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Influence of Heat Transfer on Thin Film Flow of a Third Grade Fluid with Variable Viscosity

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A Dissertation Submitted in the Partial Fulfillment of the Requirements

for the

Degree of

MASTER OF PHILOSOPHY

IN

MATHEMATICS

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CERTIFICATE

A DISSERTATION SUBMITTED IN THE PARTIAL FULFILLMENT OF THE REQUIREMENTS FOR THE DEGREE OF THE MASTER OF PHILOSOPHY

We accept this dissertation as conforming to the required standard

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2008

Dedicated to

My loving mother
Whose prayers are source of my every achievement

And to

My father
Whose devotions always inspired me

Acknowledgements

*All praise to **ALLAH** Almighty, Creator of all of us. Worthy of all persons, who always guides in glooms and obscurities and subsists of assistance in difficulties, when all other sourced channel's upper frontier ends,*

*My all admiration to the last Holy Prophet **Hazart Muhammad (SAW)** for whom it can be said with full confidence that the world has never produced and will produce such a personality. Who by his acts and deeds have taught how to lead this world?*

*I would like to pay the deepest gratitude to my honorable supervisor **Dr. Sohail Nadeem,** who guided me through my difficulties and given me critical views. In addition, he was always accessible and willing to help his students with their research. As a result, research life became smooth and rewarding for me.*

*I am grateful to chairman department of Mathematics **Prof. Dr. Muhammad Ayub** for providing me proper atmosphere to complete the research work.*

*I am thankful to my uncle **Dr. Malik Muhammad Yousaf** for his patience, guidance encouragement and excellent advice throughout this study.*

*Very special thanks go to **Rana Qumar Hussain,** without whose motivation and encouragement I would not be able to complete my dissertation.*

*The acknowledgement will surly remain incomplete if I don't express my deep indebtedness and cordial thanks to my loving Parents, brothers (**Malik Tamoor & Malik Waseem**) and sisters (**Shumaila & Anam**) for their prayers, encouragements. Without their support I would have been standing no where in my life.*

I especially deem to express my unbound thanks to my all friends and roommate for their valuable suggestions, cooperation and fruitful discussions during my research work.

Ansa Rafique

08-08-2008

Contents

0.1 Introduction

Various applications of heat transfer inside thin films in several industrial manufacturing processes have led to renewed interest among the researchers. Some of the typical applications of such flows are wire and fiber coating, reactor fluidization, polymer processing, food stuff processing, transpiration cooling, microchip production and lining of mamalian lungs. Several authors have considered flow inside a thin films like Siddiqui et. al. [1-3], Sajid and Hayat [4], Hayat and Sajid [5] and Sajid et. al. [6].

One of the important aspect in this theoretical study is the investigation of non-Newtonian fluid with variable viscosity. This is due to the fact that typical Navier-Stokes theory becomes insufficient to, when fluid description of some complex rehological fluids such as shampoo, blood, paints, polymer solutions and plastic films. In view of this Massoudi and Christie [7] investigated the effects of variable viscosity and description on the flow of a third grade fluid in a pipe. Using the idea given by Massoudi and Christie [6], Pabdermirli and Yilbas [8-9] have presented the analytical solution by using perturbation technique.

Our aim here is to extend the work of Sajid et al [6], in two directions. These two directions are the viscosity is now temperature dependent so energy equation is also solved the second extension is consider the magnetic field effects. Due to large number of scientific and engineering applications of MHD flows of non-Newtonian fluids, number of researchers have focussed their attention on this subject [10-12]. With these importances in mind this dissertatic is arranged in the following manner.

In chapter one, we have given some basic definition and the governing equations of motion and energy are derived.

Chapter two describe the thin film flow of a third grade fluid with partial slip boundary conditions.

Chapter three deals with the influence of MHD and heat transfer on the thin film flow of a third grade fluid with variable viscosity. Analytical solutions are presented to by means of Homotopy analysis method.

Chapter 1

Some basic definitions and governing equations

1.1 Fluid mechanics

Fluid mechanics is the study of how fluids move and the forces act on them. Fluids include liquids and gases. Fluid mechanics can be divided into fluid statics, the study of fluids at rest and fluid dynamics, the study of fluids in motion.

1.2 Fluid

A fluid is a substance that deforms continuously under the application of a shear (tangential) stress no matter how small the shear stress may be.

1.3 Flow

A material goes under deformation when different forces act upon it. If the deformation continuously increases without limit, then the phenomenon is known as flow.

1.4 Types of flows

1.4.1 Uniform flows

If the flow velocity is the same magnitude and direction at every point in the fluid it is said to be uniform.

Mathematically the definition of uniform flow is

$$\frac{\partial v}{\partial s} = 0,$$

where s represents the displacement in any direction.

1.4.2 Non-uniform flows

If at a given instant, the velocity is not the same at every point, the flow is non-uniform. (In practice, by this definition, every fluid that flows near a solid boundary will be non-uniform - as the fluid at the boundary must take the speed of the boundary, usually zero.

Mathematically the definition of non-uniform flow is

$$\frac{\partial v}{\partial s} \neq 0.$$

1.4.3 Steady flows

A steady flow is one in which the conditions (velocity, pressure and cross-section) may differ from point to point but do not change with time.

Mathematically flow is steady, when

$$\frac{\partial \eta}{\partial t} = 0,$$

where η is any fluid property.

1.4.4 Unsteady flows

If at any point in the fluid, the conditions change with time, the flow is described as unsteady. (In practice there is always slight variations in velocity and pressure, but if the average values are constant, the flow is considered steady).

Mathematically flow is unsteady when

$$\frac{\partial \eta}{\partial t} \neq 0.$$

1.4.5 Incompressible flow

If there is no change in density with respect to space coordinates and time. Then the flow is called incompressible and is denoted by

$$\rho\,(x, y, z, t) = constant.$$

Liquids are considered as incompressible fluids.

1.4.6 Compressible flow

If there is a change in density with respect to space coordinates and time. Then the flow is called compressible and is denoted by

$$\rho\,(x, y, z, t) \neq constant.$$

1.5 Pressure

Pressure is the force or an area applied to an object in a direction perpendicular to the surface. The gradient of pressure is called the force density.

Mathematically, pressure p at a point P may be defined as

$$p = \lim_{\delta s \longrightarrow 0} \frac{\delta F}{\delta S},$$

where δS is an elementary area around P and δF is the normal force due to fluid on δS.

1.6 Viscosity

Viscosity is the measure of the resistance of a fluid to being deformed by either shear stress or extensional stress. It is commonly perceived as "thickness", or resistance to flow.

Thus water is "thin" having a lower viscosity while vegetable oil is "thick" having higher viscosity.

Mathematically, the ratio of shear stress to the shear strain is called as viscosity. This can be written as

$$\text{viscosity } (\mu) = \frac{\text{shear stress}}{\text{rate of shear strain}}.$$

1.6.1 Kinematic viscosity(ν)

The ratio of absolute viscosity μ to density. i.e.

$$\nu = \frac{\mu}{\rho}.$$

For gasses, viscosity increases with temperature while for liquids, viscosity decreases with increasing temperature.

1.7 Types of forces

1.7.1 Body forces

Forces developed without physical contact and distributed over the volume of the fluid, are termed as body forces. Gravitational and electromagnetic forces are examples of body forces arising in a fluid.

1.7.2 Surface forces

Surface forces include all forces acting on the boundaries of a medium through direct contact. Pressure is an example of surface force.

1.8 Stress

A measure of the internal reactions of a body subjected to an external force. A system of internal forces is set up, in equilibrium and stress is expresses as the force per unit area i.e.

$$\mathrm{T} = \frac{\mathrm{F}}{A},$$

where $\mathbf{F}$ is a force and A is the area.

1.8.1 Shear stress

Shear stress is a stress state where the stress is parallel or tangential to a face of the materials opposed normal stress when the stress is perpendicular to the face. The variable used to denote shear stress is τ.

1.8.2 Normal stress

The components of stress, when the fluid is in motion or static, perpendicular to the area, are known as normal stresses.

1.8.3 Laminar flow

A type of non-turbulent flow where the movement of each part of the fluid (gaseous, liquid, or plastic) has the same velocity, with no mixing between adjacent 'layers' of the fluid in which the fluid travels smoothly or in regular paths. It may be seen at low velocities in a smooth, straight river channel and at some glacier snouts.

1.8.4 Turbulent flow

It is well known that the motion of fluids may occur in irregular fluctuations. Such motions are called the turbulent flows.

1.8.5 Reynolds number

In fluid mechanics, the Reynolds number may be described as the ratio of inertial forces ($v s \rho$) to viscous forces (μ/L).

1.8.6 Magnetic Reynold number

It is the ratio of inertial forces to the magnetic forces. In mathematical form, it is given as

$$R_m = \frac{\text{inertial forces}}{\text{magnetic forces}}. \tag{1.1}$$

1.9 Types of fluids

1.9.1 Ideal fluids

Fluids with negligible viscosity are known as ideal fluids. Ideal fluids do not exists in nature because they do not offer resistance to shearing forces. However, in engineering the gases are treated as ideal fluids.

1.9.2 Real fluid

A fluid which possess some viscosity, is known as real fluid. All the fluids, in actual practice, are real fluids. Further real fluids have been classified into Newtonian and non-Newtonian fluids.

1.9.3 Newtonian fluid

Any fluid exhibiting a linear relation between the applied shear stress and the rate of deformation. Water, air and gasoline are example of Newtonian fluid. For such fluids, Newton's law of viscosity hold. Mathematically,

$$\tau_{yx} \propto \frac{du}{dy},$$

or

$$\tau_{yx} = \mu\frac{du}{dy}. \tag{1.2}$$

1.9.4 Non Newtonian fluid

A non-Newtonian fluid is a fluid in which the viscosity changes with the applied shear stress. Non Newtonian fluids may not have a well-defined fluid viscosity. Many polymer solutions, tooth paste, blood, lucite paint are the examples of non-Newtonian fluids. For such fluids, Power law model hold. Mathematically, it is defined by

$$\tau_{yx} \propto \left(\frac{du}{dy}\right)^n, \; n \neq 1, \tag{1.3}$$

or

$$\tau_{yx} = \eta \frac{du}{dy}, \; \eta = k\left(\frac{du}{dy}\right)^{n-1}, \tag{1.4}$$

where n is flow behaviour index, η is apparent viscosity and k is the consistency index.

