# Non-Newtonian Flow With Variable Suction

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*Supervised By*

*Dr. Tasawar Hayat*

**DEPARTMENT OF MATHEMATICS**
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# CERTIFICATE

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A DISSERTATION SUBMITTED IN THE PARTIAL FULFILMENT OF THE REQUIREMENT FOR THE DEGREE OF MASTER OF PHILOSOPHY

We accept this dissertation as conforming to the required standard

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**DEPARTMENT OF MATHEMATICS**
**QUAID-I-AZAM UNIVERSITY**
**ISLAMABAD, PAKISTAN**
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Dedicated To

My

Dear Dada Jani

**Haji Falak Sher**

&

**Loving Parents**

# Acknowledgement

To begin with the name of Almighty *ALLAH*, the creator of the universe, who bestowed his blessing on me to complete this dissertation. I offer my humblest, sincerest and million Darood to the Holy Prophet Hazrat *MUHAMMAD* (peace be upon him), who exhorts his followers to seek for knowledge from cradle to grave.

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# Preface

The study of non-Newtonian fluids has attracted much attention, because of their practical applications in engineering and industry particularly in extraction of crude oil from petroleum products, food processing and construction engineering. Due to complexity of fluids, various models have been proposed. The equations of motion of non-Newtonian fluids are highly non-linear and one order higher than the Navier-Stokes equations. Finding accurate analytic solutions to such equations is not easy. There is a particular class of non-Newtonian fluids namely the second grade fluids for which one can reasonably hope to obtain an analytic solution. Important studies of second grade fluids in various contexts have been given by Straughan [1, 2], Rajagopal [3-5], Bandelli [6], Gupta [7], Liu [8], Dolapci [9], Siddheshwar [10] and Hayat et. al [11-15].

Since the pioneering work of Lighthill [16] there has been a considerable amount of research undertaken on the time-dependent flow problems dealing with the response of the boundary layer to external unsteady fluctuations about a mean value. Important contributions to the topic with constant and variable suction include the work of Stuart [17], Messiha [18], Kelley [19], Soundalgekar and Puri [20] and Hayat et. al [21].

The primary purpose of the present dissertation is to make an investigation of the combined effects of rotation, and heat transfer characteristics on the flow of a second grade fluid past a porous plate. With this fact in view, this dissertation is organized as follows:

In chapter one, some basic definitions and properties of the fluid are given. Equations of motion and perturbation method are also included.

In chapter two, the work of Soundalgekar and Puri [20] has been reviewed. This work deals with the flow of an elastico-viscous fluid past a porous plate with variable suction.

In chapter three, the simultaneous effects of rotation and heat transfer characteristics are discussed on the flow of a second grade fluid past a porous plate having variable suction. The analytic solutions for the velocity field and temperature distribution are obtained. Special attention is given in finding the solutions and to describe the physical nature. In order to see the variations of different emerging parameters, the graphs are sketched and discussed.

# Contents

# Chapter 1

# Some definitions and equations

## 1.1 Introduction

We begin our study of fluids in motion by presenting some basic definitions, concepts and equations. The equations of continuity for unsteady compressible and incompressible fluids are given. The equation of motion for any fluid satisfying the continuum is also presented. Futher, the perturbation treatment is explained with the help of an example.

## 1.2 Definitions

Flow

A material that deforms continuously under the application of shear stress. If the deformation continuously increases without limit, then the phenomenon is known as flow.

**Velocity Field**

In dealing with fluids in motion, we shall necessarily be concerned with the description of a velocity field. At a given instant the velocity field $\mathbf{V}$ is a function of the space coordinates $x$, $y$, $z$. The velocity at any point in the flow field might vary from one instant to another. Thus the complete representation of velocity field is given by

$$\mathbf{V} = \mathbf{V}(x, y, z, t) . \tag{1.1}$$

The velocity vector $\mathbf{V}$ can be written in terms of its three scalar components. Denoting the components in the $x$, $y$ and $z$ directions by $u$, $v$ and $w$ we have

$$\mathbf{V} = (u, v, w)\,. \tag{1.2}$$

**Steady Flow**

A flow in which the properties at each point in a flow field do not change with time. Mathematically

$$\frac{\partial \eta}{\partial t} = 0,$$

where $\eta$ represents any fluid property.

**Unsteady Flow**

A flow in which the fluid properties do not remain constant with time at any point.

**Compressible Flow**

A flow in which the volume and thus the density of the flowing fluid changes during the flow. All the gases are, generally considered as a compressible flow.

**Incompressible Flow**

A flow in which the volume and thus the density of the flowing fluid does not change during the flow. All the liquids are, generally considered as a incompressible flow.

**Fluid Rotation**

It is the average angular velocity of any two mutually perpendicular line elements of the fluid particles. Rotation is a vector quantity. A particle moving in general three-dimensional flow field may rotates about all three coordinates axes. In general

$$\Omega = \Omega_x i + \Omega_y j + \Omega_z k,$$

in which $\Omega_x$, $\Omega_y$ and $\Omega_z$ are the rotations about the $x$, $y$ and $z$-axes, respectively.

**Viscosity**

It is a physical property of fluids associated with shearing deformation of fluid particles subjected to the action of applied forces. In other words it is the resistance of a fluid to its motion. It is also defined as the ratio of shear stress to the rate of shear strain, i.e.

$$\text{Viscosity} = \mu = \frac{\text{Shear stress}}{\text{rate of shear strain}}.$$

**Kinematic Viscosity**

It is the ratio of absolute viscosity $\mu$ to density. It is denoted by $\nu$, i.e.

$$\nu = \frac{\mu}{\rho}.$$

## 1.3 Types of Fluids

**Ideal Fluid**

An ideal fluid is one which is incompressible and has zero density. With zero viscosity, the fluid offers no resistance to shearing forces and hence during flow and deformation of the fluid all shear forces are zero. Many flow problems are simplified by assuming that the fluid is ideal.

**Real Fluid**

All real fluids have finite viscosity, and in most cases of flow in ducts and immersed bodies it is necessary to consider the viscosity and the related shearing stresses associated with deformation of the fluid. Real fluids are futher subdivided into two main classes.

a. Newtonian fluid

b. Non-Newtonian fluid

### 1.3.1 Newtonian Fluid

For such fluid shear stress is directly proportional to the deformation rate. Mathematically

$$\tau_{yx} \propto \frac{du}{dy},$$

or

$$\tau_{yx} = \mu \frac{du}{dy}. \tag{1.3}$$

In above equation $\tau_{yx}$ is the shear stress acting on the plane normal to $y - axis$ and in the direction parallel to $x - axis$ and $\mu$ is the constant of proportionality, known as absolute or dynamic viscosity. Eq. (1.3) represents Newton's law of viscosity. Most common fluids such as water, air and gasoline are Newtonian under normal conditions.

### 1.3.2 Non-Newtonian Fluid

For such fluid shear stress in not directly proportional to deformation rate. Toothpaste and Lucite paint are examples of these fluids. Numerous empirical equations have been proposed to model the observed relations between $\tau_{yx}$ and $du/dy$ for time dependent fluids. They may be adequately represented for many engineering applications by the power law model, which for one-dimensional flow becomes

$$\tau_{yx} = k\left(\frac{du}{dy}\right)^{n}, \tag{1.4}$$

where the exponent, $n$, is called the flow behavior index and $k$, the consistency index. This equation reduces to Newton's law of viscosity for $n = 1$ with $k = \mu$. Eq. (1.4) is written in the form

$$\tau_{yx} = k\left(\frac{du}{dy}\right)^{n-1}\frac{du}{dy} = \eta\frac{du}{dy}, \tag{1.5}$$

in which $\eta = k\,(du/dy)^{n-1}$ is referred to as the apparent viscosity. Most non-Newtonian fluids have apparent viscosities that are relatively high compared to the viscosity of water. There are many models of non-Newtonian fluids. In the present dissertation, we consider elastico-viscous and second grade fluids.

## 1.4 Equation of Continuity

Let us consider a differential control volume $\Delta x\Delta y\Delta z$ in a cube. Here we take density and velocity as function of position in time and space. We compute the flux of mass per second through each face of the cube to get, for the three directions

$$-\left[\frac{\partial(\rho u)}{\partial x}\Delta x\right]\Delta y\Delta z, \quad -\left[\frac{\partial(\rho u)}{\partial y}\Delta y\right]\Delta x\Delta z, \quad -\left[\frac{\partial(\rho u)}{\partial z}\Delta z\right]\Delta x\Delta y. \tag{1.6}$$

Due to matter conservation principle the sum of these must be equal to the time rate of change of mass

$$\frac{\partial}{\partial t}\left(\rho\Delta x\Delta y\Delta z\right). \tag{1.7}$$

It is noted that fixed control volume $\Delta x \Delta y \Delta z$ is independent of time. Combination of Eqs. (1.6) and (1.7) yields

$$\frac{\partial \rho}{\partial t}+\frac{\partial (\rho u)}{\partial x}+\frac{\partial (\rho v)}{\partial y}+\frac{\partial (\rho w)}{\partial z}=0, \tag{1.8}$$

or

$$\frac{\partial \rho}{\partial t}+\nabla \cdot \rho \mathbf{V}=0, \tag{1.9}$$

which is called equation of continuity.

For an incompressible fluid, we take $\rho$ =constant. Therefore, Eq. (1.9) becomes

$$\nabla \cdot \mathbf{V}=0, \tag{1.10}$$

where

$$\nabla=\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right).$$

Eq. (1.10) can also be written as

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=0, \tag{1.11}$$

or

$$div\mathbf{V}=0. \tag{1.12}$$

## 1.5 The Momentum Equation

The equation of motion in vector form is

$$\rho \frac{d\mathbf{V}}{dt}=\rho \mathbf{b}+div\mathbf{T}, \tag{1.13}$$

in which $\mathbf{b}$ are the body forces per unit mass. The Cauchy stress $\mathbf{T}$ is

$$\mathbf{T}=\begin{bmatrix} \tau_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \tau_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \tau_{zz} \end{bmatrix},$$

where $\tau_{xx}$, $\tau_{yy}$ and $\tau_{zz}$ are the normal stresses and $\tau_{xy}$, $\tau_{xz}$, $\tau_{yx}$, $\tau_{yz}$, $\tau_{zx}$ and $\tau_{zy}$ are shear stresses.

## 1.6 Equation of Motion in Rotating System

To include the rotating in the equation of motion (1.13). Consider the figure (1.1) showing two coordinates system

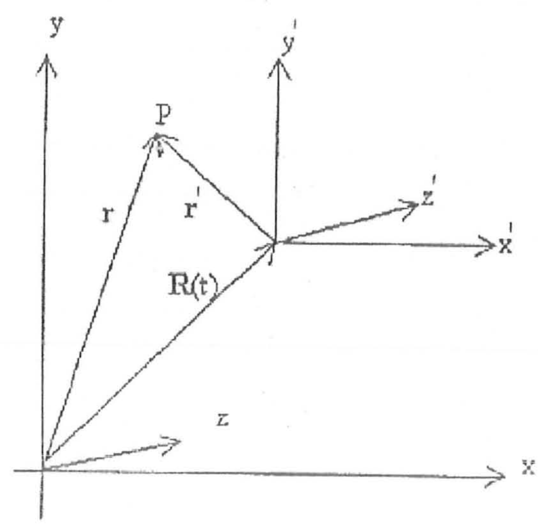


**Fig. 1.1.**

The coordinates with prime are in rotating system and those without promes are in an inertial system. Here $\mathbf{r}$ is the position vector in the inertial system, $\mathbf{r}'$ is the position vector in the rotating system, and $\mathbf{R}$ is the position vector of the origin of the rotating system in the inertial system. Obviously

$$\mathbf{r} = \mathbf{R} + \mathbf{r}', \qquad \frac{d\mathbf{r}}{dt} = \frac{d\mathbf{R}}{dt} + \frac{d\mathbf{r}'}{dt}, \tag{1.14}$$

and

$$\mathbf{r}' = x_{i'} e_{i'}, \qquad \frac{d\mathbf{r}'}{dt} = \frac{dx_{i'}}{dt} e_{i'} + x_{i'} \frac{de_{i'}}{dt}, \tag{1.15}$$

where $e_{i'}$ is a unit vector having magnitude one. If the primed coordinate system does not rotate,the derivative $de_{i'}/dt$ are all zero. If the primed system is rotating with angular velocity $\Omega$ then

$$\frac{de_{i'}}{dt} = \Omega \times e_{i'}. \tag{1.16}$$

Using Eqs. (1.15) and (1.16) in Eq. (1.14) we obtain

$$\frac{d\mathbf{r}}{dt} = \frac{d\mathbf{R}}{dt} + \frac{dx_{i'}}{dt} e_{i'} + \Omega \times \mathbf{r}'. \tag{1.17}$$

The first term on the right represents the translation of the primed system with respect to the inertial system, the second term is the translation of the point $P$ with respect to the primed system, and the third term represents the rotation of the primed system.

Differentiating Eq. (1.17) with respect to "$t$" we get

$$\frac{d^2\mathbf{r}}{dt^2} = \frac{d^2\mathbf{R}}{dt^2} + \frac{d^2x_{i'}}{dt^2} e_{i'} + \frac{dx_{i'}}{dt}\frac{de_{i'}}{dt} + \frac{d\Omega}{dt} \times \mathbf{r}' + \Omega \times \frac{d\mathbf{r}'}{dt}. \tag{1.18}$$

Since we are going to apply the equation to a fluid, taking the acceleration of a particle in the primed system as

$$\frac{d\mathbf{V}'}{dt} = \frac{d^2x_{i'}}{dt^2} e_{i'}. \tag{1.19}$$

Also

$$\frac{d\mathbf{r}'}{dt} = \mathbf{V}' + \Omega \times \mathbf{r}', \qquad \frac{dx_{i'}}{dt}\frac{de_{i'}}{dt} = \mathbf{V}' \times \Omega. \tag{1.20}$$

Making use of Eqs. (1.19) and (1.20) in Eq. (1.18) we get

$$\frac{d^2\mathbf{r}}{dt^2} = \frac{d\mathbf{V}'}{dt} = \frac{d^2\mathbf{R}}{dt^2} + \frac{d\mathbf{V}'}{dt} + \left(\mathbf{V}' \times \Omega\right) + \frac{d\Omega}{dt} \times \mathbf{r}' + \Omega \times \left(\mathbf{V}' + \Omega \times \mathbf{r}'\right),$$

or

$$\frac{d^2\mathbf{r}}{dt^2} = \frac{d^2\mathbf{R}}{dt^2} + \frac{d\mathbf{V}'}{dt} + 2\Omega \times \mathbf{V}' + \Omega \times \left(\Omega \times \mathbf{r}'\right) + \frac{d\Omega}{dt} \times \mathbf{r}', \tag{1.21}$$

where

$\frac{d^2\mathbf{R}}{dt^2}$ = translational acceleration of the primed system,

$\frac{d\mathbf{V}'}{dt}$ = particle acceleration with respect to the primed system,

$2\Omega \times \mathbf{V}'$ = Coriolis acceleration,

$\Omega \times \left(\Omega \times \mathbf{r}'\right)$ = centrifugal acceleration with respect to the primed system,

$\frac{d\Omega}{dt} \times \mathbf{r}'$ = angular acceleration in the primed system.

For equations to the rotating system, the primed coordinate system is fixed with respect to the rotating system, it does not translate, it does not undergo translational acceleration, and the rotation is constant, so

$$\frac{d\mathbf{R}}{dt} = 0, \quad \frac{d^2\mathbf{R}}{dt^2} = 0, \quad \frac{d\Omega}{dt} = 0. \tag{1.22}$$

Thus the equation of motion (1.13) for rotating system becomes

$$\rho \left[ \frac{d\mathbf{V}'}{dt} + 2\Omega \times \mathbf{V}' + \Omega \times \left(\Omega \times \mathbf{r}'\right) \right] = \rho \mathbf{b} + div\mathbf{T}. \tag{1.23}$$

Neglecting the primes, the above equation can be written as

$$\rho \left[ \frac{d\mathbf{V}}{dt} + 2\Omega \times \mathbf{V} + \Omega \times (\Omega \times \mathbf{r}) \right] = \rho \mathbf{b} + div\mathbf{T}. \tag{1.24}$$

## 1.7 Perturbation Method

This method yields approximate solutions for a large class of initial and boundary value problems for partial differential equations. It is used when a small parameter (or a large parameter) occurs in the given equation or data or problem. The assumed solution is expanded in a series of powers (or inverse powers) of the parameter and the expansion is inserted into the equation. By equating like powers of the parameter, a collection of the problems result, whose solution is expected to be simpler than that of the problem. Now, if the series expansion of the solution converges, or is expected to converge the technique is referred as perturbation technique method.

**Example**

To apply the idea of perturbation technique, we consider the following equation

$$u_{xx} + u_{yy} + \epsilon^2 u = 0, \tag{1.25}$$

in the unit disk $x^2+y^2<1$, with the Dirichlet boundary condition

$$u(x,y)=1, \quad x^2+y^2=1, \tag{1.26}$$

where $\epsilon^2$ is small so that solution of Eqs. (1.25) and (1.26) is unique.

We find solution of Eq. (1.25) in terms of a power series of $\epsilon$ i.e.

$$u(x,y)=\sum_{n=0}^{\infty} u_n(x,y)\,\epsilon^{2n}=u_0+u_1\epsilon^2+u_2\epsilon^4+\,\ldots\ldots\ldots \tag{1.27}$$

Substituting Eq. (1.27) into Eqs. (1.25) and (1.26) we obtain

$$\nabla^2\left[\sum_{n=0}^{\infty} u_n\epsilon^{2n}\right]+\epsilon^2\sum_{n=0}^{\infty} u_n\epsilon^{2n}=0,$$

or

$$\nabla^2 u_0+\nabla^2\sum_{n=1}^{\infty} u_n\epsilon^{2n}+\sum_{n=0}^{\infty} u_n\epsilon^{2n+2}=0,$$

or

$$\nabla^2 u_0+\nabla^2\sum_{n=1}^{\infty} u_n\epsilon^{2n}+\sum_{n=1}^{\infty} u_{n-1}\epsilon^{2n}=0,$$

or

$$\nabla^2 u_0+\sum_{n=1}^{\infty}\left[\nabla^2 u_n+u_{n-1}\right]\epsilon^{2n}=0. \tag{1.28}$$

From Eqs. (1.27) and (1.26) we can write

$$u(x,y)=u_0+\sum_{n=1}^{\infty} u_n\epsilon^{2n}=1, \quad x^2+y^2=1. \tag{1.29}$$

Comparing the coefficients of $\epsilon^2$ in Eqs. (1.28) and (1.29) one obtains
$o\left(\epsilon^0\right)$

$$\nabla^2 u_0=0, \tag{1.30}$$

$$u_0(x,y)=1, \quad x^2+y^2=1. \tag{1.30a}$$

$o\left(\epsilon^{2n}\right)$

$$\nabla^2 u_n + u_{n-1} = 0, \;\; n \geq 1, \tag{1.31}$$

$$u_n\left(x, y\right) = 0, \;\; x^2 + y^2 = 1, \;\; n \geq 1. \tag{1.31a}$$

Equation (1.30) can be written as

$$\frac{\partial^2 u_0}{\partial r^2} + \frac{1}{r}\frac{\partial u_0}{\partial r} = 0, \tag{1.32}$$

Using

$$r = e^t,$$

the solution of above equation is

$$u_0 = A + Bt,$$

or

$$u_0 = A + B \ln r. \tag{1.33}$$

After using Eq. (1.30$a$) in Eq. (1.33) we get

$$u_0 = 1. \tag{1.34}$$

For $n = 1$, Eqs. (1.31), (1.31$a$) and (1.34) give

$$\nabla^2 u_1 = -u_0,$$

$$\nabla^2 u_1 = -1, \tag{1.35}$$

$$u_1\left(x, y\right) = 0. \tag{1.35a}$$

In polar coordinates Eq. (1.35) becomes

$$\frac{\partial^2 u_1}{\partial r^2} + \frac{1}{r}\frac{\partial u_1}{\partial r} = -1,$$

or

$$r^2\frac{\partial^2 u_1}{\partial r^2}+r\frac{\partial u_1}{\partial r}=-r^2. \tag{1.36}$$

The solution of above equation is

$$u_1=A+B\ln r-\frac{e^{2t}}{4}. \tag{1.37}$$

Using boundary conditions in above equation, we have

$$u_1=\frac{1}{4}\left(1-r^2\right). \tag{1.38}$$

From Eqs. (1.34), (1.38) and (1.27) we obtain

$$u=1+\frac{\epsilon^2}{4}\left(1-r^2\right)+.......... \tag{1.39}$$

# Chapter 2

# Flow of an elastico-viscous fluid with variable suction

## 2.1 Introduction

This chapter deals with the flow of an elastico viscous fluid past an infinite plate with time dependent suction. Expressions for velocity and skin frictions are given. The suction velocity is taken as an oscillating function of time. The present work is a review of a paper by Soundalgekar and Puri [20]. However, detail of mathematical calculations is incorporated properly in the presented analysis.

## 2.2 Mathematical Formulation

We consider the unsteady, two-dimensional incompressible elastico-viscous (Walter's liquid $B'$) fluid flow parallel to an infinite plane porous plate. The $x'$-axis is chosen along the plate and $y'$-axis perpendicular to it. We also assume that the flow is independent of the distance parallel to the plate. Then the continuity equation requires that $v$ is at most a function of time and retains its value at the plate throughout the flow.

