Solutions of some unsteady flows over a stretching sheet using homotopy analysis method

By

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DEPARTMENT OF MATHEMATICS
QUAID-I-AZAM UNIVERSITY
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Supervised By

Prof. Dr. Muhammad Ayub

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DEPARTMENT OF MATHEMATICS
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CERTIFICATE

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We accept this dissertation as conforming to the required standard

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Preface

During the last few years the computation of non-Newtonian fluids has been on the leading edge of research in fluid mechanics. Such fluids are now acknowledged as more appropriate in technological applications than Newtonian fluids. Ideally speaking they are used in flow problems arising in the study of non-Newtonian fluids and pose a challenge to applied mathematicians, numerical analysts and computer simulationists. These stem from the fact that the rheological fluid parameter introduces some extra term in the momentum equation. Because of fluids diversity many constitutive equations have been proposed. One of the important classes of non-Newtonian fluids is viscoelastic fluids. The constitutive equation of even the simplest subclass of viscoelastic fluids namely the second grade is such that the momentum equation give rise to problems in which the order of the differential system is greater than the number of available boundary conditions. In this situation the researchers found it convenient to obtain the perturbation solution. Such solution always requires small or large parameter in the differential system. It is not necessary to have such parameter in every differential system. Therefore, the main theme of the present thesis is to develop HAM solutions for some non-linear flow problems. Note that the HAM does not require any small or large parameter in the differential system.

The boundary layer flows on a moving surface are very important due their occurrence in many engineering processes. Such flows encounter in several processes of thermal and moisture treatment of materials, particularly, in processes involving continuous pulling of a sheet through a reaction zone, as in metallurgy in textile and paper industries, in the manufacture of polymeric sheets, sheet glass and crystalline materials. As example on stretched sheets, many metallurgical processes involve the coding of continuous strip or filament by drawing them through a quiescent fluid and that in the process of drawing, when these strips are stretched.

The work on unsteady stretching flow problems is very scarce in the literature. Much attention has been given to the steady flow problems. Few attempts have been made regarding the unsteady flows. Motivated by the aforementioned facts, the entire work in this thesis is divided into nine chapters. Chapter 1 consists of some introductory remarks. The basic of differential type fluids, governing laws and homotopy analysis method (HAM) are presented in chapter 2.

Unsteady axisymmetric flows of viscous and second grade fluids over a radially stretching sheet are analyzed in chapters 3 and 4 respectively. It is concluded that an increase in time increases the velocity and magnitude of skin friction. It is further found that the obtained solution is valid for all values of the dimensionless time. The problem regarding the unsteady boundary layer flow of second grade fluid due to planar stretching is studied in chapter 5. It is noted that velocity increases by increasing the material parameter of second grade fluid.

Chapters 6-8 are devoted to the heat transfer analysis of the flow problems considered in chapters 3-5, respectively. Expressions for temperature profiles are obtained for the two heating processes namely the prescribed surface temperature (PST) case and prescribed surface heat flux (PHF) case. The influence of sundry parameters in the heat transfer analysis is highlighted. The conclusions are synthesized in chapter 9.

Contents

Chapter 1

Introduction

It is now generally acknowledged that in industrial applications non-Newtonian fluids are more suited than Newtonian fluids. According to Newton's law of viscosity the shear stress is proportional to the velocity gradient. The fluids obeying Newton's law of viscosity are known as Newtonian fluids. Such fluids include water, benzene,ethyl alcohol, hexane and most solutions of low molecular weight. There are many fluids for which Newton's law of viscosity does not hold. These are termed as non-Newtonian fluids. Such fluids exhibit a non-linear relationship between the stresses and the rate of strain. Many materials such as slurries, pastes, gels, drilling mud, clay coating, polymer melts, elastomers etc. are examples of non-Newtonian fluids. They exhibit various behaviors: time-independent behaviors (Bingham-plastic, pseudo-plastic and dilatant fluids), time-dependent behaviors (thixotropic and rheopectic fluids), Visco-plastic fluids (e.g, egg white).

Due to large variety of non-Newtonian fluids, it is not possible to have constitutive equation by which all the non-Newtonian fluids can be described. In the literature many constitutive equations are suggested. Some of them are the empirical or semi-empirical. The method of continuum mechanics is needed for more general three dimensional representation. Undoubtedly, the equation of motion of non-Newtonian fluid, in general, is of higher order than the Navier-Stokes equations. The adherence boundary condition is reasonable for a viscous fluid but it is inadequate when flows of non-Newtonian fluids are taken into account. For unique solution in such flows, one needs an extra condition. This issue of extra conditions has been discussed in detail by Rajagopal [1, 2], Rajagopal and Gupta [3], Rajagopal et al. [4] and Rajagopal and

Kaloni [5].

Among the several models of non-Newtonian fluids, there is a subclass of viscoelastic fluids namely the second grade for which one can reasonable hope to obtain an analytic solution. With this fact in mind, we also consider the second grade fluid in the present thesis. Several researcher have already discussed the flows of second grade fluid in various situations. Rajagopal [6] discussed the unsteady unidirectional flows of a second grade fluid. The flows are induced either due to the application of pressure gradient or through the motion of the boundary. In another paper, Rajagopal [7] examined the creeping flow of a second grade fluid. In continuation, Rajagopal [8] studied longitudinal and torsional oscillations of a rod in a non-Newtonian fluid. Bandelli et al. [9] obtained some unsteady solutions in second grade fluids. Erdogan et al. [10] discussed the comparison of two different solutions in the form of series of the governing equation of an unsteady flow of a second grade fluid. Then Erdogan [11] considered the unsteady motions of a second order fluid over a plane wall. Erdogan et al. [12] also studied the diffusion of line vortex in a second grade fluid. In [13] Erdogan et al. investigated the effects of the side walls on the unsteady flow of a second grade fluid in a duct of uniform cross-section. Fetecau et al. [14] obtained the starting solution for the motion of second grade fluid due to longitudinal and torsional oscillations of a circular cylinder. Fetecau et al. [15] discussed the starting solutions for some simple oscillating motions of second grade fluids. Fetecau et al. [16] also analyzed the starting solutions for some unsteady unidirectional flows of second grade fluids. In [17] Fetecau et al. solved some axial Couette flows of non-Newtonian fluids. Also Fetecau et al. [18] examined the decay of potential vortex and propagation of heat wave in second grade fluid. Tan et al. [19] discussed the Stokes first problem for second grade fluid in a porous half space. Tan et al. [20] also examined the impulsive motion of flat plate in generalized second grade fluid. Tan et al. [21] solved the unsteady flows of a generalized second grade fluid with the fractional derivative model between two parallel plates. Hayat et al. [22] discussed Hall effects on the unsteady hydromagnetic oscillatory flow of a second grade fluid. Hayat et al. [23] studied the unsteady hydromagnetic rotating flow of a conducting second grade fluid. Transient flows of a second grade fluid has been examined by Hayat et al. in [24]. Flow induced by non-coaxial rotation of a porous disk executing non-torsional oscillations and a second grade fluid rotating at infinity is also investigated by Hayat et al. [25]. In [26] Hayat et al. discussed the unsteady

Couette flow of a second grade fluid. Chen et al. [27] examined the unsteady unidirectional flow of second grade fluid between the parallel plates with different given volume flow rate conditions.

Boundary layer behavior over a moving solid surface is an important type of flow occurring in several engineering processes. The aerodynamic extrusion of plastic sheets, the cooling of an infinite metallic plate in a cooling bath, the boundary layer along a liquid film in condensation process and a polymer sheet or filament extruded continuously from a die are few examples of practical applications of a continuous flat surface. Many metallurgical processes involve the cooling of continuous strips or filaments by drawing them through a quiescent fluid. The heat transfer analysis of such non-Newtonian fluids further have many applications in a number of technological processes including production of polymer film or thin sheets. Especially heat transfer analysis plays a vital role during the handling and processing of non-Newtonian fluids. Such analysis in boundary layer flows of non-Newtonian fluids arises in the design of thrust bearing and radial diffusers, transpiration cooling, drag reduction and thermal recovery of oil. Extensive work in the literature have been performed for the boundary layer flow and heat transfer in viscous and second grade fluids over the stretching surface. Sakiadis [28] was the first who studied the boundary layer flow of an incompressible fluid on a moving solid surface, which turns out to be different from the Blasius flow past a flat plate, McCormack and Crane [29] studied the boundary layer flow of a Newtonian fluid caused by stretching of an elastic flat sheet, which moves in its own plane with a velocity from a fixed point due to the application of uniform force, and this work has been extended by many researchers for permeable plates such as Gupta and Gupta [30], Erickson et al. [31], Chen and Char [32], Magyari and Keller [33] and for impermeable plates by Crane [34], Banks [35], Ali [36] , Hayat et al. [37 – 41] and Sajid at el. [42 – 45].

The work on unsteady boundary layer flow due to stretching surface in a viscous fluid [46 – 48] has received much less attention. Nazar et al. [49] solved the unsteady boundary layer flow due to an impulsively stretching surface in a rotating fluid by means of transformations found by William and Rhyne [50] . They obtained a first-order perturbation approximation. Seshadri et al. [51] solved the unsteady mixed convection flow in a stagnation region of a heated vertical plate due to impulsive motion. Liao [52] discussed homotopy analysis method (HAM) solution of unsteady boundary layer flow caused by a impulsively stretching plate. Takhar et al. [53] discussed the unsteady MHD rotating flow over a stretching surface. Kumari et al. [54] studied the unsteady free convection flow of a continuous moving vertical surface. Nazar et al. [55] investigated the unsteady boundary layer flow in the region of the stagnation point on a stretching sheet. Lok et al. [56] examined the boundary layer flow of a micropolar fluid near the forward stagnation point of plane surface. Lakshmisha et al. [57] studied the three dimensional unsteady flow with heat and mass transfer over a continuous stretching surface. Perturbation techniques was applied by the researchers and the corresponding solutions are valid for small time [58 − 60]. The stretching sheet problems with and without heat transfer analysis was studied by many investigators [61 − 68] . All the work mentioned above regarding the stretching surface includes the linear stretching. Little information is available regarding the flow over a radially stretching sheets. Axisymmetric flow of second grade fluid past a stretching sheet has been examined by Ariel [69]. Sajid et al. [70] obtained series solution for the axisymmetric flow of a third grade fluid over a radially stretching sheet using HAM.

To the best of our knowledge, the unsteady flow over a stretching surface is not discussed in Newtonian and non-Newtonian fluids for the case of radially stretching sheets. The main objective of this thesis is to consider such flow problems and develop solutions for them. Throughout the thesis, problems are nonlinear and it is difficult to obtain exact solutions. In particular, it is often more difficult to get an analytic approximation than a numerical one of a given nonlinear problem. The numerical and analytic methods of nonlinear problems have their own advantages and limitations. Generally, one delights in giving analytic solutions of a nonlinear problem. For this purpose the useful technique for the nonlinear problems, the homotopy analysis method (HAM) proposed by Liao [71, 72] is used. HAM itself provides us with a convenient way to control the convergence of the approximation series and adjust the convergence region when necessary. Thus, this technique is valid for nonlinear problems with strong nonlinearity. Furthermore, the HAM logically contains some previous perturbation and non-perturbation techniques. Thus, it can be regarded as a generalized theory of these previous techniques.

The contents of chapter three and eight have been accepted and now are available online in "**Communications in Non-Linear Science and Numerical Simulations.**" The research material of chapter four has been accepted for publication in "**Computers and Mathematics with Applications.**" The work done in chapter 6 has been accepted for publication in "**J. Porous Media.**" The research work in chapters 5 and 7 is submitted for publication in "**Physics Letter A**" and " **Applied Mathematics and Computation**" respectively.

Chapter 2

Preliminaries

This chapter includes some basic equations regarding Newtonian and second grade fluids, conservation laws of mass and momentum, the energy equation and homotopy analysis method (HAM).

2.1 Constitutive equations of Newtonian and Second grade fluids

Since the stress at any point in the fluid is an expression of the mutual reaction of adjacent points of fluids near that point, it is natural to consider the connection between the stress and the local properties of the fluid. For stationary fluid, the stress is wholly calculated due to the static pressure. For fluid in motion, the connection between the stress and the local fluid properties is complicated. However in such cases the stress depends upon the velocity in the neighborhood of the element. Such distribution can be given in terms of the velocity gradient. Therefore, the constitutive equation for the Cauchy stress tensor $\boldsymbol{\sigma}$ in a Newtonian fluid is expressed in terms of velocity gradient as

$$\sigma = -p\mathbf{I} + \mu \left[(\text{grad } \mathbf{V}) + (\text{grad } \mathbf{V})^T \right], \tag{2.1}$$

in which p is the pressure, μ the dynamic viscosity, $\mathbf{I}$ the identity tensor and T in the superscript is the matrix transpose.

For second grade fluid, the constitutive equation is [73]

$$\sigma = -p\mathbf{I} + \mu\mathbf{A}_1 + \alpha_1\mathbf{A}_2 + \alpha_2\mathbf{A}_1^2. \tag{2.2}$$

In the above expression $\alpha_i\,(i = 1, 2)$ are the material moduli and the first two Rivilin-Ericksen tensors are

$$\mathbf{A}_1 = (\text{grad }\mathbf{V}) + (\text{grad }\mathbf{V})^T, \tag{2.3}$$

$$\mathbf{A}_2 = \frac{d\mathbf{A}_1}{dt} + \mathbf{A}_1(\text{grad }\mathbf{V}) + (\text{grad }\mathbf{V})^T\mathbf{A}_1, \tag{2.4}$$

in which d/dt signifies the material derivative. A comprehensive discussion regarding the sign of α_1 and α_2 is made by Fosdick and Rajagopal [74] . In order to satisfy the Clausius-Duhem inequality.

$$\mu \geq 0, \quad \alpha_1 + \alpha_2 = 0 \tag{2.5}$$

and if the free energy is minimum in equilibrium then

$$\alpha_1 \geq 0. \tag{2.6}$$

Note that for $\alpha_1 < 0$ the fluid model shows the anomalous behavior.

2.2 Basic equations

Analysis of any problem in fluid mechanics necessarily includes statement of the basic laws governing the fluid motion. The basic laws applicable to any fluid are:

1. Conservation of mass,
2. Newton second law of motion,
3. The principle of angular momentum,
4. The first law of thermodynamics,
5. The second law of thermodynamics.

Note that not all basic laws are required to solve any one problem. On the other hand, in many problems it is necessary to bring into the analysis additional relations that describe the behavior of physical properties of fluids under given conditions.

2.2.1 Equation of continuity

Let us consider a three dimensional unsteady flow. A control volume $\widetilde{V}$ in space is superimposed on the flow and consider the system that instantaneously occupies the control volume. Assume that it and its surface $\widetilde{S}$ remain fixed in space. The surface is permeable so that fluid can freely enter in and leave. Equation of continuity or conservation of mass stems from the principle that mass can neither be created nor destroyed within the control volume. Thus the mass conserved in the control volume $\widetilde{V}$ is given by

$$\frac{d}{dt}\int_{\widetilde{V}} \rho d\widetilde{V} = 0. \tag{2.7}$$

Here ρ is the fluid density field at time t. By Reynold's transport theorem we have

$$\int_{\widetilde{V}} \left(\frac{\partial \rho}{\partial t} + \operatorname{div}(\rho \mathbf{V})\right) d\widetilde{V} = 0. \tag{2.8}$$

Since the control volume $\widetilde{V}$ is being arbitrary for conservation of mass a necessary and sufficient condition is

$$\frac{\partial \rho}{\partial t} + \operatorname{div}(\rho \mathbf{V}) = 0 \tag{2.9}$$

which for incompressible fluid reduces to

$$\operatorname{div}\mathbf{V} = 0. \tag{2.10}$$

2.2.2 Law of conservation of linear momentum

In differential form, the law of conservation of momentum is

$$\rho\frac{d\mathbf{V}}{dt} = \rho\mathbf{B} + \operatorname{div}\boldsymbol{\sigma}, \tag{2.11}$$

in which $\rho\mathbf{B}$ is the body force per unit mass, $\boldsymbol{\sigma}$ is the Cauchy stress. The Navier Stokes equations for an incompressible fluid are given in component form as

$$\rho\left[\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}\right]=-\frac{\partial \widehat{p}}{\partial x}+\mu\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}\right)+\rho B_x, \tag{2.12}$$

$$\rho\left[\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+w\frac{\partial v}{\partial z}\right]=-\frac{\partial \widehat{p}}{\partial y}+\mu\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}+\frac{\partial^2 v}{\partial z^2}\right)+\rho B_y, \tag{2.13}$$

$$\rho\left[\frac{\partial w}{\partial t}+u\frac{\partial w}{\partial x}+v\frac{\partial w}{\partial y}+w\frac{\partial w}{\partial z}\right]=-\frac{\partial \widehat{p}}{\partial z}+\mu\left(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}+\frac{\partial^2 w}{\partial z^2}\right)+\rho B_z, \tag{2.14}$$

where B_x, B_y and B_z are the components of the body force in the x, y and $z-$directions, respectively.

2.3 Energy equation

By law of conservation of energy one can write

$$\rho\frac{de}{dt}=\boldsymbol{\sigma}\cdot\mathbf{L}-\boldsymbol{\nabla}\cdot\mathrm{q}+\rho\widehat{r}, \tag{2.15}$$

where e is the internal energy, $\mathbf{q}$ is the heat flux vector, $\widehat{r}$ is the radiant heating and $\mathbf{L}=\operatorname{grad}\mathbf{V}$. In the absence of radiant heating the above equation takes the following form

$$\rho c_p\frac{dT}{dt}=\boldsymbol{\sigma}\cdot\mathbf{L}+k\boldsymbol{\nabla}^2 T, \tag{2.16}$$

where $e=c_pT$, $\mathrm{q}=-k\boldsymbol{\nabla}T$, k is the thermal conductivity, c_p is the specific heat and T is the temperature.

2.4 Boundary layer flow

Before this century and towards the end of 19th century the science of fluid mechanics began to develop in two directions which had practically no points in common. On the one side there was the science of theoretical hydrodynamics which was evolved from Euler's equation of motion for a frictionless, non-viscous fluid and which achieved a higher degree of completeness. The results of this so-called classical science of hydrodynamics stood in glaring contradiction to experimental results in particular as regards the very important problem of pressure losses in pipes and channels, as well as with regard to the drag of a body which moves through a mass of fluid. Due to the rapid development in the technology, engineers developed their own highly empirical science of hydraulics. The science of hydraulics was based on a large number of data and differed greatly in its methods and its objects from the science of hydrodynamics.

At the beginning of the 20th century L. Prandtl has given a new dimension to fluid mechanics by introducing viscosity in the fluid and unifying the hydraulics and theoretical hydrodynamics. He noted that in the thin region near the solid boundary, the viscous interactions have a significant effects on fluid motion, however far away from the solid boundary, viscous interactions were not that significant in order to determine the flow field. Before this the viscosity effects were completely ignored in ideal flow solutions and the equations describing viscous interaction were very complex. The Navier-Stokes equations behave well for small Reynold's number whereas for higher values of Reynold's number the non-linear term gains insignificance and the situation is quit different and there may be more than one possible solution. Laminar flows may become unstable and turbulence may occur and steady symmetric may becomes unsteady and asymmetric. Also singular region may develop, especially near the solid boundaries.

It became known that the flow past a body can be divided into a thin region very near to the body called the boundary layer where the viscosity is important and the remaining portion (region) where one can ignore the viscosity. The most important application of a boundary layer can be seen as friction drag of bodies in a flow. The boundary layer has its application in lift of an airfoil and heat transfer between a body and fluid around it. Moreover, the complete equations of motion for flows with friction (the Navier-Stokes equations) had been known for a long time. The great mathematical difficulties connected with the solution of these equations with the exception of a small number of particular cases. These equations are highly non-linear, second order and elliptic in space. Solutions of full Navier-Stokes equation are expensive. Inviscid solutions are very cheap as compared to the Navier-Stokes equations. By assuming that all of the viscosity in the flow field resides in a thin boundary layer, viscous boundary layer, we are free to solve the rest of the flow field using invisid solution. The solution of the flow inside the boundary layer is cheap as well. By assuming a thin boundary layer, several terms negligible and the elliptic equation become parabolic. The boundary layer concepts retains for several reasons. The boundary layer solutions are less expensive, full Navier-Stokes equations are unnecessary in these situations and these solutions are accurate enough for many purposes. Boundary layer theory is extended to compressible turbulent boundary layer as well. Fundamental approach on boundary layers can be seen from the book by Schlichting et al. [75] . Modern investigations in the field of fluid dynamics in general, as well as in the field of boundary layer research, are characterized by a very close relation between theory and experiment. The derivation of boundary layer equations for a viscous fluid are given in [76] .

2.5 Maxwell's equations

In this section we describe the behavior of electric and magnetic fields, **E** and **B** through the following differential equations

$$\nabla\times\frac{\mathbf{B}}{\mu_2} = \mathbf{J}+\frac{\partial}{\partial t}(\varepsilon\mathbf{E}), \tag{2.17}$$

$$\nabla\times\mathbf{E} = -\frac{\partial\mathbf{B}}{\partial t}, \tag{2.18}$$

$$\nabla\cdot\mathbf{B} = 0, \tag{2.19}$$

$$\nabla\cdot\mathbf{E} = \frac{\rho_c}{\varepsilon}, \tag{2.20}$$

in which the constants μ_2 and ε are magnetic permeability and dielectric constant, respectively, $\mathrm{D} = (\varepsilon\mathrm{E})$ is the dielectric displacement and ρ_c is the charge density. The total magnetic field B often referred to as the magnetic field is related to the magnetic field **H** as $\mathbf{B} = \mu_2\mathbf{H}$.

According to Ohm's law

$$\mathbf{J} = \sigma_1(\mathbf{E}+\mathbf{V}\times\mathbf{B}), \tag{2.21}$$

where σ_1 is the electrical conductivity of the fluid. The polarization effects here are negligible $(\mathbf{E}=\mathbf{0})$ and magnetic Reynolds number is taken very small, i.e., induced magnetic field is negligible.

Under the aforesaid assumptions, the Lorentz force becomes

$$\frac{1}{\rho}(\mathbf{J}\times\mathbf{B}) = -\frac{\sigma B_0^2}{\rho}\mathbf{V}, \tag{2.22}$$

in which B_0 is the magnitude of constant applied magnetic field.

2.6 Porosity and porous media

Most important geometrical property of the porous medium is the porosity. Because of the rheological properties of fluids often change with the geometry, it is important to measure those properties in a geometry as similar to the applications as possible. Porosity is defined as the percentage of a volume of medium that is empty space that contributes to the fluid flow.