1.10 Maxwell's equations

In order to describe the behavior of electric and magnetic fields, E and B respectively, we consider the following differentials equations

$$\nabla \times \frac{\mathbf{B}}{\mu_1} = \mathbf{J} + \frac{\partial}{\partial t}(\varepsilon \mathbf{E}), \tag{1.5}$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}, \tag{1.6}$$

$$\nabla \cdot \mathbf{B} = 0, \tag{1.7}$$

$$\nabla \cdot \mathbf{E} = \frac{\rho_c}{\varepsilon}. \tag{1.8}$$

The above equations are known as Maxwell's equations. Here μ_1 and ε are the magnetic permeability and dielectric constants respectively, $\mathbf{D} = (\varepsilon \mathbf{E})$ is the dielectric displacement and ρ_c is the charge density. The total magnetic field $\mathbf{B} = (\mathbf{B}_0 + \mathbf{b})$ in terms of H is defined by $\mathbf{B} = \mu_1 \mathbf{H}$.

By Ohm's law, we have

$$\mathbf{J} = \sigma(\mathbf{E} + \mathbf{V} \times \mathbf{B}). \tag{1.9}$$

Here σ is the electrical conductivity of the fluid. In the present dissertation, there is no applied electric field. The induced magnetic field is also neglected under the assumption of a small magnetic Reynolds number. Therefore, the Lorentz force under these assumptions reduces to

$$\frac{1}{\rho}(\mathbf{J} \times \mathbf{B}) = -\frac{\sigma B_0^2}{\rho}\mathbf{V}, \tag{1.10}$$

in which $\mathbf{B}_0$ is the applied magnetic field, $\mathbf{E}$ is the electric field, ρ is the charge density, c is the speed of light and $\mathbf{J}$ is the vector current density.

1.11 Magnetohydrodynamics

It is a branch of engineering in which materials are discussed under the influence of an electromagnetic field.

1.11.1 Magnetic field

Magnetic field is a field that permeates space and which exerts a magnetic force on moving electric charges and magnetic dipoles. Magnetic fields surround electric currents, magnetic dipoles and changing electric fields.

1.12 Magnetic permeability

A measure of the ability of a substance to sustain a magnetic field, equal to the ratio between magnetic flux density $\mathbf{B}$ and magnetic field strength $\mathbf{H}$. Mathematically, it is defined as

$$\mu_m = \frac{\mathrm{B}}{\mathrm{H}}. \tag{1.11}$$

1.13 Thermodynamic properties

1.14 Heat

Heat may be defined as energy in transit form a high temperature object to a lower temperature object.

1.15 Temperature

Temperature can be defined as degree of hotness and coldness of a body. It is usually denoted by T and its unit in SI system is Kelvin.

1.16 Flux

Consider a flow of a certain physical quantity (such as mass, energy, heat, etc.). The flux is defined as a vector in the direction of the flow whose magnitude is given by the amount of quantity crossing a unit area normal to the flow in unit time.

1.17 Heat flux

Heat flux is defined as rate of heat transfer per unit cross-sectional area and is denoted by Q.

1.18 Thermal conductivity

Thermal conductivity k is the property of material that indicates its ability to conduct heat. It is defined by

$$k = \frac{\Delta \mathrm{Q}}{\Delta t} \times \frac{1}{A} \times \frac{\mathrm{x}}{\Delta \mathrm{T}}, \tag{1.12}$$

where ΔQ is the quantity of heat transmitted during time Δt through a thickness x, in a direction normal to a surface of area A due to temperature difference ΔT.

1.19 Thermal diffusivity

A measure of the rate at which a temperature disturbance at one point in a body travels to another point. It is expressed by the relationship

$$\sigma = \frac{k}{dc_p}, \tag{1.13}$$

where d is the density and c_p is the specific heat at constant pressure.

1.20 Prandtl number

The prandtl number is a dimensionless number approximating the ratio of momentum diffusivity and thermal diffusivity. Mathematically, it is expressed by the following relation

$$\mathrm{Pr} = \frac{\mu c_p}{k} = \frac{\text{Viscous diffusion rate}}{\text{Thermal diffusion rate}}. \tag{1.14}$$

In heat transfer problem, the Prandtl number controls the relative thickness of the momentum and thermal boundary layer. When Pr is small it means that the heat diffuses very quickly compared to the velocity (momentum).

1.21 Brinkman number

The Brinkman number is a dimensionless group related to heat conduction from a wall to a flowing viscous fluid, commonly used in polymer processing. Mathematically, it is defined as

$$\Gamma = NB_r = \frac{\eta U^2}{k\left(T_w - T_o\right)}, \tag{1.15}$$

where Γ is the Brinkman number, η is the fluid viscosity, U is the fluid velocity, T_w is the wall temperature and T is the bulk fluid temperature.

1.22 Internal energy

Internal energy of a system is the energy content of the system due to its thermodynamic properties such as pressure and temperature. The change of internal energy of a system depends only on the initial and final states of the system and not in any way by the path or manner of the change. This concept is used to define the first law of thermodynamic.

1.23 Enthalpy

It is defined as the sum of the internal energy E, plus the product of volume V and pressure P. It is represented by H and is defined as

$$H = E + PV. \tag{1.16}$$

1.24 Eckert number

The Eckert number is dimensionless number used in flow calculation. It expresses the relationship between a flow's kinetic energy and enthalpy and is used to characterize dissipation. It is defined as

$$E_c = \frac{v^2}{c_p \triangle T} = \frac{\text{kinetic energy}}{\text{Enthalpy}}, \tag{1.17}$$

where v is a characteristic velocity of the flow.

1.25 Entropy

A measure of the disorder or randomness in a closed system. Mathematically, it is defined as

$$dS = \frac{dQ}{dt}. \tag{1.18}$$

1.26 Specific internal energy

The specific internal energy is the internal energy per unit mass. The internal energy indicates the energy contained in random molecular motions and intermolecular forces.

1.27 No slip condition

When a fluid flow is bounded by a solid surface, molecular interaction cause the fluid in contact with the surface to seek momentum and energy equilibrium with the surface. Or we can say that all fluids at a point of contact with a solid take on the velocity of that solid, i.e,

$$V_{fluid} = V_{wall}.$$

1.28 Partial slip

Navier was the one who proposed that a liquid may slip on the solid surface and this slipping would be opposed by a frictional force proportional to the velocity of the fluid relative to the solid. He introduced the idea of 'slip-length', which is now a days the most commonly used concept of quantify the slip of a liquid at a solid interface, i.e,

$$v_r = b\frac{\partial v_b}{\partial z},$$

where v_r is the velocity of the fluid at the wall, v_b is the velocity of the fluid in the bulk and z is the axis perpendicular to the wall.

1.29 Thin film flow

Thin film flow, as the name suggests, can simply be defined as the flow of a fluid in the form of a thin film. More precisely it is a flow that consist of an expanse of liquid partially bounded by a solid substrate with a (free) surface when the liquid is exposed to another fluid (usually a gas and a most often air in applications). One can observe thin film flows even in daily life happenings. For example, the formation of tear in the eye and the flow of rain water down the window glass, etc. There are three agents that are mainly considered to be responsible for the formation of thin film. These are gravity, centrifugal forces and surface tension. Gravity form the thin films whenever the fluid flows down some inclined plane. While during the rotation of the fluid these are the centrifugal forces that are responsible for the formation of thin films. Thin films have numerous applications in industry. Spin coating is used in the manufacture of CDs and DVDs and computer disks. Any coating process, e.g, painting and manufacture of coated products are all examples of this thin film technology.

1.30 Law of conservation of mass

Let $\widetilde{V}$ be a control volume in space. We assume that it and its surface $\widetilde{S}$ remain fixed in space. The surface is permeable so that fluid can freely enter in and leave. The equation of continuity or conservation of mass stems from the principle that mass cannot be created or destroyed inside the control volume. Thus the mass in the control volume $\widetilde{V}$ is conserved at all time. In mathematical form this can be expressed as

$$\frac{d}{dt}\int_{\widetilde{V}} \rho d\widetilde{V} = 0, \tag{1.19}$$

where ρ is the density field at time t.

According to Reynolds transport result

$$\frac{d}{dt}\int_{\widetilde{V}} \Phi d\widetilde{V} = \int_{\widetilde{V}} \left(\frac{d\Phi}{dt} + \Phi\left(\nabla \cdot \mathbf{V}\right)\right) d\widetilde{V}, \tag{1.20}$$

or

$$\frac{d}{dt}\int_{\widetilde{V}} \Phi d\widetilde{V} = \int_{\widetilde{V}} \left(\frac{\partial\Phi}{\partial t} + \operatorname{div}\left(\Phi\mathbf{V}\right)\right) d\widetilde{V}, \tag{1.21}$$

where Φ is a field (scalar, vector or tensor) $\mathbf{V}$ is the velocity of the surface $\widetilde{S}$ and d/dt is the material time derivative.

By setting $\Phi = \rho$, with ρ the fluid density, Eqs. (1.19) and (1.21) give

$$\int_{\widetilde{V}} \left(\frac{\partial\rho}{\partial t} + \operatorname{div}\left(\rho\mathbf{V}\right)\right) d\widetilde{V} = 0. \tag{1.22}$$

Since the control volume $\widetilde{V}$ is arbitrary, a necessary and sufficient condition for conservation of mass is

$$\frac{\partial\rho}{\partial t} + \operatorname{div}\left(\rho\mathbf{V}\right) = 0. \tag{1.23}$$

For $\rho \neq \rho(x, y, z, t)$, the above expressions becomes

$$\operatorname{div}\mathbf{V} = 0, \tag{1.24}$$

which is known as the equation of continuity for incompressible fluids.

1.31 The equation of motion for third grade fluid

In this section, we are interested in constructing the governing equation for incompressible third grade fluid in cylindrical coordinate system. The equation of motion in the presence of body forces is

$$\rho\frac{d\mathbf{V}}{dt} = \operatorname{div}(\tau) + \mathbf{J}\times\mathbf{B}, \tag{1.25}$$

where $\mathbf{V}$ is the velocity vector, d/dt is the total derivative and τ is the cauchy stress tensor and $\mathbf{J}\times\mathbf{B}$ is defined in eq. (1.10).