The equation of state for the liquid $B'$ can be written in the form

$$p_{ik} = -pg_{ik} + p'_{ik}, \tag{2.1}$$

$$p^{'ik}(x,t) = 2\int_{-\infty}^{t} \psi\left(t-t^{'}\right) \frac{\partial x^i}{\partial x^{'m}} \frac{\partial x^k}{\partial x^{'r}} e^{(1)mr}\left(x^{'},t^{'}\right) dt^{'}, \tag{2.2}$$

where $p_{ik}$ is the stress tensor, $p$ an arbitrary isotropic pressure, $g_{ik}$ the metric tensor of a fixed co-ordinates system $x^i$, $x^{'i}$ the position at time $t^{'}$ of the element which is instantaneously at the point $x^i$ at time $t$, $e_{ik}^{(1)}$ the rate of strain tensor and where

$$\psi\left(t-t^{'}\right) = \int_0^{\infty} \frac{N(\tau)}{\tau} \exp\left[-\left(t-t^{'}\right)/\tau\right] d\tau,$$

in the above expression $N(\tau)$ is the distribution function of relaxation times $\tau$. If attention is restricted to liquids with short memories (i.e. short relaxation times), Beard and Walters [22] has shown that the equation of state can be written in a simplified form

$$p^{'ik} = 2\eta_0 e^{(1)ik} - 2k_0 \frac{\delta}{\delta t} e^{(1)ik}, \tag{2.3}$$

where

$$\eta_0 = \int_0^{\infty} N(\tau)\, d\tau,$$

is the limiting viscosity at small rates of shear

$$k_0 = \int_0^{\infty} \tau N(\tau)\, d\tau,$$

and $\delta/\delta t$ signifies the convected differentiation of a tensor quantity, which for any contravariant tensor $b^{ik}$ is given as

$$\frac{\delta b^{ik}}{\delta t} = \frac{\partial b^{ik}}{\partial t} + v^m \frac{\partial b^{ik}}{\partial x^m} - b^{im} \frac{\partial v^k}{\partial x^m} - b^{mk} \frac{\partial v^i}{\partial x^m}, \tag{2.4}$$

where $v^i$ is the velocity vector.

The above equations along with momentum and continuity equations for the problem under considerations yield

$$\rho^{'}\left(\frac{\partial u^{'}}{\partial t^{'}} + v^{'}\frac{\partial u^{'}}{\partial y^{'}}\right) = -\frac{\partial p^{'}}{\partial x^{'}} + \eta_0 \frac{\partial^2 u^{'}}{\partial y^{'2}} \tag{2.5}$$

$$-k_0\begin{pmatrix} \frac{\partial^3 u'}{\partial y'^2 \partial t'} + v'\frac{\partial^3 u'}{\partial y'^3} \\ -3\frac{\partial u'}{\partial y'}\frac{\partial^2 v'}{\partial y'^2} - 2\frac{\partial v'}{\partial y'}\frac{\partial^2 u'}{\partial y'^2} \end{pmatrix},$$

$$\rho'\left(\frac{\partial v'}{\partial t'} + v'\frac{\partial v'}{\partial y'}\right) = -\frac{\partial p'}{\partial y'} + 2\eta_0\frac{\partial^2 v'}{\partial y'^2} -2k_0\left(\frac{\partial^3 v'}{\partial y'^2 \partial t'} + v'\frac{\partial^3 v'}{\partial y'^3} - 3\frac{\partial v'}{\partial y'}\frac{\partial^2 v'}{\partial y'^2}\right), \tag{2.6}$$

$$\frac{\partial v'}{\partial y'} = 0. \tag{2.7}$$

Equation (2.7) shows that $v'$ is a function of time only. However following Messiha [18] we take

$$v' = -v_0'\left(1 + \epsilon A e^{i\omega' t'}\right), \tag{2.8}$$

where $v_0'$ is a non-zero constant mean suction velocity, $\epsilon$ is small and $A$ is a real positive constant such that $\epsilon A \leq 1$. The negative sign in (2.8) indicates that the suction velocity normal to the plate is directed towards the plate.

Using Eq. (2.8), the Eqs. (2.5) and (2.6) become

$$\frac{\partial u'}{\partial t'} - v_0'\left(1 + \epsilon A e^{i\omega' t'}\right)\frac{\partial u'}{\partial y'} = -\frac{1}{\rho'}\frac{\partial p'}{\partial x'} + \nu\frac{\partial^2 u'}{\partial y'^2} -k_0^*\left[\frac{\partial^3 u'}{\partial y'^2 \partial t'} - v_0'\left(1 + \epsilon A e^{i\omega' t'}\right)\frac{\partial^3 u'}{\partial y'^3}\right], \tag{2.9}$$

$$\frac{\partial v'}{\partial t'} = -\frac{1}{\rho'}\frac{\partial p'}{\partial y'}, \tag{2.10}$$

where

$$\nu = \frac{\eta_0}{\rho'} \qquad \text{and} \qquad k_0^* = \frac{k_0}{\rho'}.$$

Also from Eqs. (2.8) and (2.10) as $\partial p'/\partial y'$ is small in the boundary layer, it can be neglected. Hence the pressure is taken to be constant along any normal and is given by its value outside the boundary layer. If $U'(t')$ is the free-stream velocity, then from Eq. (2.9) we get

$$-\frac{1}{\rho'}\frac{\partial p'}{\partial x'} = \frac{dU'}{dt'}. \tag{2.11}$$

With the use of above equation, Eq. (2.9) gives

$$\begin{aligned} \frac{\partial u'}{\partial t'} - v_0'\left(1+\epsilon A e^{i\omega' t'}\right)\frac{\partial u'}{\partial y'} &= \frac{dU'}{dt'} + \nu\frac{\partial^2 u'}{\partial y'^2} \\ &\quad -k_0^*\left[\frac{\partial^3 u'}{\partial y'^2 \partial t'} - v_0'\left(1+\epsilon A e^{i\omega' t'}\right)\frac{\partial^3 u'}{\partial y'^3}\right]. \end{aligned} \tag{2.12}$$

The boundary conditions are

$$u' = 0 \qquad \text{at} \quad y' = 0, \tag{2.13}$$

$$u' = U'(t') \qquad \text{as} \quad y' \to \infty,$$

in which

$$U'(t') = U_0'\left(1+\epsilon e^{i\omega' t'}\right), \tag{2.14}$$

where $U_0'$ is a mean of $U'$.

After using Eq. (2.14), Eq. (2.12) and boundary conditions (2.13) take the form

$$\begin{aligned} \frac{\partial u'}{\partial t'} - v_0'\left(1+\epsilon A e^{i\omega' t'}\right)\frac{\partial u'}{\partial y'} &= U_0' i\omega' \epsilon e^{i\omega' t'} + \nu\frac{\partial^2 u'}{\partial y'^2} \\ &\quad -k_0^*\left[\frac{\partial^3 u'}{\partial y'^2 \partial t'} - v_0'\left(1+\epsilon A e^{i\omega' t'}\right)\frac{\partial^3 u'}{\partial y'^3}\right], \end{aligned} \tag{2.15}$$

$$u' = 0 \qquad \text{at} \quad y' = 0, \tag{2.16}$$

$$u' = U'(t') = U_0'\left(1+\epsilon e^{i\omega' t'}\right) \qquad \text{as} \quad y' \to \infty.$$

Introducing the following non-dimensional variables

$$\eta = \frac{y' v_0'}{\nu}, \qquad t = \frac{v_0'^2 t'}{4\nu}, \qquad \omega = \frac{4\nu\omega'}{v_0'^2}, \tag{2.17}$$

$$U = \frac{U'}{U_0'}, \qquad u = \frac{u'}{U_0'}, \qquad k = \frac{k_0^* v_0'^2}{\nu^2},$$

Eq. (2.15) and boundary conditions (2.16) can be written as

$$\begin{aligned} \frac{1}{4}\frac{\partial u}{\partial t} - \left(1 + \epsilon A e^{i\omega t}\right)\frac{\partial u}{\partial \eta} &= \frac{i\omega}{4}\epsilon e^{i\omega t} + \frac{\partial^2 u}{\partial \eta^2} \\ &\quad - k\left[\frac{\partial^3 u}{\partial \eta^2 \partial t} - \left(1 + \epsilon A e^{i\omega t}\right)\frac{\partial^3 u}{\partial \eta^3}\right], \end{aligned} \tag{2.18}$$

$$u = 0 \qquad \text{at} \qquad \eta = 0, \tag{2.19}$$

$$u = \left(1 + \epsilon e^{i\omega t}\right) \qquad \text{as} \qquad \eta \to \infty.$$

## 2.3 Analytic Solution

It is appropriate to write $u'(y', t')$ of the form

$$u'(y', t') = U_0'(f_1(y') + \epsilon e^{i\omega' t'} f_2(y')), \tag{2.20}$$

in which $\omega'$ is the frequency of the fluctuating stream, $\epsilon U_0'$ is the amplitude of the free-stream fluctuation, $f_1 U_0'$ is the mean velocity in the boundary layer, $\epsilon U_0' f_2$ is the amplitude of the velocity fluctuation in the boundary layer.

In terms of non-dimensional variables Eq. (2.20) is

$$u(\eta, t) = f_1(\eta) + \epsilon e^{i\omega t} f_2(\eta). \tag{2.21}$$

Substituting Eq. (2.21) into the Eq. (2.18) and boundary conditions (2.19) and then comparing the non-hormonic and hormonic terms and neglecting the coefficients of $\epsilon^2$, we respectively get

$$k\frac{d^3 f_1}{d\eta^3}+\frac{d^2 f_1}{d\eta^2}+\frac{df_1}{d\eta}=0, \tag{2.22}$$

$$f_1 = 0 \qquad \text{at} \qquad \eta = 0, \tag{2.23}$$

$$f_1 = 1 \qquad \text{as} \qquad \eta \rightarrow \infty,$$

$$k\frac{d^3 f_2}{d\eta^3}+\left(1-\frac{1}{4}ki\omega\right)\frac{d^2 f_2}{d\eta^2}+\frac{df_2}{d\eta}-\frac{1}{4}i\omega f_2=-\frac{1}{4}i\omega-A\frac{df_1}{d\eta}-kA\frac{d^3 f_1}{d\eta^3}, \tag{2.24}$$

$$f_2 = 0 \qquad \text{at} \qquad \eta = 0, \tag{2.25}$$

$$f_2 = 1 \qquad \text{as} \qquad \eta \rightarrow \infty.$$

Before proceeding with the solution of the above problem it would be interesting to remark here that although in the classical viscous case ($k = 0$), we encounter differential equation of order two. The presence of the elasticity of the fluid increases the order to three. It would therefore seem that an additional boundary condition must be imposed in order to get a unique solution. The difficulty in the present case is removed by seeking a solution of the form [20 – 23]

$$f_1 = f_{01} + kf_{11} + o\left(k^2\right), \tag{2.26}$$

$$f_2 = f_{02} + kf_{12} + o\left(k^2\right), \tag{2.27}$$

which is valid for small value of $k$ only.

Putting Eqs. (2.26) and (2.27) into Eqs. (2.22) – (2.25) and equating the coefficients of $k^0$ and $k^1$ we have from equations (2.22) and (2.23)

$o\left(k^{0}\right)$

$$\frac{d^2 f_{01}}{d\eta^2} + \frac{df_{01}}{d\eta} = 0, \tag{2.28}$$

$$f_{01} = 0 \qquad \text{at} \qquad \eta = 0, \tag{2.29}$$

$$f_{01} = 1 \qquad \text{as} \qquad \eta \to \infty,$$

$o\left(k^{1}\right)$

$$\frac{d^2 f_{11}}{d\eta^2} + \frac{df_{11}}{d\eta} = -\frac{d^3 f_{01}}{d\eta^3}, \tag{2.30}$$

$$f_{11} = 0 \qquad \text{at} \qquad \eta = 0, \tag{2.31}$$

$$f_{11} = 0 \qquad \text{as} \qquad \eta \to \infty,$$

and from Eqs. (2.24) and (2.25) we have

$o\left(k^{0}\right)$

$$\frac{d^2 f_{02}}{d\eta^2} + \frac{df_{02}}{d\eta} - \frac{1}{4} i\omega f_{02} = -\frac{1}{4} i\omega - A\frac{df_{01}}{d\eta}, \tag{2.32}$$

$$f_{02} = 0 \qquad \text{at} \qquad \eta = 0, \tag{2.33}$$

$$f_{02} = 1 \qquad \text{as} \qquad \eta \to \infty,$$

$o\left(k^{1}\right)$

$$\frac{d^2 f_{12}}{d\eta^2} + \frac{df_{12}}{d\eta} - \frac{1}{4} i\omega f_{12} = -\frac{d^3 f_{02}}{d\eta^3} + \frac{1}{4} i\omega \frac{d^2 f_{02}}{d\eta^2} - A\frac{df_{11}}{d\eta} - A\frac{d^3 f_{01}}{d\eta^3}, \tag{2.34}$$

$$f_{12} = 0 \qquad \text{at} \qquad \eta = 0, \tag{2.35}$$

$$f_{12} = 0 \qquad \text{as} \qquad \eta \to \infty.$$

The solutions of the above systems are

$$f_{01} = 1 - e^{-\eta}, \tag{2.36}$$

$$f_{11} = \eta e^{-\eta}, \tag{2.37}$$

$$f_{02} = 1 - Se^{-h\eta} - (1 - S)\, e^{-\eta}, \tag{2.38}$$

$$f_{12} = (1 - S)\left(e^{-h\eta} - (1 - \eta)\, e^{-\eta}\right) + L\eta e^{-h\eta}, \tag{2.39}$$

where

$$S = 1 - \frac{4iA}{\omega}, \tag{2.40}$$

$$h = h_r + ih_i = \frac{1}{2}\left[1 + \sqrt{(1 + i\omega)}\right], \tag{2.41}$$

$$h_r = \frac{1}{2} + \frac{a}{2} = \frac{1}{2} + \frac{1}{2}\left(\frac{1}{2}\left[1 + \sqrt{(1 + \omega^2)}\right]\right)^{\frac{1}{2}},$$

$$h_i = \frac{b}{2} = \frac{1}{2}\left(\frac{1}{2}\left[-1 + \sqrt{(1 + \omega^2)}\right]\right)^{\frac{1}{2}},$$

$$L = L_r + iL_i = \frac{h^2\left(h + \frac{1}{4}i\omega\right)\left(1 - \frac{4iA}{\omega}\right)}{\sqrt{(1 + i\omega)}}, \tag{2.42}$$

$$L_r = r^{-\frac{1}{2}}\left[\begin{array}{c} \frac{r^{\frac{1}{2}}}{2} + \frac{r^{\frac{1}{2}}}{2}\left(\frac{r+1}{2}\right)^{\frac{1}{2}} - \frac{r^2-1}{16}\left(\frac{r+1}{2r}\right)^{\frac{1}{2}} \\ + \left(\frac{4A}{\omega}\right)\left(\frac{\omega r^{\frac{1}{2}}}{4} + \frac{r^{\frac{1}{2}}}{2}\left(\frac{r-1}{2}\right)^{\frac{1}{2}} + \frac{r^2-1}{16}\left(\frac{r-1}{2r}\right)^{\frac{1}{2}}\right) \end{array}\right],$$

$$L_i = r^{-\frac{1}{2}} \begin{bmatrix} \frac{\omega r^{\frac{1}{2}}}{4} + \frac{r^{\frac{1}{2}}}{2}\left(\frac{r-1}{2}\right)^{\frac{1}{2}} + \frac{r^2-1}{16}\left(\frac{r-1}{2r}\right)^{\frac{1}{2}} \\ -\frac{4A}{\omega}\left(\frac{r^{\frac{1}{2}}}{2} + \frac{r^{\frac{1}{2}}}{2}\left(\frac{r+1}{2}\right)^{\frac{1}{2}} - \frac{r^2-1}{16}\left(\frac{r+1}{2r}\right)^{\frac{1}{2}}\right) \end{bmatrix},$$

$$r^2 = 1 + \omega^2.$$

With the help of Eqs. (2.36) – (2.39), Eqs. (2.26) and (2.27) become

$$f_1 = 1 - e^{-\eta} + k\eta e^{-\eta}, \tag{2.43}$$

$$\begin{aligned} f_2 &= 1 - Se^{-h\eta} - (1 - S)\, e^{-\eta} \\ &\quad + k\left[(1 - S)\left(e^{-h\eta} - (1 - \eta)\, e^{-\eta}\right) + L\eta e^{-h\eta}\right]. \end{aligned} \tag{2.44}$$

Now the Eq. (2.21) gives

$$\begin{aligned} u(\eta, t) &= 1 - e^{-\eta} + k\eta e^{-\eta} \\ &\quad + \epsilon e^{i\omega t} \begin{bmatrix} 1 - Se^{-h\eta} - (1 - S)\, e^{-\eta} \\ +k\left\{(1 - S)\left(e^{-h\eta} - (1 - \eta)\, e^{-\eta}\right) + L\eta e^{-h\eta}\right\} \end{bmatrix}. \end{aligned} \tag{2.45}$$

The above equation can also be written after neglecting the imaginary part as

$$u(\eta, t) = 1 - e^{-\eta} + k\eta e^{-\eta} + \epsilon\left(M_r \cos\omega t - M_i \sin\omega t\right), \tag{2.46}$$

where $M_r$, $M_i$ are the fluctuating parts of the velocity profile and are given by

$$M_r = 1 + e^{-h_r\eta} \begin{bmatrix} \left(\frac{4A}{\omega}\right)(1 + k)\sin h_i\eta - \cos h_i\eta \\ +\eta k\left\{L_r \cos h_i\eta + L_i \sin h_i\eta\right\} \end{bmatrix}, \tag{2.47}$$

$$\begin{aligned} M_i &= \left[\left\{\left(\frac{4A}{\omega}\right)(1 + k) + \eta k L_i\right\}\cos h_i\eta + (1 - \eta k L_r)\sin h_i\eta\right] e^{-h_r\eta} \\ &\quad - \left(\frac{4A}{\omega}\right)\left[1 + k\,(1 - \eta)\right] e^{-\eta}. \end{aligned} \tag{2.48}$$

From Eqs. (2.1) and (2.3) the expression for the shearing stress is

$$p'_{x'y'} = \eta_0 \frac{\partial u'}{\partial y'} - k_0 \left( \frac{\partial^2 u'}{\partial y' \partial t'} + v' \frac{\partial^2 u'}{\partial y'^2} \right), \tag{2.49}$$

which in term of non-dimensional variables become

$$p_{xy} = \frac{p'_{x'y'}}{U'_0 v'_0 \rho'} = \frac{\partial u}{\partial \eta} - \frac{1}{4} k \left[ \frac{\partial^2 u}{\partial \eta \partial t} - 4 \left(1 + \epsilon A e^{i\omega t}\right) \frac{\partial^2 u}{\partial \eta^2} \right]. \tag{2.50}$$

Substitution of Eq. (2.45) into Eq. (2.50) yields

$$p_{xy} = D_1 - \frac{1}{4} k \left[ D_2 - 4 \left(1 + \epsilon A e^{i\omega t}\right) D_3 \right], \tag{2.51}$$

where

$$D_1 = e^{-\eta} + k e^{-\eta} - k \eta e^{-\eta} + \epsilon e^{i\omega t} \left[ \begin{array}{c} h S e^{-h\eta} + (1 - S) e^{-\eta} \\ +k \left\{ \begin{array}{c} (1 - S) \left(-h e^{-h\eta} + e^{-\eta} + (1 - \eta) e^{-\eta}\right) \\ -L h \eta e^{-h\eta} + L e^{-h\eta} \end{array} \right\} \end{array} \right],$$

$$D_2 = \epsilon i \omega e^{i\omega t} \left\{ \begin{array}{c} h S e^{-h\eta} + (1 - S) e^{-\eta} \\ +k \left( \begin{array}{c} (1 - S) \left(-h e^{-h\eta} + e^{-\eta} + (1 - \eta) e^{-\eta}\right) \\ -L h \eta e^{-h\eta} + L e^{-h\eta} \end{array} \right) \end{array} \right\},$$

$$D_3 = \begin{array}{c} e^{-\eta} - 2k e^{-\eta} + k \eta e^{-\eta} \\ +\epsilon e^{i\omega t} \left[ \begin{array}{c} -h^2 S e^{-h\eta} - (1 - S) e^{-\eta} \\ +k \left( \begin{array}{c} (1 - S) \left( \begin{array}{c} h^2 e^{-h\eta} - 2 e^{-\eta} \\ - (1 - \eta) e^{-\eta} \end{array} \right) \\ +L h^2 \eta e^{-h\eta} - 2 L e^{-h\eta} \end{array} \right) \end{array} \right] \end{array},$$

and now Eq. (2.51) becomes

$$(p_{xy})_{\eta\to 0} = 1 + \epsilon e^{i\omega t}\left[\begin{array}{c}(1-S)\left(1+k-\frac{1}{4}i\omega k\right) \\ +h\left(S-k\left(1-S-\frac{1}{4}i\omega kS\right)\right) \\ +k\left(L-A-Sh^2\right)\end{array}\right]. \tag{2.52}$$

From above equation we can write

$$p_{xy} = 1 + \epsilon\left|B\right|\cos\left(\omega t+\beta\right), \tag{2.53}$$

where

$$B = B_r + iB_i = \left[\begin{array}{c}(1-S)\left(1+k-\frac{1}{4}i\omega k\right)+h\left(S-k\left(1-S-\frac{1}{4}i\omega kS\right)\right) \\ +k\left(L-A-Sh^2\right)\end{array}\right], \tag{2.54}$$

$$\beta = \tan^{-1}\left(\frac{B_i}{B_r}\right), \tag{2.55}$$

$$\begin{aligned} B_r &= (1-kA)\,h_r + \left(\frac{4A}{\omega}(1+k)+\frac{k\omega}{4}\right)h_i + kL_r \\ &\quad -k\left(h_r^2-h_i^2\right) - \frac{8Akh_rh_i}{\omega}, \end{aligned} \tag{2.56}$$

$$\begin{aligned} B_i &= \frac{4A}{\omega}(1+k) - \left(\frac{4A}{\omega}(1+k)+\frac{k\omega}{4}\right)h_r + (1-kA)\,h_i \\ &\quad +kL_i + \frac{4Ak}{\omega}\left(h_r^2-h_i^2\right) - 2kh_rh_i. \end{aligned} \tag{2.57}$$

