \begin{array}{c} U_0\left[-a_1^2B_2^* + 2a_1b_1B_1^* + b_1^2B_1 + 9a_1^2B_4 + 18a_1b_1B_3^* - 9b_1^2B_4 + c_1^2B_6\right. \\ +2c_1c_2B_5 - c_2^2B_6 - a_4^2B_8 + 2a_4b_4B_7 + b_4^2B_8 + 9a_4^2B_{10} + 18a_4b_4B_9 \\ \left. -9b_4^2B_{10} + c_7^2B_{12} + 2c_7c_8B_{11} - c_8^2B_{12}\right], \end{array}$$

$$K_{10} = U_0\left[a_1^2B_1^* + 2a_1b_1B_2^* - b_1^2B_1 + 9a_1^2B_3 - 18a_1b_1B_4 - 9b_1^2B_3^* + c_1^2B_5 - 2c_1c_2B_6 - c_2^2B_5 - a_4^2B_7 - 2a_4b_4B_8 + b_4^2B_7 - 9a_4^2B_9 + 18a_4b_4B_{10} + 9b_4^2B_9 - c_7^2B_{11} + 2c_7c_8B_{12} + c_8^2B_{11}\right],$$

$$K_{11} = U_0\left[\delta\left\{\begin{array}{c} -a_1B_1^* - b_1B_2^* - 3a_1B_3^* + 3b_1B_4 - c_1B_5 + c_2B_6 \\ +a_4B_7 + b_4B_8 + 3a_4B_9 - 3b_4B_{10} + c_7B_{11} - c_8B_{12} \end{array}\right\}\right],$$

$$K_{12} = U_0 \left[\delta \left\{ \begin{array}{c} -a_1 B_2^* + b_1 B_1^* + 3a_1 B_4 + 3b_1 B_3^* + c_1 B_6 + c_2 B_5 \\ -a_4 B_8 + b_4 B_7 + 3a_4 B_{10} + 3b_4 B_9 + c_7 B_{12} + c_8 B_{11} \end{array} \right\} \right],$$

$$K_{13} = U_0 \left[\begin{array}{c} -a_2 B_{13} - b_2 B_{14} + 3a_2 B_{15} + 3b_2 B_{16} - c_3 B_{17} + c_4 B_{18} - a_4 B_{19} \\ -b_4 B_{20} - 3a_4 B_9 + 3b_4 B_{10} - c_9 B_{21} + c_{10} B_{22} \end{array} \right],$$

$$K_{14} = U_0 \left[\begin{array}{c} -a_2 B_{14} + b_2 B_{13} + 3a_2 B_{16} + 3b_2 B_{15} + c_3 B_{18} + c_4 B_{17} + a_4 B_{20} \\ -b_4 B_{19} - 3a_4 B_{10} + 3b_4 B_9 - c_9 B_{22} - c_{10} B_{21} \end{array} \right],$$

$$\begin{aligned} K_{15} &= U_0 \left[a_2^2 B_{13} + 2a_2 b_2 B_{14} - b_2^2 B_{13} + 9a_2^2 B_{15} - 18a_2 b_2 B_{16} - 9b_2^2 B_{15} \right. \\ &\quad + c_3^2 B_{17} - 2c_3 c_4 B_{18} - c_4^2 B_{17} + a_4^2 B_{19} + 2a_4 b_4 B_{20} - b_4^2 B_{19} + 9a_4^2 B_9 \\ &\quad \left. - 18a_4 b_4 B_{10} - 9b_4^2 B_9 + c_9^2 B_{21} - 2c_9 c_{10} B_{22} - c_{10}^2 B_{21} \right], \end{aligned}$$

$$\begin{aligned} K_{16} &= U_0 \left[a_2^2 B_{14} - 2a_2 b_2 B_{13} - b_2^2 B_{14} - 9a_2^2 B_{16} - 18a_2 b_2 B_{15} + 9b_2^2 B_{16} - c_3^2 B_{18} \right. \\ &\quad - 2c_3 c_4 B_{17} + c_4^2 B_{18} - a_4^2 B_{20} + 2a_4 b_4 B_{19} + b_4^2 B_{20} + 9a_4^2 B_{10} + 18a_4 b_4 B_9 \\ &\quad \left. - 9b_4^2 B_{10} + c_9^2 B_{22} - 2c_9 c_{10} B_{21} - c_{10}^2 B_{22} \right], \end{aligned}$$

$$K_{17} = U_0 \left[\delta \left\{ \begin{array}{c} -a_2 B_{14} + b_2 B_{13} + 3a_2 B_{16} + 3b_2 B_{15} + c_3 B_{18} + c_4 B_{17} \\ +a_4 B_{20} - b_4 B_{19} - 3a_4 B_{10} - 3b_4 B_9 - c_9 B_{22} - c_{10} B_{21} \end{array} \right\} \right],$$

$$K_{18} = U_0 \left[\delta \left\{ \begin{array}{c} a_2 B_{13} + b_2 B_{14} + 3a_2 B_{15} - 3b_2 B_{16} + c_3 B_{17} - c_4 B_{18} \\ +a_4 B_{19} + b_4 B_{20} + 3a_4 B_9 - 3b_4 B_{10} + c_9 B_{21} - c_{10} B_{22} \end{array} \right\} \right],$$

$$K_{19} = U_0 \left[\begin{array}{c} a_2 B_{14} - b_2 B_{13} - 3a_2 B_{16} - 3b_2 B_{15} - c_3 B_{18} - c_4 B_{17} + a_4 B_{20} \\ -b_4 B_{19} - 3a_4 B_{10} - 3b_4 B_9 - c_9 B_{22} - c_{10} B_{21} \end{array} \right],$$

$$K_{20} = U_0 \left[\begin{array}{c} -a_2 B_{13} - b_2 B_{14} - 3a_2 B_{15} + 3b_2 B_{16} - c_3 B_{17} + c_4 B_{18} + a_4 B_{19} \\ +b_4 B_{20} + 3a_4 B_9 - 3b_4 B_{10} + c_9 B_{21} - c_{10} B_{22} \end{array} \right],$$

$$K_{21} = \begin{array}{l} U_0\left[-a_2^2B_{14} + 2a_2b_2B_{13} + b_2^2B_{14} + 9a_2^2B_{16} + 18a_2b_2B_{15} - 9b_2^2B_{16}\right. \\ +c_3^2B_{18} + 2c_3c_4B_{17} - c_4^2B_{18} - a_4^2B_{20} + 2a_4b_4B_{19} + b_4^2B_{20} + 9a_4^2B_{10} \\ \left.+18a_4b_4B_9 - 9b_4^2B_{10} + c_9^2B_{22} + 2c_9c_{10}B_{21} - c_{10}^2B_{22}\right], \end{array}$$

$$\begin{aligned} K_{22} &= U_0\left[a_2^2B_{13} + 2a_2b_2B_{14} - b_2^2B_{13} + 9a_2^2B_{15} - 18a_2b_2B_{16} - 9b_2^2B_{15}\right. \\ &\quad +c_3^2B_{17} - 2c_3c_4B_{18} - c_4^2B_{17} - a_4^2B_{19} - 2a_4b_4B_{20} + b_4^2B_{19} \\ &\quad \left.-9a_4^2B_9 + 18a_4b_4B_{10} + 9b_4^2B_9 - c_9^2B_{21} + 2c_9c_{10}B_{22} + c_{10}^2B_{21}\right], \end{aligned}$$

$$K_{23} = U_0\left[\delta\left\{\begin{array}{c} -a_2B_{13} - b_2B_{14} - 3a_2B_{15} + 3b_2B_{16} - c_3B_{17} + c_4B_{18} \\ -a_4B_{19} + b_4B_{20} + 3a_4B_9 - 3b_4B_{10} + c_9B_{21} - c_{10}B_{22} \end{array}\right\}\right],$$

$$K_{24} = U_0\left[\delta\left\{\begin{array}{c} -a_2B_{14} + b_2B_{13} + 3a_2B_{16} + 3b_2B_{15} + c_3B_{18} + c_4B_{17} \\ -a_4B_{20} + b_4B_{19} + 3a_4B_{10} + 3b_4B_9 + c_9B_{22} + c_{10}B_{21} \end{array}\right\}\right],$$

$$K_{25} = U_0\left[\begin{array}{c} -a_3B_{23} - b_3B_{24} + 3a_3B_{25} + 3b_3B_{26} - c_5B_{27} + c_6B_{28} - a_4B_{29} \\ -b_4B_{30} - 3a_4B_9 + 3b_4B_{10} - c_{11}B_{31} + c_{12}B_{32} \end{array}\right],$$

$$K_{26} = U_0\left[\begin{array}{c} -a_3B_{24} + b_3B_{23} + 3a_3B_{26} + 3b_3B_{25} + c_5B_{28} + c_6B_{27} + a_4B_{30} \\ -b_4B_{29} - 3a_4B_{10} - 3b_4B_9 - c_{11}B_{32} - c_{12}B_{31} \end{array}\right],$$

$$\begin{aligned} K_{27} &= U_0\left[a_3^2B_{23} + 2a_3b_3B_{24} - b_3^2B_{23} + 9a_3^2B_{25} - 18a_3b_3B_{26} - 9b_3^2B_{25}\right. \\ &\quad +c_5^2B_{27} - 2c_5c_6B_{28} - c_6^2B_{27} + a_4^2B_{29} + 2a_4b_4B_{30} - b_4^2B_{29} \\ &\quad \left.+9a_4^2B_9 - 18a_4b_4B_{10} - 9b_4^2B_9 + c_{11}^2B_{31} - 2c_{11}c_{12}B_{32} + c_{12}^2B_{31}\right], \end{aligned}$$

$$\begin{aligned} K_{28} &= U_0\left[-a_3^2B_{24} - 2a_3b_3B_{23} - b_3^2B_{24} - 9a_3^2B_{26} - 18a_3b_3B_{25} + 9b_3^2B_{26}\right. \\ &\quad -c_5^2B_{28} - 2c_5c_6B_{27} + c_6^2B_{28} - a_4^2B_{30} + 2a_4b_4B_{29} + b_4^2B_{30} \\ &\quad \left.+9a_4^2B_{10} + 18a_4b_4B_9 - 9b_4^2B_{10} + c_{11}^2B_{32} + 2c_{11}c_{12}B_{31} - c_{12}^2B_{32}\right], \end{aligned}$$

$$K_{29} = U_0 \left[\delta \left\{ \begin{array}{c} -a_3 B_{24} + b_3 B_{23} + 3a_3 B_{26} + 3b_3 B_{25} + c_5 B_{28} + c_6 B_{27} \\ +a_4 B_{30} - b_4 B_{29} - 3a_4 B_{10} - 3b_4 B_9 - c_{11} B_{32} - c_{12} B_{31} \end{array} \right\} \right],$$

$$K_{30} = U_0 \left[\delta \left\{ \begin{array}{c} a_3 B_{23} + b_3 B_{24} + 3a_3 B_{25} - 3b_3 B_{26} + c_5 B_{27} - c_6 B_{28} \\ +a_4 B_{29} + b_4 B_{30} + 3a_4 B_9 - 3b_4 B_{10} + c_{11} B_{31} - c_{12} B_{32} \end{array} \right\} \right],$$

$$K_{31} = U_0 \left[\begin{array}{c} a_3 B_{24} - b_3 B_{23} - 3a_3 B_{26} - 3b_3 B_{25} - c_5 B_{28} - c_6 B_{27} + a_4 B_{30} \\ -b_4 B_{29} - 3a_4 B_{10} - 3b_4 B_9 - c_{11} B_{32} - c_{12} B_{31} \end{array} \right],$$

$$K_{32} = U_0 \left[\begin{array}{c} -a_3 B_{23} - b_3 B_{24} - 3a_3 B_{25} + 3b_3 B_{26} - c_5 B_{27} + c_6 B_{28} + a_4 B_{29} \\ +b_4 B_{30} - 3a_4 B_9 - 3b_4 B_{10} + c_{11} B_{31} - c_{12} B_{32} \end{array} \right],$$

$$\begin{aligned} K_{33} &= U_0 \left[-a_3^2 B_{24} + 2a_3 b_3 B_{23} + b_3^2 B_{24} + 9a_3^2 B_{26} + 18a_3 b_3 B_{25} - 9b_3^2 B_{26} \right. \\ &\quad + c_5^2 B_{28} + 2c_5 c_6 B_{27} - c_6^2 B_{28} - a_4^2 B_{30} + 2a_4 b_4 B_{29} + b_4^2 B_{30} \\ &\quad \left. + 9a_4^2 B_{10} + 18a_4 b_4 B_9 - 9b_4^2 B_{10} + c_{11}^2 B_{32} + 2c_{11} c_{12} B_{31} - c_{12}^2 B_{32} \right], \end{aligned}$$

$$\begin{aligned} K_{34} &= U_0 \left[a_3^2 B_{23} + 2a_3 b_3 B_{24} - b_3^2 B_{23} + 9a_3^2 B_{25} - 18a_3 b_3 B_{26} - 9b_3^2 B_{25} \right. \\ &\quad + c_5^2 B_{27} - 2c_5 c_6 B_{28} - c_6^2 B_{27} - a_4^2 B_{29} - 2a_4 b_4 B_{30} + b_4^2 B_{29} - 9a_4^2 B_9 \\ &\quad \left. + 18a_4 b_4 B_{10} + 9b_4^2 B_9 - c_{11}^2 B_{31} + 2c_{11} c_{12} B_{32} - c_{12}^2 B_{31} \right], \end{aligned}$$

$$K_{35} = U_0 \left[\delta \left\{ \begin{array}{c} -a_3 B_{23} - b_3 B_{24} - 3a_3 B_{25} + 3b_3 B_{26} - c_5 B_{27} + c_6 B_{28} \\ +a_4 B_{29} + b_4 B_{30} + 3a_4 B_9 - 3b_4 B_{10} + c_{11} B_{31} - c_{12} B_{32} \end{array} \right\} \right],$$

$$K_{36} = U_0 \left[\delta \left\{ \begin{array}{c} -a_3 B_{24} + b_3 B_{23} + 3a_3 B_{26} + 3b_3 B_{25} + c_5 B_{28} + c_6 B_{27} \\ -a_4 B_{30} + b_4 B_{29} + 3a_4 B_{10} + 3b_4 B_9 + c_{11} B_{32} + c_{12} B_{31} \end{array} \right\} \right].$$

4.3 Blowing Solution

In this case $W_0 < 0$. The final solution satisfying the boundary conditions are given by for $\delta > \Omega_1$

$$u = U_0 \begin{bmatrix} e^{-a_5\eta_1}\left\{J_1 \cos\left(\delta\tau - b_5\eta_1\right) + J_2 \sin\left(\delta\tau - b_5\eta_1\right)\right\} \\ +e^{-3a_5\eta_1}\left\{J_3 \cos\left(\delta\tau - 3b_5\eta_1\right) - J_4 \sin\left(\delta\tau - 3b_5\eta_1\right)\right\} \\ +e^{-f_1\eta_1}\left\{J_5 \cos\left(\delta\tau - f_2\eta_1\right) - J_6 \sin\left(\delta\tau - f_2\eta_1\right)\right\} \\ +e^{-a_7\eta_1}\left\{J_7 \cos\left(\delta\tau + b_7\eta_1\right) - J_8 \sin\left(\delta\tau + b_7\eta_1\right)\right\} \\ +e^{-3a_7\eta_1}\left\{J_9 \cos\left(\delta\tau + 3b_7\eta_1\right) + J_{10} \sin\left(\delta\tau + 3b_7\eta_1\right)\right\} \\ +e^{-f_7\eta_1}\left\{J_{11} \cos\left(\delta\tau + f_8\eta_1\right) + J_{12} \sin\left(\delta\tau + f_8\eta_1\right)\right\} \end{bmatrix}, \quad (4.78)$$

$$v = U_0 \begin{bmatrix} e^{-a_5\eta_1}\left\{J_1 \sin\left(\delta\tau - b_5\eta_1\right) - J_2 \cos\left(\delta\tau - b_5\eta_1\right)\right\} \\ +e^{-3a_5\eta_1}\left\{J_3 \sin\left(\delta\tau - 3b_5\eta_1\right) + J_4 \cos\left(\delta\tau - 3b_5\eta_1\right)\right\} \\ +e^{-f_1\eta_1}\left\{J_5 \sin\left(\delta\tau - f_2\eta_1\right) + J_6 \cos\left(\delta\tau - f_2\eta_1\right)\right\} \\ -e^{-a_7\eta_1}\left\{J_7 \sin\left(\delta\tau + b_7\eta_1\right) + J_8 \cos\left(\delta\tau + b_7\eta_1\right)\right\} \\ -e^{-3a_7\eta_1}\left\{J_9 \sin\left(\delta\tau + 3b_7\eta_1\right) - J_{10} \cos\left(\delta\tau + 3b_7\eta_1\right)\right\} \\ -e^{-f_7\eta_1}\left\{J_{11} \sin\left(\delta\tau + f_8\eta_1\right) - J_{12} \cos\left(\delta\tau + f_8\eta_1\right)\right\} \end{bmatrix}, \quad (4.79)$$

for $\delta < \Omega_1$

$$u = U_0 \begin{bmatrix} e^{-a_6\eta_1}\left\{J_{13} \cos\left(\delta\tau - b_6\eta_1\right) + J_{14} \sin\left(\delta\tau - b_6\eta_1\right)\right\} \\ +e^{-3a_6\eta_1}\left\{J_{15} \cos\left(\delta\tau - 3b_6\eta_1\right) - B_{16} \sin\left(\delta\tau - 3b_6\eta_1\right)\right\} \\ +e^{-f_3\eta_1}\left\{J_{17} \cos\left(\delta\tau - f_4\eta_1\right) - J_{18} \sin\left(\delta\tau - f_4\eta_1\right)\right\} \\ +e^{-a_8\eta_1}\left\{J_{19} \cos\left(\delta\tau + b_8\eta_1\right) - J_{20} \sin\left(\delta\tau + b_8\eta_1\right)\right\} \\ +e^{-3a_7\eta_1}\left\{J_9 \cos\left(\delta\tau + 3b_7\eta_1\right) + J_{10} \sin\left(\delta\tau + 3b_7\eta_1\right)\right\} \\ +e^{-f_9\eta_1}\left\{J_{21} \cos\left(\delta\tau + f_{10}\eta_1\right) + J_{22} \sin\left(\delta\tau + f_{10}\eta_1\right)\right\} \end{bmatrix}, \quad (4.80)$$

$$v = U_0 \begin{bmatrix} e^{-a_6\eta_1}\left\{J_{13}\sin\left(\delta\tau - b_6\eta_1\right) - J_{14}\cos\left(\delta\tau - b_6\eta_1\right)\right\} \\ +e^{-3a_6\eta_1}\left\{J_{15}\sin\left(\delta\tau - 3b_6\eta_1\right) + J_{16}\cos\left(\delta\tau - 3b_6\eta_1\right)\right\} \\ +e^{-f_3\eta_1}\left\{J_{17}\sin\left(\delta\tau - f_4\eta_1\right) + J_{18}\cos\left(\delta\tau - f_4\eta_1\right)\right\} \\ -e^{-a_8\eta_1}\left\{J_{19}\sin\left(\delta\tau + b_8\eta_1\right) + J_{20}\cos\left(\delta\tau + b_8\eta_1\right)\right\} \\ -e^{-3a_7\eta_1}\left\{J_9\sin\left(\delta\tau + 3b_7\eta_1\right) - J_{10}\cos\left(\delta\tau + 3b_7\eta_1\right)\right\} \\ -e^{-f_9\eta_1}\left\{J_{21}\sin\left(\delta\tau + f_{10}\eta_1\right) - J_{22}\cos\left(\delta\tau + f_{10}\eta_1\right)\right\} \end{bmatrix}, \quad (4.81)$$

for $\delta = \Omega_1$

$$u = U_0 \begin{bmatrix} e^{-a_3\eta_1}\left\{J_{23}\cos\left(\delta\tau - b_3\eta_1\right) + J_{24}\sin\left(\delta\tau - b_3\eta_1\right)\right\} \\ +e^{-3a_3\eta_1}\left\{J_{25}\cos\left(\delta\tau - 3b_3\eta_1\right) - J_{26}\sin\left(\delta\tau - 3b_3\eta_1\right)\right\} \\ +e^{-f_5\eta_1}\left\{J_{27}\cos\left(\delta\tau - f_6\eta_1\right) - J_{28}\sin\left(\delta\tau - f_6\eta_1\right)\right\} \\ +e^{-a_7\eta_1}\left\{J_{29}\cos\left(\delta\tau + b_7\eta_1\right) - J_{30}\sin\left(\delta\tau + b_7\eta_1\right)\right\} \\ +e^{-3a_7\eta_1}\left\{J_9\cos\left(\delta\tau + 3b_7\eta_1\right) + J_{10}\sin\left(\delta\tau + 3b_7\eta_1\right)\right\} \\ +e^{-f_{11}\eta_1}\left\{J_{31}\cos\left(\delta\tau + f_{12}\eta_1\right) + J_{32}\sin\left(\delta\tau + f_{12}\eta_1\right)\right\} \end{bmatrix}, \quad (4.82)$$

$$v = U_0 \begin{bmatrix} e^{-a_3\eta_1}\left\{J_{23}\sin\left(\delta\tau - b_3\eta_1\right) - J_{24}\cos\left(\delta\tau - b_3\eta_1\right)\right\} \\ +e^{-3a_3\eta_1}\left\{J_{25}\sin\left(\delta\tau - 3b_3\eta_1\right) + J_{26}\cos\left(\delta\tau - 3b_3\eta_1\right)\right\} \\ +e^{-f_5\eta_1}\left\{J_{27}\sin\left(\delta\tau - f_6\eta_1\right) + J_{28}\cos\left(\delta\tau - f_6\eta_1\right)\right\} \\ -e^{-a_7\eta_1}\left\{J_{29}\sin\left(\delta\tau + b_7\eta_1\right) + J_{30}\cos\left(\delta\tau + b_7\eta_1\right)\right\} \\ -e^{-3a_7\eta_1}\left\{J_9\sin\left(\delta\tau + 3b_7\eta_1\right) - J_{10}\cos\left(\delta\tau + 3b_7\eta_1\right)\right\} \\ -e^{-f_{11}\eta_1}\left\{J_{31}\sin\left(\delta\tau + f_{12}\eta_1\right) - J_{32}\cos\left(\delta\tau + f_{12}\eta_1\right)\right\} \end{bmatrix}, \quad (4.83)$$

$$J_1 = a_R + \in^* \left[-a_R\left(H_{4R} + H_{5R}\right) + a_I\left(H_{4I} + H_{5I}\right)\right],$$

$$J_2 = -a_I + \in^* \left[a_I\left(H_{4R} + H_{5R}\right) + a_R\left(H_{4I} + H_{5I}\right)\right],$$

$$J_3 = \in^* \left[a_R H_{4R} - a_I H_{4I}\right], \quad J_4 = \in^* \left[a_R H_{4I} + a_I H_{4R}\right],$$

$$J_5 = \in^* \left[a_R H_{5R} - a_I H_{5I}\right], \quad J_6 = \in^* \left[a_R H_{5I} + a_I H_{5R}\right],$$

$$J_7 = b_R + \in^* \left[b_R \left(-H_{22R} - H_{23R} \right) + b_I \left(H_{22I} + H_{23I} \right) \right],$$

$$J_8 = -b_I + \in^* \left[b_I \left(H_{22R} + H_{23R} \right) + b_R \left(H_{22I} + H_{23I} \right) \right],$$

$$J_9 = \in^* \left[b_R H_{22R} - b_I H_{22I} \right], \quad J_{10} = \in^* \left[b_R H_{22I} + b_I H_{22R} \right],$$

$$J_{11} = \in^* \left[b_R H_{23R} - b_I H_{23I} \right], \quad J_{12} = \in^* \left[b_R H_{23I} + b_I H_{23R} \right],$$

$$J_{13} = a_R + \in^* \left[-a_R \left(H_{10R} + H_{11R} \right) + a_I \left(H_{10I} + H_{11I} \right) \right],$$

$$J_{14} = -a_I + \in^* \left[a_I \left(H_{10R} + H_{11R} \right) + a_R \left(H_{10I} + H_{11I} \right) \right],$$

$$J_{15} = \in^* \left[a_R H_{10R} - a_I H_{10I} \right], \quad J_{16} = \in^* \left[a_R H_{10I} + a_I H_{10R} \right],$$

$$J_{17} = \in^* \left[a_R H_{11R} - a_I H_{11I} \right], \quad J_{18} = \in^* \left[a_R H_{11I} + a_I H_{11R} \right],$$

$$J_{19} = b_R + \in^* \left[b_R \left(-H_{22R} - H_{26R} \right) + b_I \left(H_{22I} + H_{26I} \right) \right],$$

$$J_{20} = -b_I + \in^* \left[b_I \left(H_{22R} + H_{26R} \right) + b_R \left(H_{22I} + H_{26I} \right) \right],$$

$$J_{21} = \in^* \left[b_R H_{26R} - b_I H_{26I} \right], \quad J_{22} = \in^* \left[b_R H_{26I} + b_I H_{26R} \right],$$

$$J_{23} = a_R + \in^* \left[-a_R \left(H_{16R} + H_{17R} \right) + a_I \left(H_{16I} + H_{17I} \right) \right],$$

$$J_{24} = -a_I + \in^* \left[a_I \left(H_{16R} + H_{17R} \right) + a_R \left(H_{16I} + H_{17I} \right) \right],$$

$$J_{25} = \in^* \left[a_R H_{16R} - a_I H_{16I} \right], \quad J_{26} = \in^* \left[a_R H_{16I} + a_I H_{16R} \right],$$

$$J_{27} = \in^* \left[a_R H_{17R} - a_I H_{17I} \right], \quad J_{28} = \in^* \left[a_R H_{17I} + a_I H_{17R} \right],$$

$$J_{29} = b_R + \in^* \left[b_R \left(-H_{22R} - H_{29R} \right) + b_I \left(H_{22I} + H_{29I} \right) \right],$$

$$J_{30} = -b_I + \in^* \left[b_I \left(H_{22R} + H_{29R} \right) + b_R \left(H_{22I} + H_{29I} \right) \right],$$

$$J_{31} = \in^* \left[b_R H_{29R} - b_I H_{29I} \right], \quad J_{32} = \in^* \left[b_R H_{29I} - b_I H_{29R} \right].$$

$$a_5 = d_{0R} - \lambda_2 d_{1R} - \lambda_2^2 d_{2R}, \quad a_6 = p_{0R} - \lambda_2 p_{1R} - \lambda_2^2 p_{2R},$$

$$a_7 = s_{0R} - \lambda_2 s_{1R} - \lambda_2^2 s_{2R},$$

$$b_5 = c_{0I} - \lambda_2 d_{1I} - \lambda_2^2 d_{2I},\ b_6 = n_{0I} - \lambda_2 p_{1I} - \lambda_2^2 p_{2I},$$

$$b_7 = k_{0I} - \lambda_2 s_{1I} - \lambda_2^2 s_{2I},$$

$$f_1 = a_5 + 2a_7,\quad f_2 = b_5 + 2b_7,\quad f_3 = a_6 + 2a_7,\quad f_4 = b_6 + 2b_7,$$

$$f_5 = a_3 + 2a_7,\quad f_6 = b_3 + 2b_7,\quad f_7 = 2a_5 + a_7,\quad f_8 = 2b_5 + b_7,$$

$$f_9 = 2a_5 + a_8,\quad f_{10} = 2b_6 + b_8,\quad f_{11} = 2a_3 + a_7,\quad f_{12} = 2b_3 + b_7,$$

$$d_{0R} = \frac{\sigma_{11} - 1}{2},\quad p_{0R} = \frac{\sigma_{22} - 1}{2},\quad s_{0R} = \frac{\sigma_{33} - 1}{2},$$

$$d_{1R} = \frac{\left[\begin{array}{c} (1 + 2d_{0R})(3d_{0R}c_{0I}^2 - d_{0R}^3 + 2\delta d_{0R}c_{0I}) \\ +2c_{0I}\left((d_{0I}^3 - 3d_{0R}^2 c_{0I}) - \delta\left(d_{0R}^2 - c_{0I}^2\right)\right) \end{array}\right]}{[(1 + 2d_{0R})^2 + 4c_{0I}^2]},$$

$$d_{1I} = \frac{\left[\begin{array}{c} 2c_{0I}\left((3d_{0R}c_{0I}^2 - d_{0R}^3) + 2\delta d_{0R}c_{0I}\right) \\ +\left(c_{0I}^3 - 3d_{0R}^2 c_{0I} - \delta\left(d_{0R}^2 - c_{0I}^2\right)\right)(1 + 2d_{0R}) \end{array}\right]}{[(1 + 2d_{0R})^2 + 4c_{0I}^2]},$$

$$d_{2R} = \frac{\left[\begin{array}{c} \left\{\begin{array}{c} 3(d_{0R}^2 - c_{0I}^2)d_{1R} - 6d_{0R}c_{0I}d_{1I} \\ -2\delta c_{0I}d_{1R} - 2\delta d_{0R}d_{1I} - (d_{1R}^2 - d_{1I}^2) \end{array}\right\}(1 + 2d_{0R}) \\ +\left\{\begin{array}{c} 6d_{0R}c_{0I}d_{1R} + 3d_{1I}(d_{0R}^2 - c_{0I}^2) \\ +2d_{1R}d_{1I} + 2\delta d_{0R}d_{1R} - 2\delta c_{0I}d_{1I} \end{array}\right\}2c_{0I} \end{array}\right]}{[(1 + 2d_{0R})^2 + 4c_{0I}^2]},$$

$$d_{2I} = \frac{\left[\begin{array}{c} -2c_{0I}\left\{\begin{array}{c} 3(d_{0R}^2 - c_{0I}^2)d_{1R} - 6d_{0R}c_{0I}d_{1I} \\ +(d_{1R}^2 - d_{1I}^2)\} - 2\delta d_{0I}d_{1R} - 2\delta d_{0R}d_{1I} \end{array}\right\} \\ +(1 + 2d_{0R})\left\{\begin{array}{c} 6d_{0R}c_{0I}d_{1R} + 3d_{1I}(d_{0R}^2 - c_{0I}^2) \\ +2d_{1R}d_{1I} + 2\delta d_{1R}d_{0R} - 2\delta c_{0I}d_{1I} \end{array}\right\} \end{array}\right]}{[(1 + 2d_{0R})^2 + 4c_{0I}^2]},$$

$$p_{1R} = \frac{\left[\begin{array}{c} (1+2p_{0R})(3p_{0R}n_{0I}^2 - p_{0R}^3 + 2\delta p_{0R}n_{0I}) \\ +2n_{0I}\left((n_{0I}^3 - 3p_{0R}^2 n_{0I}) - \delta\left(p_{0R}^2 - n_{0I}^2\right)\right) \end{array}\right]}{[(1+2p_{0R})^2 + 4n_{0I}^2]},$$

$$p_{1I} = \frac{\left[\begin{array}{c} -2n_{0I}\left((3p_{0R}n_{0I}^2 - p_{0R}^3) + 2\delta p_{0R}n_{0I}\right) \\ +\left(n_{0I}^3 - 3p_{0R}^2 n_{0I} - \delta\left(p_{0R}^2 - n_{0I}^2\right)\right)(1+2p_{0R}) \end{array}\right]}{[(1+2p_{0R})^2 + 4n_{0I}^2]},$$

$$p_{2R} = \frac{\left[\begin{array}{c} \left\{\begin{array}{c} 3(p_{0R}^2 - n_{0I}^2)p_{1R} - 6p_{0R}n_{0I}p_{1I} \\ -2\delta n_{0I}p_{1R} - 2\delta p_{0R}p_{1I} - (p_{1R}^2 - p_{1I}^2) \end{array}\right\}(1+2p_{0R}) \\ +\left\{\begin{array}{c} 6p_{0R}n_{0I}p_{1R} + 3p_{1I}(p_{0R}^2 - n_{0I}^2) \\ -2p_{1R}p_{1I} + 2\delta n_{0I}p_{0R} - 2\delta p_{0I}p_{1I} \end{array}\right\}2n_{0I} \end{array}\right]}{[(1+2p_{0R})^2 + 4n_{0I}^2]},$$

$$p_{2I} = \frac{\left[\begin{array}{c} -2n_{0I}\left\{\begin{array}{c} 3(p_{0R}^2 - n_{0I}^2)p_{1R} - 6p_{0R}n_{0I}p_{1I} \\ -(p_{1R}^2 - p_{1I}^2)\} - 2\delta n_{0I}p_{1R} - 2\delta p_{0R}p_{1I} \end{array}\right\} \\ +(1+2p_{0R})\left\{\begin{array}{c} 6p_{0R}n_{0I}p_{1R} + 3p_{1I}(p_{0R}^2 - n_{0I}^2) \\ -2p_{1R}p_{1I} + 2\delta n_{0I}p_{0R} - 2\delta n_{0I}p_{1I} \end{array}\right\} \end{array}\right]}{[(1+2p_{0R})^2 + 4n_{0I}^2]},$$

$$s_{1R} = \frac{\left[\begin{array}{c} (1+2s_{0R})(3s_{0R}k_{0I}^2 - s_{0R}^3 + 2\delta s_{0R}k_{0I}) \\ +2k_{0I}\left((s_{0I}^3 - 3s_{0R}^2 k_{0I}) - \delta\left(s_{0R}^2 - k_{0I}^2\right)\right) \end{array}\right]}{[(1+2s_{0R})^2 + 4k_{0I}^2]},$$

$$s_{1I} = \frac{\left[\begin{array}{c} -2k_{0I}\left((3s_{0R}k_{0I}^2 - s_{0R}^3) + 2\delta s_{0R}k_{0I}\right) \\ +\left(k_{0I}^3 - 3s_{0R}^2 k_{0I} - \delta\left(s_{0R}^2 - k_{0I}^2\right)\right)(1+2s_{0R}) \end{array}\right]}{[(1+2s_{0R})^2 + 4k_{0I}^2]},$$

$$s_{2R} = \frac{\left[\begin{array}{c} \left\{\begin{array}{c} 3(s_{0R}^2 - k_{0I}^2)s_{1R} + 6s_{0R}k_{0I}s_{1I} \\ -2\delta k_{0I}s_{1R} - 2\delta s_{0R}s_{1I} \end{array}\right\}(1+2s_{0R}) \\ +\left\{\begin{array}{c} 6s_{0R}k_{0I}s_{1R} + 3s_{1I}(s_{0R}^2 - k_{0I}^2) \\ -2s_{1R}s_{1I} + 2\delta k_{0I}s_{0R} - 2\delta k_{0I}s_{1I} \end{array}\right\}2s_{0I} \end{array}\right]}{[(1+2s_{0R})^2 + 4k_{0I}^2]},$$

$$s_{2I} = \frac{\left[\begin{array}{c} -2s_{0I}\left\{\begin{array}{c} 3(s_{0R}^2 - k_{0I}^2)s_{1R} - 6s_{0R}k_{0I}s_{1I} \\ -2\delta k_{0I}s_{1R} + 2\delta s_{0R}s_{1I} \end{array}\right\} \\ +(1+2s_{0R})\left\{\begin{array}{c} 6s_{0R}k_{0I}s_{1R} + 3s_{1I}(s_{0R}^2 - k_{0I}^2) \\ +2\delta s_{1R}s_{0R} - 2\delta k_{0I}s_{1I} \end{array}\right\} \end{array}\right]}{[(1+2s_{0R})^2 + 4k_{0I}^2]},$$

$$\begin{aligned} H_{1R} &= -2\left(b_R^2 - b_I^2\right)\left[\begin{array}{c} \{a_5(b_7^2 - a_7^2) + 2b_5a_7b_7\}(f_1^2 - f_2^2) \\ +2\{b_5(a_7^2 - b_7^2) - 2a_5a_7b_7\}(f_1f_2) \end{array}\right] \\ &\quad +4b_Rb_I\left[\begin{array}{c} \{b_5(a_7^2 - b_7^2) - 2a_5a_7b_7\}(f_1^2 - f_2^2) \\ +2\{a_5(b_7^2 - a_7^2) + 2b_5a_7b_7\}(f_1f_2) \end{array}\right], \end{aligned}$$

$$\begin{aligned} H_{1I} &= 2\left(b_R^2 - b_I^2\right)\left[\begin{array}{c} \{b_5(a_7^2 - b_7^2) - 2a_5a_7b_7\}(f_1^2 - f_2^2) \\ +2\{a_5(b_7^2 - a_7^2) + 2b_5a_7b_7\}(f_1f_2) \end{array}\right] \\ &\quad -4b_Rb_I\left[\begin{array}{c} \{a_5(b_7^2 - a_7^2) + 2b_5a_7b_7\}(f_1^2 - f_2^2) \\ +2\{b_5(a_7^2 - b_7^2) - 2a_5a_7b_7\}(f_1f_2) \end{array}\right], \end{aligned}$$

$$\begin{aligned} H_{2R} &= 3b_5\left[27\lambda_2b_5^3 - 81a_5^2b_5\lambda_2 - 9\delta\lambda_2\left(a_5^2 - b_5^2\right) + 18a_5b_5 + 3b_5 - (\delta - \Omega_1)\right] \\ &\quad -3a_5\left[-27\lambda_2a_5^3 + 81a_5b_5^2\lambda_2 - 9\left(a_5^2 - b_5^2\right) + 18\delta\lambda_2a_5b_5 + 3a_5 - \Omega_2\right], \end{aligned}$$