The constitutive equation describing the third grade fluid is

$$\tau = -p\mathbf{I} + \mu\mathbf{A}_1 + \alpha_1\mathbf{A}_2 + \alpha_2\mathbf{A}_1^2 + \beta_1\mathbf{A}_3 + \beta_2(\mathbf{A}_1\mathbf{A}_2 + \mathbf{A}_2\mathbf{A}_1) + \beta_3\left(tr\mathbf{A}_1^2\right)\mathbf{A}_1, \tag{1.26}$$

where p is the pressure, I is the identity tensor, μ is the coefficient of viscosity, $\alpha_i\,(i=1,2)\,,\beta_i\,(i=1-3)$ are material constants, tr is the trace and $\mathbf{A}_i\,(i=1-3)$ are Rivlin Erickson tensors defined by

$$\begin{aligned}\mathbf{A}_1 &= \operatorname{grad}\mathbf{V} + (\operatorname{grad}\mathbf{V})^t, & (1.27)\\ \mathbf{A}_n &= \frac{d}{dt}(\mathbf{A}_{n-1}) + \mathbf{A}_{n-1}(\operatorname{grad}\mathbf{V}) + (\operatorname{grad}\mathbf{V})^t\mathbf{A}_{n-1} \quad n\geq 1. & (1.28)\end{aligned}$$

Fosdick and Rajagopal [13] showed that if such to be consistent with thermodynamics, it is necessary that

$$\begin{aligned}\mu &\geq 0,\ \alpha_1\geq 0,\ |\alpha_1+\alpha_2|\leq\sqrt{24\mu\beta_3}, & (1.29)\\ \beta_1 &= 0,\ \beta_2 = 0,\ \beta_3\geq 0. & (1.30)\end{aligned}$$

The velocity field for the problem under consideration is

$$\mathbf{V} = [0,\ 0,\ v(r)],$$

where $v(r)$ is the z component of velocity. With the help of above velocity field we can compute

$$\mathbf{A}_1 = \begin{bmatrix} 0 & 0 & \frac{dv}{dr} \\ 0 & 0 & 0 \\ \frac{dv}{dr} & 0 & 0 \end{bmatrix}, \tag{1.31}$$

$$\mathbf{A}_2 = \begin{bmatrix} 2\left(\frac{dv}{dr}\right)^2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \tag{1.32}$$

$$\mathbf{A}_1^2 = \begin{bmatrix} \left(\frac{dv}{dr}\right)^2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \left(\frac{dv}{dr}\right)^2 \end{bmatrix}, \tag{1.33}$$

$$\left(tr\mathbf{A}_1^2\right)\mathbf{A}_1 = \begin{bmatrix} 0 & 0 & \left(\frac{dv}{dr}\right)^3 \\ 0 & 0 & 0 \\ \left(\frac{dv}{dr}\right)^3 & 0 & 0 \end{bmatrix}. \tag{1.34}$$

Making use of Eqs. (1.27) to (1.34), the component form of Eq. (1.26) can be written as

$$\tau_{rr} = -p + (2\alpha_1 + \alpha_2)\left(\frac{dv}{dr}\right)^2, \tag{1.35}$$

$$\tau_{r\theta} = \tau_{\theta r} = \tau_{\theta z} = \tau_{z\theta} = 0, \tag{1.36}$$

$$\tau_{\theta\theta} = -p, \tag{1.37}$$

$$\tau_{zr} = \tau_{rz} = \mu\frac{dv}{dr} + \beta_3\left(\frac{dv}{dr}\right)^3, \tag{1.38}$$

$$\tau_{zz} = -p + \alpha_2\left(\frac{dv}{dr}\right)^2. \tag{1.39}$$

With the help of Eqs. (1.35) to (1.39), the component form of Eq. (1.25) can be written as

$$\frac{\partial p}{\partial r} = \frac{1}{r}\frac{d}{dr}\left[(2\alpha_1 + \alpha_2)\, r\left(\frac{dv}{dr}\right)^2\right], \tag{1.40}$$

$$\frac{\partial p}{\partial \theta} = 0, \tag{1.41}$$

$$\frac{\partial p}{\partial z}=\frac{1}{r}\frac{d}{dr}\left[r\left(\mu+2\beta_3\left(\frac{dv}{dr}\right)^2\right)\left(\frac{dv}{dr}\right)\right]-\sigma B_0^2 v. \tag{1.42}$$

Eq. (1.42) can be written as

$$\frac{d\mu}{dr}\frac{dv}{dr}+\frac{\mu}{r}\frac{dv}{dr}+\mu\frac{d^2v}{dr^2}+\frac{2\beta_3}{r}\left(\frac{dv}{dr}\right)^3+6\beta_3\left(\frac{dv}{dr}\right)^2\frac{d^2v}{dr^2}-\sigma B_0^2 v=\frac{\partial p}{\partial z}. \tag{1.43}$$

1.32 The energy equation for third grade fluid

In this section, we are interested in constructing the energy equation for third grade fluid. In general the energy equation can be defined as

$$\frac{de}{dt}=\tau.\mathrm{L}-\operatorname{div}\mathrm{Q}+\rho\mathrm{r}, \tag{1.44}$$

where $\mathbf{Q}$ is the heat flux vector, e is the specific internal energy and r is the radiant heating. In absence of radiant heating the above equation takes the form

$$\frac{de}{dt}=\tau.\mathrm{L}-\operatorname{div}\mathrm{Q}. \tag{1.45}$$

According to Fourier's Law

$$\mathrm{Q}=-k\operatorname{grad}\theta, \tag{1.46}$$

where k is the constant of thermal conductivity and θ is the temperature. We take the temperature field as

$$\theta=\theta(r) \tag{1.47}$$

In view of Eq. (1.47) , we have

$$\operatorname{div}\mathrm{Q}=-k\left[\frac{1}{r}\frac{d\theta}{dr}+\frac{d^2\theta}{d^2r}\right], \tag{1.48}$$

$$\tau.\mathrm{L}=tr(\tau\mathrm{L}), \tag{1.49}$$

$$\tau.\mathrm{L}=\left(\mu+2\beta_3\left(\frac{dv}{dr}\right)^2\right)\left(\frac{dv}{dr}\right)^2. \tag{1.50}$$

With the help of these equations, Eq. (1.44) can be written as

$$\mu\left(\frac{dv}{dr}\right)^2+2\beta_3\left(\frac{dv}{dr}\right)^4+k\left[\frac{1}{r}\frac{d}{dr}\left(r\frac{d\theta}{dr}\right)\right]=0. \tag{1.51}$$

If $\beta_3=0$, we obtain the energy equation for viscous fluid.

1.33 Homotopy analysis method (HAM)

Since the governing equation of fluid mechanics are in general non-linear and have higher order than the number of available boundary conditions and therefore, offer challenges to mathematician, physicists and numerical simulationists. Many researchers are using different analytical and numerical techniques for the solution of such non-linear equations. Recently, a powerful analytical tool namely the homotopy analysis method (HAM) is used by the researchers for the series solutions of highly non-linear problems. This method has been introduced by Liao and does not require any small or large parameter. It is therefore, can be applied to most non-linear problems, especially those without small or large parameters. In addition, this method provides us a great freedom to use different types of base functions to approximate the solutions of the non-linear problems. It also provides us a simple way to control and adjust the convergence region. While perturbation methods work nicely for slightly non-linear problems, the homotopy analysis technique addresses nonlinear problems in a more general manner. Through this method, a nonlinear problem that normally has a unique solution can have an infinite number of different solution expressions whose convergence region and rate are dependent on an auxiliary parameter. The method provides for ways to control and adjust the convergence region. This makes the method particularly suited for problems with strong nonlinearity. The application of the method is defined in the coming chapter.

Chapter 2

Analytical solution for a thin film flow of a third grade fluid down a vertical cylinder with partial slip

2.1 Introduction

In this chapter an analytical solution is developed for an incompressible, thin film flow of a third grade fluid down a vertical cylinder. The thin film is produced on the outer side of a vertical cylinder between radius R and $R+\delta$. At $r=R$ the disturbance is due to partial slip where as at $r=R+\delta$, the surface is being free. The solution of the governing problem has been solved by perturbation method, homotopy perturbation method and homotopy analysis method. Several known results of interest are found to follow as a special case of the solution of the problem examined. The graphical results are shown for various physical quantities.

2.2 Mathematical formulation

The physical situation considered here is of an incompressible third grade fluid lying on the outer surface of an infinitely long vertical cylinder. The flow is in the form of thin film having thickness δ on the outer side of vertical cylinder having radius R from the origin. The fluid motion in the film arises due to gravity and partial slip at $r=R$. The velocity field in the thin film is governed by [6].

$$r\frac{d^2u}{dr^2}+\frac{du}{dr}+2\left(\frac{\beta_2+\beta_3}{\mu}\right)\left[3r\left(\frac{du}{dr}\right)^2\left(\frac{d^2u}{dr^2}\right)+\left(\frac{du}{dr}\right)^3\right]+\frac{\rho g}{\mu}r=0, \tag{2.1}$$

where g is the constant of gravity.

The corresponding boundary conditions are

$$u\left(r\right)=\lambda\left[\frac{du}{dr}+2\left(\frac{\beta_2+\beta_3}{\mu}\right)\left(\frac{du}{dr}\right)^3\right]\ at\ r=R, \tag{2.2}$$

$$\frac{du}{dr}=0,\ at\ r=R+\delta, \tag{2.3}$$

the first being partial slip condition at $r=R$ and the second one again coming from τ_{rz}.

Eq (2.1), subject to boundary conditions (2.2) and (2.3), can be solved by the following methods.

1. Perturbation method.
2. Homotopy perturbation method.
3. Homotopy analysis method.

2.3 Perturbation method

Let us consider $\epsilon=\left(\frac{\beta_2+\beta_3}{\mu}\right)$ be a small parameter in Eq. (2.1), then we assume that u can be expressed in powers of ϵ as follows

$$u\left(r,\epsilon\right)=u_0\left(r\right)+\epsilon u_1\left(r\right)+\epsilon^2u_2\left(r\right)+.... \tag{2.4}$$

Substitution of Eq.(2.4) into Eqs. (2.1) to (2.3) and equating the coefficients of like powers of ϵ, we get the following problems.

