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Some Properties of Non-Commutative Multiplication Semi-groups

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A Dissertation

Submitted in the partial fulfillment of the

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CERTIFICATE

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A DISSERTATION SUBMITTED IN THE PARTIAL FULFILMENT OF THE REQUIREMENT FOR THE DEGREE OF MASTER OF PHILOSOPHY

We accept this dissertation as conforming to the required standard

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Dedicated To

My Dear Parents

Having no substitute

And

My Beloved Sisters

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NAZLI SANAM

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Abstract

Multiplication rings play an important role in ring theory. Multiplication rings have been extensively studied by Gilmer and Mott [4], Griffin [5], Mott [10] etc, and have been characterized using powerful ideal theoretic techniques. In this dissertation non-commutative multiplication semigroups without identity are characterized. This dissertation consists of three chapters.

Chapter one is devoted to preliminaries, it provides introductory concepts for semigroups its ideals, prime and semiprime ideals, regular, simple and semisimple semigroups.

In Chapter two we have reviewed the paper [4]. In this chapter we have studied the conditions when a multiplication semigroup containing identity is a regular semigroup or a principal ideal semigroup.

Chapter three is about non-commutative multiplication semigroups without identity. In this chapter we have determined conditions when the different classes of semisimple semigroups are right multiplication semigroups. We have also provided sufficient conditions for right multiplication semigroups to be right regular or regular or union of groups. Also we have characterized left cancellative right multiplication semigroups.

Contents

Chapter 1

INTRODUCTORY IDEAS

In this introductory chapter we shall give some basic concepts of semi groups and review some of the background material that will be of value for our later pursuits. For undefined terms and notations, we refer to [1], [2] and [6].

1.1 Basic Concepts in Semigroups

In 1916 O. J. Schmidt introduces the term semigroup in his book "Abstract Group Theory" (in Russian) and means semigroups which are cancellative from both sides and possibly without identity. After that again more then ten years passed before semigroups became a direct object of investigation. In the first papers various different names were used like "group", "kernelgroup", "abstract composition system", "Ubergroupe", "Schief", "abstract transmutation system", "Mischgruppe". Today we can say that the beginning of the theory of semigroups is marked by A. K. Suskewic, D. Rees, and P. Dubreil.

In some respects the theory of semigroups has very close relation with group theory and in some other respects with ring theory. It is quite understandable that the early major contributions to the theory were strongly motivated by comparisons with groups and with rings. In more recent years the subject has developed its own characteristic problems, methods and results.

1.1.1 Definition

A *groupoid* is a system $(S,.)$ consisting of a non-empty set S together with a binary operation"." on S. A *semigroup* is a groupoid $(S,.)$ such that the binary operation "." is associative. We shall write xy instead of $x.y$, and usually refer to the binary operation as multiplication "." on S. If a semigroup S has the property that for all x, y in S, $xy = yx$, we shall say that S is a *commutative semigroup.*

1.1.2 Definition

An element a of a semigroup S is said to be *left* (similarly *right*) *cancellative* if, for any x, y in S, $ax = ay$ $(xa = ya)$ implies that $x = y$. A semigroup S is called *left (right) cancellative* if every element of S is left (right) cancellative. We say that S is *cancellative* if it is both left and right cancellative.

1.1.3 Proposition

A finite cancellative semi group is a group.

1.1.4 Definition

An element of a semigroup which commutes with every element of S is called a *central element* of S. The set of all central elements of S is either empty or is a sub semigroup of S and in the later case is called *center* of S.

1.1.5 Definition

. An element e of a semigroup S is called a *left (right) identity element* of S if $ea = a(ae = a)$ for all a in S. An element e in S is called $two - sided\ identity$ (or simply *identity*) element of S if it is both left and right identity of S.

If a semigroup S contains an identity element, then it is unique and we say that S is a *semigroup with identity* or S is a *monoid.*

1.1.6 Definition

An element z of a semigroup S is called a *left (right) zero element* if $za = z(az = z)$ for all a belonging to S. An element z in S is called a *zero element* of S if it is both a left and right zero element.

The semigroup S has at most one identity element. If S has no identity element then, it is very easy to adjoin an extra element 1 to S to form a monoid by defining $1x = x1 = x$ for all x in S and $1.1 = 1$. Then $S \cup \{1\}$ becomes a semigroup with an identity element 1. We shall use the notation S^1 with the following meaning.

$$S^1 = \begin{cases} S, & \textit{if } S \textit{ has an identity element} \\ S \cup \{1\}, & \textit{otherwise} \end{cases}$$

We refer to S^1as the monoid obtained from S by adjoining an identity element if necessary. If a semigroup S with at least two elements contains a *zero element 0* i.e., for all x in S, $0x = x0 = 0$, then S is called a *semigroup with zero.*

Again, if S has no zero, it is easy to adjoine an extra element 0 to S to form a semigroup by defining $0t = t0 = 0$, for all t in S and $0 \cdot 0 = 0$. This makes the set $S \cup \{0\}$ a semigroup with zero element 0 .We shall use notation S^0 with the following meaning.

$$S^0 = \begin{cases} S, & \textit{if } S \textit{ has a zero element} \\ S \cup \{0\}, & \textit{otherwise} \end{cases}$$

and refer to S^0 as the semigroup obtained from S by adjoining a zero if necessary. A semigroup with a zero element 0 will be called *zero or null semigroup* if $ab = 0$ for all a, b in S. An element e of a semigroup S is called *idempotent* if $ee = e$. If every element of a semigroup S is idempotent we shall say that S itself is idempotent or that S is a *band.*

1.2 Examples of Semigroups

1.2.1 Example

N, the set of natural numbers is a semigroup with respect to the usual operation of addition and also with respect to the usual operation of multiplication.

1.2.2 Example

Let N be the set of natural numbers and $M_{n\times n}$ be the set of all $n \times n$ matrices with entries from N. Then $M_{n\times n}$ is a semigroup with respect to the usual addition of matrices. Also $M_{n\times n}$ is a semigroup with respect to the usual multiplication of matrices.

1.2.3 Example

Let $S = [0,1]$, then S is a semigroup with respect to the operations

$$a * b = \min\{a,b\} \ for\ all\ a,b \in S$$

$$a \circ b = \max\{a,b\} \ for\ all\ a,b \in S$$

1.2.4 Example

Let X be a $non-empty$ set. Define

$$a * b = a \ for\ all\ a,b \in X$$

$$a \circ b = b \ \ for\ all\ a,b \in X$$

Then $(X,*)$ and $(X,\circ)$ are semigroups.

Every element in $(X,*)$ is a left zero and right identity . This semigroup contains no $two-sided$ identity and also no $two-sided$ zero.

Similarly, in $(X,\circ)$ every element is a right zero and left identity. This semigroup contains no $two-sided$ identity and also no $two-sided$ zero

1.2.5 Example

Let X be a $non-empty$ set. τ_X be the set of all transformations of X (set of all functions from X to X), Then τ_X is a semigroup with respect to composition of functions. This semigroup is called "full transformation semigroup on X".

1.2.6 Example

If A, B are subsets of a semigroup S then $AB = \{ab : a \in A, b \in B\}$, it is easy to verify that this product is associative. Thus $P(S)$ (the collection of all *non − empty* subsets of S) is semigroup.

1.2.7 Example

Let X be a *non − empty* set and $P(X)$ be the collection of all subsets of X.

Then $(P(X), \cap)$ is a commutative monoid with identity X. The empty subset of X, ϕ is the zero element of $(P(X), \cap)$.

1.2.8 Example

Let X be a *non − empty* set and $P(X)$ be the collection of all subsets of X.

Then $(P(X), \cup)$ is a commutative monoid with identity ϕ, X is the zero element of $(P(X), \cup)$.

1.3 Subsemigroup

1.3.1 Definition

A non empty subset T of a groupoid S is called a *subgroupoid* of S if $ab \in T$ for all $a, b \in T$. If S is a Semigroup, then any subgroupoid of S is also a semigroup and we shall use the term *subsemigroup* rather than subgroupoid.

1.3.2 Definition

The intersection of any set of subsemigroups of S is either empty or a subsemigroup of S. If A is any *non − empty* subset of S, the intersection of all subsemigroups of S containing A (S itself being one self) is a subsemigroup of S containing A and is contained in very other subsemigroup of S containing A. We denote this subsemigroup by $< A >$ and is called the subsemigroup of *S generated by A*.

The subsemigroup $< A >$ can also be described as the set of all elements of S expressible as finite product of elements of A. If $< A > = S$ then A is called the set of generators for S or a generating set of S. Particular interest attaches to the case where A is finite, say

$A = \{a_1, a_2, ..., a_n\}$ we shall write $< A >$ as $< a_1, a_2, ..., a_n >$. Especially interesting is the case where $A = \{a\}$, where $< a > = \{a, a^2, a^3, ...\}$. we refer to $< a >$ as the monogenic or cyclic subsemigroup of S generated by the element a. The order of a is defined as the order of the semigroup $< a >$. S is called cyclic if $S = < a >$ for some $a \in S$.

If A, B are subsets of a semigroup S then $AB = \{ab : a \in A, b \in B\} = \bigcup\{Ab : b \in B\} = \bigcup\{aB : a \in A\}$. If a is an element of a semigroup S without identity element, then aS or Sa will not in general, contain a. In this situation, we use the following notations

$$\begin{aligned} S^1a &= Sa \cup \{a\}, \\ aS^1 &= aS \cup \{a\}, \\ S^1aS^1 &= SaS \cup Sa \cup aS \cup \{a\}. \end{aligned}$$

Note that S^1a, $aS^1$and S^1aS^1 are all subsets of S (which do not contain 1).

1.4 Semigroup homomorphism

1.4.1 Definition

Let $(S, .)$ and $(T, *)$ be two semigroups. A function $f : S \longrightarrow T$ is called a *semigroup homomorphism* if $f(a.b) = f(a) * f(b)$, for all a, b in S. Semigroup monomorphism, epimorphism, isomorphism and automorphism are defined as usual.

1.5 Ideals in Semigroups

1.5.1 Definition

By a *left(right) ideal* of a semigroup S we mean a $non-empty$ subset A of S such that SA is contained in A (AS is contained in A). A is called $two-sided$ *ideal* or an *ideal* of S if it is both left and right ideal of S. Evidently every ideal (whether right, left or $two-sided$) is a subsemigroup but the converse is not true. Among the ideals of S are S itself and (if S has a zero element) $\{0\}$. An ideal I such that $\{0\} \subset I \subset S$ (Strictly) is called proper.

1.5.2 Proposition

Let S be a semigroup. Then

(1) Intersection of any number of right (left) ideals of S is either empty or a right (left) ideal of S.

(2) Union of any number of right (left) ideals of S is a right (left) ideal of S.

(3) If I, J are ideals of S, then $IJ = \{ab : a \in I, b \in J\}$ is an ideal of S. Also $IJ \subseteq I \cap J$.

1.5.3 Definition

If a is an element of a semigroup S, the smallest right (left) ideal of S containing a is $aS \cup \{a\}$ $(Sa \cup \{a\}), it$ is convenient to denote it by $aS^1(S^1a)$. We shall call it the *principal right ideal (principal left ideal) generated by a.*

1.5.4 Definition

If a is an element of a semigroup S, the smallest $two - sided$ ideal of S containing a is $SaS \cup Sa \cup aS \cup \{a\}, it$ is conveniently denoted by S^1aS^1. We shall call it the *principal ideal generated by a.*

1.5.5 Definition

An ideal A of a semigroup S is called *idempotent* if $A^2 = A$.

