588

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Certificate

We accept this thesis as conforming to the required standard.

A THESIS SUBMITTED IN THE PARTIAL FULFILLMENT OF REQUIREMENTS FOR THE DEGREE OF THE MASTER OF PHILOSOPHY

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Dedicated
To
My Grand Father (Late)

Who was the one, who take care of my initial education in difficult circumstances.

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All praises to Almighty Allah, the Merciful and Compassionate, without Whose Will and Consent we cannot proceed a single step. Many thanks to Him. All and every respect for the Holy Prophet Muhammad (S.A.W.) who is forever a source of guidance and knowledge for humanity.

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Introduction

The analysis of effects of rotation and magnetic field in fluid flows has been an active area of research because of its geophysical and technological importance. It is well known that number of astronomical bodies (e.g. the Sun, Earth, Jupiter, Magnetic Stars, and Pulsars) possesses fluid interiors and (at least surface) magnetic fields. Changes in the rotation rate of such objects suggest the possible importance of hydro magnetic spin-up. Hide and Roberts [1] had made a steady state investigation of the hydro magnetic rotating an infinite rigid wall. Chandrasekhar [2-4] has also made significant contributions to the theory of hydrodynamic and hydro magnetic flow phenomenon. He pointed out the significant role of the coriolis force on problems of thermal instability and on stability of a viscous hydro magnetic flow. In order to make some applications to solar physics, Lehnert [5,6] has presented a steady state analysis of the magneto hydrodynamic waves in a Newtonian incompressible rotating fluid with the same predicted certain significant effects of the coriolis force on the properties of the magneto hydrodynamic waves in the sun.

Interest in flows of viscoelastic liquids has increased substantially over the past decades due to the occurrence of these liquids in industrial processes. The governing equations for viscoelastic fluids are in general of higher order and much complicated than the Navier-Stokes equation [7-9]. The lack of boundary conditions as well as the non-linearity of the governing equations limits the solutions of the flows involving viscoelastic fluids. One of the viscoelastic fluid models which is most popular recently is called the Burger's model [10-12]. This model is usually used for modeling asphalt concrete. There are numerous examples of the use of Burger's model to study asphalt mixes (see for example [13,14]).

The Burgers model has been used to characterized food products such as Cheese [15], Soil [16], in the modeling of high temperature viscoelasticity of fine-grained polycrystalline Olvine [17,18], in calculating the transient creep properties of the earth mantle and specifically related to the post-glacial uplift [19-22]. More recently, Hayat [23] has discussed the exact solutions to rotating flows of a Burgers fluid.

The object of the present thesis is discuss some unsteady, MHD rotating flows of Burgers fluid with heat transfer. The thesis is arranged in the following manner.

In Chapter one, some basic definitions are given, the energy and momentum equation for Burger fluid have been derived.

Chapter two is devoted for the study of unsteady unidirectional flows of second grade fluids in domain with heated boundaries.

In Chapter three, the exact solutions of unsteady incompressible MHD rotating flows of Burger fluid have been discussed in the presence of heat transfer. Three problems namely (i) MHD rotating flow of Burgers' fluid due to heated rigid plate oscillating in its own plane (ii) MHD rotating flow of Burgers' fluid due to heated parallel plates (iii) Time periodic Poiseuille flow.

Contents

Chapter 1

Basic Definitions of Fluid Mechanics

1.1 Introduction

This chapter deals with some basic definitions of fluid mechanics, different kinds of flows and fluids are defined. The equation of continuity and energy equation are given. The governing equations for flow of a Burger fluid is also derived.

1.2 Definitions

1.2.1 Fluid

Fluids are substances which are capable of flowing and which conform to the shape of containing vessels.

1.2.2 Flow

It is a material that goes under deformation when different forces act upon it. If the deformation continuously increases without limit then the phenomenon is known as flow.

1.2.3 Fluid Mechanics

The branch of engineering that examines the nature and properties of fluids, both in motion and at rest.

1.3 Types of Flow

1.3.1 Steady Flow

Flow is steady if velocity of the fluid remains same at successive periods of time i.e.

$$\frac{\partial \mathrm{V}}{\partial t} = 0.$$

1.3.2 Unsteady Flow

Flow is unsteady if velocity of the fluid changes with time i.e.

$$\frac{\partial \mathrm{V}}{\partial t} \neq 0.$$

1.3.3 Compressible Flow

Flow is compressible if the density of the fluid changes during the flow i.e.

$$\rho = \rho(x, y, z, t) \neq \text{constant}.$$

1.3.4 Incompressible Flow

A flow in which the density of the fluid particles does not change during the flow i.e.

$$\rho = \rho(x, y, z, t) = \text{constant}.$$

1.3.5 Inviscid Flow

Flows in which fluid friction is negligible are called Inviscid.

1.3.6 Newtonian Fluids

Fluids in which the shear stress is directly proportional the rate of deformation are Newtonian fluids. Most common fluids such as water, air, and gasoline are Newtonian fluids.

In other words we can say that the fluids which obey the Newton's law of viscosity are referred to as the Newtonian fluids.

$$\tau_{yx} = \mu(\frac{du}{dy}),$$

where τ_{yx} is the shear stress acting on the plane normal to the $y-$axis and μ is the constant of proportionality, called absolute or dynamic viscosity, u is the $x-$component of velocity. Water and gasoline are examples of Newtonian fluids under normal conditions.

1.3.7 Non-Newtonian Fluids

For non-Newtonian fluids the shear stress is directly proportional to the rate of angular deformation in a non linear manner i.e.

$$\tau_{yx} = \mu_1 (\frac{du}{dy})^n, (n \neq 1),$$

where n and μ_1 denote the flow behaviour index and apparent viscosity, respectively. Toothpaste, ketchup, shampoo, etc are non-Newtonian fluids. Because of complexity of fluids, there are many models for non-Newtonian fluids.

1.4 Some Properties

1.4.1 Density

The density of fluid is defined as the mass per unit volume. Mathematically, the density ρ at a point p may be defined as

$$\rho = \lim_{\delta V \to 0} \frac{\delta m}{\delta \text{V}},$$

where δV is the total volume element around the point p and δm is the mass of the fluid within δV.

1.4.2 Viscosity

Viscosity of the fluid is that property which determines the amount of its resistance to a shearing force. It is denoted by μ.

1.4.3 Kinematic Viscosity

The ratio of dynamic viscosity to the fluid density is known as kinematic viscosity and is given as

$$\nu = \frac{\mu}{\rho}.$$

1.4.4 Pressure

Force per unit area is known as pressure. Mathematically, the pressure p_1 at a point p may be defined as

$$p_1 = \lim_{\delta s \to 0} \frac{|\delta F|}{\delta s},$$

in which δs is an elementary area around point p and δF is the normal force due to fluid on δs.

1.4.5 Heat

Heat is the form of energy that is transferred between two bodies as a result of a difference in their temperature.

1.4.6 Temperature

Hotness or coldness of an object is expressed in terms of a quantity called temperature.

1.4.7 Specific Heat

Specific heat is the amount of heat required to raise the temperature of a unit mass of a substance through one degree.

1.4.8 Thermal Conductivity

It is the conduction of heat through the medium due to a thermal gradient in the medium. Thermal conductivity K of a substance depends upon the material of the substance.

1.4.9 Prandtl Number

A dimensionless number used in the study of diffusion in flowing systems, equal to the kinematic viscosity divided by the molecular diffusivity

$$P_r = \frac{\mu c_p}{K}$$

1.5 Equation of Continuity

The equation of continuity is a mathematical expression of law of conservation of mass. Consider a fluid flowing parallel to $x-$axis such that the mass flow through a cubical element of edges parallel to x, y, and $z-$axes. The equation of continuity for one dimensional flow is given by

$$-u\frac{\partial \rho}{\partial x} - \rho\frac{\partial u}{\partial x} = \frac{\partial \rho}{\partial t}, \tag{1.1}$$

or

$$\frac{\partial (\rho u)}{\partial x} = -\frac{\partial \rho}{\partial t}. \tag{1.2}$$

Using similar analysis the continuity equation for flow in three dimensions is

$$u\frac{\partial \rho}{\partial x} + v\frac{\partial \rho}{\partial y} + w\frac{\partial \rho}{\partial z} + \rho\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right) = -\frac{\partial \rho}{\partial t}, \tag{1.3}$$

or

$$\frac{\partial \rho}{\partial t} + \frac{\partial (\rho u)}{\partial x} + \frac{\partial (\rho v)}{\partial y} + \frac{\partial (\rho w)}{\partial z} = 0, \tag{1.4}$$

where u, v and w are the x, y and $z-$components of velocity respectively. In vector form the Eq.(1.4) can be written as

$$\frac{\partial \rho}{\partial t} + (\nabla.\rho \mathrm{V}) = 0, \tag{1.5}$$

where

$$\nabla = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right). \tag{1.6}$$

For steady flow Eq.(1.4) becomes

$$\frac{\partial (\rho u)}{\partial x} + \frac{\partial (\rho v)}{\partial y} + \frac{\partial (\rho w)}{\partial z} = 0. \tag{1.7}$$

If a fluid is compressible the density will vary in space, so the Eq.(1.7) applies for the steady state flow of a compressible fluid. For the steady state flow of an incompressible fluid the density is constant and the continuity equation becomes

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0,$$

or

$$\nabla.\mathrm{V} = 0. \tag{1.8}$$

1.6 Energy Equation

According to the law of conservation of energy, we have

$$\rho \frac{de}{dt} = \mathrm{T}_1 \cdot \mathrm{L} - \nabla \cdot \mathrm{q} + \rho r, \tag{1.9}$$

where ρ is the density, e is the specific internal energy, q is the heat flux vector, r is the radiant heating and **L** is the gradient of the velocity vector. In the absence of the radiant heating Eq (1.9) takes the form

$$\rho \frac{de}{dt} = \mathrm{T}_1 \cdot \mathrm{L} - \nabla \cdot \mathrm{q}, \tag{1.10}$$

where

$$e = C_p T'$$

and

$$q = -K\nabla T', \tag{1.11}$$

where K is the thermal conductivity, C_p is the specific heat at constant pressure and T' is the temperature. We take the temperature field of the form

$$T' = T'(z, t), \tag{1.12}$$

in view of Eq (1.12) we write Eq (1.11) as follows

$$divq = -K\frac{\partial^2 T'}{\partial z^2}, \tag{1.13}$$

with the help of Eq (1.13) and $e = C_p T'$, Eq (1.10) takes the following form

$$\rho C_p \frac{dT'}{dt} = \mathrm{T}_1 \cdot \mathrm{L} + K\frac{\partial^2 T'}{\partial z^2}, \tag{1.14}$$

where

$$\frac{d}{dt} = \frac{\partial}{\partial t} + (\mathrm{V} \cdot \nabla), \tag{1.15}$$

is the material time derivative,in which $\frac{\partial}{\partial t}$ is the local part of material time derivative and $(\mathrm{V} \cdot \nabla)$ is called the convective part of material time derivative.

The Cauchy stress for an incompressible viscous fluid is characterized by the following constitutive equation

$$\mathrm{T} = -p\mathrm{I} + \mu \mathrm{A}_1, \tag{1.16}$$

where μ is the coefficient of viscosity, $p\mathbf{I}$ denotes the indeterminate spherical stress and $\mathbf{A}_1$ is the kinematical stress tensor defined by

$$\mathbf{A}_1 = grad\mathbf{V} + (grad\mathbf{V})^T = \mathbf{L} + \mathbf{L}^T, \tag{1.17}$$

where "$grad$" denotes the gradient operator. We seek the velocity field of the form

$$\mathbf{V} = [u(z,t), v(z,t), 0]. \tag{1.18}$$

In view of Eq.(1.18) $\mathbf{L}$ can be written as

$$\mathbf{L} = grad\mathbf{V} = \begin{bmatrix} 0 & 0 & \frac{\partial u}{\partial z} \\ 0 & 0 & \frac{\partial v}{\partial z} \\ 0 & 0 & 0 \end{bmatrix}, \tag{1.19}$$

and

$$\mathbf{L}^T = grad\mathbf{V} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \frac{\partial u}{\partial z} & \frac{\partial v}{\partial z} & 0 \end{bmatrix}. \tag{1.20}$$

With the help of Eqs.(1.19) and (1.20) ,Eq. (1.17) can be written as

$$\mathbf{A}_1 = \mathbf{L} + \mathbf{L}^T = \begin{bmatrix} 0 & 0 & \frac{\partial u}{\partial z} \\ 0 & 0 & \frac{\partial v}{\partial z} \\ \frac{\partial u}{\partial z} & \frac{\partial v}{\partial z} & 0 \end{bmatrix}, \tag{1.21}$$

$$\mathbf{A}_1\mathbf{L} = \begin{bmatrix} 0 & 0 & \frac{\partial u}{\partial z} \\ 0 & 0 & \frac{\partial v}{\partial z} \\ \frac{\partial u}{\partial z} & \frac{\partial v}{\partial z} & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & \frac{\partial u}{\partial z} \\ 0 & 0 & \frac{\partial v}{\partial z} \\ \frac{\partial u}{\partial z} & \frac{\partial v}{\partial z} & 0 \end{bmatrix}, \tag{1.22}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & (\frac{\partial u}{\partial z})^2 + (\frac{\partial v}{\partial z})^2 \end{bmatrix}. \tag{1.23}$$

