Flows of Third Grade Fluid in a Rotating Frame

By

Muhammad Mudassar Gulzar

Department of Mathematics
Quaid-i-Azam University, Islamabad
PAKISTAN
2006

Flows of Third Grade Fluid in a Rotating Frame

By

Muhammad Mudassar Gulzar

Supervised by

Prof. Dr. Saleem Asghar
Prof. Dr. Muhammad Ayub

Department of Mathematics
Quaid-i-Azam University, Islamabad
PAKISTAN

2006

Flows of Third Grade Fluid in a Rotating Frame

By

Muhammad Mudassar Gulzar

A Thesis
Submitted in the Partial Fulfillment of the
Requirements for the Degree of

DOCTOR OF PHILOSOPHY
IN
MATHEMATICS

Supervised by

Prof. Dr. Saleem Asghar
Prof. Dr. Muhammad Ayub

Department of Mathematics
Quaid-i-Azam University, Islamabad
PAKISTAN
2006

Certificate

Flows of Third Grade Fluid in a Rotating Frame

By

Muhammad Mudassar Gulzar

A THESIS SUBMITTED IN THE PARTIAL FULFILLMENT OF THE REQUIREMENTS FOR THE DEGREE OF DOCTOR OF PHILOSOPHY
IN
MATHEMATICS

We accept this thesis as conforming to the required standard.

Dr. Muhammad Yaqub Nasir (Chairman)	Prof. Dr. Saleem Asghar (Supervisor)
Prof. Dr. Muhammad Ozair Ahmad (External Examiner)	Prof. Dr. Muhammad Ayub (Supervisor)
Dr. Muhammad Aslam Awan (External Examiner)	

Department of Mathematics
Quaid-i-Azam University, Islamabad
PAKISTAN
2006

Dedicated to

My father
Gulzar Muhammad Chaudhary (Late)
My mother
My family
&
Loving sons Abeer & Mouiz

Acknowledgements

In the name of the Creator of the universe, the most powerful, compassionate, kind and merciful. The completion of my thesis marks the culmination of one of the most cherished moments of my life with the blessings of Almighty Allah. I express my deepest and profoundest gratitude to my supervisor **Prof. Dr. Saleem Asghar, TI** for his continued support, guidance and encouragement. In fact, it could not have been possible for me to complete this work without his constant supervision, ever-available help and professional guidance. It has been an excellent learning experience for me in working under his guidance, which gives me a sense of great honour and achievement being a member of his team. In deed, it is a matter of pride and privilege for me to be his Ph. D. student. He is a role model for me for the last many years in the manner he pursues his research assignments, takes his lectures and the way he conducts himself in his routine affairs of life.

I owe a special debt of gratitude to my supervisor **Prof. Dr. Muhammad Ayub** as well for his valuable guidance, encouragement and deep concern for the accomplishment of my research work. I have always found him ready to spare his precious time for discussions patiently and willingly.

I want to extend my heartfelt thanks to **Dr. Tasawar Hayat, TI** who has not only given his valuable input and contributions to improve my thesis but has the unique credit to take the Fluid Mechanics Group to such a distinctive position in the country. He plays a pivotal role for the entire group and his sole dedication and devotion to his work has brought many laurels to the Department and the University.

I am thankful to Brig. Dr. Muhammad Rafique, TI (M) for his continued appreciation, cooperation and encouragement at College of Electrical and Mechanical Engineering (NUST), Islamabad. I am equally indebted to my friends at Quaid-i-Azam University Islamabad, especially Dr. Khalid Hanif, Dr. Masood Khan, Muhammad Mushtaq, Dr. Sohail Nadeem, Dr. Rehmat Ellahi and Dr. Muhammad Sajid Qureshi for their moral support and helping attitude. It is a joy and a privilege for me to express my gratitude to my mother, brothers, sisters, wife and our children for their love and endless support throughout the course of study.

Muhammad Mudassar Gulzar

Preface

There are many models which are used to investigate different types of fluid mechanics problems. It is difficult to characterize in general way all necessary requirements since each problem is unique. However, we can broadly classify many of the problems on the basis of the general nature of the flow and the fluid and subsequently develop some general characteristics of model designs in each of these classifications. Amongst these models, the model of Newtonian fluid is the simplest one for which the Navier-Stokes equations can describe the flow problem. However, there are many fluids with complex microstructure such as biological fluids, as well as polymeric liquids, suspensions, liquid crystals which are used in current industrial processes and show non-linear viscoelastic behaviour that cannot be characterized by Navier-Stokes equations. Because of the fluids complexity, many constitutive equations have been proposed. The non-Newtonian models that have been developed to describe the other rheological characteristics can be classified under the following three categories: fluids of differential type, rate type and integral type. Amongst these types, the differential fluids have received the special attention from the recent researchers in order to describe the several non-standard features such as normal stress effects, rod climbing, shear thinning and shear thickening. The governing equations for such fluids are more non-linear and higher order than the Navier-Stokes equations.

In the literature much attention has been focused on the flows of second grade fluid which is simplest subclass of differential type fluids. The second grade fluid model is able to predict the normal stress differences but it does not take into account the shear thinning or shear thickening phenomena that many fluids show. The third grade fluid model represents a further, although inconclusive attempt toward a more comprehensive description of the behaviour of non-Newtonian fluids. Due to this fact in mind, the model in the present thesis is a third grade.

Another aspect in the study of non-Newtonian fluids is the slip boundary condition. Although there are rigorous mathematical researches on flows of Newtonian fluids with slip condition but due attention has not been given to flows of non-Newtonian fluids with slip condition. The non-Newtonian fluids such as polymer melts often exhibit macroscopic wall slip governed by a non-linear and non-monotone relation between the slip velocity and the traction. The fluids that exhibit boundary slip are important from technological point of view for example, the polishing of artificial heart valves.

Keeping the above facts in view, the present thesis is organized as follows: Chapter zero provides the introduction of the thesis. Basic preliminaries relevant to non-Newtonian fluids, governing laws and techniques are given in Chapter one. Equation which governs the rotating flow of a third grade fluid over a porous surface is also modeled here. Chapter two describes the steady flow of a third grade fluid in a rotating frame by using no-slip condition. The same problem has been solved employing another set of dimensionless variables for the influence of dynamic viscosity. Later, this problem is solved using partial slip boundary condition. Chapter three describes the oscillatory rotating flow of a third grade fluid passed a porous plate. An asymptotic solution has been obtained. Two cases of no-slip and partial slip have been considered. Homotopy analysis method is used to obtain the analytic solutions for the problems in chapters two and three. Convergence of the obtained solutions developed in these chapters is also ensured. Chapter four has been prepared for the numerical solutions of the two partial slip boundary value problems. A reasonable agreement between the HAM and numerical solutions is presented through graphs. The concluding remarks are made at the end of each Chapter. However, a brief summary of the important results from the thesis has been included in Chapter five.

Contents

Nomenclature

$\mathbf{T}$	Cauchy stress tensor
p_1	Pressure (scalar function)
$\mathbf{I}$	Unit matrix
t	Time variable
$\mathbf{V}$	Velocity
U_0	Uniform / Free stream velocity
$\mathbf{A}_1, ..., \mathbf{A}_n$	Rivlin-Ericksen tensors
d/dt	Material time derivative
du/dt	Rate of deformation
$\tau_{yx} = \mu \frac{du}{dy}$	Shear stress
div $\mathbf{T}$	Surface forces
ρb	Body forces per unit mass
μ	Absolute or dynamic viscosity
α_1, α_2	Material moduli
β_1, β_2, β_3	Material constants for third grade fluid
$W_0 > 0$	Suction velocity
$W_0 < 0$	Blowing velocity
$\boldsymbol{\Omega}$	Constant angular velocity
u, v, w	Velocity components in the x, y and $z-$ directions
ρ	Fluid density
ν	Kinematic viscosity
$\widehat{\mathbf{k}}$	Unit vector
$\mathbf{r}$	Radial coordinate
$\rho\,(2\boldsymbol{\Omega} \times \mathbf{V})$ and $\rho\,[\boldsymbol{\Omega}\times (\boldsymbol{\Omega} \times \mathbf{r})]$	Coriolis and centripetal accelerations

λ, λ_1, λ_2	Slip parameters
U_w, V_w	Wall velocities
s	Strained coordinate
δ	Oscillating frequency
h, l	Constant grid spacing
$\mathcal{A}$	Non-linear operator
$\mathcal{L}$, $\mathcal{L}_1$, $\mathcal{L}_2$, $\mathcal{L}_3$, $\mathcal{L}_4$	Auxiliary linear operators
p	Embedding parameter
$\hbar_0$, $\hbar_1$, $\hbar_2$, $\hbar_3$, $\hbar_4$, $\hbar_5$	Non-zero auxiliary parameters

a_0, a_n, $c_0 - c_6$, $\widetilde{a}_1 - \widetilde{a}_3$, $\widetilde{b}_1 - \widetilde{b}_3$, $\widetilde{c}_1 - \widetilde{c}_3$, k_0, k_1, $m_1 - m_4$, $M_1 - M_{18}$, G_i,
H_i, J_i, K_i, L_i, $X_0 - X_2$ Constants in calculation

Chapter 0

Introduction

One of the most fascinating and beautiful subjects of science is fluid mechanics. The broad range of interesting phenomena and our daily interaction with fluids such as water and air in the environment, make this as one of the most exciting topics for the researchers. Fluids play a very vital role in many aspects of our life. We drink them, breath them, it runs through our bodies and it controls the weather. The study of motion of fluids is a complex phenomena. Fortunately we can analyze many important situations using simple idealized models and familiar principles such as Newton's laws and conservation of energy. The most famous form of equations of motion which is widely studied and applied in fluid mechanics is probably the Navier-Stokes equations. These are non-linear partial differential equations. For this reason, there exist only a limited number of exact solutions in which the non-linear terms do not disappear automatically. Analytic solutions of Navier-Stokes equations are very important not only because these are solutions of some fundamental flows but also because they serve as accuracy checks for experimental, numerical and asymptotic methods.

The important and essential branch of fluid mechanics is concerned with liquids that are often referred as complex, in recognition of the fact that these materials exhibit much more complicated behaviour. Examples of complex fluids can be found in any kitchen, bathroom, playroom or garage. These include egg white, cake batter, silly putty, proprietary oil additives, blood, mucous and many others. Most of these fluids either consist entirely of large molecules or have large molecules floating in them as well as particles or droplets. Most plastics in their liquid state fall in this category. This branch of fluid mechanics is often called non-Newtonian to distinguish it from the classical work on small-molecules or Newtonian fluids. Although this class of fluids is common in nature, in a variety of technologies and as the liquid-state precursors of many important types of advanced materials. The status of our understanding of their behaviour and our ability to predict their motion is at a very early stage of development. In general terms, the difference between complex fluids and the single component Newtonian fluids is that, in the latter case the mathematical formulation is known but the macroscopic physical processes are complex and often not well understood, especially for turbulent flow conditions and for complex fluids even the appropriate governing equations and conditions at the boundaries are still not well understood. To compound the difficulty, the model equations that have been proposed are extremely difficult to be solved and even standard methods of computational fluid dynamics generally do not work for this class of problems. These fluids are quite difficult from mathematical point of view for non-linear differential equations particularly for unsteady flows. Due to complexity of fluids, there are many models describing the properties, but not all, of non-Newtonian fluids. These models however, cannot predict all the behaviour of non-Newtonian fluids, for example normal stress differences, shear thinning or shear thickening, shear relaxation, elastic and memory effects etc. Thus, where the study of these fluids is difficult, it is important from a practical point of view and understanding the non-Newtonian fluids itself. The literature is not rich enough with the solutions of non-Newtonian fluid problems. Even we find dearth of studies on viscous problems whereas the solutions to non-Newtonian problems is a rare commodity. Among these models, the fluids of differential type, for example fluids of second and third grades have acquired special status due to their elegance [1]. For some contributions, we refer the reader to the studies $[1-19]$ and several references therein.

It has always been interesting to carry out the study of fluids which are rotating. Examples of such flows are weather patterns, atmospheric fronts, and ocean currents. The geophysical flows are strongly influenced by the diurnal rotation of the earth, which is manifested in the equations of motion as the Coriolis force. The geophysical fluid dynamics may be considered to be the study of rotating and stratified fluids. The first of the two distinguishing attributes of geophysical fluid dynamics is the effect of the earth's rotation. Because geophysical flows are relatively slow and spread over long distances, the time taken by a fluid particle (be it a parcel of air in the atmosphere or water in the ocean) to traverse the region occupied by a certain flow structure is comparable to, and often longer than a day. Thus, the earth rotates significantly during the travel time of the fluid and rotational effects enter the dynamics. Fluid flows viewed in a rotating framework of reference are subject to two additional types of forces, namely the centrifugal force and the Coriolis force. Since earth is rotating, so any flow we observe on earth is actually in a rotating frame. As it turns out, sometimes the effect of earth's rotation is negligible, but for large scale motions like the flows in oceans and atmosphere this is never true. Indeed rotation dominates most of these flows. The effects of the Coriolis force due to earth's rotation is found to be significant as compared to the inertial and viscous forces in the equation of motion. It is not only the earth which rotates, in fact all the planets of the solar system are in rotation, most notably the Jovian planets because they rotate quickly and they are mostly fluids. The rotating flows are very useful in the solar physics. The sun rotates and more massive stars rotate even faster, they are all fluids. The great spiral galaxies are defined by their rotations. The study of rotation of fluids has also attracted the mathematicians. Few studies dealing with the rotating flows are given in the investigations [20 – 26].

In all the above mentioned studies, the partial slip effects have not been discussed. The need for the development of boundary conditions has not received the attention that it deserves. The pioneers of the field such as Coulomb, Navier, Girad, Poisson, Stokes, St. Venant and others recognized that boundary conditions are constitutive equations that are determined by the material on either side of the boundary. The usual prescription of Dirichlet and Neumann conditions are often times not suitable for a realistic physical problem, for example the flow of polymers that stick-slip on the boundary. Recently non-standard boundary conditions have been considered from a rigorous mathematical perspective by Rao and Rajagopal [27]. The study is also motivated by recent experiments which suggest that gas nano-bubbles may form on solid walls and may be responsible for the appearance of partial slip boundary conditions for liquid flows. The partial slip boundary conditions cause a reduction in the velocity at the boundary. The solution to the problems with partial slip boundary has been studied by a number of authors $[27-37]$.

There are various analytical techniques to solve non-linear differential equations arising in Newtonian and non-Newtonian fluid mechanics. However, all these techniques have their limitations in application. Recently a newly developed homotopy analysis method by Liao [38] could answer some questions to very complex and intriguing non-Newtonian fluid mechanics problems. This method has been successfully applied by many researchers $[26, 38-62]$. This method is very useful for the solutions of the problems with strong non-linearity and has the following advantages:

- It is independent of the choice of any large/small parameters in the non-linear problems.
- It is helpful to control the convergence of approximation series in a convenient way and also for the adjustment of convergence regions where necessary.
- It can be employed to efficiently approximate a non-linear problem by choosing different sets of base functions.

In tradition, perturbation techniques are widely applied to give analytic approximations of non-linear problems. Homotopy analysis method is rather a general and useful method for finding the solutions of non-linear ordinary and partial differential equations for many different types. It provides great freedom and flexibility to select linear approximations.

Rotation plays a significant role in several important phenomena in cosmical fluid dynamics. Similarly a great deal of meteorology depends upon the dynamics of a revolving fluid. The large and moderate scale motions of the atmosphere are greatly affected by the vorticity of the earth's rotation. In the case of an infinitely extended space occupied by a fluid, rotating as a rigid body about an axis, the amount of energy possessed by the liquid is infinitely large and it is of great interest to know how small disturbances propagate in such a liquid. With these facts in view the present thesis has been organized in this direction. In the first chapter some definitions, derivation of the governing differential equations and the techniques/methods applied to the problems in the succeeding chapters have been presented. The second chapter deals with steady flow of an incompressible third grade fluid past a porous plate. Both fluid and plate are in a state of rigid body rotation with a constant angular velocity. This chapter has been further divided into two sections. In first section, flow problem with no-slip boundary conditions and in the second with partial slip boundary conditions has been addressed. The former problem is redimensionalised for viscous parameter as well and its solution has been included in this chapter. In the later part of the chapter, a secular term which appears in the first order solution of the partial slip boundary value problem has been removed using Lighthill technique for strained coordinates. These investigations of the problem with no-slip boundary conditions and partial slip boundary conditions have been published in **Applied Mathematics and Computation 165, 213-221 (2005)** and

Acta Mech Sinica 22, 195-198 (2006) respectively.

Chapter three is concerned with the rotating flow of a third grade fluid past the oscillating porous plate. Both no-slip and partial slip boundary conditions have been taken into account. Homotopy analysis method is employed in obtaining the analytic solutions. The contents of this chapter have been submitted for publication in **Acta Mech Sinica**.

In chapter four, we develop numerical scheme to obtain the solution of the partial slip boundary value problems discussed in chapters two and three. Finite difference method and Crank Nicholson scheme is employed for this purpose. The solutions are not only important in their own right but also provide support to the accuracy of analytical solutions and vice versa.

Chapter five provides a brief summary of the chief results and suggests extension of further research work from the thesis.

Chapter 1

Preliminaries

This chapter includes some basic concepts and definitions, the continuity equation and information about the methods/techniques that have been employed to obtain analytical and numerical solutions of the arising non-linear flow problems in the subsequent chapters. Flow modelling for rotating flow of a third grade fluid for a porous boundary is also included.

1.1 Non-Newtonian fluids

Navier-Stokes equations are the governing non-linear partial differential equations which describe the flow of Newtonian fluids. The Navier-Stokes theory is valid for the fluids of low molecular weight only. In many fields, such as food industry, drilling operations and bio-engineering, the fluids either synthetic or natural , are mixtures of different constituents such as water, particle, oils, red cells and other long chain molecules; this combination imparts strong non-Newtonian characteristics to the resulting liquids; the viscosity function varies non-linearly with the shear rate; elasticity is felt through elongational effects and time-dependent effects. In these cases, the fluids have been treated as non-Newtonian fluids. These complex fluids cannot be described by the Navier-Stokes equations. This inadequacy of Navier-Stokes theory to describe fluids such as polymer solutions, blood, certain oils and greases has led to the development of theories of non-Newtonian fluids. Fluids in which shear stress is not linearly proportional to deformation rate are known as non-Newtonian fluids [64]. There are many fluids which manifest such behaviour, one of the example is of the Lucite paint which is very thick when it is inside the can, it becomes thin when it is sheared by brushing. Toothpaste is another example, it behaves as a fluid when it is squeezed out from the tube. It does not run out by itself when the cap is removed. Non-Newtonian fluids are commonly classified as having time-independent and time-dependent behaviour. Many empirical equations have been proposed to model the observed relations between the shear stress τ_{yx} and rate of shearing strain du/dy for time-independent fluids. They are represented by the power law model as

$$\tau_{yx} = k_0 \left(\frac{du}{dy}\right)^n \tag{1.1}$$

where exponent n is called the flow behaviour index, the coefficient k_0 is the flow consistency index and du/dy is the shear rate. This equation reduces to Newton's law of viscosity for $n = 1$ with $k_0 = \mu$. To ensure that τ_{yx} has same sign as du/dy, equation (1.1) is rewritten in the form

$$\tau_{yx} = k_0 \left|\frac{du}{dy}\right|^n = \eta \frac{du}{dy}, \tag{1.1a}$$

where the term $\eta = k_0 \ |du \ / \ dy|^{n-1}$ is referred as to the apparent viscosity or affected viscosity.

A non-Newtonian fluid is a fluid in which viscosity changes with the applied shear force. Therefore, non-Newtonian fluids may not have a well-defined viscosity. The role of non-Newtonian fluid dynamics is important as it relates to plastic manufacture, performance of lubricants, clay suspensions, drilling muds, paints, processing of food and moment of biological fluids which contain higher molecular weight components. The study of non-Newtonian fluids is further complicated by the fact that the apparent viscosity may be time-dependent. Thixotropic fluids show a decrease in apparent viscosity with time under a constant applied shear stress. Many paints are thixotropic. Rheopectic fluids show an increase in apparent viscosity with time. After deformation some fluids partially return to their original shape when the applied stress is released, such fluids are called viscoelastic. Because of the difficulty to suggest a single model which exhibits all properties of viscoelastic fluids, they cannot be described as simply as Newtonian fluids. For this reason, many models or constitutive equations have been proposed and most of them are empirical or semi-empirical. For more general three-dimensional representation, the method of continuum mechanics is needed.

1.2 Equation of continuity

The continuity equation in fluid flow is based on conservation of mass. The law of conservation of mass states that the mass is neither created nor destroyed inside a control volume region. If we consider a differential control volume system enclosed by a surface fixed in the space, then the mass inside the fixed control volume system will remain conserved. For such a system, equation of continuity can be expressed as

$$\frac{\partial \rho}{\partial t} + \nabla . (\rho \mathrm{V}) = 0. \tag{1.2}$$

If density ρ is temporally constant and spatially uniform, then equation of continuity becomes

$$\nabla \cdot \mathrm{V} = 0 \tag{1.3}$$

which is applicable for incompressible fluids.

1.3 The momentum equation

The differential of equation of motion describing the flow of a fluid is

$$\rho \frac{d\mathrm{V}}{dt} = \rho \mathbf{b} + div \mathrm{T} \tag{1.4}$$

where $d\mathrm{V}/dt$ is the material time derivative, $\rho\mathbf{b}$ are body forces per unit mass and $div\mathrm{T}$ are the surface forces, T is the Cauchy stress tensor which in matrix form can be written as

$$\mathrm{T} = \begin{bmatrix} \sigma_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_{zz} \end{bmatrix} \tag{1.4a}$$

where σ_{xx}, σ_{yy}, σ_{zz} denote the normal stresses and τ_{xy}, τ_{xz}, τ_{yx}, τ_{yz}, τ_{zx}, τ_{zy} denote the shear stresses. The scalar forms of the momentum equation in terms of its velocity components u, v and w in x, y and z direction are given by

$$\rho\left(\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}\right)=\rho b_x+\frac{\partial \sigma_{xx}}{\partial x}+\frac{\partial \tau_{xy}}{\partial y}+\frac{\partial \tau_{xz}}{\partial z}, \tag{1.5}$$

$$\rho\left(\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+w\frac{\partial v}{\partial z}\right)=\rho b_y+\frac{\partial \tau_{yx}}{\partial x}+\frac{\partial \sigma_{yy}}{\partial y}+\frac{\partial \tau_{yz}}{\partial z}, \tag{1.6}$$

$$\rho\left(\frac{\partial w}{\partial t}+u\frac{\partial w}{\partial x}+v\frac{\partial w}{\partial y}+w\frac{\partial w}{\partial z}\right)=\rho b_z+\frac{\partial \tau_{zx}}{\partial x}+\frac{\partial \tau_{zy}}{\partial y}+\frac{\partial \sigma_{zz}}{\partial z}. \tag{1.7}$$

1.4 Constitutive equation for third grade fluid

A constitutive equation is a relation between stress and local properties of the fluid. For a fluid at rest the stress is determined wholly by the static pressure. Although in the case of a fluid in relative motion, the relation between the stress and the local properties of the fluid is more complicated, some modifications may be made such as the stress being dependent only on the instantaneous distribution of fluid velocity in the neighborhood of the element. This distribution may be expressed only in terms of velocity gradient components such as for a Newtonian fluid. However, non-Newtonian fluids cannot be described as simple as Newtonian fluids. One of the popular subclass of differential type non-Newtonian fluids is the model that is called the third grade fluid. The Cauchy stress tensor $\mathbf{T}$ in an incompressible third grade fluid is

$$\mathbf{T}=-p_1\mathbf{I}+\mu\mathbf{A}_1+\alpha_1\mathbf{A}_2+\alpha_2\mathbf{A}_1^2+\beta_1\mathbf{A}_3+\beta_2\left(\mathbf{A}_1\mathbf{A}_2+\mathbf{A}_2\mathbf{A}_1\right)+\beta_3\left(tr\mathbf{A}_1^2\right)\mathbf{A}_1, \tag{1.8}$$

in which $-p_1\mathbf{I}$ is the spherical part of the stress due to constraint of incompressibility, p_1 is the pressure, $\mathbf{I}$ is the identity tensor, μ is the dynamic viscosity, α_1 and α_2 are normal stresses and β_1, β_2 and β_3 are material constants. The Rivlin-Ericksen tensors $\mathbf{A}_1, \mathbf{A}_2, ...,$ are defined as

$$\mathbf{A}_1 = (grad\mathbf{V}) + (grad\mathbf{V})^T,$$

$$\mathbf{A}_{n+1} = (\frac{\partial}{\partial t} + \mathbf{V}\cdot\nabla)\mathbf{A}_n + (grad\mathbf{V})^T\mathbf{A}_n + \mathbf{A}_n(grad\mathbf{V}), \quad n > 1 \tag{1.9}$$

where $\mathbf{V}$ is the velocity and t is the time.

Dunn and Fosdick [15] have considered second grade fluid to be an exact model. The constitutive equation for a second grade fluid can easily be obtained by setting the values of material constants β_1, β_2 and β_3 equal to zero in equation (1.8). They studied the thermodynamics and stability of such a fluid in general and concluded that for the consistency of second grade fluid with thermodynamics, it is necessary that

$$\mu \geq 0, \quad \alpha_1 \geq 0, \quad \text{and} \quad \alpha_1 + \alpha_2 = 0. \tag{1.10}$$

Dunn and Rajagopal [1] have given their judgement on the status of the fluids of differential type. They state that if the material parameter α_1 is negative, the fluid exhibits undesirable stability properties. Fosdick and Rajagopal [14] showed that for a third grade fluid (1.8) to be consistent for thermodynamical consideration, the following constraints on material constants must satisfy

$$\mu \geq 0, \quad \alpha_1 \geq 0, \quad |\alpha_1 + \alpha_2| \leq \sqrt{24\mu\beta_3},$$

$$\beta_1 = 0, \qquad \beta_2 = 0, \qquad \beta_3 \geq 0. \tag{1.11}$$

Much work has been carried out to discuss thermodynamics and stability of non-Newtonian fluids by Joseph [16], Renardy [17] and Dundwoody [18]. An interesting consequence of Dunwoody's analysis is that the criterion which Renardy [17] has set to acquire instability for fluids of grade n can never be attained in any thermodynamically fluid of differential type. However, fluids of grade n which Renardy showed have an unstable rest state are incompatible with thermodynamics. This study establishes that thermodynamic incompatibility implies stability. Thus the mathematical content of their linear stability analysis and the thermodynamic results justify the frequent and intimate connection between thermodynamics and stability. We consider thermodynamic third grade model throughout this thesis.

1.5 The governing equation for a third grade fluid in a rotating system

In a rotating frame, equation (1.4) is

$$\rho\left[\frac{d\mathbf{V}}{dt}+2\boldsymbol{\Omega}\times\mathbf{V}+\boldsymbol{\Omega}\times(\boldsymbol{\Omega}\times\mathbf{r})\right]=\rho\mathbf{b}+div\mathbf{T}, \tag{1.12}$$

where ρ is the density, $\boldsymbol{\Omega}=\Omega\widehat{\mathbf{k}}$, $\widehat{\mathbf{k}}$ is a unit vector parallel to z-axis i.e. the axis of rotation, $\boldsymbol{\Omega}$ is the angular velocity, $\rho\mathbf{b}$ are the body forces, $d\mathbf{V}/dt$ denotes the substantial (material) derivative, $\rho\,(2\boldsymbol{\Omega}\times\mathbf{V})$ and $\rho\,[\boldsymbol{\Omega}\times(\boldsymbol{\Omega}\times\mathbf{r})]$ are the Coriolis and centripetal accelerations and $\mathbf{r}$ the radial coordinate given by

$$r^2=x^2+y^2. \tag{1.12a}$$

Coleman and Noll [63] defined the incompressible fluid of differential type of grade n as the simple fluid obeying the constitutive equation

$$\mathbf{T}\,(t)=-p_1\mathbf{I}+\sum_{j=1}^{n}\mathbf{Q}_j. \tag{1.13}$$

For $n = 3$, we have

$$\mathbf{Q}_1 = \mu \mathbf{A}_1, \tag{1.14}$$

$$\mathbf{Q}_2 = \alpha_1 \mathbf{A}_2 + \alpha_2 \mathbf{A}_1^2, \tag{1.15}$$

$$\mathbf{Q}_3 = \beta_1 \mathbf{A}_3 + \beta_2(\mathbf{A}_1\mathbf{A}_2 + \mathbf{A}_2\mathbf{A}_1) + \beta_3(tr\mathbf{A}_2)\mathbf{A}_1. \tag{1.16}$$

The Cauchy stress tensor $\mathbf{T}$ is given by equation (1.8) for a third grade fluid with thermodynamic constraints given in equation (1.11). We seek the velocity field in the following form

$$\mathbf{V}(z,t) = [u(z,t), v(z,t), w(z,t)]. \tag{1.17}$$

In view of equation (1.8), we have

$$\mathbf{A}_1 = \begin{pmatrix} 0 & 0 & \frac{\partial u}{\partial z} \\ 0 & 0 & \frac{\partial v}{\partial z} \\ \frac{\partial u}{\partial z} & \frac{\partial v}{\partial z} & 2\frac{\partial w}{\partial z} \end{pmatrix}, \tag{1.18}$$

$$\mathbf{A}_1^2 = \begin{pmatrix} \left(\frac{\partial u}{\partial z}\right)^2 & \frac{\partial u}{\partial z}\frac{\partial v}{\partial z} & 2\frac{\partial u}{\partial z}\frac{\partial w}{\partial z} \\ \frac{\partial u}{\partial z}\frac{\partial v}{\partial z} & \left(\frac{\partial v}{\partial z}\right)^2 & 2\frac{\partial v}{\partial z}\frac{\partial w}{\partial z} \\ 2\frac{\partial u}{\partial z}\frac{\partial w}{\partial z} & 2\frac{\partial v}{\partial z}\frac{\partial w}{\partial z} & 4\left(\frac{\partial w}{\partial z}\right)^2 \end{pmatrix}, \tag{1.19}$$

$$\mathbf{A}_2 = \begin{bmatrix} 0 & 0 & \frac{\partial^2 u}{\partial z \partial t} + w\frac{\partial^2 u}{\partial z^2} + \frac{\partial u}{\partial z}\frac{\partial w}{\partial z} \\ 0 & 0 & \frac{\partial^2 v}{\partial z \partial t} + w\frac{\partial^2 v}{\partial z^2} + \frac{\partial v}{\partial z}\frac{\partial w}{\partial z} \\ \frac{\partial^2 u}{\partial z \partial t} + w\frac{\partial^2 u}{\partial z^2} + \frac{\partial u}{\partial z}\frac{\partial w}{\partial z} & \frac{\partial^2 v}{\partial z \partial t} + w\frac{\partial^2 v}{\partial z^2} + \frac{\partial v}{\partial z}\frac{\partial w}{\partial z} & \begin{array}{c} 2\frac{\partial^2 w}{\partial z \partial t} + 2w\frac{\partial^2 w}{\partial z^2} \\ +2\left[\left(\frac{\partial u}{\partial z}\right)^2 + \left(\frac{\partial v}{\partial z}\right)^2\right] \end{array} \end{bmatrix}, \tag{1.20}$$

$$(tr\mathbf{A}_2)\,\mathbf{A}_1 = \begin{bmatrix} \widetilde{a}_1 & \widetilde{a}_2 & \widetilde{a}_3 \\ \widetilde{b}_1 & \widetilde{b}_2 & \widetilde{b}_3 \\ \widetilde{c}_1 & \widetilde{c}_2 & \widetilde{c}_3 \end{bmatrix}, \tag{1.21}$$

whence

$$\widetilde{a}_1 = 0,\ \ \widetilde{a}_2 = 0,\ \ \widetilde{a}_3 = \frac{\partial u}{\partial z}\left[\begin{array}{c} 2\frac{\partial^2 w}{\partial z \partial t} + 2w\frac{\partial^2 w}{\partial z^2} + 4(\frac{\partial w}{\partial z})^2 \\ +2\left[\left(\frac{\partial u}{\partial z}\right)^2 + \left(\frac{\partial v}{\partial z}\right)^2\right] \end{array}\right], \tag{1.22}$$

$$\widetilde{b}_1 = 0,\ \ \widetilde{b}_2 = 0,\ \ \widetilde{b}_3 = \frac{\partial v}{\partial z}\left[\begin{array}{c} 2\frac{\partial^2 w}{\partial z \partial t} + 2w\frac{\partial^2 w}{\partial z^2} + 4(\frac{\partial w}{\partial z})^2 \\ +2\left[\left(\frac{\partial u}{\partial z}\right)^2 + \left(\frac{\partial v}{\partial z}\right)^2\right] \end{array}\right], \tag{1.23}$$

$$\widetilde{c}_1 = \frac{\partial u}{\partial z}\left[\begin{array}{c} 2\frac{\partial^2 w}{\partial z \partial t} + 2w\frac{\partial^2 w}{\partial z^2} + 4(\frac{\partial w}{\partial z})^2 \\ +2\left[\left(\frac{\partial u}{\partial z}\right)^2 + \left(\frac{\partial v}{\partial z}\right)^2\right] \end{array}\right], \tag{1.24}$$

$$\widetilde{c}_2 = \frac{\partial v}{\partial z}\left[\begin{array}{c} 2\frac{\partial^2 w}{\partial z \partial t} + 2w\frac{\partial^2 w}{\partial z^2} + 4(\frac{\partial w}{\partial z})^2 \\ +2\left[\left(\frac{\partial u}{\partial z}\right)^2 + \left(\frac{\partial v}{\partial z}\right)^2\right] \end{array}\right], \tag{1.25}$$

$$\widetilde{c}_3 = 2\frac{\partial w}{\partial z}\left[\begin{array}{c} 2\frac{\partial^2 w}{\partial z \partial t} + 2w\frac{\partial^2 w}{\partial z^2} + 4(\frac{\partial w}{\partial z})^2 \\ +2\left[\left(\frac{\partial u}{\partial z}\right)^2 + \left(\frac{\partial v}{\partial z}\right)^2\right] \end{array}\right]. \tag{1.26}$$

Substituting above equations into scalar forms of equation (1.12) and neglecting the body forces we have

$$\begin{aligned} &\rho\left[\frac{\partial u}{\partial t} + w\frac{\partial u}{\partial z} - 2v\Omega - x\Omega^2\right] \\ = &-\frac{\partial p_1}{\partial x} + \mu\frac{\partial^2 u}{\partial z^2} \\ &+\alpha_1\left(\frac{\partial^3 u}{\partial z^2 \partial t} + w\frac{\partial^3 u}{\partial z^3} + 2\frac{\partial^2 u}{\partial z^2}\frac{\partial w}{\partial z} + \frac{\partial^2 w}{\partial z^2}\frac{\partial u}{\partial z}\right) \\ &+2\alpha_2\left(\frac{\partial^2 u}{\partial z^2}\frac{\partial w}{\partial z} + \frac{\partial^2 w}{\partial z^2}\frac{\partial u}{\partial z}\right) \\ &+\beta_3\frac{\partial}{\partial z}\left[\frac{\partial u}{\partial z}\left[\begin{array}{c} 2\frac{\partial^2 w}{\partial z \partial t} + 2w\frac{\partial^2 w}{\partial z^2} + 4(\frac{\partial w}{\partial z})^2 \\ +2\left[\left(\frac{\partial u}{\partial z}\right)^2 + \left(\frac{\partial v}{\partial z}\right)^2\right] \end{array}\right]\right], \end{aligned} \tag{1.27}$$

$$
\begin{aligned}
&\rho\left[\frac{\partial v}{\partial t}+w\frac{\partial v}{\partial z}+2u\Omega-y\Omega^2\right]\\
=\;&-\frac{\partial p_1}{\partial y}+\mu\frac{\partial^2 v}{\partial z^2}\\
&+\alpha_1\left(\frac{\partial^3 v}{\partial z^2\partial t}+w\frac{\partial^3 v}{\partial z^3}+\frac{\partial^2 v}{\partial z^2}\frac{\partial w}{\partial z}+\frac{\partial^2 w}{\partial z^2}\frac{\partial v}{\partial z}\right)\\
&+2\alpha_2\left(\frac{\partial^2 v}{\partial z^2}\frac{\partial w}{\partial z}+\frac{\partial^2 w}{\partial z^2}\frac{\partial v}{\partial z}\right)\\
&+\beta_3\frac{\partial}{\partial z}\left[\frac{\partial v}{\partial z}\left[\begin{array}{c}2\frac{\partial^2 w}{\partial z\partial t}+2w\frac{\partial^2 w}{\partial z^2}+4(\frac{\partial w}{\partial z})^2\\ +2\left[\left(\frac{\partial u}{\partial z}\right)^2+\left(\frac{\partial v}{\partial z}\right)^2\right]\end{array}\right]\right],
\end{aligned}
\tag{1.28}
$$

$$
\begin{aligned}
&\rho\left[\frac{\partial w}{\partial t}+w\frac{\partial w}{\partial z}\right]\\
=\;&-\frac{\partial p_1}{\partial z}+2\mu\frac{\partial^2 w}{\partial z^2}\\
&+2\alpha_1\left(\frac{\partial^3 w}{\partial z^2\partial t}+w\frac{\partial^3 w}{\partial z^3}+\frac{\partial^2 v}{\partial z^2}\frac{\partial w}{\partial z}\right)\\
&+(2\alpha_1+\alpha_2)\frac{\partial}{\partial z}\left[\left(\frac{\partial u}{\partial z}\right)^2+\left(\frac{\partial v}{\partial z}\right)^2\right]+8\alpha_2\left(\frac{\partial w}{\partial z}\right)^2\frac{\partial w}{\partial z}\\
&+\beta_3\frac{\partial}{\partial z}\left[4\frac{\partial w}{\partial z}\left[\begin{array}{c}2\frac{\partial^2 w}{\partial z\partial t}+2w\frac{\partial^2 w}{\partial z^2}+4(\frac{\partial w}{\partial z})^2\\ +2\left[\left(\frac{\partial u}{\partial z}\right)^2+\left(\frac{\partial v}{\partial z}\right)^2\right]\end{array}\right]\right].
\end{aligned}
\tag{1.29}
$$

For the case of uniform suction, equation of continuity gives $w=-W_0$ and equations (1.27) to (1.29) reduce to

$$
\begin{aligned}
&\rho\left[\frac{\partial u}{\partial t}-W_0\frac{\partial u}{\partial z}-2v\Omega\right]\\
=\;&-\frac{\partial p^*}{\partial x}+\mu\frac{\partial^2 u}{\partial z^2}\\
&+\alpha_1\left(\frac{\partial^3 u}{\partial z^2\partial t}-W_0\frac{\partial^3 u}{\partial z^3}\right)\\
&+\beta_3\frac{\partial}{\partial z}\left[2\frac{\partial u}{\partial z}\left[\left(\frac{\partial u}{\partial z}\right)^2+\left(\frac{\partial v}{\partial z}\right)^2\right]\right],
\end{aligned}
\tag{1.30}
$$

$$
\begin{aligned}
&\rho\left[\frac{\partial v}{\partial t}-W_0\frac{\partial v}{\partial z}+2u\Omega\right]\\
=\;&-\frac{\partial p^*}{\partial y}+\mu\frac{\partial^2 v}{\partial z^2}\\
&+\alpha_1\left(\frac{\partial^3 v}{\partial z^2\partial t}-W_0\frac{\partial^3 v}{\partial z^3}\right)\\
&+\beta_3\frac{\partial}{\partial z}\left[2\frac{\partial v}{\partial z}\left[\left(\frac{\partial u}{\partial z}\right)^2+\left(\frac{\partial v}{\partial z}\right)^2\right]\right],
\end{aligned}
\tag{1.31}
$$

$$
\frac{\partial p^*}{\partial z}=0, \tag{1.32}
$$

where $W_0 < 0$ is the blowing velocity, the modified pressure

$$
p^*=p_1-\frac{1}{2}\rho r^2\Omega^2-(2\alpha_1+\alpha_2)\frac{\partial}{\partial z}\left[\left(\frac{\partial u}{\partial z}\right)^2+\left(\frac{\partial v}{\partial z}\right)^2\right] \tag{1.33}
$$

and equation (1.32) indicates that p^* does not depend on z. In deriving equations (1.30) to (1.32), equation (1.11) has been used.

