Unsteady Flow Generated due to Non-coaxial Rotations of a Porous moving disk and a Jeffrey fluid

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A Thesis
Submitted in the Partial Fulfillment of the
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IN
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Certificate

We accept this thesis as conforming to the required standard.

A THESIS SUBMITTED IN THE PARTIAL FULFILLMENT OF THE REQUIREMENTS FOR THE DEGREE OF THE MASTER OF PHILOSOPHY

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Dedication

I dedicate this task to

- **My Parents and Prof. Faiz (late)**
- **My Brothers and Sister**

Acknowledgement

A Dissertation, like this cannot be accomplished without a great deal of support and encouragement, both from the institutions and individuals. I would like to take this opportunity to thank those who took part in this project as it progressed from conception to completion.

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Abdul Qayyum
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PREFACE

The exact solutions of the Navier-Stokes equations for the flow due to non-coaxial rotations of a disk and a fluid at infinity in different situations have been implied by various workers e.g. Berker [1], Coirier [2], Erdogan [3-5], Rajagopal [6], Rao and Kasiviswanathan [7], Kasiviswanathan and Rao [8] and Hayat et al. [9-11]. As is known, the Navier-Stokes equations seem to be an inappropriate model for a class of real fluids, called non- Newtonian fluids. Recently, interest in the studies of non Newtonian fluids has been increased substantially. Because of complexity of fluids in nature, the non Newtonian fluids have been categorized into various models. Amongst these, there is a subclass namely the fluids of Jeffrey type for which one can hope to obtain an analytic solution. With this fact in view, the present dissertation has been arranged as follows:

Chapter one includes some basic definitions and equations. The contents of this chapter provide relevant material for the succeeding chapters.

Chapter two describes a review of a most recent research paper of Hayat et al. [11]. Here, the magnetohydrodynamic viscous flow due to non-coaxial rotations of a porous oscillating disk and a fluid at infinity has been analyzed.

In chapter three, the analysis of chapter two is extended from Newtonian fluid to the Jeffrey fluid. The exact solutions have been constructed for suction and blowing in resonant and non-resonant cases. Comparison has also been made with the previous studies.

Contents

Chapter 1

Basic definitions and laws of fluid mechanics

1.1 Introduction

This chapter deals with some basic definitions and fundamental equations. The contents of this chapter provide the background for the succeeding chapters.

1.1.1 Fluid

A fluid is an isotropic substance the individual pieces of which continue to deform as the result of applied surface stresses.

1.1.2 Continuum model of a fluid

Fluid matter, whether liquid or gaseous, is discrete on the microscopic, i.e., the molecular level. When one is dealing with problem in which the dimensions are very large compared with molecular distances, however, it is convenient to think of lumps of fluid containing many molecules and to work with the average statistical properties of such large numbers of molecules. The detailed molecular structure is thus washed out completely and is replaced by a continuous model of matter having appropriate continuum properties so defined as to ensure that on the macroscopic scale the behavior of the real fluid.

1.1.3 Steady flow

If the physical properties of the fluid are independent upon time then the flow is treated as steady flow.

1.1.4 Unsteady flow

If the fluid properties at a given position in space vary with time then flow is unsteady.

1.1.5 Shear force

The force which is tangent to the surface are the shear force.

1.1.6 Shear stress

Shear stress at a point is the limiting value of shear force to the area as the area is reduced to a point.

1.1.7 Density

At a given temperature and pressure, the mass per unit volume is known as density. It is written as

$$\rho = \frac{m}{v_1} \tag{1.1}$$

where m is the mass and v_1 is the volume.

1.1.8 One dimensional flows

These are the flows for which the stream lines may be described as the straight lines. It is because of the reason that a straight line, being a mathematical line posses one dimension only i.e., $x-$axis, $y-$axis and $z-$axis directions.

1.1.9 Two dimensional flows

For these flows the stream lines may be represented by a curve. It is because of the reason that a curved stream line will be along any two mutually perpendicular directions.

1.1.10 Three dimensional flow

For such flows the stream lines may be represented in space.

1.1.11 Rotational flows

For such flows the curl of the velocity field does not vanish.

1.1.12 Irrotational flows

For such flows the curl of the velocity field is always zero.

1.1.13 Fluid rotation

The average angular velocity of two mutually perpendicular line elements is known as fluid rotation.

Mathematically

$$w = w_x\mathbf{i} + w_y\mathbf{j} + w_z\mathbf{k} \tag{1.2}$$

where w_x is the rotation about the $x-$axis, w_y is the rotation about the $y-$axis and w_z is the rotation about the $z-$axis. The positive sense of rotation is given by the right hand rule. In Eq. (1.2), $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$ are the unit vectors in the x, y and z directions, respectively.

1.1.14 Magnetohydrodynamics

It is the subject in which one studies the flow of an electrically conducting fluid in the presence of magnetic field. The fluid under consideration is known as MHD fluid. The term magnetohydrodynamics was first introduced by Hanne's Alfven.

Maxwell's equations

$$
\begin{aligned}
curl\mathbf{B} &= \mu_1\mathbf{J} + \frac{1}{c^2}\frac{\partial \mathbf{E}}{\partial t}, && (1.3)\\
div\mathbf{B} &= 0, && (1.4)\\
curl\mathbf{E} &= -\frac{\partial \mathbf{B}}{\partial t}, && (1.5)\\
div\mathbf{E} &= \frac{\rho^*}{\varepsilon}, && (1.6)
\end{aligned}
$$

where **B** is the total magnetic field, c is the velocity of light, **E** is the electric field, ρ^* is the charge density, **J** is the current density, μ_1 is the magnetic permeability and ε is the permittivity.

1.1.15 The substantial derivative

Let b represent the value of some fluid property (e.g., pressure, density, velocity, entropy) for a particle of fixed identity. Employing the Eulerian formulation, it is desired to calculate the time rate of change of the value of b associated with this particular particle of unchanging identity. Using Cartesian coordinates, $b = b(x, y, z)$ implies that for arbitrary and independent increments dx, dy, dz, the increment db is

$$db = \frac{\partial b}{\partial x}dx + \frac{\partial b}{\partial y}dy + \frac{\partial b}{\partial z}dz + \frac{\partial b}{\partial t}dt. \tag{1.7}$$

Passing now from arbitrary increments to the increments perceived while following in time a particle of fixed identity, the increments dx, dy and dz are no longer independent but rather related to dt by

$$dx = udt, \ dy = vdt, \ dz = wdt, \tag{1.8}$$

in which u, v and w are the components of velocity vector **V** along $x-$, $y-$ and $z-$axis respectively. Accordingly, the special value of $\frac{db}{dt}$ associated with a material particle of fixed identity, to which is assigned the special symbol $\frac{Db}{Dt}$, is given by

$$\frac{Db}{Dt} = u\frac{\partial b}{\partial x} + v\frac{\partial b}{\partial y} + w\frac{\partial b}{\partial z} + \frac{\partial b}{\partial t}. \tag{1.9}$$

In vector form

$$\frac{Db}{Dt} = (\mathbf{V} \cdot \nabla) b + \frac{\partial b}{\partial t} \tag{1.10}$$

The various name given to $\frac{Db}{Dt}$ include substantial derivative, material derivative and particle derivative.

1.2 Equation of continuity

Let $\tilde{\mathbf{V}}$ be a control volume in space. We assume that it and its surface $\bar{\mathbf{S}}$ remain fixed in space. The surface is permeable so that fluid can freely enter in and leave. Equation of continuity or conservation of mass stems from the principle that mass cannot be created nor destroyed inside the control volume. Thus the mass in the control volume $\tilde{\mathbf{V}}$ is conserved at all time, i.e.,

$$\frac{D}{Dt} \int_{\tilde{\mathbf{V}}} \rho d\tilde{\mathbf{V}} = 0, \tag{1.10 a}$$

where ρ is the density field at time t. Reynolds transport theorem states that if $\mathbf{\Phi}$ be a field (scalar, vector or tensor) associated with the fluid, then

$$\frac{D}{Dt} \int_{\tilde{\mathbf{V}}} \rho d\tilde{\mathbf{V}} = \int_{\tilde{\mathbf{V}}} \left(\frac{D\mathbf{\Phi}}{Dt} + \mathbf{\Phi} \nabla . \mathbf{V} \right) d\tilde{\mathbf{V}} = \int_{\tilde{\mathbf{V}}} \left(\frac{\partial \mathbf{\Phi}}{\partial t} + div\,(\mathbf{\Phi} . \mathbf{V}) \right) d\tilde{\mathbf{V}}, \tag{1.10 b}$$

where $\frac{D}{Dt}$ is the material time derivative.

By setting $\mathbf{\Phi} = \rho$, with ρ the fluid density, Eqs. (1.10 a) and (1.10 b) give

$$\int_{\tilde{\mathbf{V}}} \left(\frac{\partial \rho}{\partial t} + div\,(\rho \mathbf{V}) \right) d\tilde{\mathbf{V}} = 0.$$

Since the control volume $\tilde{\mathbf{V}}$ is arbitrary, a necessary and sufficient condition for conservation of mass is

$$\frac{\partial \rho}{\partial t} + div\,(\rho \mathbf{V}) = 0.$$

For an incompressible fluid, the density is constant everywhere, and the conservation of mass demands that

$$divV = 0.$$

Acceleration at a point

By Newton's law

$$\frac{D\mathbf{V}}{Dt} = \mathbf{a} = (\mathbf{V}\cdot\nabla)\mathbf{V} + \frac{\partial \mathbf{V}}{\partial t} \tag{1.11}$$

which, by means of vector identities, may also be expressed as

$$\frac{D\mathbf{V}}{Dt} = \nabla\left(\frac{\mathbf{V}^2}{2}\right) - \mathbf{V}\times(\nabla\times\mathbf{V}) + \frac{\partial \mathbf{V}}{\partial t}. \tag{1.12}$$

In scalar form we have

$$a_x = \frac{Du}{Dt} = u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} + \frac{\partial u}{\partial t},$$

$$a_y = \frac{Dv}{Dt} = u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} + \frac{\partial v}{\partial t},$$

$$a_z = \frac{Dw}{Dt} = u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} + \frac{\partial w}{\partial t},$$

where a_x, a_y and a_z are the components of the acceleration in the x, y and $z-$directions respectively.

1.3 The Navier -Stokes equations

According to the law of the conservation of momentum

$$\rho\frac{D\mathbf{V}}{Dt} = div\mathbf{T} + \rho\mathbf{b}_1, \tag{1.13}$$

where ρ is the density, $\mathbf{T}$ is the Cauchy stress tensor, $\mathbf{b}_1$ is the body force and $\mathbf{V}$ is the velocity. For viscous fluid

$$\mathbf{T} = -p\mathbf{I} + \mu\mathbf{A}_1, \tag{1.14}$$

where $p\mathbf{I}$ is indeterminate part of the stress, μ is the coefficient of viscosity and $\mathbf{A}_1$ is first kinematic tensor defined by

$$\mathbf{A}_1 = grad\mathbf{V} + (grad\mathbf{V})^T. \tag{1.15}$$

In the component form Eq.(1.13) give

$$\rho\frac{Du}{Dt} = \frac{\partial}{\partial x}(\tau_{xx}) + \frac{\partial}{\partial y}(\tau_{xy}) + \frac{\partial}{\partial z}(\tau_{xz}) + \rho b_{1x},$$

$$\rho\frac{Dv}{Dt} = \frac{\partial}{\partial x}(\tau_{yx}) + \frac{\partial}{\partial y}(\tau_{yy}) + \frac{\partial}{\partial z}(\tau_{yz}) + \rho b_{1y},$$

$$\rho\frac{Dw}{Dt} = \frac{\partial}{\partial x}(\tau_{zx}) + \frac{\partial}{\partial y}(\tau_{zy}) + \frac{\partial}{\partial z}(\tau_{zz}) + \rho b_{1z},$$

where from Eq. (1.14)

$$\tau_{xy} = \tau_{yx} = \mu\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right),$$

$$\tau_{yz} = \tau_{zy} = \mu\left(\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}\right),$$

$$\tau_{zx} = \tau_{xz} = \mu\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right),$$

$$\tau_{xx} = -p - \frac{2}{3}\mu\nabla\cdot\mathbf{V} + 2\mu\frac{\partial u}{\partial x},$$

$$\tau_{yy} = -p - \frac{2}{3}\mu\nabla\cdot\mathbf{V} + 2\mu\frac{\partial v}{\partial y},$$

$$\tau_{zz} = -p - \frac{2}{3}\mu\nabla\cdot\mathbf{V} + 2\mu\frac{\partial w}{\partial z}.$$

With the help of above equations, we have

$$\begin{aligned}\rho\left[\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}\right] &= -\frac{\partial p}{\partial x} - \frac{\partial}{\partial x}\left(\frac{2}{3}\mu\nabla\cdot\mathbf{V} - 2\mu\frac{\partial u}{\partial x}\right) + \\ &\frac{\partial}{\partial y}\left[\mu\left((\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right)\right] + \rho b_{1x} + \frac{\partial}{\partial z}\left[\mu\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right)\right],\end{aligned}$$

$$\rho\left[\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+w\frac{\partial v}{\partial z}\right] = -\frac{\partial p}{\partial y}-\frac{\partial}{\partial y}\left(\frac{2}{3}\mu\nabla\cdot\mathbf{V}-2\mu\frac{\partial \mathbf{V}}{\partial x}\right)+ \frac{\partial}{\partial x}\left[\mu\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)\right]+\rho b_{1y}+\frac{\partial}{\partial z}\left[[\mu\left(\frac{\partial v}{\partial z}+\frac{\partial w}{\partial y}\right)\right],$$

$$\rho\left[\frac{\partial w}{\partial t}+u\frac{\partial w}{\partial x}+v\frac{\partial w}{\partial y}+w\frac{\partial w}{\partial z}\right] = -\frac{\partial p}{\partial z}-\frac{\partial}{\partial z}\left(\frac{2}{3}\mu\nabla\cdot\mathbf{V}-2\mu\frac{\partial w}{\partial x}\right)+ \frac{\partial}{\partial y}\left[\mu\left(\frac{\partial v}{\partial z}+\frac{\partial w}{\partial y}\right)\right]+\rho b_{1z}+\frac{\partial}{\partial z}\left[\mu\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}\right)\right].$$

The above set of the equations is referred to as the Navier-Stokes equations of motion.

For in compressible flow $\nabla\cdot\mathbf{V}=\mathbf{0}$, and the above equations become

$$\rho\left[\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}\right]=-\frac{\partial p}{\partial x}+\mu\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}\right)+\rho b_{1x}, \tag{1.16}$$

$$\rho\left[\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+w\frac{\partial v}{\partial z}\right]=-\frac{\partial p}{\partial y}+\mu\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}+\frac{\partial^2 v}{\partial z^2}\right)+\rho b_{1y}, \tag{1.17}$$

$$\rho\left[\frac{\partial w}{\partial t}+u\frac{\partial w}{\partial x}+v\frac{\partial w}{\partial y}+w\frac{\partial w}{\partial z}\right]=-\frac{\partial p}{\partial z}+\mu\left(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}+\frac{\partial^2 w}{\partial z^2}\right)+\rho b_{1z}. \tag{1.18}$$

Where b_{1x} , b_{1y} and b_{1z} are the x , y and $z-$components of $\mathbf{b}_1$ and for $\mu=0$, the above equations reduce to well-known equations given by Euler which hold for the ideal fluids.

1.4 Transform technique

Transform techniques play an important role in obtaining the solution of the partial differential equations, especially when the boundary conditions include the infinite or the semi infinite domain. In the present thesis we will use the Laplace transform.

1.4.1 Laplace transform

Given a function $f(t)$ defined for all $t\geq 0$, the Laplace transform of f is the function F defined by

$$F(s)=L[f(t)]=\int_0^\infty e^{-st}f(t)dt \tag{1.19}$$

for all values of s (where s is a Laplace parameter) for which the improper integral converges. The inverse Laplace transform is given by

$$f(t) = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} F(s)e^{st}ds, \gamma > 0. \tag{1.20}$$

1.4.2 The error function

The error function, abbreviated as "erf", is defined by

$$erf(x) = \frac{2}{\sqrt{\pi}}\int_0^x \exp(-\beta^2)d\beta. \tag{1.21}$$

1.4.3 The complementary error function

It is denoted by $erfc(x)$ and is defined by

$$erfc(x) = 1 - erf(x) = \frac{2}{\sqrt{\pi}}\int_x^\infty \exp(-\beta^2)d\beta. \tag{1.22}$$

Chapter 2

Magnetohydrodynamic flow due to non-coaxial rotations of a porous oscillating disk and a fluid at infinity

2.1 Introduction

The time-dependent viscous flow due to non-coaxial rotations of a porous oscillating disk and a fluid at infinity has been studied. The fluid is conducting in the presence of a transverse magnetic field. The disk is non-conducting. The exact expressions for velocity profile have been given for the three cases i.e. when the angular velocity is greater than, smaller than and equal to oscillating frequency. The analytic solutions have been obtained using Laplace transform method. The contents of this chapter is basically a review of a recent work by Hayat et al. [11].

2.2 Mathematical formulation

Let us consider the unsteady flow of an electrically conducting fluid ($z > 0$) bounded by a porous disk at $z = 0$. A uniform magnetic field of strength $\mathbf{B}_0$ is applied transversely to the flow. Both the fluid and the disk have common angular velocity $\mathbf{\Omega} = \Omega\mathbf{k}$ ($\mathbf{k}$ is a unit vector parallel to the $z-$axis). Also, the axes of rotation of both the disk and the fluid are in the plane

$x = 0$ with the distance between the axes being l. The disk and the fluid are initially rotating about $z-$axis and suddenly sets in motion; the disk rotating about the $z-$axis and fluid about $z^{/}-$axis. The appropriate boundary and initial conditions are [5]:

$$u = -\Omega y + U\cos nt, \ v = \Omega x \text{ at } z = 0 \ for \ t > 0, \tag{2.1}$$

$$u = -\Omega y + U\sin nt, \ v = \Omega x \text{ at } z = 0 \ for \ t > 0, \tag{2.2}$$

$$u = -\Omega(y - l), \ v = \Omega x \text{ as } z \to \infty, \quad for \ all \ t, \tag{2.3}$$

$$u = -\Omega(y - l), \ v = \Omega x \text{ at } t = 0, \ z > 0. \tag{2.4}$$

where U is the reference velocity. The velocity field is defined as [4].

$$u = -\Omega y + f(z,t), \ v = \Omega x + g(z,t), \tag{2.5}$$

in which u and v are the components of the velocity in the directions of x and y respectively.

The governing equations are

$$\frac{\partial \mathbf{V}}{\partial t} + (\mathbf{V}\cdot\nabla)\mathbf{V} = -\frac{1}{\rho}\nabla p + \nu\nabla^2\mathbf{V} + \frac{1}{\rho}\mathbf{J}\times\mathbf{B}, \tag{2.6}$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0, \tag{2.7}$$

where $\mathbf{V} = (u, v, w)$ is the velocity field and

$p =$ pressure

$\rho =$ density of the fluid

$\mathbf{J} =$ electric current density.

$\mathbf{B} =$ Total magnetic field.

$\mathbf{B} = \mathbf{B}_0 + \mathbf{b}$

$\mathbf{B}_0 =$ imposed magnetic field, $\mathbf{b} =$ induced magnetic field.

$\nu =$ kinematic viscosity.

The Maxwell equations and the generalized Ohm's law are

$$div\mathbf{B} = 0,\ curl\mathbf{B} = \mu_1\mathbf{J}\ ,\ curl\mathbf{E} = -\frac{\partial\mathbf{B}}{\partial t}, \tag{2.8}$$

$$\mathbf{J} = \sigma(\mathbf{E} + \mathbf{V}\times\mathbf{B}), \tag{2.9}$$

where

μ = magnetic permeability

$\mathbf{E}$ = electric field.

σ = electric conductivity of the fluid.

Let us consider the following assumptions:

- The quantities ρ, ν, μ and σ are all constants throughout the flow field.
- The magnetic field $\mathbf{B}$ is perpendicular to the velocity field $\mathbf{V}$.
- Induced magnetic field is negligible compared with the imposed field so that the magnetic Reynolds number is small.
- The electric field is assumed to be zero.