1.5.6 Definition

An ideal M of a semigroup S is called a *maximal ideal* of S if J is any ideal of S such that $M \subseteq J \subseteq S$ then either $J = M$ or $J = S$

1.5.7 Definition

A right (left) ideal M of a semigroup S is called a *maximal right (left) ideal* of S if for any right (left) ideal J of S such that $M \subseteq J \subseteq S$ then either $J = M$ or $J = S$

1.6 Prime ideals

1.6.1 Definition

An ideal I of a semigroup S is called a *prime ideal* of S if $AB \subseteq I$ implies that either$A \subseteq I$ or $B \subseteq I$ for ideals A, B of S.

1.6.2 Proposition

The following conditions on an ideal I of a semigroup S are equivalent:

(1) I is prime;

(2) $aS^1b \subseteq I$ if and only if $a \in I$ or $b \in I$;

(3) If a, b are elements of a semigroup S satisfying $< a >< b > \subseteq I$ then either $a \in I$ or $b \in I$.(Where $< a >$ is the ideal generated by a and $< a >= S^1aS^1$)

Proof. $(1) \Longrightarrow (2)$: Let $a, b \in S$ and consider the set $I' = aS^1b$.

If $a \in I$ or $b \in I$ then $I' \subseteq I$ since I is an ideal.

Conversely, let S^1aS^1 and S^1bS^1 be the ideals of S generated by a and b respectively.

if $aS^1b \subseteq I$ then

$$S^1aS^1bS^1 \subseteq I$$

Hence,

$$(S^1aS^1)(S^1bS^1) \subseteq I$$

.

By (1), either $S^1aS^1 \subseteq I$ or $S^1bS^1 \subseteq I$,

this implies either $a \in I$ or $b \in I$.

$(2) \Longrightarrow (3)$: Let a, b be elements of S such that $< a >< b > \subseteq I$.

As,

$$aS^1b \subseteq < a >< b >$$

so,

$$aS^1b \subseteq I.$$

Thus by (2) either $a \in I$ or $b \in I$.

$(3) \Longrightarrow (1)$: Let A and B be ideals of S such that $AB \subseteq I$, suppose that A $\quad$ I and $a \in A$ such that $a \notin I$.

Let b be any arbitrary element of B then

$$<a><b> \subseteq AB \subseteq I.$$

By (3)

either $a \in I$ or $b \in I$.

As $a \notin I$ so $b \in I$, that is $B \subseteq I$ ■

1.6.3 Corollary

If a and b are elements of a semigroup S then the following conditions on a prime ideal I of S are equivalent:

(1) If $ab \in I$ then $a \in I$ or $b \in I$.

(2) if $ab \in I$ then $ba \in I$.

Proof. $(1) \Longrightarrow (2)$: Suppose that $ab \in I$ then $b(ab)a \in I$, since I is an ideal of S.

Thus

$$(ba)(ba) \in I.$$

By (1) $ba \in I$.

$(2) \Longrightarrow (1)$: If $ab \in I$.

This implies that $(ab)S^1 \subseteq I$

$\Longrightarrow a(bS^1) \subseteq I$

By (2) $bS^1a \subseteq I$.

Since I is prime, so $b \in I$ or $a \in I$. ■

1.6.4 Proposition

An ideal I of a commutative semigroup S is prime if and only if $ab \in I$ implies that $a \in I$ or $b \in I$ for all a and b in S.

Proof. Note that by commutativity, $ab \in I$ if and only if $aS^1b \subseteq I$.

The result follows from the above Proposition. ■

1.6.5 Definition

A *non − empty* subset M of a semigroup S is called an *m − system* if and only if $a, b \in M$ implies that there exists an element $x \in S^1$ such that $axb \in M$.

1.6.6 Corollary

An ideal I of a semigroup S is prime if and only if $S\backslash I$ is an *m − system*.

Proof. Suppose that I is a prime ideal of S. Let $a, b \in S\backslash I$.

Suppose that there does not exist $x \in S^1$ such that $axb \in S\backslash I$, this implies that $aS^1b \subseteq I$.

By Proposition 1.6.2 , either $a \in I$ or $b \in I$, which is a contradiction. Hence there exists an $x \in S^1$ such that $axb \in S\backslash I$.

Conversely, assume that $S\backslash I$ is an *m − system*. Let $a, b \in S$ such that $aS^1b \subseteq I$. If $a \notin I$ and $b \notin I$ then $a, b \in S\backslash I$ and as $S\backslash I$ is an *m − system* so there exists an $x \in S^1$ such that $axb \in S\backslash I$ that is $aS^1b * I$, which is a contradiction.

Hence either $a \in I$ or $b \in I$. ■

1.6.7 Proposition

Let S be a monoid, then every maximal ideal of S is a prime ideal.

Proof. Let P be a maximal ideal of S. Let A, B be ideals of S such that $AB \subseteq P$.

Suppose that .$A * P$

then,

$$A \cup P = S.$$

As $1 \in S$, so $1 \in A \cup P$. Since $1 \notin P$, so $1 \in A$.

Thus,

$$A = S.$$

Now,

$$B = SB = AB \subseteq P.$$

■

1.6.8 Proposition

If I is an ideal of a semigroup S and H is an ideal of S minimal among those ideals of S properly containing I then $K = \{x \in S : xH \subseteq I\}$ is a prime ideal of S.

Proof. First we show that K is an ideal of S. Let $x \in K$ and $s \in S$ then $xH \subseteq I$.

Now,

$$(sx)H \subseteq sI \subseteq I,$$

implies that, $sx \in K$.

Also,

$$(xs)H = x(sH) \subseteq xH \subseteq I,$$

implies that $as \in K$, so $K$is an ideal of S.

Let A, B be ideals of S such that $AB \subseteq K$. Suppose that $B \ * K$. We have $ABH \subseteq I$ and $BH \ * I$.

Therefore,

$$I \subset I \cup BH \subseteq H$$

and by the minimality of H, we have

$$I \cup BH = H.$$

Therefore,

$$AI \cup ABH = AH \subseteq H$$

and so $A \subseteq K$. ■

1.7 Semiprime Ideals

1.7.1 Definition

An ideal I of a semigroup S is called a *semiprime* ideal if $A^2 \subseteq I$ implies that $A \subseteq I$ for all ideals A of S.

1.7.2 Proposition

The following conditions on an ideal I of a semigroup S are equivalent:

(1) I is semiprime;

(2) $aS^1a \subseteq I$ if and only if $a \in I$.

Proof. (1) $\Longrightarrow$ (2) : Let $a \in S$ and set $I' = aS^1a$. If $a \in I$ then $I' \subseteq I$, since I is an ideal. Conversely, let S^1aS^1 be the ideal of S generated by a. If $aS^1a \subseteq I$

then,

$$(S^1aS^1)(S^1aS^1) = S^1aS^1aS^1 \subseteq S^1IS^1 = I.$$

By (1), $S^1aS^1 \subseteq I \Rightarrow a \in I$.

(2) $\Longrightarrow$ (1) : Let A be an ideal of S such that $A^2 \subseteq I$. Let $a \in A$ then $a \in S^1aS^1$ and $S^1aS^1 \subseteq A$.

Also,

$$aS^1a \subseteq (S^1aS^1)(S^1aS^1) \subseteq A^2 \subseteq I.$$

By (2) $a \in I$. Hence $A \subseteq I$. ∎

1.7.3 Definition

A *non − empty* subset A of S is called a *p − system* if and only if $a \in A$ implies that there exists an element $x \in S^1$ such that $axa \in A$.

1.7.4 Corollary

An ideal I of a semigroup S is semiprime if and only if $S\backslash I$ is a *p − system*.

Proof. Suppose that I is a semiprime ideal of S and $a \in S\backslash I$. If there does not exist $x \in S^1$ such that $axa \in S\backslash I$, then $aS^1a \subseteq I$. By above Proposition $a \in I$ which is a contradiction.

Hence there exists $x \in S^1$ such that $axa \in S\backslash I$.

Conversely, assume that $S\backslash I$, is a *p − system*. Let $a \in S$ such that $aS^1a \subseteq I$. If $a \notin I$ then $a \in S\backslash I$, so there exist $x \in S^1$ such that $axa \in S\backslash I$, this implies that $aS^1a \ * I$, which is a contradiction. Hence $a \in I$. ∎

1.7.5 Remark

(1) Clearly every prime ideal of a semigroup S is a semiprime ideal of S, but the converse is not necessarily true;

(2) Every $m-system$ in a semigroup S is a $p-system$ but the converse may not be true;

(3) Intersection of prime ideals of a semigroup S is a semiprime ideal.

1.8 Prime Right (Left) Ideals

1.8.1 Definition

A right (left) ideal I of a semigroup S is called a *prime right (prime left) ideal* of S if $AB \subseteq I$ implies that either $A \subseteq I$ or $B \subseteq I$ for all right (left) ideals A and B of S.

1.8.2 Proposition

The following conditions on a right ideal I of a semigroup S are equivalent:

(1) I is a prime right ideal;

(2) $aS^1b \subseteq I \Rightarrow a \in I$ or $b \in I$;

(3) If a, b are elements of S satisfying $(a)_r(b)_r \subseteq I$ then either $a \in I$ or $b \in I$.

Proof. $(1) \Longrightarrow (2)$: Let $a, b \in S$ such that $aS^1b \subseteq I$, then $aS^1bS^1 \subseteq IS^1 = I$.

By (1), either $aS^1 \subseteq I$ or $bS^1 \subseteq I$.

This implies that either $a \in I$ or $b \in I$.

$(2) \Longrightarrow (3)$: Let a, b be elements of S such that $(a)_r(b)_r \subseteq I$.

As

$$aS^1b \subseteq (a)_r(b)_r \subseteq I,$$

so by (2) either $a \in I$ or $b \in I$.

$(3) \Longrightarrow (1)$: Let A and B be right ideals of S such that $AB \subseteq I$. Suppose that $A * I$, then there exists $a \in A$ such that $a \notin I$.

Let b be any arbitrary element of B then,

$$(a)_r(b)_r \subseteq AB \subseteq I.$$

By (3), either $a \in I$ or $b \in I$. As $a \notin I$ so $b \in I$,

that is $B \subseteq I$. ■

1.8.3 Proposition

Let P be a prime right ideal of S. Then $s(P) = \{s \in S : S^1 s \subseteq P\}$ is the largest two sided ideal of S contained in P. Also $s(P)$ is a prime right ideal of S.

Proof. First we show that $s(P) = \{s \in S : S^1 s \subseteq P\}$ is the largest two sided ideal of S contained in P.

Let $s \in s(P)$ and $x \in S$,then

$$S^1(sx) = (S^1 s)x \subseteq Px \subseteq P$$

$\Longrightarrow sx \in s(P)$ and

$$S^1(xs) = (S^1 x)s \subseteq S^1 s \subseteq P$$

$\Longrightarrow xs \in s(P)$.

So $s(P)$ is a two-sided ideal of S.

Clearly $s(P) \subseteq P$.

Let J be a two-sided ideal of S such that $J \subseteq P$.

Let $x \in J$, then

$$Sx \subseteq J \subseteq P$$

$\Longrightarrow x \in s(P)$.

Hence $J \subseteq s(P)$.

Thus $s(P)$ is the largest two-sided ideal of S contained in P.

Let A, B be right ideals of S such that $AB \subseteq s(P) \subseteq P$. Now,

$$\begin{aligned} S^1(AB) &\subseteq S^1(s(P)) \subseteq s(P) \subseteq P \\ (S^1 A)(S^1 B) &= S^1(AS^1)B \subseteq S^1 AB \subseteq s(P) \subseteq P \end{aligned}$$

As P is a prime right ideal, so either $S^1 A \subseteq P$ or $S^1 B \subseteq P$.