Mathematically it is the ratio between the unit volume of void space $\widetilde{V}_v$ to the unit volume of the medium V_m i.e.

$$\phi = \frac{100\widetilde{V}_v}{V_m}, \tag{2.23}$$

where $0 < \phi < 1$. If $\widetilde{V}_v = V_m$ we have the case of free fluid. Also porous medium is that medium for which the permeability is non-zero. The permeability is the most important property of a porous medium that measures quantitatively the ability of a porous medium to conduct fluid flow.

2.7 Flow induced by a stretching sheet

The flow produced due to the stretching of elastic flat sheet which moves in its plan with velocity varying with the distance from a fixed point due to the application of a stress are known as stretching flow. The production of sheeting material arises in a number of industrial manufacturing processes and includes both metal and polymer sheets. In the manufacturing of the latter, the material is in a molten phase when thrust through an extrusion die and then cools and solidifies some distance away from the die before arriving at the cooling stage. The tangential velocity imported by the sheet induces motion in the surrounding fluid, which alters the convection of the sheet. Similar situation prevails during the manufacture of plastic and rubber sheets where it is often necessary to blow a gaseous medium through the not-yet solidified material, and where the stretching force depends upon time. Another example that belongs to this class of problems is the cooling of a large metallic plate in a bath, which may be an electrolyte. In this class the fluid flow is induced due to shrinking of the plate. Glass blowing, continuous casting and spinning of fibers also involve the flow due to stretching surface. Due to the very high viscosity of the fluid near the sheet, one can assume that the fluid is affected by the sheet but not vice versa. Thus the fluid problems can be idealized to the case of a fluid disturbed by a tangential moving boundary. Experiments show that the velocity of the boundary is approximately proportional to the distance to the orifice (Vleggaar [77]). The quality of the resulting sheeting material, as well as the cost of production, is affected by the speed of collection and the heat transfer rate, and knowledge of the flow properties of the ambient fluid is clearly desirable.

Fig. 2.2 (a) Fig. 2.2 (b)

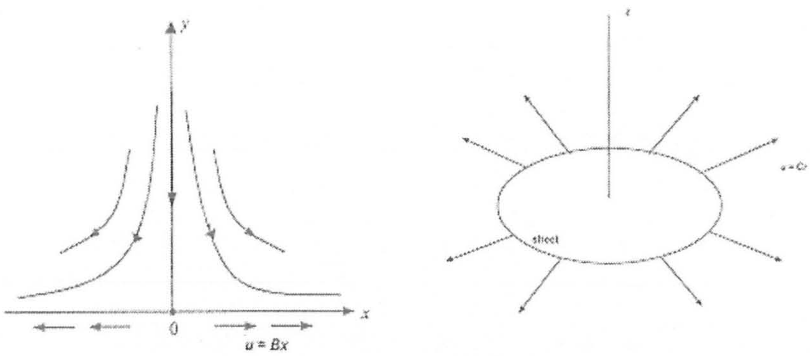


Fig. 2.2. (a) Physical model for planar stretching sheet, (b) Physical model for radial stretching sheet.

2.8 Homotopy

Definition: Homotopy is a continuous transformation from one function to another. A homotopy $\mathbb{H}$ between two continuous functions a and b from a topological space X to a topological space Y is define by a continuous mapping

$$\mathbb{H} : X \times I = [0, 1] \rightarrow Y,$$

where

$$\mathbb{H}(x, t) = h_t(x), \qquad (x \in X, t \in I)$$

is continuous then we call a family of maps $h_t : X \rightarrow Y$ $(0 \leq t \leq 1)$, indexed by the real numbers $t \in I$ is called a homotopy with h_0 and h_1 are initial and terminal map of h_t.

Two maps $f : X \rightarrow Y$ and $g : X \rightarrow Y$ are said to be homotopic if there exist a homotopy $h_t : X \rightarrow Y$ such that $h_0 = f$ and $h_1 = g$. So h_t is the homotopy connecting f and g, as $h_t : f \simeq g$. Thus, f and g are homotopic if and only if each can be changed continuously into other.

Further if f is homotopic to g then there exist a

$$[\mathbb{H}_p : p \in (0, 1)]$$

of continuous functions such that $\mathbb{H}_p : R \times [0, 1] \rightarrow R$ defined by

$$\mathbb{H}_p(x) = (1 - p) f(x) + pg(x) \quad \text{for all } x \in R \text{ and } p \in [0, 1].$$

2.9 Homotopy analysis method (HAM)

Non-linearity plays a crucial role in applied mathematics. Most of the problems arising are non-linear it is important to develop efficient tools to solve them. Since the advent of modern computers numerical techniques for nonlinear partial differential equation (PDEs) have been developing rapidly. However, it is still difficult to obtain analytic approximations of nonlinear partial differential equation, even though there exist high performance super computers and high quality computation software such as Mathematica, Maple etc. In the past, perturbation technique were applied to solve such problems but such technique requires large or small parameter. It is not possible that every problem has such parameter. Unlike the perturbation technique the homotopy analysis method is valid even for nonlinear problems whose governing equation and /or boundary conditions don't contain small or large parameter at all. Thus, it can be applied to more nonlinear problems in science and engineering.

The homotopy analysis method is rather general and valid for many different types of nonlinear ordinary differential equations and partial differential equations. It has been successfully applied to many non-linear problems such as boundary layer flows, heat transfer, MHD flows of non-Newtonian fluids and many others. It is an analytic method to approximate the solution of non-linear with strong nonlinearity, solution expressions of a non-linear problem are determined by the type of nonlinear equation and the employed analytic technique, and the convergence regions of series solution are strong dependent on physical parameters. Due to existence of strong non-linearities in the governing flow equations numerical techniques or perturbation techniques are widely used. Throughout this thesis, the HAM is used to solve the two dimensional flow of a Newtonian and non-Newtonian fluids over a stretching surface and complete form of analytic solutions are obtained. Recently, HAM is successfully applied to many non-linear flow problems [78 – 91] . The developed HAM solutions in this thesis are quite new and have been never reported in the literature.

Chapter 3

Unsteady axisymmetric flow of a viscous fluid over a radially stretching sheet

The problem of unsteady axisymmetric flow of a viscous fluid over a radially stretching sheet is considered in this chapter. The axisymmetric flow equations are given. By means of similarity transformations, the modeled non-linear partial differential equations in three independent variables are reduced to a single partial differential equation in two independent variables. The HAM solution governing the flow is developed. The convergence theorem for the present problem is established and the reliability of the convergence on the auxiliary parameter is explained. Finally, the influence of various emerging flow parameters are plotted and discussed..

3.1 Mathematical formulation

Consider the unsteady laminar flow of a viscous fluid over a stretching sheet which is placed in the plan $z = 0$; the flow being confined to $z > 0$ and is stretched in the radial direction. The sheet is stretched with the speed proportional to the radial distance from the origin. Here for mathematical modelling, we take the cylindrical polar coordinates (r, θ, z). All the physical quantities are independent of θ because of rotational symmetry of the flow i.e. $\partial/\partial\theta = 0$.

Also the azimuthal component of velocity v vanishes identically. Under these assumptions the governing equations (2.10) and (2.11) in the absence of body forces together with Eqs. (2.2) – (2.7) with $\alpha_i = 0$

$$\frac{\partial u}{\partial r} + \frac{u}{r} + \frac{\partial w}{\partial z} = 0, \tag{3.1}$$

$$\rho\left[\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial r} + w\frac{\partial u}{\partial z}\right] = -\frac{\partial \widehat{p}}{\partial r} + \mu\left[\frac{\partial^2 u}{\partial r^2} + \frac{\partial^2 u}{\partial z^2} + \frac{1}{r}\frac{\partial u}{\partial r} - \frac{u}{r^2}\right], \tag{3.2}$$

$$\rho\left[\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial r} + w\frac{\partial w}{\partial z}\right] = -\frac{\partial \widehat{p}}{\partial z} + \mu\left[\frac{\partial^2 w}{\partial z^2} + \frac{\partial^2 w}{\partial r^2} + \frac{1}{r}\frac{\partial w}{\partial r}\right], \tag{3.3}$$

where u and w are the velocities in r and z directions, respectively.

The appropriate boundary conditions for the problem under consideration are

$$\begin{aligned} u &= ar, \quad w = 0 \quad at \ z = 0, \\ u &\rightarrow 0 \qquad\qquad as \quad z \rightarrow \infty. \end{aligned} \tag{3.4}$$

Introducing the similarity transformations

$$\begin{aligned} u &= arf'(\eta, \xi), \quad w = -2\sqrt{a\upsilon\xi}f, \\ \xi &= 1 - e^{-\tau}, \quad \eta = \sqrt{\frac{a}{\upsilon\xi}}z, \quad \tau = at. \end{aligned} \tag{3.5}$$

The continuity equation (3.1) is satisfied automatically and Eqs. (3.2) and (3.3) become

$$\frac{\partial \widehat{p}}{\partial r} = \rho a^2 r\left[\frac{\eta(1-\xi)}{2\xi}f'' - (1-\xi)\frac{\partial f'}{\partial \xi} - f'^2 + 2ff'' + \frac{1}{\xi}f'''\right], \tag{3.6}$$

$$\frac{\partial \widehat{p}}{\partial \eta} = \rho a\nu\left[2(1-\xi)\xi\frac{\partial f}{\partial \xi} - (1-\xi)\eta f' + (1-\xi)f - 4\xi ff' - 2f''\right], \tag{3.7}$$

where prime denotes differentiation with respect to η and a is the stretching constant.

Eliminating pressure from Eqs.(3.6) and (3.7), we obtain

$$f^{iv} + \frac{\eta(1-\xi)}{2}f''' + 2\xi ff''' + \frac{(1-\xi)}{2}f'' - \xi(1-\xi)\frac{\partial f''}{\partial \xi} = 0. \tag{3.8}$$

The boundary conditions (3.4) now read as

$$f(0,\xi)=0,\quad f'(0,\xi)=1,\quad f'(\infty,\xi)=0. \tag{3.9}$$

In the next section, we will find the analytic solution of Eq. (3.8) subject to boundary conditions (3.9).

3.2 Analytic solution

In order to obtain the solution of a problem consisting of Eqs.(3.8) and (3.9) we use HAM. For that the initial guess

$$f_0(\eta,\xi)=1-\exp(-\eta), \tag{3.10}$$

and auxiliary linear operator

$$\mathcal{L}_1[f(\eta,\xi;p)]=\frac{\partial^3 f}{\partial\eta^3}-\frac{\partial f}{\partial\eta}, \tag{3.11}$$

are chosen and the operator $\mathcal{L}_1$ satisfies

$$\mathcal{L}_1[C_1+C_2\exp(-\eta)+C_3\exp(\eta)]=0, \tag{3.12}$$

in which C_1, C_2 and C_3 are arbitrary constants.

Zero-order deformation problem

Following the HAM procedure we can write the zeroth order deformation problem as

$$(1-p)\mathcal{L}_1\left[\widehat{f}(\eta,\xi;p)-f_0(\eta,\xi)\right]=p\hbar_1\mathcal{N}_1\left[\widehat{f}(\eta,\xi;p)\right], \tag{3.13}$$

$$\widehat{f}(0,\xi;p)=0,\quad \frac{\partial\widehat{f}(\eta,\xi;p)}{\partial\eta}|_{\eta=0}=1\frac{\partial\widehat{f}(\eta,\xi;p)}{\partial\eta}|_{\eta=+\infty}=0, \tag{3.14}$$

where the non-linear differential operator $\mathcal{N}_1$ is

$$\begin{aligned}\mathcal{N}_1\left[\widehat{f}(\eta,\xi;p)\right] &= \frac{\partial^4\widehat{f}(\eta,\xi;p)}{\partial\eta^4}+\frac{\eta\xi(1-\xi)}{2}\frac{\partial^3\widehat{f}(\eta,\xi;p)}{\partial\eta^3}+2\xi\widehat{f}(\eta,\xi;p)\frac{\partial^3\widehat{f}(\eta,\xi;p)}{\partial\eta^3}\\ &+\frac{(1-\xi)}{2}\frac{\partial^2\widehat{f}(\eta,\xi;p)}{\partial\eta^2}-\xi(1-\xi)\frac{\partial^3\widehat{f}(\eta,\xi;p)}{\partial\xi\partial\eta^2},\end{aligned} \tag{3.15}$$

where $p \in [0,1]$ is the embedding parameter, $\hbar_1$ is non-zero auxiliary parameter. Obviously for $p = 0$ and $p = 1$, we have, respectively

$$\widehat{f}(\eta, \xi; 0) = f_0(\eta, \xi), \quad \widehat{f}(\eta, \xi; 1) = f(\eta, \xi). \tag{3.16}$$

As p increases from zero to unity, $f(\eta, \xi; p)$ varies from the initial guess $f_0(\eta, \xi)$ to the exact solution $f(\eta, \xi)$ of the considered problem. Then by Taylor's theorem and Eq. (3.16) we have

$$\widehat{f}(\eta, \xi; p) = f_0(\eta, \xi) + \sum_{m=1}^{+\infty} f_m(\eta, \xi) p^m, \tag{3.17}$$

where

$$f_m(\eta, \xi) = \frac{1}{m!} \frac{\partial^m f(\eta, \xi; p)}{\partial p^m}\bigg|_{p=0}. \tag{3.18}$$

The convergence of the series (3.17) depends upon $\hbar_1$. Assume that $\hbar_1$ is chosen in such a way that the series (3.17) is convergent at $p = 1$ then due to Eq. (3.16) we have

$$f(\eta, \xi) = f_0(\eta, \xi) + \sum_{m=1}^{\infty} f_m(\eta, \xi), \tag{3.19}$$

mth-order deformation problem

Differentiating the zeroth-order deformation Eq. (3.13) mth-time with respect to p and the dividing by $m!$ and finally setting $p = 0$ we have

$$\mathcal{L}_1 [f_m(\eta, \xi) - \chi_m f_{m-1}(\eta, \xi)] = \hbar_1 \mathcal{R}_{m1}(\eta, \xi), \tag{3.20}$$

$$f_m(0, \xi; p) = 0, \quad \frac{\partial f_m(\eta, \xi; p)}{\partial \eta}|_{\eta=0} = 0, \quad \frac{\partial f_m(\eta, \xi; p)}{\partial \eta}|_{\eta=+\infty} = 0, \tag{3.21}$$

$$\begin{aligned} \mathcal{R}_{m1} &= \frac{\partial^4 f_{m-1}}{\partial \eta^4} + \frac{\eta\xi(1-\xi)}{2} \frac{\partial^3 f_{m-1}}{\partial \eta^3} + \frac{(1-\xi)}{2} \frac{\partial^2 f_{m-1}}{\partial \eta^2} \\ &\quad -\xi(1-\xi) \frac{\partial^3 f_{m-1}}{\partial \xi \partial \eta^2} + 2\xi \sum_{k=0}^{m-1} f_{m-1-k} f_k''', \end{aligned} \tag{3.22}$$

where

$$\chi_m = \left\{ \begin{array}{ll} 0, & m \leq 1 \\ 1, & m > 1 \end{array} \right\}. \tag{3.23}$$

To obtain the solution of above system of non-homogeneous equations upto first few order of approximations, the symbolic computation software MATHEMATICA is used and the following series solution is found

$$f_m(\eta, \xi) = \sum_{n=0}^{m+1} \sum_{q=0}^{2(m+1-n)} \sum_{r=0}^{m} a_{m,n}^{q,r} \xi^r \eta^q \exp[-n\eta]. \tag{3.24}$$

Detailed procedure of obtaining the recurrence formulas for the coefficients involved in Eq. (3.24) is presented in next section.

3.2.1 Derivation of Coefficient appearing in Equation (3.24)

$$f'_m(\eta, \xi) = \sum_{n=0}^{m+1} \sum_{r=0}^{m} \left[\sum_{q=1}^{2(m+1-n)} a_{m,n}^{q,r} q\eta^{q-1} - n \sum_{q=0}^{2(m+1-n)} a_{m,n}^{q,r} \eta^q \right] \xi^r \exp[-n\eta],$$

$$= \sum_{n=0}^{m+1} \sum_{r=0}^{m} \left[\sum_{q=0}^{2(m+1-n)} a_{m,n}^{q+1,r} (q+1)\eta^q - n \sum_{q=0}^{2(m+1-n)} a_{m,n}^{q,r} \eta^q \right] \xi^r \exp[-n\eta],$$

$$= \sum_{n=0}^{m+1} \sum_{q=0}^{2(m+1-n)} \sum_{r=0}^{m} \left[(q+1) a_{m,n}^{q+1,r} - n a_{m,n}^{q,r} \right] \eta^q \xi^r \exp[-n\eta],$$

$$= \sum_{n=0}^{m+1} \sum_{q=0}^{2(m+1-n)} \sum_{r=0}^{m} a1_{m,n}^{q,r} \eta^q \xi^r \exp[-n\eta], \tag{3.25}$$

where

$$a1_{m,n}^{q,r} = (q+1) a_{m,n}^{q+1,r} - n a_{m,n}^{q,r}. \tag{3.26}$$

Through a similar procedure, the other derivatives are

$$f''_m(\eta, \xi) = \sum_{n=0}^{m+1} \sum_{q=0}^{2(m+1-n)} \sum_{r=0}^{m} a2_{m,n}^{q,r} \eta^q \xi^r \exp[-n\eta], \tag{3.27}$$

$$f_m'''(\eta,\xi) = \sum_{n=0}^{m+1} \sum_{q=0}^{2(m+1-n)} \sum_{r=0}^{m} a3_{m,n}^{q,r} \eta^q \xi^r \exp[-n\eta], \tag{3.28}$$

$$f_m^{iv}(\eta,\xi) = \sum_{n=0}^{m+1} \sum_{q=0}^{2(m+1-n)} \sum_{r=0}^{m} a4_{m,n}^{q,r} \eta^q \xi^r \exp[-n\eta]. \tag{3.29}$$

The coefficients $a2_{m,n}^{q,r}$, $a3_{m,n}^{q,r}$ and $a4_{m,n}^{q,r}$ are

$$a2_{m,n}^{q,r} = (q+1)\, a1_{m,n}^{q+1,r} - na1_{m,n}^{q,r}, \tag{3.30}$$

$$a3_{m,n}^{q,r} = (q+1)\, a2_{m,n}^{q+1,r} - na2_{m,n}^{q,r}, \tag{3.31}$$

$$a4_{m,n}^{q,r} = (q+1)\, a3_{m,n}^{q+1,r} - na3_{m,n}^{q,r}, \tag{3.32}$$

and

$$\begin{aligned}
\frac{\partial f_m''(\eta,\xi)}{\partial \xi} &= \sum_{n=0}^{m+1} \sum_{q=0}^{2(m+1-n)} a2_{m,n}^{q,r} \left[\sum_{r=1}^{m} r\xi^{r-1}\right] \eta^q \exp[-n\eta], \\
&= \sum_{n=0}^{m+1} \sum_{q=0}^{2(m+1-n)} \sum_{r=0}^{m} a2_{m,n}^{q,r+1} (r+1)\, \eta^q \xi^r \exp[-n\eta], \\
&= \sum_{n=0}^{m+1} \sum_{q=0}^{2(m+1-n)} \sum_{r=0}^{m} a5_{m,n}^{q,r} \eta^q \xi^r \exp[-n\eta],
\end{aligned} \tag{3.33}$$

where

$$a5_{m,n}^{q,r} = a2_{m,n}^{q,r+1} (r+1). \tag{3.34}$$

Now, for the product term $f_{m-1-k} f_k'$ we have

$$\begin{aligned}
f_{m-1-k} f_k' &= \sum_{i_1=0}^{m-k} \sum_{j_1=0}^{2(m-k-i_1)} \sum_{l_1=0}^{m-1-k} a_{m-1-k,i_1}^{j_1,l_1} \eta^{j_1} \xi^{l_1} \exp[-i_1\eta] \\
&\times \sum_{i=0}^{k+1} \sum_{j=0}^{2(k+1-i)} \eta a_{k,i}^{j,l} \xi^l \exp[-i\eta],
\end{aligned}$$

$$f_{m-1-k}f_k'' = \sum_{i_1=0}^{m-k}\sum_{i=0}^{k+1} \exp -(i_1+i)\,\eta \sum_{j_1=0}^{2(m-k-i_1)}\sum_{j=0}^{2(k+1-i)} \sum_{l_1=0}^{m-1-k}\sum_{l=0}^{k}\left(a_{m-1-k,i_1}^{j_1,l_1}a_{k,i}^{j,l}\right)\xi^{(l_1+l)}\eta^{(j_1+j)}, \tag{3.35}$$

which further simplifies to

$$\sum_{k=0}^{m-1} f_{m-1-k}f_k'' = \sum_{n=0}^{m+1}\sum_{q=0}^{2(m+1-n)}\sum_{r=0}^{m-1} \left[\sum_{k=0}^{m-1}\sum_{i=\max\{0,n-m+k\}}^{i=\min\{n,k+1\}}\sum_{j=\max\{0,q-2m+2k+2n-2i\}}^{j=\min\{q,2k-2i+2\}}\sum_{l=\max\{0,r-m+1+k\}}^{l=\min\{r,k\}} a_{m-i-k,n-i}^{q-j,r-l}a_{k,i}^{j,l}\right]\eta^q\xi^r\exp\left[-n\eta\right],$$

$$= \sum_{n=0}^{m+1}\sum_{q=0}^{2(m+1-n)}\sum_{r=0}^{m} \left[\sum_{k=0}^{m-1}\sum_{i=\max\{0,n-m+k\}}^{i=\min\{n,k+1\}}\sum_{j=\max\{0,q-2m+2k+2n-2i\}}^{j=\min\{q,2k-2i+2\}}\sum_{l=\max\{0,r-m+1+k\}}^{l=\min\{r,k\}} \chi_{m-r+1}a_{m-i-k,n-i}^{q-j,r-l}a_{k,i}^{j,l}\right]\eta^q\xi^r\exp\left[-n\eta\right],$$

$$\sum_{k=0}^{m-1} f_{m-1-k}f_k'' = \sum_{n=0}^{m+1}\sum_{q=0}^{2(m+1-n)}\sum_{r=0}^{m}\delta_{m,n}^{q,r}\eta^q\xi^r\exp\left[-n\eta\right], \tag{3.36}$$

whence

$$\delta_{m,n}^{q,r} = \sum_{k=0}^{m-1}\sum_{i=\max\{0,n-m+k\}}^{i=\min\{n,k+1\}} \sum_{j=\max\{0,q-2m+2k+2n-2i\}}^{j=\min\{q,2k-2i+2\}}\sum_{l=\max\{0,r-m+1+k\}}^{l=\min\{r,k\}} \chi_{m-r+1}a_{m-i-k,n-i}^{q-j,r-l}a_{k,i}^{j,l}. \tag{3.37}$$