Under the above assumption we have

$$\begin{aligned}\text{The electromagnetic force} &= \frac{1}{\rho}(\mathbf{J}\times\mathbf{B})\\ &= \frac{\sigma}{\rho}[(\mathbf{V}\times\mathbf{B})\times\mathbf{B}]\\ &= \frac{\sigma}{\rho}[(\mathbf{V}\cdot\mathbf{B})\mathbf{B} - (\mathbf{B}\cdot\mathbf{B})\mathbf{V}]\\ &= \frac{\sigma}{\rho}[-(\mathbf{B}_0\cdot\mathbf{B}_0)\mathbf{V}]\\ &= -(\frac{\sigma B_0^2}{\rho})\mathbf{V}\end{aligned}$$

$$\text{The electromagnetic force} = -N\mathbf{V}, \tag{2.10}$$

where $N = \frac{\sigma B_0^2}{\rho}$ has the same dimension as Ω.

Using Eq. (2.10), Eq. (2.6) becomes

$$\frac{\partial \mathbf{V}}{\partial t} + (\mathbf{V} \cdot \nabla)\mathbf{V} = -\frac{1}{\rho}\nabla p + \nu \nabla^2 \mathbf{V} - N\mathbf{V}. \tag{2.11}$$

From Eqs. (2.5) and (2.7) one can write for uniform porous disk that

$$w = \text{constant} = -w_0, \tag{2.12}$$

where $w_0 > 0$ is the suction velocity and $w_0 < 0$ is the blowing velocity and thus the velocity field is

$$\mathbf{V} = [u, v, w] = [-\Omega y + f(z,t), \Omega x + g(z,t), -w_0]. \tag{2.13}$$

Substituting Eq. (2.13) into Eq. (2.11) we get the following scalar equations

$$\frac{\partial f}{\partial t} - \Omega(\Omega x + g) - w_0 \frac{\partial f}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu \frac{\partial^2 f}{\partial z^2} - N\{-\Omega y + f(z,t)\}, \tag{2.14}$$

$$\frac{\partial g}{\partial t} + \Omega(-\Omega y + f) - w_0 \frac{\partial g}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial y} + \nu \frac{\partial^2 g}{\partial z^2} - N\{\Omega x + g(z,t)\}, \tag{2.15}$$

$$\frac{1}{\rho}\frac{\partial p}{\partial z} = N w_0. \tag{2.16}$$

The above equations can also be written as

$$\frac{\partial f}{\partial t} - \Omega g - w_0 \frac{\partial f}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \Omega^2 x + \nu \frac{\partial^2 f}{\partial z^2} - N\{-\Omega y + f(z,t)\}, \tag{2.17}$$

$$\frac{\partial g}{\partial t} + \Omega f - w_0 \frac{\partial g}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial y} + \Omega^2 y + \nu \frac{\partial^2 g}{\partial z^2} - N\{\Omega x + g(z,t)\}, \tag{2.18}$$

$$\frac{1}{\rho}\frac{\partial p}{\partial z} = N w_0. \tag{2.19}$$

Since

$$r^2 = x^2 + y^2,$$

$$x = \frac{1}{2}\frac{\partial r^2}{\partial x}, \; y = \frac{1}{2}\frac{\partial r^2}{\partial y},$$

and thus Eqs. (2.17) to Eq. (2.19) give

$$\frac{\partial f}{\partial t} - \Omega g - w_0 \frac{\partial f}{\partial z} = -\frac{1}{\rho}\frac{\partial}{\partial x}\left(p - \frac{\rho \Omega^2 r^2}{2}\right) + \nu \frac{\partial^2 f}{\partial z^2} - N\{-\Omega y + f(z,t)\}, \tag{2.20}$$

$$\frac{\partial g}{\partial t} + \Omega f - w_0 \frac{\partial g}{\partial z} = -\frac{1}{\rho}\frac{\partial}{\partial y}\left(p - \frac{\rho \Omega^2 r^2}{2}\right) + \nu \frac{\partial^2 g}{\partial z^2} - N\{\Omega x + g(z,t)\}, \tag{2.21}$$

$$\frac{\partial p}{\partial z} = \rho N w_0. \tag{2.22}$$

Defining the modified pressure

$$\widehat{p} = p - \frac{\rho \Omega^2 r^2}{2} \tag{2.23}$$

we can write

$$\frac{\partial f}{\partial t} - \Omega g - w_0 \frac{\partial f}{\partial z} = -\frac{1}{\rho}\frac{\partial \widehat{p}}{\partial x} + \nu \frac{\partial^2 f}{\partial z^2} - N\{-\Omega y + f(z,t)\}, \tag{2.24}$$

$$\frac{\partial g}{\partial t} + \Omega f - w_0 \frac{\partial g}{\partial z} = -\frac{1}{\rho}\frac{\partial \widehat{p}}{\partial y} + \nu \frac{\partial^2 g}{\partial z^2} - N\{\Omega x + g(z,t)\}, \tag{2.25}$$

$$\frac{\partial \widehat{p}}{\partial z} = \rho N w_0. \tag{2.26}$$

The boundary and initial conditions for f and g are

$$\left.\begin{array}{c} f(0,t) = U\cos nt \text{ or } U\sin nt,\ g(0,t) = 0, t > 0, \\ f(\infty,0) = \Omega l,\ g(\infty,t) = 0 \text{ for all } t, \\ f(z,0) = \Omega l,\ z > 0,\ g(z,0) = 0,\ z > 0. \end{array}\right\} \tag{2.27}$$

Differentiating Eqs.(2.24) and (2.25) with respect to z, and using Eq.(2.26) we get

$$\frac{\partial}{\partial z}\left[\frac{\partial f}{\partial t} - \Omega g - w_0 \frac{\partial f}{\partial z}\right] = \frac{\partial}{\partial z}\left[\nu \frac{\partial^2 f}{\partial z^2} - N f(z,t)\right], \tag{2.28}$$

$$\frac{\partial}{\partial z}\left[\frac{\partial g}{\partial t} + \Omega f - w_0 \frac{\partial g}{\partial z}\right] = \frac{\partial}{\partial z}\left[\nu \frac{\partial^2 g}{\partial z^2} - N g(z,t)\right]. \tag{2.29}$$

Integration of Eqs. (2.28) and (2.29) yields

$$\frac{\partial f}{\partial t} - \Omega g - w_0 \frac{\partial f}{\partial z} = \nu \frac{\partial^2 f}{\partial z^2} - N f(z,t) + c_1(t), \tag{2.30}$$

$$\frac{\partial g}{\partial t}+\Omega f-w_0\frac{\partial g}{\partial z}=\nu\frac{\partial^2 g}{\partial z^2}-Ng(z,t)+c_2(t), \tag{2.31}$$

where the functions of integration $c_1(t)$ and $c_2(t)$ can be calculated by using the boundary condition in Eq. (2.27) and are

$$c_1(t)=N\Omega l,$$

$$c_2(t)=\Omega^2 l,$$

which upon using in Eqs. (2.30) and (2.31) yields

$$\frac{\partial f}{\partial t}-\Omega g-w_0\frac{\partial f}{\partial z}=\nu\frac{\partial^2 f}{\partial z^2}-Nf(z,t)+N\Omega l, \tag{2.32}$$

$$\frac{\partial g}{\partial t}+\Omega f-w_0\frac{\partial g}{\partial z}=\nu\frac{\partial^2 g}{\partial z^2}-Ng(z,t)+\Omega^2 l. \tag{2.33}$$

Multiplying Eq.(2.33) by i and adding in Eq.(2.32), we get

$$\frac{\partial}{\partial t}(f+ig)+i\Omega(f+ig)-w_0\frac{\partial}{\partial z}(f+ig)=\nu\frac{\partial^2}{\partial z^2}(f+ig)-N(f+ig)+\Omega l(N+i\Omega),$$

or

$$\frac{\partial}{\partial t}(f+ig)-w_0\frac{\partial}{\partial z}(f+ig)-\nu\frac{\partial^2}{\partial z^2}(f+ig)+(N+i\Omega)(f+ig)-\Omega l(N+i\Omega)=0$$

or

$$\frac{\partial}{\partial t}(f+ig)-w_0\frac{\partial}{\partial z}(f+ig)-\nu\frac{\partial^2}{\partial z^2}(f+ig)+(N+i\Omega)(f+ig-\Omega l)=0,$$

or

$$\left.\begin{array}{c}0=\frac{\partial}{\partial t}\left(\frac{f}{\Omega l}+i\frac{g}{\Omega l}\right)-w_0\frac{\partial}{\partial z}\left(\frac{f}{\Omega l}+i\frac{g}{\Omega l}\right)-\nu\frac{\partial^2}{\partial z^2}\left(\frac{f}{\Omega l}+i\frac{g}{\Omega l}\right)\\+(N+i\Omega)\left(\frac{f}{\Omega l}+i\frac{g}{\Omega l}-1\right)\end{array}\right\},$$

or

$$\left.\begin{array}{c}0=\frac{\partial}{\partial t}\left(\frac{f}{\Omega l}+i\frac{g}{\Omega l}-1\right)-w_0\frac{\partial}{\partial z}\left(\frac{f}{\Omega l}+i\frac{g}{\Omega l}-1\right)-\nu\frac{\partial^2}{\partial z^2}\left(\frac{f}{\Omega l}+i\frac{g}{\Omega l}-1\right)\\+(N+i\Omega)\left(\frac{f}{\Omega l}+i\frac{g}{\Omega l}-1\right)\end{array}\right\},$$

or

$$\nu\frac{\partial^2 G}{\partial z^2}-\frac{\partial G}{\partial t}+w_0\frac{\partial G}{\partial z}-(N+i\Omega)G=0, \tag{2.34}$$

where

$$G(z,t) = \frac{f(z,t)}{\Omega l} + i\frac{g(z,t)}{\Omega l} - 1. \tag{2.35}$$

The boundary conditions (2.27) in terms of G are

$$G(0,t) = \frac{U}{\Omega t}\cos nt - 1 \; or \; G(0,t) = \frac{U}{\Omega t}\sin nt - 1, \; G(\infty,t) = 0, \; G(z,0) = 0. \tag{2.36}$$

Using

$$G(z,t) = H(z,t)e^{-i\Omega t} \tag{2.37}$$

equation (2.34) and boundary conditions (2.36) give

$$\left.\begin{array}{c} \nu\frac{\partial^2 H}{\partial z^2} - \frac{\partial H}{\partial t} + w_0\frac{\partial H}{\partial z} - NH = 0, \\ H(0,t) = \left(\frac{U}{\Omega l}\cos nt - 1\right)e^{i\Omega t} \; or \; H(0,t) = \left(\frac{U}{\Omega l}\sin nt - 1\right)e^{i\Omega t}, \\ H(\infty,t) = 0, \; H(z,0) = 0. \end{array}\right\} \tag{2.38}$$

2.3 Solution of the problem

We will find the solution by Laplace transform treatment. For that we define the Laplace transform pair as

$$\overline{H}(z,s) = L\{H(z,t)\} = \int_0^\infty H(z,t)e^{-st}dt,$$

and

$$H(z,t) = L^{-1}\{\overline{H}(z,s)\} = \frac{1}{2\pi i}\int_{\lambda-i\infty}^{\lambda+i\infty} \overline{H}(z,s)e^{st}ds.$$

We first solve the problem which involve cosine oscillations in the boundary condition. The transformed problem for $n > \Omega$ is

$$\nu\frac{d^2\overline{H}}{dz^2} + w_0\frac{d\overline{H}}{dz} - (s+N)\overline{H} = 0, \tag{2.39}$$

$$\overline{H}(0,s) = \frac{U}{2\Omega l}\left[\frac{1}{s-i(n+\Omega)} + \frac{1}{s+i(n+\Omega)}\right] - \frac{1}{(n-\Omega)}, \tag{2.40}$$

$$\overline{H}(\infty,s) = 0. \tag{2.41}$$

The solution of Eq.(2.39) is

$$\overline{H}(z,s) = Ae^{(-\frac{w_0}{2\nu}+\sqrt{\frac{w_0^2}{4\nu^2}+\frac{s}{\nu}+\frac{N}{\nu}})z} + Be^{(-\frac{w_0}{2\nu}-\sqrt{\frac{w_0^2}{4\nu^2}+\frac{s}{\nu}+\frac{N}{\nu}})z}, \tag{2.42}$$

where the constants A and B will be determined using the boundary conditions (2.40) and (2.41) and are

$$A = 0,$$

$$B = \frac{U}{2\Omega l}\left[\frac{1}{s-i(n+\Omega)} + \frac{1}{s+i(n-\Omega)}\right] - \frac{1}{s-\Omega},$$

which upon using in Eq. (2.42) yields

$$\overline{H}(z,s) = \left[-\frac{1}{(s-i\Omega)} + \frac{U}{2\Omega l}\frac{1}{s-i(n+\Omega)} + \frac{U}{2\Omega l}\frac{1}{s+i(n+\Omega)}\right] e^{\left(-\frac{w_0}{2\nu}-\sqrt{\frac{w_0^2}{4\nu^2}+\frac{s}{\nu}+\frac{N}{\nu}}\right)z}.$$

Taking inverse Laplace transform we obtain

$$H(z,t) = e^{-\frac{w_0}{2\nu}z}\left[-I_1 + \frac{U}{2\Omega l}I_2 + \frac{U}{2\Omega l}I_3\right], \tag{2.43}$$

where

$$I_1 = L^{-1}\left\{\frac{e^{-\sqrt{\frac{w_0^2}{4\nu^2}+\frac{s}{\nu}+\frac{N}{\nu}}z}}{(s-i\Omega)}\right\}, I_2 = L^{-1}\left\{\frac{e^{-\sqrt{\frac{w_0^2}{4\nu^2}+\frac{s}{\nu}+\frac{N}{\nu}}z}}{s-i(n+\Omega)}\right\}, I_3 = L^{-1}\left\{\frac{e^{-\sqrt{\frac{w_0^2}{4\nu^2}+\frac{s}{\nu}+\frac{N}{\nu}}z}}{s+i(n-\Omega)}\right\}. \tag{2.44}$$

First we find the solution of I_1 i.e.

$$I_1 = L^{-1}\left\{\frac{e^{-\sqrt{\frac{w_0^2}{4\nu^2}+\frac{s}{\nu}+\frac{N}{\nu}}z}}{(s-i\Omega)}\right\}$$

$$= \frac{1}{2\pi i}\int \frac{e^{st-\sqrt{\frac{w_0^2}{4\nu^2}+\frac{s}{\nu}+\frac{N}{\nu}}z}}{(s-i\Omega)}ds.$$

Putting

$$s - i\Omega = \theta$$

I_1 becomes

$$I_1 = \frac{1}{2\pi i} \int \frac{e^{(\theta+i\Omega)t - \sqrt{\frac{w_0^2}{4\nu^2} + \frac{\theta+i\Omega}{\nu} + \frac{N}{\nu}}\, z}}{\theta} d\theta,$$

or

$$I_1 = \frac{e^{i\Omega t}}{2\pi i} \int \frac{e^{\theta t - \frac{z}{\sqrt{\nu}} \sqrt{\left(\frac{w_0^2}{4\nu} + N + i\Omega\right) + \theta}}}{\theta} d\theta.$$

Using the formula

$$\frac{1}{2\pi i} \int \frac{e^{xt - b\sqrt{a^2+x}}}{x} dx = \frac{1}{2}\left[e^{ab} \operatorname{erf} c\left(\frac{b+2at}{2\sqrt{t}}\right) - e^{-ab} \operatorname{erf} c\left(\frac{b-2at}{2\sqrt{t}}\right)\right],$$

we get

$$I_1 = \frac{e^{i\Omega t}}{2} \left[\begin{array}{c} e^{\sqrt{\frac{w_0^2}{4\nu} + N + i\Omega}\frac{z}{\sqrt{\nu}}} erfc\left(\frac{\frac{z}{\sqrt{\nu}} + 2\sqrt{\frac{w_0^2}{4\nu} + N + i\Omega t}}{2\sqrt{t}}\right) + \\ e^{-\sqrt{\frac{w_0^2}{4\nu} + N + i\Omega}\frac{z}{\sqrt{\nu}}} erfc\left(\frac{\frac{z}{\sqrt{\nu}} - 2\sqrt{\frac{w_0^2}{4\nu} + N + i\Omega t}}{2\sqrt{t}}\right) \end{array} \right],$$

or

$$I_1 = \frac{e^{i\Omega t}}{2} \left[\begin{array}{c} e^{\sqrt{\frac{w_0^2}{4\nu} + N + i\Omega}\frac{z}{\sqrt{\nu}}} erfc\left(\frac{\frac{z}{\sqrt{\nu}} + 2\sqrt{\frac{w_0^2}{4\nu} + N + i\Omega t}}{2\sqrt{t}}\right) + \\ e^{-\sqrt{\frac{w_0^2}{4\nu} + N + i\Omega}\frac{z}{\sqrt{\nu}}} erfc\left(\frac{\frac{z}{\sqrt{\nu}} - 2\sqrt{\frac{w_0^2}{4\nu} + N + i\Omega t}}{2\sqrt{t}}\right) \end{array} \right],$$

or

$$I_1 = \frac{e^{i\Omega t}}{2} \left[\begin{array}{c} e^{\sqrt{\frac{w_0^2}{4\nu^2} + \frac{N}{\nu} + \frac{i\Omega}{\nu}}\, z} erfc\left(\frac{z}{2\sqrt{\nu t}} + \sqrt{\left(\frac{w_0^2}{4\nu^2} + \frac{N}{\nu} + \frac{i\Omega}{\nu}\right)\nu t}\right) \\ + e^{-\sqrt{\frac{w_0^2}{4\nu^2} + \frac{N}{\nu} + \frac{i\Omega}{\nu}}\, z} erfc\left(\frac{z}{2\sqrt{\nu t}} - \sqrt{\left(\frac{w_0^2}{4\nu^2} + \frac{N}{\nu} + \frac{i\Omega}{\nu}\right)\nu t}\right) \end{array} \right]. \quad (2.45)$$

In a similar way we get

$$I_2 = \frac{e^{i(n+\Omega)t}}{2} \left[\begin{array}{c} e^{\sqrt{\frac{w_0^2}{4\nu^2} + \frac{i(n+\Omega)}{\nu} + \frac{N}{\nu}}\, z} \operatorname{erf} c\left(\frac{z}{2\sqrt{\nu t}} + \sqrt{\left(\frac{w_0^2}{4\nu^2} + \frac{i(n+\Omega)}{\nu} + \frac{N}{\nu}\right)\nu t}\right) \\ + e^{-\sqrt{\frac{w_0^2}{4\nu^2} + \frac{i(n+\Omega)}{\nu} + \frac{N}{\nu}}\, z} \operatorname{erf} c\left(\frac{z}{2\sqrt{\nu t}} - \sqrt{\left(\frac{w_0^2}{4\nu^2} + \frac{i(n+\Omega)}{\nu} + \frac{N}{\nu}\right)\nu t}\right) \end{array} \right], \quad (2.46)$$

and

$$I_3 = \frac{e^{-i(n+\Omega)t}}{2}\left[\begin{array}{c} e^{\sqrt{\frac{w_0^2}{4\nu}-i(n-\Omega)+N}\frac{z}{\sqrt{\nu}}}\operatorname{erf}c\left(\frac{\frac{z}{\sqrt{\nu}}+2\sqrt{\frac{w_0^2}{4\nu}-i(n-\Omega)+N t}}{2\sqrt{t}}\right)+e^{-\sqrt{\frac{w_0^2}{4\nu}-i(n-\Omega)+N}\frac{z}{\sqrt{\nu}}} \\ \operatorname{erf}c\left(\frac{\frac{z}{\sqrt{\nu}}-2\sqrt{\frac{w_0^2}{4\nu}-i(n-\Omega)+N t}}{2\sqrt{t}}\right)\end{array}\right]$$

or

$$I_3 = \frac{e^{-i(n-\Omega)t}}{2}\left[\begin{array}{c} e^{\sqrt{\frac{w_0^2}{4\nu^2}+\frac{N}{\nu}-\frac{i(n-\Omega)}{\nu}}z}\operatorname{erf}c\left\{\frac{z}{2\sqrt{\nu t}}+\sqrt{\left(\frac{w_0^2}{4\nu^2}+\frac{N}{\nu}-\frac{i(n-\Omega)}{\nu}\right)\nu t}\right\} \\ +e^{-\sqrt{\frac{w_0^2}{4\nu^2}+\frac{N}{\nu}-\frac{i(n-\Omega)}{\nu}}z}\operatorname{erf}c\left\{\frac{z}{2\sqrt{\nu t}}-\sqrt{\left(\frac{w_0^2}{4\nu^2}+\frac{N}{\nu}-\frac{i(n-\Omega)}{\nu}\right)\nu t}\right\}\end{array}\right]. \tag{2.47}$$