Finally $\mathbf{T}_1 \cdot \mathbf{L}$ can be written as

$$\mathbf{T}_1 \cdot \mathbf{L} = \mu \mathrm{trace}(\mathbf{A}_1 \mathbf{L}) = \mu(\frac{\partial u}{\partial z})^2 + \mu(\frac{\partial v}{\partial z})^2, \tag{1.24}$$

In view of Eq.(1.24),Eq.(1.14) takes the following form

$$\rho C_p \frac{dT'}{dt} = \mu(\frac{\partial u}{\partial z})^2 + \mu(\frac{\partial v}{\partial z})^2 + K \frac{\partial^2 T'}{\partial z'^2}, \tag{1.25}$$

or

$$C_p \frac{dT'}{dt} = \frac{\mu}{\rho}[(\frac{\partial u}{\partial z})^2 + (\frac{\partial v}{\partial z})^2] + \frac{K}{\rho} \frac{\partial^2 T'}{\partial z'^2}, \tag{1.26}$$

Eq.(1.26) can also be written as

$$C_p \frac{dT'}{dt} = \frac{\mu}{\rho} \left[\frac{\partial u}{\partial z} + i \frac{\partial v}{\partial z} \right] \left[\frac{\partial u}{\partial z} - i \frac{\partial v}{\partial z} \right] + \frac{K}{\rho} \frac{\partial^2 T'}{\partial z'^2}, \tag{1.27}$$

or

$$C_p \frac{dT'}{dt} = \frac{\mu}{\rho} \left[\frac{\partial}{\partial z}(u + iv) \right] \left[\frac{\partial}{\partial z}(u - iv) \right] + \frac{K}{\rho} \frac{\partial^2 T'}{\partial z^2}, \tag{1.28}$$

$$C_p \frac{dT'}{dt} = \nu \left[\frac{\partial F}{\partial z} \frac{\partial \bar{F}}{\partial z} \right] + \frac{K}{\rho} \frac{\partial^2 T'}{\partial z^2}, \tag{1.29}$$

where $F = u + iv$,$\bar{F} = u - iv$.

Introducing the following non-dimensional variables

$$\eta = \frac{U}{\nu} z, \ \tau = \omega t, \ F{=}UG, \theta = \frac{T' - T_\infty}{T_\circ - T_\infty} \tag{1.30}$$

Making use of Eq.(1.30),Qq.(1.29) takes the following form

$$A\frac{\partial\theta}{\partial\tau}=\frac{1}{\Pr}\frac{\partial^2\theta}{\partial\eta^2}+Ec\frac{\partial G}{\partial\eta}\frac{\partial\bar{G}}{\partial\eta} \tag{1.31}$$

where

$$A=\frac{\omega\nu}{U^2},\Pr=\frac{\rho\nu C_p}{K},Ec=\frac{U^2}{\nu C_p(T_o-T_\infty)} \tag{1.32}$$

1.7 Equation of Motion for Burgers' Fluid

In a rotating system the governing equations can be written as [23]

$$\rho\left[\frac{d\mathrm{V}}{dt}+2\Omega\times\mathrm{V}+\Omega\times(\Omega\times\mathrm{r})\right]=-\nabla\hat{p}+\operatorname{div}\mathrm{S}-\sigma\mathrm{B}_0^2\mathrm{v}, \tag{1.33}$$

where $\mathbf{V}$ denotes the velocity vector, t the time, ρ the density, σ the finite electrical conductivity of the fluid, $\frac{d}{dt}$ the material derivative and the modified pressure $\hat{p}$ including the centrifugal term is given by

$$\hat{p}=p-\frac{\rho\Omega^2r^2}{2},$$

in which p is the pressure and $r^2=x^2+y^2$.

In a Burgers' fluid, the constitutive equation for the extra stress S is given by [23]

$$\left(1+\lambda\frac{\partial}{\partial t}+\beta\frac{\partial^2}{\partial t^2}\right)\mathrm{S}=\mu(1+\lambda_r\frac{\partial}{\partial t})\mathrm{A}, \tag{1.34}$$

In the above equation μ denotes the dynamic viscosity, $\mathbf{A}$ the Rivlin Ericksen tensor, λ and β are relaxation times, $\lambda_r(<\lambda)$ is the retardation time and the upper convective derivative is

$$\frac{\partial\mathrm{S}}{\partial t}=\frac{d\mathrm{S}}{dt}-(\operatorname{grad}\mathrm{V})\mathrm{S}-\mathrm{S}(\operatorname{grad}\mathrm{V})^{T_0}, \tag{1.35}$$

Where T_0 denotes the matrix transpose It should be noted that the Burgers' model reduces to that of an Oldroyd-B fluid for $\beta = 0$. For $\beta = \lambda_r = 0$ and $\beta = \lambda = \lambda_r = 0$, we are left with the Maxwell and classical viscous fluid models, respectively. In some special flow, this model resembles to that of second grade fluid model when $\beta = \lambda = 0$. The extra stress tensor and velocity field is assumed to be

$$\mathrm{S}(z,t) = \begin{bmatrix} S_{xx} & S_{xy} & S_{xz} \\ S_{yx} & S_{yy} & S_{yz} \\ S_{zx} & S_{zy} & S_{zz} \end{bmatrix}, \tag{1.36}$$
$$\mathbf{V}(z,t) = (u, v, 0).$$

Making use of Eq. (1.36),Eqs. (1.33) – (1.35) in component form of equations can be written as

$$\rho\left[\frac{\partial u}{\partial t} + 2\Omega v\right] = -\frac{\partial \hat{p}}{\partial x} + \frac{\partial}{\partial z}S_{xz} - \sigma \mathrm{B}_0^2 u, \tag{1.37}$$

$$\rho\left[\frac{\partial v}{\partial t} + 2\Omega u\right] = -\frac{\partial \hat{p}}{\partial y} + \frac{\partial}{\partial z}S_{yz} - \sigma \mathrm{B}_0^2 v, \tag{1.38}$$

$$0 = -\frac{\partial \hat{p}}{\partial z} + \frac{\partial}{\partial z}S_{zz} \tag{1.39}$$

$$\left(1 + \lambda\frac{\partial}{\partial t} + \beta\frac{\partial^2}{\partial t^2}\right)S_{xx} - 2\begin{bmatrix} \left(\lambda + \beta\frac{\partial}{\partial t}\right)\frac{\partial u}{\partial z}S_{xz} + \\ \beta\frac{\partial u}{\partial z}\left(\frac{\partial S_{xz}}{\partial t} - \frac{\partial u}{\partial z}S_{zz}\right) \end{bmatrix} = -2\mu\lambda_r\left(\frac{\partial u}{\partial z}\right)^2, \tag{1.40}$$

$$\left(1 + \lambda\frac{\partial}{\partial t} + \beta\frac{\partial^2}{\partial t^2}\right)S_{xy} - \begin{bmatrix} \left(\lambda + \beta\frac{\partial}{\partial t}\right)\left(\frac{\partial u}{\partial z}S_{yz} + \frac{\partial v}{\partial z}S_{xz}\right) \\ +\beta\frac{\partial u}{\partial z}\left(\frac{\partial S_{yz}}{\partial t} - \frac{\partial v}{\partial z}S_{zz}\right) \\ +\beta\frac{\partial v}{\partial z}\left(\frac{\partial S_{xz}}{\partial t} - \frac{\partial u}{\partial z}S_{zz}\right) \end{bmatrix} = \mu\left(1 + \lambda_r\frac{\partial}{\partial t}\right)\frac{\partial u}{\partial z}, \tag{1.41}$$

$$\left(1+\lambda\frac{\partial}{\partial t}+\beta\frac{\partial^2}{\partial t^2}\right)S_{yy}-2\left[\left(\lambda+\beta\frac{\partial}{\partial t}\right)\frac{\partial v}{\partial z}S_{yz}+\beta\frac{\partial v}{\partial z}\left(\frac{\partial S_{yz}}{\partial t}-\frac{\partial v}{\partial z}S_{zz}\right)\right]=-2\mu\lambda_r\left(\frac{\partial v}{\partial z}\right)^2, \tag{1.42}$$

$$\left(1+\lambda\frac{\partial}{\partial t}+\beta\frac{\partial^2}{\partial t^2}\right)S_{yz}-\left[\left(\lambda+\beta\frac{\partial}{\partial t}\right)\frac{\partial v}{\partial z}S_{zz}+\beta\frac{\partial v}{\partial z}\frac{\partial S_{zz}}{\partial t}\right]=2\mu\left(1+\lambda_r\frac{\partial}{\partial t}\right)\frac{\partial v}{\partial z}, \tag{1.43}$$

$$\left(1+\lambda\frac{\partial}{\partial t}+\beta\frac{\partial^2}{\partial t^2}\right)S_{zz}=0. \tag{1.44}$$

If we take that the fluid is at rest upto moment $t=0$, we get $S_{zz}=0$ and thus Eq. (1.39) indicates that $\hat{p}$ is independent of z. Moreover, Eqs. (1.40) to (1.43) gives

$$\left(1+\lambda\frac{\partial}{\partial t}+\beta\frac{\partial^2}{\partial t^2}\right)S_{xy}-\left[\begin{array}{c}\left(\lambda+\beta\frac{\partial}{\partial t}\right)\left(\frac{\partial u}{\partial z}S_{yz}+\frac{\partial v}{\partial z}S_{xz}\right)\\ +\beta\frac{\partial u}{\partial z}\frac{\partial S_{yz}}{\partial t}+\beta\frac{\partial v}{\partial z}\frac{\partial S_{xz}}{\partial t}\end{array}\right]=-2\mu\lambda_r\frac{\partial v}{\partial z}, \tag{1.45}$$

$$\left(1+\lambda\frac{\partial}{\partial t}+\beta\frac{\partial^2}{\partial t^2}\right)S_{xz}=\mu\left(1+\lambda_r\frac{\partial}{\partial t}\right)\frac{\partial u}{\partial z}, \tag{1.46}$$

$$\left(1+\lambda\frac{\partial}{\partial t}+\beta\frac{\partial^2}{\partial t^2}\right)S_{yy}-2\left[\left(\lambda+\beta\frac{\partial}{\partial t}\right)\frac{\partial v}{\partial z}S_{yz}+\beta\frac{\partial v}{\partial z}\frac{\partial S_{yz}}{\partial t}\right]=-2\mu\lambda_r\left(\frac{\partial v}{\partial z}\right)^2, \tag{1.47}$$

$$\left(1+\lambda\frac{\partial}{\partial t}+\beta\frac{\partial^2}{\partial t^2}\right)S_{yz}=2\mu\left(1+\lambda_r\frac{\partial}{\partial t}\right)\frac{\partial v}{\partial z}. \tag{1.48}$$

With the help of Eqs. (1.37) to (1.39) and Eqs. (1.45) to (1.48), we obtain

$$\left(1+\lambda\frac{\partial}{\partial t}+\beta\frac{\partial^2}{\partial t^2}\right)\left[\frac{\partial F}{\partial t}+2i\Omega F+\frac{1}{\rho}\left(\frac{\partial\widehat{\mathrm{P}}}{\partial x}+i\frac{\partial\widehat{\mathrm{P}}}{\partial y}\right)+\frac{\sigma B_o^2}{\rho}F\right]=\nu\left(1+\lambda r\frac{\partial}{\partial t}\right)\frac{\partial^2 F}{\partial t^2}, \tag{1.49}$$

where $F=u+iv,\ \bar{F}=u-iv.$

Chapter 2

Unsteady Unidirectional Flows of Second Grade Fluids in Domain with Heated Boundaries

In this chapter, the exact solutions of unidirectional, incompressible second grade fluid with heated boundaries have been discussed. Three problems namely (i) Flow Due to a Heated Rigid Plate Oscillating in its Own Plane (ii) Flow Between Two Infinite Parallel Plates One of Which is Oscillating (iii) Time Periodic Poiseuille Flow in a Slot, have been discussed and found an exact solution of the problems. This Chapter is due to Bandelli [24]. The essential details missing in the paper [24] are also incorporated.