$$\begin{aligned} H_{2I} &= -3a_5\left[27\lambda_2b_5^3 - 81a_5^2b_5\lambda_2 - 9\delta\lambda_2\left(a_5^2 - b_5^2\right) + 18a_5b_5 + 3b_5 - (\delta - \Omega_1)\right] \\ &\quad -3b_5\left[-27\lambda_2a_5^3 + 81a_5b_5^2\lambda_2 + 9\left(a_5^2 - b_5^2\right) + 18a_5b_5\delta\lambda + 3a_5 - \Omega_2\right], \end{aligned}$$

$$
\begin{aligned}
H_{3R} &= -f_1\left[-\lambda_2 f_1^3 + 3\lambda_2 f_1 f_2^2 + 2f_1 f_2\delta\lambda_2 + f_1 - \Omega_2 - \left(f_1^2 - f_2^2\right)\right] \\
&\quad + f_2\left[\lambda_2 f_2^3 - 3f_1^2 f_2\lambda_2 - \delta\lambda_2\left(f_1^2 - f_2^2\right) + 2f_1 f_2 + f_2 - \left(\delta - \Omega_1\right)\right],
\end{aligned}
$$

$$
\begin{aligned}
H_{3I} &= -f_1\left[\lambda_2 f_2^3 - 3f_1^2 f_2\lambda_2 - \delta\lambda_2\left(f_1^2 - f_2^2\right) + 2f_1 f_2 + f_2 - \left(\delta - \Omega_1\right)\right] \\
&\quad - f_2\left[-\lambda_2 f_1^3 + 3\lambda f_1 f_2^2 + 2f_1 f_2\delta\lambda_2 + f_1 - \Omega_2 + \left(f_1^2 - f_2^2\right)\right],
\end{aligned}
$$

$$
H_{4R} = \frac{H_R H_{2R} + H_I H_{2I}}{H_{2R}^2 + H_{2I}^2}, \quad H_{4I} = \frac{H_I H_{2R} - H_R H_{2I}}{H_{2R}^2 + H_{2I}^2},
$$

$$
H_{5R} = \frac{H_{1R} H_{3R} + H_{1I} H_{3I}}{H_{3R}^2 + H_{3I}^2}, \quad H_{5I} = \frac{H_{1I} H_{3R} - H_{1R} H_{3I}}{H_{3R}^2 + H_{3I}^2},
$$

$$
\begin{aligned}
H_{6R} &= -\left(a_R^2 - a_I^2\right)\left[9\left(a_6^2 - b_6^2\right)\left(3a_6 b_6^2 - a_6^3\right) + 18a_6 b_6\left(3a_6^2 b_6 - b_6^3\right)\right] \\
&\quad + 4a_R a_I\left[9\left(a_6^2 - b_6^2\right)\left(b_6^3 - 3a_6^2 b_6\right) - 18a_6 b_6\left(a_6^3 - 3a_6 b_6^2\right)\right],
\end{aligned}
$$

$$
\begin{aligned}
H_{6I} &= -\left(a_R^2 - a_I^2\right)\left[9\left(a_6^2 - b_6^2\right)\left(b_6^3 - 3a_6^2 b_6\right) - 18a_6 b_6\left(a_6^3 - 3a_6 b_6^2\right)\right] \\
&\quad - 4a_R a_I\left[9\left(a_6^2 - b_6^2\right)\left(3a_6 b_6^2 - a_6^3\right) + 18a_6 b_6\left(3a_6^2 b_6 - b_6^3\right)\right],
\end{aligned}
$$

$$
\begin{aligned}
H_{7R} &= -2\left(b_R^2 - b_I^2\right)\left[\begin{array}{c} \left\{a_6\left(b_7^2 - a_7^2\right) + 2b_6 a_7 b_7\right\}\left(f_3^2 - f_4^2\right) \\ +2\left\{b_6\left(a_7^2 - b_7^2\right) - 2a_6 a_7 b_7\right\}\left(f_3 f_4\right) \end{array}\right] \\
&\quad + 4b_R b_I\left[\begin{array}{c} \left\{b_6\left(a_7^2 - b_7^2\right) + 2a_6 a_7 b_7\right\}\left(f_3^2 - f_4^2\right) \\ +2\left\{a_6\left(b_7^2 - a_7^2\right) + 2b_6 a_7 b_7\right\}\left(f_3 f_4\right) \end{array}\right],
\end{aligned}
$$

$$
\begin{aligned}
H_{7I} &= 2\left(b_R^2 - b_I^2\right)\left[\begin{array}{c} \left\{b_6\left(a_7^2 - b_7^2\right) + 2a_6 a_7 b_7\right\}\left(f_3^2 - f_4^2\right) \\ +2\left\{a_6\left(b_7^2 - a_7^2\right) + 2b_6 a_7 b_7\right\}\left(f_3 f_4\right) \end{array}\right] \\
&\quad - 4b_R b_I\left[\begin{array}{c} \left\{a_6\left(b_7^2 - a_7^2\right) + 2b_6 a_7 b_7\right\}\left(f_3^2 - f_4^2\right) \\ -2\left\{b_6\left(a_7^2 - b_7^2\right) + 2a_6 a_7 b_7\right\}\left(f_3 f_4\right) \end{array}\right],
\end{aligned}
$$

$$
\begin{aligned}
H_{8R} &= 3b_6\left[27\lambda_2 b_6^3 - 81a_6^2 b_6\lambda_2 - 9\delta\lambda_2\left(a_6^2 - b_6^2\right) + 18a_6 b_6 + 3b_6 - (\delta - \Omega_1)\right] \\
&\quad -3a_6\left[-27\lambda_2 a_6^3 + 81a_6 b_6^2\lambda_2 + 9\left(a_6^2 - b_6^2\right) + 18a_6 b_6\delta\lambda_2 + 3a_6 - \Omega_2\right],
\end{aligned}
$$

$$
\begin{aligned}
H_{8I} &= -3a_6\left[27\lambda_2 b_6^3 - 81a_6^2 b_6\lambda_2 - 9\delta\lambda_2\left(a_6^2 - b_6^2\right) + 18a_6 b_6 + 3b_6 - (\delta - \Omega_1)\right] \\
&\quad -3b_6\left[-27\lambda_2 a_6^3 + 81a_6 b_6^2\lambda_2 + 9\left(a_6^2 - b_6^2\right) + 18a_6 b_6\delta\lambda_2 + 3a_6 - \Omega_2\right],
\end{aligned}
$$

$$
\begin{aligned}
H_{9R} &= -f_3\left[-\lambda_2 f_3^3 + 3\lambda_2 f_3 f_4^2 + 2f_3 f_4\delta\lambda_2 + f_3 - \Omega_2 + \left(f_3^2 - f_4^2\right)\right] \\
&\quad +f_4\left[\lambda_2 f_4^3 - 3f_3^2 f_4\lambda_2 - \delta\lambda_2\left(f_3^2 - f_4^2\right) + 2f_3 f_4 + f_4 - (\delta - \Omega_1)\right],
\end{aligned}
$$

$$
\begin{aligned}
H_{9I} &= -f_3\left[\lambda_2 f_4^3 - 3f_3^2 f_4\lambda_2 - \delta\lambda_2\left(f_3^2 - f_4^2\right) + 2f_3 f_4 + f_4 - (\delta - \Omega_1)\right] \\
&\quad -f_4\left[-\lambda_2 f_3^3 + 3\lambda_2 f_3 f_4^2 + 2f_3 f_4\delta\lambda_2 + f_3 - \Omega_2 + \left(f_3^2 - f_4^2\right)\right],
\end{aligned}
$$

$$
H_{10R} = \frac{H_{6R}H_{8R} + H_{6I}H_{8I}}{H_{8R}^2 + H_{8I}^2}, \quad H_{10I} = \frac{H_{6I}H_{8R} - H_{6R}H_{8I}}{H_{8R}^2 + H_{8I}^2},
$$

$$
H_{11R} = \frac{H_{7R}H_{9R} + H_{7I}H_{9I}}{H_{9R}^2 + H_{9I}^2}, \quad H_{11I} = \frac{H_{7I}H_{9R} - H_{7R}H_{9I}}{H_{9R}^2 + H_{9I}^2},
$$

$$
\begin{aligned}
H_{12R} &= -\left(a_R^2 - a_I^2\right)\left[9\left(a_3^2 - b_3^2\right)\left(3a_3 b_3^2 - a_3^3\right) + 18a_3 b_3\left(3a_3^2 b_3 - b_3^3\right)\right] \\
&\quad +4a_R a_I\left[9\left(a_3^2 - b_3^2\right)\left(b_3^3 - 3a_3^2 b_3\right) - 18a_3 b_3\left(a_3^3 - 3a_3 b_3^2\right)\right],
\end{aligned}
$$

$$
\begin{aligned}
H_{12I} &= -\left(a_R^2 - a_I^2\right)\left[9\left(a_3^2 - b_3^2\right)\left(b_3^3 - 3a_3^2 b_3\right) - 18a_3 b_3\left(a_3^3 - 3a_3 b_3^2\right)\right] \\
&\quad -4a_R a_I\left[9\left(a_3^2 - b_3^2\right)\left(3a_3 b_3^2 - a_3^3\right) + 18a_3 b_3\left(3a_3^2 b_3 - b_3^3\right)\right],
\end{aligned}
$$

$$
\begin{aligned}
H_{13R} &= -4\left(b_R^2 - b_I^2\right)\left[\begin{array}{c}\left\{a_3\left(b_7^2 - a_7^2\right) + 2b_3 a_7 b_7\right\}\left(f_5^2 - f_6^2\right) \\ -2\left\{b_3\left(a_7^2 - b_7^2\right) + 2a_3 a_7 b_7\right\}\left(f_5 f_6\right)\end{array}\right] \\
&\quad +4b_R b_I\left[\begin{array}{c}\left\{b_3\left(a_7^2 - b_7^2\right) + 2a_3 a_7 b_7\right\}\left(f_5^2 - f_6^2\right) \\ -2\left\{a_3\left(b_7^2 - a_7^2\right) - 2b_3 a_7 b_7\right\}\left(f_5 f_6\right)\end{array}\right],
\end{aligned}
$$

$$H_{13I} = -2\left(b_R^2 - b_I^2\right)\begin{bmatrix} \{b_3\left(a_7^2 - b_7^2\right) + 2a_3a_7b_7\}\left(f_5^2 - f_6^2\right) \\ -2\{a_3\left(b_7^2 - a_7^2\right) - 2b_3a_7b_7\}\left(f_5f_6\right) \end{bmatrix}$$
$$-4b_Rb_I\begin{bmatrix} \{a_3\left(b_7^2 - a_7^2\right) + 2b_3a_7b_7\}\left(f_5^2 - f_6^2\right) \\ -2\{b_3\left(a_7^2 - b_7^2\right) + 2a_3a_7b_7\}\left(f_5f_6\right) \end{bmatrix},$$

$$H_{14R} = 3b_3\left[27\lambda_2b_3^3 - 81a_7^2b_7\lambda_2 - 9\delta\lambda_2\left(a_3^2 - b_3^2\right) + 18a_3b_3 + 3b_7\right]$$
$$-3a_7\left[-27\lambda_2a_3^3 + 81a_3b_3^2\lambda_2 + 9\left(a_3^2 - b_3^2\right) + 18a_3b_3\delta\lambda_2 + 3a_3 - \Omega_2\right],$$

$$H_{14I} = -3a_7\left[27\lambda_2b_3^3 - 81a_3^2b_3\lambda_2 - 9\delta\lambda_2\left(a_3^2 - b_3^2\right) + 18a_3b_3 + 3b_3\right]$$
$$-3b_7\left[-27\lambda_2a_3^3 + 81a_3b_3^2\lambda_2 + 9\left(a_3^2 - b_3^2\right) + 18a_3b_3\delta\lambda_2 + 3a_3 - \Omega_2\right],$$

$$H_{15R} = -f_5\left[-\lambda_2f_5^3 + 3\lambda_2f_5f_6^2 + 2f_5f_6\delta\lambda_2 + f_5 - \Omega_2 + \left(f_5^2 - f_6^2\right)\right]$$
$$+f_6\left[\lambda_2f_6^3 - 3f_5^2f_6\lambda_2 - \delta\lambda_2\left(f_5^2 - f_6^2\right) + 2f_5f_6 + f_6\right],$$

$$H_{15I} = -f_5\left[\lambda_2f_6^3 - 3f_5^2f_6\lambda_2 - \delta\lambda_2\left(f_5^2 - f_6^2\right) + 2f_5f_6 + f_6\right]$$
$$-f_6\left[-\lambda_2f_5^3 + 3\lambda_2f_5f_6^2 + 2f_5f_6\delta\lambda_2 + f_5 - \Omega_2 + \left(f_5^2 - f_6^2\right)\right],$$

$$H_{16R} = \frac{H_{12R}H_{14R} + H_{12I}H_{14I}}{H_{14R}^2 + H_{14I}^2}, \quad H_{16I} = \frac{H_{12I}H_{14R} - H_{12R}H_{14I}}{H_{14R}^2 + H_{14I}^2},$$
$$H_{17R} = \frac{H_{13R}H_{15R} + H_{13I}H_{15I}}{H_{15R}^2 + H_{15I}^2}, \quad H_{17I} = \frac{H_{13I}H_{15R} - H_{13R}H_{15I}}{H_{15R}^2 + H_{15I}^2},$$

$$H_{18R} = -\left(b_R^2 - b_I^2\right)\left[9\left(a_7^2 - b_7^2\right)\left(3a_7b_7^2 - a_7^3\right) - 18a_7b_7\left(3a_7^2b_7 - b_7^3\right)\right]$$
$$-4a_Ra_I\left[9\left(a_7^2 - b_7^2\right)\left(b_7^3 - 3a_7^2b_7\right) - 18a_7b_7\left(a_7^3 - 3a_7b_7^2\right)\right],$$

$$H_{18I} = \left(b_R^2 - b_I^2\right)\left[9\left(a_7^2 - b_7^2\right)\left(b_7^3 - 3a_7^2b_7\right) - 18a_7b_7\left(a_7^3 - 3a_7b_7^2\right)\right]$$
$$+4a_Ra_I\left[9\left(a_7^2 - b_7^2\right)\left(3a_3b_3^2 - a_7^3\right) + 18a_7b_7\left(3a_7^2b_7 - b_7^3\right)\right],$$

$$H_{19R} = 2\left(a_R^2 - a_I^2\right)\begin{bmatrix} \{b_7\left(b_5^2 - a_5^2\right) - 2b_5a_5b_7\}\left(f_7^2 - f_8^2\right) \\ -2\{b_7\left(a_5^2 - b_5^2\right) + 2a_5a_7b_5\}\left(c_7c_8\right) \end{bmatrix}$$
$$-4a_Ra_I\begin{bmatrix} \{b_7\left(b_5^2 - a_5^2\right) - 2b_5a_5b_7\}\left(f_7f_8\right) \\ -\{b_7\left(a_5^2 - b_5^2\right) + 2a_5a_7b_5\}\left(f_7^2 - f_8^2\right) \end{bmatrix},$$

$$H_{19I} = 2\left(a_R^2 - a_I^2\right)\begin{bmatrix} \{b_7\left(b_5^2 - a_5^2\right) - 2b_5a_5b_7\}\left(f_7f_8\right) \\ +\{b_7\left(a_5^2 - b_5^2\right) + 2a_5a_7b_5\}\left(f_7^2 - f_8^2\right) \end{bmatrix}$$
$$+4b_Rb_I\begin{bmatrix} \{b_7\left(b_5^2 - a_5^2\right) - 2b_5a_5b_7\}\left(f_7^2 - f_8^2\right) \\ +2\{b_7\left(a_5^2 - b_5^2\right) + 2a_5a_7b_5\}\left(f_7f_8\right) \end{bmatrix},$$

$$H_{20R} = 3b_7\left[27\lambda_2b_7^3 - 81a_7^2b_7\lambda_2 - 9\delta\lambda_2\left(a_7^2 - b_7^2\right) + 18a_7b_7 + 3b_7 + \left(\delta + \Omega_1\right)\right]$$
$$-3a_7\left[-27\lambda_2a_7^3 + 81a_7b_7^2\lambda_2 + 9\left(a_7^2 - b_7^2\right) + 18a_7b_7\delta\lambda_2 + 3a_7 - \Omega_2\right],$$

$$H_{20I} = -3b_7\left[-27\lambda_2b_7^3 + 81a_7b_7^2\lambda_2 + 9\left(a_7^2 - b_7^2\right) + 18a_7b_7\delta\lambda_2 - 3a_7 - \Omega_2\right]$$
$$-3a_7\left[27\lambda_2b_7^3 - 81a_7^2b_7\lambda_2 - 9\delta\lambda_2\left(a_7^2 - b_7^2\right) + 18a_7b_7 + 3b_7 + \left(\delta + \Omega_1\right)\right],$$

$$H_{21R} = -f_7\left[\lambda_2f_7^3 - 3\lambda_2f_7f_8^2 + 2f_7f_8\delta\lambda_2 + f_7 - \Omega_2 + \left(f_7^2 - f_8^2\right)\right]$$
$$+f_8\left[\lambda_2f_8^3 - 3f_7^2f_8\lambda_2 - \delta\lambda_2\left(f_7^2 - f_8^2\right) + 2f_7f_8 + f_8 + \left(\delta + \Omega_1\right)\right],$$

$$H_{21I} = -f_7\left[\lambda_2f_8^3 - 3f_7^2f_8\lambda_2 - \delta\lambda_2\left(f_7^2 - f_8^2\right) + 2f_7f_8 + f_8 + \left(\delta + \Omega_1\right)\right]$$
$$-f_8\left[\lambda_2f_7^3 - 3\lambda_2f_7f_8^2 + 2f_7f_8\delta\lambda_2 + f_7 - \Omega_2 + \left(f_7^2 - f_8^2\right)\right],$$

$$H_{22R} = \frac{H_{18R}H_{20R} + H_{18I}H_{20I}}{H_{20R}^2 + H_{20I}^2}, \quad H_{22I} = \frac{H_{18I}H_{20R} - H_{18R}H_{20I}}{H_{20R}^2 + H_{20I}^2},$$
$$H_{23R} = \frac{H_{19R}H_{21R} + H_{19I}H_{21I}}{H_{21R}^2 + H_{21I}^2}, \quad H_{23I} = \frac{H_{19I}H_{21R} - H_{19R}H_{21I}}{H_{21R}^2 + H_{21I}^2},$$

$$H_{24R} = 2\left(a_R^2 - a_I^2\right)\begin{bmatrix} \{b_7\left(b_6^2 - a_6^2\right) - 2b_6a_6b_7\}\left(f_9^2 - f_{10}^2\right) \\ -2\{b_7\left(a_6^2 - b_6^2\right) + 2a_6a_7b_6\}\left(f_9f_{10}\right) \end{bmatrix} + 4a_Ra_I\begin{bmatrix} \{b_7\left(b_6^2 - a_6^2\right) - 2b_6a_6b_7\}\left(f_9f_{10}\right) \\ +\{b_7\left(a_6^2 - b_6^2\right) + 2a_6a_7b_6\}\left(f_9^2 - f_{10}^2\right) \end{bmatrix},$$

$$H_{24I} = 2\left(a_R^2 - a_I^2\right)\begin{bmatrix} \{b_7\left(b_6^2 - a_6^2\right) - 2b_6a_6b_7\}\left(f_9f_{10}\right) \\ -\{b_7\left(a_6^2 - b_6^2\right) + 2a_6a_7b_6\}\left(f_9^2 - f_{10}^2\right) \end{bmatrix} + 4b_Rb_I\begin{bmatrix} \{b_7\left(b_6^2 - a_6^2\right) - 2b_6a_6b_7\}\left(f_9^2 - f_{10}^2\right) \\ +2\{b_7\left(a_6^2 - b_6^2\right) + 2a_6a_7b_6\}\left(f_9f_{10}\right) \end{bmatrix},$$

$$\begin{aligned} H_{25R} &= -f_9\left[-\lambda_2f_9^3 + 3\lambda_2f_9f_{10}^2 + 2f_9f_{10}\delta\lambda_2 + f_9 - \Omega_2 + \left(f_9^2 - f_{10}^2\right)\right] \\ &\quad + f_{10}\left[\lambda_2f_{10}^3 - 3f_9^2f_{10}\lambda_2 - \delta\lambda_2\left(f_9^2 - f_{10}^2\right) + 2f_9f_{10} + f_{10} + (\delta + \Omega_1)\right], \end{aligned}$$

$$\begin{aligned} H_{25I} &= -f_{10}\left[-\lambda_2f_9^3 + 3\lambda_2f_9f_{10}^2 + 2f_9f_{10}\delta\lambda_2 + f_9 - \Omega_2 + \left(f_9^2 - f_{10}^2\right)\right] \\ &\quad - f_9\left[\lambda_2f_{10}^3 - 3f_9^2f_{10}\lambda_2 - \delta\lambda_2\left(f_9^2 - f_{10}^2\right) + 2f_9f_{10} + f_{10} + (\delta + \Omega_1)\right], \end{aligned}$$

$$H_{26R} = \frac{H_{24R}H_{25R} + H_{24I}H_{25I}}{H_{25R}^2 + H_{25I}^2}, \quad H_{26I} = \frac{H_{24I}H_{25R} - H_{24R}H_{25I}}{H_{25R}^2 + H_{25I}^2},$$

$$H_{27R} = 2\left(a_R^2 - a_I^2\right)\begin{bmatrix} \{b_7\left(b_7^2 - a_7^2\right) - 2b_3a_3b_7\}\left(f_{11}^2 - f_{12}^2\right) \\ +2\{b_7\left(a_3^2 - b_3^2\right) + 2a_3a_7b_3\}\left(f_{11}f_{12}\right) \end{bmatrix} - 4a_Ra_I\begin{bmatrix} \{b_7\left(b_3^2 - a_3^2\right) - 2b_3a_3b_7\}\left(f_{11}f_{12}\right) \\ +\{b_7\left(a_3^2 - b_3^2\right) + 2a_3a_7b_3\}\left(f_{11}^2 - f_{12}^2\right) \end{bmatrix},$$

$$H_{27I} = 2\left(a_R^2 - a_I^2\right)\begin{bmatrix} \{b_7\left(b_3^2 - a_3^2\right) - 2b_3a_3b_7\}\left(c_{11}c_{12}\right) \\ -\{b_7\left(a_3^2 - b_3^2\right) + 2a_3a_7b_3\}\left(c_{11}^2 - c_{12}^2\right) \end{bmatrix} + 4b_Rb_I\begin{bmatrix} \{b_7\left(b_3^2 - a_3^2\right) - 2b_3a_3b_7\}\left(f_{11}^2 - f_{12}^2\right) \\ -2\{b_7\left(a_3^2 - b_3^2\right) + 2a_3a_7b_3\}\left(f_{11}f_{12}\right) \end{bmatrix},$$

$$
\begin{aligned}
H_{28R} &= -f_{11}\left[-\lambda_2 f_{11}^3 + 3\lambda_2 f_{11} f_{12}^2 + 2 f_{11} f_{12}\delta\lambda_2 + f_{11} - \Omega_2 + \left(f_{11}^2 - f_{12}^2\right)\right] \\
&\quad + f_{12}\left[\lambda_2 f_{12}^3 - 3 f_{11}^2 f_{12}\lambda_2 - \delta\lambda_2\left(f_{11}^2 - f_{12}^2\right) + 2 f_{11} f_{12} + f_{12} + 2\Omega_1\right],
\end{aligned}
$$

$$
\begin{aligned}
H_{28I} &= -f_{12}\left[-\lambda f_{11}^3 + 3\lambda f_{11} f_{12}^2 + 2 f_{11} f_{12}\delta\lambda + f_{11} - \Omega_2 + \left(f_{11}^2 - f_{12}^2\right)\right] \\
&\quad - f_{11}\left[\lambda_2 f_{12}^3 - 3 f_{11}^2 f_{12}\lambda_2 - \delta\lambda_2\left(f_{11}^2 - f_{12}^2\right) + 2 f_{11} f_{12} + f_{12} + 2\Omega_1\right],
\end{aligned}
$$

$$
H_{29R} = \frac{H_{27R}H_{28R} + H_{27I}H_{28I}}{H_{28R}^2 + H_{28I}^2}, \quad H_{29I} = \frac{H_{27I}H_{28R} - H_{27R}H_{28I}}{H_{28R}^2 + H_{28I}^2}.
$$

4.4 Numerical Method

In this section we present the solution of the problem numerically using Crank Nicolson method. Here, we could like to remark that in finding the analytical solution and to eliminate the pressure gradient we had to take an additional derivative first which rendered the equation to fourth order. The additional constant appearing because of the additional derivative is then determined to be equal to zero using boundary layer condition. Using the dimensionless quantities from equation (3.11a) into equation (4.7) and boundary condition (4.9) and (4.10) we obtain

$$
\lambda_3\frac{\partial^3 q}{\partial z^2\partial t} - \lambda_3 S_1\frac{\partial^3 q}{\partial z^3} + \Gamma\frac{\partial^2 q}{\partial z^2} + S_1\frac{\partial q}{\partial z} - \frac{\partial q}{\partial t} - (2i + N_1)\, q + \lambda_4\frac{\partial}{\partial z}\left(\left(\frac{\partial q}{\partial z}\right)^2\frac{\partial q^*}{\partial z}\right) = 0, \tag{4.84}
$$

$$
\begin{aligned}
q &= (ae^{i\sigma_1 t} + be^{-i\sigma_1 t}), \quad at\ z = 0, \quad t > 0, \qquad (4.85) \\
q &\to 0, \quad as \quad z \to \infty, \ t > 0,
\end{aligned}
$$

where

$$
N_1 = \frac{\sigma B_0^2}{\rho\Omega}, S_1 = \frac{W_0 U_0}{\nu\Omega}, \lambda_3 = \frac{\alpha U_0^2}{\nu^2}, \Gamma = \frac{U_0^2}{\nu\Omega}, \quad \lambda_4 = \frac{2\beta_3 U_0^6}{\rho\Omega\nu^4}. \tag{4.86}
$$

The two equations (4.7) and (4.84) are essentially the same in the boundary layer approximation, thus having the same solution.

We note that equation (4.84) is third order partial differential equation. The general solution of equation (4.84) in closed form is not possible. Therefore, we discuss the numerical solution here. The equation (4.84) is parabolic in t and thus has only one characteristic direction. The information at any point influences the entire region on one side of vertical characteristics and contained within the two boundaries. Therefore, lend itself to marching solution. Starting with an arbitrary initial condition consistent with boundary condition (4.85) has been chosen. Its effects disappear after few time steps, and solutions appears to be independent of it. The nonlinearity in the equation is suppressed by the Von Neumann stability analysis. This is done by solution dependent coefficients multiplying derivatives as being temporarily frozen.

As the problem is time dependent and has mixed derivative with respect to time and space coordinates so we use the implicit scheme. To convert equation (4.84) to a system of algebraic equation a number of choices are available. In general the partial differential equation (4.84) can be rewritten as

$$\frac{\partial \phi}{\partial t} = L(\phi), \tag{4.87}$$

where $\phi = (q, S_1, \lambda_3, \lambda_4)$ and L is the operator for spatial terms. The partial differential equation is thus split into subsequent differential problems consisting first in discretizing the spatial operator L with an accurate and efficient method, and then integrating an ordinary differential equation in time with a low order finite difference method. Due to stronger dependence on space variables, the higher order and the nonlinearities, the spatial operator L requires a different treatment with respect to time operator. Since we are dealing with boundary value problem with an arbitrary initial condition consistent with the boundary condition a Crank-Nicolson implicit scheme is used and is given below:

4.4.1 Discritization of the Problem

We introduce a mesh define by

$$\begin{aligned} z_j &= jh, \ \triangle t = t^{\gamma+1} - t^{\gamma}, \\ h &= \frac{z_{M_1} - z_1}{M_1 - 1}, \ (j = 1, 2, 3, ..., M_1) \end{aligned} \tag{4.88}$$

where h is a mesh size, γ is the time level and M_1 is a suitable large number such that z_∞ can be reasonably approximated by $M_1 h$. The central, forward formulae for derivatives of various orders are

$$\frac{\partial q}{\partial t} = \frac{q_j^{\gamma+1} - q_j^{\gamma}}{\triangle t}, \tag{4.89}$$

$$\frac{\partial q}{\partial z} = \frac{q_{j+1}^{\gamma} - q_{j-1}^{\gamma}}{2h}, \tag{4.90}$$

$$\frac{\partial^2 q}{\partial z^2} = \frac{q_{j+1}^{\gamma} - 2q_j^{\gamma} + q_{j-1}^{\gamma}}{h^2}, \tag{4.91}$$

$$\frac{\partial^3 q}{\partial z^2 \partial \tau} = \frac{q_{j+1}^{\gamma+1} - 2q_j^{\gamma+1} + q_{j-1}^{\gamma+1}}{\triangle t h^2} - \frac{q_{j+1}^{\gamma} - 2q_j^{\gamma} + q_{j-1}^{\gamma}}{\triangle t h^2}, \tag{4.92}$$

$$\frac{\partial^3 q}{\partial z^3} = \frac{q_{j+2}^{\gamma} - 3q_{j+1}^{\gamma} + 3q_j^{\gamma} - q_{j-1}^{\gamma}}{h^3}. \tag{4.93}$$

Due to modified Crank Nicolson scheme we modify the term in equation (4.91) as

$$\frac{\partial^2 q}{\partial z^2} = \frac{q_{j+1}^{\gamma+1} - 2q_j^{\gamma+1} + q_{j-1}^{\gamma+1}}{2h^2} + \frac{q_{j+1}^{\gamma} - 2q_j^{\gamma} + q_{j-1}^{\gamma}}{2h^2} \tag{4.94}$$

and in the nonlinear part of equation (4.84) we define derivatives as

$$\frac{\partial^2 q}{\partial z^2}=\frac{q^{\gamma}_{j+2}-2q^{\gamma}_{j+1}+q^{\gamma}_{j}}{2h^2},\quad \frac{\partial^2 \overline{q}}{\partial z^2}=\frac{\overline{q}^{\gamma}_{j+2}-2\overline{q}^{\gamma}_{j+1}+\overline{q}^{\gamma}_{j}}{2h^2}. \tag{4.95}$$

Substitutions of the derivatives from equation (4.89) – (4.95) in equation (4.84), leads to

$$\begin{aligned}
&\lambda_3\left[\frac{q^{\gamma+1}_{j+1}-2q^{\gamma+1}_{j}+q^{\gamma+1}_{j-1}}{\triangle t h^2}-\frac{q^{\gamma}_{j+1}-2q^{\gamma}_{j}+q^{\gamma}_{j-1}}{\triangle\tau h^2}\right]\\
&-\lambda_3 S_1\left[\frac{q^{\gamma}_{j+2}-3q^{\gamma}_{j+1}+3q^{\gamma}_{j}-q^{\gamma}_{j-1}}{h^3}\right]\\
&+\Gamma\left[\frac{q^{\gamma+1}_{j+1}-2q^{\gamma+1}_{j}+q^{\gamma+1}_{j-1}}{2h^2}-\frac{q^{\gamma}_{j+2}-2q^{\gamma}_{j+1}+q^{\gamma}_{j}}{2h^2}\right]\\
&+S_1\left[\frac{q^{\gamma}_{j+1}-q^{\gamma}_{j-1}}{2h}\right]-\frac{q^{\gamma+1}_{j}-q^{\gamma}_{j}}{\triangle t}-(2i+N_1)\,q^{\gamma}_{j}\\
&+\frac{\lambda_4}{4h^4}\left[\begin{array}{c}2\left(q^{\gamma}_{j+1}-q^{\gamma}_{j-1}\right)\left(q^{\gamma}_{j+2}-2q^{\gamma}_{j+1}+q^{\gamma}_{j}\right)\left(q^{*\gamma}_{j+1}-q^{*\gamma}_{j-1}\right)\\+\left(q^{\gamma}_{j+1}-q^{\gamma}_{j-1}\right)^2\left(q^{*\gamma}_{j+2}-2q^{*\gamma}_{j+1}+q^{*\gamma}_{j}\right)\end{array}\right].
\end{aligned} \tag{4.96}$$

In above equation the unknown $q^{\gamma+1}_{j}$ is not only expressed in terms of the known quantities at time level γ, namely, $q^{\gamma}_{j+2}, q^{\gamma}_{j+1}, q^{\gamma}_{j}$ and q^{γ}_{j-1} but also in terms of other unknown quantities at time level $\gamma+1$, namely $q^{\gamma+1}_{j+1}$ and $q^{\gamma+1}_{j-1}$. The equation (4.96) has unknowns $q^{\gamma+1}_{j+1}, q^{\gamma+1}_{j}$ and $q^{\gamma+1}_{j-1}$. Thus equation (4.96) when applied to given grid point i does not stand alone, it cannot give a solution for $q^{\gamma+1}_{j}$. Hence equation (4.96) must be written all interior grid points which results a system of algebraic equations. Using an implicit approach the $q^{\gamma+1}_{j}$ for all i can be found simultaneously. Such approach is unconditionally stable unless non-linear effects cause instability, which is controlled by suitable choice of $\triangle t$ and h. We can rewrite equation (4.96) in the following form

$$X_j^\gamma q_{j-1}^{\gamma+1} + Y_j^\gamma q_j^{\gamma+1} + Z_j^\gamma q_{j+1}^{\gamma+1} = -d_j^\gamma. \tag{4.97}$$

where

$$\begin{aligned} X_j^\gamma &= \frac{\lambda_3}{h^2} + \frac{\triangle t\Gamma}{h^2}, \\ Y_j^\gamma &= \frac{-2\lambda_3}{h^2} - \frac{\triangle t\Gamma}{h^2} - 1, \\ Z_j^\gamma &= \frac{\lambda_3}{h^2} + \frac{\triangle t\Gamma}{h^2}, \end{aligned}$$

$$\begin{aligned} d_j^\gamma &= \frac{\lambda_3}{h^2}\left(-q_{j+1}^\gamma + 2q_j^\gamma - q_{j-1}^\gamma\right) - \frac{\triangle t\lambda_3 S_1}{h^3}\left(q_{j+2}^\gamma - 3q_{j+1}^\gamma + 3q_j^\gamma - q_{j-1}^\gamma\right) \\ &\quad -\frac{\Gamma\triangle t}{2h^2}\left(q_{j+2}^\gamma - 2q_{j+1}^\gamma + q_j^\gamma\right) + \frac{\triangle t S_1}{2h}\left(q_{j+1}^\gamma - q_{j-1}^\gamma\right) + q_j^\gamma - \triangle t\left(2i + N_1\right) q_j^\gamma \\ &\quad +\frac{\triangle t\lambda_4}{4h^4}\left[\begin{array}{c} 2\left(q_{j+1}^\gamma - q_{j-1}^\gamma\right)\left(q_{j+2}^\gamma - 2q_{j+1}^\gamma + q_j^\gamma\right)\left(q_{j+1}^{*\gamma} - q_{j-1}^{*\gamma}\right) \\ +\left(q_{j+1}^\gamma - q_{j-1}^\gamma\right)^2\left(q_{j+2}^{*\gamma} - 2q_{j+1}^{*\gamma} + q_j^{*\gamma}\right) \end{array}\right]. \end{aligned}$$

Now, using the previous step analysis, the d_j^γ, in equation (4.97) is known and we find a computational solution for the dependent variable on the left hand side of equation (4.97). For that we apply equation (4.97) sequentially to grid point 2 through $M_1 - 1$.

At grid point 2:

$$X_2^\gamma q_1^{\gamma+1} + Y_2^\gamma q_2^{\gamma+1} + Z_2^\gamma q_3^{\gamma+1} = -d_2^\gamma, \tag{4.98}$$

At grid point 3:

$$X_3^\gamma q_1^{\gamma+1} + Y_3^\gamma q_2^{\gamma+1} + Z_3^\gamma q_3^{\gamma+1} = -d_3^\gamma, \tag{4.99}$$

and so on

At grid point $M_1 - 2$:

$$X^{\gamma}_{M_1-2} q^{\gamma+1}_{M_1-3} + Y^{\gamma}_{M_1-2} q^{\gamma+1}_{M_1-2} + Z^{\gamma}_{M_1-2} q^{\gamma+1}_{M_1-1} = -d^{\gamma}_{M_1-2}, \tag{4.100}$$

At grid point $M_1 - 1$:

$$X^{\gamma}_{M_1-1} q^{\gamma+1}_{M_1-2} + Y^{\gamma}_{M_1-1} q^{\gamma+1}_{M_1-1} + Z^{\gamma}_{M_1-1} q^{\gamma+1}_{M_1} = -d^{\gamma}_{M_1-1}. \tag{4.101}$$

The above system of algebraic equations in matrix form gives

$$\begin{bmatrix} Y_2 & Z_2 & & & & & & \\ X_3 & Y_3 & Z_3 & & & & & \\ & X_4 & Y_4 & Z_4 & & & & \\ & & \cdot & \cdot & \cdot & & & \\ & & & X_j & Y_j & Z_j & & \\ & & & & \cdot & \cdot & \cdot & \\ & & & & & X_{M_1-2} & Y_{M_1-2} & Z_{M_1-2} \\ & & & & & & Y_{M_1-1} & Z_{M_1-1} \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \\ q_3 \\ \\ q_j \\ \\ q_{M_1-2} \\ q_{M_1-1} \end{bmatrix} = \begin{bmatrix} -d_2 \\ -d_3 \\ -d_4 \\ \\ -d_j \\ \\ -d_{M_1-2} \\ -d_{M_1-1} \end{bmatrix}. \tag{4.102}$$

Nearly every mathematical packages (such as IMSL, Matlab, Mathematica, Fortran) boasts of an efficient solver of linear banded system of equations.