Now using Eqs (2.45), (2.46) and (2.47) in Eq.(2.43) we obtain

$$H(z,t) = e^{-\frac{w_0}{2\nu}z}\left[\begin{array}{c} -\frac{e^{i\Omega t}}{2}\left\{\begin{array}{c} e^{\sqrt{\frac{w_0^2}{4\nu^2}+\frac{N}{\nu}+\frac{i\Omega}{\nu}}z} erfc\left(\frac{z}{2\sqrt{\nu t}}+\sqrt{\left(\frac{w_0^2}{4\nu^2}+\frac{N}{\nu}+\frac{i\Omega}{\nu}\right)\nu t}\right)+ \\ e^{-\sqrt{\frac{w_0^2}{4\nu^2}+\frac{N}{\nu}+\frac{i\Omega}{\nu}}z} erfc\left(\frac{z}{2\sqrt{\nu t}}-\sqrt{\left(\frac{w_0^2}{4\nu^2}+\frac{N}{\nu}+\frac{i\Omega}{\nu}\right)\nu t}\right)\end{array}\right\} \\ +\left(\frac{U}{2\Omega l}\frac{e^{i(n+\Omega)t}}{2}\right) \\ \times\left\{\begin{array}{c} e^{\sqrt{\frac{w_0^2}{4\nu^2}+\frac{i(n+\Omega)}{\nu}+\frac{N}{\nu}}z}\operatorname{erf}c\left(\frac{z}{2\sqrt{\nu t}}+\sqrt{\left(\frac{w_0^2}{4\nu^2}+\frac{i(n+\Omega)}{\nu}+\frac{N}{\nu}\right)\nu t}\right) \\ +e^{-\sqrt{\frac{w_0^2}{4\nu^2}+\frac{i(n+\Omega)}{\nu}+\frac{N}{\nu}}z}\operatorname{erf}c\left(\frac{z}{2\sqrt{\nu t}}-\sqrt{\left(\frac{w_0^2}{4\nu^2}+\frac{i(n+\Omega)}{\nu}+\frac{N}{\nu}\right)\nu t}\right)\end{array}\right\} \\ +\left(\frac{U}{2\Omega l}\frac{e^{-i(n-\Omega)t}}{2}\right) \\ \times\left\{\begin{array}{c} e^{\sqrt{\frac{w_0^2}{4\nu^2}+\frac{N}{\nu}-\frac{i(n-\Omega)}{\nu}}z}\operatorname{erf}c\left(\frac{z}{2\sqrt{\nu t}}+\sqrt{\left(\frac{w_0^2}{4\nu^2}+\frac{N}{\nu}-\frac{i(n-\Omega)}{\nu}\right)\nu t}\right)+ \\ e^{-\sqrt{\frac{w_0^2}{4\nu^2}+\frac{N}{\nu}-\frac{i(n-\Omega)}{\nu}}z}\operatorname{erf}c\left(\frac{z}{2\sqrt{\nu t}}-\sqrt{\left(\frac{w_0^2}{4\nu^2}+\frac{N}{\nu}-\frac{i(n-\Omega)}{\nu}\right)\nu t}\right)\end{array}\right\}\end{array}\right]. \tag{2.48}$$

Since

$$H(z,t) = G(z,t)e^{i\Omega t},$$

so

$$G(z,t)=\frac{e^{-\frac{w_0}{2\nu}z}}{2}\left[\begin{array}{c}-\left\{\begin{array}{c}e^{\sqrt{\frac{w_0^2}{4\nu^2}+\frac{N}{\nu}+\frac{i\Omega}{\nu}}z}erfc\left(\frac{z}{2\sqrt{\nu t}}+\sqrt{\left(\frac{w_0^2}{4\nu^2}+\frac{N}{\nu}+\frac{i\Omega}{\nu}\right)\nu t}\right)+\\ e^{-\sqrt{\frac{w_0^2}{4\nu^2}+\frac{N}{\nu}+\frac{i\Omega}{\nu}}z}erfc\left(\frac{z}{2\sqrt{\nu t}}-\sqrt{\left(\frac{w_0^2}{4\nu^2}+\frac{N}{\nu}+\frac{i\Omega}{\nu}\right)\nu t}\right)\end{array}\right\}\\ +\left(\frac{U}{2\Omega l}\frac{e^{int}}{2}\right)\\ \times\left\{\begin{array}{c}e^{\sqrt{\frac{w_0^2}{4\nu^2}+\frac{i(n+\Omega)}{\nu}+\frac{N}{\nu}}z}\operatorname{erf}c\left(\frac{z}{2\sqrt{\nu t}}+\sqrt{\left(\frac{w_0^2}{4\nu^2}+\frac{i(n+\Omega)}{\nu}+\frac{N}{\nu}\right)\nu t}\right)+\\ e^{-\sqrt{\frac{w_0^2}{4\nu^2}+\frac{i(n+\Omega)}{\nu}+\frac{N}{\nu}}z}\operatorname{erf}c\left(\frac{z}{2\sqrt{\nu t}}-\sqrt{\left(\frac{w_0^2}{4\nu^2}+\frac{i(n+\Omega)}{\nu}+\frac{N}{\nu}\right)\nu t}\right)\end{array}\right\}\\ +\left(\frac{U}{2\Omega l}\frac{e^{-int}}{2}\right)\\ \times\left\{\begin{array}{c}e^{\sqrt{\frac{w_0^2}{4\nu^2}+\frac{N}{\nu}-\frac{i(n-\Omega)}{\nu}}z}\operatorname{erf}c\left(\frac{z}{2\sqrt{\nu t}}+\sqrt{\left(\frac{w_0^2}{4\nu^2}+\frac{N}{\nu}-\frac{i(n-\Omega)}{\nu}\right)\nu t}\right)+\\ e^{-\sqrt{\frac{w_0^2}{4\nu^2}+\frac{N}{\nu}-\frac{i(n-\Omega)}{\nu}}z}\operatorname{erf}c\left(\frac{z}{2\sqrt{\nu t}}-\sqrt{\left(\frac{w_0^2}{4\nu^2}+\frac{N}{\nu}-\frac{i(n-\Omega)}{\nu}\right)\nu t}\right)\end{array}\right\}\end{array}\right]. \tag{2.49}$$

Using

$$G(z,t)=\frac{f(z,t)}{\Omega l}+i\frac{g(z,t)}{\Omega l}-1$$

we have

$$\left.\begin{array}{c}\frac{f(z,t)}{\Omega l}+i\frac{g(z,t)}{\Omega l}=1+\frac{e^{-\frac{w_0}{2\nu}z}}{2}\\ \times\left[\begin{array}{c}-\left\{\begin{array}{c}e^{\sqrt{\frac{w_0^2}{4\nu^2}+\frac{N}{\nu}+\frac{i\Omega}{\nu}}z}erfc\left(\frac{z}{2\sqrt{\nu t}}+\sqrt{\left(\frac{w_0^2}{4\nu^2}+\frac{N}{\nu}+\frac{i\Omega}{\nu}\right)\nu t}\right)\\ +e^{-\sqrt{\frac{w_0^2}{4\nu^2}+\frac{N}{\nu}+\frac{i\Omega}{\nu}}z}erfc\left(\frac{z}{2\sqrt{\nu t}}-\sqrt{\left(\frac{w_0^2}{4\nu^2}+\frac{N}{\nu}+\frac{i\Omega}{\nu}\right)\nu t}\right)\end{array}\right\}\\ +\left(\frac{U}{2\Omega l}e^{int}\right)\\ \times\left\{\begin{array}{c}e^{\sqrt{\frac{w_0^2}{4\nu^2}+\frac{i(n+\Omega)}{\nu}+\frac{N}{\nu}}z}\operatorname{erf}c\left(\frac{z}{2\sqrt{\nu t}}+\sqrt{\left(\frac{w_0^2}{4\nu^2}+\frac{i(n+\Omega)}{\nu}+\frac{N}{\nu}\right)\nu t}\right)\\ +e^{-\sqrt{\frac{w_0^2}{4\nu^2}+\frac{i(n+\Omega)}{\nu}+\frac{N}{\nu}}z}\operatorname{erf}c\left(\frac{z}{2\sqrt{\nu t}}-\sqrt{\left(\frac{w_0^2}{4\nu^2}+\frac{i(n+\Omega)}{\nu}+\frac{N}{\nu}\right)\nu t}\right)\end{array}\right\}\\ +\left(\frac{U}{2\Omega l}e^{-int}\right)\\ \times\left\{\begin{array}{c}e^{\sqrt{\frac{w_0^2}{4\nu^2}+\frac{N}{\nu}-\frac{i(n-\Omega)}{\nu}}z}\operatorname{erf}c\left(\frac{z}{2\sqrt{\nu t}}+\sqrt{\left(\frac{w_0^2}{4\nu^2}+\frac{N}{\nu}-\frac{i(n-\Omega)}{\nu}\right)\nu t}\right)+\\ e^{-\sqrt{\frac{w_0^2}{4\nu^2}+\frac{N}{\nu}-\frac{i(n-\Omega)}{\nu}}z}\operatorname{erf}c\left(\frac{z}{2\sqrt{\nu t}}-\sqrt{\left(\frac{w_0^2}{4\nu^2}+\frac{N}{\nu}-\frac{i(n-\Omega)}{\nu}\right)\nu t}\right)\end{array}\right\}\end{array}\right]\end{array}\right\} \tag{2.50}$$

Defining the following non-dimensional variables

$$\xi = \sqrt{\frac{\Omega}{2\nu}}z, k = \frac{n}{\Omega}, \epsilon = \frac{U}{2\Omega l}, \tau = \Omega t, N_1 = \frac{\sigma B_0^2}{\rho\Omega}, S = \frac{w_0}{2\sqrt{\nu\Omega}}, \tag{2.51}$$

we have

$$e^{-\frac{w_0}{2\nu}z} = e^{-\frac{w_0}{2\nu\Omega}\Omega z} = e^{-\frac{w_0}{2\sqrt{\nu\Omega}}\frac{\Omega}{\sqrt{\nu\Omega}}z} = e^{-S\sqrt{\frac{\Omega}{\nu}}z} = e^{-S\sqrt{2}\sqrt{\frac{\Omega}{2\nu}}z} = e^{-\sqrt{2}S\xi}e^{-\frac{w_0}{2\nu}z},$$

$$e^{-\frac{w_0}{2\nu}z} = e^{-\sqrt{2}S\xi}, \tag{2.52}$$

$$\begin{aligned}\sqrt{\frac{w_0^2}{4\nu^2} + \frac{N}{\nu} + \frac{i\Omega}{\nu}}z &= \sqrt{\frac{4\nu\Omega S^2}{4\nu^2} + \frac{\sigma B_0^2}{\nu\rho} + \frac{i\Omega}{\nu}}z = \sqrt{\frac{\Omega S^2}{\nu} + \frac{\Omega}{\nu}\frac{\sigma B_0^2}{\rho\Omega} + \frac{i\Omega}{\nu}}z \\ &= \sqrt{S^2 + N_1 + i}\sqrt{2}\sqrt{\frac{\Omega}{2\nu}}z = \sqrt{S^2 + N_1 + i}\sqrt{2}\xi,\end{aligned}$$

$$\sqrt{\frac{w_0^2}{4\nu^2} + \frac{N}{\nu} + \frac{i\Omega}{\nu}}z = (\alpha + i\beta)\xi, \tag{2.53}$$

where

$$\alpha + i\beta = \sqrt{2}\sqrt{S^2 + N_1 + i}, \tag{2.54}$$

$$\alpha = \sqrt{\sqrt{(S^2 + N_1)^2 + 1} + (S^2 + N_1)}, \beta = \sqrt{\sqrt{(S^2 + N_1)^2 + 1} - (S^2 + N_1)}, \tag{2.55}$$

$$\frac{z}{2\sqrt{\nu t}} = \frac{1}{2\sqrt{\nu t}}\sqrt{\frac{2\nu}{\Omega}}\xi = \frac{1}{\sqrt{2}\sqrt{t\Omega}}\xi = \frac{1}{\sqrt{2\tau}}\xi,$$

$$\frac{z}{2\sqrt{\nu t}} = \frac{1}{\sqrt{2\tau}}\xi, \tag{2.56}$$

$$\begin{aligned}\sqrt{\left(\frac{w_0^2}{4\nu^2} + \frac{N}{\nu} + \frac{i\Omega}{\nu}\right)\nu t} &= \sqrt{\left(\frac{w_0^2}{4\nu^2} + \frac{N}{\nu} + \frac{i\Omega}{\nu}\right)}z\frac{\sqrt{\nu t}}{z} \\ &= (\alpha + i\beta)\xi\sqrt{\frac{\Omega}{2\nu}}\frac{1}{\xi}\sqrt{\nu t} = (\alpha + i\beta)\sqrt{\frac{\tau}{2}},\end{aligned}$$

$$\sqrt{\left(\frac{w_0^2}{4\nu^2} + \frac{N}{\nu} + \frac{i\Omega}{\nu}\right)\nu t} = (\alpha + i\beta)\sqrt{\frac{\tau}{2}}, \tag{2.57}$$

$$e^{int} = e^{ik\Omega t} = e^{ik\tau},$$

$$e^{int} = e^{ik\tau}, \tag{2.58}$$

$$\begin{aligned}
\sqrt{\frac{w_0^2}{4\nu^2} + \frac{i(n+\Omega)}{\nu} + \frac{N}{\nu}}z &= \sqrt{\frac{4S^2\nu\Omega}{4\nu^2} + \frac{i(k\Omega+\Omega)}{\nu} + \frac{1}{\nu}\frac{\sigma B_0^2}{\rho}}z \\
&= \sqrt{\frac{S^2\Omega}{\nu} + \frac{i(k+1)\Omega}{\nu} + \frac{\Omega}{\nu}\frac{\sigma B_0^2}{\rho\Omega}}z \\
&= \sqrt{S^2 + i(k+1) + N_1}\sqrt{\frac{\Omega}{\nu}}z \\
&= \sqrt{S^2 + i(k+1) + N_1}\sqrt{2}\sqrt{\frac{\Omega}{2\nu}}z \\
&= \sqrt{2}\sqrt{S^2 + i(k+1) + N_1}\xi = (\alpha_2 + i\beta_2)\xi, \qquad (2.59)
\end{aligned}$$

$$\alpha_2 + i\beta_2 = \sqrt{2}\sqrt{S^2 + i(k+1) + N_1}\xi, \tag{2.60}$$

$$\alpha_2 = \sqrt{\sqrt{(S^2+N_1)^2 + (k+1)} + (S^2+N_1)}, \beta_2 = \sqrt{\sqrt{(S^2+N_1)^2 + (k+1)} - (S^2+N_1)},$$

$$\sqrt{\frac{w_0^2}{4\nu^2} + \frac{i(n+\Omega)}{\nu} + \frac{N}{\nu}\nu t} = (\alpha_2 + i\beta_2)\sqrt{\frac{\tau}{2}}, \tag{2.61}$$

$$\sqrt{\frac{w_0^2}{4\nu^2} + \frac{N}{\nu} - \frac{i(n-\Omega)}{\nu}}z = (\alpha_1 + i\beta_1)\xi, \tag{2.62}$$

$$\alpha_1 + i\beta_1 = \sqrt{2}\sqrt{S^2 + N_1 - i(k-1)}, \tag{2.63}$$

$$\begin{aligned}
\alpha_1 &= \sqrt{\sqrt{(S^2+N_1)^2 + (k-1)^2} + (S^2+N_1)}, \\
\beta_1 &= \sqrt{\sqrt{(S^2+N_1)^2 + (k-1)^2} - (S^2+N_1)}
\end{aligned}$$

$$\sqrt{\frac{w_0^2}{4\nu^2} + \frac{N}{\nu} - \frac{i(n-\Omega)}{\nu}\nu t} = (\alpha_1 + i\beta_1)\sqrt{\frac{\tau}{2}}.$$

Equation (2.50) in non-dimensional variables becomes

$$\frac{f(z,t)}{\Omega l}+i\frac{g(z,t)}{\Omega l}=1+\frac{e^{-\sqrt{2}S\xi}}{2}\left[\begin{array}{l} -\left\{\begin{array}{c} e^{(\alpha+i\beta)\xi}erfc\left(\frac{\xi}{\sqrt{2\tau}}+(\alpha+i\beta)\sqrt{\frac{\tau}{2}}\right) \\ +e^{-(\alpha+i\beta)\xi}erfc\left(\frac{\xi}{\sqrt{\nu\tau}}-(\alpha+i\beta)\sqrt{\frac{\tau}{2}}\right) \end{array}\right\} \\ +\epsilon e^{-ik\tau}\left\{\begin{array}{c} e^{(\alpha_1+i\beta_1)\xi}\operatorname{erf}c\left(\frac{\xi}{\sqrt{2\tau}}+(\alpha_1+i\beta_1)\sqrt{\frac{\tau}{2}}\right) \\ +e^{-(\alpha_1+i\beta_1)\xi}\operatorname{erf}c\left(\frac{\xi}{\sqrt{2\tau}}-(\alpha_1+i\beta_1)\sqrt{\frac{\tau}{2}}\right) \end{array}\right\} \\ +\epsilon e^{ik\tau}\left\{\begin{array}{c} e^{(\alpha_2+\beta_2)\xi}\operatorname{erf}c\left\{\frac{\xi}{\sqrt{2\tau}}+(\alpha_2+i\beta_2)\sqrt{\frac{\tau}{2}}\right\} \\ +e^{-(\alpha_2+i\beta_2)\xi}\operatorname{erf}c\left\{\frac{\xi}{\sqrt{2\tau}}-(\alpha_2+i\beta_2)\sqrt{\frac{\tau}{2}}\right\} \end{array}\right\} \end{array}\right]. \tag{2.64}$$

where

$$\begin{aligned} \alpha &= \sqrt{\sqrt{(S^2+N_1)^2+1}+(S^2+N_1)}, \\ \beta &= \sqrt{\sqrt{(S^2+N_1)^2+1}-(S^2+N_1)}, \\ \alpha_1 &= \sqrt{\sqrt{(S^2+N_1)^2+(k-1)^2}+(S^2+N_1)}, \\ \beta_1 &= \sqrt{\sqrt{(S^2+N_1)^2+(k-1)^2}-(S^2+N_1)}, \\ \alpha_2 &= \sqrt{\sqrt{(S^2+N_1)^2+(k+1)^2}+(S^2+N_1)}, \\ \beta_2 &= \sqrt{\sqrt{(S^2+N_1)^2+(k+1)^2}-(S^2+N_1)}. \end{aligned} \tag{2.65}$$