1.6 Partial slip boundary conditions

The pioneers of the fluid mechanics such as Coulomb, Navier, Girad, Poisson, Stokes, St. Venant and others recognized that boundary conditions are constitutive equations that are determined by the material on either side of the boundary. The usual prescription of Dirichlet and Neumann conditions are not suitable for a realistic physical problem, for example the flow of polymers that stick-slip on the boundary. Recently non-standard boundary conditions have been considered from a rigorous mathematical perspective by Rao and Rajagopal [27]. The solution to the Stokes problem under vibrating wall condition that satisfies non-slip conditions at the wall has been studied in depth by number of authors. The unsteady Couette flow problem has been considered by several workers containing various effects with no-slip condition. However, the literature lacks studies that take into account the possibility of fluid slippage at the walls under vibrating conditions. This problem appears in some applications such as in micro-channels and in applications where a thin film of light oil is attached to the moving plates or when the surface is coated with special coatings such as a thick monolayer of hydrophobic octadecytrichlorosilane. The wall slip can occur if the working fluid contains concentrated suspensions.

In this thesis, the partial slip conditions have been defined in terms of the shear stresses. The general formulas for τ_{xz} and τ_{yz} are

$$
\begin{aligned}
\tau_{xz} &= \left(-p_1\mathbf{I}+\mu\mathbf{A}_1+\alpha_1\mathbf{A}_2+\alpha_2\mathbf{A}_1^2++\beta_3\left(tr\mathbf{A}_1^2\right)\mathbf{A}_1\right)_{xz} \\
&= \mu\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}\right)+\alpha_1\left[\frac{\partial}{\partial t}\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}\right)+u\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}\right)\right. \\
&+v\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}\right)+w\frac{\partial}{\partial z}\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}\right)+3\frac{\partial u}{\partial x}\frac{\partial u}{\partial z} \\
&\left.+\frac{\partial v}{\partial z}\frac{\partial u}{\partial y}+2\frac{\partial v}{\partial z}\frac{\partial v}{\partial x}+\frac{\partial w}{\partial z}\frac{\partial u}{\partial z}+3\frac{\partial w}{\partial x}\frac{\partial w}{\partial z}+\frac{\partial u}{\partial x}\frac{\partial w}{\partial x}+\frac{\partial v}{\partial x}\frac{\partial w}{\partial y}\right] \\
&+\alpha_2\left[2\frac{\partial u}{\partial x}\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}\right)+\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial y}\right)\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)\right. \\
&\left.+2\frac{\partial w}{\partial z}\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}\right)\right]+\beta_3\left[4\left(\frac{\partial u}{\partial x}\right)^2\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}\right)\right. \\
&+4\left(\frac{\partial v}{\partial y}\right)^2\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}\right)+4\left(\frac{\partial w}{\partial z}\right)^2\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}\right) \\
&+\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)^2\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}\right)+2\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}\right)^3
\end{aligned}
$$

$$+2\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}\right)\left(\frac{\partial v}{\partial z}+\frac{\partial w}{\partial y}\right)\Bigg], \tag{1.34}$$

$$\tau_{yz}=\left(-p_1\mathbf{I}+\mu\mathbf{A}_1+\alpha_1\mathbf{A}_2+\alpha_2\mathbf{A}_1^2++\beta_3\left(tr\mathbf{A}_1^2\right)\mathbf{A}_1\right)_{yz}$$

$$=\mu\left(\frac{\partial v}{\partial z}+\frac{\partial w}{\partial y}\right)+\alpha_1\left[\frac{\partial}{\partial t}\left(\frac{\partial v}{\partial z}+\frac{\partial w}{\partial y}\right)+u\frac{\partial}{\partial x}\left(\frac{\partial v}{\partial z}+\frac{\partial w}{\partial y}\right)\right.$$

$$+v\frac{\partial}{\partial y}\left(\frac{\partial v}{\partial z}+\frac{\partial w}{\partial y}\right)+w\frac{\partial}{\partial z}\left(\frac{\partial v}{\partial z}+\frac{\partial w}{\partial y}\right)+3\frac{\partial u}{\partial y}\frac{\partial u}{\partial z}$$

$$\left.+\frac{\partial u}{\partial z}\frac{\partial v}{\partial x}+3\frac{\partial v}{\partial y}\frac{\partial v}{\partial z}+\frac{\partial w}{\partial z}\frac{\partial v}{\partial z}+3\frac{\partial w}{\partial y}\frac{\partial w}{\partial z}+\frac{\partial w}{\partial x}\frac{\partial u}{\partial y}+\frac{\partial v}{\partial y}\frac{\partial w}{\partial y}\right]$$

$$+\alpha_2\left[\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}\right)+2\frac{\partial v}{\partial y}\left(\frac{\partial v}{\partial z}+\frac{\partial w}{\partial y}\right)\right.$$

$$\left.+2\frac{\partial w}{\partial z}\left(\frac{\partial v}{\partial z}+\frac{\partial w}{\partial y}\right)\right]+\beta_3\left[4\left(\frac{\partial u}{\partial x}\right)^2\left(\frac{\partial v}{\partial z}+\frac{\partial w}{\partial y}\right)\right.$$

$$+4\left(\frac{\partial v}{\partial y}\right)^2\left(\frac{\partial v}{\partial z}+\frac{\partial w}{\partial y}\right)+4\left(\frac{\partial w}{\partial z}\right)^2\left(\frac{\partial v}{\partial z}+\frac{\partial w}{\partial y}\right)$$

$$+2\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)^2\left(\frac{\partial v}{\partial z}+\frac{\partial w}{\partial y}\right)+2\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}\right)^2\left(\frac{\partial v}{\partial z}+\frac{\partial w}{\partial y}\right)$$

$$\left.+2\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}\right)^3\right] \tag{1.35}$$

which for the velocity field under consideration yield

$$\tau_{xz}=\mu\frac{\partial u}{\partial z}-\alpha_1W_0\frac{\partial^2 u}{\partial z^2}+2\beta_3\frac{\partial u}{\partial z}\left[\left(\frac{\partial u}{\partial z}\right)^2+\left(\frac{\partial v}{\partial z}\right)^2\right], \tag{1.36}$$

$$\tau_{yz}=\mu\frac{\partial v}{\partial z}-\alpha_1W_0\frac{\partial^2 v}{\partial z^2}+2\beta_3\frac{\partial v}{\partial z}\left[\left(\frac{\partial u}{\partial z}\right)^2+\left(\frac{\partial v}{\partial z}\right)^2\right]. \tag{1.37}$$

1.7 Mathematical techniques

1.7.1 Homotopy analysis method (HAM)

HAM is an approximate analytical technique for solving non-linear problems which is introduced by Liao [38]. This technique overcomes the limitations and restrictions of perturbation techniques. This method is based on the fundamental concept of 'homotopy of topology' [66]. It does not depend on small parameter assumptions of perturbation technique and can be applied to even those problems whose governing equations and boundary conditions do not contain any small parameters. In fact, this method provides great flexibility to select auxiliary linear operators and initial approximations.

Homotopy analysis method is a kind of linear property of homotopy, which transforms a non-linear problem into infinite number of linear sub-problems. This transformation is independent of small parameters. This is an effective and simple method. For more clarity, we consider the following differential equation

$$\mathcal{A}\left[x(t)\right] = 0, \tag{1.38}$$

where $\mathcal{A}$ is a non-linear operator, t is time and $x(t)$ is an unknown variable and $\mathcal{L}$ denotes an auxiliary linear operator with the property

$$\mathcal{L}F = 0 \qquad \text{when} \qquad F = 0, \tag{1.39}$$

in which F is the solution of the linear part of the non-linear differential equation.

Introducing a non-zero auxiliary parameter $\hbar$ to construct the homotopy as

$$\mathcal{H}\left[\overline{x}(t;p);p,\hbar\right] = (1-p)\ \mathcal{L}\overline{x}(t;p) - x_0(t) + p\hbar\mathcal{A}\left[\overline{x}(t;p)\right]. \tag{1.40}$$

Here $x_0(t)$ is an initial approximation of $x(t)$, $p \in [0,1]$ is an embedding parameter and $\overline{x}(t;p)$ is a function of t and p. For $p=0$ and $p=1$, we have the following results

$$\mathcal{H}\left[\overline{x}(t;p);p,\hbar\right]\Big|_{p=0} = \mathcal{L}\left[\overline{x}(t;p) - x_0(t)\right] \tag{1.40a}$$

and

$$\mathcal{H}\left[\overline{x}(t;p);p,\hbar\right]\Big|_{p=1} = \hbar\mathcal{A}\left[\overline{x}(t;p)\right]. \tag{1.40b}$$

Employing equation (1.39)

$$\overline{x}(t;0) = x_0(t) \tag{1.40c}$$

is a solution of the equation

$$\mathcal{H}\left[\overline{x}(t;p);p,\hbar\right]\Big|_{p=0} = 0, \tag{1.40d}$$

and

$$\overline{x}(t;1) = x(t) \tag{1.40e}$$

is a solution of the equation

$$\mathcal{H}\left[\overline{x}(t;p);p,\hbar\right]\Big|_{p=1} = 0. \tag{1.40f}$$

When p increases from 0 to 1, the solution $\overline{x}(t;p)$ of the equation

$$\mathcal{H}\left[\overline{x}(t;p);p,\hbar\right] = 0 \tag{1.40g}$$

depends upon the embedding parameter p and varies from the initial approximation $x_0(t)$ to the solution $x(t)$ of equation (1.38). In topology, such a kind of continuous variation is called deformation.

In order to illustrate HAM, we consider the following example

$$\frac{d^2F}{dt^2}+2F=F^2, \qquad t\geq 0, \tag{1.41}$$

subject to following boundary conditions

$$F(0)=1, \quad F(\infty)=0. \tag{1.42}$$

We select the following auxiliary linear operator

$$\mathcal{L}=\frac{d^2}{dt^2}+2. \tag{1.43}$$

The initial guess approximation of the problem is obtained by applying the auxiliary operator (1.43) on unknown function $F_0(t)$ along with boundary conditions (1.42). This gives

$$F_0(t)=e^{-k_1 t} \tag{1.44}$$

where

$$k_1=\sqrt{2}i,$$

$F_0(t)$ is an initial guess approximation of $F(t;p)$. Now introducing a non-zero auxiliary parameter $\hbar_0$ to construct the so-called zeroth-order deformation problem as

$$(1-p)\mathcal{L}\left[\overline{F}(t;p)-F_0(t)\right]=p\hbar_0\left[\frac{\partial^2\overline{F}}{\partial t^2}+\overline{F}-\overline{F}^2\right], \tag{1.45}$$

$$\overline{F}(0;p)=1 \quad \text{and} \quad \overline{F}(\infty;p)=0, \tag{1.46}$$

where $\hbar_0$ is an auxiliary parameter. For $p=0$ and $p=1$, we have

$$\overline{F}(t;0)=F_0(t) \quad \text{and} \quad \overline{F}(t;1)=F(t). \tag{1.47}$$

Assume that the deformation $\overline{F}(t;p)$ is smooth enough. If the auxiliary parameter $\hbar_0$ is properly selected such that the zero-order deformation problem (1.45) and (1.46) has solution for all $p \in [0,1]$ and that there exists the mth-order derivative as

$$F_m(t) = \frac{1}{m!} \frac{\partial^m \overline{F}(t;p)}{\partial p^m}\bigg|_{p=0} \qquad (m \geq 1) \tag{1.48}$$

then by Taylor's theorem

$$\overline{F}(t;p) = F_0(t) + \sum_{m=1}^{+\infty} F_m(t) p^m.$$

Furthermore, assuming that $\hbar_0$ is properly chosen that the power series is convergent at $p = 1$, then

$$F(t) = F_0(t) + \sum_{m=1}^{+\infty} F_m(t) \tag{1.49}$$

Differentiating the zero-order deformation problem (1.45) and (1.46) m-times with respect to p, dividing it by $m!$ and setting $p = 0$, we have the mth-order deformation problem as

$$\mathcal{L}\left[F_m(t) - \chi_m F_{m-1}(t)\right] = \hbar_0 \left[\overset{\circ\circ}{F}_{m-1} + 2F_{m-1} - \sum_{i=0}^{m-1} F_{m-1} F_i\right], \tag{1.50}$$

$$F_m(0) = 0,$$

and

$$F_m(t) \longrightarrow 0 \text{ as } t \longrightarrow \infty \tag{1.51}$$

in which dot denotes the derivatives with respect to t.

Solving the problem consisting of equations (1.50) and (1.51) up to second-order of approximation, the three terms solution is

$$F(t) = F_0(t) + F_1(t) + F_2(t).$$

For $\hbar_0 = -1$, the perturbation solution can be recovered from the above equation.

1.7.2 Numerical technique

The mathematical formulation of most of the physical problems in science that involve rate of change with respect to two or more independent variables representing time, length or angle leads either to partial differential equations or to a set of such equations. For these problems approximation methods whether analytical or numerical are the only means of solution. Analytical approximation methods often provide extremely useful information concerning the character of the solution for critical values of the dependent variables but tend to be more difficult to apply than the numerical methods. Amongst the numerical approximation methods available for solving differential equations, finite difference and finite element methods are more frequently used and more universally applicable than any other [67].

Finite difference method is an approximate method in the sense that derivatives at a point are approximated by difference quotients over a small interval. Finite difference method generally give solutions that are as accurate as the data warrants or as is necessary for the technical purpose for which the solutions are required.

Assume that a function F and its derivatives are single-valued, finite and continuous functions of z, then by Taylor's theorem:

$$F(z+h) = F(z) + hF'(z) + \frac{h^2}{2}F''(z) + \frac{h^3}{6}F'''(z) + O(h^4) \qquad (1.52)$$

and

$$F(z-h)=F(z)-hF'(z)+\frac{h^2}{2}F''(z)-\frac{h^3}{6}F'''(z)+O(h^4), \qquad (1.53)$$

where $O(h^4)$ denotes terms containing fourth and higher powers of h. These expansions give

$$\left(\frac{dF}{dz}\right)_{z=z} \simeq \frac{F(z+h)-F(z)}{h} \qquad (1.54)$$

and

$$\left(\frac{dF}{dz}\right)_{z=z} \simeq \frac{F(z)-F(z-h)}{h} \qquad (1.55)$$

with an error of order h. We assume that the second and higher powers of h are negligible. Equations (1.54) and (1.55) are called forward difference formula and backward difference formula respectively.

Subtraction of the equation (1.53) from (1.52) gives

$$\left(\frac{dF}{dz}\right)_{z=z} \simeq \frac{F(z+h)-F(z-h)}{2h} \qquad (1.56)$$

with a leading error on the right-hand side of order h^2. This approximation is called a central difference formula.

The addition of equations (1.52) and (1.53) leads to approximation of second-order derivatives

$$\left(\frac{d^2F}{dz^2}\right)_{z=z} \simeq \frac{F(z+h)-2F(z)+F(z-h)}{h^2} \qquad (1.57)$$

with an error of order h^2on the right-hand.

Similarly to approximate the third-order derivative and higher order derivatives, we make use of the previous terms. Adopting this procedure, we get the approximation for the third derivative as

$$\left(\frac{d^3F}{dz^3}\right)_{z=z} \simeq \frac{F(z+2h)-2F(z+h)+2F(z-h)-F(z-2h)}{2h^3}. \qquad (1.58)$$

Now to approximate partial derivatives of a function F of two independent continuous variables z and t, such as $\partial F/\partial t$, $\partial F/\partial z$, $\partial^2 F/\partial z^2$ and so on by finite difference method, the continuous variables z and t are discretized. For that $F(z,\, t)$ is evaluated only at the intersections i.e. the mesh points of the grid lines parallel to z and t- axes. The coordinates (z, t) of the mesh point are defined as

$$z = ih, \qquad t = jl, \tag{1.58a}$$

where $i,\ j$ are the integers, and $h,\ l$ are the constant grid spacings in the z and t direction respectively. The value of F at the mesh point is defined as

$$F_i^j = F(ih,\ jl). \tag{1.58b}$$

Using the same concept introduced for ordinary derivatives, we get the following centre-difference approximations for partial derivatives

$$\begin{aligned}(F_z)_i^j &\simeq \frac{1}{2h}\{F((i+1)h, jl) - F((i-1)h, jl)\}\\ &\simeq \frac{1}{2h}\left\{F_{i+1}^j - F_{i-1}^j\right\},\end{aligned} \tag{1.59}$$

$$(F_{zz})_i^j \simeq \frac{1}{h^2}\left\{F_{i+1}^j - 2F_i^j + F_{i-1}^j\right\}, \tag{1.60}$$

$$(F_{zzz})_i^j \simeq \frac{1}{2h^3}\left\{F_{i+2}^j - 2F_{i+1}^j + 2F_{i-1}^j - F_{i-2}^j\right\}. \tag{1.61}$$

The forward-difference approximation for F_t and $(F)_{zzz}$ at the same mesh point will be

$$F_t \simeq \frac{F_i^{j+1} - F_i^j}{l} \tag{1.62}$$

and

$$(F_{zzz})_i^j \simeq \frac{1}{h^3}\left\{F_{i+2}^j - 3F_{i+1}^j + 3F_i^j - F_{i-1}^j\right\}. \tag{1.63}$$

The Crank-Nicolson method provides an alternative implicit scheme for the approximations of partial derivatives. For this purpose, the approximations of partial derivatives are developed at the mid-point of time increment region, the function F is also approximated at the mid-point of time increment as

$$F_i^{j+\frac{1}{2}} \simeq \frac{1}{2}\left\{F_i^{j+1} + F_i^j\right\}. \tag{1.64}$$

The first partial derivative with respect to time can be approximated at $j+\frac{1}{2}$ time level

$$(F_t)_i^{j+\frac{1}{2}} \simeq \frac{1}{l}\left\{F_i^{j+1} - F_i^j\right\} \tag{1.65}$$

and the spatial derivative can be approximated at the mid-point by averaging the difference approximations at the jth and $(j+1)$th time levels

$$(F_z)_i^{j+\frac{1}{2}} \simeq \frac{1}{4h}\left[\left\{F_{i+1}^{j+1} - F_{i-1}^{j+1}\right\} + \left\{F_{i+1}^{j} - F_{i-1}^{j}\right\}\right]. \tag{1.66}$$

Similarly, the second and third spatial derivatives can be approximated at the mid-point by means of averaging as

$$\begin{aligned}(F_{zz})_i^{j+\frac{1}{2}} &\simeq \frac{1}{2h^2}\left[\left\{F_{i+1}^{j+1} - 2F_i^{j+1} + F_{i-1}^{j+1}\right\}\right. \\ &\quad \left. + \left\{F_{i+1}^{j} - 2F_i^{j} + F_{i-1}^{j}\right\}\right]\end{aligned} \tag{1.67}$$

and

$$\begin{aligned}(F_{zzz})_i^{j+\frac{1}{2}} &\simeq \frac{1}{4h^3}\left\{F_{i+2}^{j+1} - 2F_{i+1}^{j+1} + 2F_{i-1}^{j+1} - F_{i-2}^{j+1}\right. \\ &\quad \left. + F_{i+2}^{j} - 2F_{i+1}^{j} + 2F_{i-1}^{j} - F_{i-2}^{j}\right\}.\end{aligned} \tag{1.68}$$

The partial derivative of the form $\partial^3 F/\partial t \partial z^2$ is approximated at the mid-point as

$$
\begin{aligned}
& (F_{zzt})_i^{j+\frac{1}{2}} \\
\simeq\ & \frac{1}{l}\left\{(F_{zz})_i^{j+1} - (F_{zz})_i^j\right\} \\
=\ & \frac{1}{lh^2}\left[\left\{F_{i+1}^{j+1} - 2F_i^{j+1} + F_{i-1}^{j+1}\right\} - \left\{F_{i+1}^j - 2F_i^j + F_{i-1}^j\right\}\right]. \quad (1.69)
\end{aligned}
$$

1.7.3 Lighthill technique

The method of strained coordinates is a technique for dealing with certain types of non-uniformities which occur in asymptotic expansions. Various methods have been devised to overcome this difficulty and to determine a uniformly valid expansion. To deal with such problems, two related methods are known as Lindstedt-Poincare technique and Lighthill technique [65]. The former applies to systems which are periodic, where the period of motion is changed by a perturbation. It can be applied to various oscillators such as mechanical spring and mass systems, electrical systems and planetary motion.

Lighthill's method is a generalization of the Lindstedt-Poincare method which enables strained coordinates to be applied to a far wider class of problems. The method has been found to be of particular value in the solution of the partial differential equations which occurs in fluid dynamics. This method has been applied to many branches of continuum mechanics including flow past aero-foils and wave propagation in solids and fluids. It is applied to partial differential equations where one or more of the independent variables are strained. The straining may be applied to space and/or time variables or to the combination known as characteristic variables in the theory of hyperbolic partial differential equations.

Consider the application of Lighthill's technique to the study of flow past thin aero-foils. We denote the original independent variable by z and the strained coordinate by s.

Lighthill uses the straining transformation

$$z \sim s + \varepsilon f_1(s) + \varepsilon^2 f_2(s) + \cdots, \tag{1.70}$$

where the coefficients of ε^n are functions of the strained coordinates. The Lindstedt-Poincare transformation is of the form

$$\frac{s}{z} \sim 1 + \varepsilon\omega_1 + \varepsilon^2\omega_2 + \cdots \tag{1.71}$$

thus

$$z \sim \frac{s}{1 + \varepsilon\omega_1 + \varepsilon^2\omega_2 + \cdots} = s(1 - \varepsilon\omega_1 - \varepsilon^2\omega_2 - \cdots + \varepsilon^2\omega_1 + \cdots). \tag{1.72}$$

This shows that the Lindstedt-Poincare transformation is a special case of Lighthill's with $f_1 = -\omega_1 s$, $f_2 = (\omega_1 - \omega_2)s$ and in general $f_n(s) = a_n s$ where the a_n are constants.

The standard procedure with Lighthill's technique is to introduce the new variable into the governing equation and boundary or initial conditions using the relation

$$\begin{aligned}\frac{d}{dz} &= \frac{ds}{dz}\frac{d}{ds} = \left(\frac{dz}{ds}\right)^{-1}\frac{d}{ds} \sim \frac{1}{1 + \varepsilon\frac{df_1}{ds} + \varepsilon^2\frac{df_2}{ds} + \cdots}\frac{d}{ds} \\ &\sim \left\{1 - \varepsilon\frac{df_1}{ds} - \varepsilon^2\left[\frac{df_2}{ds} - \left(\frac{df_1}{ds}\right)^2\right] + \cdots\right\}\frac{d}{ds}.\end{aligned} \tag{1.73}$$

The procedure is analogous to the Lindstedt-Poincare technique in that an expansion of the form

$$u(s, \varepsilon) \sim u_0(s) + \varepsilon u_1(s) + \cdots \tag{1.74}$$

is assumed for the dependent variable. This is substituted into the transformed governing equation and order equations generated for u_0, u_1 etc. with associated boundary/initial conditions.

Chapter 2

Steady flow of a third grade fluid in a rotating frame

This chapter deals with the steady flow of an incompressible third grade fluid past a porous plate. The whole system is in a rotating frame of reference with and without slip. Analytic solutions of the non-linear problems are based on the homotopy analysis method. Recurrence formulas for the coefficients arising in the series solutions are presented. Convergence of the series solutions is explicitly analyzed. Finally, attention is focused on the effects of suction/blowing, rotation, second grade, third grade and slip parameters.

2.1 Mathematical problem for no-slip case

We consider a Cartesian coordinate system rotating uniformly with an angular velocity $\boldsymbol{\Omega}$ about the $z-$axis, taken positive in the vertically upward direction and the plate coinciding with the plane $z = 0$. The fluid flowing past a porous plate is third grade and incompressible. All material parameters of the fluid are assumed to be constants. For a rotating frame, the momentum equation (1.12) is considered in which $\mathbf{T}$ is the Cauchy stress tensor for third grade fluid as given in equation (1.8). From thermodynamical considerations, the material constants in equation (1.8) must be satisfied by (1.11). Under thermodynamical considerations, the Cauchy stress tensor of a third grade fluid is

$$\mathbf{T} = -p_1\mathbf{I} + \mu\mathbf{A}_1 + \alpha_1\mathbf{A}_2 + \alpha_2\mathbf{A}_1^2 + \beta_3(tr\mathbf{A}_1^2)\mathbf{A}_1. \tag{2.1}$$

For a uniform porous boundary, the continuity equation is satisfied if

$$\mathbf{V} = [u(z), v(z), -W_0], \tag{2.2}$$

where u and v are $x-$ and $y-$components of velocity and $W_0 > (<)\ 0$ corresponds to suction (blowing) velocity respectively.

In view of equations (1.3), (1.11), (1.12), (2.1) and (2.2), we have from equation (1.8) as

$$\rho\left[-W_0\frac{du}{dz} - 2v\Omega\right] = \mu\frac{d^2u}{dz^2} - \alpha_1 W_0\frac{d^3u}{dz^3} + 2\beta_3\frac{d}{dz}\left[\frac{du}{dz}\left\{\left(\frac{du}{dz}\right)^2 + \left(\frac{dv}{dz}\right)^2\right\}\right], \tag{2.3}$$

$$\begin{aligned}\rho\left[-W_0\frac{dv}{dz} + 2u\Omega\right] &= 2\Omega U_0\rho + \mu\frac{d^2v}{dz^2} - \alpha_1 W_0\frac{d^3v}{dz^3} \\ &\quad + 2\beta_3\frac{d}{dz}\left[\frac{dv}{dz}\left\{\left(\frac{du}{dz}\right)^2 + \left(\frac{dv}{dz}\right)^2\right\}\right],\end{aligned} \tag{2.4}$$

where U_0 denotes the uniform velocity outside the layer which is caused by the pressure gradient.

The appropriate boundary conditions are

$$u = v = 0 \text{ at } z = 0, \quad u \longrightarrow U_0, \quad v \longrightarrow 0 \text{ as } z \longrightarrow \infty. \tag{2.5}$$

Defining

$$F = \frac{u+iv}{U_0} - 1, \quad F^* = \frac{u-iv}{U_0} - 1 \tag{2.6}$$

equations (2.3) to (2.5) can be combined as

$$2i\Omega F - W_0\frac{dF}{dz} = \frac{1}{\rho}\left[\frac{d^2F}{dz^2} - \alpha W_0\frac{d^3F}{dz^3} + 2\beta_3\frac{d}{dz}\left\{\left(\frac{dF}{dz}\right)^2\frac{dF^*}{dz}\right\}\right] \tag{2.7}$$

subject to following boundary conditions

$$F(z) = -1 \text{ at } z = 0, \quad F(z) \longrightarrow 0 \text{ as } z \longrightarrow \infty \tag{2.8}$$

where F^* is conjugate of F.

It is convenient to introduce the following dimensionless quantities

$$\begin{aligned} \widehat{z} &= \frac{\rho U_0 z}{\mu}, \quad \widehat{F} = \frac{F}{U_0}, \quad \widehat{W_0} = \frac{W_0}{U_0}, \\ \widehat{\Omega} &= \frac{\Omega\mu}{\rho U_0^2}, \quad \widehat{\beta} = \frac{\beta_3\rho^2U_0^4}{\mu^3}, \quad \widehat{\alpha} = \frac{\alpha_1\rho U_0^2}{\mu^2}. \end{aligned} \tag{2.9}$$

After dropping hats, the resulting problem consists of conditions (2.8) and the following differential equation

$$\frac{d^2F}{dz^2} - 2i\Omega F + W_0\left[\frac{dF}{dz} - \alpha\frac{d^3F}{dz^3}\right] + 2\beta\frac{d}{dz}\left[\left(\frac{dF}{dz}\right)^2\frac{dF^*}{dz}\right] = 0. \tag{2.10}$$

Since equation (2.10) is a third-order differential equation which is higher than the governing equation of the Newtonian fluid. Therefore, we need one more condition. The flow under consideration is in an unbounded domain , so by augmentation of the boundary conditions (2.8), we have

$$\frac{dF}{dz} \longrightarrow 0 \text{ as } z \longrightarrow \infty. \tag{2.11}$$

2.1.1 Solution of the problem for no-slip case

Here, we give an analytic and uniformly valid solution by homotopy analysis method. For that we use the initial guess approximation

$$F_0(z) = -e^{-z} \tag{2.12}$$

and the auxiliary linear operator

$$\mathcal{L}_1 = \frac{d^2}{dz^2} - 1 \tag{2.13}$$

satisfying the property

$$\mathcal{L}_1\left[c_1 e^{z} + c_2 e^{-z}\right] = 0, \tag{2.14}$$

where c_1 and c_2 are arbitrary constants.

The deformation problem at the zeroth-order satisfies

$$(1-p)\mathcal{L}_1\left[\overline{F}(z;p) - F_0(z)\right]$$
$$= p\hbar_1 \left[\begin{array}{c} \frac{\partial^2 \overline{F}(z;p)}{\partial z^2} - 2i\Omega \overline{F}(z;p) + W_0\left(\frac{\partial \overline{F}(z;p)}{\partial z} - \alpha \frac{\partial^3 \overline{F}(z;p)}{\partial z^3}\right) \\ +2\beta \frac{\partial}{\partial z}\left\{\left(\frac{\partial \overline{F}(z;p)}{\partial z}\right)^2 \frac{\partial \overline{F}^*(z;p)}{\partial z}\right\} \end{array}\right], \tag{2.15}$$

where $\hbar_1$ is an auxiliary parameter and $p \in [0,1]$ is an embedding parameter. The boundary conditions take the form

$$\overline{F}(0;p) = -1 \text{ as } z \longrightarrow 0, \quad \overline{F}(z;p) \longrightarrow 0 \text{ as } z \longrightarrow \infty,$$
$$\frac{\partial \overline{F}(z;p)}{\partial z} \longrightarrow 0 \text{ as } z \longrightarrow \infty. \tag{2.16}$$

For $p = 0$ and $p = 1$, we have from equation (2.15)

$$\overline{F}(z;0) = F_0(z), \qquad \overline{F}(z;1) = F(z). \tag{2.17}$$

We note from the above equations that the variation of p from 0 to 1 continuously varies $\overline{F}(z;p)$ from the initial guess $F_0(z)$ to the exact solution $F(z)$. Due to Taylor's theorem and equation (2.17) one obtains

$$\overline{F}(z;p) = F_0(z) + \sum_{m=1}^{\infty} F_m(z)p^m \tag{2.18}$$

in which

$$F_m(z) = \frac{1}{m!}\frac{\partial^m \overline{F}(z;p)}{\partial p^m}\bigg|_{p=0} \quad (m \geq 1). \tag{2.19}$$

Assuming that $\hbar_1$ is properly chosen such that the series (2.18) is convergent at $p = 1$, we have from equation (2.17) that

$$F(z) = F_0(z) + \sum_{m=1}^{+\infty} F_m(z). \tag{2.20}$$

Differentiating m-times the zero-order deformation equations (2.15) and (2.16) with respect to p and then dividing them by $m!$ and finally setting $p = 0$, one obtains the following problem at mth-order

$$\mathcal{L}_1\left[F_m(z) - \chi_m F_{m-1}(z)\right] = \hbar_1 \mathcal{R}_m^1(z), \tag{2.21}$$

$$F_m(0) = 0, \quad F_m(z) \longrightarrow 0 \ \text{as} \ z \longrightarrow \infty, \quad \frac{d^m F}{dz^m} \longrightarrow 0 \ \text{as} \ z \longrightarrow \infty, \tag{2.22}$$

$$\mathcal{R}_m^1(z) = \begin{array}{l} F''_{m-1}(z) - 2i\Omega F_{m-1}(z) + W_0\left\{F'_{m-1}(z) - \alpha F'''_{m-1}(z)\right\} \\ +2\beta \sum_{n=0}^{m-1} F'_{m-1-n}(z) \sum_{i=0}^{n} \left\{F'_{n-i}(z) F_i''^{*}(z) + 2F''_{n-i}(z) F_i'^{*}(z)\right\}, \end{array} \tag{2.23}$$

$$\chi_m = \begin{vmatrix} 0, & m \leq 1, \\ 1, & m \geq 2 \end{vmatrix} \tag{2.24}$$

in which prime denotes the derivatives with respect to z.

Now solving equations (2.21) subject to boundary conditions (2.22) up to first few order of approximations, the mth-order solution can be expressed by

$$F_m(z) = \sum_{n=0}^{2m+1} \sum_{q=0}^{2m+1-n} A_{m,n}^{q} z^q e^{-nz}, \qquad m \geq 0. \tag{2.25}$$

Substituting equation (2.25) into equation (2.21), we obtain the following recurrence formulae for the coefficients $A_{m,n}^{q}$ of $F_m(z)$ for $m \geq 1$, $0 \leq n \leq 2m+1$ and $0 \leq q \leq 2m+1-n$:

$$A_{m,1}^{0} = \chi_m \chi_{2m} A_{m-1,1}^{0} - \sum_{n=2}^{2m+1} \sum_{q=0}^{2m+1-n} \theta_{m,n}^{q} \phi_{n,0}^{q}, \tag{2.26}$$

$$A_{m,1}^{k} = \chi_m \chi_{2m-k} A_{m-1,1}^{k} + \sum_{q=k-1}^{2m+1} \theta_{m,1}^{q} \phi_{1,k}^{q}, \quad 1 \leq k \leq 2m+1, \tag{2.27}$$

$$\begin{aligned} A_{m,n}^{k} &= \chi_m \chi_{2m+1-n-k} A_{m-1,n}^{k} + \sum_{q=k}^{2m+1-n} \theta_{m,n}^{q} \phi_{n,k}^{q}, \\ 2 &\leq n \leq 2m+1, \ 0 \leq k \leq 2m+1-n, \end{aligned} \tag{2.28}$$

$$\phi_{1,k}^{q} = \frac{q! 2^{q+2-k}}{k!}, \qquad 0 \leq k \leq 2q+2, \ q \geq 0, \tag{2.29}$$

$$\begin{aligned} \phi_{n,k}^{q} &= \sum_{p=0}^{q+1-k} \frac{q!}{k!(n-1)^{p+1}(n+1)^{q+1-k-p}}, \\ 0 &\leq k \leq 2q+2-n, \ q \geq 0, \ n \geq 2, \end{aligned} \tag{2.30}$$

$$\theta_{m,n}^{q} = \hbar_1 \left[\begin{array}{c} \chi_{2m+1-n-q} \left(C_{m-1,n}^{q} + 2i\Omega A_{m-1,n}^{q} \right) \\ +W_0 \left(B_{m-1,n}^{q} - \alpha D_{m-1,n}^{q} \right) + 2\beta \left(2\kappa_{m,n}^{q} + \Sigma_{m,n}^{q} \right) \end{array} \right]. \tag{2.31}$$

The coefficients $B_{m,n}^{q}$, $C_{m,n}^{q}$, $D_{m,n}^{q}$, $\kappa_{m,n}^{q}$ and $\Sigma_{m,n}^{q}$ where $m \geq 1$, $0 \leq n \leq 2m+1$, $0 \leq q \leq 2m+1-n$ are defined by

$$B_{m,n}^{k} = (k+1) A_{m,n}^{k+1} - n A_{m,n}^{k}, \tag{2.32}$$

$$C_{m,n}^{k} = (k+1)\, B_{m,n}^{k+1} - nB_{m,n}^{k}, \tag{2.33}$$

$$D_{m,n}^{k} = (k+1)\, C_{m,n}^{k+1} - nC_{m,n}^{k}, \tag{2.34}$$

$$\begin{aligned} K_{m,n}^{q} &= \sum_{k=0}^{m-1} \sum_{l=0}^{k} \sum_{p=\max\{0,n-2m+2k+1\}}^{\min\{n,2k+2\}} \sum_{t=\max\{0,q-2m+2k+1+n-j\}}^{\min\{q,2k+2-j\}} \sum_{j=\max\{0,p-2k+2l-1\}}^{\min\{p,2l+1\}} \\ &\times \sum_{i=\max\{0,t-2k+2l-1+p-j\}}^{\min\{t,2l+1-j\}} B_{l,j}^{i} C_{k-l,p-j}^{t-i} B_{m-1-k,n-p}^{*q-t}, \end{aligned} \tag{2.35}$$

$$\begin{aligned} \Sigma_{m,n}^{q} &= \sum_{k=0}^{m-1} \sum_{l=0}^{k} \sum_{p=\max\{0,n-2m+2k+1\}}^{\min\{n,2k+2\}} \sum_{t=\max\{0,q-2m+2k+1+n-j\}}^{\min\{q,2k+2-j\}} \sum_{j=\max\{0,p-2k+2l-1\}}^{\min\{p,2l+1\}} \\ &\times \sum_{i=\max\{0,t-2k+2l-1+p-j\}}^{\min\{t,2l+1-j\}} B_{l,j}^{i} B_{k-l,p-j}^{t-i} C_{m-1-k,n-p}^{*q-t}. \end{aligned} \tag{2.36}$$

For detailed procedure of the derivation of above relations, the reader is referred to [40]. All coefficients $A_{m,n}^{k}$ can be obtained using above recurrence formulas and

$$A_{0,0}^{0} = 0, \qquad A_{0,1}^{0} = -1, \tag{2.37}$$

given by the initial guess approximation as equation (2.12). The corresponding Mth-order approximation of equations (2.15) and (2.16) is

$$\sum_{m=0}^{M} F_m(z) = \sum_{n=1}^{2M+1} e^{-nz} \left(\sum_{m=n-1}^{2M} \sum_{k=0}^{2m+1-n} A_{m,n}^{k} z^{k} \right) \tag{2.38}$$

and the final solution series is

$$F(z) = \sum_{m=0}^{\infty} F_m(z) = \lim_{M\to\infty} \left[\sum_{n=1}^{2M+1} e^{-nz} \left(\sum_{m=n-1}^{2M} \sum_{k=0}^{2m+1-n} A_{m,n}^{k} z^{k} \right) \right]. \tag{2.39}$$

2.1.2 Convergence of the solution

The explicit, analytic expressions (2.39) contains the auxiliary parameter $\hbar_1$. As pointed out by Liao [39], the convergence region and the rate of approximations of the series given by the homotopy analysis method strongly depends upon $\hbar_1$.

$$\alpha = \beta = 0.1,\ W_0 = 0.2,\ \Omega = 1$$

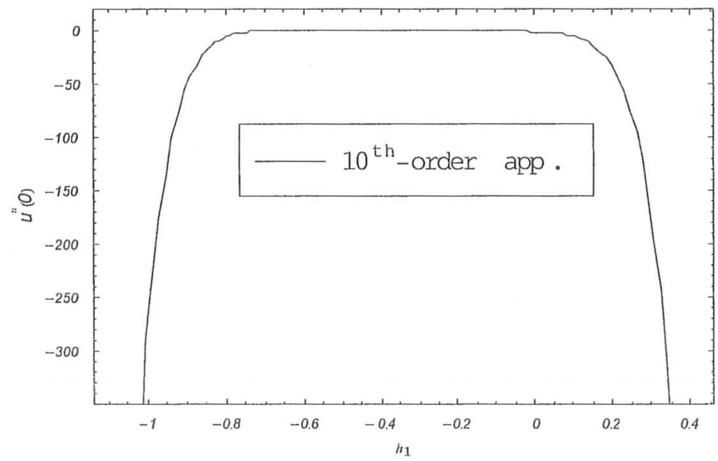


Figure 2.1(a)

$$\alpha = \beta = 0.1,\ W_0 = 0.2, \Omega = 1$$

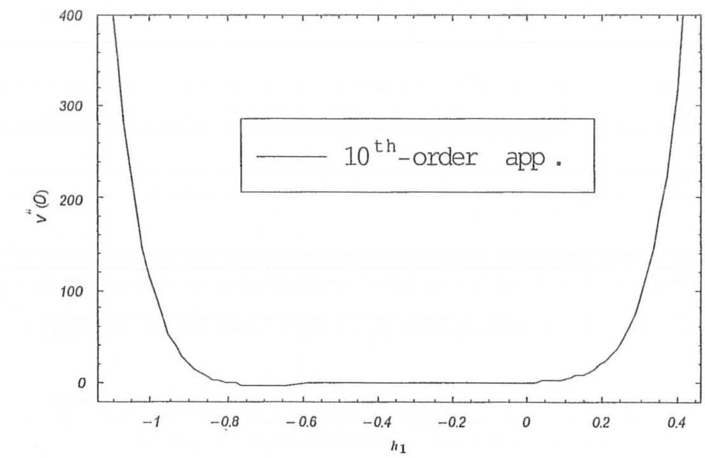


Figure 2.1(b)

Figures 2.1 : $\hbar_1$-curves for the 10th-order of approximations.