4.5 Results and Discussion

Here, the solution of equation (4.84) with the following boundary conditions

$$\begin{aligned} a &= b = \frac{1}{2} \qquad q = \cos\left(\frac{\pi}{4}\right), \\ a &= -b = \frac{1}{2} \qquad q = i\sin\left(\frac{\pi}{4}\right), \end{aligned} \tag{4.103}$$

is given using a modified Crank-Nicolson scheme and an excellent results are obtained (maximum error 10^{-3}) when $\triangle t = 0.001$ and $h = 0.1$. In order to examine the flow response and its sensitivity to a variation of physical geometrical and rheological features, with an initial guess, the convergence is reached with a small number of iterations. Of particular interest here are the variations in $\lambda_3, \lambda_4, N_1$ and S_1 at $\sigma_1 t = \pi/4$ and is discussed as follows:

1. **Effects of $\boldsymbol{\lambda_3}$** : In Figs. 4.1(a,b), the plots of u and v are against η for various values of second grade parameter (λ_3). It appears from the figures that velocities does satisfy the boundary condition at infinity in equation (4.96) in the sense that velocities (u and v) becomes zero at some values of z say z_∞ called the numerical infinity by numerical analysis. It is also shown from the figures that for cosine oscillation the velocities near the plate increases with the increase in λ_3 but away from the plate it decreases and finally reaches to zero. The variation of sine oscillation for various values of λ_3 are calculated in Figs. $4.2(a, b)$. We see that in these figures the behavior of the velocities are in reverse order, near the plate the velocity decreases but away from the plate it increases and attains steady state. It is observed from these figures that with same out put data, the velocities for cosine and sine oscillations are not same.

2. **Effects of $\boldsymbol{\lambda_4}$** : Figures $4.3(a, b)$ illustrate the velocity for sine oscillation for various values of third grade parameter (λ_4). It is noted from the graphs that u decreases with the increase in λ_4 whereas v is almost same for $\lambda_4 = 2$ and $\lambda_4 = 3$, but increases when we increase the value of λ_4 up to $\lambda_4 = 4$.

3. **Effects of N_1** : The effects of magnetic parameter (N_1) for non zero λ_3 are given in Figs. 4.4(a, b) and 4.5(a, b) for both cosine and sine oscillations respectively. It is depicted from the figures that the increase in N_1 results the decrease in layer thickness. Further, the layer thickness in case of $\lambda_4 \neq 0$ is found to decrease for large values of N_1 (Figs. 4.6(a, b) and 4.7 (a, b)).

4. **Effects of Suction/Blowing:** The real and imaginary parts of the velocity for various values of suction/blowing are displayed in Figs. 4.8(a, b) – 4.10 (a, b). It is observed that suction always causes a reduction in the layer thickness and layer thickness is large with the increase in blowing.

5. **Effects of Resonance:** In the resonance case ($\sigma_1 \rightarrow 2$) it is known that no steady asymptotic hydrodynamic solution is possible for case of resonance and blowing. But the present analysis exhibits the striking difference between the structures of hydrodynamic and hydromagnetic boundary layers (Figs. 4.11 and 4.12). Figs. 4.13(a, b) indicate the variation of u and v respectively against z for cosine oscillations for different t. The respective time required for steady state for u and v are $t = 0.1$ and $t = 0.2$, respectively. Figures 4.14(a, b) shows the variation in u and v against z for sine oscillation with different t . The steady state situation for u and v are obtained at $t = 0.5$ and $t = 0.7$, respectively. From this analysis it is noted that time to reach steady state for u and v in sine oscillation is larger than cosine oscillation in general and also for resonant case in particular.

4.6 Concluding Remarks

We have presented results for the rotating flow of a third grade fluid on a porous plate oscillating in its own plane. The velocity fields are governed by third order non-linear partial differential equation. Since we are considering third grade fluid some remarks on symmetric and asymmetric solution will not be out of place.

The flow of a viscous fluid bounded by two infinite parallel disks has been firstly examined by Berker [76]. The two considered disks are rotating with the same angular velocity about a common axis. He observed that asymmetric solutions are possible if the locus of the centre of rotation of the fluid changes with respect to the depth of the fluid bounded by two disks. He further noted that the symmetric solutions arise for the case of rigid body rotation (i.e. if the axis of rotation does not change) and symmetric solutions are subclass of asymmetric solution. Parter and Rajagopal [77] considered non-Newtonian fluids and provide a complete answer to this question. They showed that for non-Newtonian fluids, asymmetric solutions exist.

In the present problem, the fluid and the plate are in a rigid body rotation and the symmetric solutions appear. However, in the case of third grade fluid if the plate and the fluid are not in a rigid body rotation asymmetric solutions are possible as predicted by Parter and Rajagopal [77].

Also solutions (4.62)−(4.67) reveal that the structures of the associated magnetohydrodynamic boundary layers strongly depend on magnetic parameter i.e. the thickness of the boundary layers (in equations (4.62)−(4.67)) are in the combinations of the hydrodynamic and the hydromagnetic boundary layer. From the presented analysis, it is observed that the boundary layer tends to become thinner with increase of magnetic field and third grade fluid parameters. Further the thickness of the hydromagnetic boundary layers remain bounded for all frequencies. A striking difference of the presented analysis is that in resonant case the hydromagnetic steady solution exist. Another distinguishing feature of the results is that the magnetic field has the same influence on the velocity field as the material parameters of the third grade fluid.

Chapter 5

On Fluctuating Flow of a Third grade Fluid Past an Infinite Plate with Variable Suction

An analytic solution is obtained for the flow of a third grade fluid past an infinite porous plate when the suction velocity normal to the plate, as well as the external flow velocity, varies periodically with time. Expression for the velocity has been obtained in a dimensionless form. The response of the velocity to the fluctuating stream and suction velocity is studied for variations in the suction parameter, the material parameter of the third grade fluid and the frequency parameter. Also, the results of Newtonian and second grade fluids can be recovered as the special cases of the presented problem.

5.1 The Constitutive Model

We consider an incompressible third grade fluid flow along an infinite porous plate, taking the $X - axis$ along the plate and $Y - axis$ perpendicular to it. We also assume that the flow is independent of the distance parallel to the plate. The suction velocity normal to the plate is directed towards it and varies periodically with time about a non-zero constant. Thus for the problem under consideration, we seek a velocity field of the form

$$\mathbf{V} = [U_3(Y,t), U_4, 0], \tag{5.1}$$

where $U_4 < 0$ is the suction velocity.

From equation (5.1) and the continuity equation (1.9) we obtain

$$\frac{\partial U_4}{\partial Y} = 0. \tag{5.2}$$

It is evident from equation(5.2) that U_4 is a function of time only. Hence we consider U_4 in the form [45]

$$U_4 = -V_0(1+ \in A' e^{i\omega t}). \tag{5.3}$$

The negative sign in equation (5.3) indicates that the suction velocity normal to the plate is directed towards the plate. In view of equations (5.1)–(5.3), momentum equation gives

$$\begin{aligned} \frac{\partial U_3}{\partial t} - V_0(1+ \in A' e^{i\omega t})\frac{\partial U_3}{\partial Y} &= -\frac{1}{\rho}\frac{\partial P}{\partial X} + \nu\frac{\partial^2 U_3}{\partial Y^2} \\ &+\alpha\left[\frac{\partial^3 U_3}{\partial Y^2 \partial t} - V_0\left(1+ \in A' e^{i\omega' t'}\right)\frac{\partial^3 U_3}{\partial Y^3}\right] \\ &+\frac{6\beta_3}{\rho}\left(\frac{\partial U_3}{\partial Y}\right)^2 \frac{\partial^2 U_3}{\partial Y^2}, \end{aligned} \tag{5.4}$$

$$\frac{\partial U_4}{\partial t} = -\frac{1}{\rho}\frac{\partial P}{\partial Y}, \tag{5.5}$$

$$P = p - (2\alpha_1 + \alpha_2)\left(\frac{\partial U_3}{\partial Y}\right)^2.$$

From equations (5.3) and (5.5), it is clear that $\partial P/\partial Y$ is small in boundary layer and can be neglected [45]. Hence the pressure is taken to be constant along any normal and is given by its value outside the boundary layer. If $U_5(t)$ is the free stream velocity then equation (5.4) for the free stream becomes

$$-\frac{1}{\rho}\frac{\partial P}{\partial X} = \frac{dU_5}{dt}. \tag{5.6}$$

The equations (5.4) and (5.6) give

$$\begin{aligned}\frac{\partial U_3}{\partial t'} - V_0(1+ \in A' e^{i\omega t})\frac{\partial U_3}{\partial Y} &= \frac{dU_5}{dt} + \nu\frac{\partial^2 U_3}{\partial Y^2} \\ &+\alpha\left[\frac{\partial^3 U_3}{\partial Y^2 \partial t} - V_0\left(1+ \in A' e^{i\omega t}\right)\frac{\partial^3 U_3}{\partial Y^3}\right] \\ &+\frac{6\beta_3}{\rho}\left(\frac{\partial U_3}{\partial Y}\right)^2\frac{\partial^2 U_3}{\partial Y^2}. \end{aligned} \tag{5.7}$$

The boundary conditions are

$$U_3 = 0 \text{ at } Y = 0 \text{ and } U_3 = U_5(t) \; as \; Y \longrightarrow \infty. \tag{5.8}$$

The periodic free stream has the following form

$$U_5(t) = U_0\left(1+ \in e^{i\omega t}\right). \tag{5.9}$$

We introduce dimensionless quantities defined by

$$\begin{aligned} y &= \frac{YV_0}{\nu},\; \tau_1 = \frac{V_0^2 t}{4\nu},\; \omega_1 = \frac{4\nu\omega}{V_0^2}, \\ \lambda_5 &= \frac{\alpha V_0^2}{\nu^2},\; G = \frac{U_3}{U_0},\; U_2 = \frac{U_5}{U_0},\; \varepsilon = \frac{6\beta_3}{\rho\nu^3}U_0^2 V_0^2. \end{aligned} \tag{5.10}$$

With the help of equation (5.10), equations (5.7) – (5.9) take the following form

$$\frac{1}{4}\frac{\partial G}{\partial \tau_1} - (1 + \in A' e^{i\omega_1 \tau_1})\frac{\partial G}{\partial y} = \frac{1}{4}\frac{dU_2}{d\tau_1} + \frac{\partial^2 G}{\partial y^2}$$
$$+\lambda_5 \left[\frac{1}{4}\frac{\partial^3 G}{\partial y^2 \partial \tau_1} - \left(1 + \in A' e^{i\omega_1 \tau_1}\right)\frac{\partial^3 G}{\partial y^3}\right] + \varepsilon \left(\frac{\partial G}{\partial y}\right)^2 \frac{\partial^2 G}{\partial y^2}, \tag{5.11}$$

$$G = 0 \; at \; y = 0 \;\; and \;\; G \longrightarrow U_2(\tau_1) \;\; as \; y \longrightarrow \infty, \tag{5.12}$$

where

$$U_2 = 1 + \in e^{i\omega_1 \tau_1}. \tag{5.13}$$

5.2 Perturbation Solution

Writing $G = G_0 + \varepsilon G_1$ into equation (5.11) and the boundary conditions (5.12), and then collecting terms of like powers of ε, one obtains the following systems of partial differential equations along with the boundary conditions.

5.2.1 Zeroth-Order System

$$\frac{1}{4}\frac{\partial G_0}{\partial \tau_1} - (1 + \in A' e^{i\omega_1 \tau_1})\frac{\partial G_0}{\partial y} = \frac{i\omega_1}{4} \in e^{i\omega_1 \tau_1} + \frac{\partial^2 G_0}{\partial y^2}$$
$$+\lambda_5 \left[\frac{1}{4}\frac{\partial^3 G_0}{\partial y^2 \partial \tau_1} - \left(1 + \in A' e^{i\omega_1 \tau_1}\right)\frac{\partial^3 G_0}{\partial y^3}\right], \tag{5.14}$$

$$G_0 = 0 \; at \; y = 0 \;\; and \;\; G_0 \longrightarrow 1 + \in e^{i\omega_1 \tau_1} \;\; as \; y \longrightarrow \infty. \tag{5.15}$$

5.2.2 First-Order System

$$\begin{aligned} & \frac{1}{4}\frac{\partial G_1}{\partial \tau_1} - \left(1 + \in A' e^{i\omega_1 \tau_1}\right)\frac{\partial G_1}{\partial y} \\ = & \frac{\partial^2 G_1}{\partial y^2} + \lambda_5 \left[\frac{1}{4}\frac{\partial^3 G_1}{\partial y^2 \partial \tau_1} - \left(1 + \in A' e^{i\omega_1 \tau_1}\right)\frac{\partial^3 G_1}{\partial y^3}\right] \\ & + \left(\frac{\partial G_0}{\partial y}\right)^2 \frac{\partial^2 G_0}{\partial y^2}, \end{aligned} \tag{5.16}$$

$$G_1 = 0 \; at \; y = 0 \;\; and \;\; G_1 \longrightarrow 0 \; as \; y \longrightarrow \infty. \tag{5.17}$$

5.2.3 Zeroth-Order Solution

We note that the zeroth order mathematical problem is same as that of Soundalgekar and Puri [51] except that $(-k)$ is replaced by λ_5 in equation (5.14). Thus, in order to avoid repetition, the details of calculations are omitted and the solution is directly given by

$$\begin{aligned} G_0(y,\tau_1) \;\; = \;\; & 1 - e^{-y} - \lambda_5 y e^{-y} + \in e^{i\omega_1 \tau_1} \\ & \left[\begin{array}{c} 1 - S_2 e^{-h_1 y} - (1 - S_2)e^{-y} + Lye^{-h_1 y} \\ -\lambda_5 \left((1 - S_2)\, e^{-h_1 y} - (1 - y)\, e^{-y}\right) \end{array}\right], \end{aligned} \tag{5.18}$$

where

$$h_1 = \left[\frac{\sqrt{1 + i\omega_1} + 1}{2}\right], \tag{5.19}$$

$$L = \frac{h_1^2 (h_1 + \frac{i\omega_1}{4})(1 - \frac{4iA'}{\omega_1})}{\sqrt{1 + i\omega_1}}, \tag{5.20}$$

$$S_2 = 1 - \frac{4iA'}{\omega_1}. \tag{5.21}$$

5.2.4 First-Order Solution

Now, let

$$G_1(y,\tau_1) = F_1(y) + \in e^{i\omega_1\tau_1} F_2(y). \tag{5.22}$$

Substituting equations (5.18) and (5.22) in equation (5.16) and boundary conditions (5.17), and equating non-harmonic and harmonic terms and neglecting coefficients of $\in^2$, we have

$$\lambda_5 \frac{d^3F_1}{dy^3} - \frac{d^2F_1}{dy^2} - \frac{dF_1}{dy} = e^{-3y}(1+\lambda_5 y), \tag{5.23}$$

$$\begin{aligned} & \lambda_5 \frac{d^3F_2}{dy^3} - (1+\frac{i\omega_1\lambda_5}{4})\frac{d^2F_2}{dy^2} - \frac{dF_2}{dy} + \frac{i\omega_1}{4}F_2 \\ = \; & A'\frac{dF_1}{dy} + B^*_{11} - \lambda_5\left[A'\frac{d^3F_1}{dy^3} + B^*_{12}\right], \end{aligned} \tag{5.24}$$

$$F_1 = F_2 = 0, \qquad at \qquad y = 0 \; and \; y \longrightarrow \infty, \tag{5.25}$$

where

$$B^*_{11} = e^{-3y}\left[\frac{A'}{2} - 3(1-S_2)\right] + \frac{A'}{6}e^{-y} + (h_1^2 S_2 - 2h_1 S_2)e^{-(h_1+2)y}, \tag{5.26}$$

$$\begin{aligned} B^*_{12} = \; & e^{-y}\left[\frac{A'}{12}(9+2y) + 2 - \frac{A'}{6}\right] \\ & -e^{-3y}\left[\frac{A'}{4}(9+y^2-2y) - \frac{A'}{6}(y-1) - \frac{9A'}{2} - 1 + 3y + S_2 - 2S_2 y\right] \\ & -e^{-(h_1+2)y}\left[2h_1(1-S_2) + 4h_1 L - 2L - h_1^2 - h_1^2 L y\right]. \end{aligned} \tag{5.27}$$

There have been several investigations devoted to the study of the existence and uniqueness of the solutions to the equations governing the flows of fluids of differential type [11,13,16]. These equations are higher order partial differential equations than the Navier-Stokes equations. Hence the issue of whether the "no slip" boundary condition is sufficient to have a well posed problem is very important. This question cannot be answered in general for fluids of differential type of complexity n, for arbitrary n. However Rajagopal [13] provided some definite answers for fluids of grade 2 or grade 3.

Before proceeding with the solution of equations (5.23) and (5.24): it would be interesting to remark here that although in the classical viscous case ($\lambda_5 = 0$), we encounter differential equations of order two [44-47], the presence of the material parameter of the second grade fluid increases the order to three. It would therefore, seem that an additional boundary condition must be imposed in order to get a unique solution. In order to remove such difficulty, several workers have studied an acceptable additional condition. Fosdick and Berstein [78] have studied the flow in the annular region between two porous rotating cylinders. They set one of the constants in the solutions to be zero. However there is no apparent reason for such a choice. Frater [79] has studied the asymptotic suction flow. Since only two of the coefficients in the solution can be found by the no-slip condition, he imposes an extra condition that the solution tends to the Newtonian value as the coefficient of the higher derivative in the equation approaches to zero. However perturbation expansion has been also used to give correct results under certain conditions (Erdogan [80]). Thus following Kaloni [50] and Beard and Walters [71], we overcome the difficulty in the present study using perturbation expansion for small material parameter λ_5 and write the solution in the form as follows:

$$\begin{aligned} F_1 &= F_{01} + \lambda_5 F_{11} + O(\lambda_5^2), \\ F_2 &= F_{02} + \lambda_5 F_{12} + O(\lambda_5^2), \end{aligned} \tag{5.28}$$

which is valid for small values of λ_5 only. Putting equation (5.28) in equations (5.23) and (5.24) and equating the coefficient of λ_5 we obtain

$$\frac{d^2F_{01}}{dy^2} + \frac{dF_{01}}{dy} = -e^{-3y}, \tag{5.29}$$

$$\frac{d^2F_{11}}{dy^2} + \frac{dF_{11}}{dy} = -\frac{dF_{01}^3}{dy^3} + ye^{-3y}, \tag{5.30}$$

$$\frac{d^2F_{02}}{dy^2} + \frac{dF_{02}}{dy} - \frac{i\omega_1}{4}F_{02} = B_{11}^*, \tag{5.31}$$

$$\frac{d^2F_{12}}{dy^2} + \frac{dF_{12}}{dy} - \frac{i\omega_1}{4}F_{12} = -\frac{d^3F_{02}}{dy^3} - \frac{i\omega_1}{4}\frac{dF_{02}}{dy} - B_{12}^*, \tag{5.32}$$

$$\begin{aligned} F_{01} &= F_{11} = F_{02} = F_{12} = 0 \; at \; y = 0, \\ F_{01} &= F_{11} = F_{02} = F_{12} = 0 \; as \; y \longrightarrow \infty. \end{aligned} \tag{5.33}$$

Solving equations (5.29)–(5.32) under the boundary conditions (5.33), we have, in view of equation (5.28),

$$F_1 = \frac{1}{12}\left[e^{-y}\left\{2 + \lambda_5\left(9 + 2y\right)\right\} - e^{-3y}\left\{\lambda_5\left(9 + y^2 - 2y\right) + 2\right\}\right], \tag{5.34}$$

$$\begin{aligned} F_2 &= M_2\left(e^{-3y} - e^{-h_1 y}\right) + N_2\left(e^{-y} - e^{-h_1 y}\right) + P_1\left(e^{-(h_1+2)y} - e^{-h_1 y}\right) \\ &\quad -\lambda_5 \begin{pmatrix} M_3\left(e^{-3y} - e^{-h_1 y}\right) + N_3\left(e^{-y} - e^{-h_1 y}\right) \\ +P_2\left(e^{-(h_1+2)y} - e^{-h_1 y}\right) \\ +\frac{i}{3\omega_1}e^{-3y}\left(36y - 24S_2 y + 3A' y^2 - 2A' y\right) \end{pmatrix}, \end{aligned} \tag{5.35}$$

in which

$$M_2 = \frac{2\left[A' - 6\left(1 - S_2\right)\right]}{24 - i\omega_1}, \quad N_2 = \frac{2iA'}{3\omega_1}, \quad P_1 = \frac{\left(h_1^2 - 2h_1\right)S_2}{h_1^2 + 3h_1 + 2 - \frac{i\omega_1}{4}},$$

$$M_3 = \frac{i}{\omega}\left(36M_2 + 9A' + 8 - 8S_2\right), \quad N_3 = \frac{-4}{3\omega^2}\left(12N_2 - 13A' - 24\right),$$

$$P_2 = \frac{2h_1(1 - S_2) + 4h_1L - 2L - h_1^2 - h_1^2L}{h_1^2 + 3h_1 - 2 - \frac{i\omega_1}{4}}.$$

With the help of equations (5.22), (5.28), (5.34) and (5.35), we get

$$\begin{aligned} G_1 &= \frac{1}{12}\left[e^{-y}\left\{2 + \lambda_5\left(9 + 2y\right)\right\} - e^{-3y}\left\{\lambda_5\left(9 + y^2 - 2y\right) + 2\right\}\right] \\ &+ \in e^{i\omega_1\tau_1}\begin{pmatrix} M_2\left(e^{-3y} - e^{-h_1y}\right) + N_2\left(e^{-y} - e^{-h_1y}\right) \\ +P_1\left(e^{-(h_1+2)y} - e^{-h_1y}\right) \\ -\lambda_5\begin{pmatrix} M_3\left(e^{-3y} - e^{-h_1y}\right) + N_3\left(e^{-y} - e^{-h_1y}\right) \\ +P_2\left(e^{-(h_1+2)y} - e^{-h_1y}\right) \\ +\frac{i}{3\omega_1}e^{-3y}\left(36y - 24S_2y + 3A'y^2 - 2A'y\right) \end{pmatrix} \end{pmatrix}. \end{aligned} \tag{5.36}$$

Now from equations (5.18) and (5.36), the velocity field is given by

$$\begin{aligned} \frac{U_3}{U_0} &= 1 - e^{-y} - \lambda_5 y e^{-y} + \in e^{i\omega_1\tau}\begin{pmatrix} 1 - S_2e^{-h_1y} - (1 - S_2)e^{-y} \\ -\lambda_5\left\{\left(1 - S_2\right)e^{-h_1y} - \left(1 - y\right)e^{-y}\right\} \\ +Lye^{-h_1y} \end{pmatrix} \\ &+ \frac{\varepsilon}{12}\left[e^{-y}\left\{2 + \lambda_5\left(9 + 2y\right)\right\} - e^{-3y}\left\{\lambda_5\left(9 + y^2 - 2y\right) + 2\right\}\right] \\ &+\varepsilon \in e^{i\omega_1\tau_1}\begin{pmatrix} M_2\left(e^{-3y} - e^{-h_1y}\right) + N_2\left(e^{-y} - e^{-h_1y}\right) \\ +P_1\left(e^{-(h_1+2)y} - e^{-h_1y}\right) \end{pmatrix} \end{aligned}$$

$$-\varepsilon \in e^{i\omega_1\tau_1}\lambda_5 \left(\begin{array}{c} M_3\left(e^{-3y}-e^{-h_1y}\right)+N_3\left(e^{-y}-e^{-h_1y}\right) \\ +P_2\left(e^{-(h_1+2)y}-e^{-h_1y}\right) \\ +\frac{i}{3\omega_1}e^{-3y}\left(36y-24S_2y+3A'y^2-2A'y\right) \end{array} \right). \quad (5.37)$$

The real, u_R, and the imaginary u_I, parts of above expression, respectively, yield

$$\begin{aligned} u_R &= 1-e^{-y}\left(1+\lambda_5 y\right)+\in\left(M_R\cos\omega_1\tau_1-M_I\sin\omega_1\tau_1\right) \\ &\quad +\frac{\varepsilon}{12}\left[e^{-y}\left(2+\lambda_5\left(2y+9\right)\right)-e^{-3y}\left(\lambda_5\left(y^2-2y+9\right)+2\right)\right], \end{aligned} \quad (5.38)$$

$$u_I = \in\left(M_I\cos\omega_1\tau_1-M_R\sin\omega_1\tau_1\right), \quad (5.39)$$

where

$$\begin{aligned} M_R &= m_{R_{10}}+\varepsilon m_{R_{11}}, \\ M_I &= m_{I_{10}}+\varepsilon m_{I_{11}}. \end{aligned}$$

The parameter functions $m_{R_{10}}, m_{I_{10}}, m_{R_{11}}$ and $m_{I_{11}}$ involving in u_R, u_I and M_R, M_I are explicitly computed, and are given by

$$\begin{aligned} m_{R_1} &= \sqrt{\frac{1+\sqrt{1+\omega_1^2}}{2}},\ m_{I_1}=\sqrt{\frac{-1+\sqrt{1+\omega_1^2}}{2}}, \\ m_{R_2} &= \frac{1}{2}+\frac{1}{2}m_{R_1},\ m_{I_2}=\frac{1}{2}m_{I_1}, \\ m_{R_3} &= \frac{m_{R_1}}{m_{R_1}^2+m_{I_1}^2},\ m_{I_3}=\frac{m_{I_1}}{m_{R_1}^2+m_{I_1}^2}, \\ m_{R_4} &= m_{R_3}\left(R_4+4B'R_3\right)-m_{R_3}\left(R_3-4B'R_4\right), \\ m_{I_4} &= m_{r_3}\left(R_3-4B'R_4\right)-m_{I_3}\left(R_4+4B'R_3\right), \end{aligned}$$

$$
\begin{aligned}
m_{R_5} &= \frac{96A'}{R_5},\ m_{I_5} = \frac{2A'\left(\omega_1 - \frac{(24)^2}{\omega_1}\right)}{R_5}, \\
m_{R_6} &= \frac{R_6\left(R_7 + 4B'R_8\right) + R_9\left(R_8 - 4B'R_7\right)}{\left(R_6^2 + R_9^2\right)}, \\
m_{I_6} &= \frac{R_6\left(R_8 - 4B'R_7\right) + R_9\left(R_7 + 4B'R_8\right)}{\left(R_6^2 + R_9^2\right)},
\end{aligned}
$$

$$
\begin{aligned}
m_{R_7} &= -\frac{1}{\omega_1}\left(36m_{I_5} + 32B'\right),\ m_{R_7} = \frac{1}{\omega_1}\left(36m_{R_5} + 9A'\right) \\
m_{R_8} &= \frac{4}{3\omega_1^2}\left(13A' + 24\right),\ m_{I_8} = -\frac{32A'}{3\omega_1^3},
\end{aligned}
$$

$$
\begin{aligned}
m_{R_9} &= \frac{R_{16}\left(4R_{10} - 8B'm_{I_2} - R_{11} - R_{12}\right) + R_9\left(8B'm_{R_2}4R_{13} - 2R_{14} - R_{15}\right)}{R_9^2 + R_{16}^2}, \\
m_{I_9} &= \frac{R_{16}\left(8B'm_{R_2}4R_{13} - 2R_{14} - R_{15}\right) - R_9\left(4R_{10} - 8B'm_{I_2} - R_{11} - R_{12}\right)}{R_9^2 + R_{16}^2},
\end{aligned}
$$

$$
\begin{aligned}
m_{R_{10}} &= 1 - e^{-m_{R2}y}\left(\cos m_{I_2}y - 4B'\sin m_{I_2}y\right) \\
&\quad -\lambda_5\left(4B'e^{-m_{R_2}y}\sin m_{I_2}y - (1-y)\,e^{-y}\right) \\
&\quad +ye^{-m_{R_2}y}\left(m_{R_4}\cos m_{I_2}y + m_{I_4}\sin m_{I_2}y\right),
\end{aligned}
$$

$$
\begin{aligned}
m_{I_{10}} &= e^{-m_{R_2}y}\left(4B'\cos m_{I_2}y + \sin m_{I_2}y\right) - 4B'e^{-y} - 4B'\lambda_5 e^{-m_{R2}y}\cos m_{I_2}y \\
&\quad +ye^{-m_{R_2}y}\left(m_{I_4}\cos m_{I_2}y - m_{R_4}\sin m_{I_2}y\right),
\end{aligned}
$$

$$
\begin{aligned}
m_{R_{11}} &= \left(m_{R_5} - \lambda_5 m_{R_7}\right)\left(e^{-3y} - e^{-m_{R_2}y}\cos m_{I_{2y}}\right) \\
&\quad -\left(m_{I_2} - \lambda_5 m_{I_7}\right)e^{-m_{R_2}y}\sin m_{I_2}y - \lambda_5 m_{R_8}\left(e^{-y} - e^{-m_{R_2}y}\cos m_{I_2}y\right) \\
&\quad -\left(\frac{2B'}{3} + m_{I_8}\right)e^{-m_{R_2}y}\sin m_{I_2}y + \frac{32\lambda_5 A'e^{-3y}}{\omega_1^2} \\
&\quad +\left(m_{R_6} - \lambda_5 m_{R_9}\right)e^{-2y},
\end{aligned}
$$

$$
\begin{aligned}
m_{I_{11}} &= \left(m_{R_5} - \lambda_5 m_{R_7}\right) e^{-m_{R_2} y} \sin m_{I_2} y \\
&\quad + \left(m_{I_5} - \lambda_5 m_{I_7}\right)\left(e^{-3y} - e^{-m_{R_2} y} \cos m_{I_{2y}}\right) \\
&\quad - \lambda_5 m_{R_8} e^{-m_{R_2} y} \sin m_{I_2} y + \left(\frac{2B^{'}}{3} + m_{I_8}\right)\left(e^{-y} - e^{-m_{R_2} y} \cos m_{I_2} y\right) \\
&\quad - \left(\frac{12y + 3A^{'} y^2 - 2A^{'} y}{3\omega_1}\right) \lambda_5 e^{-3y} + \left(m_{I_6} - \lambda_5 m_{I_9}\right) e^{-2y},
\end{aligned}
$$

$$
\begin{aligned}
R_1 &= m_{R_2}^2 - m_{I_2}^2,\ R_2 = m_{I_2} + \frac{\omega_1}{4},\ R_3 = 2m_{R_2}^2 m_{I_2} + R_1 R_2, \\
R_4 &= m_{R_2} R_1 - 2m_{R_2} m_{I_2} R_2,\ R_5 = (24)^2 + \omega_1^2,\ R_6 = R_1 + 3m_{R_2} + 2, \\
R_7 &= R_1 - 2m_{R_2},\ R_8 = 2m_{R_2} m_{I_2} - 2m_{I_2},\ R_9 = 2m_{R_2} m_{I_2} + 3m_{I_2} - \frac{\omega_1}{4}, \\
R_{10} &= m_{R_2} m_{R_4} - m_{I_2} m_{I_4},\ R_{11} = R_1 + 2m_{R_4},\ R_{12} = R_1 m_{R_4} - 2m_{R_2} m_{I_2} m_{I_4}, \\
R_{13} &= m_{I_2} m_{R_4} - m_{R_2} m_{I_4},\ R_{14} = m_{I_4} + m_{R_2} m_{I_2},\ R_{15} = m_{I_4} R_1 + 2m_{R_2} m_{I_4}, \\
R_{16} &= R_1 + 3m_{R_2} - 2,\ B^{'} = \frac{A^{'}}{\omega_1}.
\end{aligned}
$$

The parameter functions $h_1, L, S_2, M_2, M_3, N_2, N_3, P_1$ and P_2 can be expressed in terms of these m_R's and m_I's as follows

$$
\begin{aligned}
h_1 &= \frac{1}{2} + \frac{1}{2} m_{R_1} + i\frac{1}{2} m_{I_1} = m_{R_2} + i m_{I_2},\ L = m_{R_4} + i m_{I_4},\ S_2 = 1 - 4iB^{'}, \\
M_2 &= m_{R_5} + i m_{I_5},\ M_3 = m_{R_7} + i m_{I_7},\ N_2 = \frac{2iB^{'}}{3}, \\
N_3 &= m_{R_8} + i m_{I_8},\ P_1 = m_{R_6} + i m_{I_6},\ P_2 = m_{R_9} + i m_{I_9}.
\end{aligned}
$$

5.3 Numerical Discussion

In order to investigate the effects of third grade fluid on the velocity profile near the plate (both in case of constant and variable suction), we have plotted u_R against y in Figs. 5.1 to 5.4 for different values of $\in, \varepsilon, A^{'}, \omega_1, \lambda_5$ and

$\omega_1\tau_1 = \pi/2$. From Figs. 5.1 and 5.2 we observe that the velocity profile increases with fixed ω and large values of ε. Fig. 5.3 is prepared to bring out the effect of variable suction velocity on the separation of the fluid at the plate for large frequency. It is evident from this figure that velocity increases with an increase in ω_1, A' and ε. Further, for fixed ε, increase in $\in, A'$ and ω_1 increases the velocity and then the two velocities coincide (see Fig. 5.4).

In Figs. 5.5–5.9 the fluctuating parts are plotted for different values of $\in, \varepsilon, \omega_1, \lambda_5, A'$ and for $\omega_1\tau = \pi/2$. For $A' = 0$, it is noted that an increase in ε with fixed $\in$ and ω_1 ($Fig.$ 5.5) leads to a decrease in M_R, but with increase in ε and $\in = 0.2$ and $\omega_1 = 10$, M_R is almost the same. Figure 5.6 shows the effect of ε in case of variable suction. In this case it is noted that increase in ε leads to a decrease in M_R first and then the curve tend to coincide. Further, it is clear from Fig. 5.7 that for $\varepsilon = 0.7$ and increase in A' and ω_1, results a decrease in M_R, and then the curves are almost the same. In case of third grade fluid at large ω_1 and increase in ε there is a fall of M_I (Fig. 5.8), which is not observed in Newtonian fluids. From Fig. 5.9, one can conclude that an increase in A' and ω_1 lead to an increase in M_I first; then there arises a decrease, then increase and finally it reaches to zero level.

5.4 Conclusions

In this chapter the unsteady flow past an infinite porous plate is studied under the following conditions:

- the suction velocity normal to the plate oscillates in magnitude but not in direction about a non-zero mean.

- the free stream velocity oscillates in time about a constant mean.

The solution obtained is the sum of steady and unsteady parts. The main findings of the analysis are summarized as:

1. There is a decrease and increase in the fluctuating parts M_R and M_I with the increase of third grade parameter ε and $A' \neq 0$.
2. Increase of variable suction velocity, ε and A' leads to increase in the velocity profile.
3. The velocity increases as the third grade fluid parameter increases.
4. The results for constant suction can be obtained by taking $A' = 0$.
5. The solution of second grade fluid for variable suction can be obtained as a special case of this problem by taking $\varepsilon = 0$.

As far as the authors are aware, no attempt has been made to examine the effects of variable suction velocity for second grade fluids. However, a second grade fluid exhibits normal stresses but is not shear thinning; the shear viscosity is constant. The third grade approximation of a simple fluid exhibits shear dependent viscosity. Keeping this fact in view, the problem considered for the third grade fluid in this chapter is more general.

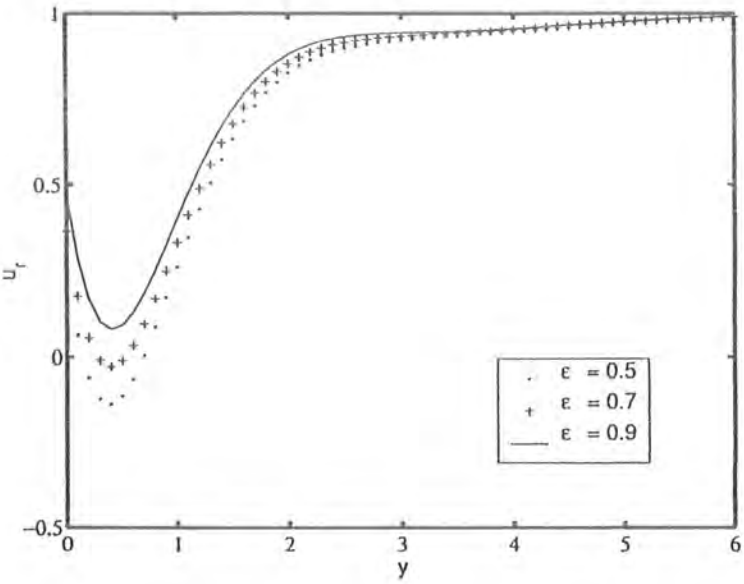

Fig. 5.1. Graphs for parameter values

$\lambda_5 = 0.7, \in= 0.5, \omega_1\tau_1 = \pi/2, A' = 0, \omega_1 = 10.$

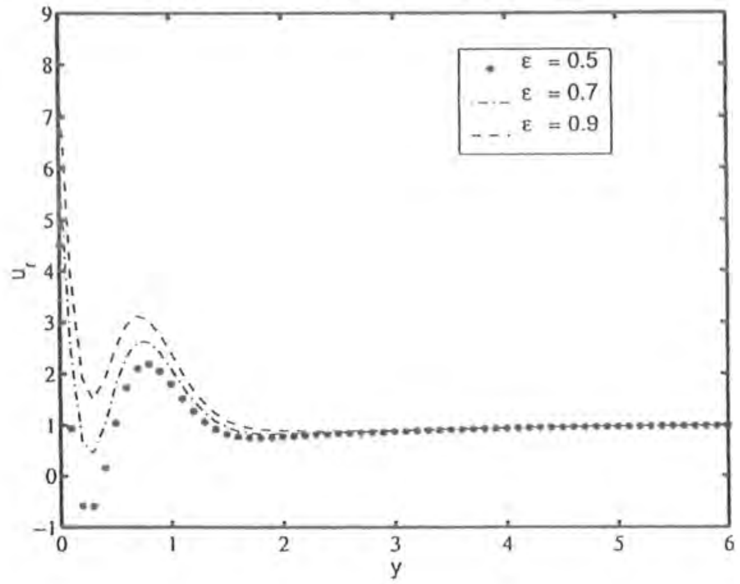

Fig. 5.2. Graphs for parameter values

$\lambda_5 = 0.8, \in= 0.5, \omega_1\tau_1 = \pi/2, A' = 0, \omega_1 = 100.$

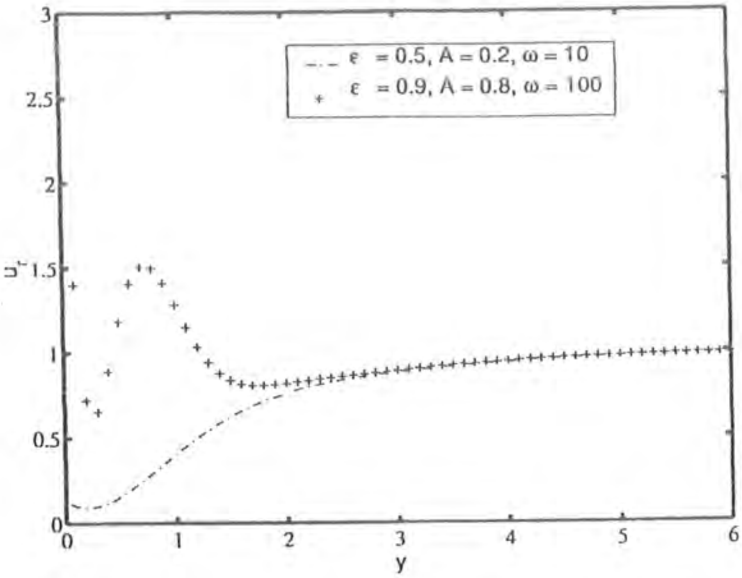

Fig. 5.3. Graphs for parameter values $\lambda_5 = 0.8, \in= 0.2, \omega_1\tau_1 = \pi/2$.