Now for $n<\Omega$ we have from Eqs. (2.38), (2.39) and (2.40) as

$$\nu\frac{d^2\overline{H}}{dz^2}+w_0\frac{d\overline{H}}{dz}-(s+N)\overline{H}=0$$

$$\overline{H}(0,S)=\frac{U}{2\Omega l}\left[\frac{1}{s-i(\Omega+n)}+\frac{1}{s-i(\Omega-n)}\right]-\frac{1}{s-i\Omega}$$

$$\overline{H}(\infty,s)=0.$$

Employing the same method of solution as for $n>\Omega$ we have

$$\overline{H}(z,s)=\left[-\frac{1}{(s-i\Omega)}+\frac{U}{2\Omega l}\frac{1}{s-i(n-\Omega)}+\frac{U}{2\Omega l}\frac{1}{s-i(n+\Omega)}\right]e^{-\left(\frac{w_0}{2\nu}+\sqrt{\frac{w_0^2}{4\nu^2}+\frac{s}{\nu}+\frac{N}{\nu}}\right)z},$$

and

$$H(z,t) = e^{-\frac{w_0}{2\nu}z}\left[-I_1^* + \frac{U}{2\Omega l}I_2^* + \frac{U}{2\Omega l}I_3^*\right], \tag{2.66}$$

where

$$I_1^* = \frac{e^{i\Omega t}}{2}\left[\begin{array}{c} e^{\sqrt{\frac{w_0^2}{4\nu^2}+\frac{N}{\nu}+\frac{i\Omega}{\nu}}z} erfc\left(\frac{z}{2\sqrt{\nu t}} + \sqrt{\left(\frac{w_0^2}{4\nu^2}+\frac{N}{\nu}+\frac{i\Omega}{\nu}\right)\nu t}\right) + \\ e^{-\sqrt{\frac{w_0^2}{4\nu^2}+\frac{N}{\nu}+\frac{i\Omega}{\nu}}z} erfc\left(\frac{z}{2\sqrt{\nu t}} - \sqrt{\left(\frac{w_0^2}{4\nu^2}+\frac{N}{\nu}+\frac{i\Omega}{\nu}\right)\nu t}\right)\end{array}\right], \tag{2.67}$$

$$I_2^* = \frac{e^{i(\Omega-n)t}}{2}\left[\begin{array}{c} e^{\sqrt{\frac{w_0^2}{4\nu^2}+\frac{i(\Omega-n)}{\nu}+\frac{N}{\nu}}z} \operatorname{erf}c\left(\frac{z}{2\sqrt{\nu t}} + \sqrt{\left(\frac{w_0^2}{4\nu^2}+\frac{i(\Omega-n)}{\nu}+\frac{N}{\nu}\right)\nu t}\right) + \\ e^{-\sqrt{\frac{w_0^2}{4\nu^2}+\frac{i(\Omega-n)}{\nu}+\frac{N}{\nu}}z} \operatorname{erf}c\left(\frac{z}{2\sqrt{\nu t}} - \sqrt{\left(\frac{w_0^2}{4\nu^2}+\frac{i(\Omega-n)}{\nu}+\frac{N}{\nu}\right)\nu t}\right)\end{array}\right], \tag{2.68}$$

$$I_3^* = \frac{e^{i(n+\Omega)t}}{2}\left[\begin{array}{c} e^{\sqrt{\frac{w_0^2}{4\nu^2}+\frac{i(n+\Omega)}{\nu}+\frac{N}{\nu}}z} \operatorname{erf}c\left(\frac{z}{2\sqrt{\nu t}} + \sqrt{\left(\frac{w_0^2}{4\nu^2}+\frac{i(n+\Omega)}{\nu}+\frac{N}{\nu}\right)\nu t}\right) + \\ e^{-\sqrt{\frac{w_0^2}{4\nu^2}+\frac{i(n+\Omega)}{\nu}+\frac{N}{\nu}}z} \operatorname{erf}c\left((\frac{z}{2\sqrt{\nu t}} - \sqrt{\left(\frac{w_0^2}{4\nu^2}+\frac{i(n+\Omega)}{\nu}+\frac{N}{\nu}\right)\nu t)}\right)\end{array}\right]. \tag{2.69}$$

Using I_1^*, I_2^* and I_3^* in (2.66), we get

$$H(z,t) = e^{-\frac{w_0}{2\nu}z}\left[\begin{array}{c} -\frac{e^{i\Omega t}}{2} \\ \times\left\{\begin{array}{c} e^{\sqrt{\frac{w_0^2}{4\nu^2}+\frac{N}{\nu}+\frac{i\Omega}{\nu}}z} erfc\left(\frac{z}{2\sqrt{\nu t}} + \sqrt{\left(\frac{w_0^2}{4\nu^2}+\frac{N}{\nu}+\frac{i\Omega}{\nu}\right)\nu t}\right) \\ +e^{-\sqrt{\frac{w_0^2}{4\nu^2}+\frac{N}{\nu}+\frac{i\Omega}{\nu}}z} erfc\left(\frac{z}{2\sqrt{\nu t}} - \sqrt{\left(\frac{w_0^2}{4\nu^2}+\frac{N}{\nu}+\frac{i\Omega}{\nu}\right)\nu t}\right)\end{array}\right\} \\ +\left(\frac{U}{2\Omega l}\frac{e^{i(\Omega-n)t}}{2}\right) \\ \times\left\{\begin{array}{c} e^{\sqrt{\frac{w_0^2}{4\nu^2}+\frac{i(\Omega-n)}{\nu}+\frac{N}{\nu}}z} \operatorname{erf}c\left(\frac{z}{2\sqrt{\nu t}} + \sqrt{\left(\frac{w_0^2}{4\nu^2}+\frac{i(\Omega-n)}{\nu}+\frac{N}{\nu}\right)\nu t}\right) \\ +e^{-\sqrt{\frac{w_0^2}{4\nu^2}+\frac{i(\Omega-n)}{\nu}+\frac{N}{\nu}}z} \operatorname{erf}c\left(\frac{z}{2\sqrt{\nu t}} - \sqrt{\left(\frac{w_0^2}{4\nu^2}+\frac{i(\Omega-n)}{\nu}+\frac{N}{\nu}\right)\nu t}\right)\end{array}\right\} \\ +\left(\frac{U}{2\Omega l}\frac{e^{i(n+\Omega)t}}{2}\right) \\ \times\left\{\begin{array}{c} \{e^{\sqrt{\frac{w_0^2}{4\nu^2}+\frac{N}{\nu}+\frac{i(n+\Omega)}{\nu}}z} \operatorname{erf}c\left(\frac{z}{2\sqrt{\nu t}} + \sqrt{\left(\frac{w_0^2}{4\nu^2}+\frac{N}{\nu}+\frac{i(n+\Omega)}{\nu}\right)\nu t}\right) \\ +e^{-\sqrt{\frac{w_0^2}{4\nu^2}+\frac{N}{\nu}+\frac{i(n+\Omega)}{\nu}}z} \operatorname{erf}c\left(\frac{z}{2\sqrt{\nu t}} - \sqrt{\left(\frac{w_0^2}{4\nu^2}+\frac{N}{\nu}+\frac{i(n+\Omega)}{\nu}\right)\nu t}\right)\end{array}\right\}\end{array}\right], \tag{2.70}$$

or

$$
\left.\begin{array}{c}
\frac{f(z,t)}{\Omega l}+i\frac{g(z,t)}{\Omega l}=1+\frac{e^{-\frac{w_0}{2\nu}z}}{2} \\
\left[\begin{array}{c}
-\left\{\begin{array}{c}
e^{\sqrt{\frac{w_0^2}{4\nu^2}+\frac{N}{\nu}+\frac{i\Omega}{\nu}}z} erfc\left(\frac{z}{2\sqrt{\nu t}}+\sqrt{\left(\frac{w_0^2}{4\nu^2}+\frac{N}{\nu}+\frac{i\Omega}{\nu}\right)\nu t}\right) \\
+e^{-\sqrt{\frac{w_0^2}{4\nu^2}+\frac{N}{\nu}+\frac{i\Omega}{\nu}}z} erfc\left(\frac{z}{2\sqrt{\nu t}}-\sqrt{\left(\frac{w_0^2}{4\nu^2}+\frac{N}{\nu}+\frac{i\Omega}{\nu}\right)\nu t}\right)
\end{array}\right\} \\
+\left(\frac{U}{2\Omega l}e^{-int}\right) \\
\times\left\{\begin{array}{c}
e^{\sqrt{\frac{w_0^2}{4\nu^2}+\frac{i(\Omega-n)}{\nu}+\frac{N}{\nu}}z} \operatorname{erf}c\left(\frac{z}{2\sqrt{\nu t}}+\sqrt{\left(\frac{w_0^2}{4\nu^2}+\frac{i(\Omega-n)}{\nu}+\frac{N}{\nu}\right)\nu t}\right) \\
+e^{-\sqrt{\frac{w_0^2}{4\nu^2}+\frac{i(\Omega-n)}{\nu}+\frac{N}{\nu}}z} \operatorname{erf}c\left(\frac{z}{2\sqrt{\nu t}}-\sqrt{\left(\frac{w_0^2}{4\nu^2}+\frac{i(\Omega-n)}{\nu}+\frac{N}{\nu}\right)\nu t}\right)
\end{array}\right\} \\
+\left(\frac{U}{2\Omega l}e^{int}\right) \\
\times\left\{\begin{array}{c}
e^{\sqrt{\frac{w_0^2}{4\nu^2}+\frac{N}{\nu}+\frac{i(\Omega+n)}{\nu}}z} \operatorname{erf}c\left(\frac{z}{2\sqrt{\nu t}}+\sqrt{\left(\frac{w_0^2}{4\nu^2}+\frac{N}{\nu}+\frac{i(\Omega+n)}{\nu}\right)\nu t}\right)+ \\
e^{-\sqrt{\frac{w_0^2}{4\nu^2}+\frac{N}{\nu}+\frac{i(\Omega+n)}{\nu}}z} \operatorname{erf}c\left(\frac{z}{2\sqrt{\nu t}}-\sqrt{\left(\frac{w_0^2}{4\nu^2}+\frac{N}{\nu}+\frac{i(\Omega+n)}{\nu}\right)\nu t}\right)
\end{array}\right\}
\end{array}\right]\times .
\end{array}\right\} \tag{2.71}
$$

In terms of non-dimensional variables, Eq. (2.71) is

$$
\frac{f(z,t)}{\Omega l}+i\frac{g(z,t)}{\Omega l}=1+\frac{e^{-\sqrt{2}S\xi}}{2}\left[\begin{array}{c}
-\left\{\begin{array}{c}
e^{(\alpha+i\beta)\xi} erfc\left(\frac{\xi}{\sqrt{2\tau}}+(\alpha+i\beta)\right)\sqrt{\frac{\tau}{2}} \\
+e^{-(\alpha+i\beta)\xi} erfc\left(\frac{\xi}{\sqrt{\nu\tau}}-(\alpha+i\beta)\right)\sqrt{\frac{\tau}{2}}
\end{array}\right\} \\
+\epsilon e^{-ik\tau}\left\{\begin{array}{c}
e^{(\alpha_3+i\beta_3)\xi} \operatorname{erf}c\left(\frac{\xi}{\sqrt{2\tau}}+(\alpha_3+i\beta_3)\right)\sqrt{\frac{\tau}{2}} \\
+e^{-(\alpha_3+i\beta_3)\xi} \operatorname{erf}c\left(\frac{\xi}{\sqrt{2\tau}}-(\alpha_3+i\beta_3)\right)\sqrt{\frac{\tau}{2}}
\end{array}\right\} \\
+\epsilon e^{ik\tau}\left\{\begin{array}{c}
e^{(\alpha_2+\beta_2)\xi} \operatorname{erf}c\left(\frac{\xi}{\sqrt{2\tau}}+(\alpha_2+i\beta_2)\right)\sqrt{\frac{\tau}{2}} \\
+e^{-(\alpha_2+i\beta_2)\xi} \operatorname{erf}c\left(\frac{\xi}{\sqrt{2\tau}}-(\alpha_2+i\beta_2)\right)\sqrt{\frac{\tau}{2}}
\end{array}\right\}
\end{array}\right], \tag{2.72}
$$

where

$$\alpha_3+i\beta_3=\sqrt{2}\sqrt{S^2+N_1+i(1-k)}, \tag{2.73}$$

$$\alpha_3=\sqrt{\sqrt{(S^2+N_1)^2+(1-k)^2}+(S^2+N_1)}, \tag{2.74}$$

$$\beta_3=\sqrt{\sqrt{(S^2+N_1)^2+(1-k)^2}-(S^2+N_1)}. \tag{2.75}$$

Now for sine oscillations we have the following respective results for $n < \Omega$ and $n > \Omega$

$$\left.\begin{array}{c} \frac{f(z,t)}{\Omega l} + i\frac{g(z,t)}{\Omega l} = 1 + \frac{e^{-\frac{w_0}{2\nu}z}}{2} \\ \times\left[\begin{array}{c} -\left\{\begin{array}{c} e^{\sqrt{\frac{w_0^2}{4\nu^2}+\frac{N}{\nu}+\frac{i\Omega}{\nu}}z} erfc\left(\frac{z}{2\sqrt{\nu t}} + \sqrt{\left(\frac{w_0^2}{4\nu^2}+\frac{N}{\nu}+\frac{i\Omega}{\nu}\right)\nu t}\right) \\ +e^{-\sqrt{\frac{w_0^2}{4\nu^2}+\frac{N}{\nu}+\frac{i\Omega}{\nu}}z} erfc\left(\frac{z}{2\sqrt{\nu t}} - \sqrt{\left(\frac{w_0^2}{4\nu^2}+\frac{N}{\nu}+\frac{i\Omega}{\nu}\right)\nu t}\right) \end{array}\right\} \\ -\left(\frac{iU}{2\Omega l}e^{-int}\right) \\ \times\left\{\begin{array}{c} e^{\sqrt{\frac{w_0^2}{4\nu^2}-\frac{i(n-\Omega)}{\nu}+\frac{N}{\nu}}z} \operatorname{erf}c\left(\frac{z}{2\sqrt{\nu t}} + \sqrt{\left(\frac{w_0^2}{4\nu^2}-\frac{i(n-\Omega)}{\nu}+\frac{N}{\nu}\right)\nu t}\right) \\ +e^{-\sqrt{\frac{w_0^2}{4\nu^2}-\frac{i(n-\Omega)}{\nu}+\frac{N}{\nu}}z} \operatorname{erf}c\left(\frac{z}{2\sqrt{\nu t}} - \sqrt{\left(\frac{w_0^2}{4\nu^2}-\frac{i(n-\Omega)}{\nu}+\frac{N}{\nu}\right)\nu t}\right) \end{array}\right\} \\ -\left(\frac{iU}{2\Omega l}e^{int}\right) \\ \times\left\{\begin{array}{c} e^{\sqrt{\frac{w_0^2}{4\nu^2}+\frac{N}{\nu}+\frac{i(n+\Omega)}{\nu}}z} \operatorname{erf}c\left(\frac{z}{2\sqrt{\nu t}} + \sqrt{\left(\frac{w_0^2}{4\nu^2}+\frac{N}{\nu}+\frac{i(n+\Omega)}{\nu}\right)\nu t}\right) \\ +e^{-\sqrt{\frac{w_0^2}{4\nu^2}+\frac{N}{\nu}+\frac{i(n+\Omega)}{\nu}}z} \operatorname{erf}c\left(\frac{z}{2\sqrt{\nu t}} - \sqrt{\left(\frac{w_0^2}{4\nu^2}+\frac{N}{\nu}+\frac{i(n+\Omega)}{\nu}\right)\nu t}\right) \end{array}\right\} \end{array}\right] \end{array}\right\}. \tag{2.76}$$

$$\frac{f(z,t)}{\Omega l} + i\frac{g(z,t)}{\Omega l} = 1 + \frac{e^{-\frac{w_0}{2\nu}z}}{2} \left[\begin{array}{c} -\left\{\begin{array}{c} e^{\sqrt{\frac{w_0^2}{4\nu^2}+\frac{N}{\nu}+\frac{i\Omega}{\nu}}z} \operatorname{erf}c\left(\frac{z}{2\sqrt{\nu t}} + \sqrt{\left(\frac{w_0^2}{4\nu^2}+\frac{N}{\nu}+\frac{i\Omega}{\nu}\right)\nu t}\right) \\ +e^{-\sqrt{\frac{w_0^2}{4\nu^2}+\frac{N}{\nu}+\frac{i\Omega}{\nu}}z} \operatorname{erf}c\left(\frac{z}{2\sqrt{\nu t}} - \sqrt{\left(\frac{w_0^2}{4\nu^2}+\frac{N}{\nu}+\frac{i\Omega}{\nu}\right)\nu t}\right) \end{array}\right\} \\ +\left(\frac{iU}{2\Omega l}e^{-int}\right) \\ \times\left\{\begin{array}{c} e^{\sqrt{\frac{w_0^2}{4\nu^2}-\frac{i(n-\Omega)}{\nu}+\frac{N}{\nu}}z} \operatorname{erf}c\left(\frac{z}{2\sqrt{\nu t}} + \sqrt{\left(\frac{w_0^2}{4\nu^2}-\frac{i(n-\Omega)}{\nu}+\frac{N}{\nu}\right)\nu t}\right) \\ +e^{-\sqrt{\frac{w_0^2}{4\nu^2}-\frac{i(n-\Omega)}{\nu}+\frac{N}{\nu}}z} \operatorname{erf}c\left(\frac{z}{2\sqrt{\nu t}} - \sqrt{\left(\frac{w_0^2}{4\nu^2}-\frac{i(n-\Omega)}{\nu}+\frac{N}{\nu}\right)\nu t}\right) \end{array}\right\} \\ -\left(\frac{iU}{2\Omega l}e^{int}\right) \\ \times\left\{\begin{array}{c} e^{\sqrt{\frac{w_0^2}{4\nu^2}+\frac{N}{\nu}+\frac{i(n+\Omega)}{\nu}}z} \operatorname{erf}c\left(\frac{z}{2\sqrt{\nu t}} + \sqrt{\left(\frac{w_0^2}{4\nu^2}+\frac{N}{\nu}+\frac{i(n+\Omega)}{\nu}\right)\nu t}\right) \\ +e^{-\sqrt{\frac{w_0^2}{4\nu^2}+\frac{N}{\nu}+\frac{i(n+\Omega)}{\nu}}z} \operatorname{erf}c\left(\frac{z}{2\sqrt{\nu t}} - \sqrt{\left(\frac{w_0^2}{4\nu^2}+\frac{N}{\nu}+\frac{i(n+\Omega)}{\nu}\right)\nu t}\right) \end{array}\right\} \end{array}\right] \tag{2.77}$$

The above expressions in terms of non-dimensional form are

$$\frac{f(z,\tau)}{\Omega l}+i\frac{g(z,\tau)}{\Omega l}=1+\frac{e^{-\sqrt{2}S\xi}}{2}\left[\begin{array}{l} -\left\{\begin{array}{c} e^{(\alpha+i\beta)\xi}\mathrm{erfc}\left(\frac{\xi}{\sqrt{2\tau}}+(\alpha+i\beta)\sqrt{\frac{\tau}{2}}\right) \\ +e^{-(\alpha+i\beta)\xi}\mathrm{erfc}\left(\frac{\xi}{\sqrt{\nu\tau}}-(\alpha+i\beta)\sqrt{\frac{\tau}{2}}\right) \end{array}\right\} \\ +i\epsilon e^{-ik\tau}\left\{\begin{array}{c} e^{(\alpha_1+i\beta_1)\xi}\,\mathrm{erf}\,c\left(\frac{\xi}{\sqrt{2\tau}}+(\alpha_1+i\beta_1)\sqrt{\frac{\tau}{2}}\right) \\ +e^{-(\alpha_1+i\beta_1)\xi}\,\mathrm{erf}\,c\left(\frac{\xi}{\sqrt{2\tau}}-(\alpha_1+i\beta_1)\sqrt{\frac{\tau}{2}}\right) \end{array}\right\} \\ -i\epsilon e^{ik\tau}\left\{\begin{array}{c} e^{(\alpha_2+i\beta_2)\xi}\,\mathrm{erf}\,c\left(\frac{\xi}{\sqrt{2\tau}}+(\alpha_2+i\beta_2)\sqrt{\frac{\tau}{2}}\right) \\ +e^{-(\alpha_2+i\beta_2)\xi}\,\mathrm{erf}\,c\left(\frac{\xi}{\sqrt{2\tau}}-(\alpha_2+i\beta_2)\sqrt{\frac{\tau}{2}}\right) \end{array}\right\} \end{array}\right], \tag{2.78}$$

$$\frac{f(z,t)}{\Omega l}+i\frac{g(z,t)}{\Omega l}=1+\frac{e^{-\sqrt{2}S\xi}}{2}\left[\begin{array}{l} -\left\{\begin{array}{c} e^{(\alpha+i\beta)\xi}\mathrm{erfc}\left(\frac{\xi}{\sqrt{2\tau}}+(\alpha+i\beta)\sqrt{\frac{\tau}{2}}\right) \\ +e^{-(\alpha+i\beta)\xi}\mathrm{erfc}\left(\frac{\xi}{\sqrt{\nu\tau}}-(\alpha+i\beta)\sqrt{\frac{\tau}{2}}\right) \end{array}\right\} \\ +i\epsilon e^{-ik\tau}\left\{\begin{array}{c} \{e^{(\alpha_3+i\beta_3)\xi}\,\mathrm{erf}\,c\left(\frac{\xi}{\sqrt{2\tau}}+(\alpha_3+i\beta_3)\sqrt{\frac{\tau}{2}}\right) \\ +e^{-(\alpha_3+i\beta_3)\xi}\,\mathrm{erf}\,c\left(\frac{\xi}{\sqrt{2\tau}}-(\alpha_3+i\beta_3)\sqrt{\frac{\tau}{2}}\right) \end{array}\right\} \\ -i\epsilon e^{ik\tau}\left\{\begin{array}{c} e^{(\alpha_2+\beta_2)\xi}\,\mathrm{erf}\,c\left(\frac{\xi}{\sqrt{2\tau}}+(\alpha_2+i\beta_2)\sqrt{\frac{\tau}{2}}\right) \\ +e^{-(\alpha_2+i\beta_2)\xi}\,\mathrm{erf}\,c\left(\frac{\xi}{\sqrt{2\tau}}-(\alpha_2+i\beta_2)\sqrt{\frac{\tau}{2}}\right) \end{array}\right\} \end{array}\right]. \tag{2.79}$$

