Analytic solution for thin film flow of a fourth order fluid

By

Muhammad Awais

Department of Mathematics
Quaid-i-Azam University, Islamabad
PAKISTAN

2007

Analytic solution for thin film flow of a fourth order fluid

QUAID-I-AZAM UNIVERSITY

By

Muhammad Awais

Supervised

By

DR. SOHAIL NADEEM

Department of Mathematics
Quaid-i-Azam University, Islamabad
PAKISTAN

2007

Analytic solution for thin film flow of a fourth order fluid

By

Muhammad Awais

A Dissertation Submitted in the Partial Fulfillment of the Requirements for the Degree of

MASTER OF PHILOSOPHY
IN
MATHEMATICS

Department of Mathematics
Quaid-i-Azam University, Islamabad
PAKISTAN

2007

Analytic Solution for Thin Film Flow of a Fourth Order Fluid

Certificate

We accept this thesis as conforming to the required standard.

A THESIS SUBMITTED IN THE PARTIAL FULFILLMENT OF REQUIREMENTS FOR THE DEGREE OF THE MASTER OF PHILOSOPHY

1. ____________________

Dr.Muhammad Yaqub Nasir

(Chairman)

2. ____________________

Dr. Sohail Nadeem

(Supervisor)

3. ____________________

(Prof. Dr. Saleem Asghar)

(External Examiner)

Department of Mathematics
Quaid-i-Azam University, Islamabad
PAKISTAN

2007

Dedicated

To

Abbu & Ammi

*Who are the most precious gems
of my life.
Who've always given me perpetual love, care,
and cheers. Whose prayers have always
been a source of great inspiration for
me and whose sustained hope in me
led me to where I stand today.*

&

*Then to my lovely brothers
and sisters, especially my brothers
whose love, care and guidance is
unmatchable and simply priceless for me.*

Acknowledgement

All praises to Almighty **Allah**, the creator of all the creatures in the universe, Who created us in the structure of human beings as the best creature. Many thanks to Him, Who created us as a Muslim and blessed us with knowledge to differentiate between right and wrong. Many many thanks to Him as He blessed us with the Holy Prophet, **Hazrat Muhammad (PBUH)** for **Whom** the whole universe is created and **Who** enabled us to worship only to one God. **He (PBUH)** brought us out of darkness and enlightened the way to heaven.

I express my heart–felt gratitude to my supervisor **Dr. Sohail Nadeem**, for his passionate interest, superb guidance and inexhaustible inspiration through out this investigation. His textural and verbal criticism enabled me in formatting this manuscript.

I am thankful to the Chairman, Department of Mathematics **Dr. Muhammad Yaqub Nasir**, Quaid-i-Azam University, Islamabad and to **Dr. Tasawar Hayat** for providing necessary research facilities. Thanks to all faculty members, especially to all teachers of Fluid Mechanics for their moral boosting and encouraging behaviour.

I especially deem to express my unbound thanks to my classmates; **Muhammad Ali, Muhammad Asif Farooq, Rab Nawaz** and **Maryam Saleem** for their pleasant company and cooperation during my work.

I am also very thankful to **Mr. Tariq Javed** and **Dr. Muhammad Sajid** for their valuable suggestions and fruitful discussions during my research work.

The acknowledgement will surely remain incomplete if I don't express my deep indebtedness and cordial thanks to **Shomaila Noreen** for her valuable suggestions, unending cooperation and unforgettable company, not only during my M.Phil but ever since I joined her, as a class fellow, in Quaid-i-Azam University. Surely **Shomaila**, without your cooperation and assistance I wouldn't have been able to cross the finishing line.

Last but surely surely not the least, my immeasurable and most special thanks to **Prof. Dr. Mubashara Munir** for all of her love, care, guidance, and what not and also to the two little angels **Qasim** and **Fatima**.

Deepest gratitude and cordial thanks for those, who have always stood by me, my loving parents, brothers and sisters for their prayers, encouragement and support. Especially my dearest **Dad, Mother** and **Brothers**, without you all I would have been standing no where in my life.

May Almighty **Allah** shower His choicest blessings and prosperity on all those who assisted me in any way during completion of my thesis.

Muhammad Awais

Preface

It is a known fact that the flows of thin films have wide applications in industry. To be more specific, such flows have relevance in engineering (microchip production), biology (lining of mammalian lungs) and chemistry (flow of surface active materials). These flows are derived through gravitational (flow down an inclined plane) force. Very often, the thin film flows have been discussed by taking viscous fluids. But in industrial and technological applications, there are non-Newtonian fluids for which the Navier-Stokes equations are inadequate. Such fluids exhibit a nonlinear stress strain relation and thus the resulting different systems are highly nonlinear and complicated. By keeping in view all these challenges, some investigators are recently engaged in obtaining solutions for flows of non-Newtonian fluids [1-24]. But literature survey indicates that much attention is not given to the thin film flows of non-Newtonian fluids. To the best of our knowledge only few such studies [11-16] are yet available.

In all the above mentioned investigations [11-16] the effects of slip condition have not been taken into account. Such effects are very important for non-Newtonian fluids (polymer melts) which exhibit wall slip. The fluids exhibiting slip are important in technological applications, for example, the polishing of artificial heart valves and internal cavities in a variety of manufactured parts is achieved by imbedding such fluids with abrasives.

Keeping all this importance in mind the following dissertation has been arranged in the following manner:

The first chapter includes some basic definitions of Fluid mechanics, derivation of equation of motion, and introduction to Perturbation method, Homotopy perturbation method and Homotopy analysis method.

The second chapter contains the review of [11]. Further we have non-dimensionalized the solution and presented the graph for the velocity field.

In chapter three, we have discussed the thin film flow of a fourth grade fluid down a vertical cylinder with partial slip boundary conditions. The problem is solved by Perturbation method, Homotopy perturbation method and Homotopy analysis method. It is found that Homotopy Analysis Method is a powerful technique for solving non-linear problems [24-30]. The graphs for the velocity field have been presented for various physical parameters involved in the chapter.

Contents

Chapter 1

Basic definitions

1.1 Introduction

In this chapter we include some basic definitions of Fluid mechanics, derivation of equation of motion, and introduction to Perturbation method, Homotopy perturbation method and Homotopy analysis method.

1.2 Fluid mechanics

Fluid mechanics is the study of gases and liquids at rest and in motion. This area of physics is divided into two types. Fluid statics, the study of the behavior of stationary fluids, and fluid dynamics, the study of the behavior of moving, or flowing, fluids. Fluid dynamics is further divided into hydrodynamics, or the study of water flow, and aerodynamics, the study of airflow. Applications of fluid mechanics include a variety of machines, ranging from the water-wheel to the airplane.

1.3 Fluid dynamics

Fluid dynamics is the sub-discipline of fluid mechanics dealing with fluids (liquids and gases) in motion. It has several subdisciplines itself, including aerodynamics (the study of gases in motion) and hydrodynamics (the study of liquids in motion).

1.4 Fluid

By definition, a fluid is a material continuum that is unable to withstand a static shear stress. Unlike an elastic solid which responds to a shear stress with a recoverable deformation, a fluid responds with an irrecoverable flow. Examples of fluids include gases and liquids.

1.5 Flow

Motion of a fluid subjected to unbalanced forces or stresses is called flow of the fluid. The motion continues as long as unbalanced forces are applied.

1.6 Types of fluids

1.6.1 Newtonian fluids

A Newtonian fluid (named for Isaac Newton) is a fluid that flows like water. Or a simple fluid in which the state of stress at any point is proportional to the time rate of strain at that point; the proportionality factor is the viscosity coefficient. Its (stress / rate of strain) curve is linear and passes through the origin. The constant of proportionality is known as the viscosity. A simple equation to describe Newtonian fluid behavior is

$$\tau_{xy} = \mu \frac{du}{dy}, \tag{1.1}$$

where τ_{xy} is the shear stress, μ is the fluid viscosity, x is the direction of flow and y is the direction perpendicular to the flow. Fluids like water, air and gasoline are Newtonian fluids.

1.6.2 Non-Newtonian fluids

A non-Newtonian fluid is a fluid in which the viscosity changes with the applied strain rate. As a result, non-Newtonian fluids may not have a well-defined viscosity. Examples are toothpaste, shampoo, ketchup etc.

1.6.3 Incompressible fluids

In fluid mechanics or more generally continuum mechanics, an incompressible flow is solid or such a fluid flow in which the divergence of velocity is zero. This is more precisely termed as isochoric flow.

These are the fluids with negligible changes in density. No fluid is truly incompressible, since even liquids can have their density increased through application of sufficient pressure.

1.6.4 Compressible fluids

A compressible flow is a situation in which the density of the flow cannot be assumed to be constant.

1.7 Types of flows

1.7.1 Uniform flow

In a flow that is uniform at a given cross section, the velocity is constant across any section normal to the flow.

1.7.2 Laminar flow

A flow, in which each liquid particle has a definite path and the path of individual particles do not cross each other. Or the flow for which the flow structure is characterized by smooth motion in laminae, or layer is called the laminar flow.

1.7.3 Turbulent flow

A flow, in which each liquid particle does not have a definite path and the path of individual particles also cross each other. Or the flow structure in the turbulent regime is characterized by random, three-dimensional motions of fluid particles in addition to the mean motion.

1.7.4 Steady flow

A flow in which the quantity of liquid flowing per second is constant. A steady flow may be uniform or non-uniform.

1.7.5 Unsteady flow

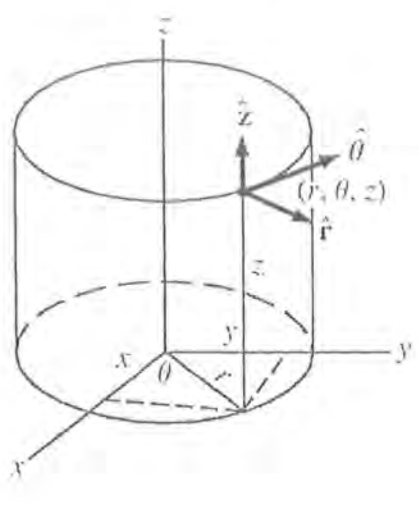
A flow in which the quantity of liquid flowing per second is not constant.

1.7.6 Force of gravity

Gravity is one of the universal forces of nature. It is an attractive force between all matter, and is very weak as compared to the other forces of nature. The gravitational force between two objects is dependent on their masses, which is why we can only see gravity in action when at least one of the objects is very large (like the Earth).

1.8 Cylindrical coordinates

In cylindrical coordinates system a point in three-dimensional space is represented by the order triple (r, θ, z), where r and θ are polar coordinates of the projection of point on the $xy-plane$.



To convert from cylindrical coordinates to rectangular coordinates we use the equations

$$x = r\cos\theta, \quad y = r\sin\theta, \quad z = z, \tag{1.2}$$

whereas to convert form rectangular coordinates to cylindrical coordinates, we use

$$r^2 = x^2 + y^2, \quad \tan\theta = \frac{y}{x}, \quad z = z. \tag{1.3}$$

1.9 Divergence

Divergence of a vector field

$$\vec{v}_{el} = [u, v, w], \tag{1.4}$$

in cylindrical coordinates is given as

$$div(\vec{v}_{el}) = \frac{1}{h_1 h_2 h_3}\left[\frac{\partial}{\partial u_1}(h_2 h_3 u) + \frac{\partial}{\partial u_2}(h_1 h_3 v) + \frac{\partial}{\partial u_3}(h_1 h_2 w)\right]. \tag{1.5}$$

where $h_1 = 1,\ h_2 = r,\ h_3 = 1$ and $u_1 = r,\ u_2 = \theta,\ u_3 = z$.

