Non-Newtonian fluid flows in rotating and non-rotating frames

By

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QUAID-I-AZAM UNIVERSITY
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Supervised By

Prof. Dr. Muhammad Ayub

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Certificate

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We accept this thesis as conforming to the required standard.

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Dedicated to

My wife

Who has given many sacrifices and bearings for the entire course of my research work

Preface

The flows of non-Newtonian fluids present some interesting and exciting challenges to researchers in engineering, applied mathematics and computer science. Engineers can design effective viscometers and other instruments to measure the non-Newtonian fluid parameters. The constitutive equations of non-Newtonian fluids are very complex involving a number of parameters. Such equations give rise to problems which are far from trivial. Typically, these equations lead to problems in which the order of differential equations exceeds the number of available boundary conditions. Therefore mathematicians can derive the proofs for existence of unique or multiple solutions. Computer Scientists can design efficient algorithms for computing the flows.

With the advent of computer and corresponding development of software for numerical integration of differential equations, the task of computing the flows in fluid dynamics became much simpler. The numerical solution of ordinary differential equations (ODE) has reached a state of art where given almost any ODE with appropriate boundary conditions, it is possible to obtain its accurate numerical solution. Nevertheless, the problem arising in the study of flow of non-Newtonian fluids still pose a challenge to applied mathematicians, numerical analysts and computer simulationists. These stem from the fact that the viscoelasticity of the fluid introduces some extra terms in the momentum equations which include in particular terms that have the higher order derivative than the number of available boundary conditions. The investigators accordingly have avoided the problem of getting the numerical solution and found it convenient to obtain the perturbation solution.

In the present thesis, our concern is to investigate the HAM (homotopy analysis method) and numerical solutions for some highly non-linear flow problems of the third and fourth order fluids. Due to these facts in mind the layout of the thesis is as follows: Chapter 1 is introductory in nature, and chapter 2 includes all basic definitions and equations which are used in the subsequent chapters.

Chapter 3 is devoted for the flow of a micropolar fluid. Here two-dimensional equations are first modelled and then solved for a geological problem. Lie group method has been employed in obtaining the analytic solution. In order to see the variation of velocity, various graphs are sketched and analyzed.

In chapter 4 the flow of third grade fluid in a porous space is considered. A modified Darcy's law for a third grade fluid has been introduced. The well known Stokes' first problem has been studied. Numerical simulations have been performed using Newton's method. The results show that for large values of time the behavior of non-Newtonian fluids is similar to that of Newtonian fluid.

The steady flow of a third grade fluid over a jerked plate is discussed in chapter 5. The third grade fluid fills the porous half space. Explicit analytic solutions are obtained using homotopy analysis method (HAM). Recurrence formulas are obtained and convergence of the results is discussed. Various graphs are plotted in order to see the behavior of the involved parameters on the velocity profile. It is noted that here the velocity decreases by increasing the porosity parameter.

The flow of the third grade fluid in a rotating frame of reference is carried out in chapter 6. The Stokes' first problem has been addressed. The effects of various emerging parameters including the rheological constants are seen. It is found that with the increase in third grade parameter the real part of velocity increases and imaginary part decreases.

Chapter 7 describes the analysis for the Stokes' first problem for a fourth order fluid in a porous space. Flow analysis is given using modified Darcy's law. The problem is solved using Newton's method. Different graphs are sketched just to see the behavior of the velocity. It is revealed that increase in the fourth order parameter depicts the decrease in the velocity.

In chapter 8 numerical solution of an oscillatory flow over a porous plate is considered. The constitutive equation for fourth order fluid is used. The governing non-linear partial differential equation is first modelled and then solved using Newton's method. The variation of various parameters of interest is shown on the velocity. Numerical simulation indicates that the boundary layer thickness increases owing to an increase in the suction parameter.

Chapter 9 describes the unidirectional steady flow of a Johnson-Segalman fluid bounded by two plates. The flow is induced due to motion of the upper plate. The general solution of the governing non-linear ordinary differential equation is developed. Numerical solution for the Couette flow is further included. The effects of Weissenberg number, Hartmann number and ratio of viscosities on the velocity are discussed. It is noted that the velocity increases by increasing the Hartmann number.

Contents

Chapter 1

Introduction

Mechanics of non-linear fluids present a special challenge to engineers, physicists and mathematicians. The non linearity can manifest itself in a variety of ways. The formulation of the shear stress for non-Newtonian fluids is a difficult problem which has not progressed very far from a theoretical point of view. However, there is no single model available in the literature which clearly exhibits all the properties of the non-Newtonian fluids. For a more fundamental understanding several empirical descriptions have established rheological models. One of the simplest ways in which the viscoelastic fluids have been classified is the methodology given by Rivilin and Ericken [1] and Truesdell and Noll [2], who present constitutive relations for the stress tensor as a function of the symmetric part of the velocity gradient, and its higher (objective) derivatives. Another class of models are the rate type fluid models, such as the Oldroyd model [3]. A discussion of the various differential, rate-type and integral models can be found in the books by Schowalter [4] and Huilgol [5], and the survey article by Rajagopal [6].

The theory of microfluids, a subclass of generalized fluids, was first time introduced by Eringen [7] in 1964 and has become very popular in the recent years. These are the fluids which exhibit certain microscopic effects, arising from the local structure and micromotions of the fluid elements. These fluids can support stress and body moments and are influenced by the spin inertia. The stress tensor for such fluid is non-symmetric. Eringen's theory has provided a good model to study a number of complicated fluids, including the flow of low concentration suspensions, liquid crystals, blood and turbulent shear flows. In 1966, Eringen [8] introduced the subclass of microfluids named as the theory of micropolar fluids, which exhibit micro-rotational inertia. This class of fluids possess a certain simplicity and elegance in their mathematical formulation and are more easily amenable to solution, which has a great attraction for mathematicians.

Recently the studies of micropolar fluids have acquired the special status due to their industrial applications. Such applications include the extrusion of polymer fluids, solidification of liquid crystals, cooling of metallic plate in a bath, animal bloods, exotic lubricants and colloidal and suspension solutions. Undoubtedly, the classical Navier-Stokes theory is inadequate for such fluids. Several workers in the field have made the useful investigations that involve micropolar fluid. For example, Srinivasacharya and Rajyalakshmi [9] studied the creeping flow of a micropolar fluid past a porous sphere. Iyengar and Vani [10] examined the flow of micropolar fluid between two concentric spheres, induced by their rotary oscillations. Kasiviswanathan and Gandhi [11] discussed the Hartman steady flow of a micropolar fluid between two infinite, parallel non-coaxially rotating disks. Al-Bary [12] developed the exponential solution of the problem of two dimensional motion of micropolar fluid in a half-plane. Dubey et al. [13] analyzed the flow of a micropolar fluid between two parallel plates rotating about two non-coincident axes under variable surfaces charges. Gorla et al. [14] studied the heat transfer analysis on the boundary layer flow of a micropolar fluid. Ibrahem et al. [15] presented the non-classical heat conduction effects in Stokes' second problem for unsteady micropolar fluids flow. Seedek [16] studied the Hartman flow of a micropolar fluid past a continuously moving plate. Kim and Lee [17] made an interesting study for Hartman oscillatory flow problem of a micropolar fluid. Agarwal [18] presented finite element solution of unsteady three dimensional micropolar fluid flow at a stagnation point. Abo-Eldahab and Ghonaim [19] discussed the numerical solution in order to see the radiation effect on heat transfer of a micropolar fluid.

The study of physics of the flows through porous media have many applications. Such flows are important because of their applications in geothermal fields, soil pollution, fibrous insulation, nuclear-waste disposal in agriculture engineering, seepage of water in river beds, in petroleum technology for the study of the movement of natural gas, oil and water through the oil reservoirs, in chemical engineering for filtration and purification process. In the geophysical context, Raptis et al. [20 – 23] presented a series of investigations for flow through a porous medium bounded by an infinite porous plate. Nield et al. [24] has discussed the convection in porous media. Vafai [25] has explained the applications of porous media. In recent studies, Fang et al. [26] has presented the solution for the incompressible Couette flow with porous walls and Hooman [27] has discussed the forced convection in a fluid saturated porous medium tube with isoflex walls.

Because of its practical applications, the Stokes' problem for the flat plate has been the subject of numerous theoretical studies. Such studies for Navier-Stokes fluid and different types of non-Newtonian fluids include the work of Zierep [28], Soundalgekar [29], Rajagopal and Na [30], Puri [31], Bandelli et al.[32], Tigoiu [33], Fetecau and Zierep [34] and Fetecau and Fetecau [35, 36]. More recently, Tan and Masuoka [37, 38] discussed the Stokes' first problem for second grade and Oldroyd-B fluid models using modified Darcy's law. They obtained the solution analytically. The second grade and Oldroyd-B fluids for steady unidirectional flow do not exhibit the rheological characteristics. The third grade fluid model for steady flow exhibits such characteristics even in steady state situation. Moreover, the viscoelastic flows in porous space are quite prevalent in many engineering fields such as enhanced oil recovery, paper and textile coating and composite manufacturing processes. Also the modeling of polymeric flow in porous space has essential focus on the numerical simulation of viscoelastic flows in a specific pore geometry model, for example, capillary tubes, indulating tubes, packs of spheres or cylinders.

Rotation plays a significant role in several important phenomenon in cosmical fluid dynamics. Similarly, a great deal of meteorology depends upon the dynamics of a revolving fluid. The large scale and the moderate motions of the atmosphere are greatly affected by the vorticity of the earth's rotation. In the case of infinite fluid rotating as a rigid body about an axis, the amount of energy possessed by the fluid is infinite and it is of great interest to know how small disturbances propagate in such a fluid. Recently, the study of rotating flows has gained considerable importance due to their applications in cosmical and geophysical fluid dynamics. Several workers have been engaged to the rotating viscous flows in various directions. Extensive literature is available on the topic dealing with the time-dependent and time-independent flows in the rotating frame. But there is yet another area of such flows in which no considerable attention has been given. This is the area of the rotating flows in non-Newtonian fluid dynamics. Little work seems to have been done in this area. Recently Hayat et al. $[39-43,88]$ presented a series of investigations for the non-Newtonian fluid in the rotating frame. Rajagopal et al. [44] has given the existence theorem for the flow of a non-Newtonian fluid past an infinite porous plate.

Extensive research has been undertaken for unidirectional flows of a second grade fluid (simplest subclass of a differential type fluids). This is perhaps due to the fact that in second grade fluid, the governing equation for unidirectional flow is linear whereas it is non-linear in third and fourth order fluids. But the steady unidirectional flows of a second grade fluid over rigid boundaries do not include the rheological characteristics in the solution. Because of this fact the third and fourth order models have gained much importance. Such models include the rheological properties even for the steady unidirectional flows over rigid boundaries. Important contributions regarding the unidirectional flows of third and fourth grade fluids are given in the studies $[45-49]$. Chen et al. $[83-85]$ has discussed the unsteady unidirectional flows with different given volume flow rate conditions. Siddiqui et al. $[87,89-91]$ has done series of investigations for the steady and unsteady flows of non-Newtonian fluids. It is known that in general the governing equations for the non-Newtonian fluids are of higher order than the Navier-Stokes equations and thus the adherence conditions become insufficient. The critical review regarding the boundary conditions, the existence and uniqueness of the solution has been given by Rajagopal $[50,51]$, Rajagopal et al. [44] and Rajagopal and Kaloni [45].

It is generally recognized that non-Newtonian fluids are more important and appropriate in technological applications than Newtonian fluids. Polymer solutions and polymer melts provide the most common examples of non-Newtonian fluids. Using the Newtonian fluid model to analyze, predict and simulate the behavior of the non-Newtonian fluid have been widely adopted in industries. However, the flow characteristics of a non-Newtonian fluids have been found to be quite different from those of a Newtonian fluids. Thus we cannot replace non-Newtonian fluid by a Newtonian fluid for practical applications. Hence, it is necessary to study the flow behavior of non-Newtonian fluids in order to obtain a thorough cognition and to improve the utilization in various manufactures. Due to variety of fluids, several non-Newtonian fluid models have been proposed. Amongst these there is a Johnson-Segalman fluid model. This model is developed to allow for non-affine deformation [52]. Some researchers [53, 54] used this model to explain the phenomenon of "spurt": in which there is a large increase in the volume throughout at a critical pressure gradient for a small increase in the driving pressure gradient. Experimentalists usually associate "spurt" with slip at the wall and there have been a number of experiments [55 – 62] to support this hypothesis. Rao and Rajagopal [63] and Rao [64] have made advances towards explaining this phenomenon. However, no attempt has been made to discuss the flow of the Johnson-Segalman fluid in the context of magnetohydrodynamics (MHD). Examples of non-Newtonian fluids which might be conductors of electricity are given by Sarpkaya [65], e.g., flow of nuclear slurries and of mercury amalgams, and lubrication with heavy oils and greases.

Due to all the afore mentioned facts in mind, the present thesis is arranged in the following form. Chapter 2 includes the basic definitions and equations which are quite helpful for the succeeding chapters. In chapter 3 the analytic solution for the flow of a micropolar fluid is developed using Lie group method. The translation type symmetry has been taken into account. The various graphs are plotted to see the variation of velocity profile for the various values of the involved parameters. The contents of this chapter are published in **Acta Mechanica, 188**, $93-102$ (2007).

In chapter 4 we have modeled the differential equation for the third grade fluid in the porous half space using modified Darcy's law. Stokes' first problem has been discussed using Newton's method. Variation of the various emerging parameters is seen on the velocity profile. To the best of our knowledge the modified Darcy's law has been introduced first time in the literature. It is found that for $\tau \geq 6\pi$ the non-Newtonian effects become weak and the flow field behaves as if it is a Newtonian fluid. The contents of this chapter have been accepted for publication in **Communications in Non-Linear Science and Numerical Simulations.**

Chapter 5 has been prepared just to provide an analytic solution for the steady flow of the third grade fluid in a porous medium. Expression for velocity has been obtained using a newly developed method namely the homotopy analysis method (HAM). The non-linear problem has been solved for the series solution. Recurrence formulas are obtained. Convergence of the obtained solution is discussed. The influence of various parameters of interest are first sketched and then discussed for the velocity profile. The contents of this chapter are published in **Applied Mathematical Modelling, 31** (11), $2424-2432$ (2007).

Chapter 6 provides the modelling for the rotating flow of a third grade fluid.

Numerical solution has been presented for Stokes' first problem using Newton's method. Various graphs are plotted in order to see the behavior of the involved parameters on the velocity distribution. The contents of this chapter have been accepted for publication in **Nonlinear Analysis Real World Applications Series B.**

In chapter 7 we have modeled the differential equation for the flow of a fourth order fluid in the porous half space using modified Darcy's law. The governing equation is solved for the Stokes' first problem using Newton's method. Effects of material parameters are shown on the velocity. It is worth mentioning that modified Darcy's law for the fourth order fluid has been introduced first time in the literature here. The contents of this chapter are published in **Acta Mechanica Sinica, 23,** $17-21$ (2007).

Chapter 8 contains the numerical solution for oscillatory flow of a fourth order fluid. The effects of Newtonian and non-Newtonian fluid parameters are anlyzed on the velocity distribution. The contents of this chapter have been submitted for publication in **Meccanica.**

In chapter 9 the Couette flow of a Johnson-Segalman fluid is discussed in the presence of the uniform magnetic field. One-dimensional, steady and incompressible flow of a Johnson-Segalman fluid is studied. The flow is created due to motion of the upper plate. The combined effects of viscoelascity and magnetic field are considered. The governing equation of the problem is first reduced to a non-linear ordinary differential equation and then solved for a general solution. The Couette flow has been also discussed numerically using Newton's method. The influence of the Weissenberg number, Hartmann number and ratio of viscosities upon the velocity have been explained. The contents of this chapter have been submitted for publication in **Mathematical Methods in the Applied Sciences**.

Chapter 2

Preliminaries

This chapter contains some basic definitions and equations. Newton's and homotopy analysis methods are also included in this chapter.

2.1 Non-Newtonian fluids

A non-Newtonian fluid is a fluid in which the viscosity changes with the applied shear force. As a result, non-Newtonian fluids may not have a well defined viscosity. Although the concept of viscosity is commonly used to characterize a material, it can be inadequate to describe the mechanical behavior of the substance, particularly non-Newtonian fluids. They are best studied through several other rheological properties which relate the relations between the stress and strain tensors under many different flow conditions, such as oscillatory shear, or extensional flow which are measured using different devices or rheometers. The rheological properties are better studied using tensor-valued constitutive equations, which are common in the field of continuum mechanics.

An inexpensive, non-toxic sample of a non-Newtonian fluid can be made easily. Just add corn starch to a cup of water. Add the starch in small portions and stir in slowly. When the suspension nears the critical concentration, then so called "shear thickening" property of this non-Newtonian fluid becomes apparent. The application of force from the spoon, your fingers etc causes the fluid to behave in a more solid like fashion. If left at rest it will recover its liquid like behavior. Shear thickening fluids of this sort are being researched for bullet resistant body armor, useful for their ability to absorb the energy of a high velocity projectile impact but remain soft and flexible when struck at low velocities.

A familiar example of the opposite, a shear thinning fluid, is paint. One wants the paint to flow readily off the brush when it is being applied to the surface being painted, but not to drip excessively.

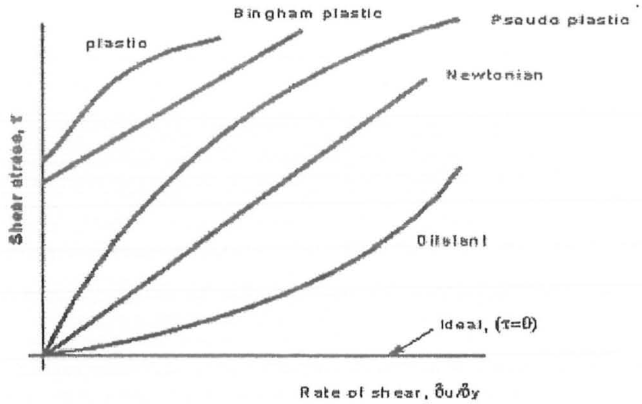

Fig. 2.1 The relation of shear rate with shear stress.

2.2 Differential type fluids

Due to complexity of fluids, there are several models of non-Newtonian fluids. One of these is a class of differential type fluids. The detail review on the topic is given by Dunn and Rajagopal [66]. The most general subclass of differential type fluids is a fourth order fluid.

The constitutive relation for the fourth order fluid is

$$\mathbf{T} = -p\mathbf{I} + \mu\mathbf{A}_1 + \alpha_1\mathbf{A}_2 + \alpha_2\mathbf{A}_1^2 + \mathbf{S}_1 + \mathbf{S}_2, \tag{2.1}$$

where

$$\mathbf{S}_1 = \beta_1\mathbf{A}_3 + \beta_2\left(\mathbf{A}_2\mathbf{A}_1 + \mathbf{A}_1\mathbf{A}_2\right) + \beta_3\left(tr\mathbf{A}_1^2\right)\mathbf{A}_1, \tag{2.2}$$

$$\begin{aligned}\mathbf{S}_2 &= \gamma_1\mathbf{A}_4 + \gamma_2\left(\mathbf{A}_3\mathbf{A}_1 + \mathbf{A}_1\mathbf{A}_3\right) + \gamma_3\mathbf{A}_2^2 + \gamma_4\left(\mathbf{A}_2\mathbf{A}_1^2 + \mathbf{A}_1^2\mathbf{A}_2\right) + \gamma_5\left(tr\mathbf{A}_2\right)\mathbf{A}_2 \\ &\quad + \gamma_6\left(tr\mathbf{A}_2\right)\mathbf{A}_1^2 + \left(\gamma_7 tr\mathbf{A}_3 + \gamma_8 tr\left(\mathbf{A}_2\mathbf{A}_1\right)\right)\mathbf{A}_1. \end{aligned} \tag{2.3}$$

In the above equations $\mathbf{T}$ is the Cauchy stress tensor, p is the hydrostatic pressure, $\mathbf{I}$ is the identity tensor, μ is the coefficient of viscosity called dynamic viscosity and α_i $(i = 1, 2)$, β_j $(j = 1$ to $3)$, γ_k $(k = 1$ to $8)$ are material constants. Note that for $\gamma_k = 0$ $(k = 1$ to $8)$ the fourth order fluid model reduces to the third order model, while when $\beta_j = 0$ $(j = 1$ to $3)$ and $\gamma_k = 0$ $(k = 1$ to $8)$ the model (2.1) reduces to a second order fluid and if $\alpha_i = 0$ $(i = 1, 2)$, $\beta_j = 0$ $(j = 1$ to $3)$ and $\gamma_k = 0$ $(k = 1$ to $8)$ it becomes the classical Navier-Stokes model. The kinematical tensors $\mathbf{A}_1$ to $\mathbf{A}_4$ are defined through the following expressions

$$\mathbf{A}_1 = \mathbf{L} + \mathbf{L}^\top, \tag{2.4}$$

$$\mathbf{A}_n = \frac{d\mathbf{A}_{n-1}}{dt} + \mathbf{A}_{n-1}\mathbf{L} + \mathbf{L}^\top\mathbf{A}_{n-1}, \quad (n > 1), \tag{2.5}$$

$$\mathbf{L} = \nabla \mathbf{V}, \tag{2.6}$$

in which $\mathbf{V}$ denotes the velocity field, ∇ is the gradient operator and d/dt is the material time derivative given by

$$\frac{d}{dt} = \frac{\partial}{\partial t} + (\mathbf{V} \cdot \nabla), \tag{2.7}$$

in which the first and second term on the right hand side indicate the local and convective parts of the derivative.

For third grade fluids, physical considerations were taken into account by Fosdick and Rajagopal [67]. They obtained that μ, α_1, α_2 and β_3 must satisfy the following hypothesis

$$\mu \geq 0, \quad \alpha_1 \geq 0, \quad |\alpha_1 + \alpha_2| \leq \sqrt{24\mu\beta_3}, \quad \beta_1 = \beta_2 = 0, \quad \beta_3 \geq 0, \tag{2.8}$$

and for second grade fluids, physical considerations were discussed by Dunn and Fosdick [75]

$$\mu \geq 0, \quad \alpha_1 \geq 0, \quad \alpha_1 + \alpha_2 \geq 0, \tag{2.9}$$

and hence the constitutive equation for second and third grade fluids are

$$\mathbf{T} = -p\mathbf{I} + \mu\mathbf{A}_1 + \alpha_1\mathbf{A}_2 + \alpha_2\mathbf{A}_1^2, \tag{2.10}$$

$$\mathbf{T} = -p\mathbf{I} + \mu\mathbf{A}_1 + \alpha_1\mathbf{A}_2 + \alpha_2\mathbf{A}_1^2 + \beta_3\left(tr\mathbf{A}_1^2\right)\mathbf{A}_1. \tag{2.11}$$

respectively.

2.3 Equation of continuity

At any point in the fluid, the continuity equation is defined as

$$\frac{\partial \rho}{\partial t} + \nabla.(\rho \mathbf{V}) = 0. \tag{2.12}$$

This is also called the "consevation of mass equation".

For an incompressible fluid the density ρ is constant and Eq. (2.12) may be simplified as

$$\nabla.\mathbf{V} = 0. \tag{2.13}$$

2.4 The momentum equation

The fundamental equation describing the flow of an incompressible fluid is

$$\rho\left(\frac{\partial}{\partial t} + (\mathbf{V}.\nabla)\right)\mathbf{V} = \operatorname{div}\mathbf{T}{+}\rho\mathbf{b}, \tag{2.14}$$

where $\rho\mathbf{b}$ are the body forces per unit mass and matrix form of Cauchy stress tensor is

$$\mathbf{T} = \begin{bmatrix} \tau_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \tau_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \tau_{zz} \end{bmatrix},$$

where τ_{xx}, τ_{yy} and τ_{zz} are the normal stresses and τ_{xy}, τ_{xz}, τ_{yx}, τ_{yz}, τ_{zx} and τ_{zy} are the shear stresses.

The scalar form of Eq. (2.14) may be written as

$$\rho\left[\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}\right] = \frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{xz}}{\partial z} + \rho b_x, \tag{2.15}$$

$$\rho\left[\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z}\right] = \frac{\partial \tau_{yx}}{\partial x} + \frac{\partial \tau_{yy}}{\partial y} + \frac{\partial \tau_{yz}}{\partial z} + \rho b_y, \tag{2.16}$$

$$\rho\left[\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z}\right] = \frac{\partial \tau_{zx}}{\partial x} + \frac{\partial \tau_{zy}}{\partial y} + \frac{\partial \tau_{zz}}{\partial z} + \rho b_z, \tag{2.17}$$

where u, v and w are the velocity components in the x, y and z directions. Here ρb_x, ρb_y and ρb_z are the body forces per unit mass in the x, y and z directions, respectively.

2.5 Porous medium

A porous medium is a continuous solid phase with intervening void or gas pockets. Natural porous media include soil, sand, mineral salts, sponge, wood and others. Synthetic porous media include paper, cloth filters, chemical reaction catalysts, and membranes. Porous medium is also used in geology, building science and hydrogeology. Porous medium is also defined as a medium that has numerous interstices, whether connected or isolated. Further porous medium is that medium for which the permeability is non-zero.

2.6 Couette flow

The term Couette flow refers to the laminar flow of a viscous fluid in the space between two surfaces, one of which is moving relative to the other. The flow is driven by virtue of viscous drag forces acting on the fluid. This type of flow is named in honor of Maurice Frédéric Alfred Couette, a Professor of Physics at the French University of Angers in the late 19th century.

Most commonly, the term "Couette flow" refers to the flow between two planes moving relative to one another (but with constant separation between the two planes). Other examples include the flow between two concentric spheres with a common axis of rotation, or the flow between two coaxial cylinders with one of the cylinders rotating at some angular velocity relative to the other. This latter type of flow is usually referred to as Taylor-Couette flow, which honors the work of G. I. Taylor on the theoretical hydrodynamic stability of this flow.

2.7 Numerical technique and discritization process

Non-linearity is a big problem for engineers, physicists and mathematicians for a long time. It is not always possible to obtain the analytic solution for the non-linear problems. Therefore, the solution by numerical techniques in such cases has got its importance. The process of obtaining the computational solution of certain problem requires the following two steps.

2.7.1 Step 1

Convert the non-linear partial differential equation and the auxiliary conditions into discrete system of algebraic equations by using the following formulas

$$\frac{\partial f}{\partial \tau} = \frac{1}{k}\left(f_{i,j} - f_{i,j-1}\right), \tag{2.18}$$

$$\frac{\partial^2 f}{\partial \tau^2} = \frac{1}{k^2}\left(f_{i,j+1} - 2f_{i,j} + f_{i,j-1}\right), \tag{2.19}$$

$$\frac{\partial f}{\partial \eta} = \frac{1}{2h}\left(f_{i+1,j} - f_{i-1,j}\right), \tag{2.20}$$

$$\frac{\partial^2 f}{\partial \eta^2} = \frac{1}{h^2}\left(f_{i+1,j} - 2f_{i,j} + f_{i-1,j}\right), \tag{2.21}$$

$$\frac{\partial^3 f}{\partial \eta^3} = \frac{1}{2h^3}\left(f_{i+2,j} - 2f_{i+1,j} + 2f_{i-1,j} - f_{i-2,j}\right), \tag{2.22}$$

$$\frac{\partial^4 f}{\partial \eta^4} = \frac{1}{h^4}\left(f_{i+2,j} - 4f_{i+1,j} + 6f_{i,j} - 4f_{i-1,j} - f_{i-2,j}\right), \tag{2.23}$$

$$\frac{\partial^5 f}{\partial \eta^5} = \frac{1}{2h^5}\left(f_{i+3,j} - 4f_{i+2,j} - 3f_{i+1,j} - 5f_{i-1,j} + 4f_{i-2,j} - 2f_{i-3,j}\right), \tag{2.24}$$

$$\frac{\partial^3 f}{\partial \eta^2 \partial \tau} = \frac{1}{h^2 k}\left(f_{i+1,j} - f_{i+1,j-1} - 2f_{i,j} + 2f_{i,j-1} + f_{i-1,j} - f_{i-1,j-1}\right), \tag{2.25}$$

$$\frac{\partial^4 f}{\partial \eta^3 \partial \tau} = \frac{1}{2h^3 k}\left(\begin{array}{c} f_{i+2,j} - f_{i+2,j-1} - 2f_{i+1,j} + 2f_{i+1,j-1} \\ +2f_{i-1,j} - 2f_{i-1,j-1} - f_{i-2,j} + f_{i-2,j-1} \end{array}\right), \tag{2.26}$$

$$\frac{\partial^5 f}{\partial\eta^2\partial\tau^3} = \frac{1}{h^2k^3}\begin{pmatrix} f_{i+1,j} - 3f_{i+1,j-1} + f_{i+1,j-2} - f_{i+1,j-3} - 2f_{i,j} + 4f_{i,j-1} \\ -4f_{i,j-1} - 4f_{i,j-2} + 2f_{i,j-1} + f_{i-1,j-2} - f_{i-1,j-3} \end{pmatrix}, \tag{2.27}$$

$$\frac{\partial^4 f}{\partial\eta^2\partial\tau^2} = \frac{1}{h^2k^2}\begin{pmatrix} f_{i+1,j} - 2f_{i+1,j-1} - 2f_{i,j} + f_{i+1,j-2} \\ +4f_{i,j-1} - 2f_{i,j-2} + f_{i-1,j} - 2f_{i-1,j-1} + f_{i-1,j-2} \end{pmatrix}. \tag{2.28}$$

etc.

This process is known as discritization.

2.7.2 Step 2

The solution process requires a numerical method for the solution of the system of algebraic equations. For the solution of a linear system of algebraic equations we have Gauss-Seidal, Gauss-Jordan and S.O.R methods. For the solution of a non-linear system of algebraic equations, we can use Newton's method.

2.8 Newton's method

The discritization process gives us a system of algebraic equations which can be written in the following form

$$\mathbf{AX} = \mathbf{B}, \tag{2.29}$$

where $\mathbf{X}$ is a column matrix of unknown nodal values. $\mathbf{A}$ contains algebraic coefficients arising from discritization and $\mathbf{B}$ is made up of known values.

The Eq. (2.29) can also be written as

$$\mathbf{R} = \mathbf{AX} - \mathbf{B} = 0. \tag{2.30}$$

Note that $\mathbf{R}$ is known as residual. This $\mathbf{R} \longrightarrow 0$ as the computational solution will tend to the exact solution of Eq. (2.29). The Newton's method can be written as

$$\mathbf{X}^{k+1} = \mathbf{X}^k - \left[\mathbf{J}^k\right]^{-1} \mathbf{R}^k, \tag{2.31}$$

where k is the iteration level and $\mathbf{J}^k$ is the Jacobian. An element of $\mathbf{J}^k$ is

$$J_{ij} = \frac{\partial R_i^k}{\partial x_j^k}. \tag{2.32}$$

Eq. (2.31) can also be written as

$$\mathbf{J}^k \Delta \mathbf{X}^k = -\mathbf{R}^k, \tag{2.33}$$

in which

$$\Delta \mathbf{X}^k = \mathbf{X}^{k+1} - \mathbf{X}^k.$$

Newton's method demonstrate quadratic convergence, if the current iteration $\mathbf{X}^k$ is sufficiently close to the converge solution $\mathbf{X}_c$.

Quadratic convergence implies that

$$\left\|\mathbf{X}^{k+1} - \mathbf{X}_c\right\| \approx \left\|\mathbf{X}^k - \mathbf{X}_c\right\|^2. \tag{2.34}$$

The criterion for the convergence of Newton's method can be developed as follows:

1. $\mathbf{J}^0$ has an inverse with its norm bounded by $\breve{a}$, i.e.

$$\left\|\left(\mathbf{J}^0\right)^{-1}\right\| \leq \breve{a}, \tag{2.35}$$

2. $\Delta \mathbf{X}^0$ has a norm bounded by $\breve{b}$, i.e.

$$\left\|\Delta \mathbf{X}^0\right\| = \left\|-\left(\mathbf{J}^0\right)^{-1} \mathbf{R}^0\right\| \leq \breve{b}, \tag{2.36}$$

3. $\mathbf{R}$ has continuous second order derivative satisfying

$$\sum_{j=1}^{N}\left|\frac{\partial^2 R_n}{\partial x_i \partial x_j}\right| \leq \frac{\breve{c}}{N}, \qquad \text{for all } \mathbf{X} \text{ in } \left\|\Delta \mathbf{X}^0\right\| < 2\breve{b}. \tag{2.37}$$

If $\breve{a}\breve{b}\breve{c} \leq 0.5$, the Newton's method will converge to the solution

$$\lim_{k \longrightarrow \infty} \mathbf{X}^k = \mathbf{X}_c,$$

at which

$$\mathbf{R}(\mathbf{X}_c) = 0,$$

and

$$\left\|\mathbf{X}^k - \mathbf{X}_c\right\| \leq \frac{\breve{b}}{2^{k-1}}.$$

The vector norms are maximum norm i.e.

$$\|\mathbf{X}\| = \max_i |x_i|.$$

The matrix norm are maximum natural norms, i.e.

$$\|\mathbf{J}\| = \max_i \sum_{i=1}^{N} |J_{ij}|.$$

The main difficulty with the Newton's method is that the radius of convergence $\breve{b}$ decreases so that $\mathbf{X}^0$ must be close to $\mathbf{X}_c$ to ensure convergence.

The main contribution to the execution time in using Newton's method is the factorization of $\mathbf{J}^k$ in the solution of Eq. (2.33). It is possible to reduce the execution time by freezing the value of $\mathbf{J}^k$ for a number of steps Δk, i.e., $\mathbf{J}^k$ need only be factorized once every Δk steps. However, more iterations are required to reach the convergent solution.

2.8.1 Advantages of Newton's method

The Newton's method has the following advantages:

1. Rapid convergence, i.e., few iterations are required.

2. Can be modified to overcome many explicit disadvantages.

3. Approximate solution can be exploited.

2.8.2 Disadvantages of Newton's method

The Newton's method has the following disadvantages:

1. Small radius of convergence if we have large number of unknowns.

2. Factorization of **J** at each iteration is computationally expensive.

3. Fails to converge if **J** becomes ill-conditioned.

2.9 Homotopy

The homotopy comes from topology. Two continuous functions or two mathematical objects are said to be homotopic if one can be continuously deformed into the other.

2.9.1 Definition of homotopy

A family of maps $h_t : X \to Y$, indexed by the real numbers, is called a homotopy if the function $\mathcal{H} : X \times [0,1] \to Y$, defined by

$$\mathcal{H}(x,t) = h_t(x), \qquad (x \in X,\, t \in I)$$

is continuous. Here h_0 and h_1 are called respectively the initial map and the terminal map of the homotopy h_t.

Two maps $\widetilde{f} : X \to Y$ and $\widetilde{g} : X \to Y$ are said to be homotopic (notation: $\widetilde{f} \simeq \widetilde{g}$), if there exists a homotopy, $h_t : X \to Y$, $(0 \le t \le 1)$, such that $h_0 = \widetilde{f}$ and $h_1 = \widetilde{g}$. In this case h_t is called the homotopy connecting $\widetilde{f}$ and $\widetilde{g}$, and is denoted by

$$h_t : \widetilde{f} \simeq \widetilde{g}.$$

Intuitively, $\widetilde{f}$ and $\widetilde{g}$ are homotopic if and only if each can be changed continuously into the other. Some special cases of homotopies are of importance. Let us suppose that X is a subspace of Y. Then the homotopy $h_t : X \to Y$ is said to be a deformation of X in Y if h_0 is the inclusion map $\tilde{\imath} : X \subset Y$. In this case we say that X is deformable into Y.

Further if $\widetilde{f}$ is homotopic to $\widetilde{g}$, then there exist a parametric family

$$\{\mathcal{H}_p : \tilde{p} \in [0, 1]\}$$

of continuous functions such that $H_p : \mathbb{R} \times [0, 1] \to \mathbb{R}$ defined by

$$\mathcal{H}_p(x) = (1 - \tilde{p})\, \widetilde{f}(x) + \tilde{p}\widetilde{g}(x), \quad \text{for all } x \in \mathbb{R} \text{ and } \tilde{p} \in [0, 1].$$

Such a homotopy is usually called as linear homotopy.

2.9.2 Homotopy analysis method (HAM)

The homotopy analysis method as proposed by Liao [68, 76 − 82] is successfully applied to obtain the analytic solution of the differential equations. The details of the application algorithm of homotopy analysis method is given in chapter 5 and therefore omitted here.

Chapter 3

Analytic solution for flow of a micropolar fluid

This chapter looks at the analytic solution for the flow of an incompressible micropolar fluid. The governing two-dimensional equations are first modeled and then solved for a geological problem. Lie group method has been used in obtaining the solution. The graphs are displayed and discussed.

3.1 Equations of motion

In tensorial notation, the basic equations which govern the flow of a micropolar fluid are:

Conservation of mass

$$\frac{\partial \rho}{\partial t} + (\rho v_k)_{,k} = 0, \tag{3.1}$$

where "," denotes the partial derivative and repeated indices means the Einstien summation convention.

Balance of momentum

$$t_{kl,k} + \rho\left(f_l - \dot{v}_l\right) = 0. \tag{3.2}$$

Balance of first stress moments

$$t_{ml} - s_{ml} + \lambda_{klm,k} + \rho\left(l_{lm} - \dot{\sigma}_{lm}\right) = 0, \tag{3.3}$$

where ρ is the mass density, v_k is the velocity vector, t_{kl} is the stress tensor, f_l is the body force per unit mass, s_{ml} is the micro-stress tensor, λ_{klm} is the first stress moments, l_{lm} is the first body moments per unit mass and $\dot{\sigma}_{lm}$ is the inertial spin.

The stress tensor t and the micro-stress tensor s are defined as [8]

$$t = \left[-\pi + \lambda trd + \lambda_0 tr\left(b - d\right)\right] I + 2\mu d + 2\mu_0\left(b - d\right) + 2\mu_1\left(b^T - d\right), \tag{3.4}$$

$$s = \left[-\pi + \lambda trd + \eta_0 tr\left(b - d\right)\right] I + 2\mu d + \xi_1\left(b - b^T - 2d\right), \tag{3.5}$$

in which I is the unit tensor, λ, λ_0, μ, μ_0, μ_1, η_0 and ξ_1 are the viscosity coefficients. Also tr denotes the trace and a superscript T indicates the transpose.

Furthermore, the rate of deformation tensor is

$$d_{kl} = \frac{1}{2}\left(v_{k,l} + v_{l,k}\right) \tag{3.6}$$

and micro-deformation rate tensor of second order is

$$b_{kl} = v_{k,l} + \sigma_{kl}. \tag{3.7}$$

For micropolar fluids, we have

$$\lambda_{klm} = -\lambda_{klm}, \qquad \sigma_{kl} = -\sigma_{kl}. \tag{3.8}$$

Using Eqs. (3.4) – (3.8) in Eqs. (3.1) – (3.3) the two dimensional equations for an incompressible micropolar fluid become [70]

$$\frac{\partial \overline{u}}{\partial \overline{x}} + \frac{\partial \overline{v}}{\partial \overline{y}} = 0, \tag{3.9}$$

$$\rho\left(\overline{u}\frac{\partial \overline{u}}{\partial \overline{x}} + \overline{v}\frac{\partial \overline{u}}{\partial \overline{y}}\right) = (\mu + k_1)\left(\frac{\partial^2 \overline{u}}{\partial \overline{x}^2} + \frac{\partial^2 \overline{u}}{\partial \overline{y}^2}\right) + k_1\frac{\partial \overline{\sigma}}{\partial \overline{y}} - \frac{\partial \overline{p}}{\partial \overline{x}}, \tag{3.10}$$

$$\rho\left(\overline{u}\frac{\partial \overline{v}}{\partial \overline{x}} + \overline{v}\frac{\partial \overline{v}}{\partial \overline{y}}\right) = (\mu + k_1)\left(\frac{\partial^2 \overline{v}}{\partial \overline{x}^2} + \frac{\partial^2 \overline{v}}{\partial \overline{y}^2}\right) - k_1\frac{\partial \overline{\sigma}}{\partial \overline{x}} - \frac{\partial \overline{p}}{\partial \overline{y}}, \tag{3.11}$$

$$\rho\overline{j}\left(\overline{u}\frac{\partial \overline{\sigma}}{\partial \overline{x}} + \overline{v}\frac{\partial \overline{\sigma}}{\partial \overline{y}}\right) = G_1\left(\frac{\partial^2 \overline{\sigma}}{\partial \overline{x}^2} + \frac{\partial^2 \overline{\sigma}}{\partial \overline{y}^2}\right) - 2k_1\overline{\sigma} + k_1\left(\frac{\partial \overline{v}}{\partial \overline{x}} - \frac{\partial \overline{u}}{\partial \overline{y}}\right), \tag{3.12}$$

where $\overline{u}$ and $\overline{v}$ are the components of the velocity field in the $\overline{x}$ and $\overline{y}$ direction, $\overline{\sigma}\,(\overline{x}, \overline{y})$ is the micro-rotation component and $\overline{p} = \overline{p}(\overline{x}, \overline{y})$ is the pressure distribution. Here μ, k_1, $G_1$and $\overline{j}$ are coefficient of viscosity, coupling constant, micro-rotation constant and local micro inertia.