Resonant suction case

If the angular velocity is equal to the frequency of oscillations ($n=\Omega$) then the problem is

$$\nu\frac{\partial^2 H}{\partial z^2}-\frac{\partial H}{\partial t}+w_0\frac{\partial H}{\partial z}-NH=0 \tag{2.80}$$

$$\begin{aligned} H(0,t) &= \left(\frac{U}{\Omega l}\cos\Omega t-1\right)e^{i\Omega t}, \\ H(\infty,t) &= 0,\ H(z,t)=0. \end{aligned} \tag{2.81}$$

The above problem in the transformed $s-$plane is of the following form:

$$\nu\frac{d^2\overline{H}}{dz^2}+w_0\frac{d\overline{H}}{dz}-(s+N)\overline{H}=0, \tag{2.82}$$

$$\overline{H}(0,s) = -\frac{1}{s-i\Omega} + \frac{U}{2\Omega l}\frac{1}{s-2i\Omega} + \frac{U}{2\Omega l}\frac{1}{s}, \tag{2.83}$$

$$\overline{H}(\infty,s) = 0. \tag{2.84}$$

The solution of the above problem in $s-$plane is

$$\overline{H}(z,s) = \left[-\frac{1}{s-i\Omega} + \frac{U}{2\Omega l}\frac{1}{s-2i\Omega} + \frac{U}{2\Omega l}\frac{1}{s}\right] e^{-\left(\frac{w_0}{2\nu}+\sqrt{\frac{w_0^2}{4\nu^2}+\frac{s}{\nu}+\frac{N}{\nu}}\right)z},$$

which upon taking Laplace inversion becomes

$$H(z,t) = e^{-\frac{w_0}{2\nu}z}\left[-\widehat{I}_1 + \frac{U}{2\Omega l}\widehat{I}_2 + \frac{U}{2\Omega l}\widehat{I}_3\right], \tag{2.85}$$

where

$$\widehat{I}_1 = L^{-1}\left\{\frac{e^{-\sqrt{\frac{w_0^2}{4\nu^2}+\frac{s}{\nu}+\frac{N}{\nu}}z}}{s-i\Omega}\right\} = I_1 = \frac{1}{2\pi i}\int \frac{e^{st-\sqrt{\frac{w_0^2}{4\nu^2}+\frac{s}{\nu}+\frac{N}{\nu}}z}}{(s-i\Omega)}ds,$$

$$\widehat{I}_2 = L^{-1}\left\{\frac{e^{-\sqrt{\frac{w_0^2}{4\nu^2}+\frac{s}{\nu}+\frac{N}{\nu}}z}}{s-2i\Omega}\right\} = \int \frac{e^{st-\sqrt{\frac{w_0^2}{4\nu^2}+\frac{s}{\nu}+\frac{N}{\nu}}z}}{s-2i\Omega}ds,$$

$$\widehat{I}_3 = L^{-1}\left\{\frac{e^{-\sqrt{\frac{w_0^2}{4\nu^2}+\frac{s}{\nu}+\frac{N}{\nu}}z}}{s}\right\} = \int \frac{e^{st-\sqrt{\frac{w_0^2}{4\nu^2}+\frac{s}{\nu}+\frac{N}{\nu}}z}}{s}ds.$$

Evaluating the above integrals we can write

$$\widehat{I}_1 = \frac{e^{i\Omega t}}{2}\left[\begin{array}{c} e^{\sqrt{\frac{w_0^2}{4\nu^2}+\frac{N}{\nu}+\frac{i\Omega}{\nu}}z} erfc\left(\frac{z}{2\sqrt{\nu t}} + \sqrt{\left(\frac{w_0^2}{4\nu^2}+\frac{N}{\nu}+\frac{i\Omega}{\nu}\right)\nu t}\right) \\ +e^{-\sqrt{\frac{w_0^2}{4\nu^2}+\frac{N}{\nu}+\frac{i\Omega}{\nu}}z} erfc\left(\frac{z}{2\sqrt{\nu t}} - \sqrt{\left(\frac{w_0^2}{4\nu^2}+\frac{N}{\nu}+\frac{i\Omega}{\nu}\right)\nu t}\right)\end{array}\right], \tag{2.86}$$

$$\widehat{I}_2 = \frac{e^{2i\Omega t}}{2}\left[\begin{array}{c} e^{\sqrt{\frac{w_0^2}{4\nu^2}+\frac{N}{\nu}+\frac{2i\Omega}{\nu}}z} \operatorname{erf}c\left(\frac{z}{2\sqrt{\nu t}} + \sqrt{\left(\frac{w_0^2}{4\nu^2}+\frac{N}{\nu}+\frac{2i\Omega}{\nu}\right)\nu t}\right) \\ +e^{-\sqrt{\frac{w_0^2}{4\nu^2}+\frac{N}{\nu}+\frac{2i\Omega}{\nu}}z} \operatorname{erf}c\left(\frac{z}{2\sqrt{\nu t}} - \sqrt{\left(\frac{w_0^2}{4\nu^2}+\frac{N}{\nu}+\frac{2i\Omega}{\nu}\right)\nu t)}\right)\end{array}\right], \tag{2.87}$$

$$\widehat{I_3} = \frac{1}{2}\left[\begin{array}{c} e^{\sqrt{\frac{w_0^2}{4\nu^2}+\frac{N}{\nu}}z}\operatorname{erf}c\left(\frac{z}{2\sqrt{\nu t}}+\sqrt{\left(\frac{w_0^2}{4\nu^2}+\frac{N}{\nu}\right)\nu t}\right) \\ +e^{-\sqrt{\frac{w_0^2}{4\nu^2}+\frac{N}{\nu}}z}\operatorname{erf}c\left(\frac{z}{2\sqrt{\nu t}}-\sqrt{\left(\frac{w_0^2}{4\nu^2}+\frac{N}{\nu}\right)\nu t)}\right) \end{array}\right]. \tag{2.88}$$

Making use of above values of $\widehat{I_1}$ $\widehat{I_2}$ and $\widehat{I_3}$ in Eq.(2.85) we have for cosine oscillation

$$H(z,t) = e^{-\frac{w_0}{2\nu}z}\left[\begin{array}{c} -\frac{e^{i\Omega t}}{2}\left\{\begin{array}{c} e^{\sqrt{\frac{w_0^2}{4\nu^2}+\frac{N}{\nu}+\frac{i\Omega}{\nu}}z}erfc\left(\frac{z}{2\sqrt{\nu t}}+\sqrt{\left(\frac{w_0^2}{4\nu^2}+\frac{N}{\nu}+\frac{i\Omega}{\nu}\right)\nu t}\right) \\ +e^{-\sqrt{\frac{w_0^2}{4\nu^2}+\frac{N}{\nu}+\frac{i\Omega}{\nu}}z}erfc\left(\frac{z}{2\sqrt{\nu t}}-\sqrt{\left(\frac{w_0^2}{4\nu^2}+\frac{N}{\nu}+\frac{i\Omega}{\nu}\right)\nu t}\right) \end{array}\right\} \\ +\frac{U}{2\Omega l}\frac{e^{2i\Omega t}}{2}\left\{\begin{array}{c} e^{\sqrt{\frac{w_0^2}{4\nu^2}+\frac{N}{\nu}+\frac{2i\Omega}{\nu}}z}\operatorname{erf}c\left(\frac{z}{2\sqrt{\nu t}}+\sqrt{\left(\frac{w_0^2}{4\nu^2}+\frac{N}{\nu}+\frac{2i\Omega}{\nu}\right)\nu t}\right) \\ +e^{-\sqrt{\frac{w_0^2}{4\nu^2}+\frac{N}{\nu}+\frac{2i\Omega}{\nu}}z}\operatorname{erf}c\left(\frac{z}{2\sqrt{\nu t}}-\sqrt{\left(\frac{w_0^2}{4\nu^2}+\frac{N}{\nu}+\frac{2i\Omega}{\nu}\right)\nu t}\right) \end{array}\right\} \\ +\frac{U}{2\Omega l}\frac{1}{2}\left\{\begin{array}{c} \{e^{\sqrt{\frac{w_0^2}{4\nu^2}+\frac{N}{\nu}}z}\operatorname{erf}c(\frac{z}{2\sqrt{\nu t}}+\sqrt{(\frac{w_0^2}{4\nu^2}+\frac{N}{\nu})\nu t})+ \\ e^{-\sqrt{\frac{w_0^2}{4\nu^2}+\frac{N}{\nu}}z}\operatorname{erf}c(\frac{z}{2\sqrt{\nu t}}-\sqrt{(\frac{w_0^2}{4\nu^2}+\frac{N}{\nu})\nu t})\}] \end{array}\right\} \end{array}\right], \tag{2.89}$$

which helps us in writing

$$\left.\begin{array}{c} \frac{f(z,t)}{\Omega l}+i\frac{g(z,t)}{\Omega l} = 1+e^{-\frac{w_0}{2\nu}z} \\ \times\left[\begin{array}{c} -\frac{e^{i\Omega t}}{2} \\ \times\left\{\begin{array}{c} e^{\sqrt{\frac{w_0^2}{4\nu^2}+\frac{N}{\nu}+\frac{i\Omega}{\nu}}z}erfc\left(\frac{z}{2\sqrt{\nu t}}+\sqrt{\left(\frac{w_0^2}{4\nu^2}+\frac{N}{\nu}+\frac{i\Omega}{\nu}\right)\nu t}\right) \\ +e^{-\sqrt{\frac{w_0^2}{4\nu^2}+\frac{N}{\nu}+\frac{i\Omega}{\nu}}z}erfc\left(\frac{z}{2\sqrt{\nu t}}-\sqrt{\left(\frac{w_0^2}{4\nu^2}+\frac{N}{\nu}+\frac{i\Omega}{\nu}\right)\nu t}\right) \end{array}\right\} \\ +\left(\frac{U}{2\Omega l}\frac{e^{2i\Omega t}}{2}\right) \\ \times\left\{\begin{array}{c} e^{\sqrt{\frac{w_0^2}{4\nu^2}+\frac{N}{\nu}+\frac{2i\Omega}{\nu}}z}\operatorname{erf}c\left(\frac{z}{2\sqrt{\nu t}}+\sqrt{\left(\frac{w_0^2}{4\nu^2}+\frac{N}{\nu}+\frac{2i\Omega}{\nu}\right)\nu t}\right) \\ +e^{-\sqrt{\frac{w_0^2}{4\nu^2}+\frac{N}{\nu}+\frac{2i\Omega}{\nu}}z}\operatorname{erf}c\left(\frac{z}{2\sqrt{\nu t}}-\sqrt{\left(\frac{w_0^2}{4\nu^2}+\frac{N}{\nu}+\frac{2i\Omega}{\nu}\right)\nu t}\right) \end{array}\right\} \\ +\left(\frac{U}{2\Omega l}\frac{1}{2}\right) \\ \times\left\{\begin{array}{c} e^{\sqrt{\frac{w_0^2}{4\nu^2}+\frac{N}{\nu}}z}\operatorname{erf}c\left(\frac{z}{2\sqrt{\nu t}}+\sqrt{\left(\frac{w_0^2}{4\nu^2}+\frac{N}{\nu}\right)\nu t}\right) \\ +e^{-\sqrt{\frac{w_0^2}{4\nu^2}+\frac{N}{\nu}}z}\operatorname{erf}c\left(\frac{z}{2\sqrt{\nu t}}-\sqrt{\left(\frac{w_0^2}{4\nu^2}+\frac{N}{\nu}\right)\nu t}\right) \end{array}\right\} \end{array}\right] \end{array}\right\}. \tag{2.90}$$

For sine oscillations we have

$$
\left.\begin{array}{c}
\frac{f(z,t)}{\Omega l}+i\frac{g(z,t)}{\Omega l}=1+\frac{e^{-\frac{w_0}{2\nu}z}}{2} \\
\times\left[\begin{array}{c}
-\left\{\begin{array}{c}
e^{\sqrt{\frac{w_0^2}{4\nu^2}+\frac{N}{\nu}+\frac{i\Omega}{\nu}}z} erfc\left(\frac{z}{2\sqrt{\nu t}}+\sqrt{\left(\frac{w_0^2}{4\nu^2}+\frac{N}{\nu}+\frac{i\Omega}{\nu}\right)\nu t}\right) \\
+e^{-\sqrt{\frac{w_0^2}{4\nu^2}+\frac{N}{\nu}+\frac{i\Omega}{\nu}}z} erfc\left(\frac{z}{2\sqrt{\nu t}}-\sqrt{\left(\frac{w_0^2}{4\nu^2}+\frac{N}{\nu}+\frac{i\Omega}{\nu}\right)\nu t}\right)
\end{array}\right\} \\
+\left(\frac{iUe^{-i\Omega t}}{2\Omega l}\right) \\
\times\left\{\begin{array}{c}
e^{\sqrt{\frac{w_0^2}{4\nu^2}+\frac{N}{\nu}}z} \operatorname{erf}c\left(\frac{z}{2\sqrt{\nu t}}+\sqrt{\left(\frac{w_0^2}{4\nu^2}+\frac{N}{\nu}\right)\nu t}\right) \\
+e^{-\sqrt{\frac{w_0^2}{4\nu^2}+\frac{N}{\nu}}z} \operatorname{erf}c\left(\frac{z}{2\sqrt{\nu t}}-\sqrt{\left(\frac{w_0^2}{4\nu^2}+\frac{N}{\nu}\right)\nu t}\right)
\end{array}\right\} \\
-\left(\frac{iUe^{i\Omega t}}{2\Omega l}\right) \\
\times\left\{\begin{array}{c}
e^{\sqrt{\frac{w_0^2}{4\nu^2}+\frac{N}{\nu}+\frac{2i\Omega}{\nu}}z} \operatorname{erf}c\left(\frac{z}{2\sqrt{\nu t}}+\sqrt{\left(\frac{w_0^2}{4\nu^2}+\frac{N}{\nu}+\frac{2i\Omega}{\nu}\right)\nu t}\right) \\
+e^{-\sqrt{\frac{w_0^2}{4\nu^2}+\frac{N}{\nu}+\frac{2i\Omega}{\nu}}z} \operatorname{erf}c\left(\frac{z}{2\sqrt{\nu t}}-\sqrt{\left(\frac{w_0^2}{4\nu^2}+\frac{N}{\nu}+\frac{2i\Omega}{\nu}\right)\nu t}\right)
\end{array}\right\}
\end{array}\right]
\end{array}\right\}. \tag{2.91}
$$

Now in terms of non- dimensional variables the Eqs. (2.90) and (2.91) yield

$$
\frac{f(z,t)}{\Omega l}+i\frac{g(z,t)}{\Omega l}=1+\frac{e^{-\sqrt{2}S\xi}}{2}\left[\begin{array}{c}
-\left\{\begin{array}{c}
e^{(\alpha+i\beta)\xi} erfc\left(\frac{\xi}{\sqrt{2\tau}}+(\alpha+i\beta)\sqrt{\frac{\tau}{2}}\right) \\
+e^{-(\alpha+i\beta)\xi} erfc\left(\frac{\xi}{\sqrt{2\tau}}-(\alpha+i\beta)\sqrt{\frac{\tau}{2}}\right)
\end{array}\right\} \\
+\varepsilon e^{-i\tau}\left\{\begin{array}{c}
e^{(\alpha_4+i\beta_4)\xi} \operatorname{erf}c\left(\frac{\xi}{\sqrt{2\tau}}+(\alpha_4+i\beta_4)\sqrt{\frac{\tau}{2}}\right) \\
+e^{-(\alpha_4+i\beta_4)\xi} \operatorname{erf}c\left(\frac{\xi}{\sqrt{2\tau}}-(\alpha_4+i\beta_4)\sqrt{\frac{\tau}{2}}\right)
\end{array}\right\} \\
+\varepsilon e^{i\tau}\left\{\begin{array}{c}
e^{(\alpha_5+i\beta_5)z} \operatorname{erf}c\left(\frac{\xi}{\sqrt{2\tau}}+(\alpha_5+i\beta_5)\sqrt{\frac{\tau}{2}}\right) \\
+e^{(\alpha_5+i\beta_5)z} \operatorname{erf}c\left(\frac{\xi}{\sqrt{2\tau}}+(\alpha_5+i\beta_5)\sqrt{\frac{\tau}{2}}\right)
\end{array}\right\}
\end{array}\right], \tag{2.92}
$$

$$
\left[\begin{array}{c}
\alpha_4=\sqrt{\sqrt{(S^2+N_1)^2}+(S^2+N_1)} \\
\beta_4=-\sqrt{\sqrt{(S^2+N_1)^2}-(S^2+N_1)} \\
\alpha_5=\sqrt{\sqrt{(S^2+N_1)^2+4}+(S^2+N_1)} \\
\beta_5=-\sqrt{\sqrt{(S^2+N_1)^2+4}-(S^2+N_1)}
\end{array}\right], \tag{2.93}
$$

$$\frac{f(z,t)}{\Omega l} + i\frac{g(z,t)}{\Omega l} = 1 + \frac{e^{-\sqrt{2}S\xi}}{2}\left[\begin{array}{l} -\left\{\begin{array}{l} e^{(\alpha+i\beta)\xi} erfc\left(\frac{\xi}{\sqrt{2\tau}} + (\alpha+i\beta)\sqrt{\frac{\tau}{2}}\right) \\ +e^{-(\alpha+i\beta)\xi} erfc\left(\frac{\xi}{\sqrt{2\tau}} - (\alpha+i\beta)\sqrt{\frac{\tau}{2}}\right) \end{array}\right\} \\ +i\varepsilon e^{-i\tau}\left\{\begin{array}{l} e^{(\alpha_4+i\beta_4)\xi} \operatorname{erf} c\left(\frac{\xi}{\sqrt{2\tau}} + (\alpha_4+i\beta_4)\sqrt{\frac{\tau}{2}}\right) \\ +e^{-(\alpha_4+i\beta_4)\xi} \operatorname{erf} c\left(\frac{\xi}{\sqrt{2\tau}} - (\alpha_4+i\beta_4)\sqrt{\frac{\tau}{2}}\right) \end{array}\right\} \\ -i\varepsilon e^{i\tau}\left\{\begin{array}{l} e^{(\alpha_5+i\beta_5)z} \operatorname{erf} c\left(\frac{\xi}{\sqrt{2\tau}} + (\alpha_5+i\beta_5)\sqrt{\frac{\tau}{2}}\right) \\ e^{-(\alpha_5+i\beta_5)z} \operatorname{erf} c\left(\frac{\xi}{\sqrt{2\tau}} - (\alpha_5+i\beta_5)\sqrt{\frac{\tau}{2}}\right) \end{array}\right\} \end{array}\right]. \tag{2.94}$$