1.10 Gradient

Where as for Eq. (1.4), gradient in cylidrical coordinates is given as

$$L = \text{grad}\,\vec{v}_{el} = \begin{bmatrix} \frac{\partial u}{\partial r} & \frac{1}{r}\frac{\partial u}{\partial \theta} - \frac{v}{r} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial r} & \frac{1}{r}\frac{\partial v}{\partial \theta} - \frac{u}{r} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial r} & \frac{1}{r}\frac{\partial w}{\partial \theta} & \frac{\partial w}{\partial z} \end{bmatrix}. \tag{1.6}$$

1.11 Equation of continuity

Suppose we have a fluid with local density $\rho(t, x, y, , z)$ and local velocity $\vec{v}(t, x, y, z)$. Consider a control volume V (not necessarily small, not necessarily rectangular) which has boundary S. The total mass in this volume is

$$M_{mass} = \int \rho dV. \tag{1.7}$$

The rate of change of this mass is

$$\frac{\partial M_{mass}}{\partial t}=\int\frac{\partial\rho}{\partial t}dV. \tag{1.8}$$

The only way such change can occur is by stuff flowing across the boundary, so

$$\frac{\partial M_{mass}}{\partial t}=\int\rho\vec{v}.dS. \tag{1.9}$$

We can change the surface integral into a volume integral using Green's theorem, to obtain

$$\frac{\partial M_{mass}}{\partial t}=\int\nabla.(\rho\vec{v})dV. \tag{1.10}$$

Comparing Eqs. (1.8) and (1.9). They are equal no matter what volume V we choose, so the integrands must be point wise equal. This gives us an expression for the local conservation of mass

$$\frac{\partial\rho}{\partial t}+\nabla.(\rho\vec{v})=0, \tag{1.11}$$

called the Equation of continuity. Equation of continuity for an incompressible flow becomes

$$div(\vec{v})=0. \tag{1.12}$$

1.12 Equation of motion

In this section, we are interested in constructing the governing equation for fourth grade fluid in cylindrical coordinates. The equation of motion in the presence of body forces is

$$\rho\frac{d\mathrm{v}}{dt}=div(\tau)+\rho\mathrm{B}, \tag{1.13}$$

where ρ is the density of the fluid , $\frac{d}{dt}$ is the total derivative, τ is the Cauchy stress tensor and $\vec{\mathrm{B}}$ is the body force. An incompressible simple fluid is defined as a material whose state of present stress is determined by the history of the deformation gradient without a preferred reference configuration.

For fourth grade fluid the Cauchy stress tensor is defined as [13, 15]

$$\begin{aligned}\tau &= -PI+\mu A_1+\alpha_1 A_2+\alpha_2 A_1^2+\beta_1 A_3+\beta_2[A_1A_2+A_2A_1]+\beta_3[tr(A_2)A_1]+\gamma_1 A_4\\ &+\gamma_2[A_3A_1+A_1A_3]+\gamma_3[A_2^2]+\gamma_4[A_2A_1^2+A_1^2A_2]+\gamma_5[(tr(A_2)A_2)]\\ &+\gamma_6[(tr(A_2)A_1^2)]+[\gamma_7(tr(A_3))+\gamma_8(tr(A_2A_1))]A_1, \end{aligned} \tag{1.14}$$

where P is the pressure, μ is the coefficient of viscosity and α_i $(i=1-2)$, β_j $(j=1-3)$ and γ_k $(k=1-8)$ are material constants.

$$A_1=\text{grad}\,\vec{v}+[\text{grad}\,\vec{v}]^T, \tag{1.15}$$

$$A_n=\frac{dA_{n-1}}{dt}+A_{n-1}(\text{grad}\,\vec{v})+(\text{grad}\,\vec{v})A_{n-1}. \quad n>1 \tag{1.16}$$

We seek the velocity field of the following form

$$\vec{v}=[0,0,u(r)], \tag{1.17}$$

where $u(r)$ is the $z-component$ of velocity.

In view of Eq. (1.17), we have

$$A_1=\begin{bmatrix}0 & 0 & \frac{du}{dr}\\ 0 & 0 & 0\\ \frac{du}{dr} & 0 & 0\end{bmatrix}, \tag{1.18}$$

$$\begin{aligned}A_2 &= \frac{dA_1}{dt}+A_1(\text{grad}\,\vec{v})+(\text{grad}\,\vec{v})^T A_1\\ &= \frac{\partial A_1}{\partial t}+(\vec{v}\cdot\vec{\nabla})A_1+A_1L+L^TA_1,\end{aligned} \tag{1.19}$$

or

$$A_2=\begin{bmatrix}2(\frac{du}{dr})^2 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}, \tag{1.20}$$

$$A_3 = \frac{dA_2}{dt} + A_2(\text{grad}\,\vec{v}) + (\text{grad}\,\vec{v})^T A_2 = \frac{\partial A_2}{\partial t} + (\vec{v}\cdot\vec{\nabla})A_2 + A_2 L + L^T A_2, \tag{1.21}$$

or

$$A_3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \tag{1.22}$$

$$A_4 = \frac{dA_3}{dt} + A_3(\text{grad}\,\vec{v}) + (\text{grad}\,\vec{v})^T A_3 = \frac{\partial A_3}{\partial t} + (\vec{v}\cdot\vec{\nabla})A_3 + A_3 L + L^T A_3, \tag{1.23}$$

or

$$A_4 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \tag{1.24}$$

$$A_1^2 = \begin{bmatrix} (\frac{du}{dr})^2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & (\frac{du}{dr})^2 \end{bmatrix}, \tag{1.25}$$

$$A_1A_2 + A_2A_1 = \begin{bmatrix} 0 & 0 & 2(\frac{du}{dr})^3 \\ 0 & 0 & 0 \\ 2(\frac{du}{dr})^3 & 0 & 0 \end{bmatrix}, \tag{1.26}$$

$$tr(A_2)A_1 = \begin{bmatrix} 0 & 0 & 2(\frac{du}{dr})^3 \\ 0 & 0 & 0 \\ 2(\frac{du}{dr})^3 & 0 & 0 \end{bmatrix}, \tag{1.27}$$

$$A_3A_1 + A_1A_3 = 0, \tag{1.28}$$

$$A_2^2 = \begin{bmatrix} 4(\frac{du}{dr})^4 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \tag{1.29}$$

$$A_2A_1^2 + A_1^2A_2 = \begin{bmatrix} 4(\frac{du}{dr})^4 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \tag{1.30}$$

$$tr(A_2)A_2 = \begin{bmatrix} 4(\frac{du}{dr})^4 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \tag{1.31}$$

$$(tr(A_2))A_1^2 = \begin{bmatrix} 2(\frac{du}{dr})^4 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 2(\frac{du}{dr})^4 \end{bmatrix}, \tag{1.32}$$

$$tr(A_2A_1) = 0, \tag{1.33}$$

$$tr(A_3) = 0. \tag{1.34}$$

On substituting the values of Eqs. (1.14) – (1.34) in Eq. (1.13) , we get the component form of Equation of motion in the absence of body forces as

$$\frac{\partial P}{\partial r} = (2\alpha_1 + \alpha_2)\frac{1}{r}\frac{\partial}{\partial r}\left[r(\frac{du}{dr})^2\right] + \frac{4}{r}(\gamma_3 + \gamma_4 + \gamma_5 + \frac{\gamma_6}{2})\frac{\partial}{\partial r}\left[r(\frac{du}{dr})^4\right], \tag{1.35}$$

$$\frac{\partial P}{\partial \theta} = 0, \tag{1.36}$$

$$\frac{\partial P}{\partial z} = \frac{\mu}{r}\frac{\partial}{\partial r}\left[r(\frac{du}{dr})\right] + \frac{2}{r}(\beta_2 + \beta_3)\frac{\partial}{\partial r}\left[r(\frac{du}{dr})^3\right]. \tag{1.37}$$

1.13 No slip condition

When a fluid flow is bounded by a solid surface, molecular interactions cause the fluid in contact with the surface to seek momentum and energy equilibrium with the surface. Or we can say that all fluids at a point of contact with a solid take on the velocity of that solid. i.e.,

$$\mathrm{V}_{fluid} = \mathrm{V}_{wall}. \tag{1.38}$$

1.14 Partial slip

Navier was the one who proposed that a liquid may slip on the solid surface, and this slipping would be opposed by a frictional force proportional to the velocity of the fluid relative to the solid. He introduced the idea of 'slip-lengh', which is now a days the most commonly used concept to quantify the slip of a liquid at a solid interface. i.e.,

$$v_r = b\frac{\partial v_b}{\partial z}, \tag{1.39}$$

where v_r is the velocity of the fluid at the wall, v_b is the velocity of the fluid in the bulk and z is the axis perpendicular to the wall.

1.15 Thin film flow

Thin film flow, as the name suggests, can simply be defined as the flow of a fluid in the form of a thin film. More precisely it is a flow that consists of an expanse of liquid partially bounded by a solid substrate with a (free) surface where the liquid is exposed to another fluid (usually a gas and most often air in applications).

One can observe thin film flows even in daily life happenings. For example, the formation of tear in the eye and the flow of rain water down the window glass.

There are three agents that are mainly considered to be responsible for the formation of thin films. These are gravity, centrifugal forces and surface tension. Gravity forms the thin films whenever the fluid flows down some inclined plane. While during the rotation of the fluid these are the centrifugal forces that are responsible for the formation of thin films.

Thin films have numerous applications in industry. Spin coating is used in the manufacture of CDs and DVDs and computer disks. Any coating process, e.g. painting and manufacture of coated products are all examples of this thin film technology.

1.16 Homotopy

In topology, two continuous functions from one topological space to another are called homotopic (Greek homos = identical and topos = place) if one can be "continuously deformed" into the other, such a deformation being called a homotopy between the two functions.

Formally, a homotopy between two continuous functions f and g from a topological space X to a topological space Y is defined to be a continuous function $H : X \times [0,1] \to Y$ from the product of the space X with the unit interval $[0,1]$ to Y such that, for all points x in X, $H(x,0) = f(x)$ and $H(x,1) = g(x)$.

If we think of the second parameter of H as "*time*", then H describes a "*continuous deformation*" of f into g. At time 0, we have the function f, at time 1 we have the function g.

1.17 Perturbation method

Let we have the equation

$$\dot{g}(t) + g^2(t) = 1, \quad t \geq 0, \tag{1.40}$$

subject to the initial condition

$$g(0) = 0. \tag{1.41}$$

To give perturbation approximation, we assume that time t is a small variable (called perturbation quantity) and then express $g(t)$ in a power series

$$g(t) = a_0 + a_1 t + a_2 t^2 + a_3 t^3 + \cdots, \tag{1.42}$$

using the initial condition (1.41) we obtain $a_0 = 0$. Then, substituting the above expression into Eq. (1.40), we have

$$a_1 = 1, \tag{1.43}$$

$$a_{k+1} = -\frac{1}{k+1} \sum_{j=0}^{k} a_j a_{k-j}, \quad k \geq 1, \tag{1.44}$$

we therefore have the perturbation solution

$$g_{pert}(t) = t - \frac{1}{3}t^3 + \frac{2}{15}t^5 - \frac{17}{315}t^7 + \cdots. \tag{1.45}$$

1.18 Homotopy perturbation method

We consider the nonlinear differential equation

$$L(u) + N(u) = f(r), \qquad r \in \Omega, \tag{1.46}$$

with boundary conditions

$$B(u, \partial u \; / \; \partial n) = 0, \qquad r \in \Gamma, \tag{1.47}$$

where L is the linear operator, while N is nonlinear operator, B is the boundary operator, Γ is the boundary of the domain Ω and $f(r)$ is a known analytic function.

The He's homotopy pertubation technique defines the homotopy $v(r,p) : \Omega \times [0,1] \longrightarrow R$ which satisfies

$$H(v,p) = (1-p)[L(v) - L(u_0)] + p[L(v) + N(v) - f(r)] = 0, \tag{1.48}$$

or

$$H(v,p) = L(v) - L(u_0) + pL(u_0) + p[N(v) - f(r)] = 0, \tag{1.49}$$

where $r \in \Omega$ and $p \in [0,1]$ is an embedding parameter, u_0 is an initial approximation which satisfies the boundary conditions. Obviously, from Eqs. (1.48) and (1.49), we have

$$H(v,0) = L(v) - L(u_0) = 0. \tag{1.50}$$

$$H(v,1) = L(v) + N(v) - f(r) = 0. \tag{1.51}$$

The changing process of p from zero to unity is just that of $v(r,p)$ from u_0 to $u(r)$. In topology, this is called deformation, $L(v) - L(u_0)$ and $L(v) + N(v) - f(r)$ are homotopic. The basic assumption is that the solution of Eqs. (1.48) and (1.49) can be expressed as a power series in p:

$$v = v_0 + pv_1 + p^2 v_2 + \cdots, \tag{1.52}$$

The approximate solution of Eq. (1.46), therefore, can be readily obtained

$$u = \lim_{p \to 1} v = v_0 + v_1 + v_2 + \cdots \tag{1.53}$$

1.19 Homotopy analysis method (HAM)

Let us now give the basic idea of HAM for the same illustrative example given in Eq. (1.40).