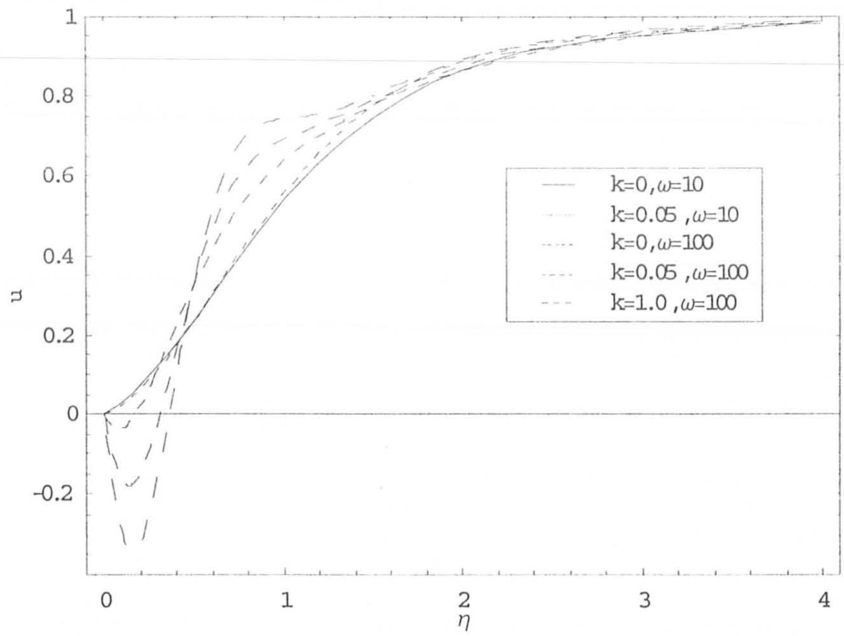


**Fig. 2.1.** Velocity profiles against $\eta$ at $\omega t = \pi/2$, $A = 0$, $\epsilon = 0.5$.

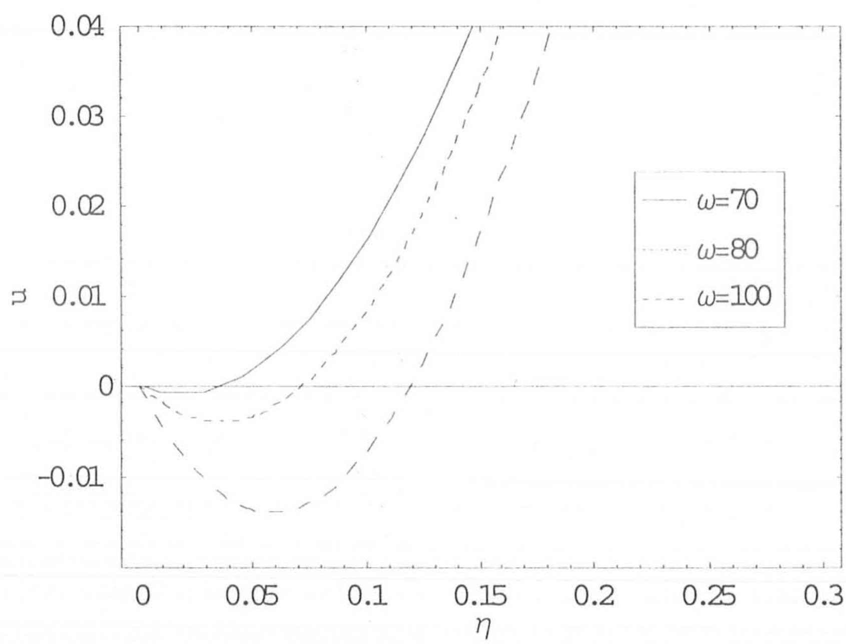


**Fig. 2.2.** Velocity profiles against $\eta$ at $\omega t = \pi/2$, $A = 0$, $\epsilon = 0.2$, $k = 0.05$.

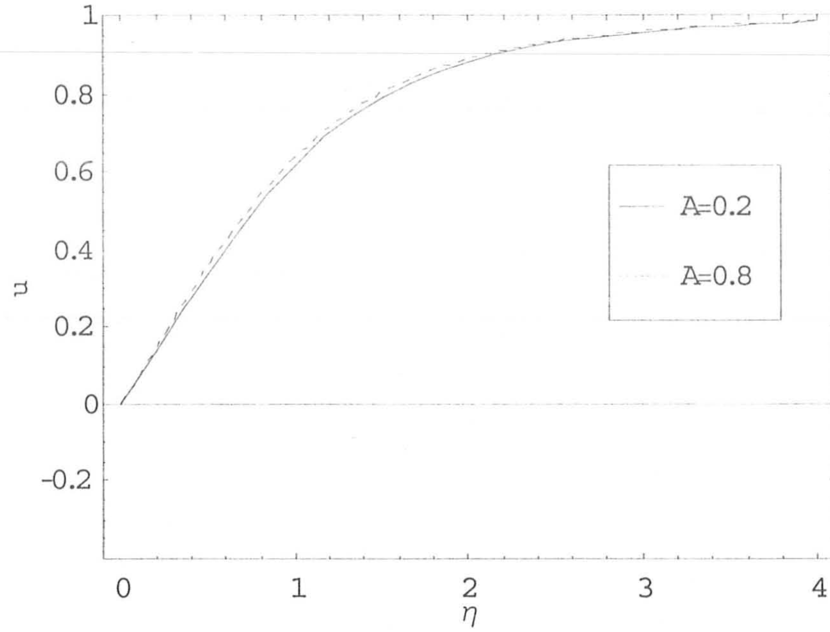


**Fig. 2.3.** Velocity profiles against $\eta$ at $\omega t = \pi/2$, $\omega = 10$, $\epsilon = 0.2$, $k = 0.05$.

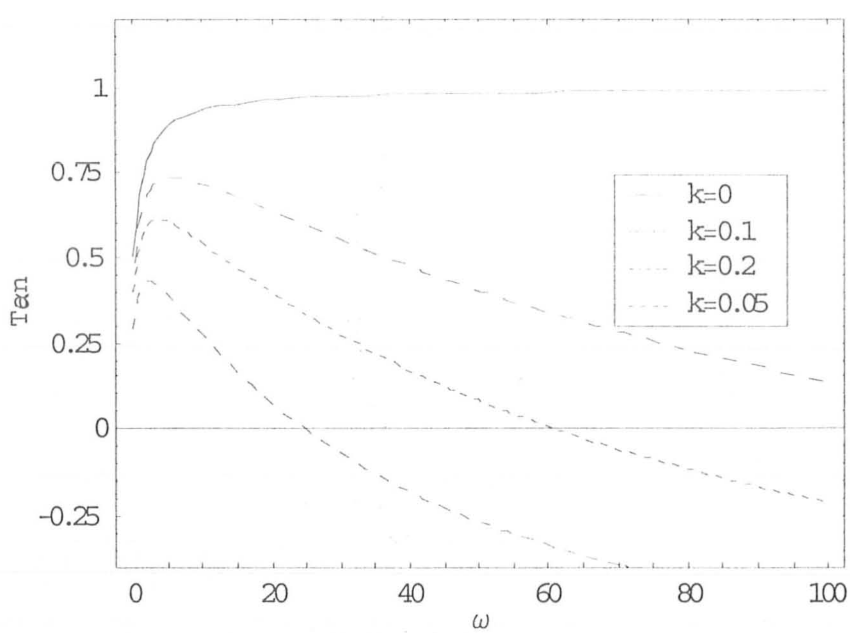


**Fig. 2.4.** Skin-friction phase against $\omega$ at $A = 0$.

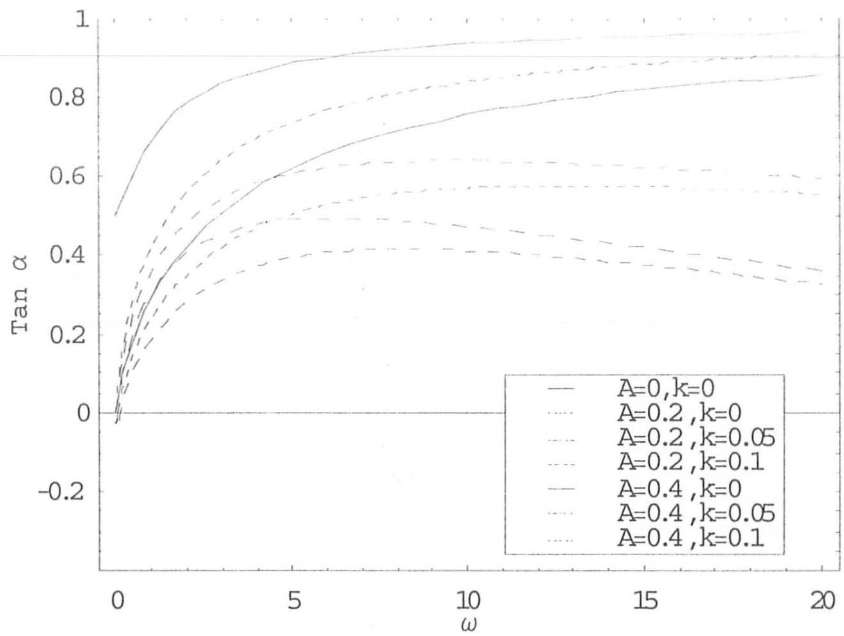


**Fig. 2.5.** Skin-friction phase against $\omega$.

## 2.4 Discussion

To see the variation of the elastic property of the elstico-viscous fluid on the velocity profiles near the wall, both in case of constant and variable suction, we have plotted $u$ against $\eta$ in figures 2.1 to 2.3 for various values of $A$, $\omega$, $k$ and $\epsilon$. Stuart [17] found that for $\epsilon = 0.5$ and $\omega = 100$ ($\frac{1}{4}\omega = \lambda$ in Stuart's case) the velocity is negative near the wall, which is also shown in figure 2.1 for $k = 0$. Also the graphs in figure 2.1 for non-zero $k$ are particularly interesting in the sense that, with the increase in k, the velocity becomes still more negative near the wall for $\epsilon = 0.5$ and $\omega = 100$. This leads us to study the nature of the velocity profiles for smaller values of $\epsilon$ and $\omega$. In figure 2.2, it can be seen for elastico viscous fluids, (for very small values of $k$) that the velocity is negative even for smaller values of $\epsilon$ and $\omega$ i.e. $\epsilon = 0.2$ and $\omega = 80$. Hence in the case of constant suction velocity, the separation occurs at the wall even for small values of $\epsilon$ and $\omega$. Figure 2.3 is prepared to bring out the effects of the variable suction velocity on the separation of the fluid at the wall. This is Messiha's case. Messiha [18] has not discussed the nature of the velocity profiles at large $\omega$, in the presence of the variable suction velocity.

Figures 2.4 and 2.5 illustrate the effects of $k$ and $A$ on the skin-friction phase. It is observed by Stuart that the skin-friction phase rises from zero at zero frequency to $\frac{1}{4}\pi$ at very high frequencies. This is shown in figure 2.4, where $k = 0$, $A = 0$ corresponds to Stuart's case. The other three curves show the effect of $k$ on the phase of the skin-friction. It is interesting to note that an increase in $k$ leads to a decrease in the phase of the skin-friction at large $\omega$. It is also noted from this figure that $\tan\alpha = 0$ when $\omega = 57$ and $k = 0.1$, from which we can conclude that the skin-friction oscillates in phase with the on-coming fluctuating main-stream. For $\omega > 57$, the phase of the skin-friction is negative. Figure 2.5 is made to compare the results with Messiha. It is seen that phase of skin friction decreases with an increase in $A$ and increases with an increase in $\omega$. The phase of the skin friction is negative for small values of $\omega$. The same is true for elastico-viscous fluids (liquid $B'$). As increase in k leads to a decrease in the phase as in the case of constant suction velocity. At large $\omega$, the trend is again towards a decrease.

# Chapter 3

# Heat transfer analysis on rotating flow of a second grade fluid past a porous plate with variable suction

## 3.1 Introduction

The present chapter deals with the study of momentum and heat transfer characteristics in a second garde rotating flow past a porous plate. The analysis is performed when the suction velocity normal to the plate, as well as the external flow velocity, varies periodically with time. The plate is assumed at a higher temperature than the fluid. Analytic solutions for velocity, shear stresses and temperature are derived. The effects of various parameters of physical interest on the velocity, shear stresses and temperature are shown and discussed in detail.

## 3.2 Mathematical Formulation

Let us consider an incompressible second grade fluid past a porous plate. The plate and the fluid rotate in unison with an angular velocity $\Omega$ about the $z'-$axis normal to the plate. The plate is located at $z' = 0$ having temperature $T_0$. The flow far away from the plate is uniform and temperature of the fluid is $T_\infty$.

For the problem under question, we consider the velocity and temperature fields as

$$\mathbf{V} = \left(u^{'}(z^{'},t^{'}), v^{'}(z^{'},t^{'}), w^{'}(z^{'},t^{'})\right), \tag{3.1}$$

$$T = T(z^{'},t^{'}), \tag{3.2}$$

in which $u^{'}$,$v^{'}$ and $w^{'}$ are the velocity components in $x^{'}$, $y^{'}$, and $z^{'}$directions respectively and $T$ indicates the temperature.

The governing equations in absence of body forces and radiant heating are

$$div\mathbf{V} = 0, \tag{3.3}$$

$$\rho^{'}\left[\frac{d\mathbf{V}}{dt^{'}} + 2\boldsymbol{\Omega}\times\mathbf{V} + \boldsymbol{\Omega}\times(\boldsymbol{\Omega}\times\mathbf{r})\right] = div\mathbf{T}, \tag{3.4}$$

$$\rho^{'}\frac{de}{dt^{'}} = \mathbf{T}\cdot\mathbf{L} - div\mathbf{q}. \tag{3.5}$$

In above equations $d/dt^{'}$, $\rho^{'}$, $e$, $\mathbf{L}$, and $\mathbf{q}$ are respectively the material derivative, density, the specific internal energy, the gradient of velocity, the heat flux vector and the radial distance $r^2 = x^2 + y^2$. The Cauchy stress $\mathbf{T}$ in an incompressible homogeneous fluid of second grade is of the form

$$\mathbf{T} = -p\mathbf{I} + \mu\mathbf{A}_1 + \alpha_1\mathbf{A}_2 + \alpha_2\mathbf{A}_1^2, \tag{3.6}$$

$$\mathbf{A}_1 = (\text{grad}\mathbf{V}) + (\text{grad}\mathbf{V})^{\top}, \tag{3.7}$$

$$\mathbf{A}_2 = \frac{d\mathbf{A}_1}{dt} + \mathbf{A}_1(\text{grad}\mathbf{V}) + (\text{grad}\mathbf{V})^{\top}\mathbf{A}_1, \tag{3.8}$$

where $\mu$, $-p\mathbf{I}$, $\alpha_j$ $(j = 1, 2)$, $\mathbf{A}_1$and $\mathbf{A}_2$ are respectively the dynamic viscosity, spherical stress, normal stress moduli and first two Rivlin-Ericksen tensors. The thermodynamic analysis of model (3.6) has been discussed in detail by Dunn and Fosdick [24]. The Clausius-Duhem inequality and the assumption that the Helmholtz free energy is a minimum in equilbrium provide the following restrictions [25]

$$\mu \geq 0, \qquad \alpha_1 \geq 0, \qquad \alpha_1 + \alpha_2 = 0. \tag{3.9}$$

It is evident from Eqs. (3.1) and (3.3) that

$$\frac{\partial w'}{\partial z'} = 0.$$

The above equation shows that $w'$ is a function of time. Following Messiha [18] and Soundalgekar and Puri [20] we take

$$w' = -W_0' \left(1 + \epsilon A e^{i\omega' t'}\right). \tag{3.10}$$

In above equation $W_0'$ is non-zero constant mean suction velocity, $A$ is real positive constant, $\epsilon$ is small such that $\epsilon A \leq 1$ and negative sign indicates that suction velocity normal to the plate is directed towards the plate.

Now the gradient of the velocity is given by

$$\text{grad}\mathbf{V} = \begin{bmatrix} \frac{\partial u'}{\partial x'} & \frac{\partial u'}{\partial y'} & \frac{\partial u'}{\partial z'} \\ \frac{\partial v'}{\partial x'} & \frac{\partial v'}{\partial y'} & \frac{\partial v'}{\partial z'} \\ \frac{\partial w'}{\partial x'} & \frac{\partial w'}{\partial y'} & \frac{\partial w'}{\partial z'} \end{bmatrix}. \tag{3.11}$$

From Eqs. (3.1) and (3.11) we can write

$$\text{grad}\mathbf{V} = \begin{bmatrix} 0 & 0 & \frac{\partial u'}{\partial z'} \\ 0 & 0 & \frac{\partial v'}{\partial z'} \\ 0 & 0 & 0 \end{bmatrix}, \qquad (\text{grad}\mathbf{V})^* = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \frac{\partial u'}{\partial z'} & \frac{\partial v'}{\partial z'} & 0 \end{bmatrix}. \tag{3.12}$$

in which $(*)$ is the matrix transpose.

With the help of Eq. (3.12), Eqs. (3.7) and (3.8) become

$$\mathbf{A}_1 = \begin{bmatrix} 0 & 0 & \frac{\partial u'}{\partial z'} \\ 0 & 0 & \frac{\partial v'}{\partial z'} \\ \frac{\partial u'}{\partial z'} & \frac{\partial v'}{\partial z'} & 0 \end{bmatrix}, \tag{3.13}$$

$$\mathbf{A}_2 = \begin{bmatrix} 0 & 0 & \frac{\partial^2 u'}{\partial z' \partial t'} + w' \frac{\partial^2 u'}{\partial z'^2} \\ 0 & 0 & \frac{\partial^2 v'}{\partial z' \partial t'} + w' \frac{\partial^2 v'}{\partial z'^2} \\ \frac{\partial^2 u'}{\partial z' \partial t'} + w' \frac{\partial^2 u'}{\partial z'^2} & \frac{\partial^2 v'}{\partial z' \partial t'} + w' \frac{\partial^2 v'}{\partial z'^2} & 2\left(\left(\frac{\partial u'}{\partial z'}\right)^2 + \left(\frac{\partial v'}{\partial z'}\right)^2\right) \end{bmatrix}, \tag{3.14}$$

$$\mathbf{A}_1^2 = \begin{bmatrix} \left(\frac{\partial u'}{\partial z'}\right)^2 & \left(\frac{\partial u'}{\partial z'}\right)\left(\frac{\partial v'}{\partial z'}\right) & 0 \\ \left(\frac{\partial u'}{\partial z'}\right)\left(\frac{\partial v'}{\partial z'}\right) & \left(\frac{\partial v'}{\partial z'}\right)^2 & 0 \\ 0 & 0 & \left(\frac{\partial u'}{\partial z'}\right)^2 + \left(\frac{\partial v'}{\partial z'}\right)^2 \end{bmatrix}. \tag{3.15}$$

Using Eqs. (3.13)−(3.15) into Eq. (3.6) and then taking the divergence of resulting equation we have

$$(divT)_{x'} = -\frac{\partial p}{\partial x'} + \mu \frac{\partial^2 u'}{\partial z'^2} + \alpha_1 \left(\frac{\partial^3 u'}{\partial z'^2 \partial t'} + w' \frac{\partial^3 u'}{\partial z'^3}\right), \tag{3.16}$$

$$(divT)_{y'} = -\frac{\partial p}{\partial y'} + \mu \frac{\partial^2 v'}{\partial z'^2} + \alpha_1 \left(\frac{\partial^3 v'}{\partial z'^2 \partial t'} + w' \frac{\partial^3 v'}{\partial z'^3}\right),$$

$$(divT)_{z'} = -\frac{\partial p}{\partial z'} + (2\alpha_1 + \alpha_2)\left(\left(\frac{\partial u'}{\partial z'}\right)^2 + \left(\frac{\partial v'}{\partial z'}\right)^2\right).$$

Now

$$\boldsymbol{\Omega} \times \mathbf{V} = \begin{vmatrix} i & j & k \\ 0 & 0 & \Omega \\ u' & v' & w' \end{vmatrix} = \left(-\Omega v', \Omega u', 0\right), \tag{3.17}$$

$$\boldsymbol{\Omega} \times \mathbf{r} = \begin{vmatrix} i & j & k \\ 0 & 0 & \Omega \\ x' & y' & 0 \end{vmatrix} = \left(-\Omega y', \Omega x', 0\right), \tag{3.18}$$

$$\boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \mathbf{r}) = \begin{vmatrix} i & j & k \\ 0 & 0 & \Omega \\ -y'\Omega & \Omega x' & 0 \end{vmatrix} = \left(-x'\Omega^2, -\Omega^2 y', 0\right), \tag{3.19}$$

$$2(\boldsymbol{\Omega} \times \mathbf{V}) + \boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \mathbf{r}) = \left(-2v'\Omega - x'\Omega^2, 2u'\Omega - y'\Omega^2, 0\right). \tag{3.20}$$