Since $s(P)$ is the largest *two − sided* ideal of S contained in P, so either $S^1A \subseteq s(P)$ or $S^1B \subseteq s(P)$.

This implies that

$$A \subseteq S^1A \subseteq s(P) \text{ or } B \subseteq S^1B \subseteq s(P)$$

Hence $s(P)$ is a prime right ideal of S.

■

1.8.4 Proposition

If I is a prime ideal of S with zero, then $(I : x) = \{s \in S : xs \in I\}$ is also a prime right ideal of S for any $x \in S$.

Proof. Clearly $(I : x) \neq \Phi$ because $0 \in (I : x)$.

Also if $s \in (I : x)$ and $t \in S$ then $st \in (I : x)$. Hence $(I : x)$ is a right ideal of S.

Let A, B be any right ideals of the semigroup S such that $AB \subseteq (I : x)$,

then,

$$xAB \subseteq I \Rightarrow x(AS)B \subseteq I \Rightarrow (xA)(SB) \subseteq I.$$

$\Longrightarrow$ $(xA)(xB) \subseteq I$,

$\Longrightarrow$ $xA \subseteq I$ or $xB \subseteq I$. $\because$ xA and xB are right ideals of S,

and I is a prime right ideal of S.

$\Longrightarrow$ $A \subseteq (I : x)$ or $B \subseteq (I : x)$. ■

1.9 Regular semigroup

1.9.1 Definition

An element x of a semigroup S is said to be regular if there exists an element $x' \in S$ such that $xx'x = x$, S is called regular if every element of S is regular.

1.9.2 Definition

An element x of a semigroup S is said to be right (left) regular if there exist an element $a \in S$ such that $x = x^2a$ $(x = ax^2)$; S is called right (left) regular if every element of S is right (left) regular.

Let a be a regular element of a semigroup S and $a = axa$ for some $x \in S$, then

$$e = ax = axa.x = ax.ax = e^2.$$

and

$$f = xa = x.axa = xa.xa = f^2$$

.

that is , $e = ax$ and $f = xa$ are idempotent elements in S with the properties $ea = axa = a = af$

Conversely, assume that for an element a there exist elements e, f and b in S such that

$a = ea = af$

and $ab = e$, $ba = f$

then $a = af = aba$

thus a is a regular element. So we have the proposition.

1.9.3 Proposition

An element x of a semigroup S is regular if and only if there exist elements t, u and y in S such that $a = ta = au$ and $t = xy$, $u = yx$.

1.9.4 Remark

If x is a regular element of a semigroup S, then the above property implies that the principal left (principal right ideal) of S generated by x has the form Sx (xS).

1.9.5 Proposition

The following conditions for a semigroup S are equivalent:

(1) S is regular;

(2) For every right ideal R and left ideal L of S,

$$RL = R \cap L$$

Proof. (1) $\Longrightarrow$ (2) : Let R and L be right and left ideals of S respectively. Then $RL \subseteq R \cap L$. For the reverse inclusion, let $x \in R \cap L$. Since S is regular, so there exist $y \in S$ such that,

$$x = xyx = (xy)x \in RL$$

$\because R$ is a right ideal.

Therefore we have $R \cap L \subseteq RL$.

Hence $RL = R \cap L$.

(2) $\Longrightarrow$ (1) : Let $a \in S$. Let R be a right ideal generated by a and L be the left ideal generated by a.

Then $R = aS^1$ and $L = S^1a$.

By hypothesis,

$$aS^1 \cap S^1a = (aS^1)(S^1a).$$

As $a \in aS^1 \cap S^1a$,

$$a \in (aS^1)(S^1a) = aS^1a$$

i.e. $a = axa$ for some $x \in S$.

Hence a is a regular element. ■

1.10 Simple Semigroups

1.10.1 Definition

A semigroup without zero is called simple if it has no proper ideal.

1.10.2 Definition

If S has no right (left) ideals other than itself , then S is said to be a right simple (left simple)

1.10.3 Definition

Let S be a semigroup with zero. Then S is a *zero − simple* or *0 − simple* if the following conditions hold:

(1) $S^2 \neq \{0\}$

(2) S has no ideal except $\{0\}$ and S itself.

1.10.4 Proposition

A semigroup S is $0-simple$ if and only if $SaS = S$ for every $a \neq 0$ in S, that is if and only if for every a, b in $S \backslash \{0\}$ there exist x, y in S such that $xay = b$.

Proof. Suppose first that S is $0-simple$. Then S^2, being an ideal of S, and being by definition distinct from $\{0\}$, must coincide with S, and it follows that $S^3 = S^2.S = S.S = S$ also. Let a be a $non-zero$ element of S. Then SaS is an ideal of S and so either $SaS = S$ or $SaS = \{0\}$. If $SaS = \{0\}$ the set $I = \{x \in S : SxS = \{0\}\}$ contains the $non-zero$ element a. Since I is easily seen to be an ideal of S it follows that $I = S$, and so $SxS = \{0\}$ for every x in S. But this implies that $S^3 = \{0\}$, in contradiction to the already noted fact that $S^3 = S$. Hence $SaS = S$ as required.

Conversely, suppose that $SaS = S$ for all $a \neq 0$ in S. Then certainly $S^2 \neq \{0\}$. If A is an ideal of S containing a $non-zero$ element a then

$$S = SaS \subseteq SAS \subseteq A,$$

and so $A = S$. Thus S is $0-simple$ ∎

1.10.5 Corollary

A Semigroup S is simple if and only if $SaS = S$ for all a in S, that is, if and only if for every a, b in S there exist x, y in S such that $xay = b$.

1.10.6 Proposition

Let S be a semigroup. if S is simple and $a \in aS$ for all $a \in S$, then every right ideal of S is prime.

Proof. Let S be a simple semigroup and $a \in aS$ for all $a \in S$.

Let I be a right ideal of S and suppose that A and B are right ideals of S such that $AB \subseteq I$. As SB is an ideal of S and so $SB = S$.

Hence,

$$A = AS = ASB = AB \subseteq I.$$

Thus I is a prime right ideal of S. ∎

1.11 Semi-simple semigroups

1.11.1 Definition

A semigroup S is called semisimple if all its ideals are idempotent that is $I^2 = I$ for every *two − sided* ideal I of S.

1.11.2 Definition

A Semigroup S is called right (left) semisimple if all of its right (left) ideals are idempotent.

1.11.3 Proposition

The following assertions for a semigroup S are equivalent:

(1) S is semisimple;

(2) for each pair of ideals I, J of S, $I \cap J = IJ$;

(3) for each right ideal R and *two − sided* ideal I, $R \cap I \subseteq IR$;

(4) for each left ideal L and *two − sided* ideal I, $L \cap I \subseteq LI$.

Proof. $(1) \Longrightarrow (2)$: Let I, J be any ideals of S, then $IJ \subseteq I \cap J$. As $I \cap J$ is an ideal of S, so

$$I \cap J = (I \cap J)^2 \subseteq IJ.$$

Thus,

$$IJ = I \cap J.$$

$(2) \Longrightarrow (3)$: Let R be a right ideal and I be an ideal of S, then S^1R is a *two − sided* ideal of S. Now

$$R \cap I \subseteq S^1R \cap I = I(S^1R) = (IS^1)R \subseteq IR.$$

Thus,

$$R \cap I \subseteq IR.$$

$(3) \Longrightarrow (4)$: Let L be a left ideal and I be an ideal of S, then LS^1 is a $two - sided$ ideal of S.

Hence,

$$L \cap I \subseteq LS^1 \cap I \subseteq (LS^1)I = L(S^1I) = LI.$$

$(4) \Longrightarrow (1)$: Let I be any ideal of S then $I \cap I \subseteq I.I$ i.e. $I \subseteq I^2$. But $I^2 \subseteq I$ always. Hence $I = I^2$. ∎

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Chapter 2

MULTIPLICATION SEMIGROUPS

In this chapter we review the paper [8].

The semigroups considered in this chapter are commutative which are not groups.

2.1 Important Concepts in Semigroups

Throughout the chapter semigroups are semigroups with identity having a unique maximal ideal M. For any ideal A of S we denote $\bigcap_{i=1}^{\infty} A^i$ by A^{ω}. Z denotes the set of all *non − cancellative* elements which is a prime ideal.

Since if $a, b \in S$, such that $ab \in Z$, then since ab is non-cancellative.Suppose a, $b \notin Z$ and a and b are cancellative elements of S. We show that ab is also cancellative.Let $(ab)x = (ab)y$ $\Longrightarrow a(bx) = a(by) \Longrightarrow bx = by \Longrightarrow x = y$. Thus $ab \notin Z$, a contradiction. Hence $a \in Z$ or $b \in Z$, that is Z is a prime ideal.

2.1.1 Definition

A commutative semigroup S is called a *multiplication semigroup* if whenever A and B are ideals of S with A contained in B, there is an ideal C of S such that $A = BC$.

2.1.2 Definition

A prime ideal Q containing an ideal A is called a *minimal prime divisor* of A, whenever P is a prime ideal with $A \subseteq P \subseteq Q$ then $P = Q$.

If P is a prime ideal containing an ideal A, using Zorn's lemma it is very easy to verify that P contains a minimal prime divisor of A.

2.1.3 Definition

The *Radical* of an ideal A is the intersection of all its minimal prime divisors, denoted by $\sqrt{A}$.

Equivalently, $\sqrt{A} = \{x \in S \mid x^n \in A$ for some natural number $n\}$.

2.1.4 Definition

An ideal Q of a semigroup S is called a *primary ideal* if A and B are ideals of S such that $AB \subseteq Q$ then either $A \subseteq Q$ or $B \subseteq \sqrt{Q}$.

2.1.5 Definition

Equivalently, an ideal Q is a *primary ideal* if $a, b \in S$ such that $ab \in Q$ then either $a \in Q$ or $b \in \sqrt{Q}$.

2.1.6 Definition

An ideal Q of a semigroup S is called $P-primary$ for any prime ideal P, if Q is primary with $\sqrt{Q} = P$

We note that the radical of any primary ideal is a prime ideal

2.1.7 Definition

For any two ideals A and B of S, we define $(A : B) = \{x \in S : xB \subseteq A\}$.

For any prime ideal P of S we define the semigroup of fractions $S_p = \{a/b : a \in S, b \in S \backslash P$ *is cancellative element*$\}$. By defining map $\eta : S \to S_p$ by $\eta(s) = s/1$ $(s \in S)$, S can be identified with a subsemigroup of S_p. For any ideal A of S, the ideal generated by $\eta(A)$ in S_p is denoted by AS_p. $AS_p \cap S$ is an ideal of S; moreover, if $A \subseteq P$ then $AS_p \cap S$ is $P-primary$ in S.

2.1.8 Definition

We say *dimension* of S is n, and write $\dim S = n$, if there is a chain of prime ideals $P_0 \subset P_1 \subset \ldots \subset P_n = M$ of length n, and no chain of prime ideals of length greater than n exists, where

M is the unique maximal ideal of S.

2.1.9 Definition

If P is a minimal prime devisor of an ideal A, then the intersection of all $P-primary$ ideals containing A is called *Isolated* $P-primary$ *component* of A.

2.1.10 Definition

The intersection of all Isolated $P-primary$ components of A, as P runs through all minimal prime divisors of A, is called $KerA$.

2.1.11 Definition

If every ideal of a semigroup S is principal, then S is called a *principal ideal semigroup* .