2.3.1 Zeroth-order problem

$$r\frac{d^2u_0}{dr^2}+\frac{du_0}{dr}+\frac{\rho g}{\mu}r=0, \tag{2.5}$$

$$u_0(r) = \lambda \left[\frac{du_0}{dr}\right] \ at\ r = R, \tag{2.6}$$

$$\frac{du_0}{dr} = 0 \ at\ r = R + \delta. \tag{2.7}$$

2.3.2 First order problem

$$r\frac{d^2u_1}{dr^2} + \frac{du_1}{dr} + 6r\left(\frac{du_0}{dr}\right)^2\left(\frac{d^2u_o}{dr^2}\right) + 2\left(\frac{du_0}{dr}\right)^3 = 0, \tag{2.8}$$

$$u_1(r) = \lambda\left[\frac{du_1}{dr} + 2\left(\frac{du_0}{dr}\right)^3\right] \ at\ r = R, \tag{2.9}$$

$$\frac{du_1}{dr} = 0 \ at\ r = R + \delta. \tag{2.10}$$

2.3.3 Second order problem

$$r\frac{d^2u_2}{dr^2} + \frac{du_2}{dr} + 6r\left(\frac{du_0}{dr}\right)^2\left(\frac{d^2u_1}{dr^2}\right) + 12r\left(\frac{du_0}{dr}\right)\left(\frac{d^2u_0}{dr^2}\right)\left(\frac{du_1}{dr}\right) + 6\left(\frac{du_0}{dr}\right)^2\left(\frac{du_1}{dr}\right) = 0, \tag{2.11}$$

$$u_2(r) = \lambda\left[\frac{du_2}{dr} + 6\left(\frac{du_0}{dr}\right)^2\left(\frac{du_1}{dr}\right)\right] \ at\ r = R, \tag{2.12}$$

$$\frac{du_2}{dr} = 0 \ at\ r = R + \delta. \tag{2.13}$$

2.3.4 Zeroth-order solution

Eq.(2.5) is non-homogeneous second order Cauchy-Euler differential equation. The solution of Eq. (2.5) is

$$u_0(r) = A_3 + B_3 \ln r - \frac{\rho g}{4\mu} r^2, \tag{2.14}$$

where A_3 and B_3 are constants.

With the help of boundary conditions (2.6) and (2.7), the solution (2.14) can be written as

$$u_0(r) = \frac{\rho g}{4\mu}\left[R^2 - r^2 + \frac{2\lambda (R+\delta)^2}{R} - 2\lambda R + 2(R+\delta)^2 \ln\frac{r}{R}\right]. \tag{2.15}$$

The solution of no slip can be easily recovered by taking $\lambda = 0$.

2.3.5 First-order solution

Using Eq. (2.15), Eq. (2.8) can be written as

$$r^2\frac{d^2u_1}{dr^2}+r\frac{du_1}{dr}=\frac{3\rho^3g^3}{4\mu^3}\left[\frac{2(R+\delta)^6}{3r^2}-2r^2(R+\delta)^2+\frac{4}{3}r^4\right]. \tag{2.16}$$

The solution of the above non-homogenous equation satisfying the boundary condition (2.9) and (2.10) is

$$u_1(r)=\frac{3\rho^3g^3}{4\mu^3}\left[\frac{(R+\delta)^2(R^2-r^2)}{2}+\frac{r^4-R^4}{12}+\frac{(R+\delta)^6}{6}\left(\frac{1}{r^2}-\frac{1}{R^2}\right)+(R+\delta)^4\ln\left(\frac{r}{R}\right)\right]. \tag{2.17}$$

2.3.6 Second-order solution

With the help of Eqs. (2.15) and (2.17), the solution of Eq. (2.11) satisfying the boundary conditions (2.12) and (2.13) can be written as

$$u_2(r)=\frac{\rho^5g^5}{8\mu^5}\left[\begin{array}{c}\frac{3(R+\delta)^{10}}{4}\left(\frac{1}{R^4}-\frac{1}{r^4}\right)+\frac{15(R+\delta)^8}{2}\left(\frac{1}{r^2}-\frac{1}{R^2}\right)\\+\frac{1}{2}(R^6-r^6)+15(R+\delta)^4(R^2-r^2)+\\\frac{15}{4}(R+\delta)^2(r^4-R^4)+30(R+\delta)^6\ln\left(\frac{r}{R}\right)\end{array}\right]. \tag{2.18}$$

Using Eqs. (2.15), (2.17) and (2.18) finally, the solution (2.14) can be written

$$\begin{aligned}u(r)&=\frac{\rho g}{4\mu}\left[R^2-r^2+\frac{2M(R+\delta)^2}{R}+2MR+2(R+\delta)^2\ln\frac{r}{R}\right]\\&+\epsilon^3\frac{\rho^3g^3}{4\mu^3}\left[\begin{array}{c}\frac{(R+\delta)^2(R^2-r^2)}{2}+\frac{r^4-R^4}{12}+\frac{(R+\delta)^6}{6}\left(\frac{1}{r^2}-\frac{1}{R^2}\right)\\+(R+\delta)^4\ln\left(\frac{r}{R}\right)\end{array}\right]\\&+\epsilon^2\frac{\rho^5g^5}{8\mu^5}\left[\begin{array}{c}\frac{3(R+\delta)^{10}}{4}\left(\frac{1}{R4}-\frac{1}{r^4}\right)+\frac{15(R+\delta)^8}{2}\left(\frac{1}{r^2}-\frac{1}{R^2}\right)\\+\frac{1}{2}(R^6-r^6)+15(R+\delta)^4(R^2-r^2)\\+\frac{15}{4}(R+\delta)^2(r^4-R^4)+30(R+\delta)^6\ln\left(\frac{r}{R}\right)\end{array}\right].\end{aligned} \tag{2.19}$$

If $\lambda=0$, the solution for no-slip can be recovered and for $\lambda=\varepsilon=0$, we obtain the Newtonian solution.

2.4 Homotopy perturbation method

The homotopy perturbation method for Eq.(2.1) , can be defined as

$$H(\upsilon,q)=(1-q)\left[L(\upsilon)-L(u_0)\right]+\left[L(\upsilon)+2\left(\frac{\beta_1+\beta_2}{\mu}\right)\frac{d}{dr}\left\{r\left(\frac{d\upsilon}{dr}\right)^3\right\}+\frac{\rho g}{\mu}r\right]=0, \quad (2.20)$$

or

$$H(\upsilon,q)=L(\upsilon)-L(u_0)+qL(u_0)+q\left[2\left(\frac{\beta_1+\beta_2}{\mu}\right)\frac{d}{dr}\left\{r\left(\frac{d\upsilon}{dr}\right)^3\right\}+\frac{\rho g}{\mu}r\right]=0. \quad (2.21)$$

For our convenience, we have taken $L=r\frac{d^2}{dr^2}+\frac{d}{dr}$ as the linear operator and

$$u_0(r)=\frac{\rho g}{4\mu}\left[R^2-r^2+\frac{2\lambda(R+\delta)^2}{R}-2\lambda R+2(R+\delta)^2\ln\frac{r}{R}\right], \quad (2.22)$$

as the initial guess.

Let us define

$$\upsilon(r,q)=\upsilon_0+q\upsilon_1+q^2\upsilon_2+.... \quad (2.23)$$

Substituting Eq. (2.23) into Eq. (2.21) and then collecting the like powers of q, we get the following problems

2.4.1 Zeroth-order problem

$$L(\upsilon_0)-L(u_0)=0, \quad (2.24)$$

$$\upsilon_0(r)=\lambda\left[\frac{d\upsilon_0}{dr}\right] \ at\ r=R, \quad (2.25)$$

$$\frac{d\upsilon_0}{dr}=0\ at\ r=R+\delta. \quad (2.26)$$

2.4.2 First-order problem

$$L(\upsilon_1)+L(u_0)+2\left(\frac{\beta_2+\beta_3}{\mu}\right)\frac{d}{dr}\left[r\left(\frac{d\upsilon_0}{dr}\right)^3\right]+\frac{\rho g}{\mu}r=0, \quad (2.27)$$

$$\upsilon_1(r) = \lambda\left[\frac{d\upsilon_1}{dr} + 2\left(\frac{\beta_2+\beta_3}{\mu}\right)\left(\frac{d\upsilon_0}{dr}\right)^3\right] \; at \; r = R, \tag{2.28}$$

$$\frac{d\upsilon_1}{dr} = 0 \; at \; r = R + \delta. \tag{2.29}$$

2.4.3 Second-order problem

$$L(\upsilon_2) + \left(\frac{\beta_2+\beta_3}{\mu}\right)\frac{d}{dr}\left[6r\left(\frac{d\upsilon_0}{dr}\right)^2\frac{d\upsilon_1}{dr}\right] = 0, \tag{2.30}$$

$$\upsilon_2(r) = \lambda\left[\frac{d\upsilon_2}{dr} + 6\left(\frac{\beta_2+\beta_3}{\mu}\right)\left(\frac{d\upsilon_0}{dr}\right)^2\left(\frac{d\upsilon_1}{dr}\right)\right] \; at \; r = R, \tag{2.31}$$

$$\frac{d\upsilon_2}{dr} = 0 \; at \; r = R + \delta. \tag{2.32}$$

2.4.4 Zeroth-order solution

The solution of Eq. (2.24) satisfying the boundary conditions (2.25) and (2.26) is

$$\upsilon_0(r) = \frac{\rho g}{4\mu}\left[R^2 - r^2 + \frac{2\lambda(R+\delta)^2}{R} - 2\lambda R + 2(R+\delta)^2\ln\frac{r}{R}\right]. \tag{2.33}$$

2.4.5 First-order solution

The solution of Eq. (2.24) satisfying the boundary conditions (2.25) and (2.26) can be written as

$$\upsilon_1(R) = A_4 + B_4\ln R + \frac{3\rho^3 g^3}{4\mu^3}\left(\frac{\beta_2+\beta_3}{\mu}\right)\left[\frac{(R+\delta)^6}{6r^2} - \frac{r^2(R+\delta)^2}{2} + \frac{1}{12}r^4\right], \tag{2.34}$$

where A_4 and B_4 are constants and are defined as

$$A_4 = \frac{3\rho^3 g^3}{4\mu^3}\left(\frac{\beta_2+\beta_3}{\mu}\right)\left[-\frac{(R+\delta)^6}{6R^2} + \frac{R^2(R+\delta)^2}{2} - \frac{1}{12}R^4 - \frac{\lambda(R+\delta)^4}{R} - (R+\delta)^4\ln R\right], \tag{2.35}$$

$$B_4 = \frac{3\rho^3 g^3}{4\mu^3}\left(\frac{\beta_2+\beta_3}{\mu}\right)(R+\delta)^4. \tag{2.36}$$

2.4.6 Second-order solution

To avoid the repetition the solution of the Eq. (2.30) satisfying the boundary conditions (2.31) and (2.32), yields

$$v_2(r) = A_5 + B_5 \ln r + \frac{\rho^5 g^5}{16\mu^5}\left(\frac{\beta_2+\beta_3}{\mu}\right)^2 \left[\begin{array}{c} -\frac{3(R+\delta)^{10}}{2r^4} + \frac{15(R+\delta)^8}{r^2} - 30r^2(R+\delta)^4 \\ +\frac{15}{2}r^4(R+\delta)^2 - r^6 \end{array}\right], \quad (2.37)$$

where A_5 and B_5 are constants and are defined as

$$A_5 = \frac{\rho^5 g^5}{16\mu^5}\left(\frac{\beta_2+\beta_3}{\mu}\right)^2 \left[\begin{array}{c} \frac{3(R+\delta)^{10}}{2R^4} - \frac{15(R+\delta)^8}{R^2} + 30R^2(R+\delta)^4 \\ +\frac{15}{2}R^4(R+\delta)^2 + R^6 - 60(R+\delta)^6 \ln R \end{array}\right],$$

$$B_5 = \frac{\rho^5 g^5}{8\mu^5}\left(\frac{\beta_2+\beta_3}{\mu}\right)^2 \left[30(R+\delta)^6\right] \quad (2.38)$$

Finally, we write the solution as

$$u(r) = \lim_{q\to 1}\left(v_0 + qv_1 + q^2 v_2 + \ldots\right), \quad (2.39)$$

which is equivalent to

$$u(r) = v_0 + v_1 + v_2 + \ldots., \quad (2.40)$$

where v_0, v_1 and v_2 are define in Eqs. (2.33), (2.34) and (2.37) respectively.