Making use of Eqs. (3.27) – (3.29), (3.33) and (3.37) into Eq. (3.22) one may write

$$\hbar_1 \mathcal{R}_{m1}(\eta,\xi) = \sum_{n=0}^{m}\sum_{q=0}^{2(m-n)}\sum_{r=0}^{m-1} \hbar \begin{bmatrix} a4^{q,r}_{m-1,n} + \chi_{r+1}\chi_{q+1} a3^{q-1,r-1}_{m-1,n} \\ +\frac{1}{2}\chi_{q+1}\chi_r a3^{q-1,r-2}_{m-1,n} + \frac{1}{2} a2^{q,r}_{m-1,n} \\ -\frac{1}{2}\chi_{r+1} a2^{q,r-1}_{m-1,n} - \chi_{r+1} a5^{q,r-1}_{m-1,n} \\ +\chi_r a5^{q,r-2}_{m-1,n} + 2\delta^{q,r-1}_{m-1,n} \end{bmatrix} \eta^q \xi^r \exp[-n\eta],$$

$$= \sum_{n=0}^{m}\sum_{q=0}^{2(m-n)}\sum_{r=0}^{m-1} \Delta^{q,r}_{m,n}\eta^q\xi^r \exp[-n\eta], \tag{3.38}$$

$$\Delta^{q,r}_{m,n} = \hbar_1 \begin{bmatrix} \chi_{2m-2n-q+2}(a4^{q,r}_{m-1,n} + \chi_{r+1}\chi_{q+1} a3^{q-1,r-1}_{m-1,n} \\ +\frac{1}{2}\chi_{q+1}\chi_r a3^{q-1,r-2}_{m-1,n} + \frac{1}{2} a2^{q,r}_{m-1,n} \\ -\frac{1}{2}\chi_{r+1} a2^{q,r-1}_{m-1,n} - \chi_{r+1} a5^{q,r-1}_{m-1,n} \\ +\chi_r a5^{q,r-2}_{m-1,n}) + 2\delta^{q,r-1}_{m-1,n} \end{bmatrix}. \tag{3.39}$$

Using Eq. (3.38), Eq. (3.20) takes the following form

$$\mathcal{L}_1 [f_m(\eta,\xi) - \chi_m f_{m-1}(\eta,\xi)] = \sum_{n=0}^{m}\sum_{q=0}^{2(m-n)}\sum_{r=0}^{m-1} \Delta^{q,r}_{m,n}\eta^q\xi^r \exp[-n\eta]. \tag{3.40}$$

In order to obtaining the solution of Eq. (3.40) we must have

$$Y''' - Y' = \eta^q \exp[-n\eta]. \tag{3.41}$$

Integration of Eq. (3.41) involves two cases

Case (1) when $n = 1$, we have

$$Y = \sum_{k=0}^{q+1}\sum_{p=0}^{q+1-k} \frac{q!}{k! 2^{q+2-k-p}} \eta^k e^{-\eta},$$

$$= \sum_{k=0}^{q+1} \mu^q_{1,k} \eta^k e^{-\eta}, \tag{3.42}$$

where

$$\mu_{1,k}^{q}=\sum_{p=0}^{q+1-k}\frac{q!}{k!2^{q+2-k-p}}\qquad 0\leq k\leq q+1,q\geq 0 \tag{3.43}$$

Case (2) when $n\geq 2$ we have

$$Y=\sum_{k=0}^{q}\sum_{r=0}^{q-k}\sum_{p=0}^{q-k-r}\frac{-q!}{k!n^{r+1}\left(n-1\right)^{q+1-r-p-k}\left(n+1\right)^{p+1}}\eta^{k}e^{-n\eta},$$

$$Y=\sum_{k=0}^{q}\mu_{n,k}^{q}\eta^{k}e^{-n\eta}, \tag{3.44}$$

in which

$$\mu_{n,k}^{q}=\sum_{r=0}^{q-k}\sum_{p=0}^{q-k-r}\frac{-q!}{k!n^{r+1}\left(n-1\right)^{q+1-r-p-k}\left(n+1\right)^{p+1}}\qquad 0\leq k\leq q,q\geq 0,n\geq 2. \tag{3.45}$$

The general solution of Eq. (3.40) is

$$\begin{aligned}f_m\left(\eta,\xi\right)-\chi_m f_{m-1}\left(\eta,\xi\right) &= -\left(\sum_{r=0}^{m}\Delta_{m,0}^{0,r}\xi^{r}\right)\eta+\\ &\sum_{r=0}^{m}\{e^{-\eta}\left[\sum_{q=0}^{2m}\Delta_{m,1}^{q,r}\mu_{1,0}^{q}+\sum_{k=1}^{2m+1}\eta^{k}\left(\sum_{q=k-1}^{2m}\Delta_{m,1}^{q,r}\mu_{1,k}^{q}\right)\right]\\ &-\sum_{n=2}^{m+1}e^{-n\eta}\left[\sum_{k=0}^{2(m+1-n)}\eta^{k}\left(\sum_{q=k}^{2(m+1-n)}\Delta_{m,n}^{q,r}\mu_{n,k}^{q}\right)\right]\}\xi^{r}\\ &+C_1^m+C_2^m e^{-\eta}+C_3^m e^{\eta},\end{aligned} \tag{3.46}$$

where C_1^m, C_2^m and C_3^m are the integral constants. In order to determine these constants we use the boundary conditions (3.21) and get

$$\begin{aligned}C_1^m &= \sum_{r=0}^{m}\Delta_{m,0}^{0,r}\xi^{r}+\sum_{r=0}^{m}\{-\sum_{q=0}^{2m}\Delta_{m,1}^{q,r}\mu_{1,1}^{q}+\\ &\sum_{n=2}^{m+1}\left[\sum_{q=0}^{2(m+1-n)}\left(-1+n\right)\Delta_{m,n}^{q,r}\mu_{n,0}^{q}-\sum_{q=1}^{2(m+1-n)}\Delta_{m,n}^{q,r}\mu_{n,1}^{q}\right]\}\xi^{r},\end{aligned} \tag{3.47}$$

$$C_2^m = -\sum_{r=0}^{m} \Delta_{m,0}^{0,r}\xi^r$$
$$+\sum_{r=0}^{m}\{-\sum_{q=0}^{2m}\Delta_{m,1}^{q,r}\mu_{1,0}^q + \sum_{q=0}^{2m}\Delta_{m,1}^{q,r}\mu_{1,1}^q +$$
$$\sum_{n=2}^{m+1}\left[n\sum_{q=0}^{2(m+1-n)}\Delta_{m,n}^{q,r}\mu_{n,0}^q - \sum_{q=1}^{2(m+1-n)}\Delta_{m,n}^{q,r}\mu_{n,1}^q\right]\}\xi^r, \tag{3.48}$$

$$C_3^m = 0. \tag{3.49}$$

From Eqs. (3.46) , (3.24) and (3.47) – (3.49) one can write

$$\sum_{n=0}^{m+1}\sum_{q=0}^{2(m+1-n)}\sum_{r=0}^{m} a_{m,n}^{q,r}\xi^r\eta^q e^{-n\eta} = \sum_{n=0}^{m+1}\sum_{q=0}^{2(m+1-n)}\sum_{r=0}^{m}\chi_m\chi_{2m-2n-q+2}a_{m-1,n}^{q,r}\xi^r\eta^q e^{-n\eta} - \sum_{r=0}^{m}\Delta_{m,0}^{0,r}\xi^r\eta$$
$$+\sum_{r=0}^{m}\left[\begin{array}{c} e^{-\eta}\{\sum_{k=1}^{2m+1}\eta^k\left(\sum_{q=1}^{2m}\Delta_{m,1}^{q,r}\mu_{1,k}^q\right)\} \\ -\sum_{n=2}^{m+1}e^{-n\eta}\{\sum_{k=0}^{2(m+1)}\eta^k\left(\sum_{q=k}^{2(m+1-n)}\Delta_{m,n}^{q,r}\mu_{n,k}^q\right)\}\end{array}\right]\xi^r$$
$$+\sum_{r=0}^{m}\left[\begin{array}{c} e^{-\eta}\{-\Delta_{m,0}^{0,r} - \sum_{q=0}^{2m}\Delta_{m,1}^{q,r}\mu_{1,1}^q \\ +\sum_{n=2}^{m+1}\left(\begin{array}{c} n\sum_{q=0}^{2(m+1-n)}\Delta_{m,n}^{q,r}\mu_{n,0}^q \\ -\sum_{q=1}^{2(m+1-n)}\Delta_{m,n}^{q,r}\mu_{n,1}^q\end{array}\right)\}\end{array}\right]\xi^r$$
$$+\sum_{r=0}^{m}\left[\begin{array}{c} \Delta_{m,0}^{0,r} - \sum_{q=0}^{2m}\Delta_{m,1}^{q,r}\mu_{1,1}^q + \\ \sum_{n=2}^{m+1}\{\sum_{q=0}^{2(m+1-n)}(n-1)\Delta_{m,n}^{q,r}\mu_{n,0}^q \\ -\sum_{q=1}^{2(m+1-n)}\Delta_{m,n}^{q,r}\mu_{n,1}^q\}\end{array}\right]\xi^r. \tag{3.50}$$

Now, comparing the coefficients of like powers in the above equation, the following recurrence formulas for the coefficients $a_{m,n}^{q,r}$ of $f_m(\eta,\xi)$ for $0 \le n \le m+1,\ 0 \le q \le 2(m+1-n)$ and

$0 \leq r \leq m$ are obtained

$$a_{m,0}^{0,r} = \chi_m \chi_{2m+2} a_{m-1,0}^{0,r} + \Delta_{m,0}^{0,r} - \sum_{q=0}^{2m} \Delta_{m,1}^{q,r} \mu_{1,1}^{q} + \sum_{n=2}^{m+1} \left[\sum_{q=0}^{2(m+1-n)} (n-1) \Delta_{m,n}^{q,r} \mu_{n,0}^{q} - \sum_{q=1}^{2(m+1-n)} \Delta_{m,n}^{q,r} \mu_{n,1}^{q} \right], \quad (3.51)$$

$$a_{m,0}^{1,r} = \chi_m \chi_{2m+1} a_{m-1,0}^{1,r} - \Delta_{m,0}^{0,r}, \quad (3.52)$$

$$a_{m,1}^{0,r} = \chi_m \chi_{2m} a_{m-1,1}^{0,r} - \Delta_{m,1}^{0,r} - \sum_{q=0}^{2m} \Delta_{m,1}^{q,r} \mu_{1,1}^{q} + \sum_{n=2}^{m+1} \left[n \sum_{q=0}^{2(m+1-n)} \Delta_{m,n}^{q,r} \mu_{n,0}^{q} - \sum_{q=1}^{2(m+1-n)} \Delta_{m,n}^{q,r} \mu_{n,1}^{q} \right], \quad (3.53)$$

$$a_{m,1}^{q,r} = \chi_m \chi_{2m-q} a_{m-1,1}^{q,r} + \sum_{q=1}^{2m} \Delta_{m,1}^{q,r} \mu_{1,k}^{q}, \quad (3.54)$$

$$a_{m,n}^{q,r} = \chi_m \chi_{2m-2n-q+2} a_{m-1,n}^{q,r} + \sum_{q=k}^{2(m+1-n)} \Delta_{m,n}^{q,r} \mu_{n,k}^{q}. \quad (3.55)$$

Now it is pointed out that $f_0(\eta,\xi)$ the initial guess has the same structure and in this subsection we proved that, if the first $(m-1)$ solutions $f_{\mathrm{i}}(\eta,\xi), \{\mathrm{i} = 0,1,2,...,m-1\}$ have the same structure as $f_0(\eta,\xi)$, then mth order solution $f_m(\eta,\xi)$ have the same structure. Utilizing the above recurrence formulas, all coefficients $a_{m,n}^{q,r}$ can be computed using only the following two coefficients

$$a_{0,0}^{0,0} = 1, \qquad a_{0,1}^{0,0} = -1 \quad (3.56)$$

and the mth-order approximation is

$$\sum_{m=0}^{M} f_m(\eta,\xi) = \sum_{m=0}^{M} a_{m,0}^{0,0} + \sum_{n=1}^{M+1} e^{-n\eta} \left(\sum_{m=n-1}^{M} \sum_{q=0}^{2(m+1-n)} \sum_{r=0}^{m} a_{m,n}^{q,r} \eta^q \xi^r \right). \quad (3.57)$$

In fact we obtain the following analytic solution of the flow

$$f(\eta,\xi)=\sum_{m=0}^{\infty}f_m(\eta,\xi)=\lim_{M\to\infty}\left[\sum_{m=0}^{M}a_{m,0}^{0,0}+\sum_{n=1}^{M+1}e^{-n\eta}\left(\sum_{m=n-1}^{M}\sum_{q=0}^{2(m+1-n)}\sum_{r=0}^{m}a_{m,n}^{q,r}\eta^q\xi^r\right)\right]. \tag{3.58}$$

3.3 Skin Friction

The shear stress τ on the surface of the stretching sheet is

$$\tau=\sigma_{rz|_{z=0}},$$

and the local skin friction coefficient or frictional drag coefficient is

$$C_f=\frac{\tau}{\rho\,(ar)^2}.$$

We have

$$C_f=\mathrm{Re}_r^{-\frac{1}{2}}\left[f''(\eta,\xi)\right],$$

where

$$\mathrm{Re}_r=\frac{a\xi}{\nu}r^2.$$

3.4 Convergence of the analytic solution

In this section we discuss the convergence of the series solution in Eq. (3.58). For this we first show that if the series (3.58) converges it will converge to the solution of the problem given in Eq. (3.8) and conditions (3.9). Suppose that the auxiliary parameter $\hbar_1$ is chosen in such a way that the series in Eq. (3.58) converges, then

$$\lim_{M\to\infty}f_M(\eta,\xi)=0. \tag{3.59}$$

From Eqs.(3.20) and (3.21)

$$\begin{aligned}
\lim_{M\to\infty}\left[\hbar_1\sum_{m=1}^{M}\mathcal{R}_{m1}(\eta,\xi)\right] &= \lim_{M\to\infty}\sum_{m=1}^{M}\mathcal{L}_1\left[f_m(\eta,\xi)-\chi_m f_{m-1}(\eta,\xi)\right],\\
&= \lim_{M\to\infty}\mathcal{L}_1\left[\sum_{m=1}^{M}f_m(\eta,\xi)-\sum_{m=1}^{M}\chi_m f_{m-1}(\eta,\xi)\right],\\
&= \lim_{M\to\infty}\mathcal{L}_1 f_M(\eta,\xi),\\
&= \mathcal{L}_1\lim_{M\to\infty}f_M(\eta,\xi)=0,\qquad \eta\in(0,\infty).
\end{aligned}\tag{3.60}$$

The above equation implies that the infinite sequence $r_1, r_2, r_3,$ ... converges to zero where

$$r_M=\sum_{m=1}^{M}\mathcal{R}_{m1}(\eta,\xi).$$

Now

$$\begin{aligned}
\sum_{i=1}^{M}\mathcal{R}_{i1}(\eta,\xi) &= \sum_{i=1}^{M}\{\frac{\partial^4 f_{i-1}}{\partial\eta^4}+\frac{\eta\xi(1-\xi)}{2}\frac{\partial^3 f_{i-1}}{\partial\eta^3}+\frac{(1-\xi)}{2}\frac{\partial^2 f_{i-1}}{\partial\eta^2}\\
&\quad -\xi(1-\xi)\frac{\partial^3 f_{i-1}}{\partial\xi\partial\eta^2}+2\xi\sum_{k=0}^{i-1}f_{i-1-k}f_k'''\},
\end{aligned}\tag{3.61}$$

$$\begin{aligned}
\lim_{M\to\infty}\left[\sum_{k=1}^{M}\mathcal{R}_{k1}(\eta,\xi)\right] &= \sum_{i=1}^{\infty}\{\frac{\partial^4 f_{i-1}}{\partial\eta^4}+\frac{\eta\xi(1-\xi)}{2}\frac{\partial^3 f_{i-1}}{\partial\eta^3}+\frac{(1-\xi)}{2}\frac{\partial^2 f_{i-1}}{\partial\eta^2}\\
&\quad -\xi(1-\xi)\frac{\partial^3 f_{i-1}}{\partial\xi\partial\eta^2}+2\xi\sum_{k=0}^{i-1}f_{i-1-k}f_k'''\},
\end{aligned}\tag{3.62}$$

$$\begin{aligned}
&= \frac{\partial^4}{\partial\eta^4}\sum_{i=1}^{\infty}f_{i-1}+\frac{\eta\xi(1-\xi)}{2}\frac{\partial^3}{\partial\eta^3}\sum_{i=1}^{\infty}f_{i-1}+\frac{(1-\xi)}{2}\frac{\partial^2}{\partial\eta^2}\sum_{i=1}^{\infty}f_{i-1}\\
&\quad -\xi(1-\xi)\frac{\partial^3}{\partial\xi\partial\eta^2}\sum_{i=1}^{\infty}f_{i-1}+2\xi\sum_{i=1}^{\infty}\sum_{k=0}^{i-1}f_{i-1-k}\frac{\partial^3 f_{i-1}}{\partial\eta^3},
\end{aligned}\tag{3.63}$$

$$= \frac{\partial^4}{\partial \eta^4}\sum_{k=0}^{\infty} f_k + \frac{\eta\xi(1-\xi)}{2}\frac{\partial^3}{\partial \eta^3}\sum_{k=0}^{\infty} f_k + \frac{(1-\xi)}{2}\frac{\partial^2}{\partial \eta^2}\sum_{k=0}^{\infty} f_k$$
$$-\xi(1-\xi)\frac{\partial^3}{\partial\xi\partial\eta^2}\sum_{k=0}^{\infty} f_k + 2\xi\left[\sum_{k=0}^{\infty} f_k\right]\left[\frac{\partial^3}{\partial\eta^3}\sum_{k=0}^{\infty} f_k\right], \qquad (3.64)$$

and, therefore, the above equation after using Eq. (3.60) reduces to

$$\left[\begin{array}{c} \frac{\partial^4}{\partial \eta^4}\sum_{k=0}^{\infty} f_k + \frac{\eta\xi(1-\xi)}{2}\frac{\partial^3}{\partial \eta^3}\sum_{k=0}^{\infty} f_k + \frac{(1-\xi)}{2}\frac{\partial^2}{\partial \eta^2}\sum_{k=0}^{\infty} f_k \\ -\xi(1-\xi)\frac{\partial^3}{\partial\xi\partial\eta^2}\sum_{k=0}^{\infty} f_k + 2\xi\left[\sum_{k=0}^{\infty} f_k\right]\left[\frac{\partial^3}{\partial\eta^3}\sum_{k=0}^{\infty} f_k\right] \end{array}\right] = 0. \qquad (3.65)$$

From Eq. (3.21), we have

$$\sum_{k=0}^{\infty} f_k(0,\xi) = 0, \quad \sum_{k=0}^{\infty} f_k'(0,\xi) = 0, \quad \sum_{k=0}^{\infty} f_k'(\infty,\xi) = 0. \qquad (3.66)$$

From Eq. (3.64) and (3.65), if the series given in (3.58) converges, it must be a solution of the problem given in Eq. (3.8) subject to the boundary conditions (3.9).

The convergence region and the rate of convergence of the series (3.58) strongly depends upon the value of the auxiliary parameter $\hbar_1$. To investigate the range of admissible values of auxiliary parameter $\hbar_1$ for which the series (3.58) is convergent, we first consider the convergence of the related series such as $f'(0)$, $f''(0)$, $f'''(0)$ and so on. If we plot these series against the parameter $\hbar_1$ the curve obtained in this way is called the $\hbar$-curve. We draw the curve for the series of $f''(0)$. If it gives a straight line parallel to $\hbar$-curve and it will not give any information about the valid values of $\hbar_1$. Then it is necessary to plot the series $f'''(0)$ and so on unless we get a curve other than a straight line. The portion of the $\hbar$-curve which is parallel to the $\hbar$-axis will give the region for the admissible values of $\hbar_1$. The $\hbar$-curves are sketched in Fig. 3.1 for two different orders of approximations. Fig. 3.1 clearly indicates that the range for the admissible values of $\hbar_1$ is $0.1 \leq \hbar_1 \leq 0.5$. Thus, by means of choosing $\hbar_1 = 0.1$, our result shows that the series (3.58) converges and we obtain an accurate analytic solution valid for all time $0 \leq \tau < \infty$

in whole region $0 \le \eta < \infty$.

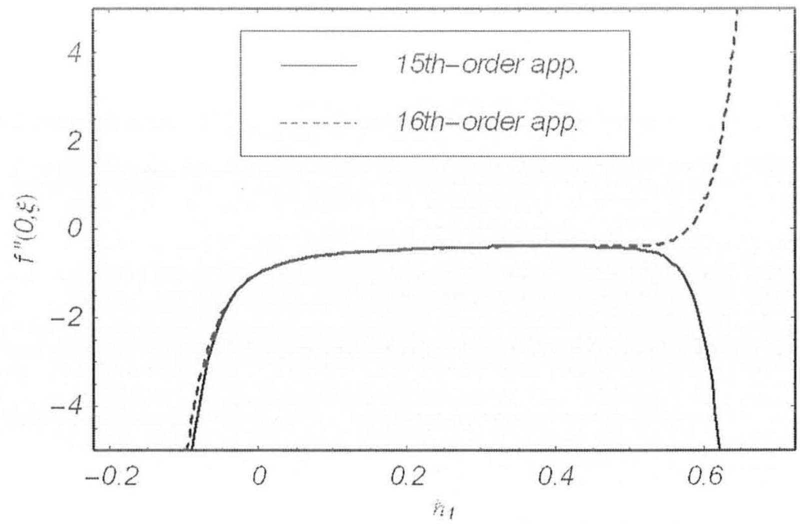


Fig. 3.1. $\hbar$-curves for different orders of approximations when $\xi = 0.5$.