2.1.12 Definition

If every ideal of a semigroup S is finitely generated, then S is called a *Noetherian semigroup.*

2.1.13 Proposition

For a multiplication semigroup S with unique maximal ideal M, the following are true:

(1) If P is a prime ideal and A is any ideal such that $P \subset A$, then $P = PA$; and $P = A^{\omega}$ or $P = PA^{\omega}$.

(2) Every ideal with prime radical is a primary ideal; in particular if P is a prime ideal, then P^n is a $P-primary$ ideal for any positive integer n.

(3) If P is a prime ideal with $P^n \neq P^{n+1}$for any positive integer n, then P^{ω} is a prime ideal.

(4) Every prime ideal is a power of its radical.

(5) If P is a proper prime ideal and A is any ideal such that $A \subseteq P^n$, $A \ * \ P^{n+1}$ for some positive integer n, then $P^n = (A : yS)$ for some $y \in S \backslash P$.

(6) If A is an idempotent ideal, then A is a multiplication semigroup not necessarily containing identity . In particular, for any $x \in A$ we have $x = xy$ for some $y \in A$.

(7) Every homomorphic image of S is a multiplication semigroup.

(8) If $dim\ S \leq 1$, then $A = \ker A$ for every ideal A.

(9) If P is a $non-idempotent$ prime ideal, then there is at most one prime ideal $Q \subset P$ with the property that there are no prime ideals between P and Q.

(10) If prime ideals are linearly ordered, then every $non-idempotent$ prime ideal P is principal.

Proof. (1) By the multiplication property $P = AB$ for some ideal B of S.

Since P is a prime ideal with $A \; * P$, so $P = AB \Rightarrow B \subseteq P$

Now if $B \subset P$ then,

$$AB \subseteq B \subset P$$

Which is a contradiction, as $P = AB$,

so $B = P$ and thus $P = AP$.

We have then,

$$P = AF = A^2P \; etc.$$

Now since A^i is an ideal of S for each i,

so $A^iP \subseteq A^i$ for each i and also $P = A^iP$, for each i

Thus $P \subseteq A^i$ for each i.

so that,

$$P \subseteq \bigcap_{i=1}^{\infty} A^i = A^\omega.$$

If $P \neq A^\omega$, then $A^\omega \; * P$ then it follows from above that,

$$P = PA^\omega.$$

(2) Let Q be any ideal with $\sqrt{Q} = P$, where P is a prime ideal.

If $P = S$, trivially Q is primary.

We assume $P \neq S$. Now suppose Q is not primary. Then for some $p \in P \backslash Q$ and $r \in S \backslash P$ we can have $pr \in Q$.

Let $A = Q \cup (pS)P$. We note that $p \notin A$. For otherwise, $p = pp'$ for some $p' \in P$. Now $p'^n \in Q$ for some n, so that we have $p = pp'^n \in Q$, which is a contradiction.

Now,

$$A \cup pS \subseteq P, \text{ because } Q \subseteq \sqrt{Q}$$

so by multiplication property $A \cup pS = PL$ for some ideal L of S;

and $P \subset P \cup rS$ implies $P = P(P \cup rS)$ by (1).

Therefore, it follows that

$$A \cup pS = P(P \cup rS)L = (A \cup pS)(P \cup rS) \subseteq A,$$

i.e., $p \in A$,

which is a contradiction. Hence Q is a primary ideal.

Now since, for any prime ideal P,

$$\sqrt{P^n} = \{a \in S : \exists\ m \in N \ such\ that\ a^m \in P^n\} = P$$

Thus from above P^n is $P - primary$ for any n.

(3) Let $x, y \notin P^\omega$. If $x, y \notin P$, since P is prime

therefore $xy \notin P$

i.e. $xy \notin \bigcap_{i=1}^{\infty} P^i = P^\omega$.

So it suffices to consider the case when $x \notin P$ and $y \in P$.

Now since $P^n \neq P^{n+1}$

so

$$P \supset P^2 \supset ... \supset P^n \supset P^{n+1} \supset ...$$

Therefore $y \in P^r \backslash P^{r+1}$ for some positive integer r.

Now $xy \notin P^{r+1}$, since P^{r+1} is $P - primary$ by (2).

Thus $xy \notin \bigcap_{i=1}^{\infty} P^i = P^\omega$. .

Finally, if $x, y \in P$ then $x \in P^r \backslash P^{r+1}$ and $y \in P^s \backslash P^{s+1}$ for some positive integers r and s.

Now since $xS \subseteq P^r$ and $yS \subseteq P^s$ so by multiplication property

$$xS = P^r A \text{ and } yS = P^s B \text{ with ideals } A, B \ * P.$$

If,

$$xy \in P^{r+s+1},$$

then,

$$xyS \subseteq P^{r+s+1}.$$

therefore,

$$xyS = P^{r+s}AB \Rightarrow P^{r+s}AB \subseteq P^{r+s+1}.$$

But P^{r+s+1} is $P-primary$ by (2), so that $P^{r+s} \subseteq P^{r+s+1}$ since $AB \ * P$.

Thus

$$P^{r+s} = P^{r+s+1},$$

Which is not true.

Hence $xy \notin P^{r+s+1}$, i.e., $xy \notin \bigcap_{i=1}^{\infty} P^i = P^{\omega}$. Thus P^{ω} is a prime ideal..

(4) Let Q be a primary ideal with $\sqrt{Q} = P$. If $P = S$, clearly $Q = S$.

We may assume $P \neq S$.

Suppose $Q \subseteq P^{\omega}$.

If $P^n \neq P^{n+1}$ for any n, then by (3) P^{ω} is a prime ideal.

This implies,

$$P = \sqrt{Q} \subseteq \sqrt{P^{\omega}} = P^{\omega},$$

which is a contradiction.

For if $P \subseteq P^{\omega}$, since $P^{\omega} \subseteq P^2$ therefore $P \subseteq P^2$ but $P^2 \subseteq P$ so $P = P^2$ which is contrary to our supposition that $P^n \neq P^{n+1}$ for any n.

Hence we have the following possibilities: either $Q \subseteq P^{\omega}$ with $P^n = P^{n+1}$ for some positive integer n, or $Q \subseteq P^n$, $Q \ * P^{n+1}$ for some positive integer n.

In the first case, for any $x \in P^n$,

since $xS \subseteq P^n$ so by the multiplication property we have $xS = P^nA$ for some ideal A.

Then,

$$xS = P^nA = P^{n+1}A = P(P^nA) = P(xS),$$

so that $x = tx$ for some $t \in P$. Since P is radical of Q so $t^k \in Q$ for some positive integer k, now since Q is an ideal of S so we have $x = t^kx \in Q$ i.e. $P^n \subseteq Q$, but by our hypothesis $Q \subseteq P^n$ so $Q = P^n$.

In the second case since $Q \subseteq P^n$ so by the multiplication property we have

$Q = P^nC$ for some ideal C of S where $C \ * P$.

Since Q is $P-primary$, clearly $P^n \subseteq Q$; because $C \ * P = \sqrt{Q}$, but by our hypothesis

$Q \subseteq P^n$so $Q = P^n$.

(5) We have $A = P^n B$ with $B * P$.

Let $y \in B \backslash P$.

Then,

$$P^n yS \subseteq A,$$

so that,

$$P^n \subseteq (A : yS)$$

If $x \in (A : yS)$, then

$$xy \in A \subseteq P^n.$$

But P^n is $P - primary$ by (2), so that $x \in P^n$ since $y \notin P$..

Thus

$$(A : yS) \subseteq P^n.$$

(6) If B is any ideal of A, then since A is an ideal of S so $BS \subseteq A$ by the multiplication property,

$$BS = AC$$

for some ideal C of S.

Now we show that B is an ideal of S.

To show this we have to show that $BS \subseteq B$

Now since $BS = AC$ and A is an idempotent ideal of S, so

$$BS = AC = AAC = ABS = ASB \subseteq AB \subseteq B$$

it is proved that B is an ideal of S.

Hence $B = AD$ for some ideal D of $S, \sin ce$ A is idempotent, so

$$B = AD = AAD = AB.$$

Let B_1 and B_2 be any two ideals of A with $B_1 \subseteq B_2$.

Then B_1 and B_2 are also ideals of S, so by the multiplication property, $B_1 = B_2 L$ for some ideal L of S.

Now since $B_2 = AB_2 = B_2A$ so $B_1 = B_2AL$, where AL is clearly an ideal of A.

Hence A is a multiplication semigroup.

If $x \in A$, then since $1 \in S$ so,

$$xS = (xS)A = x(SA) = xA$$

so that $x = xy$ for some $y \in A$.

(7) Let $\phi : S \longrightarrow T$ be a semigroup homomorphism, then $\phi(S)$ is a subsemigroup of T.

Let A and B be ideals of $\phi(S)$ such that $A \subseteq B$ then there exist ideals C and D of S such that $\phi(C) = A$ and $\phi(D) = B$

and $\phi(C) \subseteq \phi(D)$ which implies that $C \subseteq D$.

Now since S is a multiplication semigroup so by the multiplication property $C = ED$ for some ideal E of S.

This implies that,

$$\phi(C) = \phi(ED) = \phi(E)\phi(D)$$

Now since $\phi(E)$ is an ideal of $\phi(S)$, therefore $\phi(S)$ is a multiplication semigroup.

(8) Let A be any ideal .

Suppose $A \neq Ker\ A$. Let $a \in Ker\ A \backslash A$. Set $B = (A : aS)$.

Let P be a minimal prime divisor of B. Clearly

$$A \subseteq B \subseteq P,$$

so that P contains a minimal prime divisor Q of A.

Suppose $Q = P$. Now $AS_Q \cap S$ is an ideal of S with radical Q and thus is a $Q-primary$ ideal in view of (2), so that $a \in AS_Q \cap S$.

Now we can write $a/1 = b/x$ where $b \in A, x \in S\backslash Q$.

Then $ax = b \in A$, from which we obtain

$$x \in (A : aS) = B \subseteq Q,$$

which is a contradiction.

Therefore, $Q \subset P$. Now if $\dim S < 1$, we must have $Q = P$ and thus $A = Ker\ A$.

If $\dim S = 1$, then $P = M$.

By (2) and (4), $B = M^k$ for some k. Since $a \in Q$, we have $aS = QC$ for some ideal C; moreover $Q = QM^k$ by (1).

Thus

$$aS = M^k(aS) = B(aS) \subseteq A,$$

i.e., $a \in A$, which is a contradiction. Hence $A = Ker\ A$.

(9) Suppose Q and Q' are prime ideals such that $Q \subset P$ and $Q' \subset P$.

Now assume that there are no prime ideals between Q and P, and Q' and P.

Clearly $Q \cup Q' = P$; because

$$Q \subset Q \cup Q' \subset P \textit{ and } Q' \subset Q \cup Q' \subset P$$

Now since Q and Q' are prime so $Q \cup Q'$ is also a prime ideal,

but since there are no prime ideals between Q and P, and Q' and P, and also $Q \neq Q'$ so,

$$Q \cup Q' = P$$

moreover $Q = PQ$ and $Q' = PQ'$ by (1).

Now it follows that,

$$P = Q \cup Q' = PQ \cup PQ' = P(Q \cup Q') = PP = P^2,$$

i.e., $P = P^2$,which leads to a contradiction and hence $Q = Q'$.

(10) Let $x \in P \backslash P^{2}$.

Since prime ideals are linearly ordered, P is the only minimal prime divisor of xS,

that is P is radical of xS so that by (2) xS is $P - primary$ and by (4), $xS = P^r$ for some positive integer r, and thus $xS = P$. ∎

2.1.14 Proposition

If S is a multiplication semigroup with unique maximal ideal M, then the following are true:

(1) For every ideal A of S either $A = M^n$ for some $non-$negative integer n or $A \subseteq M^\omega$; in particular, if $M^\omega = \phi$ then the set of ideals of S is dually $Well - ordered.$

(2) If x and y are cancellable and $non - cancellable$ elements respectively then $yS \subset xS$; consequently $Z \subset xS$.