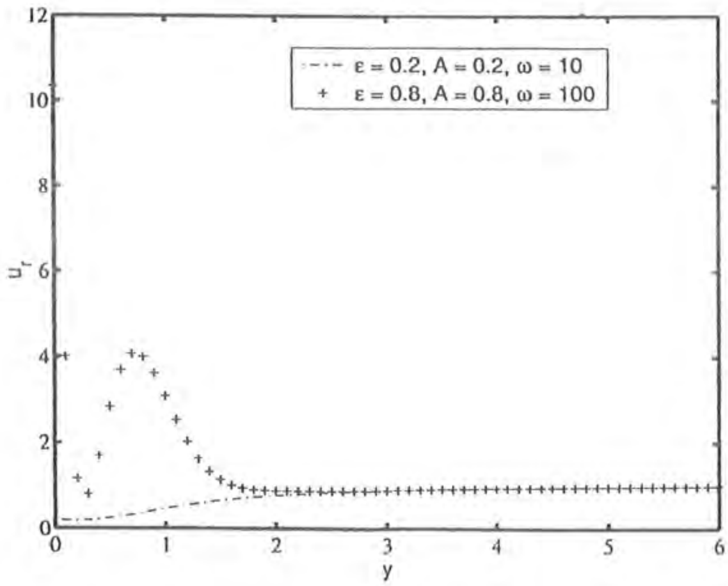

Fig. 5.4. Graphs for parameter values $\lambda_5 = 0.8, \in= 0.2, \omega_1\tau_1 = \pi/2$.

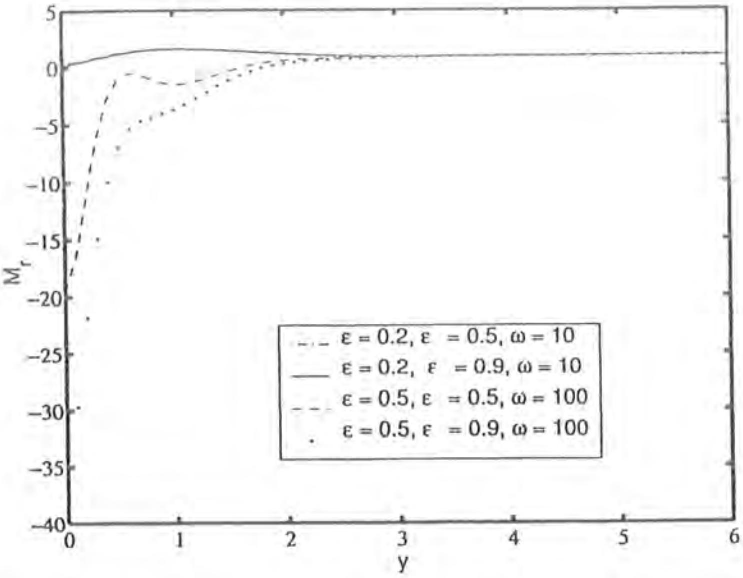

Fig. 5.5. Graphs for parameter values $\lambda_5 = 0.6, A' = 0, \omega_1\tau_1 = \pi/2$.

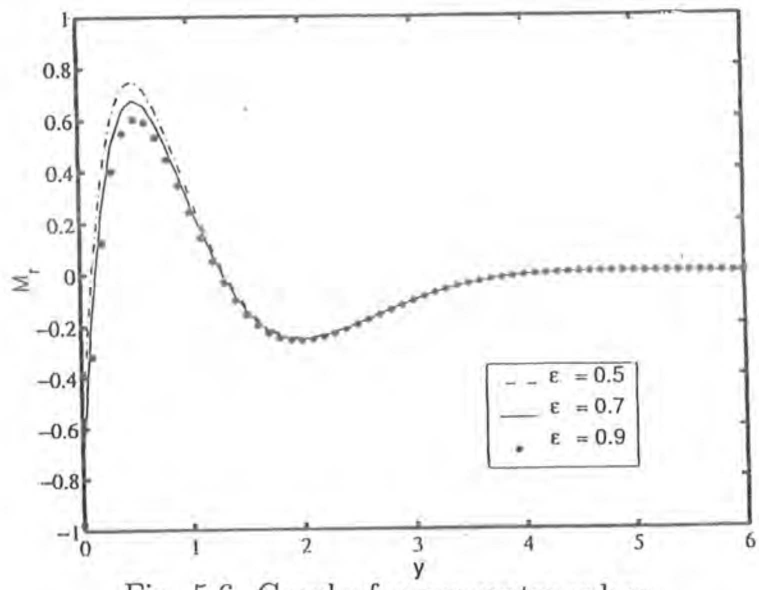

Fig. 5.6. Graphs for parameter values

$\lambda_5 = 0.7, \in= 0.2, A' = 0.4, \omega = 10, \omega_1\tau_1 = \pi/2$.

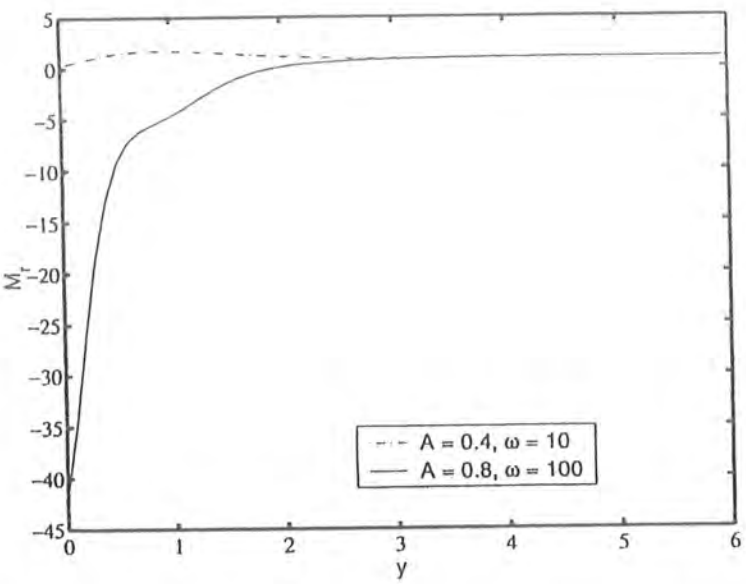

Fig. 5.7. Graphs for parameter values $\lambda_5 = 0.9, \in= 0.2, \varepsilon = 0.7, \omega_1\tau_1 = \pi/2$.

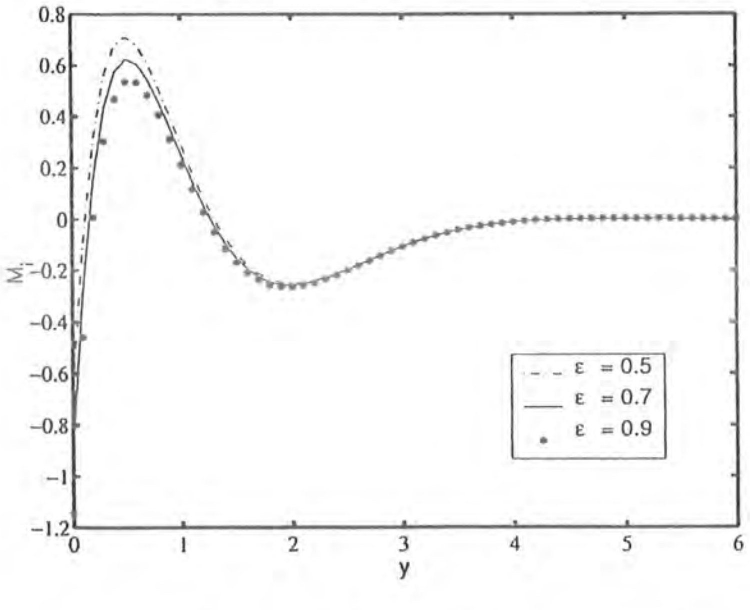

Fig. 5.8. Graphs for parameter values

$\lambda_5 = 0.8, \in= 0.2, A' = 0.4, \omega = 10, \omega_1\tau_1 = \pi/2$.

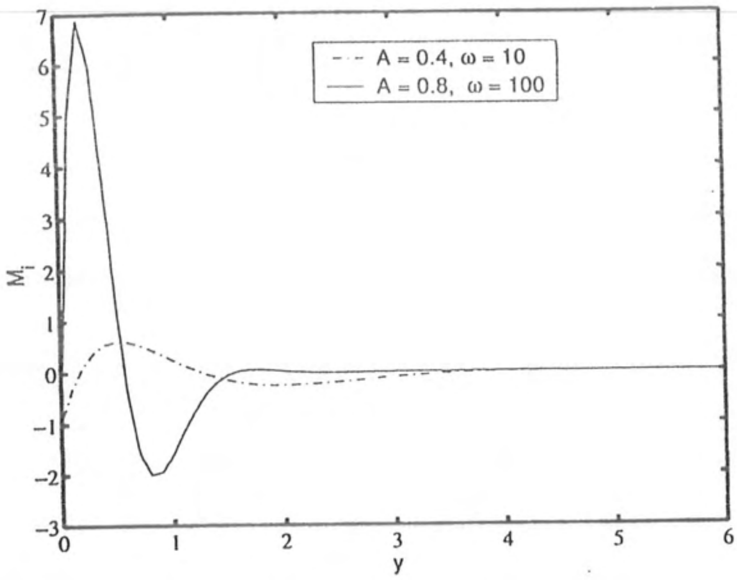

Fig. 5.9. Graphs for parameter values $\lambda_5 = 0.7, \in= 0.2, \varepsilon = 0.9, \omega_1\tau_1 = \pi/2$.

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An Oscillating Hydromagnetic Non-Newtonian Flow in a Rotating System

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Abstract—An exact solution of an oscillatory boundary layer flow bounded by two horizontal flat plates, one of which is oscillating in its own plane and the other at rest, is developed. The fluid and the plates are in a state of solid body rotation with constant angular velocity about the z-axis normal to the plates. The fluid is assumed to be second grade, incompressible, and electrically conducting. A uniform transverse magnetic field is applied. During the mathematical analysis, it is found that the steady part of the solution is identical to that of viscous fluid. The structure of the boundary layers is also discussed. Several known results of interest are found to follow as particular cases of the solution of the problem considered.

Keywords—Second grade fluid, Parallel plates, Resonance, Magnetohydrodynamic fluid, Oscillation flow.

1. INTRODUCTION

The analysis of the effects of rotation and magnetic field in fluid flows has been an active area of research because of its geophysical and technological importance. Interest in MHD flow began in 1918, when Hartmann [1] invented the electromagnetic pump. The study of magnetic field effects on the laminar flow of an incompressible electrically conducting fluid is an important problem that is related to many practical applications, such as the MHD power generator and boundary layer flow control. Historically, Rossow [2] was the first to study the hydrodynamic behavior of the

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boundary layer on a semi-infinite flat plate in the presence of a uniform transverse magnetic field. Since then, a large amount of literature has been developed on this subject. A review of this topic today can be found in [3]. The effects of a uniform transverse magnetic field were investigated by Gupta [4], Debnath [5], Soundalgekar and Pop [6], Mazumder *et al.* [7]. In another paper, Mazumder [8] investigated the unsteady oscillatory coquette flow of a viscous incompressible fluid between two parallel plates, one of which is oscillating and the other at rest. Later, Ganapthy [9] proposed an alternative solution for the problem in [8]. More recently, Singh [10] extended this analysis by including the magnetic field effects. Despite the above studies, attention has hardly been given to the study of hydromagnetic flows of non-Newtonian fluids. The study of non-Newtonian fluid dynamics is important in connection with plastics manufacture, performance of lubricants, applications of paints, processing of food, and movement of biological fluids. Most biologically important fluids contain higher molecular weight components and are, therefore, non-Newtonian.

Keeping in view the importance of non-Newtonian fluids, the main objective of this communication is to present an exact solution to the study of oscillatory flow between two parallel plates. The fluid considered is electrically conducting and second grade. The entire system rotates about an axis perpendicular to the planes of plates. The study of hydromagnetic flow of second grade fluid between two horizontal plates responds to oscillations in one plate in a rotating system has remained attented. For $\beta_1 = 0$, the problem reduces to the one discussed by Singh [10]. Moreover, for $\beta_1 = 0 = M$, the problem reduces to that of Ganapathy [9]. Similar to Singh [10] it is noted that the claim of Ganapathy [9] that the solution of Mazumder [8] explains resonance phenomenon in rotating system is incorrect.

2. GOVERNING EQUATIONS

The physical situation considered is that of the unsteady hydromagnetic flow of a second grade, incompressible, and electrically conducting fluid bounded by infinite-parallel plates, distant d apart, when both the fluid and plate rotate with a constant angular velocity Ω about the z-axis taken normal to the plates. It is assumed that the plates are electrically nonconducting and an applied uniform magnetic field $\mathbf{B}_0$ is acting parallel to the z-axis. The lower plate is at rest and the upper plate oscillating in its own plane with a velocity $U(t) = U_0(1+ \in \cos\omega t)$ about a nonzero constant mean velocity U_0. The origin is taken on the lower plate and the x-axis parallel to the direction of motion of the upper plate. Since the plates are infinite in extent, all the physical quantities, except the pressure, depend on z and t only. In a coordinate system rotating with the fluid, the governing equations of continuity and motion and Maxwell's equations are

$$\nabla.\mathbf{V} = 0, \tag{1}$$

$$\rho\left[\frac{\partial \mathbf{V}}{\partial t} + (\mathbf{V}.\nabla)\,\mathbf{V} + 2\mathbf{\Omega}\times\mathbf{V} + \mathbf{\Omega}\times(\mathbf{\Omega}\times\mathbf{r})\right] = \nabla.\mathbf{T} + \mathbf{J}\times\mathbf{B}, \tag{2}$$

$$\nabla\times\mathbf{B} = \mu_e\mathbf{J}, \tag{3}$$

$$\nabla\times\mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}, \tag{4}$$

$$\nabla.\mathbf{B} = 0, \tag{5}$$

$$\mathbf{j} = \sigma\,(\mathbf{E} + \mathbf{V}\times\mathbf{B})\,, \tag{6}$$

where $\mathbf{V} = (u, v, w)$ is the velocity vector, $\Omega = \Omega\mathbf{k}$, $\mathbf{k}$, the unit vector in the z-direction, t, the time, ρ, the density, $\mathbf{J}$, the current density, $\mathbf{B}$, the magnetic induction, $\mathbf{E}$, the electric field, μ_e, the magnetic permeability, σ, the electrical conductivity, $\mathbf{T}$, the Cauchy stress, and $\mathbf{r}$, the radial coordinate given by

$$r^2 = x^2 + y^2. \tag{7}$$

The constitutive equation for Cauchy stress tensor $\mathbf{T}$ is [11,12]

$$\mathbf{T} = -p\mathbf{I} + \mu\mathbf{A}_1 + \alpha_1\mathbf{A}_2 + \alpha_2\mathbf{A}_1^2. \tag{8}$$

Here p is the dynamic pressure function, $\mathbf{I}$ the unit tensor, μ the constant dynamic viscosity, and α_1, α_2 the normal stress moduli, $\mathbf{A}_1$ and $\mathbf{A}_2$ are the Rivlin-Eriksen tensors [13] and are given by

$$\mathbf{A}_1 = (\operatorname{grad}\mathbf{V}) + (\operatorname{grad}\mathbf{V})^{\top}, \tag{9}$$

$$\mathbf{A}_2 = \frac{d\mathbf{A}_1}{dt} + (\operatorname{grad}\mathbf{V})^{\top}\mathbf{A}_1 + \mathbf{A}_1(\operatorname{grad}\mathbf{V}), \tag{10}$$

where $\frac{d}{dt}$ is the material time derivative. According to Dunn and Fosdick [14], the second grade fluid model is compatible with thermodynamics when the Helmholtz free energy of the fluid is a minimum for the fluid in equilibrium. The fluid model then has general and pleasant boundedness and stability properties. The aforementioned and the Clausius-Duhem inequality imply that the coefficients μ, α_1, and α_2 must satisfy

$$\mu \geq 0, \quad \alpha_1 \geq 0, \qquad \alpha_1 + \alpha_2 = 0. \tag{11}$$

Since we are dealing with second grade fluid flow, the strict inequality holds true. Fosdick and Rajagopal [15] showed that when $\alpha_1 < 0$, the fluid exhibits anomalous behavior that is incompatible with any fluid of rheological interest, and so results in a fluid that is unstable. Further, we assume that the magnetic Reynolds number is so small that the induced magnetic field can be neglected in comparison with the applied one [16], so that

$$\mathbf{B} = (0, 0, B_0), \tag{12}$$

where B_0 is a constant. It also assumed that no applied and polarization voltage exists (i.e., $\mathbf{E} = 0$). This then corresponds to the case when no energy is added to or extracted from the fluid by the electric field. Since the plates are infinite in extent, all physical variables (except pressure) are functions of z and t only. Thus, equation (1) becomes $\frac{\partial w}{\partial z} = 0$ which, because $w = 0$ on the boundaries, implies that $w = 0$ every where in the fluid. Now, the equation for the conservation of electric charge, $\nabla.\mathbf{J} = 0$, leads to $J_z =$ constant, where $\mathbf{J} = (J_x, J_y, J_z)$. As in the case of vertical velocity, we immediately see that $J_z = 0$. Thus, equation (6) yields

$$J_x = \sigma B_0^2 v, \qquad J_y = -\sigma B_0^2 u. \tag{13}$$

In view of the above consideration, equation (2) can be rewritten in the component form

$$\frac{\partial u}{\partial t} = -\frac{1}{\rho}\frac{\partial p^*}{\partial x} + \left(\nu + \beta_1\frac{\partial}{\partial t}\right)\frac{\partial^2 u}{\partial z^2} + 2\Omega v - \frac{\sigma B_0^2 u}{\rho}, \tag{14}$$

$$\frac{\partial v}{\partial t} = -\frac{1}{\rho}\frac{\partial p^*}{\partial y} + \left(\nu + \beta_1\frac{\partial}{\partial t}\right)\frac{\partial^2 v}{\partial z^2} - 2\Omega u - \frac{\sigma B_0^2 v}{\rho}, \tag{15}$$

where ν is the kinematic viscosity, $\beta_1 = \alpha_1/\rho$ and the modified pressure

$$p^* = p - \frac{\rho}{2}r^2\Omega^2 - (2\alpha_1 + \alpha_2)\left[\left(\frac{\partial u}{\partial z}\right)^2 + \left(\frac{\partial v}{\partial z}\right)^2\right]. \tag{16}$$

The boundary conditions for the present problem are

$$\begin{aligned} u = v = 0, &\qquad \text{at } z = 0, \\ u = U(t) = U_0(1 + \epsilon\cos\omega t), \quad v = 0, &\qquad \text{at } z = d, \end{aligned} \tag{17}$$

where ω is the frequency of oscillations and ϵ is a constant.

Now eliminating the modified pressure gradient, under the boundary layer approximation, the resulting equation of (14) and (15) can be combined as

$$\frac{\partial q}{\partial t} = \left(\nu + \beta_1 \frac{\partial}{\partial t}\right) \frac{\partial^2 q}{\partial z^2} + \frac{\partial U}{\partial t} - 2i\Omega\,(q-U) - \frac{\sigma B_0^2}{\rho}\,(q-U)\,, \tag{18}$$

and the corresponding boundary conditions (17) are

$$\begin{aligned} q &= 0, &\quad &\text{at } z = 0, \\ q &= U(t), &\quad &\text{at } z = d, \end{aligned} \tag{19}$$

where

$$q = u + iv \tag{20}$$

is the fluid velocity in the complex form. It should be noted that equation (18) includes the Navier-Stokes fluid as a special case for $\beta_1 = 0$. If $\Omega = 0$, the equation reduces to that of second grade fluid in an inertial frame. Moreover, if $B_0 = 0$, the equation governing the flow of a nonconducting second grade fluid is obtained.

3. SOLUTION OF THE PROBLEM

In order to solve equation (18) subject to the boundary conditions (19), we look for the solution of the form [17]

$$q\,(\eta,t) = U_0\left[q_0\,(\eta) + \frac{\epsilon}{2}\left\{q_1\,(\eta)\,e^{i\omega t} + q_2\,(\eta)\,e^{-i\omega t}\right\}\right], \tag{21}$$

where

$$\eta = \frac{z}{d}, \qquad q_0\,(\eta) = u_0(\eta) + iv_0(\eta) \quad \text{and} \quad q_1\,(\eta)\,e^{i\omega t} + q_2\,(\eta)\,e^{-i\omega t} = u_1\,(\eta,t) + iv_1\,(\eta,t). \tag{22}$$

Using equation (21) into equation (18) and boundary condition (19) and then collecting harmonic and nonharmonic terms, we obtain

$$\frac{d^2 q_0}{d\eta^2} - \left(2iK + M^2\right) q_0 = -\left(2iK + M^2\right), \tag{23}$$

$$\frac{d^2 q_1}{d\eta^2} - \left(1 + \frac{i\omega\beta_1}{\upsilon}\right)^{-1}\left[2iK + M^2 + i\lambda\right] q_1 = -\left(1 + \frac{i\omega\beta_1}{\upsilon}\right)^{-1}\left[2iK + M^2 + i\lambda\right], \tag{24}$$

$$\frac{d^2 q_2}{d\eta^2} - \left(1 + \frac{i\omega\beta_1}{\upsilon}\right)^{-1}\left[2iK + M^2 - i\lambda\right] q_2 = -\left(1 + \frac{i\omega\beta_1}{\upsilon}\right)^{-1}\left[2iK + M^2 - i\lambda\right], \tag{25}$$

$$\begin{aligned} q_0 = q_1 = q_2 &= 0, &\quad &\text{at } \eta = 0, \\ q_0 = q_1 = q_2 &= 1, &\quad &\text{at } \eta = 1. \end{aligned} \tag{26}$$

In the above equations, $K = \Omega d^2/\nu$ is the rotation parameter, $\lambda = \omega d^2/\nu$ is an oscillatory Reynolds number and $M = B_0 d(\sigma/\mu)^{1/2}$ is the Hartman number. Solving equations (23)–(25) under boundary conditions (26), we get

$$q_0\,(\eta) = 1 - \frac{\sinh l\,(1-\eta)}{\sinh l}, \tag{27}$$

$$q_1\,(\eta) = 1 - \frac{\sinh m\,(1-\eta)}{\sinh m}, \tag{28}$$

$$q_2\,(\eta) = 1 - \frac{\sinh n\,(1-\eta)}{\sinh n}, \tag{29}$$

where

$$l = \left(2iK + M^2\right)^{1/2}, \qquad m = \left[\frac{2iK + M^2 + i\lambda}{(1 + i\omega\beta_1/\upsilon)}\right]^{1/2}, \qquad n = \left[\frac{2iK + M^2 - i\lambda}{(1 + i\omega\beta_1/\upsilon)}\right]^{1/2}.$$

We note from equations (27)–(29) that if $\beta_1 = 0$, then results of Singh [10] are recovered.

4. RESULTS AND DISCUSSION

We note that steady solution (27) is independent on β_1. It means that primary and secondary velocity components u_0 and v_0, respectively, for present steady flow do not depend upon the nature of the fluid. The amplitudes and phase differences in terms of u_0 and v_0 are given by

$$R_0 = \sqrt{u_0^2 + v_0^2}, \qquad \theta_0 = \tan^{-1}\frac{v_0}{u_0}.$$

From equations (27), we have for large K

$$\begin{aligned} u_0 &\approx 1 - e^{-l_r\eta}\cos l_i\eta, \\ v_0 &\approx e^{-l_r\eta}\sin l_i\eta, \end{aligned} \tag{30}$$

where

$$l_r = \frac{1}{\sqrt{2}}\left[M^2 + \sqrt{M^4 + 4K^2}\right]^{1/2}, \qquad l_i = \frac{1}{\sqrt{2}}\left[-M^2 + \sqrt{M^4 + 4K^2}\right]^{1/2}. \tag{31}$$

Equations (30) represent spiral distribution. The thickness of boundary layer is of $O(l_r^{-1})$ in the neighborhood of plates. We conclude with the help of equation (31) that the thickness is reduced as M or K is increased.

Solutions (28) and (29) together give the unsteady part of the flow. These solutions depend on β_1. For large K, the primary and secondary velocity components u_1 and v_1, respectively, for the fluctuating flow are given by

$$u_1(\eta, t) \approx 2\cos\omega t - e^{-m_r\eta}\cos(m_i\eta - \omega t) - e^{-n_r\eta}\cos(n_i\eta + \omega t), \tag{32}$$

$$v_1(\eta, t) \approx e^{-m_r\eta}\sin(m_i\eta - \omega t) + e^{-n_r\eta}\sin(n_i\eta + \omega t), \tag{33}$$

where

$$m_r = (C_1)^{-1}\left[\sqrt{\sqrt{A_1^2 + B_1^2} + A_1}\right],$$

$$m_i = (C_1)^{-1}\left[\sqrt{\sqrt{A_1^2 + B_1^2} - A_1}\right],$$

$$n_r = (C_1)^{-1}\left[\sqrt{\sqrt{A_2^2 + B_2^2} + A_2}\right],$$

$$n_i = (C_1)^{-1}\left[\sqrt{\sqrt{A_2^2 + B_2^2} - A_2}\right],$$

$$A_1 = M^2 + \frac{(2K + \lambda)\,\omega\beta_1}{\nu},$$

$$B_1 = (2K + \lambda) - \frac{M^2\omega\beta_1}{\nu},$$

$$A_2 = M^2 + \frac{(2K - \lambda)\,\omega\beta_1}{\nu},$$

$$B_2 = (2K - \lambda) - \frac{M^2\omega\beta_1}{\nu},$$

$$C_1 = \sqrt{2\left[1 + \frac{\omega^2\beta_1^2}{\nu^2}\right]}.$$

Expressions (32) and (33) for u_1 and v_1 show the emergence of a boundary layer of thickness of order $O(m_r^{-1})$ superimposed with a boundary layer of thickness of order $O(n_r^{-1})$. These boundary layers which are a direct consequence of the cyclonic and anticyclonic components of the impressed harmonic oscillations decrease with increase in M, β_1, and K. It may be noted that in second grade fluid the boundary layer thickness decrease more rapidly than for viscous fluid. Also, the present analysis exhibits a striking difference between the structure of hydrodynamic and the hydromagnetic boundary layers.

In case of resonance ($2\Omega - \omega = 0$ or $2K - \lambda = 0$), the solution of equation (25) is

$$q_2(\eta) = 1 - \frac{\sinh \tilde{M}\,(1-\eta)}{\sinh \tilde{M}}, \tag{34}$$

where

$$\tilde{M}^2 = M^2 \left(1 + \frac{i\omega\beta_1}{\upsilon}\right)^{-1}. \tag{35}$$

We note that when $M = 0$, then $q_2(\eta)$ for viscous and second grade fluids is the same and is given by

$$q_2(\eta) = \eta.$$

This above solution is the same as obtained by Singh [10]. Even in the case of resonance, differential equation (5) in [9] yields $q_2(\eta) = \eta \neq 0$. This is a contradiction to the claim of Ganapathy that $q_2(\eta) = 0$ and the solution

$$q(\eta, t) = U_0 \left[q_0(\eta) + \frac{\epsilon}{2} q_1(\eta)\, e^{i\omega t}\right],$$

of Mazumder is valid for the special case.

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Unsteady hydromagnetic rotating flow of a conducting second grade fluid
T. Hayat

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Unsteady Hydromagnetic Rotating Flow of a Conducting Non-Newtonian Fluid

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Summary. The purpose of this work is to investigate the hydromagnetic oscillatory flow of a fluid bounded by a porous plate, when the entire system rotates about an axis normal to the plate. The fluid is assumed to be non-Newtonian (second grade), incompressible and electrically conducting. The magnetic field is applied transversely to the direction of the flow. Such a flow model has great significance not only of its theoretical interest, but also for applications to geophysics and engineering. The resulting initial value problem has been solved analytically by applying Laplace transform technique and explicit expression for the velocity for steady and unsteady cases has been constructed. The analysis of the obtained results showed that the flow field is appreciably influenced by the material parameter of the second grade fluid, the applied magnetic field, the imposed frequency, rotation and suction and blowing parameters. It is observed in a second grade fluid that a steady asymptotic hydromagnetic solution exists for blowing and resonance which is different from the hydrodynamic situation.

1. Introduction

Flows of fluids through porous media are of principle interest because these are quite prevalent in nature. Such flows have attracted the attention of a number of scholars due to their applications in many branches of science and technology, viz. in the field of agriculture engineering to study the under ground water resources, seepage of water in river beds, in petroleum technology to study the movement of natural gas, oil and water through the oil reservoirs, in chemical engineering for filtration and purification processes. In view of the geophysical applications of the flows through a porous medium, Ratins [1] and Raptis and Perdikis [2] studied the unsteady flow through a porous medium bounded by an infinite porous plate.

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Moreover, the increasing number of technical applications using magnetohydrodynamic (MHD) effects has made it desirable to extend many of the available hydrodynamic solutions to include the effects of magnetic fields for those cases when the fluid is electrically conducting . For example, liquid-metal MHDs take their roots in conventional hydrodynamics of incompressible media, which become important in the metallurgical industry, in nuclear reactor sodium cooling system and in the energy storage and electrical power generation [3-5]. The greatest advantage of induction-type pumps over other types of MHD devices is the absence of electrodes [6]. Induction pumps have been used to pump coolents in nuclear reactors. These designs are also being considered in MHD power generation [7].

The scientific research of fluid systems in rotating environments has considerable bearing on problems of geophysical and astrophysical interest and fluid engineering applications. An extensive literature exists on the flow of viscous fluids in a rotating frame. Recently, Singh [8] has studied the oscillatory hydromagnetic Couette flow in a rotating system. The unsteady hydromagnetic flows of a viscous fluid in a rotating frame have been investigated by a number of workers. Mention may be made of works of Singh and Sathi [9], Gupta [10], Soundalgekar and Pop [11], Mazumder [12], Ganapathy [13] and Debnath [14].

In all these studies, however, only limited attention has been devoted to flows of non-Newtonian fluids in a rotating frame. Few studies illustrating the flows of non-Newtonian fluids in a rotating frame have been reported, see Puri [15], Puri and Kulshrestha [16] and Hayat et al. [17]. Nevertheless the study of non-Newtonian fluid dynamics is important in connection with plastic manufacture, performance of lubricants, application of paints, processing of food, and movement of biological and geophysical fluids. Most biologically important fluids contain higher molecular weight components and are, therefore, non-Newtonian. The unusual properties of polymer melts and solutions, together with the desirable attributes of many polymeric solids, have given rise to the world-wide industry of polymer processing. Alternatively, geophysical applications concern ice flow, magna flow, flows of the lowes lithsphase which are based on non-Newtonian constitutive behaviors.

In this paper, we consider the unsteady motion of an electrically, rotating, second grade fluid, initially at rest, occupying a half space and bounded by an infinite porous plate, also initially at rest. The fluid and the plate are in a state of rigid body rotation with constant angular velocity Ω about the $z-axis$ normal to the plate. Additionally, from $t = 0^+$, the plate executes small amplitude non-torsional oscillations with given frequency in its own plane. The equation of motion and constitutive equations, together with the inequalities satisfied by the material constants, are stated in section 2. The present problem is also formulated in section 2, along with the initial and boundary conditions. In section 3, the method of Laplace transform is applied to obtain the solution for the velocity. The difficulty that arises is the solution of the integrals occurring in the inverse Laplace transform. These integrals are normally difficult to handle. However, the integrals in the inverse Laplace transform are solved analytically and the expression for the velocity field is obtained. The structure of the associated hydromagnetic multiple boundary layers is examined in section 4. For the case of blowing and resonance, it is once again found in a second grade fluid that the hydromagnetic steady solution satisfies the boundary condition at infinity unlike the hydrodynamic solution. Lastly, the results are also compared to the viscous fluid [18].

2. Governing Equations

Consider an unsteady hydromagnetic flow of an electrically conducting, incompressible, rotating second order fluid bounded by a porous plate at $z = 0$. At time $t < 0$, the fluid and the plate are assumed to be at rest. A uniform magnetic field B_0 is applied parallel to the $z-axis$. In the undisturbed state, both the fluid and the plate are in a state of solid body rotation with a uniform angular velocity $\boldsymbol{\Omega}$ about the $z - axis$ normal to the plate. The hydromagnetic flow is generated in the uniformly rotating fluid system by non-torsional oscillations of frequency ω_1 of the plate in its own plane at $t = 0^+$.

The hydromagnetic equations of motion and continuity in a rotating frame of reference for an electrically conducting, second order incompressible fluid are

$$\frac{\partial \mathbf{V}}{\partial t} + (\mathbf{V} \cdot \nabla)\mathbf{V} + 2\Omega\mathbf{k} \times \mathbf{V} + \Omega\mathbf{k} \times (\Omega\mathbf{k} \times \mathbf{r}) = \frac{1}{\rho}\nabla \cdot \mathbf{T} + \frac{1}{\rho}\mathbf{J} \times \mathbf{B}, \qquad (1)$$

$$\begin{aligned} \nabla.\mathbf{V} &= 0, & (2)\\ \nabla \times \mathbf{B} &= \mu_e \mathbf{J}, & (3)\\ \nabla \times \mathbf{E} &= -\frac{\partial \mathbf{B}}{\partial t}, & (4)\\ \nabla.\mathbf{B} &= 0, & (5) \end{aligned}$$

and Ohm's law for a moving conductor

$$\mathbf{J} = \sigma[\mathbf{E} + \mathbf{V} \times \mathbf{B}], \qquad (6)$$

where $\mathbf{V}$ is the velocity vector, $\mathbf{B}$ is the magnetic induction, $\mathbf{J}$ is the current density, $\mathbf{E}$ is the electric field relative to the rotating frame, ρ is the density, μ_e is the magnetic permeability, σ is the electric conductivity, $\mathbf{k}$ is the unit vector along the $z-axis$, $\mathbf{r}$ is the position vector from the axis of rotation and $\mathbf{T}$ the Cauchy stress tensor. The last term on the right hand side of Eq. (1) represent the pondermotive force on the conducting fluid due to the interaction of $\mathbf{J}$ and $\mathbf{B}$, known as Lorentz force.

Since the plate is infinite along the x and y directions, all physical quantities except the pressure will be functions of z and t only. It may be noted that the following assumptions are compatible with Eqs. (1) to (5) :

$$\begin{aligned} \mathbf{V} &= (u,\ v,\ w)\,,\ \mathbf{B} = (B_x, B_y, 0)\,, \\ \mathbf{E} &= (E_x, E_y, E_z)\,,\ \mathbf{J} = (J_x, J_y, 0)\,. \end{aligned} \tag{7}$$

We shall assume that the induced magnetic field produced by the motion of the electrically conducting fluid is negligible because the magnetic Reynolds number is very small for metallic or slightly ionised fluids so that we can take $\mathbf{B} = (0, 0, B_0)$. In the present problem, it is assumed that no applied or polarization voltages exist i.e. $\mathbf{E}$ =0. This then corresponds to the case where no energy is added or extracted from the fluid by the electric field.