For resonant blowing we have $w_0 = -w_0$ and thus the respective expressions (2.92) and (2.94) for cosine and sine oscillations become

$$\frac{f(z,t)}{\Omega l} + i\frac{g(z,t)}{\Omega l} = 1 + \frac{e^{-\sqrt{2}S_1\xi}}{2}\left[\begin{array}{l} -\left\{\begin{array}{l} e^{(\overline{\alpha}+i\overline{\beta})\xi} erfc\left(\frac{\xi}{\sqrt{2\tau}} + (\overline{\alpha}+i\overline{\beta})\sqrt{\frac{\tau}{2}}\right) \\ +e^{-(\overline{\alpha}+i\overline{\beta})\xi} erfc\left(\frac{\xi}{\sqrt{2\tau}} - (\overline{\alpha}+i\overline{\beta})\sqrt{\frac{\tau}{2}}\right) \end{array}\right\} \\ +\varepsilon e^{-i\tau}\left\{\begin{array}{l} e^{(\overline{\alpha_4}+i\overline{\beta_4})\xi} \operatorname{erf} c\left(\frac{\xi}{\sqrt{2\tau}} + (\overline{\alpha_4}+i\overline{\beta_4})\sqrt{\frac{\tau}{2}}\right) \\ +e^{-(\overline{\alpha_4}+i\overline{\beta_4})\xi} \operatorname{erf} c\left(\frac{\xi}{\sqrt{2\tau}} - (\overline{\alpha}_4+i\overline{\beta}_4)\sqrt{\frac{\tau}{2}}\right) \end{array}\right\} \\ +\varepsilon e^{i\tau}\left\{\begin{array}{l} e^{(\overline{\alpha}_5+i\overline{\beta}_5)z} \operatorname{erf} c\left(\frac{\xi}{\sqrt{2\tau}} + (\overline{\alpha}_5+i\overline{\beta}_5)\sqrt{\frac{\tau}{2}}\right) \\ +e^{(\overline{\alpha}_5+i\overline{\beta}_5)z} \operatorname{erf} c\left(\frac{\xi}{\sqrt{2\tau}} + (\overline{\alpha}_5+i\overline{\beta}_5)\sqrt{\frac{\tau}{2}}\right) \end{array}\right\} \end{array}\right], \tag{2.95}$$

$$\frac{f(z,t)}{\Omega l} + i\frac{g(z,t)}{\Omega l} = 1 + \frac{e^{-\sqrt{2}S_1\xi}}{2}\left[\begin{array}{l} -\left\{\begin{array}{l} e^{(\overline{\alpha}+i\overline{\beta})\xi} erfc\left(\frac{\xi}{\sqrt{2\tau}} + (\overline{\alpha}+i\overline{\beta})\sqrt{\frac{\tau}{2}}\right) \\ +e^{-(\overline{\alpha}+i\overline{\beta})\xi} erfc\left(\frac{\xi}{\sqrt{2\tau}} - (\overline{\alpha}+i\overline{\beta})\sqrt{\frac{\tau}{2}}\right) \end{array}\right\} \\ +i\varepsilon e^{-i\tau}\left\{\begin{array}{l} e^{(\overline{\alpha}_4+i\overline{\beta}_4)\xi} \operatorname{erf} c\left(\frac{\xi}{\sqrt{2\tau}} + (\overline{\alpha}_4+i\overline{\beta}_4)\sqrt{\frac{\tau}{2}}\right) \\ +e^{-(\overline{\alpha}_4+i\overline{\beta}_4)\xi} \operatorname{erf} c\left(\frac{\xi}{\sqrt{2\tau}} - (\overline{\alpha}_4+i\overline{\beta}_4)\sqrt{\frac{\tau}{2}}\right) \end{array}\right\} \\ -i\varepsilon e^{i\tau}\left\{\begin{array}{l} e^{(\overline{\alpha}_5+i\overline{\beta}_5)z} \operatorname{erf} c\left(\frac{\xi}{\sqrt{2\tau}} + (\overline{\alpha}_5+i\overline{\beta}_5)\sqrt{\frac{\tau}{2}}\right) \\ +e^{-(\overline{\alpha}_5+i\overline{\beta}_5)z} \operatorname{erf} c\left(\frac{\xi}{\sqrt{2\tau}} - (\overline{\alpha}_5+i\overline{\beta}_5)\sqrt{\frac{\tau}{2}}\right) \end{array}\right\} \end{array}\right], \tag{2.96}$$

$$\begin{aligned} \overline{\alpha} &= \sqrt{\sqrt{(S_1^2+N_1)^2} + (S_1^2+N_1)}, \\ \overline{\beta} &= \sqrt{\sqrt{(S_1^2+N_1)^2+1} - (S_1^2+N_1)}, \\ \overline{\alpha}_4 &= \sqrt{\sqrt{(S_1^2+N_1)^2} + (S_1^2+N_1)}, \\ \overline{\beta}_4 &= -\sqrt{\sqrt{(S_1^2+N_1)^2} - (S^2+N_1)}, \\ \overline{\alpha}_5 &= \sqrt{\sqrt{(S_1^2+N_1)^2+4} + (S_1^2+N_1)}, \\ \overline{\beta}_5 &= \sqrt{\sqrt{(S_1^2+N_1)^2+4} - (S_1^2+N_1)}. \end{aligned} \tag{2.97}$$

Chapter 3

Time-dependent flow induced by non-coaxial rotations of a porous moving disk and a Jeffrey fluid at infinity

3.1 Introduction

The work in this chapter is concerned with deriving and solution of an equation which describes the flow due to non-coaxial rotations of a porous moving disk and a Jeffrey fluid at infinity. The unsteady situation is considered. The analysis is performed using a Laplace transform technique. Analytical result is determined for the velocity. The results of Navier-Stokes fluid are obtained as a special case.

3.2 Problem formulation

Let us introduce a Cartesian coordinate system with the $z-$axis normal to the porous disk, which lies in the plane $z = 0$. The axes of rotation of both the disk and the fluid, are assumed to be in the plane $x = 0$ with the distance between the axes being l. Additionally the disk moves with uniform acceleration. The fluid is non-Newtonian and is taken as a Jeffrey fluid.

The common angular velocity of the disk and the fluid is taken as $\boldsymbol{\Omega} = \Omega\mathbf{k}$ ($\mathbf{k}$ is a unit vector parallel to the $z-$axis). The fluid is electrically conducting and magnetic Reynolds number is very small so that the induced magnetic field may be neglected.

The unsteady motion of the conducting fluid is governed by the following laws:

$$div\mathbf{V} = 0, \tag{3.1}$$

$$\rho\frac{D\mathbf{V}}{Dt} = div\mathbf{T} + \mathbf{J} \times \mathbf{B}, \tag{3.2}$$

$$\mathbf{T} = -p\mathbf{I} + \mathbf{S}, \tag{3.3}$$

where ρ is the fluid density, $\mathbf{T}$ is the Cauchy stress tensor, $\mathbf{J}$ is the current density, $\mathbf{B}$ is the total magnetic field, $\frac{D}{Dt}$ is the material derivative, $\mathbf{V}$ is the velocity, and $\mathbf{S}$ is the extra stress which for Jeffrey fluid satisfies the following expression

$$\mathbf{S} = \frac{\mu}{1+\lambda_1}\left[\mathbf{A}_1 + \lambda_2\frac{D\mathbf{A}_1}{Dt}\right], \tag{3.4}$$

$$\mathbf{A}_1 = \mathbf{L} + \mathbf{L}^T = grad\mathbf{V} + (grad\mathbf{V})^T,$$

in which μ is the dynamic viscosity and λ_1 and λ_2 are the material parameters of the Jeffery fluid.

For the problem in question, the velocity field is defined by

$$u = -\Omega y + f(z,t),\ v = \Omega x + g(z,t) \tag{3.5}$$

which together with Eq. (3.1) gives for uniform porous disk that

$$w = -w_0 \tag{3.6}$$

where $w_0 > 0$ corresponds to the suction velocity and $w_0 < 0$ is the injection blowing velocity.

Substituting Eq. (3.3) into Eq. (3.2) one obtains

$$\rho\frac{D\mathbf{V}}{Dt} = -\boldsymbol{\nabla}p + \frac{\mu}{1+\lambda_1}div\left[\mathbf{A}_1 + \lambda_2\frac{D\mathbf{A}_1}{Dt}\right] + \mathbf{J} \times \mathbf{B},$$

or

$$\rho\left[\frac{\partial V}{\partial t}+(\mathbf{V}\cdot\boldsymbol{\nabla})\mathbf{V}\right]=-\boldsymbol{\nabla}p+\frac{\mu}{1+\lambda_1}div\left[\mathbf{A}_1+\lambda_2\frac{D\mathbf{A}_1}{Dt}\right]+\mathbf{J}\times\mathbf{B}. \tag{3.7}$$

From Eqs.(3.4) to (3.6) we have

$$\mathbf{A}_1=\begin{bmatrix} 0 & 0 & \frac{\partial f}{\partial z} \\ 0 & 0 & \frac{\partial g}{\partial z} \\ \frac{\partial f}{\partial z} & \frac{\partial g}{\partial z} & 0 \end{bmatrix},$$

$$\frac{D\mathbf{A}_1}{Dt}=\begin{bmatrix} 0 & 0 & \frac{\partial^2 f}{\partial t\partial z}-w_0\frac{\partial^2 f}{\partial z^2} \\ 0 & 0 & \frac{\partial^2 g}{\partial t\partial z}-w_0\frac{\partial^2 g}{\partial z^2} \\ \frac{\partial^2 g}{\partial t\partial z}-w_0\frac{\partial^2 g}{\partial z^2} & \frac{\partial^2 g}{\partial t\partial z}-w_0\frac{\partial^2 g}{\partial z^2} & 0 \end{bmatrix},$$

$$\begin{aligned}\mathbf{A}_1+\lambda_2\frac{D\mathbf{A}_1}{Dt} &= \begin{bmatrix} 0 & 0 & \frac{\partial f}{\partial z} \\ 0 & 0 & \frac{\partial g}{\partial z} \\ \frac{\partial f}{\partial z} & \frac{\partial g}{\partial z} & 0 \end{bmatrix}+ \\ & \lambda_2\begin{bmatrix} 0 & 0 & \frac{\partial^2 f}{\partial t\partial z}-w_0\frac{\partial^2 f}{\partial z^2} \\ 0 & 0 & \frac{\partial^2 g}{\partial t\partial z}-w_0\frac{\partial^2 g}{\partial z^2} \\ \frac{\partial^2 g}{\partial t\partial z}-w_0\frac{\partial^2 g}{\partial z^2} & \frac{\partial^2 g}{\partial t\partial z}-w_0\frac{\partial^2 g}{\partial z^2} & 0 \end{bmatrix},\end{aligned}$$

or

$$\mathbf{A}_1+\lambda_2\frac{D\mathbf{A}_1}{Dt}=\begin{bmatrix} 0 & 0 & \frac{\partial f}{\partial z}+\lambda_2\left(\frac{\partial^2 f}{\partial t\partial z}-w_0\frac{\partial^2 f}{\partial z^2}\right) \\ 0 & 0 & \frac{\partial g}{\partial z}+\lambda_2\left(\frac{\partial^2 g}{\partial t\partial z}-w_0\frac{\partial^2 g}{\partial z^2}\right) \\ \frac{\partial f}{\partial z}+\lambda_2\left(\frac{\partial^2 g}{\partial t\partial z}-w_0\frac{\partial^2 g}{\partial z^2}\right) & \frac{\partial g}{\partial z}+\lambda_2\left(\frac{\partial^2 g}{\partial t\partial z}-w_0\frac{\partial^2 g}{\partial z^2}\right) & 0 \end{bmatrix},$$

$$\begin{aligned} & div\left(\mathbf{A}_1+\lambda_2\frac{D\mathbf{A}_1}{Dt}\right) \\ = \; & div\begin{bmatrix} 0 & 0 & \frac{\partial f}{\partial z}+\lambda_2\left(\frac{\partial^2 f}{\partial t\partial z}-w_0\frac{\partial^2 f}{\partial z^2}\right) \\ 0 & 0 & \frac{\partial g}{\partial z}+\lambda_2\left(\frac{\partial^2 g}{\partial t\partial z}-w_0\frac{\partial^2 g}{\partial z^2}\right) \\ \frac{\partial f}{\partial z}+\lambda_2\left(\frac{\partial^2 g}{\partial t\partial z}-w_0\frac{\partial^2 g}{\partial z^2}\right) & \frac{\partial g}{\partial z}+\lambda_2\left(\frac{\partial^2 g}{\partial t\partial z}-w_0\frac{\partial^2 g}{\partial z^2}\right) & 0 \end{bmatrix},\end{aligned}$$

$$\left[div\left(\mathbf{A}_1+\lambda_2\frac{D\mathbf{A}_1}{Dt}\right)\right]_x=\frac{\partial^2 f}{\partial z^2}+\lambda_2\frac{\partial}{\partial z}\left(\frac{\partial^2 f}{\partial t\partial z}-w_0\frac{\partial^2 f}{\partial z^2}\right),$$

$$\left[div\left(\mathbf{A}_1+\lambda_2\frac{D\mathbf{A}_1}{Dt}\right)\right]_y=\frac{\partial g}{\partial z}+\lambda_2\frac{\partial}{\partial z}\left(\frac{\partial^2 g}{\partial t\partial z}-w_0\frac{\partial^2 g}{\partial z^2}\right),$$

$$\left[div\left(\mathbf{A}_1+\lambda_2\frac{D\mathbf{A}_1}{Dt}\right)\right]_z=0,$$

where x, y and z in the subscripts indicates the component notation.

Using the definition of velocity and above equations, the x, y and $z-$components of Eq. (3.7) are

$$\left.\begin{array}{c}\frac{\partial f}{\partial t}-\Omega(\Omega x+g)-w_0\frac{\partial f}{\partial z}=-\frac{1}{\rho}\frac{\partial p}{\partial x}+\frac{\nu}{1+\lambda_1}\left[\frac{\partial^2 f}{\partial z^2}+\lambda_2\frac{\partial}{\partial z}\left(\frac{\partial^2 f}{\partial t\partial z}-w_0\frac{\partial^2 f}{\partial z^2}\right)\right]\\ -N\{-\Omega y+f(z,t)\}\end{array}\right\}, \tag{3.8}$$

$$\left.\begin{array}{c}\frac{\partial g}{\partial t}+\Omega(-\Omega y+f)-w_0\frac{\partial g}{\partial z}=-\frac{1}{\rho}\frac{\partial p}{\partial y}+\frac{\nu}{1+\lambda_1}\left[\frac{\partial^2 g}{\partial z^2}+\lambda_2\frac{\partial}{\partial z}\left(\frac{\partial^2 g}{\partial t\partial z}-w_0\frac{\partial^2 g}{\partial z^2}\right)\right]\\ -N\{\Omega x+g(z,t)\}\end{array}\right\}, \tag{3.9}$$

$$\frac{\partial p}{\partial z}=Nw_0, \tag{3.10}$$

where

$$N=\frac{\sigma B_0}{\rho}.$$

Since

$$r^2=x^2+y^2,$$

so

$$x=\frac{1}{2}\frac{\partial r^2}{\partial x},\ y=\frac{1}{2}\frac{\partial r^2}{\partial y}.$$

With the help of above expressions (3.8), (3.9) and (3.10) can be rewritten as

$$\begin{aligned}\frac{\partial f}{\partial t}-\Omega g-w_0\frac{\partial f}{\partial z} &= -\frac{1}{\rho}\frac{\partial}{\partial x}(p-\frac{\rho\Omega^2 r^2}{2})+\frac{\nu}{1+\lambda_1}\left[\frac{\partial^2 f}{\partial z^2}+\lambda_2\frac{\partial}{\partial z}\left(\frac{\partial^2 f}{\partial t\partial z}-w_0\frac{\partial^2 f}{\partial z^2}\right)\right]\\ &\quad -N\{-\Omega y+f(z,t)\},\end{aligned} \tag{3.11}$$

$$\frac{\partial g}{\partial t}+\Omega f-w_0\frac{\partial g}{\partial z} = -\frac{1}{\rho}\frac{\partial}{\partial y}(p-\frac{\rho\Omega^2 r^2}{2})+\frac{\nu}{1+\lambda_1}\left[\frac{\partial^2 g}{\partial z^2}+\lambda_2\frac{\partial}{\partial z}\left(\frac{\partial^2 g}{\partial t\partial z}-w_0\frac{\partial^2 g}{\partial z^2}\right)\right] -N\{\Omega x+g(z,t)\}, \quad (3.12)$$

$$\frac{\partial p}{\partial z}=\rho N w_0. \quad (3.13)$$

Introducing

$$\widehat{p}=p-\frac{\rho\Omega^2 r^2}{2}, \quad (3.14)$$

equations (3.11) to (3.13) become

$$\frac{\partial f}{\partial t}-\Omega g-w_0\frac{\partial f}{\partial z} = -\frac{1}{\rho}\frac{\partial \widehat{p}}{\partial x}+\frac{\nu}{1+\lambda_1}\left[\frac{\partial^2 f}{\partial z^2}+\lambda_2\frac{\partial}{\partial z}\left(\frac{\partial^2 f}{\partial t\partial z}-w_0\frac{\partial^2 f}{\partial z^2}\right)\right] -N\{-\Omega y+f(z,t)\}, \quad (3.15)$$

$$\frac{\partial g}{\partial t}+\Omega f-w_0\frac{\partial g}{\partial z} = -\frac{1}{\rho}\frac{\partial \widehat{p}}{\partial y}+\frac{\nu}{1+\lambda_1}\left[\frac{\partial^2 g}{\partial z^2}+\lambda_2\frac{\partial}{\partial z}\left(\frac{\partial^2 g}{\partial t\partial z}-w_0\frac{\partial^2 g}{\partial z^2}\right)\right] -N\{\Omega x+g(z,t)\}, \quad (3.16)$$

$$\frac{\partial \widehat{p}}{\partial z}=\rho N w_0. \quad (3.17)$$

Taking partial derivative of Eqs. (3.15) and (3.16) with respect to z and then using Eq.(3.17) in the resulting equations we have

$$\frac{\partial}{\partial z}\left[\frac{\partial f}{\partial t}-\Omega g-w_0\frac{\partial f}{\partial z}\right]=\frac{\partial}{\partial z}\left[\begin{array}{c}\frac{\nu}{1+\lambda_1}\left\{\frac{\partial^2 f}{\partial z^2}+\lambda_2\frac{\partial}{\partial z}\left(\frac{\partial^2 f}{\partial t\partial z}-w_0\frac{\partial^2 f}{\partial z^2}\right)\right\}\\ -Nf(z,t)\end{array}\right],$$

$$\frac{\partial}{\partial z}\left[\frac{\partial g}{\partial t}+\Omega f-w_0\frac{\partial g}{\partial z}\right]=\frac{\partial}{\partial z}\left[\begin{array}{c}\frac{\nu}{1+\lambda_1}\left\{\frac{\partial^2 g}{\partial z^2}+\lambda_2\frac{\partial}{\partial z}\left(\frac{\partial^2 g}{\partial t\partial z}-w_0\frac{\partial^2 g}{\partial z^2}\right)\right\}\\ -Ng(z,t)\end{array}\right].$$

Integration of above equations gives

$$\frac{\partial f}{\partial t}-\Omega g-w_0\frac{\partial f}{\partial z}=\frac{\nu}{1+\lambda_1}\left\{\frac{\partial^2 f}{\partial z^2}+\lambda_2\frac{\partial}{\partial z}\left(\frac{\partial^2 f}{\partial t\partial z}-w_0\frac{\partial^2 f}{\partial z^2}\right)\right\}-Nf(z,t)+c_1(t), \quad (3.18)$$

$$\frac{\partial g}{\partial t}+\Omega f-w_0\frac{\partial g}{\partial z}=\frac{\nu}{1+\lambda_1}\left\{\frac{\partial^2 g}{\partial z^2}+\lambda_2\frac{\partial}{\partial z}\left(\frac{\partial^2 g}{\partial t\partial z}-w_0\frac{\partial^2 g}{\partial z^2}\right)\right\}-Ng(z,t)+c_2(t), \quad (3.19)$$

where $c_1(t)$ and $c_2(t)$ are functions of integration.