From Eqs. (3.4), (3.10), (3.16) and (3.20) we obtain

$$\begin{aligned} \frac{\partial u'}{\partial t'} - W_0'\left(1 + \epsilon A e^{i\omega' t'}\right)\frac{\partial u'}{\partial z'} - 2\Omega v' &= -\frac{1}{\rho'}\frac{\partial \widehat{p}}{\partial x'} + \upsilon\frac{\partial^2 u'}{\partial z'^2} \\ &\quad + \alpha^* \frac{\partial^3 u'}{\partial z'^2 \partial t'} \\ &\quad - \alpha^* W_0'\left(1 + \epsilon A e^{i\omega' t'}\right)\frac{\partial^3 u'}{\partial z'^3}, \end{aligned} \tag{3.21}$$

$$\begin{aligned} \frac{\partial v'}{\partial t'} - W_0'\left(1 + \epsilon A e^{i\omega' t'}\right)\frac{\partial v'}{\partial z'} + 2\Omega u' &= -\frac{1}{\rho'}\frac{\partial \widehat{p}}{\partial y'} + \upsilon\frac{\partial^2 v'}{\partial z'^2} \\ &\quad + \alpha^* \frac{\partial^3 v'}{\partial z'^2 \partial t'} \\ &\quad - \alpha^* W_0'\left(1 + \epsilon A e^{i\omega' t'}\right)\frac{\partial^3 v'}{\partial z'^3}, \end{aligned} \tag{3.22}$$

$$\frac{\partial w'}{\partial t'} = -\frac{1}{\rho'}\frac{\partial \widehat{p}}{\partial z'}, \tag{3.23}$$

subject to the boundary conditions

$$u' = v' = 0 \qquad \text{at} \qquad z' = 0, \tag{3.24}$$

$$u' \to U'(t'),\ v' \to 0 \quad \text{as} \quad z' \to \infty,$$

where $U'(t')$ is the free stream velocity and the modified pressure is

$$\widehat{p} = p - \frac{1}{2}\rho'\Omega^2 r^2 - (2\alpha_1 + \alpha_2)\left[\left(\frac{\partial u'}{\partial z'}\right)^2 + \left(\frac{\partial v'}{\partial z'}\right)^2\right], \tag{3.25}$$

and

$$\upsilon = \frac{\mu}{\rho'}, \qquad \alpha^* = \frac{\alpha_1}{\rho'}.$$

In view of Eqs. (3.10) and (3.23), $\partial\widehat{p}/\partial z'$ is small in the boundary and hence can be ignored [18, 20, 21]. The modified pressure $\widehat{p}$ is assumed constant along any normal and is given by its value outside the boundary layer. The equations (3.21) and (3.22) for the free stream yields

$$-\frac{1}{\rho'}\frac{\partial\widehat{p}}{\partial x'} = \frac{dU'}{dt'}, \tag{3.26}$$

$$-\frac{1}{\rho'}\frac{\partial\widehat{p}}{\partial y'} = 2\Omega U'.$$

Making use of Eq. (3.26) into Eqs. (3.21) and (3.22) we have

$$\begin{aligned}\frac{\partial u'}{\partial t'} - W_0'\left(1 + \epsilon A e^{i\omega' t'}\right)\frac{\partial u'}{\partial z'} - 2\Omega v' &= \frac{dU'}{dt'} + \upsilon\frac{\partial^2 u'}{\partial z'^2} + \alpha^*\frac{\partial^3 u'}{\partial z'^2\partial t'} \\ &\quad -\alpha^* W_0'\left(1 + \epsilon A e^{i\omega' t'}\right)\frac{\partial^3 u'}{\partial z'^3},\end{aligned} \tag{3.27}$$

$$\begin{aligned}\frac{\partial v'}{\partial t'} - W_0'\left(1 + \epsilon A e^{i\omega' t'}\right)\frac{\partial v'}{\partial z'} + 2\Omega u' &= 2\Omega U' + \upsilon\frac{\partial^2 v'}{\partial z'^2} + \alpha^*\frac{\partial^3 v'}{\partial z'^2\partial t'} \\ &\quad -\alpha^* W_0'\left(1 + \epsilon A e^{i\omega' t'}\right)\frac{\partial^3 v'}{\partial z'^3},\end{aligned} \tag{3.28}$$

where $U'$ is periodic free stream velocity given by

$$U'(t') = U'_0\left(1 + \epsilon e^{i\omega' t'}\right). \tag{3.29}$$

The appropriate boundary conditions for the problem are

$$\begin{aligned} u' &= v' = 0 \quad \text{at} \quad z' = 0, \\ u' &\longrightarrow U'(t'), \quad v' \longrightarrow 0 \quad \text{as} \quad z' \longrightarrow \infty. \end{aligned} \tag{3.30}$$

The Eqs. (3.27) and (3.28) after using Eq. (3.29) can be written as

$$\begin{aligned} \frac{\partial u'}{\partial t'} - W'_0\left(1 + \epsilon A e^{i\omega' t'}\right)\frac{\partial u'}{\partial z'} - 2\Omega v' &= U'_0 i\omega' \epsilon e^{i\omega' t'} + \alpha^* \frac{\partial^3 u'}{\partial z'^2 \partial t'} \\ &+ \upsilon \frac{\partial^2 u'}{\partial z'^2} - \alpha^* W'_0\left(1 + \epsilon A e^{i\omega' t'}\right)\frac{\partial^3 u'}{\partial z'^3}, \end{aligned} \tag{3.31}$$

$$\begin{aligned} \frac{\partial v'}{\partial t'} - W'_0\left(1 + \epsilon A e^{i\omega' t'}\right)\frac{\partial v'}{\partial z'} + 2\Omega u' &= 2\Omega U'_0\left(1 + \epsilon e^{i\omega' t'}\right) + \alpha^* \frac{\partial^3 v'}{\partial z'^2 \partial t'} \\ &+ \upsilon \frac{\partial^2 v'}{\partial z'^2} - \alpha^* W'_0\left(1 + \epsilon A e^{i\omega' t'}\right)\frac{\partial^3 v'}{\partial z'^3}. \end{aligned} \tag{3.32}$$

Multiplying Eq. (3.32) by $i$ and then adding to Eq. (3.31) we get

$$\begin{aligned} \frac{\partial F'}{\partial t'} - W'_0\left(1 + \epsilon A e^{i\omega' t'}\right)\frac{\partial F'}{\partial z'} + 2i\Omega F' &= U'_0 i\omega' \epsilon e^{i\omega' t'} + \upsilon \frac{\partial^2 F'}{\partial z'^2} \\ &+ 2i\Omega U'_0\left(1 + \epsilon e^{i\omega' t'}\right) + \alpha^* \frac{\partial^3 F'}{\partial z'^2 \partial t'} \\ &- \alpha^* W'_0\left(1 + \epsilon A e^{i\omega' t'}\right)\frac{\partial^3 F'}{\partial z'^3}. \end{aligned} \tag{3.33}$$

The boundary conditions in terms of $F'$ can be written as

$$\begin{aligned} F' &= 0 \quad \text{at} \quad z' = 0, \\ F' &= U'_o\left(1 + \epsilon e^{i\omega' t'}\right) \quad \text{as} \quad z' \longrightarrow \infty, \end{aligned} \tag{3.34}$$

where

$$F' = u' + iv'. \qquad (3.35)$$

Introducting the non-dimensional variables the boundary value problem consisting of Eq. (3.33) and conditions (3.34) yields

$$\begin{aligned}
\eta &= \frac{z'W_o'}{\upsilon}, \quad t = \frac{W_o'^2 t'}{4\upsilon}, \quad \omega = \frac{4\upsilon\omega'}{W_o'^2}, \quad U = \frac{U'}{U_o'}, \\
u &= \frac{u'}{U_o'}, \quad v = \frac{v'}{U_o'}, \quad F = \frac{F'}{U_o'},
\end{aligned}$$

$$\begin{aligned}
\frac{1}{4}\frac{\partial F}{\partial t} - \left(1 + \epsilon A e^{i\omega t}\right)\frac{\partial F}{\partial \eta} + 2i\Omega\frac{\upsilon}{W_0'^2}F &= \frac{1}{4}\left(i\omega\epsilon e^{i\omega t}\right) \\
&+2i\Omega\frac{\upsilon}{W_0'^2}\left(1 + \epsilon e^{i\omega t}\right) + \frac{\partial^2 F}{\partial \eta^2} \\
&+\alpha\left(\frac{1}{4}\frac{\partial^3 F}{\partial \eta^2 \partial t} - \left(1 + \epsilon A e^{i\omega t}\right)\frac{\partial^3 F}{\partial \eta^3}\right),
\end{aligned} \qquad (3.36)$$

$$\begin{aligned}
F &= 0 \quad \text{at} \quad \eta = 0, \\
F &\longrightarrow 1 + \epsilon e^{i\omega t} \quad \text{as} \quad \eta \longrightarrow \infty,
\end{aligned} \qquad (3.37)$$

where

$$\alpha = \frac{\alpha^* W_o'^2}{\upsilon^2}.$$

## 3.3 Analytic Solution

The solution of Eq. (3.36) subject to conditions (3.37) is written as

$$F'(z', t') = U_0'(\phi_1(z') + \epsilon e^{i\omega' t'}\phi_2(z')). \qquad (3.38)$$

With the help of non-dimensional quantities Eq. (3.38) can be written as

$$F(\eta, t) = \phi_1(\eta) + \epsilon e^{i\omega t}\phi_2(\eta). \tag{3.39}$$

Using above equation into Eqs. (3.36), (3.37) and separating the harmonic and non-harmonic terms by neglecting the coefficient of $\epsilon^2$ we obtain

$$\alpha\frac{d^3\phi_1}{d\eta^3} - \frac{d^2\phi_1}{d\eta^2} - \frac{d\phi_1}{d\eta} + iN\phi_1 = iN, \tag{3.40}$$

$$\alpha\frac{d^3\phi_2}{d\eta^3} - \left(1 + \frac{i\alpha\omega}{4}\right)\frac{d^2\phi_2}{d\eta^2} - \frac{d\phi_2}{d\eta} + iN_1\phi_2 = iN_1 + A\frac{d\phi_1}{d\eta} - \alpha A\frac{d^3\phi_1}{d\eta^3}. \tag{3.41}$$

The corresponding boundary conditions are

$$\begin{aligned} \phi_1 &= 0 \quad \text{at} \quad \eta = 0, \\ \phi_1 &\to 1 \quad \text{as} \quad \eta \longrightarrow \infty, \end{aligned} \tag{3.42}$$

$$\begin{aligned} \phi_2 &= 0 \quad \text{at} \quad \eta = 0, \\ \phi_2 &\longrightarrow 1 \quad \text{as} \quad \eta \longrightarrow \infty, \end{aligned} \tag{3.43}$$

where

$$N = \frac{2\Omega\nu}{W_0'^2}, \qquad N_1 = N + \frac{\omega}{4}.$$

It is worth emphasizing that the equations (3.40) and (3.41) for second grade fluid are third order (one order higher than the Navier-stokes equation). Thus, one needs three conditions for the unique solution where as two conditions are prescribed. One possible way to overcome this difficulty is to employ a perturbation analysis [20 – 23] and write the solution as follows

$$\phi_1 = \phi_{01} + \alpha\phi_{11} + o\left(\alpha^2\right), \tag{3.44}$$

$$\phi_2 = \phi_{02} + \alpha\phi_{12} + o\left(\alpha^2\right). \tag{3.45}$$

Substituting Eqs. (3.44) and (3.45) into Eqs. (3.40) – (3.43) and equating the coefficients of $\alpha^0$, $\alpha^1$ and neglecting the coefficient of $\alpha^2$ we get

$o\left(\alpha^0\right)$

$$\frac{d^2\phi_{01}}{d\eta^2} + \frac{d\phi_{01}}{d\eta} - iN\phi_{01} = -iN, \tag{3.46}$$

$$\phi_{01}\left(0\right) = 0, \qquad \phi_{01}\left(\infty\right) = 1,$$

$o\left(\alpha^1\right)$

$$\frac{d^2\phi_{11}}{d\eta^2} + \frac{d\phi_{11}}{d\eta} - iN\phi_{11} = \frac{d^3\phi_{01}}{d\eta^3}, \tag{3.47}$$

$$\phi_{11}\left(0\right) = 0, \qquad \phi_{11}\left(\infty\right) = 0,$$

$o\left(\alpha^0\right)$

$$\frac{d^2\phi_{02}}{d\eta^2} + \frac{d\phi_{02}}{d\eta} - iN_1\phi_{02} = -iN_1 - A\frac{d\phi_{01}}{d\eta}, \tag{3.48}$$

$$\phi_{02}\left(0\right) = 0, \qquad \phi_{02}\left(\infty\right) = 1,$$

$o\left(\alpha^1\right)$

$$\frac{d^2\phi_{12}}{d\eta^2} + \frac{d\phi_{12}}{d\eta} - iN_1\phi_{12} = \frac{d^3\phi_{02}}{d\eta^3} - \frac{i\omega}{4}\frac{d^2\phi_{02}}{d\eta^2} + A\frac{d^3\phi_{01}}{d\eta^3} - A\frac{d\phi_{11}}{d\eta}, \tag{3.49}$$

$$\phi_{12}\left(0\right) = 0, \qquad \phi_{12}\left(\infty\right) = 0.$$

The solutions of the above systems are

$$\phi_{01} = 1 - e^{-h_1\eta}, \tag{3.50}$$

$$\phi_{11} = -\eta L_1 e^{-h_1\eta}, \tag{3.51}$$

$$\phi_{02} = 1 - S_1 e^{-g\eta} - (1 - S_1)\, e^{-h_1\eta}, \tag{3.52}$$

$$\phi_{12} = c_5 e^{-g\eta} - \eta M_1 e^{-g\eta} - (\eta c_{3+} c_5)\, e^{-h_1\eta}. \tag{3.53}$$

In above equations

$$h_1 = h_{1r} + ih_{1i} = \frac{1 + \sqrt{1 + 4iN}}{2}, \tag{3.54}$$

$$h_{1r} = \frac{1}{2} + \frac{1}{2}\left[\frac{1}{2}\left(1 + \sqrt{1 + 16N^2}\right)\right]^2, \quad h_{1i} = \frac{1}{2}\left[\frac{1}{2}\left(-1 + \sqrt{1 + 16N^2}\right)\right]^2,$$

$$a_1 = \left[\frac{1}{2}\left(1 + \sqrt{1 + 16N^2}\right)\right]^2, \quad b_1 = \left[\frac{1}{2}\left(-1 + \sqrt{1 + 16N^2}\right)\right]^2,$$

$$r_1 = a_1^2 + b_1^2 = \sqrt{1 + 16N^2},$$

$$g = g_r + ig_i = \frac{1 + \sqrt{1 + 4iN_1}}{2}, \tag{3.55}$$

$$g_r = \frac{1}{2} + \frac{1}{2}\left[\frac{1}{2}\left(1 + \sqrt{1 + 16N_1^2}\right)\right]^2, \quad g_i = \frac{1}{2}\left[\frac{1}{2}\left(-1 + \sqrt{1 + 16N_1^2}\right)\right]^2,$$

$$a_2 = \left[\frac{1}{2}\left(1 + \sqrt{1 + 16N_1^2}\right)\right]^2, \quad b_2 = \left[\frac{1}{2}\left(-1 + \sqrt{1 + 16N_1^2}\right)\right]^2,$$

$$r_2 = a_2^2 + b_2^2 = \sqrt{1 + 16N_1^2},$$

$$S_1 = S_{1r} + iS_{1i} = 1 - \frac{4iAh_1}{\omega}, \tag{3.56}$$

$$S_{1r} = 1 + \frac{4Ah_{1i}}{\omega} \qquad S_{1i} = \frac{4Ah_{1r}}{\omega},$$

$$L_1 = L_{1r} + iL_{1i} = \frac{h_1^3}{\sqrt{1+4iN}}, \tag{3.57}$$

$$L_{1r} = \frac{1}{r_1}\left[\begin{array}{c} \frac{1}{4}\left(\frac{r_1+1}{2}\right)^{\frac{1}{2}}\left(1+\left(\frac{r_1+1}{2}\right)\right)+\frac{1}{2}-2N^2 \\ -\frac{7}{4}\left(\frac{r_1-1}{2}\right)\left(\frac{r_1+1}{2}\right)^{\frac{1}{2}} \end{array}\right],$$

$$L_{1i} = \frac{1}{r_1}\left[\begin{array}{c} \frac{1}{4}\left(\frac{r_1-1}{2}\right)^{\frac{1}{2}}\left(1-\left(\frac{r_1-1}{2}\right)\right)+\frac{5N}{2} \\ +\frac{3}{2}\left(\frac{r_1+1}{2}\right)\left(\frac{r_1-1}{2}\right)^{\frac{1}{2}} \end{array}\right],$$

$$M_1 = M_{1r} + iM_{1i} = \frac{g^2\left(g+\frac{i\omega}{4}\right)\left(1-\frac{4iAh_1}{\omega}\right)}{\sqrt{1+4iN_1}}, \tag{3.58}$$

$$M_r = \frac{1}{r_2}\left[\begin{array}{c} \frac{1}{4}\left(\frac{r_2+1}{2}\right)^{\frac{1}{2}}+\frac{1}{8}\left(r_2+1\right)\left(\frac{r_2+1}{2}\right)^{\frac{1}{2}}+\frac{r_2}{2} \\ +\frac{3}{8}\left(r_2-1\right)\left(\frac{r_2+1}{2}\right)^{\frac{1}{2}}-N_1^2\left(\frac{r_2+1}{2}\right)^{\frac{1}{2}} \\ +\frac{Ah_{1i}}{\omega}\left(\begin{array}{c} \left(\frac{r_2+1}{2}\right)^{\frac{1}{2}}+\left(\frac{r_2+1}{2}\right)\left(\frac{r_2+1}{2}\right)^{\frac{1}{2}} \\ +2r_2+\frac{3}{2}\left(r_2-1\right)\left(\frac{r_2+1}{2}\right)^{\frac{1}{2}} \\ -4N_1^2\left(\frac{r_2+1}{2}\right)^{\frac{1}{2}} \end{array}\right) \\ +\frac{Ah_{1r}}{\omega}\left(\begin{array}{c} 2N_1\left(r_2+1\right)+\frac{3}{2}\left(r_2+1\right)\left(\frac{r_2-1}{2}\right)^{\frac{1}{2}} \\ -\left(\frac{r_2-1}{2}\right)^{\frac{1}{2}}+\left(\frac{r_2-1}{2}\right)\left(\frac{r_2-1}{2}\right)^{\frac{1}{2}} \\ +4N_1^2\left(\frac{r_2-1}{2}\right)^{\frac{1}{2}} \end{array}\right) \end{array}\right],$$

$$M_i = \frac{1}{r_2}\left[\begin{array}{c} 2N_1 r_2 - \frac{1}{4}\left(\frac{r_2-1}{2}\right)^{\frac{1}{2}} + \frac{3}{8}\left(r_2+1\right)\left(\frac{r_2+1}{2}\right)^{\frac{1}{2}} \\ +\frac{3}{8}\left(r_2-1\right)\left(\frac{r_2-1}{2}\right)^{\frac{1}{2}} + N_1^2\left(\frac{r_2-1}{2}\right)^{\frac{1}{2}} \\ +\frac{Ah_{1i}}{\omega}\left(\begin{array}{c} 8N_1 + 4N_1 r_2 + \left(\frac{r_2-1}{2}\right)^{\frac{1}{2}} \\ +\left(\frac{r_2-1}{2}\right)\left(\frac{r_2-1}{2}\right)^{\frac{1}{2}} + 4N_1^2\left(\frac{r_2-1}{2}\right)^{\frac{1}{2}} \\ +\frac{5}{2}\left(r_2+1\right)\left(\frac{r_2-1}{2}\right)^{\frac{1}{2}} \end{array}\right) \\ +\frac{Ah_{1r}}{\omega}\left(\begin{array}{c} -3N_1\left(r_2-1\right) - 2r_2 \\ -\left(\frac{r_2+1}{2}\right)^{\frac{1}{2}}\left(1+\left(\frac{r_2+1}{2}\right)\right) \\ +4N_1^2\left(\frac{r_2+1}{2}\right)^{\frac{1}{2}} \end{array}\right) \end{array}\right],$$

$$c_5 = c_{5r} + \iota c_{5i} = \left(c_{1r} + c_{2r} + c_{4r}\right) + i\left(c_{1i} + c_{2i} + c_{4i}\right), \tag{3.59}$$

$$c_1 = c_{1r} + ic_{1i} = \frac{4h_1^3}{i\omega}\left(A - \left(1 - S_1\right)\right),$$

$$c_{1r} = 4\left(h_{1i}^3 - 3h_{1r}^2 h_{1i}\right)\left(A + \frac{4Ah_{1i}}{\omega}\right) - 16\frac{Ah_{1r}}{\omega}\left(h_{1r}^3 - 3h_{1i}^2 h_{1r}\right),$$

$$c_{1i} = -4\left(h_{1r}^3 - 3h_{1i}^2 h_{1r}\right)\left(A + \frac{4Ah_{1i}}{\omega}\right) - 16\frac{Ah_{1r}}{\omega}\left(h_{1r}^3 - 3h_{1r}^2 h_{1i}\right),$$

$$c_2 = c_{2r} + ic_{2i} = \frac{4A\left(h_1^3 + L_1\right)}{i\omega},$$

$$c_{2r} = \frac{4A}{\omega}\left(L_{1i}\left(h_{1r}^3 - 3h_{1i}^2 h_{1r}\right) - L_{1r}\left(h_{1i}^3 - 3h_{1r}^2 h_{1i}\right)\right),$$

$$c_{2i} = -\frac{4A}{\omega}\left(L_{1r}\left(h_{1r}^3 - 3h_{1i}^2 h_{1r}\right) + L_{1i}\left(h_{1i}^3 - 3h_{1r}^2 h_{1i}\right)\right),$$

$$c_3 = c_{3r} + ic_{3i} = \frac{4Ah_1L_1}{i\omega}, \quad c_{3r} = \frac{4A}{\omega}\left(h_{1i}L_{1r} + h_{1r}L_{1i}\right),$$

$$c_{3i} = -\frac{4A}{\omega}\left(h_{1r}L_{1r} - h_{1i}L_{1i}\right), \quad c_4 = c_{4r} + ic_{4i} = \frac{16Ah_1L_1\left(1-2h_1\right)}{\omega^2},$$

$$c_{4r} = \frac{16A}{\omega^2}\left(\left(1-2h_{1r}\right)\left(h_{1r}L_{1r} - h_{1i}L_{1i}\right) + 2h_{1i}\left(h_{1r}L_{1i} + h_{1i}L_{1r}\right)\right),$$

$$c_{4i} = \frac{16A}{\omega^2}\left(\left(1-2h_{1r}\right)\left(h_{1r}L_{1i} + h_{1i}L_{1r}\right) - 2h_{1i}\left(h_{1r}L_{1r} - h_{1i}L_{1i}\right)\right).$$

From Eqs. (3.44), (3.45) and (3.50) – (3.53) we get

$$\phi_1 = 1 - \left(1 + \alpha\eta L_1\right)e^{-h_1\eta}, \tag{3.60}$$

$$\phi_2 = 1 - S_1e^{-g\eta} - \left(1 - S_1\right)e^{-h_1\eta} + \alpha\left[c_5e^{-g\eta} - \eta M_1e^{-g\eta} - \left(\eta c_{3+}c_5\right)e^{-h_1\eta}\right], \tag{3.61}$$

and so from Eq. (3.39)

$$F = 1 - \left(1 + \alpha\eta L_1\right)e^{-h_1\eta} + \epsilon e^{i\omega t}\left[\begin{array}{c} 1 - S_1e^{-g\eta} - \left(1 - S_1\right)e^{-h_1\eta} \\ +\alpha\left\{c_5e^{-g\eta} - \eta M_1e^{-g\eta} - \left(\eta c_{3+}c_5\right)e^{-h_1\eta}\right\}\end{array}\right]. \tag{3.62}$$