Defining

$$\begin{aligned} u &= \frac{\overline{u}}{U}, \quad v = \frac{\overline{v}}{U}, \quad x = \frac{\overline{x}}{L}, \quad y = \frac{\overline{y}}{L}, \\ p &= \frac{\overline{p}}{P}, \quad \sigma = \frac{\overline{\sigma}}{\sigma *}, \quad j = \frac{\overline{j}}{J}, \end{aligned} \tag{3.13}$$

the Eqs. (3.9) – (3.12) reduce to

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \tag{3.14}$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = (\epsilon_1 + \epsilon_2)\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) + \epsilon_3\frac{\partial \sigma}{\partial y} - \epsilon_4\frac{\partial p}{\partial x}, \tag{3.15}$$

$$u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = (\epsilon_1 + \epsilon_2)\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right) - \epsilon_3\frac{\partial \sigma}{\partial x} - \epsilon_4\frac{\partial p}{\partial y}, \tag{3.16}$$

$$u\frac{\partial \sigma}{\partial x} + v\frac{\partial \sigma}{\partial y} = \epsilon_5\left(\frac{\partial^2 \sigma}{\partial x^2} + \frac{\partial^2 \sigma}{\partial y^2}\right) - \epsilon_6\sigma + \epsilon_7\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right), \tag{3.17}$$

where

$$\begin{aligned}\epsilon_1 &= \frac{\mu}{\rho L U}, \quad \epsilon_2 = \frac{k_1}{\rho L U}, \quad \epsilon_3 = \frac{k_1 \sigma^*}{\rho U^2}, \quad \epsilon_4 = \frac{P}{\rho U^2}, \\ \epsilon_5 &= \frac{G_1 J}{\rho L U \bar{j}}, \quad \epsilon_6 = \frac{2 k_1 L J}{\rho U \bar{j}}, \quad \epsilon_7 = \frac{k_1 J}{\rho \sigma^* \bar{j}},\end{aligned} \tag{3.18}$$

and ϵ_1 and ϵ_2 are the reciprocal Reynolds numbers.

3.2 Symmetry analysis

In order to obtain the analytical solution, we use Lie group theory to Eqs. (3.14) – (3.17). For this we write

$$\begin{aligned} x^* &= x + \varepsilon \xi_1 (x, y, u, v, p) + O(\varepsilon^2), \\ y^* &= y + \varepsilon \xi_2 (x, y, u, v, p) + O(\varepsilon^2), \\ u^* &= u + \varepsilon \eta_1 (x, y, u, v, p) + O(\varepsilon^2), \\ v^* &= v + \varepsilon \eta_2 (x, y, u, v, p) + O(\varepsilon^2), \\ p^* &= p + \varepsilon \eta_3 (x, y, u, v, p) + O(\varepsilon^2), \\ \sigma^* &= \sigma + \varepsilon \eta_4 (x, y, u, v, p) + O(\varepsilon^2) \end{aligned} \tag{3.19}$$

as the infinitesimal Lie point transformations. We have assumed that the Eq. (3.14) to (3.17) are invariant under the transformations given in Eq. (3.19). The corresponding infinitesimal generator is

$$X = \xi_1 \frac{\partial}{\partial x} + \xi_2 \frac{\partial}{\partial y} + \eta_1 \frac{\partial}{\partial u} + \eta_2 \frac{\partial}{\partial v} + \eta_3 \frac{\partial}{\partial p} + \eta_4 \frac{\partial}{\partial \sigma}, \tag{3.20}$$

where ξ_1, ξ_2, η_1, η_2, η_3 and η_4 are the infinitesimals corresponding to x, y, u, v, p and σ respectively. Since our equations are atmost of order two, therefore, we need second order prolongation of the generator in Eq. (3.20) and then apply the invariance condition to get the following infinitesimals [69 – 72].

$$\begin{aligned}\xi_1 &= b, \quad \xi_2 = c, \\ \eta_1 &= 0, \quad \eta_2 = 0, \quad \eta_3 = d, \quad \eta_4 = e. \end{aligned} \tag{3.21}$$

Therefore, equations admits four parameter Lie group of transformations. Parameters b, c, d and e correspond to translations in the x, y, p and σ coordinates, respectively. By considering the translations in x, y directions and choosing $d, e = 0$ and solving the corresponding characteristic equation the similarity variables and functions are given as

$$\begin{aligned}\xi &= y - mx, \quad u = f(\xi), \quad v = g(\xi), \quad p = h(\xi), \\ \sigma &= N(\xi), \end{aligned} \tag{3.22}$$

where $m = c/b$ be an arbitrary parameter. In view of variables and functions in Eq. (3.22), Eqs. (3.14) – (3.17) become

$$-mf' + g' = 0, \tag{3.23}$$

$$(-mff' + gf') = (\epsilon_1 + \epsilon_2)\left(1 + m^2\right) f'' + \epsilon_3 N' + \epsilon_4 m h', \tag{3.24}$$

$$(-mfg' + gg') = (\epsilon_1 + \epsilon_2)\left(1 + m^2\right) g'' + \epsilon_3 m N' - \epsilon_4 h', \tag{3.25}$$

$$(-mfN' + gN') = \epsilon_5\left(1 + m^2\right) N'' - \epsilon_6 N - \epsilon_7\left(mg' + f'\right). \tag{3.26}$$

Integration of Eq. (3.23) yields

$$g = mf + C_1. \tag{3.27}$$

Eliminating $h(\xi)$ from Eqs. (3.24) and (3.25) and making use of Eq. (3.26) we get

$$\left(1 + m^2\right) C_1 f' = (\epsilon_1 + \epsilon_2)\left(1 + m^2\right)^2 f'' + \epsilon_3\left(1 + m^2\right) N'. \tag{3.28}$$

From Eqs. (3.26) and (3.27), one can write

$$C_1 N' = \epsilon_5 \left(1+m^2\right) N'' - \epsilon_6 N - \epsilon_7 \left(1+m^2\right) f'. \tag{3.29}$$

Now Integrating Eq. (3.25) and then using Eqs. (3.27) and (3.28), we obtain

$$h = \frac{C_2}{\epsilon_4}, \tag{3.30}$$

in which C_1 and C_2 are any arbitrary constants. Eliminating $f(\xi)$ between Eqs. (3.28) and (3.29), we have

$$N^{iv} - AN''' + BN'' + CN' = 0, \tag{3.31}$$

where

$$\begin{aligned} A &= \frac{C_1 \left(\epsilon_5 + \epsilon_1 + \epsilon_2\right)}{\epsilon_5 \left(1+m^2\right)\left(\epsilon_1+\epsilon_2\right)}, \\ B &= \frac{\left[C_1^2 - \left(1+m^2\right)\left\{\epsilon_6\left(\epsilon_1+\epsilon_2\right) + \epsilon_3\epsilon_7\right\}\right]}{\epsilon_5\left(1+m^2\right)^2\left(\epsilon_1+\epsilon_2\right)}, \\ C &= \frac{C_1\epsilon_6}{\epsilon_5\left(1+m^2\right)^2\left(\epsilon_1+\epsilon_2\right)}. \end{aligned} \tag{3.32}$$

The solution of Eq. (3.31) is given by

$$N(\xi) = C_3 e^{\alpha_1 \xi} + C_4 e^{\alpha_2 \xi} + C_5 e^{\alpha_3 \xi} + C_6, \tag{3.33}$$

where C_3, C_4, C_5 and C_6 are any arbitrary constants and $\widetilde{\alpha}_i$ $(i = 1, 2, 3)$ are the roots of the following equation

$$\widetilde{\alpha}^3 - A\widetilde{\alpha}^2 + B\widetilde{\alpha} + C = 0. \tag{3.34}$$

From Eqs. (3.29) and (3.33), the expression for $f(\xi)$ is

$$f(\xi) = \widetilde{\beta}_1 e^{\alpha_1 \xi} + \widetilde{\beta}_2 e^{\alpha_2 \xi} + \widetilde{\beta}_3 e^{\alpha_3 \xi} - \frac{\epsilon_6 C_6 \xi}{\epsilon_7\left(1+m^2\right)} + C_7, \tag{3.35}$$

in which C_7 is any arbitrary constant and $\widetilde{\beta}_i$ $(i = 1, 2, 3)$ are given through the following

$$\widetilde{\beta}_i = \frac{C_{i+2}\left[\epsilon_5\left(1+m^2\right)\alpha_i^2 - \epsilon_6 - C_1\alpha_i\right]}{\epsilon_7\left(1+m^2\right)\alpha_i}. \tag{3.36}$$

In the form of original variable we have

$$\begin{aligned} u(x,y) &= \widetilde{\beta}_1 e^{\alpha_1(y-mx)} + \widetilde{\beta}_2 e^{\alpha_2(y-mx)} + \widetilde{\beta}_3 e^{\alpha_3(y-mx)} \\ &\quad -\frac{\epsilon_6 C_6\left(y-mx\right)}{\epsilon_7\left(1+m^2\right)} + C_7, \end{aligned} \tag{3.37}$$

$$v(x,y) = m\begin{pmatrix} \widetilde{\beta}_1 e^{\alpha_1(y-mx)} + \widetilde{\beta}_2 e^{\alpha_2(y-mx)} \\ +\widetilde{\beta}_3 e^{\alpha_3(y-mx)} - \frac{\epsilon_6 C_6(y-mx)}{\epsilon_7(1+m^2)} + C_7 \end{pmatrix} + C_1, \tag{3.38}$$

$$\sigma(x,y) = C_3 e^{\alpha_1(y-mx)} + C_4 e^{\alpha_2(y-mx)} + C_5 e^{\alpha_3(y-mx)} + C_6, \tag{3.39}$$

$$p(x,y) = \frac{C_2}{\epsilon_4}. \tag{3.40}$$

Eqs. (3.37) – (3.40) give the solution of Eqs. (3.14) – (3.17) that involve seven unknown constants. For determining the values of these constants we consider a problem that occur in geology. Consider a magmatic micropolar fluid and a plate over it. The plate occupies the position $y = 0$. The positive y goes deep into the fluid beneath the plate. The relevant boundary conditions are of the form:

$$\begin{aligned} u(x,0) &= U_0, \quad u(x,\infty) = 0, \quad \frac{\partial u}{\partial x}(0,y) = 0, \quad v(x,0) = -V_0, \\ \sigma(x,0) &= 0, \quad \sigma(x,\infty) = 0, \quad p(x,\infty) = p_0. \end{aligned} \tag{3.41}$$

The expressions (3.37) to (3.40) subject to the conditions in the above equation become

$$u(x,y) = \frac{-U_0}{\gamma_2 - \gamma_1}\left(\gamma_1 e^{-\alpha y} - \gamma_2 e^{-\beta y}\right), \tag{3.42}$$

$$v(x,y) = m\left(\frac{-U_0}{\gamma_2 - \gamma_1}\left(\gamma_1 e^{-\alpha y} - \gamma_2 e^{-\beta y}\right)\right) - mU_0 - V_0, \tag{3.43}$$

$$\sigma(x,y) = \frac{U_0}{\gamma_2 - \gamma_1}\left(e^{-\alpha y} - e^{-\beta y}\right), \tag{3.44}$$

$$p(x,y) = p_0, \tag{3.45}$$

where

$$\gamma_1 = \frac{\epsilon_5\left(1+m^2\right)\alpha^2 - \epsilon_6 + C_1\alpha}{-\epsilon_7\left(1+m^2\right)\alpha}, \tag{3.46}$$

$$\gamma_2 = \frac{\epsilon_5\left(1+m^2\right)\beta^2 - \epsilon_6 + C_1\beta}{-\epsilon_7\left(1+m^2\right)\beta} \tag{3.47}$$

and $-\alpha$ and $-\beta$ are the negative roots of Eq. (3.34).

3.3 Discussion

This section deals with the interpretation of the translational parameter m and the magmatic fluid penetrating parameter V_0 on the x and y components of the velocity and on the angular velocity σ. Figs. 3.1, 3.2 and 3.4 – 3.7 have been prepared for the velocity components where as Fig. 3.3 holds for the angular velocity. It is found from Figs. 3.1 and 3.2 that velocity components u and v are decreasing functions of m. It is also evident from Fig. 3.3 that the behavior of m on the angular velocity is opposite to that of u and v.

From Eqs. (3.42) – (3.44), we note that the magmatic fluid penetrating parameter only enters into the y-component of the velocity. The x-component of the velocity u and of σ are independent of V_0. It is found from Fig. 3.4 and 3.5 that the x-component of velocity increases by increasing the value of V_0 for either $V_0 > 0$ or $V_0 < 0$. It is clear from the Figs. 3.6 and 3.7 that the behavior of V_0 on the y-component of the velocity is opposite to that of the x-component of the velocity distribution.

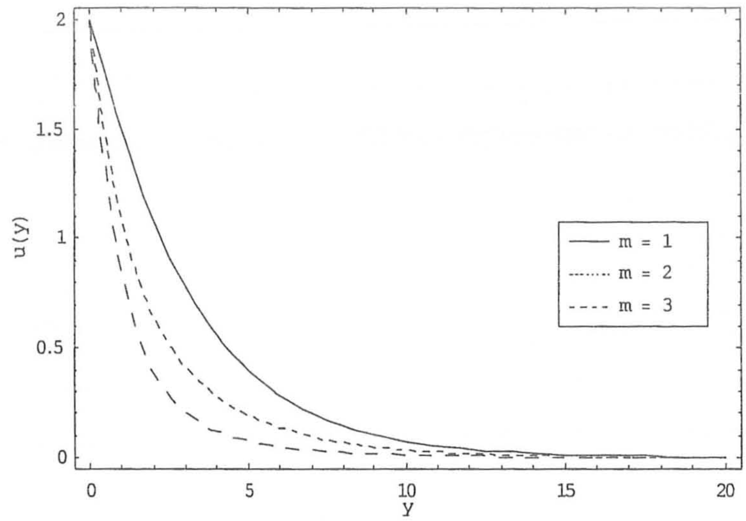

Fig. 3.1 Variation of dimensionless velocity distribution along x-axis with the value of m ($\epsilon_1 \rightarrow \epsilon_4, \epsilon_6, \epsilon_7 = 0.5$; $\epsilon_5 = 2$; $U_0 = V_0 = 2$).

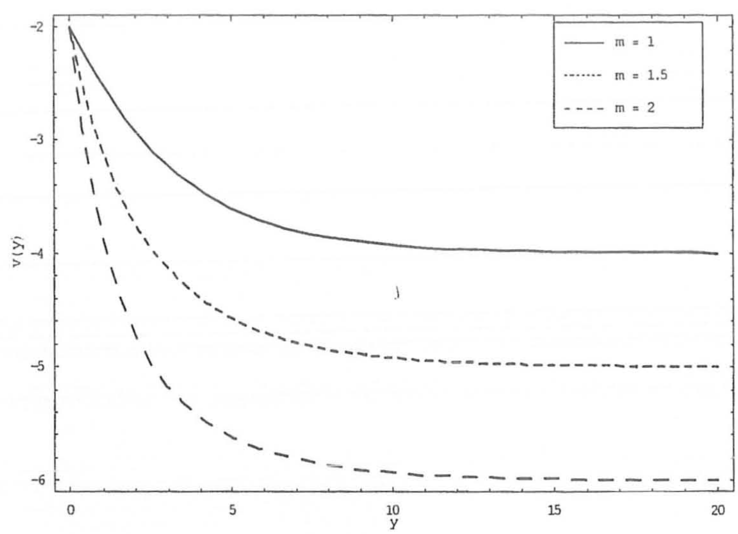

Fig. 3.2 Variation of dimensionless velocity distribution along y-axis with the value of m ($\epsilon_1 \rightarrow \epsilon_4, \epsilon_6, \epsilon_7 = 0.5$; $\epsilon_5 = 2$; $U_0 = V_0 = 2$).

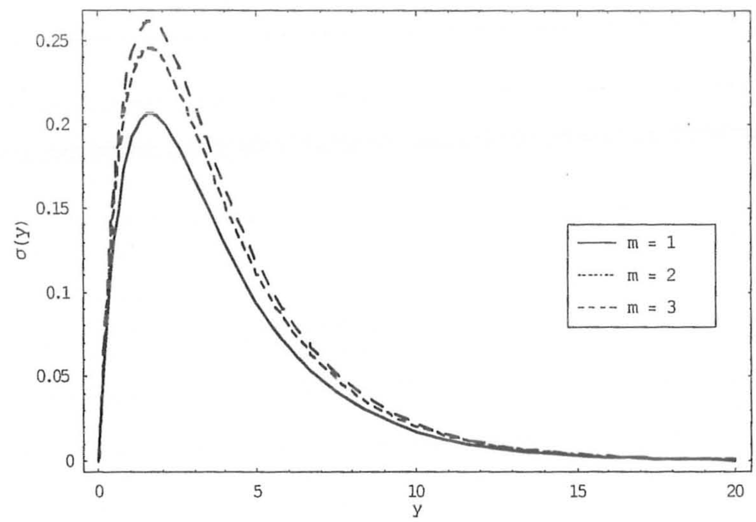

Fig. 3.3 Variation of dimensionless angular velocity with

the value of m ($\epsilon_1 \rightarrow \epsilon_4, \epsilon_6, \epsilon_7 = 0.5$; $\epsilon_5 = 2$; $U_0 = V_0 = 2$).

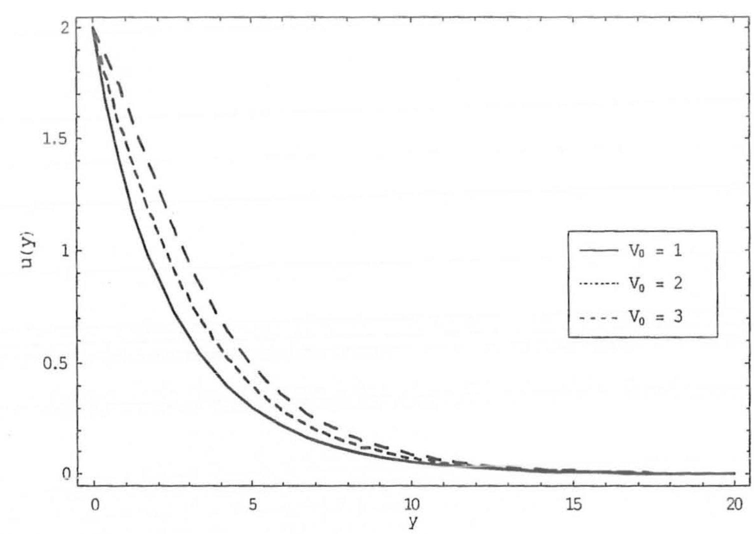

Fig. 3.4 Variation of dimensionless velocity distribution along x-axis

with the value of V_0 ($\epsilon_1 \rightarrow \epsilon_4, \epsilon_6, \epsilon_7 = 0.5$; $\epsilon_5 = 2$; $U_0 = 2$).

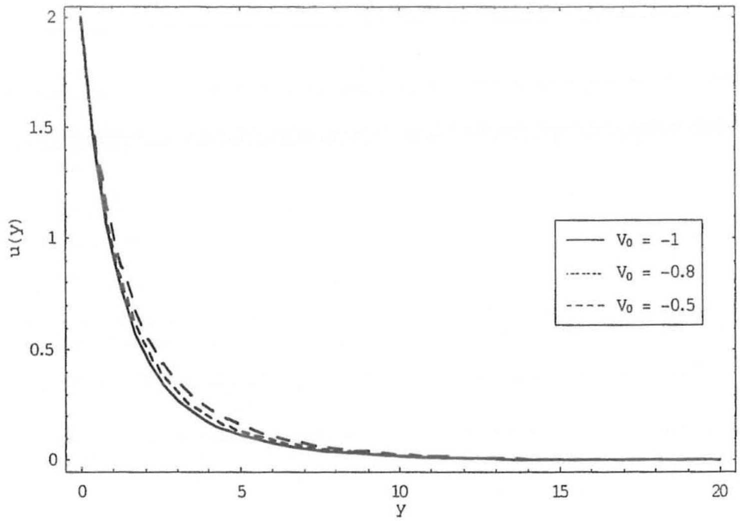

Fig. 3.5 Variation of dimensionless velocity distribution along x-axis with the value of V_0 ($\epsilon_1 \rightarrow \epsilon_4, \epsilon_6, \epsilon_7 = 0.5$; $\epsilon_5 = 2$; $U_0 = 2$).

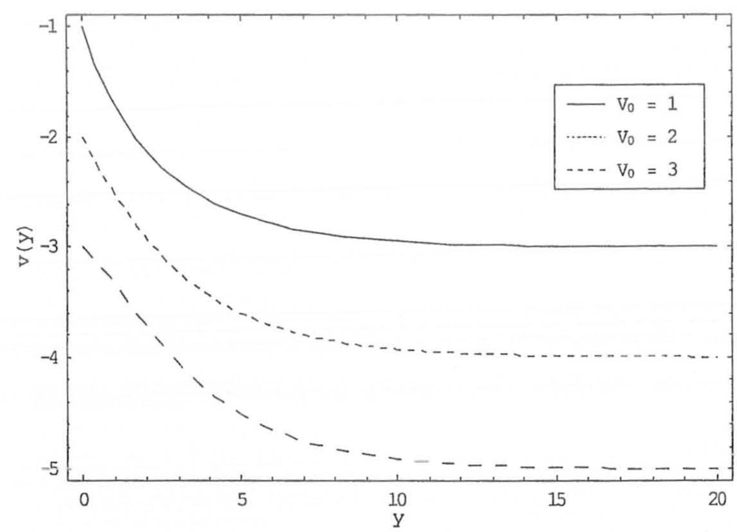

Fig. 3.6 Variation of dimensionless velocity distribution along y-axis with the value of V_0 ($\epsilon_1 \rightarrow \epsilon_4, \epsilon_6, \epsilon_7 = 0.5$; $\epsilon_5 = 2$, $U_0 = 2$.

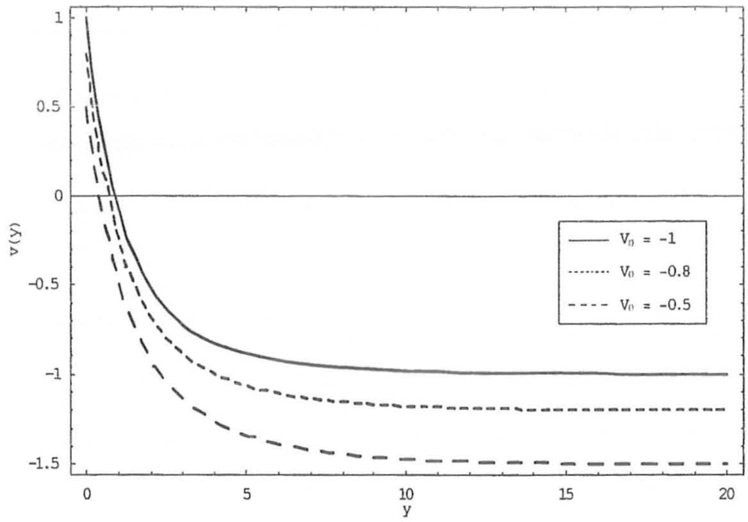

Fig. 3.7 Variation of dimensionless velocity distribution along y-axis

with the value of V_0 ($\epsilon_1 \rightarrow \epsilon_4, \epsilon_6, \epsilon_7 = 0.5$; $\epsilon_5 = 2$, $U_0 = 2$.

3.4 Concluding remarks

In this chapter, we have presented the analytical solution for the steady two dimensional equations of a micropolar fluid. Lie group analysis has been employed and the solutions corresponding to the translational symmetry are developed. The results are also sketched graphically. These results show the similar behaviour as that of numerical solution [19]. The contents of this chapter are published in "**Acta Mechanica**" **188**, $93 - 102$ (2007).

Chapter 4

Stokes' first problem for a third grade fluid in a porous half space

This chapter investigates the flow of a third grade fluid in a porous space. A modified Darcy's law for a third grade fluid has been introduced. Stokes' first problem has been studied. Numerical simulations have been performed using Newton's method. The numerical solution indicates that for a short time non-Newtonian effect is present in the velocity field. However, for a long time the velocity field becomes a Newtonian one.

4.1 Governing equations

In a porous space, the equations describing the flow of an incompressible third grade fluid are Eqs. (2.1) – (2.8) and

$$\rho\left(\frac{\partial}{\partial t}+(\mathbf{V}.\nabla)\right)\mathbf{V}=-\nabla p+\operatorname{div}\mathbf{T}+\mathbf{r}, \tag{4.1}$$

where r is the Darcy resistance for a third grade fluid in a porous space.

4.2 Problem formulation

Consider a Cartesian coordinate system $OXYZ$ with y-axis in the upward direction. The incompressible third grade fluid flows through a porous space $y > 0$ and in contact with an infinite flat plate at $y = 0$. Initially both fluid and plate are at rest. At $t = 0^+$, the plate is impulsively brought to the constant velocity U_0. Under the stated assumptions, we may write the velocity in the following form :

$$\mathbf{V} = u(y,t)\,\widehat{\imath}, \tag{4.2}$$

where $\widehat{\imath}$ and u are respectively the unit vector and velocity in the x-direction. The above equation automatically satisfies the continuity equation. Further Eqs. (2.4) – (2.6) and (2.10) give

$$\tau_{xx} = \alpha_2 \left(\frac{\partial u}{\partial y}\right)^2, \tag{4.3}$$

$$\tau_{xy} = \mu\frac{\partial u}{\partial y} + \alpha_1 \frac{\partial^2 u}{\partial y \partial t} + 2\beta_3 \left(\frac{\partial u}{\partial y}\right)^3, \tag{4.4}$$

$$\tau_{yy} = 2\alpha_1 \left(\frac{\partial u}{\partial y}\right)^2 + \alpha_2 \left(\frac{\partial u}{\partial y}\right)^2, \tag{4.5}$$

$$\tau_{xz} = \tau_{zz} = 0, \quad \tau_{xy} = \tau_{yx}, \quad \tau_{yz} = \tau_{zy}, \quad \tau_{xz} = \tau_{zx}. \tag{4.6}$$

In an unbounded porous medium the Darcy's law holds for viscous fluid flows, having low speed. This law relates the pressure drop induced by the frictional drag and velocity and ignores the boundary effects on the flow (i.e., invalid where there are boundaries of the porous medium). According to this law the induced pressure drop is directly proportional to the velocity. For the porous medium with boundaries, Brinkman proposed an equation describing the locally averaged flow. Although the equation proposed by Brinkman holds only for steady viscous flows but there are several modified Darcy's laws available in the literature for viscous flows in a porous medium. Much attention has not been given to mathematical macroscopic filtration models concerning viscoelastic flows in a porous medium. On the basis of Oldroyd constitutive equation, the following law for describing both relaxation and retardation phenomenon in an unbounded porous medium has been suggested [38]:

$$\left(1+\lambda\frac{\partial}{\partial t}\right)\nabla p = -\frac{\mu\phi}{k}\left(1+\lambda_r\frac{\partial}{\partial t}\right)\mathbf{V}, \tag{4.7}$$

where k is the permeability, λ and λ_r are the constant relaxation and retardation times respectively and ϕ is the porosity of the porous medium. Note that for $\lambda = \lambda_r = 0$, Eq. (4.7) reduces to well-known Darcy's law of viscous fluids. By analogy with Maxwell's constitutive relationship the following phenomenological model has been available in the literature [73]:

$$\left(1+\lambda\frac{\partial}{\partial t}\right)\nabla p = -\frac{\mu\phi}{k}\mathbf{V}. \tag{4.8}$$

For unidirectional flow of second grade fluid the constitutive equation can be obtained from that of an Oldroyd-B fluid by taking $\lambda = 0$ [36, 74]. Thus, in a porous medium, the relationship between ∇p and $\mathbf{V}$ for unidirectional flow of a second grade fluid can be written from Eq. (4.7) as follows:

$$\frac{\partial p}{\partial x} = -\left[\mu + \alpha_1\frac{\partial}{\partial t}\right]\frac{\phi u}{k}, \tag{4.9}$$

where

$$\mu\lambda_r = \alpha_1.$$

Employing the same idea as in Eqs. (4.7) – (4.9), we propose the following constitutive relationship between the pressure drop and velocity for unidirectional flow of a third grade fluid:

$$\frac{\partial p}{\partial x} = -\left[\mu + \alpha_1 \frac{\partial}{\partial t} + 2\beta_3 \left(\frac{\partial u}{\partial y}\right)^2\right] \frac{\phi u}{k}. \tag{4.10}$$

The pressure gradient in the above equation can also be interpreted as a measure of the resistance to flow in the bulk of the porous medium and r_x (x-component of $\mathbf{r}$) is a measure of the flow resistance offered by the solid matrix. Thus $\mathbf{r}$ can be inferred from Eq. (4.10) to satisfy the following equation:

$$r_x = -\left[\mu + \alpha_1 \frac{\partial}{\partial t} + 2\beta_3 \left(\frac{\partial u}{\partial y}\right)^2\right] \frac{\phi u}{k}. \tag{4.11}$$

Substituting Eqs. (2.4) – (2.6), (4.2) and (4.11) in Eq. (2.10) and then neglecting $\partial p/\partial x$, we obtain

$$\rho \frac{\partial u}{\partial t} = \mu \frac{\partial^2 u}{\partial y^2} + \alpha_1 \frac{\partial^3 u}{\partial y^2 \partial t} + 6\beta_3 \left(\frac{\partial u}{\partial y}\right)^2 \frac{\partial^2 u}{\partial y^2} - \left[\mu + \alpha_1 \frac{\partial}{\partial t} + 2\beta_3 \left(\frac{\partial u}{\partial y}\right)^2\right] \frac{\phi u}{k}. \tag{4.12}$$

The relevant boundary and initial conditions are

$$u(0,t) = U_0\,, \qquad u(y,t) \longrightarrow 0 \text{ as } y \longrightarrow \infty\,; \qquad u(y,0) = 0. \tag{4.13}$$

Introducing the following non-dimensional variables

$$\eta = \frac{U_0}{\nu} y, \qquad \tau = \frac{U_0^2}{\nu} t, \qquad f = \frac{u}{U_0}, \tag{4.14}$$

the problem becomes

$$(1+d)\frac{\partial f}{\partial \tau} = \frac{\partial^2 f}{\partial \eta^2} + a\frac{\partial^3 f}{\partial \eta^2 \partial \tau} + 6b\left(\frac{\partial f}{\partial \eta}\right)^2 \frac{\partial^2 f}{\partial \eta^2} - \left[c + 2e\left(\frac{\partial f}{\partial \eta}\right)^2\right] f, \tag{4.15}$$

$$f(0,\tau) = 1, \qquad f(\eta,\tau) \longrightarrow 0 \quad \text{as} \quad \eta \longrightarrow \infty, \qquad f(\eta,0) = 0, \tag{4.16}$$

where

$$
\begin{aligned}
a &= \frac{\alpha_1 U_0^2}{\rho \nu^2}, \qquad b = \frac{\beta_3 U_0^4}{\rho \nu^3}, \qquad c = \frac{\nu^2 \phi}{k U_0^2}, \qquad d = \frac{\alpha_1 \phi}{\rho k}, \\
e &= \frac{\beta_3 \phi U_0^2}{\rho k \nu}.
\end{aligned} \tag{4.17}
$$

4.3 Numerical results and discussion

We note that Eq.(4.15) is a third order partial differential equation. It is perhaps not possible to obtain the exact analytic solution. Due to this, we seek the numerical solution. For this purpose the Eq. (4.15) is transformed into system of algebraic equations by substituting the approximations to the derivatives given in section 2.7 as

$$
\begin{aligned}
& \left(\frac{1+d}{k}\right)(f_{i,j} - f_{i,j-1}) - \frac{1}{h^2}(f_{i+1,j} - 2f_{i,j} + f_{i-1,j}) \\
& -\frac{a}{h^2 k}(f_{i+1,j} - f_{i+1,j-1} - 2f_{i,j} + 2f_{i,j-1} + f_{i-1,j} - f_{i-1,j-1}) \\
& -\frac{6b}{4h^4}\left[(f_{i+1,j} + f_{i-1,j})^2 (f_{i+1,j} - 2f_{i,j} + f_{i-1,j})\right] \\
& +cf_{i,j} + \frac{2e}{4h^2}(f_{i+1,j} - f_{i-1,j})^2 f_{i,j} \\
= \;& 0.
\end{aligned} \tag{4.18}
$$

The above system of algebraic equations also gives

$$
\begin{aligned}
& R_i = Af_{i,j} + Bf_{i+1,j} + Cf_{i-1,j} + K_1 f_{i+1,j}^3 + K_2 f_{i+1,j}^2 f_{i,j} + K_3 f_{i+1,j}^2 f_{i-1,j} \\
& +K_4 f_{i-1,j}^2 f_{i+1,j} + K_5 f_{i-1,j}^2 f_{i,j} + K_6 f_{i-1,j}^3 + K_7 f_{i+1,j} f_{i-1,j} f_{i,j} + Ff_{i,j-1} \\
& +Gf_{i+1,j-1} + Hf_{i-1,j-1},
\end{aligned} \tag{4.19}
$$

where

$$
\begin{aligned}
A &= \left[\left(\frac{1+d}{k}\right)+\frac{2}{h^2}+\frac{2a}{h^2k}+c\right], \quad B=-\left[\frac{1}{h^2}+\frac{a}{h^2k}\right], \\
C &= -\left[\frac{1}{h^2}+\frac{a}{h^2k}\right], \quad K_1=-\frac{3b}{2h^4}, \quad K_2=\frac{3b}{h^4}+\frac{e}{2h^2}, \\
K_3 &= -K_1, \quad K_4=-K_1, \quad K_5=K_2, \\
K_6 &= K_1, \quad K_7=-\frac{6b}{h^4}-\frac{e}{h^2}, \quad F=-\left(\frac{1+d}{k}\right)-\frac{2a}{h^2k}, \\
G &= \frac{a}{h^2k}, \quad H=G.
\end{aligned} \tag{4.20}
$$

Now the initial and boundary conditions can be written in the following form

$$f_{0,j}=1, \qquad f_{M,j}=0, \qquad f_{i,0}=0, \qquad i=0,1,2,...,M \qquad j=0,1,2,3.... \tag{4.21}$$

Here M denotes an integer large enough such that Mh approximates infinity. Since our Eq. (4.15) is of third order while given boundary conditions are two, therefore, we introduce an augmented boundary condition

$$\frac{\partial f(\infty,\tau)}{\partial\eta}=0, \tag{4.22}$$

and consequently the problem becomes well-posed. This boundary condition is discretized to give

$$\frac{f_{M+1,j}-f_{M,j}}{h}=0,$$

i.e.,

$$f_{M+1,j}=f_{M,j}. \tag{4.23}$$

The problem consisting of Eq.(4.15) and conditions given in Eq.(4.16) has been solved numerically by employing the Newton method. Solutions for the non-Newtonian fluid models are obtained for $\tau=2\pi$. From the numerical solution f is used to express the non-dimensional velocity profile parallel to x-axis. Results for the flow are obtained for various values of the parameters a, b, c, d and e. The discussion of emerging parameters on the velocity is as follows:

Fig. 4.1 presents the velocity profile f for various values of a and d. This figure shows that increasing the parameter a or d decreases the velocity and the boundary layer thickness. Fig. 4.2 elucidates the influence of b and e on the velocity profile f. It is evident from the figure that an increase in these parameters results in a decrease of the velocity profile. The effect of porosity parameter on f is displayed in Fig. 4.3. It is clear that both velocity and boundary layer thickness decrease by increasing the porosity parameter. Fig. 4.4 shows how the velocity changes with the value of the second grade parameter in a non-porous space. It is found that here the velocity increases by increasing a.

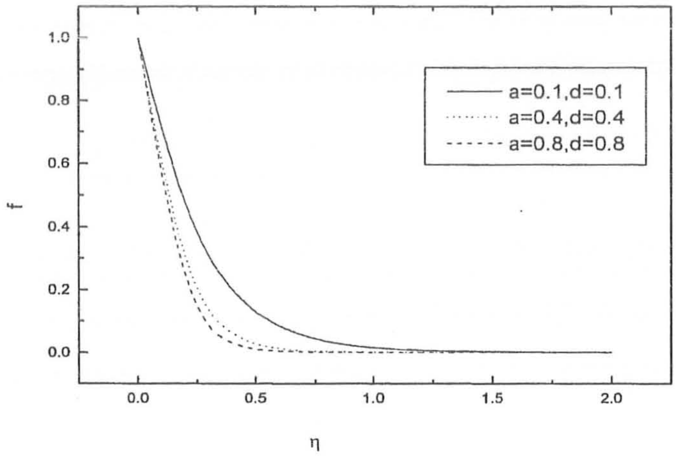

Fig. 4.1 Variation of second grade and porosity parameters on f

for $b = 0$, $c = 0$, $e = 0$ at $\tau = 2\pi$.

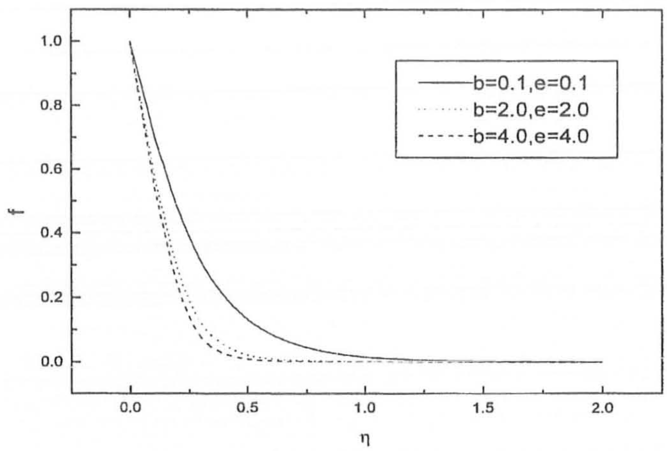

Fig. 4.2 Variation of third grade and porosity parameters on f

for $c = 0$, $a = 0.1$, $d = 0.1$ at $\tau = 2\pi$.

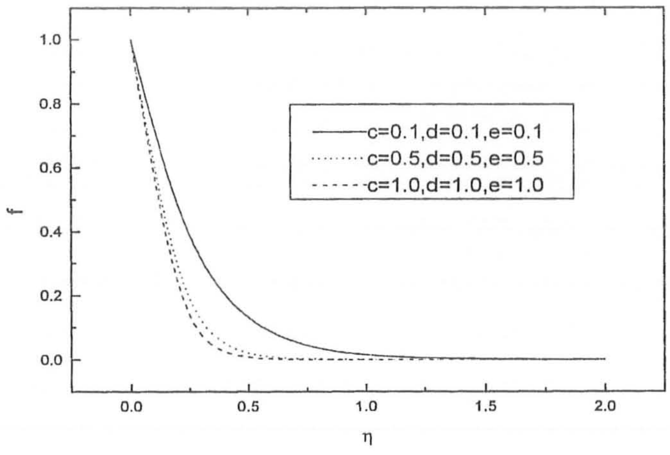

Fig. 4.3 Influence of porosity parameter on f for

$a = 0.1$, $b = 0.1$ at $\tau = 2\pi$.

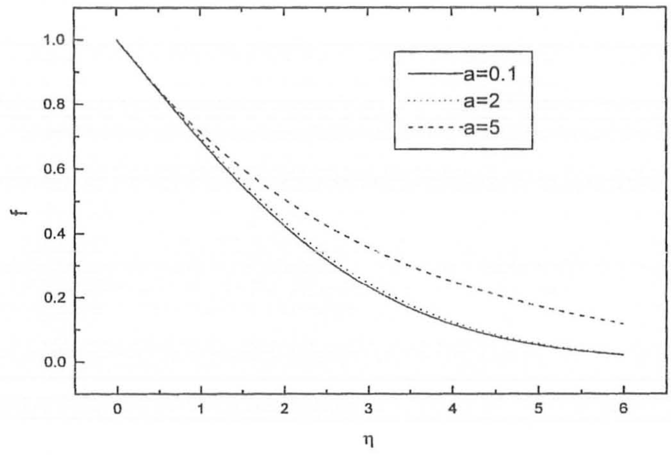

Fig. 4.4 Variation of second grade parameter on f with

$b = c = d = e = 0$ at $\tau = 2\pi$.

4.4 Concluding remarks

In the present chapter, Stokes' first problem is analyzed for the third grade fluid in a porous space. The governing constitutive relationship for modified Darcy's law in a third grade fluid has been proposed. To the best of our knowledge such relationship is not available in the literature. It is noted that modified Darcy's law in unidirectional flow of a third grade fluid yields non-linear expression in terms of velocity whereas it is linear for Newtonian, Oldroyd-B, Maxwell and second grade fluids. The governing non-linear problem that comprised the balance laws of mass and momentum has been solved numerically. Results for velocity are presented. It is important to note that variation of second grade parameter on the velocity in porous and non-porous space is quite different. It is further found that for $\tau \geq 6\pi$ the non-Newtonian effects become weak and the flow field behaves like a Newtonian fluid.

Chapter 5

Analytical solution for the steady flow of a third grade fluid in a porous half space

This chapter deals with the homotopy analysis method (HAM) solution for steady flow of a third grade fluid over a jerked plate. The solution is developed when the fluid fills the porous half space. Recurrence formulas are given. Convergence of the obtained results is analyzed. The graphs for velocity are sketched and influence of various parameters of interest is seen.