2.0.1 Governing Equations

The incompressible flow of second grade fluid is characterized by the following constitutive equation [25]

$$\mathrm{T} = -p\mathrm{I}+\mu \mathrm{A}_1 + \alpha_1 \mathrm{A}_2 + \alpha_2 \mathrm{A}_1^2, \tag{2.1}$$

where μ is the coefficient of viscosity, α_1 and α_2 are normal stress moduli, $-p\mathrm{I}$ denotes the indeterminate stress and $\mathrm{A}_1$and A_2 are kinematic tensors defined by

$$\mathrm{A}_1 = (\mathrm{grad}\,\mathrm{V}) + (\mathrm{grad}\,\mathrm{V})^{\mathrm{T}}, \tag{2.2}$$

$$\mathrm{A}_2 = \frac{d}{dt}\mathrm{A}_1 + (\mathrm{A}_1\,\mathrm{grad}\,\mathrm{V}) + (\mathrm{grad}\,\mathrm{V})\mathrm{A}_1, \tag{2.3}$$

where V is the velocity, *grad* the gradient operator and $\frac{d}{dt}$ denotes the material time derivative which is defined as

$$\frac{d}{dt} = \frac{d}{\partial t} + (\mathrm{V}.\nabla). \tag{2.4}$$

2.0.2 Momentum Equation

The balance of linear momentum in absence of body forces can be written as

$$\rho\frac{d\mathrm{V}}{dt} = \mathrm{div}\,\mathrm{T}. \tag{2.5}$$

On substituting Eqs.(2.1) to(2.4) into Eq.(2.5), we obtain

$$\begin{aligned} & \mu\nabla\mathrm{V}+\alpha_1\Delta\mathrm{V}_t+\alpha_1(\Delta\omega\mathrm{XV})+(\alpha_1+\alpha_2) \\ & \{\mathrm{A}_1\Delta\mathrm{V}+2\operatorname{div}[(\operatorname{grad}\mathrm{V})(\operatorname{grad}\mathrm{V})^T\}-\rho(\omega\mathrm{XV})-\rho\mathrm{V}_t \\ = \; & \operatorname{grad}\mathrm{P}, \end{aligned} \tag{2.6}$$

where

$$\mathrm{P}=p-\alpha_1(\mathrm{V}-\nabla\mathrm{V})-\frac{2\alpha_1+\alpha_2}{4}\left|\mathrm{A}_1\right|^2+\frac{1}{2}\rho\left|\mathrm{V}\right|^2+\rho\phi,$$

Δ denotes the Laplacean, the subscript t denotes the partial derivative with respect to time, $|\mathrm{A}_1|$ is the trace norm of $\mathrm{A}_1$and $\omega=\operatorname{curl}\mathrm{V}$. We seek the velocity field of the form

$$\mathrm{V}=(u(y,t),0,0) \tag{2.7}$$

with the help of Eq.(2.7), the Eq.(2.6) after using the thermodynamic model [25] can be written as

$$\mu\frac{\partial^2 u}{\partial y^2}+\alpha_1\frac{\partial^3 u}{\partial y^2}\partial t-\rho\frac{2u}{\partial t}=\frac{\partial P}{\partial x}, \tag{2.8}$$

$$(2\alpha_1+\alpha_2)\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right)^2=\frac{\partial P}{\partial y}, \tag{2.9}$$

$$0=\frac{\partial P}{\partial z}. \tag{2.10}$$

Setting

$$\widehat{\mathrm{P}}=p-(2\alpha_1+\alpha_2)\left(\frac{\partial u}{\partial y}\right)^2.$$

The above system of equations can be written as

$$\mu\frac{\partial^2 u}{\partial y^2}+\alpha_1\frac{\partial^3 u}{\partial y^2\partial t}-\rho\frac{\partial u}{\partial t}=\frac{\partial\widehat{\mathrm{P}}}{\partial x}. \tag{2.11}$$

$$\frac{\partial\widehat{\mathrm{P}}}{\partial y}=\frac{\partial\widehat{\mathrm{P}}}{\partial z}=0. \tag{2.12}$$

Eqs.(2.11)and (2.12) implies that

$$\mu\frac{\partial^3 u}{\partial y^3}+\alpha_1\frac{\partial^4 u}{\partial y^3\partial t}-\rho\frac{\partial^2 u}{\partial y\partial t}=0. \tag{2.13}$$

2.0.3 Energy Equations

According to law of conservation of energy, we have

$$\rho\frac{de}{dt}=\mathrm{T.L}+\operatorname{div}q+\rho r, \tag{2.14}$$

where ρ is the density of the fluid, e is the specific internal energy, q the heat flux vector and r the radiant heating.

In absence of radiant heating (According to [24]), Eq.(2.14) takes the following form

$$\rho\frac{de}{dt}=\mathrm{T.L}+\operatorname{div}q. \tag{2.15}$$

According to Bandelli [24] for second grade fluid

$$\mathrm{T.L}=\left(\mu\frac{\partial u}{\partial y}\right)^2, \tag{2.16}$$

$$\operatorname{div}\overline{q}=-K\left(\frac{\partial T_1}{\partial y^2}\right), \tag{2.17}$$

$$e = \rho C_p, \tag{2.18}$$

where C_p is the specific heat, K is the conductivity and $T_1(y,t)$ the temperature. In view of Eqs.(2.16) – (2.18) Eq.(2.15) can be written as

$$\rho C_p \frac{dT_1}{dt} = \mu \left(\frac{\partial u}{\partial y}\right)^2 + K \frac{\partial^2 T_1}{\partial y^2}. \tag{2.19}$$

The above results implies the existence of the following flows in case of second grade fluid in the presence of heat analysis.

2.1 Flow Due to a Rigid Plate Oscillating in its Own Plane

Let us consider a semi-infinite second grade, incompressible fluid is bounded by an infinite rigid plate at $y = 0$ due to the cosine oscillations of the plate in its own plane with the frequency ω. The fluid above the plate is at rest. Let T_o and T_∞ denote respectively the temperatures of the plate and the fluid at infinity. The governing equation of motion and energy takes the following form Eq.(2.13) & Eq.(2.19),

$$\mu \frac{\partial^3 u}{\partial y^3} + \alpha_1 \frac{\partial^4 u}{\partial y^3 \partial t} - \rho \frac{\partial^4 u}{\partial y \partial t} = 0, \tag{2.20}$$

$$\rho C_p \frac{\partial \theta}{\partial t} = \frac{\mu}{T_o - T_\infty} \left(\frac{\partial u}{\partial y}\right)^2 + K \frac{\partial^2 \theta}{\partial y^2}, \tag{2.21}$$

where

$$\theta = \frac{T_1 - T_\infty}{T_o - T_\infty}.$$

With the corresponding boundary conditions are

$$\begin{aligned} u(y,t) &= U\ \cos\omega t,\ \theta = 1 \text{ at } \ y = 1, \\ u(y,t) &\rightarrow 0,\ \theta \rightarrow 0 \text{ as } y \rightarrow \infty. \end{aligned} \tag{2.22}$$

2.1.1 Solution of the Problem

For solution of Eq.(2.8), we assume that

$$u(y,t) = UF(y)\cos\omega t,$$

or

$$u(y,t) = U\mathrm{Re}[F(y)e^{i\omega t}]. \tag{2.23}$$

Substituting Eq.(2.23), into Eq.(2.20) and the boundary conditions (2.22), we obtain

$$\mu\frac{d^3F}{dy^3} + i\omega\alpha_1\frac{d^3F}{dy^3} - i\omega\frac{\rho dF}{dy} = 0. \tag{2.24}$$

$$\begin{aligned} F(0) &= 1 \\ F &\rightarrow 0 \text{ as } y \rightarrow \infty. \end{aligned} \tag{2.25}$$

The solution of Eq.(2.24) corresponding to the boundary conditions (2.25) can be written as

$$F(y) = e^{-(m+in)y}, \tag{2.26}$$

where

$$
\begin{aligned}
m^2 &= \frac{1}{2}\frac{\rho\omega}{[\mu^2+(\alpha_1\omega)^2]}\{[\mu^2+(\alpha_1\omega)^2]^{1/2}+\alpha_1\omega\}, \\
n^2 &= \frac{1}{2}\frac{\rho\omega}{[\mu^2+(\alpha_1\omega)^2]}\{[\mu^2+(\alpha_1\omega)^2]^{1/2}-\alpha_1\omega\}.
\end{aligned}
$$

From Eqs.(2.25) and(2.23) , we have

$$u(y,t)=Ue^{-my}\cos(\omega t-ny). \tag{2.27}$$

For the solution of Eq.(2.21), we take the temperature field of the form

$$\theta(y,t)=\theta(y)e^{2i\omega t}. \tag{2.28}$$

Substituting Eq.(2.28) into energy equation, we find, that $\theta(y)$ satisfies the linear ordinary differential equation

$$2i\omega\rho C\theta-K\theta''=Af(y). \tag{2.29}$$

Subject to the boundary conditions

$$\theta(0)=1,\quad \theta(\infty)=0, \tag{2.30}$$

where

$$A=\frac{\mu[U(m+in)]^2}{T_o-T_\infty},$$

$$f(y)=\exp[-2y(m+in)], \tag{2.31}$$

and prime denotes the derivative with respect to y. On substituting the solution of Eqs.(2.29) to (2.30) into Eq.(2.28), we obtain

$$\begin{aligned}\theta(y,t) &= e^{\frac{-ay}{\sqrt{2}}}\left[(1-D]\cos(2\omega t-\frac{ay}{\sqrt{2}})+\left[\sin(2\omega t-\frac{ay}{\sqrt{2}}\right]+\right. \\ & e^{-2my}[D\cos 2(\omega-n)t-E\sin 2(\omega-n)t], \qquad (2.32)\end{aligned}$$

where

$$\begin{aligned} a &= \sqrt{\frac{2\rho\omega c}{K}}, \\ D &= \frac{\mu U^2}{T_o-T_\infty}\left[\frac{4(m^2-n^2)^2-2mn(a^2-\rho mn)}{(a^2-\rho mn)^2+16(m^2-n^2)^2}\right], \\ E &= \frac{\mu U^2}{T_o-T_\infty}\left[\frac{a^2(m^2-n^2)}{(a^2-\rho mn)^2+16(m^2-n^2)^2}\right].\end{aligned}$$

2.2 Flow Between Two Infinite, Parallel Plates One of Which is Oscillating

Let us consider an incompressible unsteady second grade fluid bounded by infinite parallel plates at temperatures T_1 and T_2. The distance between the plates are considered to be d. Consider the lower plate at $y=0$ oscillating with velocity $U\cos\omega t$ and the upper plate is at rest. According to Rajagopal [25] using the similar procedure as discussed in previous section the solution of the boundary value problem can be written as

$$u(y,t)=\mathrm{Re}\left[\frac{U\sinh b(d-y)e^{iwt}}{\sinh bd}\right], \qquad (2.33)$$

where

$$b^2 = \frac{\rho\omega(\alpha_1\omega + i\mu)}{[\mu^2 + (\alpha_1\omega)^2]}.$$

On substituting Eq.(2.33) into Eq.(2.21), the solution of resultant equation can be easily written as

$$\theta(y,t) = \mathrm{Re}\left[\left\{\begin{array}{c} D_1 \cosh py + E_1 \sinh py \\ -\frac{A}{2p^2}\left(1 + \frac{P^2}{p^2-4b^2}\cosh 2b(d-y)\right) + \frac{y}{d} \end{array}\right\} e^{2i\omega t}\right], \qquad (2.34)$$

where

$$\begin{aligned} p &= \sqrt{\frac{2i\rho C\omega}{K}}, A = \frac{1}{T_1 - T_2}\left(\frac{bU}{\sinh bd}\right)^2, \\ D_1 &= \frac{A}{2p^2}\left(1 + \frac{p^2\cosh 2bd}{p^2 - 4b^2}\right), \\ E_1 &= \frac{A}{2p^2 \sinh bd}\left[1 + \frac{p^2}{p^2 - 4b^2}\left(1 + \frac{p^2\cosh 2bd}{p^2 - 4b^2}\right)\cosh pd\right]. \end{aligned}$$

2.3 Time Period Poiseuille Flow in a Slot

Let consider the flow is between two infinite parallel plates, at $y = 0$ and $y = h$ respectively. Assume that both the plates are at rest and let the pressure gradient along the $x-$direction, $\frac{\partial P}{\partial x}$ is given by

$$\frac{\partial P}{\partial x} = -\rho(P_o + Q_o \cos\omega t). \qquad (2.35)$$

To avoid the repetition using the same procedure as discussed in previous sections, we can write the solution for the velocity field as

$$u(y,t)=\frac{\rho h^2 P_o}{2\mu}\left[1-\left(\frac{y}{h}\right)^2\right]+\frac{2Q_o}{\omega}\,\mathrm{Re}\left[e^{i\omega t}\left(\frac{\cosh\beta(1+i)-\cosh\beta(ix)y/h}{2i\cosh\beta(1+i)}\right)\right],$$

where

$$\beta=\frac{1}{\sqrt{2}}\frac{\left[\frac{\rho h^2\omega}{2\mu}\right]}{\left[1+\left(\frac{\alpha_1\omega}{\mu}\right)^2\right]^{1/2}}\left[\left\{\left[1+\left(\frac{\alpha_1\omega}{\mu}\right)^2\right]^{1/2}+1\right\}^{1/2}-\left\{i\left[1+\left(\frac{\alpha_1\omega}{\mu}\right)^2\right]^{1/2}-1\right\}\right].$$

By taking the derivative of complex form of Eq.(2.35) and substituting into energy Eq.(2.21), we have

$$\rho C\frac{\partial\theta}{\partial t}=K\frac{\partial^2\theta}{\partial y^2}+Ay^2+B\sinh^2 b_1 y e^{2i\omega t}+Cy\sinh b_1 y e^{i\omega t}, \tag{2.36}$$

where

$$\begin{aligned}A &= \frac{\mu}{T_1-T_2}\left(\frac{\rho h^2 P_o}{2\mu}\right)^2,\\ B &= \frac{\mu}{T_1-T_2}\left(\frac{Q_o b_1}{i\omega\cosh b_1 h}\right)^2,\\ C &= 2\sqrt{AB}, b_1=\frac{\beta(1+i)}{h}.\end{aligned}$$