3.5 Results and discussion

In this section, the influence of time is discussed on the velocity components and the skin friction coefficient. Such effects have been discussed through Figs. 3.2, 3.3 and Table 3.1. Fig. 3.1 shows that the r-component of velocity and the thermal boundary layer thickness increase when dimensionless time τ is increased. However with the increase in dimensionless time τ, the z-component of velocity decreases and the thermal boundary layer thickness increases as shown in Fig. 3.3. It also depicts that one can obtain a velocity profile for all the times and the desired solution is valid for all times. The values of skin friction coefficient are tabulated in Table 3.1. It is found that magnitude of skin friction increases by increasing time τ and figure 3.4 are given for the skin friction in the case of viscous fluid.

τ	$C_f \operatorname{Re}_r^{\frac{1}{2}}$
0.01	−0.558759
0.10	−0.671632
0.25	−0.734321
0.50	−0.840094
1.00	−0.926639
10.0	−0.958742

Table 3.1. Skin friction coefficient $C_f \operatorname{Re}_r^{\frac{1}{2}}$ for different values of time when $\hbar = 0.1$.

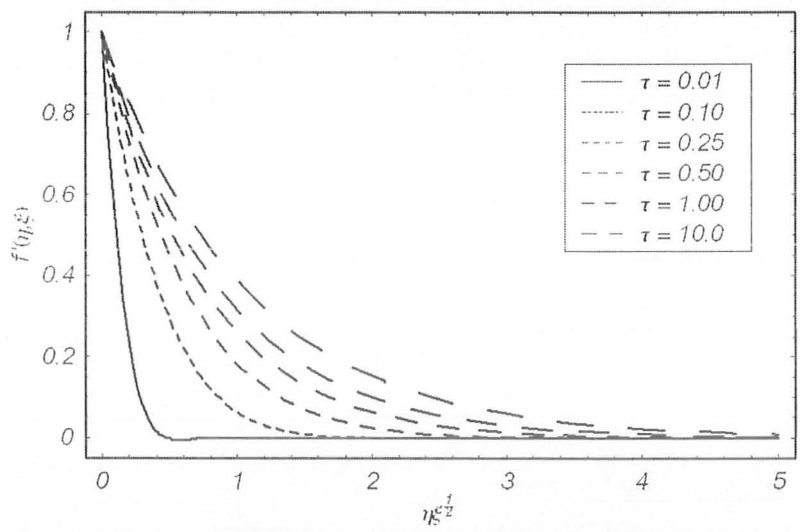


Fig. 3.2. The velocity profile $f'(\eta, \xi)$ at different dimensionless time $\tau = at$ when $\hbar_1 = 0.1$.

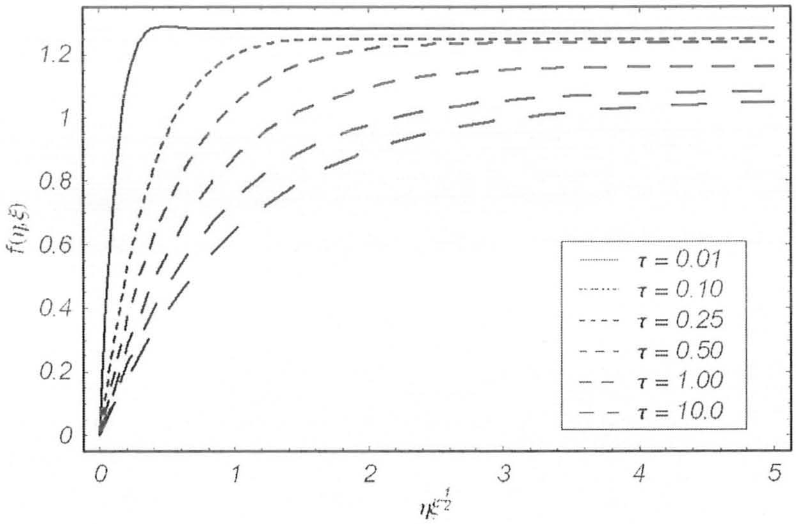


Fig. 3.3. The velocity profile $f(\eta, \xi)$ at different dimensionless time $\tau = at$ when $\hbar_1 = 0.1$.

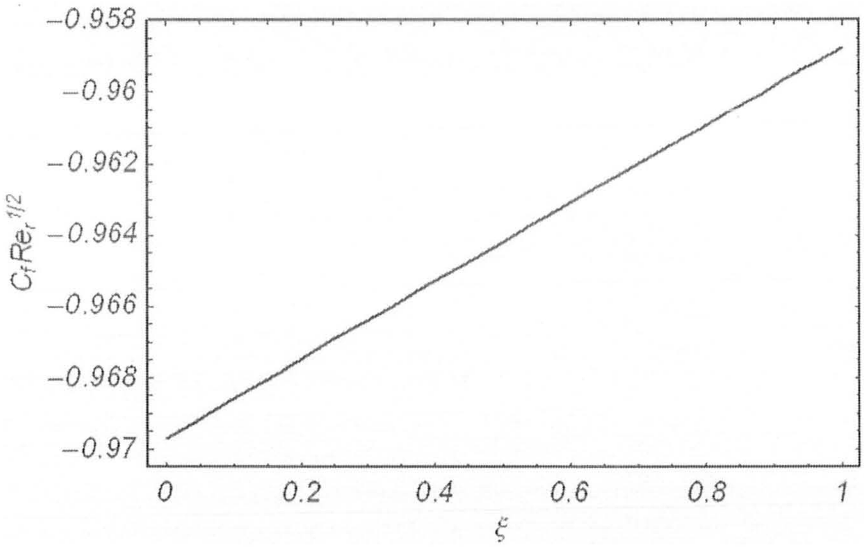


Fig. 3.4. The variations $\mathrm{Re}_x^{1/2} C_f$ for the viscous case .

Chapter 4

Unsteady axisymmetric flow of a second grade fluid over a radially stretching sheet

The aim of this chapter is to analyze the flow problem of the previous chapter for a second grade fluid. An analytic solution by HAM is presented for computing the axisymmetric flow induced by a radially stretching sheet. The obtained analysis is valid for all values of rheological parameters and time. Also the convergence of the solution is discussed and the effects of material moduli is highlighted.

4.1 Mathematical description of the problem

The geometry of the problem is same as in the previous chapter. The difference lies in the consideration of the constitutive equation of a second grade fluid. The equations which govern the flow are the incompressibility condition in Eq. (3.1) . The constitutive equations for a second grade fluid is defined in Eq.(2.2) . The incompressibility condition is automatically satisfied and from Eqs.(2.2) and (2.11) we have:

$$
\begin{aligned}
\rho\left(\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial r}+w\frac{\partial u}{\partial z}\right) &= -\frac{\partial \widehat{p}}{\partial r}+\mu\left(\frac{\partial^2 u}{\partial r^2}+\frac{\partial^2 u}{\partial z^2}+\frac{1}{r}\frac{\partial u}{\partial r}-\frac{u}{r^2}\right) \\
&+\alpha_1\begin{bmatrix}
\frac{2}{r}\frac{\partial^2 u}{\partial r\partial t}-\frac{2}{r^2}\frac{\partial u}{\partial t}+\frac{2u^2}{r^3} \\
-\frac{2w}{r^2}\frac{\partial u}{\partial z}-\frac{1}{r}\left(\frac{\partial u}{\partial z}\right)^2-\frac{\partial u}{\partial z}\frac{\partial^2 w}{\partial z^2} \\
+w\frac{\partial^3 u}{\partial z^3}-\frac{2u}{r^2}\frac{\partial u}{\partial r}+\frac{\partial u}{\partial r}\frac{\partial^2 u}{\partial z^2} \\
+\frac{\partial w}{\partial r}\frac{\partial^2 w}{\partial z^2}\frac{1}{r}\left(\frac{\partial w}{\partial r}\right)^2+\frac{2w}{r}\frac{\partial^2 u}{\partial r\partial z} \\
+\frac{\partial w}{\partial r}\frac{\partial^2 u}{\partial r\partial z}+2\frac{\partial w}{\partial z}\frac{\partial^2 w}{\partial r\partial z}-\frac{\partial u}{\partial r}\frac{\partial^2 w}{\partial r\partial z} \\
+u\frac{\partial^3 u}{\partial r\partial z^2}+w\frac{\partial^3 w}{\partial r\partial z^2}+\frac{2u}{r}\frac{\partial^2 u}{\partial r^2} \\
+2\frac{\partial u}{\partial r}\frac{\partial^2 u}{\partial r^2}+\frac{\partial u}{\partial z}\frac{\partial^2 w}{\partial r^2}+2\frac{\partial w}{\partial r}\frac{\partial^2 w}{\partial r^2} \\
+2w\frac{\partial^3 u}{\partial r^2\partial z}+u\frac{\partial^3 w}{\partial r^2\partial z}+2u\frac{\partial^3 u}{\partial r^3}
\end{bmatrix},
\end{aligned}
\tag{4.1}
$$

$$
\begin{aligned}
\rho\left(\frac{\partial w}{\partial t}+u\frac{\partial w}{\partial r}+w\frac{\partial w}{\partial z}\right) &= -\frac{\partial \widehat{p}}{\partial z}+\mu\left(\frac{\partial^2 w}{\partial r^2}+\frac{\partial^2 w}{\partial z^2}+\frac{1}{r}\frac{\partial w}{\partial r}\right) \\
&+\alpha_1\begin{bmatrix}
\frac{1}{r}\frac{\partial^2 u}{\partial z\partial t}+\frac{1}{r}\frac{\partial^2 w}{\partial r\partial t}-\frac{1}{r}\frac{\partial u}{\partial z}\frac{\partial w}{\partial z} \\
+\frac{w}{r}\frac{\partial^2 u}{\partial z^2}+2\frac{\partial u}{\partial z}\frac{\partial^2 u}{\partial z^2}+2\frac{\partial w}{\partial z}\frac{\partial^2 w}{\partial z^2} \\
+2w\frac{\partial^3 w}{\partial z^3}+\frac{1}{r}\frac{\partial u}{\partial z}\frac{\partial u}{\partial r}+\frac{1}{r}\frac{\partial w}{\partial z}\frac{\partial w}{\partial r} \\
+\frac{\partial w}{\partial r}\frac{\partial^2 u}{\partial z^2}-\frac{1}{r}\frac{\partial u}{\partial r}\frac{\partial w}{\partial r}+\frac{u}{r}\frac{\partial^2 u}{\partial r\partial z} \\
-\frac{\partial w}{\partial z}\frac{\partial^2 u}{\partial r\partial z}+2\frac{\partial u}{\partial r}\frac{\partial^2 u}{\partial r\partial z}+\frac{w}{r}\frac{\partial^2 w}{\partial r\partial z} \\
+\frac{\partial u}{\partial z}\frac{\partial^2 w}{\partial r\partial z}+w\frac{\partial^3 u}{\partial r\partial z^2}+2u\frac{\partial^3 w}{\partial r\partial z^2} \\
+\frac{\partial u}{\partial z}\frac{\partial^2 u}{\partial r^2}-\frac{\partial w}{\partial r}\frac{\partial^2 u}{\partial r^2}+\frac{u}{r}\frac{\partial^2 w}{\partial r^2} \\
+\frac{\partial w}{\partial z}\frac{\partial^2 w}{\partial r^2}+u\frac{\partial^3 u}{\partial r^2\partial z} \\
+w\frac{\partial^3 w}{\partial r^2\partial z}+u\frac{\partial^3 w}{\partial r^3}
\end{bmatrix}.
\end{aligned}
\tag{4.2}
$$

Note that the above equation reduces to Eqs. (3.2) and (3.3) of Newtonian fluid when $\alpha_1 = 0$. Furthermore, the boundary conditions of the problem are given in Eq.(3.4).

Upon making use of transformations (3.5) and then eliminating pressure from the resulting equations we arrive at

$$
f^{iv}+\frac{1-\xi}{2}\left(f''+\eta f'''\right)-\xi\left(1-\xi\right)\frac{\partial f''}{\partial \xi}+2\xi f f'''-2\alpha f f^{v}=0. \tag{4.3}
$$

The boundary conditions in terms of f are given in (3.9) and

$$\alpha = \frac{a\alpha_1}{\mu}.$$

4.2 Solution by HAM

4.2.1 Zeroth-order deformation problems

The velocity distribution $f(\eta, \xi)$ can be expressed by the set of base functions of the form

$$\left\{ \eta^k \xi^j \exp(-n\eta) \middle| k \geq 0, j \geq 0, n \geq 0 \right\} \tag{4.4}$$

in the form of the following series

$$f(\eta, \xi) = \sum_{m=0}^{\infty} f_m(\eta, \xi)$$

where

$$f_m(\eta, \xi) = b_{0,0}^{0,0} + \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} b_{m,n}^{k,j} \eta^k \xi^j \exp(-n\eta), \tag{4.5}$$

in which $b_{m,n}^{k,j}$ are coefficients. Invoking the so-called *Rule of solution expressions* for $f(\eta, \xi)$ and Eqs. (4.3) and (3.9). We have chosen the same initial guess as given in Eq. (3.10) and auxiliary linear operator $\mathcal{L}_1$ given in Eq. (3.11). Equations (4.3) show that the nonlinear operators here is:

$$\begin{aligned} \mathcal{N}_2\left[\overline{f}(\eta, \xi; p)\right] &= \frac{\partial^4 \overline{f}(\eta, \xi, p)}{\partial \eta^4} + \frac{\eta \xi (1-\xi)}{2} \frac{\partial^3 \overline{f}(\eta, \xi, p)}{\partial \eta^3} \\ &+ 2\xi \overline{f}(\eta, \xi, p) \frac{\partial^3 \overline{f}(\eta, \xi, p)}{\partial \eta^3} + \frac{(1-\xi)}{2} \frac{\partial^2 \overline{f}(\eta, \xi, p)}{\partial \eta^2} \\ &- \xi(1-\xi) \frac{\partial^3 \overline{f}(\eta, \xi, p)}{\partial \xi \partial \eta^2} - 2\alpha \overline{f}(\eta, \xi, p) \frac{\partial^5 \overline{f}(\eta, \xi, p)}{\partial \eta^5}. \end{aligned} \tag{4.6}$$

If $\hbar$ is the auxiliary nonzero parameter then the zero order deformation problem satisfies

$$(1-p)\mathcal{L}_1\left[\overline{f}(\eta, \xi, p) - f_0(\eta)\right] = p\hbar_2 \mathcal{N}_2\left[\overline{f}(\eta, \xi, p)\right], \tag{4.7}$$

For $p = 0$ and $p = 1$, we have

$$\overline{f}(\eta, \xi, 0) = f_0(\eta), \qquad \overline{f}(\eta, \xi, 1) = f(\eta, \xi). \tag{4.8}$$

The initial guesses $f_0(\eta)$ approaches $f(\eta, \xi)$ as p varies from 0 to 1. Through Taylor's series expansion we can write

$$\overline{f}(\eta, \xi, p) = f_0(\eta) + \sum_{m=1}^{\infty} f_m(\eta, \xi)\; p^m, \tag{4.9}$$

where

$$f_m(\eta, \xi) = \frac{1}{m!} \frac{\partial^m \overline{f}(\eta, \xi, p)}{\partial p^m}\bigg|_{p=0} \tag{4.10}$$

and the convergence of the series (4.9) depends upon $\hbar_2$. The values of $\hbar_2$ are chosen in such a way that the series (4.9) are convergent at $p = 1$. Then by using Eq. (4.8) one obtains

$$f(\eta, \xi) = f_0(\eta) + \sum_{m=1}^{\infty} f_m(\eta, \xi), \tag{4.11}$$

4.2.2 mth-order deformation problems

Here we first differentiate Eq. (4.7) m times with respect to p then divide by $m!$ and setting $p = 0$ we get

$$\mathcal{L}_2 \left[f_m(\eta, \xi) - \chi_m f_{m-1}(\eta, \xi) \right] = \hbar_2 \mathcal{R}_{m2}(\eta, \xi), \tag{4.12}$$

subject to conditions (3.21) and

$$\begin{aligned} \mathcal{R}_{m2} \;&=\; \frac{\partial^4 f_{m-1}}{\partial \eta^4} + \frac{\eta \xi (1-\xi)}{2} \frac{\partial^3 f_{m-1}}{\partial \eta^3} + \frac{(1-\xi)}{2} \frac{\partial^2 f_{m-1}}{\partial \eta^2} - \xi (1-\xi) \frac{\partial^3 f_{m-1}}{\partial \xi \partial \eta^2} \\ &\quad + 2\xi \sum_{k=0}^{m-1} f_{m-1-k} f_k^{'''} - 2\alpha \sum_{k=0}^{m-1} f_{m-1-k} f_k^{v}. \end{aligned} \tag{4.13}$$

The general solutions of Eqs. (4.12) subject to (3.21) are

$$f_m(\eta, \xi) = f_m^{\star}(\eta, \xi) + C_1 \exp(-\eta) + C_2 \exp(\eta) + C_3, \tag{4.14}$$

where $f_m^\star(\eta,\xi)$ are the particular solutions and the constants are determined by the boundary conditions (3.21) which are given by

$$C_2 = 0, \quad C_1 = \left.\frac{\partial f_m^\star(\eta,\xi)}{\partial \eta}\right|_{\eta=0}, \quad C_3 = -C_1 - f_m^\star(0,\xi). \tag{4.15}$$

The linear non-homogeneous Eqs. (4.9) – (4.13) are solved using MATHEMATICA in the order $m = 1, 2, 3, ...$ given by the initial guess.

4.3 Skin Friction

The shear stress τ on the surface of the stretching sheet is

$$\tau = \sigma_{rz|_{z=0}}, \tag{4.16}$$

and the local skin friction coefficient or frictional drag coefficient is

$$C_f = \frac{\tau}{\rho\,(ar)^2}. \tag{4.17}$$

Equation (4.17) can be written as

$$C_f = \mathrm{Re}_r^{-\frac{1}{2}}\left[f''(\eta,\xi) + 2\alpha\left(f'(\eta,\xi)\,f''(\eta,\xi) - f(\eta,\xi)\,f''(\eta,\xi)\right)\right], \tag{4.18}$$

where

$$\mathrm{Re}_r = \frac{a\xi}{\nu}r^2.$$

4.4 Convergence of the analytic solution

As long as a solution series given by the homotopy method converges, it must be one of the solution. So, it is important to ensure the convergence of the solution series. In this section, we discuss the convergence of the series which contains the auxiliary parameter $\hbar_2$. The values of $\hbar_2$ determines the convergence region and rate of approximation for the homotopy analysis method. The auxiliary parameter $\hbar_2$ provides us with a simple way to ensure the convergence of the series solution. For this purpose $\hbar$-curves are sketched in Fig. 4.1 for two different orders of approximations. Fig. 4.1 clearly indicates that the range for the admissible values of $\hbar_2$ is $0 \leq \hbar_2 \leq 0.6$. Thus, by means of choosing $\hbar_2 = 0.2$, we obtain an accurate analytic solution valid for all time $0 \leq \tau < \infty$ in whole region $0 \leq \eta < \infty$.

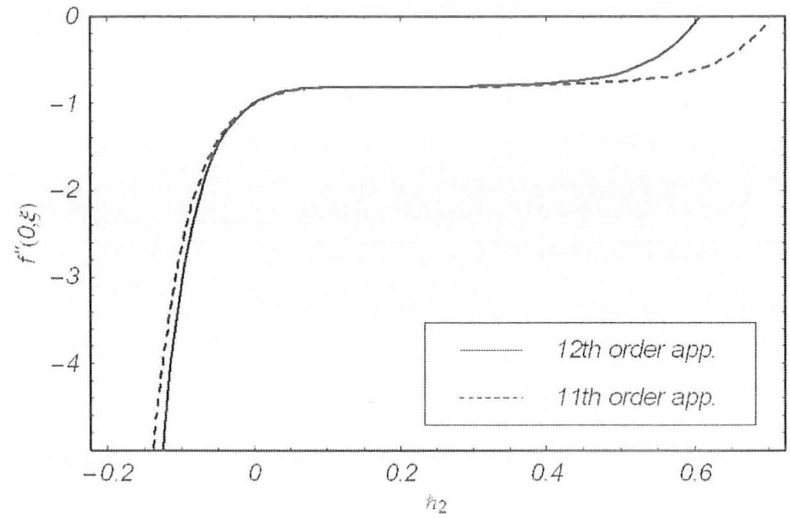


Fig. 4.1. $\hbar-$curve for different order of approximations when $\xi = 0.5$.

4.5 Results and discussion

In this section, the influence of dimensionless time τ and non-Newtonian parameter α is seen on the velocity components. For this purpose, we plotted the graphs $4.2 - 4.5$. It is observed that when dimensionless time τ increases, the r-component of velocity and the boundary layer thickness increases. However the z-component of velocity decreases and the boundary layer thickness increases with an increase in τ. The behavior of velocity profile for different values of second grade parameter α is displayed in Figs.4.4 and 4.5. Figure 4.4 indicates that the r-component of velocity increases and the boundary layer thickness increases with an increase in α.

However the z-component of velocity decreases and layer thickness increases by increasing the values of the second grade parameter α. Fig. 4.6 display the effects of skin frictions for different order of second grade parameter α.

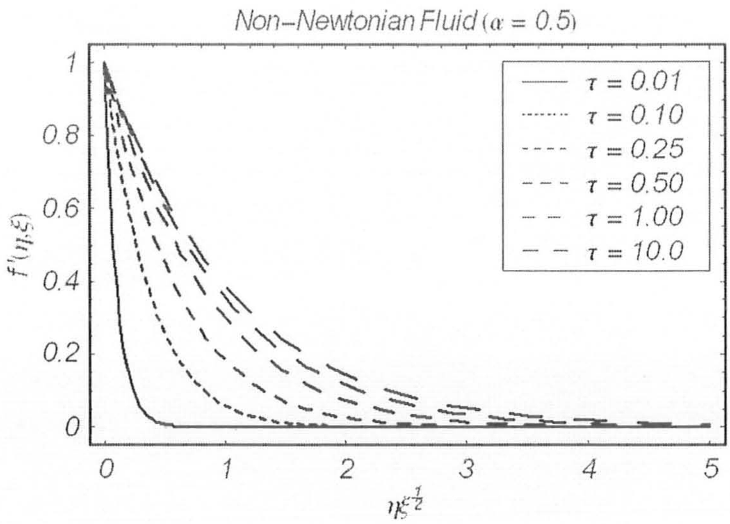


Fig. 4.2. Variations of velocity field f' with increasing non-dimensional time τ.