Using Eq. (3.35), the above equation gives

$$u = u_1 + \epsilon e^{i\omega t}u_2, \tag{3.63}$$

$$v = v_1 + \epsilon e^{i\omega t}v_2. \tag{3.64}$$

Substitution of Eqs. (3.60) – (3.62) and (3.35) into Eqs. (3.63) and (3.64) yields

$$\begin{aligned} u &= u_1 + \epsilon e^{i\omega t} u_2 = 1 - e^{-h_{1r}\eta}\left((1+\alpha\eta L_{1r})\cos h_{1i}\eta + \alpha\eta L_{1i}\sin h_{1i}\eta\right) \\ &+\epsilon e^{i\omega t}\begin{bmatrix} 1 - e^{-g_r\eta}\left(\left(1-\frac{4Ah_{1i}}{\omega}\right)\cos g_i\eta + \frac{4Ah_{1r}}{\omega}\sin g_i\eta\right) \\ +e^{-h_{1r}\eta}\left(\frac{4Ah_{1i}}{\omega}\cos h_{1i}\eta - \frac{4Ah_{1r}}{\omega}\sin h_{1i}\eta\right) \\ +\alpha e^{-g_r\eta}\left(c_{5r}\cos g_i\eta + c_{5i}\sin g_i\eta\right) \\ -\alpha\eta e^{-g_r\eta}\left(M_{1r}\cos g_i\eta + M_{1i}\sin g_i\eta\right) \\ -\alpha e^{-h_{1r}\eta}\left((\eta c_{3r}+c_{5r})\cos h_{1i}\eta + (\eta c_{3i}+c_{5i})\sin h_{1i}\eta\right) \end{bmatrix}, \end{aligned} \tag{3.65}$$

$$\begin{aligned} v &= v_1 + \epsilon e^{i\omega t} v_2 = e^{-h_{1r}\eta}\left((1+\alpha\eta L_{1r})\sin h_{1i}\eta + \alpha\eta L_{1i}\cos h_{1i}\eta\right) \\ &+\epsilon e^{i\omega t}\begin{bmatrix} e^{-g_r\eta}\left(\left(1-\frac{4Ah_{1i}}{\omega}\right)\sin g_i\eta - \frac{4Ah_{1r}}{\omega}\cos g_i\eta\right) \\ -e^{-h_{1r}\eta}\left(\frac{4Ah_{1i}}{\omega}\sin h_{1i}\eta + \frac{4Ah_{1r}}{\omega}\cos h_{1i}\eta\right) \\ +\alpha e^{-g_r\eta}\left(c_{5i}\cos g_i\eta - c_{5r}\sin g_i\eta\right) \\ -\alpha\eta e^{-g_r\eta}\left(M_{1i}\cos g_i\eta - M_{1r}\sin g_i\eta\right) \\ -\alpha e^{-h_{1r}\eta}\left((\eta c_{3i}+c_{5i})\cos h_{1i}\eta - (\eta c_{3r}+c_{5r})\sin h_{1i}\eta\right) \end{bmatrix}. \end{aligned} \tag{3.66}$$

The drag $T_{x'z'}$ and lateral stress $T_{y'z'}$ at the plate are

$$T_{x'z'} = \frac{\partial u'}{\partial z'} + \alpha^*\left(\frac{\partial^2 u'}{\partial z'\partial t'} + w'\frac{\partial^2 u'}{\partial z'^2}\right), \tag{3.67}$$

$$T_{y'z'} = \frac{\partial v'}{\partial z'} + \alpha^*\left(\frac{\partial^2 v'}{\partial z'\partial t'} + w'\frac{\partial^2 v'}{\partial z'^2}\right), \tag{3.68}$$

which in non-dimensional form can be written as

$$T_{xz} = \frac{P'_{x'z'}}{U'_0 W'_0 \rho'} = \frac{\partial u}{\partial \eta} - \frac{\alpha}{4}\left[\frac{\partial^2 u}{\partial\eta\partial t} - 4\left(1+\epsilon A e^{i\omega t}\right)\frac{\partial^2 u}{\partial \eta^2}\right], \tag{3.69}$$

$$T_{yz} = \frac{P'_{y'z'}}{U'_0 W'_0 \rho'} = \frac{\partial v}{\partial \eta} - \frac{\alpha}{4}\left[\frac{\partial^2 v}{\partial\eta\partial t} - 4\left(1+\epsilon A e^{i\omega t}\right)\frac{\partial^2 v}{\partial \eta^2}\right]. \tag{3.70}$$

The above equations (3.69) and (3.70) after using Eqs. (3.65) and (3.66) give

$$T_{xz} = \alpha\left(h_{1i}^2 - h_{1r}^2\right) - h_{1r} - \alpha L_{1r} + \epsilon e^{i\omega t}\left[\begin{array}{c} g_r - \frac{4A}{\omega}\left(h_{1i}g_r + h_{1r}g_i\right) - \frac{8Ah_{1i}h_{1r}}{\omega} \\ -\alpha\left(g_r c_{5r} - g_i c_{5i}\right) + \alpha A\left(h_{1i}^2 - h_{1r}^2\right) \\ -\alpha M_{1r} + \alpha\left(h_{1r}c_{5r} - h_{1i}c_{5i}\right) - \alpha c_{3r} \\ -i\alpha\omega\left(\frac{g_r}{4} - \frac{A}{\omega}\left(h_{1i}g_r + h_{1r}g_i\right) - \frac{2Ah_{1i}h_{1r}}{\omega}\right) \\ +\alpha\left(\begin{array}{c} \left(g_i^2 - g_r^2\right)\left(1 - \frac{4Ah_{1i}}{\omega}\right) + \frac{8Ah_{1r}g_i g_r}{\omega} + \\ \frac{12Ah_{1i}h_{1r}^2}{\omega} - \frac{4Ah_{1i}^3}{\omega} \end{array}\right) \end{array}\right], \tag{3.71}$$

$$T_{yz} = h_{1i} - \alpha L_{1i} - 2\alpha h_{1i}h_{1r} + \epsilon e^{i\omega t}\left[\begin{array}{c} g_i + \frac{4A}{\omega}\left(h_{1r}g_r - h_{1i}g_i\right) + \frac{4A}{\omega}\left(h_{1r}^2 - h_{1i}^2\right) \\ -\alpha\left(g_r c_{5i} + g_i c_{5r}\right) - 2\alpha A h_{1i}h_{1r} \\ -\alpha M_{1i} + \alpha\left(h_{1r}c_{5i} + h_{1i}c_{5r}\right) - \alpha c_{3i} \\ -i\alpha\omega\left(g_i + \frac{4A}{\omega}\left(h_{1r}g_r - h_{1i}g_i\right) + \frac{4A}{\omega}\left(h_{1r}^2 - h_{1i}^2\right)\right) \\ +\alpha\left(\begin{array}{c} -2g_i g_r + \frac{8Ah_{1i}g_i g_r}{\omega} - \frac{4Ah_{1r}}{\omega}\left(g_r^2 - g_i^2\right) \\ +\frac{12Ah_{1r}h_{1i}^2}{\omega} - \frac{4Ah_{1r}^3}{\omega} \end{array}\right) \end{array}\right]. \tag{3.72}$$

The Eqs. (3.71) and (3.72) can br written as

$$T_{xz} = \alpha\left(h_{1i}^2 - h_{1r}^2\right) - h_{1r} - \alpha L_{1r} + \epsilon\left|B_1\right|\cos\left(\omega t + \gamma\right), \tag{3.73}$$

$$T_{yz} = h_{1i} - \alpha L_{1i} - 2\alpha h_{1i}h_{1r} + \epsilon\left|B_2\right|\cos\left(\omega t + \delta\right), \tag{3.74}$$

where

$$\gamma = \tan^{-1}\left(\frac{B_{1i}}{B_{1r}}\right), \quad \delta = \tan^{-1}\left(\frac{B_{2i}}{B_{2r}}\right),$$

$$B_1 = B_{1r} + iB_{1i} = \left[ \begin{array}{c} g_r - \frac{4A}{\omega}\left(h_{1i}g_r + h_{1r}g_i\right) - \frac{8Ah_{1i}h_{1r}}{\omega} \\ -\alpha\left(g_r c_{5r} - g_i c_{5i}\right) + \alpha A\left(h_{1i}^2 - h_{1r}^2\right) \\ -\alpha M_{1r} + \alpha\left(h_{1r}c_{5r} - h_{1i}c_{5i}\right) - \alpha c_{3r} \\ -i\alpha\omega\left(\frac{g_r}{4} - \frac{A}{\omega}\left(h_{1i}g_r + h_{1r}g_i\right) - \frac{2Ah_{1i}h_{1r}}{\omega}\right) \\ +\alpha\left( \begin{array}{c} \left(g_i^2 - g_r^2\right)\left(1 - \frac{4Ah_{1i}}{\omega}\right) + \frac{8Ah_{1r}g_i g_r}{\omega} + \\ \frac{12Ah_{1i}h_{1r}^2}{\omega} - \frac{4Ah_{1i}^3}{\omega} \end{array} \right) \end{array} \right], \tag{3.1}$$

$$B_{1r} = \left[ \begin{array}{c} g_r - \frac{4A}{\omega}\left(h_{1i}g_r + h_{1r}g_i\right) - \frac{8Ah_{1i}h_{1r}}{\omega} \\ -\alpha\left(g_r c_{5r} - g_i c_{5i}\right) + \alpha A\left(h_{1i}^2 - h_{1r}^2\right) \\ -\alpha M_{1r} + \alpha\left(h_{1r}c_{5r} - h_{1i}c_{5i}\right) - \alpha c_{3r} \\ +\alpha\left( \begin{array}{c} \left(g_i^2 - g_r^2\right)\left(1 - \frac{4Ah_{1i}}{\omega}\right) + \frac{8Ah_{1r}g_i g_r}{\omega} + \\ \frac{12Ah_{1i}h_{1r}^2}{\omega} - \frac{4Ah_{1i}^3}{\omega} \end{array} \right) \end{array} \right], \tag{3.76a}$$

$$B_{1i} = \alpha\omega\left(\frac{g_r}{4} - \frac{A}{\omega}\left(h_{1i}g_r + h_{1r}g_i\right) - \frac{2Ah_{1i}h_{1r}}{\omega}\right), \tag{3.76b}$$

$$B_2 = B_{2r} + iB_{2i} = \left[ \begin{array}{c} g_i + \frac{4A}{\omega}\left(h_{1r}g_r - h_{1i}g_i\right) + \frac{4A}{\omega}\left(h_{1r}^2 - h_{1i}^2\right) \\ -\alpha\left(g_r c_{5i} + g_i c_{5r}\right) - 2\alpha A h_{1i}h_{1r} \\ -\alpha M_{1i} + \alpha\left(h_{1r}c_{5i} + h_{1i}c_{5r}\right) - \alpha c_{3i} \\ -i\alpha\omega\left(g_i + \frac{4A}{\omega}\left(h_{1r}g_r - h_{1i}g_i\right) + \frac{4A}{\omega}\left(h_{1r}^2 - h_{1i}^2\right)\right) \\ +\alpha\left( \begin{array}{c} -2g_i g_r + \frac{8Ah_{1i}g_i g_r}{\omega} - \frac{4Ah_{1r}}{\omega}\left(g_r^2 - g_i^2\right) \\ +\frac{12Ah_{1r}h_{1i}^2}{\omega} - \frac{4Ah_{1r}^3}{\omega} \end{array} \right) \end{array} \right], \tag{3.2}$$

$$B_{2r} = \left[ \begin{array}{c} g_i + \frac{4A}{\omega}\left(h_{1r}g_r - h_{1i}g_i\right) + \frac{4A}{\omega}\left(h_{1r}^2 - h_{1i}^2\right) \\ -\alpha\left(g_r c_{5i} + g_i c_{5r}\right) - 2\alpha A h_{1i}h_{1r} \\ -\alpha M_{1i} + \alpha\left(h_{1r}c_{5i} + h_{1i}c_{5r}\right) - \alpha c_{3i} \\ +\alpha\left( \begin{array}{c} -2g_i g_r + \frac{8Ah_{1i}g_i g_r}{\omega} - \frac{4Ah_{1r}}{\omega}\left(g_r^2 - g_i^2\right) \\ +\frac{12Ah_{1r}h_{1i}^2}{\omega} - \frac{4Ah_{1r}^3}{\omega} \end{array} \right) \end{array} \right], \tag{3.78a}$$

$$B_{2i} = \alpha\omega\left(g_i + \frac{4A}{\omega}\left(h_{1r}g_r - h_{1i}g_i\right) + \frac{4A}{\omega}\left(h_{1r}^2 - h_{1i}^2\right)\right). \tag{3.78b}$$

We now proceed to derive the energy equation appropriate for the problem under consideration. We start with the energy equation (3.5). It follows from Eqs. (3.5) – (3.9) and $\mathbf{L}$ =grad$\mathbf{V}$

that

$$\mathbf{T}\cdot\mathbf{L} = \mu\left[\left(\frac{\partial u'}{\partial z'}\right)^2+\left(\frac{\partial v'}{\partial z'}\right)^2\right] \tag{3.79}$$
$$+\alpha\left[\begin{array}{c}\frac{\partial u'}{\partial z'}\left(\frac{\partial^2 u'}{\partial t'\partial z'}+w'\frac{\partial^2 u'}{\partial z'^2}\right)\\ +\frac{\partial v'}{\partial z'}\left(\frac{\partial^2 v'}{\partial t'\partial z'}+w'\frac{\partial^2 v'}{\partial z'^2}\right)\end{array}\right].$$

Following the thermodynamical considerations given in Dunn and Fosdick [24] for fluids of second grade and representing $\mathbf{q}$ by Fourier's law with a constant thermal conductivity $k$, Eq. (3.5) reduces to

$$\rho' c\left[\frac{\partial T}{\partial t'}+w'\frac{\partial T}{\partial z'}\right]-k\frac{\partial^2 T}{\partial z'^2} = \mu\left[\left(\frac{\partial u'}{\partial z'}\right)^2+\left(\frac{\partial v'}{\partial z'}\right)^2\right] \tag{3.80}$$
$$+\alpha_1\left[\begin{array}{c}\frac{\partial u'}{\partial z'}\left(\frac{\partial^2 u'}{\partial t'\partial z'}+w'\frac{\partial^2 u'}{\partial z'^2}\right)\\ +\frac{\partial v'}{\partial z'}\left(\frac{\partial^2 v'}{\partial t'\partial z'}+w'\frac{\partial^2 v'}{\partial z'^2}\right)\end{array}\right],$$

where $c$ is the specific heat. The boundary conditions for the temperature are

$$T=T_0 \quad \text{at} \quad z'=0, \tag{3.81}$$

$$T\to T_\infty \quad \text{as} \quad z'\to\infty.$$

Using

$$\theta=\frac{T-T_0}{T_\infty-T_0}, \tag{3.82}$$

equation (3.80) and boundary conditions (3.81) become

$$-\frac{\partial^2\theta}{\partial\eta^2}-P_r\left(1+\epsilon A e^{i\omega t}\right)\frac{\partial\theta}{\partial\eta}+\frac{P_r}{4}\frac{\partial\theta}{\partial t} = E_c\left[\left(\frac{\partial u}{\partial\eta}\right)^2+\left(\frac{\partial v}{\partial\eta}\right)^2\right] \tag{3.83}$$

$$
+P\left[\begin{array}{c}\frac{\partial u}{\partial \eta}\frac{\partial^2 u}{\partial \eta \partial t}+\frac{\partial v}{\partial \eta}\frac{\partial^2 v}{\partial \eta \partial t}\\ -\left(1+\epsilon A e^{i\omega t}\right)\frac{\partial u}{\partial \eta}\frac{\partial^2 u}{\partial \eta^2}\\ -\left(1+\epsilon A e^{i\omega t}\right)\frac{\partial v}{\partial \eta}\frac{\partial^2 v}{\partial \eta^2}\end{array}\right],
$$

$$
\theta = 0 \qquad \text{at} \qquad \eta = 0, \tag{3.84}
$$

$$
\theta \to 1 \qquad \text{at} \qquad \eta \to \infty,
$$

in which

$$
\begin{aligned}
\text{the Prandtl number } P_r &= \frac{\mu c}{k}, \text{ the Eckert number } E_c = \frac{k^* U_0'^2}{(T_\infty - T_0)} \\
\text{and } P &= \frac{\alpha U_0'^2 \mu}{k\,(T_\infty - T_0)}.
\end{aligned}
$$

To solve Eq. (3.83) with boundary conditions (3.84) we write

$$
\theta = \theta_0 + \epsilon e^{i\omega t}\theta_1. \tag{3.85}
$$

Substituting Eq. (3.85) into Eq. (3.83) and boundary conditions (3.84) and equating the coefficients of the harmonic and non-harmonic terms after neglecting the coefficients of $\epsilon^2$ we get

$$
\begin{aligned}
\frac{d^2\theta_0}{d\eta^2} + P_r\frac{d\theta_0}{d\eta} &= -E_c\left[\left(\frac{du_1}{d\eta}\right)^2 + \left(\frac{dv_1}{d\eta}\right)^2\right] \\
&\quad + P\left[\frac{du_1}{d\eta}\frac{d^2u_1}{d\eta^2} + \frac{dv_1}{d\eta}\frac{d^2v_1}{d\eta^2}\right],
\end{aligned} \tag{3.86}
$$

$$
\theta_0 = 0 \qquad \text{at} \qquad \eta = 0, \tag{3.87}
$$

$$
\theta_0 \to 1 \qquad \text{as} \qquad \eta \to \infty,
$$

$$\frac{d^2\theta_1}{d\eta^2} + P_r\frac{d\theta_1}{d\eta} - \frac{P_r}{4}i\omega\theta_1 = -P_rA\frac{d\theta_0}{d\eta} - 2E_c\left[\frac{du_1}{d\eta}\frac{du_2}{d\eta} + \frac{dv_1}{d\eta}\frac{dv_2}{d\eta}\right] \tag{3.88}$$

$$-P\left[\begin{array}{l} i\omega\left(\frac{du_1}{d\eta}\frac{du_2}{d\eta} + \frac{dv_1}{d\eta}\frac{dv_2}{d\eta}\right) \\ -\left(\frac{du_1}{d\eta}\frac{d^2u_2}{d\eta^2} + \frac{dv_1}{d\eta}\frac{d^2v_2}{d\eta^2}\right) \\ -A\left(\frac{du_1}{d\eta}\frac{d^2u_1}{d\eta^2} + \frac{dv_1}{d\eta}\frac{d^2v_1}{d\eta^2}\right) \\ -\left(\frac{du_2}{d\eta}\frac{d^2u_1}{d\eta^2} + \frac{dv_2}{d\eta}\frac{d^2v_1}{d\eta^2}\right) \end{array}\right],$$

$$\theta_1 = 0 \qquad \text{at} \qquad \eta = 0, \tag{3.89}$$

$$\theta_1 \to 0 \qquad \text{as} \qquad \eta \to \infty.$$

Solving Eqs. (3.86) and (3.88) along with the boundary conditions (3.87) and (3.89) we obtain

$$\theta_0 = 1 - (1 + d_7)\, e^{-P_r\eta} + (d_7 + d_8\eta)\, e^{-2h_r\eta}, \tag{3.90}$$

$$\begin{aligned} \theta_1 = {} & -m_{16}e^{-f\eta} + (m_7 + m_9 + m_8\eta)\, e^{-2h_r\eta} \\ & + (m_{10} + m_{14} + m_{12}\eta)\, e^{-(h_r+g_r)\eta}\cos(h_i - g_i)\,\eta \\ & + (m_{11} + m_{15} + m_{13}\eta)\, e^{-(h_r+g_r)\eta}\sin(h_i - g_i)\,\eta, \end{aligned} \tag{3.91}$$

where

$$d_1 = -E_c\left(h_{1r}^2 + h_{1i}^2 - 2\alpha L_{1i}h_{1i}\right), \quad d_2 = -2E_c\alpha L_{1r}\left(h_{1r}^2 + h_{1i}^2\right),$$

$$d_3 = P\left(-h_{1r}^3 + 3\alpha L_{1r}h_{1r}^2 - h_{1r}h_{1i}^2 + 2\alpha L_{1i}h_{1i}h_{1r} + \alpha L_{1r}h_{1i}^2\right),$$

$$d_4 = -2P\alpha L_{1r}h_{1r}\left(h_{1r}^2 + h_{1i}^2\right), \quad d_5 = d_1 + d_3, \quad d_6 = d_2 + d_4,$$

$$d_7 = \frac{d_5}{4h_{1r}^2 - 2P_r h_{1r}} - \frac{d_6\left(4h_{1r} - P_r\right)}{\left(4h_{1r}^2 - 2P_r h_{1r}\right)^2}, \quad d_8 = \frac{d_6}{\left(4h_{1r}^2 - 2P_r h_{1r}\right)},$$

$$d_9 = \begin{bmatrix} -\frac{4Ah_{1r}}{\omega}\left(h_{1r}g_r + h_{1i}^2\right) + \left(1 - \frac{4Ah_{1i}}{\omega}\right)\left(h_{1i}g_r - h_{1r}g_i\right) \\ +\alpha\left(h_{1r}\left(g_r c_{5i} + g_i c_{5r}\right) - h_{1i}\left(g_r c_{5r} - g_i c_{5i}\right)\right) \\ +\alpha\left(h_{1r}M_{1i} - h_{1i}M_{1r}\right) + \frac{4\alpha Ah_{1r}}{\omega}\left(g_r L_{1r} - g_i L_{1i}\right) \\ +\alpha\left(1 - \frac{4Ah_{1i}}{\omega}\right)\left(g_i L_{1r} - g_r L_{1i}\right) \end{bmatrix},$$

$$d_{10} = \begin{bmatrix} -\frac{4\alpha Ah_{1r}^2}{\omega}\left(L_{1r}g_r + L_{1i}g_i\right) - \alpha h_{1r}\left(1 - \frac{4Ah_{1i}}{\omega}\right)\left(L_{1r}g_i - L_{1i}g_r\right) \\ -\alpha\left(h_{1r}\left(g_r M_{1i} + g_i M_{1r}\right) - h_{1i}\left(g_r M_{1r} - g_i M_{1i}\right)\right) \\ -\frac{4\alpha Ah_{1r}h_{1i}}{\omega}\left(g_i L_{1r} - g_r L_{1i}\right) + \alpha h_{1i}\left(1 - \frac{4Ah_{1i}}{\omega}\right)\left(g_r L_{1r} - g_i L_{1i}\right) \end{bmatrix},$$