We invoke the boundary conditions for the temperature field as

$$\theta(0,t)=0,\quad \theta(h,t)=1. \tag{2.37}$$

We seek the solution of Eq.(2.36) as

$$\theta(y,t)=\theta_1(y)+\theta_2(y)e^{2i\omega t}+\theta_3(y)e^{i\omega t}. \tag{2.38}$$

Substituting Eq.(2.38) into Eqs.(2.36) and (2.37), we obtain three linear boundary value problems. The solution of those problems straightforward can be written as

$$\theta_1(y) = y(\frac{1}{h} + \frac{A}{12K}(y^3 - h^3), \tag{2.39}$$

$$\theta_2(y) = \frac{B}{2P^2}\left(1 - \frac{P^2 \cos 2b_1 y}{P^2 - 4b_1^2}\right) + k_1 \cosh py + k_2 \sinh py, \tag{2.40}$$

$$\theta_3(y) = -\frac{C}{P^2 - b_1^2}\left[y \sinh b_1 y + \frac{2b_1}{P^2 - b_1^2}\cosh b_1 y + C_1 \sinh py + C_2 \cosh py\right], \tag{2.41}$$

where

$$\begin{aligned} K_1 &= -\frac{B}{2P^2}\left(\frac{P^2}{P^2 - 4b_1^2}\right), \\ K_2 &= -\frac{B}{2p^2 \sinh ph}\left[\left(\frac{p^2}{p^2 - 4b_1^2}\right)\cosh ph - \left(1 - \frac{p^2 \cos 2b_1}{p^2 - 4b_1^2}\right)\right], \end{aligned}$$

$$\begin{aligned} C_1 &= \frac{C}{(p^2 - b_1^2)\sinh ph}\left[\frac{2b_1}{p^2 - b_1^2}(\cosh bh - \cosh ph + h \sinh ph)\right], \\ C_2 &= \frac{2Cb_1}{(p^2 - b_1^2)^2}. \end{aligned}$$

Chapter 3

Magnetohydrodynamic Rotating Flows of Burgers Fluid With Heat Transfer

3.1 Introduction

This chapter deals with the exact solution of unsteady, magnetohydrodynamics, rotating, incompressible flows of burgers fluid with heated boundaries. The entire system is assumed to rotate about the axis normal to the plate. The governing equations for this investigation are solved analytically for three physical problems namely (i) Flow Due to Heated Rigid Plate Oscillating in its Own Plane (ii) Flow Between Two Infinite Parallel Plates One of Which is Oscillating (iii) Time Periodic Poiseuille Flow in a Slot. The graphical results are shown for various physical parameters.

3.1.1 Problem Formulation

In a coordinate system rotating with the fluid, the governing equations of continuity and motion are (1.8) and (1.49). If we take the velocity field of the form

$$\mathrm{V} = (u(z,t), v(z,t), 0). \tag{3.1}$$

The condition of incompressibility is satisfied. Referred to the rotating frame of reference to unsteady motion of electrically conducting, incompressible Burgers' fluid is governed by Eqs. (1.37) to (1.39) and Eqs. (1.45) to (1.49). For our convenience we can write Eq.(1.49) as

$$\left(1+\lambda\frac{\partial}{\partial t}+\beta\frac{\partial^2}{\partial t^2}\right)\left[\frac{\partial \mathrm{F}}{\partial t}+2i\Omega F+\frac{1}{\rho}\left(\frac{\partial\widehat{\mathrm{P}}}{\partial x}+i\frac{\partial\widehat{\mathrm{P}}}{\partial y}\right)+\frac{\sigma B_o^2}{\rho}F\right]=\nu\left(1+\lambda r\frac{\partial}{\partial t}\right)\frac{\partial^2 F}{\partial t^2}. \tag{3.2}$$

Problem I:

3.2 Flow Due to Rigid Plate Oscillating in its Own Plane

Let us introduce the Cartesian coordinates system (x, y, z) and consider the motion of a conducting impressible Burgers fluid bounded by a plate $z = 0$. The plate at $z = 0$ is oscillating with velocity $U_o \cos \omega t$.The fluid occupies the space $z > 0$. The fluid and the plate are in a state of rigid body rotation with the constant angular velocity $\Omega = \Omega k$ (k is the unit vector in z-direction). A uniform magnetic field B_0 fixed relative to the fluid is acting parallel to the

z-axis. It is assumed that no applied voltage is applied, which implies the absence of an electric field. The magnetic Reynold's number is assumed to be very small.

The governing equation for the flow is Eq.(3.2). In the absence of pressure gradient Eq.(3.2) takes the following form

$$\left(1+\lambda\frac{\partial}{\partial t}+\beta\frac{\partial^2}{\partial t^2}\right)\left[\frac{\partial F}{\partial t}+2i\Omega F+\frac{\sigma B_o^2}{\rho}F\right]=\nu\left(1+\lambda r\frac{\partial}{\partial t}\right)\frac{\partial^2 F}{\partial z^2}. \quad (3.3)$$

The corresponding boundary conditions are

$$F(0,t)=U\cos\omega t, \quad (3.4)$$

$$F(\infty,t)\to 0 \text{ as } z\to\infty. \quad (3.5)$$

Introducing the following non-dimensional variables

$$\begin{aligned}\eta_1 &= \frac{U}{\nu}z, \quad \tau=\omega t, \quad F{=}UG_1\\ A &= \frac{\omega\nu}{U^2}, \quad \beta\omega^2=R_1, \quad \frac{\Omega}{\omega}=R_3\\ \omega\lambda &= R_2, \quad M=\frac{\sigma B_o^2}{\rho\omega}, \quad \omega\lambda r=R_{4.}\end{aligned} \quad (3.6)$$

Making use of Eq.(3.6), Eqs.(3.3) to (3.5) takes the following form

$$\begin{aligned}& AR_1\frac{\partial^3 G_1}{\partial\tau^3}+A\left[R_2+R_1(2iR_3+M)\right]\frac{\partial^2 G_1}{\partial\tau^2}\\ & +A\left[1+R_2(2iR_3+M)\right]\frac{\partial G_1}{\partial\tau}+A\left[2iR_3+M\right]G_1\\ =\ & \frac{\partial^2 G_1}{\partial\eta^2}+R_4\frac{\partial^3 G_1}{\partial\eta^2\partial\tau},\end{aligned} \quad (3.7)$$

$$\begin{aligned} G_1(0,\tau) &= \cos\tau, \\ G_1(\infty,\tau) &\rightarrow 0. \end{aligned} \tag{3.8}$$

Solution of the Problem

For the solution of the above boundary value problem we take the assumed form of the solution of the form

$$G_1(\eta_1,\tau) = Re[G_o(\eta_1)e^{i\tau}]. \tag{3.9}$$

With the help of Eq.(3.9), Eq.(3.7) and the boundary conditions (3.8) takes the following form

$$(1+iR_4)\frac{d^2G_o}{d\eta_1^2} - (a_1+ib_1)G_o = 0, \tag{3.10}$$

$$\begin{aligned} G_o(0) &= 1, \\ G_o &\rightarrow 0 \ \ as \ \ \eta_1 \rightarrow \infty. \end{aligned} \tag{3.11}$$

where

$$\begin{aligned} a_1 &= AM - AR_2 - AR_1M - 2AR_2R_3, \\ b_1 &= A - AR_2M - 2AR_3 - 2AR_1R_3. \end{aligned}$$

The solution of above differential equation with the help of boundary condition (3.11), can be written as

$$G_o(\eta_1) = e^{-(m_1 + in_1)\eta_1}, \tag{3.12}$$

where

$$\begin{aligned}
m_1 &= \sqrt{\frac{a_1 + b_1 R_4 + \sqrt{(a_1^2 + b_1^2)(1 + R_4^2)}}{2(1 + R_4^2)}}, \\
n_1 &= \frac{b_1 - a_1 R_4}{\sqrt{2(1 + R_4^2)}\sqrt{a_1 + b_1 R_4 + \sqrt{(a_1^2 + b_1^2)(1 + R_4^2)}}}.
\end{aligned}$$

Using Eq.(3.12), the solution (3.9), can be written as

$$G_1(\eta_1, \tau) = e^{-m_1\eta_1} \cos(\tau - n_1\eta_1). \tag{3.13}$$

Problem 2:

3.3 Flow Between Two Infinite Parallel Plates One of Which is Oscillating

Let us consider an incompressible rotating, Burgers fluid bounded by two infinite parallel plates. A constant magnetic field B_o is applied along z-direction. The distance between the plates is d. The lower plate at $z = 0$ oscillates with velocity $U\cos\omega t$ while the upper plate is at rest. The governing equation for the problem is Eq.(3.3) and the boundary condition are defined as

$$\begin{aligned}
F(0,t) &= U\cos\omega t, \\
F(d,t) &= 0.
\end{aligned} \tag{3.14}$$

Making use of Eq.(3.6) and $\eta_2 = \frac{z}{d}$, $A_1 = \frac{wd^2}{\nu}$, $G_2 = \frac{\mathrm{F}}{\mathrm{U}}$, into Eqs.(3.3) and (3.14), we obtain

$$\begin{aligned} & A_1 R_1 \frac{\partial^3 G_2}{\partial \tau^3} + A_1 \left[R_2 + R_1(2iR_3 + M) \right] \frac{\partial^2 G_2}{\partial \tau^2} \\ & + A_1 \left[1 + R_2(2iR_3 + M) \right] \frac{\partial G_2}{\partial \tau} + A_1(2iR_3 + M) G_2 \\ = \; & \frac{\partial^3 G_2}{\partial \eta_2^2} + R_4 \frac{\partial^3 G_2}{\partial \eta_2^2 \partial \tau^2}, \end{aligned} \tag{3.15}$$

$$\begin{aligned} G_2(0,\tau) &= \cos\tau, \\ G_2(1,\tau) &= 0. \end{aligned} \tag{3.16}$$

Solution of the Problem

To avoid the repetition the solution of this problem can be obtained by similar procedure as discussed in previous section the solution thus can be written as

$$G_2(\eta_2,\tau) = \frac{e^{-p_1\eta_2} \sin p_2(1-\eta_2) \cos\tau}{\sin p_2}, \tag{3.17}$$

where

$$\begin{aligned} p_1 &= \sqrt{\frac{a_1 + b_1 R_4 + \sqrt{(a_1^2 + b_1^2)\left(1 + R_4^2\right)}}{2\left(1 + R_4^2\right)}}, \\ p_2 &= \frac{b_1 - a_1 R_4}{\sqrt{2\left(1 + R_4^2\right)}\sqrt{a_1 + b_1 R_4 + \sqrt{(a_1^2 + b_1^2)\left(1 + R_4^2\right)}}}. \end{aligned}$$

Problem 3:

3.4 Time Periodic Plane Poiseuille Flow

Let us consider two infinite parallel plates at $z = \pm h_1$ and the modified pressure gradient is assumed to be of the following form

$$C = \left(\frac{\partial \widehat{\mathrm{P}}}{\partial x} + i\frac{\partial \widehat{\mathrm{P}}}{\partial y}\right) = -(P+Q\cos\omega t). \tag{3.18}$$

The governing equation for the problem is Eq.(3.2) and the boundary conditions are

$$\begin{aligned} F(h_1, \tau) &= 0, \\ F(-h_1, \tau) &= 0. \end{aligned} \tag{3.19}$$

Introducing the following non-dimensional variables

$$\tau = \omega t, \quad \eta_2 = \frac{Z}{h_1}, \quad G_3 = \frac{F}{U}. \tag{3.20}$$

Making use of Eq.(3.20), Eqs.(3.2) and (3.19) take the following form

$$\begin{aligned} & A_1 R_1 \frac{\partial^3 G_3}{\partial \tau^3} + A_1\left[R_2 + R_1(2iR_3 + M)\right]\frac{\partial^2 G_3}{\partial \tau^2} \\ & + A_1\left[1 + R_2(2iR_3 + M)\right]\frac{\partial G_3}{\partial \tau} + A_1(2iR_3 + M)G_3 \\ & + \left(1 + R_1\frac{\partial}{\partial \tau} + R_2\frac{\partial^2}{\partial \tau^2}\right)(P + Q\cos\tau) \\ = & \frac{\partial^2 G_3}{\partial \eta_3^2} + R_4\frac{\partial^3 G_3}{\partial \eta_3^2 \partial \tau}, \end{aligned} \tag{3.21}$$

$$
\begin{aligned}
G_3(1,\tau) &= 0, \\
G_3(-1,\tau) &= 0.
\end{aligned} \tag{3.22}
$$