In Figures 2.1, the $\hbar_1$-curves are plotted to see the range of admissible values for the parameter $\hbar_1$. It is clear from these figures that $\hbar_1$ is $-0.8 \leq \hbar_1 < 0$ and our calculations show that the series given in equation (2.39) converges in the whole region of z for $\hbar_1 = -0.4$. For different values of $\hbar_1$, we have a family of solutions as given in equation (2.39). Now to address the question of convergence of the series for the appropriate choice of $\hbar_1$, we concentrate on the values of physical quantities on the plate. Since the series (2.39) must converge to a unique value, we will be looking at all values of $\hbar_1$ for which the family of solutions remain the same. This can be obtained by drawing the graph of the physical quantity versus $\hbar_1$ and the region of $\hbar_1$ for which this quantity remains constant will be the appropriate range of the value of $\hbar_1$.

2.1.3 Graphs and discussion

In this section the graphs for velocity components u and v are sketched to see the influence of suction/blowing parameter, rotation, second and third grade fluid parameters. For these cases, figures a and b corresponds to velocity components u and v respectively.

Figures 2.2 are prepared to see the influence of suction. It is noted that real and imaginary parts of velocity increases by increasing W_0. In Figures 2.3, we have shown the variation of blowing parameter. The effects of rotation are illustrated in Figures 2.4. The graphs reveal that an increase in rotation increases the velocity and decreases the boundary layer thickness. Figures 2.5 show the variation of second grade fluid parameter α on the velocity. Here u increases for large α. Figures 2.6 show the effects of material parameter of third grade fluid on the velocity. It is interesting to note that as β increases from 0 to 1, the velocity increases.

To see the convergence of the series for all values of z, we draw our attention to the fact that the most important physical quantity to be determined is the skin friction coefficient. This implies that the series must converge for $z = 0$, this convergence has been established by finding appropriate choice of $\hbar_1$. The same connotation is applicable for the graphs presented in the succeeding chapters.

It is important to comment that the values of various parameters for which calculations are made have been adopted from the literature which has been extensively used. We are relying on these values, while not being very sure that these values have been supported by experimental evidence.

$\hbar_1 = -0.4,\ \alpha = \beta = 0.5,\ \Omega = 1.5$

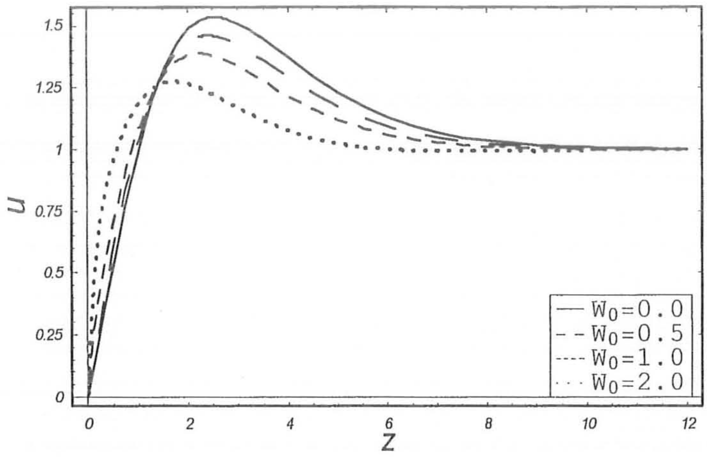


Figure 2.2(a)

$\hbar_1 = -0.4,\ \alpha = \beta = 0.5,\ \Omega = 1.5$

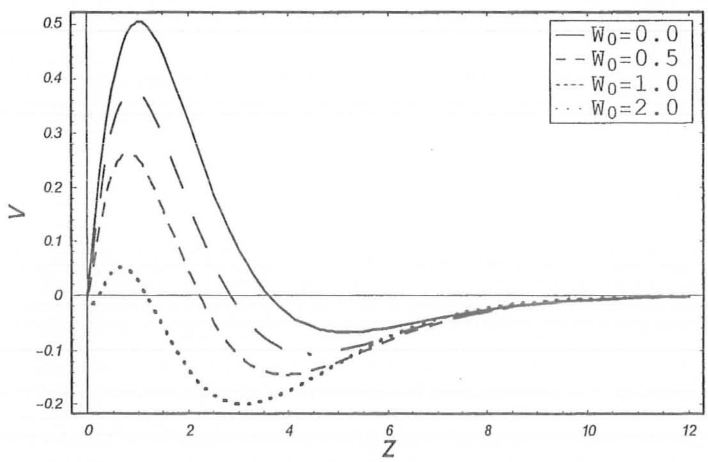


Figure 2.2(b)

Figures 2.2 : Influence of suction velocity $W_0 > 0$ on the velocity components u and v.

$\hbar_1 = -0.4,\ \alpha = \beta = 0.5,\ \Omega = 1.5$

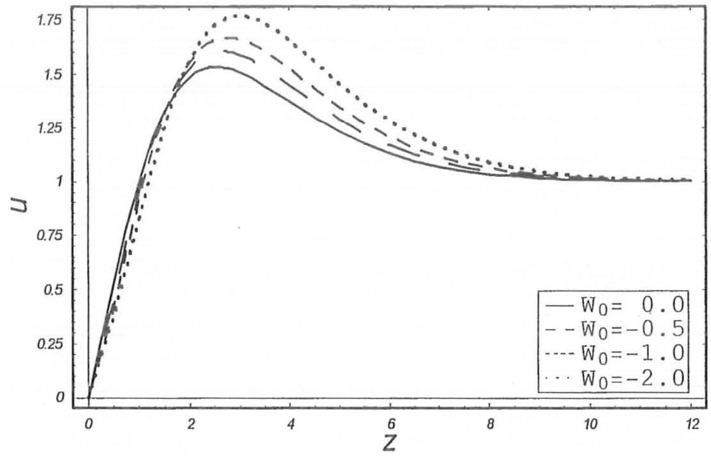


Figure 2.3(a)

$\hbar_1 = -0.4,\ \alpha = \beta = 0.5,\ \Omega = 1.5$

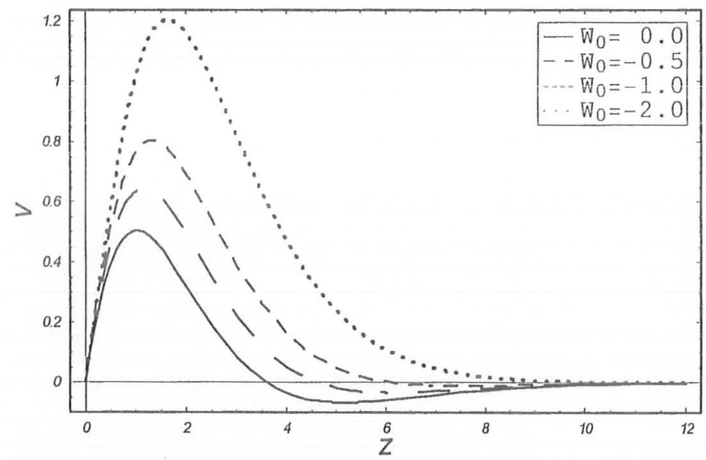


Figure 2.3(b)

Figures 2.3 : Influence of suction velocity $W_0 < 0$ on the velocity components u and v.

$\hbar_1 = -0.4,\ \alpha = \beta = W_0 = 0.5$

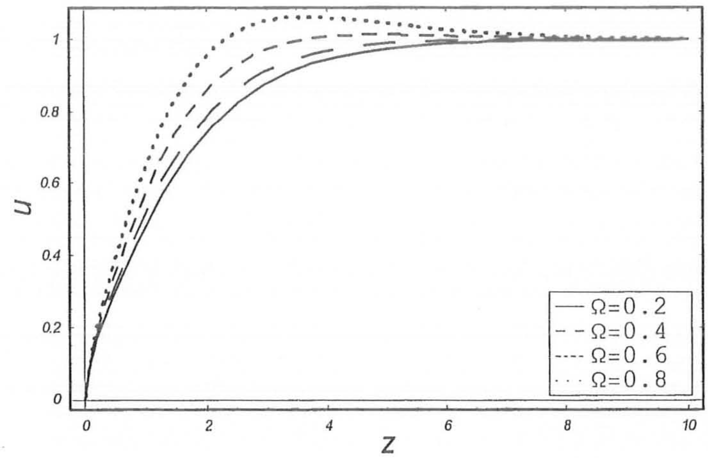


Figure 2.4(a)

$\hbar_1 = -0.4,\ \alpha = \beta = W_0 = 0.5$

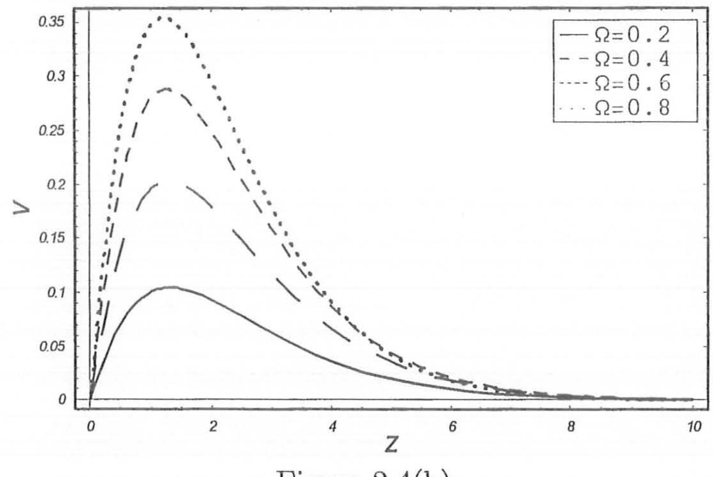


Figure 2.4(b)

Figures 2.4 : Influence of rotation parameter Ω on the velocity components u and v.

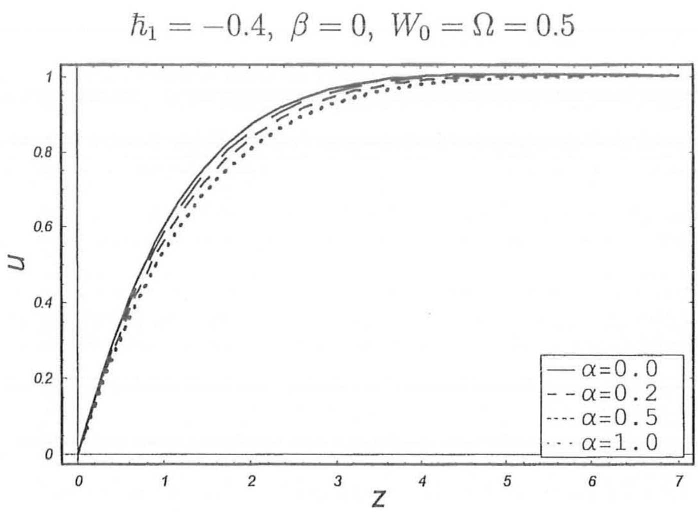


Figure 2.5(a)

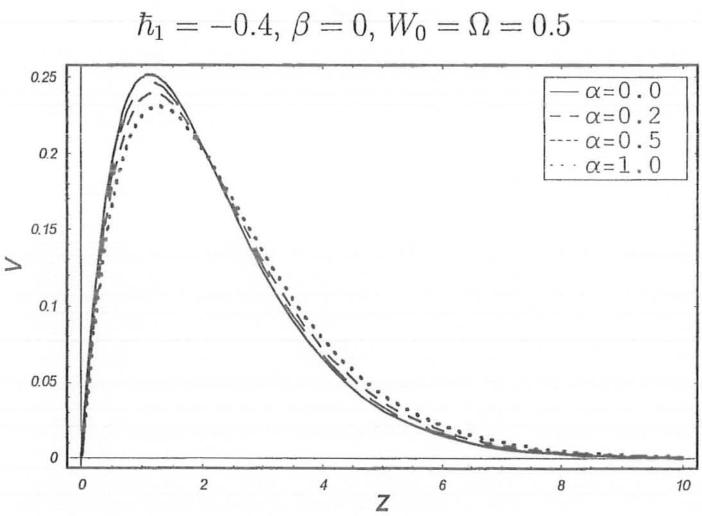


Figure 2.5(b)

Figures 2.5 : Influence of second grade parameter α on the velocity components u and v.

$\hbar_1 = -0.4,\ \alpha = W_0 = \Omega = 0.5$

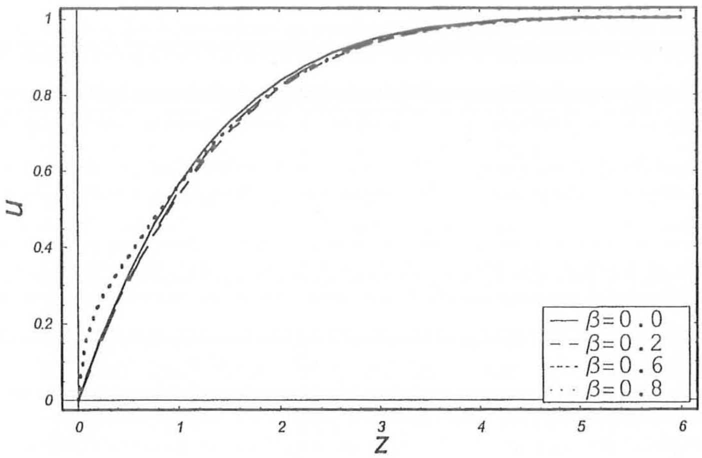


Figure 2.6(a)

$\hbar_1 = -0.4,\ \alpha = W_0 = \Omega = 0.5$

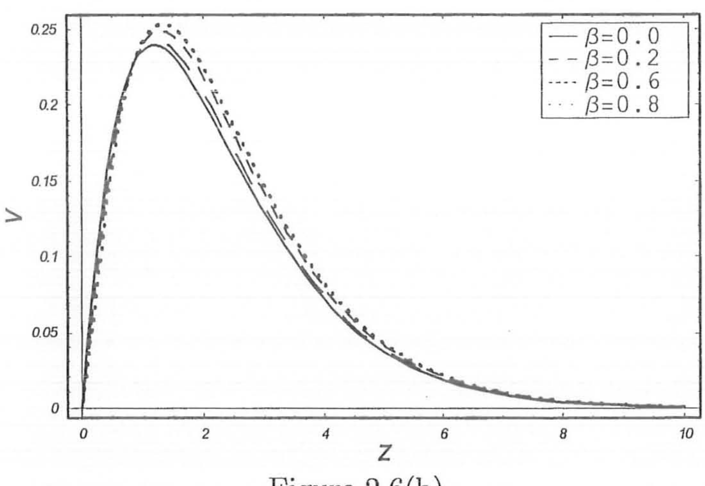


Figure 2.6(b)

Figures 2.6 : Influence of third grade parameter β on the velocity components u and v.

2.1.4 Redimensionalization of the problem for viscous parameter

In this section we reconsider the problem for the viscous parameter μ by introducing new dimensionless quantities in equation (2.7). Introduction of present dimensionless parameters for viscous parameter leads to a differential equation in which the viscous term is more significant. The behaviour of such a fluid is investigated by an analytical solution and graphical solution for suction, blowing, viscosity and third grade fluid parameters have been shown. For this purpose the following dimensionless parameters are introduced:

$$\widehat{z} = \frac{z\Omega}{U_0}, \quad \widehat{\mu} = \frac{\Omega\mu}{\rho U_0^2}, \quad \widehat{\beta} = \frac{\beta_3\Omega^3}{\rho U_0^2}, \quad \widehat{\alpha} = \frac{\alpha_1\Omega^2}{\rho U_0^2} \tag{2.40}$$

and other parameters are the same as given in equation (2.9). After dropping the hats, the governing momentum equation (2.7) can be rewritten in the following form

$$\mu\frac{d^2F}{dz^2} - 2iF + W_0\left[\frac{dF}{dz} - \alpha\frac{d^3F}{dz^3}\right] + 2\beta\frac{d}{dz}\left[\left(\frac{dF}{dz}\right)^2\frac{dF^\star}{dz}\right] = 0. \tag{2.41}$$

This equation is solved subject to boundary conditions (2.8) and (2.11).

2.1.5 Solution of the problem

The analytical solution of this problem is obtained using the same procedure as discussed in section 2.1. Here we use the initial guess as

$$F_0(z) = -e^{-m_1 z} \tag{2.42}$$

and take the auxiliary linear operator

$$\mathcal{L}_2 = \mu\frac{d^2}{dz^2} - 2i \tag{2.43}$$

satisfying the property

$$\mathcal{L}_2\left[c_3 e^{m_1 z} + c_4 e^{-m_1 z}\right] = 0, \tag{2.44}$$

where c_3 and c_4 are arbitrary constants and

$$m_1 = \sqrt{\frac{2i}{\mu}}, \tag{2.44a}$$

where the hats have been suppressed in μ.

We construct similar homotopy equation and use the same homotopy relations as used in section 2.1. For zeroth-order and mth-order deformation problems, equation (2.41) give

$$(1-p)\mathcal{L}_2\left[\overline{F}(z;p) - F_0(z)\right]$$
$$= p\hbar_2 \left[\begin{array}{c} \mu\frac{\partial^2 \overline{F}(z;p)}{\partial z^2} - 2i\overline{F}(z;p) + W_0\left(\frac{\partial \overline{F}(z;p)}{\partial z} - \alpha\frac{\partial^3 \overline{F}(z;p)}{\partial z^3}\right) \\ +2\beta\frac{\partial}{\partial z}\left\{\left(\frac{\partial \overline{F}(z;p)}{\partial z}\right)^2 \frac{\partial \overline{F}^*(z;p)}{\partial z}\right\} \end{array}\right], \tag{2.45}$$

$$\mathcal{L}_2\left[F_m(z) - \chi_m F_{m-1}(z)\right]$$
$$= \hbar_2 \left[\begin{array}{c} \mu F''_{m-1}(z) - 2iF_{m-1}(z) + W_0\left\{F'_{m-1}(z) - \alpha F'''_{m-1}(z)\right\} \\ +2\beta\sum_{n=0}^{m-1} F'_{m-1-n}(z)\sum_{i=0}^{n}\left\{F'_{n-i}(z)\, F_i''^{*}(z) + 2F''_{n-i}(z) F_i'^{*}(z)\right\} \end{array}\right], \tag{2.46}$$

with the boundary conditions (2.22).

Now solving equations (2.46) subject to boundary conditions (2.22) up to second-order approximations, we obtain three terms solution of the problem (2.41) and (2.16) as follows:

$$F(z) = F_0(z) + F_1(z) + F_2(z), \tag{2.47}$$

where

$$F_1(z) = \frac{2\hbar_2\beta m_1^{2m_1^*}(2m_1+m_1^*)}{(2m_1+m_1^*)^2-2i}(e^{-m_1 z}-e^{-(2m_1+m_1^*)z}), \tag{2.48}$$

$$F_2(z) = [M_1+M_2]\,(e^{-m_1 z}-e^{-(2m_1+m_1^*)z})+M_3(e^{-m_1 z}-e^{-(3m_1+m_1^*)z}), \tag{2.49}$$

in which

$$M_0 = \frac{2\hbar_2\beta m_1^{2m_1^*}(2m_1+m_1^*)}{(2m_1+m_1^*)^2-2i},$$

$$M_1 = 2M_0\left[1+\hbar_2\left\{1+W_0[(3+i)\alpha-\frac{1}{10}(3-i)\mu]\right\}\right],$$

$$M_2 = -\frac{8\hbar_2\beta}{\mu}(M_0^*+2M_0)\left(\frac{2-i}{3-4i}\right),$$

$$M_3 = \frac{8\hbar_2\beta}{\mu\{\mu+(2i-5)\}}[M_0^*(1-2i)+2M_0(4-7i)],$$

where m_1^* and M_0^* are the conjugates of m_1 and M_0 respectively.

2.1.6 Graphs and discussion

The following graphs have been drawn to see the behaviour of velocity components u and v. We make use of equation (2.6) for displaying the graphs.

Figures 2.7 and 2.8 describe the effects on the velocity profile for different values of suction and blowing parameters keeping $\hbar_2$, α, β, μ fixed and varying W_0. It has been observed that the velocity decreases with an increase of suction parameter and it increases with the increase of blowing parameter. This behaviour of the velocity profile is quite satisfactory as it is in accordance with the real physical situation. Figures 2.9 show the effects of material parameter of third grade fluid β on the velocity parts when $\hbar_2$, α, μ and W_0 are fixed. For both the components of velocity u and v, the boundary layer thickness decreases with the increase of third grade fluid parameter. However, in Figure 2.9(b), we notice initially increase in the velocity near the plate and then decrease in the velocity away from the plate. In Figures 2.10, we have shown the variation of viscosity parameters keeping $\hbar_2$, α, β and W_0 fixed. It is found that increase in viscosity parameter is responsible to decrease the velocity and the fluid flow seems to be more smooth and uniform for higher values of viscosity parameter. Figures 2.11 is drawn to choose the best suitable value for the homotopy parameter $\hbar_2$.

$\hbar_2 = 0.1,\ \alpha = \beta = \mu = 0.5$

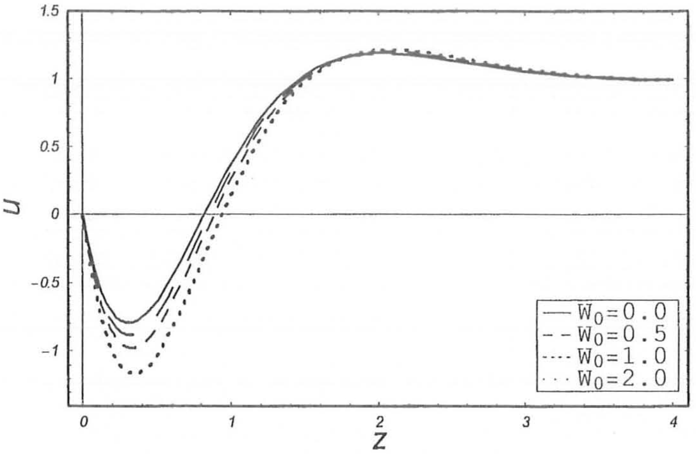


Figure 2.7(a)

$\hbar_2 = 0.1,\ \alpha = \beta = \mu = 0.5$

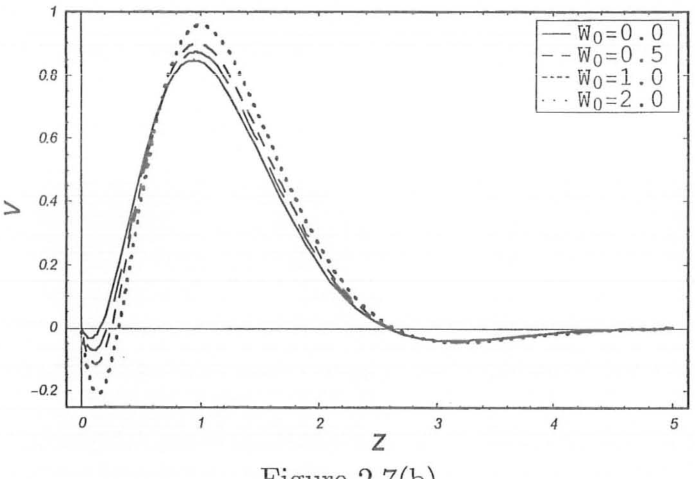


Figure 2.7(b)

Figures 2.7 : The variation of velocity components for various values of suction parameter W_0 with fixed $\hbar_2$, α, β and μ.

$\hbar_2 = 0.1,\ \alpha = \beta = \mu = 0.5$

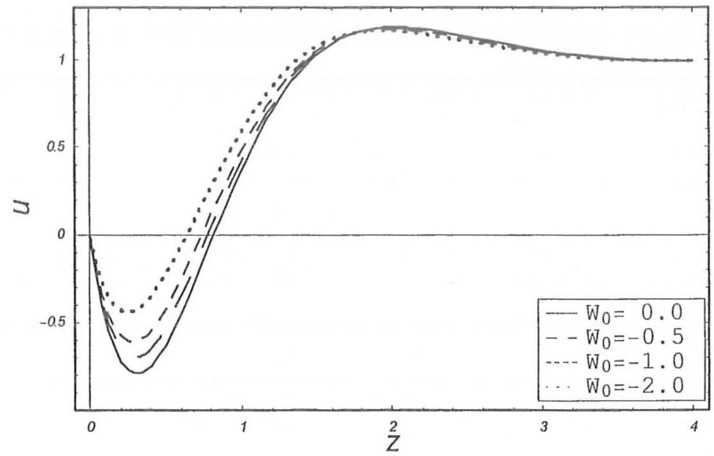


Figure 2.8(a)

$\hbar_2 = 0.1,\ \alpha = \beta = \mu = 0.5$

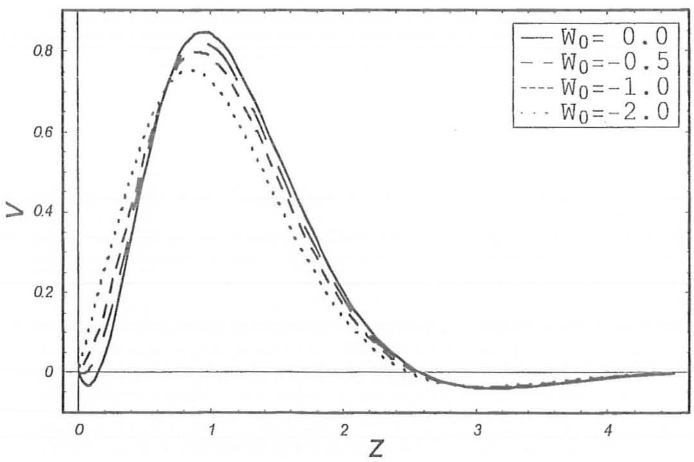


Figure 2.8(b)

Figures 2.8 : The variation of velocity components for various values of blowing parameter W_0 with fixed $\hbar_2$, α, β and μ.

$\hbar_2 = 0.1,\ \alpha = \mu = W_0 = 0.5$

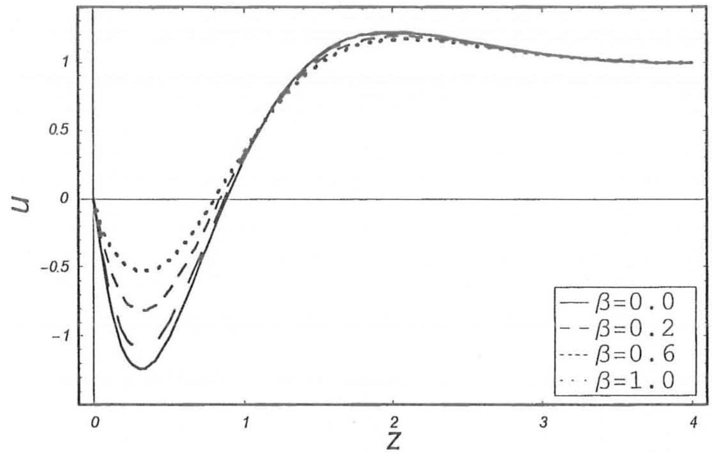


Figure 2.9(a)

$\hbar_2 = 0.1,\ \alpha = \mu = W_0 = 0.5$

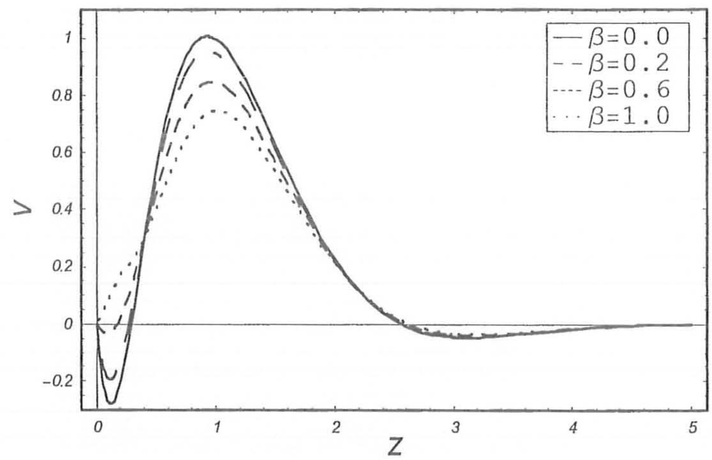


Figure 2.9(b)

Figures 2.9 : The variation of velocity components for various values of non-Newtonian material parameter β with fixed $\hbar_2$, α, μ and W_0.

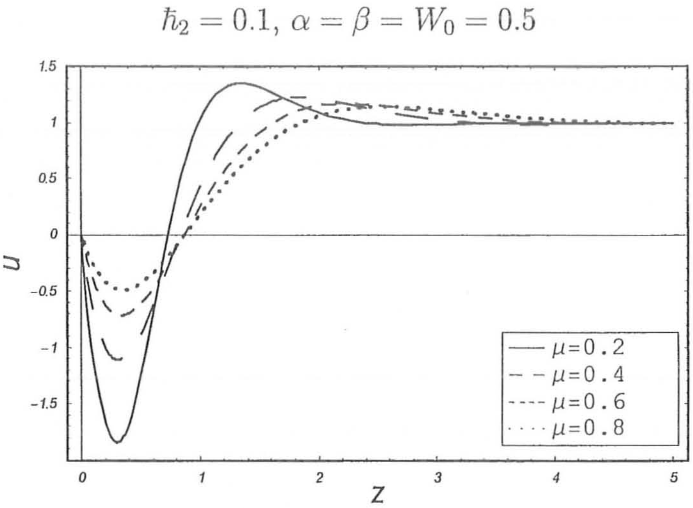


Figure 2.10(a)

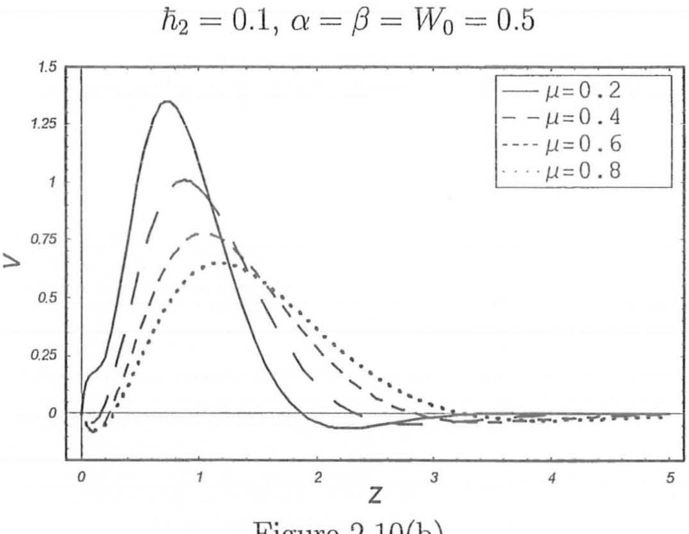


Figure 2.10(b)

Figures 2.10 : The variation of velocity components for various values of viscous parameter μ with fixed $\hbar_2$, α, β and W_0.

$\mu = W_0 = 0.2,\ \alpha = \beta = 0.1$

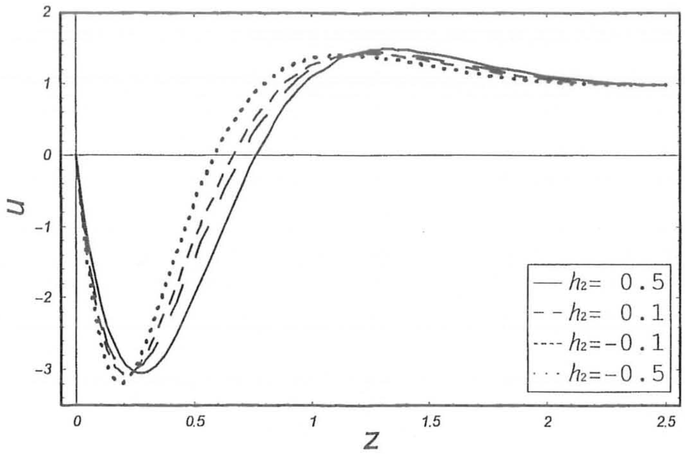


Figure 2.11(a)

$\mu = W_0 = 0.2,\ \alpha = \beta = 0.1$

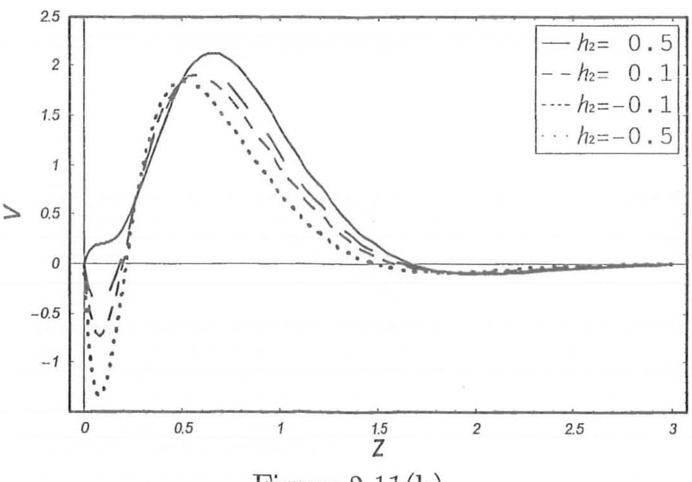


Figure 2.11(b)

Figures 2.11 : The variation of velocity components for various values of $\hbar_2$ with fixed μ, α, β and W_0.

2.2 Mathematical problem for the partial slip case

Let an infinite porous plate at $z = 0$ bound a semi-infinite expanse $(z > 0)$ of a third grade fluid which is assumed to be incompressible. Both the plate and the fluid does not rotate as a solid body with constant angular velocity Ω about an axis normal to the plate. We take Cartesian axes (x, y, z) such that the z-axis is parallel to the common axes of rotation of the fluid. Since the plate is infinite in extent, all the physical quantities, except the pressure, depend on z only for steady flow. Furthermore, the fluid adheres to the plate partially and thus motion of the fluid exhibits the slip condition.

Making a reference to our work of section 2.1, the dimensionless governing problem is

$$\frac{d^2F(z)}{dz^2} - 2i\Omega F(z) + W_0\left[\frac{dF(z)}{dz} - \alpha\frac{d^3F(z)}{dz^3}\right]$$
$$= -2\beta\frac{d}{dz}\left[\left(\frac{dF(z)}{dz}\right)^2\frac{dF^*(z)}{dz}\right], \tag{2.50}$$

$$F(0) + 1 = \lambda_1\left[\frac{dF(z)}{dz} - \alpha W_0\frac{d^2F(z)}{dz^2} + 2\beta\left(\frac{dF(z)}{dz}\right)^2\frac{dF^*(z)}{dz}\right], \tag{2.51}$$

$$F(z) = 0 \text{ as } z \longrightarrow \infty. \tag{2.52}$$

In equation (2.51)

$$\lambda_1 = \frac{\rho U_0}{\mu}\tilde{\lambda} \tag{2.52a}$$

is non-dimensional partial slip coefficient, $\tilde{\lambda} = \lambda\mu$ is slip length. In writing conditions (2.51), we have used the following partial slip condition

$$(u, v) - (U_w, V_w) = \lambda(\tau_{xz}, \tau_{yz}), \tag{2.52b}$$

where U_w, V_w (the wall velocities) are zero in this case and F and F^* are defined in equation (2.6).

2.2.1 Solution of the problem for the partial slip case

We note that equations (2.50) and (2.51) are highly non-linear and are not amenable to exact solutions. To solve this problem, we use homotopy analysis method in the same fashion as discussed in section 2.1. For HAM solution we take the initial guess of $F(z)$ as

$$F_0(z) = -\frac{e^{-m_2 z}}{1+m_2\lambda_1} \tag{2.53}$$

and the auxiliary linear operator

$$\mathcal{L}_3 = \frac{d^2}{dz^2} - 2i\Omega \tag{2.53a}$$

satisfying the property (2.44) and

$$m_2 = \sqrt{2i\Omega}. \tag{2.53b}$$

Employing the same procedure as in the no-slip case, the zeroth-order deformation problem is

$$\begin{aligned}(1-p)\mathcal{L}_3\left[\overline{F}(z;p) - F_0(z)\right] \\ = p\hbar_3\left[\begin{array}{c}\frac{d^2\overline{F}(z;p)}{dz^2} - 2i\Omega\overline{F}(z;p) + W_0\left\{\frac{d\overline{F}(z;p)}{dz} - \alpha\frac{d^3\overline{F}(z;p)}{dz^3}\right\} \\ +2\beta\frac{d}{dz}\left\{\left(\frac{d\overline{F}(z;p)}{dz}\right)^2\frac{d\overline{F}^*(z;p)}{dz}\right\}\end{array}\right]\end{aligned} \tag{2.54}$$

subject to following boundary conditions

$$(1-p)\left[\overline{F}(0;p)+1-\lambda_1\frac{d\overline{F}(0;p)}{dz}\right]$$

$$=p\hbar_3\left[\overline{F}(z;p)+1-\lambda_1\left\{\begin{array}{c}\frac{d\overline{F}(z;p)}{dz}-\alpha W_0\frac{d^2\overline{F}(z;p)}{dz^2}\\+2\beta\left(\frac{d\overline{F}(z;p)}{dz}\right)^2\frac{d\overline{F}^*(z;p)}{dz}\end{array}\right\}\right]\Bigg|_{z=0}, \tag{2.55}$$

$$\overline{F}(z;p)\longrightarrow 0 \text{ as } z\longrightarrow\infty, \tag{2.56}$$

where $\hbar_3$ is an auxiliary parameter and $p\in[0,1]$ is an embedding parameter.