$$d_{11} = \begin{bmatrix} -\frac{4Ah_{1r}h_{1i}}{\omega}\left(h_{1r} - g_r\right) + \alpha\left(1 - \frac{4Ah_{1i}}{\omega}\right)\left(L_{1r}g_r - L_{1i}g_i\right) \\ -\alpha\left(h_{1r}\left(g_r c_{5r} - g_i c_{5i}\right) - h_{1i}\left(g_r c_{5i} + g_i c_{5r}\right)\right) \\ +\frac{4\alpha Ah_{1r}}{\omega}\left(g_i L_{1r} - g_r L_{1i}\right) + \left(1 - \frac{4Ah_{1i}}{\omega}\right)\left(g_r h_{1r} + g_i h_{1i}\right) \\ -\alpha\left(M_{1r}h_{1r} + M_{1i}h_{1i}\right) \end{bmatrix},$$

$$d_{12} = \begin{bmatrix} -\frac{4\alpha Ah_r^2}{\omega}\left(L_r g_i - L_i g_r\right) + \alpha h_r\left(1 - \frac{4Ah_i}{\omega}\right)\left(L_r g_r + L_i g_i\right) \\ -\alpha\left(h_r\left(g_r M_r + g_i M_i\right) - h_i\left(g_r M_i + g_i M_r\right)\right) \\ -\frac{4\alpha Ah_r h_i}{\omega}\left(g_r L_r - g_i L_i\right) + \alpha h_i\left(1 - \frac{4Ah_i}{\omega}\right)\left(g_i L_r - g_r L_i\right) \end{bmatrix},$$

$$d_{13} = \begin{bmatrix} \frac{-8Ah_{1r}h_{1i}}{\omega}\left(h_{1r} - \alpha L_{1r}\right) + \alpha c_{5r}\left(h_{1r}^2 + h_{1i}^2\right) \\ +\frac{4Ah_{1i}}{\omega}\left(h_{1r}^2 - h_{1i}^2\right) - \frac{4\alpha AL_{1i}}{\omega}\left(h_{1r}^2 - h_{1i}^2\right) - \alpha\left(h_{1r}c_{3r} + h_{1i}c_{3i}\right) \end{bmatrix},$$

$$d_{14} = \begin{bmatrix} \frac{-8Ah_{1r}h_{1i}}{\omega}\left(h_{1r}L_{1r} - h_{1i}L_{1i}\right) + \frac{4\alpha A}{\omega}\left(h_{1r}^2 - h_{1i}^2\right)\left(h_{1r}L_{1i} - h_{1i}L_{1r}\right) \\ +\alpha c_{3r}\left(h_{1r}^2 + h_{1i}^2\right) \end{bmatrix},$$

$$d_{15} = \begin{bmatrix} \frac{4Ah_{1r}}{\omega}\left(g_r^2 - g_i^2\right)\left(h_{1r} - \alpha L_{1r}\right) + 2g_r g_i\left(1 - \frac{4Ah_{1i}}{\omega}\right)\left(h_{1r} - \alpha L_{1r}\right) \\ -2\alpha\left(h_{1r}\left(g_r M_{1i} - g_i M_{1r}\right) - h_{1i}\left(g_r M_{1r} - g_i M_{1i}\right)\right) \\ -\alpha\left(g_r^2 - g_i^2\right)\left(h_{1r}c_{5i} - h_{1i}c_{5r}\right) \\ -\left(1 - \frac{4Ah_{1i}}{\omega}\right)\left(g_r^2 - g_i^2\right)\left(h_{1i} - \alpha L_{1i}\right) \\ -2\alpha g_r g_i\left(h_{1r}c_{5r} + h_{1i}c_{5i}\right) + \frac{8Ah_{1r}g_r g_i}{\omega}\left(h_{1i} - \alpha L_{1i}\right) \end{bmatrix},$$

$$d_{16} = \begin{bmatrix} \frac{4\alpha Ah_{1r}}{\omega}\left(g_r^2 - g_i^2\right)\left(h_{1r}L_{1r} - h_{1i}L_{1i}\right) + 2\alpha g_r g_i\left(h_{1r}M_{1r} + g_i M_{1i}\right) \\ +\alpha\left(g_r^2 - g_i^2\right)\left(h_{1r}M_{1i} - h_{1i}M_{1r}\right) + \frac{8\alpha Ah_{1r}g_r g_i}{\omega}\left(h_{1r}L_{1i} + h_{1i}L_{1r}\right) \\ -\alpha\left(1 - \frac{4Ah_{1i}}{\omega}\right)\left(g_r^2 - g_i^2\right)\left(h_{1r}L_{1i} + h_{1i}L_{1r}\right) \\ +2\alpha g_r g_i\left(1 - \frac{4Ah_{1i}}{\omega}\right)\left(h_{1r}L_{1r} - h_{1i}L_{1i}\right) \end{bmatrix},$$

$$d_{17} = \begin{bmatrix} -\frac{4Ah_{1r}}{\omega}\left(g_r^2 - g_i^2\right)\left(h_{1i} - \alpha L_{1i}\right) + 2g_r g_i\left(1 - \frac{4Ah_{1i}}{\omega}\right)\left(h_{1i} - \alpha L_{1i}\right) \\ +2\alpha\left(h_{1r}\left(g_r M_{1r} - g_i M_{1i}\right) + h_{1i}\left(g_r M_{1i} - g_i M_{1r}\right)\right) \\ +\alpha\left(g_r^2 - g_i^2\right)\left(h_{1r}c_{5r} + h_{1i}c_{5i}\right) + \frac{8Ah_{1r}g_r g_i}{\omega}\left(h_{1r} + \alpha L_{1r}\right) \\ -\left(1 - \frac{4Ah_{1i}}{\omega}\right)\left(g_r^2 - g_i^2\right)\left(h_{1r} - \alpha L_{1r}\right) - 2\alpha g_r g_i\left(h_{1r}c_{5i} - h_{1i}c_{5r}\right) \end{bmatrix},$$

$$d_{18} = \begin{bmatrix} -\frac{4\alpha Ah_{1r}}{\omega}\left(g_r^2 - g_i^2\right)\left(h_{1r}L_{1i} + h_{1i}L_{1r}\right) + \frac{8\alpha Ah_{1r}g_r g_i}{\omega}\left(h_{1r}L_{1r} - h_{1i}L_{1i}\right) \\ -2\alpha g_r g_i\left(1 - \frac{4Ah_{1i}}{\omega}\right)\left(h_{1r}L_{1i} + h_{1i}L_{1r}\right) \\ +2\alpha g_r g_i\left(h_{1r}M_{1i} - h_{1i}M_{1r}\right) - \alpha\left(g_r^2 - g_i^2\right)\left(h_{1r}M_{1r} + h_{1i}M_{1i}\right) \\ -\alpha\left(1 - \frac{4Ah_{1i}}{\omega}\right)\left(g_r^2 - g_i^2\right)\left(h_{1r}L_{1r} - h_{1i}L_{1i}\right) \end{bmatrix},$$

$$d_{19}=\begin{bmatrix} \frac{8Ah_{1r}h_{1i}}{\omega}\left(h_{1r}^2+h_{1i}^2\right)-\alpha\left(h_{1r}^3c_{5r}+h_{1i}^3c_{5i}\right)+\alpha h_{1r}^2h_{1i}c_{5i} \\ -\frac{12\alpha Ah_{1r}h_{1i}}{\omega}\left(h_{1r}L_{1r}-h_{1i}L_{1i}\right)+\frac{4\alpha A}{\omega}\left(L_{1r}h_{1i}^3-L_{1i}h_{1r}^3\right) \\ +2\alpha c_{3r}\left(h_{1r}^2+h_{1i}^2\right)+\alpha h_{1r}h_{1i}\left(h_{1r}c_{5i}-h_{1i}c_{5r}\right) \end{bmatrix},$$

$$d_{20}=\begin{bmatrix} \frac{8\alpha Ah_{1r}h_{1i}L_{1r}}{\omega}\left(h_{1r}^2+h_{1i}^2\right)-\frac{4\alpha AL_{1i}}{\omega}\left(h_{1r}^4-h_{1i}^4\right) \\ -\alpha\left(h_{1r}^3c_{3r}+h_{1i}^3c_{3i}\right)+\alpha h_{1r}h_{1i}\left(h_{1r}c_{3i}-h_{1i}c_{3r}\right) \end{bmatrix},$$

$$d_{21}=\begin{bmatrix} g_i\left(1-\frac{4Ah_{1i}}{\omega}\right)\left(h_{1r}^2-h_{1i}^2\right)-\alpha h_{1r}^2\left(g_rc_{5i}+g_ic_{5r}\right)+2\alpha h_{1r}h_{1i}M_{1r} \\ +2\alpha h_{1r}h_{1i}\left(g_rc_{5r}-g_ic_{5i}\right)-2h_{1r}g_r\left(1-\frac{4Ah_{1i}}{\omega}\right)\left(h_{1i}-\alpha L_{1i}\right) \\ +\frac{4Ah_{1r}g_r}{\omega}\left(h_{1r}^2-h_{1i}^2\right)+\alpha h_{1i}^2\left(g_rc_{5i}+g_ic_{5r}\right)-\frac{8\alpha Ah_{1r}^2g_rL_{1r}}{\omega} \\ +\frac{8Ah_{1r}^2g_i}{\omega}\left(h_{1i}-\alpha L_{1i}\right)-2\alpha L_{1r}\left(1-\frac{4Ah_{1i}}{\omega}\right)\left(h_{1r}g_i-h_{1i}g_r\right) \\ -\frac{8\alpha Ah_{1r}h_{1i}}{\omega}\left(L_{1r}g_i-L_{1i}g_r\right)+2\alpha L_{1i}h_{1i}g_i\left(1-\frac{4Ah_{1i}}{\omega}\right) \end{bmatrix},$$

$$d_{22}=\begin{bmatrix} \alpha\left(g_rM_{1i}+g_iM_{1r}\right)\left(h_{1r}^2-h_{1i}^2\right) \\ -\frac{8\alpha Ah_{1r}}{\omega}\left(h_{1r}^2-h_{1i}^2\right)\left(L_{1r}g_r+L_{1i}g_i\right) \\ +2\alpha h_{1r}h_{1i}\left(g_rM_{1r}-g_iM_{1i}\right)+\frac{8\alpha Ah_{1r}h_{1i}}{\omega}\left(L_{1r}g_i-L_{1i}g_r\right) \\ +\alpha\left(1-\frac{4Ah_{1i}}{\omega}\right)\left(L_{1r}g_i-L_{1i}g_r\right)\left(h_{1r}^2g_i-h_{1i}^2\right) \\ -2\alpha h_{1r}h_{1i}\left(1-\frac{4Ah_{1i}}{\omega}\right)\left(L_{1r}g_r+L_{1i}g_i\right) \end{bmatrix},$$

$$d_{23} = \left[ \begin{array}{c} -g_r\left(1-\frac{4Ah_{1i}}{\omega}\right)\left(h_{1r}^2-h_{1i}^2\right)+\alpha g_r c_{5r}\left(h_{1r}^2-h_{1i}^2\right) \\ +\alpha g_i c_{5i}\left(h_{1r}^2+h_{1i}^2\right)-\frac{8Ah_{1r}^2 g_r}{\omega}\left(h_{1i}-\alpha L_{1i}\right) \\ +\frac{4Ah_{1r}g_i}{\omega}\left(h_{1r}^2-h_{1i}^2\right)+2\alpha h_{1r}h_{1i}M_{1i} \\ +2\alpha h_{1r}h_{1i}\left(g_r c_{5i}+g_i c_{5r}\right) \\ -2h_{1r}g_i\left(1-\frac{4Ah_{1i}}{\omega}\right)\left(h_{1i}+\alpha L_{1i}\right) \\ -\frac{8\alpha Ah_{1r}^2 g_i L_{1r}}{\omega}+\frac{8\alpha Ah_{1r}h_{1i}}{\omega}\left(L_{1r}g_r+L_{1i}h_{1i}\right) \\ +2\alpha h_{1i}\left(1-\frac{4Ah_{1i}}{\omega}\right)\left(L_{1r}g_i-L_{1i}g_r\right) \end{array} \right],$$

$$d_{24} = \left[ \begin{array}{c} -\alpha\left(1-\frac{4Ah_{1i}}{\omega}\right)\left(h_{1r}^2-h_{1i}^2\right)\left(L_{1r}g_r+L_{1i}g_i\right) \\ -\alpha\left(h_{1r}^2-h_{1i}^2\right)\left(M_{1r}g_r-M_{1i}g_i\right)-\frac{8\alpha Ah_{1r}^2 h_{1i}}{\omega}\left(L_{1r}g_r+L_{1i}g_i\right) \\ +\frac{4\alpha Ah_{1r}}{\omega}\left(L_{1r}g_i-L_{1i}g_r\right)\left(h_{1r}^2-h_{1i}^2\right) \\ -2\alpha h_{1r}h_i\left(1-\frac{4Ah_{1i}}{\omega}\right)\left(L_{1r}g_i-L_{1i}g_r\right) \\ -2\alpha h_{1r}h_{1i}\left(M_{1i}g_r+M_{1r}g_i\right) \end{array} \right],$$

$$d_{25} = \left[ \begin{array}{c} -\alpha\left(h_{1r}^3 c_{5r}+h_{1i}^3 c_{5i}\right)+\alpha c_{3r}\left(h_{1r}^2-h_{1i}^2\right)+\frac{8\alpha Ah_{1r}L_{1i}}{\omega}\left(h_{1r}^2-h_{1i}^2\right) \\ +2\alpha h_{1r}h_{1i}c_{3i}-\alpha h_{1r}h_{1i}\left(h_{1r}c_{5i}+h_{1i}c_{5r}\right)-\frac{8\alpha Ah_{1i}^3 L_{1r}}{\omega} \end{array} \right],$$

$$d_{26} = \left[ \begin{array}{c} -\frac{4\alpha AL_{1i}}{\omega}\left(h_{1r}^4+h_{1i}^4\right)-\frac{8\alpha Ah_{1i}^2 L_{1i}h_{1r}^2}{\omega} \\ -\alpha\left(h_{1r}^3 c_{3r}+h_{1i}^3 c_{3i}\right)-\alpha h_{1r}h_{1i}\left(h_{1r}c_{3i}+h_{1i}c_{3r}\right) \end{array} \right],$$

$$d_{27} = \left[-h_{1r}^3+3\alpha h_{1r}^2 L_{1r}-h_{1r}h_{1i}^2+2\alpha h_{1r}h_{1i}L_{1i}+\alpha L_r h_{1i}^2\right],$$

$$d_{28} = -2\alpha L_{1r}h_{1r}\left(h_{1r}^2+h_{1i}^2\right),$$

$$m_1 = m_{1r} + im_{1i} = \begin{bmatrix} -P_r A \left(P_r \left(1 + d_7\right) - 2h_{1r} d_7 + d_8\right) + PAd_{27} \\ -\left(2E_c + i\omega P\right) d_{13} + P\left(d_{19} + d_{25}\right) \end{bmatrix},$$

$$m_2 = m_{2r} + im_{2i} = \begin{bmatrix} 2P_r A h_{1r} d_8 - \left(2E_c + i\omega P\right) d_{14} \\ +PAd_{28} + P\left(d_{20} + d_{26}\right) \end{bmatrix},$$

$$m_3 = m_{3r} + im_{3i} = \left[-\left(2E_c + i\omega P\right) d_9 + P\left(d_{15} + d_{21}\right)\right],$$

$$m_4 = m_{4r} + im_{4i} = \left[-\left(2E_c + i\omega P\right) d_{10} + P\left(d_{16} + d_{22}\right)\right],$$

$$m_5 = m_{5r} + im_{5i} = \left[-\left(2E_c + i\omega P\right) d_{11} + P\left(d_{17} + d_{23}\right)\right],$$

$$m_6 = m_{6r} + im_{6i} = \left[-\left(2E_c + i\omega P\right) d_{12} + P\left(d_{18} + d_{24}\right)\right],$$

$$m_7 = m_{7r} + im_{7i},$$

$$m_{7r} = \frac{m_{1r}\left(4h_{1r}^2 - 2h_{1r}P_r\right) - m_{1i}\left(\frac{\omega P_r}{4}\right)}{\left(4h_{1r}^2 - 2h_{1r}P_r\right)^2 + \left(\frac{\omega P_r}{4}\right)^2}, \quad m_{7i} = \frac{m_{1i}\left(4h_{1r}^2 - 2h_{1r}P_r\right) + m_{1r}\left(\frac{\omega P_r}{4}\right)}{\left(4h_{1r}^2 - 2h_{1r}P_r\right)^2 + \left(\frac{\omega P_r}{4}\right)^2},$$

$$m_8 = m_{8r} + im_{8i},$$

$$m_{8r} = \frac{m_{2r}\left(4h_{1r}^2 - 2h_{1r}P_r\right) - m_{2i}\left(\frac{\omega P_r}{4}\right)}{\left(4h_{1r}^2 - 2h_{1r}P_r\right)^2 + \left(\frac{\omega P_r}{4}\right)^2}, \quad m_{8i} = \frac{m_{2i}\left(4h_{1r}^2 - 2h_{1r}P_r\right) + m_{2r}\left(\frac{\omega P_r}{4}\right)}{\left(4h_{1r}^2 - 2h_{1r}P_r\right)^2 + \left(\frac{\omega P_r}{4}\right)^2},$$

$$n_1 = n_{1r} + in_{1i} = \begin{bmatrix} \left(4h_{1r}^2 - 2h_{1r}P_r\right)^2 + \left(\frac{\omega P_r}{4}\right)^2 \\ -2i\left(4h_{1r}^2 - 2h_{1r}P_r\right)\left(\frac{\omega P_r}{4}\right) \end{bmatrix},$$

$$n_{1r} = \left(4h_{1r}^2 - 2h_{1r}P_r\right)^2 + \left(\frac{\omega P_r}{4}\right)^2, \quad n_{1i} = -2\left(4h_{1r}^2 - 2h_{1r}P_r\right)\left(\frac{\omega P_r}{4}\right),$$

$$m_9 = m_{9r} + im_{9i},$$

$$m_{9r} = \frac{\left(4h_{1r} - P_r\right)\left(n_{1r}m_{2r} - m_{2i}n_{1i}\right)}{n_{1r}^2 + n_{1i}^2}, \quad m_{9i} = \frac{\left(4h_{1r} - P_r\right)\left(n_{1i}m_{2r} + m_{2i}n_{1r}\right)}{n_{1r}^2 + n_{1i}^2},$$

$$n_2 = n_{2r} + in_{2i} = \begin{bmatrix} \left(h_{1r} + g_r\right)^2 - \left(h_{1i} - g_i\right)^2 - P_r\left(h_{1r} + g_r\right) \\ +i\left(-2\left(h_{1r} + g_r\right)\left(h_{1i} - g_i\right) + P_r\left(h_{1i} - g_i\right) - \frac{\omega P_r}{4}\right) \end{bmatrix},$$

$$n_{2r} = \left[\left(h_{1r} + g_r\right)^2 - \left(h_{1i} - g_i\right)^2 - P_r\left(h_{1r} + g_r\right)\right],$$

$$n_{2i} = \left[-2\left(h_{1r} + g_r\right)\left(h_{1i} - g_i\right) + P_r\left(h_{1i} - g_i\right) - \frac{\omega P_r}{4}\right],$$

$$m_{10} = m_{10r} + im_{10i}, \quad m_{10r} = \frac{\left(n_{2r}m_{5r} - m_{3r}n_{2i}\right)}{n_{2r}^2 + n_{2i}^2},$$

$$m_{10i} = \frac{\left(n_{2r}m_{5i} - m_{3i}n_{2i}\right)}{n_{2r}^2 + n_{2i}^2}, \quad m_{11} = m_{11r} + im_{11i},$$

$$m_{11r}=\frac{(n_{2i}m_{5r}+m_{3r}n_{2r})}{n_{2r}^2+n_{2i}^2},\quad m_{11i}=\frac{(n_{2i}m_{5i}+m_{3i}n_{2r})}{n_{2r}^2+n_{2i}^2},$$

$$m_{12}=m_{12r}+im_{12i},\quad m_{12r}=\frac{(n_{2r}m_{6r}-m_{4r}n_{2i})}{n_{2r}^2+n_{2i}^2},$$

$$m_{12i}=\frac{(n_{2r}m_{6i}-m_{4i}n_{2i})}{n_{2r}^2+n_{2i}^2},\quad m_{13}=m_{13r}+im_{13i},$$

$$m_{13r}=\frac{(n_{2i}m_{6r}+m_{6r}n_{2r})}{n_{2r}^2+n_{2i}^2},\quad m_{13i}=\frac{(n_{2i}m_{6i}+m_{6i}n_{2r})}{n_{2r}^2+n_{2i}^2},$$

$$n_3=n_{3r}+in_{3i}=\left[\left(2\left(h_{1r}+g_r\right)-P_r\right)+i\left(-2\left(h_{1i}-g_i\right)\right)\right],$$

$$n_{3r}=2\left(h_{1r}+g_r\right)-P_r,\quad n_{3i}=-2\left(h_{1i}-g_i\right),\quad n_4=n_{4r}+in_{4i},$$

$$n_{4r}=\begin{bmatrix}\left(h_{1r}+g_r\right)^4+\left(h_{1i}-g_i\right)^4+\frac{\omega P_r^2}{2}\left(h_{1i}-g_i\right)\\ -6\left(h_{1r}+g_r\right)^2\left(h_{1i}-g_i\right)^2+\frac{\omega^2P_r^2}{16}\\ +P_r^2\left(\left(h_{1r}+g_r\right)^2-\left(h_{1i}-g_i\right)^2\right)-\omega P_r\left(h_{1r}+g_r\right)\left(h_{1i}-g_i\right)\\ +2P_r\left(-\left(h_{1r}+g_r\right)^3+3\left(h_{1r}+g_r\right)\left(h_{1i}-g_i\right)^2\right)\end{bmatrix},$$