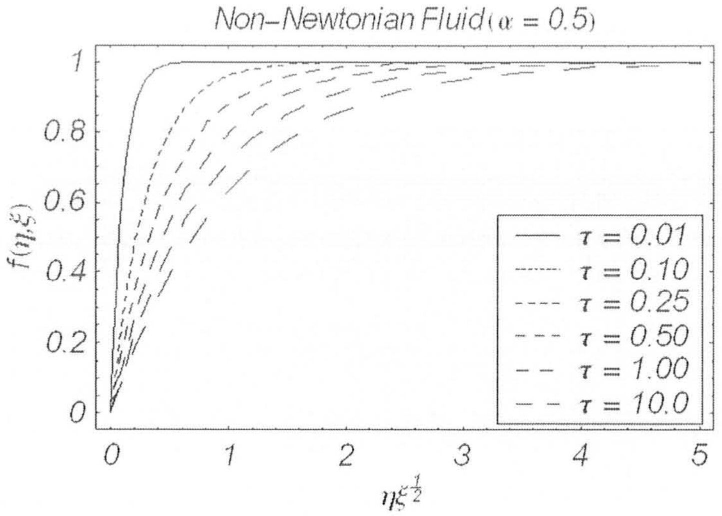


Fig. 4.3. Variations of velocity field f with increasing non-dimensional time τ.

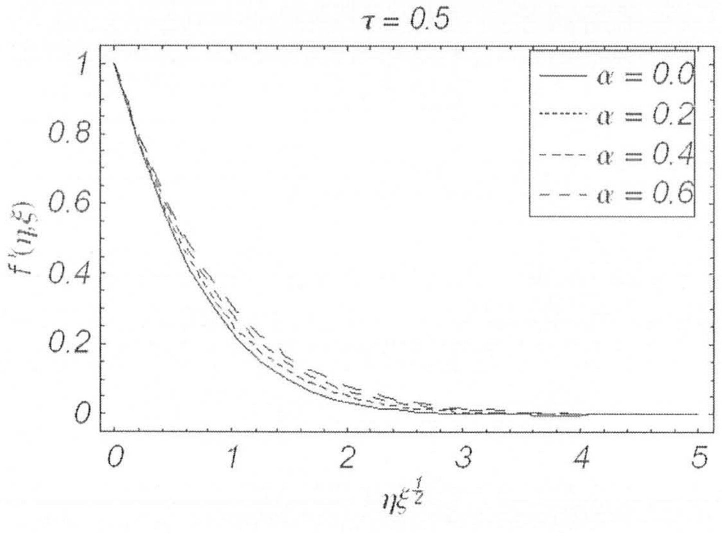


Fig. 4.4. Variations of velocity field f' with increasing non-dimensional α.

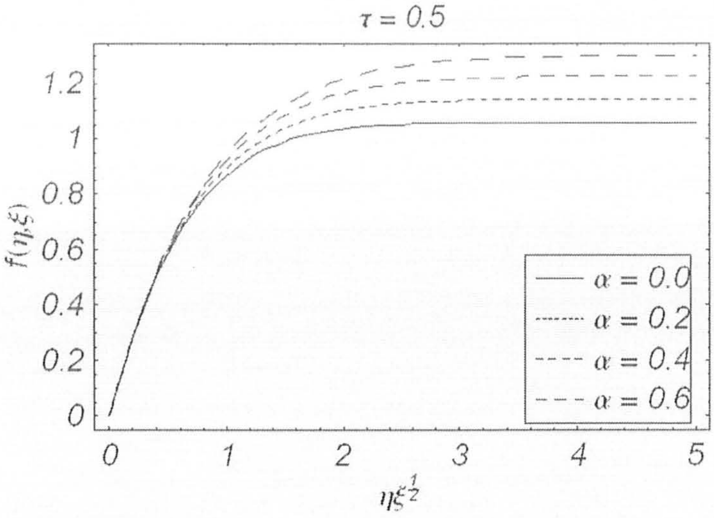


Fig. 4.5. Variations of velocity field f with increasing non-dimensional α.

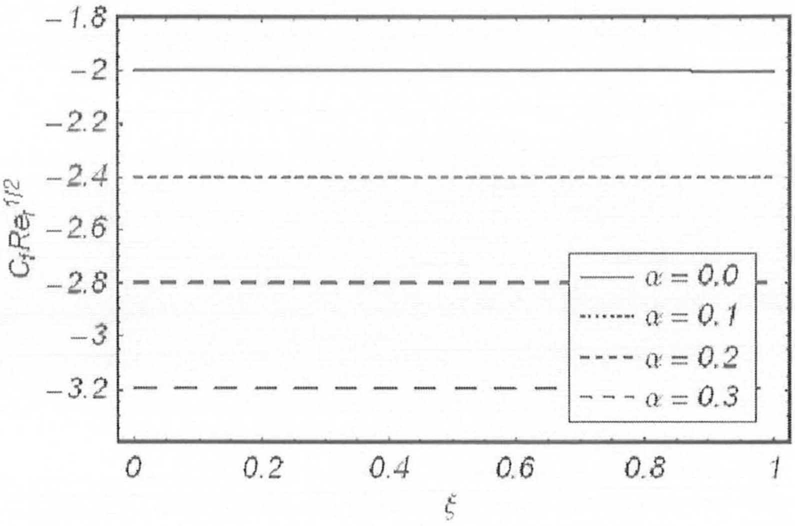


Fig. 4.6. The variations $\mathrm{Re}_x^{1/2} C_f$ for different values of α.

Chapter 5

Boundary layer flow of an unsteady second grade fluid due to a stretching sheet

The unsteady laminar flow of a second grade fluid due to a stretching sheet has been investigated in this chapter. The flow is governed by a third order boundary value problem whose analytic solution is obtained in terms of non-dimensional second grade parameter. Analytic solution valid for all time has been derived by employing HAM. Finally, the convergence is developed and influence of various parameters of interest is examined.

5.1 Flow analysis

Consider the two-dimensional unsteady boundary layer flow of a second grade fluid over a stretching sheet placed in the XOZ- plane and moving with a velocity ax in the x-direction, a being a constant. The continuity equation (2.10) and the momentum equations after using Eqs. (2.2) and (2.11) yields

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0, \tag{5.1}$$

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=\upsilon\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right)+\frac{\alpha_1}{\rho}\left\{\begin{array}{c} 5\frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial x^2}+\frac{\partial^3 u}{\partial x^2\partial t}+u\frac{\partial^3 u}{\partial x^3} \\ +v\frac{\partial^3 u}{\partial x^2\partial y}+u\frac{\partial^3 u}{\partial y^2\partial x}+\frac{\partial u}{\partial y}\frac{\partial^2 u}{\partial x\partial y} \\ +\frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial y^2}+\frac{\partial^3 u}{\partial y^2\partial t}+v\frac{\partial^3 u}{\partial y^3} \\ +\frac{\partial u}{\partial y}\frac{\partial^2 v}{\partial x^2}+2\frac{\partial v}{\partial x}\frac{\partial^2 v}{\partial x^2}\end{array}\right\}, \qquad (5.2)$$

$$\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}=\upsilon\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}\right)+\frac{\alpha_1}{\rho}\left\{\begin{array}{c} 2\frac{\partial u}{\partial y}\frac{\partial^2 u}{\partial y^2}+\frac{\partial v}{\partial x}\frac{\partial^2 u}{\partial y^2}+\frac{\partial^3 v}{\partial x^2\partial t} \\ +u\frac{\partial^3 v}{\partial x^3}+v\frac{\partial^3 v}{\partial x^2\partial y}+\frac{\partial v}{\partial x}\frac{\partial^2 v}{\partial x\partial y} \\ +u\frac{\partial^3 v}{\partial y^2\partial x}+\frac{\partial v}{\partial y}\frac{\partial^2 v}{\partial x^2}+5\frac{\partial v}{\partial y}\frac{\partial^2 v}{\partial y^2} \\ +v\frac{\partial^3 v}{\partial y^3}+\frac{\partial^3 v}{\partial y^2\partial t}\end{array}\right\}, \qquad (5.3)$$

where υ is a kinematic viscosity. It is clear that the above equations are different from those obtained for a radially stretching sheet in the previous chapter. Under the usual boundary layer arguments that u, t, x $\frac{\partial u}{\partial x}$, $\frac{\partial^2 u}{\partial x^2}$, be $O(1)$ and y, v be $O(\delta)$ yields the flow governed through Eqs. (5.1) – (5.3) as

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=\nu\frac{\partial^2 u}{\partial y^2}+\frac{\alpha_1}{\rho}\left[u\frac{\partial^3 u}{\partial x\partial y^2}+\frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial y^2}+\frac{\partial u}{\partial y}\frac{\partial^2 u}{\partial x\partial y}+v\frac{\partial^3 u}{\partial y^3}+\frac{\partial^3 u}{\partial y^2\partial t}\right]. \qquad (5.4)$$

The relevant boundary conditions of the flow are:

$$\begin{aligned} u &= ax, \quad v=0 \qquad \text{at } y=0, \\ u &\to 0 \qquad\qquad \text{as } y\to\infty. \end{aligned} \qquad (5.5)$$

Note that ν and α_1/ρ being $O(\delta^2)$ and the terms of order $O(\delta)$ are neglected (where δ being the boundary layer thickness).

Introducing

$$u=axg'(\xi,\eta), \qquad v=-\sqrt{a\nu\xi}g(\xi,\eta), \qquad \eta=\sqrt{\frac{a}{\nu\xi}}y, \quad \xi=1-e^{-at}, \qquad (5.6)$$

equations. (5.4) – (5.5) takes the following form

$$\left[\begin{array}{c}\{\xi-\alpha(1-\xi)\}g'''+\frac{\eta\xi(1-\xi)g''}{2}-\xi^2(1-\xi)\frac{\partial g'}{\partial \xi}+\alpha(1-\xi)\left\{\xi\frac{\partial g'''}{\partial\xi}-\frac{\eta}{2}g^{iv}\right\} \\ +\xi^2\left(-g'^2+gg''\right)+\alpha\xi\left(g''^2+2g'g'''-gg^{iv}\right)\end{array}\right]=0, \qquad (5.7)$$

with the boundary conditions

$$\begin{aligned} g &= 0, \quad g' = 1 \quad \text{at } \eta = 0, \\ g' &\to 0 \quad \text{as} \quad \eta \to \infty. \end{aligned} \tag{5.8}$$

5.2 Exact analytic solution

The velocity distribution $g(\eta, \xi)$ can be expressed by the set of base functions of the form

$$\left\{ \eta^k \xi^j \exp(-n\eta) \middle| \, k \geq 0, j \geq 0, n \geq 0 \right\} \tag{5.9}$$

in the form of the following series

$$g(\eta, \xi) = \sum_{m=0}^{\infty} g_m(\eta, \xi)$$

where

$$g_m(\eta, \xi) = c_{0,0}^{0,0} + \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} c_{m,n}^{k,j} \eta^k \xi^j \exp(-n\eta), \tag{5.10}$$

in which $c_{m,n}^{k,j}$ are coefficients. Invoking the so-called *Rule of solution expressions* for $g(\eta, \xi)$ and Eqs. (5.7) and (3.9) we have used the same initial guess and linear operator as in the previous chapter. Eq. (5.7) suggests that

$$\begin{aligned} \mathcal{N}_3 [\widehat{g}(\eta, \xi; p)] &= [\{\xi - \alpha(1-\xi)\} \frac{\partial^3 \widehat{g}}{\partial \eta^3} + \frac{\eta}{2} \xi (1-\xi) \frac{\partial^2 \widehat{g}}{\partial \eta^2} - \xi^2 (1-\xi) \frac{\partial^2 \widehat{g}}{\partial \eta \partial \xi} \\ &+ \alpha (1-\xi) \{\xi \frac{\partial^4 \widehat{g}}{\partial \eta^3 \partial \xi} - \frac{\eta}{2} \frac{\partial \widehat{g}}{\partial \eta^4}\} + \xi^2 \{-\left(\frac{\partial \widehat{g}}{\partial \eta}\right)^2 + \widehat{g} \frac{\partial^2 \widehat{g}}{\partial \eta^2}\} + \\ &\alpha \xi \{\left(\frac{\partial^2 \widehat{g}}{\partial \eta^2}\right)^2 + 2 \frac{\partial \widehat{g}}{\partial \eta} \frac{\partial^3 \widehat{g}}{\partial \eta^3} - \widehat{g} \frac{\partial^4 \widehat{g}}{\partial \eta^4}\}]. \end{aligned} \tag{5.11}$$

Let $\hbar_3$ is an auxiliary nonzero parameter the zero-order deformation equation

$$(1-p) \mathcal{L}_1 [\widehat{g}(\eta, \xi; p) - g_0(\eta, \xi)] = p \hbar_3 \mathcal{N}_3 [\widehat{g}(\eta, \xi; p)], \tag{5.12}$$

is subject to the following boundary conditions

$$\widehat{g}(0,\xi;p)=0, \quad \frac{\partial \widehat{g}(\eta,\xi;p)}{\partial \eta}|_{\eta=0}=1, \quad \frac{\partial \widehat{g}(\eta,\xi;p)}{\partial \eta}|_{\eta=+\infty}=0. \tag{5.13}$$

As p increases from 0 to 1, $\widehat{g}(\eta,\xi;p)$ varies from the initial guess $g_0(\eta,\xi)$ to the solution $g(\eta,\xi)$. For $p=0$ and $p=1$, one can write

$$\widehat{g}(\eta,\xi;0)=g_0(\eta,\xi), \quad \widehat{g}(\eta,\xi;1)=g(\eta,\xi). \tag{5.14}$$

According to Taylor's series

$$\widehat{g}(\eta,\xi;p)=g(\eta,\xi;0)+\sum_{m=1}^{+\infty} g_m(\eta,\xi)\,p^m, \tag{5.15}$$

where

$$g_m(\eta,\xi)=\frac{1}{m!}\frac{\partial^m \widehat{g}(\eta,\xi;p)}{\partial p^m}\bigg|_{p=0}, \tag{5.16}$$

converges at $p=1$. Then, we have

$$g(\eta,\xi)=g(\eta,\xi)+\sum_{n=1}^{+\infty} g_n(\eta,\xi). \tag{5.17}$$

mth-order deformation equations

$$\mathcal{L}_1\left[g_m(\eta,\xi)-\chi_m g_{m-1}(\eta,\xi)\right]=\hbar_3 \mathcal{R}_{m3}(\eta,\xi), \tag{5.18}$$

subject to the boundary conditions

$$g_m(0,\xi;p)=0, \quad \frac{\partial g_m(\eta,\xi;p)}{\partial \eta}|_{\eta=0}=0, \quad \frac{\partial g_m(\eta,\xi;p)}{\partial \eta}|_{\eta=+\infty}=0, \tag{5.19}$$

where

$$\mathcal{R}_{m3}(\eta,\xi) = [\{\xi-\alpha(1-\xi)\}\frac{\partial^3 g_{m-1}}{\partial\eta^3}+\frac{\eta}{2}\xi(1-\xi)\frac{\partial^2 g_{m-1}}{\partial\eta^2}-\xi^2(1-\xi)\frac{\partial^2 g_{m-1}}{\partial\eta\partial\xi}$$
$$+\alpha(1-\xi)\{\xi\frac{\partial^4 g_{m-1}}{\partial\eta^3\partial\xi}-\frac{\eta}{2}\frac{\partial^4 g_{m-1}}{\partial\eta^4}\}+\xi^2\{-\sum_{k=0}^{m-1} g''_{m-1-k}g''_k+\sum_{k=0}^{m-1} g_{m-1-k}g''_k\}+$$
$$\alpha\xi\{\sum_{k=0}^{m-1} g''_{m-1-k}g''_k+2\sum_{k=0}^{m-1} g'_{m-1-k}g'''_k-\sum_{k=0}^{m-1} g_{m-1-k}g''''_k\}]. \tag{5.20}$$

The general solutions of Eqs. (5.18) subjected to (5.19) are

$$g_m(\eta,\xi)=g_m^{\star}(\eta,\xi)+C_1\exp(-\eta)+C_2\exp(\eta)+C_3, \tag{5.21}$$

where $g_m^{\star}(\eta,\xi)$ are the particular solutions and the constants are determined by the boundary conditions (3.21) which are given by

$$C_2=0,\quad C_1=\left.\frac{\partial g_m^{\star}(\eta,\xi)}{\partial\eta}\right|_{\eta=0},\quad C_3=-C_1-g_m^{\star}(0,\xi). \tag{5.22}$$

The linear non-homogeneous Eqs. (5.18) – (5.20) are solved using MATHEMATICA in the order $m=1,2,3,...$ given by the initial guess

5.3 Skin Friction

The shear stress τ_w on the surface of stretching sheet is

$$\tau_w=\sigma_{xy}\mid_{y=0}, \tag{5.23}$$

and the local skin friction coefficient or frictional drag is

$$Cg=\frac{\tau_\omega}{\rho(ax)^2}, \tag{5.24}$$

or

$$\mathrm{Re}_x^{\frac{1}{2}}Cg=g''-\alpha\{\frac{1-\xi}{2\xi}\left(g'-\eta g'''\right)+(1-\xi)\frac{\partial g''}{\partial\xi}+3g'g''-gg'''\}\mid_{\eta=0}, \tag{5.25}$$

where

$$\mathrm{Re}_x = \frac{a\xi}{\nu}x^2.$$

5.4 Convergence

In this section, we discuss the convergence of the series which contains the auxiliary parameter $\hbar_3$.The values of $\hbar_3$ determine the convergence region and rate of approximation for the homotopy analysis method. The $\hbar$-curves are sketched in Fig. 5.1 for 10th-order of approximation. Fig. 5.1 clearly indicates that the range for the admissible values of $\hbar_3$ is $-2 \leq \hbar_3 \leq -0.5$. By means of choosing $\hbar_3 = -1.0$, we obtain an accurate analytic solution valid for all time $0 \leq \tau < \infty$ in whole region $0 \leq \eta < \infty$.

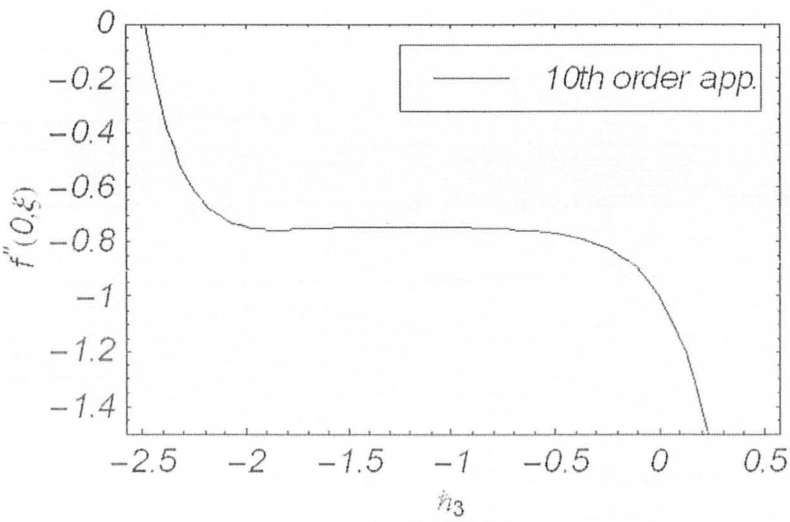


Fig. 5.1. $\hbar$-curve for 10th order of approximation when $\xi = 0.5$.

5.5 Results and discussion

This section explains the variations of dimensionless time τ and the second grade fluid parameter α on the velocity components and skin friction coefficient. Such variations have been discussed through Figs. 5.2 – 5.8. The purpose of Figs. 5.2 – 5.5 is to see the variations of τ on the velocity components f and f' in the Newtonian and second grade cases. From these figures it is noted that when τ increases the x-component of velocity and the boundary layer thickness increases for Newtonian and second grade fluids but the variation is large in magnitude in the case of second grade fluid case. This shows that the parameter α enhances the effects of the dimensionless time. However the y-component of velocity decreases and boundary layer thickness increases with an increase in τ for Newtonian and second grade fluids as shown in Figs. 5.3 – 5.5. In order to see the effects of second grade parameter α on the velocity components f and f', Figs. 5.6 and 5.7 are prepared. From these Figs. it is obvious that when we increase the values of second grade parameter α, the x component of velocity and boundary layer thickness increases. However the y component of velocity increases and the boundary layer thickness decreases. Fig. 5.8 is plotted just to see the variation of skin friction coefficient under the influence of second grade parameter and the dimensionless time. This Fig. elucidates that the magnitude of skin friction coefficient decreases with an increase in the second grade parameter.

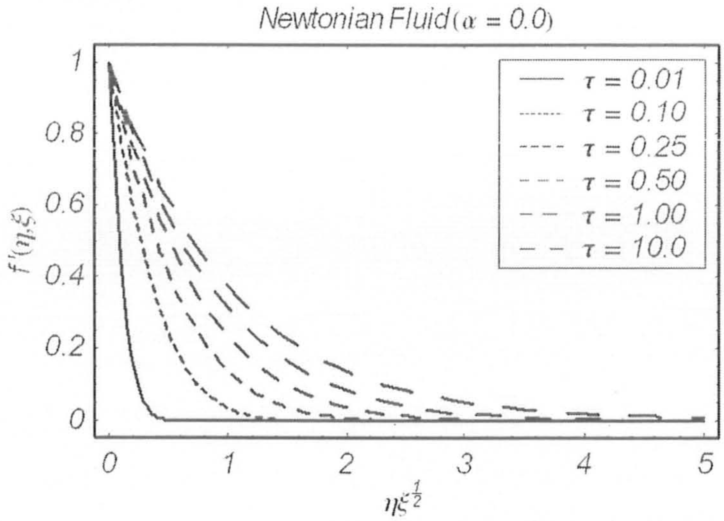


Fig. 5.2. The velocity profile $f'(\eta, \xi)$ at different dimensionless time τ.

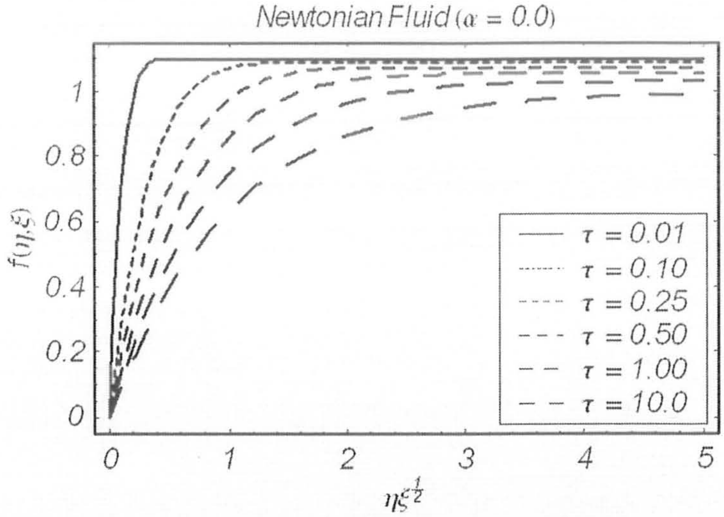


Fig. 5.3. The velocity profile $f(\eta, \xi)$ at different dimensionless time τ.