Differentiating equations (2.54) to (2.56) m-times with respect to p and letting $p=0$, we obtain for $m\geq 1$, the following mth-order deformation problem

$$\mathcal{L}_3\left[F_m(z)-\chi_m F_{m-1}(z)\right]$$
$$=\hbar_3\left[\begin{array}{c}F''_{m-1}(z)-2i\Omega F_{m-1}(z)+W_0\left(F'_{m-1}(z)-\alpha F'''_{m-1}(z)\right)\\+2\beta\sum\limits_{n=0}^{m-1}F'_{m-1-n}(z)\sum\limits_{i=0}^{n}\left(F'_{n-i}(z)F''^{*}_{i}(z)+2F''_{n-i}(z)F'^{*}_{i}(z)\right)\end{array}\right], \tag{2.57}$$

$$\left[F_m(0)+1-\lambda_1F'_m(0)\right]$$
$$=\hbar_3\left[F_{m-1}+(1-\chi_m)-\lambda_1\left\{\begin{array}{c}F'_{m-1}-\alpha W_0F''_{m-1}\\+2\beta\sum\limits_{n=0}^{m-1}F'_{m-1-n}\sum\limits_{i=0}^{n}F'_{n-i}F'^{*}_{i}\end{array}\right\}\right]\Bigg|_{z=0} \tag{2.58}$$

and

$$F_m(z)\longrightarrow 0 \text{ as } z\longrightarrow\infty. \tag{2.59}$$

The three term solution of the problem consisting of equations (2.50) to (2.52) is

$$F(z)=F_0(z)+F_1(z)+F_2(z). \tag{2.60}$$

In above expression

$$F_1(z) = [M_4 z + M_6]\, e^{-m_2 z} - M_5 e^{-(2m_2+m_2^*)z}, \tag{2.61}$$

$$F_2(z) = \left[\begin{array}{c} \frac{1}{2(1+m_2\lambda_1)}\left\{\begin{array}{c} 2\alpha W_0\lambda_1\hbar_3\left(m_2^2 M_6 - 2m_2 M_4 - (2m_2+m_2^*)^2 M_5\right) \\ -\frac{M_7\lambda_1}{4m_2} - \frac{M_8\lambda_1}{4m_2^2} \end{array}\right\} \\ +\frac{1}{8(2+i)(1+m_2\lambda_1)\Omega}\left\{\begin{array}{c} M_9(1+m_2\lambda_1) + M_{10}(1+(2m_2+m_2^*)\lambda_1) \\ +M_{11}\lambda_1 - 2M_{11}(2m_2+m_2^*)\left(1+\frac{(2m_2+m_2^*)\lambda_1)}{4(2+i)\Omega}\right) \end{array}\right\} \\ +\frac{M_{12}(1+(3m_2+2m_2^*)\lambda_1)}{16(3+i)(1+m_2\lambda_1)\Omega} \\ -\frac{4\hbar_3\lambda_1\beta}{(1+m_2\lambda_1)^2}\left\{\begin{array}{c} \frac{m_2^2}{1+m_2\lambda_1}\left(M_4^* - m_2^* M_6^*\right) \\ +\frac{4\Omega}{1+m_2^*\lambda_1}\left(M_4 - m_2 M_6\right) \\ +M_5^*\frac{2(1+3i)\Omega^{\frac{3}{2}}}{1+m_2\lambda_1} + M_5\frac{8(1+2i)\Omega^2}{1+m_2^*\lambda_1} \end{array}\right\} \\ -\frac{1}{2m_2}\left\{M_7 z + M_8\left(\frac{z}{2m_2}+\frac{z^2}{2}\right)\right\} \end{array}\right] e^{-m_2 z}$$

$$-\frac{1}{8(2+i)\Omega}\left[M_9 + M_{10} - M_{11}\left(z + \frac{(2m_2+m_2^*)}{2(2+i)\Omega}\right)\right] e^{-(2m_2+m_2^*)z}$$

$$-\left[\frac{M_{12}}{8(3+i)\Omega}\right] e^{-(3m_2+2m_2^*)z}, \tag{2.62}$$

where

$$M_4 = \frac{1}{2(1+m_2\lambda_1)}\hbar_3 W_0(\alpha m_2^2 - 1),$$

$$M_5 = \frac{2\hbar_3\beta(1+2i)\Omega}{(2+i)(1+m_2^*\lambda_1)(1+m_2\lambda_1)^2},$$

$$M_6 = \frac{1}{1+m_2\lambda_1}\left[\begin{array}{c} M_1\lambda_1 + M_2(1+(2m_2+m_2^*)\lambda_1) \\ -\frac{\hbar_3\lambda_1}{1+m_2\lambda_1}\left(\alpha m_2^2 W_0 + \frac{4\beta(1+i)\Omega^{\frac{3}{2}}}{(1+m_2\lambda_1)(1+m_2^*\lambda_1)}\right) \end{array}\right],$$

$$M_7 = -4(1+\hbar_3)m_2 M_1 + 2\hbar_3 \left[m_2 W_0 M_3(\alpha m_2^2 - 1) - W_0 M_1(3m_2^2 - 1)\right],$$

$$M_8 = 2\hbar_3 m_2 W_0 M_1(\alpha m_2^2 - 1),$$

$$M_9 = M_2\left[8(1+\hbar_3)(2+i)\Omega + 2\hbar_3 W_0(2m_2+m_2^*)(\alpha(2m_2+m_2^*)^2 - 1)\right],$$

$$M_{10} = \frac{4\hbar_3\beta}{1+m_2\lambda_1}\left[\begin{array}{c}(2m_2+m_2^*)\left\{\frac{2i\Omega}{1+m_2\lambda_1}(M_1^* - m_2^* M_3^*) + \frac{4\Omega}{1+m_2^*\lambda_1}(M_1 - m_2 M_3)\right\} \\ +2(1+i)\Omega^{\frac{3}{2}}\left(\frac{1}{1+m_2\lambda_1}M_1^* + \frac{2}{1+m_2^*\lambda_1}M_1\right)\end{array}\right],$$

$$M_{11} = \frac{8\hbar_3\beta}{1+m_2\lambda_1}\left[(2m_2+m_2^*)(1+i)\Omega^{\frac{3}{2}}\left\{\frac{1}{1+m_2\lambda_1}M_1^* + \frac{2}{1+m_2^*\lambda_1}M_1\right\}\right],$$

$$M_{12} = \frac{4\hbar_3\beta}{1+m_2\lambda_1}\left[(2m_2+m_2^*)\left\{\frac{2(1+3i)\Omega^{\frac{3}{2}}}{1+m_2\lambda_1}M_2^* + \frac{8(1+2i)\Omega}{1+m_2^*\lambda_1}M_2\right\}\right]$$

and m_2^*, M_4^*, M_5^* and M_6^* are complex conjugates of m_2, M_4, M_5 and M_6 respectively.

2.2.2 Graphs and discussion

The effects of partial slip on the velocity profiles are given using homotopy analysis method. The velocity components u and v using equation (2.6) are sketched in Figures 2.12 to 2.15 in order to see the influence of partial slip.

Figures 2.12 illustrate the partial slip on the flow of viscous fluid when $\hbar_3 = -0.1$, $\alpha = \beta = 0$ and $\Omega = W_0 = 0.5$. It is seen that u increases with the increase of partial slip parameter and v first increases and then decreases.

The variation of partial slip parameter on the flow of third grade fluid has been depicted in Figures 2.13 for fixed $\hbar_3 = -0.1$, $\alpha = \beta = 1$ and $\Omega = W_0 = 0.5$. The observations for u and v are found of similar type. However, u in case of third grade fluid parameter shows some change in the velocity profile as compared to that of viscous fluid. There is no significant change in v for both viscous and third grade fluid. In Figures 2.14, the effect of partial slip parameter for different values of rotation has been shown and it is observed that the increase in rotation parameter increases the velocity. Figures 2.15 is drawn to determine the best suitable value for the homotopy parameter $\hbar_3$ to choose better convergence for the solution.

$$\hbar_3 = -0.1,\ \alpha = \beta = 0,\ \Omega = 0.5,\ W_0 = 0.5$$

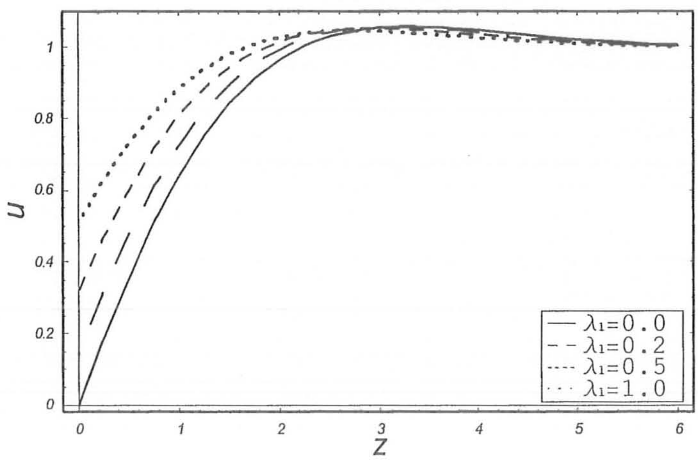


Figure 2.12(a)

$$\hbar_3 = -0.1,\ \alpha = \beta = 0,\ \Omega = 0.5,\ W_0 = 0.5$$

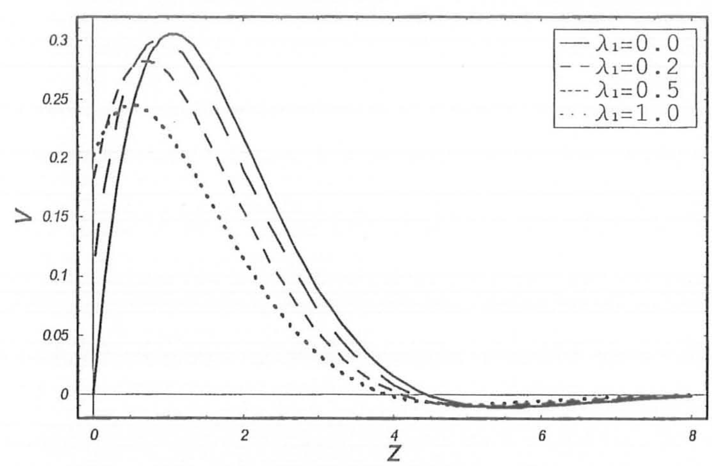


Figure 2.12(b)

Figures 2.12 : The variation of velocity components for various values of partial slip parameter λ_1 for viscous fluid with fixed $\hbar_3$, α, β, Ω and W_0.

$\hbar_3 = -0.1, \; \alpha = \beta = 1, \; \Omega = W_0 = 0.5$

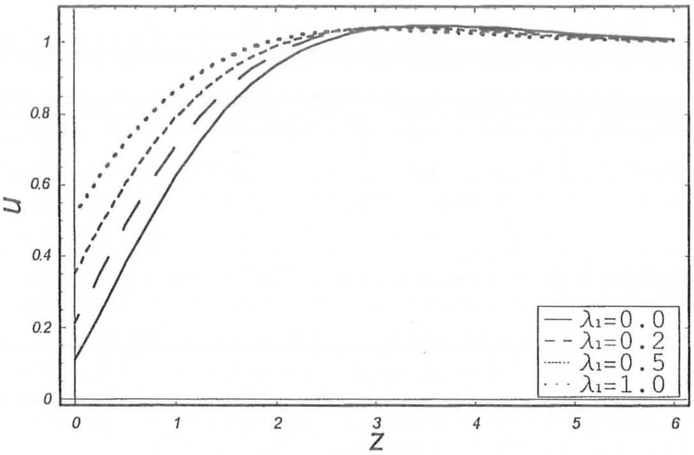


Figure 2.13(a)

$\hbar_3 = -0.1, \; \alpha = \beta = 1, \; \Omega = W_0 = 0.5$

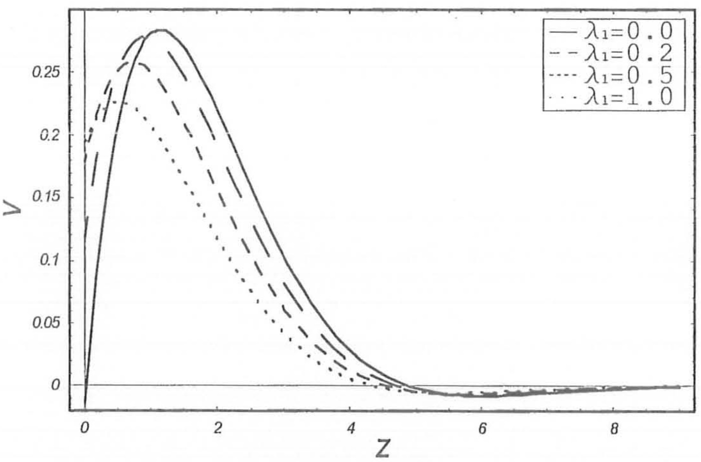


Figure 2.13(b)

Figures 2.13 : The variation of velocity components for various values of partial slip parameter λ_1 for third grade fluid with fixed $\hbar_3$, α, β, Ω and W_0.

$\hbar_3 = -0.1,\ \alpha = \beta = \lambda_1 = W_0 = 0.5$

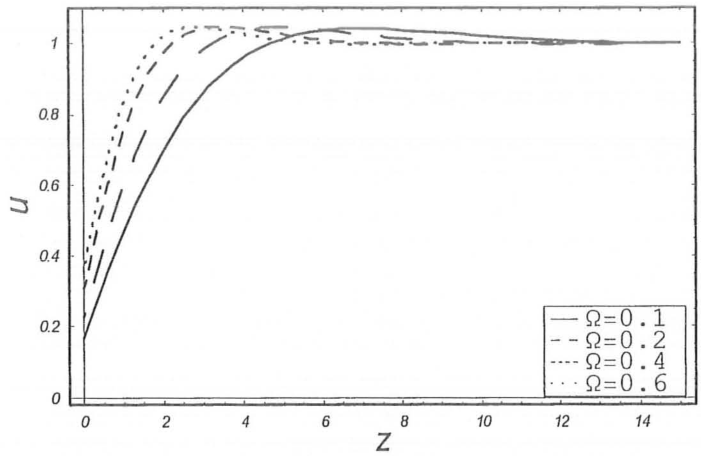


Figure 2.14(a)

$\hbar_3 = -0.1,\ \alpha = \beta = \lambda_1 = W_0 = 0.5$

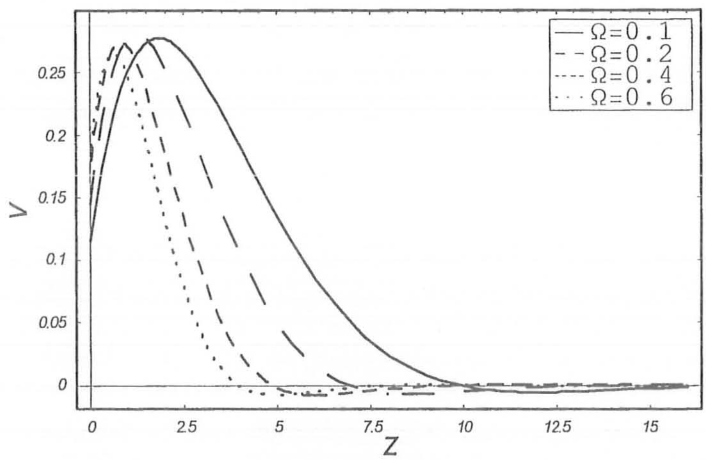


Figure 2.14(b)

Figures 2.14 : The variation of velocity components for various values of rotation Ω with fixed $\hbar_3$, α, β, λ_1 and W_0.

$\lambda_1 = \Omega = 0.2,\ \alpha = \beta = W_0 = 0.1$

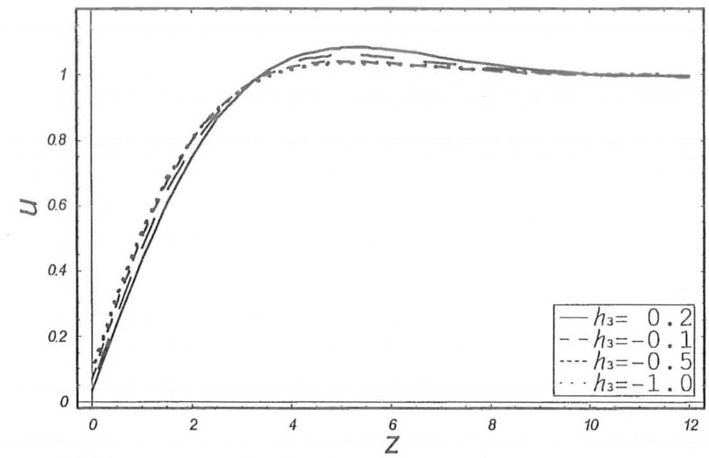


Figure 2.15(a)

$\lambda_1 = \Omega = 0.2,\ \alpha = \beta = W_0 = 0.1$

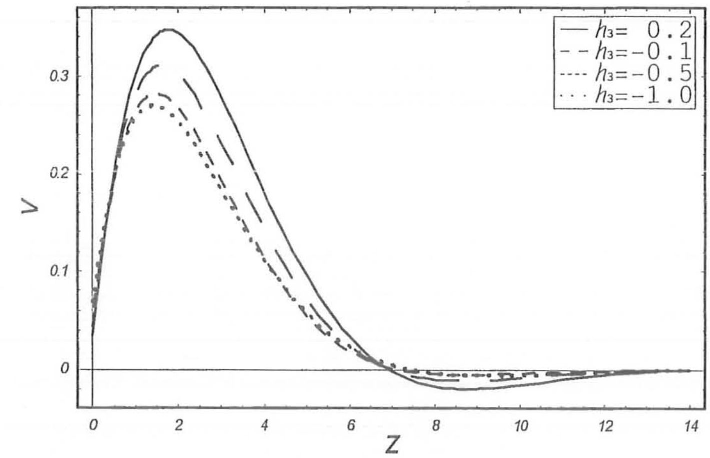


Figure 2.15(b)

Figures 2.15 : The variation of velocity components for various values of $\hbar_3$ with fixed λ_1,, α, β, Ω and W_0.

2.2.3 Removal of secular term by Lighthill technique

We observe that the term $M_4 z e^{-m_2 z}$ contains a secular or strained coordinate z in equation (2.61) in the sense that the series will converge slowly to the function $F(z)$. This secular term may add up in the next higher order terms to further reduce the rate of convergence. Therefore, it is desirable to eliminate this term. In order to do so, we use Lighthill technique. For this expanding the function $F(z)$ as a perturbation series in terms of ε as equation (1.70). This requires an appropriate transformation of the variable z to s as

$$z = s + \varepsilon f_1(s), \tag{2.63}$$

where $f(s)$ is an arbitrary function of s. Using this transformation in equation (2.61), we arrive at

$$F(s) = -\frac{1}{a_0}e^{-m_2 s} + \varepsilon\left[\left(-\frac{1}{a_0}m_2 f_1 + M_4 s + M_6\right)e^{-m_2 s} - M_5 e^{-(2m_2+m_2^{\star})s}\right] + O(\varepsilon^2) \tag{2.64}$$

where

$$a_0 = \frac{1}{1+m_2\lambda_1}.$$

To eliminate the secular term, we take

$$-\frac{1}{a_0}m_2 f_1(s) + M_4 s = 0 \tag{2.65}$$

giving

$$f_1(s) = \frac{s M_4 a_0}{m_2}. \tag{2.66}$$

Thus from equation (2.63), we have

$$s = z\left[1 - \frac{\varepsilon M_4 a_0}{m_2}\right]. \tag{2.67}$$

Using equation (2.67) in equation (2.64), we obtain

$$F(z) = \left[-\frac{1}{1+m_2\lambda_1} + \varepsilon M_6 \right] e^{-m_2(1-\frac{\varepsilon M_4 a_0}{m_2})z} - \varepsilon M_5 e^{-(2m_2+m_2^*)(1-\frac{\varepsilon M_4 a_0}{m_2})z}. \tag{2.68}$$

It is evident that, no secular term appears in equation (2.68). In order to see the validity and improvement in the result, we will compare the contributions of $F_{1s}(z)$ i.e. the secular term and $F_{1R}(z)$ i.e. the non secular term as given in equations (2.69) and (2.70) respectively, where

$$F_{1s}(z) = [M_4 z + M_6]\, e^{-m_2 z} - M_5 e^{-(2m_2+m_2^*)z} \tag{2.69}$$

and

$$F_{1R}(z) = -\frac{1}{1+m_2\lambda_1} + M_6 e^{-m_2(1-\frac{M_4 a_0}{m_2})z} - M_5 e^{-(2m_2+m_2^*)(1-\frac{M_4 a_0}{m_2})z} \tag{2.70}$$

at $\varepsilon = 1$.

The two results are graphed separately and the convergence is checked.

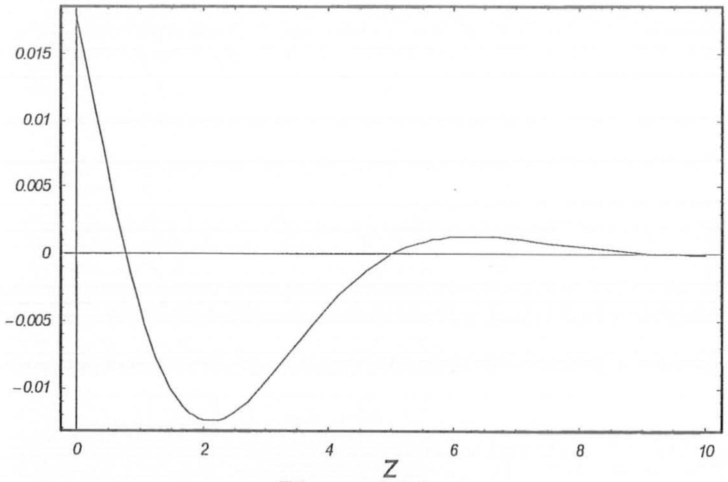


Figure 2.16

Figure 2.16: Profile of partial slip effects for $F_{1s}(z)$ with fixed values of $\hbar_3 = -0.1$, $\alpha = \beta = 1$ and $\Omega = W_0 = 0.5$.

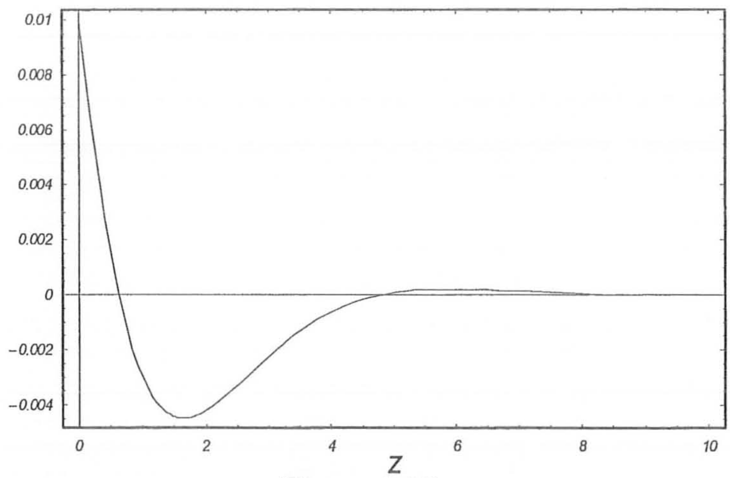


Figure 2.17

Figure 2.17: Profile of partial slip effects for $F_{1R}(z)$ with fixed values of $\hbar_3 = -0.1$, $\alpha = \beta = 1$ and $\Omega = W_0 = 0.5$.

Comparisons of figures (2.16) and (2.17) reveals that the convergence is improved by removing the so-called secular term.

2.2.4 Concluding remarks

In this chapter, the third grade fluid flow past a porous plate has been analyzed. The whole system is in a rotating frame. Two illustrative cases of no-slip and partial slip have been considered. Secular behaviour in the partial slip case has been removed. The most distinctive feature here is that; unlike the inertial frame, the steady asymptotic blowing solution exists. The physical implication of this conclusion is that rotation causes a reduction in the boundary layer thickness. Thus, if blowing is not too large, the thinning effect of rotation may just counterbalance the thickening effect of blowing so that the vorticity generated at the plate instead of being convected away from the plate by blowing remains confined near the plate and a steady solution is possible.

Chapter 3

Oscillating flows of a third grade fluid in a rotating frame

In this chapter, analytic solution of an oscillating flow is constructed in a rotating fluid. The fluid is considered as third grade. The flow is generated in the uniformly rotating fluid past a porous oscillating plate. Analytic solution for no-slip and partial slip situations are obtained employing homotopy analysis method. Convergence of the obtained explicit solutions in no-slip and partial slip conditions has been analyzed carefully. Attention is focused upon the physical nature of the solution by displaying graphs.

3.1 Problem formulation for the no-slip case

An infinite porous plate (located at $z = 0$) and the third grade fluid (which is in contact with the plate and occupies the whole of the region $z \geq 0$) are in uniform rotation. For the sake of simplicity the angular velocity $\mathbf{\Omega}$

is taken parallel to z-axis. The fluid is assumed to be homogeneous and incompressible. Referred to the rotating frame of reference, the momentum scalar equations are equations (1.30) and (1.31). The appropriate boundary conditions are

$$u(z,t) = U_0(1+\cos\delta t) \longrightarrow 0 \text{ as } z \longrightarrow 0, \;\; u(z,t) = U_0 \text{ as } z \longrightarrow \infty, \quad (3.1)$$

$$v(z,t) = U_0 \sin\delta t \longrightarrow 0 \text{ as } z \longrightarrow 0, \;\; v(z,t) = 0 \text{ as } z \longrightarrow \infty. \quad (3.2)$$

From equations (1.30), (1.31), (3.1) and (3.2), we have

$$\begin{aligned} \rho\left[\frac{\partial u}{\partial t} - W_0\frac{\partial u}{\partial z} - 2v\Omega\right] &= \mu\frac{\partial^2 u}{\partial z^2} + \alpha_1\left\{\frac{\partial^3 u}{\partial t\partial z^2} - W_0\frac{\partial^3 u}{\partial z^3}\right\} \\ &+2\beta_3\frac{\partial}{\partial z}\left[\frac{\partial u}{\partial z}\left\{\left(\frac{\partial u}{\partial z}\right)^2 + \left(\frac{\partial v}{\partial z}\right)^2\right\}\right], \end{aligned} \quad (3.3)$$

$$\begin{aligned} \rho\left[\frac{\partial v}{\partial t} - W_0\frac{\partial v}{\partial z} + 2u\Omega\right] &= 2\Omega U_0\rho + \mu\frac{\partial^2 v}{\partial z^2} + \alpha_1\left\{\frac{\partial^3 v}{\partial t\partial z^2} - W_0\frac{\partial^3 v}{\partial z^3}\right\} \\ &+2\beta_3\frac{\partial}{\partial z}\left[\frac{\partial v}{\partial z}\left\{\left(\frac{\partial u}{\partial z}\right)^2 + \left(\frac{\partial v}{\partial z}\right)^2\right\}\right], \end{aligned} \quad (3.4)$$

where U_0 indicates the free stream velocity and δ is an oscillating frequency.

Introducing

$$\widehat{t} = \frac{\rho U_0^2 t}{\mu}, \quad \widehat{\delta} = \frac{\delta\mu}{\rho U_0^2}, \quad (3.5)$$

along with the other parameters already defined in equations (2.6) and (2.9), the boundary value problem in non-dimensional variables after dropping hats becomes

$$\frac{\partial F}{\partial t} - \frac{\partial^2 F}{\partial z^2} - \alpha\frac{\partial^3 F}{\partial t\partial z^2} + 2i\Omega F - W_0\left[\frac{\partial F}{\partial z} - \alpha\frac{\partial^3 F}{\partial z^3}\right] - 2\beta\frac{\partial}{\partial z}\left[\left(\frac{\partial F}{\partial z}\right)^2\frac{\partial F^\star}{\partial z}\right] = 0, \quad (3.6)$$

$$F(0,t) = e^{i\delta t} \text{ at } z=0, \;\; F(z,t)=0 \text{ as } z \longrightarrow \infty. \tag{3.7}$$

By augmentation processes

$$\frac{\partial F(z,t)}{\partial z} = 0 \text{ as } z \longrightarrow \infty. \tag{3.8}$$

3.1.1 Analytic solution for the no-slip case

We see that equation (3.6) is highly non-linear and its analytic solution is not very simple. We use homotopy analysis method to solve this equation. For that we take the initial guess approximation as

$$F_0(z,t) = e^{-z+i\delta t} \tag{3.9}$$

and we use same auxiliary linear operator given in equation (2.13). The zeroth-order deformation problem is

$$(1-p)\mathcal{L}_1\left[\overline{F}(z,t;p) - F_0(z,t)\right]$$
$$= p\hbar_4 \left[\begin{array}{c} \frac{\partial \overline{F}(z,t;p)}{\partial t} - \frac{\partial^2 \overline{F}(z,t;p)}{\partial z^2} - \alpha \frac{\partial^3 \overline{F}(z,t;p)}{\partial t \partial z^2} + 2i\Omega \overline{F}(z,t;p) \\ -W_0 \left\{ \frac{\partial \overline{F}(z,t;p)}{\partial z} - \alpha \frac{\partial^3 \overline{F}(z,t;p)}{\partial z^3} \right\} - 2\beta \frac{\partial}{\partial z} \left\{ \left(\frac{\partial \overline{F}(z,t;p)}{\partial z} \right)^2 \frac{\partial \overline{F}^*(z,t;p)}{\partial z} \right\} \end{array} \right], \tag{3.10}$$

$$\overline{F}(0,t;p) = e^{i\delta t}, \qquad \overline{F}(\infty,t;p) = 0, \qquad \frac{\partial \overline{F}(\infty,t;p)}{\partial z} = 0 \tag{3.11}$$

in which $\hbar_4$ is an auxiliary non-zero parameter. For $p=0$ and $p=1$, we have

$$\overline{F}(z,t;0) = F_0(z,t), \qquad \overline{F}(z,t;1) = F(z,t). \tag{3.12}$$

As p increases from 0 to 1, $\overline{F}(z,t)$ varies from $F_0(z,t)$ to $F(z,t)$. By Taylor's theorem and equation (3.12)

$$\overline{F}(z,t) = F_0(z,t) + \sum_{m=1}^{\infty} F_m(z,t)p^m \tag{3.13}$$

where

$$F_m(z,t) = \frac{1}{m!}\frac{\partial^m}{\partial p^m}\overline{F}(z,t)\bigg|_{p=0} \quad (m \geq 1). \tag{3.14}$$

The convergence of the series (3.13) depends on $\hbar_4$. Suppose that $\hbar_4$ is selected in such a way that the series (3.13) is convergent at $p = 1$. Then

$$F(z,t) = F_0(z,t) + \sum_{m=1}^{\infty} F_m(z,t).$$

The mth-order deformation problem is given by

$$\mathcal{L}_1\left[F_m(z,t) - \chi_m F_{m-1}(z,t)\right] = \hbar_4 \mathcal{R}_m^2(z,t), \tag{3.15}$$

$$F_m(0,t) = 0, \quad F_m(\infty,t) = 0, \quad \frac{\partial F_m(\infty,t)}{\partial z} = 0, \tag{3.16}$$

$$\mathcal{R}_m^2(z,t) = \begin{array}{c} \frac{\partial F_{m-1}}{\partial t} - F''_{m-1} - \alpha\frac{\partial F''_{m-1}}{\partial t} + 2i\Omega F_{m-1} \\ -W_0\{F'_{m-1} - \alpha F'''_{m-1}\} - 2\beta\sum_{n=0}^{m-1} F'_{m-1-n}\sum_{i=0}^{n}\left\{F'_{n-i}F''^{*}_{i} + 2F''_{n-i}F'^{*}_{i}\right\}. \end{array} \tag{3.17}$$

Examining the solution of first few order of approximations, the solution for $F_m(z,t)$ can be expressed as

$$F_m(z,t) = \sum_{n=0}^{2m+1}\sum_{q=0}^{2m+1-n}\sum_{r=0}^{2m} b_{m,n}^{q,r} t^r z^q e^{-nz}, \qquad m \geq 0, \tag{3.18}$$

in which the involved coefficients $b_{m,n}^{q,r}$ can be obtained using the relations

$$b_{m,1}^{0,r} = \chi_m \chi_{2m} b_{m-1,1}^{0,r} - \sum_{q=0}^{2m+1-n} \Phi_{m,n}^{q,r}\Psi_{n,0}^{q}, \tag{3.19}$$

$$b_{m,1}^{q,r} = \chi_m \chi_{2m-q} b_{m-1,1}^{q,r} - \sum_{q=k-1}^{2m+1-n} \Phi_{m,1}^{q,r}\Psi_{1,k}^{q}, \tag{3.20}$$

$$b_{m,n}^{q,r} = \chi_m \chi_{2m-2n-q+2} b_{m-1,n}^{q,r} + \sum_{q=k}^{2m+1-n} \Phi_{m,n}^{q,r}\Psi_{n,k}^{q}, \tag{3.21}$$

$$\Psi^{q}_{1,k} = \frac{-q!}{k!2^{q+2-k}}, \qquad 1 \leq k \leq q+1, \tag{3.22}$$

$$\begin{aligned} \Psi^{q}_{n,k} &= \sum_{p=0}^{q-k} \frac{q!}{k!(n-1)^{q+1-k-p}(n+1)^{p+1}}, \\ 1 &\leq k \leq q,\ q \geq 0,\ n \geq 2, \end{aligned} \tag{3.23}$$

$$\Phi^{q,r}_{m,n} = \hbar_4 \left[\chi_{2m-r} \left\{ \begin{array}{c} \chi_{2m+3-n-q} \left(\begin{array}{c} \chi_{r+2} b'^{\,q,r}_{m,n} - b2^{q,r}_{m,n} - \alpha \chi_{r+2} b'2^{q,r}_{m,n} \\ +2i\Omega b^{q,r}_{m,n} - W_0 \left(b1^{q,r}_{m,n} - \alpha b3^{q,r}_{m,n} \right) \end{array} \right) \\ -2\beta \left(\Gamma 1^{q,r}_{m,n} + \Gamma 2^{q,r}_{m,n} \right) \end{array} \right\} \right], \tag{3.24}$$

$$\begin{aligned} \Gamma 1^{q,r}_{m,n} &= \sum_{k=0}^{m-1} \sum_{l=0}^{k} \sum_{n_1=\max\{0,n_3-(k-l+1)\}}^{\min\{n_3,m-k\}} \sum_{q_1=\max\{0,q_3-2(m-l+1-n_3)\}}^{\min\{q_3,2(m-k-n_1)\}} \sum_{r_1=\max\{0,r_3-2(k-l)\}}^{\min\{r_3,2(m-k-1)\}} \\ &\times \sum_{n_4=\max\{0,n-(m-l+1)\}}^{\min\{n,l+1\}} \sum_{q_4=\max\{0,q-2(m-l+1-n_3)\}}^{\min\{q,2(l+1-n_4)\}} \sum_{r_4=\max\{0,r-2(m-l+1)\}}^{\min\{r,2l\}} \\ &\times b1^{q_1,r_1}_{m-1-k,n_1} b2^{q_3-q_1,r_3-r_1}_{k-l,n_3-n_1} b^{*}1^{q-q_3,r-r_3}_{l,n-n_3}, \end{aligned} \tag{3.25}$$

$$\begin{aligned} \Gamma 2^{q,r}_{m,n} &= \sum_{k=0}^{m-1} \sum_{l=0}^{k} \sum_{n_1=\max\{0,n_3-(k-l+1)\}}^{\min\{n_3,m-k\}} \sum_{q_1=\max\{0,q_3-2(m-l+1-n_3)\}}^{\min\{q_3,2(m-k-n_1)\}} \sum_{r_1=\max\{0,r_3-2(k-l)\}}^{\min\{r_3,2(m-k-1)\}} \\ &\times \sum_{n_4=\max\{0,n-(m-l+1\}}^{\min\{n,l+1\}} \sum_{q_4=\max\{0,q-2(m-l+1-n_3\}}^{\min\{q,2(l+1-n_4)\}} \sum_{r_4=\max\{0,r-2(m-l+1)\}}^{\min\{r,2l\}} \\ &\times b^{q_1,r_1}_{m-1-k,n_1} b1^{q_3-q_1,r_3-r_1}_{k-l,n_3-n_1} b^{*}2^{q-q_3,r-r_3}_{l,n-n_3}, \end{aligned} \tag{3.26}$$

$$b'^{\,q,r}_{m,n} = b^{q,r+1}_{m,n}(r+1), \qquad b'2^{q,r}_{m,n} = b2^{q,r+1}_{m,n}(r+1), \tag{3.27}$$

$$b3^{q,r}_{m,n} = b2^{q+1,r}_{m,n}(q+1) - nb2^{q,r}_{m,n}, \tag{3.28}$$

$$b2^{q,r}_{m,n} = b1^{q+1,r}_{m,n}(q+1) - nb1^{q,r}_{m,n}. \tag{3.29}$$

The series solution of the equations (3.10) and (3.11) is

$$F(z,t)=\sum_{m=0}^{\infty}F_m(z,t)=\lim_{M\to\infty}\left[\sum_{n=1}^{2M+1}e^{-nz}\left(\sum_{m=n-1}^{2M}\sum_{k=0}^{2m+1-n}\sum_{r=0}^{2M}b_{m,n}^{q,r}z^{q}t^{r}\right)\right]. \tag{3.30}$$

3.1.2 Convergence of the solution

The explicit, analytic expressions (3.30) contains auxiliary parameter $\hbar_4$ that determines the convergence region and rate of approximations of HAM as pointed out by Liao [39].

In Figures 3.1 the $\hbar_4$-curves are plotted to see the range of admissible values for the parameter $\hbar_4$. It is clear from this figure that the range for the admissible values for $\hbar_4$ is $-1.7 \leq \hbar_4 < 0$ and the series given by equation (3.30) converges in this region of z for $\hbar_4 = -0.5$. For different values of $\hbar_4$, we have a family of solutions as given in equation (3.30). The convergence of the series is established on the similar arguments as discussed in section 2.1.2.