2.5 Homotopy analysis method

Introducing the non-dimensional variables

$$\eta = \frac{r}{R},\ f = \frac{R}{\nu}u,\ K = \frac{gR^3}{\nu^2},\ \beta = \frac{\mu(\beta_2+\beta_3)}{R^4\rho^2},\ N = \frac{\lambda}{R}. \quad (2.41)$$

With the help of Eq. (2.41), Eqs. (2.1) to (2.3) take the following form

$$\eta\frac{d^2 f}{d\eta^2} + \frac{df}{d\eta} + 2\beta\left[3\eta\left(\frac{df}{d\eta}\right)^2\left(\frac{d^2 f}{d\eta^2}\right) + \left(\frac{df}{d\eta}\right)^3\right] + K\eta = 0, \quad (2.42)$$

$$f(\eta) = N\left[\frac{df}{d\eta} + 2\beta\left(\frac{df}{d\eta}\right)^3\right] at\ \eta = 1, \tag{2.43}$$
$$\frac{df}{d\eta} = 0,\ at\ \eta = d,$$

in which

$$d = \left(1 + \frac{\delta}{R}\right).$$

Integrating Eq. (2.42) with respect to η and using Eq. (2.43), we obtain

$$f(1) = \frac{NK}{2}\left(d^2 - 1\right)\ and\ \frac{df}{d\eta} = 0,\ at\ r = d. \tag{2.44}$$

From Eq. (2.43), it is straight forward to choose the initial guess

$$f_0(\eta) = \frac{K}{4}\left[1 - \eta^2 + 2N\left(d^2 - 1\right) + 2d^2 \ln \eta\right], \tag{2.45}$$

and the auxiliary linear operator

$$\mathcal{L}(f) = \eta f'' + f'. \tag{2.46}$$

The so called zeroth-order deformation problem can be defined as

$$(1 - q^*)\mathcal{L}\left[f^*(\eta, q^*) - f_0(\eta)\right] = q^*\hbar N\left[f^*(\eta, q^*)\right], \tag{2.47}$$

where

$$N\left[f^*(\eta, q^*)\right] = \eta\frac{\partial^2 f^*(\eta, q^*)}{\partial \eta^2} + \frac{\partial f^*(\eta, q^*)}{\partial \eta} + K\eta + 2\beta\left[3\eta\left(\frac{\partial f^*(\eta, q^*)}{\partial \eta}\right)^2\left(\frac{\partial^2 f^*(\eta, q^*)}{\partial \eta^2}\right) + \left(\frac{\partial f^*(\eta, q^*)}{\partial \eta}\right)^3\right] = 0, \tag{2.48}$$

where $f^*(\eta, q^*)$ is the solution which depends only upon the initial guess $f_0(\eta)$, the auxiliary linear operator $\mathcal{L}$, the auxiliary parameter $\hbar$ and the embedding parameter $q^* \in (0, 1)$. The embedding parameter has the property that at $q^* = 0$

$$f^*(\eta, 0) = f_0(\eta), \tag{2.49}$$

and at $q^* = 1$

$$f^*(\eta, 1) = f(\eta). \tag{2.50}$$

Thus the embedding parameter has the property that as q^* varies from 0 to 1, $f^*(\eta, q^*)$ varies continuously from initial guess to the final solution. Thus

$$f(\eta) = f_0(\eta) + \sum_{m=1}^{\infty} f_m(\eta), \tag{2.51}$$

The mth-order deformation problem can be written as

$$\mathcal{L}\left[f_m(\eta) - \chi_m f_{m-1}(\eta)\right] = R_m(\eta), \tag{2.52}$$

$$f_m(1) = 0 \; and \; f_m'(d) = 0, \tag{2.53}$$

where

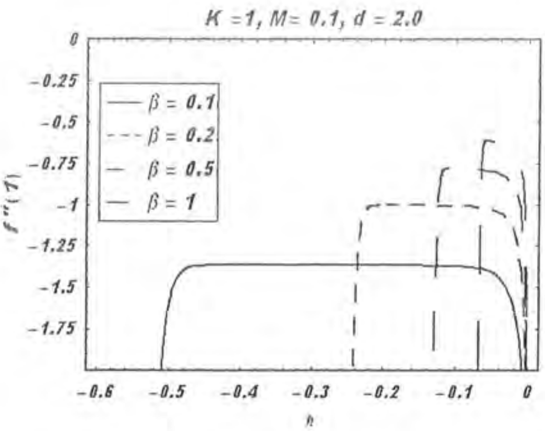
$$R_m(\eta) = \hbar \left[\begin{array}{c} \eta f_{m-1}''(\eta) + f_{m-1}'(\eta) + K\eta(1-\chi_m) - \\ 2\beta \sum_{k=0}^{m-1} f_{m-1-k}'(\eta) \sum_{l=0}^{k} f_{k-1}'(\eta) \left\{ f_l'(\eta) + 3\eta f_l''(\eta) \right\} \end{array} \right], \tag{2.54}$$

We now use the symbolic calculation software MATHEMATICA and solve the set linear differential Eq. (2.52) with the corresponding boundary conditions. It is found from the MATHEMATICA iterations that f_m can be written as

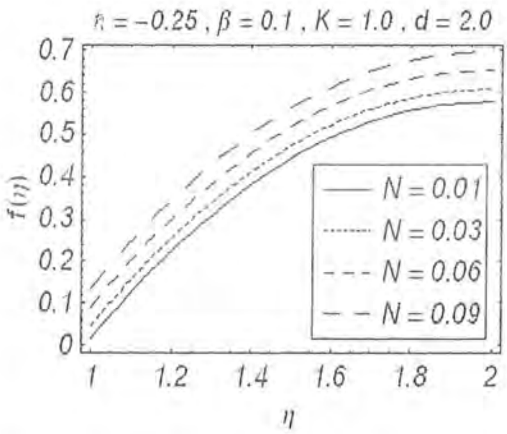
$$f_m(\eta) = \sum_{n=-2m}^{2m+2} a_{m,n}\eta^n + A_m \ln \eta, \; m \geqslant 0, \tag{2.55}$$

where $a_{m,n}$ and A_m are constants.

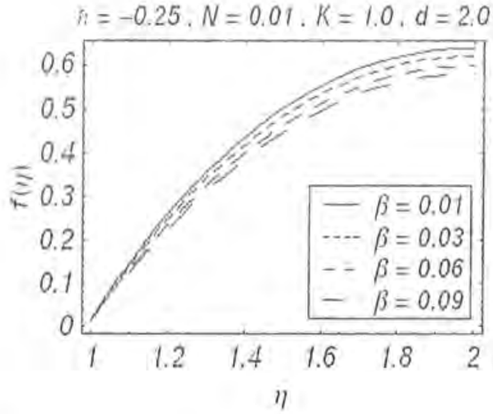
2.6 Graphical results



ħ-curves for the 26th order of approximation and for different values of fluid parameter β.



Velocity-curves for different values of slip parameter N



Velocity-curves for different values of fluid parameter β

2.7 Results and discussion

The auxiliary parameter $\hbar$, which gives us the convergence region and rate of approximation for the HAM in Eq. (2.53) which is the final explicit analytic expression. we plot the $\hbar$ curves for 25th order of approximation and for different values of β. Fig (1), as we take decreasing value of the constant β we get the more convergence region in $\hbar$ range.

In Fig (2) we consider the variation of slip parameter N, by increasing N, the velocity increases through out the given domain.

Fig (3) The variation of velocity for different values of third grade fluid parameter β for the partial slip case is considered. we can see from the figure, the velocity f increases in the given film by increasing the parameter β.

2.8 Conclusion

Analytical solutions for a thin flow of third grade fluids are presented. The governing non-linear differential equations with corresponding non- linear bounsary conditions solved by Perturbation method, Homotopy Perturbation method and Homotopy analysis method. Homotopy analysis method is the best method and the convergence region shows the validity of the solution given.

Chapter 3

Effects of heat transfer and MHD on a thin film flow of a third grade fluid with variable viscosity

3.1 Introduction

In this chapter, we have studied the MHD thin film flow of a third grade fluid with variable viscosity. The Vogel model of viscosity is taken into account. The governing nonlinear coupled differential equations are solved by Homotopy analysis method. The graphical results are presented for various physical parameters appear in the problem.