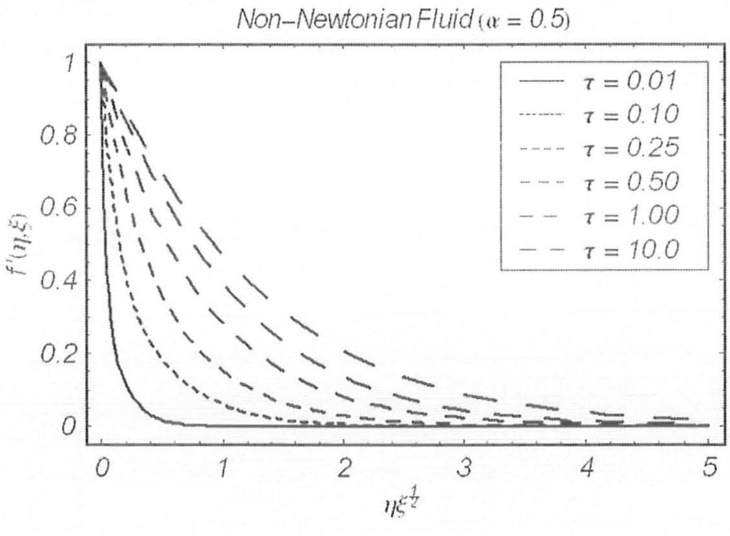


Fig. 5.4. The velocity profile $f'(\eta, \xi)$ at different dimensionless time τ.

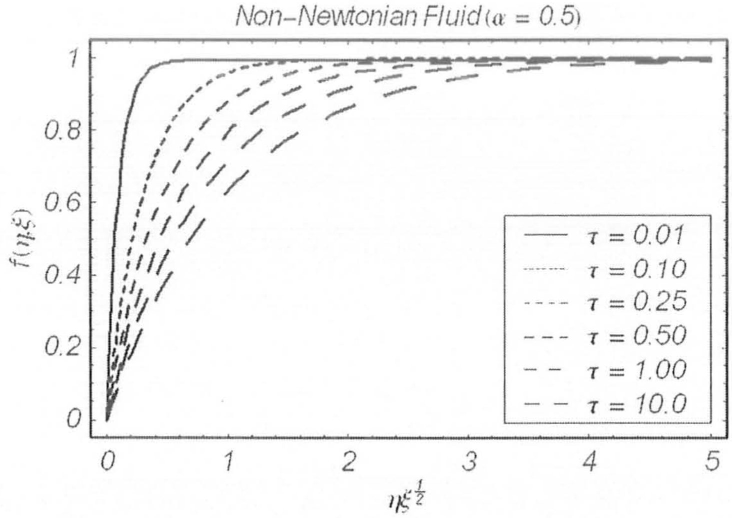


Fig. 5.5. The velocity profile $f(\eta, \xi)$ at different dimensionless time τ.

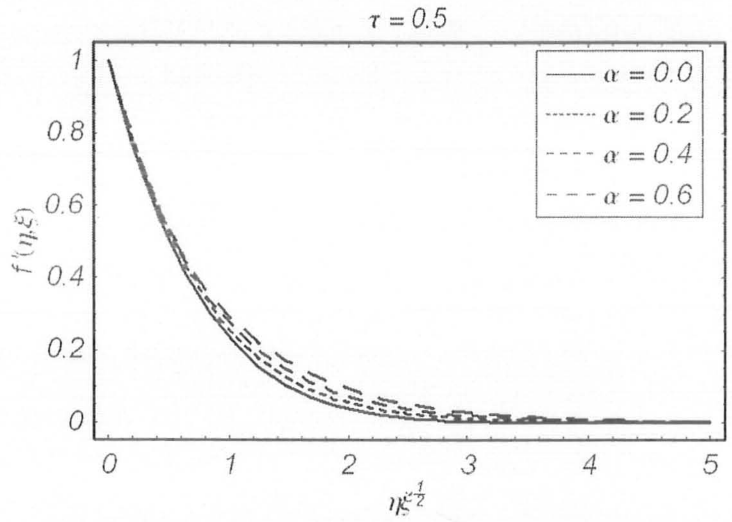


Fig. 5.6. The velocity profiles $f'(\eta, \xi)$ at different values of second grade parameter α.

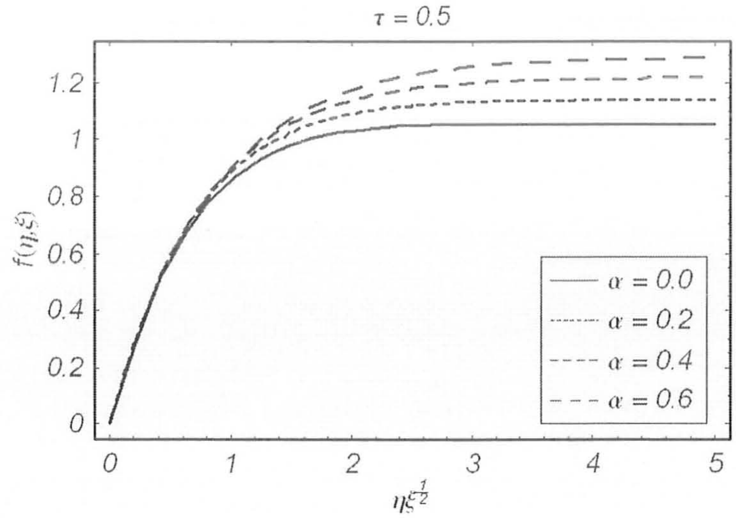


Fig. 5.7. The velocity profiles $f(\eta, \xi)$ at different values of second grade parameter α.

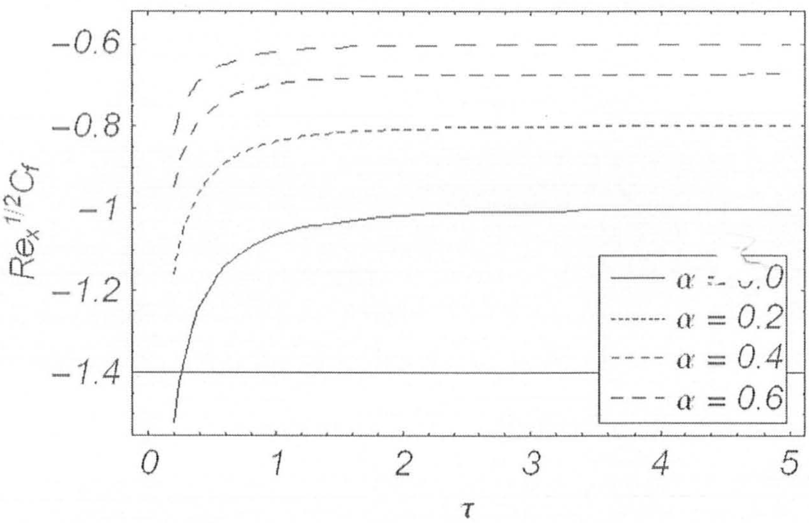


Fig. 5.8. The variationt $\mathrm{Re}_x^{1/2} C_f$ for different values α at different dimensionless time τ.

Chapter 6

Unsteady MHD axisymmetric flow and heat transfer of a viscous fluid over a radially stretching sheet in a porous medium

This chapter describe the fully developed unsteady flow and heat transfer characteristics of a viscous fluid over a radially stretching sheet. The fluid is electrically conducting and occupies the porous space. The heat transfer analysis has been carried out for the two heating processes, (i) prescribed surface temperature (PST-case) and (ii) prescribed surface heat flux (PHF-case). The analytic convergent solution of the governing non-linear partial differential equations is computed through HAM. Analytical expressions for velocity and temperature are first constructed and then shown graphically. The numerical values for the skin friction coefficient is presented in tabular form. Attention has been given to see the variations of the emerging parameters on the velocity and temperature distributions.

6.1 Flow equations

Consider an axially stretching boundary for which the lateral surface velocity is proportional to the distance r i.e. ar, a being the constant of proportionality. The fluid fills the porous space $z > 0$. The fluid is electrically conducting in the presence of uniform applied magnetic field $(0, B_0, 0)$. The induced magnetic field is assumed negligible under the assumption of small magnetic Reynolds number. Besides this no electric field is applied and the effect of polarization of the ionized fluid is neglected. The governing equations under these assumptions are

$$\begin{aligned}\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial r}+w\frac{\partial u}{\partial z} &= -\frac{1}{\rho}\frac{\partial \widehat{p}}{\partial r}+\nu\left[\frac{\partial^2 u}{\partial r^2}+\frac{\partial^2 u}{\partial z^2}+\frac{1}{r}\frac{\partial u}{\partial r}-\frac{u}{r^2}\right] \\ &\quad -\frac{\sigma B_0^2}{\rho}u-\frac{\nu\phi}{k}u, \end{aligned} \tag{6.1}$$

$$\begin{aligned}\frac{\partial w}{\partial t}+u\frac{\partial w}{\partial r}+w\frac{\partial w}{\partial z} &= -\frac{1}{\rho}\frac{\partial \widehat{p}}{\partial z}+\nu\left[\frac{\partial^2 w}{\partial z^2}+\frac{\partial^2 w}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial r}\right] \\ &\quad -\frac{\sigma B_0^2}{\rho}w-\frac{\nu\phi}{k}w, \end{aligned} \tag{6.2}$$

and the energy equation

$$\begin{aligned}c_p\left(\frac{\partial T}{\partial t}+u\frac{\partial T}{\partial r}+w\frac{\partial T}{\partial z}\right) &= \frac{k_0}{\rho}\left(\frac{\partial^2 T}{\partial r^2}+\frac{\partial^2 T}{\partial z^2}+\frac{1}{r}\frac{\partial T}{\partial r}\right) \\ &\quad +\nu\left(\begin{array}{c} 2\frac{u^2}{r^2}+\left(\frac{\partial u}{\partial z}\right)^2 \\ +2\left(\frac{\partial w}{\partial z}\right)^2+2\left(\frac{\partial u}{\partial r}\right)^2 \\ +2\frac{\partial u}{\partial z}\frac{\partial w}{\partial r}+2\left(\frac{\partial w}{\partial r}\right)^2 \end{array}\right). \end{aligned} \tag{6.3}$$

In above equations, u and w are the velocities in the r- and z- directions respectively, T is the temperature, σ is the electrical conductivity of fluid, ρ is the density, ν is the kinematic viscosity, ϕ is the porosity, k is the permeability, k_0 is the thermal conductivity, c_p is the specific heat and a is the stretching rate. The appropriate boundary conditions for flow analysis are given in Eq.(3.4). For temperature we have the following two sets of boundary conditions.

For PST case

$$T = T_w = T_\infty + A\left(\frac{r}{l}\right)^2 \quad \text{at} \quad z = 0, \quad T \to T_\infty \quad \text{as} \quad z \to \infty. \tag{6.4}$$

For PHF case

$$-k\frac{\partial T}{\partial z} = q_w = D\left(\frac{r}{l}\right)^2 \quad \text{at} \quad z = 0, \quad T \to T_\infty \quad \text{as} \quad z \to \infty. \tag{6.5}$$

in which A and D are constants. Introducing

$$\begin{aligned} u &= arf'(\eta,\xi), \quad w = -2\sqrt{a\nu\xi}f(\eta,\xi), \\ \xi &= 1 - e^{-\tau}, \quad \eta = \sqrt{\frac{a}{\nu\xi}}z, \quad \tau = at, \\ \theta(\eta,\xi) &= g(\xi,\eta) = \frac{T - T_\infty}{T_w - T_\infty}, \end{aligned} \tag{6.6}$$

and

$$\begin{aligned} T &= T_\infty + A\left(\frac{r}{l}\right)^2 \theta(\xi,\eta) \text{ for PST case}, \\ T &= T_\infty + \frac{D}{k}\left(\frac{r}{l}\right)^2 \sqrt{\frac{\nu}{a}} g(\xi,\eta) \text{ for PHF case}, \end{aligned} \tag{6.7}$$

equations (6.1) – (6.7) become

$$f^{iv} + \frac{\eta(1-\xi)}{2}f''' + 2\xi ff'' + \frac{(1-\xi)}{2}f'' - \xi(1-\xi)\frac{\partial f''}{\partial \xi} - \left(K + M^2\right)\xi f'' = 0. \tag{6.8}$$

PST case:

$$\left[\begin{array}{c} \xi\theta'' + \Pr\xi(1-\xi)\left\{\frac{\eta}{2}\theta' - \xi\frac{\partial\theta}{\partial\xi}\right\} + \frac{4\xi^2}{\delta}\theta + 2\Pr\xi^2\left(f\theta' - f'\theta\right) \\ +\Pr E\left(\xi f''^2 + \frac{12\xi^2}{\delta}f'^2\right) \end{array}\right] = 0. \tag{6.9}$$

PHF case:

$$\left[\begin{array}{c} \xi g'' + \Pr\xi(1-\xi)\left\{\frac{\eta}{2}g' - \xi\frac{\partial g}{\partial\xi}\right\} + \frac{4\xi^2}{\delta}g + 2\Pr\xi^2\left(fg' - f'g\right) \\ +\Pr E\left(\xi f''^2 + \frac{12\xi^2}{\delta}f'^2\right) \end{array}\right] = 0, \tag{6.10}$$

$$
\begin{aligned}
f(0,\xi) &= 0, \quad f'(0,\xi) = \theta(0,\xi) = 1 = -g'(0,\xi), \\
f'(\infty,\xi) &= \theta(\infty,\xi) = g(\infty,\xi) = 0,
\end{aligned} \tag{6.11}
$$

where

$$
\begin{aligned}
\delta &= \frac{ar^2}{\nu}, \quad \Pr = \frac{\mu c_p}{k}, \quad K = \frac{\phi\nu}{ka}, \quad M^2 = \frac{\sigma B_0^2}{\rho a}, \\
E &= \frac{a^2 l^2}{c_p A} \quad \text{(PST-case)}, \quad E = \frac{k_0 a^2 l^2}{D c_p}\sqrt{\frac{a}{\nu}} \quad \text{(PHF-case)}.
\end{aligned} \tag{6.12}
$$

In the next section, we are going to find the analytic solution. The non-linear partial differential equation (6.1) governing the flow has to be solved subject to the boundary conditions (3.4) by the homotopy analysis method (HAM).

6.2 HAM solutions

The velocity and temperature distributions $f(\eta,\xi)$, $\theta(\eta,\xi)$ and $g(\eta,\xi)$ can be expressed by the set of base functions of the form

$$
\left\{ \eta^k \xi^j \exp(-n\eta) \middle| \, k \geq 0, j \geq 0, n \geq 0 \right\} \tag{6.13}
$$

in the form of the following series

$$
f(\eta,\xi) = \sum_{m=0}^{\infty} f_m(\eta,\xi)
$$

$$
\theta(\eta,\xi) = \sum_{m=0}^{\infty} \theta_m(\eta,\xi)
$$

$$
g(\eta,\xi) = \sum_{m=0}^{\infty} g_m(\eta,\xi)
$$

where

$$
\begin{aligned}
f_m(\eta,\xi) &= d_{0,0}^{0,0} + \sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\sum_{j=0}^{\infty} d_{m,n}^{k,j}\eta^k\xi^j \exp(-n\eta), \\
\theta_m(\eta,\xi) &= \sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\sum_{j=0}^{\infty} e_{m,n}^{k,j}\eta^k\xi^j \exp(-n\eta), \\
g_m(\eta,\xi) &= \sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\sum_{j=0}^{\infty} h_{m,n}^{k,j}\eta^k\xi^j \exp(-n\eta),
\end{aligned}
\tag{6.14}
$$

in which $d_{m,n}^{k,j}$, $e_{m,n}^{k,j}$ and $h_{m,n}^{k,j}$ are the coefficients. Invoking the so-called *Rule of solution expressions* for $f(\eta,\xi)$, $\theta(\eta,\xi)$ and $g(\eta,\xi)$ and Eqs. (6.8) – (6.10) for the velocity we have the same initial guess and linear operator as in chapter 3. Whereas $\theta_0(\eta)$, $g_0(\eta)$ and linear operators $\mathcal{L}_2$ are

$$\theta_0(\eta) = \exp(-\eta), \quad g_0(\eta) = \exp(-\eta), \tag{6.15}$$

$$\mathcal{L}_2(\theta) = \theta'' - \theta, \tag{6.16}$$

where

$$\mathcal{L}_2[C_4\exp(-\eta) + C_5\exp(\eta)] = 0, \tag{6.17}$$

and C_4 and C_5 are the constants. Equations (6.8) – (6.10) show that the nonlinear operators are:

$$
\begin{aligned}
\mathcal{N}_4\left[\overline{f}(\eta,\xi;p)\right] &= \frac{\partial^4\overline{f}(\eta,\xi;p)}{\partial\eta^4} + \frac{\eta\xi(1-\xi)}{2}\frac{\partial^3\overline{f}(\eta,\xi;p)}{\partial\eta^3} + \\
&\quad 2\xi\overline{f}(\eta,\xi;p)\frac{\partial^3\overline{f}(\eta,\xi;p)}{\partial\eta^3} + \frac{(1-\xi)}{2}\frac{\partial^2\overline{f}(\eta,\xi;p)}{\partial\eta^2} \\
&\quad - (K+M^2)\,\xi\frac{\partial^2\overline{f}(\eta,\xi;p)}{\partial\eta^2} - \xi(1-\xi)\frac{\partial^3\overline{f}(\eta,\xi;p)}{\partial\xi\partial\eta^2},
\end{aligned}
\tag{6.18}
$$

$$\mathcal{N}_5\left[\overline{f}(\eta,\xi;p),\widetilde{\theta}(\eta,\xi;p)\right] = \xi\widetilde{\theta}''(\eta,\xi;p)+\frac{4\xi^2}{\delta}\widetilde{\theta}(\eta,\xi;p) + \Pr\xi(1-\xi)\left\{\frac{\eta}{2}\widetilde{\theta}'(\eta,\xi;p)-\xi\frac{\partial\widetilde{\theta}(\eta,\xi;p)}{\partial\xi}\right\} + 2\Pr\xi^2\left(\overline{f}(\eta,\xi;p)\widetilde{\theta}'(\eta,\xi;p)-\overline{f}(\eta,\xi;p)\widetilde{\theta}(\eta,\xi;p)\right) + \Pr E\left(\xi\overline{f}''^2(\eta,\xi;p)+\frac{12\xi^2}{\delta}\overline{f}'^2(\eta,\xi;p)\right), \tag{6.19}$$

$$\mathcal{N}_6\left[\overline{f}(\eta,\xi;p),\widetilde{g}(\eta,\xi;p)\right] = \xi\widetilde{g}''(\eta,\xi;p)+\frac{4\xi^2}{\delta}\widetilde{g}(\eta,\xi;p) + \Pr\xi(1-\xi)\left\{\frac{\eta}{2}\widetilde{g}'(\eta,\xi;p)-\xi\frac{\partial\widetilde{g}(\eta,\xi;p)}{\partial\xi}\right\} + 2\Pr\xi^2\left(\overline{f}(\eta,\xi;p)\widetilde{g}'(\eta,\xi;p)-\overline{f}(\eta,\xi;p)\widetilde{g}(\eta,\xi;p)\right) + \Pr E\left(\xi\overline{f}''^2(\eta,\xi;p)+\frac{12\xi^2}{\delta}\overline{f}'^2(\eta,\xi;p)\right). \tag{6.20}$$

Letting $\hbar_4$, $\hbar_5$ as the non-zero auxiliary parameters, the zeroth order deformation problems are

$$(1-p)\mathcal{L}_1\left[\overline{f}(\eta,\xi,p)-f_0(\eta)\right]=p\hbar_4\mathcal{N}_4\left[\overline{f}(\eta,\xi,p)\right], \tag{6.21}$$

$$(1-p)\mathcal{L}_2\left[\widetilde{\theta}(\eta,\xi,p)-\theta_0(\eta)\right]=p\hbar_5\mathcal{N}_5\left[\overline{f}(\eta,\xi,p),\widetilde{\theta}(\eta,\xi,p)\right], \tag{6.22}$$

$$(1-p)\mathcal{L}_2\left[\widetilde{g}(\eta,\xi,p)-g_0(\eta)\right]=p\hbar_5\mathcal{N}_6\left[\overline{f}(\eta,\xi,p),\widetilde{g}(\eta,\xi,p)\right], \tag{6.23}$$

$$\begin{aligned}\overline{f}(0,\xi,p) &= 0, \quad \overline{f}'(0,\xi,p)=\widetilde{\theta}(0,\xi,p)=1=-\widetilde{g}'(0,\xi,p)\\ \overline{f}'(\infty,\xi,p) &= 0, \quad \widetilde{\theta}(\infty,\xi,p)=\widetilde{g}(\infty,\xi,p)=0,\end{aligned} \tag{6.24}$$

where $p\,(\in[0,1])$ is an embedding parameter. When for $p=0$ and $p=1$, we have

$$\begin{aligned}\overline{f}(\eta,\xi,0) &= f_0(\eta), \quad \overline{f}(\eta,\xi,1)=f(\eta,\xi),\\ \widetilde{\theta}(\eta,\xi,0) &= \theta_0(\eta), \quad \widetilde{\theta}(\eta,\xi,1)=\theta(\eta,\xi),\\ \widetilde{g}(\eta,\xi,0) &= g_0(\eta), \quad \widetilde{g}(\eta,\xi,1)=g(\eta,\xi).\end{aligned} \tag{6.25}$$