1.19.1 Zeroth-order deformation equation

Let $g_0(t)$ denote the initial guess of $g(t)$, which satisfies the initial condition (1.41), i-e, [21 − 22]

$$g_0(0) = 0. \tag{1.54}$$

Let $q \in [0, 1]$ denote the so-called embedding parameter. The homotopy analysis method is based on a kind of continuous mapping $g(t) \to \Phi(t; q)$ such that, as the embedding parameter q increases from 0 to 1, $\Phi(t; q)$ varies from the initial guess $g_0\ (t)$ to the exact solution $g(t)$. To ensure this, choose such an auxiliary linear operator as

$$\mathcal{L}^*[\Phi(t; q)] = \gamma_1(t)\frac{\partial \Phi(t; q)}{\partial t} + \gamma_2(t)\Phi(t; q), \tag{1.55}$$

where $\gamma_1(t) \neq 0$ and $\gamma_2(t)$ are real functions to be determined later. From Eq. (1.40), we define the non-linear operator

$$\mathcal{N}^*[\Phi(t; q)] = \frac{\partial \Phi(t; q)}{\partial t} + \Phi^2(t; q) - 1. \tag{1.56}$$

Let $\hbar \neq 0$ and $\mathcal{H}^*(t) \neq 0$ denote the so-called auxiliary parameter and auxiliary function, respectively. Using the embedding parameter $q \in [0, 1]$, we construct a family of equations

$$(1 - q)\mathcal{L}^*[\Phi(t; q) - g_0(t)] = \hbar q \mathcal{H}^*(t)\mathcal{N}^*[\Phi(t; q)], \tag{1.57}$$

subject to the initial condition

$$\Phi(0; q) = 0. \tag{1.58}$$

It should be emphasized that we have great freedom to choose the auxiliary parameter $\hbar$, the auxiliary function $\mathcal{H}^*(t)$, the initial approximation $g_0(t)$, and the auxiliary linear operator $\mathcal{L}^*$. It is such freedom that plays important roles and establishes the cornerstone of the validity and flexibility of the homotopy analysis method.

When $q = 0$, Eq. (1.57) becomes

$$\mathcal{L}^*[\Phi(t;0) - g_0(t)] = 0, \qquad t \geq 0, \tag{1.59}$$

subject to the initial condition

$$\Phi(0;0) = 0. \tag{1.60}$$

According to the Eqs. (1.54) and (1.55), the solution of Eqs. (1.59) and (1.60) is simply

$$\Phi(t;0) = g_0(t). \tag{1.61}$$

When $q = 1$, Eq. (1.57) becomes

$$\hbar q \mathcal{H}^*(t)\mathcal{N}^*[\Phi(t;q)] = 0, \qquad t \geq 0, \tag{1.62}$$

subject to the initial condition

$$\Phi(0;1) = 0. \tag{1.63}$$

Since $\hbar \neq 0$, $\mathcal{H}^*(t) \neq 0$ and by means of the definition (1.56), Eqs. (1.62) and (1.63) are equivalent to the original Eqs. (1.40) and (1.41), provided

$$\Phi(t;1) = g(t). \tag{1.64}$$

Therefore, according to Eqs. (1.61) and (1.64), $\Phi(t;q)$ varies from initial guess $g_0(t)$ to the exact solution $g(t)$ as the embedding parameter q varies from 0 to 1. In topology, this kind of variation is called *deformation*, and Eqs. (1.57) and (1.58) are called the *zeroth-order deformation equations*.

Having the freedom to choose the auxiliary parameter $\hbar$, the auxiliary function $\mathcal{H}^*(t)$, the initial approximation $g_0(t)$, and the auxiliary linear operator $\mathcal{L}^*$, we can assume that all of them are properly chosen so that the solution $\Phi(t;q)$ of the zero-order deformation Eqs. (1.57) and (1.58) exists for $0 \le q \le 1$, and besides its mth-order derivative with respect to the embedding parameter q, i.e.,

$$g_0^{[m]}(t) = \left.\frac{\partial^m \Phi(t;q)}{\partial q^m}\right|_{q=0}, \tag{1.65}$$

exits, where $m = 1, 2, 3, \cdots$. For brevity, $g_0^{[m]}(t)$ is called the *mth-order deformation derivative.* Define

$$g_m(t) = \frac{g_0^{[m]}(t)}{m!} = \frac{1}{m!}\left.\frac{\partial^m \Phi(t;q)}{\partial q^m}\right|_{q=0}. \tag{1.66}$$

By Taylor's theorem, we expand $\Phi(t;q)$ in a power series of the embedding parameter q as follows

$$\Phi(t;q) = \Phi(t;0) + \sum_{m=1}^{+\infty} \frac{1}{m!}\frac{\partial^m \Phi(t;q)}{\partial q^m}|_{q=0}\, q^m. \tag{1.67}$$

From Eqs. (1.61) and (1.66), the above power series becomes

$$\Phi(t;q) = g_0(t) + \sum_{m=1}^{+\infty} g_m(t) q^m. \tag{1.68}$$

Assume that the auxiliary parameter $\hbar$, the auxiliary function $\mathcal{H}^*(t)$, the initial approximation $g_0(t)$, and the auxiliary linear operator $\mathcal{L}^*$ are so properly chosen that the series (1.68) converges at $q = 1$. Then, at $q = 1$, the series (1.68) becomes

$$\Phi(t;1) = g_0(t) + \sum_{m=1}^{+\infty} g_m(t). \tag{1.69}$$

Therefore, using Eq. (1.64), we have

$$g(t) = g_0(t) + \sum_{m=1}^{+\infty} g_m(t). \tag{1.70}$$

The above expression provides us with a relationship between the initial guess $g_0(t)$ and the exact solution $g(t)$ by means of the terms $g_m(t)$ $(m = 1, 2, 3, \cdots)$.

1.19.2 High-order deformation equation

Define the vector

$$\vec{g}_n = \{g_0(t), g_1(t), g_2(t), \cdots, g_n(t)\}. \tag{1.71}$$

According to the definition (1.66), the governing equation and corresponding initial condition of $g_m(t)$ can be deduced from the zero-order deformation Eqs. (1.57) and (1.58). Differentiating Eqs. (1.57) and (1.58) m times with respect to the embedding parameter q and then setting $q = 0$ and finally dividing them by $m!$, we have the so-called mth-order deformation equation

$$\mathcal{L}^*[g_m(t) - \chi_m g_{m-1}(t)] = \hbar \mathcal{H}^*(t) \mathcal{R}_m^*(\vec{g}_{m-1}), \tag{1.72}$$

subject to the initial condition

$$g_m(0) = 0, \tag{1.73}$$

where

$$\mathcal{R}_m^*(\vec{g}_{m-1}) = \frac{1}{(m-1)!} \frac{\partial \mathcal{N}^*[\Phi(t;q)]}{\partial q^m}\bigg|_{q=0}, \tag{1.74}$$

and

$$\chi_m = \begin{cases} 0 \text{ when } m \leq 1, \\ 1 \text{ otherwise.} \end{cases} \tag{1.75}$$

From Eqs. (1.56) and (1.54), we have

$$\mathcal{R}_m^*(\vec{g}_{m-1}) = \dot{g}_{m-1}(t) + \sum_{j=0}^{m-1} g_j(t) g_{m-1-j}(t) - (1 - \chi_m), \tag{1.76}$$

where $(\cdot)$ denotes the derivative w.r.t t. Notice that $\mathcal{R}_m^*(\vec{g}_{m-1})$ given by the above expression is only dependent upon

$$g_0(t),\ g_1(t),\ g_2(t), \cdots, g_{m-1}(t), \tag{1.77}$$

which are known, when solving the mth-order deformation Eqs. (1.72) and (1.73). Thus, according to the definition (1.55) of the auxiliary operator $\mathcal{L}^*$, Eq. (1.72) is a linear first-order differential equation, subject to the linear initial condition (1.73). Therefore, the solution $g_m(t)$ of high-order deformation Eqs. (1.72) and (1.73) can be easily gained, especially by means of computation software such as Mathematica, Maple, Matlab, and so on. According to (1.70), we in essence transfer the original nonlinear problem, governed by Eqs. (1.57) and (1.58), into an infinite number of linear sub-problems governed by high-order deformation Eqs. (1.72) and (1.73), and then use the sum of the solutions $g_m(t)$ of its first several sub-problems to approximate the next solution. Note that such a kind of transformation needs not the existence of any small or large parameters in governing equation and initial/boundary conditions.

The mth-order approximation of $g(t)$ is given by

$$g(t) \approx \sum_{n=0}^{m} g_n(t). \tag{1.78}$$

It should be noted that the zero-order deformation Eq. (1.57) is determined by the auxiliary linear operator $\mathcal{L}^*$, the initial approximation $g_0(t)$, the auxiliary parameter $\hbar$, and the auxiliary function $\mathcal{H}^*(t)$. Theoretically speaking, the solution $g(t)$ given by the above approach is dependent of the auxiliary linear operator $\mathcal{L}^*$, the initial approximation $g_0(t)$, the auxiliary parameter $\hbar$, and the auxiliary function $\mathcal{H}^*(t)$. Thus, unlike all previous analytic techniques, the convergence region and rate of solution series given by the above approach might not be uniquely determined.

1.19.3 Advantages of HAM

The homotopy analysis method provides us the following advantages

1. HAM provides us with great freedom to express solutions of a given non-linear problem by means of different base functions. Therefore, we can approximate a non-linear problem more efficiently by choosing a proper set of base functions.

2. HAM always provides us with a family of solution expressions in the auxiliary parameter $\hbar$. The convergence region and rate of each solution expression among the family might be determined by the auxiliary parameter $\hbar$. So, the auxiliary parameter $\hbar$ provides us with an additional way to conveniently adjust and control the convergence region and rate of solution series.

3. Homotopy analysis method is also independent of any small or large quantities.

1.19.4 Limitations

Homotopy analysis method requires that we should be able to represent the solution in a set of base functions called *the rule of solution expression*. Unfortunately, the rule of solution expression implies such an assumption that we should have, more or less, some knowledge about a given non-linear problem *a prior*. How can we get such kind of prior knowledge before we solve a problem that is completely new for us? How can we know that a set of base functions is better than others and is more efficient to approximate a non-linear problem which we know nothing? So, theoretically, this assumption impairs the homotopy analysis method.

Chapter 2

Homotopy perturbation method for thin film flow of a fourth grade fluid down a vertical cylinder

2.1 Introduction

In this chapter, the analytical solutions for an incompressible, fourth grade fluid falling on the outer surface of an infinitely long cylinder have been examined. The flow is in the form of thin film. This thin film is produced between a distance R and $R+\delta$. The governing equation is highly non-linear and has been solved by perturbation and homotopy perturbation method. It is shown from the chapter that the perturbation solution is a special case of homotopy perturbation solution. This chapter is due to Siddiqui et al [11]. The essential details missing in this paper are incorporated. The solution is also non-dimensionalized and graphical results are obtained for some physical parameter appearing in the governing equation.

2.2 Mathematical formulation

Let us consider an incompressible, steady fourth grade fluid lying on the outer surface of an infinitely long vertical cylinder. The flow is in the form of a thin, uniform axisymmetric film of thickness δ, in contact with the stationary air. We are considering cylindrical coordinates, thus we seek the velocity field of the form

$$\vec{v} = [0, 0, u(r)], \tag{2.1}$$

with the help of Eq. (2.1), Equation of continuity (1.12) is identically satisfied. For our covenience, the Eqs. (1.35) to (1.37) can be written as

$$\frac{\partial P}{\partial r} = (2\alpha_1 + \alpha_2)\frac{1}{r}\frac{\partial}{\partial r}\left[r(\frac{du}{dr})^2\right] + \frac{4}{r}(\gamma_3 + \gamma_4 + \gamma_5 + \frac{\gamma_6}{2})\frac{\partial}{\partial r}\left[r(\frac{du}{dr})^4\right], \tag{2.2}$$

$$\frac{\partial P}{\partial \theta} = 0, \tag{2.3}$$

$$\frac{\partial P}{\partial z} = \frac{\mu}{r}\frac{\partial}{\partial r}\left[r(\frac{du}{dr})\right] + \frac{2}{r}(\beta_2 + \beta_3)\frac{\partial}{\partial r}\left[r(\frac{du}{dr})^3\right]. \tag{2.4}$$

Eq. (2.3) shows that P is independent of θ. Elimination of P from Eqs. (2.2) and (2.4), gives

$$r\frac{d^2u}{dr^2} + \frac{du}{dr} + 2(\frac{\beta_{2+}\beta_3}{\mu})\left[3r(\frac{du}{dr})^2(\frac{d^2u}{dr^2}) + (\frac{du}{dr})^3\right] + \frac{\rho g}{\mu}r = 0, \tag{2.5}$$

where g is the constant of gravity.