3.2 Mathematical formulation

Let us consider an incompressible, thermodynamically compatible, steady third grade fluid lying on the outer surface of an infinitely long vertical cylinder. We assume that the fluid is subject to transverse magnetic field $\mathbf{B}$. A very small magnetic Reynolds number is assumed and hence the induced magnetic field can be neglected. When the fluid moves into the magnetic field, two major physical effects arise. The first one is that an electric field E is induced in the fluid. We will assume that there is no excess charge density and therefore, $\nabla.\mathrm{E} = 0$. Neglecting the induced magnetic field implies that $\nabla \times \mathrm{E} = 0$, and therefore, the induced electric field is negligible. The second effect is dynamical in nature, i.e, a Lorentz force ($\mathbf{J} \times \mathbf{B}$), where $\mathbf{J}$ is the current density, This force acts on the fluid and modifies its motion. This results in the transfer of energy from the electromagnetic field to the fluid. In present study, the relativistic effects are neglected and the current density $\mathbf{J}$ is given by the Ohm's law as

$$\mathbf{J} = \sigma (\mathbf{V} \times \mathbf{B}).$$

The governing equations of motion and energy in cylindrical coordinates are derived in chapter one but for the sake of convenience, we can write it as

$$\frac{\partial p}{\partial r} = \frac{1}{r}\frac{d}{dr}\left[(2\alpha_1 + \alpha_2) r \left(\frac{dv}{dr}\right)^2\right], \tag{3.1}$$

$$\frac{\partial p}{\partial \theta} = 0, \tag{3.2}$$

$$\frac{\partial p}{\partial z} = \frac{1}{r}\frac{d}{dr}\left[r\left(\mu + 2\beta_3 \left(\frac{dv}{dr}\right)^2\right)\left(\frac{dv}{dr}\right)\right] - \frac{\sigma B_0^2 v}{\rho}, \tag{3.3}$$

$$\mu\left(\frac{dv}{dr}\right)^2 + 2\beta_3 \left(\frac{dv}{dr}\right)^4 + k\left[\frac{1}{r}\frac{d}{dr}\left(r\frac{d\theta}{dr}\right)\right] = 0. \tag{3.4}$$

Eq (3.2) shows that p is independent of θ, elimination of p from (3.1) and (3.3) gives

$$\frac{d\mu}{dr}\frac{dv}{dr} + \frac{\mu}{r}\frac{dv}{dr} + \mu\frac{d^2v}{dr^2} + \frac{2\beta_3}{r}\left(\frac{dv}{dr}\right)^3 + 6\beta_3\left(\frac{dv}{dr}\right)^2\frac{d^2v}{dr^2} = C_1, \tag{3.5}$$

where C_1 is constant. The relevant boundary conditions for the flow problems are defined as

$$\begin{aligned} v(r) &= 0,\ \theta = \theta_w,\ at\ r = R, \\ \frac{dv}{dr} &= 0,\ \frac{d\theta}{dr} = 0,\ at\ r = R + \delta. \end{aligned} \tag{3.6}$$

Introducing the nondimensional variables.

$$\bar{r} = \frac{r}{R},\ \bar{v} = \frac{v}{v_0},\ \bar{\mu} = \frac{\mu}{\mu^*},\ \bar{\theta} = \frac{\theta - \theta_w}{\theta_m - \theta_w}. \tag{3.7}$$

Making use of Eq. (3.7), Eqs. (3.4), (3.5) and boundary conditions (3.6) after dropping the bars can be written as

$$\frac{d\mu}{dr}\frac{dv}{dr}+\frac{\mu}{r}\left[\frac{dv}{dr}+r\frac{d^2v}{dr^2}\right]+\frac{\Lambda}{r}\left(\frac{dv}{dr}\right)^2\left[\frac{dv}{dr}+3r\frac{d^2v}{dr^2}\right]-Mv=C, \tag{3.8}$$

$$\frac{d^2\theta}{dr^2}+\frac{1}{r}\frac{d\theta}{dr}+\Gamma\left(\frac{dv}{dr}\right)^2\left[\mu+\Lambda\left(\frac{dv}{dr}\right)^2\right]=0, \tag{3.9}$$

$$v(r)=0,\ \theta(r)=0,\ at\ r=1, \tag{3.10}$$

$$\frac{dv}{dr}=\frac{d\theta}{dr}=0,\ at\ r=d,\ d=1+\frac{\delta}{R}. \tag{3.11}$$

where

$$\Lambda = \frac{2\beta_3 v_0^2}{R^2\mu^*},\ C=\frac{C_1R^2}{\mu^* v_0}, \tag{3.12}$$

$$\Gamma = \frac{\mu^* v_0^2}{k(\theta_m-\theta_w)},\ M=\frac{\sigma B_0^2R^2}{\mu}. \tag{3.13}$$

To simplify the Eq (3.8) and (3.9), we have used the Vogels viscosity model.

3.3 Vogel's viscosity model

The Vogels viscosity model is defined as

$$\mu=\mu^*\exp\left[\frac{A}{B+\theta}-\theta_w\right], \tag{3.14}$$

where A, B and θ_w are Vogels constants. Using the Maclaurin series expansion, we can write

$$\mu=\frac{C}{S}\left(1-\frac{\theta A}{B^2}\right), \tag{3.15}$$

where

$$S=\frac{C}{\mu^*\exp\left[\frac{A}{B}-\theta_w\right]}. \tag{3.16}$$

Substitution of equation (3.15), (3.8)and (3.9) take the form

$$-\frac{AC}{B^2S}\frac{d\theta}{dr}\frac{dv}{dr}+\frac{C}{rS}\frac{dv}{dr}-\frac{AC}{rSB^2}\theta\frac{dv}{dr}+\frac{C}{S}\frac{d^2v}{dr^2}$$

$$-\frac{C}{S}\frac{A\theta}{B^2}\frac{d^2v}{dr^2}+\frac{\Lambda}{r}\left(\frac{dv}{dr}\right)^3+3\Lambda\frac{d^2v}{dr^2}\left(\frac{dv}{dr}\right)^2-Mv=C, \tag{3.17}$$

$$\frac{d^2\theta}{dr^2}+\frac{1}{r}\frac{d\theta}{dr}+\frac{\Gamma C}{S}\left(\frac{dv}{dr}\right)^2-\frac{\Gamma C}{S}\frac{A\theta}{B^2}\left(\frac{dv}{dr}\right)^2+\Lambda\Gamma\left(\frac{dv}{dr}\right)^4=0. \tag{3.18}$$

3.4 Solution of the problem by means of HAM

For the solution of the above coupled nonlinear differential equations, we have used homotopy analysis method. According to HAM, it is straightforward to choose

$$v_0(r) = \frac{S}{2}\left(r^2-1\right)-Sd\left(r-1\right), \tag{3.19}$$

$$\theta_0(r) = \frac{\Gamma cS}{12}\left(\left(1-d\right)^4-\left(r-d\right)^4\right), \tag{3.20}$$

as the initial guesses (for velocity and temperature) and the auxiliary linear operators for velocity and temperature are

$$\mathcal{L}_v(v) = v''+\frac{1}{r}v', \tag{3.21}$$

$$\mathcal{L}_\theta(\theta) = \theta''+\frac{1}{r}\theta', \tag{3.22}$$

satisfying

$$\mathcal{L}_v\left(A_1+B_1\ln r\right)=0, \tag{3.23}$$

$$\mathcal{L}_\theta\left(A_2+B_2\ln r\right)=0, \tag{3.24}$$

where A_1, A_2, B_1 *and* B_2 are constants. The so-called zeroth order deformation for velocity temperature equations are defined as

$$
\begin{aligned}
(1-p) L_v \left[\bar{v}(r,p) - v_0(r)\right] &= p h_v N_v \left[\bar{v}(r,p), \bar{\theta}(r,p)\right], \quad (3.25)\\
(1-p) L_\theta \left[\bar{\theta}(r,p) - \theta_0(r)\right] &= p h_\theta N_\theta \left[\bar{v}(r,p), \bar{\theta}(r,p)\right], \quad (3.26)\\
\bar{v}(r,p) &= \bar{\theta}(r,p) = 0, \ r = 1,\\
\frac{\partial \bar{v}(r,p)}{\partial r} &= \frac{\partial \bar{\theta}(r,p)}{\partial r} = 0, \ r = d, \quad (3.27)
\end{aligned}
$$

where

$$
\begin{aligned}
N_v \left[\bar{v}(r,p), \bar{\theta}(r,p)\right] &= -\frac{AC}{B^2 S}\frac{d\theta}{dr}\frac{dv}{dr} + \frac{C}{rS}\frac{dv}{dr} - \frac{AC}{rSB^2}\theta\frac{dv}{dr} + \frac{C}{S}\frac{d^2v}{dr^2}\\
&\quad -\frac{CA\theta}{SB^2}\frac{d^2v}{dr^2} + \frac{\Lambda}{r}\left(\frac{du}{dr}\right)^3 + 3\Lambda\frac{d^2u}{dr^2}\left(\frac{du}{dr}\right)^2 - Mu - C, \quad (3.28)\\
N_\theta \left[\bar{v}(r,p), \bar{\theta}(r,p)\right] &= \frac{d^2\theta}{dr^2} + \frac{1}{r}\frac{d\theta}{dr} + \frac{\Gamma C}{S}\left(\frac{dv}{dr}\right)^2 - \frac{\Gamma C}{S}\frac{A\theta}{B^2}\left(\frac{dv}{dr}\right)^2 + \Lambda\Gamma\left(\frac{dv}{dr}\right)^4 . \quad (3.29)
\end{aligned}
$$

In above equations $p \in [0,1]$ is an embedding parameter, h_u and h_θ are the auxiliary non-zero parameters. HAM defines

$$
\begin{aligned}
\bar{v}(r,0) &= v_0(r),\\
\bar{\theta}(r,0) &= \theta_0(r),\\
\bar{v}(r,1) &= v(r),\\
\bar{\theta}(r,1) &= \theta(r),
\end{aligned}
$$

when $p = 0$ and $p = 1$, respectively.

By expanding $\bar{v}(r,p)$ and $\bar{\theta}(r,p)$ in Taylor's power series at $p = 0$, we have

$$
\begin{aligned}
\bar{v}(r,p) &= \bar{v}(r,0) + \sum_{m=1}^{\infty} v_m(r) p^m, \quad (3.30)\\
\bar{\theta}(r,p) &= \bar{\theta}(r,0) + \sum_{m=1}^{\infty} \theta_m(r) p^m, \quad (3.31)
\end{aligned}
$$

where

$$v_m(r) = \frac{1}{m!}\frac{\partial^m v(r,p)}{\partial p^m}, \quad (3.32)$$

$$\theta_m(r) = \frac{1}{m!}\frac{\partial^m \theta(r,p)}{\partial p^m}. \quad (3.33)$$

Differentiating equations (3.25) and (3.26), m times with respect to p and put $p=0$, the mth order deformation problem is defined as

$$\mathcal{L}_v\left[v_m(r) - \chi_m v_{m-1}(r)\right] = \hbar_v R_v(r), \quad (3.34)$$

$$\mathcal{L}_\theta\left[\theta_m(r) - \chi_m \theta_{m-1}(r)\right] = \hbar_\theta R_\theta(r), \quad (3.35)$$

where

$$\begin{aligned} R_v = {} & \frac{ACr}{B^2S}\sum_{k=0}^{m-1} v'_{m-1-k}\theta'_k + \frac{C}{S}v'_{m-1} - \frac{AC}{B^2S}\sum_{k=0}^{m-1} v'_{m-1-k}\theta_k \\ & + \frac{rC}{S}v''_{m-1} - \frac{rCA}{SB^2}\sum_{k=0}^{m-1} v'_{m-1-k}\theta_k + \Lambda\sum_{k=0}^{m-1} v'_{m-1-k}\sum_{l=0}^{k} v'_{k-l}v'_l \\ & + 3r^2\sum_{k=0}^{m-1} v'_{m-1-k}\sum_{l=0}^{k} v'_{k-l}v''_l - Cr(1-\chi_m) - Mrv_k(1-\chi_m), \end{aligned} \quad (3.36)$$

$$\begin{aligned} R_\theta = {} & r\theta''_{m-1} + \theta'_{m-1} + \frac{\Gamma Cr}{S}\sum_{k=0}^{m-1} v'_{m-1-k}v'_k - \frac{\Gamma Cr}{S}\frac{A}{B^2}\sum_{k=0}^{m-1} v'_{m-1-k}\sum_{l=0}^{k} v'_{k-l}\theta_l \\ & + \Gamma\Lambda r\sum_{k=0}^{m-1} v'_{m-1-k}\sum_{k=0}^{m-1} v'_{k-l}\sum_{s=0}^{k} v'_{l-s}v'_s. \end{aligned} \quad (3.37)$$

We now use the symbolic calculations software Mathematica and solve the set of linear differential equations (3.34)and (3.35) with the corresponding boundary conditions, up to first few order of approximations. It is found that $v_m(r)$ and $\theta_m(r)$ can be written as

$$v_m(r) = \sum_{n=0}^{3m+4} a_{m,n}r^n, m \geq 0, \quad (3.38)$$

$$\theta_m(r) = \sum_{n=0}^{5m+4} b_{m,n}r^{5n+4}, m \geq 0, \quad (3.39)$$

where $a_{m,n}$ and $b_{m,n}$ are constants.