$\alpha = \beta = 0.1,\ \Omega = 1,\ W_0 = 0.2,\ \delta = 1$

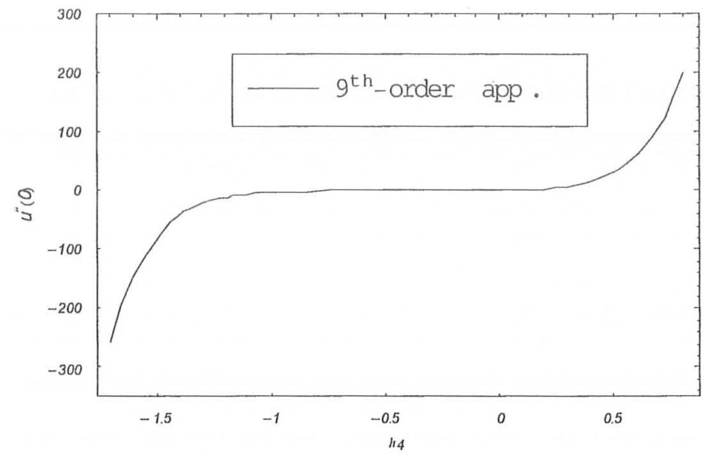


Figure 3.1(a)

$\alpha = \beta = 0.1,\ \Omega = 1,\ W_0 = 0.2,\ \delta = 1$

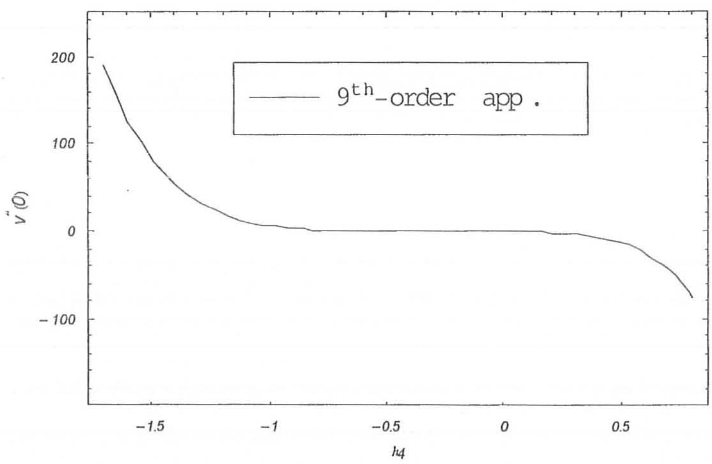


Figure 3.1(b)

Figures 3.1 : $\hbar_4$-curves for the 9th-order of approximations.

3.1.3 Results and discussion

We draw several graphs of the velocity field for velocity components u and v for an oscillating flow past a porous plate. The values of u and v have been sketched using equation (2.6). The controlling parameters are suction and blowing, rotation and third grade material parameter.

Figures 3.2 show a decrease in velocity for u component of velocity with the increase of suction parameter with fixed values of $\hbar_4$, α, β, t, Ω and δ. The increase in the suction parameter also increases the boundary layer thickness. Whereas, the behaviour of velocity profile in v is almost opposite to the first case. Figures 3.3 are drawn to observe the effects of blowing parameter on the velocity profile with fixed values of $\hbar_4$, α, β, Ω, t and δ. The increase in velocity with the increase of blowing parameter is significant in the real part u of velocity. The variation in the velocity profile for different values of rotation is depicted in Figures 3.4 with fixed values of $\hbar_4$, α, β, W_0, t and δ. In the velocity component u, a decrease in the velocity and increase in the boundary layer thickness with an increase in rotation parameter is observed. In the velocity component v, some changes in the velocity profile away from the plate are visible due to an increase in rotation parameter. Figures 3.5 are drawn for various values of third grade fluid parameter with fixed $\hbar_4$, α, Ω, W_0, t and δ. A decrease in the velocity near the plate is seen for the increasing values of β. Figures 3.6 show the variation in the velocity profile for different values of time when $\hbar_4$, α, β, Ω, W_0 and δ are fixed. The increase in time decreases the velocity near the plate for velocity component u. In the case of velocity component v, the fluid flow becomes uniform and smooth for large times.

$\hbar_4 = -0.5,\ \alpha = \beta = 0.5,\ t = \Omega = \delta = 1$

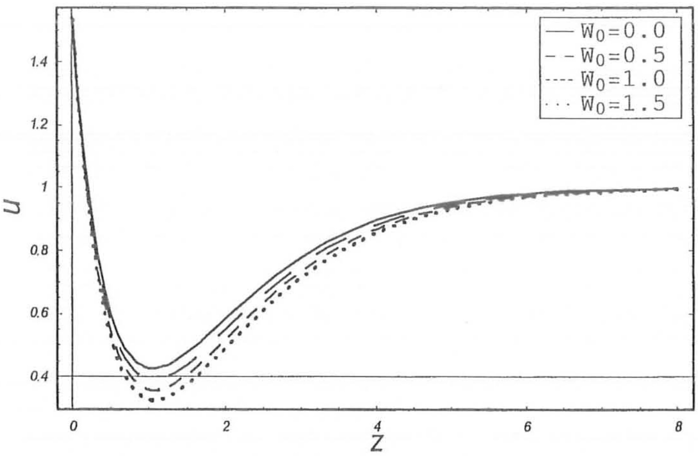


Figure 3.2(a)

$\hbar_4 = -0.5,\ \alpha = \beta = 0.5,\ t = \Omega = \delta = 1$

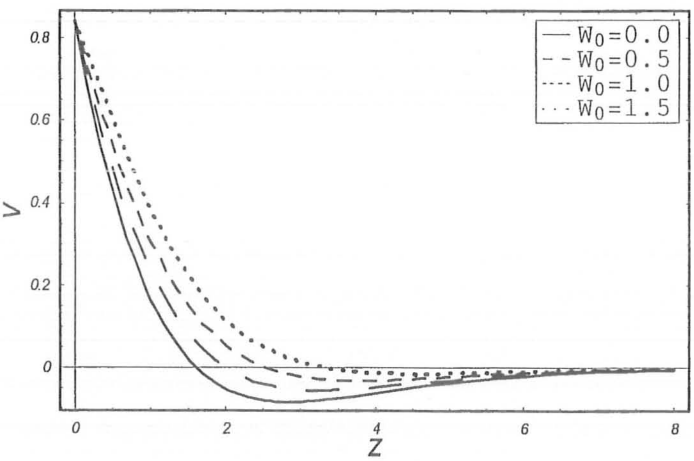


Figure 3.2(b)

Figures 3.2 : The variation of velocity components for various values of suction parameter W_0.

$\hbar_4 = -0.5,\ \alpha = \beta = 0.5,\ t = \Omega = \delta = 1$

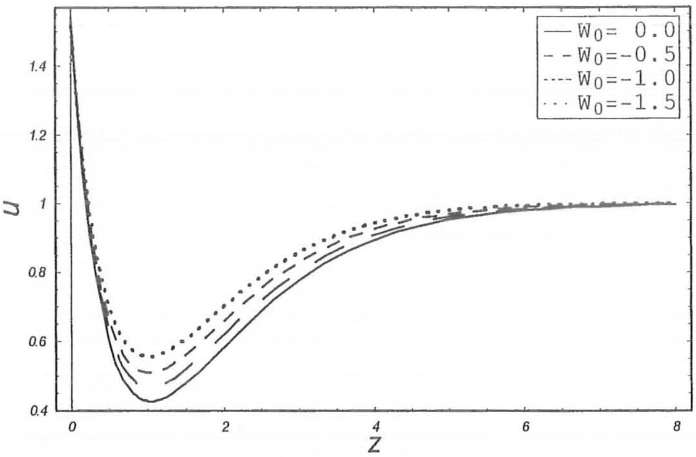


Figure 3.3(a)

$\hbar_4 = -0.5,\ \alpha = \beta = 0.5,\ t = \Omega = \delta = 1$

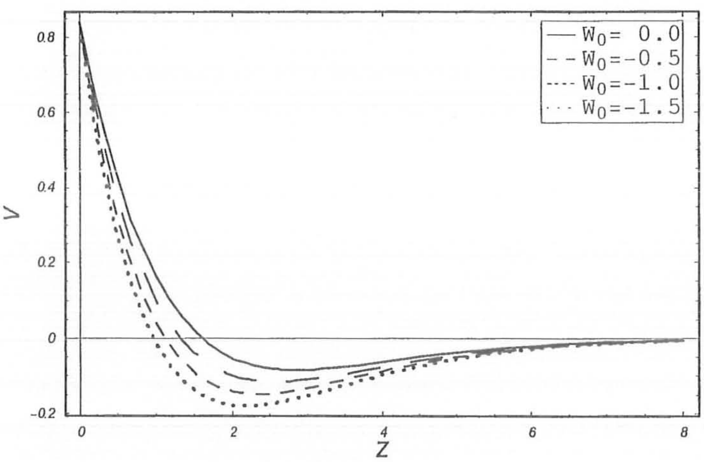


Figure 3.3(b)

Figures 3.3 : The variation of velocity components for various values of blowing parameter W_0.

$\hbar_4 = -0.5, \ \alpha = \beta = W_0 = 0.5, \ t = \delta = 1$

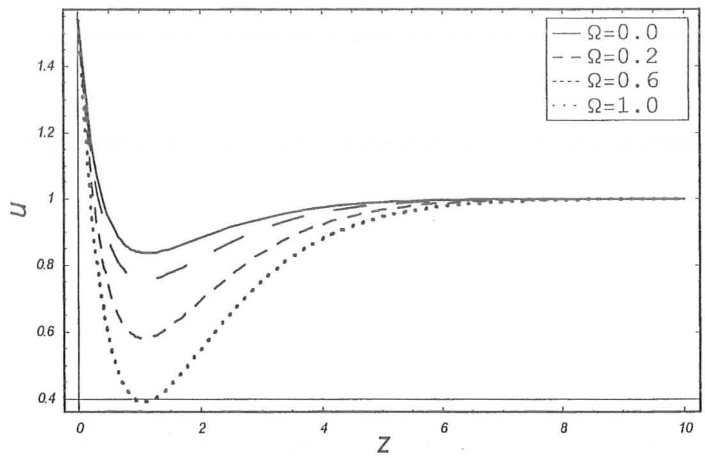


Figure 3.4(a)

$\hbar_4 = -0.5, \ \alpha = \beta = W_0 = 0.5, \ t = \delta = 1$

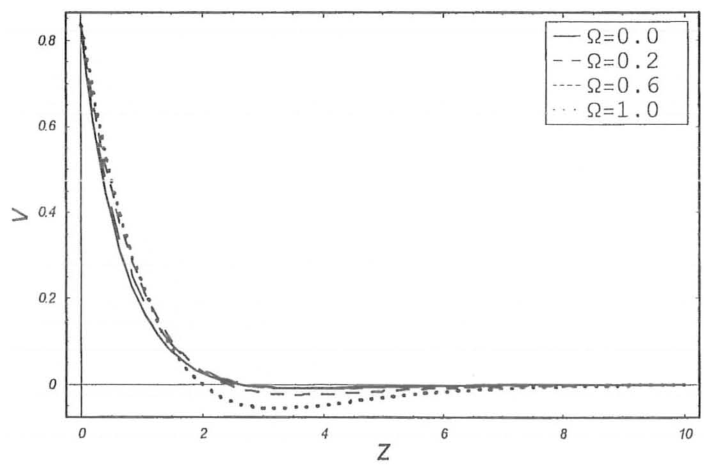


Figure 3.4(b)

Figures 3.4 : The variation of velocity components for various values of rotation Ω.

$\hbar_4 = -0.5,\ \alpha = \Omega = W_0 = 0.5,\ t = \delta = 1$

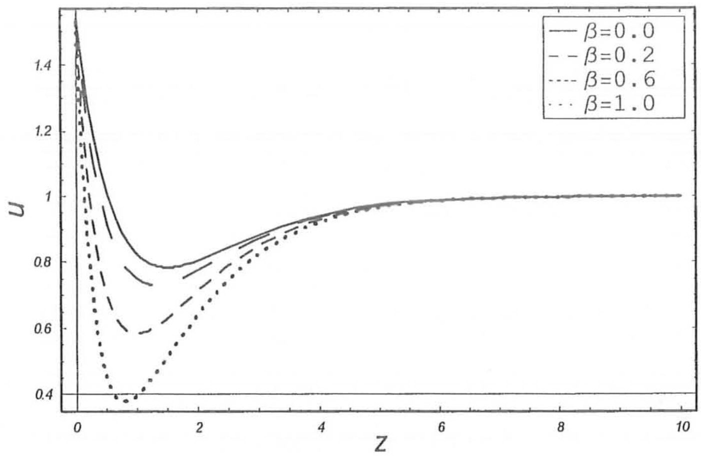


Figure 3.5(a)

$\hbar_4 = -0.5,\ \alpha = \Omega = W_0 = 0.5,\ t = \delta = 1$

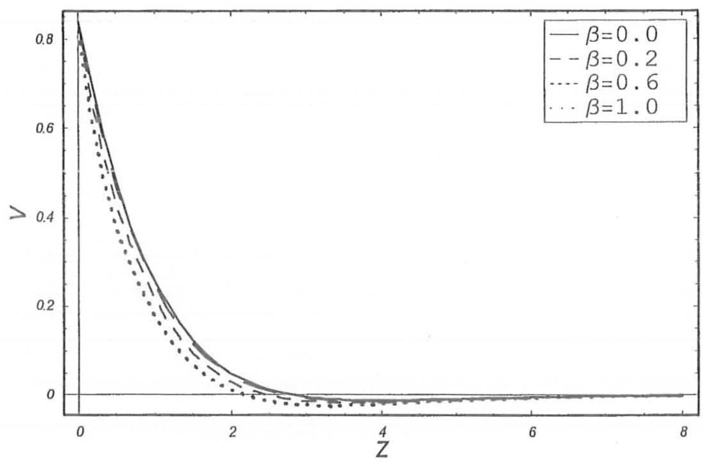


Figure 3.5(b)

Figures 3.5 : The variation of velocity components for various values of third grade fluid parameter β.

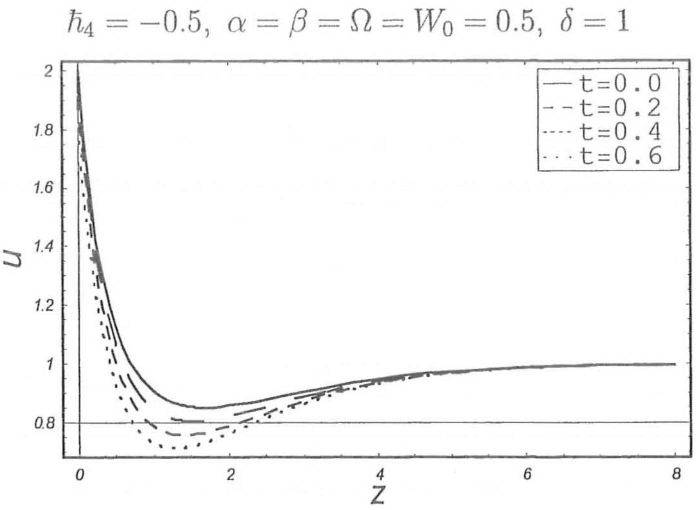


Figure 3.6(a)

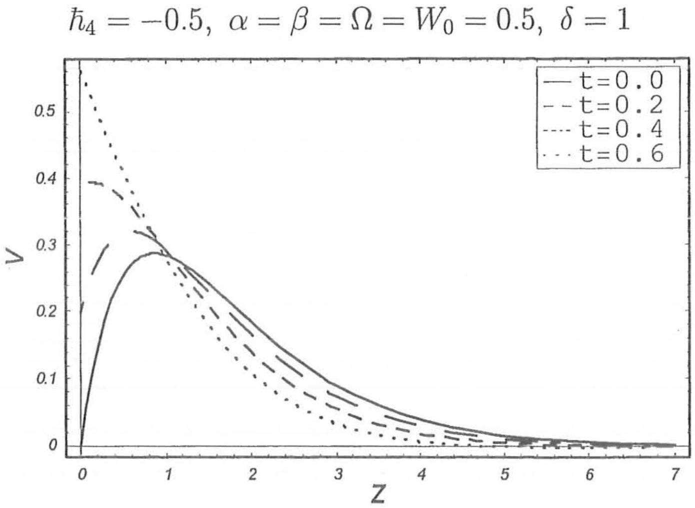


Figure 3.6(b)

$\hbar_4 = -0.5, \ \alpha = \beta = \Omega = W_0 = 0.5, \ \delta = 1$

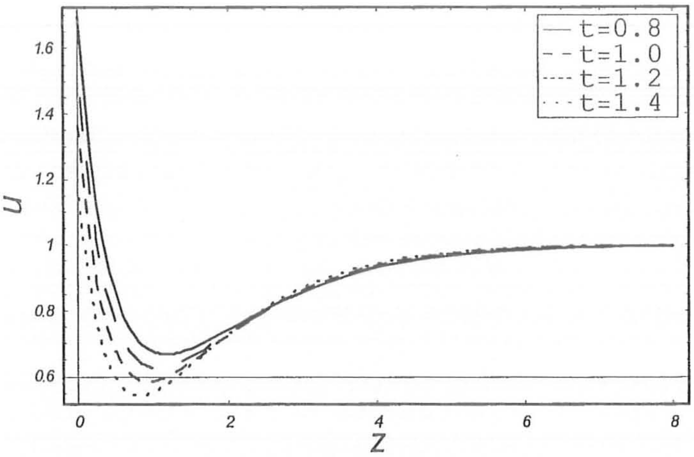


Figure 3.6(c)

$\hbar_4 = -0.5, \ \alpha = \beta = \Omega = W_0 = 0.5, \ \delta = 1$

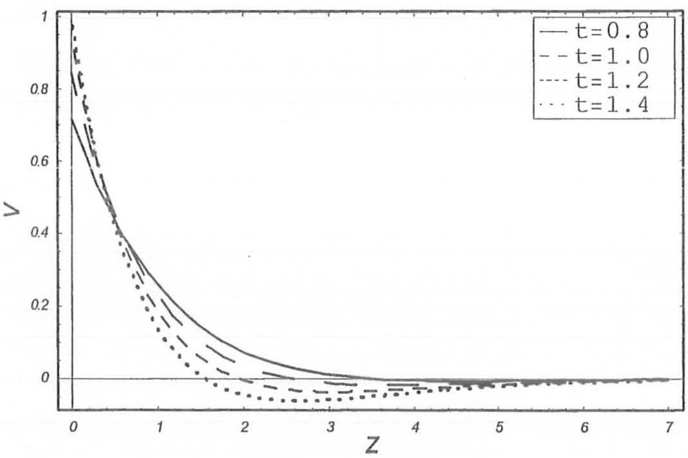


Figure 3.6(d)

Figures 3.6 : The variation of velocity components for various values of time t.

3.2 Problem formulation for the partial slip case

In this section, the physical model is the same as in section 3.1 except the partial slip condition replaces the no-slip condition. Thus the partial slip condition at the porous plate is defined as

$$u - U_0 e^{i\delta t} = \frac{\lambda_2}{\mu}\tau_{xz},$$

$$v = \frac{\lambda_2}{\mu}\tau_{yz},$$

where λ_2 is the partial slip coefficient and the expressions for τ_{xz} and τ_{yz} are defined through equations (1.36) and (1.37) as

$$\frac{\tau_{xz}}{\mu} = \frac{\partial u}{\partial z} + \frac{\alpha_1}{\mu}\left\{\frac{\partial^2 u}{\partial t \partial z} - W_0 \frac{\partial^2 u}{\partial z^2}\right\} + 2\frac{\beta_3}{\mu}\frac{\partial u}{\partial z}\left\{\left(\frac{\partial u}{\partial z}\right)^2 + \left(\frac{\partial v}{\partial z}\right)^2\right\}, \quad (3.31)$$

$$\frac{\tau_{yz}}{\mu} = \frac{\partial v}{\partial z} + \frac{\alpha_1}{\mu}\left\{\frac{\partial^2 v}{\partial t \partial z} - W_0 \frac{\partial^2 v}{\partial z^2}\right\} + 2\frac{\beta_3}{\mu}\frac{\partial v}{\partial z}\left\{\left(\frac{\partial u}{\partial z}\right)^2 + \left(\frac{\partial v}{\partial z}\right)^2\right\}. \quad (3.32)$$

The other boundary conditions are

$$u(z,t) = U_0, \quad v(z,t) = 0 \text{ as } z \longrightarrow \infty,$$

$$\frac{\partial u}{\partial z} \longrightarrow 0, \quad \frac{\partial v}{\partial z} \longrightarrow 0 \text{ as } z \longrightarrow \infty. \quad (3.33)$$

By equations (2.6), (2.9) and (3.5), the dimensionless boundary conditions after dropping the hats are

$$F(0,t) = e^{i\delta t} + \lambda_2 \left[\begin{array}{c} \frac{\partial F(z,t)}{\partial z} + \alpha\left\{\frac{\partial^2 F(z,t)}{\partial t \partial z} - W_0 \frac{\partial^2 F(z,t)}{\partial z^2}\right\} \\ +2\beta\left(\frac{\partial F(z,t)}{\partial z}\right)^2 \frac{\partial F^\star(z,t)}{\partial z} \end{array} \right], \quad (3.34)$$

$$F(z,t) = 0 \text{ as } z \longrightarrow \infty, \quad \frac{\partial F(z,t)}{\partial z} = 0 \text{ as } z \longrightarrow \infty. \quad (3.35)$$

3.2.1 Analytic solution for the partial slip case

To solve this problem, we use homotopy analysis method in the same fashion as discussed in section 3.1. Here we use initial guess approximation as

$$F_0(z,t) = e^{-(m_3+im_4)z+i\delta t} \tag{3.36}$$

and the auxiliary linear operator

$$\mathcal{L}_4 = \frac{\partial}{\partial t} - \frac{\partial^2}{\partial z^2} - \alpha\frac{\partial^3}{\partial t\partial z^2} + 2i\Omega \tag{3.37}$$

satisfying the property

$$\mathcal{L}_4[c_5e^{-(m_3+im_4)z} + c_6e^{(m_3+im_4)z}]e^{i\delta t} = 0$$

where c_5 and c_6 are arbitrary constants and

$$m_3 = \left[\frac{\sqrt{\alpha^2\delta^2c_0^2 + 4c_0^2} + \alpha\delta c_0}{2}\right]^{\frac{1}{2}}, \tag{3.38}$$

$$m_4 = \left[\frac{\sqrt{\alpha^2\delta^2c_0^2 + 4c_0^2} - \alpha\delta c_0}{2}\right]^{\frac{1}{2}} \tag{3.39}$$

in which

$$c_0 = \frac{\delta + 2\Omega}{2(1+\alpha^2\delta^2)}.$$

We construct the similar homotopy for equation (3.6) as already have been carried out in section 3.1 and also use the same homotopy relations. However, for zeroth-order deformation problem, the partial slip boundary condition and the other boundary conditions at infinity take the form

$$\begin{aligned}&(1-p)\left[\overline{F}(0,t;p) - e^{i\delta t}\right]\\ &= p\hbar_5\left[\overline{F}(z,t;p) - e^{i\delta t} - \lambda_2\left\{\begin{array}{c}\frac{\partial\overline{F}(z,t;p)}{\partial z} - \alpha\left[\frac{\partial^2\overline{F}(z,t;p)}{\partial t\partial z} - W_0\frac{\partial^2\overline{F}(z,t;p)}{\partial z^2}\right]\\ -2\beta\left(\frac{d\overline{F}(zt,;p)}{dz}\right)^2\frac{d\overline{F}^*(z,t;p)}{dz}\end{array}\right\}\right],\end{aligned} \tag{3.40}$$

$$\overline{F}(0,t;p) = e^{i\delta t}, \quad \overline{F}(\infty,t;p) = 0, \quad \frac{\partial \overline{F}}{\partial z}(\infty,t;p) = 0. \tag{3.41}$$

Differentiating m-times the zero-order deformation equations (3.40) and (3.41) with respect to p and letting $p = 0$, we obtain for $m \geq 1$, the following equation

$$F_m(0,t) - e^{i\delta t}$$

$$= \hbar_5 \left[F_{m-1} - (1-\chi_m)e^{i\delta t} - \lambda_2 \left\{ \begin{array}{l} F'_{m-1} - \alpha \left[\frac{\partial F'_{m-1}}{\partial t} - W_0 F''_{m-1} \right] \\ +2\beta \sum\limits_{n=0}^{m-1} F'_{m-1-n} \sum\limits_{i=0}^{n} F'_{n-i} F'^{*}_{i} \end{array} \right\} \right] \Bigg|_{z=0}, \tag{3.42}$$

$$F_m(z,t) \longrightarrow 0 \text{ as } z \longrightarrow \infty, \quad \frac{\partial^m F}{\partial z^m}(z,t) \longrightarrow 0 \text{ as } z \longrightarrow \infty. \tag{3.43}$$

Now solving equation (3.10) subject to boundary conditions (3.42) and (3.43) up to second-order approximations, we obtain the three terms solution of the problem (3.6), (3.34) and (3.35) as follows

$$F(z,t) = F_0(z,t) + F_1(z,t) + F_2(z,t), \tag{3.44}$$

where

$$F_1(z,t) = \left[(M_{13}z + M_{14})\, e^{-(m_3+im_4)z} - M_{15} e^{-(3m_3+im_4)z} \right] e^{i\delta t}, \tag{3.45}$$

$$F_2(z,t) = \frac{1}{2(1+i\alpha\delta)} \left[\begin{array}{c} \left[\begin{array}{c} (1+2\hbar_5)\{M_{14}-M_{15}\} - 2\hbar_5\lambda_2[(1-i\alpha\delta)\{M_{13} \\ -M_{14}(m_3+im_4) + M_{15}(3m_3+im_4)\} \\ +W_0\{2M_{13}(m_3+im_4) - M_{14}(m_3+im_4)^2 \\ +M_{15}(3m_3+im_4)^2\} - 2\beta\{(m_3+im_4)^2(M_{13}^* \\ -M_{14}^*(m_3-im_4) + M_{15}^*(3m_3-im_4)) \\ +2(m_3^2+m_4^2)(M_{13} - M_{14}(m_3+im_4) \\ +M_{15}(3m_3+im_4)\}] \\ +\frac{1}{2(m_3+im_4)}\left\{M_{16}z + M_{17}\left(\frac{z^2}{2} + \frac{z}{2(m_3+im_4)}\right)\right\} \end{array} \right] e^{-(m_3+im_4)z} \\ +\frac{M_{19}}{4(2m_3^2+im_3m_4)} z e^{-(3m_3+im_4)z} \\ +\frac{1}{4(2m_3^2+im_3m_4)}\left(M_{18} - \frac{3m_3+im_4}{2(2m_3^2+im_3m_4)}M_{19}\right)\left(e^{-(m_3+im_4)z} - e^{-(3m_3+im_4)z}\right) \\ +\frac{M_{20}}{8(3m_3^2+im_3m_4)}\left(e^{-(m_3+im_4)z} - e^{-(5m_3+im_4)z}\right) \end{array} \right] e^{i\delta t}$$

in which

$$M_{13} = \frac{\hbar_5}{2(1+i\alpha\delta)(m_3+im_4)} \left[\begin{array}{c} i(2\Omega+\delta) - (1+i\alpha\delta)(m_3+im_4)^2 \\ +W_0(m_3+im_4)\{1-\alpha(m_3+im_4)^2\} \end{array} \right],$$

$$M_{14} = \hbar_5 \left[\begin{array}{c} \frac{-\beta}{2(1+i\alpha\delta)(2m_3^2+im_3m_4)}[(m_3-im_4)(3m_3+im_4)(m_3+im_4)^2] \\ +\lambda_2(m_3+im_4)[(1+i\alpha\delta) + \alpha W_0(m_3+im_4) + 2\beta(m_3^2+m_4^2)] \end{array} \right],$$

$$M_{15} = \frac{-\hbar_5\beta}{2(1+i\alpha\delta)(2m_3^2+im_3m_4)}\left[(m_3-im_4)(3m_3+im_4)(m_3+im_4)^2\right],$$

$$M_{16} = 2(1+\hbar_5) \left[\begin{array}{c} i(\delta+2\Omega)M_{14} + (m_3+im_4)\{2M_{13} - M_{14}(m_3+im_4)\} \\ +i\alpha\delta(m_3+im_4)\{2M_{13} - M_{14}(m_3+im_4)\} - W_0\{M_{13} - M_{14}(m_3+im_4)\} \\ +\alpha W_0(m_3+im_4)^2\{3M_{13} - M_{14}(m_3+im_4)\} \end{array} \right],$$

$$M_{17} = 2(1+\hbar_5) \left[\begin{array}{c} i(\delta+2\Omega)M_{13} - M_{13}(m_3+im_4)^2\{1+i\alpha\delta\} \\ +W_0M_1(m_3+im_4)\{1-\alpha(m_3+im_4)^2\} \end{array} \right],$$

$$M_{18} = 2(1+\hbar_5)\left[\begin{array}{c} M_{14}[-i(\delta+2\Omega)+(3m_3+im_4)^2\{1+i\alpha\delta(3m_3+im_4)\} \\ -W_0(3m_3+im_4)\{1+\alpha(3m_3+im_4)^2\}] \end{array}\right]$$

$$+4\hbar_5\beta\left[\begin{array}{c} (3m_3+im_4)\left\{\begin{array}{c} M_{13}^*(m_3+im_4)^2+2M_{14}(m_3^2+m_4^2) \\ -M_{14}^*(m_3-im_4)(m_3+im_4)^2 \\ -2M_{14}(m_3^2+m_4^2)(m_3+im_4) \end{array}\right\} \\ +\{M_{13}^*(m_3-im_4)(m_3+im_4)^2+2M_{13}(m_3^2+m_4^2)(m_3+im_4)\} \end{array}\right],$$

$$M_{19} = 4\hbar_5\beta(m_3+im_4)(3m_3+im_4)(m_3^2+m_4^2)\left[M_{13}^*+2M_{13}\right],$$

$$M_{20} = 4\hbar_5\beta(5m_3+im_4)\left[M_{15}^*(m_3+im_4)^2(3m_3-im_4)+2M_{15}(m_3^2+m_4^2)(3m_3+im_4)\right]$$

in which M_{13}^*, M_{14}^* and $M_{15}^*$are complex conjugates of M_{13}, M_{14} and M_{15} respectively.

3.2.2 Discussion of results

In order to study the partial slip effects for various values of λ_2, β, Ω and t for the oscillating flow, graphs are sketched using equation (2.6) and explained as follows

Figures 3.7 give the effects of partial slip parameter λ_2 on the velocity components u and v. For the viscous fluid, with the increase in λ_2, the velocity decreases near the boundary however, it starts increasing earlier in the case of v away from the plate. Figures 3.8 show that in the presence of third grade parameter β, the variation in the velocity profile is much more significant for various values of λ_2 in both the real and the imaginary parts of the velocity as compared to the viscous fluid. Figures 3.9 show that the velocity increases both in u and v with increase in time t and the velocities attain stability earlier for larger times. Figures 3.10 are sketched to determine best suitable value for the homotopy parameter $\hbar_5$ for the better convergence of the solution.

$$\hbar_5 = -0.2,\ \alpha = \beta = 0,\ W_0 = \Omega = t = 0.5,\ \delta = 1$$

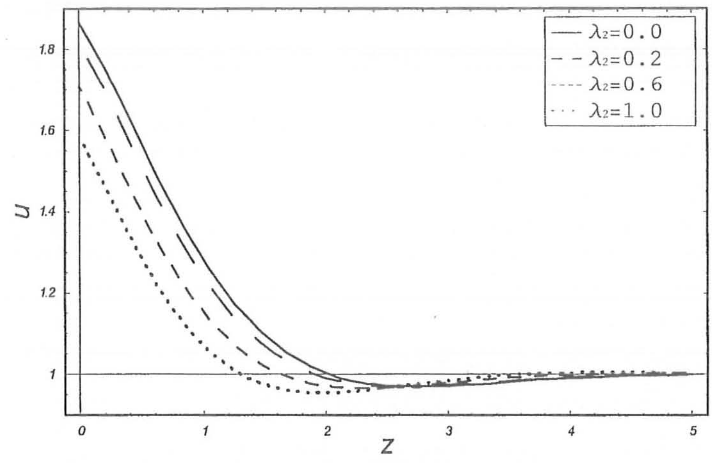


Figure 3.7(a)

$$\hbar_5 = -0.2,\ \alpha = \beta = 0,\ W_0 = \Omega = t = 0.5,\ \delta = 1$$

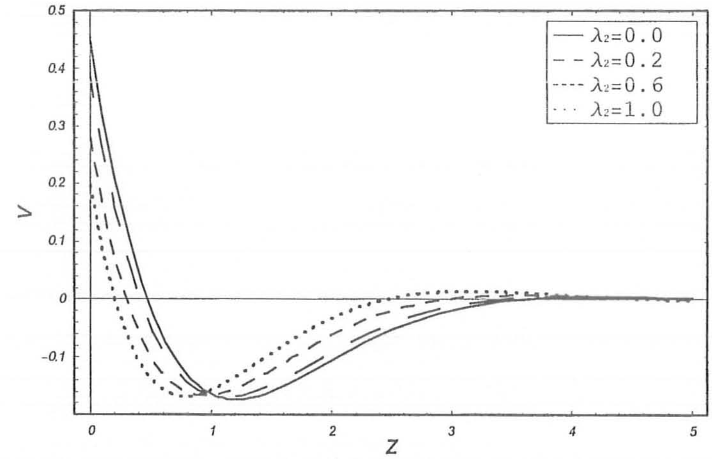


Figure 3.7(b)

Figures 3.7 : The variation of velocity components for various values of partial slip parameter λ_2 for viscous fluid with fixed $\hbar_5$, α, β, W_0, Ω, t and δ.

$\hbar_5 = -0.2,\ \alpha = \beta = W_0 = \Omega = t = 0.5,\ \delta = 1$

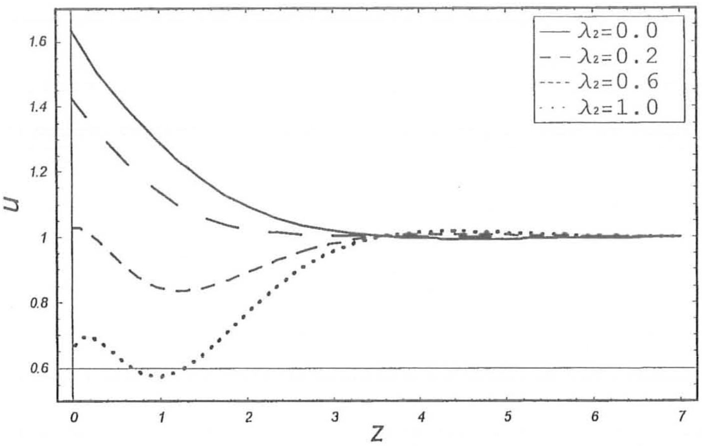


Figure 3.8(a)

$\hbar_5 = -0.2,\ \alpha = \beta = W_0 = \Omega = t = 0.5,\ \delta = 1$

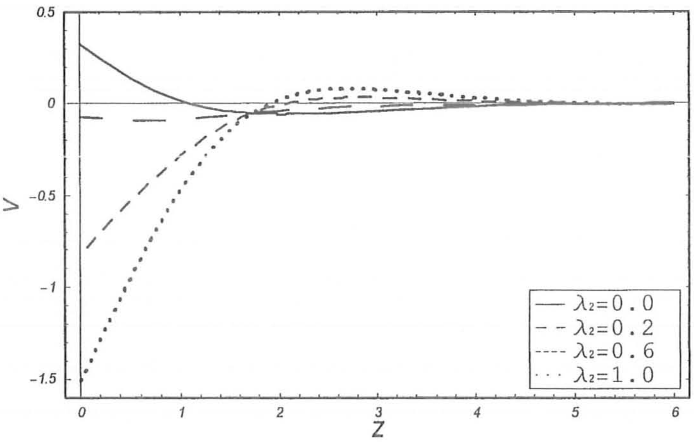


Figure 3.8(b)

Figures 3.8 : The variation of velocity components for various values of partial slip parameter λ_2 for third grade fluid with fixed $\hbar_5$, α, β, W_0, Ω, t and δ.

$\hbar_5 = -0.2,\ \alpha = \beta = W_0 = \Omega = \lambda_2 = 0.5,\ \delta = 1$

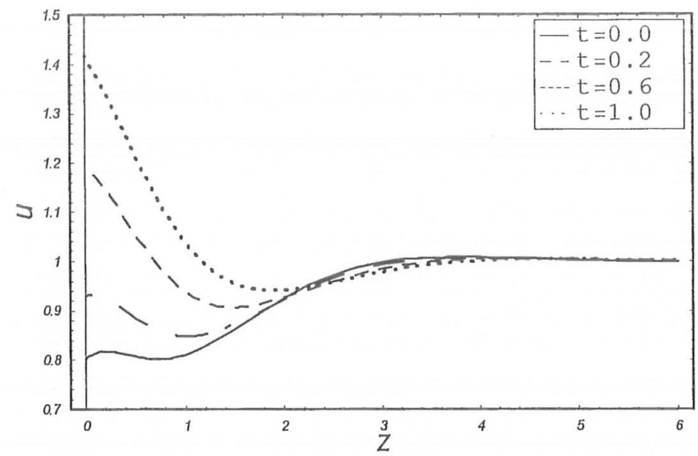


Figure 3.9(a)

$\hbar_5 = -0.2,\ \alpha = \beta = W_0 = \Omega = \lambda_2 = 0.5,\ \delta = 1$

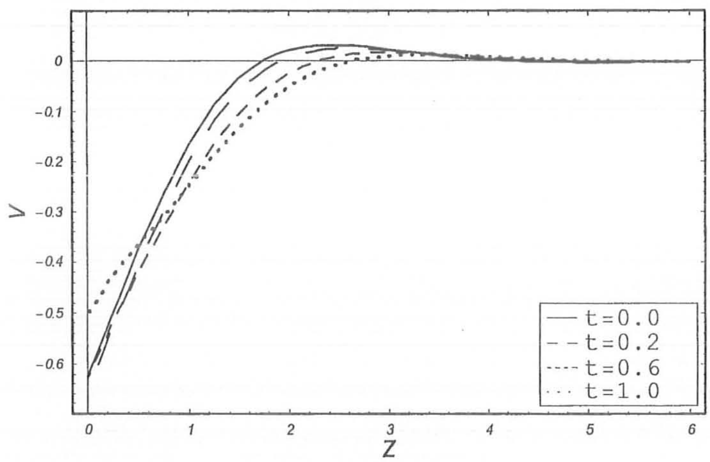


Figure 3.9(b)

Figures 3.9 : The variation of velocity components for various values of time t with fixed $\hbar_5$, λ_2, α, β, Ω, and δ.