The appropriate boundary conditions are

$$u(0,t) = -\Omega y + ct,\ v = \Omega x,\ t > 0,$$

$$u(\infty,t) = -\Omega(y-l),\ v = \Omega x,\ \text{for all } t,$$

$$u(z,0) = -\Omega(y-l),\ v = \Omega x,\ z > 0,$$

which in terms of f and g take the following form

$$\left.\begin{array}{c} f(0,t) = ct,\ g(0,t) = 0,\ t > 0, \\ f(\infty,t) = \Omega l,\ g(\infty,t) = 0,\ \text{for all } t, \\ f(z,0) = \Omega l,\ g(z,0) = 0, z > 0, \end{array}\right\} \tag{3.20}$$

where c is the constant acceleration.

From Eqs. (3.18), (3.19) and conditions (3.20) we obtain

$$c_1(t) = N\Omega l,\ c_2(t) = \Omega^2 l,$$

which after using in Eqs.(3.18) and (3.19), we get

$$\left.\begin{array}{c} \frac{\partial f}{\partial t} - \Omega g - w_0\frac{\partial f}{\partial z} = \frac{\nu}{1+\lambda_1}\left\{\frac{\partial^2 f}{\partial z^2} + \lambda_2\frac{\partial}{\partial z}\left(\frac{\partial^2 f}{\partial t\partial z} - w_0\frac{\partial^2 f}{\partial z^2}\right)\right\} \\ -Nf(z,t) + N\Omega l \end{array}\right\}, \tag{3.21}$$

$$\left.\begin{array}{c} \frac{\partial g}{\partial t} + \Omega f - w_0\frac{\partial g}{\partial z} = \frac{\nu}{1+\lambda_1}\left\{\frac{\partial^2 g}{\partial z^2} + \lambda_2\frac{\partial}{\partial z}\left(\frac{\partial^2 g}{\partial t\partial z} - w_0\frac{\partial^2 g}{\partial z^2}\right)\right\} \\ -Ng(z,t) + \Omega^2 l \end{array}\right\}. \tag{3.22}$$

The above two equations can be combined as

$$\left.\begin{array}{c} \frac{\partial F}{\partial t} - w_0\frac{\partial F}{\partial z} = \frac{\nu}{1+\lambda_1}\left\{\frac{\partial^2 F}{\partial z^2} + \lambda_2\frac{\partial}{\partial z}\left(\frac{\partial^2 F}{\partial t\partial z} - w_0\frac{\partial^2 F}{\partial z^2}\right)\right\} \\ -\left(\frac{\sigma B_0^2}{\rho} + i\Omega\right)F \end{array}\right\}, \tag{3.23}$$

where

$$F(z,t) = \frac{f(z,t)}{\Omega l} + i\frac{g(z,t)}{\Omega l} - 1. \tag{3.24}$$

Now the conditions in terms of F are

$$\left.\begin{array}{l} F(0,t) = \frac{ct}{\Omega l} - 1, \ t > 0, \\ F(\infty,t) = 0, \ t > 0, \\ F(z,0) = 0, \ z > 0. \end{array}\right\} \tag{3.25}$$

Introducing the non-dimensional variables

$$\xi = \sqrt{\frac{\Omega}{2\nu}}z, \ \tau = \Omega t, \tag{3.26}$$

we have

$$\left.\begin{array}{r} \frac{\partial F}{\partial t} = \Omega\frac{\partial F}{\partial \tau}, \\ \frac{\partial F}{\partial z} = \sqrt{\frac{\Omega}{2\nu}}\frac{\partial F}{\partial \xi}, \\ \frac{\partial^2 F}{\partial z^2} = \frac{\Omega}{2\nu}\frac{\partial^2 F}{\partial \xi^2}, \\ \frac{\partial^2 F}{\partial t \partial z} = \Omega\sqrt{\frac{\Omega}{2\nu}}\frac{\partial^2 F}{\partial \tau \partial \xi}, \\ \frac{\partial^2 F}{\partial t \partial z} - w_0\frac{\partial^2 F}{\partial z^2} = \Omega\sqrt{\frac{\Omega}{2\nu}}\frac{\partial^2 F}{\partial \tau \partial \xi} - w_0\frac{\Omega}{2\nu}\frac{\partial^2 F}{\partial \xi^2}, \\ \frac{\partial}{\partial z}\left\{\frac{\partial^2 F}{\partial t \partial z} - w_0\frac{\partial^2 F}{\partial z^2}\right\} = \sqrt{\frac{\Omega}{2\nu}}\frac{\partial}{\partial \xi}\left\{\Omega\sqrt{\frac{\Omega}{2\nu}}\frac{\partial^2 F}{\partial \tau \partial \xi} - w_0\frac{\Omega}{2\nu}\frac{\partial^2 F}{\partial \xi^2}\right\}. \end{array}\right\} \tag{3.27}$$

Making use of Eqs.(3.27) one can write Eq.(3.23) and boundary conditions (3.25) in non-dimensional variables as

$$\begin{aligned} \frac{\partial F}{\partial \tau} - w_0\sqrt{\frac{\Omega}{2\nu}}\frac{\partial F}{\partial \xi} &= \frac{\nu}{1+\lambda_1}\left\{\frac{\Omega}{2\nu}\frac{\partial^2 F}{\partial \xi^2} + \lambda_2\frac{\Omega}{2\nu}\frac{\partial}{\partial \xi}\left(\Omega\frac{\partial^2 F}{\partial \tau \partial \xi} - w_0\sqrt{\frac{\Omega}{2\nu}}\frac{\partial^2 F}{\partial \xi^2}\right)\right\} \\ &\quad - \left(\frac{\sigma B_0^2}{\rho} + i\Omega\right)F, \end{aligned} \tag{3.28}$$

$$\left.\begin{array}{r} F(0,\tau) = \frac{c\tau}{\Omega^2 l} - 1, \ \tau > 0, \\ F(\infty,\tau) = 0, \ \tau > 0, \\ F(\xi,\tau) = \frac{f(\xi,\tau)}{\Omega l} + i\frac{g(\xi,\tau)}{\Omega l} - 1. \end{array}\right\} \tag{3.29}$$

The equation (3.29) can also be written as

$$\frac{\partial F}{\partial \tau} - S\frac{\partial F}{\partial \xi} = \frac{\nu}{2(1+\lambda_1)}\left\{\frac{\partial^2 F}{\partial \xi^2} + \lambda_2\Omega\left(\frac{\partial^3 F}{\partial \tau \partial \xi^2} - S\frac{\partial^3 F}{\partial \xi^3}\right)\right\} - (N + i\Omega)F, \tag{3.30}$$

where

$$S = \frac{w_0}{\sqrt{2\Omega \upsilon}}$$

is the porosity parameter in non-dimensional variable.

3.3 Solution of the problem

For the solution of the problem, we define the Laplace transform pair as

$$\overline{F}(\xi, s) = L\{F(\xi, \tau)\} = \int_0^\infty F(\xi, \tau)e^{-s\tau}d\tau, \tag{3.31}$$

$$F(\xi, \tau) = L^{-1}\{\overline{F}(\xi, s)\} = \frac{1}{2\pi i}\int_{\lambda - i\infty}^{\lambda + i\infty} \overline{F}(\xi, s)e^{s\tau}ds. \tag{3.32}$$

In the transformed $s-$plane, the governing problem becomes

$$\left.\begin{array}{c}(1+s\beta)\frac{d^2\overline{F}(\xi,p)}{d\xi^2} - \beta S\frac{d^3\overline{F}(\xi,p)}{d\xi^3} + 2S(1+\lambda_1)\frac{d\overline{F}(\xi,p)}{d\xi} = 2(1+\lambda_1)(N+s)\overline{F} \\ +2(1+\lambda_1)i\overline{F}\end{array}\right\}, \tag{3.33}$$

$$\left.\begin{array}{c}\overline{F}(0, s) = \frac{c}{\Omega^2 s^2 l} - \frac{1}{s}, \\ \overline{F}(\infty, s) = 0.\end{array}\right\} \tag{3.34}$$

In general, for flow of non-Newtonian fluids, the equations of motion are of higher order than the Navier-Stokes equations. The adherence boundary conditions is insufficient for determinacy. The standard method used to overcome this difficulty is to resort to perturbation that lowers the order of the equation. With this fact in mind we write

$$\overline{F} = \overline{F_0} + \beta\overline{F_1} + O(\beta^2). \tag{3.35}$$

where $\beta = \lambda_2\Omega$ is the perturbation parameter.

Substituting Eq. (3.35) into Eqs.(3.33) and (3.34) and then equating the coefficients of like power of β we get the following systems:

3.4 System of zero order

$$\frac{d^2\overline{F}_0}{d\xi^2} + 2S(1+\lambda_1)\frac{d\overline{F}_0}{d\xi} - 2(1+\lambda_1)(N+s+i)\overline{F}_0 = 0, \tag{3.36}$$

$$\left.\begin{array}{c} \overline{F}_0(0,s) = \frac{c}{\Omega^2 s^2 l} - \frac{1}{s}, \\ \overline{F}_0(\infty,s) = 0. \end{array}\right\}. \tag{3.37}$$

3.5 System of first order

$$\frac{d^2\overline{F}_1}{d\xi^2} + 2(1+\lambda_1)S\frac{d\overline{F}_1}{d\xi} - 2(1+\lambda_1)(N+s+i)\overline{F}_1 = S\frac{d^3\overline{F}_0}{d\xi^3} - s\frac{d^2\overline{F}_0}{d\xi^2}, \tag{3.38}$$

$$\overline{F}_1(\infty,s) = 0, \overline{F}_1(0,s) = 0. \tag{3.39}$$

3.6 Solution for zero order system

The general solution of Eq.(3.36) is

$$\overline{F}_0(\xi,s) = A_1 e^{(1+\lambda_1)(-S+\lambda_0)} + B_1 e^{-(1+\lambda_1)(S+\lambda_0)}, \tag{3.40}$$

where

$$\lambda_0 = \sqrt{S^2 + \frac{2(N+s+i)}{1+\lambda_1}} \tag{3.41}$$

and A_1, B_1 are arbitrary constants. Using boundary condition (3.37), A_1, B_1 have values

$$A_1 = 0, \; B_1 = \frac{c}{\Omega^2 s^2 l} - \frac{1}{s}, \tag{3.42}$$

which upon substituting in Eq. (3.40) yields

$$\overline{F}_0(\xi,s) = \left(\frac{c}{\Omega^2 s^2 l} - \frac{1}{s}\right) e^{-(1+\lambda_1)(S+\lambda_0)},$$

or

$$\overline{F}_0(\xi,p) = \left(\frac{c}{\Omega^2 s^2 l} - \frac{1}{s}\right) e^{-\left((1+\lambda_1)S+\sqrt{(1+\lambda_1)^2 S^2 + 2(1+\lambda_1)(n+s+i)}\right)\xi}. \tag{3.43}$$

Laplace inversion of above expression is

$$F_0(\xi,\tau) = L^{-1}\left\{\overline{F}_0(\xi,s)\right\},$$

or

$$F_0(\xi,\tau) = \frac{1}{2\pi i}\int_{\gamma_1 - i\infty}^{\gamma_1 + i\infty} \overline{F}_0(\xi,s) e^{s\tau} ds,$$

or

$$F_0(\xi,\tau) = \frac{1}{2\pi i}\int_{\gamma_1 - i\infty}^{\gamma_1 + i\infty} \left(\frac{c}{\Omega^2 s^2 l} - \frac{1}{s}\right) e^{-\left((1+\lambda_1)S + \sqrt{\begin{array}{c}(1+\lambda_1)^2 S^2 \\ +2(1+\lambda_1)(n+s+i)\end{array}}\right)\xi} e^{s\tau} ds$$

or

$$F_0(\xi,\tau) = \frac{e^{-(1+\lambda_1)S\xi}}{2\pi i}\int_{\gamma_1 - i\infty}^{\gamma_1 + i\infty} \left(\frac{c}{\Omega^2 s^2 l} - \frac{1}{s}\right) e^{-\xi\sqrt{\begin{array}{c}(1+\lambda_1)^2 S^2 \\ +2(1+\lambda_1)(n+s+i)\end{array}} + s\tau} ds,$$

or

$$F_0(\xi,\tau) = \frac{e^{-(1+\lambda_1)S\xi}}{2\pi i}\left[\begin{array}{c} \frac{c}{\Omega^2 l}\int_{\gamma_1 - i\infty}^{\gamma_1 + i\infty} \frac{e^{s\tau - \xi\sqrt{(1+\lambda_1)^2 S^2 + 2(1+\lambda_1)(n+s+i)}}}{s^2} ds + \\ \int_{\gamma_1 - i\infty}^{\gamma_1 + i\infty} \frac{e^{s\tau - \xi\sqrt{(1+\lambda_1)^2 S^2 + 2(1+\lambda_1)(n+s+i)}}}{s} ds \end{array}\right]. \tag{3.44}$$

Now using residue theory

$$\int_{\gamma_1 - i\infty}^{\gamma_1 + i\infty} \frac{e^{s\tau - \xi\sqrt{(1+\lambda_1)^2 S^2 + 2(1+\lambda_1)(n+s+i)}}}{s^2} ds = 2\pi i\left[\tau - \frac{\xi(1+\lambda_1)}{\sqrt{(1+\lambda_1)^2 S^2 + 2(1+\lambda_1)(n+i)}}\right] \times e^{-\xi\sqrt{(1+\lambda_1)^2 S^2 + 2(1+\lambda_1)(n+i)}},$$

$$\int_{\gamma_1 - i\infty}^{\gamma_1 + i\infty} \frac{e^{s\tau - \xi\sqrt{(1+\lambda_1)^2 S^2 + 2(1+\lambda_1)(N+s+i)}}}{s} ds$$

$$= \pi i \begin{bmatrix} e^{\xi 2(1+\lambda_1)\sqrt{\frac{1+\lambda_1}{2}+N+i}} \operatorname{erf} c\left(\frac{\xi\sqrt{2(1+\lambda_1)}+2\sqrt{\frac{1+\lambda_1}{2}+N+i\tau}}{2\sqrt{\tau}}\right) \\ e^{-\xi 2(1+\lambda_1)\sqrt{\frac{1+\lambda_1}{2}+N+i}} \operatorname{erf} c\left(\frac{\xi\sqrt{2(1+\lambda_1)}-2\sqrt{\frac{1+\lambda_1}{2}+N+i\tau}}{2\sqrt{\tau}}\right) \end{bmatrix}$$

$$= \pi i \begin{bmatrix} e^{\sqrt{2}\xi(1+\lambda_1)\sqrt{1+\lambda_1+2(N+i)}} \operatorname{erf} c\left(\frac{\xi\sqrt{1+\lambda_1}+\sqrt{1+\lambda_1+2(N+i)\tau}}{\sqrt{2\tau}}\right) \\ e^{-\sqrt{2}\xi(1+\lambda_1)\sqrt{1+\lambda_1+2(N+i)}} \operatorname{erf} c\left(\frac{\xi\sqrt{1+\lambda_1}-\sqrt{1+\lambda_1+2(N+i)\tau}}{\sqrt{2\tau}}\right) \end{bmatrix},$$

and so

$$F_0(\xi,\tau) = \frac{e^{-(1+\lambda_1)S\xi}}{2\pi i} \begin{bmatrix} \frac{c}{\Omega^2 l} 2\pi i \left\{\tau - \frac{\xi(1+\lambda_1)}{\sqrt{(1+\lambda_1)^2 S^2 + 2(1+\lambda_1)(N+p+i)}}\right\} e^{-\xi\sqrt{(1+\lambda_1)^2 S^2 + 2(1+\lambda_1)(N+p+i)}} \\ -\pi i \left\{ \begin{array}{c} e^{\sqrt{2}\xi(1+\lambda_1)\sqrt{1+\lambda_1+2(N+i)}} \operatorname{erf} c\left(\frac{\xi\sqrt{1+\lambda_1}+\sqrt{1+\lambda_1+2(N+i)\tau}}{\sqrt{2\tau}}\right) \\ e^{-\sqrt{2}\xi(1+\lambda_1)\sqrt{1+\lambda_1+2(N+i)}} \operatorname{erf} c\left(\frac{\xi\sqrt{1+\lambda_1}-\sqrt{1+\lambda_1+2(N+i)\tau}}{\sqrt{2\tau}}\right) \end{array} \right\} \end{bmatrix},$$

or

$$F_0(\xi,\tau) = \frac{e^{-(1+\lambda_1)S\xi}}{2} \begin{bmatrix} \frac{2c}{\Omega^2 l} \left\{\tau - \frac{\xi(1+\lambda_1)}{\sqrt{(1+\lambda_1)^2 S^2 + 2(1+\lambda_1)(N+i)}}\right\} e^{-\xi\sqrt{(1+\lambda_1)^2 S^2 + 2(1+\lambda_1)(N+i)}} \\ -e^{\sqrt{2}\xi(1+\lambda_1)\sqrt{1+\lambda_1+2(N+i)}} \operatorname{erf} c\left(\frac{\xi\sqrt{1+\lambda_1}+\sqrt{1+\lambda_1+2(N+i)\tau}}{\sqrt{2\tau}}\right) + \\ e^{-\sqrt{2}\xi(1+\lambda_1)\sqrt{1+\lambda_1+2(N+i)}} \operatorname{erf} c\left(\frac{\xi\sqrt{1+\lambda_1}-\sqrt{1+\lambda_1+2(N+i)\tau}}{\sqrt{2\tau}}\right) \end{bmatrix}. \tag{3.45}$$

3.7 Solution for system of order one

Substituting Eq. (3.43) into Eq. (3.38) we arrive at

$$\frac{d^2\overline{F}_1}{d\xi^2} + 2(1+\lambda_1)S\frac{d\overline{F}_1}{d\xi} - 2(1+\lambda_1)(P+i)\overline{F}_1 = -(1+\lambda_1)^2(S+\lambda_0)^2[S(1+\lambda_1)(S+\lambda_0) + s]\left(\frac{c}{\Omega^2 s^2 l} - \frac{1}{s}\right) e^{-(1+\lambda_1)(S+\lambda_0)\xi}. \tag{3.46}$$

The above equation is second order non homogeneous equation. Its general solution is sum of complementary function and the particular integral. The complementary function and the particular integral of Eq.(3.46) are

$$\overline{F}_{1c} = B_2 e^{(1+\lambda_1)(S-\lambda_0)\xi} + B_3 e^{-(1+\lambda_1)(S+\lambda_0)\xi},$$

$$\left.\begin{array}{l} \overline{F}_{1p} = [S(1+\lambda_1)^3(S+\lambda_0)^3 + s(1+\lambda_1)^2(S+\lambda_0)^2]\left(\frac{c}{\Omega^2 s^2 l} - \frac{1}{s}\right) \\ \quad \times \left(\frac{\xi}{2\lambda_0(1+\lambda_1)} e^{-(1+\lambda_1)(S+\lambda_0)\xi}\right) \end{array}\right\}$$

and thus the general solution is

$$\begin{aligned} \overline{F}_1 &= B_2 e^{(1+\lambda_1)(S-\lambda_0)\xi} + B_3 e^{-(1+\lambda_1)(S+\lambda_0)\xi} + \frac{1}{2\lambda_0}\left(\begin{array}{c} S(1+\lambda_1)^2(S+\lambda_0)^3 \\ +s(1+\lambda_1)(S+\lambda_0)^2 \end{array}\right) \\ &\left(\frac{c}{\Omega^2 s^2 l} - \frac{1}{s}\right)\xi e^{-(1+\lambda_1)(S+\lambda_0)\xi}, \end{aligned} \tag{3.47}$$

where B_2 and B_3 are arbitrary constants. Using the boundary conditions (3.39), we get $B_2 = 0 = B_3$ and so Eq.(3.48) reduces to

$$\overline{F}_1(\xi, s) = \frac{\xi(1+\lambda_1)e^{-(1+\lambda_1)(S+\lambda_0)\xi}}{2}\left(\frac{c}{\Omega^2 s^2 l} - \frac{1}{s}\right)\left[\begin{array}{c} 2sS + S^2\left\{(1+\lambda_1)S^2 + s\right\}\frac{1}{\lambda_0} + \\ \left\{3S^2(1+\lambda_1) + s\right\}\lambda_0 + \\ 3S^3(1+\lambda_1) + S(1+\lambda_1)\lambda_0^2 \end{array}\right]. \tag{3.48}$$