The corresponding boundary conditions for the thin film flow are

$$u(r) = 0, \quad \text{at } r = R, \tag{2.6}$$

$$\frac{du}{dr} = 0, \quad \text{at } r = R + \delta, \tag{2.7}$$

the first being no slip condition at $r = R$ and the second coming from τ_{rz}.

Eq. (2.5), subject to boundary conditions (2.6) and (2.7), can be solved by the following methods.

1. Perturbation method
2. Homotopy perturbation method

2.3 Perturbation method

Let us consider $\epsilon = \frac{(\beta_2+\beta_3)}{\mu}$ be a small parameter in Eq. (2.5), then we assume that u can be expressed in powers of ϵ as follows

$$u(r,\epsilon) = u_0(r) + \epsilon u_1(r) + \epsilon^2 u_2(r) + \cdots \tag{2.8}$$

Substitution Eq. (2.8) into Eqs. (2.5) to (2.7) and equating the coefficients of like powers of ϵ, we get the following systems

2.3.1 Zeroth-order system

$$r\frac{d^2u_0}{dr^2} + \frac{du_0}{dr} + \frac{\rho g}{\mu}r = 0, \tag{2.9}$$

$$u_0(r) = 0 \text{ at } r = R, \tag{2.10}$$

$$\frac{du_0}{dr} = 0 \text{ at } r = R + \delta. \tag{2.11}$$

2.3.2 First-order system

$$r\frac{d^2u_1}{dr^2} + \frac{du_1}{dr} + 6r(\frac{du_0}{dr})^2(\frac{d^2u_0}{dr^2}) + 2(\frac{du_0}{dr})^3 = 0, \tag{2.12}$$

$$u_1(r) = 0 \text{ at } r = R, \tag{2.13}$$

$$\frac{du_1}{dr} = 0 \text{ at } r = R + \delta. \tag{2.14}$$

2.3.3 Second-order system

$$r\frac{d^2u_2}{dr^2} + \frac{du_2}{dr} + 6r(\frac{du_0}{dr})^2(\frac{d^2u_1}{dr^2}) + 12r(\frac{du_0}{dr})(\frac{d^2u_0}{dr^2})(\frac{du_1}{dr}) + 6(\frac{du_0}{dr})^2(\frac{du_1}{dr}) = 0, \tag{2.15}$$

$$u_2(r) = 0 \text{ at } r = R, \tag{2.16}$$

$$\frac{du_2}{dr} = 0 \text{ at } r = R + \delta. \tag{2.17}$$

2.3.4 Zeroth-order solution

Eq. (2.9) is non-homogeneous second order Cauchy-Euler differential equation. Its solution can be easily written as

$$u_0(r) = A_1 + B_1 \ln r - \frac{\rho g}{4\mu} r^2, \tag{2.18}$$

where A_1 and B_1 are constants.

Differentiation of Eq. (2.18) gives

$$\frac{du_0}{dr} = \frac{B_1}{r} - \frac{\rho g}{2\mu} r,$$

with the help of boundary conditions (2.10) and (2.11), we get

$$A_1 = \frac{\rho g}{4\mu}\left[R^2 - 2(R+\delta)^2 \ln(R)\right],$$

$$B_1 = \frac{\rho g}{2\mu}(R+\delta)^2.$$

Substituting the values of A_1 and B_1 in solution (2.18), we obtain

$$u_0(r) = \frac{\rho g}{4\mu}\left[R^2 - r^2 + 2(R+\delta)^2 \ln\left(\frac{r}{R}\right)\right]. \tag{2.19}$$

which is essentially the Newtonian solution by [33].

2.3.5 First-order solution

With the help of Eq. (2.19), Eq. (2.12) can be written as

$$r^2 \frac{d^2 u_1}{dr^2} + r\frac{du_1}{dr} = \frac{3\rho^3 g^3}{4\mu^3}\left[\frac{2(R+\delta)^6}{3r^2} - 2r^2(R+\delta)^2 + \frac{4}{3}r^4\right]. \tag{2.20}$$

The solution of the above non-homogeneous equation can be expressed as

$$u_1(r) = A_2 + B_2 \ln r + \frac{3\rho^3 g^3}{4\mu^3}\left[\frac{(R+\delta)^6}{6r^2} - \frac{r^2(R+\delta)^2}{2} + \frac{1}{12}r^4\right], \tag{2.21}$$

where A_2 and B_2 are constants.

The differentiation of the above gives

$$\frac{du_1}{dr} = \frac{B_2}{r} + \frac{3\rho^3 g^3}{4\mu^3}\left[-\frac{(R+\delta)^6}{3r^3} - r(R+\delta)^2 + \frac{1}{3}r^3\right],$$

using the boundary conditions (2.13) and (2.14) into Eq. (2.21), we get

$$A_2 = -\frac{3\rho^3 g^3}{4\mu^3}(R+\delta)^4 \ln R - \frac{3\rho^3 g^3}{4\mu^3}\left[\frac{(R+\delta)^6}{6R^2} - \frac{R^2(R+\delta)^2}{2} + \frac{1}{12}R^4\right], \tag{2.22}$$

$$B_2 = \frac{3\rho^3 g^3}{4\mu^3}(R+\delta)^4. \tag{2.23}$$

Substituting the values of A_2 and B_2 into Eq. (2.21), we get

$$\begin{aligned} u_1(r) &= \frac{3\rho^3 g^3}{4\mu^3}\left[\frac{(R+\delta)^2(R^2-r^2)}{2} + \frac{r^4-R^4}{12} + \frac{(R+\delta)^6}{6}\left(\frac{1}{r^2}-\frac{1}{R^2}\right)\right. \\ &\left. +(R+\delta)^4 \ln\left(\frac{r}{R}\right)\right]. \end{aligned} \tag{2.24}$$

2.3.6 Second-order solution

To avoid the repetition the solution of Eq. (2.15) satisfying the boundary conditions (2.16) and (2.17), yields

$$\begin{aligned} u_2(r) &= \frac{\rho^5 g^5}{8\mu^5}\left[\frac{3(R+\delta)^{10}}{4}\left(\frac{1}{R^4}-\frac{1}{r^4}\right) + \frac{15(R+\delta)^8}{2}\left(\frac{1}{r^2}-\frac{1}{R^2}\right) + 15(R+\delta)^4(R^2-r^2)\right. \\ &\left. +\frac{15}{4}(R+\delta)^2(r^4-R^4) + \frac{1}{2}(R^6-r^6) + 30(R+\delta)^6 \ln\left(\frac{r}{R}\right)\right]. \end{aligned} \tag{2.25}$$

Finally, with the help of Eqs. (2.19), (2.24) and (2.25), the solution (2.8) can be written as

$$\begin{aligned} u(r) &= \frac{\rho g}{4\mu}\left[R^2 - r^2 + 2(R+\delta)^2 \ln\left(\frac{r}{R}\right)\right] \\ &+\epsilon\frac{3\rho^3 g^3}{4\mu^3}\left[\frac{(R+\delta)^2(R^2-r^2)}{2} + \frac{r^4-R^4}{12} + \frac{(R+\delta)^6}{6}\left(\frac{1}{r^2}-\frac{1}{R^2}\right) + (R+\delta)^4 \ln\left(\frac{r}{R}\right)\right] \\ &+\epsilon^2\frac{\rho^5 g^5}{8\mu^5}\left[\frac{3(R+\delta)^{10}}{4}\left(\frac{1}{R^4}-\frac{1}{r^4}\right) + \frac{15(R+\delta)^8}{2}\left(\frac{1}{r^2}-\frac{1}{R^2}\right) + \frac{1}{2}(R^6-r^6)\right. \\ &\left. +15(R+\delta)^4(R^2-r^2) + \frac{15}{4}(R+\delta)^2(r^4-R^4) + 30(R+\delta)^6 \ln\left(\frac{r}{R}\right)\right]. \end{aligned} \tag{2.26}$$

If $\epsilon = 0$, the above solution reduces to the Newtonian fluid [33].

2.4 Homotopy perturbation method

The homotopy perturbation method for Eq. (2.5), can be defined as by [31 − 32]

$$H(v,q) = (1-q)[L(v) - L(u_0)]$$

$$+q\left[L(v) + 2\left(\frac{\beta_1+\beta_2}{\mu}\right)\frac{d}{dr}\left\{r\left(\frac{dv}{dr}\right)^3\right\} + \frac{\rho g}{\mu}r\right] = 0, \tag{2.27}$$

or

$$H(v,q) = L(v) - L(u_0) + qL(u_0) + q\left[2\left(\frac{\beta_1+\beta_2}{\mu}\right)\frac{d}{dr}\left\{r\left(\frac{dv}{dr}\right)^3\right\} + \frac{\rho g}{\mu}r\right] = 0\,. \tag{2.28}$$

For our convenience we have taken $L = r\frac{d^2}{dr^2} + \frac{d}{dr}$ as the linear operator. We can define the initial guess as

$$u_0(r) = \frac{\rho g}{4\mu}\left[R^2 - r^2 + 2(R+\delta)^2\ln\left(\frac{r}{R}\right)\right]. \tag{2.29}$$

Let us define

$$v(r,q) = v_0 + qv_1 + q^2v_2 + \cdots. \tag{2.30}$$

Substituting Eq. (2.30) into Eq. (2.28) and then comparing like powers of q, one obtains the following problems with the corresponding boundary conditions

2.4.1 Zeroth-order problem

$$L(v_0) - L(u_0) = 0, \tag{2.31}$$

$$v_0(r) = 0 \text{ at } r = R, \tag{2.32}$$

$$\frac{dv_0}{dr} = 0 \text{ at } r = R+\delta. \tag{2.33}$$

2.4.2 First-order problem

$$L(v_1) + L(u_0) + 2\left(\frac{\beta_2+\beta_3}{\mu}\right)\frac{d}{dr}\left[r\left(\frac{dv_0}{dr}\right)^3\right] + \frac{\rho g}{\mu}r = 0, \tag{2.34}$$

$$v_1(r) = 0 \text{ at } r = R, \tag{2.35}$$

$$\frac{dv_1}{dr} = 0 \text{ at } r = R + \delta. \tag{2.36}$$

2.4.3 Second-order problem

$$L(\nu_2) + \left(\frac{\beta_2 + \beta_3}{\mu}\right) \frac{d}{dr}\left[6r\left(\frac{d\nu_0}{dr}\right)^2 \frac{d\nu_1}{dr}\right] = 0, \tag{2.37}$$

$$v_2(r) = 0 \text{ at } r = R, \tag{2.38}$$

$$\frac{dv_2}{dr} = 0 \text{ at } r = R + \delta. \tag{2.39}$$

For the solution of above written problems, we follow the same procedure as discussed in the previous section, we can write the solution to these problems as

2.4.4 Zeroth-order solution

The solution of Eq. (2.31), satisfying the boundary conditions (2.32) and (2.33) is

$$v_0(r) = \frac{\rho g}{4\mu}\left[R^2 - r^2 + 2(R+\delta)^2 \ln\left(\frac{r}{R}\right)\right]. \tag{2.40}$$

2.4.5 First-order solution

Using Eq. (2.40) into Eq. (2.34), the solution of Eq. (2.34) is

$$\begin{aligned} v_1(r) &= \frac{3\rho^3 g^3}{4\mu^3}\left(\frac{\beta_2+\beta_3}{\mu}\right)\left[\frac{(R+\delta)^2(R^2-r^2)}{2} + \frac{r^4 - R^4}{12} + \frac{(R+\delta)^6}{6}\left(\frac{1}{r^2} - \frac{1}{R^2}\right)\right. \\ &\left. + (R+\delta)^4 \ln\left(\frac{r}{R}\right)\right]. \end{aligned} \tag{2.41}$$

2.4.6 Second-order solution

To avoid the repetition the solution of Eq. (2.37), satisfying the boundary conditions (2.38) and (2.39), yields

$$\begin{aligned} v_2(r) &= \frac{\rho^5 g^5}{8\mu^5}\left(\frac{\beta_2+\beta_3}{\mu}\right)^2\left[\frac{3(R+\delta)^{10}}{4}\left(\frac{1}{R^4} - \frac{1}{r^4}\right) + \frac{15(R+\delta)^8}{2}\left(\frac{1}{r^2} - \frac{1}{R^2}\right) + \frac{1}{2}(R^6 - r^6)\right. \\ &\left. + 15(R+\delta)^4(R^2 - r^2) + \frac{15}{4}(R+\delta)^2(r^4 - R^4) + 30(R+\delta)^6 \ln\left(\frac{r}{R}\right)\right]. \end{aligned} \tag{2.42}$$