$$n_{4i}=\begin{bmatrix}-4\left(h_{1r}+g_r\right)^3\left(h_{1i}-g_i\right)+4\left(h_{1r}+g_r\right)\left(h_{1i}-g_i\right)^3\\ +2P_r\left(3\left(h_{1r}+g_r\right)^2\left(h_{1i}-g_i\right)-\left(h_{1i}-g_i\right)^3\right)-\frac{\omega P_r^2}{2}\left(h_{1r}+g_r\right)\\ -2P_r^2\left(h_{1r}+g_r\right)\left(h_{1i}-g_i\right)-\frac{\omega P_r}{2}\left(\left(h_{1r}+g_r\right)^2-\left(h_{1i}-g_i\right)^2\right)\end{bmatrix},$$

$$m_{14} = m_{14r} + im_{14i},$$

$$m_{14r} = \frac{m_{6r}\left(n_{4r}n_{3r} + n_{4i}n_{3i}\right) + m_{4r}\left(n_{4r}n_{3i} - n_{4i}n_{3r}\right)}{n_{4r}^2 + n_{4i}^2},$$

$$m_{14i} = \frac{m_{6i}\left(n_{4r}n_{3r} + n_{4i}n_{3i}\right) + m_{4i}\left(n_{4r}n_{3i} - n_{4i}n_{3r}\right)}{n_{4r}^2 + n_{4i}^2},$$

$$m_{15} = m_{15r} + im_{15i},$$

$$m_{15r} = \frac{-m_{6r}\left(n_{4r}n_{3i} - n_{4i}n_{3r}\right) + m_{4r}\left(n_{4r}n_{3r} + n_{4i}n_{3i}\right)}{n_{4r}^2 + n_{4i}^2},$$

$$m_{14i} = \frac{-m_{6i}\left(n_{4r}n_{3i} - n_{4i}n_{3r}\right) + m_{4i}\left(n_{4r}n_{3r} + n_{4i}n_{3i}\right)}{n_{4r}^2 + n_{4i}^2},$$

$$m_{16} = m_{16r} + im_{16i} = m_7 + m_9 + m_{10} + m_{14},$$

$$m_{16r} = m_{7r} + m_{9r} + m_{10r} + m_{146r}, \quad m_{16i} = m_{7i} + m_{9i} + m_{10i} + m_{146i},$$

$$f = f_r + if_i = \frac{\left(P_r + \sqrt{P_r^2 + i\omega P_r}\right)}{2},$$

$$f_r = \frac{P_r}{2} + \frac{a_3}{2} = \frac{P_r}{2} + \frac{1}{2}\left[\frac{P_r^2 + \sqrt{P_r^4 + \omega^2 P_r^2}}{2}\right]^{\frac{1}{2}},$$

$$f_i = \frac{b_3}{2} = \frac{1}{2}\left[\frac{-P_r^2 + \sqrt{P_r^4 + \omega^2 P_r^2}}{2}\right]^{\frac{1}{2}},$$

$$a_3 = \left[\frac{P_r^2 + \sqrt{P_r^4 + \omega^2 P_r^2}}{2}\right]^{\frac{1}{2}}, \quad b_3 = \left[\frac{-P_r^2 + \sqrt{P_r^4 + \omega^2 P_r^2}}{2}\right]^{\frac{1}{2}},$$

$$r_3 = a_3^2 + b_3^2 = \sqrt{P_r^4 + \omega^2 P_r^2}.$$

From Eqs.(3.85), (3.90) and (3.91) we can write

$$\theta = \theta_0 + \epsilon\left(\theta_{1r}\cos\omega t - \theta_{1i}\sin\omega t\right), \tag{3.92}$$

in which

$$\begin{aligned} \theta_{1r} \;=\; & -e^{-f_r\eta}\left(m_{16r}\cos f_i\eta - m_{16i}\sin f_i\eta\right) \\ & + \left(m_{7r} + m_{9r} + m_{8r}\eta\right)e^{-2h_{1r}\eta} \\ & + \left[\begin{array}{c} \left(m_{10r} + m_{14r} + m_{12r}\eta\right)\cos\left(h_i - g_i\right)\eta \\ +\left(m_{11r} + m_{15r} + m_{13r}\eta\right)\sin\left(h_i - g_i\right)\eta \end{array}\right] e^{-(h_{1r}+g_r)\eta}, \end{aligned} \tag{3.93}$$

$$\begin{aligned} \theta_{1i} \;=\; & -e^{-f_r\eta}\left(m_{16r}\sin f_i\eta + m_{16i}\cos f_i\eta\right) \\ & + \left(m_{7i} + m_{9i} + m_{8i}\eta\right)e^{-2h_{1r}\eta} \\ & + \left[\begin{array}{c} \left(m_{10i} + m_{14i} + m_{12i}\eta\right)\cos\left(h_i - g_i\right)\eta \\ +\left(m_{11i} + m_{15i} + m_{13i}\eta\right)\sin\left(h_i - g_i\right)\eta \end{array}\right] e^{-(h_{1r}+g_r)\eta}. \end{aligned} \tag{3.94}$$

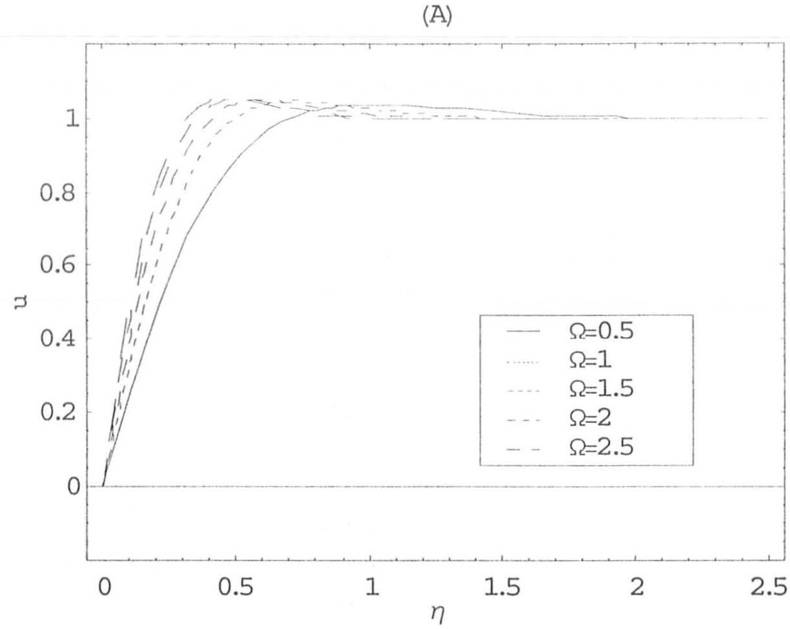


**Fig. 3.1 (A)**. Effect of $\Omega$ on real part of velocity profile $u$ vs $\eta$ for Newtonian fluid at $\alpha = 0,\ \omega t = \pi/2,\ A = 0.2,\ \epsilon = \omega = 0.5,\ W_0 = -0.1,\ \nu = 0.1.$

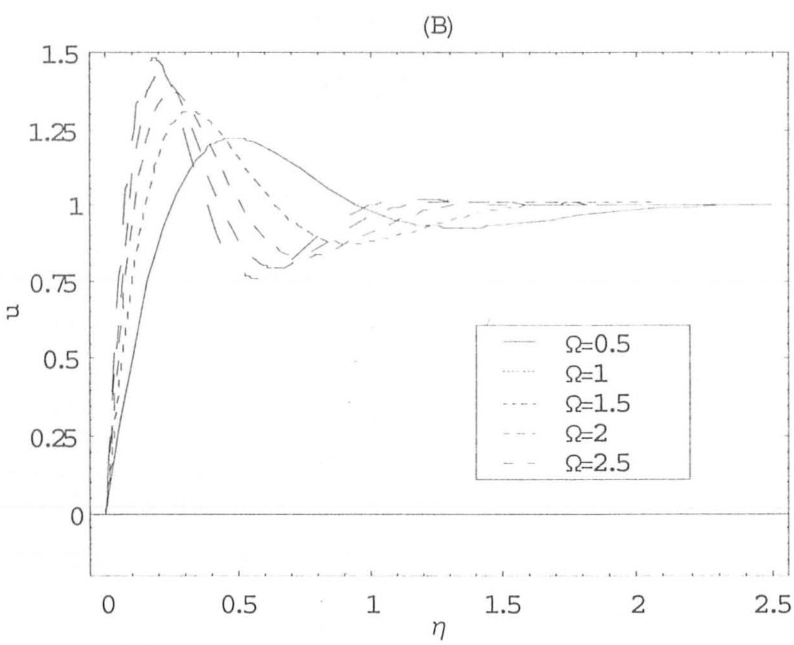


**Fig. 3.1 (B)**. Effect of $\Omega$ on real part of velocity profile $u$ vs $\eta$ for second grade fluid at $\alpha = 0.4,\ \omega t = \pi/2,\ A = 0.2,\ \epsilon = \omega = 0.5,\ W_0 = -0.1,\ \nu = 0.1.$

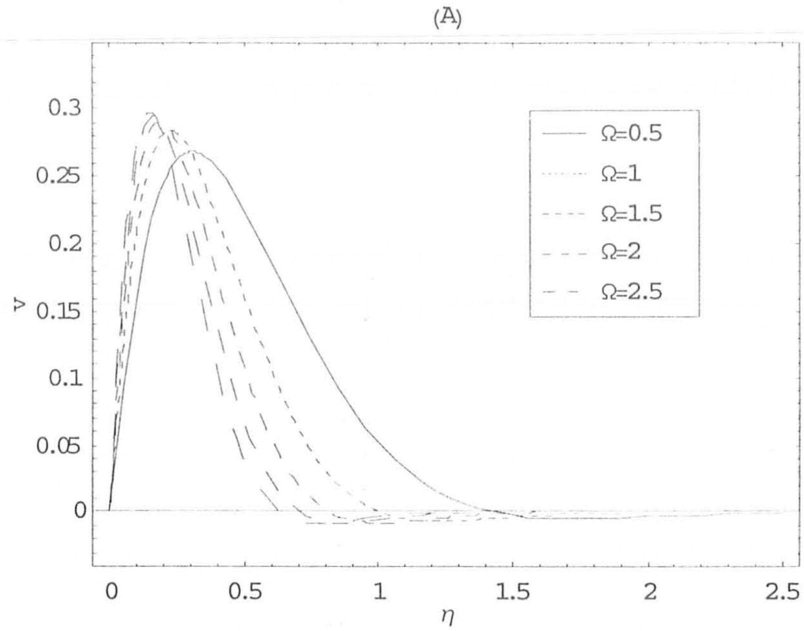


**Fig. 3.2(A).** Effect of $\Omega$ on imaginary part of velocity profile $v$ vs $\eta$ for Newtonian fluid at $\alpha = 0,\ \omega t = \pi/2,\ A = 0.2,\ \epsilon = \omega = 0.5,\ W_0 = -0.1,\ \nu = 0.1.$

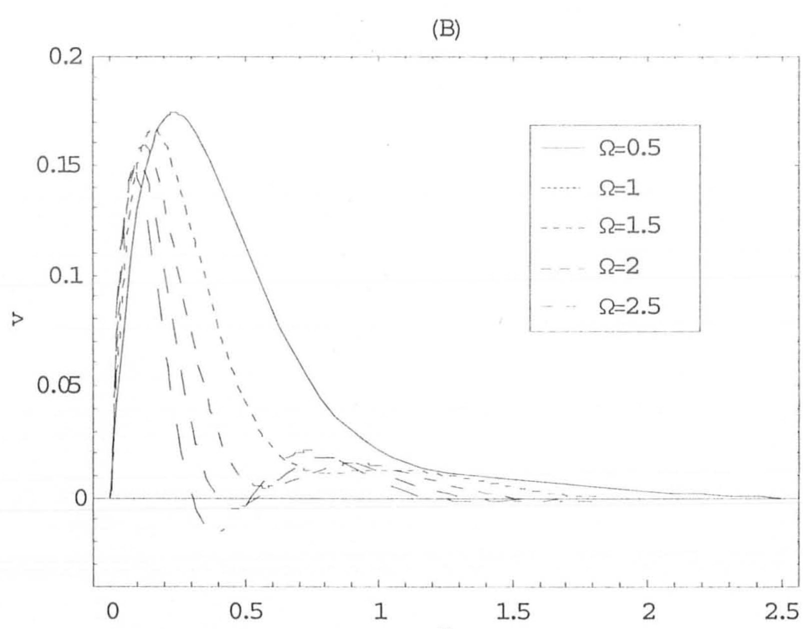


**Fig. 3.2(B).** Effect of $\Omega$ on imaginary part of velocity profile $v$ vs $\eta$ for second grade fluid at $\alpha = 0.1,\ \omega t = \pi/2,\ A = 0.2,\ \epsilon = \omega = 0.5,\ W_0 = -0.1,\ \nu = 0.1.$

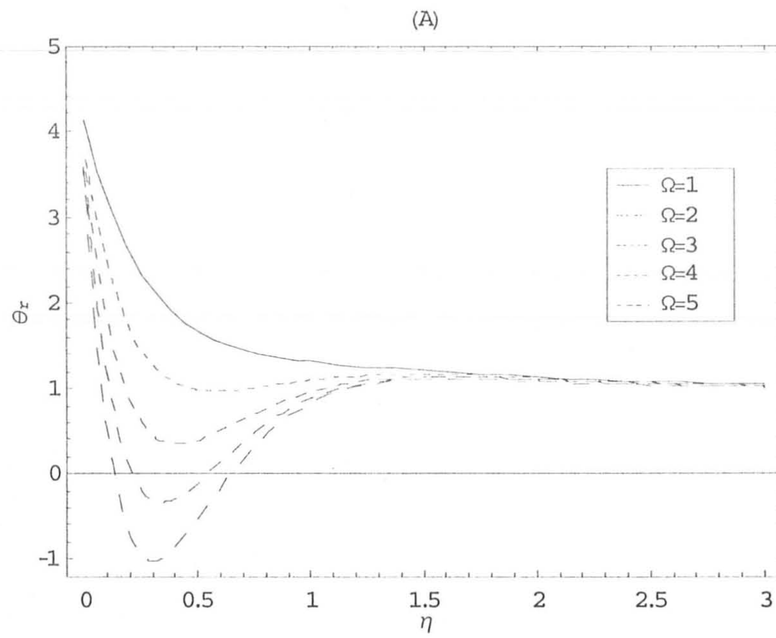


**Fig. 3.3(A).** Effect of $\Omega$ on real part of temperature profile $\theta_r$ vs $\eta$ for Newtonian fluid at $\alpha = 0$, $\omega t = \pi/2, A = \epsilon = \omega = 0.5, W_0 = -0.1, \nu = 0.1, P_r = 1.5, E_c = 5.0, k = 0.2, P = 0.3.$

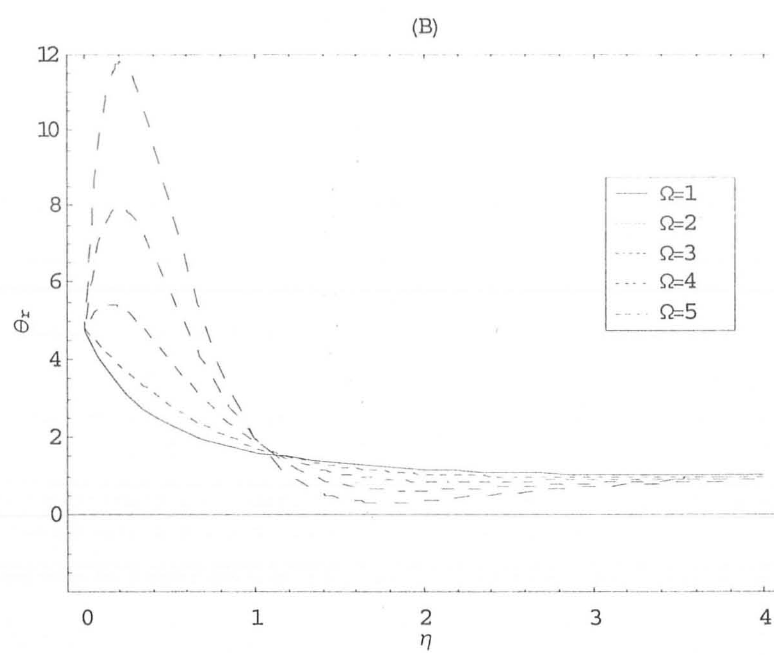


**Fig. 3.3(B).** Effect of $\Omega$ on real part of temperature profile $\theta_r$ vs $\eta$ for second grade fluid at $\alpha = 0.04$, $\omega t = \pi/2, A = \epsilon = \omega = 0.5, W_0 = -0.1, \nu = 0.1, P_r = 1.5, E_c = 5.0, k = 0.2, P = 0.3.$

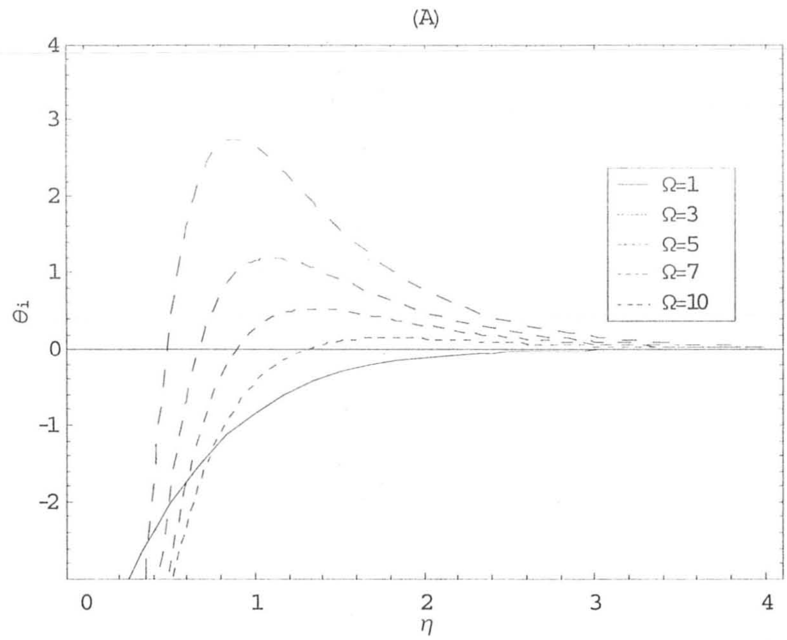


**Fig. 3.4(A)**. Effect of $\Omega$ on imaginary part of temperature profile $\theta_i$ vs $\eta$ for Newtonian fluid at $\alpha = 0$, $\omega t = \pi/2, A = \epsilon = \omega = 0.5, W_0 = -0.1, \nu = 0.1, P_r = 1.5,\ E_c = 5.0, k = 0.2, P = 0.3.$

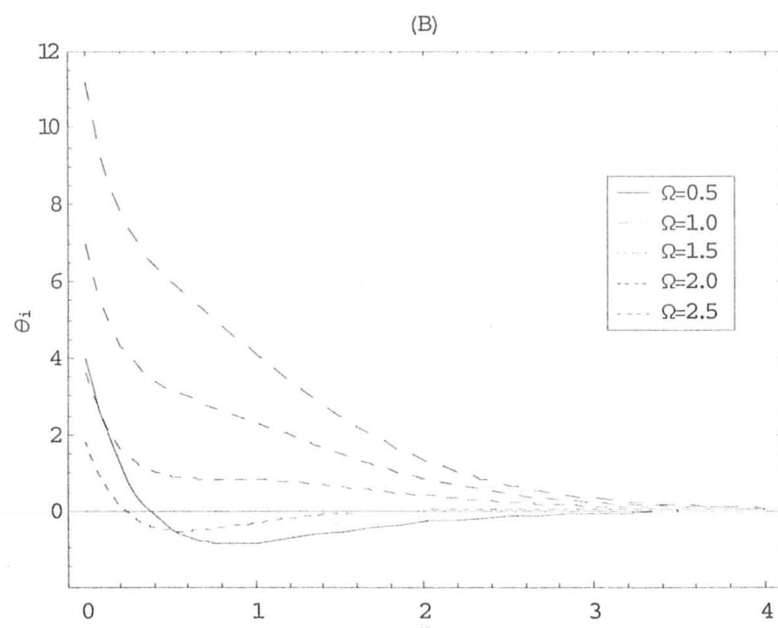


**Fig. 3.4(B)**. Effect of $\Omega$ on imaginary part of temperature profile $\theta_i$ vs $\eta$ for second grade fluid at $\alpha = 0.05$,

$\omega t = \pi/2, A = \epsilon = \omega = 0.5, W_0 = -0.1,\ \nu = 0.1, P_r = 1.5, E_c = 5.0, k = 0.2, P = 0.3.$

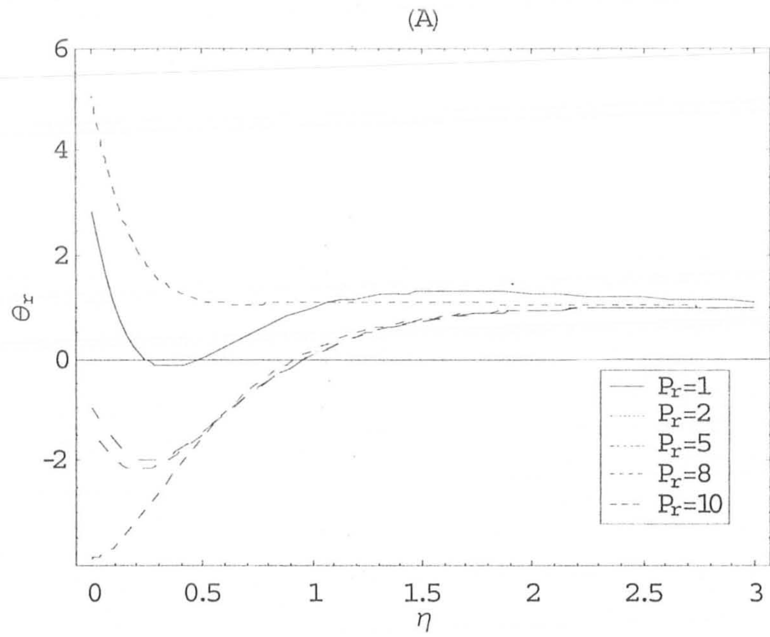


**Fig. 3.5(A).** Effect of $P_r$ on real part of temperature profile $\theta_r$ vs $\eta$ for Newtonian fluid at $\alpha = 0$, $\omega t = \pi/2, A = \epsilon = \omega = 0.5, W_0 = -0.1,\ \nu = 0.1, \Omega = 3.0, E_c = 5.0, k = 0.2, P = 0.3.$