The Cauchy stress tensor $\mathbf{T}$ for an incompressible homogeneous fluid of second grade is

$$\mathbf{T} = -p_1\mathbf{I} + \mu\mathbf{A}_1 + \alpha_1\mathbf{A}_2 + \alpha_2\mathbf{A}_1^2. \tag{8}$$

Here, $-p_1\mathbf{I}$ is the indeterminate part of the stress due to constraint of incompressibility, μ is the dynamic viscosity, α_1 and α_2 are two material constants commonly referred to as normal moduli. The kinematic tensors $\mathbf{A}_1$ and $\mathbf{A}_2$ are defined through [19]

$$\begin{aligned} \mathbf{A}_1 &= (grad\ \mathbf{V}) + (grad\ \mathbf{V})^T, \\ \mathbf{A}_2 &= \frac{d\mathbf{A}_1}{dt} + (grad\ \mathbf{V})^T\mathbf{A}_1 + \mathbf{A}_1(grad\ \mathbf{V}), \end{aligned} \tag{9}$$

where $grad$ denotes the gradient operator and d/dt the material time derivative.

Since the model (8) is invariant, it has been considered as an exact model for some fluid. If the fluid has to be compatible with thermodynamics in the sense that the Clausius-Duhem inequality is met in all motions, then the following conditions have to be satisfied [20]

$$\mu \geq 0, \; \alpha_1 + \alpha_2 = 0. \tag{10}$$

Dunn and Rajagopal [21] critically examined the status of the fluids of differential type. We shall not get involved in a lengthy discussion of the issues here and simply state that, if the material parameter α_1, is negative, it follows that the fluid exhibits undesirable stability properties [22]. In the present study we shall restrict ourselves to the case:

$$\alpha_1 \geq 0. \tag{11}$$

Using these assumptions, Eq. (2) gives $w = -W_0$ ($W_0 > 0$ for the suction and $W_0 < 0$ for blowing) while, Eq. (1) in the absence of pressure gradient reduces to

$$\frac{\partial q}{\partial t} - W_0 \frac{\partial q}{\partial z} + (2i\Omega + n^*)\, q = \nu \frac{\partial^2 q}{\partial z^2} + \beta_1 \left(\frac{\partial^3 q}{\partial z^2 \partial t} - W_0 \frac{\partial^3 q}{\partial z^3} \right), \; z \geq 0, \tag{12}$$

where

$$q = u + iv, \; n^* = \frac{\sigma B_0^2}{\rho}, \; \nu = \mu/\rho, \; \beta_1 = \frac{\alpha_1}{\rho}.$$

The initial and boundary conditions are

$$\begin{aligned} q &= 0, \; \textit{for all} \;\; z > 0 \;\; \textit{and} \;\; t \leq 0, \\ q &= -U + U^*(ae^{i\omega_1 t} + be^{-i\omega_1 t}), \;\; \textit{at} \;\; z = 0, \; t > 0, \\ q &\to 0, \;\; \textit{as} \;\; z \to \infty, \; t > 0, \end{aligned} \tag{13}$$

where U and U^* are the real constants with the dimension of velocity and a, b, are complex constants.

Using the non-dimensional variables

$$\widehat{z} = zU^*/\nu, \; \widehat{t} = \Omega t, \; \widehat{q} = q/U^*,$$

equations (12) and (13) after dropping the hats become

$$\frac{\partial^2 q}{\partial z^2} + \beta_1 \left(\nu_1 \frac{\partial^3 q}{\partial z^2 \partial t} - \nu_2 S \frac{\partial^3 q}{\partial z^3} \right) + S \frac{\partial q}{\partial z} = \frac{E}{2} \frac{\partial q}{\partial t} + (iE + n)\, q, \tag{15}$$

$$q = 0, \;\; \textit{for all} \; z > 0 \;\; \textit{and} \;\; t \leq 0, \tag{16}$$

$$
\begin{aligned}
q &= -U/U^* + (ae^{i\sigma_1 t} + be^{-i\sigma_1 t}), \quad at \;\; z=0, \; t>0, \\
q &\rightarrow 0, as \;\; z \rightarrow \infty, \; t>0,
\end{aligned} \tag{17}
$$

where

$$
S = W_0/U^*, \; E = 2\Omega\nu/U^{*2}, \; \sigma_1 = \omega_1/\Omega, \; \nu_1 = \Omega/\nu, \; \nu_2 = U^{*2}/\nu^2, \; n = \frac{n^*\nu}{U^{*2}}.
$$

Equations (15) – (17) comprise the initial boundary value problem we are now going to solve.

3. Solution of the problem

Let

$$
\overline{q}(z,p) = \int_0^\infty q(z,t)\; e^{-pt}dt, \; (\mathrm{Re}\, p > 0) \tag{18}
$$

be the Laplace transform of q. Then multiplying Eq. (15) and conditions (17), respectively by e^{-pt} and integrating between the limits 0 to ∞ and using (16) yields the following transformed boundary value problem

$$
\frac{d^2\overline{q}}{dz^2} + S\frac{d\overline{q}}{dz} - \left(\frac{E}{2}p + iE + n\right)\overline{q} + \beta_1\left(p\nu_1\frac{d^2\overline{q}}{dz^2} - \nu_2 S\frac{d^3\overline{q}}{dz^3}\right) = 0, \tag{19}
$$

$$
\begin{aligned}
\overline{q} &= -\frac{U}{U^*p} + \frac{a}{p-i\sigma_1} + \frac{b}{p+i\sigma_1}, \; at \; z=0, \\
\overline{q} &\rightarrow 0, as \; z \rightarrow \infty.
\end{aligned} \tag{20}
$$

Equation (19) is a third order ordinary differential equation when $\beta_1 \neq 0$, which for $\beta_1 = 0$ reduces to an equation governing the Newtonian fluid. Hence, the presence of the material parameter of the second grade fluid increases the order of the governing equation from two to three, and therefore three boundary conditions are needed for a unique solution. But only two boundary conditions are prescribed in (20). To overcome the difficulty, we follow Kaloni, Beard and Walters and Siddiqui et al. [23-25] and assume the solution in the form as follows

$$
\overline{q} = \overline{q}_0 + \beta_1\overline{q}_1 + O(\beta_1^2). \tag{21}
$$

Substituting relation (21) into (19) and (20), and equating the coefficient of various powers of β_1, we obtain the following systems of equations, along with the boundary conditions:

3.1. Zeroth-order equations

$$\frac{d^2\overline{q}_0}{dz^2} + S\frac{d\overline{q}_0}{dz} - \left(\frac{E}{2}p + iE + n\right)\overline{q}_0 = 0, \tag{22}$$

$$\begin{aligned} \overline{q}_0 &= -\frac{U}{U^*p} + \frac{a}{p - i\sigma_1} + \frac{b}{p + i\sigma_1}, \, at \; z = 0, \\ \overline{q}_0 &\to 0, as \; z \to \infty. \end{aligned} \tag{23}$$

3.2. First-order equations

$$\frac{d^2\overline{q}_1}{dz^2} + S\frac{d\overline{q}_1}{dz} - \left(\frac{E}{2}p + iE + n\right)\overline{q}_1 = -p\nu_1\frac{d^2\overline{q}_0}{dz^2} + \nu_2 S\frac{d^3\overline{q}_0}{dz^3}, \tag{24}$$

$$\begin{aligned} \overline{q}_1 &= 0, at \; z = 0, \\ \overline{q}_1 &\to 0, as \; z \to \infty. \end{aligned} \tag{25}$$

3.3. Zeroth-order solution

This problem is essentially the same as that for Newtonian Fluids and was solved by Debnath [18]. However, for convenience of the readers, the solution is given by

$$\overline{q}_0 = \left(\frac{a}{p - i\sigma_1} + \frac{b}{p + i\sigma_1} - \frac{U}{U^*p}\right) e^{-(S+\sqrt{S^2+4iE+2Ep+4n})z/2} \tag{26}$$

which, after Laplace inversion yields

$$\begin{aligned} q_0(z,t) &= \frac{a}{2}e^{i\sigma_1 t - Sz/2} \times \\ &\times \left\{ \begin{array}{c} e^{(a_1+ib_1)z} \operatorname{erf}c\left(\frac{\xi}{2\sqrt{t}} + (a_1+ib_1)\sqrt{\frac{2t}{E}}\right) \\ +e^{-(a_1+ib_1)z} \operatorname{erf}c\left(\frac{\xi}{2\sqrt{t}} - (a_1+ib_1)\sqrt{\frac{2t}{E}}\right) \end{array} \right\} \\ &+\frac{b}{2}e^{-i\sigma_1 t - Sz/2} \times \\ &\times \left\{ \begin{array}{c} e^{(a_2+ib_2)z} \operatorname{erf}c\left(\frac{\xi}{2\sqrt{t}} + (a_2+ib_2)\sqrt{\frac{2t}{E}}\right) \\ +e^{-(a_2+ib_2)z} \operatorname{erf}c\left(\frac{\xi}{2\sqrt{t}} - (a_2+ib_2)\sqrt{\frac{2t}{E}}\right) \end{array} \right\} \\ &-\frac{U}{2U^*}e^{-Sz/2} \left\{ \begin{array}{c} e^{(a_3+ib_3)z} \operatorname{erf}c\left(\frac{\xi}{2\sqrt{t}} + (a_3+ib_3)\sqrt{\frac{2t}{E}}\right) \\ +e^{-(a_3+ib_3)z} \operatorname{erf}c\left(\frac{\xi}{2\sqrt{t}} - (a_3+ib_3)\sqrt{\frac{2t}{E}}\right) \end{array} \right\}, \end{aligned} \tag{27}$$

where

$$\xi = z\sqrt{\frac{E}{2}},$$

$$a_1 = \frac{1}{2\sqrt{2}}\sqrt{(S^2+4n)+\sqrt{(S^2+4n)^2+4E^2(2+\sigma_1)^2}}, \quad (28)$$

$$a_2 = \frac{1}{2\sqrt{2}}\sqrt{(S^2+4n)+\sqrt{(S^2+4n)^2+4E^2(2-\sigma_1)^2}}, \quad (29)$$

$$a_3 = \frac{1}{2\sqrt{2}}\sqrt{(S^2+4n)+\sqrt{(S^2+4n)^2+16E^2}}, \quad (30)$$

$$b_1 = \frac{1}{2\sqrt{2}}\sqrt{-(S^2+4n)+\sqrt{(S^2+4n)^2+4E^2(2+\sigma_1)^2}}, \quad (31)$$

$$b_2 = \frac{1}{2\sqrt{2}}\sqrt{-(S^2+4n)+\sqrt{(S^2+4n)^2+4E^2(2-\sigma_1)^2}}, \quad (32)$$

$$b_3 = \frac{1}{2\sqrt{2}}\sqrt{-(S^2+4n)+\sqrt{(S^2+4n)^2+16E^2}} \quad (33)$$

and $\operatorname{erf} c(x+iy)$ is the complementary error function of the complex argument which can be calculated in terms of tabulated functions [26]. The tables given in [26] do not give $\operatorname{erf} c(x+iy)$ directly but an auxiliary function $W_1(x+iy)$, which is defined as

$$\operatorname{erf} c(x+iy) = W_1(-y+ix)\, e^{-(x+iy)^2}.$$

It is easily shown that some properties of $W_1(x+iy)$ are

$$W_1(-x+iy) = W_2(x+iy),$$
$$W_1(x-iy) = 2e^{-(x+iy)^2} - W_2(x+iy),$$

where $W_2(x+iy)$ is complex conjugate of $W_1(x+iy)$.

3.4. First-order solution

The solution of (24) with (25) is

$$\overline{q}_1(z,p) = ze^{-Sz/2}\left[\begin{array}{c} \nu_3 p + \nu_4 + \nu_{10}\, p\sqrt{p+\frac{S^2+4iE+4n}{2E}} \\ -\nu_{11}\, p\left(\sqrt{p+\frac{S^2+4iE+4n}{2E}}\right)^{-1} - \nu_{12}\, p^2\left(\sqrt{p+\frac{S^2+4iE+4n}{2E}}\right)^{-1} \\ +\nu_{13}\left(\sqrt{p+\frac{S^2+4iE+4n}{2E}}\right) - \nu_{14}\left(\sqrt{p+\frac{S^2+4iE+4n}{2E}}\right)^{-1} \end{array}\right] F\,. \tag{34}$$

In Eq. (34)

$$F = \left(\frac{a}{p-i\sigma_1} + \frac{b}{p+i\sigma_1} - \frac{U}{U^* p}\right) e^{-z\sqrt{E/2}\left(p+\frac{S^2+4iE+4n}{2E}\right)^{1/2}},$$

$$\begin{aligned} \nu_3 &= \frac{S}{2}\left(\nu_1 + \frac{E\nu_2}{2}\right),\ \nu_4 = \frac{S\nu_2}{2}\left(S^2+iE+n\right),\ \nu_5 = \frac{\nu_1}{2}, \\ \nu_6 &= (iE+n)\,\nu_1 + \frac{ES^2\nu_2}{4},\ \nu_7 = \frac{E\nu_1}{2},\ \nu_8 = \frac{S^2\nu_2}{2},\ \nu_9 = \frac{(iE+n)\,S^2\nu_2}{2}, \\ \nu_{10} &= \nu_5\sqrt{2E},\ \nu_{11} = \nu_6\left(\sqrt{2E}\right)^{-1},\ \nu_{12} = \nu_7\left(\sqrt{2E}\right)^{-1}, \\ \nu_{13} &= \nu_8\sqrt{2E},\ \nu_{14} = \nu_9\left(\sqrt{2E}\right)^{-1}. \end{aligned}$$

Taking the inverse Laplace transform

$$q_1(z,t) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \overline{q}_1(z,p)\, e^{pt}\, dp \tag{35}$$

and after lengthy calculations the solution can be written as

$$\begin{aligned} q_1(z,t) &= ze^{-Sz/2}\left\{\frac{\nu_3(a+b)z}{2t}\sqrt{\frac{E}{2}} + H_1\right\}\frac{e^{-Ez^2/8t-(a_3+ib_3)\sqrt{\frac{2}{E}}}}{\sqrt{\pi t}} \\ &+z\, e^{i\sigma_1 t - Sz/2}\left\{\begin{array}{c} (H_2+H_3)\, e^{-z(a_1+ib_1)}\times \\ \operatorname{erf} c\left(\frac{\xi}{2\sqrt{t}} - (a_1+ib_1)\sqrt{\frac{2t}{E}}\right) \\ +\,(H_2-H_3)\, e^{z(a_1+ib_1)}\times \\ \operatorname{erf} c\left(\frac{\xi}{2\sqrt{t}} + (a_1+ib_1)\sqrt{\frac{2t}{E}}\right) \end{array}\right\} \\ &+z\, e^{-i\sigma_1 t - Sz/2}\left\{\begin{array}{c} (H_4+H_5)\, e^{-z(a_3+ib_3)}\times \\ \operatorname{erf} c\left(\frac{\xi}{2\sqrt{t}} - (a_2+ib_2)\sqrt{\frac{2t}{E}}\right) \\ +\,(H_4-H_5)\, e^{z(a_2+ib_2)}\times \\ \operatorname{erf} c\left(\frac{\xi}{2\sqrt{t}} + (a_2+ib_2)\sqrt{\frac{2t}{E}}\right) \end{array}\right\} \end{aligned} \tag{36}$$

$$-\frac{Uz}{U^*}e^{-Sz/2}\left\{\begin{array}{c}(H_6+H_7)\ e^{-z(a_3+ib_3)}\times\\ \operatorname{erf}c\left(\frac{\xi}{2\sqrt{t}}-(a_3+ib_3)\sqrt{\frac{2t}{E}}\right)\\ +\ (H_6-H_7)\ e^{(a_3+ib_3)}\times\\ \operatorname{erf}c\left(\frac{\xi}{2\sqrt{t}}+(a_3+ib_3)\sqrt{\frac{2t}{E}}\right)\end{array}\right\},$$

where

$$\begin{aligned} H_1 &= -\frac{\nu_3 U}{U^*}+\frac{i\sigma_1\nu_{10}(a-b)}{4}-(\nu_{11}-\nu_{13})\left(a+b-\frac{U}{U^*}\right)\\ &\quad -\nu_{12}\left[\begin{array}{c} a\left\{i\sigma_1-\left(\frac{S^2+4iE+4n}{2E}\right)\right\}-b\left\{i\sigma_1+\left(\frac{S^2+4iE+4n}{2E}\right)\right\}\\ +\frac{U}{U^*}\left(\frac{S^2+4iE+4n}{2E}\right)\end{array}\right],\end{aligned}$$

$$H_2=\frac{a}{2}\left(i\sigma_1\nu_3+\nu_4\right),$$

$$H_3=\frac{a}{2}\left[\begin{array}{c}\left(\frac{i\sigma_1\nu_{10}}{4}+\nu_{13}\right)\sqrt{i\sigma_1+\frac{S^2+4iE+4n}{2E}}\\ +\ (\sigma_1^2\nu_{12}-i\sigma_1\nu_{11}-\nu_{14})\left(\sqrt{i\sigma_1+\frac{S^2+4iE+4n}{2E}}\right)^{-1}\end{array}\right],$$

$$H_4=\frac{b}{2}\left(\nu_4-i\sigma_1\nu_3\right),$$

$$H_5=\frac{b}{2}\left[\begin{array}{c}\left(\nu_{13}-\frac{i\sigma_1\nu_{10}}{4}\right)\sqrt{\frac{S^2+4iE+4n}{2E}-i\sigma_1}\\ +\ (\sigma_1^2\nu_{12}+i\sigma_1\nu_{11}-\nu_{14})\left(\sqrt{\frac{S^2+4iE+4n}{2E}-i\sigma_1}\right)^{-1}\end{array}\right],$$

$$H_6=\frac{\nu_4}{2},\ H_7=\frac{1}{2}\left[\nu_{13}\sqrt{\frac{S^2+4iE+4n}{2E}}-\nu_{14}\left(\sqrt{\frac{S^2+4iE+4n}{2E}}\right)^{-1}\right].$$

The starting solution for the case of suction is of the form

$$q(z,t)=q_0(z,t)+\beta_1 q_1(z,t). \tag{37}$$

For blowing $S<0$, and we take $-S=S_1>0$. The solution in this case is given by

$$\widetilde{q}(z,t)=\widetilde{q}_0(z,t)+\beta_1\widetilde{q}_1(z,t) \tag{38}$$

with

$$\widetilde{q}_0(z,t) = \frac{a}{2}e^{i\sigma_1 t+S_1 z/2}\times \quad (39)$$

$$\times\left\{\begin{array}{c} e^{\left(\widetilde{a}_1+i\widetilde{b}_1\right)z}\operatorname{erf}c\left(\frac{\xi}{2\sqrt{t}}+\left(\widetilde{a}_1+i\widetilde{b}_1\right)\sqrt{\frac{2t}{E}}\right) \\ +e^{-\left(\widetilde{a}_1+i\widetilde{b}_1\right)z}\operatorname{erf}c\left(\frac{\xi}{2\sqrt{t}}-\left(\widetilde{a}_1+i\widetilde{b}_1\right)\sqrt{\frac{2t}{E}}\right)\end{array}\right\}$$

$$+\frac{b}{2}e^{-i\sigma_1 t+S_1 z/2}\times$$

$$\times\left\{\begin{array}{c} e^{\left(\widetilde{a}_2+i\widetilde{b}_2\right)z}\operatorname{erf}c\left(\frac{\xi}{2\sqrt{t}}+\left(\widetilde{a}_2+i\widetilde{b}_2\right)\sqrt{\frac{2t}{E}}\right) \\ +e^{-\left(\widetilde{a}_2+i\widetilde{b}_2\right)z}\operatorname{erf}c\left(\frac{\xi}{2\sqrt{t}}-\left(\widetilde{a}_2+i\widetilde{b}_2\right)\sqrt{\frac{2t}{E}}\right)\end{array}\right\}$$

$$-\frac{U}{2U^*}e^{S_1 z/2}\left\{\begin{array}{c} e^{\left(\widetilde{a}_3+i\widetilde{b}_3\right)z}\operatorname{erf}c\left(\frac{\xi}{2\sqrt{t}}+\left(\widetilde{a}_3+i\widetilde{b}_3\right)\sqrt{\frac{2t}{E}}\right) \\ +e^{-\left(\widetilde{a}_3+i\widetilde{b}_3\right)z}\operatorname{erf}c\left(\frac{\xi}{2\sqrt{t}}-\left(\widetilde{a}_3+i\widetilde{b}_3\right)\sqrt{\frac{2t}{E}}\right)\end{array}\right\},$$

$$\widetilde{q}_1(z,t) = ze^{S_1 z/2}\left\{\widetilde{H}_1-\frac{S_1(\nu_1+\frac{E\nu_2}{2})(a+b)z}{4t}\sqrt{\frac{E}{2}}\right\}\frac{e^{-Ez^2/8t-\left(\widetilde{a}_3+i\widetilde{b}_3\right)\sqrt{\frac{2}{E}}}}{\sqrt{\pi t}} \quad (40)$$

$$+z\ e^{i\sigma_1 t+S_1 z/2}\left\{\begin{array}{c}\left(\widetilde{H}_2+\widetilde{H}_3\right)\ e^{-z\left(\widetilde{a}_1+i\widetilde{b}_1\right)}\times \\ \operatorname{erf}c\left(\frac{\xi}{2\sqrt{t}}-\left(\widetilde{a}_1+i\widetilde{b}_1\right)\sqrt{\frac{2t}{E}}\right) \\ +\left(\widetilde{H}_2-\widetilde{H}_3\right)\ e^{z\left(\widetilde{a}_2+i\widetilde{b}_2\right)}\times \\ \operatorname{erf}c\left(\frac{\xi}{2\sqrt{t}}+\left(\widetilde{a}_1+i\widetilde{b}_1\right)\sqrt{\frac{2t}{E}}\right)\end{array}\right\}$$

$$+z\ e^{-i\sigma_1 t+S_1 z/2}\left\{\begin{array}{c}\left(\widetilde{H}_4+\widetilde{H}_5\right)\ e^{-z\left(\widetilde{a}_3+i\widetilde{b}_3\right)}\times \\ \operatorname{erf}c\left(\frac{\xi}{2\sqrt{t}}-\left(\widetilde{a}_2+i\widetilde{b}_2\right)\sqrt{\frac{2t}{E}}\right) \\ +\left(\widetilde{H}_4-\widetilde{H}_5\right)\ e^{z\left(\widetilde{a}_2+i\widetilde{b}_2\right)}\times \\ \operatorname{erf}c\left(\frac{\xi}{2\sqrt{t}}+\left(\widetilde{a}_2+i\widetilde{b}_2\right)\sqrt{\frac{2t}{E}}\right)\end{array}\right\}$$

$$-\frac{Uz}{U^*}e^{S_1 z/2}\left\{\begin{array}{c}\left(\widetilde{H}_6+\widetilde{H}_7\right)\ e^{-z\left(\widetilde{a}_3+i\widetilde{b}_3\right)}\times \\ \operatorname{erf}c\left(\frac{\xi}{2\sqrt{t}}-\left(\widetilde{a}_2+i\widetilde{b}_2\right)\sqrt{\frac{2t}{E}}\right) \\ +\left(\widetilde{H}_6-\widetilde{H}_7\right)\ e^{\left(\widetilde{a}_3+i\widetilde{b}_3\right)}\times \\ \operatorname{erf}c\left(\frac{\xi}{2\sqrt{t}}+\left(\widetilde{a}_3+i\widetilde{b}_3\right)\sqrt{\frac{2t}{E}}\right)\end{array}\right\},$$

where $\widetilde{H}_r$ ($r=1$ to 7), and $\widetilde{a}_j$ and $\widetilde{b}_j$ ($j=1$ to 3) are given by H_r a_j and b_j with S replaced by $-S_1$.

4. Discussion

The solutions (37) and (38) represent the general features of the unsteady hydromagnetic boundary layer flows of a second grade fluid in the rotating system including the effects of uniform suction and blowing respectively.

In particular, when $\beta_1 = 0$, solution (37) is identical with that of Debnath [18]. When $\beta_1 = 0$ and $n = 0$, solution (37) reduces to that of Debnath and Mukherjee [27]. Further when $\beta_1 = 0, U = 0, S = 0$ and $n = 0$, solution (37) recovers to that of Thornley [28].

Finally, it can be seen that unless $S = 0$ or $n = 0$

$$\begin{aligned} Lim_{t\to\infty}\, Lim_{\sigma_1\to 2}\, q(z,t) &= Lim_{\sigma_1\to 2}\, Lim_{t\to\infty}\, q(z,t) \\ &\sim [a + 2\beta_1 z\,(H_2^* + H_3^*)]\, e^{2it-\left(\frac{S}{2}+a_1^*+ib_1^*\right)z} \\ &\quad + [b + 2\beta_1 z\,(H_4^* + H_5^*)]\, e^{-2it-\left(\frac{S}{2}+a_2^*+ib_2^*\right)z} \\ &\quad - [1 + 2\beta_1 z\,(H_6^* + H_7^*)]\, \frac{U}{U^*} e^{-\left(\frac{S}{2}+a_3^*+ib_3^*\right)z}, \end{aligned} \tag{41}$$

where H_r^* $(r = 2$ to $7)$ and a_j^*, $b_j^*\,(j = 1$ to $3)$ are the limiting values of H_r and a_j, b_j respectively as $\sigma_1 \to 2$.

In order to determine the steady state structure of the solution (37) we use the asymptotic formula for the complimentary error function

$$\operatorname{erf} c\left(\frac{\xi}{2\sqrt{t}} \pm (a_j + ib_j)\sqrt{\frac{2t}{E}}\right) \to (0,2) \ \text{as}\ t \to \infty. \tag{42}$$

Evidently, in the limit $t \to \infty$, solution (37) , yields

$$\begin{aligned} q_s(z,t) &\sim [a + 2\beta_1 z\ (H_2 + H_3)]\, e^{i\sigma_1 t-\frac{S}{2}z-(a_1+\ ib_1)z} \\ &\quad + [b + 2\beta_1 z\ (H_4 + H_5)]\, e^{-i\sigma_1 t-\frac{S}{2}z-(a_2+\ ib_2)z} \\ &\quad - [1 + 2\beta_1 z\ (H_6 + H_7)]\, \frac{U}{U^*} e^{-\frac{S}{2}z-(a_3+\ ib_3)z}. \end{aligned} \tag{43}$$

This result describes physically meaningful hydromagnetic boundary layer flow of a second grade fluid for both resonant $(\sigma_1 = 2)$ and non-resonant $(\sigma_1 \neq 2)$ frequency. It may be noted that the effects of the suction and material parameters, Coriolis and electromagnetic forces are reflected in the unsteady and the ultimate steady velocity fields. Solution (43) indicates the existence of three distinct boundary layers of thicknesses of order $f_j = \frac{\nu}{U^*\left(a_j+\frac{s}{2}\right)}$ with $f_1 < f_3 < f_2$ for suction. Clearly, the solution (43) and the associated boundary layers are modified due to the presence of S, β_1, n and σ_1. It is noted that the thicknesses of the boundary layers decrease with an increase of the external magnetic field or suction parameter and remain bounded for all values of the imposed oscillations, β_1, n and S.

In the presence of an external magnetic field ($n \neq 0$) and $S = 0$, the ultimate steady state solution does not depend on the order of the double limit operation $t \to \infty$ $\sigma_1 \to 2$. In fact, the double limiting procedure with $n \neq 0$ and $S = 0$ yields

$$\begin{aligned} Lim_{t\to\infty}\, Lim_{\sigma_1\to 2}\, q(z,t) &= Lim_{\sigma_1\to 2}\, Lim_{t\to\infty}\, q(z,t) \\ &= \left[a + 2\beta_1 z L_1\right] e^{2it-(h_1+ig_1)z} \\ &\quad + \left[b + 2\beta_1 z L_2\right] e^{-2it-h_2 z} \\ &\quad - \frac{U}{U^*} e^{-(h_3+ig_3)z}, \end{aligned} \tag{44}$$

where

$$\begin{aligned} L_1 &= \frac{a}{2}\sqrt{\frac{E}{2}}\left[\frac{i\nu_1}{2}\sqrt{4i} + \frac{4\nu_1}{\sqrt{4i}}\right]\sqrt{\frac{2n}{4iE}+1}, \\ L_2 &= \frac{i\nu_1}{4} b\sqrt{n}, \end{aligned}$$

$$h_1 = \frac{1}{\sqrt{2}}\left[n + \sqrt{n^2 + 16E^2}\right]^{1/2}, \; h_2 = \sqrt{n},$$

$$h_3 = \frac{1}{\sqrt{2}}\left[n + \sqrt{n^2 + E^2}\right]^{1/2},$$

$$g_1 = \frac{1}{\sqrt{2}}\left[-n + \sqrt{n^2 + 16E^2}\right]^{1/2}, \; g_2 = 0,$$

$$g_3 = \frac{1}{\sqrt{2}}\left[-n + \sqrt{n^2 + E^2}\right]^{1/2}$$

and solution is identical to that of Newtonian fluid for fixed viscosity and $\Omega = 0$.

It should be pointed out that the mathematical nature and physical content of the solution obtained by the three limiting procedures $\sigma_1 \to 2$ $t \to \infty$ $n \to 0$ in this case or in the reverse order t$\to \infty$ $\sigma_1 \to 2$ $n \to 0$ are radically different. In fact from (44)

$$Lim_{\sigma_1 \to 2}\, Lim_{t\to\infty}\, Lim_{n\to 0}\, q(z,t) = a\,(1+2\beta_1 zM)\, e^{2it - z\sqrt{2E}(1+i)} + be^{-2it} - \frac{U}{U^*} e^{-z\sqrt{2E}(1+i)}, \tag{45}$$

and

$$Lim_{t\to\infty}\, Lim_{\sigma_1 \to 2}\, Lim_{n\to 0}\, q(z,t) = a\,(1+2\beta_1 zM)\, e^{2it - z\sqrt{2iE}} + be^{-2it}\,\mathrm{erf}\,c\left(\frac{z}{2}\sqrt{\frac{E}{2t}}\right) - \frac{U}{U^*} e^{-z\sqrt{iE}}, \tag{46}$$

where

$$M = \frac{\sqrt{E/2}}{2}\left[\frac{i\nu_1}{2}\sqrt{4i} + \frac{4\nu_1}{\sqrt{4i}}\right].$$

Clearly, the above two results are different. Result (45) does not even qualify for solution because it does not satisfy the boundary condition at infinity unless $b = 0$. Result (46) satisfies all the conditions and is a correct solution. In case of blowing ultimate steady-state solution is obtained from (38) by taking the limit $t \to \infty$ and has the form

$$\begin{aligned} \widetilde{q}_S(z,t) \sim & \left[a + 2\beta_1 z\,(\widetilde{H}_2 + \widetilde{H}_3)\right] e^{i\sigma t + \frac{S_1}{2}z - (\widetilde{a}_1 + i\widetilde{b}_1)z} \\ & + \left[b + 2\beta_1 z\,(\widetilde{H}_4 + \widetilde{H}_5)\right] e^{-i\sigma t + \frac{S_1}{2}z - (\widetilde{a}_2 + i\widetilde{b}_2)z} \\ & - \left[1 + 2\beta_1 z\,(\widetilde{H}_6 + \widetilde{H}_7)\right] \frac{U}{U^*} e^{\frac{S_1}{2}z - (\widetilde{a}_3 + i\widetilde{b}_3)z}. \end{aligned} \tag{47}$$

The above solution describes the hydromagnetic oscillations which decay exponentially within the three boundary layers of thicknesses of the order $\widetilde{f}_j = \frac{\nu}{U^*\left(\widetilde{a}_j + \frac{s}{2}\right)}$ with $\widetilde{f}_1 < \widetilde{f}_3 < \widetilde{f}_2$. It is worth noting that the thicknesses of these boundary layers are significantly modified by the magnetic field. The most important feature of (47) is that unlike the hydrodynamic situation for the resonant case, (47) satisfies the boundary condition at infinity for all values of σ_1 including the resonant frequency. Consequently, the associated boundary layers remain bounded for the resonant case. In contrast to the hydrodynamic solution for the case of blowing and resonance where the blowing promotes the spreading of the shear oscillations far away from the plate, the hydromagnetic solution (47) represents the oscillatory boundary layer flow confined to the ultimate boundary layers for the frequencies including the resonant frequency. The physical implication of this conclusion is that for the case of resonance, the unbounded spreading of oscillations away from the plate is controlled by the magnetic field.

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MHD rotating flow of a third-grade fluid on an oscillating porous plate

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Summary. The rotating flow of a third-grade fluid on an oscillating porous plate in the presence of a transverse magnetic field is considered. An analytic solution of the governing nonlinear boundary layer equation is obtained. Expressions for the velocity profile are established. It is found that an external magnetic field has the same effect on the flow as the material parameters of the fluid. Further the symmetric and asymmetric nature of the solutions is discussed.

1 Introduction

There are very few exact solutions of the Navier-Stokes equations, and these even become rare if non-Newtonian constitutive equations are considered in the equation of motion. The model in the present paper is a third-grade fluid.

The Cauchy stress S in an incompressible homogeneous fluid of third grade has the form [1]

$$S = -p\mathbf{I} + \mu \mathbf{A}_1 + \alpha_1 \mathbf{A}_2 + \alpha_2 \mathbf{A}_1{}^2 + \beta_1 \mathbf{A}_3 + \beta_2(\mathbf{A}_1\mathbf{A}_2 + \mathbf{A}_2\mathbf{A}_1) + \beta_3(\mathrm{tr}\,\mathbf{A}_1{}^2)\,\mathbf{A}_1\,, \tag{1}$$

where

$$\begin{aligned} \mathbf{A}_1 &= (\mathrm{grad}\,\mathbf{V}) + (\mathrm{grad}\,\mathbf{V})^T, \\ \mathbf{A}_i &= \frac{d\mathbf{A}_{i-1}}{dt} + \mathbf{A}_{i-1}(\mathrm{grad}\,\mathbf{V}) + (\mathrm{grad}\,\mathbf{V})^T\mathbf{A}_{i-1}, i > 1\,. \end{aligned} \tag{2}$$

In the above equations, the spherical stress $-p\mathbf{I}$ is due to the constraint of incompressibility, μ is the viscosity, $\alpha_1, \alpha_2, \beta_1, \beta_2$, and β_3 are the material constants, $\frac{d}{dt}$ denotes the material time derivative, V denotes the velocity field, and A_1, A_2 and A_3 are the first three Rivlin-Ericksen tensors.

We shall not consider the above model as an approximation to a simple fluid [2] in the sense of a retardation, but consider it to be an exact model in the sense described by Fosdick and Rajagopal in [3]. We require that the Clausius-Duhem inequality holds and that the specific Helmholtz free energy be a minimum when the fluid is locally at rest, which leads to the following restrictions on the material coefficients:

$$\begin{aligned} &\mu \geq 0\,, \qquad \alpha_1 \geq 0\,, \qquad \beta_1 = \beta_2 = 0\,, \qquad \beta_3 \geq 0\,, \\ &-\sqrt{24\mu\beta_3} \leq \alpha_1 + \alpha_2 \leq \sqrt{24\mu\beta_3}\,. \end{aligned} \tag{3}$$

Therefore, the model of (1) reduces to

$$S = -p\mathbf{I} + \mathbf{T}\,, \qquad \mathbf{T} = \mu\mathbf{A}_1 + \alpha_1\mathbf{A}_2 + \alpha_2{\mathbf{A}_1}^2 + \beta_3(\operatorname{tr}{\mathbf{A}_1}^2)\,. \tag{4}$$

In 1983, Rajagopal and Na [4] studied the flow of a third grade fluid due to an oscillating plate. Siddiqui and Kaloni [5] studied plane steady flows of a third grade fluid. Later, Erdogan [6] analyzed the flow of a third grade fluid due to a plane surface suddenly set in motion. More recently, Mollica and Rajagopal [7] examined secondary flows due to axial shearing of a third grade fluid between two eccentrically placed cylinders.

Using the viscous fluid model, the flow of a fluid near a porous oscillating infinite plane has been investigated in [8]. Debnath [9] studied the unsteady magnetohydrodynamic boundary layers in a rotating flow of a viscous fluid. Later, Foote et al. [10] discussed the flow due to an oscillating porous plate for an elastico viscous fluid. Puri [11] examined a flow of an elastico-viscous fluid on an oscillating plate. More recently, Turbatu et al. [12] generalized the viscous fluid flow problem of an oscillating flat plate in two directions. They first considered the oscillating flat plate with superimposed blowing or suction. The second generalization is concerned with an increasing or decreasing velocity amplitude of the oscillating flat plate.

As far as the authors are aware, no attempt has been made to examine the flow of a third grade fluid on an oscillating porous plate even in a non-rotating medium. In this paper we propose to study the flow of a third grade conducting fluid due to an oscillating porous plate in a rotating medium and also obtain the result for the non-conducting fluid and for the non-rotating case. The interest in magnetohydrodynamic (MHD) fluid flows stems from the fact that liquid metals that occur in nature and industry are electrically conducting. These fluids, for the most part, are of finite electrical conductivity. Also considered are MHD fluid flows of finite conductivity. These two types of fluid flow are attractive both from a mathematical as well as a physical standpoint. MHD flow of non-Newtonian fluids was probably first considered by Sarpkaya [13]. A critical analysis is made of the associated hydrodynamic and hydromagnetic boundary layers adjacent to the boundary plate. This is followed by a comparative study of these boundary layers. The significant effects of the Coriolis force and the imposed magnetic filed on the hydromagnetic flow are also investigated. It is once again found that, similar to a viscous fluid, the difficulty of the corresponding hydrodynamic problem associated with the resonant frequency $\omega = 2\Omega$ is automatically resolved in the present hydromagnetic analysis of a third grade fluid.