Solution of the Problem

The solution of the above problem satisfying the boundary conditions (3.22) can be written as

$$
\begin{aligned}
\mathrm{G}_3(\eta_3,\tau) &= \frac{-R_5 e^{-a_2\eta_3}}{A_1\left(2iR_3+M\right)}\left[\frac{\cosh a_2 \cos b_2\eta_3}{\cosh b_2}+\frac{\sinh a_2 \sin b_2\eta_3}{\sin b_2}\right] \\
&+\frac{R_5}{A_1\left(2iR_3+M\right)}+\cos\tau\left[\frac{-R_6 e^{-a_3\eta_3}}{c_5+id_5}\left\{\begin{array}{c}\frac{\cosh a_3 \cos b_3\eta_3}{\cosh b_3} \\ +\frac{\sinh a_3 \sin b_3\eta_3}{\sin b_3}\end{array}\right\}+\frac{R_6}{c_5+id_5}\right],
\end{aligned} \tag{3.23}
$$

$$
a_2=\sqrt{\frac{A_1\left(M+\sqrt{M^2+4R_3^2}\right)}{2}},
$$

$$
b_2=\frac{A_1 R_3}{\sqrt{\frac{A_1\left(M+\sqrt{M^2+4R_3^2}\right)}{2}}},
$$

$$
\begin{aligned}
c_5 &= A_1 M - A_1 R_2 - A_1 R_1 M - 2A_1 R_2 R_3, \\
d_5 &= A_1 + A_1 R_2 M + 2A_1 R_3 - 2A_1 R_1 R_3 - A_1 R,
\end{aligned}
$$

$$
a_3=\sqrt{\frac{(c_5+R_4 d_5)+\sqrt{(c_5+R_4 d_5)^2+(d_5-R_4 c_5)^2}}{2(1+R_4^2)}},
$$

$$b_3 = \sqrt{\frac{-(c_5 + R_4 d_5) + \sqrt{(c_5 + R_4 d_5)^2 + (d_5 - R_4 c_5)^2}}{2(1 + R_4^2)}},$$

$$R_5 = \frac{Q\rho d^2}{\nu}, R_6 = \frac{P\rho d^2}{\nu}.$$

3.5 Heat Transfer Analysis

3.5.1 Flow Due to Heated Rigid Plate Oscillating in its Own Plane

Let us consider the Burgers fluid occupy the space above the plate $z = 0$. The plate at $z = 0$ is oscillating with velocity $U_o \cos\omega t$. Let T_0 and T_∞ denotes the temperature of the plate and fluid at infinity. The following solution already calculated in eq.(3.13) can be written as

$$G_1(\eta_1, \tau) = \mathrm{Re}[e^{i\tau} e^{-\eta(m_1 + in_1)}]. \quad (3.24)$$

Introducing the following non-dimensional variables

$$\tau = \omega t, \ \eta_1 = \frac{UZ}{\nu}, \ \theta_1 = \frac{T' - T_\infty}{T_o - T_\infty}, F = UG_1, \quad (3.25)$$

the energy equation will be of the form

$$A\frac{\partial \theta_1}{\partial \tau} = \frac{1}{\mathrm{Pr}} \frac{\partial^2 \theta_1}{\partial \eta^2} + Ec\frac{\partial G_1}{\partial \eta} \frac{\partial \bar{G}_1}{\partial \eta},$$

Let

$$\theta(\eta_1, \tau) = \theta_1(\eta_1) e^{2i\tau}, \quad (3.26)$$

be solution of energy eq.(1.23)

$$A\frac{\partial\theta}{\partial\tau}=\frac{1}{\Pr}\frac{\partial^2\theta}{\partial\eta^2}+Ec\frac{\partial G}{\partial\eta}\frac{\partial\bar{G}}{\partial\eta},$$

using Eqs.(3.24), (3.25)and (3.26) in above energy equation,the simplified form is

$$-Ec(m_1+in_1)e^{-2\eta_1(m_1+in_1)}=\frac{1}{\Pr}\frac{d^2\theta_1}{d\eta_1^2}-2iA\theta_1(\eta_1), \tag{3.27}$$

The corresponding boundary conditions are

$$\theta_1(0)=1,\theta_1(\infty)=0. \tag{3.28}$$

The solution of Eq.(3.26) satisfying the boundary condition (3.28) are

$$\theta_1(\eta_1,\tau)=[c_1e^{-(1+i)\eta_1\sqrt{A\Pr}}+c_2e^{-2\eta_1(m_1+in_1)}]e^{2i\tau}, \tag{3.29}$$

where

$$c_1=\frac{2iA\Pr-(m_1+in_1)^2(4+Ec\Pr)}{2iA\Pr-4(m_1+in_1)^2},$$

$$c_2=\frac{(m_1+in_1)^2Ec\Pr}{2iA\Pr-4(m_1+in_1)^2}.$$

3.5.2 Flow Between Two Infinite Parallel Plates One of Which is Oscillating

Let us consider the Burgers fluid occupy the slot between two Parallel Plates. The distance between the plates is d. The lower plate at $z=0$ oscillates with velocity $U\cos\omega t$ while the upper plate is at rest. Let T_1 and T_2 denotes the temperature of the lower and upper plates respectively.

The velocity field is already computed in eq. (3.17) and is given by

$$G_2(\eta_2, \tau) = \mathrm{Re}\left[\frac{e^{-p_1\eta_2}\sin p_2(1-\eta_2)e^{i\tau}}{\sin p_2}\right] \tag{3.30}$$

With the help of the following non-dimensional parameters

$$\tau = \omega t,\ \eta_2 = \frac{Z}{d},\ \theta_2 = \frac{T' - T_2}{T_1 - T_2}, F = UG_2, \tag{3.31}$$

the energy Eq.(1.31) takes the following form

$$A\frac{\partial \theta_2}{\partial \tau} = \frac{1}{\Pr}\frac{\partial^2 \theta_2}{\partial \eta_2^2} + Ec\frac{\partial G_2}{\partial \eta_2}\frac{\partial \bar{G}_2}{\partial \eta_2}, \tag{3.32}$$

and the boundary condition takes the form

$$\theta_2(0) = 1, \theta_2(1) = 0. \tag{3.33}$$

The solution of eq.(3.32) satisfying the boundary conditions (3.33) is

$$\theta_2(\eta_2,\tau) = e^{2i\tau}\left[\begin{array}{c} e^{-\eta_2\sqrt{A_1\Pr}}\left(c_3 Cos\sqrt{A_1\Pr}\eta_2 + c_4\,\mathrm{Si}\,n\sqrt{A_1\Pr}\eta_2\right) - \frac{Ec\Pr(p_1^2+p_2^2)e^{-2p_1\eta_2}}{2\,\mathrm{Si}\,n^2 p_2(4p_1^2 - 2iA_1\Pr)} + \\ e^{-2p_1\eta_2}\left\{\frac{(4p_1^2-4p_2^2-2iA_1\Pr)Cos(2p_2+2\tan^{-1}\frac{p_2}{p_1}-2p_2\eta_2)+8p_1p_2\,\mathrm{Si}\,n(2p_2+2\tan^{-1}\frac{p_2}{p_1}-2p_2\eta_2)}{(4p_1^2-4p_2^2-2iA_1\Pr)^2+16p_1^2p_2^2}\right\}\end{array}\right]$$

where

$$c_3 = -\frac{Ec\Pr(p_1^2+p_2^2)}{2\,\mathrm{Si}\,n^2 p_2}\left[\frac{\begin{array}{c}(4p_1^2-4p_2^2-2iA_1\Pr)Cos(2p_2+2\tan^{-1}\frac{p_2}{p_1})\\ +8p_1p_2\,\mathrm{Si}\,n(2p_2+2\tan^{-1}\frac{p_2}{p_1})\end{array}}{(4p_1^2-4p_2^2-2iA_1\Pr)^2+16p_1^2p_2^2}\right]$$
$$+1+\frac{Ec\Pr(p_1^2+p_2^2)}{2\,\mathrm{Si}\,n^2 p_2(4p_1^2-2iA_1\Pr)},$$

and

$$c_4 = \frac{-Ec\,\mathrm{Pr}(p_1^2+p_2^2)}{2\,\mathrm{Si}\,n^2 p_2}\frac{e^{-(2p_1-\sqrt{A_1\mathrm{Pr}})}}{\mathrm{Si}\,n\sqrt{A_1\mathrm{Pr}}}\left[\frac{\begin{array}{c}(4p_1^2-4p_2^2-2iA_1\mathrm{Pr})Cos(2\tan^{-1}\frac{p_2}{p_1})\\ +8p_1p_2\,\mathrm{Si}\,n(2\tan^{-1}\frac{p_2}{p_1})\end{array}}{(4p_1^2-4p_2^2-2iA_1\mathrm{Pr})^2+16p_1^2p_2^2}\right]$$

$$-c_3 Cot\sqrt{A_1\mathrm{Pr}}+\frac{Ec\,\mathrm{Pr}(p_1^2+p_2^2)}{2\,\mathrm{Si}\,n^2p_2(4p_1^2-2iA_1\mathrm{Pr})}\frac{e^{-(2p_1-\sqrt{A_1\mathrm{Pr}})}}{\mathrm{Si}\,n\sqrt{A_1\mathrm{Pr}}}.$$

3.5.3 Time Periodic Plane Poiseuille Flow

Let us consider two infinite parallel plates at $z=\pm h_1$.The distance between the plates is $2h_1$ let T_1 and T_2 denotes the temperature of the lower and upper plates respectively. The solution for momentum equation is already computed and for our convenience we can write it as

$$\mathrm{G}_3(\eta_3,\tau) = \frac{-R_5e^{-a_2\eta_3}}{A_1(2iR_3+M)}\left[\frac{\cosh a_2\cos b_2\eta_3}{\cosh b_2}+\frac{\sinh a_2\sin b_2\eta_3}{\sin b_2}\right] \tag{3.34}$$

$$+\frac{R_5}{A_1(2iR_3+M)}+\mathrm{Re}\,e^{i\tau}\left[\frac{-R_6e^{-a_3\eta_3}}{c_5+id_5}\left\{\begin{array}{c}\frac{\cosh a_3\cos b_3\eta_3}{\cosh b_3}\\ +\frac{\sinh a_3\sin b_3\eta_3}{\sin b_3}\end{array}\right\}+\frac{R_6}{c_5+id_5}\right],$$

Introducing the following non-dimensional variables

$$\tau=\omega t,\eta_3=\frac{Z}{h_1},\theta_3=\frac{T'-T_1}{T_1-T_2},F=UG_3, \tag{3.35}$$

the energy equation(1.31) will be of the form

$$A\frac{\partial\theta_3}{\partial\tau}=\frac{1}{\mathrm{Pr}}\frac{\partial^2\theta_3}{\partial\eta_3^2}+Ec\frac{\partial G_3}{\partial\eta_3}\frac{\partial\bar{G}_3}{\partial\eta_3}, \tag{3.36}$$

The corresponding boundary conditions are

$$\theta_3(1)=-1,\theta_3(-1)=0. \tag{3.37}$$

By taking the derivative of terms involving $e^{i\tau}$ in Eq.(3.34) and the boundary conditions (3.37) and substituting it into energy Eq.(3.36),the solution can be written as

$$\theta_3(\eta_3,\tau) = e^{2i\tau}\left[\begin{array}{c} e^{-\eta_3\sqrt{A_1\Pr}}\left(E_1 Cos\sqrt{A_1\Pr}\eta_3 + F_1 \operatorname{Si}n\sqrt{A_1\Pr}\eta_3\right) \\ +\frac{Ec^2R_6^2e^{-2a_3\eta_3}}{d_3^2+d_4^2}\left(\begin{array}{c} \frac{E^2+F^2}{4iA_1\Pr-8a_3^2}+\frac{(E^2-F^2)(2a_3^2-2b_3^2-iA_1\Pr)}{(4a_3^2-4b_3^2-2iA_1\Pr)^2+64a_3^2b_3^2}Cos2b_3\eta_3 \\ -\frac{EF(4a_3^2-4b_3^2-2iA_1\Pr)+4(E^2-F^2)a_3b_3}{(4a_3^2-4b_3^2-2iA_1\Pr)^2+64a_3^2b_3^2}\operatorname{Si}n2b_3\eta_3 \end{array}\right)\end{array}\right], \quad (3.38)$$

where

$$E = \frac{a_3 \operatorname{Si}nha_3}{\operatorname{Si}nb_3} + \frac{b_3 Cosha_3}{Cosb_3},$$

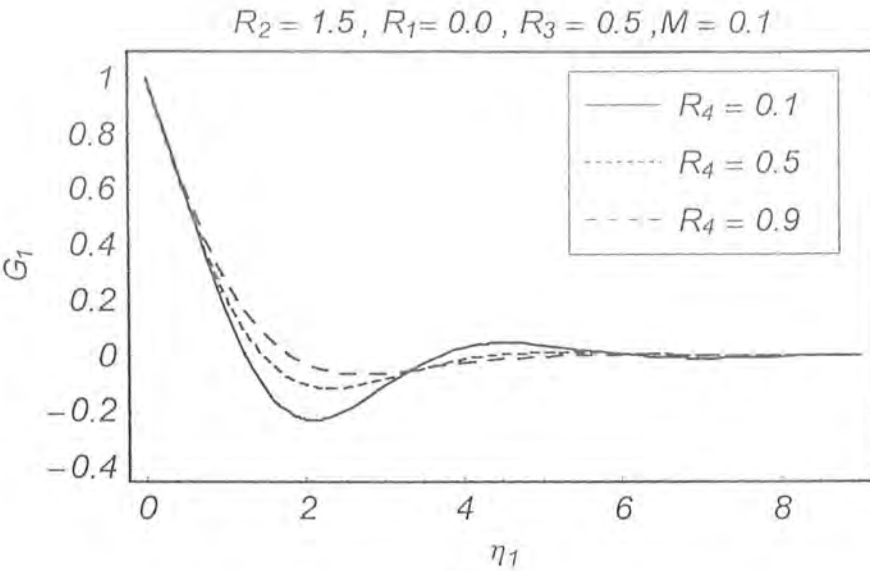
$$F = \frac{a_3 Cosha_3}{Cosb_3} - \frac{b_3 \operatorname{Si}nha_3}{\operatorname{Si}nb_3},$$