The complete analytic solutions can be defined as

$$v(r) = \lim_{z\to\infty}\left[\sum_{m=0}^{z}\left(\sum_{n=0}^{3m+4} a_{m,n}r^{n}\right)\right], \tag{3.40}$$

$$\theta(r) = \lim_{z\to\infty}\left[\sum_{m=0}^{z}\left(\sum_{n=0}^{5m+4} d_{m,n}r^{n}\right)\right]. \tag{3.41}$$

3.5 Graphical results

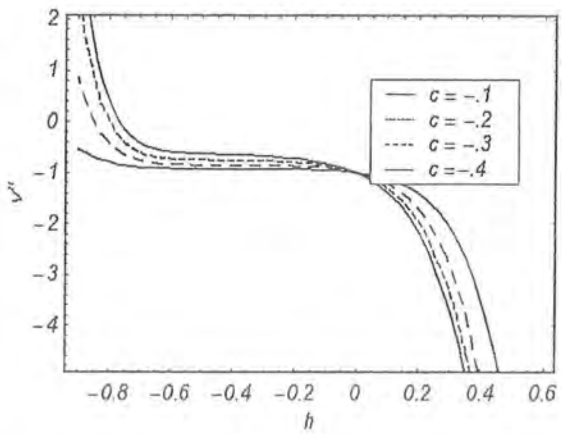


Fig.3.1 $\hbar$−curves of v'' for the 13^{th} order of approx.

for different values of pressure drop (c)

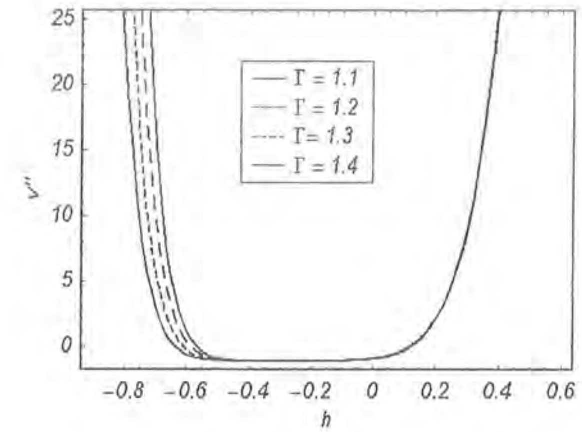


Fig.3.2 $\hbar$−curves of $v''$for the 13^{th}order of approx. for different values of Brinkman number (Γ)

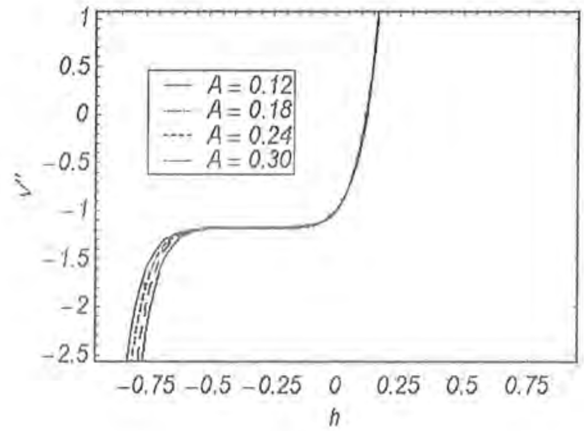


Fig.3.3 $\hbar$−curves of $v''$for the 13^{th}order of approx. for different values of parameter(A)

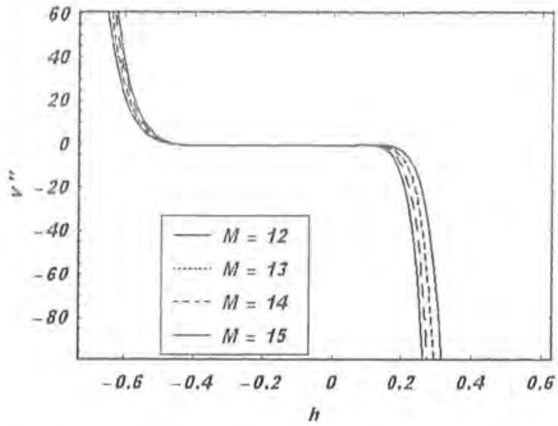


Fig.3.4 $\hbar$−curves of $v''$for the 13^{th}order of approx. for different values of (M)

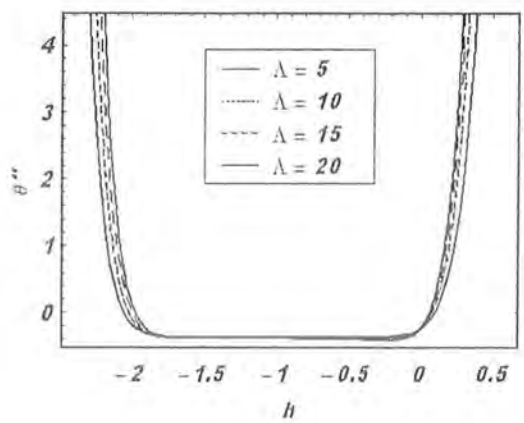


Fig.3.5 $\hbar$−curves of $\theta''$for the 13^{th}order of approx. for different values of non-Newtonian parameter (Λ)

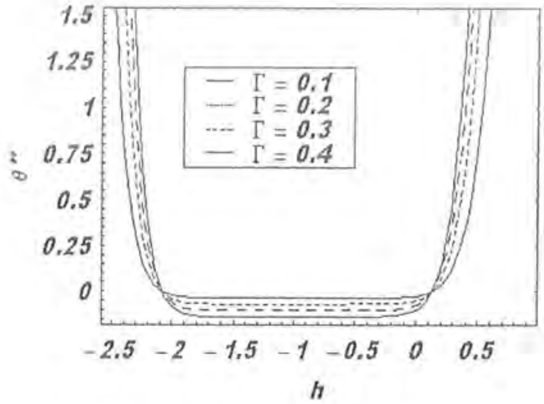


Fig.3.6 $\hbar$–curves of $\theta''$for the 13^{th}order of approx. for different values of Brinkman number (Γ)

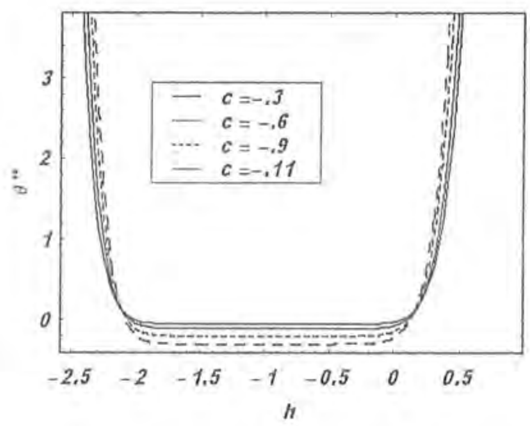


Fig.3.7 $\hbar$–curves of $\theta''$for the 13^{th}order of approx. for different values of pressure drop (c)

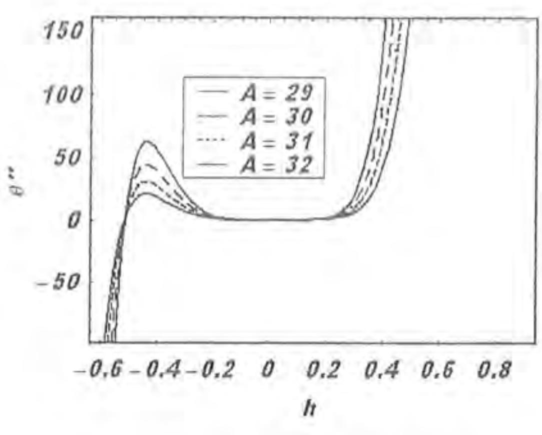


Fig.3.8 $\hbar$–curves of $\theta''$for the 13^{th}order of approx.

for different values of parameter(A)

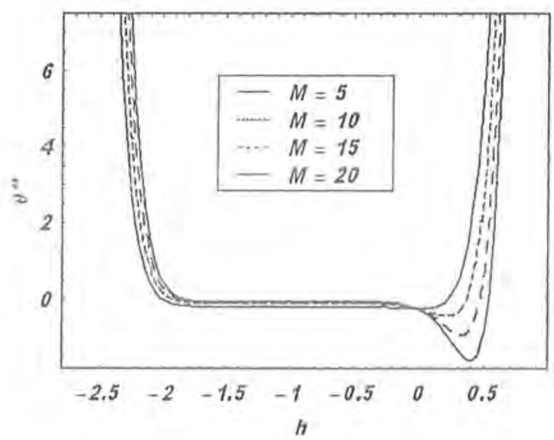


Fig.3.9 $\hbar$–curves of $\theta''$for the 13^{th}order of approx.

for different values of (M)