5.1 Problem formulation

Let us consider the steady flow of a third grade fluid in a porous half space. Taking the positive y-axis of a Cartesian coordinate system in the upward direction, let the third grade fluid fills the porous half space $y > 0$ above and in contact with a plate occupying the xz-plane. The flow is induced due to suddenly moved plate. The fluid far away from the plate is at rest. Under these conditions, no flow occurs in y and z-directions and steady flow velocity at a given point in the porous half space depends only on its y coordinate. From Eqs. (2.4) – (2.6), (2.10) and (4.11) the governing problem is

$$\frac{\mu}{\rho}\frac{d^2u}{dy^2} + \frac{6\beta_3}{\rho}\left(\frac{du}{dy}\right)^2\frac{d^2u}{dy^2} - \left[\mu + 2\beta_3\left(\frac{du}{dy}\right)^2\right]\frac{\phi u}{\rho k} = 0, \tag{5.1}$$

$$u(0) = U_0, \qquad u(y) \longrightarrow 0 \text{ as } y \longrightarrow \infty. \tag{5.2}$$

The Eq. (5.1) can also be written as

$$\mu^*\frac{d^2u}{dy^2} + b_1^*\left(\frac{du}{dy}\right)^2\frac{d^2u}{dy^2} - b_2^*\left(\frac{du}{dy}\right)^2 u - \phi_1 u = 0, \tag{5.3}$$

where

$$\begin{aligned} \mu^* &= \frac{\mu}{\rho + \alpha_1\phi/k}, & b_1^* &= \frac{6\beta_3}{\rho + \alpha_1\phi/k}, \\ b_2^* &= \frac{2\beta_3\phi/k}{\rho + \alpha_1\phi/k}, & \phi_1 &= \frac{\mu\phi/k}{\rho + \alpha_1\phi/k}. \end{aligned} \tag{5.4}$$

Introducing the non-dimensional variables as defined in Eq. (4.14) the problem becomes

$$\frac{d^2f}{dz^2} + \widetilde{b}_1\left(\frac{df}{dz}\right)^2\frac{d^2f}{dz^2} - \widetilde{b}_2 f\left(\frac{df}{dz}\right)^2 - \widetilde{c}f = 0, \tag{5.5}$$

$$f(0) = 1, \qquad f(\eta) \longrightarrow 0 \quad \text{as} \quad \eta \longrightarrow \infty, \tag{5.6}$$

in which

$$\widetilde{b}_1 = \frac{b_1^* U_0^4}{\mu^*\nu^2}, \qquad \widetilde{b}_2 = \frac{b_2^* U_0^2}{\mu^*}, \qquad \widetilde{c} = \frac{\phi_1\nu^2}{\mu^* U_0^2}. \tag{5.7}$$

The second order differential Eq. (5.5) subject to boundary conditions (5.6) can be solved using homotopy analysis method (HAM).

5.2 Analytic solution

In order to obtain the HAM solution, we choose

$$f_0(z) = e^{-\eta}, \tag{5.8}$$

$$\mathcal{L}(f) = f'' + f', \tag{5.9}$$

as initial approximation of f and auxiliary linear operator $\mathcal{L}$ satisfying

$$\mathcal{L}\left(C_1 + C_2 e^{-\eta}\right) = 0, \tag{5.10}$$

in which C_1 and C_2 are arbitrary constants. If $\tilde{p} \in [0, 1]$ is an embedding parameter and $\hbar$ is an auxiliary nonzero parameter then

$$(1 - \tilde{p}) \mathcal{L}[\theta(\eta, \tilde{p}) - f_0(\eta)] = \tilde{p}\hbar\mathcal{N}[\theta(\eta, \tilde{p})], \tag{5.11}$$

$$\theta(0, \tilde{p}) = 1, \qquad \theta(\infty, \tilde{p}) = 0, \tag{5.12}$$

where

$$\mathcal{N}[\theta(\eta, \tilde{p})] = \frac{\partial^2\theta(\eta, \tilde{p})}{\partial\eta^2} + \widetilde{b}_1\left(\frac{\partial\theta(\eta, \tilde{p})}{\partial\eta}\right)^2\frac{\partial^2\theta(\eta, \tilde{p})}{\partial\eta^2} - \widetilde{b}_2\theta(\eta, p)\left(\frac{\partial\theta(\eta, \tilde{p})}{\partial\eta}\right)^2 - \widetilde{c}\theta(\eta, \tilde{p}). \tag{5.13}$$

For $\tilde{p} = 0$ and $\tilde{p} = 1$, we have

$$\theta(\eta, 0) = f_0(\eta), \qquad \theta(\eta, 1) = f(\eta). \tag{5.14}$$

As $\tilde{p}$ increases from 0 to 1, $\theta(\eta, p)$ varies from $f_0(\eta)$ to $f(\eta)$. By Taylor's theorem and Eq. (5.11) one obtains

$$\theta(\eta, \tilde{p}) = f_0(\eta) + \sum_{m=1}^{\infty} f_m(\eta)\ \tilde{p}^m, \tag{5.15}$$

where

$$f_m(\eta) = \frac{1}{m!}\frac{\partial^m \theta(\eta,\tilde{p})}{\partial p^m}\bigg|_{\tilde{p}=0} \tag{5.16}$$

and the convergence of the series (5.15) depends upon $\hbar$. Assume that $\hbar$ is selected such that the series (5.16) is convergent at $\tilde{p}=1$, then due to Eq. (5.13) we get

$$f(\eta) = f_0(\eta) + \sum_{m=1}^{\infty} f_m(\eta). \tag{5.17}$$

For the mth order deformation problem, we differentiate Eq. (5.11) m times with respect to p, divide by $m!$ and then set $\tilde{p}=0$. The resulting deformation problem at the mth order is

$$\mathcal{L}\left[f_m(\eta) - \chi_m f_{m-1}(\eta)\right] = \hbar \mathcal{R}_m(\eta), \tag{5.18}$$

$$f_m(0) = f_m(\infty) = 0, \tag{5.19}$$

where

$$\mathcal{R}_m(\eta) = \hbar\left[\frac{d^2 f_{m-1}}{d\eta^2} - \tilde{c} f_{m-1}\right] + \hbar\sum_{k=0}^{m-1}\frac{df_{m-1-k}}{d\eta}\sum_{l=0}^{k}\left[\frac{df_{k-l}}{d\eta}\left(\tilde{b}_1\frac{d^2 f_l}{d\eta^2} - \tilde{b}_2 f_l\right)\right], \tag{5.20}$$

$$\chi_m = \begin{cases} 0, & m \le 1, \\ 1, & m > 1. \end{cases} \tag{5.21}$$

The solution of the above problem up to first few order of approximations may be obtained using the symbolic computation software MATHEMATICA. The solution of the problem can be expressed as an infinite series of the form

$$f_m(\eta) = \sum_{n=0}^{2m+1}\sum_{q=0}^{2m+1-2n} a_{m,n}^{q}\eta^q e^{-n\eta}, \qquad m \ge 0. \tag{5.22}$$

Invoking Eq. (5.18) into Eq. (5.22) we get the following recurrence formulas for the coefficient $a^q_{m,n}$ of $f_m(\eta)$ when $m \geq 1$, $0 \leq n \leq 2m+1$

$$a^0_{m,1} = \chi_m \chi_{2m-1} a^0_{m-1,1} - \sum_{n=2}^{2m+1} \sum_{q=0}^{2m+1-2n} \Gamma^q_{m,n} \mu^q_{n,0}, \tag{5.23}$$

$$a^k_{m,0} = \chi_m \chi_{2m+1-k} a^k_{m-1,0}, \qquad 0 \leq k \leq 2m+1, \tag{5.24}$$

$$a^k_{m,1} = \chi_m \chi_{2m-1-k} a^k_{m-1,1} - \sum_{q=k-1}^{2m} \Gamma^q_{m,1} \mu^q_{1,k}, \qquad 1 \leq k \leq 2m-1, \tag{5.25}$$

$$a^k_{m,n} = \chi_m \chi_{2m+1-2n-k} a^k_{m-1,n} + \sum_{q=k}^{2m+1-2n} \Gamma^q_{m,n} \mu^q_{n,k}, \quad 2 \leq n \leq 2m+1, 0 \leq k \leq 2m+1-2n, \tag{5.26}$$

where

$$\Gamma^q_{m,n} = \hbar \sum_{n=0}^{2m+1} \sum_{q=0}^{2m+1-2n} \left[\chi_{2m+1-2n-q} \left(a2^q_{m-1,n} - ca^q_{m-1,n} + b_1 \delta 3^q_{m,n} - b_2 \delta 4^q_{m,n} \right) \right], \tag{5.27}$$

$$\mu^q_{1,k} = \frac{q!}{k!}, \qquad 0 \leq k \leq 2q+1, q \geq 0, \tag{5.28}$$

$$\mu^q_{n,k} = \sum_{p=0}^{q-k} \frac{q!}{k! n^{p+1} (n-1)^{q-p+1}}, \qquad 0 \leq k \leq 2q, q \geq 0, n \geq 2, \tag{5.29}$$

$$\delta 3^q_{m,n} = \sum_{k=0}^{m-1} \sum_{l=0}^{k} \sum_{r=\max\{0,n-2k+2m-1\}}^{r=\min\{n,2k+2\}} \sum_{s=\max\{0,q-2m+2n-2r+1\}}^{s=\min\{q,2k+2-2r\}} \Pi 1^s_{k,r} a1^{q-s}_{m-1-k,n-r}, \tag{5.30}$$

$$\delta 4^q_{m,n} = \sum_{k=0}^{m-1} \sum_{l=0}^{k} \sum_{r=\max\{0,n-2k+2m-1\}}^{r=\min\{n,2k+2\}} \sum_{s=\max\{0,q-2m+2n-2r+1\}}^{s=\min\{q,2k+2-2r\}} \Pi 2^s_{k,r} a1^{q-s}_{m-1-k,n-r}, \tag{5.31}$$

$$\Pi 1^s_{k,r} = \sum_{j=\max\{0,r-2k+2l-1\}}^{j=\min\{r,2l+1\}} \sum_{i=\max\{0,s-2k+2l+2r-2j-1\}}^{i=\min\{s,2l+1-2j\}} a2^i_{l,j} a1^{s-i}_{k-l,r-j}, \tag{5.32}$$

$$\Pi 2^s_{k,r} = \sum_{j=\max\{0,r-2k+2l-1\}}^{j=\min\{r,2l+1\}} \sum_{i=\max\{0,s-2k+2l+2r-2j-1\}}^{i=\min\{s,2l+1-2j\}} a1^i_{l,j} a1^{s-i}_{k-l,r-j}, \tag{5.33}$$

$$a1^q_{m,n} = (q+1)\, a^{q+1}_{m,n} - n a^q_{m,n}, \tag{5.34}$$

$$a2^q_{m,n} = (q+1)\, a1^{q+1}_{m,n} - n a1^q_{m,n}. \tag{5.35}$$

Utilizing the above recurrence formulas, all coefficients $a_{m,n}^{k}$ can be computed using only the first two

$$a_{0,0}^{0} = 0, \qquad a_{0,1}^{0} = 1, \tag{5.36}$$

given by the initial guess approximation in Eq. (5.8). The corresponding Mth order approximation of Eqs. (5.5) and (5.6) is

$$\sum_{m=0}^{M} f_m(\eta) = \sum_{n=1}^{2M+1} e^{-n\eta} \left(\sum_{m=n-1}^{M} \sum_{k=0}^{2m+1-2n} a_{m,n}^{k} \eta^{k} \right), \tag{5.37}$$

and the explicit analytic solution of the problem is

$$f(\eta) = \sum_{m=0}^{\infty} f_m(\eta) = \lim_{M \longrightarrow \infty} \left[\sum_{n=1}^{2M+1} e^{-n\eta} \left(\sum_{m=n-1}^{M} \sum_{k=0}^{2m+1-2n} a_{m,n}^{k} \eta^{k} \right) \right]. \tag{5.38}$$

5.3 Convergence of the analytic solution

Clearly Eq. (5.38) contains the auxiliary parameter $\hbar$. As pointed out by Liao [68], the convergence region and rate of approximation given by HAM are strongly dependent upon $\hbar$. For this purpose, the $\hbar$ curve is plotted for f up to the seventeenth order approximation. It is obvious from Fig. 5.1 that the range for the admissible value for $\hbar$ is $-1 \leq \hbar \leq -0.15$. Our calculations indicate that the series of the velocity field in Eq. (5.38) converges in the whole region of z when $\hbar = -0.2$.

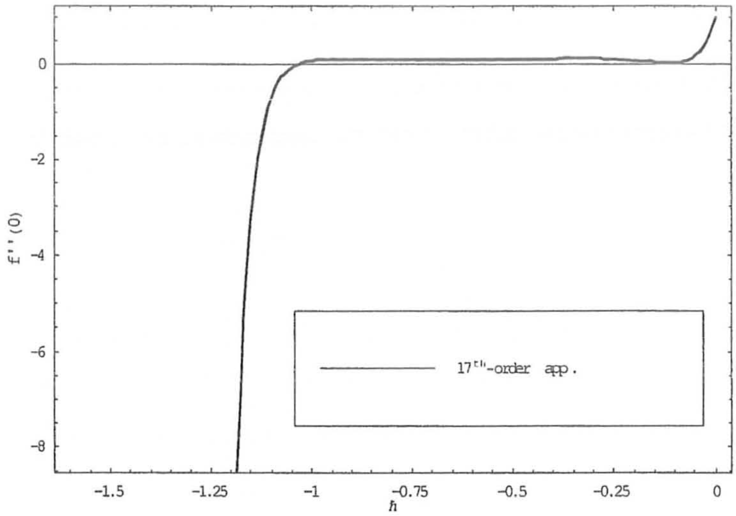

Fig. 5.1 $\hbar$-curve for the seventeenth order of the approximation for the velocity field f for $\widetilde{b}_1 = 0.5$, $\widetilde{b}_2 = 0.1$, $\widetilde{c} = 0.8$.

5.4 Results and discussion

In Fig. 5.2, the velocity field f is plotted for the different values of the parameter $\widetilde{b}_1$. It is apparent from this figure that by increasing $\widetilde{b}_1$ the velocity increases. Fig. 5.3 elucidates the effects of the parameter $\widetilde{b}_2$. It is noted from Fig. 5.3 that the velocity decreases by increasing $\widetilde{b}_2$. Fig. 5.4 shows the velocity distribution for various values of the parameter $\widetilde{c}$. Here the velocity decreases by increasing $\widetilde{c}$.

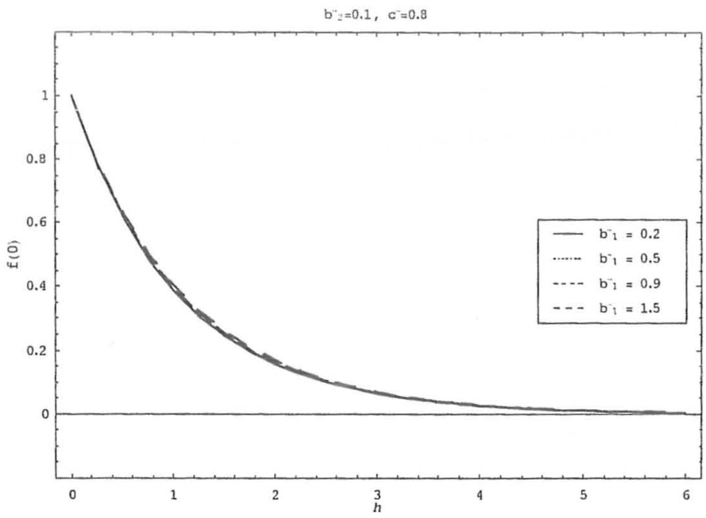

Fig. 5.2 Variation of the velocity distribution for the various values of $\widetilde{b}_1$.

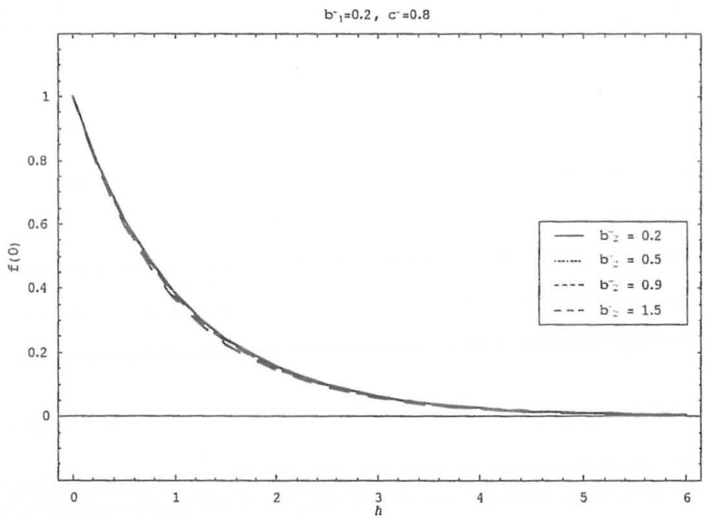

Fig. 5.3 Variation of the velocity distribution for the various values of $\widetilde{b}_2$.

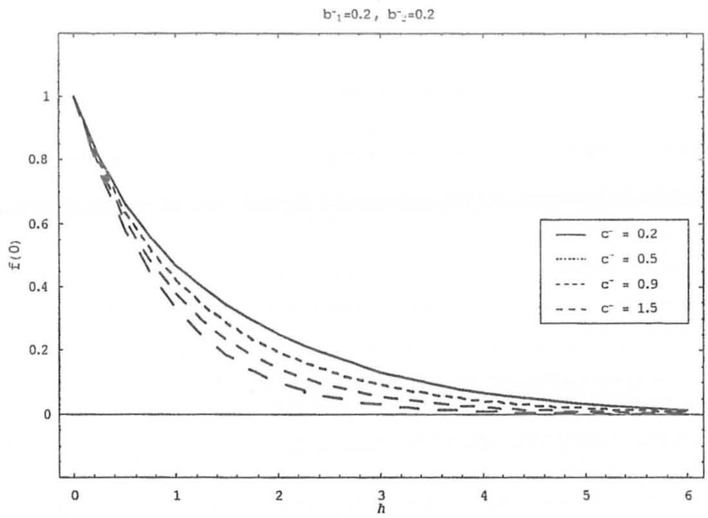

Fig. 5.4 Variation of the velocity distribution for the various values of $\widetilde{c}$.

5.5 Concluding remarks

In this chapter, HAM solution for steady flow of a third grade fluid in porous space is developed. The governing constitutive relationship for modified Darcy's law in a third grade fluid has been used. It is noted that modified Darcy's law even for unidirectional steady flow of a third grade fluid yields non-linear expression in terms of velocity whereas it is linear for Newtonian, Oldroyd-B, Maxwell and second grade fluids. It is further noted that unlike the Newtonian, Oldroyd-B, Maxwell and second grade fluids, the modified Darcy's law for third grade fluid exhibits the rheological characteristics even in steady state situation. The contents of this chapter have been accepted for publication in **Applied Mathematical Modelling**.

Chapter 6

Stokes' first problem for the rotating flow of a third grade fluid

In this chapter the non-linear rheological effects of third grade fluid over a jerked plate is addressed in a rotating frame. Numerical solution for the non-linear problem is given. The non-linear effects on the velocity is shown and discussed. This reveal that characteristics for shear thickening/shear thinning behavior of a fluid are dependent upon the rheological properties.

6.1 Flow analysis

An infinite plate (located at $z = 0$) and the third grade fluid (which is in contact with the jerked plate and occupies the whole of the region $z \geq 0$) are in uniform rotation. For the sake of simplicity, the angular velocity $\boldsymbol{\Omega}$ is taken parallel to z-axis. We examine the flow of third grade fluid described by the constitutive equation (2.10) above the plate in the rotating system. The fluid is assumed to be incompressible. Referred to the rotating frame of reference, the incompressibility condition is (2.13) and the momentum equation is

$$\rho\left[\frac{\partial \mathbf{V}}{\partial t}+(\mathbf{V}.\nabla)\,\mathbf{V}+2\boldsymbol{\Omega}\times\mathbf{V}+\boldsymbol{\Omega}\times(\boldsymbol{\Omega}\times\mathbf{r}_1)\right]=\text{div }\mathbf{T}, \tag{6.1}$$

in which $\mathbf{r}_1$ is the radial coordinate with $r_1^2=x^2+y^2$.

We assume the velocity field in the form

$$\mathbf{V}=\left[u\,(z,t)\,,\;v\,(z,t)\,,\;w\,(z,t)\right]. \tag{6.2}$$

It follows from the the incompressibility condition (2.13) that $w=0$.

Upon making use of Eq. (6.2) into Eq. (2.10), we obtain

$$\tau_{xx}=\alpha_2\left(\frac{\partial u}{\partial z}\right)^2, \tag{6.3}$$

$$\tau_{yy}=\alpha_2\left(\frac{\partial v}{\partial z}\right)^2, \tag{6.4}$$

$$\begin{aligned}\tau_{zz} &= 2\mu\left(\frac{\partial w}{\partial z}\right)+2\alpha_1\left[\begin{array}{c}\frac{\partial^2 w}{\partial z\partial t}+w\left(\frac{\partial^2 w}{\partial z^2}\right)+\left(\frac{\partial u}{\partial z}\right)^2\\ +\left(\frac{\partial v}{\partial z}\right)^2+2\left(\frac{\partial w}{\partial z}\right)^2\end{array}\right]\\ &+\alpha_2\left[\left(\frac{\partial u}{\partial z}\right)^2+\left(\frac{\partial v}{\partial z}\right)^2+4\left(\frac{\partial w}{\partial z}\right)^2\right]\\ &+4\beta_3\left[\left(\frac{\partial u}{\partial z}\right)^2+\left(\frac{\partial v}{\partial z}\right)^2+2\left(\frac{\partial w}{\partial z}\right)^2\right]\left(\frac{\partial w}{\partial z}\right),\end{aligned} \tag{6.5}$$

$$\tau_{xy}=\alpha_2\left(\frac{\partial u}{\partial z}\right)\left(\frac{\partial v}{\partial z}\right), \tag{6.6}$$

$$\begin{aligned}\tau_{xz} &= \mu\left(\frac{\partial u}{\partial z}\right)+\alpha_1\left[\frac{\partial^2 u}{\partial z\partial t}+w\left(\frac{\partial^2 u}{\partial z^2}\right)+\left(\frac{\partial u}{\partial z}\right)\left(\frac{\partial w}{\partial z}\right)\right]\\ &+2\alpha_2\left(\frac{\partial u}{\partial z}\right)\left(\frac{\partial w}{\partial z}\right)+2\beta_3\left[\begin{array}{c}\left(\frac{\partial u}{\partial z}\right)^2+\left(\frac{\partial v}{\partial z}\right)^2\\ +2\left(\frac{\partial w}{\partial z}\right)^2\end{array}\right]\left(\frac{\partial u}{\partial z}\right),\end{aligned} \tag{6.7}$$

$$\tau_{yz} = \mu\left(\frac{\partial v}{\partial z}\right)+\alpha_1\left[\frac{\partial^2 v}{\partial z\partial t}+w\left(\frac{\partial^2 v}{\partial z^2}\right)+\left(\frac{\partial v}{\partial z}\right)\left(\frac{\partial w}{\partial z}\right)\right]$$
$$+2\alpha_2\left(\frac{\partial v}{\partial z}\right)\left(\frac{\partial w}{\partial z}\right)+2\beta_3\left[\begin{array}{c}\left(\frac{\partial u}{\partial z}\right)^2+\left(\frac{\partial v}{\partial z}\right)^2\\ +2\left(\frac{\partial w}{\partial z}\right)^2\end{array}\right]\left(\frac{\partial v}{\partial z}\right),$$

and

$$\tau_{xy}=\tau_{yx},\quad \tau_{xz}=\tau_{zx},\quad \tau_{yz}=\tau_{zy}. \tag{6.8}$$

Using Eqs. (6.3) – (6.7) into Eq. (6.1), we obtain

$$\frac{\partial u}{\partial t}-2\Omega v=-\frac{1}{\rho}\frac{\partial \widehat{p}}{\partial x}+\frac{1}{\rho}\left[\mu\frac{\partial u}{\partial z}+\alpha_1\frac{\partial^2 u}{\partial z\partial t}+2\beta_3\frac{\partial}{\partial z}\left(\frac{\partial u}{\partial z}\left\{\left(\frac{\partial u}{\partial z}\right)^2+\left(\frac{\partial v}{\partial z}\right)^2\right\}\right)\right], \tag{6.9}$$

$$\frac{\partial v}{\partial t}+2\Omega u=-\frac{1}{\rho}\frac{\partial \widehat{p}}{\partial y}+\frac{1}{\rho}\left[\mu\frac{\partial v}{\partial z}+\alpha_1\frac{\partial^2 v}{\partial z\partial t}+2\beta_3\frac{\partial}{\partial z}\left(\frac{\partial v}{\partial z}\left\{\left(\frac{\partial u}{\partial z}\right)^2+\left(\frac{\partial v}{\partial z}\right)^2\right\}\right)\right], \tag{6.10}$$

$$0=-\frac{1}{\rho}\frac{\partial \widehat{p}}{\partial z}, \tag{6.11}$$

where the modified pressure

$$\widehat{p}=p-\frac{\rho}{2}\Omega^2\left(x^2+y^2\right) \tag{6.12}$$

and $\widehat{p}\neq\widehat{p}(z)$, which is obvious from Eq. (6.11)

The relevant boundary and initial conditions are:

$$\begin{aligned} u &= U_0,\quad v=0,\quad \text{at } z=0,\quad t>0,\\ u &\longrightarrow 0,\quad v\longrightarrow 0 \quad \text{as} \quad z\longrightarrow\infty \quad \text{for all } t,\\ u(z,0) &= 0,\quad v(z,0)=0,\quad z>0. \end{aligned} \tag{6.13}$$

Combining Eqs. (6.9) and (6.10) and then neglecting the pressure gradient we arrive at

$$\frac{\partial F}{\partial t}+2i\Omega F=\nu\frac{\partial^2 F}{\partial z^2}+\frac{\alpha_1}{\rho}\frac{\partial^3 F}{\partial z^2\partial t}+\frac{2\beta_3}{\rho}\frac{\partial}{\partial z}\left\{\left(\frac{\partial F}{\partial z}\right)^2\frac{\partial \overline{F}}{\partial z}\right\}, \tag{6.14}$$

where

$$F = u + iv, \quad \overline{F} = u - iv. \tag{6.15}$$

In terms of F, the conditions (6.13) reduce to the following

$$F(0,t) = U_0\,, \qquad F(z,t) \longrightarrow 0 \;\text{ as }\; z \longrightarrow \infty\,, \qquad F(z,0) = 0. \tag{6.16}$$

The emerging non-dimensional parameters are defined as

$$\eta = \frac{U_0}{\nu}z, \qquad \tau = \frac{U_0^2}{\nu}t, \qquad f = \frac{F}{U_0}, \qquad C = \frac{U_0^2}{\nu}\Omega. \tag{6.17}$$

By means of above non-dimensional parameters we can write

$$\frac{\partial f}{\partial \tau} + 2iCf = \frac{\partial^2 f}{\partial \eta^2} + a\frac{\partial^3 f}{\partial \eta^2 \partial \tau} + 2b\frac{\partial}{\partial \eta}\left\{\left(\frac{\partial f}{\partial \eta}\right)^2 \frac{\partial \overline{f}}{\partial \eta}\right\}, \tag{6.18}$$

$$f(0,\tau) = 1, \qquad f(\eta,\tau) \longrightarrow 0 \quad \text{as} \quad \eta \longrightarrow \infty, \qquad f(\eta,0) = 0, \tag{6.19}$$

in which

$$a = \frac{\alpha_1 U_0^2}{\rho \nu^2}, \qquad b = \frac{\beta_1 U_0^4}{\rho \nu^3}, \qquad C = \Omega. \tag{6.20}$$

6.2 Numerical results and discussion

Here we note that Eq. (6.18) is a third order non-linear partial differential equation and it is difficult to obtain the exact analytic solution. The governing Eq. (6.18) is transformed into an algebraic equation by substituting the approximations to the derivatives given in section 2.7 and get

$$\left(\frac{1}{k}\right)(f_{i,j}-f_{i,j-1})+2iCf_{i,j}=\frac{1}{h^2}(f_{i+1,j}-2f_{i,j}+f_{i-1,j})$$
$$+\frac{a}{h^2k}(f_{i+1,j}-f_{i+1,j-1}-2f_{i,j}+2f_{i,j-1}+f_{i-1,j}-f_{i-1,j-1})$$
$$+\frac{b}{h^4}\left[\begin{array}{c}(f_{i+1,j}+f_{i-1,j})(f_{i+1,j}-2f_{i,j}+f_{i-1,j})\left(f_{i+1,j}^{-}+f_{i-1,j}^{-}\right)\\ +\frac{1}{2}(f_{i+1,j}+f_{i-1,j})^2\left(f_{i+1,j}^{-}-2f_{i,j}^{-}+f_{i-1,j}^{-}\right)\end{array}\right]. \tag{6.21}$$

The following system of algebraic equations is obtained

$$R_i=Af_{i,j}+Bf_{i+1,j}+Cf_{i-1,j}+K_1f_{i+1,j}^2\overline{f}_{i+1,j}+K_2f_{i+1,j}^2\overline{f}_{i,j}$$
$$+K_3f_{i+1,j}^2\overline{f}_{i-1,j}+K_4f_{i-1,j}^2\overline{f}_{i+1,j}+K_5f_{i-1,j}^2\overline{f}_{i,j}+K_6f_{i-1,j}^2\overline{f}_{i-1,j}$$
$$+K_7\overline{f}_{i+1,j}f_{i+1,j}f_{i,j}+K_8\overline{f}_{i+1,j}f_{i+1,j}f_{i-1,j}+K_9\overline{f}_{i+1,j}f_{i,j}f_{i-1,j}$$
$$+K_{10}f_{i+1,j}f_{i,j}\overline{f}_{i-1,j}+K_{11}f_{i+1,j}f_{i-1,j}\overline{f}_{i-1,j}+K_{12}\overline{f}_{i-1,j}f_{i,j}f_{i-1,j}$$
$$+K_{13}f_{i+1,j}\overline{f}_{i,j}f_{i-1,j}+Ff_{i,j-1}+Gf_{i+1,j-1}+Hf_{i-1,j-1}, \tag{6.22}$$

$$f_{0,j}=1, \qquad f_{M,j}=0, \qquad f_{i,0}=0, \qquad i=0,1,2,...,M \qquad j=0,1,2,3..., \tag{6.23}$$

where

$$\begin{aligned}
A &= \left[\frac{1}{k}+\frac{2}{h^2}+\frac{2a}{h^2k}+2i\Omega\right], \quad B=-\left[\frac{1}{h^2}+\frac{a}{h^2k}\right],\\
C &= -\left[\frac{1}{h^2}+\frac{a}{h^2k}\right], \quad K_1=-\frac{3b}{2h^4}, \quad K_2=\frac{b}{h^4},\\
K_3 &= \frac{b}{2h^4{}_1}, \quad K_4=K_3, \quad K_5=K_2,\\
K_6 &= K_1, \quad K_7=\frac{2b}{h^4}, \quad K_8=K_2,\\
K_9 &= -K_7, \quad K_{10}=-K_7, \quad K_{11}=K_2,\\
K_{12} &= K_2, \quad K_{13}=-K_7, \quad F=-\left(\frac{1}{k}\right)-\frac{2a}{h^2k},\\
G &= \frac{a}{h^2k}, \quad H=G.
\end{aligned} \tag{6.24}$$

It is worth mentioning that Eq. (6.18) is third order and we have two boundary conditions. Therefore, we need a third boundary condition. Due to this fact we resolve the difficulty here through the augmentation procedure and write the augmented boundary condition as

$$\frac{\partial f(\infty, \tau)}{\partial \eta} = 0. \tag{6.25}$$

The problem now is well-posed. This boundary condition is discretized to give

$$\frac{f_{M+1,j} - f_{M,j}}{h} = 0, \tag{6.26}$$

i.e.,

$$f_{M+1,j} = f_{M,j}.$$

The non-linear differential system consisting of Eq.(6.18) and conditions (6.19) has been solved numerically by means of the Newton method. Solutions for the non-Newtonian fluid models are obtained for $\tau = 1$. From the numerical solution f is used to express the non-dimensional velocity profile. Results for the flow are obtained for various values of the parameters a, b, C and τ.

Fig.6.1 (a & b) presents the velocity profile f for various values of b. These figures indicate that increasing the parameter b increases real part of the velocity. However, imaginary part of the velocity decreases for large values of b. Fig. 6.2 (a & b) shows the influence of C on the velocity profile f. It is evident from the figure that an increase in C results in a decrease of the real and imaginary parts of the velocity. The effect of the second grade parameter on f is illustrated in Fig. 6.3. (a & b). It is noted that the velocity increases in the real part whereas in the imaginary part it first increases and then decreases by increasing the second grade parameter. Fig. 6.4 (a & b) shows how the velocity varies for different values of τ. It is found that here the real part of the velocity increases and the imaginary part of the velocity decreases by increasing τ. In Fig. 6.5 (a & b) the velocity distribution is presented in the Newtonian case for the various values of C. It is observed that the influence of C in Newtonian and third grade fluid are similar.

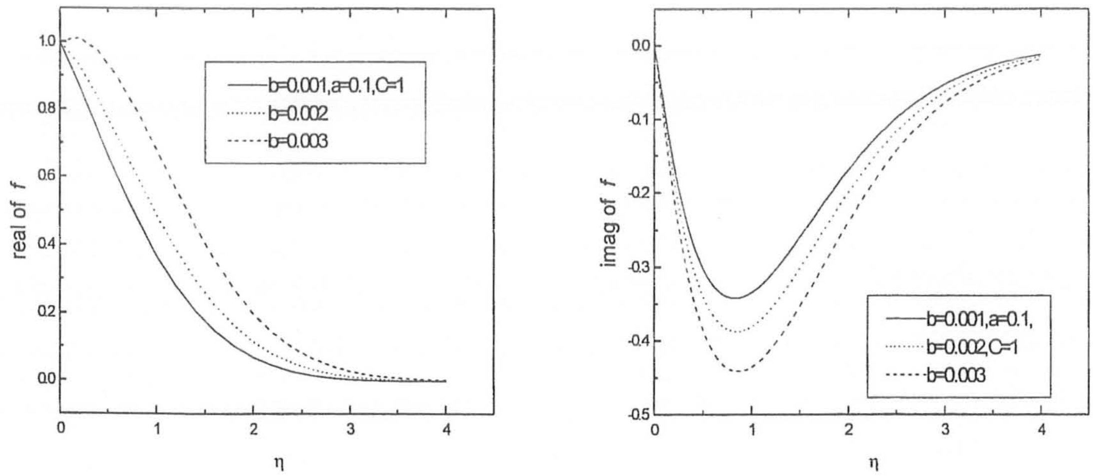

Fig. 6.1 (a) Fig. 6.1 (b)

Fig. 6.1. Influence of the velocity distribution for the various values of the third grade parameter for $\tau = 1$.

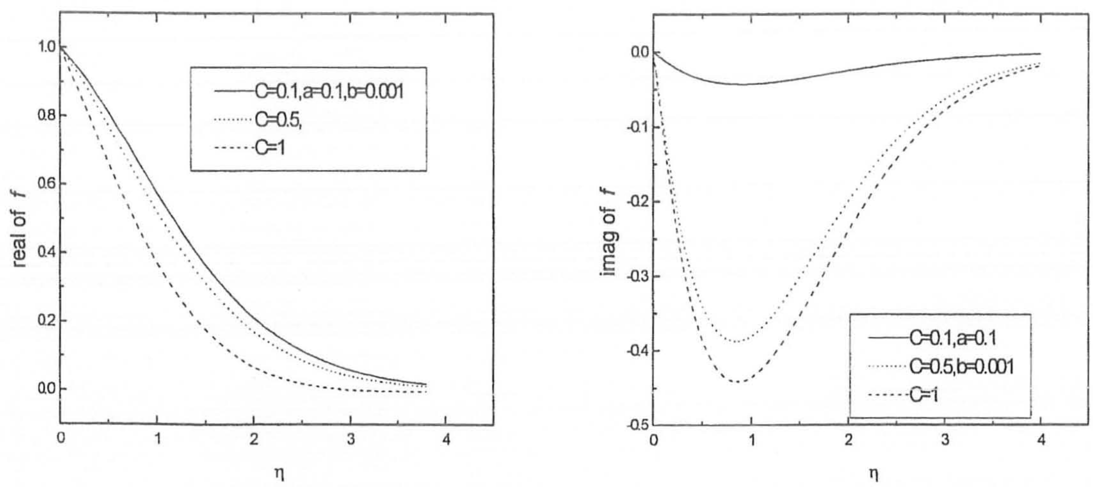

Fig. 6.2 (a) Fig. 6.2 (b)

Fig. 6.2. Influence of Ω on the velocity distribution for $\tau = 1$.

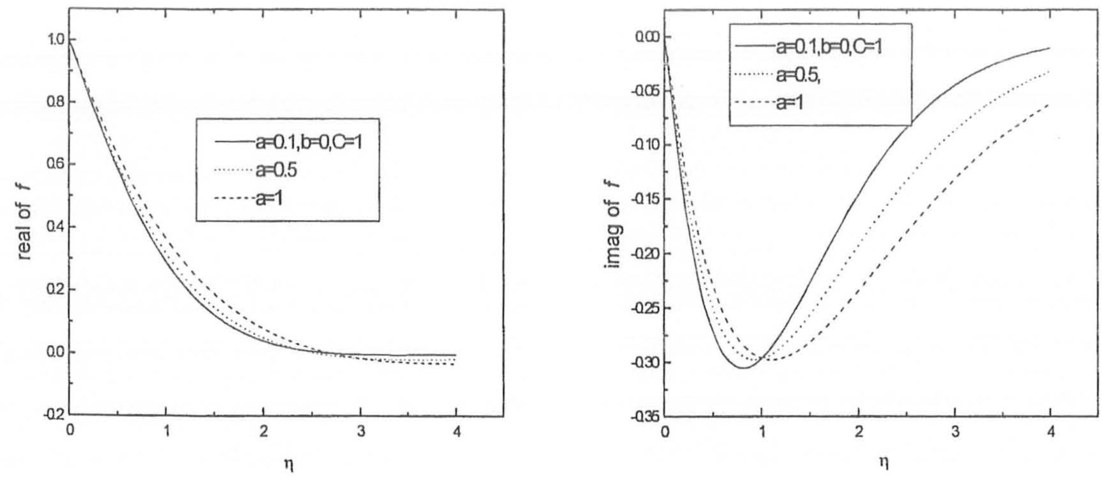

Fig. 6.3 (a) Fig. 6.3 (b)

Fig. 6.3. Influence of the various values of the second grade parameter on the velocity distribution for $\tau = 1$.

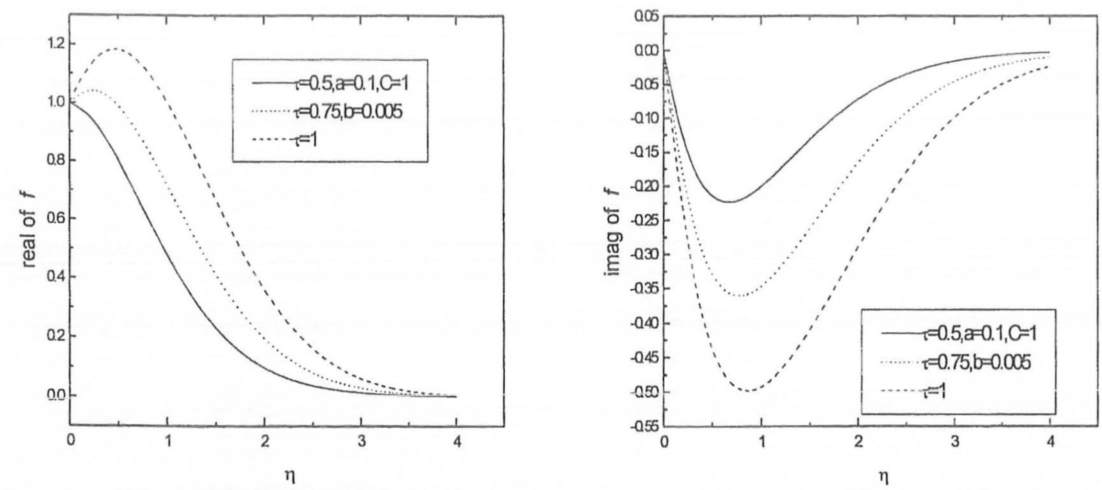

Fig. 6.4 (a) Fig. 6.4 (b)

Fig. 6.4. Influence of the velocity distribution for the various values of τ.

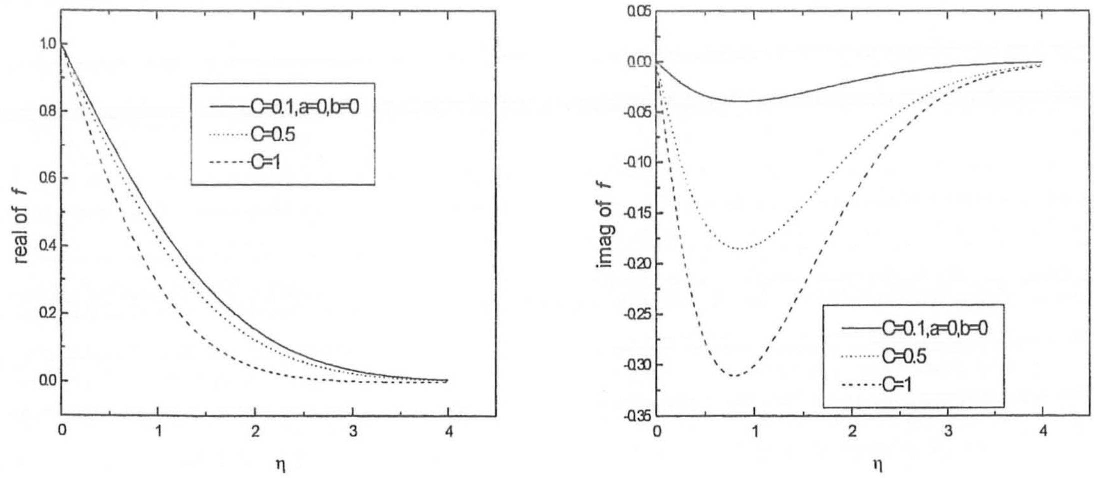

Fig. 6.5 (a) Fig. 6.5 (b)

Fig. 6.5. Influence of the velocity distribution for the various values of Ω for the Newtonian case.

6.3 Concluding remarks

The Stokes' first problem of a third grade fluid is discussed in a rotating frame of reference. The problem that comprised the balance laws of mass and momentum has been first non-dimensionalized and then solved numerically. Results for the real and imaginary parts of the velocity are presented. It is found that at $\tau = 1$ and different values of C, the flow characteristics in a third grade fluid are similar to that of Newtonian fluid.