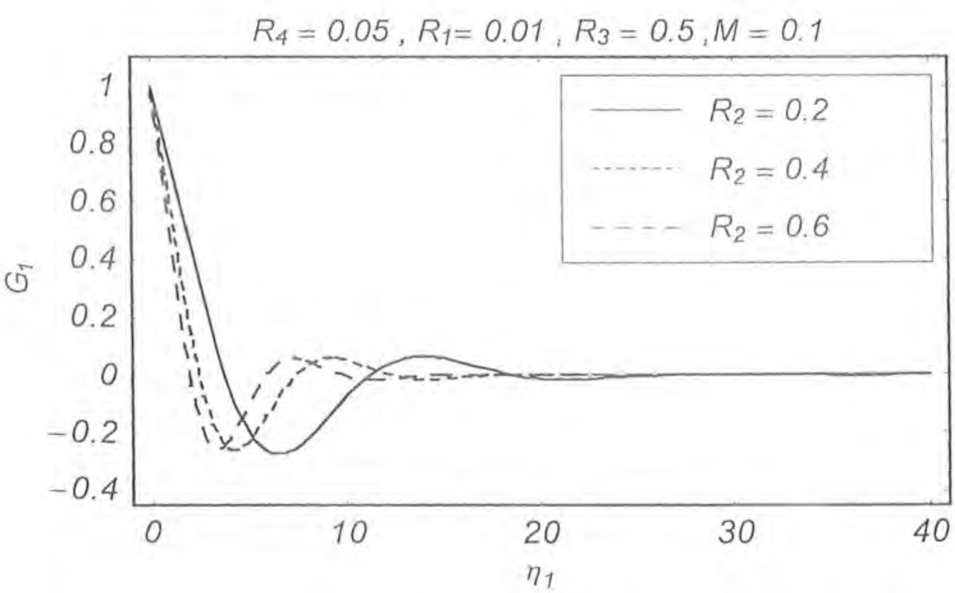
$$\begin{aligned} E_1 &= Sec\sqrt{A_1\Pr}\left[-\frac{e^{\sqrt{A_1\Pr}}}{2} - \frac{Ec^2R_6^2e^{-2a_3\eta_3}}{d_3^2+d_4^2}\left\{\left(\frac{E^2+F^2}{4iA_1\Pr-8a_3^2}\right.\right.\right. \\ &\left.+\frac{(E^2-F^2)(2a_3^2-2b_3^2-iA_1\Pr)}{(4a_3^2-4b_3^2-2iA_1\Pr)^2+64a_3^2b_3^2}Cos2b_3\right)Cosh(2a_3-\sqrt{A_1\Pr}) \\ &\left.\left.+\frac{EF(4a_3^2-4b_3^2-2iA_1\Pr)+4(E^2-F^2)a_3b_3}{(4a_3^2-4b_3^2-2iA_1\Pr)^2+64a_3^2b_3^2}\operatorname{Si}n2b_3\operatorname{Si}nh(2a_3-\sqrt{A_1\Pr})\right\}\right], \end{aligned}$$

$$\begin{aligned} F_1 &= Co\sec\sqrt{A_1\Pr}\left[\frac{1}{2} + \frac{Ec^2R_6^2e^{-2a_3\eta_3}}{d_3^2+d_4^2}\left\{\left(\frac{E^2+F^2}{4iA_1\Pr-8a_3^2}\right.\right.\right. \\ &\left.+\frac{(E^2-F^2)(2a_3^2-2b_3^2-iA_1\Pr)}{(4a_3^2-4b_3^2-2iA_1\Pr)^2+64a_3^2b_3^2}Cos2b_3\right)\operatorname{Si}nh(2a_3-\sqrt{A_1\Pr}) \\ &\left.\left.+\frac{EF(4a_3^2-4b_3^2-2iA_1\Pr)+4(E^2-F^2)a_3b_3}{(4a_3^2-4b_3^2-2iA_1\Pr)^2+64a_3^2b_3^2}\operatorname{Si}n2b_3Cosh(2a_3-\sqrt{A_1\Pr})\right\}\right]. \end{aligned}$$

3.6 Results And Discussion

We have plotted the velocity and temperature fields for three kinds of flow problems namely (1) Flow due to heated rigid plate oscillating in its own plane (2) Flow between two heated parallel plates one of which is oscillating (3) Time periodic poiseuille flow. The influence of different physical parameters such as retardation time R_4, relaxation times R_1 and R_2, MHD parameter M and rotation parameter R_3 on the velocity and temperature are investigated. Figure has been prepared to show the variation of velocity with the increase in R_4. It is seen from the figure that velocity oscillates near the plate while it achieves steady state behavior away from the plate. It is also seen that with the increase in R_4 the velocity near the plate increases. Figure 2 shows the behavior of R_2 on velocity field. In this figure the velocity behavior is opposite to that of the velocity behavior in the case of R_4. The same effects are seen for the variation of R_1 on velocity field in figure 3. The effects of M on velocity field are shown in figure 4. It is shown from the figure that with the increase in M the velocity near the plate increases but it goes to zero earlier as compared to the variation of R_2 and R_4. Thus we say that with the increase in M the layer thickness reduces. In figure 5, we have seen the effect of rotation R_3 on velocity field. It is seen from the figure that the rotation also causes the reduction of the boundary layer. Figures 6 to 10 are plotted when the flow is between the two parallel plates. From these figures it is shown that with the increase in R_2 and R_1, the velocity field increases while the increase in R_4 and M the velocity decreases. Figure 11 to 15 are plotted for the case of time periodic poiseuille flow, the similar effects are seen from these figures as we have already discussed. Figures 16 to 21 shown the behavior of Eckert number E_c and Prandtl number P_r on temperature field. In figure 16, the temperature field decreases with the increase in P_r while in figure 17, there is minor effect on temperature field with the variation of E_c. In figure 18, the temperature field increases with the increase in E_c.Figure 19 shows the decrease in temperature field with the increase in P_r. In figure 20 and 21, it is seen that the temperature field decreases with the increase in both E_c and P_r.


$R_2 = 1.5$, $R_1 = 0.0$, $R_3 = 0.5$, $M = 0.1$
$R_4 = 0.1$
$R_4 = 0.5$
$R_4 = 0.9$
G_1
1
0.8
0.6
0.4
0.2
0
−0.2
−0.4
0
2
4
6
8
η_1
Fig. 1


$R_4 = 0.05$, $R_1 = 0.01$, $R_3 = 0.5$, $M = 0.1$
$R_2 = 0.2$
$R_2 = 0.4$
$R_2 = 0.6$
G_1
1
0.8
0.6
0.4
0.2
0
−0.2
−0.4
0
10
20
30
40
η_1
Fig. 2

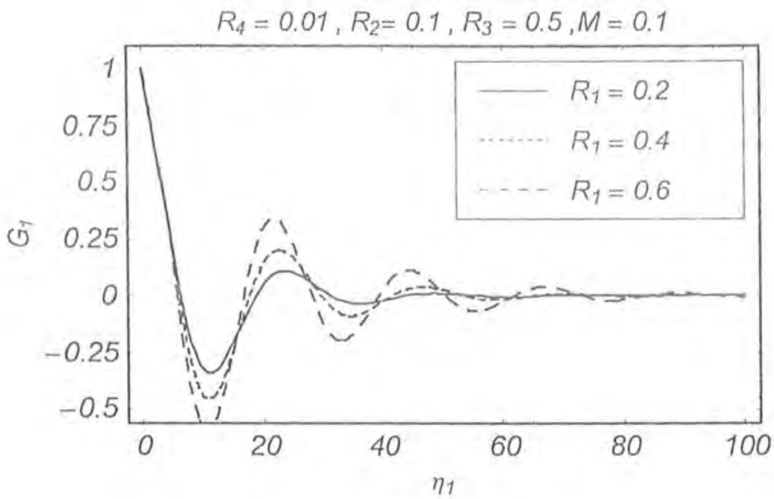

$R_4 = 0.01$, $R_2 = 0.1$, $R_3 = 0.5$, $M = 0.1$
$R_1 = 0.2$
$R_1 = 0.4$
$R_1 = 0.6$
G_1
η_1

Fig. 3

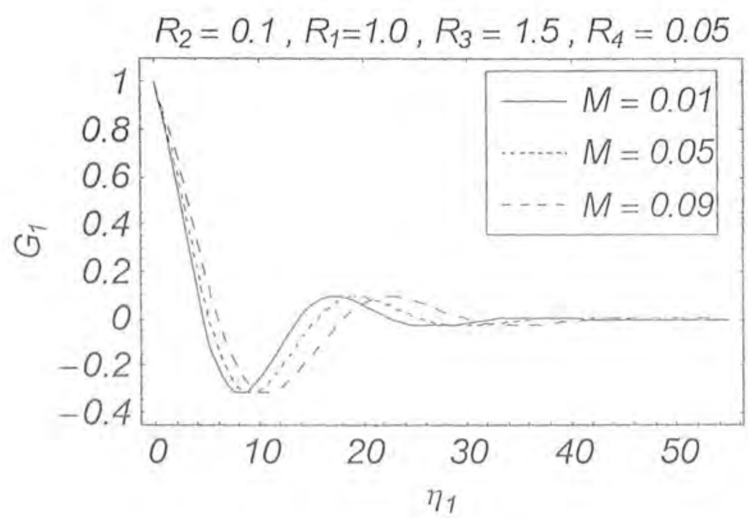
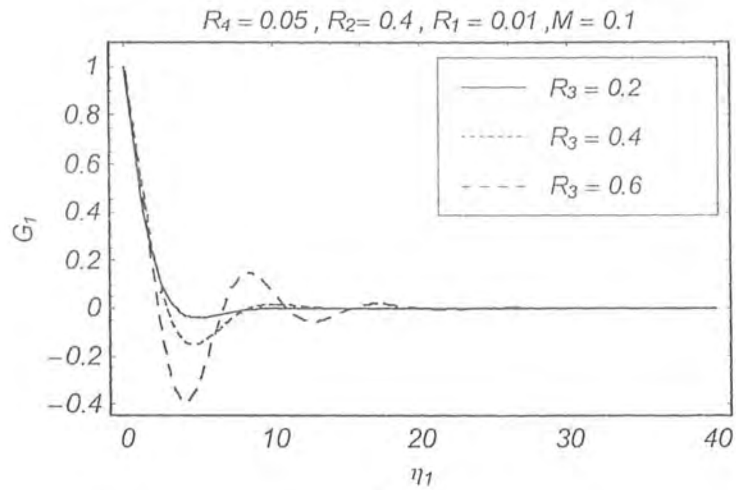
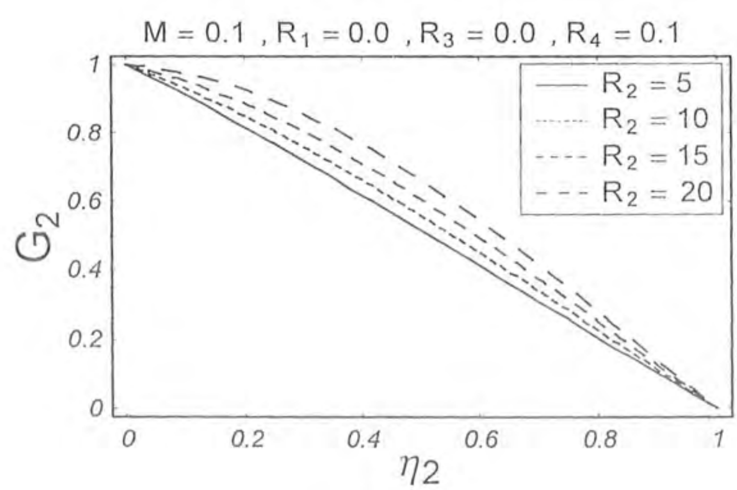

$R_2 = 0.1$, $R_1 = 1.0$, $R_3 = 1.5$, $R_4 = 0.05$
$M = 0.01$
$M = 0.05$
$M = 0.09$
G_1
η_1

Fig. 4



$R_4 = 0.05$, $R_2 = 0.4$, $R_1 = 0.01$, $M = 0.1$
$R_3 = 0.2$
$R_3 = 0.4$
$R_3 = 0.6$
G_1
η_1

Fig. 5



$M = 0.1$, $R_1 = 0.0$, $R_3 = 0.0$, $R_4 = 0.1$
$R_2 = 5$
$R_2 = 10$
$R_2 = 15$
$R_2 = 20$
G_2
$\eta 2$