$\alpha = \beta = 0.1,\ W_0 = \Omega = \lambda_2 = 0.2,\ t = \delta = 1$

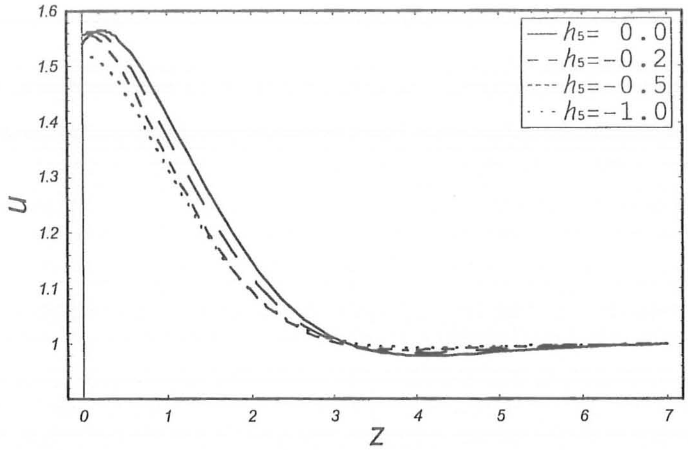


Figure 3.10(a)

$\alpha = \beta = 0.1,\ W_0 = \Omega = \lambda_2 = 0.2,\ t = \delta = 1$

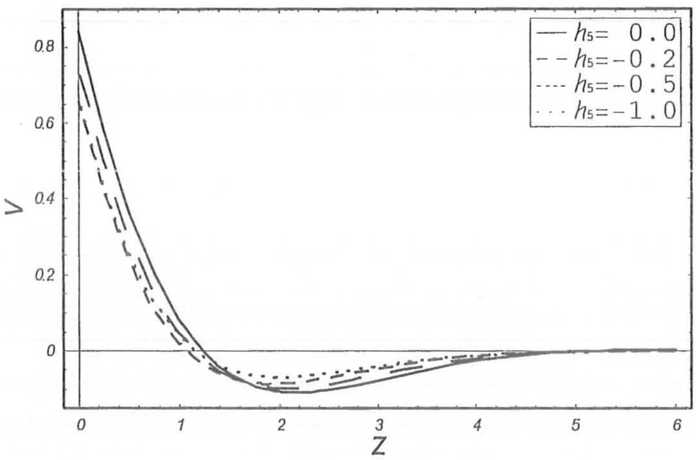


Figure 3.10(b)

Figures 3.10 : The variation of velocity components for various values of $\hbar_5$ with fixed $\lambda_2,\ \alpha,\ \beta,\ \Omega,\ W_0,\ t$ and δ.

3.2.3 Removal of secular term by Lighthill technique

We observe that there appears a secular term $M_{13}ze^{-(m_3+im_4)z}e^{ikt}$ in equation (3.45) in the sense that the series will converge slowly to the function $F(z,t)$. This secular term may also add up in the next higher order terms to further reduce the rate of convergence. Therefore, it is desirable to eliminate this term. In order to do so, we use Lighthill technique. For this we expand the function $F(z,t)$ as a perturbation series in terms of ε similarly as in equation (1.70). This requires an appropriate transformation of the variable z to s as

$$z = s + \varepsilon f_2(s) \tag{3.46}$$

where $f_2(s)$ is an arbitrary function of s. Using this transformation in equation (3.45), we arrive at

$$\begin{aligned} F(s,t) &= e^{-(m_3+im_4)s}e^{ikt} \\ &+\varepsilon\left[\begin{array}{c} (m_3+im_4)f_2e^{-(m_3+im_4)s} + M_{13}se^{-(m_3+im_4)s} \\ +M_{14}e^{-(m_3+im_4)s} - M_{15}e^{-(3m_3+im_4)s} \end{array}\right]e^{ikt} + O(\varepsilon^2). \end{aligned} \tag{3.47}$$

To eliminate the secular term s, consider that

$$(m_3+im_4)f_2 + M_{13}s = 0 \tag{3.48}$$

giving

$$f_2(s) = -\frac{M_{13}s}{m_3+im_4}. \tag{3.49}$$

Thus equation (3.46) gives

$$s = z\left[1 + \frac{\varepsilon M_{13}}{m_3+im_4}\right]. \tag{3.50}$$

Using equation (3.50) in equation (3.47), we have

$$F(z,t) = \left[(1+\varepsilon M_{14})e^{-((m_3+im_4)+\varepsilon M_{13})z} - \varepsilon M_{15}e^{-(3m_3+im_4)\left(1+\frac{\varepsilon M_{13}}{m_3+im_4}\right)z}\right]e^{ikt}. \tag{3.51}$$

We note that no secular term appears in equation (3.51). In order to see the validity and improvement in the result, we will compare the contributions of $F_{1s}(z,t)$ i.e. the secular term and $F_{1R}(z,t)$ i.e. non secular term as given in equations (3.52) and (3.53) respectively, where

$$F_{1s}(z,t) = \left[(M_{13}z + M_{14})\,e^{-(m_3+im_4)z} - M_{15}e^{-(3m_3+im_4)z}\right]e^{ikt} \tag{3.52}$$

and

$$F_{1R}(z,t) = \left[(1+M_{14})e^{[(m_3+im_4)+M_{13}]z} - M_{15}e^{-(3m_3+im_4)\left(1+\frac{M_{13}}{m_3+im_4}\right)z}\right]e^{ikt} \tag{3.53}$$

at $\varepsilon = 1$.

The two results are graphed separately and the convergence is checked.

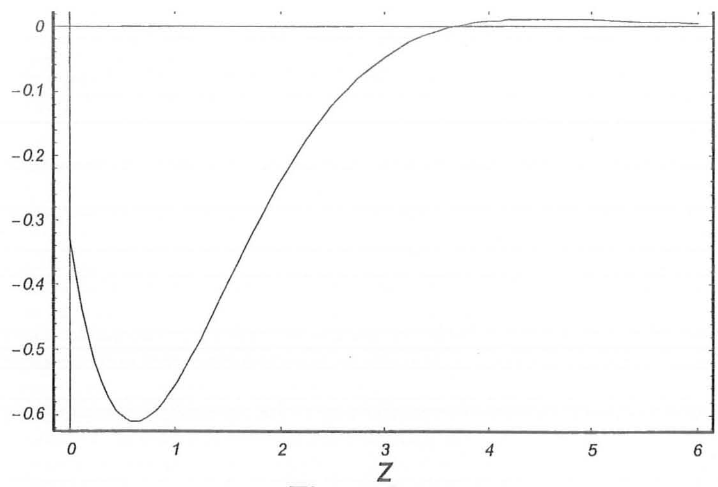


Figure 3.11

Figure 3.11 : Profile of partial slip effects for $F_{1s}(z)$ with fixed values of $\hbar_5 = -0.2$ and $t = \alpha = \beta = \delta = 1$, $\Omega = W_0 = 0.5$

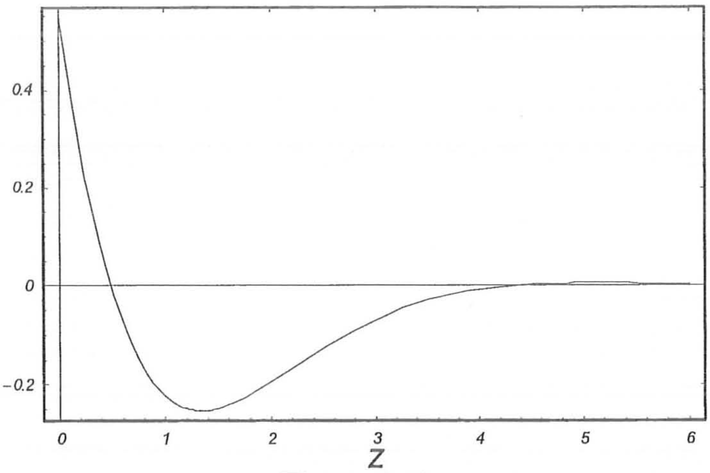


Figure 3.12

Figure 3.12 :Profile of partial slip effects for $F_{1R}(z)$ with fixed values of $\hbar_5 = -0.2$ and $t = \alpha = \beta = \delta = 1,\ \Omega = W_0 = 0.5$

Comparisons of figures (3.11) and (3.12) reveals that the convergence is improved by removing the so-called secular term

3.2.4 Concluding remarks

Two oscillating flow problems of a third grade fluid have been solved by homotopy analysis method. Convergence of the developed solutions has been checked explicitly. Specifically, non-linear equations with non-linear boundary conditions have been solved. To the best of our knowledge, such kind of analytic solution have never been reported in the past. Graphs are plotted for the influence of various pertinent parameters on the velocity components and discussed.

Chapter 4

Numerical solutions for rotating flows of a third grade fluid with partial slip

The purpose of this chapter is to provide the numerical solutions for the steady and oscillating flow problems with partial slip conditions. These two flow problems have been already solved by homotopy analysis method (HAM) in the previous two chapters. The present study is made to check the accuracy of the HAM solutions. Finite difference method is used for the numerical solution. A reasonable agreement is achieved between the graphical results of HAM and numerical solutions.

4.1 Steady flow past a porous plate

In this section the governing problem consists of equations (2.10), (2.51) and (2.52).

4.1.1 Numerical solution

By virtue of equations (1.59) to (1.61), we can discretize equation (2.10) in the following form

$$\frac{1}{h^2}\left(F_{i+1}-2F_i+F_{i-1}\right)-2i\Omega F_i$$

$$+W_0\left[\frac{1}{2h}\left(F_{i+1}-F_{i-1}\right)-\frac{\alpha}{2h^3}\left(F_{i+2}-2F_{i+1}+2F_{i-1}-F_{i-2}\right)\right]$$

$$+\frac{\beta}{2h^4}\left[\begin{array}{c} 2\left(F_{i+1}-F_{i-1}\right)\left(F_{i+2}-2F_{i+1}+F_i\right)\left(F^*_{i+1}-F^*_{i-1}\right)+ \\ \left(F_{i+1}-F_{i-1}\right)^2\left(F^*_{i+2}-2F^*_{i+1}+F^*_i\right)\end{array}\right]=0, \quad (4.1)$$

where $h=z_i-z_{i-1}$ is preferred to be 0.01 for the present calculations.

The iterative procedure applied to the non-linear part of the above equation is

$$G_i\,F^{(n+1)}_{i-2}+H_i\,F^{(n+1)}_{i-1}+I_i\,F^{(n+1)}_{i}+J_i\,F^{(n+1)}_{i+1}+K_i\,F^{(n+1)}_{i+2}=L_i \qquad (4.2)$$

and the initial guess approximation is taken to be

$$F^{(0)}_i=0, \qquad 0\leq i\leq Q, \qquad (4.3)$$

where

$$
\begin{aligned}
G_i &= \alpha W_0, && (4.4)\\
H_i &= 2h - h^2 W_0 - 2\alpha W_0, && (4.5)\\
I_i &= -4h - 4h^3 i\Omega, && (4.6)\\
J_i &= 2h + h^2 W_0 + 2\alpha W_0, && (4.7)\\
K_i &= -\alpha W_0, && (4.8)
\end{aligned}
$$

$$
L_i = -\frac{\beta}{h}\left[\begin{array}{c} 2\left(F_{i+1}^{(n)} - F_{i-1}^{(n)}\right)\left(F_{i+2}^{(n)} - 2F_{i+1}^{(n)} + F_i^{(n)}\right)\left(F_{i+1}^{*(n)} - F_{i-1}^{*(n)}\right) + \\ \left(F_{i+1}^{(n)} - F_{i-1}^{(n)}\right)^2\left(F_{i+2}^{*(n)} - 2F_{i+1}^{*(n)} + F_i^{*(n)}\right) \end{array}\right]. \tag{4.9}
$$

For $i = 1$, equation (4.2) is

$$
G_1 F_{-1}^{(n+1)} + H_1 F_0^{(n+1)} + I_1 F_1^{(n+1)} + J_1 F_2^{(n+1)} + K_1 F_3^{(n+1)} = L_1. \tag{4.10}
$$

The value of F at the fictitious point z_{-1} is approximated by means of the Langrange polynomial of third degree

$$
F_{-1}^{(n+1)} = X_0 F_0^{(n+1)} + X_1 F_1^{(n+1)} + X_2 F_2^{(n+1)} + X_3 F_3^{(n+1)}. \tag{4.11}
$$

In above equation

$$
X_0 = \left(\frac{z_{-1} - z_1}{z_0 - z_1}\right)\left(\frac{z_{-1} - z_2}{z_0 - z_2}\right)\left(\frac{z_{-1} - z_3}{z_0 - z_3}\right), \tag{4.12}
$$

$$
X_1 = \left(\frac{z_{-1} - z_0}{z_1 - z_0}\right)\left(\frac{z_{-1} - z_2}{z_1 - z_2}\right)\left(\frac{z_{-1} - z_3}{z_1 - z_3}\right), \tag{4.13}
$$

$$
X_2 = \left(\frac{z_{-1} - z_0}{z_2 - z_0}\right)\left(\frac{z_{-1} - z_1}{z_2 - z_1}\right)\left(\frac{z_{-1} - z_3}{z_2 - z_3}\right), \tag{4.14}
$$

and

$$X_3 = \left(\frac{z_{-1} - z_0}{z_3 - z_0}\right)\left(\frac{z_{-1} - z_1}{z_3 - z_1}\right)\left(\frac{z_{-1} - z_2}{z_3 - z_2}\right). \qquad (4.15)$$

Substitution of equation (4.11) into equation (4.10) yields

$$\begin{aligned} (G_1 X_0 + H_1)\, F_0^{(n+1)} + (G_1 X_1 + I_1)\, F_1^{(n+1)} & \\ + (G_1 X_2 + J_1)\, F_2^{(n+1)} + (G_1 X_3 + K_1)\, F_3^{(n+1)} &= L_1. \end{aligned} \qquad (4.16)$$

Now $F_0^{(n+1)} = F_0$ is known and fixed, so the equation (4.16) can be written as

$$I_1'\, F_1^{(n+1)} + J_1'\, F_2^{(n+1)} + K_1'\, F_3^{(n+1)} = L_1', \qquad (4.16a)$$

in which

$$\begin{aligned} I_1' &= G_1 X_1 + I_1, \;\; J_1' = G_1 X_2 + J_1, \\ K_1' &= G_1 X_3 + K_1, \;\; L_1' = L_1 - (G_1 X_0 + H_1)\, F_0. \end{aligned} \qquad (4.17)$$

For $i = 2:$

$$G_2\, F_0^{(n+1)} + H_2\, F_1^{(n+1)} + I_2\, F_2^{(n+1)} + J_2\, F_3^{(n+1)} + K_2\, F_4^{(n+1)} = L_2. \qquad (4.18)$$

Since $F_0^{(n+1)} = F_0$ is known thus from the above equation, we have

$$H_2\, F_1^{(n+1)} + I_2\, F_2^{(n+1)} + J_2 F_3^{(n+1)} + K_2\, F_4^{(n+1)} = L_2', \qquad (4.18a)$$

where

$$L_2' = L_2 - G_2\, F_0. \qquad (4.19)$$

For $3 \leq i \leq Q - 3,$ the equations are

$$G_i\, F_{i-2}^{(n+1)} + H_i\, F_{i-1}^{(n+1)} + I_i\, F_i^{(n+1)} + J_i\, F_{i+1}^{(n+1)} + K_i\, F_{i+2}^{(n+1)} = L_i. \qquad (4.19a)$$

For $i = Q - 2$, we have

$$G_{Q-2}\,F_{Q-4}^{(n+1)} + H_{Q-2}\,F_{Q-3}^{(n+1)} + I_{Q-2}\,F_{Q-2}^{(n+1)}$$

$$+J_{Q-2}\,F_{Q-1}^{(n+1)} + K_{Q-2}\,F_{Q}^{(n+1)} = L_{Q-2}. \qquad (4.20)$$

Since $F_Q^{(n+1)} = F_Q$ is known, so the equation (4.20) is

$$G_{Q-2}\,F_{Q-4}^{(n+1)} + H_{Q-2}\,F_{Q-3}^{(n+1)} + I_{Q-2}\,F_{Q-2}^{(n+1)} + J_{Q-2}\,F_{Q-1}^{(n+1)} = L'_{Q-2}, \qquad (4.20a)$$

whence

$$L'_{Q-2} = L_{Q-2} - K_{Q-2}\,F_Q. \qquad (4.21)$$

For $i = Q - 1$, we can write

$$G_{Q-1}\,F_{Q-3}^{(n+1)} + H_{Q-1}\,F_{Q-2}^{(n+1)} + I_{Q-1}\,F_{Q-1}^{(n+1)}$$

$$+ \quad J_{Q-1}\,F_{Q}^{(n+1)} + K_{Q-1}\,F_{Q+1}^{(n+1)} = L_{Q-1}. \qquad (4.22)$$

To find the value of L_{Q-1}, we must have the value of F_{Q+1}. Now augmentation of the boundary condition

$$\frac{\partial F}{\partial z} = 0 \qquad \text{as} \quad z \to \infty \qquad (4.23)$$

yields a well-posed problem. The boundary condition is discretized to give

$$F_{Q+1} = F_Q \qquad \text{i.e.} \quad F_{Q+1}^{(n+1)} = F_Q^{(n+1)} \qquad (4.24)$$

Thus for $i = Q - 1$, equation (4.22) becomes

$$G_{Q-1}\,F_{Q-3}^{(n+1)} + H_{Q-1}\,F_{Q-2}^{(n+1)} + I_{Q-1}\,F_{Q-1}^{(n+1)} = L'_{Q-1}, \qquad (4.24a)$$

where

$$L'_{Q-1} = L_{Q-1} - (J_{Q-1} + K_{Q-1})\,F_Q. \qquad (4.25)$$

It is noted that there are $Q-1$ equations in $Q-1$ unknowns and in matrix form, we have

$$\begin{bmatrix} I_1' & J_1' & K_1' & 0 & 0 & 0 & 0 & 0 & . & . & 0 \\ H_2 & I_2 & J_2 & K_2 & 0 & 0 & 0 & 0 & . & . & 0 \\ G_3 & H_3 & I_3 & J_3 & K_3 & 0 & 0 & 0 & . & . & 0 \\ . & . & . & . & . & . & . & . & . & . & . \\ 0 & . & 0 & G_i & H_i & I_i & J_i & K_i & 0 & . & 0 \\ . & . & . & . & . & . & . & . & . & . & . \\ 0 & 0 & 0 & . & . & 0 & G_{Q-3} & H_{Q-3} & I_{Q-3} & J_{Q-3} & K_{Q-3} \\ 0 & 0 & 0 & 0 & . & . & 0 & G_{Q-2} & H_{Q-2} & I_{Q-2} & J_{Q-2} \\ 0 & 0 & 0 & 0 & 0 & . & . & 0 & G_{Q-1} & H_{Q-1} & I_{Q-1} \end{bmatrix}$$

$$\times \begin{bmatrix} F_1^{(n+1)} \\ F_2^{(n+1)} \\ F_3^{(n+1)} \\ . \\ F_i^{(n+1)} \\ . \\ F_{Q-3}^{(n+1)} \\ F_{Q-2}^{(n+1)} \\ F_{Q-1}^{(n+1)} \end{bmatrix} = \begin{bmatrix} L_1' \\ L_2' \\ L_3 \\ . \\ L_i \\ . \\ L_{Q-3} \\ L_{Q-2}' \\ L_{Q-1}' \end{bmatrix}. \tag{4.26}$$

We notice that matrix involved in above equation is pentadiagonal.

We observe that the partial slip boundary condition (2.51) is also highly non-linear. Thus following the same procedure as adopted for discretization of equation (2.10), equation (2.51) may be discretized as

$$F_0^{(n)} = r_1 F_1^{(n)} + r_2 F_2^{(n)} + r_3 \left[-1 + E_1 T^{(n)}\right], \tag{4.27}$$

whence

$$r_0 = h^2 + \lambda(h + \alpha W_0), \quad (4.28)$$

$$r_1 = \lambda(h + 2\alpha W_0)/r_0, \quad (4.29)$$

$$r_2 = -\alpha W_0 \lambda / r_0, \quad (4.30)$$

$$r_3 = h^2 / r_0, \quad (4.31)$$

$$E_1 T^{(n)} = \frac{2\beta}{h^3}\lambda \left[\left(F_1^{(n)} - F_0^{(n)} \right)^2 \left(F_1^{*(n)} - F_0^{*(n)} \right) \right]. \quad (4.32)$$

To evaluate $F_0^{(n+1)}$, first we take $E_1 T^{(n+1)} = E_1 T^{(n)}$ in the system of algebraic equations and the solution of the system is sought for the unknown values of $F_i^{(n+1)}$; $i = 1, 2, 3, ..., Q-1$. Then we update $F_0^{(n+1)}$ by using iterative method as follows

$$F_{0,(k+1)}^{(n+1)} = r_1 F_1^{(n+1)} + r_2 F_2^{(n+1)} + r_3 \left[\frac{2\beta\lambda}{h^3} \left(F_1^{(n+1)} - F_{0,(k)}^{(n+1)} \right)^2 \left(F_1^{*(n+1)} - F_{0,(k)}^{*(n+1)} \right) - 1 \right] \quad (4.33)$$

where

$$F_{0,(1)}^{(n+1)} = F_0^{(n+1)}.$$

This iterative procedure is continued until $F_{0,(k+1)}^{(n+1)} \approx F_{0,(k)}^{(n+1)}$.

For $i = 1$, $i = 2$, $3 \leq i \leq Q-3$, $i = Q-2$ and $i = Q-1$, we have

$$I_1' F_1^{(n+1)} + J_1' F_2^{(n+1)} + K_1' F_3^{(n+1)} = L_1', \quad (4.33a)$$

$$H_2' F_1^{(n+1)} + I_2' F_2^{(n+1)} + J_2 F_3^{(n+1)} + K_2 F_4^{(n+1)} = L_2', \quad (4.33b)$$

$$G_i F_{i-2}^{(n+1)} + H_i F_{i-1}^{(n+1)} + I_i F_i^{(n+1)} + J_i F_{i+1}^{(n+1)} + K_i F_{i+2}^{(n+1)} = L_i \quad (4.33c)$$

$$G_{Q-2} F_{Q-4}^{(n+1)} + H_{Q-2} F_{Q-3}^{(n+1)} + I_{Q-2} F_{Q-2}^{(n+1)} + J_{Q-2} F_{Q-1}^{(n+1)} = L_{Q-2}', \quad (4.33d)$$

$$G_{Q-1}\,F_{Q-3}^{(n+1)} + H_{Q-1}\,F_{Q-2}^{(n+1)} + I_{Q-1}\,F_{Q-1}^{(n+1)} = L'_{Q-1}, \tag{4.33e}$$

in which G_i, H_i, I_i, J_i, K_i and L_i are given through equations (4.4) to (4.9).

In the above equations

$$I'_1 = G_1X_1 + I_1 + r_1\left(H_1 + X_0G_1\right), \tag{4.34}$$

$$J'_1 = G_1X_2 + J_1 + r_2\left(H_1 + X_0G_1\right), \tag{4.35}$$

$$K'_1 = G_1X_3 + K_1, \tag{4.36}$$

$$L'_1 = L_1 - \left(G_1X_0 + H_1\right)\left[r_3\left(E_1T^{(n)} - 1\right)\right], \tag{4.37}$$

$$H'_2 = H_2 + r_1G_2, \tag{4.38}$$

$$I'_2 = I_2 + r_2G_2, \tag{4.39}$$

$$L'_2 = L_2 - G_2\left[r_3(E_1T^{(n)} - 1)\right], \tag{4.40}$$

$$L'_{Q-2} = L_{Q-2} - K_{Q-2}\,F_Q, \tag{4.41}$$

$$L'_{Q-1} = L_{Q-1} - \left(J_{Q-1} + K_{Q-1}\right)F_Q. \tag{4.42}$$

The matrix form of the above set of $Q-1$ equations in $Q-1$ unknowns is a pentadiagonal.

4.1.2 Graphs and discussion

Figures 4.1 and 4.2 are sketched to see the influence of partial slip parameter for the viscous and third grade fluids. It is evident from Figures 4.1 that the velocity components u and v increase with an increase in slip parameter. The effect of slip parameter on the third grade fluid is similar to that of viscous fluid as shown in Figures 4.1. Figures 4.3 is prepared to see the effects of rotation parameter in the presence of slip parameter. Here u increases for large values of rotation parameter which is quite similar to that of suction parameter.

$\alpha = \beta = 0, \ \Omega = 0.5, \ W_0 = 0.5$

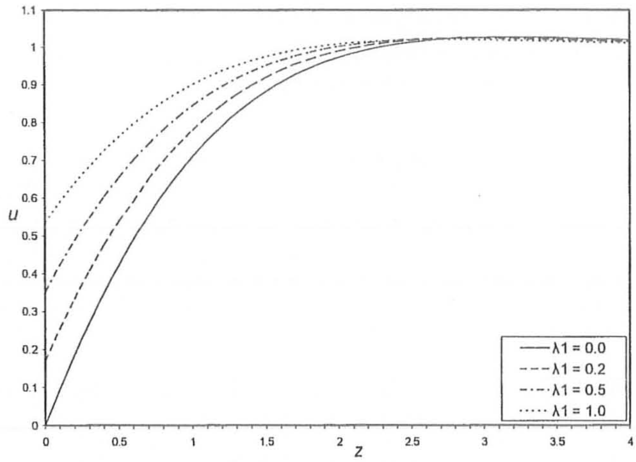


Figure 4.1(a)

$\alpha = \beta = 0, \ \Omega = 0.5, \ W_0 = 0.5$

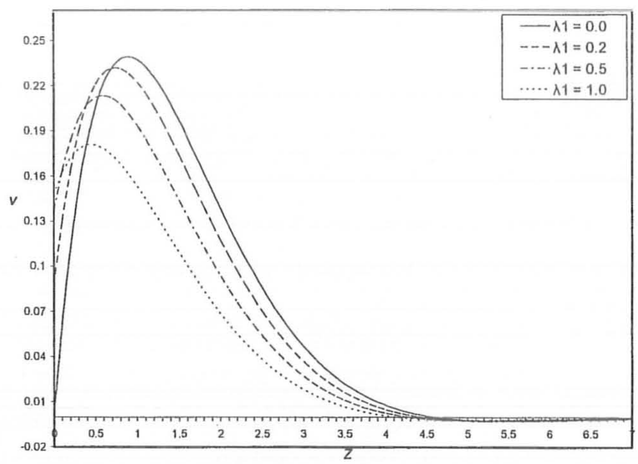


Figure 4.1(b)

Figures 4.1 : The variation of velocity components for various values of partial slip parameter λ_1 for viscous fluid with fixed α, β, Ω and W_0.

$\alpha = \beta = 1, \ \Omega = W_0 = 0.5$

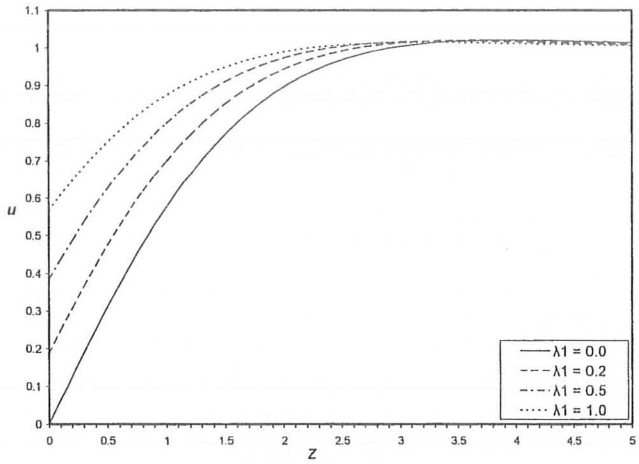


Figure 4.2(a)

$\alpha = \beta = 1, \ \Omega = W_0 = 0.5$

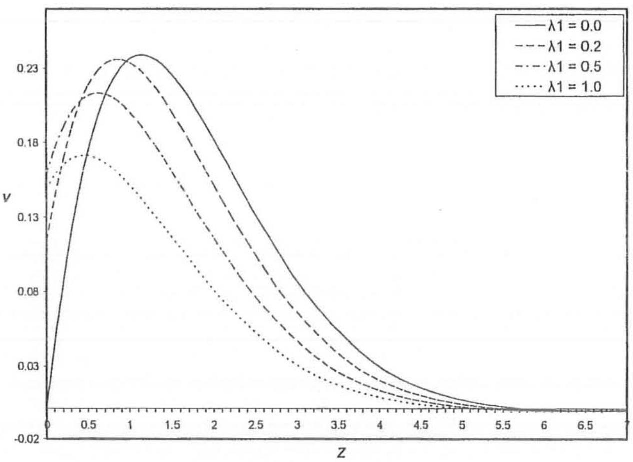


Figure 4.2(b)

Figures 4.2 : The variation of velocity components for various values of partial slip parameter λ_1 for third grade fluid with fixed α, β, Ω and W_0.

$\alpha = \beta = \lambda_1 = W_0 = 0.5$

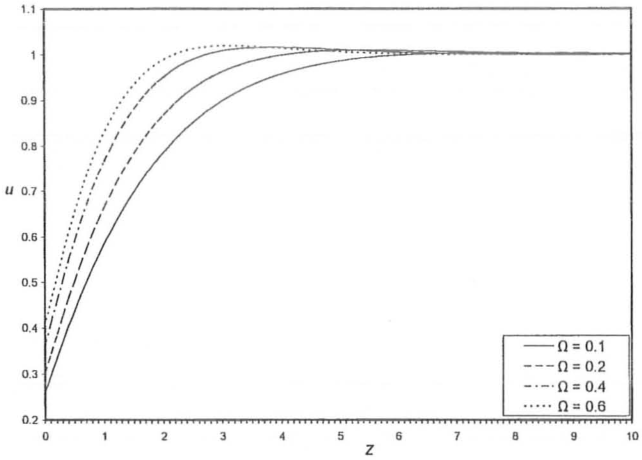


Figure 4.3(a)

$\alpha = \beta = \lambda_1 = W_0 = 0.5$

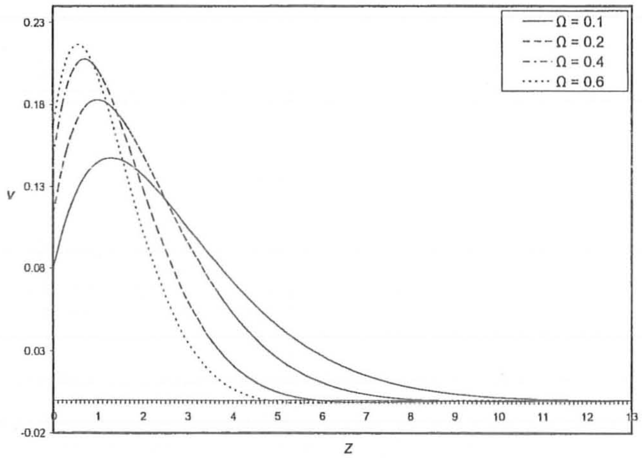


Figure 4.3(b)

Figures 4.3 : The variation of velocity components for various values of rotation Ω with fixed α, β, λ_1 and W_0.

4.2 Oscillating flow past a porous plate

In this section the problem governing the flow is described by equations (3.6), (3.34) and (3.35).

4.2.1 Numerical solution

We note that partial differential equation (3.6) and boundary conditions (3.34) are highly non-linear. Also equation (3.6) is time dependent and has mixed derivatives with respect to time and space coordinates. Thus we make use of the implicit scheme. There are a number of methods available to discretize the partial differential equations into a system of algebraic equations. To transform the partial differential equations (3.6) and (3.34) into a system of algebraic equations, we use finite difference approximations to derivatives which are centred midway in time between the known and the unknown levels and we take central differences to approximate the derivatives for unknown level and for known level, we also use central differences to approximate the derivatives except for the second-order derivative. Furthermore, we approximate the non-linear term only at the known level by using central difference for the first-order derivative and forward difference for the second-order derivative.

The equation (3.6) is transformed into algebraic equations by substituting the approximations to derivatives using equations (1.62) to (1.65), (1.67) and

(1.69) as follows

$$\frac{\alpha}{lh^2}\left[\left(F_{i+1}^{j+1}-2F_i^{j+1}+F_{i-1}^{j+1}\right)-\left(F_{i+2}^{j}-2F_{i+1}^{j}+F_i^{j}\right)\right]$$

$$-\frac{\alpha W_0}{4h^3}\left[\begin{array}{c}\left(F_{i+2}^{j+1}-2F_{i+1}^{j+1}+2F_{i-1}^{j+1}-F_{i-2}^{j+1}\right)\\ +\left(F_{i+2}^{j}-2F_{i+1}^{j}+2F_{i-1}^{j}-F_{i-2}^{j}\right)\end{array}\right]$$

$$+\frac{1}{2h^2}\left[\left(F_{i+1}^{j+1}-2F_i^{j+1}+F_{i-1}^{j+1}\right)+\left(F_{i+2}^{j}-2F_{i+1}^{j}+F_i^{j}\right)\right]$$

$$+\frac{W_0}{4h}\left[\left(F_{i+1}^{j+1}-F_{i-1}^{j+1}\right)+\left(F_{i+1}^{j}-F_{i-1}^{j}\right)\right]$$

$$-\frac{1}{l}\left[F_i^{j+1}-F_i^{j}\right]-i\Omega\left[F_i^{j+1}+F_i^{j}\right]$$

$$+\frac{\beta}{2h^4}\left[\begin{array}{c}2\left(F_{i+1}^{j}-F_{i-1}^{j}\right)\left(F_{i+2}^{j}-2F_{i+1}^{j}+F_i^{j}\right)\left(F_{i+1}^{*\,j}-F_{i-1}^{*\,j}\right)+\\ \left(F_{i+1}^{j}-F_{i-1}^{j}\right)^2\left(F_{i+2}^{*\,j}-2F_{i+1}^{*\,j}+F_i^{*\,j}\right)\end{array}\right]=0. \tag{4.43}$$

The problem consisting of above equation along with initial and boundary conditions of section 3.2 becomes

$$\widetilde{G}_i\,F_{i-2}^{j+1}+\widetilde{H}_i\,F_{i-1}^{j+1}+\widetilde{I}_i\,F_i^{j+1}+\widetilde{J}_i\,F_{i+1}^{\,j+1}+\widetilde{K}_i\,F_{i+2}^{j+1}=\widetilde{L}_i, \tag{4.43a}$$

$$F_0^j=e^{i\delta jl},\quad F_Q^j=0,\quad F_i^0=0,\quad i=0,1,2,...,Q, \tag{4.44}$$

where

$$\widetilde{G}_i = \frac{l\alpha W_0}{4h^3}, \tag{4.45}$$

$$\widetilde{H}_i = -\frac{l\alpha W_0}{2h^3}+\frac{\alpha}{h^2}+\frac{l}{2h^2}-\frac{lW_0}{4h}, \tag{4.46}$$

$$\widetilde{I}_i = -\frac{2\alpha}{h^2}-\frac{l}{h^2}-1-i\Omega l, \tag{4.47}$$

$$\widetilde{J}_i = \frac{l\alpha W_0}{2h^3}+\frac{\alpha}{h^2}+\frac{l}{2h^2}+\frac{lW_0}{4h}, \tag{4.48}$$

$$\widetilde{K}_i = -\frac{l\alpha W_0}{4h^3}, \tag{4.49}$$

$$
\begin{aligned}
\widetilde{L}_i &= \frac{\alpha}{h^2}\left[F^j_{i+2} - 2F^j_{i+1} + F^j_i\right] + \frac{l\alpha W_0}{4h^3}\left[F^j_{i+2} - 2F^j_{i+1} + 2F^j_{i-1} - F^j_{i-2}\right] \\
&\quad - \frac{l}{2h^2}\left[F^j_{i+2} - 2F^j_{i+1} + F^j_i\right] \\
&\quad - \frac{lW_0}{4h}\left[F^j_{i+1} - F^j_{i-1}\right] - F^j_i + i\Omega l F^j_i \\
&\quad - \frac{l\beta}{2h^4}\left[2\left(F^j_{i+1} - F^j_{i-1}\right)\left(F^j_{i+2} - 2F^j_{i+1} + F^j_i\right)\left(F^{*\,j}_{i+1} - F^{*\,j}_{i-1}\right)\right. \\
&\quad \left. + \left(F^j_{i+1} - F^j_{i-1}\right)^2\left(F^{*\,j}_{i+2} - 2F^{*\,j}_{i+1} + F^{*\,j}_i\right)\right]. \qquad (4.50)
\end{aligned}
$$

For $i = 1$ we have from equation (4.44) as

$$
\widetilde{G}_1\, F^{j+1}_{-1} + \widetilde{H}_1\, F^{j+1}_0 + \widetilde{I}_1\, F^{j+1}_1 + \widetilde{J}_1\, F^{j+1}_2 + \widetilde{K}_1\, F^{j+1}_3 = \widetilde{L}_1. \qquad (4.51)
$$

The value of F at the fictitious point z_{-1} is approximated by means of the Langrange polynomial of third degree

$$
F^{j+1}_{-1} = X_0 F^{j+1}_0 + X_1 F^{j+1}_1 + X_2 F^{j+1}_2 + X_3 F^{j+1}_3, \qquad (4.52)
$$

where X_0, X_1, X_2 and X_3 are determined as equations (4.12) to (4.15). Using equation (4.52) in equation (4.51), we have

$$
\begin{aligned}
&\left(\widetilde{G}_1 X_0 + \widetilde{H}_1\right) F^{j+1}_0 + \left(\widetilde{G}_1 X_1 + \widetilde{I}_1\right) F^{j+1}_1 \\
&+ \left(\widetilde{G}_1 X_2 + \widetilde{J}_1\right) F^{j+1}_2 + \left(\widetilde{G}_1 X_3 + \widetilde{K}_1\right) F^{j+1}_3 = \widetilde{L}_1. \qquad (4.53)
\end{aligned}
$$

Now F^{j+1}_0 is known, so the equation (4.53) must be written as

$$
\widetilde{I}'_1\, F^{j+1}_1 + \widetilde{J}'_1\, F^{j+1}_2 + \widetilde{K}'_1\, F^{j+1}_3 = \widetilde{L}'_1, \qquad (4.53a)
$$

in which

$$
\begin{aligned}
\widetilde{I}'_1 &= \widetilde{G}_1 X_1 + \widetilde{I}_1,\ \widetilde{J}'_1 = \widetilde{G}_1 X_2 + \widetilde{J}_1, \\
\widetilde{K}'_1 &= \widetilde{G}_1 X_3 + \widetilde{K}_1,\ \widetilde{L}'_1 = \widetilde{L}_1 - \left(\widetilde{G}_1 X_0 + \widetilde{H}_1\right) F^{j+1}_0. \qquad (4.54)
\end{aligned}
$$

For $i = 2$:

$$\widetilde{G}_2\, F_0^{j+1} + \widetilde{H}_2\, F_1^{j+1} + \widetilde{I}_2\, F_2^{j+1} + \widetilde{J}_2\, F_3^{j+1} + \widetilde{K}_2\, F_4^{j+1} = \widetilde{L}_2. \tag{4.55}$$

Since F_0^{j+1} is known thus from above equation, we have

$$\widetilde{H}_2\, F_1^{j+1} + \widetilde{I}_2\, F_2^{j+1} + \widetilde{J}_2\, F_3^{j+1} + \widetilde{K}_2\, F_4^{j+1} = \widetilde{L}'_2, \tag{4.55a}$$

where

$$\widetilde{L}'_2 = \widetilde{L}_2 - \widetilde{G}_2\, F_0^{j+1}. \tag{4.56}$$

For $3 \leq i \leq Q-3,$ the equations are

$$\widetilde{G}_i\, F_{i-2}^{j+1} + \widetilde{H}_i\, F_{i-1}^{j+1} + \widetilde{I}_i\, F_i^{j+1} + \widetilde{J}_i\, F_{i+1}^{j+1} + \widetilde{K}_i\, F_{i+2}^{j+1} = \widetilde{L}_i. \tag{4.56a}$$

For $i = Q-2$, one may write

$$\begin{aligned} &\widetilde{G}_{Q-2}\, F_{Q-4}^{j+1} + \widetilde{H}_{Q-2}\, F_{Q-3}^{j+1} + \widetilde{I}_{Q-2}\, F_{Q-2}^{j+1} \\ &+\widetilde{J}_{Q-2}\, F_{Q-1}^{j+1} + \widetilde{K}_{Q-2}\, F_Q^{j+1} = \widetilde{L}_{Q-2}. \end{aligned} \tag{4.57}$$

Since F_Q^{j+1} is known, so above equation reduces to

$$\widetilde{G}_{Q-2}\, F_{Q-4}^{j+1} + \widetilde{H}_{Q-2}\, F_{Q-3}^{j+1} + \widetilde{I}_{Q-2}\, F_{Q-2}^{j+1} + \widetilde{J}_{Q-2}\, F_{Q-1}^{j+1} = \widetilde{L}'_{Q-2}, \tag{4.57a}$$

in which

$$\widetilde{L}'_{Q-2} = \widetilde{L}_{Q-2} - \widetilde{K}_{Q-2}\, F_Q^{j+1}. \tag{4.58}$$

For $i = Q-1$, we have

$$\begin{aligned} &\widetilde{G}_{Q-1}\, F_{Q-3}^{j+1} + \widetilde{H}_{Q-1}\, F_{Q-2}^{j+1} + \widetilde{I}_{Q-1}\, F_{Q-1}^{j+1} \\ + \quad &\widetilde{J}_{Q-1}\, F_Q^{j+1} + \widetilde{K}_{Q-1}\, F_{Q+1}^{j+1} = \widetilde{L}_{Q-1}. \end{aligned} \tag{4.59}$$

To find the value of $\widetilde{L}_{Q-1}$ at the time level j, we must have the value of F_{Q+1}^{j}. The augmentation of the boundary condition

$$\frac{\partial F(\infty, t)}{\partial z} = 0 \quad \text{as} \quad z \to \infty \tag{4.60}$$

defines a well posed problem. The boundary condition is discretized to give

$$F_{Q+1}^{j} = F_{Q}^{j}. \tag{4.61}$$

For $i = Q - 1$, equation (4.59) becomes

$$\widetilde{G}_{Q-1}\, F_{Q-3}^{j+1} + \widetilde{H}_{Q-1}\, F_{Q-2}^{j+1} + \widetilde{I}_{Q-1}\, F_{Q-1}^{j+1} = \widetilde{L}'_{Q-1}, \tag{4.61a}$$

where

$$\widetilde{L}'_{Q-1} = \widetilde{L}_{Q-1} - \left(\widetilde{J}_{Q-1} + \widetilde{K}_{Q-1}\right) F_{Q}^{j+1}. \tag{4.62}$$

It is noted that there are $Q - 1$ equations in $Q - 1$ unknowns and these in matrix form give

$$\begin{bmatrix} \widetilde{I}'_1 & \widetilde{J}'_1 & \widetilde{K}'_1 & 0 & 0 & 0 & 0 & 0 & . & . & 0 \\ \widetilde{H}_2 & \widetilde{I}_2 & \widetilde{J}_2 & \widetilde{K}_2 & 0 & 0 & 0 & 0 & . & . & 0 \\ \widetilde{G}_3 & \widetilde{H}_3 & \widetilde{I}_3 & \widetilde{J}_3 & \widetilde{K}_3 & 0 & 0 & 0 & . & . & 0 \\ . & . & . & . & . & . & . & . & . & . & . \\ 0 & . & 0 & G_i & \widetilde{H}_i & \widetilde{I}_i & \widetilde{J}_i & \widetilde{K}_i & 0 & . & 0 \\ . & . & . & . & . & . & . & . & . & . & . \\ 0 & 0 & 0 & . & . & 0 & \widetilde{G}_{Q-3} & \widetilde{H}_{Q-3} & \widetilde{I}_{Q-3} & \widetilde{J}_{Q-3} & \widetilde{K}_{Q-3} \\ 0 & 0 & 0 & 0 & . & . & 0 & \widetilde{G}_{Q-2} & \widetilde{H}_{Q-2} & \widetilde{I}_{Q-2} & \widetilde{J}_{Q-2} \\ 0 & 0 & 0 & 0 & 0 & . & . & 0 & \widetilde{G}_{Q-1} & \widetilde{H}_{Q-1} & \widetilde{I}_{Q-1} \end{bmatrix}$$

$$\times \begin{bmatrix} F_1 \\ F_2 \\ F_3 \\ . \\ F_i \\ . \\ F_{Q-3} \\ F_{Q-2} \\ F_{Q-1} \end{bmatrix} = \begin{bmatrix} \widetilde{L}'_1 \\ \widetilde{L}'_2 \\ \widetilde{L}_3 \\ . \\ \widetilde{L}_i \\ . \\ \widetilde{L}_{Q-3} \\ \widetilde{L}'_{Q-2} \\ \widetilde{L}'_{Q-1} \end{bmatrix}. \tag{4.63}$$

We notice that matrix involved in the above equations is pentadiagonal.