2 Constitutive equations and boundary conditions

We consider a semi-infinite expanse of a homogeneous, incompressible, electrically conducting third grade fluid bounded by an infinite non-conducting porous plate at $z = 0$. Both the fluid and the plate are in a state of rigid body rotation with constant angular velocity $\Omega = \Omega\hat{k}$, where $\hat{k}$ is a unit vector parallel to the z-axis normal to the plate. A uniform magnetic field $\mathrm{B}_0 = \mathrm{B}_0\,\hat{k}$ is parallel to the fluid system. The plate is also subjected to simple harmonic oscillations along the x-direction, relative to the rotating coordinate system (x, y, z). We examine the boundary layer flow generated in the rotating system by harmonic oscillations of the plate. The physical variables are functions of z and t. Thus, we assume that the velocity field in the boundary layer is of the form

$$\mathrm{V} = [u(z,t), v(z,t), w(z,t)]\,, \tag{5}$$

where u, v, w, are the velocity components in the x, y, z-directions, respectively. Since the fluid is incompressible, only isochoric (i.e. volume preserving) flows are possible, i.e. the flow meets the constraint

$$\operatorname{div} \mathbf{V} = 0 \tag{6}$$

The motion of the conducting fluid in the rotating Cartesian coordinate system is governed by the conservation law of momentum which is

$$\varrho \left[\frac{\partial \mathbf{V}}{\partial t} + (\mathbf{V}.\nabla)\,\mathbf{V} + 2\Omega \times \mathbf{V} + \Omega \times (\Omega \times \mathbf{r}) \right] = \operatorname{div} \mathbf{S} + \mathbf{j} \times \mathbf{B}\,, \tag{7}$$

where ϱ is the density of the fluid, $\mathbf{j}$ is the electric current density, $\mathbf{B}$ is the total magnetic field so that $\mathbf{B} = \mathbf{B}_0 + \mathbf{b}$, $\mathbf{b}$ is the induced magnetic field, and $\mathbf{r}$ is the radial coordinate, i.e. $\mathbf{r}^2 = x^2 + y^2$. Neglecting displacement currents, the Maxwell equations and the generalized Ohm's law appropriate for the problem are

$$\operatorname{div} \mathbf{B} = 0\,, \qquad \operatorname{curl} \mathbf{B} = \mu_1 \mathbf{j}\,, \qquad \operatorname{curl} \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}\,, \tag{8}$$

$$\mathbf{j} = \sigma(\mathbf{E} + \mathbf{V} \times \mathbf{B})\,, \tag{9}$$

where μ_1 is the magnetic permeability, $\mathbf{E}$ is the electric field and σ is the electrical conductivity of the fluid.

The assumptions made by Rossow [14] are found to be physically reasonable for the boundary layer flows. Based on these assumptions particularly of small magnetic Reynolds number [15], the linearized magnetohydrodynamic force involved in (7) can be put into the form

$$\frac{1}{\varrho}\mathbf{j} \times \mathbf{B} = -\frac{\sigma}{\varrho}\,\mathbf{B}_0{}^2\mathbf{V}\,. \tag{10.1}$$

Also it follows from (6) that

$$w = -W_0 = \text{constant}\,, \tag{10.2}$$

where $W_0 > 0$ indicates suction at the plate and $W_0 < 0$ indicates blowing. Substituting (4), (5), (10.1) and (10.2) into (7), and then eliminating the pressure gradient from the resulting equations by cross differentiation, we obtain

$$\frac{\partial^2 F}{\partial z\,\partial t} - W_0\,\frac{\partial^2 F}{\partial z^2} - 2i\Omega\,\frac{\partial F}{\partial z} = \nu\,\frac{\partial^3 F}{\partial z^3} + \frac{\alpha_1}{\varrho}\left(\frac{\partial^4 F}{\partial z^3\,\partial t} - W_0\,\frac{\partial^4 F}{\partial z^4}\right)$$
$$-\frac{\sigma B_0{}^2}{\varrho}\,\frac{\partial F}{\partial z} + 2\,\frac{\beta_3}{\varrho}\,\frac{\partial^2}{\partial z^2}\left(\left(\frac{\partial F}{\partial z}\right)^2 \frac{\partial \bar{F}}{\partial z}\right), \tag{11}$$

where

$$F = u + iv\,, \qquad \bar{F} = u - iv\,, \tag{12}$$

The flow is set up in the rotating system by the elliptic harmonic oscillations of the porous plate so that relevant boundary conditions of the problem are

$$F = U_0(ae^{i\omega t} + be^{-i\omega t})\,, \quad \text{at} \quad z = 0\,, \quad t > 0\,,$$
$$F \to 0\,, \quad \text{as} \quad z \to \infty\,, \quad t > 0\,. \tag{13}$$

3 Solution of the boundary value problem

Introducing the dimensionless variables

$$\eta = \frac{zW_0}{\nu}, \qquad \tau = \frac{{W_0}^2 t}{\nu}, \qquad F = U_0 G(\eta, \tau), \tag{14}$$

Eq. (11) and the boundary conditions (13) take the following forms:

$$\frac{\partial^2 G}{\partial \eta \, \partial \tau} - \frac{\partial^2 G}{\partial \eta^2} - \frac{2i\Omega\nu}{{W_0}^2}\frac{\partial G}{\partial \eta} = \frac{\partial^3 G}{\partial \eta^3} + \frac{\alpha_1 {W_0}^2}{\varrho \nu^2}\left(\frac{\partial^4 G}{\partial \eta^3 \, \partial \tau} - \frac{\partial^4 G}{\partial \eta^4}\right)$$
$$- \frac{\sigma {B_0}^2}{\varrho {W_0}^2}\frac{\partial G}{\partial \eta} + 2\,\frac{\beta_3 {W_0}^2 {U_0}^2}{\varrho \nu^3}\frac{\partial^2}{\partial \eta^2}\left(\left(\frac{\partial G}{\partial \eta}\right)^2 \frac{\partial \bar{G}}{\partial \eta}\right), \tag{15}$$

$$G = (ae^{i\delta\tau} + be^{-i\delta\tau}), \quad \text{at} \quad \eta = 0, \quad \tau > 0,$$
$$G \to 0, \quad \text{as} \quad \eta \to \infty \quad \tau > 0, \tag{16}$$

where

$$\delta = \frac{\omega\nu}{{W_0}^2}. \tag{17}$$

We note that it is worth emphasizing that the equation for a fluid of second and third grade is in general of higher order than the Navier-Stokes equation. Thus, in general, one needs conditions in addition to the usual no slip condition to solve the flow problem of these fluids. Furthermore, the equations for second grade and Newtonian flow are linear and (15) is non-linear. As a result, it seems to be impossible to obtain the general solution in closed form for arbitrary values of all parameters arising in (15). One possible way to overcome this difficulty is to employ the solution of the problem as a power series expansion in the small parameter

$$\epsilon_1 = \frac{2\beta_3}{\varrho}. \tag{18}$$

Accordingly, we assume that G can be expanded in powers of ϵ_1 as follows:

$$G = G_0 + \epsilon_1 G_1 + \ldots \tag{19}$$

Substituting (19) into (15) and (16) and then equating terms of like powers of ϵ_1, one obtains the following systems of partial differential equations along with boundary conditions:

System of order zero

$$\frac{\partial^2 G_0}{\partial \eta \, \partial \tau} - \frac{\partial^2 G_0}{\partial \eta^2} - \frac{2i\Omega\nu}{{W_0}^2}\frac{\partial G_0}{\partial \eta} = \frac{\partial^3 G_0}{\partial \eta^3} - \frac{\sigma {B_0}^2 \nu}{\varrho {W_0}^2}\frac{\partial G_0}{\partial \eta} + \frac{\alpha_1 {W_0}^2}{\varrho \nu^2}\left(\frac{\partial^4 G_0}{\partial \eta^3 \, \partial \tau} - \frac{\partial^4 G_0}{\partial \eta^4}\right), \tag{20}$$

with the boundary conditions

$$G_0 = (ae^{i\delta\tau} + be^{-i\delta\tau}), \quad \text{at} \quad \eta = 0, \quad \tau > 0,$$
$$G_0 \to 0, \quad \text{as} \quad \eta \to \infty, \quad \tau > 0. \tag{21}$$

System of order one

$$\frac{\partial^2 G_1}{\partial\eta\,\partial\tau}-W_0\,\frac{\partial^2 G_1}{\partial\eta^2}-\frac{2i\Omega\nu}{W_0{}^2}\,\frac{\partial G_1}{\partial\eta}=\frac{\partial^3 G_1}{\partial\eta^3}+\frac{\alpha_1 W_0{}^2}{\varrho\nu^2}\left(\frac{\partial^4 G_1}{\partial\eta^3\,\partial\tau}-W_0\,\frac{\partial^4 G_1}{\partial\eta^4}\right)$$
$$-\frac{\sigma B_0{}^2}{\varrho W_0{}^2}\,\frac{\partial G_1}{\partial\eta}+\frac{W_0{}^2 U_0{}^2}{\nu^3}\,\frac{\partial^2}{\partial\eta^2}\left(\left(\frac{\partial G_0}{\partial\eta}\right)^2\frac{\partial\overline{G_0}}{\partial\eta}\right), \tag{22}$$

with the boundary conditions

$$\begin{aligned} &G_1=0\,, \quad \text{at} \quad \eta=0\,, \qquad \tau>0\,,\\ &G_1\to 0\,, \quad \text{as} \quad \eta\to\infty\,, \quad \tau>0\,. \end{aligned} \tag{23}$$

The boundary conditions suggest, for an oscillatory flow, a solution of the form

$$G_0=\phi_{01}(\eta)+a\phi_{02}(\eta)\,e^{i\delta\tau}+b\phi_{03}(\eta)\,e^{-i\delta\tau}\,, \qquad \delta>0\,, \tag{24}$$

$$G_1=\phi_{11}(\eta)+a\phi_{12}(\eta)\,e^{i\delta\tau}+b\phi_{13}(\eta)\,e^{-i\delta\tau}\,. \tag{25}$$

Substituting (24) in (20) and (21) and (25) in (22) and (23) we get the six ordinary differential equations. The solution of these ordinary differential equations has been obtained employing the procedure used by Hinch [16], and the velocity fields are directly given by

for $\delta>\Omega_1$

$$u=U_0\begin{bmatrix} e^{-a_1\eta}\{B_1\cos(\delta\tau-b_1\eta)+B_2\sin(\delta\tau-b_1\eta)\}\\ +e^{-3a_1\eta}\{B_3\cos(\delta\tau-3b_1\eta)-B_4\sin(\delta\tau-3b_1\eta)\}\\ +e^{-c_1\eta}\{B_5\cos(\delta\tau-c_2\eta)-B_6\sin(\delta\tau-c_2\eta)\}\\ +e^{-a_4\eta}\{B_7\cos(\delta\tau+b_4\eta)-B_8\sin(\delta\tau+b_4\eta)\}\\ +e^{-3a_4\eta}\{B_9\cos(\delta\tau+3b_4\eta)+B_{10}\sin(\delta\tau+3b_4\eta)\}\\ +e^{-c_7\eta}\{B_{11}\cos(\delta\tau+c_8\eta)+B_{12}\sin(\delta\tau+c_8\eta)\} \end{bmatrix}, \tag{26}$$

$$v=U_0\begin{bmatrix} e^{-a_1\eta}\{B_1\sin(\delta\tau-b_1\eta)-B_2\cos(\delta\tau-b_1\eta)\}\\ +e^{-3a_1\eta}\{B_3\sin(\delta\tau-3b_1\eta)+B_4\cos(\delta\tau-3b_1\eta)\}\\ +e^{-c_1\eta}\{B_5\sin(\delta\tau-c_2\eta)+B_6\cos(\delta\tau-c_2\eta)\}\\ -e^{-a_4\eta}\{B_7\sin(\delta\tau+b_4\eta)+B_8\cos(\delta\tau+b_4\eta)\}\\ -e^{-3a_4\eta}\{B_9\sin(\delta\tau+3b_4\eta)-B_{10}\cos(\delta\tau+3b_4\eta)\}\\ -e^{-c_7\eta}\{B_{11}\sin(\delta\tau+c_8\eta)-B_{12}\cos(\delta\tau+c_8\eta)\} \end{bmatrix}, \tag{27}$$

for $\delta<\Omega_1$

$$u=U_0\begin{bmatrix} e^{-a_2\eta}\{B_{13}\cos(\delta\tau-b_2\eta)+B_{14}\sin(\delta\tau-b_2\eta)\}\\ +e^{-3a_2\eta}\{B_{15}\cos(\delta\tau-3b_2\eta)-B_{16}\sin(\delta\tau-3b_2\eta)\}\\ +e^{-c_3\eta}\{B_{17}\cos(\delta\tau-c_4\eta)-B_{18}\sin(\delta\tau-c_4\eta)\}\\ +e^{-a_4\eta}\{B_{19}\cos(\delta\tau+b_4\eta)-B_{20}\sin(\delta\tau+b_4\eta)\}\\ +e^{-3a_4\eta}\{B_9\cos(\delta\tau+3b_4\eta)+B_{10}\sin(\delta\tau+3b_4\eta)\}\\ +e^{-c_9\eta}\{B_{21}\cos(\delta\tau+c_{10}\eta)+B_{22}\sin(\delta\tau+c_{10}\eta)\} \end{bmatrix}, \tag{28}$$

$$v = U_0 \begin{bmatrix} e^{-a_2\eta}\{B_{13}\sin(\delta\tau - b_2\eta) - B_{14}\cos(\delta\tau - b_2\eta)\} \\ + e^{-3a_2\eta}\{B_{15}\sin(\delta\tau - 3b_2\eta) + B_{16}\cos(\delta\tau - 3b_2\eta)\} \\ + e^{-c_3\eta}\{B_{17}\sin(\delta\tau - c_4\eta) + B_{18}\cos(\delta\tau - c_4\eta)\} \\ - e^{-a_4\eta}\{B_{19}\sin(\delta\tau + b_4\eta) + B_{20}\cos(\delta\tau + b_4\eta)\} \\ - e^{-3a_4\eta}\{B_9\sin(\delta\tau + 3b_4\eta) - B_{10}\cos(\delta\tau + 3b_4\eta)\} \\ - e^{-c_9\eta}\{B_{21}\sin(\delta\tau + c_{10}\eta) - B_{22}\cos(\delta\tau + c_{10}\eta)\} \end{bmatrix}, \tag{29}$$

for $\delta = \Omega_1$

$$u = U_0 \begin{bmatrix} e^{-a_3\eta}\{B_{23}\cos(\delta\tau - b_3\eta) + B_{24}\sin(\delta\tau - b_3\eta)\} \\ + e^{-3a_3\eta}\{B_{25}\cos(\delta\tau - 3b_3\eta) - B_{26}\sin(\delta\tau - 3b_3\eta)\} \\ + e^{-c_5\eta}\{B_{27}\cos(\delta\tau - c_6\eta) - B_{28}\sin(\delta\tau - c_6\eta)\} \\ + e^{-a_4\eta}\{B_{29}\cos(\delta\tau + b_4\eta) - B_{30}\sin(\delta\tau + b_4\eta)\} \\ + e^{-3a_4\eta}\{B_9\cos(\delta\tau + 3b_4\eta) + B_{10}\sin(\delta\tau + 3b_4\eta)\} \\ + e^{-c_{11}\eta}\{B_{31}\cos(\delta\tau + c_{12}\eta) + B_{32}\sin(\delta\tau + c_{12}\eta)\} \end{bmatrix}, \tag{30}$$

$$v = U_0 \begin{bmatrix} e^{-a_3\eta}\{B_{23}\sin(\delta\tau - b_3\eta) - B_{24}\cos(\delta\tau - b_3\eta)\} \\ + e^{-3a_3\eta}\{B_{25}\sin(\delta\tau - 3b_3\eta) + B_{26}\cos(\delta\tau - 3b_3\eta)\} \\ + e^{-c_5\eta}\{B_{27}\sin(\delta\tau - c_6\eta) + B_{28}\cos(\delta\tau - c_6\eta)\} \\ - e^{-a_4\eta}\{B_{29}\sin(\delta\tau + b_4\eta) + B_{30}\cos(\delta\tau + b_4\eta)\} \\ - e^{-3a_4\eta}\{B_9\sin(\delta\tau + 3b_4\eta) - B_{10}\cos(\delta\tau + 3b_4\eta)\} \\ - e^{-c_{11}\eta}\{B_{31}\sin(\delta\tau + c_{12}\eta) - B_{32}\cos(\delta\tau + c_{12}\eta)\} \end{bmatrix}, \tag{31}$$

In (26) to (31),

$$B_1 = a_R + \epsilon_1[-a_R(A_{4R} + A_{5R}) + a_I(A_{4I} + A_{5I})],$$

$$B_2 = -a_I + \epsilon_1[a_I(A_{4R} + A_{5R}) + a_R(A_{4I} + A_{5I})],$$

$$B_3 = \epsilon_1[a_R A_{4R} - a_I A_{4I}], \qquad B_4 = \epsilon_1[a_R A_{4I} + a_I A_{4R}],$$

$$B_5 = \epsilon_1[a_R A_{5R} - a_I A_{5I}], \qquad B_6 = \epsilon_1[a_R A_{5I} + a_I A_{5R}],$$

$$B_7 = b_R + \epsilon_1[b_R(-A_{22R} - A_{23R}) + b_I(A_{22I} + A_{23I})],$$

$$B_8 = -b_I + \epsilon_1[b_I(A_{22R} + A_{23R}) + b_R(A_{22I} + A_{23I})],$$

$$B_9 = \epsilon_1[b_R A_{22R} - b_I A_{22I}], \qquad B_{10} = \epsilon_1[b_R A_{22I} + b_I A_{22R}],$$

$$B_{11} = \epsilon_1[b_R A_{23R} - b_I A_{23I}], \qquad B_{12} = \epsilon_1[b_R A_{23I} + b_I A_{23R}],$$

$$B_{13} = a_R + \epsilon_1[-a_R(A_{10R} + A_{11R}) + a_I(A_{10I} + A_{11I})],$$

$$B_{14} = -a_I + \epsilon_1[a_I(A_{10R} + A_{11R}) + a_R(A_{10I} + A_{11I})],$$

$$B_{15} = \epsilon_1[a_R A_{10R} - a_I A_{10I}], \qquad B_{16} = \epsilon_1[a_R A_{10I} + a_I A_{10R}],$$

$$B_{17} = \epsilon_1[a_R A_{11R} - a_I A_{11I}], \qquad B_{18} = \epsilon_1[a_R A_{11I} + a_I A_{11R}],$$

$$B_{19} = b_R + \epsilon_1[b_R(-A_{22R} - A_{26R}) + b_I(A_{22I} + A_{26I})],$$

$$B_{20} = -b_1 + \epsilon_1[b_I(A_{22R} + A_{26R}) + b_R(A_{22I} + A_{26I})]\,,$$

$$B_{21} = \epsilon_1[b_R A_{26R} - b_I A_{26I}]\,, \qquad B_{22} = \epsilon_1[b_R A_{26I} + b_I A_{26R}]\,,$$

$$B_{23} = a_R + \epsilon_1[-a_R(A_{16R} + A_{17R}) + a_I(A_{16I} + A_{17I})]\,,$$

$$B_{24} = -a_I + \epsilon_1[a_I(A_{16R} + A_{17R}) + a_R(A_{16I} + A_{17I})]\,,$$

$$B_{25} = \epsilon_1[a_R A_{16R} - a_I A_{16I}]\,, \qquad B_{26} = \epsilon_1[a_R A_{16I} + a_I A_{16R}]\,,$$

$$B_{27} = \epsilon_1[a_R A_{17R} - a_I A_{17I}]\,, \qquad B_{28} = \epsilon_1[a_R A_{17I} + a_I A_{17R}]\,,$$

$$B_{29} = b_R + \epsilon_1[b_I(A_{22I} + A_{29I}) - b_R(A_{22R} + A_{29R})]\,,$$

$$B_{30} = -b_I + \epsilon_1[b_I(A_{22R} + A_{29R}) + b_R(A_{22I} + A_{29I})]\,,$$

$$B_{31} = \epsilon_1[b_R A_{29R} - b_I A_{29I}]\,, \qquad B_{32} = \epsilon_1[b_R A_{29I} - b_I A_{29R}]\,,$$

$$A_{4R} = \frac{A_R A_{2R} + A_I A_{2I}}{A_{2R}^2 + A_{2I}^2}\,, \qquad A_{4I} = \frac{A_I A_{2R} - A_R A_{2I}}{A_{2R}^2 + A_{2I}^2}\,,$$

$$A_{5R} = \frac{A_{1R} A_{3R} + A_{1I} A_{3I}}{A_{3R}^2 + A_{3I}^2}\,, \qquad A_{5I} = \frac{A_{1I} A_{3R} - A_{1R} A_{3I}}{A_{3R}^2 + A_{3I}^2}\,,$$

$$A_{10R} = \frac{A_{6R} A_{8R} + A_{6I} A_{8I}}{A_{8R}^2 + A_{8I}^2}\,, \qquad A_{10I} = \frac{A_{6I} A_{8R} - A_{6R} A_{8I}}{A_{8R}^2 + A_{8I}^2}\,,$$

$$A_{11R} = \frac{A_{7R} A_{9R} + A_{7R} A_{9I}}{A_{9R}^2 + A_{9I}^2}\,, \qquad A_{11I} = \frac{A_{7I} A_{9R} - A_{7R} A_{9I}}{A_{9R}^2 + A_{9I}^2}\,,$$

$$A_{16R} = \frac{A_{12R} A_{14R} + A_{12I} A_{14I}}{A_{14R}^2 + A_{14I}^2}\,, \qquad A_{16I} = \frac{A_{12I} A_{14R} - A_{12R} A_{14I}}{A_{14R}^2 + A_{14I}^2}\,,$$

$$A_{17R} = \frac{A_{13R} A_{15R} + A_{13I} A_{15I}}{A_{15R}^2 + A_{15I}^2}\,, \qquad A_{17I} = \frac{A_{13I} A_{15R} - A_{13R} A_{15I}}{A_{15R}^2 + A_{15I}^2}\,,$$

$$A_{22R} = \frac{A_{18R} A_{20R} + A_{18I} A_{20I}}{A_{20R}^2 + A_{20I}^2}\,, \qquad A_{22I} = \frac{A_{18I} A_{20R} - A_{18R} A_{20I}}{A_{20R}^2 + A_{20I}^2}\,,$$

$$A_{23R} = \frac{A_{19R} A_{21R} + A_{19I} A_{21I}}{A_{21R}^2 + A_{21I}^2}\,, \qquad A_{23I} = \frac{A_{19I} A_{21R} - A_{19R} A_{21I}}{A_{21R}^2 + A_{21I}^2}\,,$$

$$A_{26R} = \frac{A_{24R} A_{25R} + A_{24I} A_{25I}}{A_{25R}^2 + A_{25I}^2}\,, \qquad A_{26I} = \frac{A_{24I} A_{25R} - A_{24R} A_{25I}}{A_{25R}^2 + A_{25I}^2}\,,$$

$$A_{29R} = \frac{A_{27R} A_{28R} + A_{27I} A_{28I}}{A_{28R}^2 + A_{28I}^2}\,, \qquad A_{29I} = \frac{A_{27I} A_{28R} - A_{27R} A_{28I}}{A_{28R}^2 + A_{28I}^2}\,,$$

$$a_1 = c_{0R} - \lambda c_{1R} - \lambda^2 c_{2R}\,, \qquad a_2 = n_{0R} - \lambda n_{1R} - \lambda^2 n_{2R}\,,$$

$$a_3 = h_0 - \lambda h_{1R} - \lambda^2 h_{2R}\,, \qquad a_4 = k_{0R} - \lambda k_{1R} - \lambda^2 k_{2R}\,,$$

$$b_1 = c_{0I} - \lambda c_{1I} - \lambda^2 c_{2I}\,, \qquad b_2 = n_{0I} - \lambda n_{1I} - \lambda^2 n_{2I}\,,$$

$$b_3 = \lambda h_{1I} - \lambda^2 h_{2I}\,, \qquad b_4 = k_{0I} - \lambda k_{1I} - \lambda^2 k_{2I}\,,$$

$$a = a_R + i a_I\,, \qquad b = b_R + i b_I\,,$$

$$A_{1R} = 2L(b_R^2 - b_I^2)\begin{bmatrix}\{a_1(b_4^2 - a_4^2) + 2b_1a_4b_4\}(c_1^2 - c_2^2)\\ +2\{b_1(a_4^2 - b_4^2) - 2a_1a_4b_4\}(c_1c_2)\end{bmatrix} - 4Lb_Rb_I\begin{bmatrix}\{b_1(a_4^2 - b_4^2) - 2a_1a_4b_4\}(c_1^2 - c_2^2)\\ +2\{a_1(b_4^2 - a_4^2) + 2b_1a_4b_4\}(c_1c_2)\end{bmatrix},$$

$$A_{1I} = 2L(b_R^2 - b_I^2)\begin{bmatrix}\{b_1(a_4^2 - b_4^2) - 2a_1a_4b_4\}(c_1^2 - c_2^2)\\ +2\{a_1(b_4^2 - a_4^2) + 2b_1a_4b_4\}(c_1c_2)\end{bmatrix} + 4Lb_Rb_I\begin{bmatrix}\{a_1(b_4^2 - a_4^2) + 2b_1a_4b_4\}(c_1^2 - c_2^2)\\ +2\{b_1(a_4^2 - b_4^2) - 2a_1a_4b_4\}(c_1c_2)\end{bmatrix},$$

$$A_{2R} = 3b_1[27\lambda b_1^3 - 81a_1^2b_1\lambda - 9(a_1^2 - b_1^2) + 18a_1b_1\delta\lambda + 3a_1 + \Omega_2] - 3a_1[-27\lambda a_1^3 + 81a_1b_1^2\lambda - 9\delta\lambda(a_1^2 - b_1^2) - 18a_1b_1 + 3b_1 + \delta - \Omega_1],$$

$$A_{2I} = -3a_1[27\lambda b_1^3 - 81a_1^2b_1\lambda - 9\delta\lambda(a_1^2 - b_1^2) - 18a_1b_1 + 3b_1 + \delta - \Omega_1] - 3b_1[-27\lambda a_1^3 + 81a_1b_1^2\lambda - 9(a_1^2 - b_1^2) + 18a_1b_1\delta\lambda + 3a_1 + \Omega_2],$$

$$A_{3R} = -c_1[-\lambda c_1^3 + 3\lambda c_1c_2^2 + 2c_1c_2\delta\lambda + c_1 + \Omega_2 - (c_1^2 - c_2^2)] + c_2[\lambda c_2^3 - 3c_1^2c_2\lambda - \delta\lambda(c_1^2 - c_2^2) - 2c_1c_2 + c_2 + \delta - \Omega_1],$$

$$A_{3I} = -c_1[\lambda c_2^3 - 3c_1^2c_2\lambda - \delta\lambda(c_1^2 - c_2^2) - 2c_1c_2 + c_2 + \delta - \Omega_1] - c_2[-\lambda c_1^3 + 3\lambda c_1c_2^2 + 2c_1c_2\delta\lambda + c_1 + \Omega_2 - (c_1^2 - c_2^2)],$$

$$A_{6R} = L(a_R^2 - a_I^2)[9(a_2^2 - b_2^2)(3a_2b_2^2 - a_2^3) + 18a_2b_2(3a_2^2b_2 - b_2^3)] - 4La_Ra_I[9(a_2^2 - b_2^2)(b_2^3 - 3a_2^2b_2) - 18a_2b_2(a_2^3 - 3a_2b_2^2)],$$

$$A_{6I} = L(a_R^2 - a_I^2)[9(a_2^2 - b_2^2)(b_2^3 - 3a_2^2b_2) - 18a_2b_2(a_2^3 - 3a_2b_2^2)] + 4La_Ra_I[9(a_2^2 - b_2^2)(3a_2b_2^2 - a_2^3) + 18a_2b_2(3a_2^2b_2 - b_2^3)],$$

$$A_{7R} = 2L(b_R^2 - b_I^2)\begin{bmatrix}\{a_2(b_4^2 - a_4^2) + 2b_2a_4b_4\}(c_3^2 - c_4^2)\\ +2\{b_2(a_4^2 - b_4^2) - 2a_2a_4b_4\}(c_3c_4)\end{bmatrix} - 4Lb_Rb_I\begin{bmatrix}\{b_2(a_4^2 - b_4^2) - 2a_2a_4b_4\}(c_3^2 - c_4^2)\\ +2\{a_2(b_4^2 - a_4^2) + 2b_2a_4b_4\}(c_3c_4)\end{bmatrix},$$

$$A_{7I} = 2L(b_R^2 - b_I^2)\begin{bmatrix}\{b_2(a_4^2 - b_4^2) - 2a_2a_4b_4\}(c_3^2 - c_4^2)\\ +2\{a_2(b_4^2 - a_4^2) + 2b_2a_4b_4\}(c_3c_4)\end{bmatrix} + 4Lb_Rb_I\begin{bmatrix}\{a_2(b_4^2 - a_4^2) + 2b_2a_4b_4\}(c_3^2 - c_4^2)\\ +2\{b_2(a_4^2 - b_4^2) - 2a_2a_4b_4\}(c_3c_4)\end{bmatrix},$$

$$A_{8R} = 3b_2[27\lambda b_2^3 - 81a_2^2b_2\lambda - 9\delta\lambda(a_2^2 - b_2^2) - 18a_2b_2 + 3b_2 + \delta - \Omega_1] - 3a_2[-27\lambda a_2^3 + 81a_2b_2^2\lambda - 9(a_2^2 - b_2^2) + 18a_2b_2\delta\lambda + 3a_2 + \Omega_2],$$

$$A_{8I} = -3a_2[27\lambda b_2^3 - 81a_2^2b_2\lambda - 9\delta\lambda(a_2^2 - b_2^2) - 18a_2b_2 + 3b_2 + \delta - \Omega_1] - 3b_2[-27\lambda a_2^3 + 81a_2b_2^2\lambda - 9(a_2^2 - b_2^2) + 18a_2b_2\delta\lambda + 3a_2 + \Omega_2],$$

$$A_{9R} = -c_3[-\lambda c_3^3 + 3\lambda c_3 c_4^2 + 2c_3c_4\delta\lambda + c_3 + \Omega_2 - (c_3^2 - c_4^2)] + c_4[\lambda c_4^3 - 3c_3^2c_3\lambda - \delta\lambda(c_3^2 - c_4^2) - 2c_3c_4 + c_4 + \delta - \Omega_1],$$

$$A_{9I} = -c_3[\lambda c_4^3 - 3c_3^2c_4\lambda - \delta\lambda(c_3^2 - c_4^2) - 2c_3c_4 + c_4 + \delta - \Omega_1] - c_4[-\lambda c_3^3 + 3\lambda c_3c_4^2 + 2c_3c_4\delta\lambda + c_3 + \Omega_2 - (c_3^2 - c_4^2)],$$

$$A_{12R} = L(a_R^2 - a_I^2)\,[9(a_3^2 - b_3^2)\,(3a_3b_3^2 - a_3^3) + 18a_3b_3(3a_3^2b_3 - b_3^3)] - 4La_Ra_I[9(a_3^2 - b_3^2)\,(b_3^3 - 3a_3^2b_3) - 18a_3b_3(a_3^3 - 3a_3b_3^2)],$$

$$A_{12I} = L(a_R^2 - a_I^2)\,[9(a_3^2 - b_3^2)\,(b_3^3 - 3a_3^2b_3) - 18a_3b_3(a_3^3 - 3a_3b_3^2)] + 4La_Ra_I[9(a_3^2 - b_3^2)\,(3a_3b_3^2 - a_3^3) + 18a_3b_3(3a_3^2b_3 - b_3^3)],$$

$$A_{13R} = 2L(b_R^2 - b_I^2)\begin{bmatrix}\{a_3(b_4^2 - a_4^2) + 2b_3a_4b_4\}\,(c_5^2 - c_6^2)\\ + 2\{b_3(a_4^2 - b_4^2) - 2a_3a_4b_4\}\,(c_5c_6)\end{bmatrix} - 4Lb_Rb_I\begin{bmatrix}\{b_3(a_4^2 - b_4^2) - 2a_3a_4b_4\}\,(c_5^2 - c_6^2)\\ + 2\{a_3(b_4^2 - a_4^2) + 2b_3a_4b_4\}\,(c_5c_6)\end{bmatrix},$$

$$A_{13I} = 2L(b_R^2 - b_I^2)\begin{bmatrix}\{b_3(a_4^2 - b_4^2) - 2a_3a_4b_4\}\,(c_5^2 - c_6^2)\\ + 2\{a_3(b_4^2 - a_4^2) + 2b_3a_4b_4\}\,(c_5c_6)\end{bmatrix} + 4Lb_Rb_I\begin{bmatrix}\{a_3(b_4^2 - a_4^2) + 2b_3a_4b_4\}\,(c_5^2 - c_6^2)\\ + 2\{b_3(a_4^2 - b_4^2) - 2a_3a_4b_4\}\,(c_5c_6)\end{bmatrix},$$

$$A_{14R} = 3b_3[27\lambda b_3^3 - 81a_3^2b_3\lambda - 9\delta\lambda(a_3^2 - b_3^2) - 18a_3b_3 + 3b_3] - 3a_3[-27\lambda a_3^3 + 81a_3b_3^2\lambda - 9(a_3^2 - b_3^2) + 18a_3b_3\delta\lambda + 3a_3 + \Omega_2],$$

$$A_{14I} = -3a_3[27\lambda b_3^3 - 81a_3^2b_3\lambda - 9\delta\lambda(a_3^2 - b_3^2) - 18a_3b_3 + 3b_3] - 3b_3[-27\lambda a_3^3 + 81a_3b_3^2\lambda - 9(a_3^2 - b_3^2) + 18a_3b_3\delta\lambda + 3a_3 + \Omega_2],$$

$$A_{15R} = -c_5[-\lambda c_5^3 + 3\lambda c_5c_6^2 + 2c_5c_6\delta\lambda + c_5 + \Omega_2 - (c_5^2 - c_6^2)] + c_6[\lambda c_6^3 - 3c_5^2c_6\lambda - \delta\lambda(c_5^2 - c_6^2) - 2c_5c_6 + c_6],$$

$$A_{15I} = -c_5[\lambda c_6^3 - 3c_5^2c_6\lambda - \delta\lambda(c_5^2 - c_6^2) - 2c_5c_6 + c_6] - c_6[-\lambda c_5^3 + 3\lambda c_5c_6^2 + 2c_5c_6\delta\lambda + c_5 + \Omega_2 - (c_5^2 - c_6^2)],$$

$$A_{18R} = L(b_R^2 - b_I^2)\,[9(a_4^2 - b_4^2)\,(3a_4b_4^2 - a_4^3) - 18a_4b_4(3a_4^2b_4 - b_4^3)] - 4La_Ra_I[9(a_4^2 - b_4^2)\,(b_4^3 - 3a_4^2b_4) + 18a_4b_4(a_4^3 - 3a_4b_4^2)],$$

$$A_{18I} = L(b_R^2 - b_I^2)\,[9(a_4^2 - b_4^2)\,(b_4^3 - 3a_4^2b_4) + 18a_4b_4(a_4^3b - 3a_4b_4^2)] + 4La_Ra_I[9(a_4^2 - b_4^2)\,(3a_4b_4^2 - a_4^3) - 18a_4b_4(3a_4^2b_4 - b_4^3)],$$

$$A_{27R} = 2L(a_R^2 - a_I^2)\begin{bmatrix} \{b_4(b_3^2 - a_3^2) + 2b_3a_3b_4\}(c_{11}^2 - c_{12}^2) \\ +2\{b_4(a_3^2 - b_3^2) + 2a_3a_4b_3\}(c_{11}c_{12}) \end{bmatrix}$$

$$- 4La_Ra_I\begin{bmatrix} \{b_4(b_3^2 - a_3^2) + 2b_3a_3b_4\}(c_{11}c_{12}) \\ -\{b_4(a_3^2 - b_3^2) + 2a_3a_4b_3\}(c_{11}^2 - c_{12}^2) \end{bmatrix},$$

$$A_{27I} = 2L(a_R^2 - a_I^2)\begin{bmatrix} \{b_4(b_3^2 - a_3^2) + 2b_3a_3b_4\}(c_{11}c_{12}) \\ -\{b_4(a_3^2 - b_3^2) + 2a_3a_4b_3\}(c_{11}^2 - c_{12}^2) \end{bmatrix}$$

$$+ 4Lb_Rb_I\begin{bmatrix} \{b_4(b_3^2 - a_3^2) + 2b_3a_3b_4\}(c_{11}^2 - c_{12}^2) \\ +2\{b_4(a_3^2 - b_3^2) + 2a_3a_4b_3\}(c_{11}c_{12}) \end{bmatrix},$$

$$A_{28R} = c_{11}[\lambda c_{11}^3 - 3\lambda c_{11}c_{12}^2 + 2c_{11}c_{12}\delta\lambda - c_{11} - \Omega_2 + (c_{11}^2 - c_{12}^2)]$$
$$+ c_{12}[\lambda c_{12}^3 - 3c_{11}^2c_{12}\lambda + \delta\lambda(c_{11}^2 - c_{12}^2) + 2c_{11}c_{12} + c_{12} - 2\Omega_1],$$

$$A_{28I} = c_{12}[\lambda c_{11}^3 - 3\lambda c_{11}c_{12}^2 + 2c_{11}c_{12}\delta\lambda - c_{11} - \Omega_2 + (c_{11}^2 - c_{12}^2)]$$
$$- c_9[\lambda c_{12}^3 - 3c_{11}^2c_{12}\lambda + \delta\lambda(c_{11}^2 - c_{12}^2) - 2c_{11}c_{12} + c_{12} - 2\Omega_1],$$