Chapter 7

Stokes' first problem for the fourth order fluid in a porous half space

Based on modified Darcy's law, Stokes' first problem for a fourth order fluid in a porous half space is investigated here. To the best of our knowledge such modified Darcy's law has been introduced for the first time in this chapter. Numerical solution of the velocity field is obtained and discussed. Several limiting cases are deduced as the special cases of the present analysis.

7.1 Problem formulation

We consider the flow of a fourth order fluid in a porous half space, taking the positive y-axis of a Cartesian coordinate system in the upward direction. A fourth order fluid flows through a porous space $y > 0$ above and in contact with a flat plate occupying the xz-plane. Initially both the fluid and the plate are at rest. For $t > 0$, the plate suddenly starts to slide in its plane with a constant speed U_0. The velocity field is the same as the one defined in Eq. (4.2). Using Eq. (4.2), Eqs. (2.1) – (2.6) yield

$$\begin{aligned}\tau_{xx} &= \alpha_2\left(\frac{\partial u}{\partial y}\right)^2 + 2\beta_2\left(\frac{\partial u}{\partial y}\frac{\partial^2 u}{\partial t\partial y}\right) \\ &+2\gamma_2\left(\frac{\partial u}{\partial y}\frac{\partial^3 u}{\partial t^2\partial y}\right) + \gamma_3\left(\frac{\partial^2 u}{\partial t\partial y}\right)^2 \\ &+2\gamma_6\left(\frac{\partial u}{\partial y}\right)^4,\end{aligned} \tag{7.1}$$

$$\begin{aligned}\tau_{xy} &= \mu\frac{\partial u}{\partial y} + \alpha_1\frac{\partial^2 u}{\partial y\partial t} + \beta_1\frac{\partial^3 u}{\partial y\partial t^2} \\ &2\left(\beta_2+\beta_3\right)\left(\frac{\partial u}{\partial y}\right)^3 + \gamma_1\frac{\partial^4 u}{\partial y\partial t^3} \\ &+\begin{pmatrix} 6\gamma_2+2\gamma_3+2\gamma_4 \\ +2\gamma_5+6\gamma_7+2\gamma_8\end{pmatrix}\left[\left(\frac{\partial u}{\partial y}\right)^2\left(\frac{\partial^2 u}{\partial y\partial t}\right)\right],\end{aligned} \tag{7.2}$$

$$\tau_{xz} = 0, \tag{7.3}$$

$$\begin{aligned}\tau_{yy} &= 2\alpha_1\left(\frac{\partial u}{\partial y}\right)^2 + \alpha_2\left(\frac{\partial u}{\partial y}\right)^2 \\ &+6\beta_1\left[\left(\frac{\partial u}{\partial y}\right)\left(\frac{\partial^2 u}{\partial y\partial t}\right)\right] + 2\beta_2\left[\left(\frac{\partial u}{\partial y}\right)\left(\frac{\partial^2 u}{\partial y\partial t}\right)\right] \\ &+6\gamma_1\frac{\partial}{\partial t}\left[\left(\frac{\partial u}{\partial y}\right)\left(\frac{\partial^2 u}{\partial y\partial t}\right)\right] + 2\gamma_2\left[\left(\frac{\partial u}{\partial y}\right)\left(\frac{\partial^3 u}{\partial y\partial t^2}\right)\right] \\ &+4\gamma_3\left(\frac{\partial u}{\partial y}\right)^2 + 4\gamma_5\left(\frac{\partial u}{\partial y}\right)^4 + 2\gamma_6\left(\frac{\partial u}{\partial y}\right)^4,\end{aligned} \tag{7.4}$$

$$\tau_{zz} = 0, \tag{7.5}$$

$$\tau_{xy} = \tau_{yx}, \quad \tau_{yz} = \tau_{zy}, \quad \tau_{xz} = \tau_{zx}. \tag{7.6}$$

Employing the same idea as in section 4.3, we propose the following constitutive relationship between the pressure drop and velocity for unidirectional flow of a fourth order fluid:

$$\frac{\partial p}{\partial x} = -\left[\begin{array}{c}\mu + \alpha_1\frac{\partial}{\partial t} + \beta_1\frac{\partial^2}{\partial t^2} + 2\left(\beta_2+\beta_3\right)\left(\frac{\partial u}{\partial y}\right)^2 + \gamma_1\frac{\partial^3}{\partial t^3} \\ +\left(6\gamma_2+2\gamma_3+2\gamma_4+2\gamma_5+6\gamma_7+2\gamma_8\right)\left(\frac{\partial u}{\partial y}\right)\left(\frac{\partial^2 u}{\partial y\partial t}\right)\end{array}\right]\frac{\phi u}{k}. \tag{7.7}$$

The pressure gradient in above equation can also be interpreted as a measure of the resistance to flow in the bulk of the porous medium and r is a measure of the flow resistance offered by the solid matrix. Thus r_x can be inferred from Eq. (7.7) to satisfy the following equation:

$$r_x = -\left[\begin{array}{c} \mu + \alpha_1 \frac{\partial}{\partial t} + \beta_1 \frac{\partial^2}{\partial t^2} + 2\left(\beta_2 + \beta_3\right)\left(\frac{\partial u}{\partial y}\right)^2 + \gamma_1 \frac{\partial^3}{\partial t^3} \\ + \left(6\gamma_2 + 2\gamma_3 + 2\gamma_4 + 2\gamma_5 + 6\gamma_7 + 2\gamma_8\right)\left(\frac{\partial u}{\partial y}\right)\left(\frac{\partial^2 u}{\partial y \partial t}\right) \end{array}\right]\frac{\phi u}{k}. \tag{7.8}$$

Substituting Eqs. (7.1) – (7.8) in Eq.(4.1) and then neglecting $\partial p/\partial x$, we obtain

$$\begin{aligned} \rho \frac{\partial u}{\partial t} &= \mu \frac{\partial^2 u}{\partial y^2} + \alpha_1 \frac{\partial^3 u}{\partial y^2 \partial t} + \beta_1 \frac{\partial^4 u}{\partial y^2 \partial t^2} + 6\left(\beta_2 + \beta_3\right)\left(\frac{\partial u}{\partial y}\right)^2 \frac{\partial^2 u}{\partial y^2} + \gamma_1 \frac{\partial^5 u}{\partial y^2 \partial t^3} \\ &+ \left(6\gamma_2 + 2\gamma_3 + 2\gamma_4 + 2\gamma_5 + 6\gamma_7 + 2\gamma_8\right)\frac{\partial}{\partial y}\left[\left(\frac{\partial u}{\partial y}\right)^2\left(\frac{\partial^2 u}{\partial y \partial t}\right)\right] \\ &- \left[\begin{array}{c} \mu + \alpha_1 \frac{\partial}{\partial t} + \beta_1 \frac{\partial^2}{\partial t^2} + 2\left(\beta_2 + \beta_3\right)\left(\frac{\partial u}{\partial y}\right)^2 + \gamma_1 \frac{\partial^3}{\partial t^3} \\ + \left(6\gamma_2 + 2\gamma_3 + 2\gamma_4 + 2\gamma_5 + 6\gamma_7 + 2\gamma_8\right)\left(\frac{\partial u}{\partial y}\right)\left(\frac{\partial^2 u}{\partial y \partial t}\right) \end{array}\right]\frac{\phi u}{k}. \end{aligned} \tag{7.9}$$

The relevant boundary and initial conditions are

$$u(0,t) = U_0\,, \qquad u(y,t) \longrightarrow 0 \text{ as } y \longrightarrow \infty\,; \qquad u(y,0) = 0. \tag{7.10}$$

The above problem in non-dimensional form is

$$\begin{aligned} \frac{\partial f}{\partial \tau} &= \frac{\partial^2 f}{\partial \eta^2} + a \frac{\partial^3 f}{\partial \eta^2 \partial \tau} + b_1 \frac{\partial^4 f}{\partial \eta^2 \partial \tau^2} + 6\left(b_2 + b_3\right)\left(\frac{\partial f}{\partial \eta}\right)^2 \frac{\partial^2 f}{\partial \eta^2} + c_1 \frac{\partial^5 f}{\partial \eta^2 \partial \tau^3} \\ &+2\left(\begin{array}{c} 6c_2 + 2c_3 + 2c_4 \\ +2c_5 + 6c_7 + 2c_8 \end{array}\right)\left[\left(\frac{\partial f}{\partial \eta}\right)\left(\frac{\partial^2 f}{\partial \eta^2}\right)\left(\frac{\partial^2 f}{\partial \eta \partial \tau}\right) + \left(\frac{\partial f}{\partial \eta}\right)^2\left(\frac{\partial^3 f}{\partial \eta^2 \partial \tau}\right)\right] \\ &- \hat{d}\, f - \hat{e}\left(\frac{\partial f}{\partial \tau}\right) - g\left(\frac{\partial^2 f}{\partial \tau^2}\right) - 2\left(v_1 + v_2\right)\left(\frac{\partial f}{\partial \eta}\right)^2 f - L\left(\frac{\partial^3 f}{\partial \tau^3}\right) \\ &- \left(6m_2 + 2m_3 + 2m_4 + 2m_5 + 6m_7 + 2m_8\right)\left(\frac{\partial f}{\partial \eta}\right)\left(\frac{\partial^2 f}{\partial \eta \partial \tau}\right) f, \end{aligned} \tag{7.11}$$

$$f(0,\tau) = 1, \qquad f(\eta,\tau) \longrightarrow 0 \quad \text{as} \quad \eta \longrightarrow \infty, \qquad f(\eta,0) = 0, \tag{7.12}$$

in which

$$\begin{aligned}
a &= \frac{\alpha_1 U_0^2}{\rho \nu^2}, \qquad b_i = \frac{\beta_i U_0^4}{\rho \nu^3}\ (i = 1-3), \qquad c_i = \frac{\gamma_i U_0^6}{\rho \nu^4}\ (i = 1, 2, 3, 4, 5, 7, 8), \\
\hat{d} &= \frac{\phi \nu^2}{k U_0^2}, \qquad \hat{e} = \frac{\alpha_1 \phi}{\rho k}, \qquad g = \frac{\beta_1 \phi U_0^2}{\rho k \nu}, \qquad v_2 = \frac{\beta_2 \phi U_0^2}{\rho k \nu}, \\
v_3 &= \frac{\beta_3 \phi U_0^2}{\rho k \nu}, \qquad L = \frac{\gamma_1 \phi U_0^4}{\rho k \nu^2}, \qquad m_i = \frac{\gamma_i \phi U_0^4}{\rho k \nu^2}. \qquad (7.13)
\end{aligned}$$

7.2 Numerical results and discussion

Using Eqs. (2.18) – (2.28) and (7.13) we can write

$$
\begin{aligned}
&\left(\frac{1}{k}\right)(f_{i,j}-f_{i,j-1}) = \frac{1}{h^2}(f_{i+1,j}-2f_{i,j}+f_{i-1,j}) \\
&+\frac{a}{h^2k}(f_{i+1,j}-f_{i+1,j-1}-2f_{i,j}+2f_{i,j-1}+f_{i-1,j}-f_{i-1,j-1}) \\
&+\frac{b_1}{h^2k^2}\begin{pmatrix} f_{i+1,j}-2f_{i+1,j-1}+f_{i+1,j-2}-2f_{i,j}+4f_{i,j-1} \\ -2f_{i,j-2}+f_{i-1,j}-2f_{i-1,j-1}+f_{i-1,j-2} \end{pmatrix} \\
&-\frac{6(b_2+b3)}{4h^4}\left[(f_{i+1,j}+f_{i-1,j})^2(f_{i+1,j}-2f_{i,j}+f_{i-1,j})\right] \\
&+\frac{c_1}{h^2k^3}\begin{pmatrix} f_{i+1,j}-3f_{i+1,j-1}+f_{i+1,j-2}-f_{i+1,j-3}-2f_{i,j}+4f_{i,j-1} \\ -4f_{i,j-1}-4f_{i,j-2}+2f_{i,j-1}+f_{i-1,j-2}-f_{i-1,j-3} \end{pmatrix}+ \\
&\frac{(6c_2+2c_3+2c_4+2c_5+6c_7+2c_8)}{2h^4k}\left[\begin{array}{c} (f_{i+1,j}-f_{i-1,j}) \\ (f_{i+1,j}-2f_{i,j}+f_{i-1,j}) \\ (f_{i+1,j}-f_{i+1,j-1}-f_{i-1,j}+f_{i-1,j-1}) \\ +(f_{i+1,j}-f_{i-1,j})^2 \\ \begin{pmatrix} f_{i+1,j}-f_{i+1,j-1}-2f_{i,j} \\ +2f_{i,j-1}+f_{i-1,j}-f_{i-1,j-1} \end{pmatrix} \end{array}\right] \\
&-\hat{d}\, f_{i,j}-\frac{\hat{e}}{k}(f_{i,j}-f_{i,j-1})-\frac{g}{k^2}(f_{i,j}-2f_{i,j-1}+f_{i,j-2}) \\
&-\frac{(v_2+v_3)}{2h^2}f_{i,j}(f_{i+1,j}-f_{i-1,j})-\frac{L}{k^3}(f_{i,j}-3f_{i,j-1}+3f_{i,j-2}-f_{i,j-3}) \\
&-\frac{\begin{pmatrix} 6m_2+2m_3+2m_4 \\ +2m_5+6m_7+2m_8 \end{pmatrix}}{4h^2k}f_{i,j}(f_{i+1,j}-f_{i-1,j})\begin{pmatrix} f_{i+1,j}-f_{i+1,j-1} \\ -f_{i-1,j}+f_{i-1,j-1} \end{pmatrix}
\end{aligned}
\tag{7.14}
$$

and thus the resulting system is

$$R_i = Af_{i,j} + Bf_{i+1,j} + Cf_{i-1,j} + K_1 f_{i+1,j}^3 + K_2 f_{i+1,j}^2 f_{i,j} + K_3 f_{i+1,j}^2 f_{i-1,j}$$

$$+K_4 f_{i-1,j}^2 f_{i+1,j} + K_5 f_{i-1,j}^2 f_{i,j} + K_6 f_{i-1,j}^3 + K_7 f_{i+1,j} f_{i-1,j} f_{i,j} + K_8 f_{i+1,j} f_{i-1,j}^3$$

$$K_9 f_{i+1,j}^2 f_{i+1,j-1} + K_{10} f_{i+1,j} f_{i,j} f_{i+1,j-1} + K_{11} f_{i+1,j-1} f_{i-1,j} f_{i,j} + K_{12} f_{i-1,j}^3 f_{i+1,j-1}$$

$$+K_{13} f_{i-1,j}^4 + K_{14} f_{i+1,j}^2 f_{i-1,j-1} + K_{15} f_{i+1,j} f_{i-1,j-1} f_{i,j} + K_{16} f_{i,j} f_{i-1,j} f_{i-1,j-1}$$

$$+K_{17} f_{i-1,j}^3 f_{i-1,j-1} + K_{18} f_{i+1,j}^2 f_{i,j-1} + K_{19} f_{i-1,j}^2 f_{i+1,j-1} + K_{20} f_{i-1,j}^2 f_{i,j-1}$$

$$+K_{21} f_{i-1,j}^2 f_{i-1,j-1} + K_{22} f_{i+1,j} f_{i-1,j} f_{i+1,j-1} + K_{23} f_{i+1,j} f_{i-1,j} f_{i,j-1}$$

$$+K_{24} f_{i+1,j} f_{i-1,j} f_{i-1,j-1} + K_{25} f_{i,j} f_{i+1,j} + K_{26} f_{i,j} f_{i-1,j} + F f_{i,j-1}$$

$$+G f_{i+1,j-1} + H f_{i-1,j-1} + I f_{i+1,j-2} + J f_{i,j-2}$$

$$+K f_{i-1,j-2} + M f_{i+1,j-3} + N f_{i-1,j-3} + P f_{i,j-1}, \tag{7.15}$$

$$f_{0,j} = 1, \qquad f_{M,j} = 0, \qquad f_{i,0} = 0, \qquad i = 0, 1, 2, ..., M \qquad j = 0, 1, 2, 3..., \tag{7.16}$$

where

$$
\begin{aligned}
A &= \left[\frac{1}{k}+\frac{2}{h^2}+\frac{2a}{h^2k}+\frac{2b_1}{h^2k^2}+\frac{2c_1}{h^2k^3}+d\hat{\ }+\frac{e\hat{\ }}{k}+\frac{L}{k^3}+\frac{g}{k^2}\right],\\
B &= -\left[\frac{1}{h^2}+\frac{a}{h^2k}+\frac{b_1}{h^2k^2}+\frac{c_1}{h^2k^3}\right], \quad C=-\left[\frac{1}{h^2}+\frac{a}{h^2k}+\frac{b_1}{h^2k^2}\right],\\
K_1 &= -\frac{3\left(b_2+b_3\right)}{2h^4}-\frac{1}{4h^4k}-\frac{1}{2h^4k}\left(6c_2+2c_3+2c_4+2c_5+6c_7+2c_8\right),\\
K_2 &= \frac{3\left(b_2+b_3\right)}{h^4}+\frac{1}{h^4k}\left(6c_2+2c_3+2c_4+2c_5+6c_7+2c_8\right)\\
&\quad +\frac{1}{4h^4k}\left(6m_2+2m_3+2m_4+2m_5+6m_7+2m_8\right)+\frac{1}{2h^4k},\\
K_3 &= -K_1, \quad K_4=\frac{3\left(b_2+b_3\right)}{h^4}+\frac{1}{4h^4k},\\
K_5 &= \frac{3\left(b_2+b_3\right)}{h^4}+\frac{1}{h^4k}\left(6c_2+2c_3+2c_4+2c_5+6c_7+2c_8\right)\\
&\quad +\frac{1}{4h^2k}\left(6m_2+2m_3+2m_4+2m_5+6m_7+2m_8\right)+\frac{1}{2h^4k},\\
K_6 &= -\frac{3\left(b_2+b_3\right)}{2h^4}-\frac{1}{4h^4k},\\
K_7 &= -\frac{6\left(b_2+b_3\right)}{h^4}-\frac{2}{h^4k}\left(6c_2+2c_3+2c_4+2c_5+6c_7+2c_8\right)\\
&\quad -\frac{1}{2h^2k}\left(6m_2+2m_3+2m_4+2m_5+6m_7+2m_8\right)-\frac{1}{h^4k},\\
K_8 &= \frac{1}{2h^4k}\left(6c_2+2c_3+2c_4+2c_5+6c_7+2c_8\right), \quad K_9=K_8+\frac{1}{4h^4k},\\
K_{10} &= \frac{1}{h^4k}\left(6c_2+2c_3+2c_4+2c_5+6c_7+2c_8\right)\\
&\quad -\frac{1}{4h^2k}\left(6m_2+2m_3+2m_4+2m_5+6m_7+2m_8\right),\\
K_{11} &= \frac{1}{h^4k}\left(6c_2+2c_3+2c_4+2c_5+6c_7+2c_8\right)\\
&\quad +\frac{1}{4h^2k}\left(6m_2+2m_3+2m_4+2m_5+6m_7+2m_8\right),
\end{aligned}
$$

$$
\begin{aligned}
K_{12} &= -K_8, \quad K_{13} = -K_8, \\
K_{14} &= -K_8 + \frac{1}{4h^4k}, \quad K_{15} = K_{11}, \\
K_{16} &= \frac{1}{h^4k}\left(6c_2 + 2c_3 + 2c_4 + 2c_5 + 6c_7 + 2c_8\right) \\
&\quad - \frac{1}{4h^2k}\left(6m_2 + 2m_3 + 2m_4 + 2m_5 + 6m_7 + 2m_8\right), \\
K_{17} &= K_8, \quad K_{18} = -\frac{1}{2h^4k}, \quad K_{19} = \frac{1}{4h^4k}, \\
K_{20} &= -\frac{1}{2h^4k}, \quad K_{21} = -\frac{1}{h^4k}, \quad K_{22} = -\frac{1}{2h^4k}, \\
K_{23} &= \frac{1}{h^4k}, \quad K_{24} = -\frac{1}{2h^4k}, \quad K_{25} = \frac{(v_2 + v_3)}{2h^2}, \\
K_{26} &= -K_{25}, \quad F = \frac{1}{k} - \frac{2a}{h^2k} - \frac{4b_1}{h^2k^2} - \frac{6c_1}{h^2k^3} - \frac{e}{k} - \frac{3L}{k^3} - \frac{2g}{k^2}, \\
G &= \frac{a}{h^2k} + \frac{2b_1}{h^2k^2} + \frac{3c_1}{h^2k^3}, \quad H = \frac{a}{h^2k} + \frac{2b_1}{h^2k^2}, \\
I &= -\frac{b_1}{h^2k^2} - \frac{c_1}{h^2k^3}, \quad J = \frac{2b_1}{h^2k^2} + \frac{4c_1}{h^2k^3} + \frac{3L}{k^3} + \frac{g}{k^2}, \\
K &= -\frac{b_1}{h^2k^2} - \frac{c_1}{h^2k^3}, \quad M = -\frac{3(b_2 + b_3)}{2h^4}, \\
N &= -\frac{c_1}{h^2k^3}, \quad P = -\frac{L}{k^3}. \qquad (7.17)
\end{aligned}
$$

By the process of augmentation we can write

$$\frac{\partial f(\infty, \tau)}{\partial \eta} = 0, \qquad (7.18)$$

$$\frac{\partial^2 f(\infty, \tau)}{\partial \eta^2} = 0, \qquad (7.19)$$

$$\frac{\partial^3 f(\infty, \tau)}{\partial \eta^3} = 0, \qquad (7.20)$$

and consequently the problem becomes well-posed. These boundary conditions are discretized to incorporate in the numerical scheme.

The non-linear differential system consisting of Eq. (7.11) and conditions (7.12) has been solved numerically by employing the Newton method. Solutions for the non-

Newtonian fluid models are obtained for $\tau = 2\pi$. From the numerical solution f is used to express the non-dimensional velocity profile parallel to x-axis. Results for the flow are obtained for various values of the involved parameters.

Fig. 7.1 presents the velocity profile f for various values of c_1. This figure shows that increasing the fourth order parameter c_1 decreases both the velocity and the boundary layer thickness. Fig. 7.2 indicates the influence of the fourth order parameters c_i $(i = 2, 3, 4, 5, 7, 8)$ on the velocity profile f. It is evident from the figure that an increase in these parameters yields a decrease in velocity profile. The effect of porosity and fourth order parameters on f is displayed in Fig. 7.3. It is interesting to note that both velocity and boundary layer thickness increase by increasing these parameters. Fig. 7.4 shows the variation of the porosity parameter on the velocity. It is found that here the velocity decreases by increasing this parameter. Figs. 7.5 and 7.6 depict the velocity distribution for various values of the third and second order fluid parameters. The behavior of the velocity is quite similar for second and third order fluid parameters. The velocity increases for the large values of the second and third order parameters.

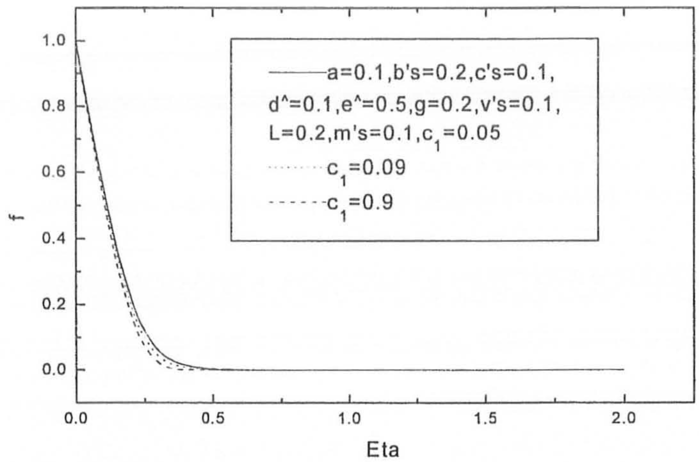

Fig. 7.1 Variation of the fourth order parameter c_1 on f at $\tau = 2\pi$.

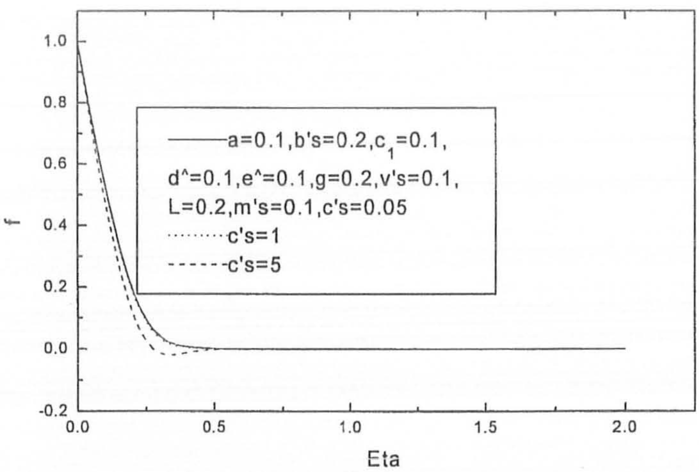

Fig. 7.2 Variation of the fourth order parameters c_i $(i = 2, 3, 4, 5, 7, 8)$ on f at $\tau = 2\pi$.

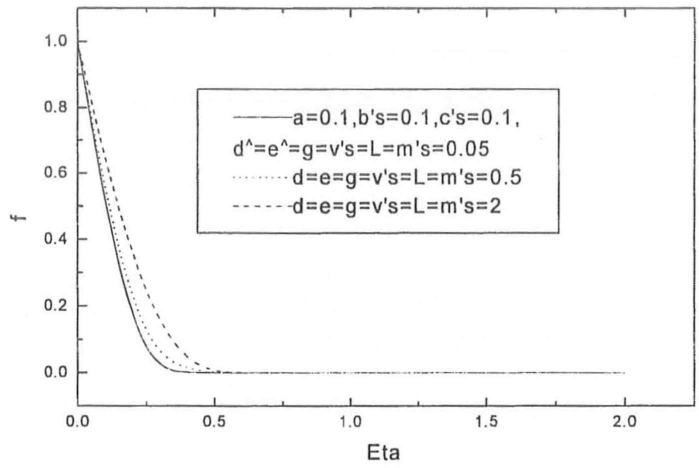

Fig. 7.3 Variation of the fourth order and porosity parameters on f at $\tau = 2\pi$.

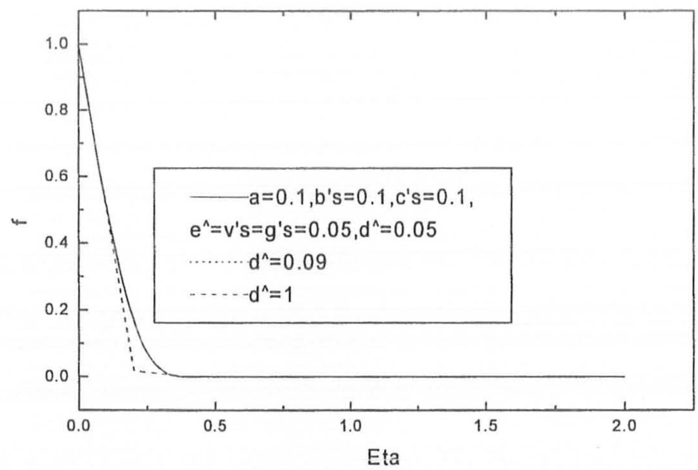

Fig. 7.4 Variation of the porosity parameter on f at $\tau = 2\pi$.

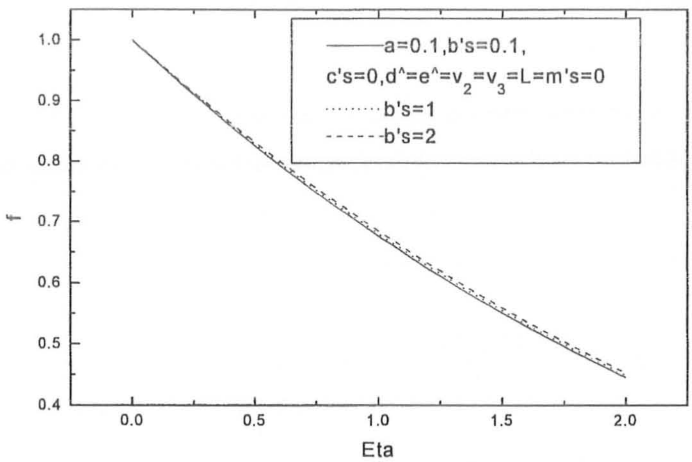

Fig. 7.5 Variation of third order parameters on f at $\tau = 2\pi$.

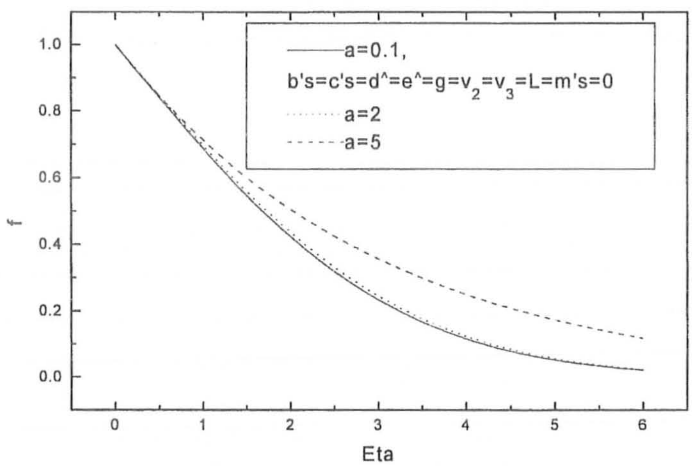

Fig. 7.6 Variation of the second order parameter on f at $\tau = 2\pi$.

7.3 Concluding remarks

In the present chapter, Stokes' first problem is generalized for the fourth order fluid in a porous space. The governing constitutive relationship for modified Darcy's law in a fourth order fluid has been proposed. To the best of our knowledge such relationship is not available in the literature. It is noted that modified Darcy's law for unidirectional flow of a fourth order fluid yields a non-linear expression in terms of the velocity while it is linear for Newtonian, Oldroyd-B, Maxwell and second grade fluids. The governing non-linear problem has been solved numerically. It is observed that for $\tau \geq 5\pi$ the fourth order fluid behaves like a Newtonian fluid. The contents of this chapter have been accepted for publication in "**Acta Mechanica Sinica**".

Chapter 8

Oscillatory flow of a fourth order fluid over a porous plate

In this chapter, a numerical solution of an oscillatory flow over a porous plate is considered. The fluid is considered as a fourth order. The governing non-linear partial differential equation is first modelled and then solved using Newton's method. The variation of various parameters of interest is shown on the velocity. The differences among the velocity fields corresponding to various fluid models are delineated.

8.1 Problem formulation

Let us consider the flow of an incompressible fourth order fluid with constant properties. The fluid is over an oscillating plate at $y = 0$. The x-axis is chosen parallel to the plate. Moreover, the plate is porous and oscillates in its own plane. The flow is independent upon x [i.e $u = u(y,t)$, u is the velocity in the x direction, $v = V_0$, $w = 0$].

Under the assumptions given in the above section we have the following stress components

$$\begin{aligned}\tau_{xx} &= -p+\alpha_2\left(\frac{\partial u}{\partial y}\right)^2+2\beta_2\frac{\partial u}{\partial y}\left(\frac{\partial^2 u}{\partial y\partial t}+V_0\frac{\partial^2 u}{\partial y^2}\right)\\ &+2\gamma_2\left(\frac{\partial u}{\partial y}\right)\left(\frac{\partial^3 u}{\partial y\partial t^2}+2V_0\frac{\partial^3 u}{\partial y^2\partial t}+V_0^2\frac{\partial^3 u}{\partial y^3}\right)\\ &+\gamma_3\left(\frac{\partial^2 u}{\partial y\partial t}+V_0\frac{\partial^2 u}{\partial y^2}\right)^2+2\gamma_6\left(\frac{\partial u}{\partial y}\right)^4, \end{aligned}\tag{8.1}$$

$$\begin{aligned}\tau_{yy} &= -p+(2\alpha_1+\alpha_2)\left(\frac{\partial u}{\partial y}\right)^2+2\left(3\beta_1+\beta_2\right)\left(\frac{\partial u}{\partial y}\right)\left(\frac{\partial^2 u}{\partial y\partial t}+V_0\frac{\partial^2 u}{\partial y^2}\right)\\ &+6\gamma_1\left[\left(\frac{\partial}{\partial t}+V_0\frac{\partial}{\partial y}\right)\left(\frac{\partial u}{\partial y}\frac{\partial^2 u}{\partial y\partial t}+V_0\frac{\partial u}{\partial y}\frac{\partial^2 u}{\partial y^2}\right)\right]\\ &+2\left(\gamma_1+\gamma_2\right)\left(\frac{\partial^3 u}{\partial y\partial t^2}+2V_0\frac{\partial^3 u}{\partial y^2\partial t}+V_0^2\frac{\partial^3 u}{\partial y^3}\right)\frac{\partial u}{\partial y}\\ &+\gamma_3\left(\frac{\partial^2 u}{\partial y\partial t}+V_0^2\frac{\partial^2 u}{\partial y^2}\right)^2+2\left(2\gamma_3+2\gamma_4+2\gamma_5+\gamma_6\right)\left(\frac{\partial u}{\partial y}\right)^4, \end{aligned}\tag{8.2}$$

$$\tau_{zz} = -p, \tag{8.3}$$

$$\begin{aligned}\tau_{xy} &= \mu\left(\frac{\partial u}{\partial y}\right)+\alpha_1\left(\frac{\partial^2 u}{\partial y\partial t}+V_0\frac{\partial^2 u}{\partial y^2}\right)+\beta_1\left(\frac{\partial^3 u}{\partial y\partial t^2}+2V_0\frac{\partial^3 u}{\partial y^2\partial t}+V_0^2\frac{\partial^3 u}{\partial y^3}\right)\\ &+2\left(\beta_2+\beta_3\right)\left(\frac{\partial u}{\partial y}\right)^3+\gamma_1\left(\frac{\partial^4 u}{\partial y\partial t^3}+3V_0\frac{\partial^4 u}{\partial y^2\partial t^2}+3V_0^2\frac{\partial^4 u}{\partial y^3\partial t}+V_0^3\frac{\partial^4 u}{\partial y^4}\right)\\ &+2\left(3\gamma_2+\gamma_3+\gamma_4+\gamma_5+3\gamma_7+\gamma_8\right)\left(\frac{\partial u}{\partial y}\right)^2\left(\frac{\partial^2 u}{\partial y\partial t}+V_0\frac{\partial^2 u}{\partial y^2}\right), \end{aligned}\tag{8.4}$$

$$\tau_{xz} = \tau_{zy} = 0, \tag{8.5}$$

where $\tau_{xy} = \tau_{yx}$, $\tau_{xz} = \tau_{zx}$, $\tau_{yz} = \tau_{zy}$ and $V_0 < 0$ corresponds to the suction case and $V_0 > 0$ indicates blowing situation.

The scalar momentum equations are

$$\rho\left[\frac{\partial u}{\partial t}+V_0\frac{\partial u}{\partial y}\right] = \frac{\partial \tau_{xx}}{\partial x}+\frac{\partial \tau_{xy}}{\partial y}+\frac{\partial \tau_{xz}}{\partial z}, \tag{8.6}$$

$$0 = \frac{\partial \tau_{yx}}{\partial x}+\frac{\partial \tau_{yy}}{\partial y}+\frac{\partial \tau_{yz}}{\partial z}, \tag{8.7}$$

$$0 = \frac{\partial \tau_{zx}}{\partial x} + \frac{\partial \tau_{zy}}{\partial y} + \frac{\partial \tau_{zz}}{\partial z}. \tag{8.8}$$

Inserting the stress components in above equations we obtain

$$\begin{aligned}
\left[\rho\frac{\partial u}{\partial t} + V_0\frac{\partial u}{\partial y}\right] &= \mu\frac{\partial^2 u}{\partial y^2} + \alpha_1\left[\frac{\partial^3 u}{\partial y^2 \partial t} + V_0\frac{\partial^3 u}{\partial y^3}\right] \\
&+\beta_1\left[\frac{\partial^4 u}{\partial y^2 \partial t^2} + 2V_0\frac{\partial^4 u}{\partial y^3 \partial t} + V_0^2\frac{\partial^4 u}{\partial y^4}\right] + 6\left(\beta_2 + \beta_3\right)\left(\frac{\partial u}{\partial y}\right)^2\left(\frac{\partial^2 u}{\partial y^2}\right) \\
&+\gamma_1\left[\frac{\partial^5 u}{\partial y^2 \partial t^3} + 3V_0\frac{\partial^5 u}{\partial y^3 \partial t^2} + 3V_0^2\frac{\partial^5 u}{\partial y^4 \partial t} + V_0^3\frac{\partial^5 u}{\partial y^5}\right] \\
&+2\begin{pmatrix} 3\gamma_2 + \gamma_3 + \gamma_4 \\ +\gamma_5 + 3\gamma_7 + \gamma_8 \end{pmatrix}\begin{bmatrix} 2\left(\frac{\partial u}{\partial y}\right)\left(\frac{\partial^2 u}{\partial y^2}\right)\left(\frac{\partial^2 u}{\partial y \partial t}\right) \\ +\left(\frac{\partial u}{\partial y}\right)^2\frac{\partial^3 u}{\partial y^2 \partial t} \\ +2V_0\left(\frac{\partial u}{\partial y}\right)\left(\frac{\partial^2 u}{\partial y^2}\right) \\ +V_0\left(\frac{\partial u}{\partial y}\right)^2\left(\frac{\partial^3 u}{\partial y^3}\right) \end{bmatrix},
\end{aligned} \tag{8.9}$$

where modified pressure gradient term has been neglected.

The boundary and initial conditions for the flow are

$$\begin{aligned}
u(0,t) &= U_0\, e^{-i\omega t}, \qquad \omega > 0,\ t > 0, \\
u(y,t) &\longrightarrow 0 \quad \text{as } y \longrightarrow \infty,\ U(y,0) = 0,\ y > 0,
\end{aligned} \tag{8.10}$$

where U_0 is the reference velocity and ω is the oscillating frequency.