By using the property of Homotopy perturbation method the original solution can be obtained by using

$$u(r) = \lim_{q \to 1}(v_0 + qv_1 + q^2 v_2 + \cdots),$$

which is equivalent to

$$u(r) = v_0 + v_1 + v_2 + \cdots. \tag{2.43}$$

Finally,with the help of Eqs. (2.40) to (2.42), Eq. (2.43) can be written as

$$\begin{aligned} u(r) &= \frac{\rho g}{4\mu}\left[R^2 - r^2 + 2(R+\delta)^2 \ln\left(\frac{r}{R}\right)\right] \\ &+ \frac{3\rho^3 g^3}{4\mu^3}\left(\frac{\beta_2+\beta_3}{\mu}\right)\left[\frac{(R+\delta)^2(R^2-r^2)}{2} + \frac{r^4-R^4}{12} + \frac{(R+\delta)^6}{6}\left(\frac{1}{r^2} - \frac{1}{R^2}\right)\right. \\ &\left. + (R+\delta)^4 \ln\left(\frac{r}{R}\right)\right] + \frac{\rho^5 g^5}{8\mu^5}\left(\frac{\beta_2+\beta_3}{\mu}\right)^2\left[\frac{3(R+\delta)^{10}}{4}\left(\frac{1}{R^4} - \frac{1}{r^4}\right)\right. \\ &+ \frac{15(R+\delta)^8}{2}\left(\frac{1}{r^2} - \frac{1}{R^2}\right) + \frac{1}{2}(R^6 - r^6) + 15(R+\delta)^4(R^2-r^2) \\ &\left. + \frac{15}{4}(R+\delta)^2(r^4 - R^4) + 30(R+\delta)^6 \ln\left(\frac{r}{R}\right)\right]. \end{aligned} \tag{2.44}$$

Introducing the non-dimensional variables

$$\eta = \frac{r}{R}, \quad f = \frac{R}{\nu}u, \quad K = \frac{gR^3}{\nu^2}, \quad \beta = \frac{\mu(\beta_2+\beta_3)}{R^4 \rho^2}, \quad d = (1 + \frac{\delta}{R}). \tag{2.45}$$

The above solution in non-dimensional form can be written as

$$\begin{aligned} u(\eta) &= v_0(\eta) + v_1(\eta) + v_2(\eta) \\ &= \frac{K}{4}\left[1 - r^2 + 2d^2 \ln(\eta)\right] + \frac{3K^3\beta}{4}\left[\frac{d^2(1-\eta^2)}{2} + \frac{\eta^4 - 1}{12} + \frac{d^6}{6}\left(\frac{1}{\eta^2} - 1\right) + d^4 \ln(\eta)\right] \\ &+ \frac{K^5\beta}{8}\left[\frac{3d^{10}}{4}\left(1 - \frac{1}{\eta^4}\right) + \frac{15d^8}{2}\left(\frac{1}{\eta^2} - 1\right) + 15d^4(1 - r^2) + \frac{15}{4}d^2(\eta^4 - 1)\right. \\ &\left. + \frac{1}{2}(1 - \eta^6) + 30d^6 \ln(\eta)\right]. \end{aligned} \tag{2.46}$$

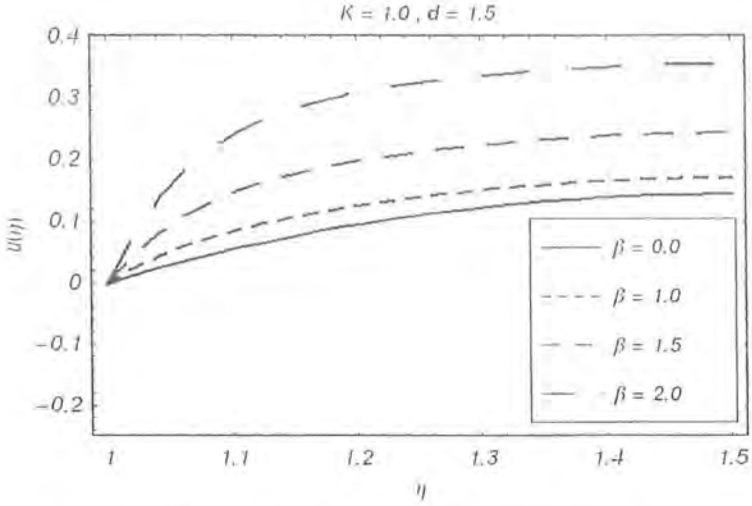


Fig.(6): *velocity – curves* are drawn for different values of fluid prameter β.

2.4.7 Flow rate and average film velocity

To determine the volume flow rate Q, we use the following formula

$$Q = \int_0^{2\pi} \int_R^{R+\delta} r u(r) dr d\theta \tag{2.47}$$

By making use of Eq. (2.44) in the above formula, we obtain

$$\begin{aligned} Q &= \frac{\pi R^4 S}{8}\left[4M^4 \ln M - \frac{\delta}{R}(1+M)\left(3M^2-1\right)\right] + \frac{3\pi S^3}{2}\left(\frac{\beta_2+\beta_3}{\mu}\right)\left[+\frac{R^6}{2}M^6(\ln M)\left(\frac{4}{3}-\frac{R^2}{3}\right)\right. \\ &\left. +R^6\left(\frac{M^4}{2}-\frac{M^2}{6}-\frac{13M^6}{36}+\frac{1}{36}\right)\right] + \frac{\pi S^5}{4}\left(\frac{\beta_2+\beta_3}{\mu}\right)^2\left[\frac{15R^4M^4}{2}+\frac{3R^8M^2}{2}-\frac{3R^8}{8}\right. \\ &\left.+\frac{3R^8M^{12}}{8}-\frac{9R^8M^{10}}{2}+\frac{15R^8M^8}{16}+\frac{15R^4M^6}{2}+\frac{15R^8M^4}{8}\right] + \frac{45\pi}{8}\left(\frac{\beta_2+\beta_3}{\mu}\right)^2 S^5R^8M^8\ln M \\ &+\frac{15\pi}{4}\left(\frac{\beta_2+\beta_3}{\mu}\right)^2 S^5R^6M^6\left(\frac{R^2}{4}-\frac{R^2M^2}{4}\right), \end{aligned} \tag{2.48}$$

where $S = \frac{\rho g}{\mu}$, $M = 1 + \frac{\delta}{R}$.

It should be noted that if $(\beta_2 + \beta_3) = 0$, the solution is identical to the Newtonian fluid.

The average film velocity $\bar{V}$ is defined by

$$\bar{V} = \frac{Q}{\pi[(R+\delta)^2 - R^2]}, \tag{2.49}$$

where Q is given by Eq. (2.48).

2.5 Conclusion

In this chapter, we present the solution by two methods namely the perturbation method and homotopy perturbation method proposed by Prof. He. [31 − 32]. The traditional perturbation method assumes the parameter $\epsilon = \left(\frac{\beta_2+\beta_3}{\mu}\right)$ as a small parameter where as homotopy perturbation method uses the embedding parameter q as a small parameter. However, the two solutions were in complete agreement. Comparison of the results shows that the homotopy perturbation method can completely overcome the limitations arising in the traditional perturbation methods.

Chapter 3

Analytic solution for a thin film flow of a fourth grade fluid down a vertical cylinder with partial slip

3.1 Introduction

In this chapter an analytic solution is developed for an incompressible, thin film flow of a fourth grade fluid down a vertical cylinder. The thin film is produced on the outer side of a vertical cylinder between radius R and $R+\delta$. At $r = R$ the disturbance is due to partial slip whereas at $r = R + \delta$ the surface is being free. The solution of the governing problem has been solved by perturbation method, homotopy perturbation method and homotopy analysis method. Several known results of interest are found to follow as a special case of the solution of the problem examined. The graphical results are shown for various physical quantities.

3.2 Mathematical formulation

The physical situation considered here is of an incompressible fourth grade fluid lying on the outer surface of an infinitely long vertical cylinder. The flow is in the form of thin film having thickness δ on the outer side of vertical cylinder having radius R from the origin.

The fluid motion in the film arises due to gravity and partial slip at $r = R$. The velocity field in the thin film is governed by [11].

With partial slip condition Eq. (2.5 − 2.7) become

$$r\frac{d^2u}{dr^2} + \frac{du}{dr} + 2\left(\frac{\beta_2+\beta_3}{\mu}\right)\left[3r(\frac{du}{dr})^2(\frac{d^2u}{dr^2}) + (\frac{du}{dr})^3\right] + \frac{\rho g}{\mu}r = 0, \tag{3.1}$$

where g is the constant of gravity.

The corresponding boundary conditions are

$$u(r) = \lambda\left[\frac{du}{dr} + 2\left(\frac{\beta_2+\beta_3}{\mu}\right)\left(\frac{du}{dr}\right)^3\right] \text{ at } r = R, \tag{3.2}$$

where λ is the slip length.

$$\frac{du}{dr} = 0, \quad \text{at } r = R + \delta, \tag{3.3}$$

the first being partial slip condition at $r = R$ and the second one again coming from τ_{rz}.

Eq. (3.1), subject to boundary conditions (3.2) and (3.3), can be solved by the following methods.

1. Perturbation method.
2. Homotopy perturbation method.
3. Homotopy analysis method.

3.3 Perturbation method

Let us consider $\epsilon = \frac{(\beta_2+\beta_3)}{\mu}$ be a small parameter in Eq. (3.1), then we assume that u can be expressed in powers of ϵ as follows

$$u(r,\epsilon) = u_0(r) + \epsilon u_1(r) + \epsilon^2 u_2(r) + \cdots. \tag{3.4}$$

Substitution of Eq. (3.4) into Eqs. (3.1) to (3.3) and equating the coefficients of like powers of ϵ, we get the following problems.

3.3.1 Zeroth-order problem

$$r\frac{d^2u_0}{dr^2} + \frac{du_0}{dr} + \frac{\rho g}{\mu}r = 0, \tag{3.5}$$

$$u_0(r) = \lambda \left[\frac{du_0}{dr}\right] \text{ at } r = R, \tag{3.6}$$

$$\frac{du_0}{dr} = 0 \text{ at } r = R + \delta. \tag{3.7}$$

3.3.2 First-order problem

$$r\frac{d^2u_1}{dr^2} + \frac{du_1}{dr} + 6r(\frac{du_0}{dr})^2(\frac{d^2u_0}{dr^2}) + 2(\frac{du_0}{dr})^3 = 0, \tag{3.8}$$

$$u_1(r) = \lambda \left[\frac{du_1}{dr} + 2\left(\frac{du_0}{dr}\right)^3\right] \text{ at } r = R, \tag{3.9}$$

$$\frac{du_1}{dr} = 0 \text{ at } r = R + \delta. \tag{3.10}$$

3.3.3 Second-order problem

$$r\frac{d^2u_2}{dr^2} + \frac{du_2}{dr} + 6r(\frac{du_0}{dr})^2(\frac{d^2u_1}{dr^2}) + 12r(\frac{du_0}{dr})(\frac{d^2u_0}{dr^2})(\frac{du_1}{dr}) + 6(\frac{du_0}{dr})^2(\frac{du_1}{dr}) = 0, \tag{3.11}$$

$$u_2(r) = \lambda \left[\frac{du_2}{dr} + 6\left(\frac{du_0}{dr}\right)^2\left(\frac{du_1}{dr}\right)\right] \text{ at } r = R, \tag{3.12}$$

$$\frac{du_2}{dr} = 0 \text{ at } r = R + \delta. \tag{3.13}$$

3.3.4 Zeroth-order solution

Eq. (3.5) is non-homogeneous second order Cauchy-Euler differential equation. The solution of Eq. (3.5) is

$$u_0(r) = A_3 + B_3 \ln r - \frac{\rho g}{4\mu} r^2, \tag{3.14}$$

where A_3 and B_3 are constants.

With the help of boundary conditions (3.6) and (3.7), the solution (3.14) can be written as

$$u_0(r) = \frac{\rho g}{4\mu}\left[R^2 - r^2 + \frac{2\lambda(R+\delta)^2}{R} - 2\lambda R + 2(R+\delta)^2 \ln\frac{r}{R}\right]. \tag{3.15}$$

The solution of no-slip can be easily recovered by taking $\lambda = 0$.