$$\sigma_1 = \left[\frac{1}{2}\left\{\sqrt{(1 + 4\Omega_2)^2 + 16(\delta - \Omega_1)^2} + (1 + 4\Omega_2)\right\}\right]^{\frac{1}{2}},$$

$$\sigma_2 = \left[\frac{1}{2}\left\{\sqrt{(1 + 4\Omega_2)^2 + 16(\Omega_1 - \delta)^2} + (1 + 4\Omega_2)\right\}\right]^{\frac{1}{2}},$$

$$\sigma_3 = \left[\frac{1}{2}\left\{\sqrt{(1 + 4\Omega_2)^2 + 16(\Omega_1 + \delta)^2} + (1 + 4\Omega_2)\right\}\right]^{\frac{1}{2}},$$

$$\xi_1 = \left[\frac{1}{2}\left\{\sqrt{(1 + 4\Omega_2)^2 + 16(\delta - \Omega_1)^2} - (1 + 4\Omega_2)\right\}\right]^{\frac{1}{2}},$$

$$\xi_2 = \left[\frac{1}{2}\left\{\sqrt{1 + 4\Omega_2 + 16(\Omega_1 - \delta)^2} - (1 + 4\Omega_2)\right\}\right]^{\frac{1}{2}},$$

$$\xi_3 = \left[\frac{1}{2}\left\{\sqrt{1 + 4\Omega_2 + 16(\Omega_1 + \delta)^2} - (1 + 4\Omega_2)\right\}\right]^{\frac{1}{2}},$$

$$\lambda = \frac{\alpha_1 W_0^2}{\varrho\nu^2}, \qquad \Omega_1 = \frac{2\Omega\nu}{W_0^2}, \qquad \Omega_2 = \frac{\sigma B_0^2\nu}{\varrho W_0^2}, \qquad L = \frac{U_0^2 W_0^2}{\nu^3},$$

$$c_1 = a_1 + 2a_4, \qquad c_2 = b_1 + 2b_4, \qquad c_3 = a_2 + 2a_4, \qquad c_4 = b_2 + 2b_4,$$

$$c_5 = a_3 + 2a_4, \qquad c_6 = b_3 + 2b_4, \qquad c_7 = 2a_1 + a_4, \qquad c_8 = 2b_1 + b_4,$$

$$c_9 = 2a_2 + a_4, \qquad c_{10} = 2b_2 + b_4, \qquad c_{11} = 2a_3 + a_4, \qquad c_{12} = 2b_3 + b_4,$$

$$c_{0R} = \frac{1 + \alpha_1}{2}, \qquad n_{0R} = \frac{1 + \alpha_2}{2}, \qquad k_{0R} = \frac{1 + \alpha_3}{2},$$

$$c_{0I} = \frac{\beta_1}{2}, \qquad n_{0I} = \frac{\beta_2}{2}, \qquad k_{0I} = \frac{\beta_3}{2},$$

$$A_{19R} = 2L(a_R^2 - a_I^2)\begin{bmatrix} \{b_4(b_1^2 - a_1^2) + 2b_1a_1b_4\}(c_7^2 - c_8^2) \\ +2\{b_4(a_1^2 - b_1^2) + 2a_1a_4b_1\}(c_7c_8) \end{bmatrix} - 4La_Ra_I\begin{bmatrix} \{b_4(b_1^2 - a_1^2) + 2b_1a_1b_4\}(c_7c_8) \\ -\{b_4(a_1^2 - b_1^2) + 2a_1a_4b_1\}(c_7^2 - c_8^2) \end{bmatrix},$$

$$A_{19I} = 2L(a_R^2 - a_I^2)\begin{bmatrix} \{b_4(b_1^2 - a_1^2) + 2b_1a_1b_4\}(c_7c_8) \\ -\{b_4(a_1^2 - b_1^2) + 2a_1a_4b_1\}(c_7^2 - c_8^2) \end{bmatrix} + 4Lb_Rb_I\begin{bmatrix} \{b_4(b_1^2 - a_1^2) + 2b_1a_1b_4\}(c_7^2 - c_8^2) \\ +2\{b_4(a_1^2 - b_1^2) + 2a_1a_4b_1\}(c_7c_8) \end{bmatrix},$$

$$A_{20R} = 3b_4[27\lambda b_4^3 - 81a_4^2b_4\lambda - 9\delta\lambda(a_4^2 - b_4^2) - 18a_4b_4 + 3b_4 - (\delta - \Omega_1)] - 3a_4[-27\lambda a_4^3 + 81a_4b_4^2\lambda - 9(a_4^2 - b_4^2) - 18a_4b_4\delta\lambda + 3a_4 + \Omega_2],$$

$$A_{20I} = 3b_4[27\lambda b_4^3 - 81a_4b_4^2\lambda + 9(a_4^2 - b_4^2) + 18a_4b_4\delta\lambda - 3a_4 - \Omega_2] - 3a_4[27\lambda b_4^3 - 81a_4^2b_4\lambda + 9\delta\lambda(a_4^2 - b_4^2) - 18a_4b_4 + 3b_4 - (\delta + \Omega_1)],$$

$$A_{21R} = -c_7[-\lambda c_7^3 + 3\lambda c_7c_8^2 + 2c_7c_8\delta\lambda + c_7 + \Omega_2 - (c_7^2 - c_8^2)] + c_8[\lambda c_8^3 - 3c_7^2c_8\lambda + \delta\lambda(c_7^2 - c_8^2) - 2c_7c_8 + c_8 - (\delta + \Omega_1)],$$

$$A_{21I} = -c_7[\lambda c_8^3 - 3c_7^2c_8\lambda + \delta\lambda(c_7^2 - c_8^2) - 2c_7c_8 + c_8 + (\delta + \Omega_1)] - c_8[-\lambda c_7^3 + 3\lambda c_7c_8^2 + 2c_7c_8\delta\lambda + c_7 + \Omega_2 - (c_7^2 - c_8^2)$$

$$A_{24R} = 2L(a_R^2 - a_I^2)\begin{bmatrix} \{b_4(b_2^2 - a_2^2) + 2b_2a_2b_4\}(c_9^2 - c_{10}^2) \\ +2\{b_4(a_2^2 - b_2^2) + 2a_2a_4b_2\}(c_9c_{10}) \end{bmatrix} - 4La_Ra_I\begin{bmatrix} \{b_4(b_2^2 - a_2^2) + 2b_2a_2b_4\}(c_9c_{10}) \\ -\{b_4(a_2^2 - b_2^2) + 2a_2a_4b_2\}(c_9^2 - c_{10}^2) \end{bmatrix},$$

$$A_{24I} = 2L(a_R^2 - a_I^2)\begin{bmatrix} \{b_4(b_2^2 - a_2^2) + 2b_2a_2b_4\}(c_9c_{10}) \\ -\{b_4(a_2^2 - b_2^2) + 2a_2a_4b_2\}(c_9^2 - c_{10}^2) \end{bmatrix} + 4Lb_Rb_I\begin{bmatrix} \{b_4(b_2^2 - a_2^2) + 2b_2a_2b_4\}(c_9^2 - c_{10}^2) \\ +2\{b_4(a_2^2 - b_2^2) + 2a_2a_4b_2\}(c_9c_{10}) \end{bmatrix},$$

$$A_{25R} = -c_9[-\lambda c_9^3 + 3\lambda c_9c_{10}^2 - 2c_9c_{10}\delta\lambda + c_9 + \Omega_2 - (c_9^2 - c_{10}^2)] + c_{10}[\lambda c_{10}^3 - 3c_9^2c_{10}\lambda + \delta\lambda(c_9^2 - c_{10}^2) + 2c_9c_{10} + c_{10} - (\delta + \Omega_1)],$$

$$A_{25I} = c_{10}[\lambda c_9^3 - 3\lambda c_9c_{10}^2 + 2c_9c_{10}\delta\lambda - c_9 - \Omega_2 + (c_9^2 - c_{10}^2)] - c_9[\lambda c_{10}^3 - 3c_9^2c_{10}\lambda + \delta\lambda(c_9^2 - c_{10}^2) + 2c_9c_{10} + c_{10} - (\delta + \Omega_1)],$$

$$c_{1R}=\frac{\left[\begin{array}{l}(1-2c_{0R})\,(3c_{0R}c_{0I}^2-c_{0R}^3+2\delta c_{0R}c_{0I})\\ -2c_{0I}\left((c_{0I}^3-3c_{0R}^2c_{0I})-\delta(c_{0R}^2-c_{0I}^2)\right)\end{array}\right]}{[(1-2c_{0R})^2+4c_{0I}^2]},$$

$$c_{1I}=\frac{\left[\begin{array}{c}2c_{0I}\left((3c_{0R}c_{01}^2-c_{0R}^3)+2\delta c_{0R}c_{0I}\right)\\ +\left(c_{0I}^3-3c_{0R}^2c_{0I}-\delta(c_{0R}^2-c_{0I}^2)\right)(1-2c_{0R})\end{array}\right]}{[(1-2c_{0R})^2+4c_{0I}^2]},$$

$$c_{2R}=\frac{\left[\begin{array}{c}\left\{\begin{array}{c}3(c_{0R}^2-c_{0I}^2)\,c_{1R}-6c_{0R}c_{0I}c_{1I}\\ -2\delta c_{0I}c_{1R}-2\delta c_{0R}c_{1I}-(c_{1R}^2-c_{1I}^2)\end{array}\right\}(1-2c_{0R})\\ -\left\{\begin{array}{c}6c_{0R}c_{0I}c_{1R}+3c_{1I}(c_{0R}^2-c_{0I}^2)\\ -2c_{1R}c_{1I}+2\delta c_{0I}c_{0R}-2\delta c_{0I}c_{1I}\end{array}\right\}2c_{0I}\end{array}\right]}{[(1-2c_{0R})^2+4c_{0I}^2]},$$

$$c_{2I}=\frac{\left[\begin{array}{l}2c_{0I}\left\{\begin{array}{c}3(c_{0R}^2-c_{0I}^2)\,c_{1R}-6c_{0R}c_{0I}c_{1I}\\ -(c_{1R}^2-c_{1I}^2)\}-2\delta c_{0I}c_{1R}-2\delta c_{0R}c_{1I}\end{array}\right\}\\ +(1-2c_{0R})\left\{\begin{array}{c}6c_{0R}c_{0I}c_{1R}+3c_{1I}(c_{0R}^2-c_{0I}^2)\\ -2c_{1R}c_{1I}+2\delta c_{0I}c_{0R}-2\delta c_{0I}c_{1I}\end{array}\right\}\end{array}\right]}{[(1-2c_{0R})^2+4c_{0I}^2]},$$

$$n_{1R}=\frac{\left[\begin{array}{l}(1-2n_{0R})\,(3n_{0R}n_{0I}^2-n_{0R}^3+2\delta n_{0R}n_{0I})\\ -2n_{0I}\left((n_{0I}^3-3n_{0R}^2n_{0I})-\delta(n_{0R}^2-n_{0I}^2)\right)\end{array}\right]}{[(1-2n_{0R})^2+4n_{0I}^2]},$$

$$n_{1I}=\frac{\left[\begin{array}{c}2n_{0I}\left((3n_{0R}n_{01}^2n_{0R}^3)+2\delta n_{0R}n_{0I}\right)\\ +\left(n_{0I}^3-3n_{0R}^2n_{0I}-\delta(n_{0R}^2-n_{0I}^2)\right)(1-2n_{0R})\end{array}\right]}{[(1-2n_{0R})^2+4n_{0I}^2]},$$

$$n_{2R}=\frac{\left[\begin{array}{c}\left\{\begin{array}{c}3(n_{0R}^2-n_{0I}^2)\,n_{1R}-6n_{0R}n_{0I}n_{1I}\\ -2\delta n_{0I}n_{1R}-2\delta n_{0R}n_{1I}-(n_{1R}^2-n_{1I}^2)\end{array}\right\}(1-2n_{0R})\\ -\left\{\begin{array}{c}6n_{0R}n_{0I}n_{1R}+3n_{1I}(n_{0R}^2-n_{0I}^2)\\ -2n_{1R}n_{1I}+2\delta n_{0I}n_{0R}-2\delta n_{0I}n_{1I}\end{array}\right\}2n_{0I}\end{array}\right]}{[(1-2n_{0R})^2+4n_{0I}^2]},$$

$$n_{2I}=\frac{\left[\begin{array}{l}2n_{0I}\left\{\begin{array}{c}3(n_{0R}^2-n_{0I}^2)\,n_{1R}-6n_{0R}n_{0I}n_{1I}\\ -(n_{1R}^2-n_{1I}^2)\}-2\delta n_{0I}n_{1R}-2\delta n_{0R}n_{1I}\end{array}\right\}\\ +(1-2n_{0R})\left\{\begin{array}{c}6n_{0R}n_{0I}n_{1R}+3n_{1I}(n_{0R}^2-n_{0I}^2)\\ -2n_{1R}n_{1I}+2\delta n_{0I}n_{0R}-2\delta n_{0I}n_{1I}\end{array}\right\}\end{array}\right]}{[(1-2n_{0R})^2+4n_{0I}^2]},$$

$$h_0=\frac{[1+\sqrt{1+4\Omega_2}]}{2},\qquad h_{1R}=\frac{h_0^3}{1+2h_0},\qquad h_{1I}=\frac{-\delta h_0^2}{1+2h_0},$$

$$h_{2R}=\frac{3h_0^2h_{1R}+2\delta h_0h_{1R}+(h_{1R}^2-h_{1I}^2)}{1+2h_0}, \qquad h_{2I}=\frac{3h_0^2h_{1I}-2\delta h_0h_{1R}-2h_{1R}h_{1I}}{1+2h_0}$$

$$k_{1R}=\frac{\begin{bmatrix}(1-2k_{0R})\,(3k_{0R}k_{0I}^2-k_{0R}^3-2\delta k_{0R}k_{0I})\\ -2k_{0I}\left((k_{0I}^3-3k_{0R}^2k_{0I})+\delta(k_{0R}^2-k_{0I}^2)\right)\end{bmatrix}}{[(1-2k_{0R})^2+4k_{0I}^2]},$$

$$k_{1I}=\frac{\begin{bmatrix}2k_{0I}\left((3k_{0R}k_{01}^2-k_{0R}^3)+2\delta k_{0R}k_{0I}\right)\\ +\left(k_{0I}^3-3k_{0R}^2k_{0I}+\delta(k_{0R}^2-k_{0I}^2)\right)(1-2k_{0R})\end{bmatrix}}{[(1-2k_{0R})^2+4k_{0I}^2]},$$

$$k_{2R}=\frac{\begin{bmatrix}\begin{Bmatrix}3(k_{0R}^2-k_{0I}^2)\,k_{1R}-6k_{0R}k_{0I}k_{1I}\\ +2\delta k_{0I}k_{1R}+2\delta k_{0R}k_{1I}\end{Bmatrix}(1-2k_{0R})\\ -\begin{Bmatrix}6k_{0R}k_{0I}k_{1R}+3k_{1I}(k_{0R}^2-k_{0I}^2)\\ -2k_{1R}k_{1I}+2\delta k_{0I}k_{0R}-2\delta k_{0I}k_{1I}\end{Bmatrix}2k_{0I}\end{bmatrix}}{[(1-2k_{0R})^2+4k_{0I}^2]},$$

$$k_{2I}=\frac{\begin{bmatrix}2k_{0I}\begin{Bmatrix}3(k_{0R}^2-k_{0I}^2)\,k_{1R}-6k_{0R}k_{0I}k_{1I}\\ +2\delta k_{0I}k_{1R}+2\delta k_{0R}k_{1I}\end{Bmatrix}\\ +(1-2k_{0R})\begin{Bmatrix}6k_{0R}k_{0I}k_{1R}+3k_{1I}(k_{0R}^2-k_{0I}^2)\\ -2\delta k_{1R}k_{0R}+2\delta k_{0I}k_{1I}\end{Bmatrix}\end{bmatrix}}{[(1-2k_{0R})^2+4k_{0I}^2]}.$$

4 Concluding remarks

We have presented results for the rotating flow of a third grade fluid on a porous plate oscillating in its own plane. The fields are governed by a third order non-linear partial differential equation. Since we are considering a third grade fluid some remarks on symmetric and asymmetric solution will not be out of place.

Berker [17] was the first to discuss the problem of two infinite parallel disks rotating with the same angular velocity Ω about a common axis. He showed, that the locus of the centre of rotation of the fluid changes with respect to the depth of the fluid between the disks, and asymmetric solutions are predicted. He [17] further pointed out, if the axis of rotation does not change (rigid body rotation) then symmetric solutions appear. Thus, he observed that symmetric solutions are the subclass of asymmetric solutions. The complete answer to this question has been given by Parter and Rajagobal [18] who showed that the asymmetric solutions exist for non-Newtonian fluids.

In the present problem, the fuid and the plate are in a rigid body rotation, and the symmetric solutions appear. However, in the case of a third grade fluid if the disk and the fluid are not in a rigid body rotation asymmetric solutions are possible as predicted by Parter and Rajagopal [18].

Solutions (26) to (31) reveal that the structures of the associated magnetohydrodynamic boundary layers on the plate are qualitatively similar to those of the hydrodynamic layers. In the present solution, the thickness of the boundary layer (in Eqs. (26) to (31)) are in the combinations of the hydrodynamic and the hydromagnetic boundary layer and are evidently smaller than those of the classical Stokes and Ekman layers. It appears that the electromagnetic force makes the hydromagnetic boundary layer thickness thinner.

It may also be observed that the thickness of the hydromagnetic boundary layer remains bounded for all frequencies. A striking difference of the presented analysis is also that in the resonance case the hydromagnetic solution exists. Further, another distinguishing feature of the results is that an increase of the magnetic field has the same influence on the field as increased material parameters of the fluid. According to (26) to (31) the main effects of material parameters and a magnetic field are to reduce the velocity within the boundary layer and also reduce the boundary layer thickness.

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Fluctuating flow of a third order fluid past an infinite plate with variable suction

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THE TWO-DIMENSIONAL flow problem of a third order incompressible fluid past an infinite porous plate is discussed when the suction velocity normal to the plate, as well as the the external flow velocity, varies periodically with time. The governing partial differential equation is of third order and nonlinear. Analytic solution is obtained using the series method. Expressions for the velocity and the skin friction have been obtained in a dimensionless form. The results of viscous and second order fluids can be recovered as special cases of this problem. Finally, several graphs are plotted and discussed.

1. Introduction

THE OSCILLATING flows play an important role in many engineering applications. The study of such flows was first initiated by LIGHTHILL [1] who studied the effects of free stream oscillations on the boundary layer flows of viscous, incompressible fluid past an infinite plate. Thereafter STUART [2] extended it to study a two-dimensional flow past an infinite, porous plate with constant suction when the free stream oscillates in time about a constant mean. After the appearance of LIGHTHILL'S [1] classic paper on the response of skin friction in laminar flow due to fluctuations in the free stream, considerable interest has been developed in the subject of boundary layers which have a regular fluctuating flow superimposed on the mean boundary flow. A large number of papers dealing with this subject have appeared, cf. for example WATSON [3], MESSIHA [4], KELLY [5] and LAL [6]. The idea has been also extended to magnetohydrodynamic flows, SURYAPRAKASARO [7], and the elastico-viscous flows, KALONI [8], SOUNDALGEKAR and PURI [9] and PURI [10]. The boundary layer suction is a very effective method for prevention of the separation. The effects of different arrangements and configurations of the suction holes and slits on the undesired phenomenon of separation have been studied extensively by various scholars, and have been compiled by LACHMAN [11]. In technological fields, the boundary layer phenomenon in non-Newtonian fluids has recently become a fascinating problem, under a wide range of geometrical, dynamical and rheological conditions.

Some experiments by BARNES *et al.* [12] confirmed that an increase in the flow rate is possible and that the phenomenon appears to be governed by the shear-dependent viscosity. In fact, in [13] WALTERS and TOWNSEND show that the mean flow rate is unaffected by second-order viscoelasticity. Although the second-order model is able to predict the normal stress differences which are characteristic of non-Newtonian liquids, it is not shear thinning or thickening, the shear viscosity is constant. Third-order model exhibits shear-dependent viscosity, for a simple-shearing motion ($u' = (\gamma y', 0, 0)$), where γ is the rate of strain. The relation between the shearing stress and the rate of strain is given by $S_{xy} = \mu\left(1 \mp T_s^2\gamma^2\right)\gamma$, where T_s is the shear relaxation time (its reciprocal is the characteristic rate of strain at which the apparent shear viscosity noticeably decreases or increases), and μ is the lower limiting viscosity. Experiments made by BRUCE [14] has shown that there are materials that exhibit: (1) strong normal stresses but are weakly shear thinning or thickening (class 1 a, b); (2) roughly equal normal and shear effects (class 2 a, b); (3) weak normal stresses, but they are strongly shear thinning or thickening (class 3 a, b).

Since many years there has been much interest in the effect of a variable suction velocity on the flow field. Regarding the elasto-viscous (Walters liquid B$'$) model, SOUNDALGEKAR and PURI [9] obtained the perturbation solution for the fluctuating flow of the elasto-viscous fluids past an infinite plate with variable suction.

As far as the authors are aware, no attempt has been made to examine the effect of the variable suction velocity on the flow fields of third-order fluids past an infinite plate. In the present work such an attempt has been considered. Literature survey revealed no previous attempts on studying this problem, even in the constant suction velocity case. The external flow velocity in the present paper is taken as $U_0'\left[1+\in e^{iw't'}\right]$ and the suction velocity is assumed to be of the form $v_0'\left[1+\in Ae^{iw't'}\right]$, where v_0' is a non-zero constant mean suction velocity, $\in$ is small and A is a positive constant such that $\in A \leq 1$. By neglecting higher powers of $\in$, approximate solutions are obtained for the velocity field in the boundary layer.

2. The constitutive model

The incompressible, homogeneous fluid of third order is a simple fluid of the differential type whose Cauchy stress tensor has the representation [15]

$$(2.1)\quad \mathbf{T} = -p\mathbf{I}+\mu\mathbf{A}_1+\alpha_1\mathbf{A}_2+\alpha_2\mathbf{A}_1^2+\beta_1\mathbf{A}_3+\beta_2\left(\mathbf{A}_1\mathbf{A}_2+\mathbf{A}_2\mathbf{A}_1\right)+\beta_3\left(\mathrm{tr}\mathbf{A}_1^2\right)\mathbf{A}_1,$$

where $-p\mathbf{I}$ is the indeterminate part of the stress due to the constraint of incompressibility, $\mu, \alpha_1, \alpha_2, \beta_1, \beta_2$ and β_3 are material constants, and the tensors A_n, $n = 1, 2, 3$ are defined through [16]

$$\text{(2.2)} \qquad \begin{aligned} \mathbf{A}_1 &= (\text{grad}\mathbf{V}) + (\text{grad}\mathbf{V})^T, \\ \mathbf{A}_n &= \left(\frac{\partial}{\partial t'} + \mathbf{V}.\nabla\right)\mathbf{A}_{n-1} + \mathbf{A}_{n-1}(\text{grad}\mathbf{V}) + (\text{grad}\mathbf{V})^T\mathbf{A}_{n-1},\ n > 1, \end{aligned}$$

where $\mathbf{V}$ is the velocity and t' is the time.

JOSEPH [17] proved that the rest state of fluids of grade n, $n \neq 1$, any is unstable in the spectral sense of linearized theory when the ratio of the coefficients of $\mathbf{A}_n$ and $\mathbf{A}_{n-1}$ in the constitutive equation is negative. Hence, if $\alpha_1 < 0$ then the above model exhibits unacceptable stability characteristics. On the other hand, Eq. (2.1) must be consistent with thermodynamics principles. The thermodynamic of fluid model by Eq. (2.1) has been the object of a detailed study by FOSDICK and RAJAGOPAL [18]. They have shown that the Eq. (2.1) to be compatible with thermodynamics, and the free energy to be minimum when the fluid is at rest, the material constants should satisfy the relations

$$\text{(2.3)} \qquad \begin{aligned} &\mu \geq 0, \qquad \alpha_1 \geq 0, \qquad \beta_1 = \beta_2 = 0, \\ &\beta_3 \geq 0, \qquad -\sqrt{24\mu\beta_3} \leq \alpha_1 + \alpha_2 \leq \sqrt{24\mu\beta_3}. \end{aligned}$$

It is easy to see that the ratio of the coefficients of A_2 and A_3 in the form of $\mathbf{T}$, i.e. the "ratio" $\dfrac{\alpha_1}{0}$, does not satisfy neither the hypothesis of JOSEPH [17] nor the hypothesis of RENARDY [19], who assumed the coefficients α_{n-1} ($n \geq 5$ and here 3) of $\mathbf{A}_n$ is non-zero for instability. We also point out that the retarded motion approximation does not lead the models. Thus subject was clearly explained by DUNN and RAJAGOPAL [20]. Therefore, the model of Eq. (2.1) reduces to:

$$\text{(2.4)} \qquad \mathbf{T} = -p\mathbf{I} + \mu\mathbf{A}_1 + \alpha_1\mathbf{A}_2 + \alpha_2\mathbf{A}_1^2 + \beta_3\left(\text{tr}\mathbf{A}_1^2\right)\mathbf{A}_1.$$

The equation of motion, in the absence of body forces, is

$$\text{(2.5)} \qquad \rho'\frac{d\mathbf{V}}{dt'} = \text{div}\mathbf{T},$$

where ρ' is the density of the fluid in the dimensional form and $\dfrac{d}{dt'}$ is the material derivative. The fluid is incompressible, thus only isochoric (i.e. volume preserving) flows are possible, i.e. the flow satisfies the constraints

$$\text{(2.6)} \qquad \text{div}\mathbf{V} = 0.$$

We consider a two-dimensional incompressible fluid flow along an infinite plane porous wall. The flow is independent of the distance parallel to the wall and the suction velocity normal to the wall is directed towards it and varies periodically with time about a non-zero constant mean value v_0'. The x'-axis is taken along the wall, y'-axis normal to the wall. Dash denotes dimensional quantities. Thus for the problem under consideration, we seek a velocity field of the form

$$\mathbf{V} = \left[u'\left(y', t'\right), v', 0\right], \tag{2.7}$$

where $v' < 0$ is the suction velocity.

From Eqs. (2.6) and (2.7)

$$\frac{\partial v'}{\partial y'} = 0. \tag{2.8}$$

It is evident from Eq. (2.8) that v' is a function of time only. Hence we consider v' in the form [4]

$$v' = -v_0'(1+ \in Ae^{i\omega' t'}). \tag{2.9}$$

The negative sign in Eq. (2.9) indicates that the suction velocity normal to the wall is directed towards the wall. In view of Eqs. (2.4), (2.7) and (2.9), Eq. (2.5) takes the form

$$\begin{aligned} \frac{\partial u'}{\partial t'} - v_0'(1+ \in Ae^{i\omega' t'})\frac{\partial u'}{\partial y'} &= -\frac{1}{\rho'}\frac{\partial P'}{\partial x'} + \nu\frac{\partial^2 u'}{\partial y'^2} \\ &+ \frac{\alpha_1}{\rho'}\left[\frac{\partial^3 u'}{\partial y'^2 \partial t'} - v_0'\left(1+ \in Ae^{i\omega' t'}\right)\frac{\partial^3 u'}{\partial y'^3}\right] + \frac{6\beta_3}{\rho'}\left(\frac{\partial u'}{\partial y'}\right)^2 \frac{\partial^2 u'}{\partial y'^2}, \end{aligned} \tag{2.10}$$

$$\frac{\partial v'}{\partial t'} = -\frac{1}{\rho'}\frac{\partial P'}{\partial y'}, \tag{2.11}$$

where

$$\nu = \frac{\mu}{\rho'},$$

$$P' = p' - (2\alpha_1 + \alpha_2)\left(\frac{\partial u'}{\partial y'}\right)^2.$$

From Eqs. (2.9) and (2.11), it is clear that $\frac{\partial P'}{\partial y'}$ is small in the boundary layer and can be neglected [9]. Hence the pressure is taken to be constant along any normal and is given by its value outside the boundary layer. If $U'(t')$ is the stream velocity parallel to the wall just outside the boundary layer, then

$$-\frac{1}{\rho'}\frac{\partial P'}{\partial x'} = \frac{dU'}{dt'}$$

and the Eq. (2.10) takes the form

$$\frac{\partial u'}{\partial t'} - v_0'\left(1+\in Ae^{i\omega' t'}\right)\frac{\partial u'}{\partial y'} = \frac{dU'}{dt'} + \nu\frac{\partial^2 u'}{\partial y'^2} + \frac{\alpha_1}{\rho'}\left[\frac{\partial^3 u'}{\partial y'^2 \partial t'} - v_0'\left(1+\in Ae^{i\omega' t'}\right)\frac{\partial^3 u'}{\partial y'^3}\right] + \frac{6\beta_3}{\rho'}\left(\frac{\partial u'}{\partial y'}\right)^2\frac{\partial^2 u'}{\partial y'^2}. \tag{2.12}$$

The boundary conditions are

$$u' = 0 \quad \text{at} \quad y' = 0 \quad \text{and} \quad u' = U'(t') \quad \text{as} \quad y' \longrightarrow \infty. \tag{2.13}$$

We introduce dimensionless quantities defined by

$$y = \frac{y' v_0'}{\nu}, \quad t = \frac{v_0'^2 t'}{4\nu}, \quad \omega = \frac{4\nu\omega'}{v_0'^2},$$
$$\alpha = \frac{\alpha_1 v_0'^2}{\rho\nu^2}, \quad u = \frac{u'}{U_0'}, \quad U = \frac{U'}{U_0'}, \quad \in_1 = \frac{6\beta_3}{\rho'\nu^3}U_0'^2 v_0'^2, \tag{2.14}$$

where U_0' is the reference velocity and ω' is the frequency. Equation (2.12) takes the dimensionless form

$$\frac{1}{4}\frac{\partial u}{\partial t} - (1+\in Ae^{i\omega t})\frac{\partial u}{\partial y} = \frac{1}{4}\frac{dU}{dt} + \frac{\partial^2 u}{\partial y^2} + \alpha\left[\frac{1}{4}\frac{\partial^3 u}{\partial y^2 \partial t} - \left(1+\in Ae^{i\omega t}\right)\frac{\partial^3 u}{\partial y^3}\right] + \in_1\left(\frac{\partial u}{\partial y}\right)^2\frac{\partial^2 u}{\partial y^2}, \tag{2.15}$$

subject to the conditions

$$u = 0 \quad \text{at} \quad y = 0 \quad \text{and} \quad u \longrightarrow U \quad \text{as} \quad y \longrightarrow \infty, \tag{2.16}$$

where

$$U = 1+\in e^{i\omega t}. \tag{2.17}$$

3. Perturbation solution

We note that the resulting equation of motion (2.15) is of the third order. Moreover, this equation is nonlinear as compared to the cases of the second order, elastic-viscous [9] and Newtonian flow [4] equations. As a result, it seems to be impossible to obtain the general solution in a closed form for arbitrary values of all parameters appearing in the nonlinear equation. Even in the case of constant suction and elastic-viscous fluid [8], all analytic solutions obtained so far are based on the assumptions that one or more of the parameters are zero or small. Therefore, we seek the solution of the problem as a power series expansion in the small parameters $\in_1$. Accordingly, we assumed that the velocity component u can be expanded in powers of $\in_1$ as follows:

$$u(y,\in_1) = u_0(y) + \in_1 u_1(y) + \ldots \tag{3.1}$$

Substituting Eq. (3.1) into Eq. (2.15) and the boundary conditions (2.16), and then collecting terms of the same powers of $\in_1$, one obtains the following systems of partial differential equations along with appropriate boundary conditions.

System of order zero

$$\frac{1}{4}\frac{\partial u_0}{\partial t} - (1+\in Ae^{i\omega t})\frac{\partial u_0}{\partial y} = \frac{i\omega}{4}\in e^{i\omega t} + \frac{\partial^2 u_0}{\partial y^2} + \alpha\left[\frac{1}{4}\frac{\partial^3 u_0}{\partial y^2\partial t} - (1+\in Ae^{i\omega t})\frac{\partial^3 u_0}{\partial y^3}\right], \tag{3.2}$$

$$u_0 = 0 \quad \text{at} \quad y = 0 \quad \text{and} \quad u_0 \longrightarrow 1+\in e^{i\omega t} \quad \text{as} \quad y \longrightarrow \infty. \tag{3.3}$$

System of order one

$$\frac{1}{4}\frac{\partial u_1}{\partial t} - (1+\in Ae^{i\omega t})\frac{\partial u_1}{\partial y} = \frac{\partial^2 u_1}{\partial y^2} + \alpha\left[\frac{1}{4}\frac{\partial^3 u_1}{\partial y^2\partial t} - (1+\in Ae^{i\omega t})\frac{\partial^3 u_1}{\partial y^3}\right] + \left(\frac{\partial u_0}{\partial y}\right)^2\frac{\partial^2 u_0}{\partial y^2}, \tag{3.4}$$

$$u_1 = 0 \quad \text{at} \quad y = 0 \quad \text{and} \quad u_1 \longrightarrow 0 \quad \text{as} \quad y \longrightarrow \infty. \tag{3.5}$$

Zeroth-order solution

We note that the zeroth order mathematical problem is same as that of SOUNDALGEKAR and PURI [9] except that $(-k)$ is replaced by α in Eq. (3.2). Thus, in order to avoid repetition, the details of calculations are omitted and the solution is directly given by

$$u_0(y,t) = 1 - e^{-y} - \alpha y e^{-y} + \in e^{i\omega t} \left[\begin{array}{c} 1 - Se^{-hy} - (1-S)e^{-y} + Lye^{-hy} \\ -\alpha \begin{pmatrix} (1-S)\, e^{-hy} \\ -(1-y)\, e^{-y} \end{pmatrix} \end{array} \right], \tag{3.6}$$

where

$$h = \left[\frac{\sqrt{1+i\omega}+1}{2} \right], \tag{3.7}$$

$$L = \frac{h^2 \left(h + \frac{i\omega}{4} \right) \left(1 - \frac{4iA}{\omega} \right)}{\sqrt{1+i\omega}}, \tag{3.8}$$

$$S = 1 - \frac{4iA}{\omega}. \tag{3.9}$$

First-order solution

Now, let

$$u_1(y,t) = f_1(y) + \in e^{i\omega t} f_2(y). \tag{3.10}$$

Substituting Eqs. (3.6) and (3.10) in Eq. (3.4) and boundary conditions (3.5), comparing nonharmonic and harmonic terms and neglecting coefficients of $\in^2$, we get

$$\alpha \frac{d^3 f_1}{dy^3} - \frac{d^2 f_1}{dy^2} - \frac{df_1}{dy} = e^{-3y}(1+\alpha y), \tag{3.11}$$

$$\alpha \frac{d^3 f_2}{dy^3} - \left(1 + \frac{i\omega\alpha}{4}\right) \frac{d^2 f_2}{dy^2} - \frac{df_2}{dy} + \frac{i\omega}{4} f_2 = A\frac{df_1}{dy} + B_1 - \alpha \left[A\frac{d^3 f_1}{dy^3} + B_2 \right], \tag{3.12}$$

$$\text{at} \quad y = 0 \quad \text{and} \quad y \longrightarrow \infty : f_1 = f_2 = 0, \tag{3.13}$$

where

$$B_1 = e^{-3y}\left[\frac{A}{2} - 3(1-S)\right] + \frac{A}{6}e^{-y} + (h^2S - 2hS)e^{-(h+2)y}, \tag{3.14}$$

$$\begin{aligned} B_2 = {} & e^{-y}\left[\frac{A}{12}(9+2y) + 2 - \frac{A}{6}\right] \\ & - e^{-3y}\left[\frac{A}{4}(9+y^2-2y) - \frac{A}{6}(y-1) - \frac{9A}{2} - 1 + 3y + S - 2Sy\right] \\ & - e^{-(h+2)y}\left[2h(1-S) + 4hL - 2L - h^2 - h^2Ly\right]. \end{aligned} \tag{3.15}$$

There have been several investigations devoted to study the existence and uniqueness of the solutions to the equations governing the flows of fluids of differential type [21–23]. These equations are usually higher order partial differential equations than the Navier–Stokes equations. Hence the issue of whether the "no-slip" boundary condition is sufficient to have a well-posed problem is very important. This question can not be answered by any generality for fluids of differential type of complexity n, for arbitrary n. However, if attention is confined to fluids of grade 2 or grade 3, one can provide some definite answers, while some partial answers are also possible for fluids of grade n [24].