We also note that the partial slip boundary condition (3.34) is highly non-linear.This problem is an extension of the problem already presented in section 4.2. Thus following the same procedure, the discretized form for the equation (3.34) has been presented. The discretized form for the partial slip boundary condition has been obtained by using forward and backward differences for the derivatives involved with respect to space and time coordinates respectively. It results in the form

$$F_0^j = \widetilde{r}_1 F_1^j + \widetilde{r}_2 F_2^j + \widetilde{r}_3 \left[F_1^{j-1} - F_0^{j-1}\right] + \widetilde{r}_4 \left[e^{i\delta jl} + E_2 T^j\right], \tag{4.64}$$

where

$$\widetilde{r}_0 = h^2 l + \lambda(hl + \alpha h + \alpha W_0 l), \tag{4.65}$$

$$\widetilde{r}_1 = \lambda(hl + \alpha h + 2\alpha W_0 l)/\widetilde{r}_0, \tag{4.66}$$

$$\widetilde{r}_2 = -\alpha W_0 \lambda l/\widetilde{r}_0, \tag{4.67}$$

$$\widetilde{r}_3 = -\alpha\lambda h/\widetilde{r}_0, \tag{4.68}$$

$$\widetilde{r}_4 = h^2 l/\widetilde{r}_0, \tag{4.69}$$

$$E_2 T^j = \frac{2\beta}{h^3}\lambda\left[\left(F_1^j - F_0^j\right)^2\left(F_1^{*\,j} - F_0^{*\,j}\right)\right]. \tag{4.70}$$

To evaluate F_0^{j+1}, first we take $E_2T^{j+1} = E_2T^j$ in the system of algebraic equations and the solution of the system is sought for the known values of F_i^{j+1}; $i = 1, 2, 3, ..., Q-1$. Then we update F_0^{j+1} by using iterative method as follows

$$\begin{aligned} F_0^{j+1,k+1} = {} & \widetilde{r}_1 F_1^{j+1} + \widetilde{r}_2 F_2^{j+1} + \widetilde{r}_3\left[F_1^j - F_0^j\right] \\ & + \widetilde{r}_4 \frac{\beta\lambda}{h^3}\left[\left(F_1^{j+1} - F_0^{j+1,k}\right)^2\left(\overline{F}_1^{j+1} - \overline{F}_0^{j+1,k}\right)\right] \\ & + \widetilde{r}_4 e^{i\delta(j+1)l}, \end{aligned} \tag{4.71}$$

in which

$$F_0^{j+1,0} = F_0^j.$$

This iterative procedure is continued until $F_0^{j+1,k+1} \approx F_0^{j+1,k}$. Furthermore, F_0^0 is evaluated by letting $F_0^{-1} = F_1^{-1} = 0$ and by using iterative method as described above with $F_0^{0,0} = 0$ as an initial guess.

For $i=1$, $i=2$, $3 \le i \le Q-3$, $i=Q-2$ and $i=Q-1$, we have

$$\widetilde{I}_1'\, F_1^{j+1} + \widetilde{J}_1'\, F_2^{j+1} + \widetilde{K}_1'\, F_3^{j+1} = \widetilde{L}_1', \tag{4.71a}$$

$$\widetilde{H}_2'\, F_1^{j+1} + \widetilde{I}_2'\, F_2^{j+1} + \widetilde{J}_2\, F_3^{j+1} + \widetilde{K}_2\, F_4^{j+1} = \widetilde{L}_2', \tag{4.71b}$$

$$\widetilde{G}_i\, F_{i-2}^{j+1} + \widetilde{H}_i\, F_{i-1}^{j+1} + \widetilde{I}_i\, F_i^{j+1} + \widetilde{J}_i\, F_{i+1}^{j+1} + \widetilde{K}_i\, F_{i+2}^{j+1} = \widetilde{L}_i \tag{4.71c}$$

$$\widetilde{G}_{Q-2}\, F_{Q-4}^{j+1} + \widetilde{H}_{Q-2}\, F_{Q-3}^{j+1} + \widetilde{I}_{Q-2}\, F_{Q-2}^{j+1} + \widetilde{J}_{Q-2}\, F_{Q-1}^{j+1} = \widetilde{L}_{Q-2}', \tag{4.71d}$$

$$\widetilde{G}_{Q-1}\, F_{Q-3}^{j+1} + \widetilde{H}_{Q-1}\, F_{Q-2}^{j+1} + \widetilde{I}_{Q-1}\, F_{Q-1}^{j+1} = \widetilde{L}_{Q-1}', \tag{4.71e}$$

where $\widetilde{G}_i$, $\widetilde{H}_i$, $\widetilde{I}_i$, $\widetilde{J}_i$, $\widetilde{K}_i$ and $\widetilde{L}_i$ are given through equations (4.45) to (4.49) and

$$\widetilde{I}_1' = \widetilde{G}_1 X_1 + \widetilde{I}_1 + \widetilde{r}_1 \left(\widetilde{H}_1 + X_0 \widetilde{G}_1 \right), \tag{4.72}$$

$$\widetilde{J}_1' = \widetilde{G}_1 X_2 + \widetilde{J}_1 + \widetilde{r}_2 \left(\widetilde{H}_1 + X_0 \widetilde{G}_1 \right), \tag{4.73}$$

$$\widetilde{K}_1' = \widetilde{G}_1 X_3 + \widetilde{K}_1, \tag{4.74}$$

$$\widetilde{L}_1' = \widetilde{L}_1 - \left(\widetilde{G}_1 X_0 + \widetilde{H}_1 \right) \left[\widetilde{r}_3 \left(F_1^j - F_0^j \right) + \widetilde{r}_4 \left(E_2 T^j + e^{i\delta(j+1)l} \right) \right], \tag{4.75}$$

$$\widetilde{H}_2' = \widetilde{H}_2 + \widetilde{r}_1 \widetilde{G}_2, \tag{4.76}$$

$$\widetilde{I}_2' = \widetilde{I}_2 + \widetilde{r}_2 \widetilde{G}_2, \tag{4.77}$$

$$\widetilde{L}_2' = \widetilde{L}_2 - \widetilde{G}_2 \left[\widetilde{r}_3 \left(F_1^j - F_0^j \right) + \widetilde{r}_4 \left(E_2 T^{j+1} + e^{i\delta(j+1)l} \right) \right], \tag{4.78}$$

$$\widetilde{L}_{Q-2}' = \widetilde{L}_{Q-2} - \widetilde{K}_{Q-2}\, F_Q^{j+1}, \tag{4.79}$$

$$\widetilde{L}_{Q-1}' = \widetilde{L}_{Q-1} - \left(\widetilde{J}_{Q-1} + \widetilde{K}_{Q-1} \right) F_{Q.}^{j+1} \tag{4.80}$$

The matrix form of the above set of $Q-1$ equations is a pentadiagonal and can be expressed as (4.63).

4.2.2 Graphs and discussion

In this subsection, we present the graphs of velocity components by varying partial slip parameter and time.

$$\alpha = \beta = 0, \; W_0 = \Omega = t = 0.5, \; \delta = 1$$

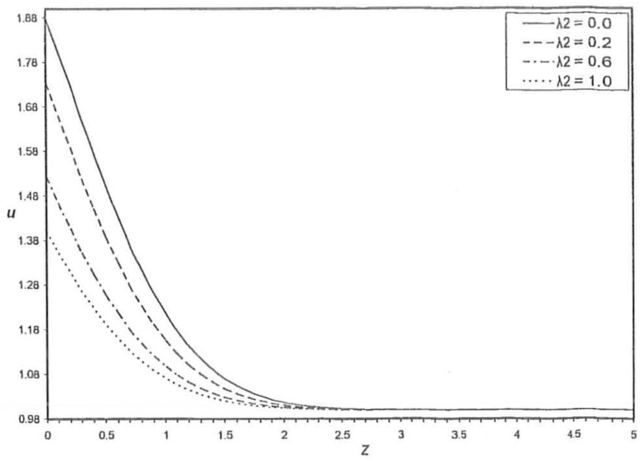


Figure 4.4(a)

$$\alpha = \beta = 0, \; W_0 = \Omega = t = 0.5, \; \delta = 1$$

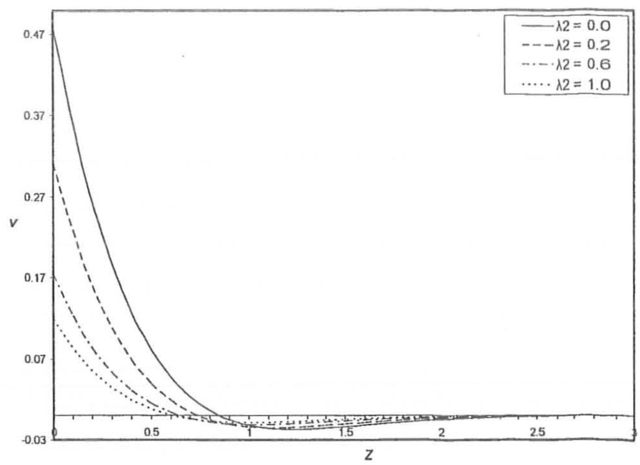


Figure 4.4(b)

Figures 4.4 : The variation of velocity components for various values of partial slip parameter λ_2 for viscous fluid with fixed α, β, W_0, Ω, t and δ.

$\alpha = \beta = W_0 = \Omega = t = 0.5, \ \delta = 1$

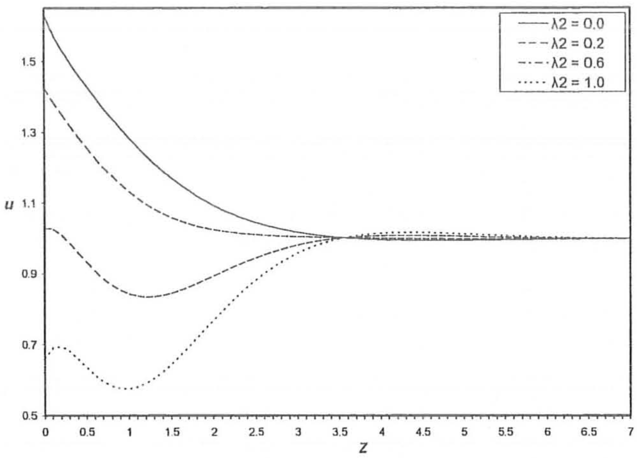


Figure 4.5(a)

$\alpha = \beta = W_0 = \Omega = t = 0.5, \ \delta = 1$

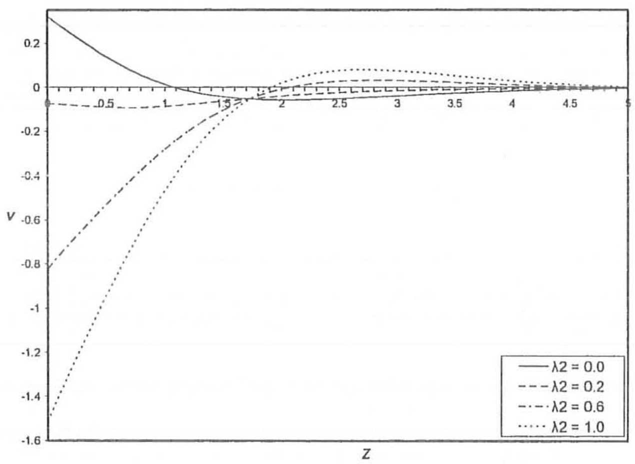


Figure 4.5(b)

Figures 4.5 : The variation of velocity components for various values of partial slip parameter λ_2 for third grade fluid with fixed α, β, W_0, Ω, t and δ.

$$\alpha = \beta = W_0 = \Omega = \lambda_2 = 0.5, \ \delta = 1$$

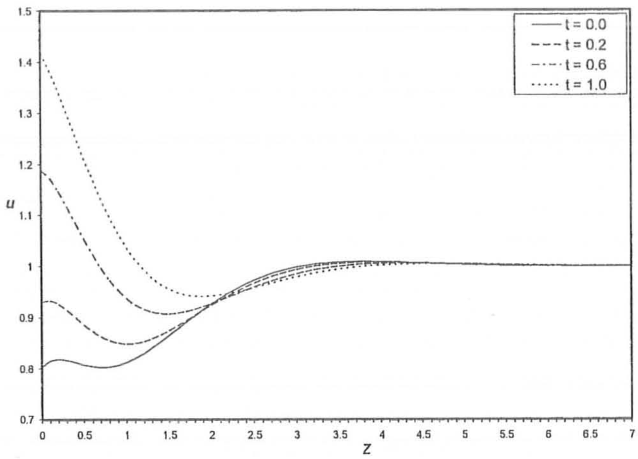


Figure 4.6(a)

$$\alpha = \beta = W_0 = \Omega = \lambda_2 = 0.5, \ \delta = 1$$

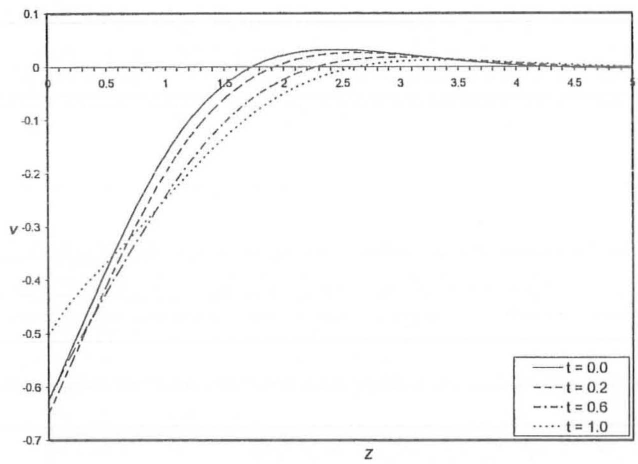


Figure 4.6(b)

Figures 4.6 : The variation of velocity components for various values of time t with fixed λ_2, α, β, Ω, and δ.

Figures 4.4 show the effects of slip parameter in the case of viscous fluid where as Figures 4.5 present the analysis for third grade fluid parameter. Figures 4.6 describe the behaviour of velocity components under the variation of time. It is observed that the velocity decreases by increasing slip parameter for viscous and non-Newtonian flows. The velocity increases by increasing time for the partial slip flow of a non-Newtonian fluid.

4.2.3 Concluding remarks

Here numerical solutions for the two flow problems with partial slip conditions have been developed. It should be pointed out that the partial slip condition for the corresponding problems in viscous and second grade fluids are linear where as in the third grade fluid it is highly non-linear. Thus it seems worthwhile to solve non-linear differential equations with non-linear boundary conditions. Moreover, the similarity in the behaviour of the velocity profile for both semi-analytic and numerical solutions show the accuracy and validity of the results. The comparison of graphs give good agreement between HAM and numerical solutions.

Chapter 5

Conclusion

The work presented in the thesis deals with some non-linear problems for steady and oscillating flows. The steady and oscillatory rotating flows of a third grade fluid using no-slip and partial slip conditions have been considered. The problems addressed are presented in chapters 2 to 4. Chapter 2 deals with the steady flow of an incompressible third grade fluid. The whole system is in a rotating frame of reference with and without slip. The third grade fluid flow past a porous plate has been analyzed. Two illustrative cases of no-slip and partial slip have been considered. The most characteristic feature of the results obtained is that; unlike the inertial frame, the steady asymptotic blowing solution exists.

In chapter 3, analytic solution of an oscillating flow is constructed in a rotating fluid. The fluid is considered as third grade. The flow is generated in the uniformly rotating fluid past a porous oscillating plate. Analytic solution for no-slip and partial slip situations are obtained employing homotopy analysis method. Two oscillating flow problems of a third grade fluid have been solved. Convergence of the developed solutions has been checked explicitly. Specifically, non-linear equations with non-linear boundary conditions have been solved.

Chapter 4 presents the numerical solutions for the steady and oscillating flow problems with partial slip conditions. The study is made to check the accuracy of the HAM solutions. Finite difference method is used for the numerical solution. A sound agreement is achieved between the graphical results of HAM and numerical solutions.

The work presented in the thesis on flows of third grade fluid in a rotating frame can further be extended in many interesting fields of fluid dynamics. It can be used to investigate the heat and mass transfer flows in which the infinite porous plate may be assumed to be at a higher temperature than the fluid or the plate may be assumed to be insulated. The combined heat and mass transfer problems in a rotating frame are of great importance because of its practical applications such as migration of moisture through the air contained in fibrous insulations, grain storage insulations and dispersion of chemical containments through water-saturated soil.

The study of flows of third grade fluid in the presence of magnetic field will also have considerable contributions in the literature due to its applications in cosmical and geophysical fluid dynamics. The order of the magnitude analysis shows that the Coriolis force is very significant as compared to inertial and viscous forces. Further, it reveals that the Coriolis and Magnetohydrodynamics (MHD) forces are of comparable magnitude. It is generally admitted that the Coriolis force due to earth's rotation has a strong effect on the hydromagnetic flow in the earth's liquid core. Thus the extension of research work in this direction will be noteworthy.

Transport flow phenomena in porous media continues to be a field which attracts intensive research activity. This is primarily due to the fact that it plays an important and practical role in large variety of diverse scientific applications. It covers wide range of the engineering and technological applications including both stable and unstable flows. Furthermore, the research into thermal convection in porous medium has also increased subsequently during the recent years. It is therefore, appropriate to explore and undertake a new critical evaluation of this major field of research with a different approach and technique like homotopy analysis method for the non-Newtonian fluids in the rotating systems.

References

1. J. E. Dunn and K. R. Rajagopal, *Fluids of differential type. Critical review and thermodynamic analysis.* Int. J. Engng. Sci., **33**, 689 – 729 (1995).

2. K. R. Rajagopal, *A note on unsteady unidirectional flows of a non-Newtonian fluid.* Int. J. Non-Linear Mech., **17**, 369 – 373 (1982).

3. K. R. Rajagopal, *On the creeping flow of the second grade fluid.* J. Non-Newtonian Fluid Mech., **15**, 239 – 246 (1984).

4. K. R. Rajagopal and A. S. Gupta, *Flow and stability of a second grade fluid between two parallel plates rotating about noncoincident axes.* Int. J. Engng. Sci., **19**, 1401 – 1409 (1981).

5. K. R. Rajagopal and A. S. Gupta, *An exact solution for a non-Newtonian fluid past an infinite porous plate.* Meccanica., **19**, 158 – 160 (1984).

6. R. Bandelli and K. R. Rajagopal, *Start-up flows of second grade fluids in domains with one finite dimension.* Int. J. Non-Linear Mech., **30**, 817 – 839 (1995).

7. R. Bandelli and K. R. Rajagopal, *On the falling of objects in non-Newtonian fluids.* Annali di Ferrara., **39**, 1 – 18 (1996).

8. T. Hayat, S. Asghar and A. M. Siddiqui, *Periodic unsteady flows of a non-Newtonian fluid.* Acta Mech., **131**, 169 – 175 (1998).

9. T. Hayat, S. Asghar and A. M. Siddiqui, *Some unsteady unidirectional flows of a non-Newtonian fluid.* Int. J. Engng. Sci., **38**, 337 – 346 (2000).

10. T. Hayat, M. Khan, A. M. Siddiqui and S. Asghar, *Transient flows of a second grade fluid.* Int. J. Non-Linear Mech., **39**, 1621 – 1633 (2004).

11. T. Hayat, Q. Abbas, M. Khan and A. M. Siddiqui, *Non-Newtonian flows over an oscillating plate with variable suction.* Arch. Mech., **55**, 327 – 344 (2003).

12. T. Hayat, Y. Wang, A. M. Siddiqui, K. Hutter and S. Asghar, *Peristaltic transport of a third-order fluid in a circular cylindrical tube.* Math. Mod. Meths. Appl. Sci., **12**, 1691 – 1706 (2002).

13. T. Hayat, K. Hutter, S. Nadeem and S. Asghar, *Unsteady hydromagnetic flow of a conducting second grade fluid.* Z. angew. Math. Phys., **55**, 625 – 641 (2004).

14. R. L. Fosdick and K. R. Rajagopal, *Thermodynamics and stability of fluids of third grade.* Proc. Roy. Soc. Lond., A **339**, 351 – 377 (1980).

15. E. Dunn and R. L. Fosdick, *Thermodynamic stability and boundedness of fluids of complexity 2 and fluids of second grade.* Arch. Rat. Mech. Anal., **56**, 191 – 252 (1974).

16. D. D. Joseph, *Instability of the rest state of fluids of arbitrary grade greater than one.* Arch. Rat. Mech. Anal., **75**, 251 – 256 (1981).

17. M. Renardy, *Local existence theorems for the first and second initial-boundary value problems for a weakly non-Newtonian fluid.* Arch. Rat. Mech., **83**, 229 – 244 (1983).

18. J. Dunwoody, *On the Stability of the rest state of fluids of complexity n.* Proc. Roy. Soc. Lond., **378**, 437 – 444 (1981).

19. D. D. Joseph, *Fluid Dynamics of Viscoelastic Liquids (Applied Mathematical Sciences).* Springer, Berlin, (1990).

20. P. Puri and P. K. Kulshrestha, *Rotating flow of non-Newtonian fluids.* Applicable Anal., **4**, 131 – 140 (1974).

21. P. Puri, *Rotating flow of an elastico-viscous fluid on an oscillating plate.* Z. angew. Math. Mech., **54**, 743 – 745 (1974).

22. C. Wang, *Stretching a surface in a rotating fluid.* J. Appl. Maths. Phys. (ZAMP)., **39**, 177 – 185 (1988).

23. H. P. Greenspan and I. N. Howard, *On a time-dependant motion of a rotating fluid.* J. Fluid. Mech., **17**, 385 – 404 (1963).

24. T. Hayat, S. Nadeem, S. Asghar and A. M. Siddiqui, *Fluctuating flow of a third grade fluid on a porous plate.* Int. J. Non-Linear Mech., **368**, 901 – 916 (2001).

25. T. Hayat, S. Nadeem, S. Asghar and A. M. Siddiqui, *MHD rotating flow of a third grade fluid on an oscillating plate.* Acta Mech., **152**, 177 – 190 (2001).

26. S. Asghar, M. Mudassar Gulzar and T. Hayat, *Rotating flow of a third grade fluid by homotopy analysis method.* Appl. Math. and Comput., **165**, 213 – 221 (2005).

27. I. J. Rao and K. R. Rajagopal, *The effect of the slip boundary condition on the flow of fluids in a channel.* Acta Mech., **135**, 113 – 126 (1999).

28. M. J. Miksis and S. H. Davis, *Slip over rough and coated surfaces.* J. Fluid Mech., **273**, 125 – 139 (1994).

29. C. Le. Roux, *Existence and uniqueness of the flows of second grade fluids with partial slip boundary conditions.* Arch. Rat. Mech. Anal., **148**, 309 (1999).

30. R. I. Tanner, *Partial wall slip in polymer flow.* Ind. Engng. Chem. Res., **33**, 2434 – 2436 (1994).

31. T. Hayat, M. Khan and M. Ayub, *On non-linear flows with slip boundary condition.* Z. angew. Math. Phys., **56**, 1012 – 1029 (2005).

32. S. Asghar, M. Mudassar Gulzar and M. Ayub, *Effects of partial slip on a flow of a third grade fluid.* Acta Mech. Sinica., **22**, 195 – 198 (2006).

33. F. J. Lim and W. R. Schowalter, *Wall slip of narrow molecular weight distribution polybutadienes.* J. Rheol., **33**, 1359 – 1382 (1989).

34. A. M. Kraynik and W. R. Schowalter, *Slip at the wall and extrudate roughness with aqueous solutions of polyvinyl alcohol and sodium borate.* J. Rheol., **25**, 95 – 114 (1981).

35. G. V. Vinogradov and L. I. Ivanova, *Wall slippage and elastic turbulence of polymers in the rubbery state.* Rheol Acta., **7**, 243 – 254 (1968).

36. J. L. White, M. H. Han, N. Nakajima and R. Brzoskowski, *The influence of materials of construction on biconical rotor and considerations of slippage.* J. Rheol., **35**, 167 – 189 (1991).

37. W. J. Sillman and L. E. Scriven, *Separating flow near a static contact line: slip at a wall and shape of a free surface.* J. Comp. Phys., **34**, 287 – 313 (1980).

38. S. J. Liao, *Beyond Perturbation: Introduction to the Homotopy Analysis Method.* CRC Press LLC (2004).

39. S. J. Liao, *The proposed homotopy analysis technique for the solution of nonlinear problems.* PhD thesis Shanghai Jiao Tong University, (1992).

40. S. J. Liao, *A uniformly valid analytic solution of 2D viscous flow past a semi-infinite flat plate.* J. Fluid Mech., **385**, 101 – 128 (1999).

41. S. J. Liao, *On the homotopy analysis method for nonlinear problems.* Appl. Math. Comput., **169**, 495 – 513 (2005).

42. S. J. Liao, *An analytic approximate technique for free oscillations of positively damped systems with algebraically decaying amplitude.* Int. J. Non-Linear Mech., **38**, 1173 – 1183 (2003).

43. S. J. Liao, *On the analytic solution of magnetohydrodynamic flows of non-Newtonian fluids over a stretching sheet.* J. Fluid Mech., **488**, 189 – 212 (2003).

44. S. J. Liao and I. Pop, *Explicit analytic solution for similarity boundary layer equations.* Int. J. Heat and Mass Trans., **47**, 75 – 85 (2004).

45. S. J. Liao and K. F. Cheung, *Homotopy analysis of nonlinear progressive waves in deep water.* J. Eng. Math., **45**, 105 – 116 (2003).

46. S. J. Liao, *An analytic solution of unsteady boundary-layer flows caused by an impulsively stretching plate.* Comm. Non-linear Sci. Numer. Simm., **11**, 326 – 339 (2006).

47. S. J. Liao, *Comparison between homotopy analysis method and homotopy perturbation method.* Appl. Math. Compt., **169**, 1186 – 1194 (2005).

48. J. Cheng, S. J. Liao and I. Pop, *Analytic series solution for unsteady mixed convection boundary layer flow near the stagnation point on a vertical surface in a porous medium.* Trans. Porous Media., **61**, 365 – 379 (2005).

49. S. Abbasbandy, *The application of homotopy analysis method to nonlinear equations arising in heat transfer.* Phys. Lett. A (in press).

50. Y. Tan and S. Abbasbandy, *Homotopy perturbation method for quadratic recati differential equation.* Comm. Non-linear Sci. Numer. Simm., **175**, 581 – 589 (2006).

51. Y. Wu, C. Wang and S. J. Liao, *Solving solitary waves with discontinuity by means of the homotopy analysis method.* Chaos Solitons & Fractals., **26**, 177 – 185 (2005).

52. M. Ayub, A. Rasheed and T. Hayat, *Exact flow of a third grade fluid past a porous plate using homotopy analysis method.* Int. J. Engng. Sci., **41**, 2091 – 2103 (2003).

53. T. Hayat, M. Khan and M. Ayub, *On the explicit analytic solutions of an Oldroyd 6-constant fluid.* Int. J. Engng. Sci., **42**, 123 – 135 (2004).

54. T. Hayat, M. Khan and S. Asghar, *Homotopy analysis of MHD flows of an Oldroyd 8-constant fluid.* Acta Mech., **168**, 213 – 232 (2004).

55. T. Hayat, Z. Abbas, M. Sajid and S. Asghar, *The influence of thermal radiation on MHD flow of a second grade fluid.* Int. J. Heat Mass Transfer (in press).

56. T. Hayat, Z. Abbas and M. Sajid, *Series solution for the upper-convected Maxwell fluid over a porous stretching plate.* Phys. Lett. A **358**, 396 – 403 (2006).

57. M. Sajid, T. Hayat and S. Asghar, *Comparison of the HAM and HPM solutions of thin film flows of non-Newtonian fluids on a moving belt.* Nonlinear Dyn. (in press).

58. T. Hayat and M. Sajid, *On analytic solution for thin film flow of a fourth grade fluid down a vertical cylinder.* Phys. Lett. A **361**, 316 – 322 (2007).

59. M. Sajid, T. Hayat and S. Asghar, *On the analytic solution of steady flow of a fourth grade fluid.* Phys. Lett. A **355**, 213 – 232 (2004).

60. Z. Abbas, M. Sajid and T. Hayat, *MHD boundary layer flow of an upper-convected Maxwell fluid in porous channel.*, Theor. Comp. Fluid Dyn. **20**, 229 – 238 (2006).

61. H. Xc, *An explicit analytic solution for free convection about a vertical flat plate embedded in a porous medium by means of homotopy analysis method.* Appl. Math. Comput. **158**, 433 – 443 (2004).

62. Q. Sun, *Solving the Klin-Gordon equation by means of homotopy analysis method.* Appl. Math. Comput. **169**, 18 – 24 (2006).

63. B. D. Coleman and W. Noll, *An approximation theorem for functionals, with applications in continuum mechanics.* Arch., Rat. Mech. Anal. **6**, 356 – 370 (1960).

64. R. W. Fox, Alan T. McDonald, Philip J. Pritchard, *Introduction to Fluid Mechanics.* John Wiley and Sons Inc (2004).

65. A. W. Bush, *Perturbation Methods for Engineers and Scientists.* CRC Press Library of Engineering Mathematics (1990).

66. P. J. Hilton, *An Introduction to Homotopy Theory.* Cambridge University Press (1953).

67. G. D. Smith, *Numerical Solutions of Partial Differential Equations: Finite Difference Methods.* Oxford University Press, New York (1985).

PUBLISHED WORK FROM THE THESIS OF THE AUTHOR

Acta Mech Sinica (2006) 22:195–198
DOI 10.1007/s10409-006-0109-3

RESEARCH PAPER

S. Asghar · M. Mudassar Gulzar · M. Ayub

Effects of partial slip on flow of a third grade fluid

Received: 20 September 2004 / Revised: 14 November 2005 / Accepted: 28 November 2005 / Published online: 30 May 2006

Abstract This paper is an analytical study of the rotating flow of a third grade fluid past a porous plate with partial slip effects. It serves as a flow model for the study of polymers. The analytic solution has been determined using homotopy analysis method (HAM).

Keywords Partial-slip · Non-Newtonian fluid · Analytic solution · Homotopy analysis method

1 Introduction

Materials such as polymer solutions or melts, drilling mud, elastomers, certain oils and greases and many other emulsions are classified as non-Newtonian fluids. Due to complexity of fluids, there are many models describing the properties, but not all of non-Newtonian fluids. These models, however, cannot predict all the behaviours of non-Newtonian fluids, for example, normal stress differences, shear thinning or shear thickening, shear relaxation, elastic and memory effects etc. Among these models, the fluids of differential type, for example, fluids of second and third grades have acquired special attentions due to their elegance [1]. Important contributions to the topic include the works of Rajagopal [2,3], Rajagopal and Gupta [4,5], Bandelli and Rajagopal [6,7] and Hayat et al. [8–11]. Also, there are a few studies which describe the flow of non-Newtonian fluids in a rotating frame of Refs. [12–16].

The English text was polished by Yunming Chen.

S. Asghar
Department of Mathematics,
COMSATS Institute of Information
Technology H-8 Islamabad, Pakistan

M. Mudassar Gulzar · M. Ayub (✉)
Department of Mathematics,
Quaid-i-Azam University,
45320, Islamabad, Pakistan
E-mail: mayub59@yahoo.com

In all the above mentioned studies, the partial slip effects have been ignored. The need for the development of boundary conditions has not received the attention that it deserves. The pioneers of the field such as Coulomb, Navier, Girad, Poisson, Stokes, St. Venant and others recognised that boundary conditions are constitutive equations that should be determined by the material on either side of the boundary. The usual prescription of Dirichlet and Neumann conditions are often unsuitable for a realistic physical problem, for example the flow of polymers that stick-slip on the boundary. Recently non-standard boundary conditions have been considered from a rigorous mathematical perspective by Rao and Rajagopal [17].

The main purpose of the present paper is to examine the effects of partial slip on the rotating flow past a uniformly porous plate. The fluid is incompressible and third grade. In view of the scarcity of the methods for the study of non-linear boundary conditions, this paper might be a reasonable addition to the literature. The analytical solution of the flow has been obtained using HAM which is already successfully applied to many problems [16,18–25].

2 Mathematical analysis

Let us consider the steady flow generated in a semi-infinite expanse of an incompressible, thermodynamic compatible third grade fluid past an infinite porous plate at $z = 0$ subjected to uniform suction or blowing. The fluid and plate are in a state of rigid body rotation having constant angular velocity Ω.

This paper is in fact a sequel of our earlier work [16] and hence it may be fair to avoid rewriting the constitutive equations. Thus making reference to our work [16], the governing problem in non-dimensional variables is of the following type:

$$\frac{d^2F(z)}{dz^2} - 2i\Omega F(z) + W_0\left[\frac{dF(z)}{dz} - \alpha\frac{d^3F(z)}{dz^3}\right.$$

$$+2\beta\frac{\mathrm{d}}{\mathrm{d}z}\left[\left(\frac{\mathrm{d}F(z)}{\mathrm{d}z}\right)^2\frac{\mathrm{d}F^*(z)}{\mathrm{d}z}\right]=0, \qquad (1)$$

$$F(0)+1=\lambda_1\left[\frac{\mathrm{d}F(z)}{\mathrm{d}z}-\alpha W_0\frac{\mathrm{d}^2F(z)}{\mathrm{d}z^2}+2\beta\left(\frac{\mathrm{d}F(z)}{\mathrm{d}z}\right)^2\frac{\mathrm{d}F^*(z)}{\mathrm{d}z}\right]\Bigg|_{z=0}, \qquad (2)$$

$$F(z)=0 \text{ as } z\to\infty. \qquad (3)$$

In Eqs. (1) and (2), $\lambda_1=\frac{\rho U}{\mu}\tilde{\lambda}$ is non-dimensional partial slip coefficient, $\tilde{\lambda}=\lambda\mu$ is slip length. λ satisfies the partial slip boundary conditions $(u,v)-(U_w,V_w)=\lambda(\tau_{xz},\tau_{yz})$ and

$$F=\frac{u+\mathrm{i}v}{U}-1,$$

$$F^*=\frac{u-\mathrm{i}v}{U}-1, \qquad (4)$$

where U denotes the reference velocity and U_w, V_w (the wall velocities) are zero in our case.