(3) S/Z is a Dedekind semigroup (with zero adjoined if $Z \neq \phi$).

(4) If $M \neq Z$, then $M = xS$ for some cancellable element x; and $Z = M^\omega$.

(5) $Z = Z^2$ or $Z = yS$ for some $non - idempotent$ element y.

2.1.15 Corollary

Let S be a semigroup with unique maximal ideal M as $M^\omega = \phi$. Then S is a multiplication semigroup if and only if ideals of S are powers of M. Furthermore, if $M \neq Z$ then the multiplication semigroup S is a dedekind semigroup.

2.1.16 Theorem

If S is a regular semigroup or a principal ideal semigroup, then S is a multiplication semigroup.

Proof. Consider any pair of ideals A and B with $A \subseteq B$.

We know that,

$$AB \subseteq A \cap B$$

If S is a regular semigroup,

then for $x \in A \cap B$, there exists a $y \in S$

such that,

$$x = xyx \in AB$$

this implies that,

$$A \cap B = AB$$

Now since $A \subseteq B$,

so $A \cap B = A$ and $A = AB$.

Therefore S is a multiplication semigroup

On the other hand, if S is a principal ideal semigroup, then A and B are of the form aS and bS respectively for some a and b where $a \in A$ and $b \in B$.

Now since $A \subseteq B$ so $a \in B = bS$, so $a = bs$ for some $s \in S$, so that $aS = (bS)(sS)$.

That is $A = B(sS)$. Now since sS is an ideal of S so S is a multiplication semigroup. ∎

The following example shows that the converse of the above theorem need not be true.

2.1.17 Example

Let $S = \{1, x, x^2, ...; e, f, ef\}$ with $ex = x = fx = efx$, $e = e^2$, $f = f^2$. It can be easily seen that S is a multiplication semigroup which is neither a regular semigroup nor a principal ideal semigroup.

2.1.18 Theorem

Let S be a multiplication semigroup in which every ideal is a product of primary ideals. Then S is regular if and only if every prime ideal is idempotent.

Proof. In a regular semigroup, every ideal is idempotent.

For, A be an ideal of the semigroup S.

For $a \in A$ there exists an $x \in S$, such that $a = a^2x = aax = a(ax) \in AA = A^2 \Longrightarrow A \subseteq A^2$, also $A^2 \subseteq A$ so $A = A^2$. Thus trivially every prime ideal is idempotent.

Conversely, if A is any ideal then $A = \prod\limits_{i=1}^{n} Q_i$, where Q_i 's are primary ideals.

By (4) of Proposition 2.1.13, $Q_i = P_i^{n_i}$ where $P_i = \sqrt{Q_i}$ is a prime ideal for every i.

Since every P_i is an idempotent ideal, we must have then,

$$A = \prod_{i=1}^{n} P_i^{n_i} = \prod_{i=1}^{n} P_i = A^2.$$

Hence S is a regular semigroup. ∎

2.1.19 Corollary

A noetherian multiplication semigroup S is regular if and only if every prime ideal is idempotent.

Proof. In view of above theorem , It suffices to show that every ideal is a product of primary ideals in a noetherian multiplication semigroup S.

As in commutative rings, in a noetherian semigroup it is easy to check that every ideal is a finite intersection of primary ideals .

Now, for any ideal A, $A = \bigcap_{i=1}^{n} Q_i$ where each Q_i is a primary ideal.

By (4) of the Proposition 2.1.13, $Q_i = P_i^{n_i}$ for some positive integer n_i, where $P_i = \sqrt{Q_i}$ is a prime ideal for every i.

We can write $A = \bigcap_{i=1}^{n} P_i^{n_i}$ with $P_i * P_j$ for $i \neq j$.

Since $\bigcap_{i=1}^{n} P_i^{n_i} \subseteq P_i^{n_i}$ for every i, by multiplication property we have

$$\bigcap_{i=1}^{n} P_i^{n_i} = .P_1^{n_1} A_1 \subseteq P_2^{n_2}$$

Where A_1 is some ideal of S. By (2) of Proposition 2.1.13, $P_2^{n_2}$ is $P_2 - primary$ so that $A_1 \subseteq P_2^{n_2}$ since $P_1^{n_1} * P_2$.

Then $A_1 = P_2^{n_2} A_2$ for some ideal A_2.

So, by induction

$$\bigcap_{i=1}^{n} P_i^{n_i} = \prod_{i=1}^{n} P_i^{n_i} B$$

where B is some ideal of S.

Hence,

$$A = \bigcap_{i=1}^{n} P_i^{n_i} = \prod_{i=1}^{n} P_i^{n_i}$$

■

2.1.20 Theorem

Let S be a multiplication semigroup satisfying any one of the following conditions:

(a) S contains no idempotent prime ideals.

(b) Every idempotent prime ideal of S contains at most a finite number of prime ideals.

(c) No idempotent prime ideal is equal to the union of all the prime ideals properly contained in it.

(d) Every idempotent prime ideal of S if finitely generated.

Then S is a principal ideal semigroup if and only if prime ideals of S are linearly ordered.

Proof. It can be easily verified that ideals are linearly ordered in a principal ideal semigroup.

Conversely, if prime ideals of S are linearly ordered, then every ideal has only one minimal prime divisor, so that the radical of every ideal is a prime ideal. Hence by (2) of Proposition 2.1.13, every ideal is primary and by (4) of Proposition 2.1.13, every ideal is a power of a prime ideal.

So, it suffices to prove that every prime ideal is principal in order to show that S is a principal ideal semigroup.

Now by (10) of Proposition 2.1.13, if prime ideals are linearly ordered, then every $non-idempotent$ prime ideal is principal.

Therefore we have to prove only that every idempotent prime ideal is principal.

Now if S satisfies (a), clearly S is a principal ideal semigroup.

Let S satisfy (b).

Let P be an idempotent prime ideal.

If P does not contain any prime ideals then for any $x \in P$, P is the only minimal prime divisor of xS.

That is P is the radical of xS, by (2) of Proposition 2.1.13, xS is $P-primary$. By (4) of the Proposition 2.1.13,

$$xS = P^n$$

for some positive integer n. Since P is idempotent so,

$$P = xS$$

Suppose P contains prime ideals. Let $Q_1, Q_2, ..., Q_n$ be the set of all prime ideals with each $Q_i \subset P$.

Then,

$$\bigcup_{i=1}^{n} Q_i \subset P.$$

Now for any $x \in P \backslash \bigcup_{i=1}^{n} Q_i$, then P is the only minimal prime divisor of xS.

That is P is the radical of xS, by (2) of Proposition 2.1.13 xS is $P-primary$. By (4) of the Proposition 2.1.13

$$xS = P^n$$

for some positive integer n. Since P is idempotent so,

$$P = xS.$$

Let S satisfy (c).

If an idempotent prime ideal P does not contain any prime ideals, then P is principal as in the preceding paragraph.

On the other hand, if P contains prime ideals and if $\{Q_\alpha\}_\alpha$ is the set of all prime ideals with each $Q_\alpha \subset P$

then,

$$\bigcup_\alpha Q_\alpha \subset P$$

and hence $P = xS$ for any $x \in P \backslash \bigcup_\alpha Q_\alpha$ as in the preceding paragraph.

Finally we show that (d) implies (c), which completes the proof.

If P is a finitely generated idempotent prime ideal then we can write

$$P = \bigcup_{i=1}^{n} e_i S$$

.

Let $\{Q_\alpha\}_\alpha$ be the set of all prime ideals with each $Q_\alpha \subset P$.

If $\bigcup_\alpha Q_\alpha = P$, then $e_i \in Q_{\alpha_i}$ for some i,

so that,

$$P = \bigcup_{i=1}^{n} e_i S \subseteq P = \bigcup_{i=1}^{n} Q_{\alpha_i} \subset P,$$

which is a contradiction.

Therefore,

$$P \neq \bigcup_\alpha Q_\alpha$$

■

The following example shows that a multiplication semigroup S is not necessarily a principal ideal semigroup if it does not satisfy any one of the conditions $(a) - (d)$ stated in the above Theorem even though its prime ideals are linearly ordered.

2.1.21 Example

Set $S = \{1, x_1, x_2, ...\}$ with $x_i x_j = x_i$ for $i \leq j$.

The ideal $\bigcup_{i=1}^{\infty} x_i S$ is not a principal ideal .

2.1.22 Theorem

For a finite dimensional multiplication semigroup S with unique maximal ideal M, the following are equivalent.

(1) S is a principal ideal semigroup.

(2) Every idempotent prime ideal of S is principal.

(3) Prime ideals of S are Linearly ordered.

Proof. (1) $\Longrightarrow$ (2) : is evident.

(2) $\Longrightarrow$ (3) : Let $\dim S = n$.

Then there exists a chain of prime ideals

$$P_0 \subset P_1 \subset ... \subset P_n = M$$

with no prime ideals between P_i and P_{i-1} for every $i \geq 1$.

If P_i is a $non - idempotent$ prime ideal,

then by (9) of Proposition 2.1.13, P_i contains properly at most one prime ideal Q with no prime ideals between P_i and Q

so that P_{i-1} is the only prime ideal properly contained in P_i with no prime ideals between P_i and P_{i-1} for $i \geq 1$.

Now if Pj is an idempotent prime ideal, $Pj = eS$ for some idempotent element e.

Suppose Q' is a prime ideal such that,

$$Q' \neq P_{j-1},$$

$Q' \subset P_j$, with no prime ideals between P_j and $Q\prime$.

Then,

$$Q' \cup P_{j-1} = P_j$$

so that $e \in Q'$or P_{j-1},

i.e., $P_j \subseteq Q'$ or $P_j \subseteq P_{j-1}$ which is not true.

Thus every idempotent prime ideal P_j also contains properly at most one prime ideal Q with no prime ideals between P_j and Q,

so that P_{j-1} is the only prime ideal properly contained in P_j with no prime ideals between P_j and P_{j-1} for $j \geq 1$.

It now follows that there are no prime ideals in S other than the P_i *'s* of the above chain.

Thus prime ideals of S are linearly ordered.

(3) $\Longrightarrow$ (1) : Since $\dim S = n$, clearly every idempotent prime ideal of S contains properly at most a finite number of prime ideals.

Now by Theorem 2.1.20 (b), the result follows. ∎

2.1.23 Theorem

If S is a $finite-dimensional$ semigroup with unique maximal ideal M, then S is a multiplication semigroup if and only if S is any one of the following types.

(1) S is a Dedekind semigroup containing only cancellable elements.

(2) S is a principal ideal semigroup.

(3) There exists an idempotent prime ideal P which is not principal, which is a multiplication *sub − semigroup* of S (P may or may not contain identity) such that every ideal of S not contained in P is principal.

Proof. If S contains only cancellable elements then from Corollary 2.1.15 it follows that S is of type (1).

Suppose S contains *non − cancellable* elements also.

If every idempotent prime ideal is principal, then S is of type (2) by the Theorem 2.1.22.

On the other hand, Suppose there is an idempotent prime ideal which is not principal.