Introducing the following non-dimensional variables

$$\eta = \sqrt{\frac{\omega}{2\nu}}y, \qquad \tau = \omega t, \qquad f = \frac{u}{U_0}, \tag{8.11}$$

we get

$$
\begin{aligned}
\left[\frac{\partial f}{\partial \tau} + \sqrt{2} d^{\wedge\wedge} \left(\frac{\partial f}{\partial \eta}\right)\right] = {} & \frac{1}{2}\frac{\partial^2 f}{\partial \eta^2} + \frac{a^{\wedge\wedge}}{2}\left(\frac{\partial^3 f}{\partial \eta^2 \partial \tau} + d\frac{\partial^3 f}{\partial \eta^3}\right) \\
& + b_1^{\wedge\wedge}\left(\frac{\partial^4 f}{\partial \eta^2 \partial \tau^2} + d\frac{\partial^4 f}{\partial \eta^3 \partial \tau} + 2d^2\frac{\partial^4 f}{\partial \eta^4}\right) \\
& + \frac{3}{2}\left(b_2^{\wedge\wedge} + b_3^{\wedge\wedge}\right)\left(\frac{\partial f}{\partial \eta}\right)^2\left(\frac{\partial^2 f}{\partial \eta^2}\right) \\
& + \frac{c_1^{\wedge\wedge}}{2}\left(\begin{array}{c} \frac{\partial^5 f}{\partial \eta^2 \partial \tau^3} + 3\sqrt{2} d \frac{\partial^5 f}{\partial \eta^3 \partial \tau^2} \\ +6d^2\frac{\partial^5 f}{\partial \eta^4 \partial \tau} + \sqrt{2} d^3 \frac{\partial^5 f}{\partial \eta^5} \end{array}\right) \\
& + \frac{1}{2}\left(\begin{array}{c} 3c_2^{\wedge\wedge} + c_3^{\wedge\wedge} + c_4^{\wedge\wedge} \\ +c_5^{\wedge\wedge} + 3c_7^{\wedge\wedge} + c_8^{\wedge\wedge} \end{array}\right)\left[\begin{array}{c} 2\left(\frac{\partial f}{\partial \eta}\right)\left(\frac{\partial^2 f}{\partial \eta^2}\right)\left(\frac{\partial^2 f}{\partial \eta \partial \tau}\right) \\ +\left(\frac{\partial f}{\partial \eta}\right)^2\left(\frac{\partial^3 f}{\partial \eta^2 \partial \tau}\right) \\ +d\left(\frac{\partial f}{\partial \eta}\right)\left(\frac{\partial^2 f}{\partial \eta^2}\right) \\ +\sqrt{2} d\left(\frac{\partial f}{\partial \eta}\right)^2\left(\frac{\partial^3 f}{\partial \eta^3}\right) \end{array}\right],
\end{aligned}
\tag{8.12}
$$

$$
f(0,\tau) = e^{-i\tau}, \qquad f(\eta,\tau) \longrightarrow 0 \quad \text{as} \quad \eta \longrightarrow \infty, \qquad f(\eta, 0) = 0, \tag{8.13}
$$

where ν is the kinematic viscosity and

$$
\begin{aligned}
a^{\wedge\wedge} &= \frac{\alpha_1 \omega}{\rho \nu}, \qquad b_1^{\wedge\wedge} = \frac{\beta_1 \omega^2}{\rho \nu}, \qquad b_2^{\wedge\wedge} = \frac{\beta_2 \omega U_0^2}{\rho \nu^2}, \qquad b_3^{\wedge\wedge} = \frac{\beta_3 \omega U_0^2}{\rho \nu^2}, \\
d^{\wedge\wedge} &= \frac{V_0}{2\sqrt{\nu \omega}}, \qquad c_1^{\wedge\wedge} = \frac{\gamma_1 \omega^3}{\rho \nu}, \\
c_2^{\wedge\wedge} &= c_3^{\wedge\wedge} = c_4^{\wedge\wedge} = c_5^{\wedge\wedge} = c_7^{\wedge\wedge} = c_8^{\wedge\wedge} = \frac{\gamma_i \omega^2 U_0^2}{\rho \nu^2}, \qquad i = 2,3,4,5,7,8.
\end{aligned}
\tag{8.14}
$$

8.2 Numerical results and discussion

The governing equation (8.12) is transformed into an algebraic equation by substituting the approximations to the derivatives given in section 2.7. Thus we have

$$
\begin{aligned}
&\left(\frac{1}{k}\right)\left(f_{i,j}-f_{i,j-1}\right)+\frac{\sqrt{2}d^{\wedge\wedge}}{2h}\left(f_{i+1,j}-f_{i-1,j}\right)=\frac{1}{2h^2}\left(f_{i+1,j}-2f_{i,j}+f_{i-1,j}\right)\\
&+\frac{a^{\wedge\wedge}}{2}\left[\begin{array}{c}\frac{1}{h^2k}\left(f_{i+1,j}-f_{i+1,j-1}-2f_{i,j}+2f_{i,j-1}+f_{i-1,j}-f_{i-1,j-1}\right)\\ +\frac{d^{\wedge\wedge}}{2h^3}\left(f_{i+2,j}-2f_{i+1,j}+2f_{i-1,j}-f_{i-2,j}\right)\end{array}\right]\\
&+\frac{b_1^{\wedge\wedge}}{2}\left[\begin{array}{c}\frac{1}{h^2k}\left(\begin{array}{c}f_{i+1,j}-2f_{i+1,j-1}+f_{i+1,j-2}-2f_{i,j}+4f_{i,j-1}\\ -2f_{i,j-2}+f_{i-1,j}-2f_{i-1,j-1}+f_{i-1,j-2}\end{array}\right)\\ +\frac{d^{\wedge\wedge}}{2kh^3}\left(\begin{array}{c}f_{i+2,j}-f_{i+2,j-1}-2f_{i+1,j}+2f_{i+1,j-1}\\ +2f_{i-1,j}-2f_{i-1,j-1}-f_{i-2,j}+f_{i-2,j-1}\end{array}\right)\\ +\frac{2d^{\wedge\wedge 2}}{h^4}\left(\begin{array}{c}f_{i+2,j}-4f_{i+1,j}+6f_{i,j}\\ -4f_{i-1,j}+f_{i-2,j}\end{array}\right)\end{array}\right]\\
&+\frac{3\left(b_2^{\wedge\wedge}+b^{\wedge\wedge}3\right)}{2}\left[\frac{1}{2h^4}\left(f_{i+1,j}+f_{i-1,j}\right)^2\left(f_{i+1,j}-2f_{i,j}+f_{i-1,j}\right)\right]\\
&+\frac{c_1^{\wedge\wedge}}{2}\left[\begin{array}{c}\frac{1}{h^2k^3}\left(\begin{array}{c}f_{i+1,j}-3f_{i+1,j-1}+f_{i+1,j-2}-f_{i+1,j-3}\\ -2f_{i,j}+4f_{i,j-1}-4f_{i,j-1}-4f_{i,j-2}+2f_{i,j-1}\\ +f_{i-1,j-2}-f_{i-1,j-3}\end{array}\right)\\ +\frac{3d^{\wedge\wedge}}{\sqrt{2}h^3k^2}\left(\begin{array}{c}f_{i+2,j}-2f_{i+2,j-1}+f_{i+2,j-2}-2f_{i+1,j}\\ +4f_{i+1,j-1}-2f_{i+1,j-2}+2f_{i-1,j}-4f_{i-1,j-1}\\ +2f_{i-1,j-2}-f_{i-2,j}+2f_{i-2,j-1}-f_{i-2,j-2}\end{array}\right)\\ +\frac{6d^{\wedge\wedge}}{h^4k}\left(\begin{array}{c}f_{i+2,j}-f_{i+2,j-1}-4f_{i+1,j}+4f_{i+1,j-1}+6f_{i,j}\\ -10f_{i,j-1}+4f_{i-1,j-1}+4f_{i-2,j}-f_{i-2,j-1}\end{array}\right)\\ +\frac{d^{\wedge\wedge 3}}{\sqrt{2}h^5}\left(\begin{array}{c}f_{i+3,j}-4f_{i+2,j}-3f_{i+1,j}\\ -5f_{i-1,j}+4f_{i-2,j}-2f_{i-3,j}\end{array}\right)\end{array}\right]
\end{aligned}
$$

$$+\frac{\left(6c_2^{\hat{}\hat{}}+2c_3^{\hat{}\hat{}}+2c_4^{\hat{}\hat{}}+2c_5^{\hat{}\hat{}}+6c_7^{\hat{}\hat{}}+2c_8^{\hat{}\hat{}}\right)}{2}\left[\begin{array}{c} \frac{1}{4h^4k}\left(f_{i+1,j}-f_{i-1,j}\right) \\ \left(f_{i+1,j}-2f_{i,j}+f_{i-1,j}\right) \\ \left(f_{i+1,j}-f_{i+1,j-1}-f_{i-1,j}+f_{i-1,j-1}\right) \\ +\frac{1}{4h^4k}\left(f_{i+1,j}-f_{i-1,j}\right)^2 \\ \left(\begin{array}{c} f_{i+1,j}-f_{i+1,j-1}-2f_{i,j} \\ +2f_{i,j-1}+f_{i-1,j}-f_{i-1,j-1} \end{array}\right) \\ +\frac{2d}{h^3}\left(f_{i+1,j}-f_{i-1,j}\right) \\ \left(f_{i+1,j}-2f_{i,j}+f_{i-1,j}\right) \\ +\frac{\sqrt{2}d^{\hat{}\hat{}}}{8h^5}\left(f_{i+1,j}-f_{i-1,j}\right)^2 \\ \left(f_{i+2,j}-2f_{i+1,j}+2f_{i-1,j}-f_{i-2,j}\right) \end{array}\right] \tag{8.15}$$

and the system of algebraic equations is

$$R_i = Af_{i,j} + Bf_{i+1,j} + Cf_{i-1,j} + Df_{i+2,j} + Ef_{i-2,j} + E_1 f_{i+3,j}$$

$$+E_2 f_{i-3,j} + K_1 f_{i+1,j}^3 + K_2 f_{i+1,j}^2 f_{i,j} + K_3 f_{i+1,j}^2 f_{i-1,j} + K_4 f_{i-1,j}^2 f_{i+1,j}$$

$$+K_5 f_{i-1,j}^2 f_{i,j} + K_6 f_{i-1,j}^3 + K_7 f_{i+1,j} f_{i-1,j} f_{i,j} + K_8 f_{i+1,j}^2 f_{i+1,j-1}$$

$$+K_9 f_{i+1,j}^2 f_{i+1,j-1} + K_{10} f_{i+1,j} f_{i,j} f_{i+1,j-1} + K_{11} f_{i+1,j} f_{i-1,j-1} f_{i,j}$$

$$+K_{12} f_{i-1,j} f_{i+1,j-1} f_{i,j} + K_{13} f_{i-1,j} f_{i,j} f_{i-1,j-1} + K_{14} f_{i+1,j}^2 f_{i,j-1}$$

$$+K_{15} f_{i-1,j}^2 f_{i,j-1} + K_{16} f_{i+1,j} f_{i-1,j} f_{i+1,j-1} + K_{17} f_{i+1,j} f_{i-1,j} f_{i,j-1}$$

$$+K_{18} f_{i+1,j} f_{i-1,j} f_{i-1,j-1} + K_{19} f_{i+1,j}^2 + +K_{20} f_{i+1,j} f_{i,j}$$

$$+K_{21} f_{i-1,j}^2 + K_{22} f_{i-1,j} f_{i,j} + K_{23} f_{i+1,j}^2 f_{i+2,j} + K_{24} f_{i+1,j}^2 f_{i-2,j}$$

$$+K_{25} f_{i-1,j}^2 f_{i+2,j} + K_{26} f_{i-1,j}^2 f_{i-2,j} + K_{27} f_{i+1,j} f_{i-1,j} f_{i+2,j}$$

$$+K_{28} f_{i+1,j} f_{i-1,j} f_{i-2,j} + F f_{i,j-1} + G f_{i+1,j-1} + H f_{i-1,j-1}$$

$$+I f_{i+2,j-2} + J f_{i-2,j-1} + K f_{i+1,j-2} + L f_{i,j-2} + M f_{i-1,j-2}$$

$$+N f_{i-1,j-3} + P f_{i+1,j-3} + Q f_{i+2,j-2} + R f_{i-2,j-2}, \tag{8.16}$$

$$f_{0,j} = 1, \quad f_{M,j} = 0, \quad f_{i,0} = 0, \quad i = 0, 1, 2, ..., M \quad j = 0, 1, 2, 3..., \tag{8.17}$$

where

$$A = \left[\frac{1}{k} + \frac{1}{h^2} + \frac{\hat{\hat{a}}}{h^2 k} + \frac{\hat{\hat{b}}_1}{h^2 k^2} - \frac{6\hat{\hat{b}}_1 d^2}{h^4} + \frac{\hat{\hat{c}}_1}{h^2 k^3}\right],$$

$$B = \left[\begin{array}{c} -\frac{\hat{\hat{d}}}{\sqrt{2}h} - \frac{1}{2h^2} - \frac{\hat{\hat{a}}}{2h^2 k} + \frac{\hat{\hat{a}}\hat{\hat{d}}}{2h^3} - \frac{\hat{\hat{b}}_1}{2h^2 k^2} + \frac{\hat{\hat{b}}_1 \hat{\hat{d}}}{2kh^3} \\ + \frac{4\hat{\hat{b}}_1 \hat{\hat{d}}^2}{h^4} - \frac{\hat{\hat{c}}_1}{2h^2 k^3} + \frac{3\hat{\hat{d}}\hat{\hat{c}}_1}{\sqrt{2}k^2 h^3} + \frac{3\hat{\hat{c}}_1 \hat{\hat{d}}^3}{2\sqrt{2}h^5} \end{array}\right],$$

$$C = \left[\begin{array}{c} -\frac{\hat{\hat{d}}}{\sqrt{2}h} - \frac{1}{2h^2} - \frac{\hat{\hat{a}}}{2h^2 k} - \frac{\hat{\hat{a}}\hat{\hat{d}}}{2h^3} - \frac{\hat{\hat{b}}_1}{2h^2 k^2} - \frac{\hat{\hat{b}}_1 \hat{\hat{d}}}{2kh^3} \\ + \frac{4\hat{\hat{b}}_1 \hat{\hat{d}}^2}{h^4} - \frac{3\hat{\hat{d}}\hat{\hat{c}}_1}{\sqrt{2}k^2 h^3} + \frac{5\hat{\hat{c}}_1 \hat{\hat{d}}^3}{2\sqrt{2}h^5} \end{array}\right],$$

$$D = \left[-\frac{\hat{\hat{a}}\hat{\hat{d}}}{4h^3} - \frac{\hat{\hat{b}}_1 \hat{\hat{d}}}{4h^3 k} - \frac{\hat{\hat{b}}_1 \hat{\hat{d}}^2}{h^4} - \frac{3\hat{\hat{d}}\hat{\hat{c}}_1}{2\sqrt{2}k^2 h^3} + \frac{\sqrt{2}\hat{\hat{c}}_1 \hat{\hat{d}}^3}{h^5}\right],$$

$$E = \left[\frac{\hat{\hat{a}}\hat{\hat{d}}}{4h^3} + \frac{\hat{\hat{b}}_1 \hat{\hat{d}}}{4h^3 k} - \frac{\hat{\hat{b}}_1 \hat{\hat{d}}^2}{h^4} + \frac{3\hat{\hat{d}}\hat{\hat{c}}_1}{2\sqrt{2}k^2 h^3} - \frac{\sqrt{2}\hat{\hat{c}}_1 \hat{\hat{d}}^3}{h^5}\right],$$

$$E_1 = -\frac{\hat{\hat{c}}_1 \hat{\hat{d}}^3}{2\sqrt{2}h^5}, \quad E_2 = \frac{\hat{\hat{c}}_1 \hat{\hat{d}}^3}{\sqrt{2}h^5},$$

$$K_1 = -\frac{3\left(\hat{\hat{b}}_2 + \hat{\hat{b}}_3\right)}{8h^4} - \frac{1}{2h^4 k}\left(6\hat{\hat{c}}_2 + 2\hat{\hat{c}}_3 + 2\hat{\hat{c}}_4 + 2\hat{\hat{c}}_5 + 6\hat{\hat{c}}_7 + 2\hat{\hat{c}}_8\right)$$
$$-\frac{\hat{\hat{d}}}{4h^5}\left(6\hat{\hat{c}}_2 + 2\hat{\hat{c}}_3 + 2\hat{\hat{c}}_4 + 2\hat{\hat{c}}_5 + 6\hat{\hat{c}}_7 + 2\hat{\hat{c}}_8\right),$$

$$K_2 = \frac{3\left(\hat{\hat{b}}_2 + \hat{\hat{b}}_3\right)}{4h^4} + \frac{1}{2h^4 k}\left(6\hat{\hat{c}}_2 + 2\hat{\hat{c}}_3 + 2\hat{\hat{c}}_4 + 2\hat{\hat{c}}_5 + 6\hat{\hat{c}}_7 + 2\hat{\hat{c}}_8\right),$$

$$K_3 = \frac{3\left(\hat{\hat{b}}_2 + \hat{\hat{b}}_3\right)}{8h^4} + \frac{1}{2h^4 k}\left(6\hat{\hat{c}}_2 + 2\hat{\hat{c}}_3 + 2\hat{\hat{c}}_4 + 2\hat{\hat{c}}_5 + 6\hat{\hat{c}}_7 + 2\hat{\hat{c}}_8\right)$$
$$-\frac{3\hat{\hat{d}}}{4h^5}\left(6\hat{\hat{c}}_2 + 2\hat{\hat{c}}_3 + 2\hat{\hat{c}}_4 + 2\hat{\hat{c}}_5 + 6\hat{\hat{c}}_7 + 2\hat{\hat{c}}_8\right),$$

$$K_4 = \frac{3\left(\hat{\hat{b}}_2 + \hat{\hat{b}}_3\right)}{8h^4} + \frac{1}{4h^4 k}\left(6\hat{\hat{c}}_2 + 2\hat{\hat{c}}_3 + 2\hat{\hat{c}}_4 + 2\hat{\hat{c}}_5 + 6\hat{\hat{c}}_7 + 2\hat{\hat{c}}_8\right)$$
$$+\frac{3\hat{\hat{d}}}{4h^5}\left(6\hat{\hat{c}}_2 + 2\hat{\hat{c}}_3 + 2\hat{\hat{c}}_4 + 2\hat{\hat{c}}_5 + 6\hat{\hat{c}}_7 + 2\hat{\hat{c}}_8\right),$$

$$K_5 = \frac{3\left(\hat{\hat{b}}_2 + \hat{\hat{b}}_3\right)}{2h^4},$$

$$K_6 = -\frac{3\left(\hat{\hat{b}}_2 + \hat{\hat{b}}_3\right)}{8h^4} + \frac{1}{4h^4 k}\left(6\hat{\hat{c}}_2 + 2\hat{\hat{c}}_3 + 2\hat{\hat{c}}_4 + 2\hat{\hat{c}}_5 + 6\hat{\hat{c}}_7 + 2\hat{\hat{c}}_8\right)$$
$$-\frac{\hat{\hat{d}}}{4h^5}\left(6\hat{\hat{c}}_2 + 2\hat{\hat{c}}_3 + 2\hat{\hat{c}}_4 + 2\hat{\hat{c}}_5 + 6\hat{\hat{c}}_7 + 2\hat{\hat{c}}_8\right),$$

$$K_7 = -\frac{3\left(\hat{\hat{b}}_2 + \hat{\hat{b}}_3\right)}{2h^4} - \frac{1}{2h^4 k}\left(6\hat{\hat{c}}_2 + 2\hat{\hat{c}}_3 + 2\hat{\hat{c}}_4 + 2\hat{\hat{c}}_5 + 6\hat{\hat{c}}_7 + 2\hat{\hat{c}}_8\right),$$

$$
\begin{aligned}
K_8 &= \frac{1}{4h^4k}\left(6c_2^{\wedge\wedge} + 2c_3^{\wedge\wedge} + 2c_4^{\wedge\wedge} + 2c_5^{\wedge\wedge} + 6c_7^{\wedge\wedge} + 2c_8^{\wedge\wedge}\right), \\
K_9 &= K_8, \quad K_{10} = -K_8, \quad K_{11} = K_8, \\
K_{12} &= -K_8, \quad K_{13} = K_8, \quad K_{14} = -K_8, \\
K_{15} &= -K_8, \quad K_{16} = K_{15}, \quad K_{17} = -2K_{15}, \\
K_{18} &= K_{15}, \quad K_{19} = \frac{d^{\wedge\wedge}}{h^3}\left(6c_2^{\wedge\wedge} + 2c_3^{\wedge\wedge} + 2c_4^{\wedge\wedge} + 2c_5^{\wedge\wedge} + 6c_7^{\wedge\wedge} + 2c_8^{\wedge\wedge}\right), \\
K_{20} &= 4K_{19}, \quad K_{21} = K_{19}, \quad K_{22} = -K_{20}, \\
K_{23} &= -\frac{d^{\wedge\wedge}}{8h^5}\left(6c_2^{\wedge\wedge} + 2c_3^{\wedge\wedge} + 2c_4^{\wedge\wedge} + 2c_5^{\wedge\wedge} + 6c_7^{\wedge\wedge} + 2c_8^{\wedge\wedge}\right), \\
K_{24} &= -\frac{d^{\wedge\wedge}}{4h^5}\left(6c_2^{\wedge\wedge} + 2c_3^{\wedge\wedge} + 2c_4^{\wedge\wedge} + 2c_5^{\wedge\wedge} + 6c_7^{\wedge\wedge} + 2c_8^{\wedge\wedge}\right), \\
K_{25} &= K_{23}, \quad K_{26} = -K_{23}, \quad K_{27} = -K_{24}, \\
K_{28} &= -K_{24}, \quad F = -\frac{1}{k} - \frac{a^{\wedge\wedge}}{h^2k} - \frac{2b_1^{\wedge\wedge}}{h^2k^2} - \frac{3c_1^{\wedge\wedge}}{h^2k^3}, \\
G &= \frac{a^{\wedge\wedge}}{2h^2k} + \frac{b_1^{\wedge\wedge}}{h^2k^2} - \frac{b_1^{\wedge\wedge}d^{\wedge\wedge}}{2kh^3} + \frac{3c_1^{\wedge\wedge}}{2h^2k^3} - \frac{3\sqrt{2}d^{\wedge\wedge}c_1^{\wedge\wedge}}{k^2h^3}, \\
H &= \frac{a^{\wedge\wedge}}{2h^2k} + \frac{b_1^{\wedge\wedge}}{h^2k^2} + \frac{b_1^{\wedge\wedge}d^{\wedge\wedge}}{2kh^3} + \frac{3\sqrt{2}d^{\wedge\wedge}c_1^{\wedge\wedge}}{k^2h^3}, \\
I &= \frac{b_1^{\wedge\wedge}d^{\wedge\wedge}}{4h^3k} + \frac{3d^{\wedge\wedge}c_1^{\wedge\wedge}}{\sqrt{2}h^3k^2}, \quad J = -\frac{b_1^{\wedge\wedge}d^{\wedge\wedge}}{4h^3k} - \frac{3d^{\wedge\wedge}c_1^{\wedge\wedge}}{\sqrt{2}h^3k^2}, \\
K &= -\frac{b_1^{\wedge\wedge}}{2h^2k^2} - \frac{c_1^{\wedge\wedge}}{2h^2k^3} + \frac{3d^{\wedge\wedge}c_1^{\wedge\wedge}}{\sqrt{2}h^3k^2}, \\
L &= \frac{b_1^{\wedge\wedge}}{h^2k^2} + \frac{2c_1^{\wedge\wedge}}{h^2k^3}, \\
M &= -\frac{b_1^{\wedge\wedge}}{2h^2k^2} - \frac{c_1^{\wedge\wedge}}{2h^2k^3} - \frac{3d^{\wedge\wedge}c_1^{\wedge\wedge}}{\sqrt{2}h^3k^2}, \\
N &= \frac{c_1^{\wedge\wedge}}{2h^2k^3}, \quad P = N, \\
Q &= -\frac{3d^{\wedge\wedge}c_1^{\wedge\wedge}}{2\sqrt{2}h^3k^2}, \quad R = -Q. \qquad (8.18)
\end{aligned}
$$

Since our equation (8.12) is of order five while given boundary conditions are two, therefore we introduce the process of augmentation and consequently the problem becomes well-posed. These boundary conditions are discretized and incorporated in the numerical scheme.

Here, the non-linear differential equation (8.12) under the boundary and initial conditions described in Eq. (8.13) is numerically solved using Newtons' method. A numerical solution is given for $\tau = 2\pi$. The numerical solution f is used to express the non-dimensional velocity profile parallel to x-axis. Results for the flow are obtained for various values of the involving parameters.

The influence of suction and blowing on the velocity f is shown in Fig. 8.1. This Fig. shows the variation of d^{**} for the case of the Newtonian fluid. Here it is noted that suction causes reduction in the boundary layer thickness whereas blowing increases the layer thickness.

In order to illustrate the influence of suction and blowing on f in the case of a fourth grade fluid, we made Fig. 8.2. This Fig. elucidates the similar characteristics as the ones in Fig. 8.1. But it is found that boundary layer thickness in case of fourth order fluid is larger than that of Newtonian fluid. Fig. 8.3. has been plotted just to see the variation of $\gamma_i\,(i = 2$ to $8)$ on f while other parameters in the fourth order fluid are fixed. It is revealed that boundary layer thickness decreases by increasing $\gamma_i\,(i = 2$ to $8)$. Fig. 8.4 shows the variation of the fourth order parameter c_1^{**} on f. Here it is observed that f increases by increasing c_1^{**}. Fig. 8.5 and 8.6 indicate the variation of f in third and second order fluids, respectively. These Figs. show that boundary layer thickness in a third order fluid is less than that of the second order fluid. However, the boundary layer in both the fluids is less when compared with fourth order fluid.

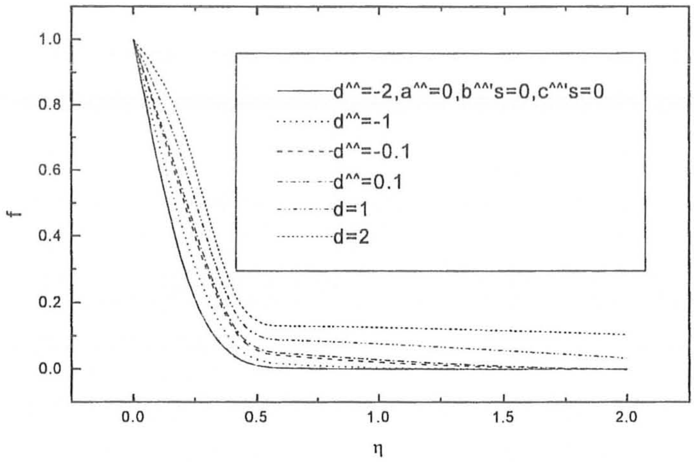

Fig. 8.1. Influence of suction/blowing on the velocity distribution for the Newtonian fluid.

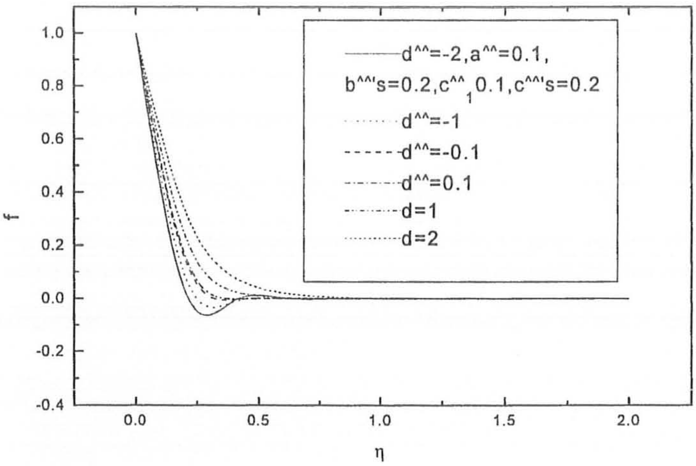

Fig. 8.2. Influence of suction/blowing on the velocity distribution for the fourth order fluid.

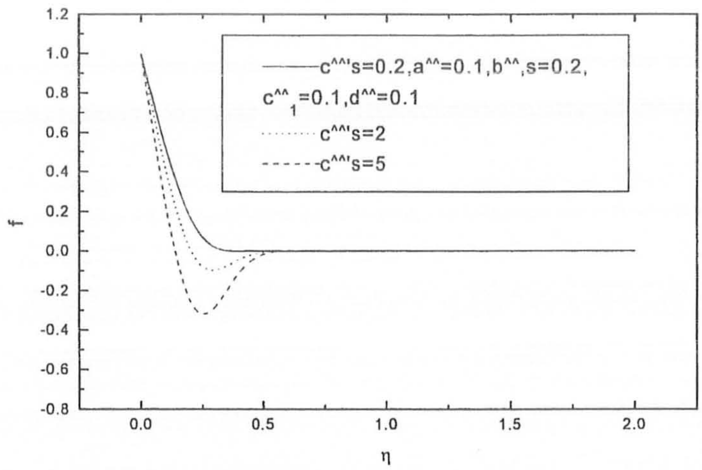

Fig. 8.3. Variation of the fourth order parameters $c_i^{\hat{}\hat{}}$ ($i = 2$ to 8) on f.

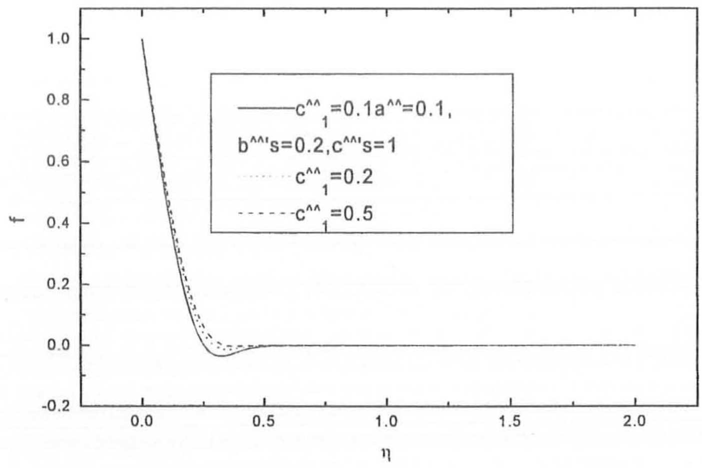

Fig. 8.4. Variation of the fourth order parameter $c_1^{\hat{}\hat{}}$ on f.

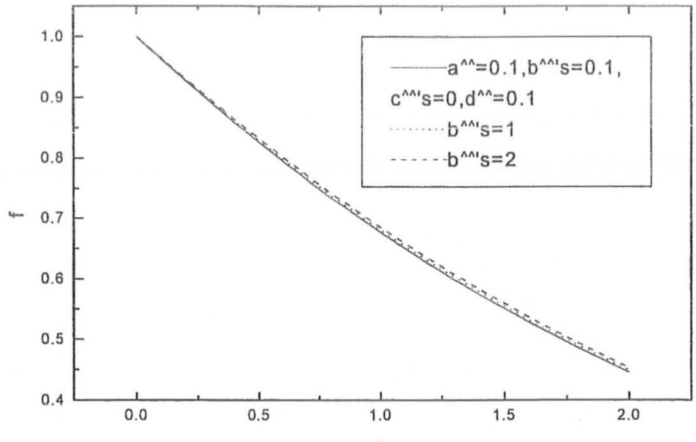

Fig. 8.5. Variation of the third grade parameters on f.

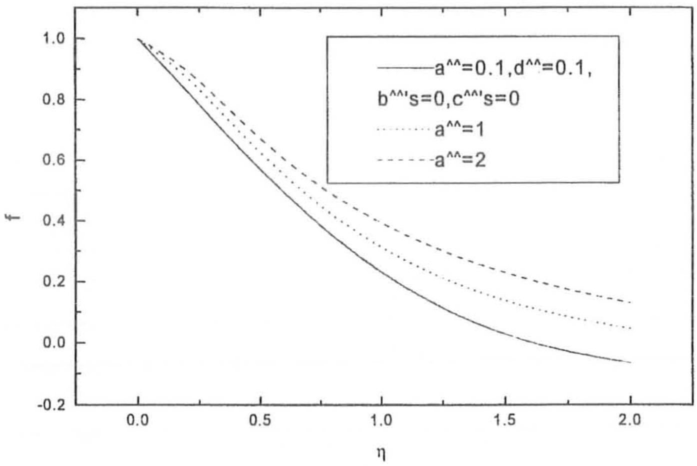

Fig. 8.6. Vriation of the second grade parameters on f .

8.3 Concluding remarks

The effects of suction and blowing on the flows of an incompressible Newtonian and non-Newtonian fluids have been studied. The governing equation with the boundary and initial conditions have been non-dimensionalized. Numerical solution of the non-linear problem has been obtained. From the present analysis, it may be concluded that:

- The boundary layer thickness decreases owing to an increase in the suction parameter.
- The boundary layer thickness in blowing case is greater than the one obtained with suction.
- The boundary layer thickness in fourth order fluid is larger than that of Newtonian fluid.
- The results for Newtonian, second grade and third grade fluid models can be recovered as the limiting cases of the present solution by taking appropriate values of the material constants.

Chapter 9

Couette flow of a Johnson-Segalman fluid in the presence of a uniform magnetic field

This chapter describes the one-dimensional, steady and incompressible flow of a Johnson-Segalman fluid between two plates. The flow is induced due to motion of the upper plate. The combined effects of viscoelascity and magnetic field are considered. The magnetic field is applied transversely to the direction of the flow. The governing equation of the problem is reduced to a non-linear ordinary differential equation and is solved analytically in general. The Couette flow has been discussed numerically using Newton's method. The influence of the Weissenberg number, Hartmann number and ratio of viscosities upon the velocity has been discussed.

9.1 Mathematical analysis

Consider the steady, unidirectional and incompressible flow of an electrically conducting Johnson-Segalman fluid past an infinite plate. A magnetic field with a constant magnetic flux density $\mathbf{B}_0$ is applied perpendicular to the plate. We assume that the induced magnetic field produced by the motion of an electrically conducting fluid is negligible. The assumption is justified since the magnetic Reynolds number is small, which is generally the case in normal aerodynamic applications. Since no external electric field is applied and the effect of polarization of the ionized fluid is negligible, we can also assume that the electric field $\mathbf{E} = 0$. The flow under consideration is governed by Eq. (2.13) and

$$\rho\frac{d\mathbf{V}}{dt} = \text{div } \mathbf{T} + \mathbf{j} \times \mathbf{B}, \tag{9.1}$$

where $\mathbf{T}$ is the Cauchy stress tensor and the third term on the right hand side of Eq. (9.1) is the Lorentz force which through the aforementioned assumptions is given by

$$\mathbf{j} \times \mathbf{B} = \sigma_1 (\mathbf{V} \times \mathbf{B}) \times \mathbf{B}. \tag{9.2}$$

Here σ_1, $\mathbf{j}$ and $\mathbf{B}$ are the electric conductivity, current density, and total magnetic field respectively. Note that $\mathbf{B} = \mathbf{B}_0 + \mathbf{b}$ ($\mathbf{B}_0$ and $\mathbf{b}$ are applied and induced magnetic fields). The constitutive equation for $\mathbf{T}$ is [52]

$$\mathbf{T} = -p\mathbf{I} + \widetilde{\mathbf{S}}, \tag{9.3}$$

$$\widetilde{\mathbf{S}} + \lambda\left(\frac{d\widetilde{\mathbf{S}}}{dt} + \widetilde{\mathbf{S}}(\mathbf{W} - a\mathbf{D}) + (\mathbf{W} - a\mathbf{D})^T\,\widetilde{\mathbf{S}}\right) = 2\eta D, \tag{9.4}$$

$$\mathbf{D} = \frac{1}{2}\left(\mathbf{L} + \mathbf{L}^T\right), \quad \mathbf{W} = \frac{1}{2}\left(\mathbf{L} - \mathbf{L}^T\right), \quad \mathbf{L} = \operatorname{grad}\mathbf{V}. \tag{9.5}$$

In Eqs. (9.3) and (9.4), η is the viscosity, λ is the relaxation time and a is the slip parameter. When $a = 1$, $\lambda = \mu = 0$ the model (9.3) reduces to the Newtonian model and when $a = 1$, $\mu = 0$ it reduces to the Maxwell fluid.

We consider the following forms for the velocity and extra stress tensor:

$$\mathbf{V} = (u(y),\ 0,\ 0), \quad \widetilde{\mathbf{S}} = \widetilde{\mathbf{S}}(y). \tag{9.6}$$

Using the assumed form of velocity, Eq. (2.13) is identically satisfied and from Eqs. (9.1) to (9.6) we have the following equations for the non-zero components of $\widetilde{\mathbf{S}}$:

$$\frac{1}{\rho}\frac{d}{dy}\widetilde{S}_{xy} + \nu\frac{d^2u}{dy^2} - \frac{\sigma_1 B_0^2}{\rho}u = \frac{1}{\rho}\frac{\partial p}{\partial x}, \tag{9.7}$$

$$\frac{d}{dy}\widetilde{S}_{yy} = \frac{1}{\rho}\frac{\partial p}{\partial y}, \tag{9.8}$$

$$0 = \frac{1}{\rho}\frac{\partial p}{\partial z}, \tag{9.9}$$

$$\widetilde{S}_{xx} - \lambda(1+a)\frac{du}{dy}\widetilde{S}_{xy} = 0, \tag{9.10}$$

$$\widetilde{S}_{xy} + \frac{\lambda}{2}\left[(1-a)\widetilde{S}_{xx} - (1+a)\widetilde{S}_{yy}\right]\frac{du}{dy} = \eta\frac{du}{dy}, \tag{9.11}$$

$$\widetilde{S}_{yy} + \lambda(1-a)\frac{du}{dy}\widetilde{S}_{xy} = 0, \tag{9.12}$$

where Eq. (9.9) indicates that p is not a function of z. Thus p is at most a function of x and y.

Defining the dimensionless quantities

$$x^* = \frac{U_0}{\nu}x, \quad y^* = \frac{U_0}{\nu}y, \quad u^* = \frac{u}{U_0}, \quad S^* = \frac{\nu\widetilde{S}}{(\mu+\eta)U_0^2}, \quad p^* = \frac{p}{\rho U_0^2} \tag{9.13}$$

and modified pressure $\widehat{p}^*$ by

$$\widehat{p}^* = p^* - S_{yy}^*, \tag{9.14}$$

Eqs. (9.7) to (9.12) becomes

$$\phi\frac{d}{dy^*}S^*_{xy} + \frac{\mu}{\eta}\frac{d^2u^*}{dy^{*2}} - Nu^* = \frac{\partial\widehat{p}^*}{\partial x^*}, \tag{9.15}$$

$$0 = \frac{\partial\widehat{p}^*}{\partial y^*}, \tag{9.16}$$

$$S^*_{xx} - W_e\,(1+a)\,S^*_{xy}\frac{du^*}{dy^*} = 0, \tag{9.17}$$

$$S^*_{xy} + \frac{W_e}{2}\left[(1-a)\,S^*_{xx} - (1+a)\,S^*_{yy}\right]\frac{du^*}{dy^*} = \frac{1}{\phi}\frac{du^*}{dy^*}, \tag{9.18}$$

$$S^*_{yy} + W_e\,(1-a)\,S^*_{xy}\frac{du^*}{dy^*} = 0. \tag{9.19}$$

We note from Eq. (9.16) that $\widehat{p}^*$ is not a function of y^* and the Weissenberg number W_e, Hartmann number N and the ratio of viscosities ϕ are given by

$$W_e = \frac{\lambda U_0^2}{\nu}, \qquad \phi = \frac{\mu+\eta}{\eta}, \qquad N = \frac{\nu\sigma_1 B_0^2}{U_0^2}. \tag{9.20}$$

From Eqs. (9.17) – (9.19) we can write

$$S^*_{xx} = \frac{W_e\,(1+a)}{\phi}\frac{\left(\frac{du^*}{dy^*}\right)^2}{\left[1+W_e^2\,(1-a^2)\left(\frac{du^*}{dy^*}\right)^2\right]}, \tag{9.21}$$

$$S^*_{xy} = \frac{1}{\phi}\frac{\left(\frac{du^*}{dy^*}\right)}{\left[1+W_e^2\,(1-a^2)\left(\frac{du^*}{dy^*}\right)^2\right]}, \tag{9.22}$$

$$S^*_{yy} = -\frac{W_e\,(1-a)}{\phi}\frac{\left(\frac{du^*}{dy^*}\right)^2}{\left[1+W_e^2\,(1-a^2)\left(\frac{du^*}{dy^*}\right)^2\right]}. \tag{9.23}$$

Now eliminating the pressure $\widehat{p}^*$ from Eqs. (9.15) and (9.16) and then using Eq. (9.22) in the resulting equation we obtain

$$\frac{\mu}{\eta}\frac{d^2u^*}{dy^{*2}} + \frac{d}{dy^*}\left\{\frac{\left(\frac{du^*}{dy^*}\right)}{\left[1+W_e^2\,(1-a^2)\left(\frac{du^*}{dy^*}\right)^2\right]}\right\} - Nu^* = k_1, \tag{9.24}$$

where $k = dp^*/dx$ is the constant pressure gradient.

9.2 General solution

We can write Eq. (9.24) as follows:

$$\frac{\mu}{\eta}\frac{d^2u^*}{dy^{*2}} + \frac{d}{dy^*}\left\{\frac{\left(\frac{du^*}{dy^*}\right)}{\left[1+\alpha\left(\frac{du^*}{dy^*}\right)^2\right]}\right\} - Nu^* = k_1, \tag{9.25}$$

where

$$\alpha = W_e^2\left(1-a^2\right).$$

We let

$$P^* = \frac{\mu}{\eta}\frac{du^*}{dy^*} + \frac{\frac{du^*}{dy^*}}{1+\alpha\left(\frac{du^*}{dy^*}\right)^2}. \tag{9.26}$$

Then Eq. (9.25) can be written as

$$\frac{dP^*}{dy^*} - Nu^* - k_1 = 0. \tag{9.27}$$

Equation (9.26) can be solved for du^*/dy^* in terms of P^*. This gives

$$\frac{du^*}{dy^*} = \frac{\eta}{\alpha\mu}\left[\sqrt[3]{\frac{B}{2}+\frac{\sqrt{B^2-4A^3}}{2}} + \sqrt[3]{\frac{B}{2}-\frac{\sqrt{B^2-4A^3}}{2}}\right] + \frac{\eta}{3\mu}P^*, \tag{9.28}$$

where

$$\begin{aligned} A &= \frac{\alpha^2}{9}p^{*2} - \frac{\alpha\mu}{3\eta} - \frac{\alpha\mu^2}{3\eta^2}, \\ B &= \frac{2}{27}\alpha^3 P^{*3} + \frac{2}{3}\frac{\alpha\mu^2}{\eta^2}P^* - \frac{\alpha^2\mu}{3\eta}P^*. \end{aligned} \tag{9.29}$$

We utilise Eq. (9.27) to obtain P^* as function of y^* with the insertion of Eq. (9.28) into Eq. (9.27). We have

$$\frac{d^2P^*}{dy^{*2}} - N\left[\frac{\eta}{\alpha\mu}\sqrt[3]{\frac{B}{2}+\frac{\sqrt{B^2-4A^3}}{2}} + \frac{\eta}{\alpha\mu}\sqrt[3]{\frac{B}{2}-\frac{\sqrt{B^2-4A^3}}{2}} + \frac{\eta}{3\mu}P^*\right] = 0. \tag{9.30}$$

The first integral of Eq. (9.30) gives

$$\frac{dP^*}{dy^*}=\pm\sqrt{2N\int\left[\frac{\eta}{\alpha\mu}\sqrt[3]{\frac{B}{2}+\frac{\sqrt{B^2-4A^3}}{2}}+\frac{\eta}{\alpha\mu}\sqrt[3]{\frac{B}{2}-\frac{\sqrt{B^2-4A^3}}{2}}+\frac{\eta}{3\mu}P^*\right]dP^*+C_1}, \tag{9.31}$$

where C_1 is an arbitrary constant. The double integration of Eq. (9.30) results in

$$\int\frac{dP^*}{\pm\sqrt{2N\int\left[\frac{\eta}{\alpha\mu}\sqrt[3]{\frac{B}{2}+\frac{\sqrt{B^2-4A^3}}{2}}+\frac{\eta}{\alpha\mu}\sqrt[3]{\frac{B}{2}-\frac{\sqrt{B^2-4A^3}}{2}}+\frac{\eta}{3\mu}P^*\right]dP^*+C_1}}=y^*+C_2, \tag{9.32}$$

where C_2 is another arbitrary constant. The substitution of Eq. (9.31) into Eq. (9.27) yields

$$Nu^*=\pm\sqrt{2N\int\left[\frac{\eta}{\alpha\mu}\sqrt[3]{\frac{B}{2}+\frac{\sqrt{B^2-4A^3}}{2}}+\frac{\eta}{\alpha\mu}\sqrt[3]{\frac{B}{2}-\frac{\sqrt{B^2-4A^3}}{2}}+\frac{\eta}{3\mu}P^*\right]dP^*+C_1}-k_1. \tag{9.33}$$

So the general solution of Eq. (9.25) is given in parametric form by Eqs. (9.32) and (9.33) up to the evaluation of an integral. Note that Eq. (9.25) is of second order and thus there are two arbitrary constants C_1 and C_2. Also the constant pressure gradient k occurs in Eq. (9.33). Simplified version of the integral in Eq. (9.32) is given in the Appendix.