The initial guesses $f_0(\eta)$, $\theta_0(\eta)$ and $g_0(\eta)$ approach $f(\eta,\xi)$, $\theta(\eta,\xi)$ and $g(\eta,\xi)$, respectively, as p varies from 0 to 1. By Taylor's series expansion:

$$\begin{aligned}
\overline{f}(\eta,\xi,p) &= f_0(\eta)+\sum_{m=1}^{\infty} f_m(\eta,\xi)\ p^m,\\
\widetilde{\theta}(\eta,\xi,p) &= \theta_0(\eta)+\sum_{m=1}^{\infty} \theta_m(\eta,\xi)\ p^m,\\
\widetilde{g}(\eta,\xi,p) &= g_0(\eta)+\sum_{m=1}^{\infty} g_m(\eta,\xi)\ p^m,
\end{aligned} \tag{6.26}$$

where

$$\begin{aligned}
f_m(\eta,\xi) &= \left.\frac{1}{m!}\frac{\partial^m \overline{f}(\eta,\xi,p)}{\partial p^m}\right|_{p=0},\\
\theta_m(\eta,\xi) &= \left.\frac{1}{m!}\frac{\partial^m \widetilde{\theta}(\eta,\xi,p)}{\partial p^m}\right|_{p=0},\\
g_m(\eta,\xi) &= \left.\frac{1}{m!}\frac{\partial^m \widetilde{g}(\eta,\xi,p)}{\partial p^m}\right|_{p=0},
\end{aligned} \tag{6.27}$$

and the convergence of the series (6.26) depends upon the values of the parameters $\hbar_4$ and $\hbar_5$. The values of $\hbar_4$ and $\hbar_5$ are chosen in such a way that the series (6.26) are convergent at $p=1$. Then by using Eq. (6.25) one obtains

$$\begin{aligned}
f(\eta,\xi) &= f_0(\eta)+\sum_{m=1}^{\infty} f_m(\eta,\xi),\\
\theta(\eta,\xi) &= \theta_0(\eta)+\sum_{m=1}^{\infty} \theta_m(\eta,\xi),\\
g(\eta,\xi) &= g_0(\eta)+\sum_{m=1}^{\infty} g_m(\eta,\xi).
\end{aligned} \tag{6.28}$$

6.2.1 mth-order deformation problems

Here we first differentiating Eqs. (6.21) – (6.23) m times with respect to p then dividing by $m!$ and setting $p=0$ we get

$$\mathcal{L}_1\left[f_m(\eta,\xi)-\chi_m f_{m-1}(\eta,\xi)\right]=\hbar_4\mathcal{R}_{m4}(\eta,\xi), \tag{6.29}$$

$$\mathcal{L}_2\left[\theta_m(\eta,\xi)-\chi_m\theta_{m-1}(\eta,\xi)\right]=\hbar_5\mathcal{R}_{m5}(\eta,\xi), \tag{6.30}$$

$$\mathcal{L}_2\left[g_m(\eta,\xi)-\chi_m g_{m-1}(\eta,\xi)\right]=\hbar_5\mathcal{R}_{m6}(\eta,\xi), \tag{6.31}$$

$$f_m(0,\xi)=f'_m(0,\xi)=f'_m(\infty,\xi)=\theta_m(0,\xi)=\theta_m(\infty,\xi)=g'_m(0,\xi)=g_m(\infty,\xi)=0, \tag{6.32}$$

whence

$$\begin{aligned}\mathcal{R}_{m4}(\eta,\xi) &= \frac{\partial^4 f_{m-1}}{\partial\eta^4}+\frac{\eta\xi(1-\xi)}{2}\frac{\partial^3 f_{m-1}}{\partial\eta^3}+\frac{(1-\xi)}{2}\frac{\partial^2 f_{m-1}}{\partial\eta^2}\\ &-\left(K+M^2\right)\xi\frac{\partial^2 f_{m-1}}{\partial\eta^2}-\xi(1-\xi)\frac{\partial^3 f_{m-1}}{\partial\xi\partial\eta^2}+2\kappa\sum_{k=0}^{m-1}f_{m-1-k}f_k''',\end{aligned} \tag{6.33}$$

$$\begin{aligned}\mathcal{R}_{m5}(\eta,\xi) &= \xi\frac{\partial^2\theta_{m-1}}{\partial\eta^2}+\frac{4}{\delta}\xi^2\theta_{m-1}\\ &+\Pr\xi(1-\xi)\left[\frac{\eta}{2}\frac{\partial\theta_{m-1}}{\partial\eta}-\xi\frac{\partial\theta_{m-1}}{\partial\xi}\right]\\ &+2\Pr\left[\sum_{k=0}^{m-1}f_{m-1-k}\frac{\partial\theta_k}{\partial\eta}-\sum_{k=0}^{m-1}\frac{\partial f_{m-1-k}}{\partial\eta}\theta_k\right]\\ &+\Pr\left[\sum_{k=0}^{m-1}\frac{\partial^2 f_{m-1-k}}{\partial\eta^2}\frac{\partial^2 f_k}{\partial\eta^2}-\frac{12\xi^2}{\delta}\sum_{k=0}^{m-1}\frac{\partial f_{m-1-k}}{\partial\eta}\frac{\partial f_k}{\partial\eta}\right],\end{aligned} \tag{6.34}$$

$$\begin{aligned}\mathcal{R}_{m6}(\eta,\xi) &= \xi\frac{\partial^2 g_{m-1}}{\partial\eta^2}+\frac{4}{\delta}\xi^2 g_{m-1}\\ &+\Pr\xi(1-\xi)\left[\frac{\eta}{2}\frac{\partial g_{m-1}}{\partial\eta}-\xi\frac{\partial g_{m-1}}{\partial\xi}\right]\\ &+2\Pr\left[\sum_{k=0}^{m-1}f_{m-1-k}\frac{\partial g_k}{\partial\eta}-\sum_{k=0}^{m-1}\frac{\partial f_{m-1-k}}{\partial\eta}g_k\right]\\ &+\Pr\left[\sum_{k=0}^{m-1}\frac{\partial^2 f_{m-1-k}}{\partial\eta^2}\frac{\partial^2 f_k}{\partial\eta^2}-\frac{12\xi^2}{\delta}\sum_{k=0}^{m-1}\frac{\partial f_{m-1-k}}{\partial\eta}\frac{\partial f_k}{\partial\eta}\right],\end{aligned} \tag{6.35}$$

$$\chi_m=\begin{cases}0, & m\le 1,\\ 1, & m>1.\end{cases} \tag{6.36}$$

The general solutions of Eqs. (6.29) – (6.36) can be written as

$$\begin{aligned} f_m(\eta,\xi) &= f_m^\star(\eta,\xi) + C_1\exp(-\eta) + C_2\exp(\eta) + C_3, \\ \theta_m(\eta,\xi) &= \theta_m^\star(\eta,\xi) + C_4\exp(-\eta) + C_5\exp(\eta), \\ g_m(\eta,\xi) &= g_m^\star(\eta,\xi) + C_4\exp(-\eta) + C_5\exp(\eta), \end{aligned} \tag{6.37}$$

in which $f_m^\star(\eta,\xi)$, $\theta_m^\star(\eta,\xi)$ and $g_m^\star(\eta,\xi)$ are the particular solutions and the constants are determined by the boundary conditions (6.32) which are given by

$$\begin{aligned} C_2 &= C_5 = 0, \quad C_1 = \left.\frac{\partial f_m^\star(\eta,\xi)}{\partial\eta}\right|_{\eta=0}, \quad C_3 = -C_1 - f_m^\star(0,\xi), \\ C_4 &= -\theta_m^\star(0,\xi) \quad \text{(PST-case)}, \quad C_4 = \left.\frac{\partial g_m^\star(\eta,\xi)}{\partial\eta}\right|_{\eta=0} \quad \text{(PHF-case)}. \end{aligned} \tag{6.38}$$

In the next section, the linear non-homogeneous Eqs. (6.29) – (6.36) are solved using MATHEMATICA in the order $m = 1, 2, 3, ...$

6.3 Convergence of the HAM solution

The explicit, analytic expressions of axisymmetric flow and heat transfer analysis contains two auxiliary parameters h_4 and h_5 respectively. The convergence region and the rate of approximation given by HAM are strongly dependent upon these auxiliary parameters. In Figs. 6.1 (a) – 6.1 (c), it is clear that the range for admissible values for h_4 and h_5 are $0 < h_4 < 0.6$ and $-1 < h_5 < 0$. The series converges in the whole region of η when $h_4 = 0.1$ and $h_5 = -0.75$ for both the prescribed surface heat flux (PHF case) and the prescribed surface temperature (PST case). It is also observed that the series $f(\eta,\xi)$ converges faster than that of $\theta(\eta,\xi)$ and $g(\eta,\xi)$ because of the fact that the non-linearity in the later case is stronger than the former. Thus, by means of choosing auxiliary parameters h_4 and h_5, we obtain an accurate analytic solution valid for all time $0 \leq \tau < \infty$ in the whole region $0 \leq \eta < \infty$.

Fig. 6.1 (a). Flow analysis

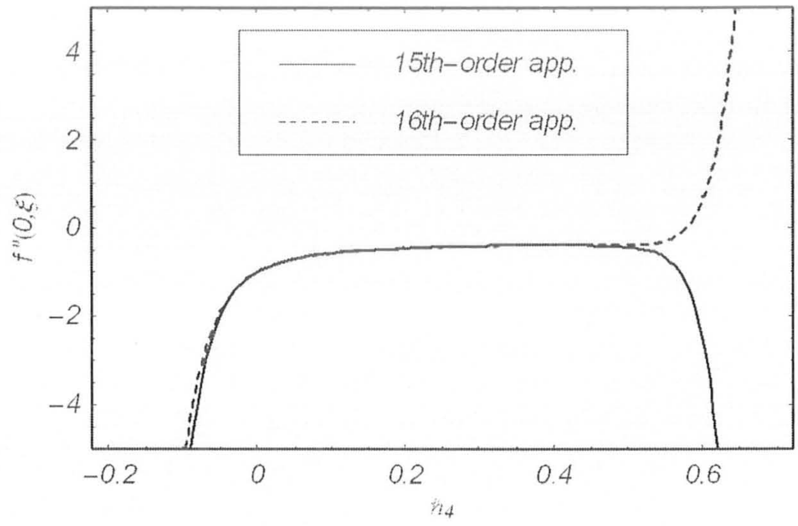


Fig. 6.1 (a) h-curve for the non-dimensional velocity when $\xi = 0.5$.

Fig. 6.1 (b). PST case

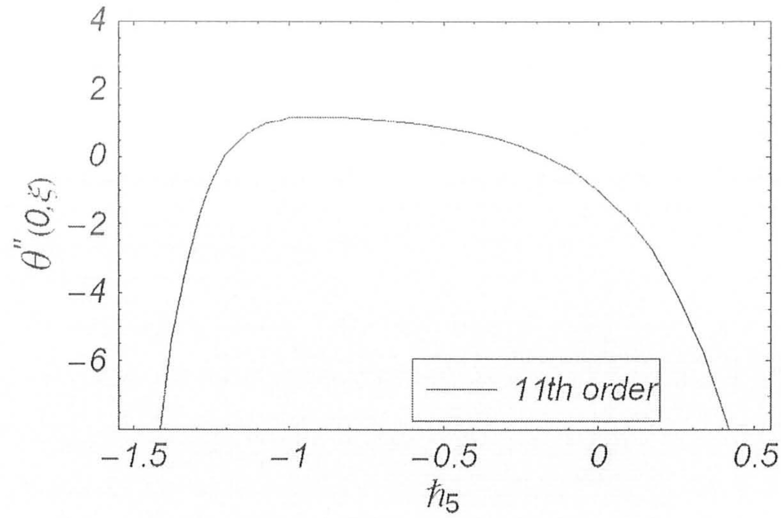


Fig. 6.1 (b) h-curve for the non-dimensional temperature θ.

Fig. 6.1 (c). PHF case

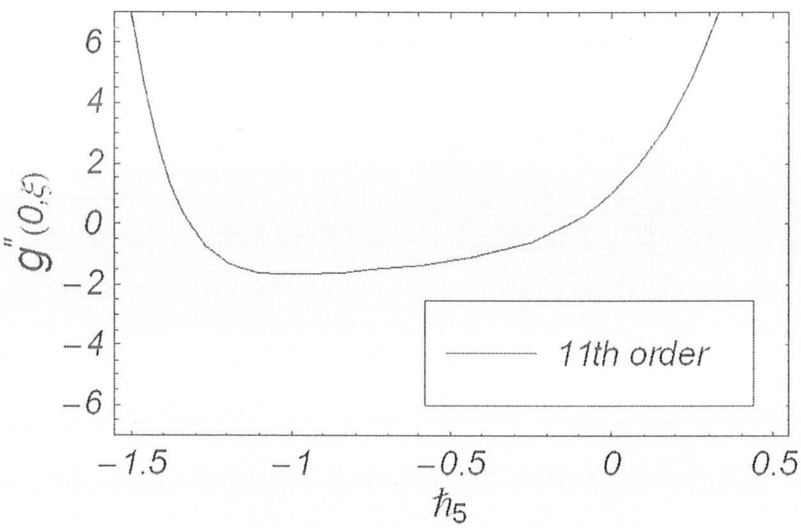


Fig. 6.1 (c) h-curve for the non-dimensional temperature g.

6.4 Results and discussion

In this section, attention has been focused on the variations of emerging parameters such as velocity profile for different values of time τ, Prandtl number Pr , Eckert number E and effects of porosity parameter k and the Hartmann number M. The behavior of velocity profile for different values of time τ is displayed in Figs. 6.2 and 6.3. Fig. 6.2 indicates that the r-component of velocity increases and also the boundary layer thickness increases with an increase in time τ. However the z-component of velocity decreases and layer thickness increases with an increase in τ as shown in Fig. 6.3. It also depicts that the solution is valid for all times. The values of skin friction coefficient are tabulated in Table 6.1. It is found that magnitude of skin friction increases with an increase in time τ. The influences of porosity parameter k and Hartmann number M are similar to τ on f and opposite to τ on f' (see Figs. 6.5 – 6.7).

τ	$C_f \mathrm{Re}_r^{\frac{1}{2}}$
0.01	−0.558759
0.10	−0.671632
0.25	−0.734321
0.50	−0.840094
1.00	−0.926639
10.0	−0.958742

Table 6.1. Skin friction coefficient $C_f \mathrm{Re}_r^{\frac{1}{2}}$ for different values of time.

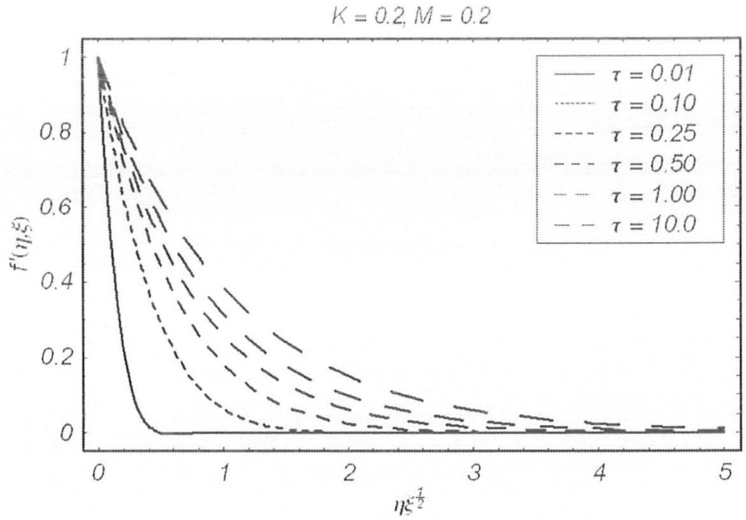


Fig. 6.2. The velocity profile $f'(\eta, \xi)$ at various non-dimensional time τ.

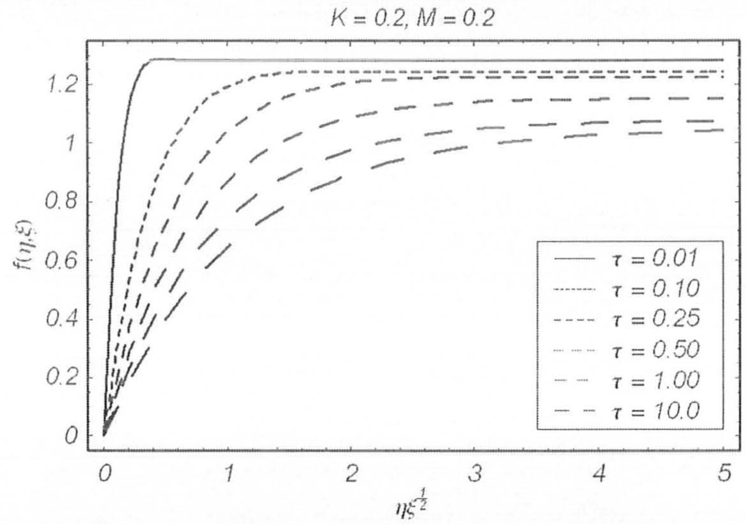


Fig. 6.3. The velocity profile $f(\eta, \xi)$ at various non-dimensional time τ.

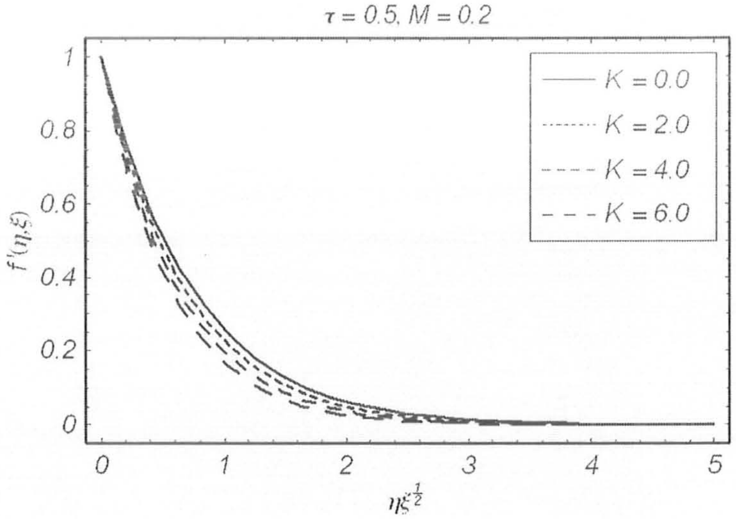


Fig. 6.4. Influence of porosity parameter on the velocity field f'.

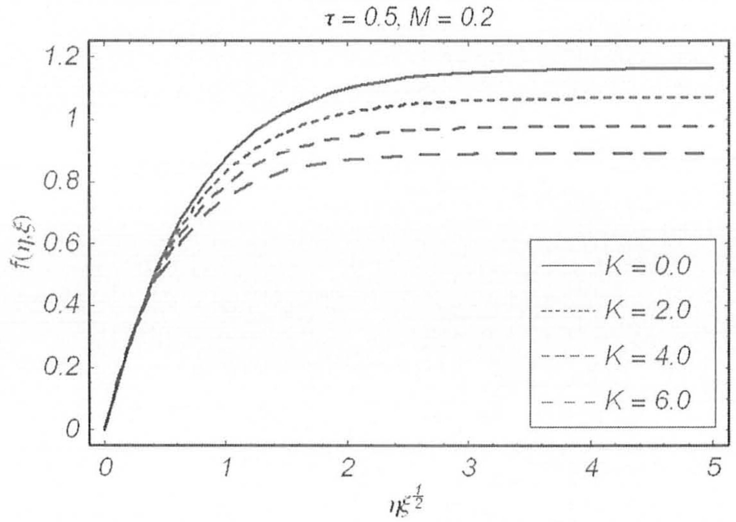
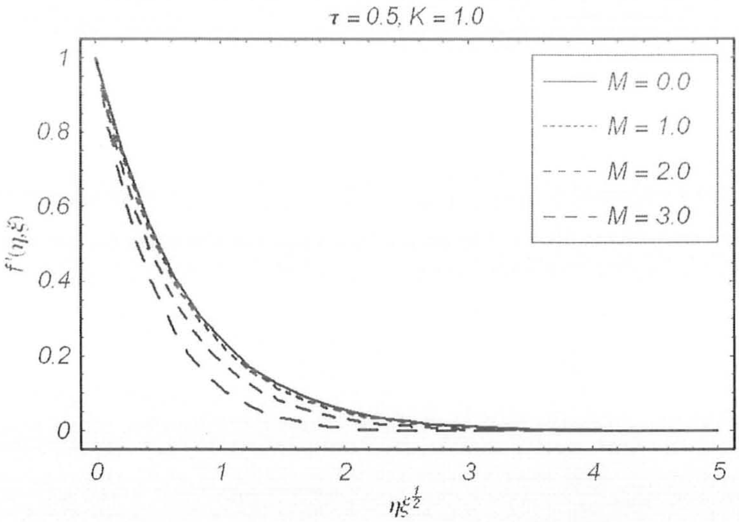
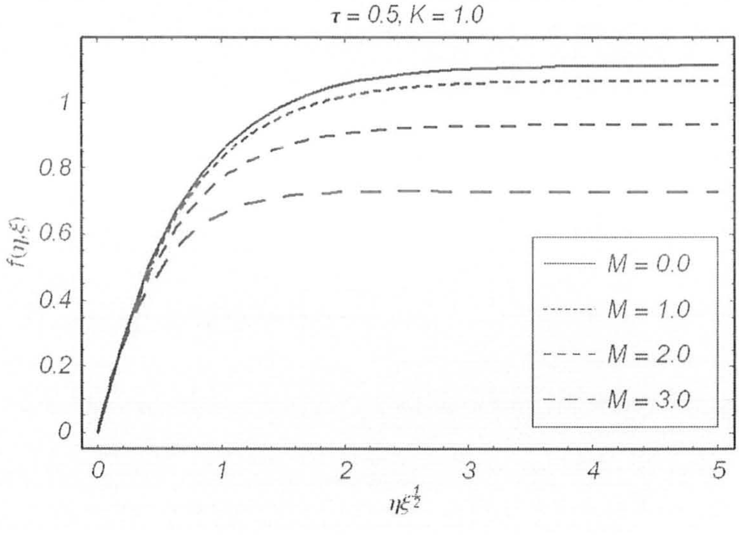


Fig. 6.5. Influence of porosity parameter on the velocity field f.



Figs. 6.6. Influence of Hartmann number on the velocity field f'.



Figs. 6.7. Influence of Hartmann number on the velocity field f.

The graphs for different values of Pr, E, k, M and τ on the temperature are displayed in Figures 6.8 – 6.17. In these Figures, the temperature $g(\eta, \xi)$ and $\theta(\eta, \xi)$ corresponds to the prescribed surface heat flux (PHF-case) and prescribed surface temperature (PST-case) respectively.