Taking the inverse Laplace transform of Eq.(3.48), we have

$$F_1(\xi, \tau) = \frac{\xi(1+\lambda_1)e^{-S(1+\lambda_1)\xi}}{2}\left[\frac{c}{\Omega^2 l} I_1 - I_2\right], \tag{3.49}$$

where

$$I_1 = \frac{1}{2\pi i}\int_{\gamma_1 - i\infty}^{\gamma_1 + i\infty}\left[\begin{array}{c} 2sS + S^2\left\{(1+\lambda_1)S^2 + s\right\}\frac{1}{\lambda_0} + \\ \left\{3S^2(1+\lambda_1) + s\right\}\lambda_0 + \\ 3S^3(1+\lambda_1) + S(1+\lambda_1)\lambda_0^2 \end{array}\right]\frac{e^{-(1+\lambda_1)\lambda_0\xi + s\tau}}{s^2}ds, \tag{3.50}$$

$$I_2 = \frac{1}{2\pi i}\int_{\gamma_1 - i\infty}^{\gamma_1 + i\infty}\left[\begin{array}{c} 2sS + S^2\left\{(1+\lambda_1)S^2 + s\right\}\frac{1}{\lambda_0} + \\ \left\{3S^2(1+\lambda_1) + s\right\}\lambda_0 + \\ 3S^3(1+\lambda_1) + S(1+\lambda_1)\lambda_0^2 \end{array}\right]\frac{e^{-(1+\lambda_1)\lambda_0\xi + s\tau}}{s}ds. \tag{3.51}$$

For the solution of I_1, let

$$\eta = \left[S^2 + \frac{2(N+s+i)}{1+\lambda_1}\right]^{\frac{1}{2}},$$

$$ds = \eta(1+\lambda_1)d\eta,$$

$$s = \frac{1+\lambda_1}{2}\left[\eta^2 - \left(S^2 + \frac{2(N+i)}{1+\lambda_1}\right)\right],$$

$$s = \frac{1+\lambda_1}{2}\left[\eta^2 - c_0^2\right], c_0^2 = \left(S^2 + \frac{2(N+i)}{1+\lambda_1}\right),$$

and so Eq. (3.50) gives

$$I_1 = \frac{e^{-\left(\frac{1+\lambda_1}{2}\right)c_0^2\tau}}{\pi i}\int_{\gamma_1-i\infty}^{\gamma_1+i\infty}\left[\begin{array}{c} \frac{\eta}{\left(\eta^2-c_0^2\right)^2} + S^2\left\{\frac{S^2}{\left(\eta^2-c_0^2\right)^2} + \frac{1}{2\left(\eta^2-c_0^2\right)}\right\} \\ +\left\{\frac{3S^2\eta^2}{\left(\eta^2-c_0^2\right)^2} + \frac{\eta^2}{2\left(\eta^2-c_0^2\right)}\right\} \\ +\left\{\frac{3S^3\eta}{\left(\eta^2-c_0^2\right)^2} + \frac{S\eta^3}{\left(\eta^2-c_0^2\right)^2}\right\} \end{array}\right] e^{-(1+\lambda_1)\eta\xi + \left(\frac{1+\lambda_1}{2}\right)\tau\eta^2} d\eta,$$

or

$$I_1 = e^{-\left(\frac{1+\lambda_1}{2}\right)c_0^2\tau}\left[I_{11} + I_{12} + I_{13} + I_{14} + I_{15} + I_{16} + I_{17}\right], \tag{3.52}$$

where

$$I_{11} = \frac{1}{\pi i}\int_{\gamma_1-i\infty}^{\gamma_1+i\infty}\frac{\eta e^{-(1+\lambda_1)\eta\xi + \left(\frac{1+\lambda_1}{2}\right)\tau\eta^2}}{\left(\eta^2 - c_0^2\right)^2}d\eta, \tag{3.53}$$

$$I_{12} = \frac{S^4}{\pi i}\int_{\gamma_1-i\infty}^{\gamma_1+i\infty}\frac{e^{-(1+\lambda_1)\eta\xi + \left(\frac{1+\lambda_1}{2}\right)\tau\eta^2}}{\left(\eta^2 - c_0^2\right)^2}d\eta, \tag{3.54}$$

$$I_{13} = \frac{S^2}{2\pi i}\int_{\gamma_1-i\infty}^{\gamma_1+i\infty}\frac{e^{-(1+\lambda_1)\eta\xi + \left(\frac{1+\lambda_1}{2}\right)\tau\eta^2}}{\eta^2 - c_0^2}d\eta, \tag{3.55}$$

$$I_{14} = \frac{3S^2}{\pi i}\int_{\gamma_1-i\infty}^{\gamma_1+i\infty}\frac{\eta^2 e^{-(1+\lambda_1)\eta\xi + \left(\frac{1+\lambda_1}{2}\right)\tau\eta^2}}{\left(\eta^2 - c_0^2\right)^2}d\eta, \tag{3.56}$$

$$I_{15} = \frac{1}{2\pi i}\int_{\gamma_1-i\infty}^{\gamma_1+i\infty}\frac{\eta^2 e^{-(1+\lambda_1)\eta\xi + \left(\frac{1+\lambda_1}{2}\right)\tau\eta^2}}{\eta^2 - c_0^2}d\eta, \tag{3.57}$$

$$I_{16} = \frac{3S^3}{\pi i}\int_{\gamma_1-i\infty}^{\gamma_1+i\infty}\frac{\eta e^{-(1+\lambda_1)\eta\xi + \left(\frac{1+\lambda_1}{2}\right)\tau\eta^2}}{\left(\eta^2 - c_0^2\right)^2}d\eta, \tag{3.58}$$

$$I_{17} = \frac{S}{\pi i} \int_{\gamma_1 - i\infty}^{\gamma_1 + i\infty} \frac{\eta^3 e^{-(1+\lambda_1)\eta\xi + \left(\frac{1+\lambda_1}{2}\right)\tau\eta^2}}{\left(\eta^2 - c_0^2\right)^2} d\eta. \tag{3.59}$$

Solving the above integrals we finally write

$$I_{11} = \frac{1+\lambda_1}{c_0} e^{\left(\frac{1+\lambda_1}{2}\right)\tau c_0^2} \left[\xi \sinh\{(1+\lambda_1)c_0\xi\} + \tau c_0 \cosh\{(1+\lambda_1)c_0\xi\}\right], \tag{3.60}$$

$$I_{12} = \frac{S^4}{2c_0^3} \begin{bmatrix} e^{c_0^2\left(\frac{1+\lambda_1}{2}\right)+(1+\lambda_1)c_0\xi} \operatorname{erf} c\left(\sqrt{\frac{1+\lambda_1}{2\tau}}(\xi + c_0\tau)\right) \\ -e^{c_0^2\left(\frac{1+\lambda_1}{2}\right)-(1+\lambda_1)c_0\xi} \operatorname{erf} c\left(\sqrt{\frac{1+\lambda_1}{2\tau}}(\xi - c_0\tau)\right) \\ -2c_0(1+\lambda_1) e^{\left(\frac{1+\lambda_1}{2}\right)\tau c_0^2} \{\xi \cosh((1+\lambda_1)c_0\xi) + \tau c_0 \sinh((1+\lambda_1)c_0\xi)\} \end{bmatrix}, \tag{3.61}$$

$$I_{13} = \frac{S^2}{4c_0} \begin{bmatrix} e^{c_0^2\left(\frac{1+\lambda_1}{2}\right)-(1+\lambda_1)c_0\xi} \operatorname{erf} c\left(\sqrt{\frac{1+\lambda_1}{2\tau}}(\xi - c_0\tau)\right) \\ -e^{c_0^2\left(\frac{1+\lambda_1}{2}\right)+(1+\lambda_1)c_0\xi} \operatorname{erf} c\left(\sqrt{\frac{1+\lambda_1}{2\tau}}(\xi + c_0\tau)\right) \end{bmatrix}, \tag{3.62}$$

$$I_{14} = \frac{3S^2}{4c_0} \begin{bmatrix} 3\begin{Bmatrix} e^{c_0^2\left(\frac{1+\lambda_1}{2}\right)-(1+\lambda_1)c_0\xi} \operatorname{erf} c\left(\sqrt{\frac{1+\lambda_1}{2\tau}}(\xi - c_0\tau)\right) \\ -e^{c_0^2\left(\frac{1+\lambda_1}{2}\right)+(1+\lambda_1)c_0\xi} \operatorname{erf} c\left(\sqrt{\frac{1+\lambda_1}{2\tau}}(\xi + c_0\tau)\right) \end{Bmatrix} \\ -2c_0(1+\lambda_1) e^{\left(\frac{1+\lambda_1}{2}\right)\tau c_0^2} \{\xi \cosh((1+\lambda_1)c_0\xi) + \tau c_0 \sinh((1+\lambda_1)c_0\xi)\} \end{bmatrix}, \tag{3.63}$$

$$I_{15} = \frac{c_0}{4} \begin{bmatrix} e^{c_0^2\left(\frac{1+\lambda_1}{2}\right)-(1+\lambda_1)c_0\xi} \operatorname{erf} c\left(\sqrt{\frac{1+\lambda_1}{2\tau}}(\xi - c_0\tau)\right) \\ -e^{c_0^2\left(\frac{1+\lambda_1}{2}\right)+(1+\lambda_1)c_0\xi} \operatorname{erf} c\left(\sqrt{\frac{1+\lambda_1}{2\tau}}(\xi + c_0\tau)\right) \end{bmatrix}, \tag{3.64}$$

$$I_{16} = \frac{3S^3(1+\lambda_1)}{c_0} e^{\left(\frac{1+\lambda_1}{2}\right)\tau c_0^2} \left[\xi \sinh\{(1+\lambda_1)c_0\xi\} + \tau c_0 \cosh\{(1+\lambda_1)c_0\xi\}\right], \tag{3.65}$$

$$I_{17} = \frac{S}{2} \begin{bmatrix} e^{c_0^2\left(\frac{1+\lambda_1}{2}\right)-(1+\lambda_1)c_0\xi} \operatorname{erf} c\left(\sqrt{\frac{1+\lambda_1}{2\tau}}(\xi - c_0\tau)\right) \\ +e^{c_0^2\left(\frac{1+\lambda_1}{2}\right)+(1+\lambda_1)c_0\xi} \operatorname{erf} c\left(\sqrt{\frac{1+\lambda_1}{2\tau}}(\xi + c_0\tau)\right) \\ +c_0(1+\lambda_1) e^{\left(\frac{1+\lambda_1}{2}\right)\tau c_0^2} \begin{bmatrix} \xi \cosh\{(1+\lambda_1)c_0\xi\} \\ +\tau c_0 \sinh\{(1+\lambda_1)c_0\xi\} \end{bmatrix} \end{bmatrix}. \tag{3.66}$$

Now the value of I_1 is

$$I_1 = e^{-\left(\frac{1+\lambda_1}{2}\right)\tau c_0^2} \begin{bmatrix} (1+\lambda_1)\, e^{\left(\frac{1+\lambda_1}{2}\right)\tau c_0^2} \begin{bmatrix} \left(\frac{2\xi - 2S^4\tau - 3c_0^2\tau S^2 + 6S^3\xi + Sc_0^3\tau}{2c_0}\right) \sinh\{(1+\lambda_1)c_0\xi\} + \\ \left(\frac{2c_0^2\tau - 2\xi S^4 - 4c_0^3\xi + 6c_0^2 S^3\tau + Sc_0^3\xi}{2c_0^2}\right) \cosh\{(1+\lambda_1)c_0\xi\} \end{bmatrix} \\ \begin{bmatrix} e^{c_0^2(\frac{1+\lambda_1}{2}) + (1+\lambda_1)c_0\xi} \left[\frac{2S^4 - 10c_0^2S^2 - c_0^4 + 2c_0^3 S}{4c_0^3}\right] \operatorname{erf} c\left(\sqrt{\frac{1+\lambda_1}{2\tau}}(\xi + c_0\tau)\right) \\ e^{c_0^2(\frac{1+\lambda_1}{2}) - (1+\lambda_1)c_0\xi} \left[\frac{-2S^4 + 10c_0^2S^2 + c_0^4 + 2Sc_0^3}{4c_0^3}\right] \operatorname{erf} c\left(\sqrt{\frac{1+\lambda_1}{2\tau}}(\xi - c_0\tau)\right) \end{bmatrix} \end{bmatrix} \tag{3.67}$$

Employing a similar procedure as for I_1, one obtains

$$I_2 = \frac{(1+\lambda_1)^2}{2} \begin{bmatrix} \left\{\frac{Sc_0^3 + 3c_0S^3 - 3S^2c_0^2 - S^4(1+\lambda_1)}{c_0}\right\} e^{(1+\lambda_1)\eta\xi c_0} \operatorname{erf} c\left(\sqrt{\frac{1+\lambda_1}{2\tau}}\,(\xi + c_0\tau)\right) \\ \left\{\frac{Sc_0^3 + 3c_0S^3 + 3S^2c_0^2 + S^4(1+\lambda_1)}{c_0}\right\} e^{-(1+\lambda_1)\eta\xi c_0} \operatorname{erf} c\left(\sqrt{\frac{1+\lambda_1}{2\tau}}\,(\xi - c_0\tau)\right) \end{bmatrix},$$

or

$$I_2 = \frac{(1+\lambda_1)^2}{2} \begin{bmatrix} L^- e^{(1+\lambda_1)\eta\xi c_0} \operatorname{erf} c\left(\sqrt{\frac{1+\lambda_1}{2\tau}}\,(\xi + c_0\tau)\right) \\ L^+ e^{-(1+\lambda_1)\eta\xi c_0} \operatorname{erf} c\left(\sqrt{\frac{1+\lambda_1}{2\tau}}\,(\xi - c_0\tau)\right) \end{bmatrix}, \tag{3.68}$$

where

$$L^{\pm} = \frac{Sc_0^3 + 3c_0S^3 \pm \left(3S^2c_0^2 + S^4(1+\lambda_1)\right)}{c_0}$$

and now the expression for F_1 may be written as

$$\left.\begin{array}{c} F_1(\xi,\tau) = \frac{\xi(1+\lambda_1)e^{-S(1+\lambda_1)\xi}}{2} \\ \begin{bmatrix} \frac{c}{\Omega^2 l} e^{-\left(\frac{1+\lambda_1}{2}\right)\tau c_0^2} \begin{bmatrix} (1+\lambda_1)\, e^{\left(\frac{1+\lambda_1}{2}\right)\tau c_0^2} \begin{bmatrix} M \sinh\{(1+\lambda_1)c_0\xi\} \\ +N\cosh\{(1+\lambda_1)c_0\xi\} \end{bmatrix} \\ + \begin{bmatrix} K^- e^{c_0^2(\frac{1+\lambda_1}{2}) + (1+\lambda_1)c_0\xi} \operatorname{erf} c\left(\sqrt{\frac{1+\lambda_1}{2\tau}}(\xi + c_0\tau)\right) \\ +K^+ e^{c_0^2(\frac{1+\lambda_1}{2}) - (1+\lambda_1)c_0\xi} \operatorname{erf} c\left(\sqrt{\frac{1+\lambda_1}{2\tau}}(\xi - c_0\tau)\right) \end{bmatrix} \end{bmatrix} \\ -\frac{(1+\lambda_1)^2}{2} \begin{bmatrix} L^- e^{(1+\lambda_1)\eta\xi c_0} \operatorname{erf} c\left(\sqrt{\frac{1+\lambda_1}{2\tau}}\,(\xi + c_0\tau)\right) \\ +L^+ e^{-(1+\lambda_1)\eta\xi c_0} \operatorname{erf} c\left(\sqrt{\frac{1+\lambda_1}{2\tau}}\,(\xi - c_0\tau)\right) \end{bmatrix} \end{bmatrix}, \end{array}\right\} \tag{3.69}$$

where

$$\left.\begin{aligned} K\pm &= \tfrac{2Sc_0^3 \pm (c_0^4 + 10c_0^2 S^2 - 2S^4)}{4c_0^3}, \\ M &= \tfrac{2\xi - 2S^4\tau - 3c_0^2\tau S^2 + 6S^3\xi + Sc_0^3\tau}{2c_0}, \\ R &= \tfrac{2c_0^2\tau - 2\xi S^4 - 4c_0^3\xi + 6c_0^2 S^3\tau + Sc_0^3\xi}{2c_0^2}. \end{aligned}\right\} \tag{3.70}$$

Inserting Eqs.(3.45) and (3.69) into Eq.(3.35) we obtain F up to $O(\beta)$ as

$$F = \tfrac{e^{-(1+\lambda_1)S\xi}}{2}\left\{\begin{array}{l} \left[\begin{array}{c} \frac{2c}{\Omega^2 l}\left\{\tau - \frac{\xi(1+\lambda_1)}{\sqrt{(1+\lambda_1)^2S^2+2(1+\lambda_1)(N+i)}}\right\} e^{-\xi\sqrt{(1+\lambda_1)^2S^2+2(1+\lambda_1)(N+i)}} \\ -e^{\sqrt{2}\xi(1+\lambda_1)\sqrt{1+\lambda_1+2(N+i)}}\operatorname{erf}c\left(\frac{\xi\sqrt{1+\lambda_1}+\sqrt{1+\lambda_1+2(N+i)}\tau}{\sqrt{2\tau}}\right) \\ +e^{-\sqrt{2}\xi(1+\lambda_1)\sqrt{1+\lambda_1+2(N+i)}}\operatorname{erf}c\left(\frac{\xi\sqrt{1+\lambda_1}-\sqrt{1+\lambda_1+2(N+i)}\tau}{\sqrt{2\tau}}\right) \end{array}\right] \\ \qquad + \left(\beta\frac{\xi(1+\lambda_1)e^{-S(1+\lambda_1)\xi}}{2}\right) \\ \times\left[\begin{array}{l} \frac{c}{\Omega^2 l}e^{-\left(\frac{1+\lambda_1}{2}\right)\tau c_0^2}\left[\begin{array}{c} (1+\lambda_1)\,e^{\left(\frac{1+\lambda_1}{2}\right)\tau c_0^2}\left[\begin{array}{c} M\sinh\{(1+\lambda_1)c_0\xi\} \\ +R\cosh\{(1+\lambda_1)c_0\xi\}\end{array}\right] \\ +\left[\begin{array}{c} K^- e^{c_0^2(\frac{1+\lambda_1}{2})+(1+\lambda_1)c_0\xi}\operatorname{erf}c\left(\sqrt{\frac{1+\lambda_1}{2\tau}}(\xi+c_0\tau)\right) \\ +K^+ e^{c_0^2(\frac{1+\lambda_1}{2})-(1+\lambda_1)c_0\xi}\operatorname{erf}c\left(\sqrt{\frac{1+\lambda_1}{2\tau}}(\xi-c_0\tau)\right)\end{array}\right]\end{array}\right] \\ -\frac{(1+\lambda_1)^2}{2}\left[\begin{array}{c} L^- e^{(1+\lambda_1)\eta\xi c_0}\operatorname{erf}c\left(\sqrt{\frac{1+\lambda_1}{2\tau}}(\xi+c_0\tau)\right) \\ +L^+ e^{-(1+\lambda_1)\eta\xi c_0}\operatorname{erf}c\left(\sqrt{\frac{1+\lambda_1}{2\tau}}(\xi-c_0\tau)\right)\end{array}\right]\end{array}\right] \end{array}\right\}. \tag{3.71}$$

3.8 Concluding remarks

In this chapter, we have used the Jeffrey fluid as a non-Newtonian fluid model. This model describes the flow of a linear viscoelastic fluid. The presented analysis is valid under restrictive conditions as in particular the magnetic Reynolds number and the parameter β are small. Any how the considered model is rigorous. The salient feature of the analysis is to obtain a meaningful blowing solution which can be obtained by using S by $-S$ in Eq.(3.71). It is also very important to note that steady state cannot be achieved from the solution for any value of τ. This is due to the fact since disk is moving with uniform acceleration for $\tau > 0$. Finally, the results for Navier-Stokes fluid can be obtained by taking λ_1 and λ_2 equal to zero.

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