3.3.5 First-order solution

Using Eq. (3.15), Eq. (3.8) can be written as

$$r^2\frac{d^2u_1}{dr^2}+r\frac{du_1}{dr}=\frac{3\rho^3g^3}{4\mu^3}\left[\frac{2(R+\delta)^6}{3r^2}-2r^2(R+\delta)^2+\frac{4}{3}r^4\right], \tag{3.16}$$

The solution of the above non-homogeneous equation satisfying the boundary conditions (3.9) and (3.10) is

$$\begin{aligned} u_1(r) &= \frac{3\rho^3g^3}{4\mu^3}\left[\frac{(R+\delta)^2(R^2-r^2)}{2}+\frac{r^4-R^4}{12}+\frac{(R+\delta)^6}{6}\left(\frac{1}{r^2}-\frac{1}{R^2}\right)\right. \\ &\quad \left.+(R+\delta)^4\ln\left(\frac{r}{R}\right)\right]. \end{aligned} \tag{3.17}$$

3.3.6 Second-order solution

With the help of Eqs. (3.15) and (3.17), the solution of Eq. (3.11) satisfying the boundary conditions (3.12) and (3.13) can be written as

$$\begin{aligned} u_2(r) &= \frac{\rho^5g^5}{8\mu^5}\left[\frac{3(R+\delta)^{10}}{4}\left(\frac{1}{R^4}-\frac{1}{r^4}\right)+\frac{15(R+\delta)^8}{2}\left(\frac{1}{r^2}-\frac{1}{R^2}\right)+\frac{1}{2}(R^6-r^6)\right. \\ &\quad \left.+15(R+\delta)^4(R^2-r^2)+\frac{15}{4}(R+\delta)^2(r^4-R^4)+30(R+\delta)^6\ln\left(\frac{r}{R}\right)\right]. \end{aligned} \tag{3.18}$$

Using Eqs. (3.15),(3.17) and (3.18) into Eq. (3.4), we obtain

$$\begin{aligned} u(r) &= \frac{\rho g}{4\mu}\left[R^2-r^2+\frac{2M(R+\delta)^2}{R}+2MR+2(R+\delta)^2\ln\frac{r}{R}\right] \\ &\quad +\epsilon\frac{3\rho^3g^3}{4\mu^3}\left[\frac{(R+\delta)^2(R^2-r^2)}{2}+\frac{r^4-R^4}{12}+\frac{(R+\delta)^6}{6}\left(\frac{1}{r^2}-\frac{1}{R^2}\right)+(R+\delta)^4\ln\left(\frac{r}{R}\right)\right] \\ &\quad +\epsilon^2\frac{\rho^5g^5}{8\mu^5}\left[\frac{3(R+\delta)^{10}}{4}\left(\frac{1}{R^4}-\frac{1}{r^4}\right)+\frac{15(R+\delta)^8}{2}\left(\frac{1}{r^2}-\frac{1}{R^2}\right)+\frac{1}{2}(R^6-r^6)\right. \\ &\quad \left.+15(R+\delta)^4(R^2-r^2)+\frac{15}{4}(R+\delta)^2(r^4-R^4)+30(R+\delta)^6\ln\left(\frac{r}{R}\right)\right]. \end{aligned} \tag{3.19}$$

If $\lambda=0$, the solution for no-slip can be recovered and for $\lambda=\varepsilon=0$, we obtain the Newtonian solution [33].

3.4 Homotopy perturbation method

The homotopy perturbation method for Eq. (3.1), defines

$$H(v,q)=(1-q)\left[L(v)-L(u_0)\right]+q\left[L(v)+2\left(\frac{\beta_1+\beta_2}{\mu}\right)\frac{d}{dr}\left\{r\left(\frac{dv}{dr}\right)^3\right\}+\frac{\rho g}{\mu}r\right]=0, \quad (3.20)$$

or

$$H(v,q)=L(v)-L(u_0)+qL(u_0)+q\left[2\left(\frac{\beta_1+\beta_2}{\mu}\right)\frac{d}{dr}\left\{r\left(\frac{dv}{dr}\right)^3\right\}+\frac{\rho g}{\mu}r\right]=0. \quad (3.21)$$

For our convenience we have taken $L=r\frac{d^2}{dr^2}+\frac{d}{dr}$ as the linear operator and

$$u_0(r)=\frac{\rho g}{4\mu}\left[R^2-r^2+\frac{2\lambda(R+\delta)^2}{R}-2\lambda R+2(R+\delta)^2\ln\frac{r}{R}\right], \quad (3.22)$$

as the initial guess.

Let us define

$$v(r,q)=v_0+qv_1+q^2v_2+\cdots. \quad (3.23)$$

Substituting Eq. (3.23) into Eq. (3.21) and then collecting the like powers of q, we get the following problems

3.4.1 Zeroth-order problem

$$L(\nu_0)-L(u_0)=0, \quad (3.24)$$

$$v_0(r)=\lambda\left[\frac{dv_0}{dr}\right] \text{ at } r=R, \quad (3.25)$$

$$\frac{dv_0}{dr}=0 \text{ at } r=R+\delta. \quad (3.26)$$

3.4.2 First-order problem

$$L(\nu_1)+L(u_0)+2\left(\frac{\beta_2+\beta_3}{\mu}\right)\frac{d}{dr}\left[r\left(\frac{d\nu_0}{dr}\right)^3\right]+\frac{\rho g}{\mu}r=0, \quad (3.27)$$

$$v_1(r)=\lambda\left[\frac{dv_1}{dr}+2\left(\frac{\beta_2+\beta_3}{\mu}\right)\left(\frac{dv_0}{dr}\right)^3\right] \text{ at } r=R\ , \quad (3.28)$$

$$\frac{dv_1}{dr} = 0 \text{ at } r = R + \delta. \tag{3.29}$$

3.4.3 Second-order problem

$$L(\nu_2) + \left(\frac{\beta_2 + \beta_3}{\mu}\right)\frac{d}{dr}\left[6r\left(\frac{d\nu_0}{dr}\right)^2\frac{d\nu_1}{dr}\right] = 0, \tag{3.30}$$

$$v_2(r) = \lambda\left[\frac{dv_2}{dr} + 6\left(\frac{\beta_2 + \beta_3}{\mu}\right)\left(\frac{dv_0}{dr}\right)^2\left(\frac{dv_1}{dr}\right)\right] \text{ at } r = R, \tag{3.31}$$

$$\frac{dv_2}{dr} = 0 \text{ at } r = R + \delta. \tag{3.32}$$

3.4.4 Zeroth-order solution

The solution of Eq. (3.24) satisfying the boundary conditions (3.25) and (3.26) is

$$v_0(r) = \frac{\rho g}{4\mu}\left[R^2 - r^2 + \frac{2\lambda(R+\delta)^2}{R} - 2\lambda R + 2(R+\delta)^2 \ln\frac{r}{R}\right]. \tag{3.33}$$

3.4.5 First-order solution

The solution of Eq. (3.27) satisfying the boundary conditions (3.28) and (3.29) can be written as

$$v_1(R) = A_4 + B_4 \ln R + \frac{3\rho^3 g^3}{4\mu^3}\left(\frac{\beta_2 + \beta_3}{\mu}\right)\left[\frac{(R+\delta)^6}{6r^2} - \frac{r^2(R+\delta)^2}{2} + \frac{1}{12}r^4\right], \tag{3.34}$$

where A_4 and B_4 are constants and are given as

$$A_4 = \frac{3\rho^3 g^3}{4\mu^3}\left(\frac{\beta_2 + \beta_3}{\mu}\right)\left[-\frac{(R+\delta)^6}{6R^2} + \frac{R^2(R+\delta)^2}{2} - \frac{1}{12}R^4 - \frac{\lambda(R+\delta)^4}{R} - (R+\delta)^4 \ln R\right], \tag{3.35}$$

$$B_4 = \frac{3\rho^3 g^3}{4\mu^3}\left(\frac{\beta_2 + \beta_3}{\mu}\right)(R+\delta)^4. \tag{3.36}$$

3.4.6 Second-order solution

To avoid the repetition the solution of Eq. (3.30) satisfying the boundary conditions Eq. (3.31) and Eq. (3.32), yields

$$\begin{aligned} v_2(r) &= A_5 + B_5 \ln r + \frac{\rho^5 g^5}{16\mu^5}\left(\frac{\beta_2+\beta_3}{\mu}\right)^2 \left[-\frac{3(R+\delta)^{10}}{2r^4} + \frac{15(R+\delta)^8}{r^2}\right. \\ &\left. -30r^2(R+\delta)^4 + \frac{15}{2}r^4(R+\delta)^2 - r^6\right], \end{aligned} \tag{3.37}$$

where A_5 and B_5 are constants and are given as

$$\begin{aligned} A_5 &= \frac{\rho^5 g^5}{16\mu^5}\left(\frac{\beta_2+\beta_3}{\mu}\right)^2 \left[\frac{3(R+\delta)^{10}}{2R^4} - \frac{15(R+\delta)^8}{R^2} + 30R^2(R+\delta)^4\right. \\ &\left. \frac{15}{2}R^4(R+\delta)^2 + R^6 - 60(R+\delta)^6 \ln R\right], \end{aligned} \tag{3.38}$$

$$B_5 = \frac{\rho^5 g^5}{8\mu^5}\left(\frac{\beta_2+\beta_3}{\mu}\right)^2 [30(R+\delta)^6]. \tag{3.39}$$

Finally, we write the solution as

$$u(r) = \lim_{q\to 1}(v_0 + qv_1 + q^2 v_2 + \cdots), \tag{3.40}$$

which is equivalent to

$$u(r) = v_0 + v_1 + v_2 + \cdots. \tag{3.41}$$

Where v_0, v_1 and v_2 are defined in Eqs. (3.33), (3.34) and (3.37) respectively.

3.5 Mathematical analysis for HAM solution

Introducing the non-dimensional variables

$$\eta = \frac{r}{R}, \quad f = \frac{R}{\nu}u, \quad K = \frac{gR^3}{\nu^2}, \quad \beta = \frac{\mu(\beta_2+\beta_3)}{R^4\rho^2}, \quad N = \frac{\lambda}{R}. \tag{3.42}$$

With the help of Eq. (3.42), Eqs. (3.1) to (3.3) take the following form

$$\eta \frac{d^2 f}{d\eta^2} + \frac{df}{d\eta} + 2\beta \left[3\eta \left(\frac{df}{d\eta}\right)^2 \left(\frac{d^2 f}{d\eta^2}\right) + \left(\frac{df}{d\eta}\right)^3 \right] + K\eta = 0, \tag{3.43}$$

$$f(\eta) = N \left[\frac{df}{d\eta} + 2\beta \left(\frac{df}{d\eta}\right)^3 \right] \text{ at } \eta = 1, \text{ and } \frac{df}{d\eta} = 0 \text{ at } \eta = d, \tag{3.44}$$

in which $d = (1 + \frac{\delta}{R})$.