Before proceeding with the solution of Eqs. (3.11) and (3.12), it would be interesting to remark here that although in the classical viscous case ($\alpha = 0$) we encounter differential equations of order two [2, 4], the presence of the material parameter of the second order fluid increases the order to three. It would therefore seem that the additional boundary condition must be imposed in order to get a unique solution. In order to overcome such a difficulty, several authors have studied an acceptable additional condition. FOSDICK and BERSTEIN [25] have studied the flow in the annular region between two porous rotating cylinders. They assumed one of the constants in the solutions to be zero. However, there is no apparent reason for such a choice. FRATER [26] has studied the asymptotic suction flow. Since only two of the coefficients in the solution can be found by the no-slip condition, he imposes an extra condition that the solution tends to the Newtonian value as the coefficient of the higher derivative in the equation approaches zero. However, the perturbation expansion may give correct results under certain conditions [27]. Thus following [8, 28], we overcome the difficulty in the present study using perturbation expansion for small material parameter α and assume the solution in the form as follows [28]:

$$\begin{aligned} f_1 &= f_{01} + \alpha f_{11} + O(\alpha^2), \\ f_2 &= f_{02} + \alpha f_{12} + O(\alpha^2), \end{aligned} \tag{3.16}$$

which is valid for small values of α only. Putting Eq. (3.16) in Eqs. (3.11) and (3.12) and equating the coefficient of α we obtain

$$\frac{d^2 f_{01}}{dy^2} + \frac{df_{01}}{dy} = -e^{-3y}, \tag{3.17}$$

$$\frac{d^2 f_{11}}{dy^2} + \frac{df_{11}}{dy} = -\frac{df_{01}^3}{dy^3} + ye^{-3y}, \tag{3.18}$$

$$\frac{d^2 f_{02}}{dy^2} + \frac{df_{02}}{dy} - \frac{i\omega}{4} f_{02} = B_1, \tag{3.19}$$

$$\frac{d^2 f_{12}}{dy^2} + \frac{df_{12}}{dy} - \frac{i\omega}{4} f_{12} = -\frac{d^3 f_{02}}{dy^3} - \frac{i\omega}{4}\frac{df_{02}}{dy} - B_2, \tag{3.20}$$

$$\begin{aligned} f_{01} = f_{11} = f_{02} = f_{12} = 0 \text{ at } y = 0, \\ f_{01} = f_{11} = f_{02} = f_{12} = 0 \text{ as } y \longrightarrow \infty. \end{aligned} \tag{3.21}$$

Solving Eqs. (3.17) to (3.20) under the boundary conditions (3.21), we have, in view of Eq. (3.16),

$$f_1 = \frac{1}{12}\left[e^{-y}\left\{2 + \alpha\left(9 + 2y\right)\right\} - e^{-3y}\left\{\alpha\left(9 + y^2 - 2y\right) + 2\right\}\right], \tag{3.22}$$

$$\begin{aligned} f_2 = M_1\left(e^{-3y} - e^{-hy}\right) + N_1\left(e^{-y} - e^{-hy}\right) + P_1\left(e^{-(h+2)y} - e^{-hy}\right) \\ - \alpha\begin{pmatrix} M_2\left(e^{-3y} - e^{-hy}\right) + N_2\left(e^{-y} - e^{-hy}\right) + P_2\left(e^{-(h+2)y} - e^{-hy}\right) \\ + \dfrac{i}{3\omega}e^{-3y}\left(36y - 24Sy + 3Ay^2 - 2Ay\right) \end{pmatrix}, \end{aligned} \tag{3.23}$$

where

$$M_1 = \frac{2\left[A - 6\left(1 - S\right)\right]}{24 - i\omega}, \quad N_1 = \frac{2iA}{3\omega}, \quad P_1 = \frac{\left(h^2 - 2h\right)S}{h^2 + 3h + 2 - \dfrac{i\omega}{4}}, \tag{3.24}$$

$$M_2 = \frac{i}{\omega}\left(36M_1 + 9A + 8 - 8S\right), \quad N_2 = \frac{-4}{3\omega^2}\Big(12N_1 - 13A - 24\Big), \tag{3.25}$$

$$P_2 = \frac{2h(1 - S) + 4hL - 2L - h^2 - h^2 L}{h^2 + 3h - 2 - \dfrac{i\omega}{4}}. \tag{3.26}$$

In view of Eqs. (3.10), (3.22) and (3.23), we have

$$u_1 = \frac{1}{12}\left[e^{-y}\left\{2+\alpha\left(9+2y\right)\right\} - e^{-3y}\left\{\alpha\left(9+y^2-2y\right)+2\right\}\right] \tag{3.27}$$

$$+ \in e^{i\omega t}\left(\begin{array}{c} M_1\left(e^{-3y}-e^{-hy}\right)+N_1\left(e^{-y}-e^{-hy}\right) \\ +P_1\left(e^{-(h+2)y}-e^{-hy}\right) \\ -\alpha\left(\begin{array}{c} M_2\left(e^{-3y}-e^{-hy}\right)+N_2\left(e^{-y}-e^{-hy}\right) \\ +P_2\left(e^{-(h+2)y}-e^{-hy}\right) \\ +\frac{i}{3\omega}e^{-3y}\left(36y-24Sy+3Ay^2-2Ay\right) \end{array}\right) \end{array}\right).$$

Now from Eqs. (3.6) and (3.27), the velocity field in the boundary layer is given by

$$u = 1-e^{-y}-\alpha y e^{-y} + \in e^{i\omega t}\left(\begin{array}{c} 1-Se^{-hy}-(1-S)e^{-y}- \\ \alpha\left\{(1-S)\,e^{-hy}-(1-y)\,e^{-y}\right\} \\ +Lye^{-hy} \end{array}\right) \tag{3.28}$$

$$+ \in_1 \frac{1}{12}\left[e^{-y}\left\{2+\alpha\left(9+2y\right)\right\} - e^{-3y}\left\{\alpha\left(9+y^2-2y\right)+2\right\}\right]$$

$$+ \in_1 \in e^{i\omega t}\left(\begin{array}{c} M_1\left(e^{-3y}-e^{-hy}\right)+N_1\left(e^{-y}-e^{-hy}\right) \\ +P_1\left(e^{-(h+2)y}-e^{-hy}\right) \\ -\alpha\left(\begin{array}{c} M_2\left(e^{-3y}-e^{-hy}\right)+N_2\left(e^{-y}-e^{-hy}\right) \\ +P_2\left(e^{-(h+2)y}-e^{-hy}\right) \\ +\frac{i}{3\omega}e^{-3y}\left(36y-24Sy+3Ay^2-2Ay\right) \end{array}\right) \end{array}\right).$$

The real, u_r, and the imaginary, u_i, parts of this expression, respectively, yield

$$u_r = 1-e^{-y}\left(1+\alpha y\right)+\frac{\epsilon_1}{12}\Big[e^{-y}\left\{2+\alpha(2y+9)\right\} \tag{3.29}$$

$$- e^{-3y}\left\{\alpha\left(y^2-2y+9\right)+2\right\}\Big] + \epsilon\left\{M_r\cos(\omega t)-M_i\sin(\omega t)\right\},$$

$$u_i = \epsilon\left(M_r\sin(\omega t)+M_i\cos(\omega t)\right), \tag{3.30}$$

where

$$M_r = m_{r_{10}} + \epsilon_1 m_{r_{11}}, \tag{3.31}$$

$$M_i = m_{i_{10}} + \epsilon_1 m_{i_{11}}. \tag{3.32}$$

The parameter functions $m_{r_{10}}, m_{i_{10}}, m_{r_{11}}$ and $m_{i_{11}}$ involved in u_r, u_i and M_r, M_i are explicitly computed, and are listed in the Appendix.

The other interesting aspect of the solution (3.28) is, however, the prediction of the shear stress near the wall. From Eq. (4) the expression for the shear stress is given by

$$(3.33) \qquad P'_{x'y'} = \mu\frac{\partial u'}{\partial y'} + \frac{\alpha_1}{\rho'}\left[\frac{\partial^2 u'}{\partial y' \partial t'} - v'_0\left(1+ \in Ae^{i\omega' t'}\right)\frac{\partial^2 u'}{\partial y'^2}\right] + 2\beta_3\left(\frac{\partial u'}{\partial y'}\right)^3,$$

which in virtue of Eq. (2.14) reduces to

$$(3.34) \qquad P_{xy} = \frac{P'_{x'y'}}{U'_0 v'_0 \rho'} = \frac{\partial u}{\partial y} + \frac{\alpha}{4}\left[\frac{\partial^2 u}{\partial y \partial t} - 4\left(1+ \in Ae^{i\omega t}\right)\frac{\partial^2 u}{\partial y^2}\right] + \frac{\in_1}{3}\left(\frac{\partial u}{\partial y}\right)^3.$$

where u is given by Eq. (3.28).

4. Discussions

In order to investigate the effects of the third order fluid on the velocity profile near the plate (both in case of constant and variable suction), we have plotted u_r against y in Figs. 1 to 4 for the different values of ϵ, ϵ_1, A, ω, α and $\omega t = \pi/2$. From Figs. 1 and 2 we observe that the velocity profile increases with fixed ω and large values of ϵ_1. Figure 3 is prepared to bring out the effects of the variable suction velocity on the separation of the fluid at the plate for large frequency. It is evident from this figure that velocity increases with an increase in ω, in A and ϵ_1, the third order fluid parameter. Further, for fixed ϵ_1, increase in ϵ, A and ω increases the velocity and then the two velocities coincide (see Fig. 4).

In Figs. 5 to 9 the fluctuating parts are plotted for different values of ϵ, ϵ_1, ω, α, A and for $\omega t = \pi/2$. For $A = 0$, it is noted that an increase in ϵ_1 with fixed ϵ and ω (Fig. 5) leads to a decrease in M_r, but with increase in ϵ_1 and for $\epsilon = 0.2$ and $\omega = 10$, M_r is almost the same. Figure 6 shows the effect of ϵ_1 in case of variable suction. In this case, it is noted that increase in ϵ_1 leads to a decrease in M_r first then the curves tend to coincide. Further, it is clear from Fig. 7 that for $\epsilon_1 = 0.7$ and increase in A and ω, results in a decrease in M_r, and ultimately the curves are almost the same. In case of non-Newtonian fluids at large ω and increase in ϵ_1 there is a fall of M_i (Fig. 8), which is not observed in Newtonian fluids. From Fig. 9, one can conclude that an increase in A and ω leads to an increase in M_i first; then there arises a decrease, then increase and finally it reaches zero level.

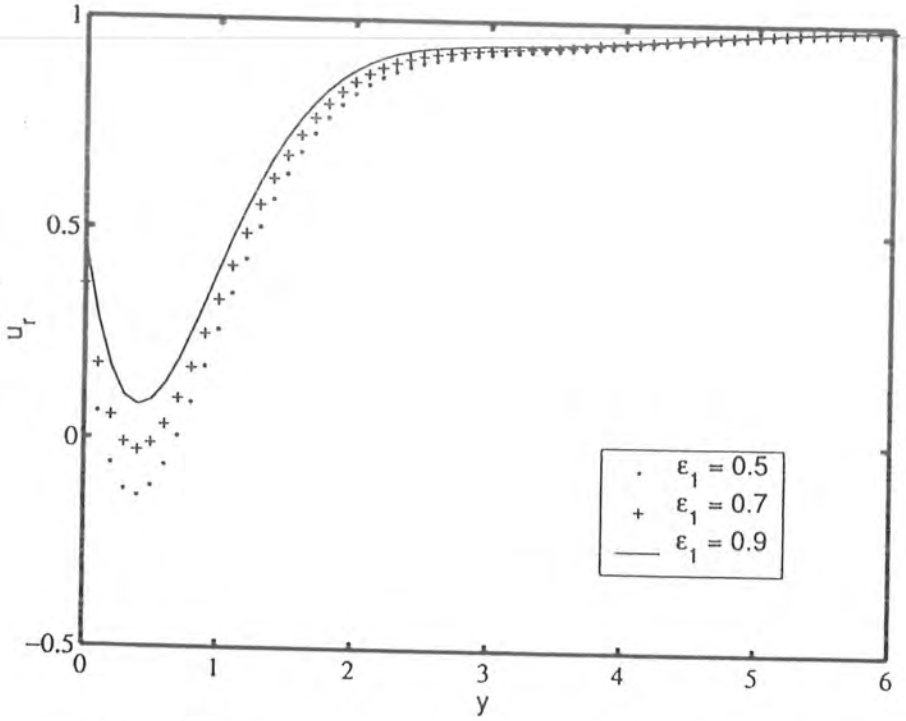

FIG. 1. Graphs for the parameter values $\alpha = 0.7, \epsilon = 0.5, \omega t = \pi/2, A = 0, \omega = 10$.

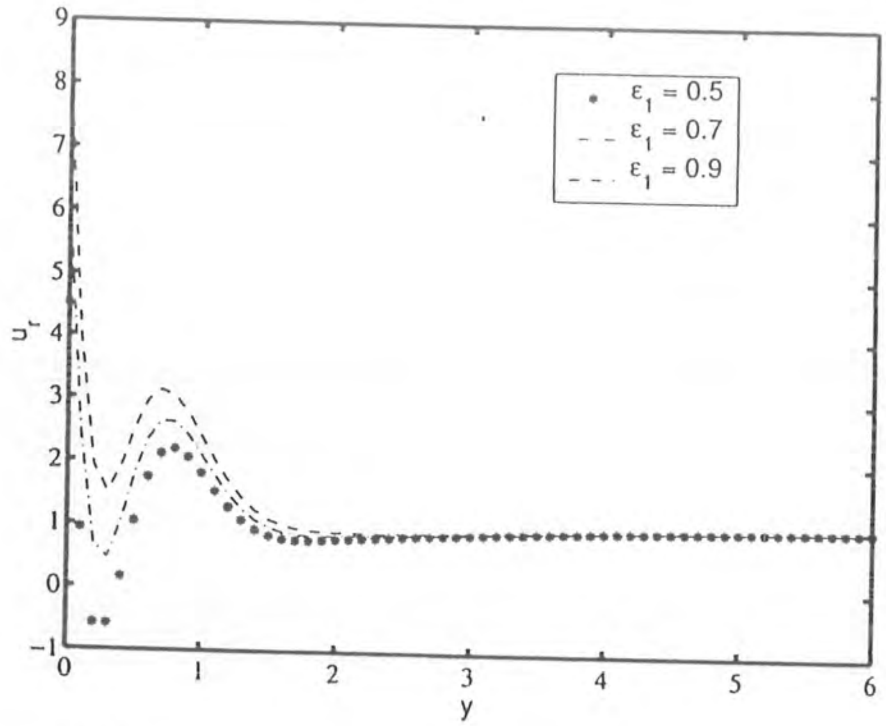

FIG. 2. Graphs for the parameter values $\alpha = 0.8, \epsilon = 0.5, \omega t = \pi/2, A = 0, \omega = 100$.

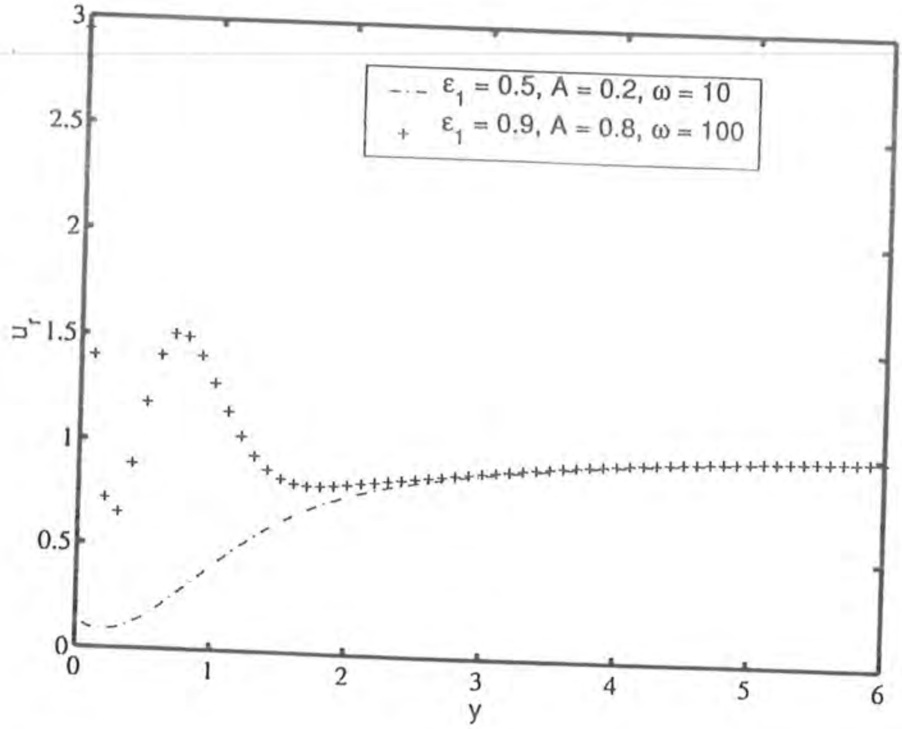

FIG. 3. Graphs for the parameter values $\epsilon = 0.2, \alpha = 0.8, \omega t = \pi/2$.

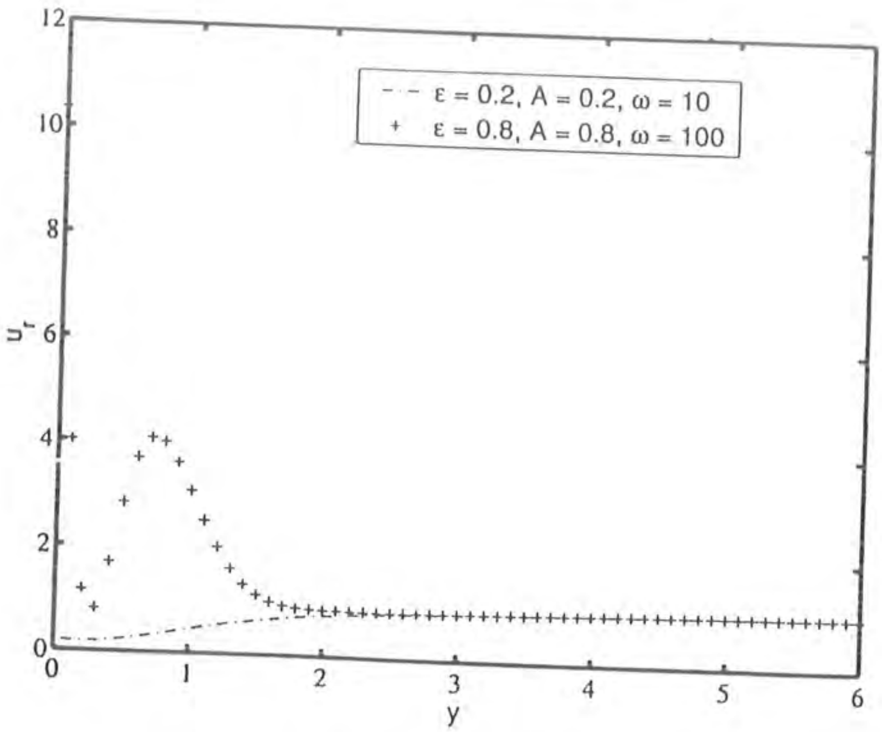

FIG. 4. Graphs for the parameter values $\epsilon_1 = 0.7, \alpha = 0.9, \omega t = \pi/2$.

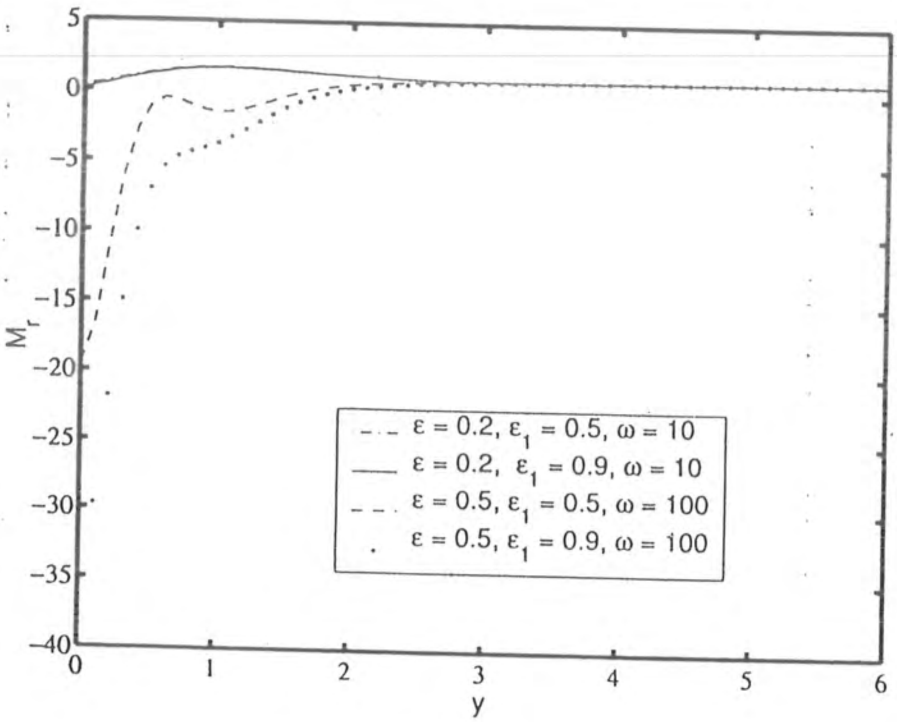

FIG. 5. Graphs for the parameter values $\alpha = 0.6, A = 0, \omega t = \pi/2$.

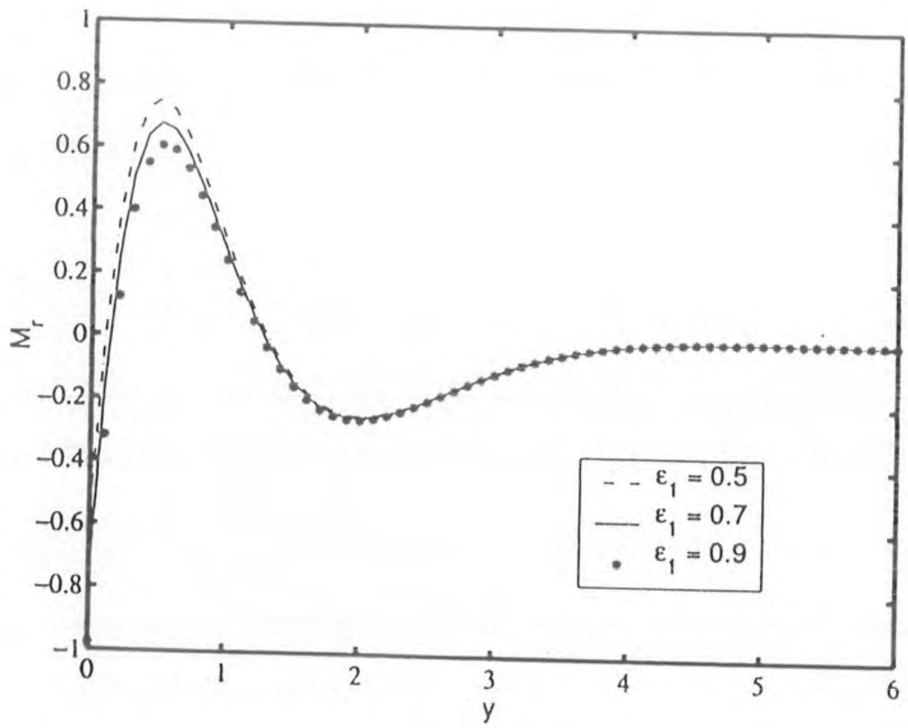

FIG. 6. Graphs for the parameter values $\alpha = 0.7, \omega t = \pi/2, \epsilon = 0.2, A = 0.4, \omega = 10$.

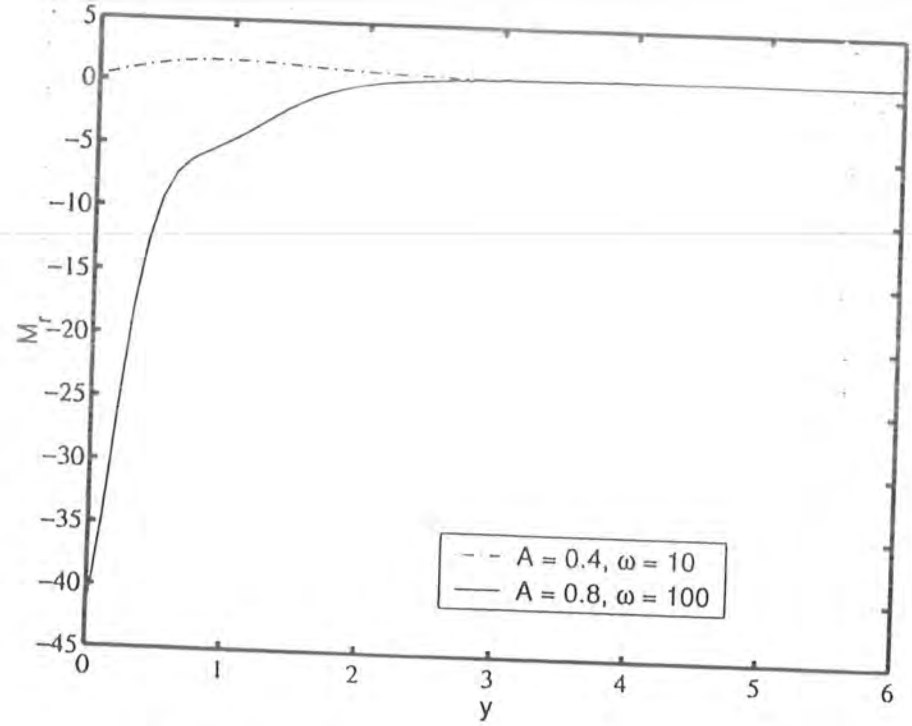

FIG. 7. Graphs for the parameter values $\alpha = 0.9, \omega t = \pi/2, \epsilon = 0.2, \epsilon_1 = 0.7$.

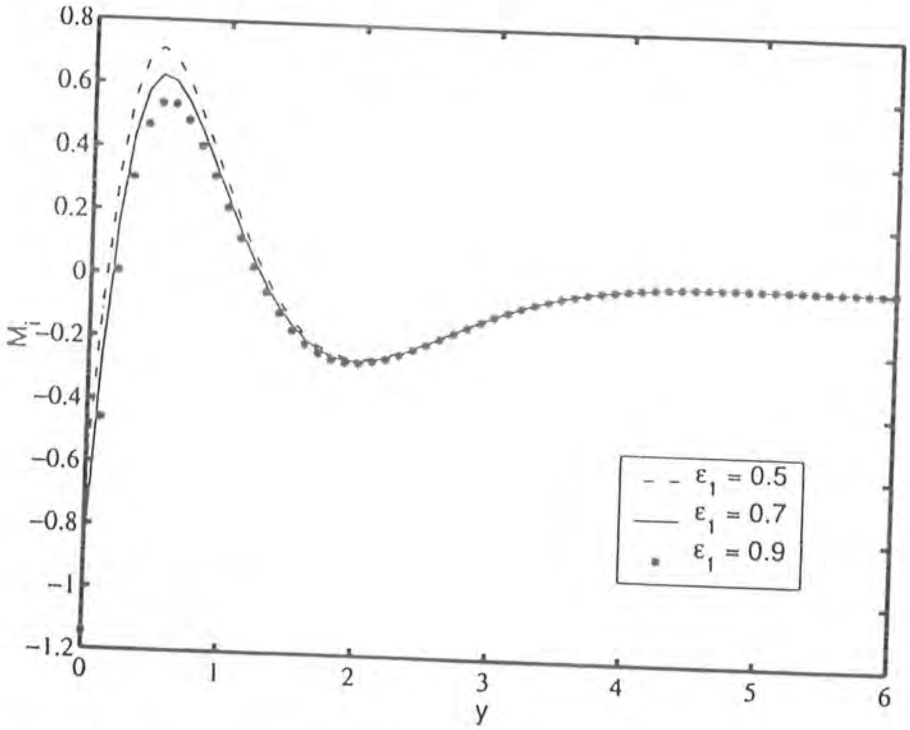

FIG. 8. Graphs for the parameter values $\alpha = 0.8, \omega t = \pi/2, \epsilon = 0.2, A = 0.4, \omega = 10$.

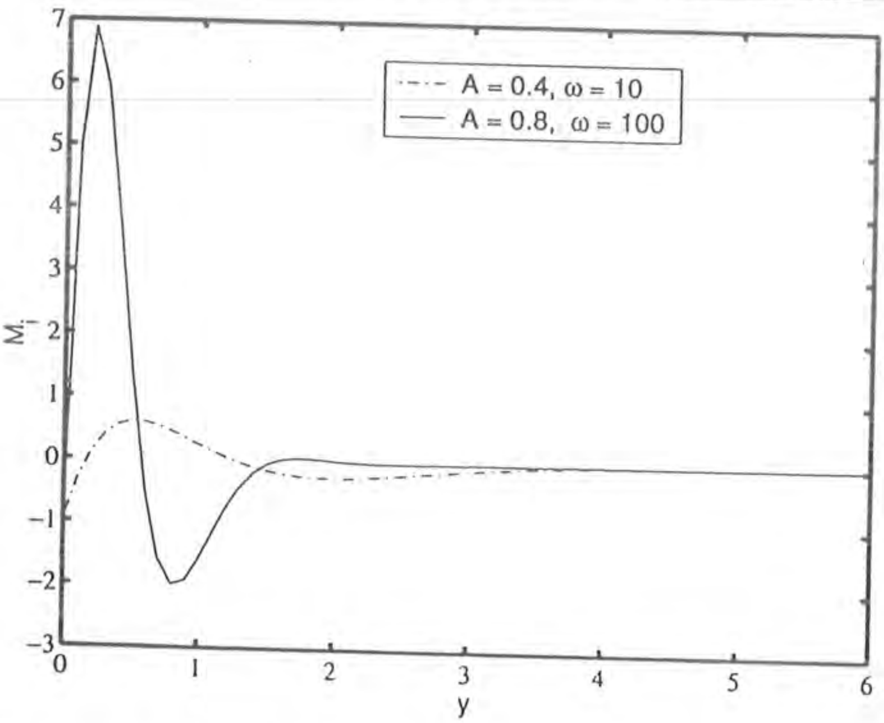

FIG. 9. Graphs for the parameter values $\alpha = 0.7, \omega t = \pi/2, \epsilon = 0.2, \epsilon_1 = 0.9$.

5. Conclusions

In this paper, the unsteady flow past an infinite porous plate is studied under the following conditions:
(i) the suction velocity normal to the plate oscillates in magnitude but not in direction about a non-zero mean value,
(ii) the free stream velocity oscillates in time about a constant mean value.
The solution obtained is the sum of steady and unsteady parts. The following results are obtained:

1. There is a decrease and increase in the fluctuating parts M_r and M_i with the increase of the third order parameter ϵ_1 and $A \neq 0$.
2. Increase of variable suction, increase in ϵ_1 and A lead to an increase in the velocity.
3. The velocity increases as the third order fluid parameter increases.
4. The results for constant suction can be obtained by taking $A = 0$.
5. The solution for second-order fluid with variable suction can be obtained as a special case of this problem by taking $\epsilon_1 = 0$.

As far as the authors are aware, no attempt has been made to examine the effect of variable suction velocity for second order fluids. However, a second order fluid exhibits normal stresses but is not shear thinning; the shear viscosity is constant. The third order approximation of a simple fluid exhibits shear-dependent viscosity. Keeping this fact in view, the problem considered for the third order fluid in this paper is more general.

Appendix

Equation (3.28) is a very complex algebraic equation. In order to split it into real and imaginary parts, for brevity, we define the following list of parameters:

$$m_{r_1} := \sqrt{\frac{1+\sqrt{1+\omega^2}}{2}}, \qquad m_{i_1} := \sqrt{\frac{-1+\sqrt{1+\omega^2}}{2}},$$

$$m_{r_2} := \frac{1}{2} + \frac{1}{2} m_{r_1}, \qquad m_{i_2} := \frac{1}{2} m_{i_1},$$

$$m_{r_3} := \frac{m_{r_1}}{m_{r_1}^2 + m_{i_1}^2}, \qquad m_{i_3} := -\frac{m_{i_1}}{m_{r_1}^2 + m_{i_1}^2},$$

$$m_{r_4} := m_{r_3}(R_4 + 4BR_3) - m_{i_3}(R_3 - 4BR_4),$$

$$m_{i_4} := m_{r_3}(R_3 - 4BR_4) + m_{i_3}(R_4 + 4BR_3),$$

$$m_{r_5} := 96A/R_5, \qquad m_{i_5} := 2A\left(\omega - (24)^2/\omega\right)/R_5,$$

$$m_{r_6} := \{R_6(R_7 + 4BR_8) + R_9(R_8 - 4BR_7)\} / \left(R_6^2 + R_9^2\right),$$

$$m_{i_6} := \{R_6(R_8 - 4BR_7) - R_9(R_7 + 4BR_8)\} / \left(R_6^2 + R_9^2\right),$$

$$m_{r_7} := -\frac{1}{\omega}(36 m_{i_5} + 32B), \quad m_{i_7} := \frac{1}{\omega}(36 m_{r_5} + 9A),$$

$$m_{r_8} := \frac{4}{3\omega^2}(13A + 24), \qquad m_{i_8} := -\frac{32A}{3\omega^3},$$

$$m_{r_9} := \{(-8Bm_{i_2} + 4R_{10} - R_{11} - R_{12}) R_{16} + (8Bm_{r_2} + 4R_{13} - 2R_{14} - R_{15}) R_9\} \div \left(R_{16}^2 + R_9^2\right),$$

$$m_{i_9} := \{(8Bm_{r_2} + 4R_{13} - 2R_{14} - R_{15}) R_{16} - (-8Bm_{i_2} + 4R_{10} - R_{11} - R_{12}) R_9\} \div \left(R_{16}^2 + R_9^2\right),$$

$$m_{r_{10}} := 1 - e^{-m_{r_2}y}(\cos(m_{i_2}y) - 4B\sin(m_{i_2}y)) - \alpha\big(4Be^{-m_{r_2}y}\sin(m_{i_2}y) - (1-y)e^{-y}\big)$$
$$+\, ye^{-m_{r_2}y}(m_{r_4}\cos(m_{i_2}y) + m_{i_4}\sin(m_{i_2}y)),$$
$$m_{i_{10}} := e^{-m_{r_2}y}(4B\cos(m_{i_2}y) + \sin(m_{i_2}y)) - 4Be^{-y} - 4B\alpha e^{-m_{r_2}y}\cos(m_{i_2}y)$$
$$+\, ye^{-m_{r_2}y}(m_{i_4}\cos(m_{i_2}y) - m_{r_4}\sin(m_{i_2}y)),$$

$$m_{r_{11}} := (m_{r_5} - \alpha m_{r_7})\left(e^{-3y} - e^{-m_{r_2}y}\cos(m_{i_2}y)\right) - (m_{i_5} - \alpha m_{i_7})\,e^{-m_{r_2}y}\sin(m_{i_2}y)$$
$$-\, \alpha m_{r_8}\left(e^{-y} - e^{-m_{r_2}y}\cos(m_{i_2}y)\right) - (2B/3 + m_{i_8})\,e^{-m_{r_2}y}\sin(m_{i_2}y)$$
$$+\, 32\alpha Ae^{-3y}/\omega^2 + (m_{r_6} - \alpha m_{r_9})\,e^{-2y},$$
$$m_{i_{11}} := (m_{r_5} - \alpha m_{r_7})\,e^{-m_{r_2}y}\sin(m_{i_2}y) + (m_{i_5} - \alpha m_{i_7})\left(e^{-3y} - e^{-m_{r_2}y}\cos(m_{i_2}y)\right)$$
$$-\, \alpha m_{r_8}e^{-m_{r_2}y}\sin(m_{i_2}y) + (2B/3 - \alpha m_{i_8})\left(e^{-y} - e^{-m_{r_2}y}\cos(m_{i_2}y)\right)$$
$$-\, \left(12y + 3Ay^2 - 2Ay\right)\alpha e^{-3y}/(3\omega) + (m_{i_6} - \alpha m_{i_9})\,e^{-2y},$$

where

$$R_1 := m_{r_2}^2 - m_{i_2}^2, \quad R_2 := m_{i_2} + \omega/4, \quad R_3 := 2m_{r_2}^2 m_{i_2} + R_1R_2,$$
$$R_4 := m_{r_2}R_1 - 2m_{r_2}m_{i_2}R_2, \quad R_5 := (24)^2 + \omega^2, \quad R_6 := R_1 + 3m_{r_2} + 2,$$
$$R_7 := R_1 - 2m_{r_2}, \quad R_8 := 2m_{r_2}m_{i_2} - 2m_{i_2}, \quad R_9 := 2m_{r_2}m_{i_2} + 3m_{i_2} - \omega/4,$$
$$R_{10} := m_{r_2}m_{r_4} - m_{i_2}m_{i_4}, \quad R_{11} := 2m_{r_4} + R_1,$$
$$R_{12} := m_{r_4}R_1 - 2m_{r_2}m_{i_2}m_{i_4}, \quad R_{13} := m_{i_2}m_{r_4} + m_{r_2}m_{i_4},$$
$$R_{14} := m_{i_4} + m_{r_2}m_{i_2}, \quad R_{15} := m_{i_4}R_1 + 2m_{r_2}m_{i_2}m_{i_4}, \quad R_{16} := R_1 + 3m_{r_2} - 2,$$
$$B := A/\omega.$$

The parameter functions $h, L, S, M_1, M_2, N_1, N_2, P_1$ and P_2 of Eqs. (3.7)–(3.9) and (3.24)–(3.26) can now be expressed in terms of these m_rs and m_is as follows:

$$h = \frac{1}{2} + \frac{1}{2}m_{r_1} + i\frac{1}{2}m_{i_1} = m_{r_2} + im_{i_2}, \qquad L = m_{r_4} + im_{i_4}, \qquad S = 1 - i4B,$$
$$M_1 = m_{r_5} + im_{i_5}, \qquad M_2 = m_{r_7} + im_{i_7}, \qquad N_1 = i2B/3,$$
$$N_2 = m_{r_8} + im_{i_8}, \qquad P_1 = m_{r_6} + im_{i_6}, \qquad P_2 = m_{r_9} + im_{i_9}.$$

Substituting the values of these parameters, Eq. (3.28) can be split into real and imaginary parts (the calculation is very lengthy and tedious but straightforward), u_r and u_i, as given in Eqs. (3.29) and (3.30), with

$$M_r = m_{r_{10}} + \epsilon_1 m_{r_{11}}, \qquad M_i = m_{i_{10}} + \epsilon_1 m_{i_{11}}.$$

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