Fig. 6

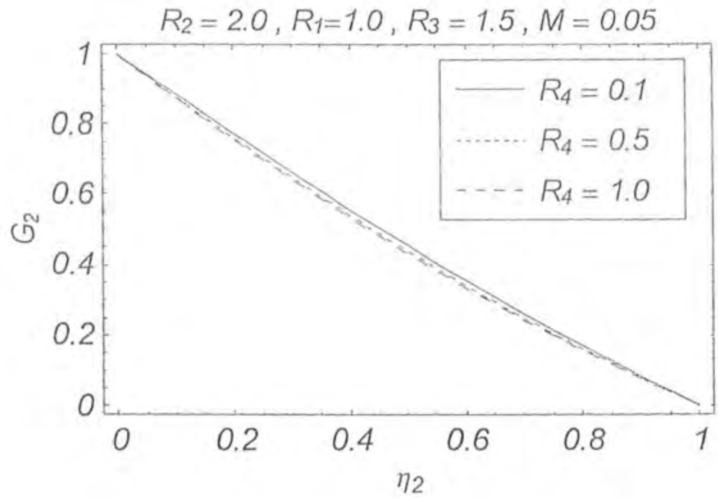

$R_2 = 2.0$, $R_1 = 1.0$, $R_3 = 1.5$, $M = 0.05$
$R_4 = 0.1$
$R_4 = 0.5$
$R_4 = 1.0$
G_2
η_2

Fig. 7

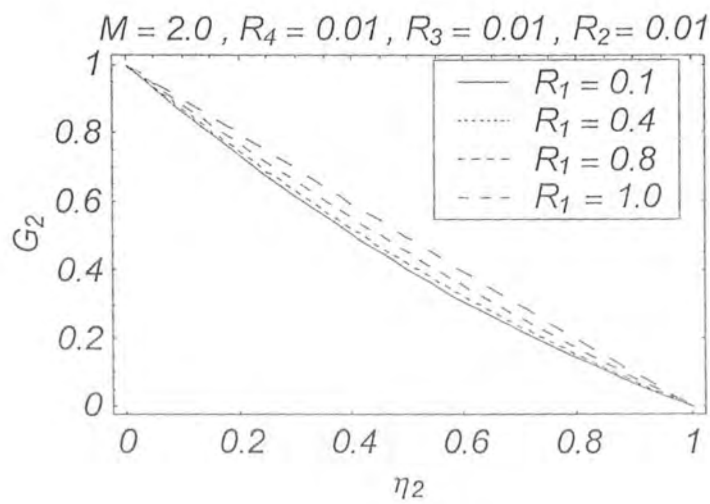

$M = 2.0$, $R_4 = 0.01$, $R_3 = 0.01$, $R_2 = 0.01$
$R_1 = 0.1$
$R_1 = 0.4$
$R_1 = 0.8$
$R_1 = 1.0$
G_2
η_2

Fig. 8

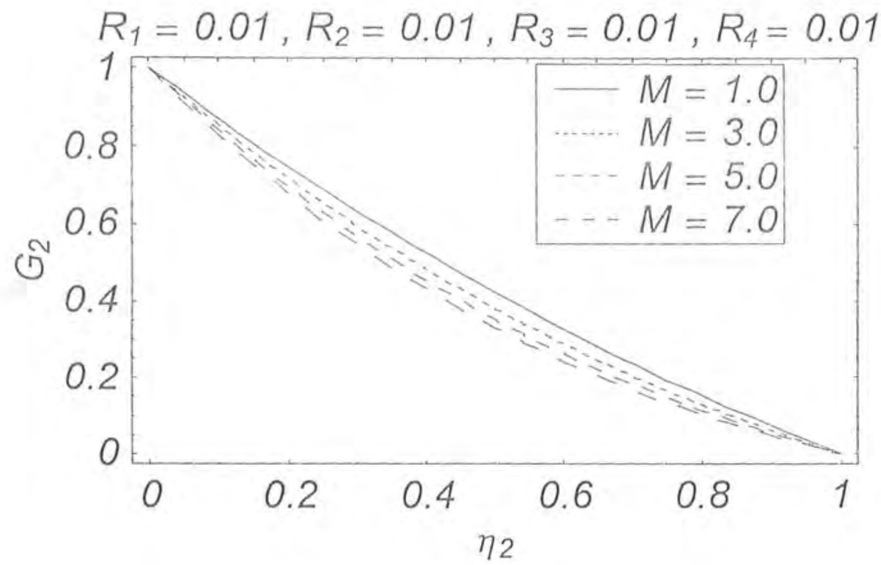

$R_1 = 0.01$, $R_2 = 0.01$, $R_3 = 0.01$, $R_4 = 0.01$
M = 1.0
M = 3.0
M = 5.0
M = 7.0
G_2
0
0.2
0.4
0.6
0.8
1
η_2

Fig. 9

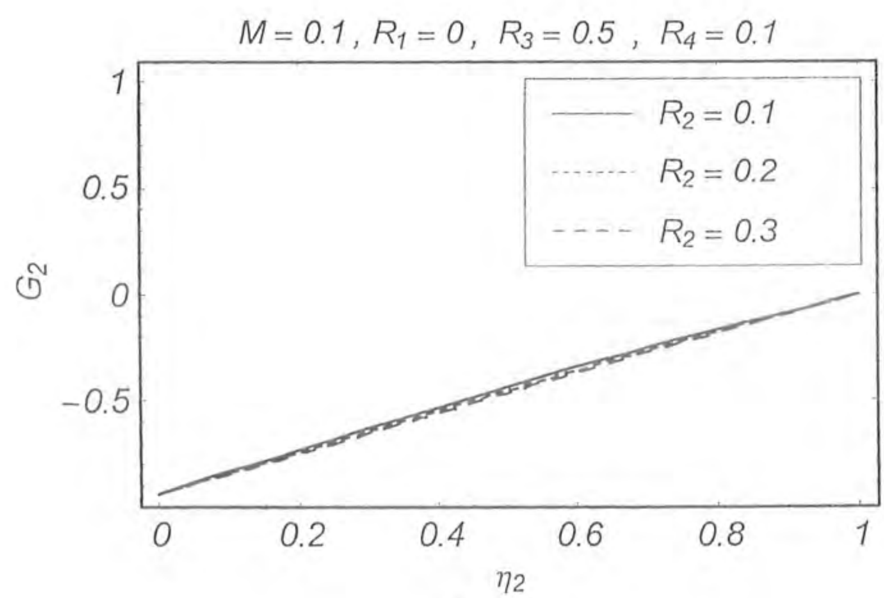
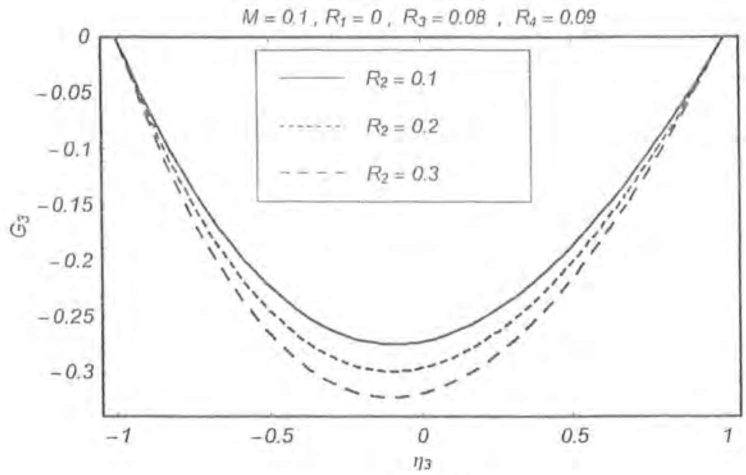
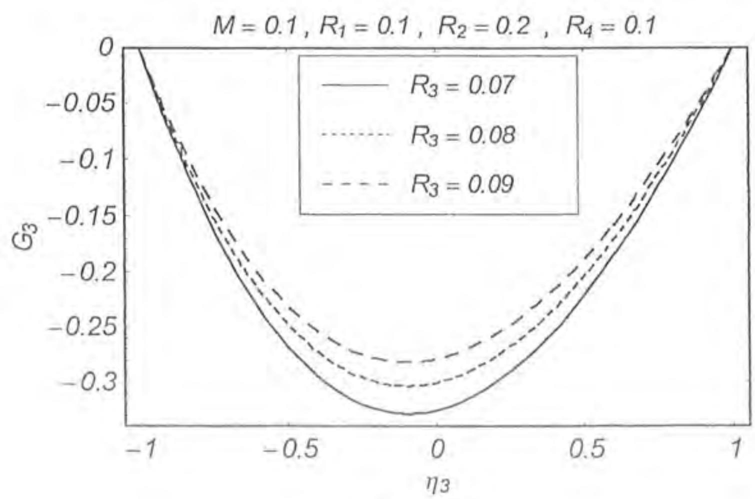

$M = 0.1$, $R_1 = 0$, $R_3 = 0.5$, $R_4 = 0.1$
$R_2 = 0.1$
$R_2 = 0.2$
$R_2 = 0.3$
G_2
1
0.5
0
−0.5
0
0.2
0.4
0.6
0.8
1
η_2

Fig. 10



M = 0.1 , R1 = 0 , R3 = 0.08 , R4 = 0.09
R2 = 0.1
R2 = 0.2
R2 = 0.3
G3
η3
0
-0.05
-0.1
-0.15
-0.2
-0.25
-0.3
-1
-0.5
0
0.5
1

Fig. 11



M = 0.1 , R1 = 0.1 , R2 = 0.2 , R4 = 0.1
R3 = 0.07
R3 = 0.08
R3 = 0.09
G3
η3
0
-0.05
-0.1
-0.15
-0.2
-0.25
-0.3
-1
-0.5
0
0.5
1