If $\{P_\alpha\}$ is the set of all idempotent prime ideals which are not principal, then

$$(\bigcup_{\alpha} P_{\alpha})^2 \subseteq \bigcup_{\alpha} P_{\alpha}$$

and if $x \in \bigcup_{\alpha} P_{\alpha}$ then $x \in P_{\alpha_o}$ for some $\alpha_{o,}$ since P_{α_o} is idempotent so $x \in P^2_{\alpha_o} \Longrightarrow x = yz$ for some $y, z \in P_{\alpha_o}$.

Thus $y, x \in \bigcup_{\alpha} P_{\alpha}$ such that $x = yz$, i.e. $x \in (\bigcup_{\alpha} P_{\alpha})^2$. Hence

$$\bigcup_{\alpha} P_{\alpha} \subseteq (\bigcup_{\alpha} P_{\alpha})^2$$

Consequently

$$\bigcup_{\alpha} P_{\alpha} = (\bigcup_{\alpha} P_{\alpha})^2$$

Clearly $\bigcup_{\alpha} P_{\alpha}$ is a prime ideal.
Let $\bigcup_{\alpha} P_{\alpha} = P$, also $\bigcup_{\alpha} P_{\alpha} = P$ is not principal.
for if P is principal then,

$$P = \bigcup_{\alpha} P_{\alpha} = xS,$$

for some $x \in S$

$\Longrightarrow x \in P_{\alpha_o}$ for some α_o

$\Longrightarrow xS \subseteq P_{\alpha_o}$

but $P_{\alpha_o} \subseteq xS$, Hence $P_{\alpha_o} = xS$, which is a contradiction since each P_{α} is not principal.

By (4) of Proposition 2.1.13, P is a multiplication subsemigroup of S.

We claim that prime ideals not contained in P are linearly ordered.

Since S is $finite-dimentional$ we can have a chain of prime ideals

$$P = Q_0 \subset Q_1 \subset ... \subset Q_t = M$$

with no prime ideals between any two consecutive prime ideals.

If $Q_i \neq Q_i^2$ in view of (9) of Proposition 2.1.13, Q_{i-1} is the only prime ideal properly contained in Q_i with no prime ideals between Q_i and Q_{i-1}.

If Q_j is an idempotent prime ideal, since Q_j is a principal ideal,

it follows from the proof of Theorem 2.1.22 that Q_{j-1} is the only prime ideal properly contained in Q_j with no prime ideals between Q_j and Q_{j-1}.

It is clear that $Q_1, Q_2, ..., Q_t = M$ are the only prime ideals not contained in P, which are linearly ordered.

Now Using a similar proof as in (10) of Proposition 2.1.13, we conclude that every *non − idempotent* prime ideal not contained in P is principal.

Also by choice of P every idempotent prime ideal not contained in P is principal.

Clearly every ideal not contained in P has only one minimal prime divisor, which is principal.

Hence in view of (2) and (4) of Proposition 2.1.13, every ideal not contained in P is principal.

Conversely a semigroup of type (1) is a multiplication semigroup by Theorem 12 of [3].

From Theorem 2.1.16 it follows that a semigroup of type (2) is a multiplication semigroup.

Now suppose S is of type (3). Let A and B be any two ideals of S with $A \subseteq B$. We have the following possibilities:

1. $A, B \subseteq P$;
2. $A \subseteq P, B \ast P$;
3. $A, B \ast P$.

In the first case $A = BC$ for some ideal C of P, since A and B are trivially ideals of P with $A \subseteq B$.

Now we can write $A = (BS)C = B(SC)$ so that SC is an ideal of S.

In the second case $B = yS$ for some y; so that it is easy to verify that $A = B(A : B)$ where $(A : B)$ is clearly an ideal of S.

Finally, in the third case since

$$A = aS \subseteq B = bS$$

we have $a = bc$ for some $c \in S$ so that,

$$A = aS = (bc)S = (bS)(cS).$$

where cS is an ideal of S.

Thus S is a multiplication semigroup. ∎

In Proposition 2.1.13 we have seen that the conditions (2) and (4) are necessary for any multiplication semigroup S, while (8) is seen to be a necessary condition when $\dim S \leq 1$. Even with this restriction on dimension of S, the above three conditions are not sufficient. However, with some additional hypothesis the sufficiency is established in the following theorem.

2.1.24 Theorem

Let $\dim S \leq 1$ and every $non-\max imal$ prime ideal be idempotent. Then the semigroup S with unique maximal ideal M is a multiplication semigroup if and only if it satisfies the following conditions:

(1) Every ideal with prime radical is primary

(2) Every primary ideal is a power of its radical.

(3) For every ideal A, $A = \ker A$.

Proof. One implication is evident.

Now, suppose S satisfies (1), (2) and (3).

Let A and B be any two ideals with $A \subseteq B$.

Since $A = AS$ for any ideal A, it suffices to consider the case when $A \subset B$ with $B \neq S$.

If M is the only minimal prime divisor of both A and B then by (1) and (2),

$$A = M^k \ and \ B = M^l$$

for some positive integers k and l where $k > l$.

Setting $C = M^{k-l}$ we have $A = BC$.

Now if $\{P_\alpha\}$ is the set of all minimal prime divisors of A where each $P_\alpha \subset M$, then it is easy to verify that $\{P_\alpha\}$ is the set of all minimal prime divisors of AB too.

By (2), for each α, P_α is the isolated $P_\alpha - primary$ component of A and AB, since every $non-\max imal$ prime ideal is idempotent.

In view of (3),

$$A = \bigcap_\alpha P_\alpha = AB.$$

Hence S is a multiplication semigroup. ∎

The following theorem shows that the conditions (1) and (2) of Theorem 2.1.24 are enough to determine a *sub − class* of *one − dimensional* multiplication semigroups.

2.1.25 Theorem

If S contains only two prime ideals which are different from S, then S with unique maximal ideal M is a multiplication semigroup if and only if it satisfies

(1) Every ideal with prime radical is primary

(2) Every primary ideal is power of its radical.

Proof. One implication is obvious.

Now suppose S satisfies (1) and (2).

If P and M are the only prime ideals, clearly $P \subset M$.

To show that S is a multiplication semigroup, as before it suffices to consider any pair of ideals A and B such that $A \subset B$ with $B \neq S$.

If A and B have the same minimal prime divisor P or M, in view of (1) and (2) the multiplication property for the pair of ideals A and B can easily be verified.

Suppose P and M are minimal prime divisors of A and B respectively, Then by (1) and (2) we can write

$$A = P^k \text{ } and \text{ } B = M^l$$

for some positive integers k and l.

Since P is the only minimal prime divisor of the ideal P^kM^l, by (1) and (2) we obtain $P^kM^l = P^n$ for some positive integer n .

By (1) P^n is $P - primary$, so that we have $P^k \subseteq P^n$ since $M^l * P$.

Thus $P^k = P^n$ and hence $P^kM^l = P^k$, *i.e.* $AB = A$. ∎

Chapter 3

$NON - COMMUTATIVE$ $MULTIPLICATION$ $SEMIGROUPS$

The semigroups considered in this chapter are non-commutative and are without identity.

3.1 Some Properties of Non-Commutative Multiplication Semigroups

3.1.1 Definition

A Semigroup S is called a right (left) *multiplication semigroup* if for any pair of right (left) ideals A and B of S with $A \subseteq B$, there exists a right (left) ideal C such that $A = CB(A = BC)$.

3.1.2 Proposition

For a right multiplication semigroup S, the following are true.

(1) $S = S^2$ and $a \in aS$ for every $a \in S$.

(2) For any right ideal A, $A \subseteq SA$.

(3) If M is a maximal right ideal, then $M = M^2$ or $M = SM$.

Proof. (1) Since $S \subseteq S$, by the multiplication property of S, we have

$$S = BS$$

for some right ideal B of S.

Now,

$$S = BS \subseteq SS = S^2$$

i.e., $S \subseteq S^2$

but $S^2 \subseteq S$

Hence $S = S^2$.

Since for every $a \in S$, $\{a\} \cup aS$ is a right ideal of S. As $\{a\} \cup aS \subseteq S$,

so by the right multiplication property of S, $\{a\} \cup aS = CS$, for some right ideal C of S.

Now since $S = S^2$, so

$$\{a\} \cup aS = CS = CS^2 = (CS)S = (\{a\} \cup aS)S = aS \cup aS = aS$$

thus $a \in aS$.

(2) Since $A \subseteq A$, so by the right multiplication property of S, we have

$$A = DA$$

for some right ideal D of S.

Now,

$$A = DA \subseteq SA$$

i.e., $A \subseteq SA$.

(3) Since $M \subseteq M$, so by the right multiplication property of S, we have

$$M = NM$$

for some right ideal N of S.

If $N \subseteq M$, then

$$M = NM \subseteq MM = M^2$$

But

$$M^2 = MM \subseteq MS \subseteq M$$

i.e., $M = M^2$.

If $N * M$, then

$$M \subset M \cup N \subseteq S$$

but since M is maximal so $M \cup N = S$

Now,

$$M^2 \cup M = M^2 \cup NM = (M \cup N)M = SM$$

But since $M^2 \subseteq M$, so $M \cup M^2 = M$.

thus $M = SM$. ■

3.1.3 Proposition

In a semigroup S if $a \in aS\ aS$ for every $a \in S$, then S is a right multiplication semigroup.

Proof. Let A and B be two right ideals of S such that $A \subseteq B$,

Let $a \in A$, then $a \in aS\ aS$ by our hypothesis .

Now since $A \subseteq B$, so

$$a \in aSaS \subseteq AA \subseteq AB$$

i.e., $A \subseteq AB$.

But since A is a right ideal of S, so

$$AB \subseteq A$$

This implies that $A = AB$.

Therefore S is a right multiplication semigroup. ■

3.1.4 Corollary

If S is a right regular semigroup or a regular semigroup, then S is a right multiplication semigroup.

Proof. Let S be a right regular semigroup, then for each $a \in S$, there exists $x \in S$ such that

$$a = a^2x = a(ax) = a^2xax = aaxax = a(ax)a(x) \in aSaS.$$

i.e. $a \in aS\ aS$.

So by the Proposition 3.1.3, S is a right multiplication semigroup.

Let S be a regular semigroup, then for each $a \in S$, there exists $x \in S$ such that

$$a = a(xa) = axaxa = a(x)a(xa) \in aSaS$$

i.e. $a \in aS\ aS$.

So by the above Proposition 3.1.3, S is a right multiplication semigroup. ∎

3.1.5 Theorem

Let S be a simple semigroup. Then S is a right multiplication semigroup if and only if $a \in aS$ for every $a \in S$.

Proof. Let S be a simple semigroup and $a \in aS$ for every $a \in S$.

Let A and B be two right ideals of S such that $A \subseteq B$,

since B is a right ideal so SB is a $two - sidded$ ideal.

But since S is a simple semigroup so it does not contain any proper $two - sided$ ideal therefore $SB = S$

Now,

$$AS = A(SB) = (AS)B$$

Since $a \in aS$ for every $a \in S$,

we have $A \subseteq AS$,

but since A is a right ideal of S so $AS \subseteq A$

i.e., $A = AS$.

Thus $A = AB$ and hence S is a right multiplication semigroup.

Conversely, let S be a right multiplication semigroup,

then by (1) of Proposition 3.1.2 $a \in aS$, for every $a \in S$. ∎

3.1.6 Theorem

The following conditions on a semigroup S are equivalent:

(1) every proper ideal is an intersection of prime ideals;

(2) $A^2 = A$ for every ideal A of S (*i.e.* S is *semi – simple*);

(3) for every $a \in S$, *there exist* x, y, z such that $a = xayaz$.

(4) the product in any order of a finite number of ideals of S is equal to their intersection.

Proof. (1) $\Longrightarrow$ (2) : Let A be any ideal of S, then A^2 is also an ideal of S. Let $A^2 = \cap Q_i$, where Q_i '*s* are prime ideals.