9.3 Numerical solution for boundary value problem

Let us now consider the flow of a Johnson-Segalman fluid between two parallel plates of infinite length at $y = 0$ and $y = d$. The flow here is maintained by setting one of the plates in motion. In this case, it is assumed that the bottom plate is moving with velocity U_0 and the top plate is at rest. The governing dimensionless boundary conditions are of the following form:

$$
\begin{aligned}
u^* &= 1 \quad \text{for} \quad y^* = 0, \\
u^* &= 0 \quad \text{for} \quad y^* = d^*,
\end{aligned}
\tag{9.34}
$$

where $\quad d^* = U_0 d/\nu.$

Note that Eq.(9.25) is the second order non-linear differential equation and it is solved numerically. The governing equation (9.25) is transformed into an algebraic equation by substituting the approximations to the derivatives given in section 2.7 and get

$$
\begin{aligned}
& \left(\frac{a}{h^4}\right)\begin{pmatrix} f_{i+1}^2 + 4f_i^2 + f_{i-1}^2 \\ +2f_{i+1}f_{i-1} - 2f_if_{i+1} - 2f_if_{i-1} \end{pmatrix} \\
& + \frac{b}{2h^4}\left(f_{i+1} - 2f_i + f_{i-1}\right)\left(f_{i+1}^2 + f_{i-1}^2 - 2f_{i+1}f_{i-1}\right) \\
& + \frac{c}{16h^6}\left(f_{i+1} - 2f_i + f_{i-1}\right)\begin{pmatrix} f_{i+1,j}^4 - 4f_{i+1}^3 f_{i-1} \\ +6f_{i+1}^2 f_{i-1} - 4f_{i+1}f_{i-1}^3 + f_{i-1}^4 \end{pmatrix} \\
& - Nf_i - \frac{d}{16h^4}f_i\begin{pmatrix} f_{i+1,j}^4 - 4f_{i+1}^3 f_{i-1} \\ +6f_{i+1}^2 f_{i-1} - 4f_{i+1}f_{i-1}^3 + f_{i-1}^4 \end{pmatrix} \\
& - \frac{e}{2h^4}f_i\left(f_{i+1,j} + f_{i-1,j}\right)^2 - k_1 \\
& - \frac{g}{16h^4}\begin{pmatrix} f_{i+1,j}^4 - 4f_{i+1}^3 f_{i-1} \\ +6f_{i+1}^2 f_{i-1} - 4f_{i+1}f_{i-1}^3 + f_{i-1}^4 \end{pmatrix} - \frac{h_1}{2h^2}\left(f_{i+1,j} + f_{i-1,j}\right)^2 \\
= {} & 0.
\end{aligned}
\tag{9.35}
$$

The system of algebraic equations can be expressed as

$$
\begin{aligned}
R_i &= Af_i + K_1 f_i^2 + K_2 f_{i+1}^2 + K_3 f_{i-1}^2 + K_4 f_{i+1}^3 + K_5 f_{i-1}^3 \\
&+ K_6 f_{i+1}^5 + K_7 f_{i-1}^5 + K_8 f_{i+1}^4 + K_9 f_{i-1}^4 + K_{10} f_{i+1} f_{i-1} \\
&+ K_{11} f_{i+1} f_i + K_{12} f_i f_{i-1} + K_{13} f_{i+1} f_{i-1}^2 + K_{14} f_{i+1}^2 f_{i-1} \\
&+ K_{15} f_{i+1}^2 f_i + K_{16} f_i f_{i-1}^2 + K_{17} f_{i+1}^4 f_{i-1} + K_{19} f_{i+1}^2 f_{i-1}^3 \\
&+ K_{20} f_{i+1} f_{i-1}^4 + K_{21} f_i f_{i+1}^4 + K_{22} f_i f_{i-1}^4 + K_{23} f_{i+1}^3 f_{i-1} \\
&+ K_{24} f_{i-1}^3 f_{i+1} + K_{25} f_i f_{i+1} f_{i-1} + K_{26} f_i f_{i+1}^3 f_{i-1} + K_{27} f_i f_{i+1}^2 f_{i-1}^2 \\
&+ K_{28} f_i f_{i+1} f_{i-1}^3 + K_{29} f_i f_{i+1}^2 f_{i-1}, \qquad (9.36)
\end{aligned}
$$

in which

$$
\begin{aligned}
A &= -N, \quad K_1 = \frac{4a}{h^4}, \quad K_2 = \frac{a}{h^4} - \frac{h_1}{2h^2}, \\
K_3 &= K_2, \quad K_4 = \frac{b}{2h^4}, \quad K_5 = K_4, \\
K_6 &= \frac{c}{16h^6}, \quad K_7 = K_6, \quad K_8 = \frac{-g}{16h^4}, \\
K_9 &= K_8, \quad K_{10} = \frac{a}{h^4} + \frac{h_1}{h^2}, \quad K_{11} = -\frac{2a}{h^4}, \\
K_{12} &= K_{11}, \quad K_{13} = -K_5, \quad K_{14} = -\frac{b}{2h^4} - \frac{6g}{16h^4}, \\
K_{15} &= -\frac{b}{h^4} - \frac{e}{2h^2}, \quad K_{15} = G, \quad K_{16} = K_{15}, \\
K_{17} &= -\frac{c}{4h^6} + \frac{c}{16h^6}, \quad K_{18} = \frac{3c}{8h^6}, \quad K_{19} = K_{17}, \\
K_{20} &= \frac{c}{16h^6} - \frac{c}{4h^6}, \quad K_{21} = -\frac{c}{8h^6} - \frac{d}{16h^4}, \\
K_{22} &= -\frac{c}{4h^6} - \frac{d}{16h^4}, \quad K_{23} = \frac{g}{4h^4}, \quad K_{24} = K_{23}, \\
K_{25} &= \frac{2b}{h^4} + \frac{e}{h^2}, \quad K_{26} = \frac{c}{2h^6} + \frac{d}{4h^4}, \quad K_{27} = -\frac{3c}{4h^6}, \\
K_{28} &= \frac{c}{2h^6} + \frac{d}{4h^4}, \quad K_{29} = -\frac{3d}{8h^4}, \qquad (9.37)
\end{aligned}
$$

The initial and boundary conditions are

$$f_0 = 1, \qquad f_M = 0, \qquad f_i = 0, \qquad i = 0, 1, 2, ..., M \qquad j = 0, 1, 2, 3.... \tag{9.38}$$

In order to obtain the numerical solution, the Newton's method has been utilized. The results of various interesting parameters including Weissenberg number, Hartmann number and ratio of viscosities are presented in the following.

Fig. 9.1 shows the influence of Weissenberg number on the velocity profile u^*. It is evident from the figure that an increase in α results in the decrease of the velocity. Fig. 9.2 depicts the variation of Hartmann number on the velocity. It is found that the velocity increases with an increase in N. The boundary layer thickness decreases. This means that the magnetic force provides a mechanism to the control boundary layer thickness. In Fig. 9.3 the velocity distribution is presented for the various values of viscosities. It is observed that the velocity decreases by increasing the influence of μ/η.

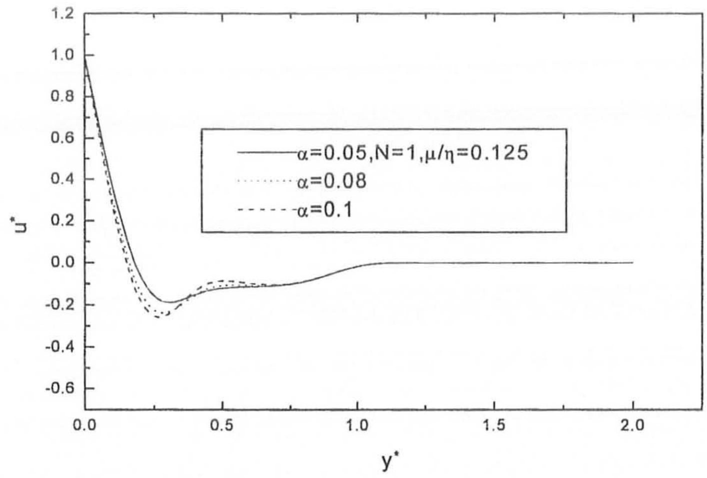

Fig. 9.1. Variation of the velocity distribution for the various values of the Weissenberg number.

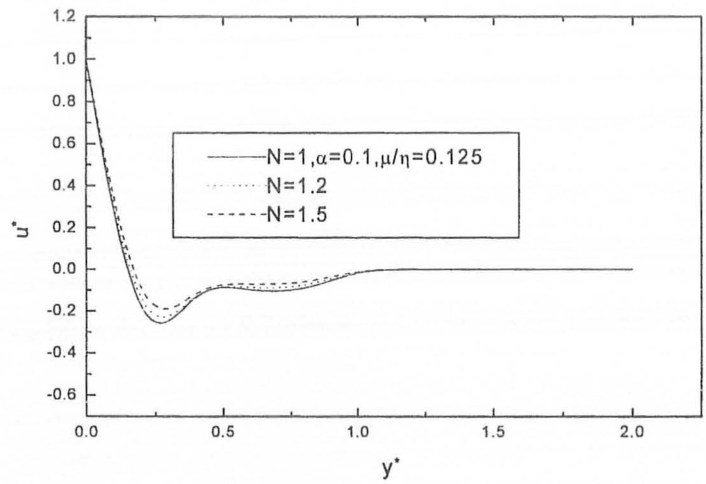

Fig. 9.2. Variation of the velocity distribution for the various values of the Hartmann number.

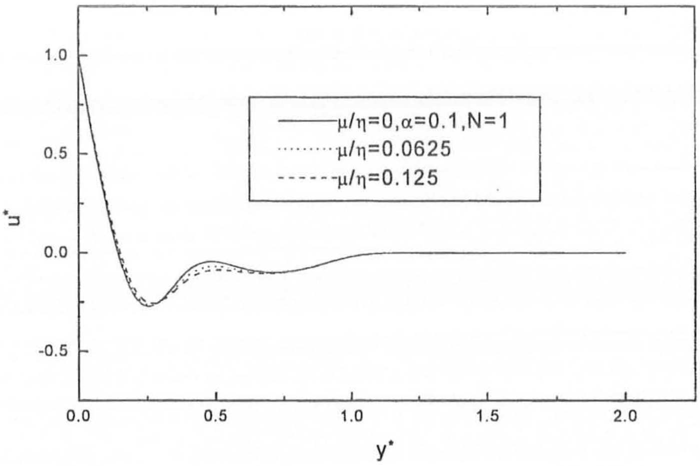

Fig. 9.3. Variation of the velocity distribution for the various values of viscosities ratio.

9.4 Conclusion

The Couette flow between two parallel plates filled with a magnetohydrodynamic Johnson-Segalman fluid is studied numerically. A non-linear constitutive model for the fluid is used. The model is substituted into the governing equations and the resulting one dimensional equation for MHD flow is derived. This equation is solved analytically in general to study the sensitivity of the flow to the parameters that are used in the constitutive model. The various dimensionless parameters seem to affect the velocity profile a lot. The velocity profile is greatly influenced by the Weissenberg and Hartmann numbers. The obtained solution is valid for all values of Weissenberg number. However, the specific Eq. (9.25) that is to be solved can be rather daunting, as it leads to the evaluations of complicated expression in Eqs. (9.32) and (9.33). Finally the boundary value problem is solved numerically and the results are presented graphically for the various values of the interesting involved parameters. It is found that the velocity of the fluid increases by increasing the Hartmann number whereas it decreases by increasing the Weissenberg number and viscosities ratio.

9.5 Appendix

We evaluate the integral appearing in Eq. (9.32) i.e.

$$I = \int \frac{dP^*}{\pm\sqrt{2N \int \left[\frac{\eta}{\alpha\mu}\sqrt[3]{\frac{B}{2}+\frac{\sqrt{B^2-4A^3}}{2}}+\frac{\eta}{\alpha\mu}\sqrt[3]{\frac{B}{2}-\frac{\sqrt{B^2-4A^3}}{2}}+\frac{\eta}{3\mu}P^*\right] dP^* + C_1}}. \quad (A_1)$$

Taking

$$\frac{\alpha}{3}P^* = q, \qquad \frac{\alpha\mu}{3\eta} = \beta, \qquad \frac{\mu}{\eta} = \frac{1}{8}, \quad (A_2)$$

from Eq. (9.29) we have

$$A = q^2 - \frac{9}{8}\beta, \qquad B = 2q^2\frac{9}{4}\beta q. \quad (A_3)$$

Substituting

$$q = \cosh 3\theta \quad (A_4)$$

and using Eq. (A_3) we have the value of the denominator as

$$\pm\sqrt{2N \int \left[\frac{\eta}{\alpha\mu}\sqrt[3]{\frac{B}{2}+\frac{\sqrt{B^2-4A^3}}{2}}+\frac{\eta}{\alpha\mu}\sqrt[3]{\frac{B}{2}-\frac{\sqrt{B^2-4A^3}}{2}}+\frac{\eta}{3\mu}P^*\right] dP^* + C_1}$$

$$= \pm\frac{2\sqrt{3N}}{\alpha}\sqrt{\cosh 6\theta + 3\cosh 4\theta + 6\cosh 2\theta + C_3}, \quad (A_5)$$

in which

$$C_3 = \frac{\alpha^2 C_1}{12N}. \tag{A$_6$}$$

From Eqs. (A_1), (A_4) and (A_5) we can write

$$I = \pm\frac{3\sqrt{3}}{2\sqrt{N}}\int\frac{\sinh 3\theta d\theta}{\sqrt{\cosh 6\theta + 3\cosh 4\theta + 6\cosh 2\theta + C_3}}. \tag{A$_7$}$$

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Analytic solution for flow of a micropolar fluid

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Summary. The time-independent equations for the two dimensional incompressible micropolar fluid have been considered. Using group method the equations have been reduced to ordinary differential equations and then solved analytically. Finally the boundary value problem has been discussed, and the graphical results are in good agreement with the numerical solution.

1 Introduction

Eringen [1], [2] developed the theory of microfluids which exhibit microscopic effects arising from the local structure and micro-motions of the fluid elements. Such fluids support stress and body moments and include the local rotary inertia. The equations based on the theory of microfluids are much more complicated even for the case of a constitutively linear situation, and the non-trivial solution in the field is not easy to obtain. There is a subclass of microfluids namely the micropolar fluids for which one can reasonably hope to obtain a non-trivial analytic solution. The micropolar fluids support couple stress, body couples, micro-rotational effects and micro-rotational inertia. The mathematical theory of equations of micropolar fluids and application of these fluids in the theory of lubrication and in the porous space is given in [3].

Recently the studies of micropolar fluids have acquired a special status due to their industrial applications. Such applications include the extrusion of polymer fluids, solidification of liquid crystals, cooling of a metallic plate in a bath, animal bloods, exotic lubricants and colloidal and suspension solutions. Undoubtedly, the classical Navier-Stokes theory is inadequate for such fluids. Several workers in the field have made the useful investigations that involve a micropolar fluid. For example, Sriniasacharya and Rajyalakshmi [4] studied the creeping flow of a micropolar fluid past a porous sphere. Iyengar and Vani [5] examined the flow of a micropolar fluid between two concentric spheres, induced by their rotary oscillations. Kasiviswanathan and Gandhi [6] discussed the Hartman steady flow of a micropolar fluid between two infinite, parallel non-coaxially rotating disks. Al-Bary [7] developed the exponential solution of the problem of two dimensional motion of a micropolar fluid in a half-plane. Dubey et al. [8] analyzed the flow of a micropolar fluid between two parallel plates rotating about two non-coincident axes under variable surfaces charges. Gorla et al. [9] studied the heat transfer analysis on the boundary layer flow of a micropolar fluid. Ibrahem et al. [10] presented the non-classical heat conduction effects in Stokes' second problem for unsteady micropolar fluids flow. Seddek [11] studied the Hartman flow of a micropolar fluid past a continuously moving plate. Kim and Lee [12] made an interesting study for the Hartman oscillatory flow problem of a micropolar fluid. Agrawal [13] presented a finite element solution of unsteady three dimensional micropolar fluid flow at a stagnation point. Abo-Eldahab and Ghonaim [14] discussed the numerical solution in order to see the radiation effect on heat transfer of a micropolar fluid.

However, most of the previous investigations deal with the numerical solution. The aim here is to provide an analytic solution for the flow problem of a micropolar fluid. The group analysis method has been extensively used for unsteady axisymmetric incompressible viscous flow by Nucci [15]. Recently, Yürüsoy et al. [16] have obtained the solution for the creeping flow of the second grade fluid using group method. They found the analytic solution for the two-dimensional flow of a micropolar fluid. The analytic solution is given using group method [17]–[19]. The translation type symmetry has been taken into account. The graphs are also plotted and discussed.

2 Equations of motion

The two dimensional equations for an incompressible micropolar fluid are [3]

$$\frac{\partial \overline{u}}{\partial \overline{x}}+\frac{\partial \overline{v}}{\partial \overline{y}}=0, \tag{1}$$

$$\rho\left(\overline{u}\frac{\partial \overline{u}}{\partial \overline{x}}+\overline{v}\frac{\partial \overline{u}}{\partial \overline{y}}\right)=(\mu+k_1)\left(\frac{\partial^2 \overline{u}}{\partial \overline{x}^2}+\frac{\partial^2 \overline{u}}{\partial \overline{y}^2}\right)+k_1\frac{\partial \overline{\sigma}}{\partial \overline{y}}-\frac{\partial \overline{p}}{\partial \overline{x}}, \tag{2}$$

$$\rho\left(\overline{u}\frac{\partial \overline{v}}{\partial \overline{x}}+\overline{v}\frac{\partial \overline{v}}{\partial \overline{y}}\right)=(\mu+k_1)\left(\frac{\partial^2 \overline{v}}{\partial \overline{x}^2}+\frac{\partial^2 \overline{v}}{\partial \overline{y}^2}\right)-k_1\frac{\partial \overline{\sigma}}{\partial \overline{x}}-\frac{\partial \overline{p}}{\partial \overline{y}}, \tag{3}$$

$$\rho\overline{j}\left(\overline{u}\frac{\partial \overline{\sigma}}{\partial \overline{x}}+\overline{v}\frac{\partial \overline{\sigma}}{\partial \overline{y}}\right)=G_1\left(\frac{\partial^2 \overline{\sigma}}{\partial \overline{x}^2}+\frac{\partial^2 \overline{\sigma}}{\partial \overline{y}^2}\right)-2k_1\overline{\sigma}+k_1\left(\frac{\partial \overline{v}}{\partial \overline{x}}-\frac{\partial \overline{u}}{\partial \overline{y}}\right), \tag{4}$$

where $\overline{u}$ and $\overline{v}$ are the components of the velocity field in the $\overline{x}$ and $\overline{y}$ direction, $\overline{\sigma}(\overline{x},\overline{y})$ is the micro-rotation component, and $\overline{p}=\overline{p}(\overline{x},\overline{y})$ is the pressure distribution. Here ρ, μ, k_1, G_1 and $\overline{j}$ are mass density, coefficient of viscosity, coupling constant, micro-rotation constant and local micro inertia.

Defining

$$u=\frac{\overline{u}}{U},\quad v=\frac{\overline{v}}{U},\quad x=\frac{\overline{x}}{L},\quad y=\frac{\overline{y}}{L},$$

$$p=\frac{\overline{p}}{P},\quad \sigma=\frac{\overline{\sigma}}{\sigma^*},\quad j=\frac{\overline{j}}{J}, \tag{5}$$

Eqs. (1)–(4) reduce to

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0, \tag{6}$$

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=(\epsilon_1+\epsilon_2)\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right)+\epsilon_3\frac{\partial \sigma}{\partial y}-\epsilon_4\frac{\partial p}{\partial x}, \tag{7}$$

$$u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}=(\epsilon_1+\epsilon_2)\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}\right)-\epsilon_3\frac{\partial \sigma}{\partial x}-\epsilon_4\frac{\partial p}{\partial y}, \tag{8}$$

$$u\frac{\partial \sigma}{\partial x}+v\frac{\partial \sigma}{\partial y}=\epsilon_5\left(\frac{\partial^2\sigma}{\partial x^2}+\frac{\partial^2\sigma}{\partial y^2}\right)-\epsilon_6\sigma+\epsilon_7\left(\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}\right), \tag{9}$$

where

$$\epsilon_1=\frac{\mu}{\rho LU},\quad \epsilon_2=\frac{k_1}{\rho LU},\quad \epsilon_3=\frac{k_1\sigma^*}{\rho U^2},\quad \epsilon_4=\frac{P}{\rho U^2},$$
$$\epsilon_5=\frac{G_1J}{\rho LU\bar{j}},\quad \epsilon_6=\frac{2k_1LJ}{\rho U\bar{j}},\quad \epsilon_7=\frac{k_1J}{\rho\sigma^*\bar{j}} \tag{10}$$

and ϵ_1 and ϵ_2 are the reciprocal Reynolds numbers.

3 Symmetry analysis

In order to obtain the analytical solution, we use Lie group theory to Eqs. (6)–(9). For this we write

$$\begin{aligned}
x^* &= x+\varepsilon\xi_1(x,y,u,v,p)+O(\varepsilon^2),\\
y^* &= y+\varepsilon\xi_2(x,y,u,v,p)+O(\varepsilon^2),\\
u^* &= u+\varepsilon\eta_1(x,y,u,v,p)+O(\varepsilon^2),\\
v^* &= v+\varepsilon\eta_2(x,y,u,v,p)+O(\varepsilon^2),\\
p^* &= p+\varepsilon\eta_3(x,y,u,v,p)+O(\varepsilon^2),\\
\sigma^* &= \sigma+\varepsilon\eta_4(x,y,u,v,p)+O(\varepsilon^2)
\end{aligned} \tag{11}$$

as the infinitesimal Lie point transformations. We have assumed that Eqs. (6)–(9) are invariant under the transformations given in Eq. (11). The corresponding infinitesimal generator is

$$X=\xi_1\frac{\partial}{\partial x}+\xi_2\frac{\partial}{\partial y}+\eta_1\frac{\partial}{\partial u}+\eta_2\frac{\partial}{\partial v}+\eta_3\frac{\partial}{\partial p}+\eta_4\frac{\partial}{\partial \sigma}, \tag{12}$$

where ξ_1, ξ_2, η_1, η_2, η_3 and η_4 are the infinitesimals corresponding to x, y, u, v, p and σ, respectively. Since our equations are at most of order two, therefore, we need second order prolongation of the generator in Eq. (12) and then apply the invariance condition to get the following infinitesimals [17]–[19]:

$$\xi_1=b,\quad \xi_2=c,$$
$$\eta_1=0,\quad \eta_2=0,\quad \eta_3=d,\quad \eta_4=e. \tag{13}$$

Therefore, the equations admit a four parameter Lie group of transformations. Parameters b, c, d and e correspond to translations in the x, y, p and σ coordinates, respectively. By considering the translations in x, y directions and choosing $d, e=0$ and solving the corresponding characteristic equation the similarity variables and functions are given as

$$\xi=y-mx,\quad u=f(\xi),\quad v=g(\xi),\quad p=h(\xi),$$
$$\sigma=N(\xi), \tag{14}$$

where $m=c/b$ is an arbitrary parameter. In view of variables and functions in Eq. (14), Eqs. (6)–(9) become

$$-mf'+g'=0, \tag{15}$$

$$(-mff' + gf') = (\epsilon_1 + \epsilon_2)(1 + m^2) f'' + \epsilon_3 N' + \epsilon_4 mh', \tag{16}$$

$$(-mfg' + gg') = (\epsilon_1 + \epsilon_2)(1 + m^2) g'' + \epsilon_3 mN' - \epsilon_4 h', \tag{17}$$

$$(-mfN' + gN') = \epsilon_5 (1 + m^2) N'' - \epsilon_6 N - \epsilon_7 (mg' + f'). \tag{18}$$

Integration of Eq. (15) yields

$$g = mf + C_1. \tag{19}$$

Eliminating $h(\xi)$ from Eqs. (16) and (17) and making use of Eq. (18) we get

$$(1 + m^2) C_1 f' = (\epsilon_1 + \epsilon_2)(1 + m^2)^2 f'' + \epsilon_3 (1 + m^2) N'. \tag{20}$$

From Eqs. (18) and (19) one can write

$$C_1 N' = \epsilon_5 (1 + m^2) N'' - \epsilon_6 N - \epsilon_7 (1 + m^2) f'. \tag{21}$$

Now integrating Eq. (17) and then using Eqs. (19) and (20) we obtain

$$h = \frac{C_2}{\epsilon_4} \tag{22}$$

in which C_1 and C_2 are any arbitrary constants. Eliminating $f(\xi)$ between Eqs. (20) and (21) we have

$$N^{iv} - AN''' + BN'' + CN' = 0, \tag{23}$$

where

$$A = \frac{C_1(\epsilon_5 + \epsilon_1 + \epsilon_2)}{\epsilon_5 (1 + m^2)(\epsilon_1 + \epsilon_2)},$$
$$B = \frac{\left[C_1^2 - (1 + m^2)\{\epsilon_6(\epsilon_1 + \epsilon_2) + \epsilon_3 \epsilon_7\}\right]}{\epsilon_5 (1 + m^2)^2 (\epsilon_1 + \epsilon_2)}, \tag{24}$$
$$C = \frac{C_1 \epsilon_6}{\epsilon_5 (1 + m^2)^2 (\epsilon_1 + \epsilon_2)}.$$

The solution of Eq. (23) is given by

$$N(\xi) = C_3 e^{\alpha_1 \xi} + C_4 e^{\alpha_2 \xi} + C_5 e^{\alpha_3 \xi} + C_6, \tag{25}$$

where C_3, C_4, C_5 and C_6 are any arbitrary constants and α_i $(i = 1, 2, 3)$ are the roots of the following equation:

$$\alpha^3 - A\alpha^2 + B\alpha + C = 0. \tag{26}$$

From Eqs. (21) and (25) the expression for $f(\xi)$ is

$$f(\xi) = \beta_1 e^{\alpha_1 \xi} + \beta_2 e^{\alpha_2 \xi} + \beta_3 e^{\alpha_3 \xi} - \frac{\epsilon_6 C_6 \xi}{\epsilon_7 (1 + m^2)} + C_7, \tag{27}$$

in which C_7 is any arbitrary constant and β_i $(i = 1, 2, 3)$ are given through the following expression:

$$\beta_i = \frac{C_{i+2}\left[\epsilon_5 (1 + m^2)\alpha_i^2 - \epsilon_6 - C_1 \alpha_i\right]}{\epsilon_7 (1 + m^2)\alpha_i}. \tag{28}$$

In the form of the original variable we have

$$u(x,y)=\beta_1 e^{\alpha_1(y-mx)}+\beta_2 e^{\alpha_2(y-mx)}+\beta_3 e^{\alpha_3(y-mx)}-\frac{\epsilon_6 C_6(y-mx)}{\epsilon_7(1+m^2)}+C_7, \tag{29}$$

$$v(x,y)=m\begin{pmatrix} \beta_1 e^{\alpha_1(y-mx)}+\beta_2 e^{\alpha_2(y-mx)} \\ +\beta_3 e^{\alpha_3(y-mx)}-\dfrac{\epsilon_6 C_6(y-mx)}{\epsilon_7(1+m^2)}+C_7 \end{pmatrix}+C_1, \tag{30}$$

$$\sigma(x,y)=C_3 e^{\alpha_1(y-mx)}+C_4 e^{\alpha_2(y-mx)}+C_5 e^{\alpha_3(y-mx)}+C_6, \tag{31}$$

$$p(x,y)=\frac{C_2}{\epsilon_4}. \tag{32}$$

Equations (29)–(32) give the solution of Eqs. (6)–(9) that involve seven unknown constants. For determining the values of these constants we consider a problem that occurs in geology. Consider a magmatic micropolar fluid and a plate over it. The plate occupies the position $y=0$. The positive y goes deep into the fluid beneath the plate. The relevant boundary conditions are of the form:

$$\begin{aligned} &u(x,0)=U_0, \quad u(x,\infty)=0, \quad \frac{\partial u}{\partial x}(0,y)=0, \quad v(x,0)=-V_0, \\ &\sigma(x,0)=0, \quad \sigma(x,\infty)=0, \quad p(x,\infty)=p_0, \end{aligned} \tag{33}$$

where U_0 is the velocity of the plate, V_0 is the magmatic fluid penetrating into the plate and p_0 is the pressure deep in the magmatic fluid. The expressions (29) to (32) after using conditions (33) give

$$u(x,y)=\frac{-U_0}{\gamma_2-\gamma_1}(\gamma_1 e^{-\alpha y}-\gamma_2 e^{-\beta y}), \tag{34}$$

$$v(x,y)=m\left(\frac{-U_0}{\gamma_2-\gamma_1}(\gamma_1 e^{-\alpha y}-\gamma_2 e^{-\beta y})\right)-mU_0-V_0, \tag{35}$$

$$\sigma(x,y)=\frac{U_0}{\gamma_2-\gamma_1}(e^{-\alpha y}-e^{-\beta y}), \tag{36}$$

$$p(x,y)=p_0, \tag{37}$$

where

$$\gamma_1=\frac{\epsilon_5(1+m^2)\alpha^2-\epsilon_6+C_1\alpha}{-\epsilon_7(1+m^2)\alpha}, \tag{38}$$

$$\gamma_2=\frac{\epsilon_5(1+m^2)\beta^2-\epsilon_6+C_1\beta}{-\epsilon_7(1+m^2)\beta} \tag{39}$$

and $-\alpha$ and $-\beta$ are the negative roots of Eq. (26).

4 Discussion

This Section deals with the interpretation of the translational parameter m and the magmatic fluid penetrating parameter V_0 on the x and y components of the velocity and on the angular velocity σ. Figures 1, 2 and 4–7 have been prepared for the velocity components whereas Fig. 3 holds for the angular velocity. It is found from Figs. 1 and 2 that the velocity components u and

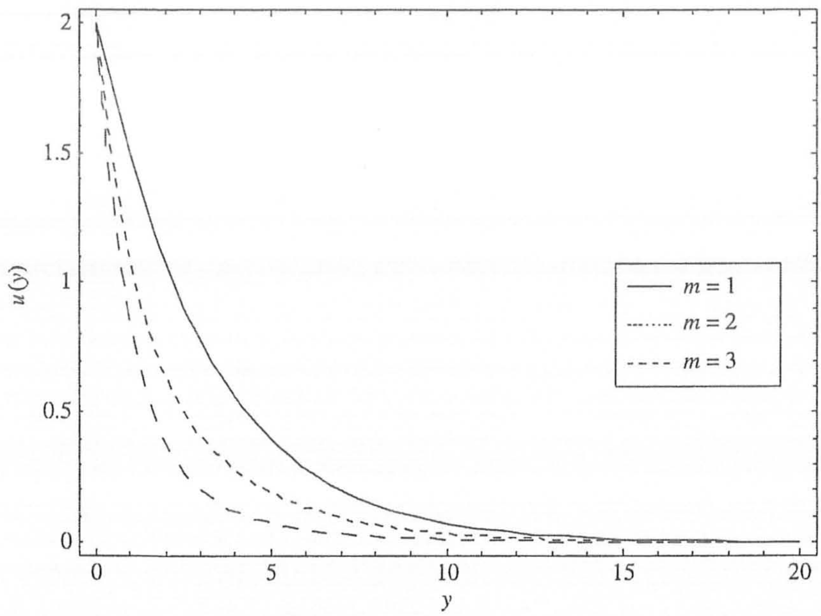

Fig. 1. Variation of the dimensionless velocity distribution along the x-axis with the value of m ($\epsilon_1 \rightarrow \epsilon_4, \epsilon_6, \epsilon_7 = 0.5$; $\epsilon_5 = 2$; $U_0 = V_0 = 2$)

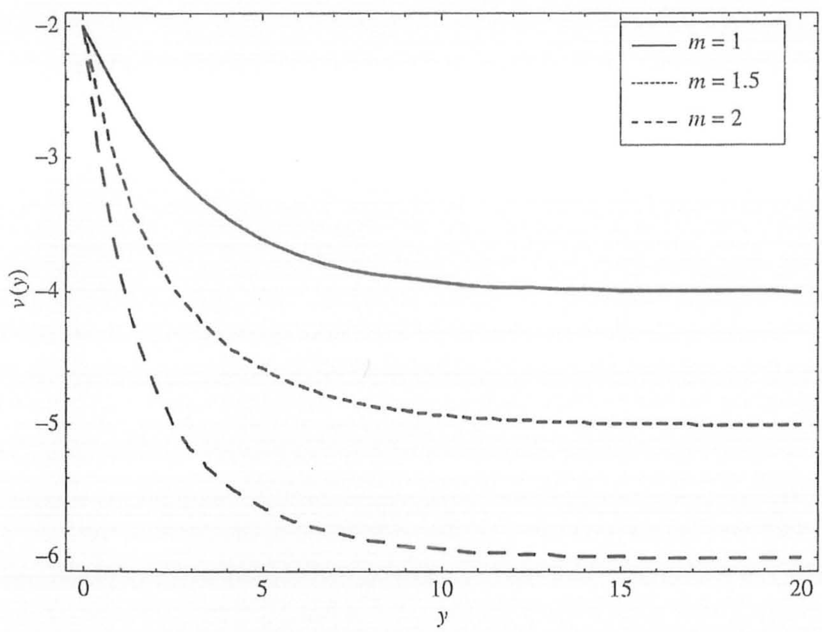

Fig. 2. Variation of the dimensionless velocity distribution along the y-axis with the value of m ($\epsilon_1 \rightarrow \epsilon_4, \epsilon_6, \epsilon_7 = 0.5$; $\epsilon_5 = 2$; $U_0 = V_0 = 2$)

v are decreasing functions of m. It is also evident from Fig. 3 that the behavior of m on the angular velocity is opposite to that of u and v.

From Eqs. (34) to (36) we note that the magmatic fluid penetrating parameter only enters into the y-component of velocity. The x-component of velocity u and σ are independent of

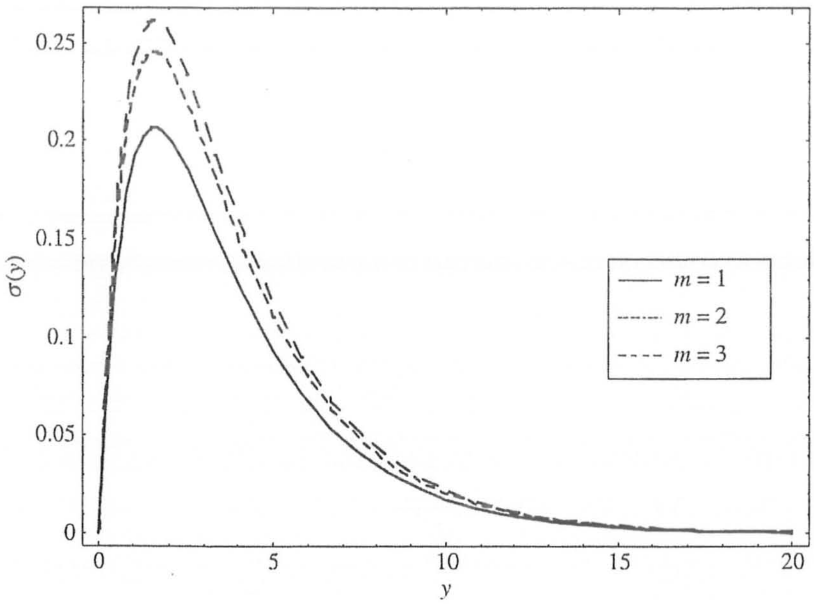

Fig. 3. Variation of the dimensionless angular velocity with the value of m ($\epsilon_1 \rightarrow \epsilon_4, \epsilon_6, \epsilon_7 = 0.5$; $\epsilon_5 = 2$; $U_0 = V_0 = 2$)

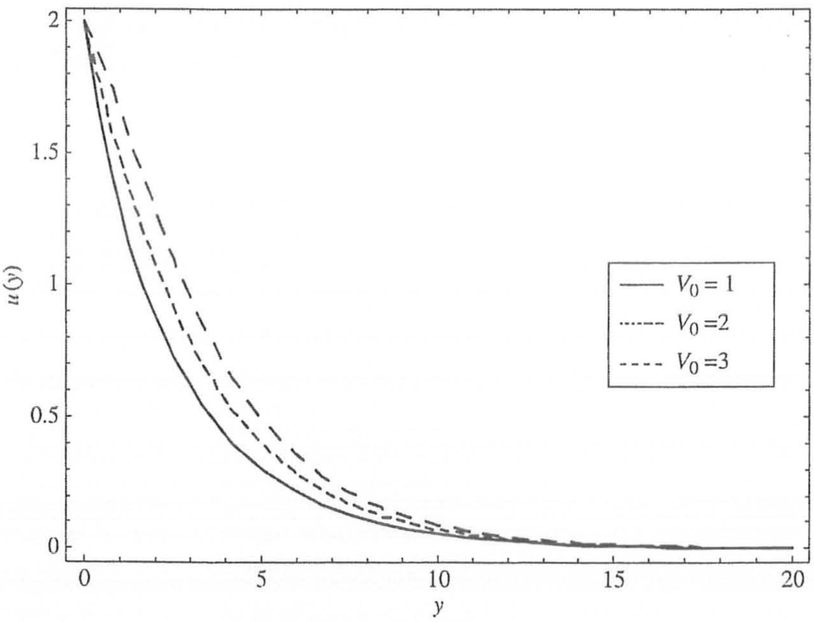

Fig. 4. Variation of the dimensionless velocity distribution along the x-axis with the value of V_0 ($\epsilon_1 \rightarrow \epsilon_4, \epsilon_6, \epsilon_7 = 0.5$; $\epsilon_5 = 2$; $U_0 = 2$)

V_0. It is found from Figs. 4 and 5 that the x-component of velocity increases by increasing the value of V_0 either $V_0 > 0$ or $V_0 < 0$. It is clear from Figs. 6 and 7 that the behavior of V_0 on the y-component of the velocity is opposite to that of the x-component of the velocity distribution.

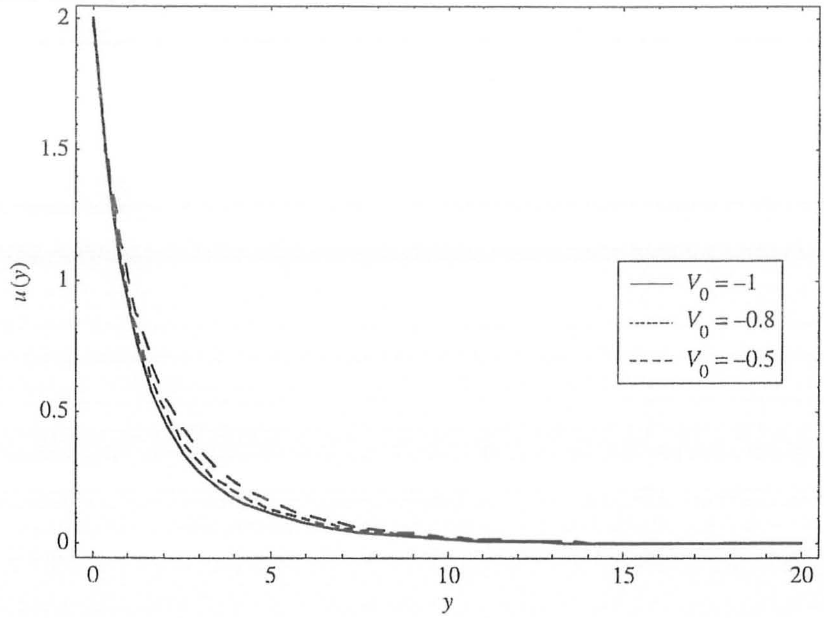

Fig. 5. Variation of the dimensionless velocity distribution along the x-axis with the value of V_0 ($\epsilon_1 \to \epsilon_4, \epsilon_6, \epsilon_7 = 0.5$; $\epsilon_5 = 2$; $U_0 = 2$)

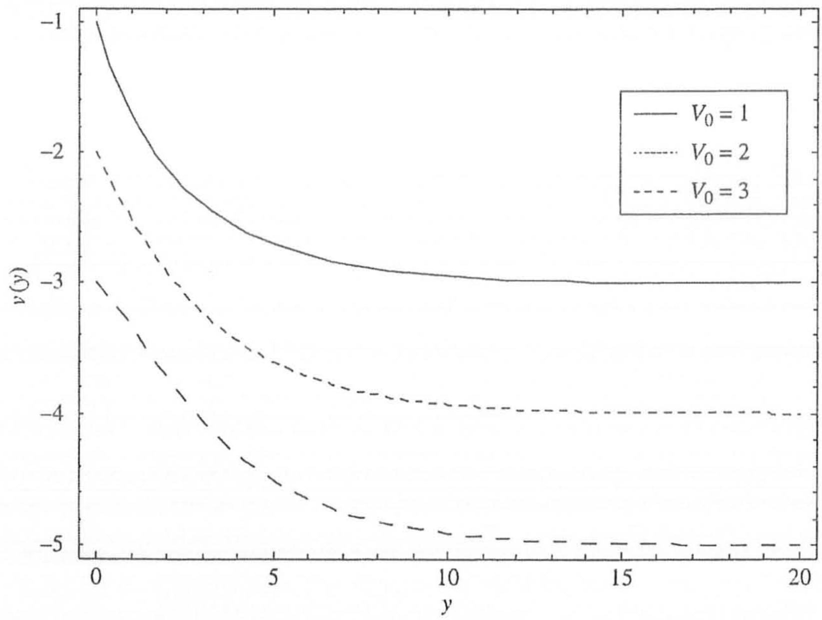

Fig. 6. Variation of the dimensionless velocity distribution along the y-axis with the value of V_0 ($\epsilon_1 \to \epsilon_4, \epsilon_6, \epsilon_7 = 0.5$; $\epsilon_5 = 2$, $U_0 = 2$)

5 Concluding remarks

In this communication we presented the analytical solution for the steady two dimensional equations of a micropolar fluid. Lie group analysis has been employed and the solutions

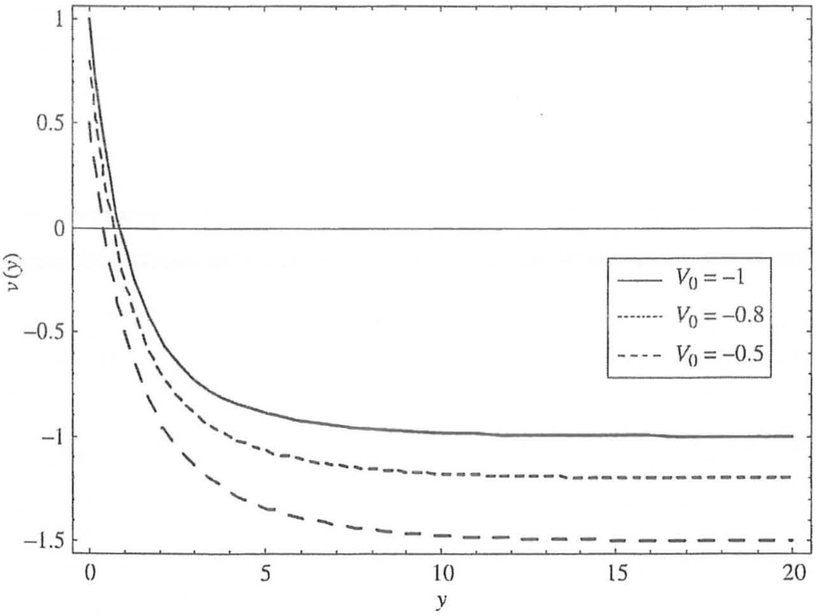

Fig. 7. Variation of the dimensionless velocity distribution along the y-axis with the value of V_0 ($\epsilon_1 \to \epsilon_4, \epsilon_6, \epsilon_7 = 0.5$; $\epsilon_5 = 2$, $U_0 = 2$)

corresponding to the translational symmetry are developed. The results are also sketched graphically and show the similar behaviour of the numerical solution [14].