We see that Eqs. (1) and (2) are highly non-linear and not amenable to exact solutions. Thus, we use the homotopy analysis method to solve non-linear governing equations of third grade fluid. For that, we take

$$\mathcal{L}=\frac{\mathrm{d}^2}{\mathrm{d}z^2}-2\mathrm{i}\Omega \qquad (5)$$

as linear auxiliary operator. Using Eq. (6), we construct the so-called zeroth order deformation problem as

$$(1-p)\mathcal{L}[\bar{F}(z;p)-F_0(z)]=p\hbar\left\{\frac{\mathrm{d}^2\bar{F}(z;p)}{\mathrm{d}z^2}-2\mathrm{i}\Omega\bar{F}(z;p)+W_0\left[\frac{\mathrm{d}\bar{F}(z;p)}{\mathrm{d}z}-\alpha\frac{\mathrm{d}^3\bar{F}(z;p)}{\mathrm{d}z^3}\right]+2\beta\frac{\mathrm{d}}{\mathrm{d}z}\left[\left(\frac{\mathrm{d}\bar{F}(z;p)}{\mathrm{d}z}\right)^2\frac{\mathrm{d}\bar{F}^*(z;p)}{\mathrm{d}z}\right]\right\}, \qquad (6)$$

$$(1-p)\left[\bar{F}(0;p)+1-\lambda_1\frac{\mathrm{d}\bar{F}(0;p)}{\mathrm{d}z}\right]=p\hbar\left\{\bar{F}(z;p)+1-\lambda_1\left[\frac{\mathrm{d}\bar{F}(z;p)}{\mathrm{d}z}-\alpha W_0\frac{\mathrm{d}^2\bar{F}(z;p)}{\mathrm{d}z^2}+2\beta\left(\frac{\mathrm{d}\bar{F}(z;p)}{\mathrm{d}z}\right)^2\frac{\mathrm{d}\bar{F}^*(z;p)}{\mathrm{d}z}\right]\right\}\Bigg|_{z=0}, \qquad (7)$$

$$\bar{F}(z;p)\to 0 \text{ as } z\to\infty, \qquad (8)$$

where $\hbar$ is an auxiliary parameter and $p\in[0,1]$ is an embedding parameter.

Let us take

$$F_0(z)=-\frac{\mathrm{e}^{-mz}}{1+m\lambda_1},\quad m=\sqrt{2\mathrm{i}\Omega} \qquad (9)$$

as the initial guess approximation of $\bar{F}(z;p)$.

Now, it is clearly seen that

$$\bar{F}(z;0)=F_0(z), \qquad (10)$$

$$\bar{F}(z;1)=F(z), \qquad (11)$$

when $p=0$ and $p=1$, respectively.

With the help of Taylor's theorem $\bar{F}(z;p)$ can be expanded in power series of p as follows

$$\bar{F}(z;p)=F_0(z)+\sum_{k=1}^{+\infty}F_k(z)p^k, \qquad (12)$$

where

$$F_k(z)=\frac{1}{k!}\frac{\partial^k\bar{F}(z;p)}{\partial p^k}\Big|_{p=0},\quad k\geq 1, \qquad (13)$$

we assume that the deformation $\bar{F}(z;p)$ is smooth enough.

If $\hbar$ is properly selected that Eq. (7) is convergent at $p=1$, we have from Eqs. (11) and (12)

$$F(z)=F_0(z)+\sum_{k=1}^{+\infty}F_k(z). \qquad (14)$$

Differentiating Eqs. (7)–(9) k-times with respect to p and letting $p=0$, we obtain, for $k\geq 1$, the following problem

$$\mathcal{L}[F_k(z)-\chi_kF_{k-1}(z)]=\hbar\left[F''_{k-1}(z)-2\mathrm{i}\Omega F_{k-1}(z)+W_0(F'_{k-1}(z)-\alpha F'''_{k-1}(z))+2\beta\sum_{n=0}^{k-1}F'_{k-1-n}(z)\sum_{i=0}^{n}(F'_{n-i}(z)F''^*_i(z)+2F''_{n-i}(z)F'^*_i(z))\right], \qquad (15)$$

$$[F_k(0)+1-\lambda_1F'_k(0)]=\hbar\left\{-\lambda_1\left[F_{k-1}(0)+(1-\chi_k)F'_{k-1}(0)-\alpha W_0F''_{k-1}(0)+2\beta\sum_{n=0}^{k-1}F'_{k-1-n}(0)\sum_{i=0}^{n}F'_{n-i}(0)F'^*_i(0)\right]\right\}, \qquad (16)$$

at $z=0$ and

$$F_k(z)\to 0 \text{ as } z\to\infty, \qquad (17)$$

where

$$\chi_k=\begin{cases}0, & k\leq 1,\\ 1, & k\geq 2,\end{cases}$$

and prime denotes the derivatives with respect to z.

Following the same method of solution as in Ref. [16], the three terms solution is given by

$$F(z)=F_0(z)+F_1(z)+F_2(z), \qquad (18)$$

where

$$F_1(z)=[M_1z+M_3]\mathrm{e}^{-mz}-M_2\mathrm{e}^{-(2m+m^*)z}, \qquad (19)$$

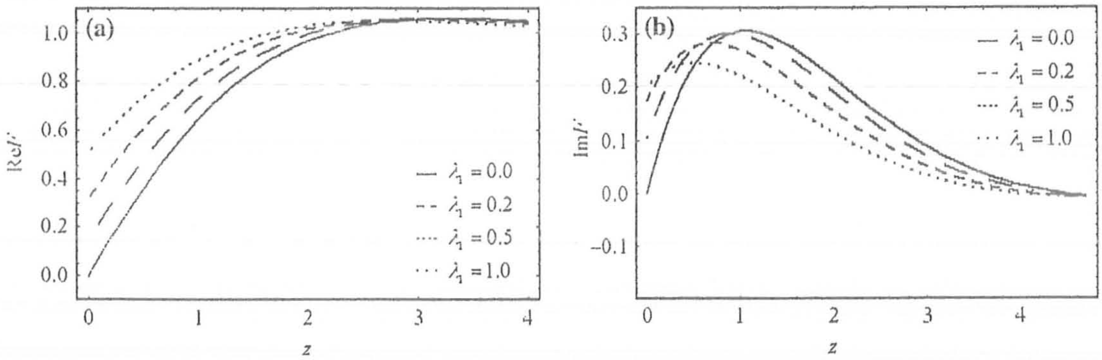


Fig. 1 The variation of velocity parts for various values of partial slip parameters λ_1 for viscous fluid with fixed $\hbar = -0.1, \alpha = \beta = 0, \Omega = 0.5$ and $W_0 = 0.5$

$$
\begin{aligned}
F_2(z) = \frac{1}{2}\Bigg\{ & \frac{1}{1+m\lambda_1}\Big[2\alpha\hbar W_0\lambda_1(m^2M_3 - 2mM_1 \\
& - (2m+m^*)^2M_2) - \frac{M_4\lambda_1}{4m} - \frac{M_5\lambda_1}{4m^2}\Big] \\
& + \frac{1}{4(2+\mathrm{i})(1+m\lambda_1)\Omega}\,[M_6(1+m\lambda_1) \\
& + M_7(1+(2m+m^*)\lambda_1) + M_8\lambda_1 \\
& - 2M_8(2m+m^*)\left(1+\frac{(2m+m^*)\lambda_1}{4(2+\mathrm{i})\Omega}\right)\Big] \\
& + \frac{M_9(1+(3m+2m^*)\lambda_1)}{8(3+\mathrm{i})(1+m\lambda_1)\Omega} - \frac{4\hbar\lambda_1\beta}{(1+m\lambda_1)^2} \\
& \times\Big[\frac{m^2}{1+m\lambda_1}(M_1^* - m^*M_3^*) \\
& + \frac{4\Omega}{1+m^*\lambda_1}(M_1 - mM_3) \\
& + M_2^*\frac{2(1+3\mathrm{i})\Omega^{\frac{3}{2}}}{1+m\lambda_1} + M_2\frac{8(1+2\mathrm{i})\Omega^2}{1+m^*\lambda_1}\Big] \\
& - \frac{1}{2m}\Big[M_4z + M_5\left(\frac{z}{2m}+\frac{z^2}{2}\right)\Big]\Bigg\}\mathrm{e}^{-mz} \\
& - \frac{1}{8(2+\mathrm{i})\Omega}\,[M_6 + M_7 \\
& - M_8\left(z + \frac{(2m+m^*)}{2(2+\mathrm{i})\Omega}\right)\Big]\mathrm{e}^{-(2m+m^*)z} \\
& - \Big[\frac{M_9}{8(3+\mathrm{i})\Omega}\Big]\mathrm{e}^{-(3m+2m^*)z},
\end{aligned} \tag{20}
$$

$$
\begin{aligned}
M_1 &= \frac{1}{2(1+m\lambda_1)}\hbar W_0(\alpha m^2 - 1), \\
M_2 &= \frac{2\hbar\beta(1+2\mathrm{i})\Omega}{(2+\mathrm{i})(1+m^*\lambda_1)(1+m\lambda_1)^2}, \\
M_3 &= \frac{1}{1+m\lambda_1}\,[M_1\lambda_1 + M_2(1+(2m+m^*)\lambda_1) \\
& \quad - \frac{\hbar\lambda_1}{1+m\lambda_1}\left(\alpha m^2W_0 + \frac{4\beta(1+\mathrm{i})\Omega^{\frac{3}{2}}}{(1+m\lambda_1)(1+m^*\lambda_1)}\right)\Big], \\
M_4 &= -4(1+\hbar)mM_1 + 2\hbar\left[mW_0M_3(\alpha m^2 - 1)\right. \\
& \quad \left. - W_0M_1(3m^2 - 1)\right], \\
M_5 &= 2\hbar mW_0M_1(\alpha m^2 - 1),
\end{aligned}
$$

$$
\begin{aligned}
M_6 &= M_2\Big[8(1+\hbar)(2+\mathrm{i})\Omega \\
& \quad + 2\hbar W_0(2m+m^*)(\alpha(2m+m^*)^2 - 1)\Big], \\
M_7 &= \frac{4\hbar\beta}{1+m\lambda_1}\Bigg\{(2m+m^*)\Big[\frac{2\mathrm{i}\Omega}{1+m\lambda_1}(M_1^* - m^*M_3^*) \\
& \quad + \frac{4\Omega}{1+m^*\lambda_1}(M_1 - mM_3)\Big] + 2(1+\mathrm{i})\Omega^{\frac{3}{2}} \\
& \quad \times\left(\frac{1}{1+m\lambda_1}M_1^* + \frac{2}{1+m^*\lambda_1}M_1\right)\Bigg\}, \\
M_8 &= \frac{8\hbar\beta}{1+m\lambda_1}\Big[(2m+m^*)(1+\mathrm{i})\Omega^{\frac{3}{2}} \\
& \quad \times\left(\frac{1}{1+m\lambda_1}M_1^* + \frac{2}{1+m^*\lambda_1}M_1\right)\Big], \\
M_9 &= \frac{4\hbar\beta}{1+m\lambda_1}\Bigg\{(2m+m^*)\Big[\frac{2(1+3\mathrm{i})\Omega^{\frac{3}{2}}}{1+m\lambda_1}M_2^* \\
& \quad + \frac{8(1+2\mathrm{i})\Omega}{1+m^*\lambda_1}M_2\Big]\Bigg\},
\end{aligned}
$$

where m^*, M_1^*, M_2^* and M_3^* are complex conjugates of m, M_1, M_2 and M_3, respectively.

3 Graphical results and discussion

The effects of partial slip on the velocity profiles are given using homotopy analysis method. The velocity components are sketched in Figs. 1 and 2 in order to see the influence of partial slip.

Figures 1 illustrate the effect of partial slip on the flow of viscous fluid when $\hbar = -0.1$, $\Omega = 0.5$ and $W_0 = 0$. It is seen that the real part of F increases with the increase of partial slip parameter. The imaginary part of F first increases and then decreases.

The influence of partial slip parameter on the flow of third grade fluid has been depicted in Fig.2 for $\hbar = -0.1$, $\alpha = \beta = 1$, $\Omega = W_0 = 0.5$. The observations for real and imaginary parts of F are found of similar type. However, the real part of F in case of third grade changes much more

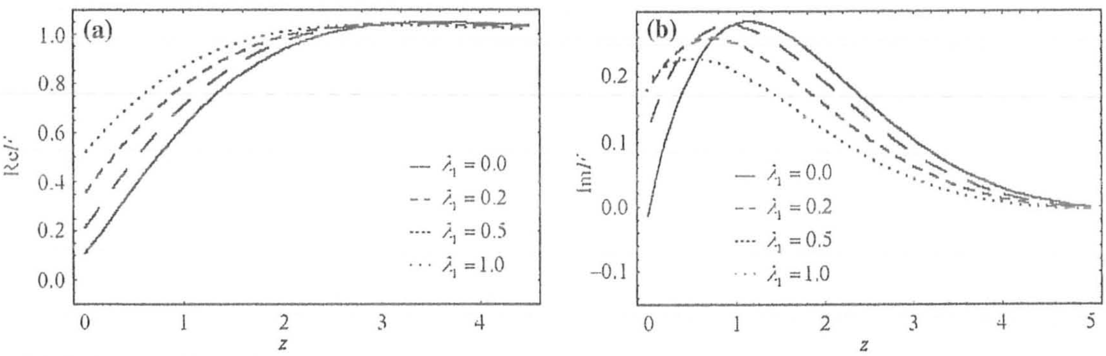


Fig. 2 The variation of velocity parts for various values of partial slip parameters λ_1 for third grade fluid with fixed $\hbar = -0.1$, $\alpha = \beta = 1$ and $\Omega = W_0 = 0.5$

when compared to that of viscous fluid. There is no significant change in imaginary part of F for both viscous and third grade fluid.

References

1. Dunn, J.E., Rajagopal, K.R.: Fluids of differential type. Critical review and thermodynamic analysis. Int. J. Engng. Sci. **33**, 689–729 (1995)
2. Rajagopal, K.R.: A note on unsteady unidirectional flows of a non-Newtonian fluid. Int. J. Non-Linear Mechs. **17**, 369–373 (1982)
3. Rajagopal, K.R.: On the creeping flow of the second grade fluid. J. Non-Newtonian Fluid Mech. **15**, 239–246 (1984)
4. Rajagopal, K.R., Gupta, A.S.: Flow and stability of a second grade fluid between two parallel plates rotating about noncoincident axes. Int. J. Engng. Sci. **19**, 1401–1409 (1981)
5. Rajagopal, K.R., Gupta, A.S.: An exact solution for the flow of a non-Newtonian fluid past an infinite plate. Mechanica. **19**, 158–160 (1984)
6. Bandelli, R., Rajagopal, K.R.: Start-up flows of second grade fluids in domains with one finite dimension. Int. J. Non-Linear Mechs. **30**, 817–839 (1995)
7. Bandelli, R., Rajagopal, K.R.: On the falling of objects in non-Newtonian fluids. Annali di Ferrara **39**, 1–18 (1996)
8. Hayat, T., Asghar, S., Siddiqui, A.M.: Periodic unsteady flows of a non-Newtonian fluid. Acta Mech. **131**, 169–175 (1998)
9. Hayat, T., Asghar, S., Siddiqui, A.M.: Some unsteady unidirectional flows of a non-Newtonian fluid. Int. J. Engng. Sci. **38**, 337–346 (2000)
10. Hayat, T., Abbas, Q., Khan, M., Siddiqui, A.M.: Non-Newtonian flows over an oscillating plate with variable suction. Arch. Mech. **55**, 327–344 (2003)
11. Hayat, T., Masood Khan, Siddiqui, A.M., Asghar, S.: Transient flows of a second grade fluid. Int. J. Non-Linear Mech. **39**, 1621–1633 (2004)
12. Puri, P., Kulshrestha, P.K.: Rotating flow of non-Newtonian fluids. Applicable Anal. **4**, 131–140 (1974)
13. Hayat, T., Nadeem, S., Asghar, S., Siddiqui, A.M.: Fluctuating flow of a third grade fluid on a porous plate in a rotating medium. Int. J. Non-Linear Mech. **38**, 901–916 (2001)
14. Hayat, T., Hutter, K., Nadeem, S., Asghar, S.: Unsteady hydromagnetic flow of a conducting second grade fluid. Z. Angew. Math. Phys. **55**, 625–641 (2004)
15. Puri, P.: Rotating flow of an elastico-viscous fluid on an oscillating plate. Z. Angew. Math. Mech. **54**, 743–745 (1974)
16. Asghar, S., Mudassar Gulzar, M., Hayat, T.: Rotating flow of a third grade fluid by homotopy analysis method. Appl. Math. and Computation. **165**, 213–221 (2005)
17. Rao, I.J., Rajagopal, K.R.: The effect of the slip boundary condition on the flow of fluids in a channel. Acta Mech. **135**, 113–126 (1999)
18. Liao, S.J.: The proposed homotopy analysis technique for the solution of nonlinear problems. [Ph. D. thesis]. Shanghai Jiao Tong University, 1992
19. Liao, S.J.: A uniformly valid analytic solution of 2D viscous flow past a semi-infinite flat plate. J. Fluid Mech. **385**, 101–128 (1999)
20. Liao, S.J.: An analytic approximate technique for free oscillations of positively damped systems with algebraically decaying amplitude. Int. J. Non-Linear Mech. **38**, 1173–1183 (2003)
21. Liao, S.J.: On the analytic solution of magnetohydrodynamic flows of non-Newtonian fluids over a stretching sheet. J. Fluid Mech. **488**, 189–212 (2003)
22. Ayub, M., Rasheed, A., Hayat, T.: Exact flow of a third grade fluid past a porous plate using homotopy analysis method. Int. J. Engng. Sci. **41**, 2091–2103 (2003)
23. Liao, S.J., Pop, I.: Explicit analytic solution for similarity boundary layer equations. Int. J. Heat and Mass Trans. **47**, 75–85 (2004)
24. Hayat, T., Khan, M., Ayub, M.: On the explicit analytic solutions of an Oldroyd 6-constant fluid. Int. J. Engng. Sci. **42**, 123–135 (2004)
25. Hayat, T., Khan, M., Asghar, S.: Homotopy analysis of MHD flows of an Oldroyd 8-constant fluid. Acta Mech. **168**, 213–232 (2004)

Available online at www.sciencedirect.com
SCIENCE DIRECT®
ELSEVIER Applied Mathematics and Computation 165 (2005) 213–221
APPLIED MATHEMATICS AND COMPUTATION
www.elsevier.com/locate/amc

Rotating flow of a third grade fluid by homotopy analysis method

S. Asghar [a], M. Mudassar Gulzar [b], T. Hayat [b,*]

[a] *COMSATS Institute of Information Technology Abbottabad, Pakistan*
[b] *Department of Mathematics, Quaid-i-Azam University, 45320 Islamabad, Pakistan*

Abstract

The steady flow of a rotating third grade fluid past a porous plate has been analyzed. The resulting nonlinear boundary value problem has been solved using homotopy analysis method. Explicit expression for the velocity field has been obtained. The variations of velocity with respect to rotation, suction, blowing and non-Newtonian parameters are shown and discussed.

Keywords: Rotating flow; Third grade fluid; Homotopy analysis method

1. Introduction

The analysis of the effects of rotation in fluid flows has been an interesting area because of its geophysical and technological importance. The involved equations are nonlinear and thus to understand specific aspects of the fluid flow simplified models have been taken into account. In this work, the steady-state flow of an incompressible fluid past a porous plate is considered. The fluid is

* Corresponding author.
E-mail address: t_pensy@hotmail.com (T. Hayat).

0096-3003/$ - see front matter
doi:10.1016/j.amc.2004.04.047

third grade and the whole system is in a rotating frame. Both (analytical and graphical) solutions of the governing nonlinear differential equation is given. Analytic solution of the problem is given by a newly developed method known as homotopy analysis method by Liao [1]. This method has already been successfully applied by various workers [2–8]. Briefly, the homotopy analysis method has the following advantages:

- It is independent of the choice of any large/small parameters in the nonlinear problem.
- It is helpful to control the convergence of approximation series in a convenient way and also for the adjustment of convergence regions where necessary.
- It can be employed to efficiently approximate a nonlinear problem by choosing different sets of base functions.

The layout of the paper is:

In Section 2, the problem is formulated. The solution of the problem is given in Section 3. Section 4 deals with the discussion of several graphs and in Section 5, concluding remarks are presented.

2. Mathematical formulation

We consider a Cartesian coordinate system rotating uniformly with an angular velocity Ω about the z-axis, taken positive in the vertically upward direction, with the plate coinciding with the plane $z = 0$. The fluid past a porous plate is third grade and incompressible. All material parameters of the fluid are assumed constant. In rotating frame, the momentum equation is

$$\rho\left[\frac{\partial \mathbf{V}}{\partial t} + (\mathbf{V}\cdot\nabla)\mathbf{V} + 2\Omega\times\mathbf{V} + \Omega\times(\Omega\times\mathbf{r})\right] = \mathrm{div}\mathbf{T}. \tag{1}$$

In above equation ρ is the density of the fluid, $\mathbf{r}$ is the radial coordinate and $\mathbf{V}$ is the velocity. The Cauchy stress tensor $\mathbf{T}$ for third grade fluid is [9]

$$\mathbf{T} = -p_1\mathbf{I} + \mu\mathbf{A}_1 + \alpha_1\mathbf{A}_2 + \alpha_2\mathbf{A}_1^2 + \beta_1\mathbf{A}_3 + \beta_2(\mathbf{A}_1\mathbf{A}_2 + \mathbf{A}_2\mathbf{A}_1) + \beta_3(\mathrm{tr}\mathbf{A}_1^2)\mathbf{A}_2 \tag{2}$$

in which p_1 is the pressure, $\mathbf{I}$ is the identity tensor, μ is the dynamic viscosity, α_i $(i = 1,2)$, β_i $(i = 1,2,3)$ are the material constants and the Rivlin–Erickson tensors are defined by

$$\mathbf{A}_1 = (\mathrm{grad}\mathbf{V}) + (\mathrm{grad}\mathbf{V})^{\mathrm{T}}, \tag{3}$$

$$\mathbf{A}_{n+1} = \left(\frac{\partial}{\partial t} + \mathbf{V}\cdot\nabla\right)\mathbf{A}_n + (\mathrm{grad}\mathbf{V})^{\mathrm{T}}\mathbf{A}_n + \mathbf{A}_n(\mathrm{grad}\mathbf{V}), \quad n > 1. \tag{4}$$

For thermodynamical considerations, the material constants must satisfy [10]

$$\mu \geqslant 0, \quad \alpha_1 \geqslant 0, \quad \beta_1 = \beta_2 = 0, \quad \beta_3 \geqslant 0, \quad |\alpha_1 + \alpha_2| \leqslant \sqrt{24\mu\beta_3} \tag{5}$$

and hence Eq. (2) gives

$$\mathbf{T} = -p_1\mathbf{I} + \mu\mathbf{A}_1 + \alpha_1\mathbf{A}_2 + \alpha_2\mathbf{A}_1^2 + \beta_3(\mathrm{tr}\mathbf{A}_1^2)\mathbf{A}_2. \tag{6}$$

The equation of continuity is

$$\mathrm{div}\mathbf{V} = 0. \tag{7}$$

For steady flow and uniformly porous plate, it follows from Eq. (7) that

$$\mathbf{V} = [u(z),\ v(z),\ -W_0], \tag{8}$$

where u and v are x- and y-components of velocity and $W_0 > (<)\ 0$ corresponds to suction (blowing) velocity, respectively.

In view of Eqs. (2)–(4) and (6)–(8) we have from Eq. (1) as

$$\rho\left[-W_0\frac{du}{dz} - 2v\Omega\right] = \mu\frac{d^2u}{dz^2} - \alpha_1 W_0\frac{d^3u}{dz^3} + 2\beta\frac{d}{dz}\left[\frac{du}{dz}\left\{\left(\frac{du}{dz}\right)^2 + \left(\frac{dv}{dz}\right)^2\right\}\right], \tag{9}$$

$$\rho\left[-W_0\frac{dv}{dz} + 2u\Omega\right] = 2\Omega U\rho + \mu\frac{d^2v}{dz^2} - \alpha_1 W_0\frac{d^3v}{dz^3} + 2\beta\frac{d}{dz}\left[\frac{dv}{dz}\left\{\left(\frac{du}{dz}\right)^2 + \left(\frac{dv}{dz}\right)^2\right\}\right], \tag{10}$$

where U denotes the uniform velocity outside the layer which is caused by the pressure gradient.

Defining

$$F = \frac{u + iv}{U} - 1, \tag{11}$$

Eqs. (9) and (10) can be combined into the following equation

$$-W_0\frac{dF}{dz} + 2i\Omega F = \frac{1}{\rho}\left[\frac{d^2F}{dz^2} - W_0\alpha\frac{d^3F}{dz^3} + 2\beta\frac{d}{dz}\left\{\left(\frac{dF}{dz}\right)^2\frac{dF^\star}{dz}\right\}\right], \tag{12}$$

where $F^\star$ is the conjugate of F.

Using the following dimensionless parameters

$$\hat{z} = \frac{\rho U z}{\mu}, \quad \widehat{F} = \frac{F}{U}, \quad \widehat{W}_0 = \frac{W_0}{U}, \quad \widehat{\Omega} = \frac{\Omega\mu}{\rho U^2}, \quad \widehat{\beta} = \frac{\beta_3\rho^2U^4}{\mu^3},$$
$$\widehat{\alpha} = \frac{\alpha_1\rho U^2}{\mu^2} \tag{13}$$

and dropping hats, Eq. (12) can be written as

$$\frac{d^2F}{dz^2} - 2i\Omega F + W_0\left[\frac{dF}{dz} - \alpha\frac{d^3F}{dz^3}\right] + 2\beta\frac{d}{dz}\left[\left(\frac{dF}{dz}\right)^2\frac{dF^\star}{dz}\right] = 0. \tag{14}$$

Eq. (14) must be solved subject to the following boundary conditions:

$$u = v = 0 \text{ at } z = 0, \quad u \to U \text{ as } z \to \infty, \quad v \to 0 \text{ as } z \to \infty \tag{15}$$

which on using Eqs. (11) and (13) can be written as

$$F(z) = -1 \text{ at } z = 0, \quad F(z) \to 0 \text{ as } z \to \infty. \tag{16}$$

Since Eq. (14) is third order and is higher than the governing equation of the Newtonian fluid and thus we need one more condition. The flow under consideration is in an unbounded domain, so by augmentation of boundary conditions [11] we have

$$\frac{dF}{dz} \to 0 \quad \text{as } z \to \infty. \tag{17}$$

3. Solution of the problem

Here, we give the analytic and uniformly valid solution by homotopy analysis method. For that we use

$$\mathscr{L} = \frac{d^2}{dz^2} - 2i\Omega \tag{18}$$

as linear auxiliary operator. Using Eq. (18), the deformation problem at the zeroth order satisfies

$$(1-p)\mathscr{L}[\overline{F}(z;p) - F_0(z)] = p\hbar\left[\frac{\partial^2\overline{F}(z;p)}{\partial z^2} - 2i\Omega\overline{F}(z;p)\right.$$
$$\left. + W_0\left(\frac{\partial\overline{F}(z;p)}{\partial z} - \alpha\frac{\partial^3\overline{F}(z;p)}{\partial z^3}\right) + 2\beta\frac{\partial}{\partial z}\left\{\left(\frac{\partial\overline{F}(z;p)}{\partial z}\right)^2\frac{\partial\overline{F}^\star(z;p)}{\partial z}\right\}\right], \tag{19}$$

where $\hbar$ is an auxiliary parameter and $p \in [0,1]$ is an embedding parameter. The boundary conditions take the form as

$$\overline{F}(0;p) = -1 \text{ as } z \to 0, \quad \overline{F}(z;p) \to 0 \text{ as } z \to \infty,$$
$$\frac{\partial\overline{F}(z;p)}{\partial z} \to 0 \text{ as } z \to \infty. \tag{20}$$

For $p = 0$ and $p = 1$, we have from Eq. (19) as

$$\overline{F}(z;0) = F_0(z), \tag{21}$$

$$\overline{F}(z;1) = F(z). \tag{22}$$

We note from the above equations that the variation of p from 0 to 1 is continuous variation of $\overline{F}(z;p)$ from $F_0(z)$ to $F(z)$. The initial approximation $F_0(z)$ is taken as

$$F_0(z) = -\mathrm{e}^{-\lambda z}, \tag{23}$$

where

$$\lambda = \sqrt{2\mathrm{i}\Omega}.$$

We assume that the deformation $\overline{F}(z;p)$ is smooth enough, so that

$$F^{[k]}(z) = \left.\frac{\partial^k \overline{F}(z;p)}{\partial p^k}\right|_{p=0} \quad (k \geqslant 1) \tag{24}$$

exists. Thus with the help of Eq. (21), the expansion $\overline{F}(z;p)$ can be written as

$$\overline{F}(z;p) = F_0(z) + \sum_{k=1}^{+\infty} F_k(z) p^k \tag{25}$$

in which

$$F_k(z) = \frac{1}{k!}\left.\frac{\partial^k \overline{F}(z;p)}{\partial p^k}\right|_{p=0} \quad (k \geqslant 1). \tag{26}$$

Differentiating k-times the zero-order deformation Eqs. (19) and (20) with respect to p and then dividing them by $k!$ and finally setting $p = 0$, we have, due to definition (22), the kth-order deformation problem

$$\mathscr{L}[F_k(z) - \mathscr{X}_k F_{k-1}(z)] = \hbar \Bigg[F''_{k-1}(z) - 2\mathrm{i}\Omega F_{k-1}(z) + W_0\{F'_{k-1}(z) - \alpha F''''_{k-1}(z)\}$$
$$+ 2\beta \sum_{n=0}^{k-1} F'_{k-1}(z) \sum_{i=0}^{n} \{F'_{n-i}(z) F''^{\star}_{i}(z) + 2F''_{n-i}(z) F'^{\star}_{i}(z)\}\Bigg] \tag{27}$$

with the boundary conditions

$$F_k(0) = 0, \quad F_k(z) \to 0 \text{ as } z \to \infty, \quad \frac{\mathrm{d}^k F}{\mathrm{d}z^k} \to 0 \text{ as } z \to \infty, \tag{28}$$

where

$$\mathscr{X}_k = \begin{cases} 0, & k \leqslant 1, \\ 1, & k \geqslant 2 \end{cases}$$

and prime denotes the derivative with respect to z.

Applying homotopy analysis method, the four term solution of above problem is given as:

$$F(z) = F_0(z) + F_1(z) + F_2(z) + F_3(z), \tag{29}$$

where

$$F_1(z) = \frac{2}{5}[\hbar\beta(4+3i)\Omega]\left(e^{-\lambda z} - e^{-(2\lambda+\lambda^*)z}\right), \tag{30}$$

$$F_2(z) = \begin{bmatrix} M_1\left\{1+\hbar+\frac{7-i}{20\sqrt{\Omega}}\hbar W_0[\alpha(8+6i)\Omega-1]\right\} \\ -\frac{2}{5}\hbar\beta(2M_1+M_1^*)(4+3i)\Omega \end{bmatrix}\left(e^{-\lambda z} - e^{-(2\lambda+\lambda^*)z}\right)$$
$$+\hbar\beta\left[(5+i)\Omega M_1 + \frac{1}{10}(11+23i)\Omega M_1^*\right]\left(e^{-\lambda z} - e^{-(3\lambda+2\lambda^*)z}\right), \tag{31}$$

$$F_3(z) = \frac{1}{2}\begin{bmatrix} M_2\left\{1+\hbar+\frac{1}{20\sqrt{\Omega}}(7-i)\hbar W_0[\alpha(8+6i)\Omega-1]\right\} \\ +\frac{2}{5}\hbar\beta(2-i)\Omega\{(4+6i)M_1M_1^* \\ +(1+2i)(2M_1^2-2M_2-2M_3-M_2^*-M_3^*)\} \end{bmatrix}\left(e^{-\lambda z} - e^{-(2\lambda+\lambda^*)z}\right)$$
$$+\frac{1}{2}\begin{bmatrix} M_3\left\{1+\hbar+\frac{1}{5\sqrt{\Omega}}(1-\frac{i}{8})\hbar W_0[\alpha(24+10i)\Omega-1]\right\} \\ -\frac{1}{10}\hbar\beta(3-i)\Omega\{(32+40i)M_1M_1^* \\ +(3-2i)[(2+4i)(2M_1^2-M_3)+(1-2i)M_3^*]\} \end{bmatrix}\left(e^{-\lambda z} - e^{-(3\lambda+2\lambda^*)z}\right)$$
$$+\frac{1}{51}\hbar\beta(4-i)\Omega\begin{bmatrix} 50M_1^2+(30+18i)M_3 \\ +(60+70i)M_1M_1^*+(5+12i)M_3^* \end{bmatrix}\left(e^{-\lambda z} - e^{-(4\lambda+3\lambda^*)z}\right), \tag{32}$$

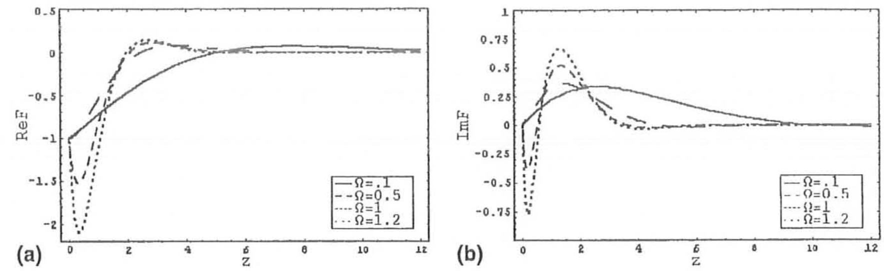
in which

$$M_1 = \frac{2}{5}\hbar\beta(4+3i)\Omega,$$

$$M_2 = 2M_1\left[1+\hbar+\frac{7-i}{20\sqrt{\Omega}}\hbar W_0[\alpha(8+6i)\Omega-1] - \frac{4}{5}\hbar\beta(2M_1+M_1^*)(4+3i)\Omega\right],$$

$$M_3 = 2\hbar\beta\left[(5+i)\Omega M_1 + \frac{1}{10}(11+23i)\Omega M_1^*\right]$$

and λ^*, M_1^*, M_2^* and M_3^* are the conjugates of λ, M_1, M_2 and M_3 respectively.

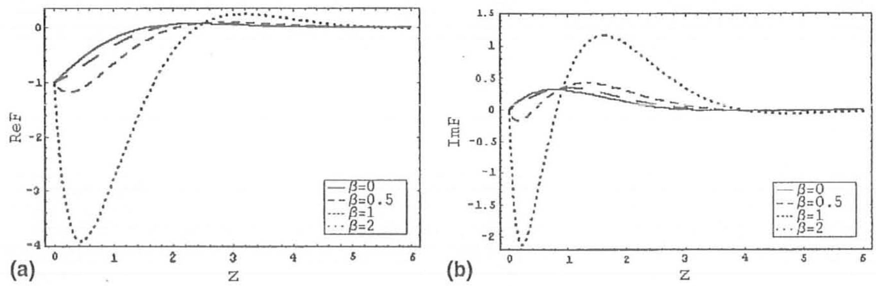


(a) (b)

Fig. 3. The variation of velocity parts for various values of rotation Ω with fixed $\hbar = -0.5$, $\alpha = \beta = 0.5$ and $W_0 = 1$.

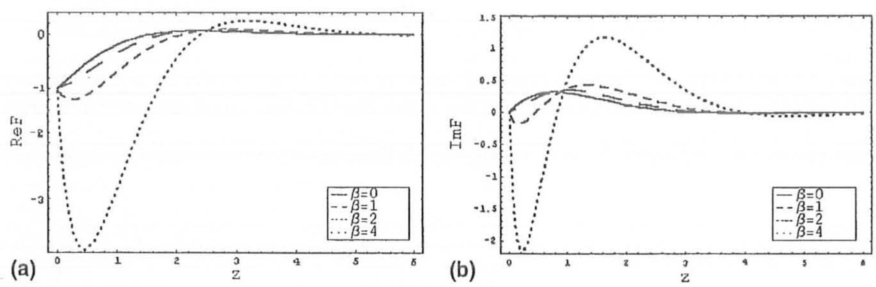
Fig. 4 show the effects of material parameter of third grade fluid on the velocity parts when $\hbar$, α, Ω and W_0 are fixed. It is interesting to note that as β increases from 0 to 2, the velocity parts near the plate increase.

The influence of $\hbar$ on the velocity profiles are given in Fig. 5. Here, it is noted that the convergence of the obtained solution is strongly dependent on the choice of $\hbar$ and the convergence region enlarges as $\hbar$ tends to zero from below.



(a) (b)

Fig. 4. The variation of velocity parts for various values of non-Newtonian material parameter β with fixed $\hbar = -0.2$, $\alpha = W_0 = 0.5$ and $\Omega = 1$.



(a) (b)

Fig. 5. The variation of velocity parts for various values of non-Newtonian material parameter β with fixed $\hbar = -0.01$, $\alpha = W_0 = 0.5$ and $\Omega = 1$.

5. Concluding remarks

In this work, the non-Newtonian flow past a porous plate has been analyzed. The whole system is in a rotating frame. The most distinctive feature here is that unlike the inertial frame, the steady asymptotic blowing solution exists. The physical implication of this conclusion is that rotation causes a reduction in the layer thickness. Thus, if blowing is not too large, the thinning effect of rotation may just counterbalance the thickening effect of blowing so that the vorticity generated at the plate instead of being converted away from the plate by blowing remains confined near the plate and a steady solution is possible.

References

[1] S.J. Liao, The proposed homotopy analysis technique for the solution of nonlinear problems, PhD thesis, Shanghai Jiao Tong University, 1992.
[2] S.J. Liao, A uniformly valid analytic solution of 2D viscous flow past a semi-infinite flat plate, J. Fluid Mech. 385 (1999) 101–128.
[3] S.J. Liao, An analytic approximate technique for free oscillations of positively damped systems with algebraically decaying amplitude, Int. J. Nonlinear Mech. 38 (2003) 1173–1183.
[4] S.J. Liao, On the analytic solution of magnetohydrodynamic flows of non-Newtonian fluids over a stretching sheet, J. Fluid Mech. 488 (2003) 189–212.
[5] M. Ayub, A. Rasheed, T. Hayat, Exact flow of a third grade fluid past a porous plate using homotopy analysis method, Int. J. Eng. Sci. 41 (2003) 2091–2103.
[6] S.J. Liao, I. Pop, Explicit analytic solution for similarity boundary layer equations, Int. J. Heat Mass Trans. 47 (2004) 75–85.
[7] H. Xc, An explicit analytic solution for free convection about a vertical flat plate embedded in a porous medium by means of homotopy analysis method, Appl. Math. Comput., in press.
[8] T. Hayat, M. Khan, M. Ayub, On the explicit analytic solutions of an Oldroyd 6-constant fluid, Int. J. Eng. Sci. 42 (2004) 123–135.
[9] T. Hayat, Y. Wang, A.M. Siddiqui, K. Hutter, S. Asghra, Perstaltic transport of a third-order fluid in a circular cylinderical tube, Math. Mod. Meth. Appl. Sci. 12 (2002) 1691–1706.
[10] R.L. Fosdick, K.R. Rajagopal, Thermodynamics and stability of fluids of third grade, Proc. Roy. Soc. London A 339 (1980) 351–377.
[11] K.R. Rajagopal, A.S. Gupta, An exact solution for a non-Newtonian fluid past an infinite porous plate, Meccanica. 19 (1984) 158–160.