Then $A^2 \subseteq Q_i$, for all i.

i.e. $AA \subseteq Q_i$, for all i

Now, since each Q_i is a prime ideal, so $A \subseteq Q_i$, for all i.

Thus,

$$A \subseteq \cap Q_i = A^2 \subseteq A.$$

This implies that $A = A^2$.

(2) $\Longrightarrow$ (3) : Let $a \in S$, then by (2)

$$S^1aS^1 = (S^1aS^1)(S^1aS^1)$$

implies that,

$$a \in (S^1aS^1)(S^1aS^1) = S^1aS^1aS^1$$

implies that there exist $x, y, z \in S^1$

such that $a = xayaz$

if $x, y, z \in S$, then we have proved (3).

If $x = 1$ then

$$a = ayaz = (ayaz)yaz = (ay)a(zy)az$$

Similarly if $y = 1$ or $z = 1$, then again $a = tasaq$ where $t, s, q \in S$.

$(3) \Longrightarrow (4)$: Let I and J be two ideals of S,

then $IJ \subseteq I \cap J$.

Let $a \in I \cap J$, this implies that, $a \in I$ and $a \in J$.

Then by (3) $a = xayaz$, for some $x, y, z \in S$

Now,

$$a = xayaz = (xay)(az) \in IJ.$$

Thus,

$$I \cap J = IJ$$

This result is true for finite number of ideals.

$(4) \Longrightarrow (2)$: Let A be any ideal of S, then by (4)

$$A^2 = AA = A \cap A = A$$

i.e. $A^2 = A$

$(3) \Longrightarrow (1)$: Let A be an arbitrary proper ideal of S.

Then for $a \in S$ such that $a \notin A$.

Let

$$\breve{A} = \{J \text{ is an ideal of } S : A \subseteq J \text{ and } a \notin J\}$$

Then $\breve{A} \neq \phi$, because $A \in \breve{A}$.

Then $(\breve{A}, \subseteq)$ is a partially ordered set.

Since every chain in $\breve{A}$ is bounded above, so by the Zorn's Lemma there exists a maximal ideal in $\breve{A}$, say M.

We claim that M is prime. Let X and Y be two ideals of S, such that $XY \subseteq M$, with X and Y * M.

Then by the maximality of M, $a \in M \cup X$ and $a \in MUY$.

But by the hypothesis $a = xayaz$, for some $x, y, z \in S$

i.e.,

$$.a = xa(yaz) \in (M \cup X)(M \cup Y) \subseteq M,$$

which is a contradiction, therefore M is a prime ideal.

Thus we have find a prime ideal which contains A but does not contain a.

Hence if $\{P_\alpha\}$ is the collection of all prime ideals that contains A,

then $A = \cap P_\alpha$. ■

3.1.7 Theorem

A semisimple semigroup S is a right multiplication semigroup if and only if $a \in aS\ aS$, for every $a \in S$.

Proof. Let S be a right multiplication semigroup .

Now by (2) of proposition 3.1.2

$$\{a\} \cup aS \subseteq S(\{a\} \cup aS)$$

but $\{a\} \cup aS = aS^1$, so

$$aS^1 \subseteq SaS^1$$

Now by the right multiplication property of S,

$$aS^1 = TSaS^1$$

for some right ideal T of S.

Since S is a semisimple semigroup and since every ideal in a semisimple semigroup is idempotent so,

$$aS^1 = TSaS^1 = (TSaS^1)SaS^1 = aS^1SaS^1 = a(S^1S)aS^1 = aSaS^1$$

Thus $a \in aS\ aS$.

Conversely, $a \in aS\ aS$, for every $a \in S$,

then by Proposition 3.1.3, S is a right multiplication semigroup. ■

3.1.8 Theorem

For a left cancellative semigroup S, he following are equivalent.

(1) S is a *semi − simple* right multiplication semigroup.

(2) S is a simple semigroup with $a \in aS$ for every a in S.

Proof. (1) $\Longrightarrow$ (2) : By Theorem 3.1.7.

$$a \in aS\ aS, \quad \text{for every } a \in S$$

i.e.,

$$a = axat, \quad \text{for some } x, t \in S$$

$a = axat \Longrightarrow as = axats$ for all s in S and since by our hypothesis S is a left cancellative semigroup so $s = xats$ for all s in S, i.e. xat is a left identity.

Thus every ideal of S contains a left identity and hence is equal to S.

Therefore S contains no proper ideals and hence is a *simple* semigroup.

Since S is a simple right multiplication semigroup so by Theorem 3.1.5, $a \in aS$ for every a in S.

(2) $\Longrightarrow$ (1) :

Since S is a simple semigroup so it contains no proper ideals so the ideal S^2 of S is equal to S.

i.e., $S^2 = S$

i.e. all ideals of S are idempotent and hence S is a *semi − simple* semigroup.

Since S is simple and by (2) $a \in aS$ for every a in S, thus by Theorem 3.1.5, S is a right multiplication semigroup. ∎

3.1.9 Theorem

Let S be a semigroup in which every right ideal is *two − sided* . Then S is right regular if and only if S is *semi − simple* right multiplication semigroup.

Proof. Let S be a *semi − simple* right multiplication semigroup, then by Theorem 3.1.7, for every $a \in S$, $a \in aS\ aS$.

Now by our hypothesis $Sa \subseteq aS^1$, so

$$a \in aSaS = a(Sa)S \subseteq a(aS^1)S = a^2(S^1S) = a^2S$$

i.e. $a \in a^2S$

this implies that $a = a^2x$, for some $x \in S$, .*i.e.* a is a right regular element.

Since a is an arbitrary element of S, so each element of S is right regular and hence S is a right regular semigroup.

Conversely, let S be a right regular semigroup, then every right ideal of S is idempotent.

So each ideal of S is idempotent. Thus S is a $semi - simple$ semigroup.

By the Corollary 3.1.4, every right regular semigroup is a right multiplication semigroup.

Hence S is a $semi - simple$ right multiplication semigroup. ■

3.1.10 Definition

A semigroup S is called a *left group* if it is left simple and right cancellative.

This is equivalent to saying that, for any elements a and b of S, there exist one and only one element x of S such that $xa = b$.

3.1.11 Lemma

Every idempotent element of a left simple semigroup S is a right identity element of S.

Proof. Let e be an idempotent element of S, and let a be any element of S. Since S is left simple, there exist x in S such that $xe = a$.

Then,

$$ae = xe^2 = xe = a$$

this completes the proof. ■

3.1.12 Theorem

The following assertions concerning a semigroup are equivalent:

(1) S is a left group.

(2) S is left simple and contains idempotents.

Proof. (1) $\Longrightarrow$ (2) : A left group S is left simple by definition .

Let $a \in S$. By left simplicity , there exists e in S such that $ea = a$.

Hence $e^2a = ea$, and by the right cancellation,$e^2 = e$.

(2) $\Longrightarrow$ (1) : S is left simple by the hypothesis, it is left to prove that S is right cancellative.

for a, b, c in S, let $ac = bc$.

Let E be the set of all idempotents in S.

Let f be an elements in E, such that $f = xc$, for some x in S.

Let $e = cx$. Then since f is an idempotent element of S, so by the Lemma 3.1.11,

$$e^2 = cxcx = c(xc)x = cfx = cx = e.$$

Hence,

$$a = ae = acx = (ac)x = (bc)x = be = b$$

Thus S is a right cancellative semigroup. and hence is a left group. ■

3.1.13 Theorem

The following are equivalent for a semigroup S

(1) S is a left simple right multiplication semigroup.

(2) S is a left simple semigroup containing idempotents.

(3) S is a left group.

Proof. (1) $\Longrightarrow$ (2) : Let $a \in S$. Since S is a right multiplication semigroup so by (1) of Proposition 3.1.2, $a \in aS$, i.e. $a = ax$ for some $x \in S$.

Now Sa is a left ideal of S, but S is left simple semigroup and so it contains no proper left ideals so $Sa = S$.

This implies that $x \in Sa$. Therefore $a = ax \in aSa$. i.e. each $a \in S$, is of the form aba,for some b in S.

Now let $c = ab$, then

$$c = ab = (aba)b = (ab)(ab) = cc = c^2.$$

This shows that S contains idempotents.

(2) $\Longrightarrow$ (3) : follows from the Theorem 3.1.12. .

(3) $\Longrightarrow$ (1) : Since S is a left group so it is a left simple semigroup. It is left to prove that S is a right multiplication semigroup.

Let $a \in S$, Sa is a left ideal of S but by the hypothesis S is left simple, so $Sa = S$.

This implies that $a \in Sa$, i.e. $a = xa$, for some $x \in S$.

Now,

$$a = xa = x(xa) = (xx)a = x^2 a$$

But S is a left group and hence is right cancellative so,

$$x = x^2$$

That is, x is idempotent. By the Lemma 3.1.11, every idempotent element in a left group is a right identity , therefore $tx = t$, for all t in S.

This implies that $a = ax$. Since $Sa = S$, then $a = ax \in aS = aSa$, $i.e. a = aba$,

for some $b \in S$. i.e. a is regular,but since a is an arbitrary element of S so every element of S is regular, that is S is a regular semigroup and by the Corollary 3.1.4, S is a right multiplication semigroup ■

3.1.14 Theorem [1]

The following conditions are equivalent on a semigroup S.

(1) S is a union of groups.

(2) S is both left and right regular.

(3) Every left and every right ideal of S is semiprime.

(4) S is left regular and regular (S is right regular and regular)

(5) S is a union of disjoint groups.

3.1.15 Theorem

If every left ideal is a *two – sided* ideal in a semigroup S, then S is the union of groups if and only if S is a *semi – simple* right multiplication semigroup.

Proof. By Theorem 3.1.7, since S is a *semi–simple* right multiplication semigroup, $a \in aS$ aS for every $a \in S$.

But by hypothesis, every left ideal is a *two – sided* ideal, so $aS \subseteq S^1 a$,

so that,

$$a \in aSaS = aS(aS) \subseteq aS(S^1 a) = a(S^1 S)a = aSa$$

i.e. $a \in aSa$, for every a in S. That is each a in S is of the form $a = axa$ .i.e. a is regular.

But since a is an arbitrary element of S so every element of S is regular.

Now by Theorem 3.1.14, S is a union of groups.

Conversely, let S be a union of groups then by Theorem 3.1.14, S is a right regular and regular semigroup.

And hence by the Corollary 3.1.4, S is a right multiplication semigroup. ∎

3.1.16 Lemma

Let S be a semigroup containing maximal left ideals. If $a \notin L^*$, where L^* is the intersection of all maximal left ideals, and if $a \in aS$, then $a \in aSa$.

Proof. Since $a \notin L^*$, there exist a maximal left ideal N not containing a.

Therefore,

$$N \subset N \cup S^1 a \subseteq S,$$

since N is maximal so $N \cup S^1 a = S$.

Now by hypothesis $a \in aS$, this implies that $a = as$, for some s in S.

Now $s \notin N$, for if $s \in N$ then $a = as \in N$. which is a contradiction.

Hence,

$$S = N \cup S^1 s = N \cup S^1 a$$

by the maximality of N.

Thus $s = a$ or $s \in Sa$.

Now if $s = a$ then $a = a^2$, this implies that $a^3 = a^2 a = aa = a^2 = a$.

This implies that $a \in aSa$.

And if $s \in Sa$, then $a = as \in aSa$. ∎

3.1.17 Theorem

Let S be a right multiplication semigroup containing maximal left ideals. Then S is a regular semigroup if $L^* = \phi$, where L^* is the intersection of all maximal left ideals.