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Analytical solution for the steady flow of the third grade fluid in a porous half space

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tract

n this study, we model the flow of a third grade fluid in a porous half space. Based on modified Darcy's law, the flow r a suddenly moved flat plate is discussed analytically by using homotopy analysis method (HAM). The influence of ous parameters of interest on the velocity profile is seen.

words: Third grade fluid; Porous space; Analytical solution; Modified Darcy's law; HAM solution

ntroduction

:t is well known that the governing equations for the non-Newtonian fluids are more non-linear and of her order [1,2] than the Navier–Stokes equations. Thus, to find the analytic solutions of such equations ot an easy task. With this fact in view several authors [3–22] are now engaged in finding the analytic solu- ı under imposed restrictions. The simplest subclass of non-Newtonian fluids for which one can reasonably ıe to obtain an analytic solution is the second grade. The second grade and Oldroyd-B fluids for steady directional flow do not exhibit the rheological characteristics. The third grade fluid models even for steady / exhibits such characteristics. For this reason the model in the present study is the third grade fluid one. reover, the viscoelastic flows in porous space are quite prevalent in many engineering fields such as anced oil recovery, paper and textile coating and composite manufacturing processes. Also the modeling ıolymeric flow in porous space has essential focus on the numerical simulation of viscoelastic flows in a ːific pore geometry model, for example, capillary tubes, undulating tubes, packs of spheres or cylinders. ː to these motivations, the layout of the paper is as follows:

n Section 2, the basic equations are presented. In Section 3 we give the problem formulation. The analytic ıtion by HAM is developed in Section 4. The convergence of the obtained series solution is analyzed in

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ion 5. The graphical results are presented and the effects of the parameters are discussed in Section 6 fol-
ed by concluding remarks in Section 7.

asic equations

he equations which govern the flow of an incompressible fluid in a porous space are

$$\operatorname{div}\mathbf{V} = 0, \tag{1}$$

$$\rho(\mathbf{V}.\nabla)\mathbf{V} = -\nabla p + \operatorname{div}\mathbf{S} + \mathbf{r}. \tag{2}$$

bove equations, $\mathbf{V}$ is the velocity, ρ the fluid density, p the hydrostatic pressure, $\mathbf{S}$ the extra stress tensor
$\mathbf{r}$ the Darcy resistance for a third grade fluid in a porous space.
he constitutive equation for $\mathbf{S}$ in a third grade fluid is [20]

$$\mathbf{S} = \mu\mathbf{A}_1 + \alpha_1\mathbf{A}_2 + \alpha_2\mathbf{A}_1^2 + \beta_1\mathbf{A}_3 + \beta_2(\mathbf{A}_2\mathbf{A}_1 + \mathbf{A}_1\mathbf{A}_2) + \beta_3(\operatorname{tr}\mathbf{A}_1^2)\mathbf{A}_1. \tag{3}$$

e μ is the dynamic viscosity, α_i $(i = 1,2)$, and β_i $(i = 1-3)$ are the material constants corresponding to
nd and third order approximations respectively. The kinematical tensors $\mathbf{A}_n$ are defined as

$$\mathbf{A}_1 = \nabla\mathbf{V} + (\nabla\mathbf{V})^{\mathrm{T}}, \tag{4}$$

$$\mathbf{A}_n = \left(\frac{\partial}{\partial t} + (\mathbf{V}.\nabla)\right)\mathbf{A}_{n-1} + \mathbf{A}_{n-1}(\nabla\mathbf{V}) + (\nabla\mathbf{V})^{\mathrm{T}}\mathbf{A}_{n-1}, \quad n = 2,3\ldots \tag{5}$$

e that Eq. (3) is compatible with thermodynamics if [23]

$$\begin{aligned} &\mu \geqslant 0; \quad \alpha_1 \geqslant 0, \quad |\alpha_1 + \alpha_2| \leqslant \sqrt{24\mu\beta_3}, \\ &\beta_1 = \beta_2 = 0, \quad \beta_3 \geqslant 0, \end{aligned} \tag{6}$$

hich case Eq. (3) becomes

$$\mathbf{S} = [\mu + \beta_3(\operatorname{tr}\mathbf{A}_1^2)]\mathbf{A}_1 + \alpha_1\mathbf{A}_2 + \alpha_2\mathbf{A}_1^2. \tag{7}$$

roblem formulation

onsider a Cartesian coordinate system $OXYZ$ with y-axis in the upward direction. The incompressible
d grade fluid flows through a porous space $y > 0$ and in contact with an infinite flat plate at $y = 0$. Initially
fluid and plate are at rest. At $t = 0^+$, the plate is impulsively brought to the constant velocity U_0.
n an unbounded porous medium the Darcy's law holds for viscous fluid flows having low speed. This law
es the pressure drop induced by the frictional drag and velocity and ignores the boundary effects on the
(i.e. invalid where there are boundaries of the porous medium). According to this law the induced pres-
drop is directly proportional to the velocity. For the porous medium with boundaries, Brinkman pro-
d an equation describing the locally averaged flow. Although the equation proposed by Brinkman
s only for steady viscous flows but there are several modified Darcy's laws available in the literature
iscous flows in a porous medium. Much attention has not been given to mathematical macroscopic filtra-
models concerning viscoelastic flows in a porous medium. On the basis of Oldroyd constitutive equation,
ollowing law for describing both relaxation and retardation phenomenon in an unbounded porous med-
has been suggested [3]:

$$\left(1 + \lambda\frac{\partial}{\partial t}\right)\nabla p = -\frac{\mu\phi}{k}\left(1 + \lambda_r\frac{\partial}{\partial t}\right)\mathbf{V}, \tag{8}$$

e k is the permeability, λ and λ_r are the constant relaxation and retardation times respectively and ϕ is the
sity of the porous medium. Note that for $\lambda = \lambda_r = 0$, Eq. (8) reduces to well-known Darcy's law of viscous
s.

y analogy with Maxwell's constitutive relationship the following phenomenological model has been avail-
in the literature [24]:

$$\left(1+\lambda\frac{\partial}{\partial t}\right)\nabla p=-\frac{\mu\phi}{k}\mathbf{V}. \tag{9}$$

unidirectional flow of second grade fluid the constitutive equation can be obtained from that of an Old-d-B fluid by taking $\lambda = 0$ [3]. Thus in a porous medium, the relationship between ∇p and $\mathbf{V}$ for unidirec-al flow of a second grade fluid can be written from Eq. (8) as follows [22]:

$$(\nabla p)_x=-\frac{\mu\phi}{k}\left(1+\lambda_r\frac{\partial}{\partial t}\right)u, \tag{10}$$

re

$$\mu\lambda_r=\alpha_1.$$

ploying the same idea as in Eqs. (8)–(10), we propose the following constitutive relationship between the ssure drop and velocity for unidirectional flow of a third grade fluid:

$$(\nabla p)_x=-\left[\mu+\alpha_1\frac{\partial}{\partial t}+2\beta_3\left(\frac{\partial u}{\partial y}\right)^2\right]\frac{\phi u}{k}. \tag{11}$$

pressure gradient in above equation can also be interpreted as a measure of the resistance to flow in the of the porous medium and r_x is a measure of the flow resistance offered by the solid matrix. Thus r_x can nferred from Eq. (11) to satisfy the following equation:

$$r_x=-\left[\mu+\alpha_1\frac{\partial}{\partial t}+2\beta_3\left(\frac{\partial u}{\partial y}\right)^2\right]\frac{\phi u}{k}. \tag{12}$$

stituting Eqs. (4), (5), (7) and (12) in Eq. (2) and then neglecting ∇p in the x-direction we have the follow-steady state problem

$$0=\frac{\mu}{\rho}\frac{d^2u}{dy^2}+\frac{6\beta_3}{\rho}\left(\frac{du}{dy}\right)^2\frac{d^2u}{dy^2}-\left[\mu+2\beta_3\left(\frac{du}{dy}\right)^2\right]\frac{\phi u}{\rho k}. \tag{13}$$

relevant boundary and initial conditions are

$$u(0)=V_0,\quad u(y)\to 0\quad \text{as } y\to\infty. \tag{14}$$

(13) can also be written as

$$0=\mu^*\frac{d^2u}{dy^2}+b_1^*\left(\frac{du}{dy}\right)^2\frac{d^2u}{dy^2}-b_2^*\left(\frac{du}{dy}\right)^2u-\phi_1 u, \tag{15}$$

re

$$\begin{aligned}\mu^*&=\frac{\mu}{\rho+\alpha_1\phi/k},\\ b_1^*&=\frac{6\beta_3}{\rho+\alpha_1\phi/k},\\ b_2^*&=\frac{2\beta_3\phi/k}{\rho+\alpha_1\phi/k},\\ \phi_1&=\frac{\mu\phi/k}{\rho+\alpha_1\phi/k}.\end{aligned} \tag{16}$$

oducing the following non-dimensional variables:

$$z=\frac{V_0}{\nu}y,\quad f=\frac{u}{V_0}, \tag{17}$$

roblem becomes

$$\frac{d^2f}{dz^2}+b_1\left(\frac{df}{dz}\right)^2\frac{d^2f}{dz^2}-b_2f\left(\frac{df}{dz}\right)^2-cf=0, \tag{18}$$

$$f(0)=1,\quad f(z)\to 0\quad \text{as } z\to\infty, \tag{19}$$

e

$$b_1=\frac{b_1^*V_0^4}{\mu^*\nu^2},\quad b_2=\frac{b_2^*V_0^2}{\mu^*},\quad c=\frac{\phi_1\nu^2}{\mu^*V_0^2}. \tag{20}$$

alytic solution

order to obtain the HAM solution, we choose

$$f_0(z)=e^{-z}, \tag{21}$$

$$\mathscr{L}(f)=f''+f', \tag{22}$$

itial approximation of f and auxiliary linear operator $\mathscr{L}$ satisfying

$$\mathscr{L}(C_1+C_2e^{-z})=0, \tag{23}$$

e C_1 and C_2 are arbitrary constants. If $p\in[0,1]$ is an embedding parameter and $\hbar$ is an auxiliary nonzero meter then

$$(1-p)\mathscr{L}[\theta(z,p)-f_0(z)]=p\hbar\mathscr{N}[\theta(z,p)], \tag{24}$$

$$\theta(0,p)=1,\quad \theta(\infty,p)=0, \tag{25}$$

e

$$\mathscr{N}[\theta(z,p)]=\frac{\partial^2\theta(z,p)}{\partial z^2}+b_1\left(\frac{\partial\theta(z,p)}{\partial z}\right)^2\frac{\partial^2\theta(z,p)}{\partial z^2}-b_2\theta(z,p)\left(\frac{\partial\theta(z,p)}{\partial z}\right)^2-c\theta(z,p). \tag{26}$$

$p=0$ and $p=1$, we have

$$\theta(z,0)=f_0(z),\quad \theta(z,1)=f(z). \tag{27}$$

increases from 0 to 1, $\theta(z,p)$ varies from $f_0(z)$ to $f(z)$. By Taylor's theorem and Eq. (27) one obtains

$$\theta(z,p)=f_0(z)+\sum_{m=1}^{\infty}f_m(z)p^m, \tag{28}$$

e

$$f_m(z)=\frac{1}{m!}\left.\frac{\partial^m\theta(z,p)}{\partial p^m}\right|_{p=0}. \tag{29}$$

ly the convergence of the series (28) depends upon $\hbar$. Assume that $\hbar$ is selected such that the series (29) is ergent at $p=1$, then due to Eq. (26) we have

$$f(z)=f_0(z)+\sum_{m=1}^{\infty}f_m(z). \tag{30}$$

he mth order deformation problem, we differentiate Eq. (24) m times with respect to p, divide by $m!$ and set $p=0$. The resulting deformation problem at the mth order is

$$\mathscr{L}[f_m(z)-\chi_mf_{m-1}(z)]=\hbar\mathscr{R}_m(z), \tag{31}$$

$$f_m(0)=f_m(\infty)=0, \tag{32}$$

ere

$$\mathscr{R}_m(z) = \left[\frac{d^2 f_{m-1}}{dz^2} - c f_{m-1}\right] + \sum_{k=0}^{m-1} \frac{d f_{m-1-k}}{dz} \sum_{l=0}^{k} \left[\frac{d f_{k-l}}{dz}\left(b_1 \frac{d^2 f_l}{dz^2} - b_2 f_l\right)\right], \tag{33}$$

$$\chi_m = \begin{cases} 0 & m \leqslant 1, \\ 1 & m > 1. \end{cases} \tag{34}$$

e solution of the above problem upto first few order of approximations may be obtained using the symbolic
nputation software MATHEMATICA. The solution of the problem can be expressed as an infinite series of
form

$$f_m(z) = \sum_{n=0}^{2m+1} \sum_{q=0}^{2m+1-2n} a^q_{m,n} z^q e^{-nz}, \quad m \geqslant 0. \tag{35}$$

oking Eq. (35) into Eq. (31) we get the following recurrence formulae for the coefficient $a^q_{m,n}$ of $f_m(z)$ when
$\geqslant 1,\ 0 \leqslant n \leqslant 2m+1$

$$a^0_{m,1} = \chi_m \chi_{2m-1} a^0_{m-1,1} - \sum_{n=2}^{2m+1} \sum_{q=0}^{2m+1-2n} \Gamma^q_{m,n} \mu^q_{n,0}, \tag{36}$$

$$a^k_{m,0} = \chi_m \chi_{2m+1-k} a^k_{m-1,0}, \quad 0 \leqslant k \leqslant 2m+1, \tag{37}$$

$$a^k_{m,1} = \chi_m \chi_{2m-1-k} a^k_{m-1,1} - \sum_{q=k-1}^{2m} \Gamma^q_{m,1} \mu^q_{1,k}, \quad 1 \leqslant k \leqslant 2m-1, \tag{38}$$

$$a^k_{m,n} = \chi_m \chi_{2m+1-2n-k} a^k_{m-1,n} + \sum_{q=k}^{2m+1-2n} \Gamma^q_{m,n} \mu^q_{n,k}, \quad 2 \leqslant n \leqslant 2m+1,\ 0 \leqslant k \leqslant 2m+1-2n, \tag{39}$$

ere

$$\Gamma^q_{m,n} = \hbar \sum_{n=0}^{2m+1} \sum_{q=0}^{2m+1-2n} \left[\chi_{2m+1-2n-q}\left(a2^q_{m-1,n} - c a^q_{m-1,n} + b_1 \delta 3^q_{m,n} - b_2 \delta 4^q_{m,n}\right)\right], \tag{40}$$

$$\mu^q_{1,k} = \frac{q!}{k!}, \quad 0 \leqslant k \leqslant 2q+1,\ q \geqslant 0, \tag{41}$$

$$\mu^q_{n,k} = \sum_{p=0}^{q-k} \frac{q!}{k! n^{p+1} (n-1)^{q-p+1}}, \quad 0 \leqslant k \leqslant 2q,\ q \geqslant 0,\ n \geqslant 2. \tag{42}$$

e coefficients $\delta 3^q_{m,n}$, $\delta 4^q_{m,n}$, are

$$\delta 3^q_{m,n} = \sum_{k=0}^{m-1} \sum_{l=0}^{k} \sum_{r=\max\{0,n-2k+2m-1\}}^{r=\min\{n,2k+2\}} \sum_{s=\max\{0,q-2m+2n-2r+1\}}^{s=\min\{q,2k+2-2r\}} \Pi 1^s_{k,r} a1^{q-s}_{m-1-k,n-r}, \tag{43}$$

$$\delta 4^q_{m,n} = \sum_{k=0}^{m-1} \sum_{l=0}^{k} \sum_{r=\max\{0,n-2k+2m-1\}}^{r=\min\{n,2k+2\}} \sum_{s=\max\{0,q-2m+2n-2r+1\}}^{s=\min\{q,2k+2-2r\}} \Pi 2^s_{k,r} a1^{q-s}_{m-1-k,n-r}, \tag{44}$$

ere

$$\Pi 1^s_{k,r} = \sum_{j=\max\{0,r-2k+2l-1\}}^{j=\min\{r,2l+1\}} \sum_{i=\max\{0,s-2k+2l+2r-2j-1\}}^{i=\min\{s,2l+1-2j\}} a2^i_{l,j} a1^{s-i}_{k-l,r-j}, \tag{45}$$

$$\Pi 2^s_{k,r} = \sum_{j=\max\{0,r-2k+2l-1\}}^{j=\min\{r,2l+1\}} \sum_{i=\max\{0,s-2k+2l+2r-2j-1\}}^{i=\min\{s,2l+1-2j\}} a1^i_{l,j} a1^{s-i}_{k-l,r-j}, \tag{46}$$

$$a1^q_{m,n} = (q+1) a^{q+1}_{m,n} - n a^q_{m,n}, \tag{47}$$

$$a2^q_{m,n} = (q+1) a1^{q+1}_{m,n} - n a1^q_{m,n}. \tag{48}$$

izing the above recurrence formulae, all coefficients $a^k_{m,n}$ can be computed using only the first two

$$a^0_{0,0} = 0, \quad a^0_{0,1} = 1, \tag{49}$$

n by the initial guess approximation in Eq. (21).The corresponding Mth order approximation of Eqs. (18) (19) is

$$\sum_{m=0}^{M} f_m(z) = \sum_{n=1}^{2M+1} e^{-nz} \left(\sum_{m=n-1}^{M} \sum_{k=0}^{2m+1-2n} a^k_{m,n} z^k \right) \tag{50}$$

the analytic solution of the problem is

$$f(z) = \sum_{m=0}^{\infty} f_m(z) = \lim_{M\to\infty} \left[\sum_{n=1}^{2M+1} e^{-nz} \left(\sum_{m=n-1}^{M} \sum_{k=0}^{2m+1-2n} a^k_{m,n} z^k \right) \right]. \tag{51}$$

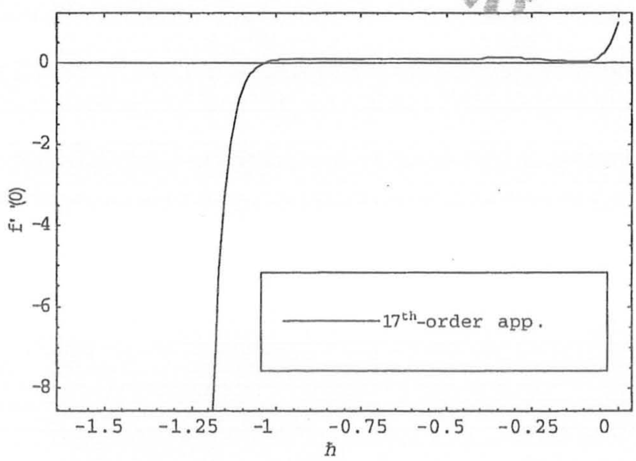

Fig. 1. $\hbar$-curve for the seventeenth order of the approximation for the velocity field f for $b_1 = 0.5$, $b_2 = 0.1$, $c = 0.8$.

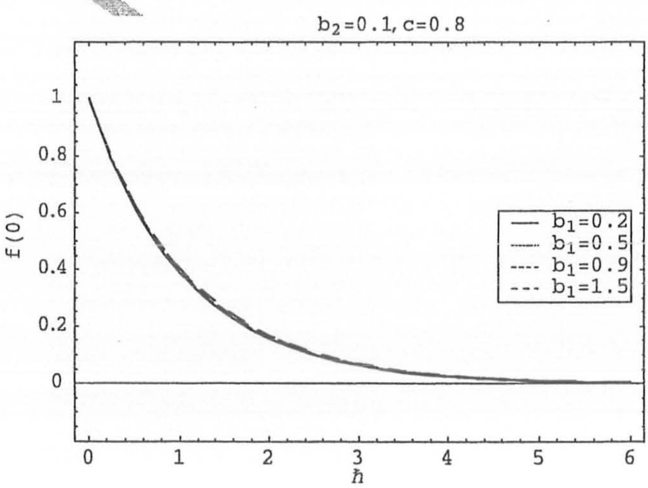

Fig. 2. Variation of the velocity distribution for the various values of b_1.

Convergence of the analytic solution

Clearly Eq. (51) contains the auxiliary parameter $\hbar$. As pointed out by Liao [4], the convergence region and ce of approximation given by HAM are strongly dependent upon this auxiliary parameter. For this purpose, e $\hbar$curve is plotted for f up to the seventeen order approximation. It is obvious from Fig. 1 that the range for e admissible value for $\hbar$ is $-1 \leqslant \hbar \leqslant -0.15$. Our calculations depict that the series of the velocity field in Eq. 1) converges in the whole region of z when $\hbar = -0.2$.

Results and discussion

In Fig. 2, the velocity field f is plotted for the different values of the parameter b_1. It is clear from this Fig at with the increase in b_1 the velocity increases. Fig. 3 elucidates the effects of the parameter b_2. It is evident om Fig. 3 that velocity decreases by increasing b_2. Fig. 4 represents the velocity distribution for the various lues of the parameter c. It is clear from Fig. 4 that the velocity also decreases with the increase in c. Figs. 5–7 present the velocity distribution for the large values of b_1, b_2 and c respectively and similar effects has been en as in case of Figs. 1–3.

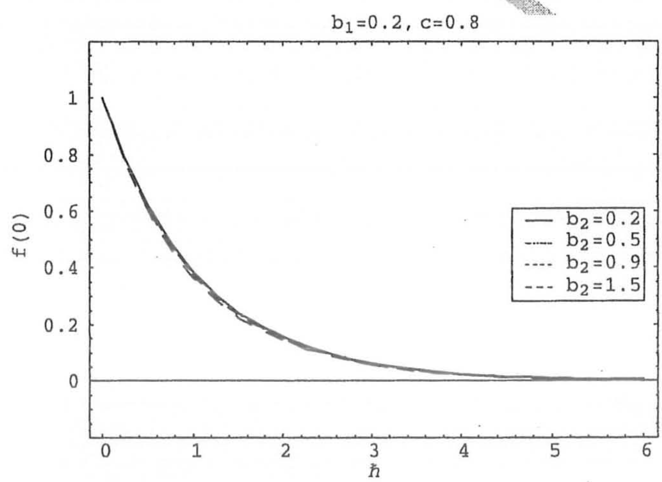

Fig. 3. Variation of the velocity distribution for the various values of b_2.

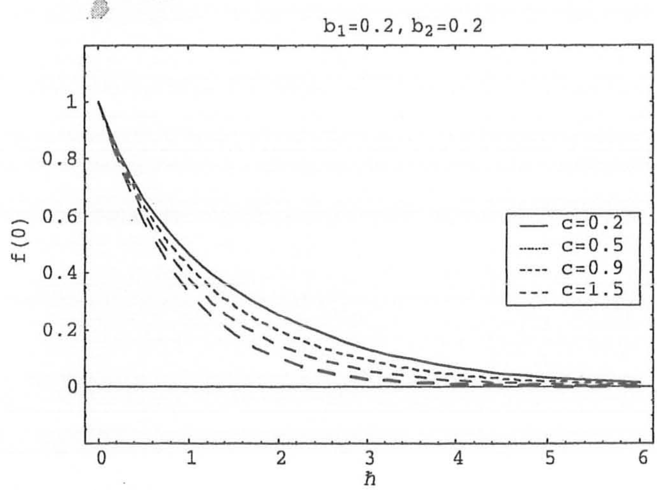

Fig. 4. Variation of the velocity distribution for the various values of c.

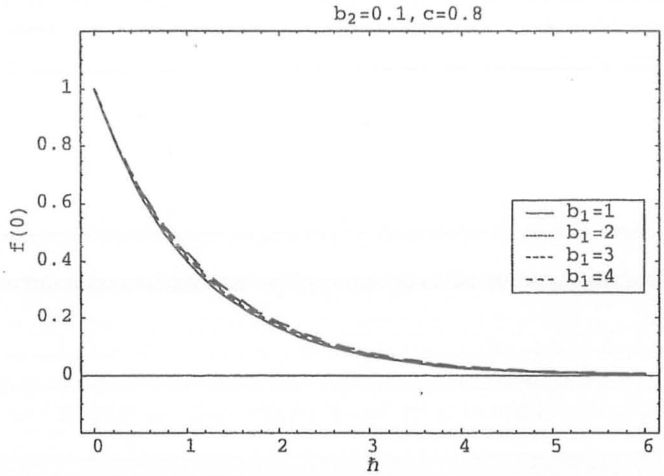

Fig. 5. Variation of the velocity distribution for the various values of b_1.

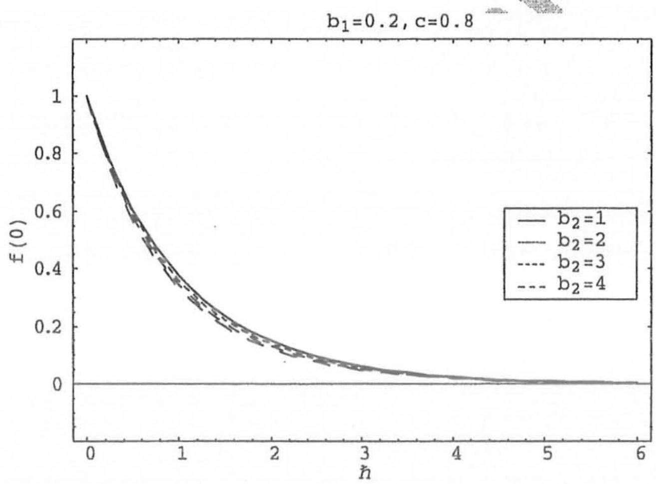

Fig. 6. Variation of the velocity distribution for the various values of b_2.

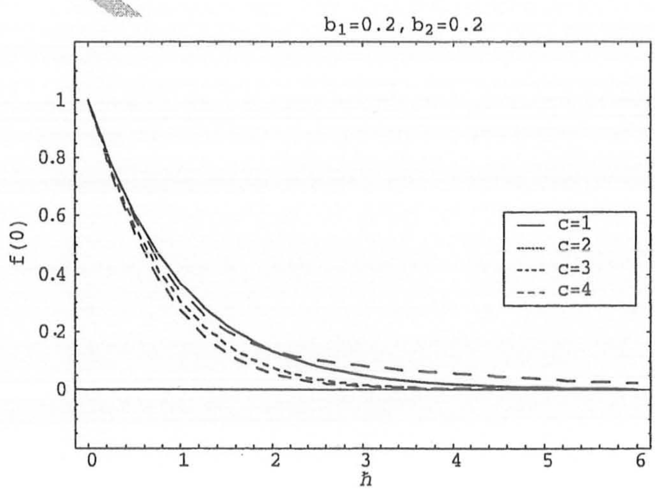

Fig. 7. Variation of the velocity distribution for the various values of c.

Concluding remarks

n the present paper, the steady third grade fluid in a porous space is considered. The governing constitutive tionship for modified Darcy's law in a third grade fluid has been proposed. It is noted that modified rcy's law for unidirectional flow of a third grade fluid yields non-linear expression in terms of velocity ereas it is linear for Newtonian, Oldroyd-B, Maxwell and second grade fluids. The governing non-linear blem that comprised the balance laws of mass and momentum has been solved using homotopy analysis hod (HAM). The significant contributions of the non-Newtonian parameters b_1, b_2 and c on the velocity pointed out.

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Stokes' first problem for a third grade fluid in a porous half space

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Abstract

In this study, we model the flow of a third grade fluid in a porous half space. Based on modified Darcy's law, the flow over a suddenly moved flat plate is discussed numerically. The influence of various parameters of interest on the velocity profile is seen.

Keywords: Third grade fluid; Porous space; Numerical solution; Modified Darcy's law

1. Introduction

Because of its practical applications, the Stokes' problem for the flat plate has been the subject of numerous theoretical studies. Such studies for Navier–Stokes fluid and different types of non-Newtonian fluids include the work of Zierep [1], Soundalgekar [2], Rajagopal and Na [3], Puri [4], Bandelli et al. [5], Tigoiu [6], Fetecau and Zierep [7] and Fetecau and Fetecau [8,9]. More recently, Tan and Masuoka [10,11] discussed the Stokes' first problem for second grade and Oldroyd-B fluid models using modified Darcy's law. They obtained the solution analytically. The second grade and Oldroyd-B fluids for steady unidirectional flow do not exhibit the rheological characteristics. The third grade fluid model even for steady flow exhibits such characteristics. For this reason the model in the present study is the third grade fluid one. Moreover, the governing equations for non-Newtonian fluids [15,16] are highly non-linear and of higher order when compared with that of the Newtonian fluid. The viscoelastic flows in porous space are quite prevalent in many engineering fields such as enhanced oil recovery, paper and textile coating and composite manufacturing processes. Also the modeling of polymeric flow in porous space has essential focus on the numerical simulation of viscoelastic flows in a specific pore geometry model, for example, capillary tubes, indulating tubes, packs of spheres or cylinders. Due to these motivations, the layout of the paper is as follows:

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In Section 2 we present the basic equations. In Section 3 we give the problem formulation. The numerical results and discussion are presented in Section 4 followed by concluding remarks in Section 5.

2. Basic equations

In a porous space, the equations governing the flow of an incompressible third grade fluid are

$$\mathrm{div}\mathbf{V} = 0, \tag{1}$$

$$\rho\left(\frac{\partial}{\partial t} + (\mathbf{V}\cdot\nabla)\right)\mathbf{V} = -\nabla p + \mathrm{div}\mathbf{S} + \mathbf{r}. \tag{2}$$

In above equations, $\mathbf{V}$ is the velocity, ρ the fluid density, t the time, p the hydrostatic pressure, $\mathbf{S}$ the extra stress tensor and $\mathbf{r}$ the Darcy resistance for a third grade fluid in a porous space.

The constitutive equation for $\mathbf{S}$ in a third grade fluid is

$$\mathbf{S} = \mu\mathbf{A}_1 + \alpha_1\mathbf{A}_2 + \alpha_2\mathbf{A}_1^2 + \beta_1\mathbf{A}_3 + \beta_2(\mathbf{A}_2\mathbf{A}_1 + \mathbf{A}_1\mathbf{A}_2) + \beta_3(tr\mathbf{A}_1^2)\mathbf{A}_1. \tag{3}$$

Here μ is the dynamic viscosity, and α_i $(i = 1,2)$, and β_i $(i = 1–3)$ are the material constants corresponding to second and third order approximations, respectively. The kinematical tensors $\mathbf{A}_n$ are defined through the following equations:

$$\mathbf{A}_1 = \nabla\mathbf{V} + (\nabla\mathbf{V})^{\mathrm{T}}, \tag{4}$$

$$\mathbf{A}_n = \left(\frac{\partial}{\partial t} + (\mathbf{V}\cdot\nabla)\right)\mathbf{A}_{n-1} + \mathbf{A}_{n-1}(\nabla\mathbf{V}) + (\nabla\mathbf{V})^{\mathrm{T}}\mathbf{A}_{n-1}, \quad n = 2,3\ldots \tag{5}$$

Note that Eq. (3) is compatible with thermodynamics if and only if [12]

$$\begin{aligned} &\mu \geqslant 0; \quad \alpha_1 \geqslant 0, \ |\alpha_1 + \alpha_2| \leqslant \sqrt{24\mu\beta_3}; \\ &\beta_1 = \beta_2 = 0, \ \beta_3 \geqslant 0 \end{aligned} \tag{6}$$

in which case Eq. (3) becomes

$$\mathbf{S} = [\mu + \beta_3(tr\mathbf{A}_1^2)]\mathbf{A}_1 + \alpha_1\mathbf{A}_2 + \alpha_2\mathbf{A}_1^2. \tag{7}$$

3. Problem formulation

Consider a Cartesian coordinate system $OXYZ$ with y-axis in the upward direction. The incompressible third grade fluid flows through a porous space $y > 0$ and in contact with an infinite flat plate at $y = 0$. Initially both fluid and plate are at rest. At $t = 0^+$, the plate is impulsively brought to the constant velocity U_0. Under the stated assumptions, we may write the velocity in the following form:

$$\mathbf{V} = u(y,t)\hat{i}, \tag{8}$$

where $\hat{i}$ and u are, respectively, the unit vector and velocity in the x-direction. The above equation automatically satisfies the continuity equation. Further Eqs. (4)–(7) give

$$S_{xx} = \alpha_2\left(\frac{\partial u}{\partial y}\right)^2, \tag{9}$$

$$S_{xy} = \mu\frac{\partial u}{\partial y} + \alpha_1\frac{\partial^2 u}{\partial y\partial t} + 2\beta_3\left(\frac{\partial u}{\partial y}\right)^3, \tag{10}$$

$$S_{yy} = 2\alpha_1\left(\frac{\partial u}{\partial y}\right)^2 + \alpha_2\left(\frac{\partial u}{\partial y}\right)^2, \tag{11}$$

$$S_{xz} = S_{zz} = 0, \quad S_{xy} = S_{yx}, \quad S_{yz} = S_{zy}, \quad S_{xz} = S_{zx}. \tag{12}$$

In an unbounded porous medium the Darcy's law holds for viscous fluid flows, having low speed. This law relates the pressure drop induced by the frictional drag and velocity and ignores the boundary effects on the flow (i.e., invalid where there are boundaries of the porous medium). According to this law the induced pressure drop is directly proportional to the velocity. For the porous medium with boundaries, Brinkman proposed an equation describing the locally averaged flow. Although the equation proposed by Brinkman holds only for steady viscous flows but there are several modified Darcy's laws available in the literature for viscous flows in a porous medium. Much attention has not been given to mathematical macroscopic filtration models concerning viscoelastic flows in a porous medium. On the basis of Oldroyd constitutive equation, the following law for describing both relaxation and retardation phenomenon in an unbounded porous medium has been suggested [11]:

$$\left(1+\lambda\frac{\partial}{\partial t}\right)\nabla p = -\frac{\mu\phi}{k}\left(1+\lambda_r\frac{\partial}{\partial t}\right)\mathbf{V}, \tag{13}$$

where k is the permeability, λ and λ_r are the constant relaxation and retardation times, respectively, and ϕ is the porosity of the porous medium. Note that for $\lambda = \lambda_r = 0$, Eq. (13) reduces to well-known Darcy's law of viscous fluids.

By analogy with Maxwell's constitutive relationship the following phenomenological model has been available in the literature [13]:

$$\left(1+\lambda\frac{\partial}{\partial t}\right)\nabla p = -\frac{\mu\phi}{k}\mathbf{V}. \tag{14}$$

For unidirectional flow of second grade fluid the constitutive equation can be obtained from that of an Oldroyd-B fluid by taking $\lambda = 0$ [9,14]. Thus in a porous medium, the relationship between ∇p and $\mathbf{V}$ for unidirectional flow of a second grade fluid can be written from Eq. (13) as follows:

$$\frac{\partial p}{\partial x} = -\frac{\mu\phi}{k}\left(1+\lambda_r\frac{\partial}{\partial t}\right)u, \tag{15}$$

where

$$\mu\lambda_r = \alpha_1.$$

Employing the same idea as in Eqs. (13)–(15), we propose the following constitutive relationship between the x-component of pressure drop and velocity for unidirectional flow of a third grade fluid:

$$\frac{\partial p}{\partial x} = -\left[\mu + \alpha_1\frac{\partial}{\partial t} + 2\beta_3\left(\frac{\partial u}{\partial y}\right)^2\right]\frac{\phi u}{k}. \tag{16}$$

The pressure gradient in above equation can also be interpreted as a measure of the resistance to flow in the bulk of the porous medium and r_x is a measure of the flow resistance offered by the solid matrix in the x-direction. Thus r_x can be inferred from Eq. (16) to satisfy the following equation:

$$r_x = -\left[\mu + \alpha_1\frac{\partial}{\partial t} + 2\beta_3\left(\frac{\partial u}{\partial y}\right)^2\right]\frac{\phi u}{k}. \tag{17}$$

Substituting Eqs. (4), (5), (7), (8) and (17) in Eq. (2) give after neglecting $\partial p/\partial x$ as

$$\rho\frac{\partial u}{\partial t} = \mu\frac{\partial^2 u}{\partial y^2} + \alpha_1\frac{\partial^3 u}{\partial y^2\partial t} + 6\beta_3\left(\frac{\partial u}{\partial y}\right)^2\frac{\partial^2 u}{\partial y^2} - \left[\mu + \alpha_1\frac{\partial}{\partial t} + 2\beta_3\left(\frac{\partial u}{\partial y}\right)^2\right]\frac{\phi u}{k}. \tag{18}$$

The relevant boundary and initial conditions are

$$u(0,t) = U_0,\quad u(y,t)\to 0 \text{ as } y\to\infty;\quad u(y,0) = 0. \tag{19}$$

Introducing the following non-dimensional variables:

$$\eta = \frac{U_0}{\nu}y, \quad \tau = \frac{U_0^2}{\nu}t, \quad f = \frac{u}{U_0}, \tag{20}$$

the problem becomes

$$(1+ca)\frac{\partial f}{\partial \tau} = \frac{\partial^2 f}{\partial \eta^2} + a\frac{\partial^3 f}{\partial \eta^2 \partial \tau} + 6b\left(\frac{\partial f}{\partial \eta}\right)^2 \frac{\partial^2 f}{\partial \eta^2} - c\left[1 + 2b\left(\frac{\partial f}{\partial \eta}\right)^2\right] f, \tag{21}$$

$$f(0,\tau) = 1, \quad f(\eta,\tau) \to 0 \text{ as } \eta \to \infty, \quad f(\eta, 0) = 0, \tag{22}$$

where

$$a = \frac{\alpha_1 U_0^2}{\rho \nu^2}, \quad b = \frac{\beta_3 U_0^4}{\rho \nu^3}, \quad c = \frac{\nu^2 \phi}{k U_0^2}. \tag{23}$$

Note that a, b and c are non-dimensional parameters defined in the above equation. Obviously, a is the non-dimensional second grade parameter, b is the non-dimensional third grade parameter and c is the non-dimensional porosity parameter.

4. Numerical results and discussion

We note that Eq. (21) is a third order partial differential equation. It is perhaps not possible to obtain the exact analytic solution. Due to this, we seek the numerical solution. For obtaining the system of algebraic equations we use the following approximations to the derivatives:

$$\frac{\partial f}{\partial \tau} = \frac{1}{k}(f_{i,j} - f_{i,j-1}), \tag{24}$$

$$\frac{\partial f}{\partial \eta} = \frac{1}{2h}(f_{i+1,j} - f_{i-1,j}), \tag{25}$$

$$\frac{\partial^2 f}{\partial \eta^2} = \frac{1}{h^2}(f_{i+1,j} - 2f_{i,j} + f_{i-1,j}), \tag{26}$$

$$\frac{\partial^3 f}{\partial \eta^2 \partial \tau} = \frac{1}{h^2 k}(f_{i+1,j} - f_{i+1,j-1} - 2f_{i,j} + 2f_{i,j-1} + f_{i-1,j} - f_{i-1,j-1}). \tag{27}$$

Eq. (21) can be written as

$$\left(\frac{1+ca}{k}\right)(f_{i,j} - f_{i,j-1}) - \frac{1}{h^2}(f_{i+1,j} - 2f_{i,j} + f_{i-1,j}) - \frac{a}{h^2 k}(f_{i+1,j} - f_{i+1,j-1} - 2f_{i,j} + 2f_{i,j-1} + f_{i-1,j} - f_{i-1,j-1})$$
$$- \frac{6b}{4h^4}[(f_{i+1,j} + f_{i-1,j})^2 (f_{i+1,j} - 2f_{i,j} + f_{i-1,j})] + cf_{i,j} + \frac{2bc}{4h^2}(f_{i+1,j} - f_{i-1,j})^2 f_{i,j} = 0. \tag{28}$$

The above system of algebraic equations also gives

$$R_i = Af_{i,j} + Bf_{i+1,j} + Cf_{i-1,j} + K_1 f_{i+1,j}^3 + K_2 f_{i+1,j}^2 f_{i,j} + K_3 f_{i+1,j}^2 f_{i-1,j} + K_4 f_{i-1,j}^2 f_{i+1,j} + K_5 f_{i-1,j}^2 f_{i,j}$$
$$+ K_6 f_{i-1,j}^3 + K_7 f_{i+1,j} f_{i-1,j} f_{i,j} + Ff_{i,j-1} + Gf_{i+1,j-1} + Hf_{i-1,j-1}, \tag{29}$$

where

$$A = \left[\left(\frac{1+ca}{k}\right) + \frac{2}{h^2} + \frac{2a}{h^2k} + c\right], \quad B = -\left[\frac{1}{h^2} + \frac{a}{h^2k}\right],$$
$$C = -\left[\frac{1}{h^2} + \frac{a}{h^2k}\right], \quad K_1 = -\frac{3b}{2h^4}, \quad K_2 = \frac{3b}{h^4} + \frac{bc}{2h^2},$$
$$K_3 = -K_1, \quad K_4 = -K_1, \quad K_5 = K_2, \tag{30}$$
$$K_6 = K_1, \quad K_7 = -\frac{6b}{h^4} - \frac{bc}{h^2}, \quad F = -\left(\frac{1+ca}{k}\right) - \frac{2a}{h^2k},$$
$$G = \frac{a}{h^2k}, \quad H = G.$$

Now the initial and boundary conditions can be written in the following form:

$$f_{0,j} = 1, \ f_{M,j} = 0, \ \dot{f}_{i,0} = 0, \quad i = 0,1,2,\ldots,M, \ j = 0,1,2,3\ldots \tag{31}$$

Here M denotes an integer large enough such that Mh approximates infinity. Since our Eq. (21) is of third order while given boundary conditions are two, therefore, we introduce an augmented boundary condition

$$\frac{\partial f(\infty,\tau)}{\partial \eta} = 0 \tag{32}$$

and consequently the problem becomes well-posed. This boundary condition is discretized to give

$$\frac{f_{M+1,j} - f_{M,j}}{h} = 0,$$

i.e.,

$$f_{M+1,j} = f_{M,j}. \tag{33}$$

The system consisting of Eqs. (29)–(33) has been solved numerically by employing the Newton's method. Solutions for the non-Newtonian fluid models are obtained for $\tau = 2\pi$. From the numerical solution f is used to express the non-dimensional velocity profile parallel to x-axis. The main emphasis has been given to the influence of second grade, third grade and porosity parameters on the velocity profile. In order to observe these effects, Figs. 1–3 have been made.