Integrating Eq. (3.43) with respect to η and using Eq. (3.44), we obtain

$$f(1) = \frac{NK}{2}(d^2 - 1), \text{ and } \frac{df}{d\eta} = 0 \text{ at } r = d. \tag{3.45}$$

From Eq. (3.44), it is straight forward to choose the initial guess

$$f_0(\eta) = \frac{K}{4}[1 - \eta^2 + 2N(d^2 - 1) + 2d^2 \ln \eta], \tag{3.46}$$

and the auxiliary linear operator

$$\mathcal{L}(f) = \eta f'' + f', \tag{3.47}$$

The so-called *zeroth-order deformation problem* can be defined as

$$(1 - q^*)\mathcal{L}[\hat{f}(\eta, q^*) - f_0(\eta)] = q^* \hbar \mathcal{N}[\hat{f}(\eta, q^*)], \tag{3.48}$$

where

$$\mathcal{N}[\hat{f}(\eta, q^*)] = \eta \frac{\partial^2 \hat{f}(\eta, q^*)}{\partial \eta^2} + \frac{\partial \hat{f}(\eta, q^*)}{\partial \eta} + K\eta$$

$$+2\beta \left[3\eta \left(\frac{\partial \hat{f}(\eta, q^*)}{\partial \eta}\right)^2 \left(\frac{\partial^2 \hat{f}(\eta, q^*)}{\partial \eta^2}\right) + \left(\frac{\partial \hat{f}(\eta, q^*)}{\partial \eta}\right)^3 \right] = 0, \tag{3.49}$$

where $\hat{f}(\eta, q^*)$ is the solution which depends only upon the initial guess $f_0(\eta)$, the auxiliary linear operator $\mathcal{L}$, the auxiliary parameter $\hbar$, and the embedding parameter $q^* \in [0, 1]$. The embedding parameter has the property that at $q^* = 0$

$$\hat{f}(\eta, 0) = f_0(\eta), \tag{3.50}$$

and at $q^* = 1$

$$\bar{f}(\eta, 1) = f(\eta), \tag{3.51}$$

Thus the embedding parameter has the property that as q^* varies from 0 to 1, $\bar{f}(\eta, q^*)$ varies continuously from initial guess to the final solution. Thus

$$f(\eta) = f_0(\eta) + \sum_{m=1}^{\infty} f_m(\eta), \tag{3.52}$$

The *mth-order deformation problem* can be written

$$\mathcal{L}[f_m(\eta) - \chi_m f_{m-1}(\eta)] = \mathcal{R}_m(\eta), \tag{3.53}$$

with boundary conditions

$$f_m(1) = 0 \quad \text{and} \quad f'_m(d) = 0, \tag{3.54}$$

where

$$\begin{aligned} \mathcal{R}_m(\eta) &= \hbar \Big[\eta f''_{m-1}(\eta) + f'_{m-1}(\eta) + K\eta(1-\chi_m) - \\ &\quad + 2\beta \sum_{k=0}^{m-1} f'_{m-1-k}(\eta) \sum_{l=0}^{k} f'_{k-l}(\eta) \{ f'_l(\eta) + 3\eta f''_l(\eta) \} \Big]. \end{aligned} \tag{3.55}$$

We now use the symbolic calculation software MATHEMATICA and solve the set of linear differential Eq. (3.53) with the corresponding boundary conditions. It is found from the MATHEMATICA iterations that f_m can be written as

$$f_m(\eta) = \sum_{n=-2m}^{2m+2} a_{m,n}\eta^n + A_m \ln \eta, \quad m \geqslant 0 \tag{3.56}$$

To calculate $\mathcal{R}_m(\eta)$, we find the following

$$f'_m(\eta) = \sum_{n=-2m}^{2m+2} b_{m,n}\eta^n, \tag{3.57}$$

where

$$b_{m,n} = (n+1)a_{m,n+1}, \tag{3.58}$$

$$f''_m(\eta) = \sum_{n=-2m}^{2m+2} c_{m,n}\eta^n, \tag{3.59}$$

in which

$$c_{m,n} = (n+1)b_{m,n+1}. \tag{3.60}$$

From here we can write

$$f'_{k-l}(\eta) = \sum_{n=-2k+2l}^{2k-2l+2} b_{k-l,n}\eta^n, \tag{3.61}$$

$$f'_{m-1-k}(\eta) = \sum_{n=-2m+2k+2}^{2m-2k} b_{m-1-k,n}\eta^n, \tag{3.62}$$

$$f'_l(\eta) = \sum_{n=-2l}^{2l+2} b_{l,n}\eta^n, \tag{3.63}$$

$$f''_l(\eta) = \sum_{n=-2l}^{2l+2} c_{l,n}\eta^n, \tag{3.64}$$

$$f'_{k-l}f'_l = \sum_{j=-2l}^{2l+2} \eta^{i+j} \sum_{i=-2k+2l}^{2k-2l+2} b_{l,\,j}b_{k-l,i}. \tag{3.65}$$

Put

$$i + j = x,$$

$$-2k \leq x \leq 2k + 4,$$

$$i = x - j,$$

$$-2k + 2l \leq x - j \leq 2k - 2l + 2,$$

or

$$x - 2k + 2l - 2 \leq j \leq x + 2k - 2l,$$

this shows that

$$j = \max\{x - 2k + 2l - 2, -2l\} \text{ to } \min\{x + 2k - 2l, 2l + 2\},$$

therefore,

$$f'_{k-l}\, f'_l = \sum_{x\,=\,-2k}^{2k\,+\,4} \sum_{j\,=\,\max\{x-2k+2l-2,-2l\}}^{\min\{x+2k-2l,2l+2\}} b_{l,\,j} b_{k-l,x-j} \eta^x, \tag{3.66}$$

now we find that

$$f'_{m-1-k} f'_{k-l} f'_l = \sum_{x\,=\,-2k}^{2k\,+\,4} \sum_{s\,=\,-2m\,+\,2k\,+\,2}^{2m\,-\,2k} \eta^{x+s} \sum_{j\,=\,\max\{x-2k+2l-2,-2l\}}^{\min\{x+2k-2l,2l+2\}} b_{l,\,j} b_{k-l,x-j} b_{m-1-k,s}, \tag{3.67}$$

put

$$x + s = n,$$

$$-2m + 2 \leq n \leq 2m + 4,$$

$$s = n - x,$$

so,

$$-2m + 2k + 2 \leq n - x \leq 2m - 2k,$$

or

$$n - 2m + 2k \leq x \leq n + 2m - 2k - 2,$$

which defines

$$x = \max\{-2k, n - 2m + 2k\} \text{ to } \min\{2k + 4, n + 2m - 2k - 2\},$$

therefore,

$$\begin{aligned} f'_{m-1-k} f'_{k-l} f'_l \; &= \sum_{n\,=\,-2m\,+\,2}^{2m\,+\,4} \eta^n \\ &\times \sum_{x\,=\,\max\{-2k,n-2m+2k\}}^{\min\{2k+4,n+2m-2k-2\}} \; \sum_{j\,=\,\max\{x-2k+2l-2,-2l\}}^{\min\{x+2k-2l,2l+2\}} b_{l,\,j} b_{k-l,x-j} b_{m-1-k,n-x}. \end{aligned} \tag{3.68}$$

We can write the following expression in simplified form as

$$\sum_{k=0}^{m-1} f'_{m-1-k} \sum_{l=0}^{k} f'_{k-l} f'_l = \sum_{n=-2m+2}^{2m+4} \delta_{m,n} \eta^n, \tag{3.69}$$

where

$$\delta_{m,n} = \sum_{k=0}^{m-1} \sum_{l=0}^{k} \sum_{x=\max\{-2k,n-2m+2k\}}^{\min\{2k+4,n+2m-2k-2\}} \sum_{j=\max\{-2l,x-2k+2l-2\}}^{\min\{2l+2,x+2k-2l\}} b_{l,j} b_{k-l,x-j} b_{m-1-k,n-x}, \tag{3.70}$$

$$\sum_{k=0}^{m-1} f'_{m-1-k} \sum_{l=0}^{k} f'_{k-l} f''_l = \sum_{n=-2m+2}^{2m+4} \Delta_{m,n} \eta^n, \tag{3.71}$$

where

$$\Delta_{m,n} = \sum_{k=0}^{m-1} \sum_{l=0}^{k} \sum_{x=\max\{-2k,n-2m+2k\}}^{\min\{2k+4,n+2m-2k-2\}} \sum_{j=\max\{-2l,x-2k+2l-2\}}^{\min\{2l+2,x+2k-2l\}} c_{l,j} b_{k-l,x-j} b_{m-1-k,n-x}, \tag{3.72}$$

using all these values in Eq. (3.55), we get

$$\begin{aligned} \mathcal{R}_m(\eta) &= \sum_{n=-2m}^{2m+2} \hbar \left[\chi_{n+2m+1} \chi_{4m+1-n} c_{m-1,n-1} + \chi_{n+2m} \chi_{4m+2-n} b_{m-1,n} \right. \\ &\quad \left. + 2\beta \left(\chi_{n+2m} \delta_{m,n} + 3\chi_{n+2m+1} \Delta_{m,n-1} \right) \right] \eta^n + K\eta(1-\chi_m), \end{aligned} \tag{3.73}$$

or

$$\mathcal{R}_m(\eta) = \sum_{n=-2m}^{2m+2} \Gamma_{m,n} \eta^n + \hbar K \eta (1-\chi_m), \tag{3.74}$$

where

$$\begin{aligned} \Gamma_{m,n} &= \hbar \left[\chi_{n+2m+1} \chi_{4m+1-n} c_{m-1,n-1} + \chi_{n+2m} \chi_{4m+2-n} b_{m-1,n} \right. \\ &\quad \left. + 2\beta \left(\chi_{n+2m} \delta_{m,n} + 3\chi_{n+2m+1} \Delta_{m,n-1} \right) \right], \end{aligned} \tag{3.75}$$

using Eq. (3.74), Eq. (3.53) becomes

$$\mathcal{L}[f_m(\eta) - \chi_m f_{m-1}(\eta)] = \sum_{n=-2m}^{2m+2} \Gamma_{m,n} \eta^n + \hbar K \eta (1-\chi_m), \tag{3.76}$$

suppose that

$$f_m(\eta) - \chi_m f_{m-1}(\eta) = Y, \tag{3.77}$$

with the help of Eq. (3.77), Eq. (3.76) can be written as

$$\mathcal{L}[Y] = \sum_{n\,=\,-2m}^{2m\,+\,2} \Gamma_{m,n}\eta^n + \hbar K\eta(1-\chi_m), \tag{3.78}$$

or

$$\eta Y'' + Y' = \sum_{n\,=\,-2m}^{2m\,+\,2} \Gamma_{m,n}\eta^n + \hbar K\eta(1-\chi_m), \tag{3.79}$$

or

$$\frac{d}{d\eta}\left(\eta Y'\right) = \sum_{n\,=\,-2m}^{2m\,+\,2} \Gamma_{m,n}\eta^n + \hbar K\eta(1-\chi_m), \tag{3.80}$$

integration of above equation gives

$$Y = \sum_{n\,=\,-2m}^{2m\,+\,2} \frac{\Gamma_{m,n}\eta^{n+1}}{(n+1)^2} + \frac{\hbar K\eta^2}{4}(1-\chi_m) + C_1\ln\eta + C_2, \tag{3.81}$$

using C_1 and $C_2$are constants.using Eq. (3.77), Eq. (3.81) takes the following form

$$f_m(\eta) - \chi_m f_{m-1}(\eta) = \sum_{n\,=\,-2m}^{2m\,+\,2} \frac{\Gamma_{m,n}\eta^{n+1}}{(n+1)^2} + \frac{\hbar K\eta^2}{4}(1-\chi_m) + C_1\ln\eta + C_2, \tag{3.82}$$

with the help of boundary conditions (3.54), we get

$$C_1 = -\sum_{n\,=\,-2m}^{2m\,+\,2} \frac{\Gamma_{m,n}d^{n+1}}{(n+1)} - \frac{\hbar K d^2}{2}(1-\chi_m), \tag{3.83}$$

$$C_2 = -\sum_{n\,=\,-2m}^{2m\,+\,2} \frac{\Gamma_{m,n}}{(n+1)^2} - \frac{\hbar K}{4}(1-\chi_m), \tag{3.84}$$

with the help of C_1 and C_2 , the *mth-order solution* (3.82) can be written as

$$\begin{aligned} f_m(\eta) &= \sum_{n=-2m}^{2m+2} \chi_m \chi_{n+2m} \chi_{4m+2-n} a_{m-1,n} \eta^n + \chi_m A_{m-1} \ln \eta \\ &+ \sum_{n=-2m}^{2m+2} \frac{\Gamma_{m,n} \eta^{n+1}}{(n+1)^2} + \frac{\hbar K \eta^2}{4}(1-\chi_m) + C_1 \ln \eta + C_2. \end{aligned} \tag{3.85}$$

From Eq. (3.56) and (3.85), we can write

$$\begin{aligned} &\sum_{n=-2m}^{2m+2} \left(a_{m,n} - \chi_m \chi_{n+2m} \chi_{4m+2-n} a_{m-1,n}\right) \eta^n \\ &+ (A_m - \chi_m A_{m-1}) \ln \eta \\ = &\sum_{n=-2m}^{2m+2} \frac{\Gamma_{m,n} \eta^{n+1}}{(n+1)^2} + \frac{\hbar K \eta^2}{4}(1-\chi_m) + C_1 \ln \eta + C_2, \end{aligned} \tag{3.86}$$

equating the like powers of η , we get for $m \geqslant 1, -2m \leqslant n \leqslant 2m+2$,

$$\begin{aligned} a_{m,0} &= \chi_m \chi_{2m} \chi_{4m+2} a_{m-1,0} \\ &- \left(\sum_{n=-2m}^{2m+2} \frac{\Gamma_{m,n}}{(n+1)^2} + \frac{\hbar K}{4}(1-\chi_m) \right), \end{aligned} \tag{3.87}$$

where $m \geqslant 1, -2m \leqslant n \leqslant 2m+2$.