Proof. Since $L^* = \phi$, so $a \notin L^*$, for every $a \in S$.

And by (1) of Proposition 3.1.2, since S is a right multiplication semigroup so $a \in aS$, for every $a \in S$.

By the Lemma 3.1.16, $a \in aSa$, for every $a \in S$.

that is, each $a \in S$ is regular or in other words S is a regular semigroup. ■

3.1.18 Theorem

If S is a *semi − simple* right multiplication semigroup such that $ab \in Sba$ for every $a, b \in S$, then S is a regular semigroup.

Proof. Since S is a *semi − simple* right multiplication semigroup so by Theorem 3.1.7, $a \in aS\ aS$, for every $a \in S$.

That is, $a = asat$, for some $s, t \in S$.

Now by hypothesis $at \in Sta \subseteq Sa$, i.e.

$$a = as(at) \subseteq asSa \subseteq aSa.$$

each $a \in S$ is of the form asa, i.e. each $a \in S$ is regular .

As a is in arbitrary element of S, so each element of S is regular i.e. S is a regular semigroup.

■

3.1.19 Definition

A semigroup S is called *intra − regular* if for any element a of S, there exist x and y in S such that $xa^2y = a$.

3.1.20 Lemma

A Semigroup S is *intra − regular* if and only if every *two − sided* ideal of S is semiprime.

Proof. Let S be *intra − regular*, and let T be a *two − sided* ideal of S.

Let $a^2 \in T$, $s \in S$. Then $a \in Sa^2S \subseteq STS \subseteq T$, because S is *intra − regular*.

Conversely assume that every ideal of S is semiprime.

Let $a \in S$, then $a^2 \in S^1a^2S^1$ implies $a \in S^1a^2S^1$.

Thus S is *intra − regular.* ■

3.1.21 Theorem [1]

The following assertions concerning a semigroup S are equivalent.

(1) S is union of simple semigroups.

(2) S is $intra-regular$.

(3) Every ideal of S is semiprime.

3.1.22 Theorem

Let S be a left regular or $intra-regular$ semigroup. Then S is a right multiplication semigroup if and only if S is a union of simple semigroups S_α with $x \in xS_\alpha$ for every $x \in S_\alpha$.

Proof. Let S be the union of simple semigroups $\{S_\alpha\}$ with $x \in xS_\alpha$ for every $x \in S_\alpha$.

Then for every $a \in S$, $a \in S_\alpha$ for some α. By the given condition $a \in aS_\alpha$.

Since each S_α is a simple semigroup so it contains no proper $two-sided$ ideal,

so the $two-sided$ ideal $S_\alpha a S_\alpha$ is equal to S_α.

Therefore $a \in aS_\alpha$ implies that,

$$a \in a(S_\alpha a S_\alpha) = aS_\alpha a S_\alpha$$

i.e. $a \in aS_\alpha a S_\alpha$.

By Theorem 3.1.7, S is a right multiplication semigroup.

The converse is proved by first asserting that left regular right multiplication semigroups are $intra-regular$.

Since by Theorem 3.1.6, left regular semigroups are semisimple,

for every $a \in S$ we have $a \in aS\ aS$ by Theorem 3.1.7.

But $a \in Sa^2$ by left regularity , $a \in Sa^2$ so that $a \in Sa^2S$. Now by Theorem 3.1.21,

the intra regular semigroup S is a union of simple semi-groups S_α.

Since $intra-regular$ semigroups are $semi-simple$ by Theorem 3.1.6 and since S is a right multiplication semigroup, for every $a \in S_\alpha$ we have $a \in aS\ aS$ so $a = axat$ for some $x, t \in S$ by Theorem 3.1.7.

Now clearly $S^1xatS^1 = S^1aS^1$.

Thus $xat \in S_\alpha$ and hence $a \in aS_\alpha$. ■

3.1.23 Proposition

Let S be a right multiplication semigroup with unique maximal ideal M such that $A \neq AM$ for every proper right ideal A and suppose that every proper right ideal is included in M. Then S is a right multiplication semigroup if and only if S satisfies the following conditions:

(1) every proper right ideal is of the form M^r,

(2) $S = S^2$,

(3) $M = SM$ and

(4) $M = MS$.

Moreover $M = xS^1$ for some $x \in S$ and every element a of M is of the form $x^r u$ where $u \notin M$.

Proof. Suppose that S is a right multiplication semigroup.

Since $M \neq MM = M^2$ by hypothesis, so by (3) of Proposition 3.1.2, $M = SM$.

Now we claim

$$M^{\omega} = \bigcap_{n=1}^{\infty} M^n = \phi.$$

If $M^{\omega} = \bigcap_{n=1}^{\infty} M^n \neq \phi$, then $M^{\omega} \subseteq M$ implies $M^{\omega} = CM$ for some right ideal C by the right multiplication property of S.

$C = S$ is inadmissible since otherwise $M^{\omega} = SM = M$ and so $M \subseteq M^n$, for all n.

This implies that $M \subseteq M^2$, but $M^2 \subseteq M$.i.e. $M = M^2$, which is a contradiction.

Then by hypothesis every proper right ideal is contained in M and as $C \neq S$, so $C \subseteq M$.

Now if $C \subseteq M^{\omega}$, then $M^{\omega} \subseteq CM$ implies that $M^{\omega} \subseteq M^{\omega}M$, but $M^{\omega}M \subseteq M^{\omega}$ i.e. $M^{\omega}M = M^{\omega}$, which is not true.

If $C \ast M^{\omega}$, then $C \subseteq M^r \backslash M^{r+1}$ for some natural number r.

This implies that $C = TM^r$ for some right ideal $T \ast M$, by the right multiplication property.

Thus $T = S$, and hence

$$C = SM^r = (SM)M^{r-1} = MM^{r-1} = M^r;$$

$M^{\omega} = M^{r+1}$ and so $M^{r+1} = M^{r+2}$, i.e. $M^{r+1} = M^{r+1}M$,which is false.

Since $M^{\omega} = \phi$, we have for every proper right ideal A, $A \subseteq M^r \backslash M^{r+1}$ for some natural number r.

This implies that $A = LM^r$, for some right ideal $L * M$, by the multiplication property.

Thus $L = S$, and hence,

$$A = SM^r = (SM)M^{r-1} = MM^{r-1} = M^r.$$

(2) and (3) are evident from Proposition 3.1.2.

Since $M \subseteq S$ we have by the multiplication property of S, $M = DS$ for some right ideal D.

Then $M = MS$ since $D \subseteq M$ and $S = S^2$.

Conversely, Let A and B be two ideals of S, such that $A \subseteq B$.

Then by our hypothesis, A and $B \subseteq M$, also $A \neq AM$ and $B \neq BM$.

By (1) $A = M^r$and $B = M^s$, for some natural numbers r and s.

$M^r \neq M^{r+1} \neq M^{r+2} \neq ...$, also

$M^s \neq M^{s+1} \neq M^{s+2} \neq ...$

Now since $A \subseteq B, i.e.\ M^r \subseteq M^s$, this implies that $s \leq r$.

Now $M^r = M^{r-s}M^s$, where M^{r-s} is a right ideal of S.

And hence S is a right multiplication semigroup.

Finally, since $M \neq M^2$, by hypothesis, we have for any $x \in M \backslash M^2$, $xS^1 \subseteq M$.

Then by the right multiplication property, $xS^1 = TM$, for some right ideal T.

If $T \subseteq M$, then $x \in M^2$,which is false.

So we must have $T = S$. Thus $xS^1 = SM = M$..

Now let $a \in M$. Then $a = xs_1$. If $s_1 \notin M$, we are done. Otherwise $s_1 = xs_2$.

Proceeding in this manner, if every $s_i \in M$, then

$a = xs_1 = x^2s_2 = ...$ and so $a \in M^\omega = \phi$.

Thus $a = x^r s_r$ for some natural number r, where $s_r \notin M$. ∎

3.1.24 Definition

A Semigroup S is called $Q^* - simple$ if it contains no proper prime ideals.

3.1.25 Lemma

If S is a left cancellative semigroup then every maximal right ideal is *two − sided* ideal.

Proof. Let M be a maximal right ideal, which is not *two − sided.* Then there exist an $x \in M$ and $s \notin M$ such that $sx \notin M$.

Therefore

$$S = M \cup \{s\} \cup sS = M \cup \{sx\} \cup sxS$$

Then $s = sx$ or $s = sxt$,for some $t \in S$.

Now if $s = sx$ then,

$$s = sx = (sx)x = s(xx) = sx^2$$

since S is left cancellative so,

$$x = x^2$$

i.e. x is idempotent. For all a in S,

$$xa = (xx)a = x(xa)$$

since S is left cancellative so,

$$a = xa$$

i.e. x is left identity.

Similarly if $s = sxt$, for some $t \in S$ then xt is idempotent and hence is left identity.

Thus M contains a left identity and so $M = S$, a contradiction.

Hence every maximal right ideal M in S is *two − sided.* ■

3.1.26 Theorem [12]

Let S be a Left Cancellative semigroup with $S = S^2$. Then the following are equivalent:

(1) S contains proper maximal right ideals;

(2) S contains idempotents and S is not right simple;

(3) S contains a proper maximal right ideal M such that every proper right ideal is contained in M

3.1.27 .Theorem

Let S be a Left Cancellative semigroup. Then S is a right multiplication semigroup if and only if S contains idempotents and S is one of the following:

(1) S is a simple semigroup with $a \in aS$ for every $a \in S$.

(2) S contains a unique maximal $two-sided$ ideal with $M = MS$ such that every proper right ideal is of the form M^r and thus $two-sided$.

In the second case $M = xS^1$ for any $x \in M \backslash M^2$ and every $a \in S$ is of the form $x^r u$, where $u \notin M$ and r is a $non-negative$ integer. Furthermore S is an extension of a $Q^* - simple$ semigroup by a right $o - simple$ semigroup.

Proof. Let S be a right multiplication semigroup.

Then by Proposition 3.1.2, $a \in aS$ for every $a \in S$.

$a = ax$, for some x in S,

$$a = ax = (ax)x = a(xx) = ax^2,$$

since x is left cancellative so $x = x^2$.

This implies that S contains idempotents.

If S is a simple semigroup, S is of type (1).

Suppose that S is not simple.

Then S is not right simple and so by Theorem 3.1.26, S contains a proper maximal right ideal M such that every proper right ideal is contained in M.

Then By Lemma 3.1.25, M is also the unique maximal $two-sided$ ideal.

Now we prove that $A \neq AM$ for every proper right ideal A.

Suppose $A = AM$. Then for $a \in A$, $aS^1 \subseteq A$, by the right multiplication property of $S, aS^1 = TA$, T being some right ideal.

Thus,

$$aS^1 = TA = T(AM) = (TA)M = aS^1M$$

This implies $a = am$, $m \in M$ and so m is an idempotent and thus a left identity by left cancellative condition.

Hence $M = S$, which is false. $S = S^2$ by (1) of Proposition 3.1.2, and $S = SM$ by (3) of Proposition 3.1.2.

Since S contains idempotents which are left identities,then by Proposition 3.1.23, and Theorem 3.1.5, except the last every statement is evident.

Since M is $two-sided$ ideal and a maximal right ideal , S/M is clearly a right $0-simple$ semigroup.

Now we claim that the subsemigroup M has no proper prime ideals.

Suppose that P is a prime ideal of M and $P \neq M$.

Since PM is a proper ideal of S, $PM = M^r \subseteq P$ and hence $M = P$.

Thus M contains no proper prime ideal and hence is a $Q^*-simple$ semigroup.

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