Fig. 1 is prepared just to see the effects of a dimensionless second grade parameter on the dimensionless velocity f. It is to be pointed out that f increases by increasing the value of a. It is also seen that the boundary layer thickness increases. The variation of the third grade parameter b on the dimensionless velocity f is given in Fig. 2. This figure elucidates that variation of b on the velocity is quite opposite to that of a, i.e., the dimensionless velocity f decreases when value of b is increased. Fig. 3 shows that how the velocity varies with respect

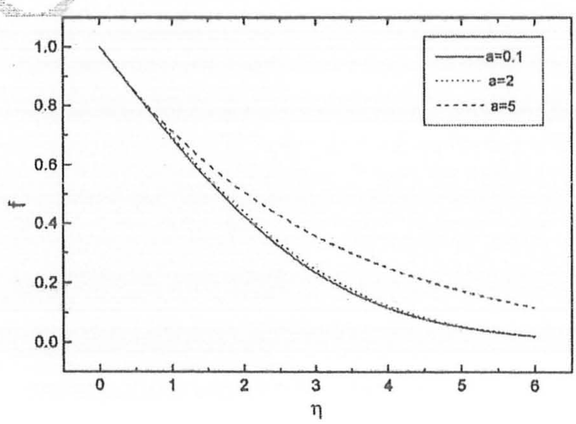

Fig. 1. Influence of second grade parameter on f with $b = c = 0.2$ at $\tau = 2\pi$.

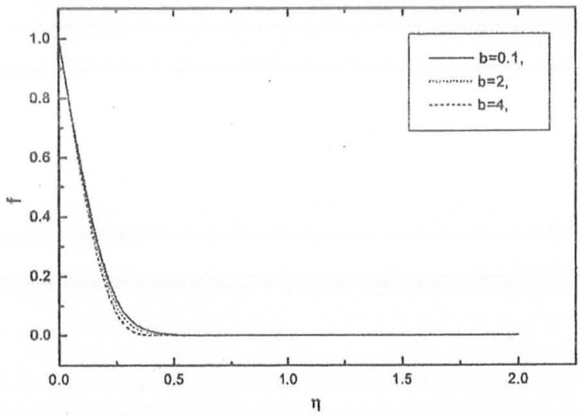

Fig. 2. Influence of third grade parameter on f with $c = 0.2$ and $a = 0.1$, at $\tau = 2\pi$.

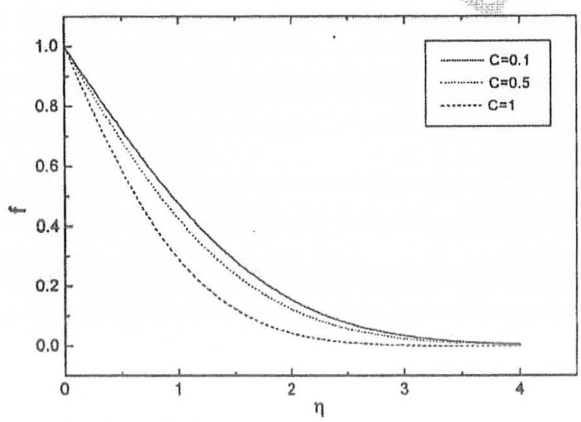

Fig. 3. Influence of porosity parameter on f with $a = 0.1$ and $b = 0.2$ at $\tau = 2\pi$.

to the porosity parameter c. It can be seen here that the dimensionless velocity decreases for large values of c. The boundary layer thickness is also found to decrease.

5. Concluding remarks

In the present work, Stoke's first problem is generalized for the third grade fluid in a porous space. The governing constitutive relationship for modified Darcy's law in a third grade fluid has been proposed. To the best of our knowledge such relationship is not available in the literature. It is noted that modified Darcy's law in unidirectional flow of a third grade fluid yields non-linear expression in terms of velocity where as it is linear for Newtonian, Oldroyd-B, Maxwell and second grade fluids. The governing non-linear problem that comprised the balance laws of mass and momentum has been solved numerically. Results for velocity are presented. It is important to note that variation of second grade parameter on the velocity in porous and non-porous space is quite different. It is further found that for $\tau \geqslant 6\pi$ the non-Newtonian effects become weak and the flow field behaves as if it is a Newtonian fluid.

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Stokes' first problem for the rotating flow of a third grade fluid

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act

s work is concerned with the unsteady rotating flow of the third grade fluid over a suddenly moving plate in its own plane. on-linear problem governing the flow has been solved numerically. The influence of material parameter of third grade fluid otation upon the velocity has been discussed.
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rds: Third grade fluid; Stokes' first problem; Rotating fluid; Numerical solution

ntroduction

e equations governing the flow of a viscous fluid namely the Navier–Stokes equations are non-linear. But there everal complicated fluids which are not well described by these equations. Due to this reason many constitutive tions have been proposed for the non-Newtonian fluids. The equations for non-Newtonian fluids are much com-ted and higher order than the Navier–Stokes equations. Even the various investigators are presently engaged in ng the solutions for such flow problems. Some recent attempts relevant to the flows of non-Newtonain fluids in otating frame are given in references [2–5,12,13].

cently, the study of rotating flows has gained considerable importance due to their applications in cosmical and hysical fluid dynamics. Several workers have been engaged to the rotating viscous flows in various directions. nsive literature is available on the topic dealing the time-dependent and time-independent flows in the rotating :. But there is yet another area of such flows in which no considerable attention has been given. This is the area : rotating flows in non-Newtonian fluid dynamics. Little work seems to have been done in this area. Mention may ade to some recent references [7–11,1] in this area.

all the above-mentioned studies, the rotating flows of non-Newtonian fluids have been studied as a boundary value em. Therefore, all the mentioned studies lack the features of unsteadiness. This study fills the gap in this area. , the main object of the present study is to discuss the unsteady flow of a non-Newtonian fluid in a rotating frame ference. For that we select the model of third grade fluid. The flow in the fluid is induced by the suddenly moving in its own plane. The governing equation for the rotating flow of a thermodynamic third grade fluid has been :led and then solved numerically using Newton's method.

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n Section 2 we present the flow analysis. The numerical results and discussion are presented in Section 3 followed concluding remarks in Section 4.

Flow analysis

onsider an incompressible third grade fluid occupying the space $z > 0$. The plate at $z = 0$ is moved suddenly with nstant velocity for $t > 0$. Both the fluid and plate are in a solid body rotation. Initially the fluid and plate are at rest. laws which govern the flow are

$$\operatorname{div} V = 0, \tag{1}$$

Q1

$$\rho\left[\frac{\partial V}{\partial t} + (V.\nabla)V + 2\Omega \times V + \Omega \times (\Omega \times r)\right] = -\nabla p + \operatorname{div} \mathbf{T}, \tag{2}$$

which V is the velocity, ρ the fluid density, t the time, p the hydrostatic pressure, $\mathbf{T}$ the extra stress tensor, Ω the stant angular velocity and r the radial coordinate with $r = x^2 + y^2$.

he extra stress tensor $\mathbf{T}$ in a third grade fluid is

$$\mathbf{T} = \mu\mathbf{A}_1 + \alpha_1\mathbf{A}_2 + \alpha_2\mathbf{A}_1^2 + \beta_1\mathbf{A}_3 + \beta_2(\mathbf{A}_2\mathbf{A}_1 + \mathbf{A}_1\mathbf{A}_2) + \beta_3(\operatorname{tr}\mathbf{A}_1^2)\mathbf{A}_1. \tag{3}$$

e μ is the dynamic viscosity, α_i $(i = 1, 2)$, and β_i $(i = 1\text{–}3)$ are the material constants. The kinematical tensors $\mathbf{A}_n$

$$\mathbf{A}_1 = \nabla V + (\nabla V)^{\mathrm{T}}, \tag{4}$$

$$\mathbf{A}_n = \left(\frac{\partial}{\partial t} + (V.\nabla)\right)\mathbf{A}_{n-1} + \mathbf{A}_{n-1}(\nabla V) + (\nabla V)^{\mathrm{T}}\mathbf{A}_{n-1}, \quad n = 2, 3, \ldots . \tag{5}$$

thermodynamics of the fluid requires that [6]

$$\mu \geqslant 0, \quad \alpha_1 \geqslant 0, \quad |\alpha_1 + \alpha_2| \leqslant \sqrt{24\mu\beta_3},$$

$$\beta_1 = \beta_2 = 0, \quad \beta_3 \geqslant 0. \tag{6}$$

refore Eq. (3) can be written as

$$\mathbf{T} = \left[\mu + \beta_3(\operatorname{tr}\mathbf{A}_1^2)\right]\mathbf{A}_1 + \alpha_1\mathbf{A}_2 + \alpha_2\mathbf{A}_1^2. \tag{7}$$

ce the plate is infinite so the velocity field V for the present flow is

$$V = [u(z, t), v(z, t), w(z, t)] \tag{8}$$

ch together with the incompressibility condition yields $w = 0$ (u, v and w are the velocities in the x, y, z directions, ectively).

ubstituting Eqs. (7) and (8) into Eq. (2) one obtains

$$\frac{\partial u}{\partial t} - 2\Omega v = -\frac{1}{\rho}\frac{\partial \widehat{p}}{\partial x} + \frac{1}{\rho}\left[\mu\frac{\partial^2 u}{\partial z^2} + \alpha_1\frac{\partial^3 u}{\partial z^2 \partial t} + 2\beta_3\frac{\partial}{\partial z}\left(\frac{\partial u}{\partial z}\left\{\left(\frac{\partial u}{\partial z}\right)^2 + \left(\frac{\partial v}{\partial z}\right)^2\right\}\right)\right], \tag{9}$$

$$\frac{\partial v}{\partial t} + 2\Omega u = -\frac{1}{\rho}\frac{\partial \widehat{p}}{\partial y} + \frac{1}{\rho}\left[\mu\frac{\partial^2 v}{\partial z^2} + \alpha_1\frac{\partial^3 v}{\partial z^2 \partial t} + 2\beta_3\frac{\partial}{\partial z}\left(\frac{\partial v}{\partial z}\left\{\left(\frac{\partial u}{\partial z}\right)^2 + \left(\frac{\partial v}{\partial z}\right)^2\right\}\right)\right], \tag{10}$$

$$0 = -\frac{1}{\rho}\frac{\partial \widehat{p}}{\partial z}, \tag{11}$$

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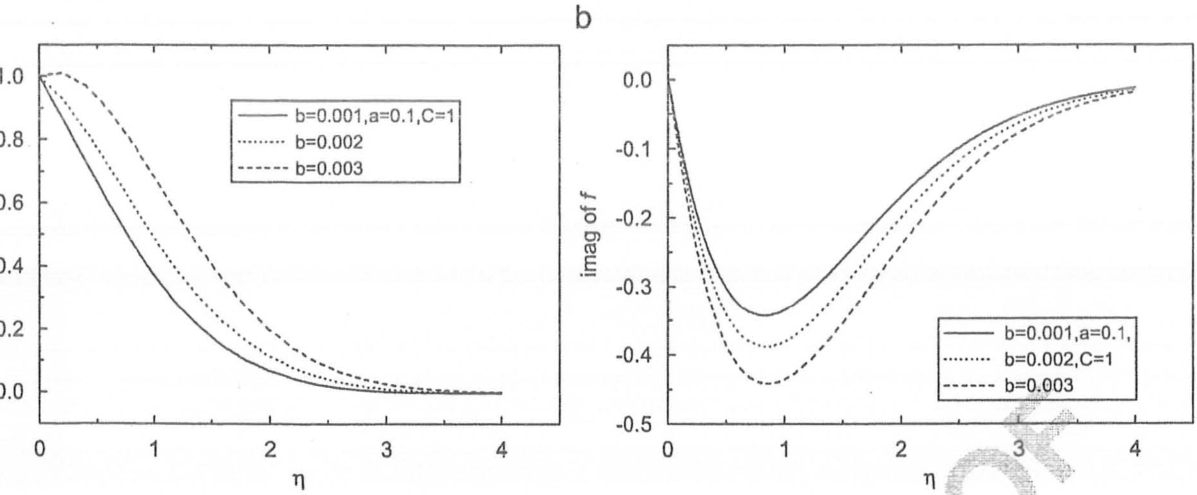

Fig. 1. Influence of the third grade parameter on the velocity distribution for $\tau = 1$.

re the modified pressure

$$\widehat{p} = p - \frac{\rho}{2}\Omega^2(x^2 + y^2) \tag{12}$$

Eq. (11) shows that $\widehat{p} \neq \widehat{p}(z)$.
ie relevant boundary and initial conditions are

$$\begin{aligned} &u = U_0, \quad v = 0 \quad \text{at } z = 0, \ t > 0, \\ &u \to 0, \quad v \to 0 \quad \text{as } z \to \infty \quad \text{for all } t, \\ &u(z, 0) = 0, \quad v(z, 0) = 0, \ z > 0. \end{aligned} \tag{13}$$

ibining Eqs. (9) and (10) and then neglecting the pressure gradient, we have

$$\frac{\partial F}{\partial t} + 2\mathrm{i}\Omega F = \nu\frac{\partial^2 F}{\partial z^2} + \frac{\alpha_1}{\rho}\frac{\partial^3 F}{\partial z^2 \partial t} + \frac{2\beta_3}{\rho}\frac{\partial}{\partial z}\left\{\left(\frac{\partial F}{\partial z}\right)^2 \frac{\partial \overline{F}}{\partial z}\right\}, \tag{14}$$

re

$$F = u + \mathrm{i}v \quad \text{and} \quad \overline{F} = u - \mathrm{i}v. \tag{15}$$

boundary and initial conditions now are

$$F(0, t) = U_0, \quad F(z, t) \to 0 \quad \text{as } z \to \infty, \ F(z, 0) = 0. \tag{16}$$

convenient to write the problem in dimensionless variables. For that we introduce the following variables:

$$\eta = \frac{U_0}{\nu}z, \quad \tau = \frac{U_0^2}{\nu}t, \quad f = \frac{F}{U_0}, \quad C = \frac{U_0^2}{\nu}\Omega. \tag{17}$$

problem becomes

$$\frac{\partial f}{\partial \tau} + 2\mathrm{i}Cf = \frac{\partial^2 f}{\partial \eta^2} + a\frac{\partial^3 f}{\partial \eta^2 \partial \tau} + 2b\frac{\partial}{\partial \eta}\left\{\left(\frac{\partial f}{\partial \eta}\right)^2 \frac{\partial \overline{f}}{\partial \eta}\right\}, \tag{18}$$

$$f(0, \tau) = 1, \quad f(\eta, \tau) \to 0 \quad \text{as } \eta \to \infty, \ f(\eta, 0) = 0, \tag{19}$$

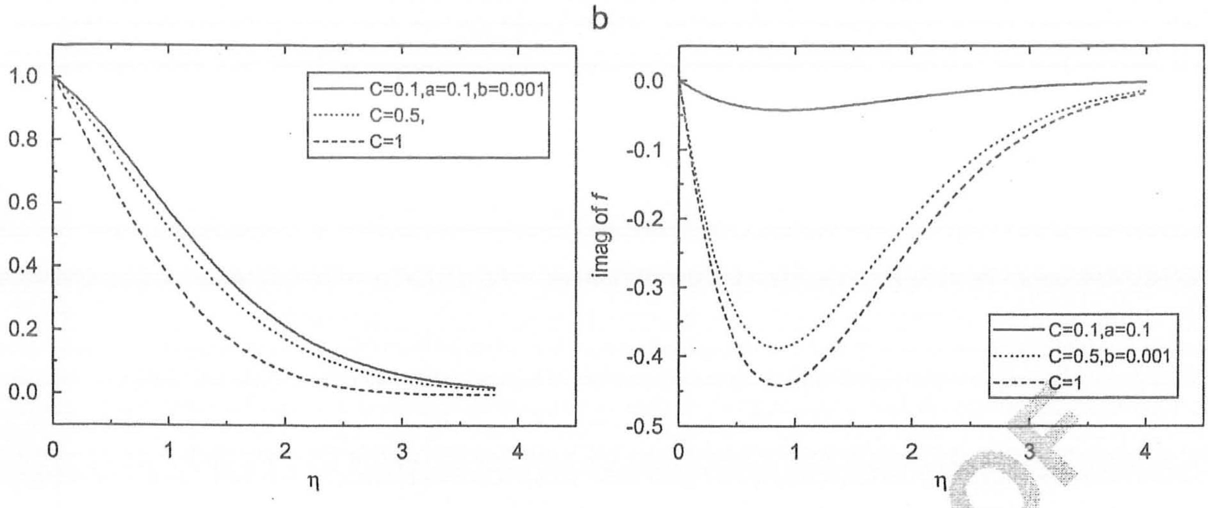

Fig. 2. Influence of Ω on the velocity distribution for $\tau = 1$.

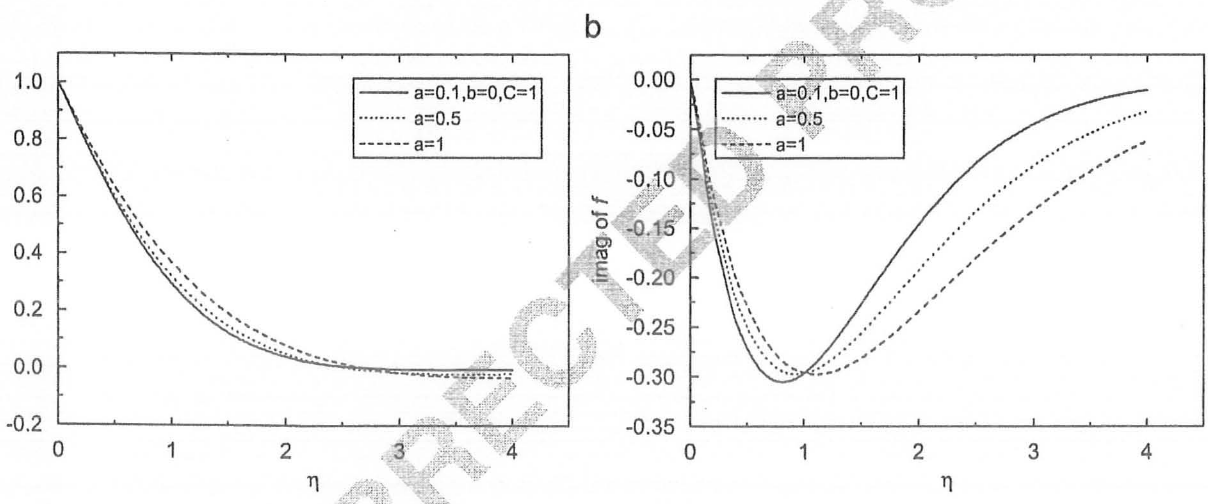

Fig. 3. Influence of the various values of the second grade parameter on the velocity distribution for $\tau = 1$.

vhich

$$a = \frac{\alpha_1 U_0^2}{\rho \nu^2}, \quad b = \frac{\beta_1 U_0^4}{\rho \nu^3}. \tag{20}$$

Numerical results and discussion

The non-linear differential system consisting of Eq. (18) and conditions (19) has been solved numerically by em-ying the Newton method. Solutions for the non-Newtonian fluid models are obtained for $\tau = 1$. From the numerical ution f is used to express the non-dimensional velocity profile. Results for the flow are obtained for various values he parameters a, b, C and τ.

ig. 1(a) and (b) presents the velocity profile f for various values of b. These figures indicate that increasing the ameter b increases real part of the velocity. However, imaginary part of the velocity decreases for large values of b. . 2(a) and (b) shows the influence of C on the velocity profile f. It is evident from the figure that increase in C results lecrease the real and imaginary parts of the velocity. The effect of the second grade parameter on f is illustrated in

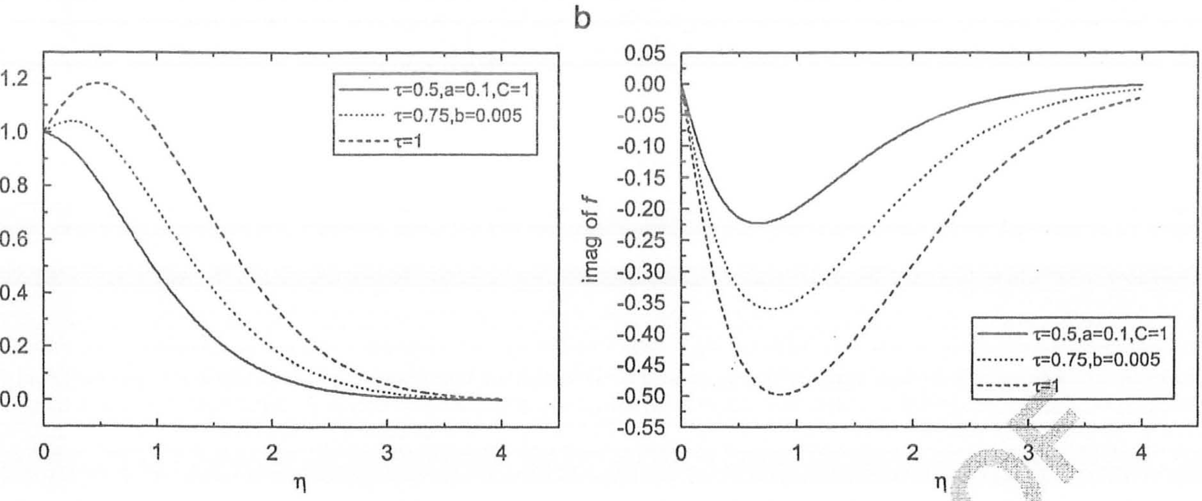

Fig. 4. Influence of various values of τ on the velocity distribution.

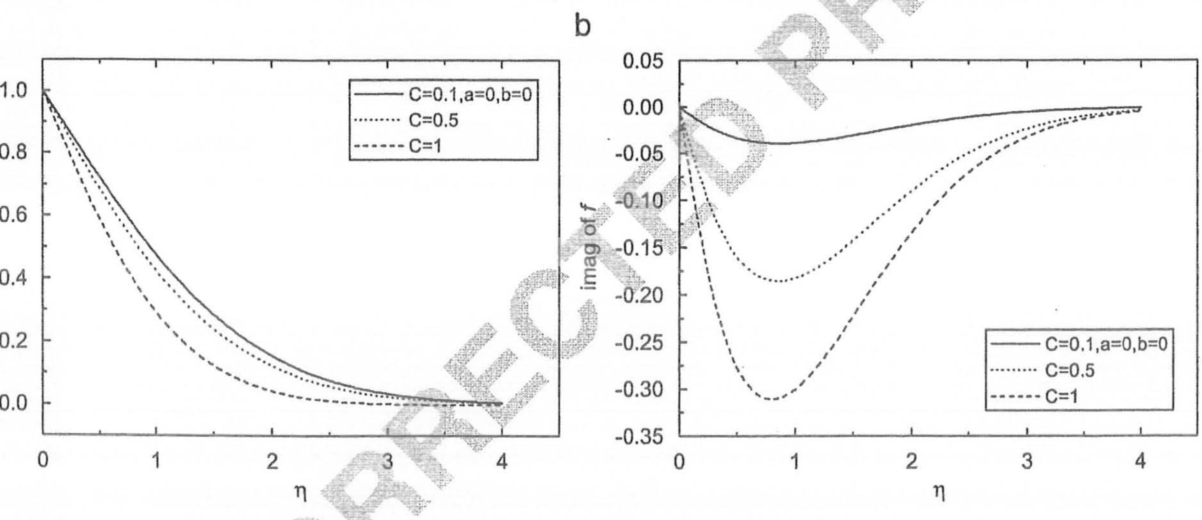

Fig. 5. Influence of Ω on the velocity distribution for the Newtonian case.

3(a) and (b). It is noted that the velocity increases in the real part whereas in the imaginary part it first increases then decreases by increasing the second grade parameter. Fig. 4(a) and (b) shows how the velocity changes for ous values of τ. It is found that here real part of velocity increases and imaginary part of velocity decreases by easing τ. In Fig. 5(a) and (b) the velocity distribution is presented in the Newtonian case for the various values of t is observed that the influence of C in Newtonian and third grade fluid is similar.

Concluding remarks

he Stoke's first problem of a third grade fluid is discussed in a rotating frame of reference. The problem that prised the balance laws of mass and momentum has been first non-dimensionalized and then solved numerically. ults for the real and imaginary parts of the velocity are presented. It is found that at $\tau = 1$ and different values of he flow characteristics in a third grade fluid are similar to that of Newtonian fluid.

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RESEARCH PAPER

Stokes' first problem for the fourth order fluid in a porous half space

T. Hayat · F. Shahazad · M. Ayub

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Abstract In this study, the flow of a fourth order fluid in a porous half space is modeled. By using the modified Darcy's law, the flow over a suddenly moving flat plate is studied numerically. The influence of various parameters of interest on the velocity profile is revealed.

Keywords Fourth order fluid · Porous space · Numerical solution · Modified Darcy's law

1 Introduction

Because of its practical applications, the Stokes' problem for the flat plate has been the subject of numerous theoretical studies. Such studies for Navier-Stokes fluid and different types of non-Newtonian fluids include the work of Zierep [1], Soundalgekar [2], Rajagopal and Na [3], Puri [4], Bandelli et al. [5], Tigoiu [6], Fetecau and Zierep [7] and Fetecau and Fetecau [8,9]. More recently, Tan and Masuoka [10,11] discussed the Stokes' first problem for the second grade and Oldroyd-B fluids using the modified Darcy's law. They obtained an analytical solution. The second grade and Oldroyd-B fluids for steady unidirectional flow do not exhibit the rheological characteristics. The third and fourth order fluids would exhibit such characteristics even for steady flow. For this reason the model in the present study is of a fourth order fluid. The viscoelastic flows in a porous space are quite prevalent in many engineering fields, such as, enhanced oil recovery, paper and textile coating and composite manufacturing processes. Also the modeling of polymeric flow in a porous space is essential for the numerical simulation of viscoelastic flows in a specific pore geometry model, for example, capillary tubes, undulating tubes, packs of spheres or cylinders. With these motivations in mind, the layout of the paper is as follows:

In Sect. 2 we present the basic equations. In Sect. 3 we give the problem formulation. The numerical results and discussion are presented in Sect. 4 followed by concluding remarks in Sect. 5.

The English text was polished by Keren Wang.

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2 Basic equations

In a porous space, the equations governing the flow of an incompressible fluid are

$$\mathrm{div}\,\mathbf{V} = 0, \tag{1}$$

$$\rho\left(\frac{\partial}{\partial t} + (\mathbf{V}\cdot\nabla)\right)\mathbf{V} = -\nabla p + \mathrm{div}\mathbf{S} + \mathbf{r}. \tag{2}$$

In the above equations, $\mathbf{V}$ is the velocity, ρ the fluid density, t the time, p the hydrostatic pressure, $\mathbf{S}$ the extra stress tensor and $\mathbf{r}$ the Darcy resistance for a third grade fluid in a porous space.

The constitutive equations for a third grade fluid are

$$\mathbf{S} = \mu \mathbf{A}_1 + \alpha_1 \mathbf{A}_2 + \alpha_2 \mathbf{A}_1^2 + \mathbf{S}_1 + \mathbf{S}_2, \tag{3}$$

$$\mathbf{S}_1 = \beta_1 \mathbf{A}_3 + \beta_2(\mathbf{A}_2\mathbf{A}_1 + \mathbf{A}_1\mathbf{A}_2) + \beta_3(\mathrm{tr}\mathbf{A}_1^2)\mathbf{A}_1, \tag{4}$$

$$\begin{aligned}\mathbf{S}_2 = {} & \gamma_1 \mathbf{A}_4 + \gamma_2(\mathbf{A}_3\mathbf{A}_1 + \mathbf{A}_1\mathbf{A}_3) + \gamma_3\mathbf{A}_2^2 \\ & + \gamma_4(\mathbf{A}_2\mathbf{A}_1^2 + \mathbf{A}_1^2\mathbf{A}_2) + \gamma_5(\mathrm{tr}\mathbf{A}_2)\mathbf{A}_2 + \gamma_6(\mathrm{tr}\mathbf{A}_2)\mathbf{A}_1^2 \\ & + (\gamma_7\mathrm{tr}\mathbf{A}_3 + \gamma_8\mathrm{tr}(\mathbf{A}_2\mathbf{A}_1))\,\mathbf{A}_1,\end{aligned} \tag{5}$$

$$= \mu \frac{\partial^2 u}{\partial y^2} + \alpha_1 \frac{\partial^3 u}{\partial y^2 \partial t} + \beta_1 \frac{\partial^4 u}{\partial y^2 \partial t^2}$$

$$+ 6(\beta_2 + \beta_3) \left(\frac{\partial u}{\partial y}\right)^2 \frac{\partial^2 u}{\partial y^2} + \gamma_1 \frac{\partial^5 u}{\partial y^2 \partial t^3}$$

$$+ (6\gamma_2 + 2\gamma_3 + 2\gamma_4 + 2\gamma_5 + 6\gamma_7 + 2\gamma_8)$$

$$\times \frac{\partial}{\partial y} \left[\left(\frac{\partial u}{\partial y}\right)^2 \left(\frac{\partial^2 u}{\partial y \partial t}\right) \right] - \left[\mu + \alpha_1 \frac{\partial}{\partial t} \right.$$

$$+ \beta_1 \frac{\partial^2}{\partial t^2} + 2(\beta_2 + \beta_3) \left(\frac{\partial u}{\partial y}\right)^2 + \gamma_1 \frac{\partial^3}{\partial t^3}$$

$$+ (6\gamma_2 + 2\gamma_3 + 2\gamma_4 + 2\gamma_5 + 6\gamma_7 + 2\gamma_8)$$

$$\left. \times \left(\frac{\partial u}{\partial y}\right) \left(\frac{\partial^2 u}{\partial y \partial t}\right) \right] \frac{\phi u}{k}. \quad (14)$$

relevant boundary and initial conditions are

$$t) = U_0, \quad u(y,t) \to 0,$$
$$\to \infty; \quad u(y,0) = 0. \quad (15)$$

oducing the following non-dimensional variables

$$\frac{U_0}{\nu} y, \quad \tau = \frac{U_0^2}{\nu} t, \quad f = \frac{u}{U_0}, \quad (16)$$

(14) and (15) become

$$= \frac{\partial^2 f}{\partial \eta^2} + a \frac{\partial^3 f}{\partial \eta^2 \partial \tau} + b_1 \frac{\partial^4 f}{\partial \eta^2 \partial \tau^2}$$

$$+ 6(b_2 + b_3) \left(\frac{\partial f}{\partial \eta}\right)^2 \frac{\partial^2 f}{\partial \eta^2} + c_1 \frac{\partial^5 f}{\partial \eta^2 \partial \tau^3}$$

$$+ 2(6c_2 + 2c_3 + 2c_4 + 2c_5 + 6c_7 + 2c_8)$$

$$\times \left[\left(\frac{\partial f}{\partial \eta}\right) \left(\frac{\partial^2 f}{\partial \eta^2}\right) \left(\frac{\partial^2 f}{\partial \eta \partial \tau}\right) \right.$$

$$\left. + \left(\frac{\partial f}{\partial \eta}\right)^2 \left(\frac{\partial^3 f}{\partial \eta^2 \partial \tau}\right) \right] - df - e \left(\frac{\partial f}{\partial \tau}\right)$$

$$- g \left(\frac{\partial^2 f}{\partial \tau^2}\right) - 2(\nu_1 + \nu_2) \left(\frac{\partial f}{\partial \eta}\right)^2 f - L \left(\frac{\partial^3 f}{\partial \tau^3}\right)$$

$$- (6m_2 + 2m_3 + 2m_4 + 2m_5 + 6m_7 + 2m_8)$$

$$\times \left(\frac{\partial f}{\partial \eta}\right) \left(\frac{\partial^2 f}{\partial \eta \partial \tau}\right) f, \quad (17)$$

$$= 1, \quad f(\eta, \tau) \to 0$$
$$\to \infty; \quad f(\eta, 0) = 0, \quad (18)$$

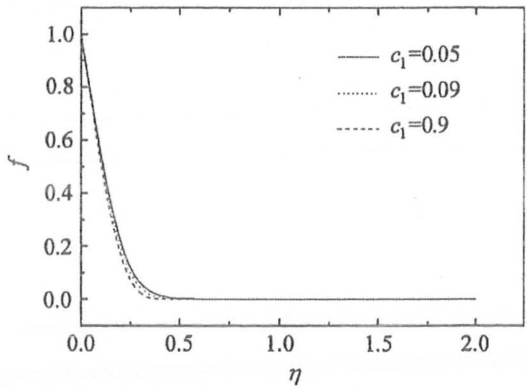


Fig. 1 Variation of the fourth order parameter c_1 on f at $\tau = 2\pi$ ($a = 0.1$, $b's = 0.2$, $c's = 0.1$, $d = 0.1$, $e = 0.5$, $g = 0.2$, $\nu's = 0.1$, $L = 0.2$, $m's = 0.1$)

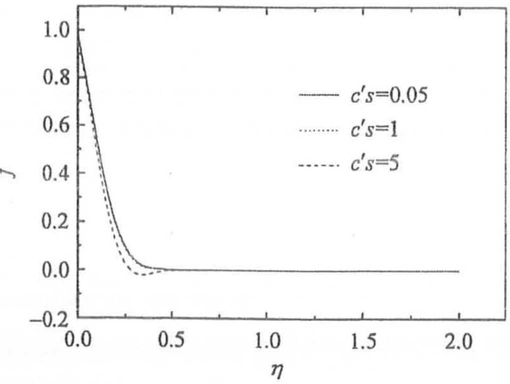
where

$$a = \frac{\alpha_1 U_0^2}{\rho \nu^2}, \quad b_i = \frac{\beta_i U_0^4}{\rho \nu^3} \quad (i = 1,2,3),$$
$$c_i = \frac{\gamma_i U_0^6}{\rho \nu^4} \quad (i = 1,2,3,4,5,7,8), \quad d = \frac{\phi \nu^2}{k U_0^2},$$
$$e = \frac{\alpha_1 \phi}{\rho k}, \quad g = \frac{\beta_1 \phi U_0^2}{\rho k \nu}, \quad (19)$$
$$\nu_2 = \frac{\beta_2 \phi U_0^2}{\rho k \nu}, \quad \nu_3 = \frac{\beta_3 \phi U_0^2}{\rho k \nu},$$
$$L = \frac{\gamma_1 \phi U_0^4}{\rho k \nu^2}, \quad m_i = \frac{\gamma_i \phi U_0^4}{\rho k \nu^2}.$$

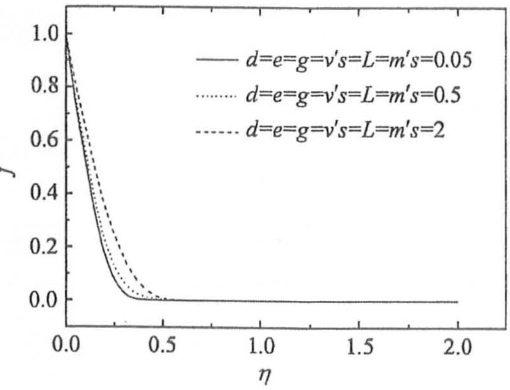
4 Numerical results and discussion

The non-linear differential equation system consisting of Eq. (17) and conditions (18) is solved numerically by employing the Newton method. Solutions for the non-Newtonian fluid models are obtained for $\tau = 2\pi$. In the numerical solution, f is used to express the non-dimensional velocity profile parallel to x-axis. Results for the flow are obtained for various values of the parameters involved.

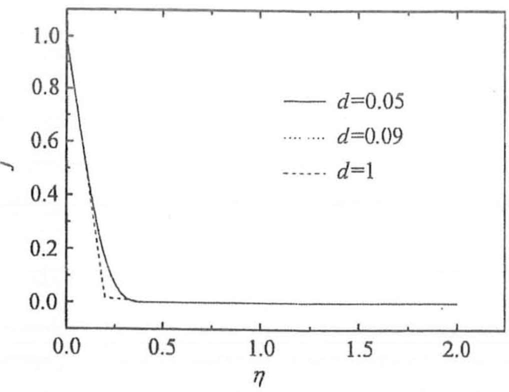
Figure 1 presents the velocity profile f for various values of c_1. The figure shows that increasing the fourth order parameter c_1 decreases both velocity and boundary layer thickness. Figure 2 elucidates the influence of the fourth order parameters c_i ($i = 2,3,4,5,7,8$) on the velocity profile f. It is evident from Fig. 2 that an increase of these parameters results in a decrease of the velocity profile. We further note that both figures hold for the fourth order parameters. The non-dimensional parameter c_1 involves only one material parameter γ_1 and the parameter c_i is the sum of γ_i ($i = 2,\ldots,8$). The effect of porosity and fourth order parameters on f is displayed in Fig. 3. It is clear that both velocity and



Variation of the fourth order parameters c_i $(i=2, 3,$ 8) on f at $\tau = 2\pi$ $(a = 0.1, b's = 0.2, c_1 = 0.1, d = 0.1,$ $1, g = 0.2, v's = 0.1, L = 0.2, m's = 0.1)$



Variation of the fourth order and porosity parameters on $= 2\pi$ $(a = 0.1, b's = 0.1, c's = 0.1)$



Variation of the porosity parameter on f at $\tau=2\pi$ $(a = 0.1,$ $1, c's=0.1, e = v's = g's = 0.05)$

dary layer thickness increase by increasing these neters. Figure 4 shows how the velocity changes the various values of the porosity parameter. It ınd that the velocity decreases by increasing this neter.

es 5 and 6 indicate the velocity distribution for vari- alues of the third and second order fluid parameters.

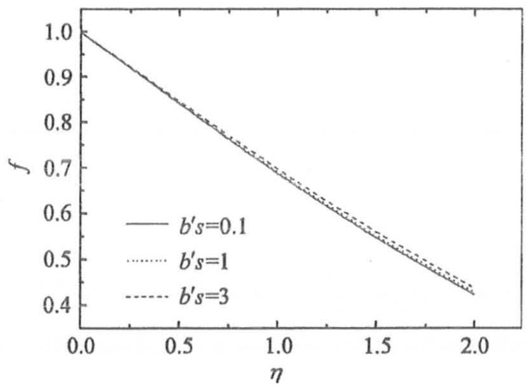


Fig. 5 Variation of third order parameters on f at $\tau=2\pi$ $(a=0.1,$ $c's = d = e = g = v_2 = v_3 = L = m's = 0)$

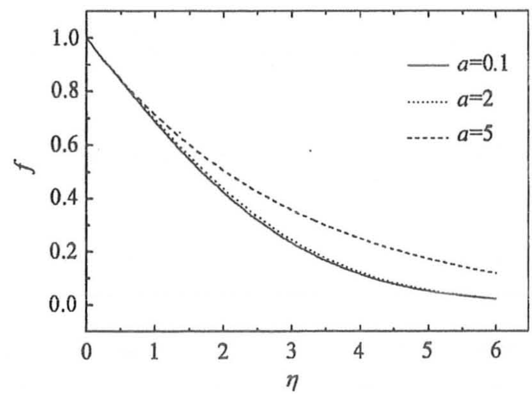


Fig. 6 Variation of the second order parameter on f at $\tau = 2\pi$ $(b's = c's = d = e = g = v_2 = v_3 = L = m's = 0)$

5 Concluding remarks

In the present work, Stoke's first problem is generalized for the fourth order fluid in a porous space. The governing constitutive relationship for the modified Darcy's law in a fourth order fluid is proposed. It is noted that the modified Darcy's law for the unidirectional flow of a fourth order fluid yields a non-linear expression with respect to the velocity whereas it is linear for Newtonian, Oldroyd-B, Maxwell and second grade fluids. The governing non-linear problem including the balance laws of mass and momentum is solved numerically. It is observed that for $\tau \geq 5\pi$ the fourth order fluid behaves like a Newtonian fluid.

References

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