$$a_{m,1} = \chi_m \chi_{2m+1} \chi_{4m+1} a_{m-1,1} + \Gamma_{m,0}, \tag{3.88}$$

$$a_{m,2} = \chi_m \chi_{2m+2} \chi_{4m} a_{m-1,2} + \frac{\Gamma_{m,1}}{4} + \frac{\hbar K}{4}(1-\chi_m), \tag{3.89}$$

$$A_m = \chi_m A_{m-1} - \sum_{n=-2m}^{2m+2} \frac{\Gamma_{m,n} d^{n+1}}{(n+1)} - \frac{\hbar K d^2}{2}(1-\chi_m), \tag{3.90}$$

$$a_{m,n} = \chi_m \chi_{n+2m} \chi_{4m+2-n} a_{m-1,n} + \frac{\Gamma_{m,n-1}}{n^2}, -2m \leqslant n < 0 \text{ and } 2 < n \leqslant 2m+2, \tag{3.91}$$

using the above recurrence formulae, one can easily calculate all coefficients $a_{m,n}$ and A_m by using

$$a_{0,0} = \frac{NK}{2}(d^2 - 1) + \frac{K}{4}, \quad a_{0,1} = 0, \quad a_{0,2} = -\frac{K}{4}, \quad A_0 = \frac{Kd^2}{2}, \tag{3.92}$$

given by the initial guess approximation in Eq. (3.46). The corresponding Zth-order approximation of Eq. (3.43) can be written as

$$\sum_{m=0}^{Z} f_m(\eta) = \sum_{m=0}^{Z} \left(\sum_{n=-2m}^{2m+2} a_{m,n}\eta^n + A_m \ln \eta \right). \tag{3.93}$$

Finally, with the help of Eq. (3.93), the complete analytic solution is given as

$$f(\eta) = \lim_{Z \to \infty} \left[\sum_{m=0}^{Z} \left(\sum_{n=-2m}^{2m+2} a_{m,n}\eta^n + A_m \ln \eta \right) \right]. \tag{3.94}$$

3.5.1 Graphical results

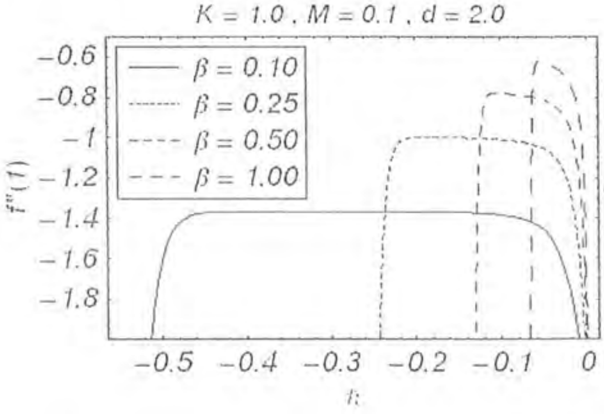


Fig. (1): $\hbar$ − *curve* for the 25^{th} order of approx. and for different values of *fluid parameter* β.

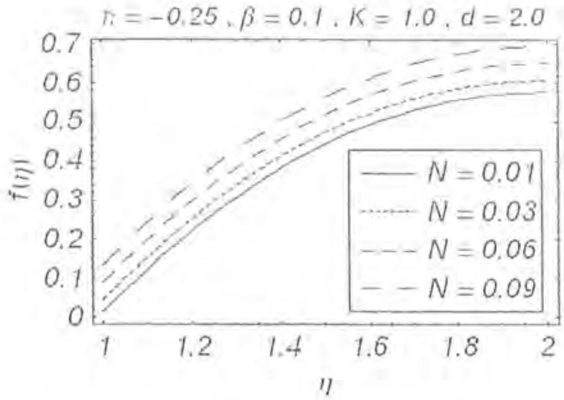


Fig. (2): $Velocity - curves$ for different values of $slip\ prameter\ N$.

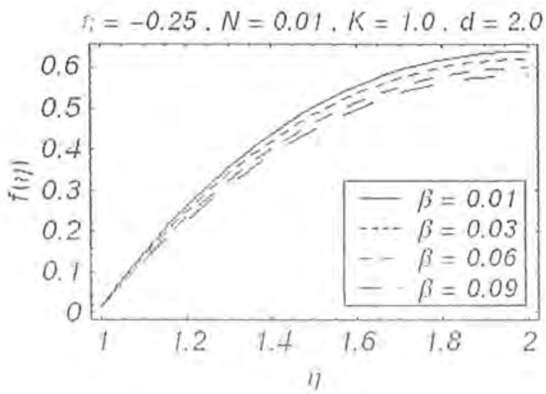


Fig. (3): $Velocity - curves$ for different values of $fluid\ parameter\ \beta$.

3.5.2 Results and discussion:

The auxiliary parameter $\hbar$, which in fact gives us the convergence region and rate of approximation for the HAM is present in Eq. (3.53), which is the final explicit analytic expression. Let us plot the $\hbar - curves$ for 26^{th} order of approximation and for three different values of β. Keenly observing Fig. (1), one finds that as you go on decreasing the value of the constant β which is indeed enclosing all of the third grade parameters, the convergence region keeps on expanding more and more. So, as we go on making β smaller and smaller we get more and more expansion in the range of $\hbar$.

In Fig. (2), the variation of slip parameter N is considered when the film thickness is assumed to be 2. It is seen from the figure that as we increase N, the velocity increases throughout the given domain.

Fig. (3) shows the variation of velocity for various values of third grade fluid parameter β for the partial slip case. It is found from the figure that as we increase the parameter β, the velocity f increases in the given film.

3.6 Conclusion

Analytical solutions are presented for the thin flim flow of fourth grade fluids. The governing non-linear differential equations with the corresponding non-linear boundary conditions are solved by Perturbation method, Homotopy perturbation method and Homotopy analysis method. It is seen from these methods that Homotopy analysis method is the best method and the convergence region shows the validity of the solution given.

Bibliography

[1] T. Hayat, Y. Wang and K. Hutter, *Hall effects on the unsteady hydromagnetic oscillatory flow of a second grade fluid, Int. J. Non-Linear Mech.* **39** (2004) 1027 – 1037.

[2] C. Fetecau T. Hayat and C. Fetecau, *Steady state solutions for more simple flows of generalized Bufger's fluids, Intl. J. Non-Linear Mech.* **41** (2006) 880 – 887.

[3] C. Fetecau and C. Fetecau, *Unsteady flows of Oldroyd-B fluids in a channel of rectangular cross-section, Intl. J. Non-Linear Mech.* **40** (2005) 1214 – 1219.

[4] C. Fetecau and C. Fetecau, *Starting solutions for some unsteady unidirectional flows of a second grade fluid, Int. J. Eng. Sci.* **43** (2005) 781 – 789.

[5] W. C. Tan and T. Masuoka, Stokes first problem for second grade fluid in a porous half space, Int. J. Non-Linear Mech. **40** (2005) 515 – 522.

[6] W. C. Tan and T. Masuoka, *Stokes first problem for Oldroyd-B fluid in a porous half space, Physics Fluids.* **17** (2005) 023101 – 023107.

[7] T. Hayat and A. H. Kara, *A variational analysis of non-Newtonian flow in a rotating system, Intl. J. Comput. Fluid Dyn.* **20** (2006) 157 – 162

[8] T. Hayat, *Oscillatory solution in rotating flow of a Johnson-Segalman fluid, Z. Angew. Math. Mech.* **85** (2005) 449 – 456.

[9] T. Hayat and N. Ali, *Peristaltically induced motion of MHD third grade fluid in deformable tube,Phys.A.* **370** (2006) 225 – 239.

[10] T. Hayat and M. Sajid, *Analytic solution for axisymmetric flow and heat transfer of a second grade fluid past a stretching sheet, Intl. J. Heat Mass Transf.* **50** (2007) 75 – 84.

[11] A. M. Siddiqui, R. Mahmood and Q. K, Ghauri, *Homotopy Perturbation Method for thin film flow of a fourth grade fluid down a vertical cylinder, Phys. Lett. A.* **352** (2006) 404 – 410.

[12] A. M. Siddiqui, M. Ahmed and Q. K, Ghauri, *Thin film flow of non-Newtonian fluids on a moving belt, Chaos, Solitons Fractal.* **33** (2007) 1006 – 1016.

[13] A. M. Siddiqui, R. Mahmood and Q. K, Ghauri, *Homotopy Perturbation Method for thin film flow of a third grade fluid down an inclined plane, Chaos, Solitons Fractal (in press).*

[14] M. Sajid and T. Hayat, *The application of homotopy analysis method for thin film flow of a third order fluid, Chaos, Solitons Fractal (in press).*

[15] T. Hayat and M. Sajid, *On analytic solution of thin film flow of a fourth grade fluid down a vertical cylinder, Phys. Lett. A.* **361** (2007) 316 – 322.

[16] M. Sajid, T. Hayat and S. Asghar, *Comparison between* HAM *and* HPM *solutions of thin film flows of non-Newtonian fluids on a moving belt, Nonlinear Dyn. (in press).*

[17] S. Asghar, S. Nadeem, K. Hanif and T. Hayat, *Analytic solution of stokes second problem for second-grade fluid, Mathematical Problems in Engineering.* **2006** (2006) 1 – 8.

[18] T. Hayat, S. Nadeem and S. Asghar, *Unsteady MHD flow due to eccentrically rotating porous disk and a third grade fluid at infinity, Int. J. of Applied Mechanics and Engineering,* **11**, No. 2, (2006) 415 – 419.

[19] T. Hayat, S. Nadeem, S. Asghar. and A.M. Siddiqui, *MHD rotating flow of a third grade fluid on an oscillating porous plate, Acta Mech.,* **152**, 177 – 190.

[20] T. Hayat, S. Nadeem, S. Asghar and A. M. Siddiqui, *Fluctuating flow of a third grade fluid on a porous plate in a rotating medium, Internat. J. Nonlinear Mech.* **36** (2001) 901 – 916.

[21] S. J. Liao, *Beyond perturbation: introduction to homotopy analysis method. Boca Raton: Chapman & Hall/CRC Press*; 2003.

[22] S. J. Liao, *On the homotopy analysis method for nonlinear problems, Appl. Math. Comput.* **147** (2004) 499 – 513.

[23] T. Hayat, M. Khan and M. Ayub, *Couette and Poiseuille flows of an Oldroyd 6-constant fluid with magnetic field. J. Math. Anal. & Appl.* **298** (2004) 225 – 244.

[24] T. Hayat, M. Khan and S. Asghar, *Homotopy analysis of* MHD *flows of an Oldroyd 8-constant fluid. Acta Mech.* **168** (2004) 213 – 232.

[25] C. Yang and S. J. Liao, *On the explicit purely analytic solution of Von Karman swirling viscous flow. Comm. Non-linear Sci. Numer. Simul.* **11** (2006) 83 – 93.

[26] S. J. Liao, *A new branch of solutions of boundary-layer flows over an impermeable stretched plate. Int. J. Heat and Mass Transfer,* 48 (2005) 2529 – 2539.

[27] S. J. Liao, *An analytic solution of unsteady boundary-layer flows caused by an impulsively stretching plate. Comm. Non-linear Sci. Numer. Simul.* **11** (2006) 326 – 339.

[28] M. Sajid, T. Hayat and S. Asghar, *On the analytic solution of steady flow of a fourth grade fluid. Phys. Lett. A.* **355** (2006) 18 – 24.

[29] Z. Abbas, M. Sajid and T. Hayat, *MHD boundary layer flow of an upper-convected Maxwell fluid in a porous channel, Theor. Comput. Fluid Dyn.* 20 (2006) 229 – 238.

[30] S. J. Liao, *On the proposed homotopy analysis technique for nonlinear problems and its applications, Ph. D. dissertation, Shanghai Jio Tong University,* 1992.

[31] J.H. He, *Homotopy perturbation technique. Comput. Methods Appl. Mech. Eng.* **178** (1999), p.257.

[32] J.H. He, *Int. J. Nonlinear Mech.* **35** (1) (2000), p. 37.

[33] L. D. Landau, E. M Lifshitz, *Fluid Mechanics, second ed., Pergamon, New York,* 1989.