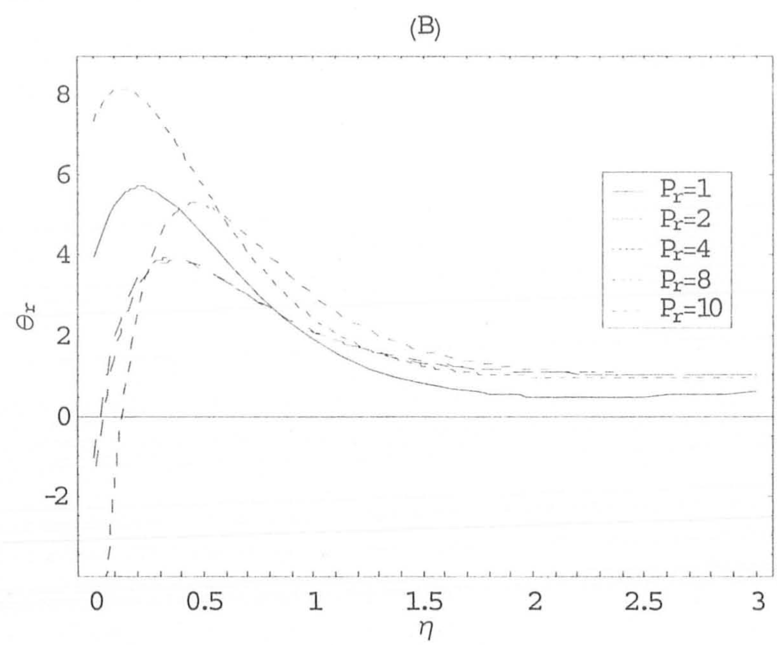


**Fig. 3.5(B).** Effect of $P_r$ on real part of temperature profile $\theta_r$ vs $\eta$ for second grade fluid at $\alpha = 0.05$, $\omega t = \pi/2, A = \epsilon = \omega = 0.5, W_0 = -0.1,\ \nu = 0.1, \Omega = 3.0, E_c = 5.0, k = 0.2, P = 0.3.$

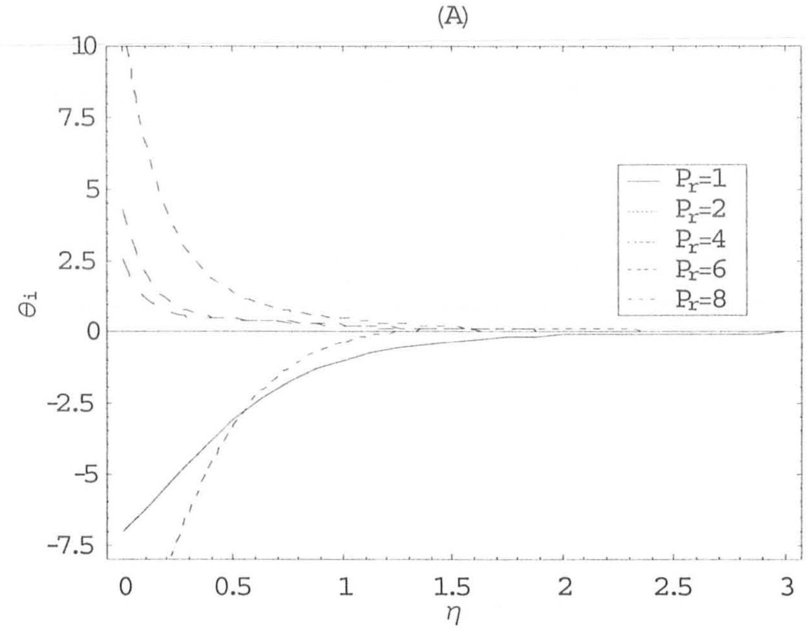


**Fig. 3.6(A).** Effect of $P_r$ on imaginary part of temperature profile $\theta_i$ vs $\eta$ for Newtonian fluid at $\alpha = 0$, $\omega t = \pi/2, A = \epsilon = \omega = 0.5, W_0 = -0.1,\ \nu = 0.1, \Omega = 2.5, E_c = 5.0, k = 0.2, P = 0.3.$

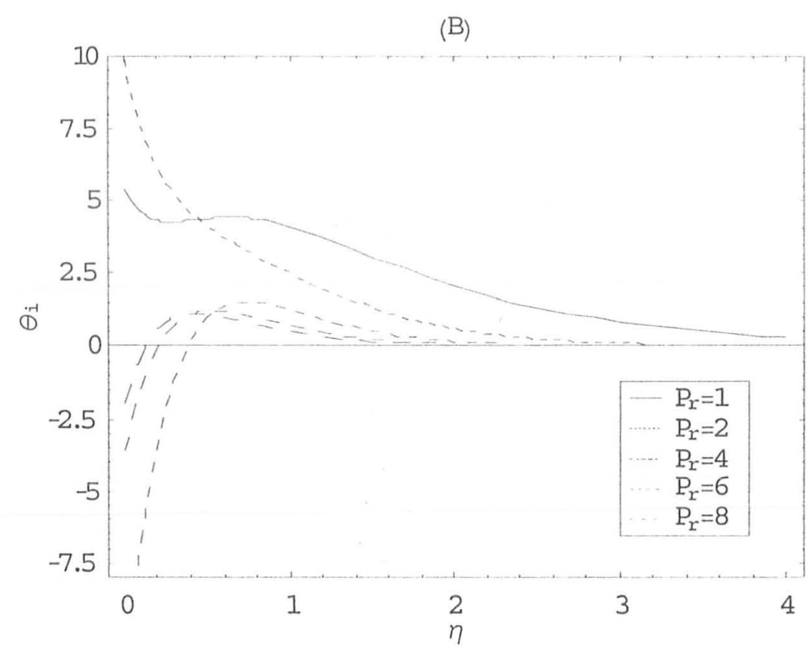


**Fig. 3.6(B).** Effect of $P_r$ on imaginary part of temperature profile $\theta_i$ vs $\eta$ for seond grade fluid at $\alpha = 0.04$,

$\omega t = \pi/2, A = \epsilon = \omega = 0.5, W_0 = -0.1,\ \nu = 0.1, \Omega = 2.5, E_c = 5.0, k = 0.2, P = 0.3.$

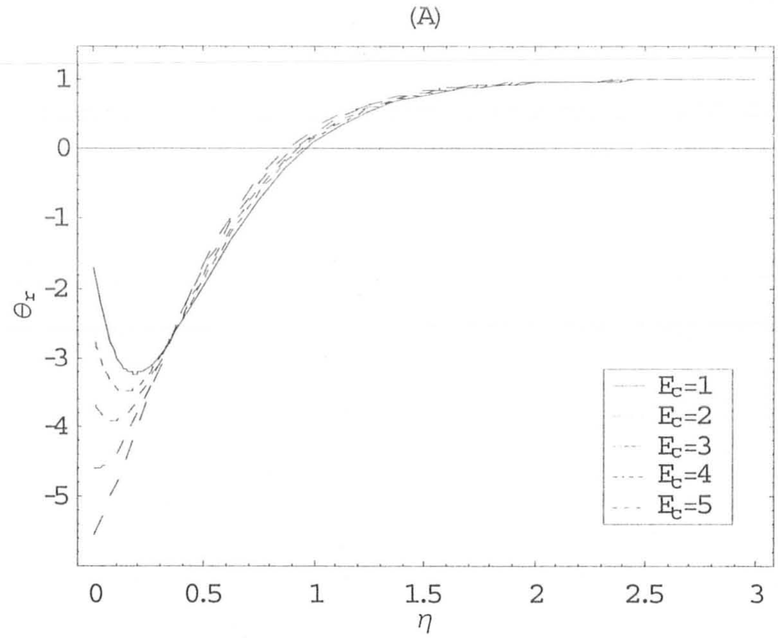


Fig. 3.7(A). Effect of $E_c$ on real part of temperature profile $\theta_r$ vs $\eta$ for Newtonian fluid at $\alpha = 0,\ \omega t = \pi/2, A = \epsilon = \omega = 0.5, W_0 = -0.1,\ \nu = 0.1, \Omega = 4.0, P_r = 5.0, k = 0.2, P = 0.3$.

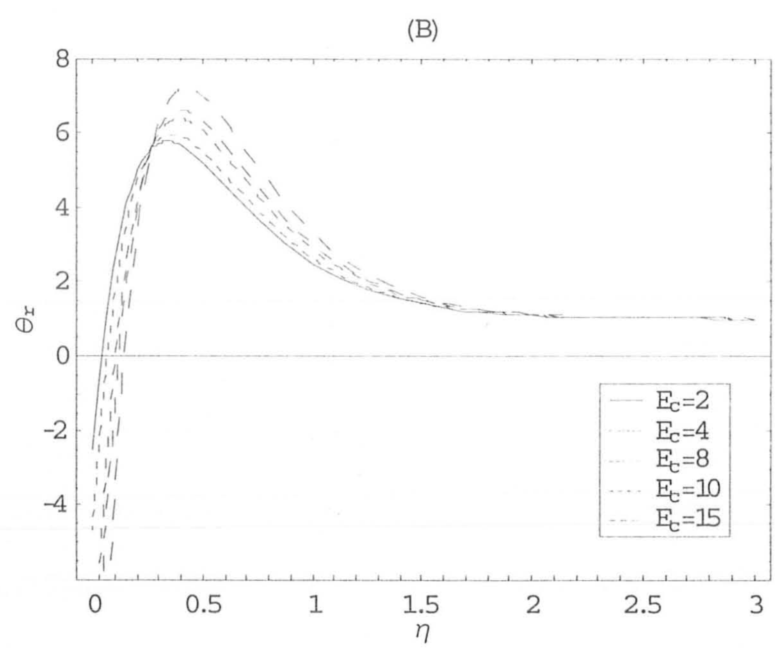


Fig. 3.7(B). Effect of $E_c$ on real part of temperature profile $\theta_r$ vs $\eta$ for second grade fluid at $\alpha = 0.04,\ \omega t = \pi/2, A = \epsilon = \omega = 0.5, W_0 = -0.1,\ \nu = 0.1, \Omega = 4.0, P_r = 5.0, k = 0.2, P = 0.3$.

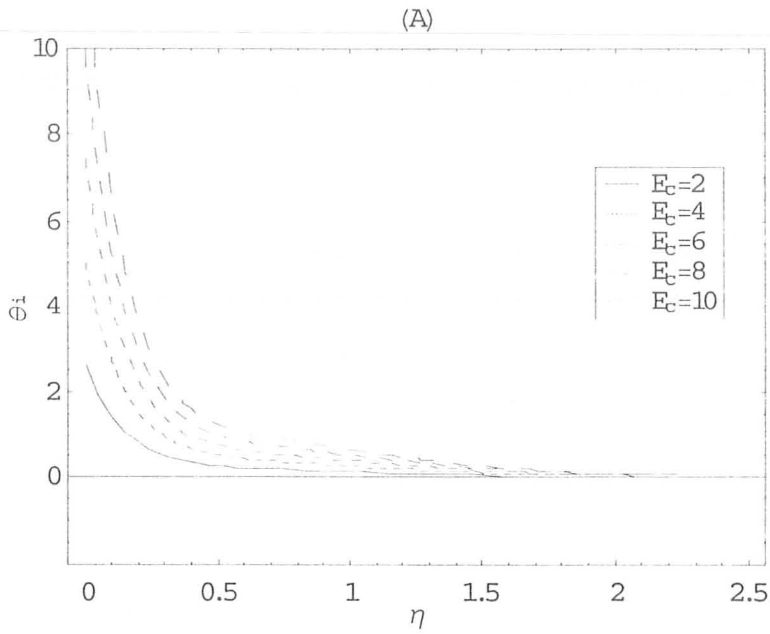


**Fig. 3.8(A)**. Effect of $E_c$ on imaginary part of temperature profile $\theta_i$ vs $\eta$ for Newtonian fluid at $\alpha = 0$, $\omega t = \pi/2, A = \epsilon = \omega = 0.5, W_0 = -0.1,\ \nu = 0.1, \Omega = 2.5, P_r = 5.0, k = 0.2, P = 0.3$.

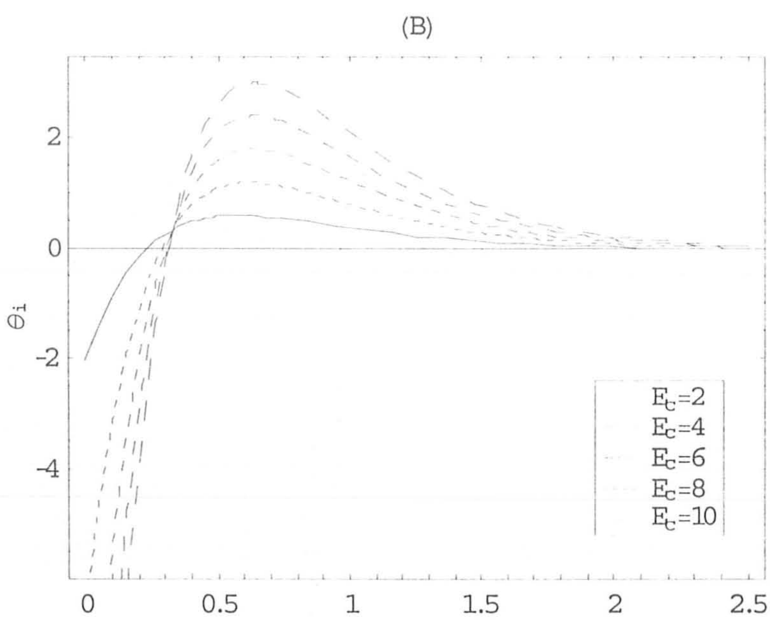


**Fig. 3.8(B)**. Effect of $E_c$ on imaginary part of temperature profile $\theta_i$ vs $\eta$ for seond grade fluid at $\alpha = 0.05$,

$\omega t = \pi/2, A = \epsilon = \omega = 0.5, W_0 = -0.1, \nu = 0.1, \Omega = 2.5, P_r = 5.0, k = 0.2, P = 0.3$.

## 3.4 Discussion

In this chapter we consider the problem of heat transfer in rotating flow of an incompressible fluid of second grade. A perturbation procedure has been used to obtain the analytic solution. The effects of various parameters such as $\Omega$, $P_r$, and $E_c$ on the real and imaginary parts of velocity $(u, v)$ and temperature $(\theta_r, \theta_i)$ distributions is studied and the results have been presented by several graphs.

To study the effect of $\Omega$ on the velocity components, we have plotted $u$ and $v$ against $\eta$ in figures 3.1 and 3.2 for Newtonian and second grade fluids. From the figure 3.1$(A)$, it is observed that near the plate $u$ increases with the increase of $\Omega$. Figure 3.1$(B)$ indicates that $u$ increases very near to the plate and then fluctuates through an increase in $\Omega$. The comparison of these two figures reveal that $u$ in case of second grade fluid is greater than that of Newtonian fluid. Also, the velocity boundary layer thickness for second grade fluid is larger than the Newtonian fluid. It is also seen from figures 3.2$(A)$ and 3.2$(B)$ that $v$ increases near the plate and then decreases for large value of $\Omega$. The fluctuations in second grade fluid are more visible than that of Newtonian fluid. Also, the value of $v$ for second grade fluid is smaller from the case of Newtonian fluid.

Figures 3.3 and 3.4 show the effect of $\Omega$ on the real $(\theta_r)$ and imaginary $(\theta_i)$ parts of temperature distributions. Figure 3.3$(A)$ shows that with the increase of $\Omega$, $\theta_r$ decreases near the wall. As shown in figure 3.3$(B)$, we can see that as $\Omega$ increases, $\theta_r$ increases near the plate and then, at a distance of $\eta = 1$, the $\theta_r$ beguns to decrease. That is, the behavior of $\theta_r$ is quite opposite for Newtonian and second grade fluid near the plate. Figure 3.4$(A)$ shows the variation of $\Omega$ on $\theta_i$. It can be seen that as $\Omega$ increases, the value of $\theta_i$ decreases at a distance of approximately $\eta = 0.8$ and then increases. Figure 3.4$(B)$ indicates that $\theta_i$ increases near the wall for $\Omega > 1$.

In order to illustrate the variation of $P_r$ on $\theta_r$ and $\theta_i$, we have prepared figures 3.5 and 3.6. Figure 3.5$(A)$ and 3.6$(A)$ explains the effect of $P_r$ on $\theta_r$ and $\theta_i$, respectively for Newtonian fluid case. From these figures it is revealed that near the plate, $\theta_r$ decreases and $\theta_i$ increases for $P_r > 2$. The thermal boundary layer thickness in $\theta_r$ increases where as for $\theta_i$ decreases. For second grade fluid, we note that from figures 3.5$(B)$ and 3.6$(B)$ that for $P_r > 2$, $\theta_r$ decreases near the wall and increases far away. Also $\theta_i$ decreases for $P_r > 2$.

Figures 3.7 and 3.8 show the effect of $E_c$ on $\theta_r$ and $\theta_i$. From figures 3.7$(A)$ and 3.7$(B)$, we observe that $\theta_r$ near the wall decreases with the increase in $E_c$ and increases far away. The thermal boundary layer thickness increases for large $E_c$. Moreover, it can be seen from figure 3.8($A$) that $\theta_i$ increases for large values of $E_c$. From figure 3.8($B$) it can be seen that with the increase in the values of $E_c$ the temperature $\theta_i$ decreases near the plate and increases far away. The thermal boundary layer thicknesses in both the fluids increases.

# Bibliography


[1] B. Straughan, Energy stability in the Benard problem for a fluid of second grade, J. Appl. Math. Phys. (ZAMP) **34,** 502-508 (1983).

[2] H. Franchi and B. Straughan, Stability and nonexistence results in the generalized theory of a fluid of second grade, J. Math. Anal. Appl. **180**, 122-137 (1993).

[3] K. R. Rajagopal, A note on unsteady unidirectional flows of a non-Newtonian fluid, Int. J. Non-linear Mech. **17**, 369-373 (1982).

[4] R. Bandelli, Unsteady unidirectional flows of second grade fluids in domains with heated boundaries, Int. J. Non-linear Mech. **30**, 263-269 (1995).

[5] G. Gupta and M. Massoudi, Flow of a generalized second grade fluid between heated plates, Acta Mech. **99**, 21-33 (1993).

[6] K. R. Rajagopal, On the creeping flow of the second grade fluid, J. Non-Newtonian Fluid Mech. **15**, 239-246 (1984).

[7] T. Hayat, S. Asghar and A. M. Siddiqui, Periodic unsteady flows of a non-Newtonian fluid, Acta Mech. **131**, 169-175 (1998).

[8] T. Hayat, S. Asghar and A. M. Siddiqui, On the moment of a plane disk in a non-Newtonian fluid, Acta Mech. **136**, 125-131 (1999).

[9] T. Hayat, S. Asghar and A. M. Siddiqui, Some unsteady unidirectional flows of a non-Newtonian fluid, Int. J. Eng. Sci. **38**, 337-346 (2000).

[10] K. R. Rajagopal, Longitudinal and torsional ocsillations of a rod in a non-Newtonian fluid, Acta Mech. **49**, 282-285 (1983).

[11] I-Chung Liu, Effect of modulation on onset of thermal convection of a second-grade fluid layer, Int. J. Non-linear Mech. **39**, 1647-1657 (2004).

[12] I. T. Dolapci and M. Pakdemirli, Approximate symmetries of creeping flow equations of a second grade fluid, Int. J. Non-linear Mech. **39**, 1603-1619 (2004).

[13] T. Hayat, Masood Khan, A. M. Siddiqui and S. Asghar, Transient flows of a second grade fluid, Int. J. Non-linear Mech. **39**, 1621-1633 (2004).

[14] T. Hayat and K. Hutter, Rotating flow of a second grade fluid on a porous plate, Int. J. Non-linear Mech. **39**, 767-777 (2004).

[15] P. G. Siddheshwar and C. V. Sri Krishna, Unsteady non-linear convection in a second grade fluid, Int. J. Non-linear Mech. **37**, 321-330 (2002).

[16] M. J. Lighthill, The response of laminar skin friction and heat transfer to fluctuations in the stream velocity, Proc. R. Soc. Lond. A224, 1-33 (1954).

[17] J. T. Stuart, A solution of the Navier-Stokes and energy equations illustrating the response of skin friction and temperature of an infinite plate thermometer fluctuations in the stream velocity, Proc. R. Soc. Lond. A231, 116-131 (1955).

[18] S. A. S. Messiha, Laminar boundary layers in oscillatory flow along an infinite flat plate with variable suction, Proc. Camb. Phil. Soc. **62**, 329-337 (1966).

[19] R. E. Kelley, The flow of a viscous fluid past a wall of infinite extent with time-dependent suction, Quart. J. Mech. Appl. Math. **18**, 287-298 (1964).

[20] V. M. Soundalgekar and P. Puri, On fluctuating flow of an elastico-viscous fluid past an infinite plate with variable suction, J. Fluid Mech. **35**, 561-573 (1969).

[21] T. Hayat, Q. Abbas, S. Asghar, A. M. Siddiqui, T. Farid and G. Murtaza, Flow of an elastico-viscous fluid past an infinite wall with time-dependent suction, Acta Mech. **153**, 133-145 (2002).

[22] D. W. Beard and K. Walters, Elastico-viscous boundary layer flows, Proc. Camb. Phil. Soc. **60**, 667-674 (1964).

[23] P. N. Kaloni, Fluctuating flow of an elastico-viscous fluid past an infinite plate with variable suction, Phys. Fluids **10**, 1344-1346 (1967).

[24] J. E. Dunn and R. L. Fosdick, Thermodynamics, stability and boundedness of fluids of complexity and fluids of second grade, Arch. Rat. Mech. Anal. **56**, 191-252 (1974).

[25] R. L. Fosdick and K. R. Rajagopal, Anomalous features in the model of second order fluids, Arch. Rat. Mech. Anal. **70**, 145-152 (1979).