Fig. 12

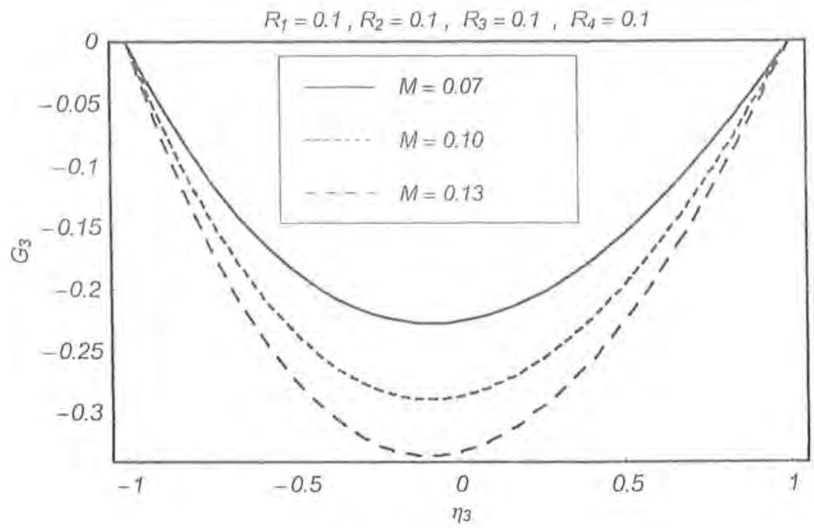

$R_1 = 0.1$, $R_2 = 0.1$, $R_3 = 0.1$, $R_4 = 0.1$
M = 0.07
M = 0.10
M = 0.13
G_3
η_3

Fig. 15

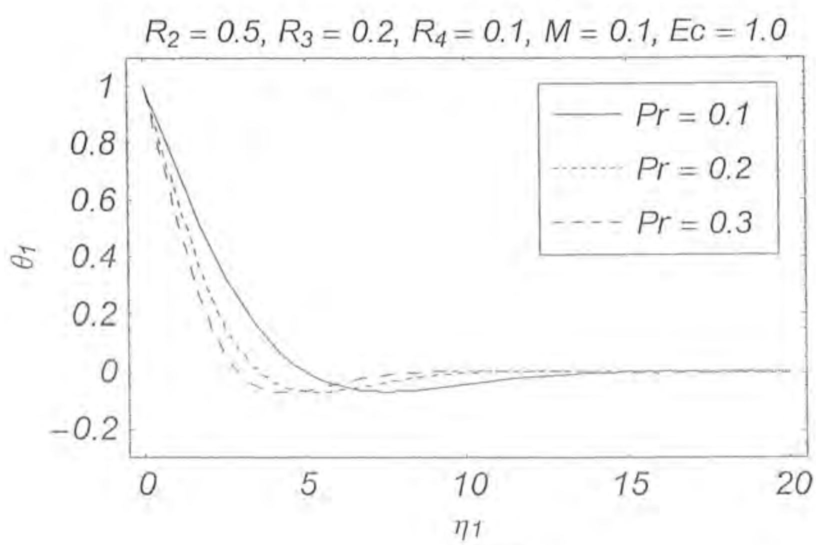
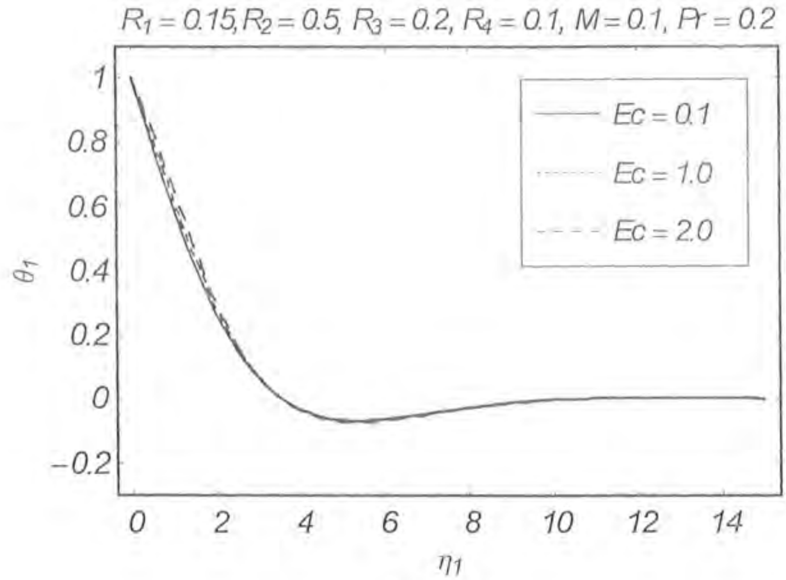
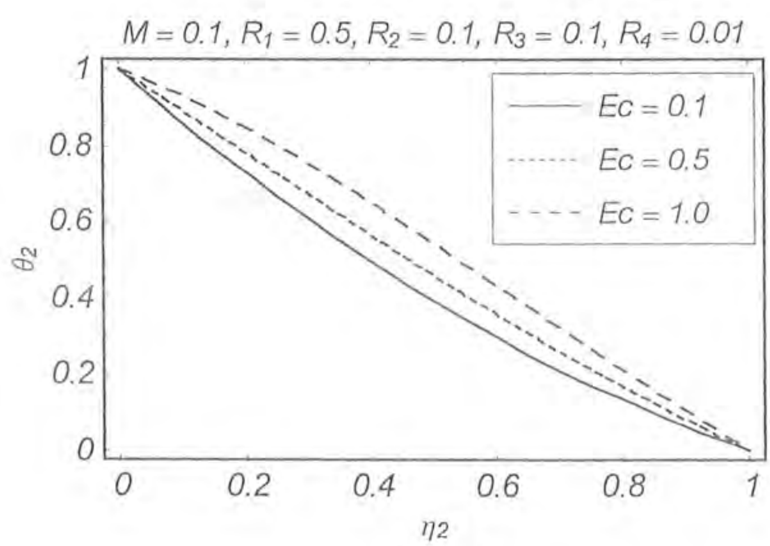

$R_2 = 0.5$, $R_3 = 0.2$, $R_4 = 0.1$, $M = 0.1$, $Ec = 1.0$
Pr = 0.1
Pr = 0.2
Pr = 0.3
θ_1
η_1

Fig. 16



R1 = 0.15, R2 = 0.5, R3 = 0.2, R4 = 0.1, M = 0.1, Pr = 0.2
Ec = 0.1
Ec = 1.0
Ec = 2.0
θ1
η1

Fig. 17



M = 0.1, R1 = 0.5, R2 = 0.1, R3 = 0.1, R4 = 0.01
Ec = 0.1
Ec = 0.5
Ec = 1.0
θ2
η2

Fig. 18

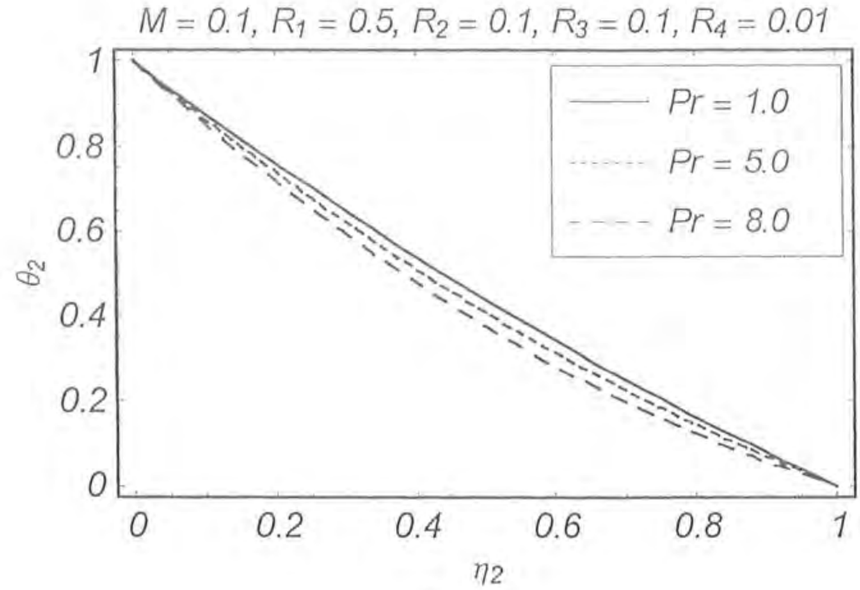

M = 0.1, R1 = 0.5, R2 = 0.1, R3 = 0.1, R4 = 0.01
Pr = 1.0
Pr = 5.0
Pr = 8.0
θ2
η2

Fig. 19

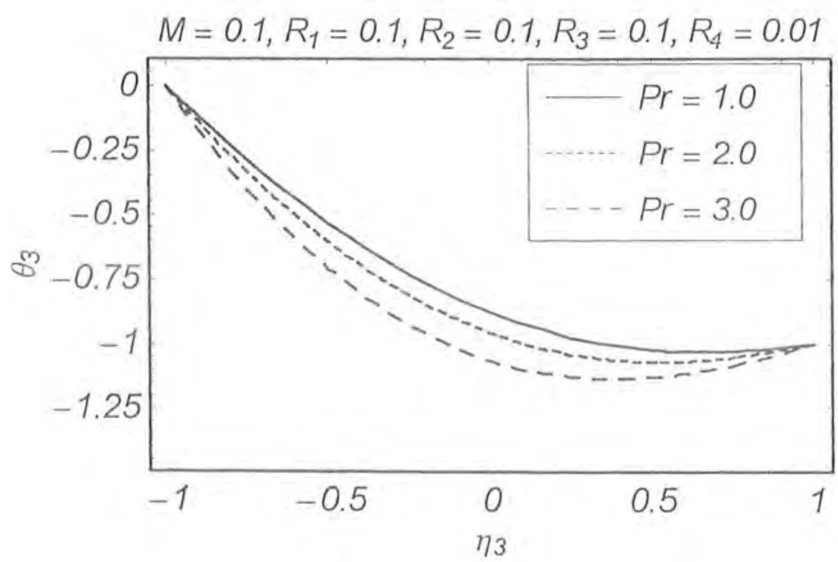

M = 0.1, R1 = 0.1, R2 = 0.1, R3 = 0.1, R4 = 0.01
Pr = 1.0
Pr = 2.0
Pr = 3.0
θ3
η3

Fig. 20

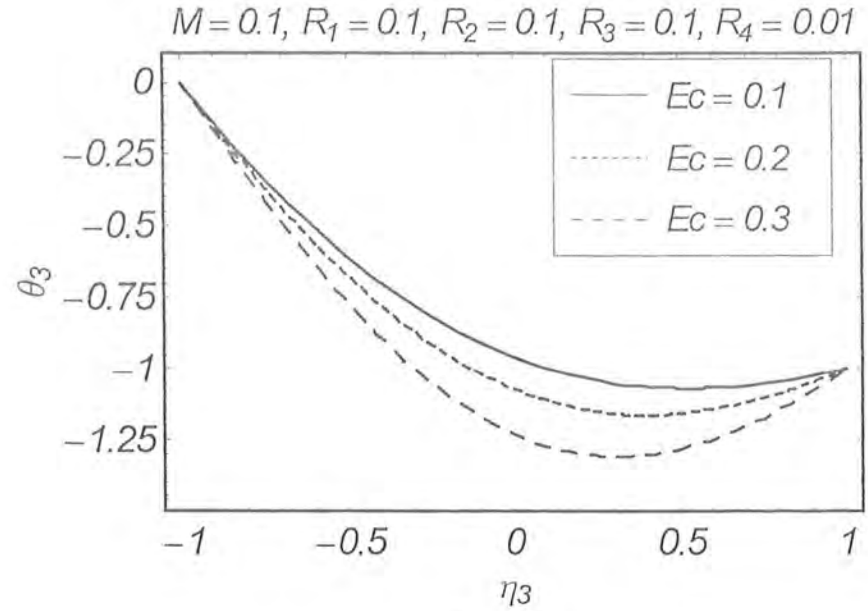

M = 0.1, R1 = 0.1, R2 = 0.1, R3 = 0.1, R4 = 0.01
Ec = 0.1
Ec = 0.2
Ec = 0.3
θ3
η3
0
−0.25
−0.5
−0.75
−1
−1.25
−1
−0.5
0
0.5
1

Fig. 21

3.7 References

1. R. Hide and P. H. Roberts, The origin of the mean geomagnetic field, in: Physics and Chemistry of the Earth, Vol. 4, Pergamon press, New York, (1961).

2. S. Chandrasekhar, The stability of viscous flow between the rotating cylinders in the presence of magnetic field, Proc. Roy. Soc. Lond. A 216, 293 (1953).

3. S. Chandrasekhar, The instability of a layer heated below and subjected to a Coriolis force, Proc. Roy. Soc. Lond. A 217, 306 (1953)

4. S. Chandrasekhar, Hydrodynamic and hydromagnetic stability, Oxford, University Press, 1961.

5. B. Lehnert, Magnetohydrodynamic waves under the action of Coriolis force, Astrophys. J. 119, 647 (1954).

6. B. Lehnert, Magnetohydrodynamic waves under the action of Coriolis force II, Astrophys. J. 121, 481 (1955).

7. K.R. Rajagopal, on boundary conditions for fluids of the differential type, In Navier-Stockes Equations and Related Non-Linear Problems, (Edited by A. Sequira), pp. 273, Plenum Press, New York, (1995).

8. K.R. Rajagopal and P. N. Kaloni, Some remarks on boundary conditions for fluids of the differential type, In Continuum Mechanics and its Applications (Edited by G. A. C. Graham,S. K. Malik), pp.935, Hemisphere, New York, (1989).

9. D. Mansutti, G. Pontrelli and K.R. Rajagopal, Steady flows of non-Newtnian fluids past a porous plate with suction or injection, Int. J. Numer. Meth. Fluids 17, 921, (1993).

10. J.M. Krishnan and K.R. Rajagopal, Thermodynamic framework for the constitutive modeling of asphalt concrete: Theory and application, J. Mater. Civil Eng. 16,155, (2004).

11. J.M. Krishnan and K.R. Rajagopal, Review of the uses and modeling of bitumen from ancient to modern times, Appl. Mech. Rev. 56,149,(2003).

12. P. Ravindran, J.M. Krishnan and K.R. Rajagopal, A note on the flow of a Burgers' fluid in an orthogonal rheometer, Int. J. Eng. Sci. 42,1973,(2004).

13. K. Majidzadeh and H. E. Schweyer,Viscoelastic response of asphalts in the vicinity of the glass transition point,Assoc. Asphalts Paving Technol. Proc. 36,80,(1967).

14. A. H. Gerritsen, C. A. P. M. van Grup,J.P.J. van der Heide,A.A.A. Molenaar and A.C.Prona, Prediction and prevention of surface cracking in asphaltic,pavements, In Proceedings of the Sixth International Conference, Structural Design of Asphalt Pavements, Volume 1,University of Michigan, (1987).

15. C.A. Tovar,C.A. Cerdeirina, L. Romani, B.Prieto and J. Carballo, Viscoelastic behavior of Arzua-Ulloa cheese, J. Texture Stud. 34,115, (2003).

16. M.C. Wang and K.Y. Lee, Creep behavior of cement stabilized soils, Highway Research Record 442,58, (1973).

17. P.N. Chopra High-temperature transient creep in olivine rocks, tectonophysics 279, 93,(1997).

18. B.H. Tan, I. Jackson and J.D.F. Gerald,High-temperature viscoelasticity of fine-grained polycrystalline olivine, Phys. Chem. Miner. 28,641, (2001).

19. W.R. Peltier, P.Wu and D.A. Yuen, The viscosities of the earth mantle, In Anelasticity in the earth, (Edited by F.D. Stacey, M.S. Paterson, A. Nicholas), American Geophysical Union, Colorado,(1981).

20. D.A. Yuen and W.R. Peltier, Normal modes of the viscoelastic earth, Geophys. J. Royal Astronom. Earth 69,495,(1982).

21. G. Muller, Generalized Maxwell bodies and estimates of mantle viscosity, Geophys. J. Royal Astronom. Soc. 87,1113,(1986).

22. G. Rumpker and D. Wolf, Viscoelastic relaxation of burgers half-space: Implications for the interpretation of the Fennoscandian uplift, Geophys. J. Int. 124, 541,(1996).

23. T. Hayat, Exact solutions of rotating flows of Burgers' fluid, Comp. and Mathematical Appl. 52, 1413 (2006).

24. R. Bandelli, Unsteady unidirectional flows of second grade fluids in domain with heated boundaries, Int. J. Non. Linear Mech. 30(3), 263 (1995).

25. K. R. Rajagopal, A note on unsteady unidirectional flows of non-Newtonian fluid, Int. J. Non-Linear Mech. 17, 369 (1982).