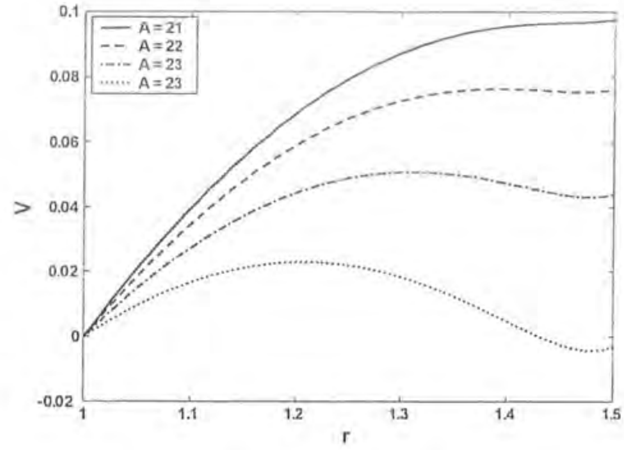


Fig.3.10 Velocity profile along the pipe for different values of parameter (A)

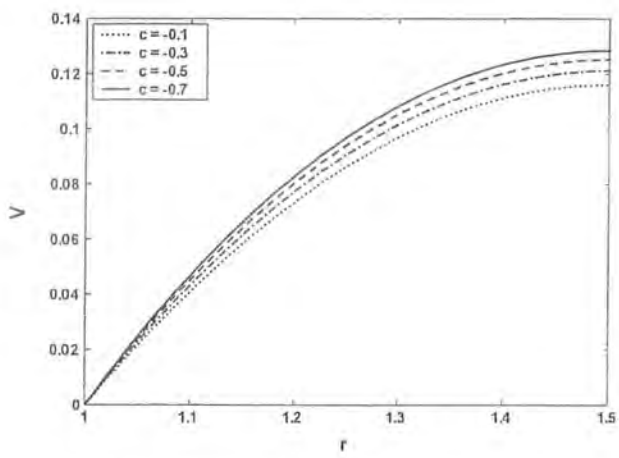


Fig.3.11 Velocity profile along the pipe for different values of pressure drop (c)

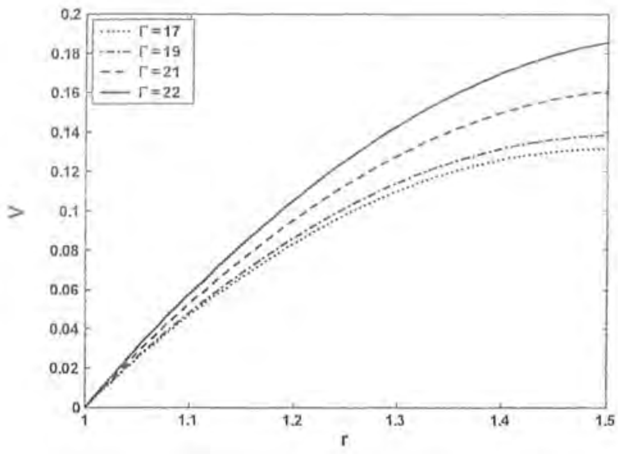


Fig.3.12 Velocity profile along the pipe for different values of non-Newtonian Brinkman number (Γ)

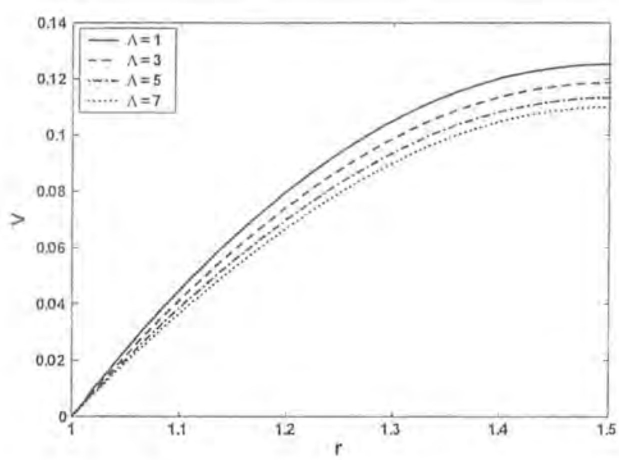


Fig.3.13 Velocity profile along the pipe for different values of non-Newtonian parameter(Λ)

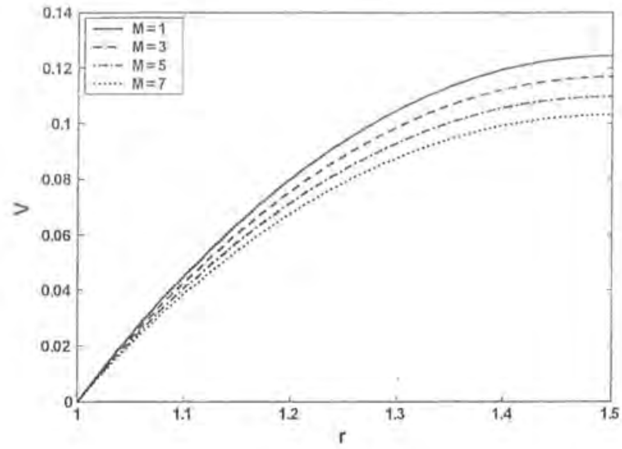


Fig.3.14 Velocity profile along the pipe for different values of parameter (M)

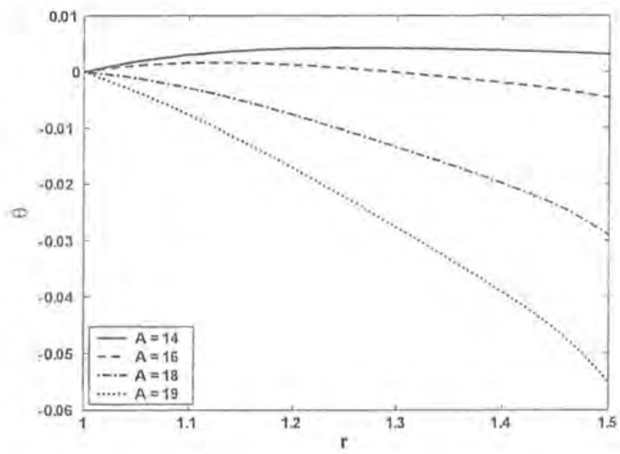


Fig.3.15 Temperature profile along the pipe for different values of parameter (A)

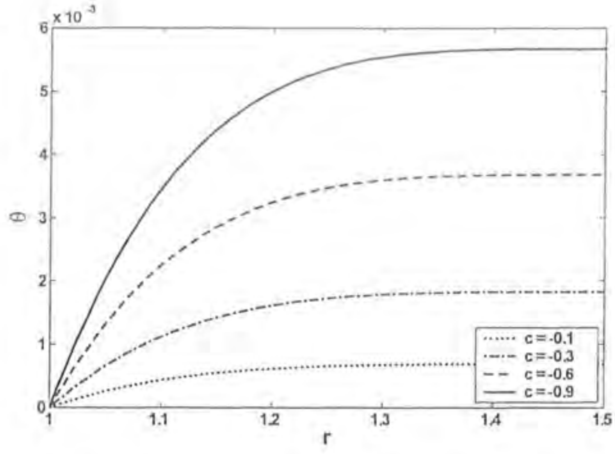


Fig.3.16 Temperature profile along the pipe for different values of pressure drop (c)

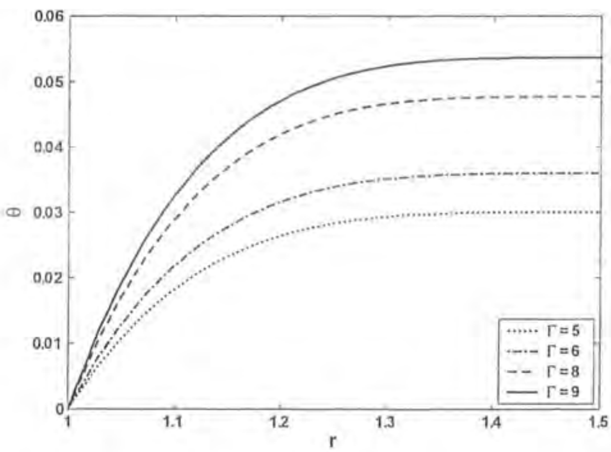


Fig.3.17 Temperature profile along the pipe for different values of non-Newtonian Brinkman number (Γ)

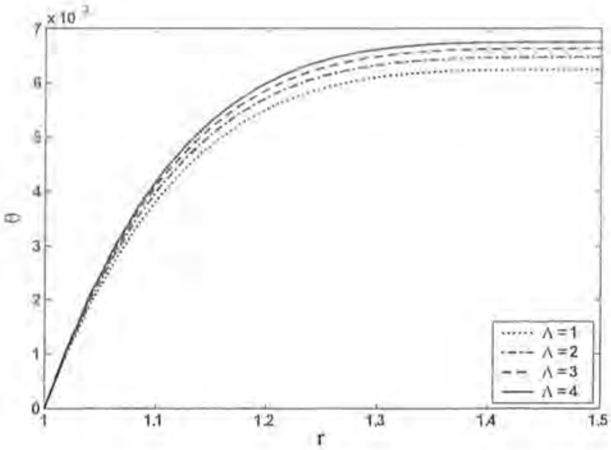


Fig.3.18 Temperature profile along the pipe for different values of non-Newtonian parameter(Λ)

3.6 Results and discussion

We have discussed the MHD thin film flow of a third grade fluid with variable, temperature dependent viscosity. The validity of the analytical solution is checked with the graphs of convergence regions which are displayed in Figs. 3.1 to 3.9 for different values of pressure gradient c, Brinkman number Γ, Vogels viscosity parameter B, magnetic parameter M and third grade parameter Λ. To check the convergence the second derivative of velocity and temperature are plotted against $\hbar$ also named as $\hbar-curves$. The region which is parallel to $\hbar-$axis represent the convergence region. For the proceeding paragraph, we must choose the value of $\hbar$ between these convergence region to plot the velocity and temperature. It is found from these figures that the convergence region for temperature is greater than the velocity field.

Fig. 3.10 is prepared for the velocity field for different values of A. It is found that the velocity field decreases with the increase in A between the film thickness, the results are prominent for large A, but for small A the results are almost similar which are not shown here.

The velocity field for various values of pressure drop c is plotted in Fig. 3.11. It is observed that the velocity field increases with the increase in pressure drop c. The velocity field increases for large values of Brinkman number Γ which is shown in Fig. 3.12. The variation of non-

Newtonian parameter Λ on the velocity field is presented in Fig. 3.13. It is seen that for large values of Λ the velocity is increases. It is observed that with the increase in magnetic parameter M the velocity field decreases (see Fig. 3.14).

Figs. 3.15 to 3.18 are prepared for the temperature field for different values of A, c, Γ and Λ. It is dipicted from the figures that with the increase in large A the temperature field decreases while with the increase in c, Γ and Λ the temperature field increases.

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