The variations of Prandtl number Pr on temperature profiles are displayed in Figs. 6.8 and 6.9. These Figs. indicate that an increase in Pr in the both cases the temperature decreases but the thermal boundary layer thickness increases. The influence of the Eckert number E on the temperature distribution are shown in Figs. 6.10 and 6.11. From these Figs. it is clear that the temperature and the thermal boundary layer thickness increases in both cases. The variations of porosity parameter k and Hartmann number M are displayed in Figs. 6.12 – 6.15. From these Figs. it is noted that in both cases the temperature and the boundary layer thickness increases with an increase in k and M respectively. It is also observed that the effects of Pr are quite opposite to that of dimensionless time τ whereas the effects of Eckert number E, porosity parameter k and Hartmann number M are similar to τ and opposite to Pr. The temperature profile for both PST and PHF cases are plotted in Figs. 6.16 and 6.17 in order to see the variations of dimensionless time τ. It can be easily seen from these Figs. that if we increase dimensionless time τ the temperature and the thermal boundary layer thickness increases.

Fig. 6.8 (PST-case)

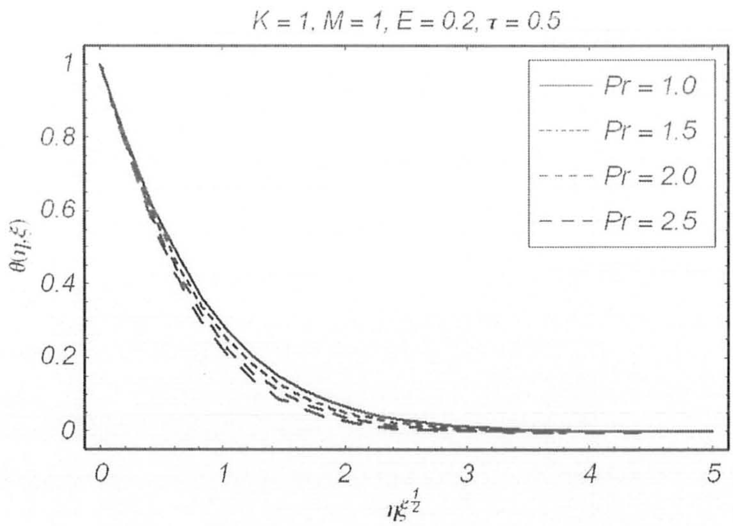


Fig. 6.8. The temperature profile $\theta(\eta, \xi)$ for various values of Pr.

Fig. 6.9 (PHF-case)

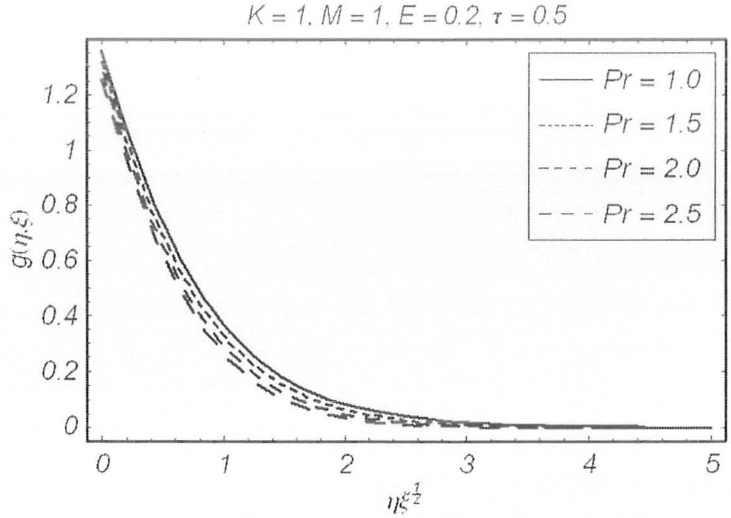


Fig. 6.9. The temperature profile $g(\eta, \xi)$ for various values of Pr.

Fig. 6.10 (PST-case)

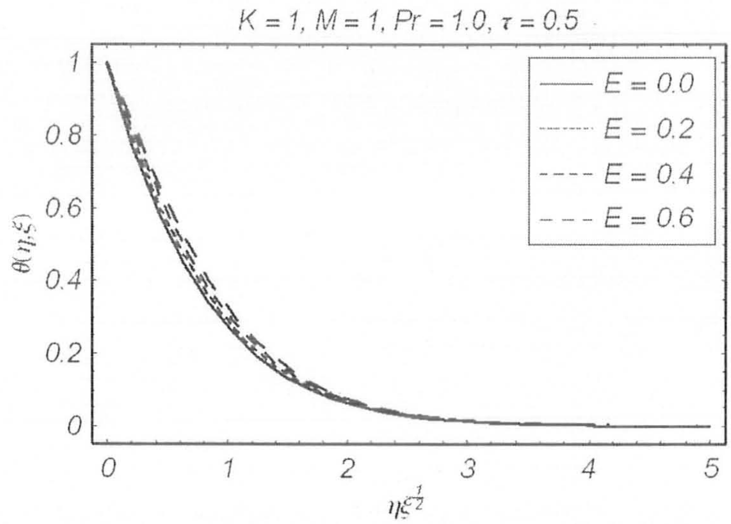


Fig. 6.10. The temperature profile $\theta(\eta, \xi)$ for various values of E.

Fig. 6.11 (PHF-case)

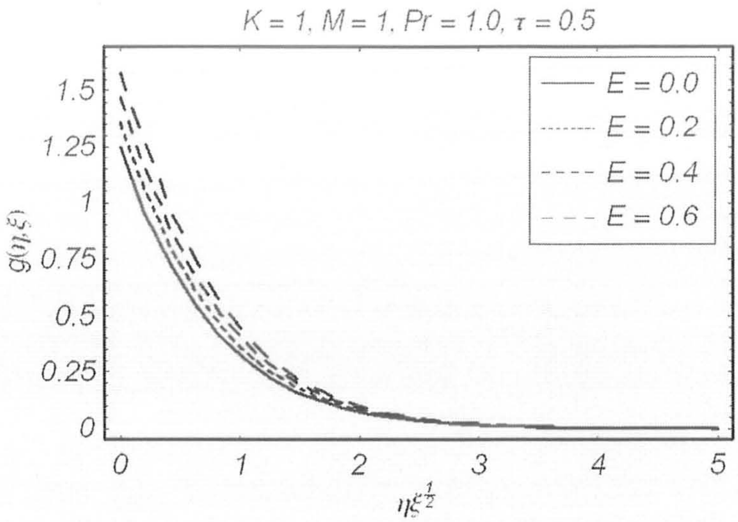


Fig. 6.11. The temperature profile $g(\eta, \xi)$ for various values of E.

Fig. 6.12 (PST-case)

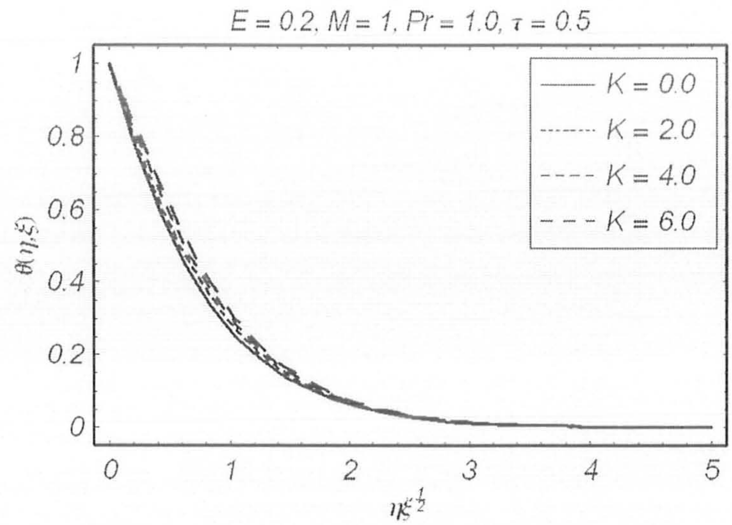


Fig. 6.12. The temperature profile $\theta(\eta, \xi)$ for various values of k.

Fig. 6.13 (PHF-case)

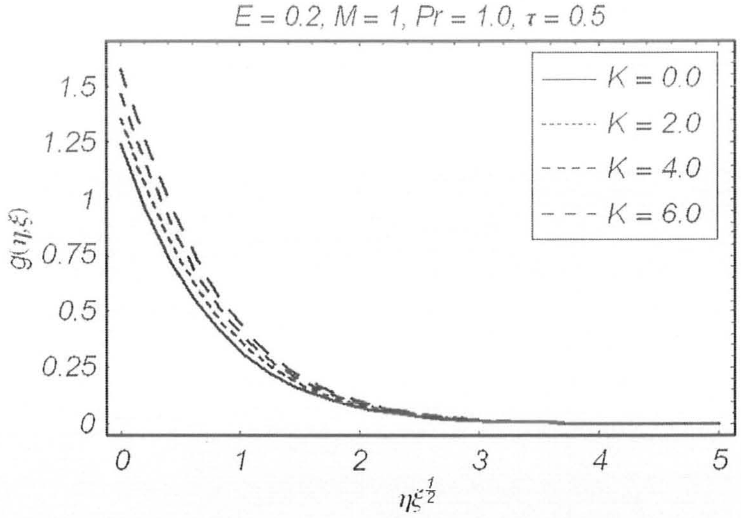


Fig. 6.13. The temperature profile $g(\eta, \xi)$ for various values of k.

Fig. 6.14 (PST-case)

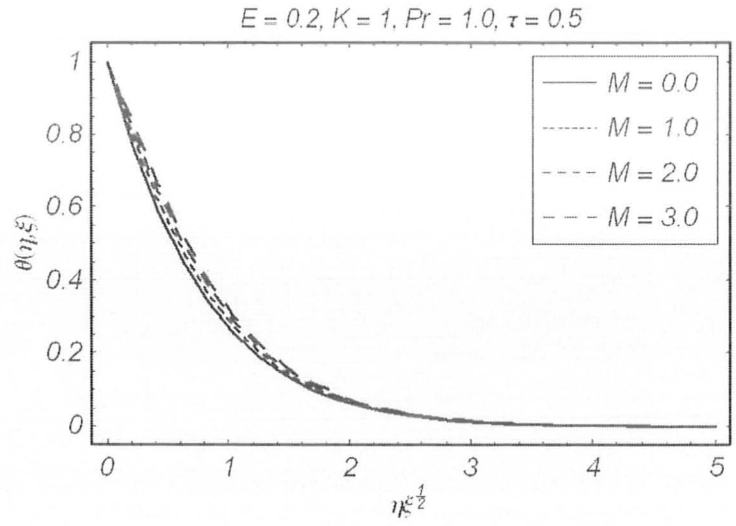


Fig. 6.14. The temperature profile $\theta(\eta, \xi)$ for various values of M.

Fig. 6.15 (PHF-case)

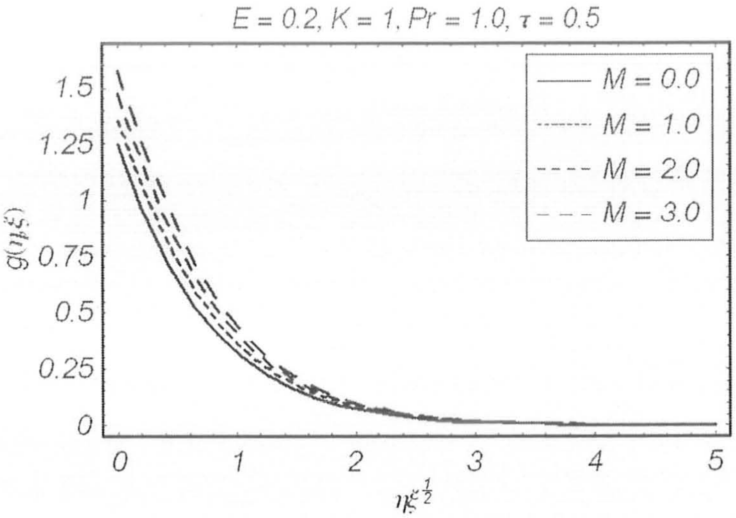


Fig. 6.15. The temperature profile $g(\eta, \xi)$ for various values of M.

Fig. 6.16 (PST-case)

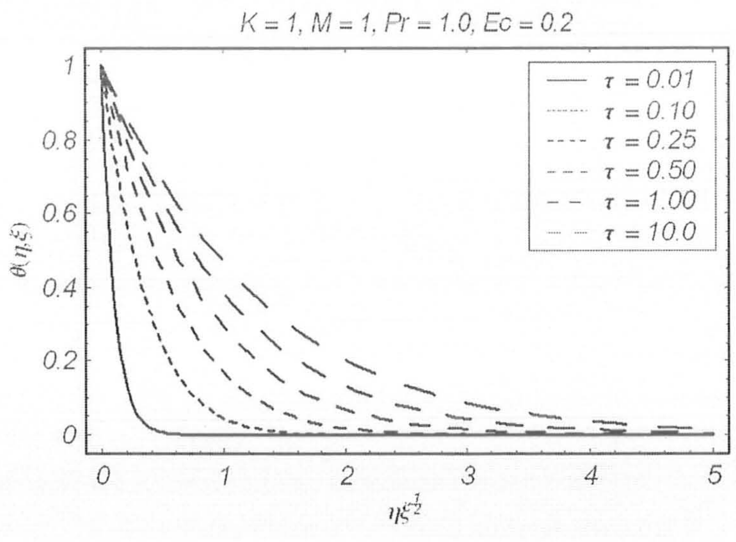


Fig. 6.16. The temperature profile $\theta(\eta, \xi)$ for various values of τ.

Fig. 6.17 (PHF-case)

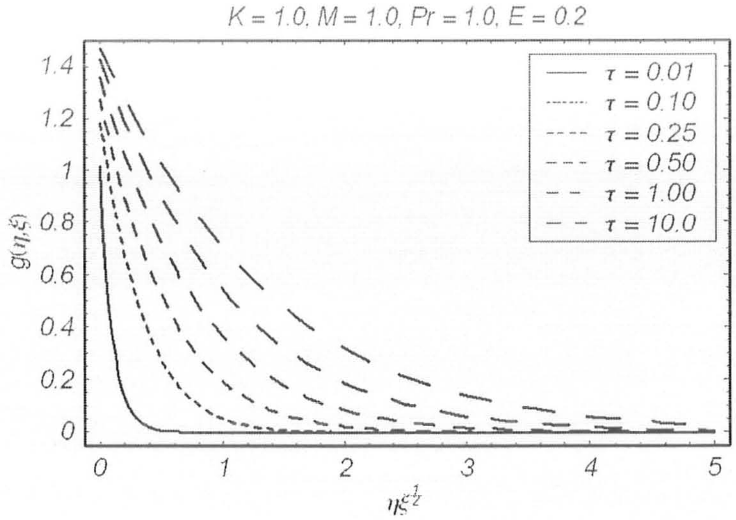


Fig. 6.17. The temperature profile $g(\eta, \xi)$ for various values of τ.

Chapter 7

Heat transfer analysis of unsteady axisymmetric flow of a second grade fluid over a radially stretching sheet

This chapter investigates the heat transfer analysis for the flow problem considered in chapter 4. In section 7.1 the equation for the heat transfer analysis of an unsteady axisymmetric flow of a second grade fluid is given. The heat transfer analysis has been analyzed for the two heating processes, namely (i) with prescribed surface temperature (PST-case) and (ii) prescribed surface heat flux (PHF-case). The convergent series solutions are constructed in both cases and discussed for the sundry parameters in the temperature distribution.

7.1 Heat transfer analysis

The energy equation, corresponding to the unsteady axisymmetric flow of a second grade fluid is given by

$$\rho c_p\left(\frac{\partial T}{\partial t}+u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial y}\right) = k\left(\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}\right)+\mu\left\{\begin{array}{c}\left(\frac{\partial u}{\partial y}\right)^2+2\left(\frac{\partial v}{\partial y}\right)^2+\left(\frac{\partial v}{\partial x}\right)^2\\ +2\left(\frac{\partial u}{\partial x}\right)^2+2\frac{\partial u}{\partial y}\frac{\partial v}{\partial x}\end{array}\right\}$$
$$+\alpha_1\left[\begin{array}{c}\frac{\partial u}{\partial y}\frac{\partial^2 u}{\partial y\partial t}+2\frac{\partial v}{\partial y}\frac{\partial^2 v}{\partial y\partial t}+2v\frac{\partial v}{\partial y}\frac{\partial^2 v}{\partial y^2}+\frac{\partial v}{\partial x}\frac{\partial^2 u}{\partial y\partial t}\\ +v\frac{\partial v}{\partial x}\frac{\partial^2 u}{\partial y^2}+2\frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial x\partial t}+\frac{\partial u}{\partial y}\frac{\partial^2 v}{\partial x\partial t}\\ +\frac{\partial v}{\partial x}\frac{\partial^2 v}{\partial x\partial t}+u\frac{\partial u}{\partial y}\frac{\partial^2 u}{\partial x\partial y}+v\frac{\partial u}{\partial y}\frac{\partial^2 u}{\partial y^2}\end{array}\right]. \quad (7.1)$$

For the appropriate boundary conditions we consider two heating processes.

7.1.1 The prescribed surface temperature (PST case)

From Eqs. (7.1) and (6.4) – (6.7), we get

$$\left[\begin{array}{c}\xi\theta''+\Pr\xi(1-\xi)\left\{\frac{\eta}{2}\theta'-\xi\frac{\partial\theta}{\partial\xi}\right\}+\frac{4\xi^2}{\delta}\theta+2\Pr\xi^2\left(f\theta'-f'\theta\right)\\ +\Pr E\left(\xi f''^2+\frac{12\xi^2}{\delta}f'^2\right)+\Pr E\alpha\left(\begin{array}{c}\xi f'f''^2-2\xi ff''f'''-\frac{24\xi^2}{\delta}ff'f''\\ -(1-\xi)\left\{\frac{6\xi\eta}{\delta}f'f''+\frac{1}{2}f''^2+\frac{\eta}{2}f''f'''\right\}\end{array}\right)\end{array}\right]=0. \quad (7.2)$$

7.1.2 The prescribed surface heat flux (PHF case)

In this case the governing equation for temperature through Eqs. (7.1) and (6.4) – (6.7) is written as

$$\left[\begin{array}{c}\xi g''+\Pr\xi(1-\xi)\left\{\frac{\eta}{2}g'-\xi\frac{\partial g}{\partial\xi}\right\}+\frac{4\xi^2}{\delta}g+2\Pr\xi^2\left(fg'-f'g\right)\\ +\Pr E\left(\xi f''^2+\frac{12\xi^2}{\delta}f'^2\right)+\Pr E\alpha\left(\begin{array}{c}\xi f'f''^2-2\xi ff''f'''-\frac{24\xi^2}{\delta}ff'f''\\ -(1-\xi)\left\{\frac{6\xi\eta}{\delta}f'f''+\frac{1}{2}f''^2+\frac{\eta}{2}f''f'''\right\}\end{array}\right)\end{array}\right]=0. \quad (7.3)$$

The non-linear equations (4.3), (7.2) and (7.3) has to be solved subject to the conditions (3.9) and (6.11) by HAM in the next section.

7.2 HAM solution

The velocity field has already been computed in chapter 4. The temperature distributions $\theta(\eta, \xi)$, $g(\eta, \xi)$ can be expressed in terms of set of base functions of the form

$$\left\{ \eta^k \xi^j \exp(-n\eta) \middle| \, k \geq 0, j \geq 0, n \geq 0 \right\}, \tag{7.4}$$

can be expressed through the following series

$$\theta(\eta, \xi) = \sum_{m=0}^{\infty} \theta_m(\eta, \xi)$$

$$g(\eta, \xi) = \sum_{m=0}^{\infty} g_m(\eta, \xi)$$

whence

$$\begin{aligned} \theta_m(\eta, \xi) &= \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} a_{1m,n}^{k,j} \eta^k \xi^j \exp(-n\eta), \\ g_m(\eta, \xi) &= \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} a_{2m,n}^{k,j} \eta^k \xi^j \exp(-n\eta), \end{aligned} \tag{7.5}$$

in which $a_{1m,n}^{k,j}$ and $a_{2m,n}^{k,j}$ are the coefficients. Invoking the so-called *Rule of solution expressions* for $f(\eta, \xi)$, $\theta(\eta, \xi)$ and $g(\eta, \xi)$ and Eqs. (4.3), (7.2) and (7.3) we have the same initial guesses and linear operators as in previous chapter and

$$\mathcal{N}_7\left[\widetilde{\theta}(\eta, \xi; p), \overline{f}(\eta, \xi; p)\right] = \left[\begin{array}{c} \xi\widetilde{\theta}'' + \Pr\xi(1-\xi)\{\frac{\eta}{2}\widetilde{\theta}' - \xi\frac{\partial\widetilde{\theta}}{\partial\xi}\} + \frac{4\xi^2}{\delta}\widetilde{\theta} + 2\Pr\xi^2\left(\overline{f}\widetilde{\theta}' - \overline{f}'\widetilde{\theta}\right) \\ +\Pr E\left(\xi\overline{f}''^2 + \frac{12\xi^2}{\delta}\overline{f}''^2\right) + \Pr E\alpha\{\xi\overline{f}'\overline{f}''^2 - 2\xi\overline{f}\overline{f}''\overline{f}''' - \frac{24\xi^2}{\delta}\overline{f}\overline{f}'\overline{f}'' \\ (1-\xi)\left[\eta\frac{6\xi}{\delta}\overline{f}'\overline{f}'' + \frac{1}{2}\overline{f}''^2 + \frac{\eta}{2}\overline{f}''\overline{f}'''\right]\} \end{array} \right], \tag{7.6}$$

$$\mathcal{N}_8\left[\widetilde{g}(\eta, \xi; p), \overline{f}(\eta, \xi; p)\right] = \left[\begin{array}{c} \xi\widetilde{g}'' + \Pr\xi(1-\xi)\{\frac{\eta}{2}\widetilde{g}' - \xi\frac{\partial\widetilde{g}}{\partial\xi}\} + \frac{4\xi^2}{\delta}\widetilde{g} + 2\Pr\xi^2\left(\overline{f}\widetilde{g}' - \overline{f}'\widetilde{g}\right) \\ +\Pr E\left(\xi\overline{f}''^2 + \frac{12\xi^2}{\delta}\overline{f}''^2\right) + \Pr E\alpha\{\xi\overline{f}'\overline{f}''^2 - 2\xi\overline{f}\overline{f}''\overline{f}''' - \frac{24\xi^2}{\delta}\overline{f}\overline{f}'\overline{f}'' \\ (1-\xi)\left[\eta\frac{6\xi}{\delta}\overline{f}'\overline{f}'' + \frac{1}{2}\overline{f}''^2 + \frac{\eta}{2}\overline{f}''\overline{f}'''\right]\} \end{array} \right]. \tag{7.7}$$

If $\hbar$ is the auxiliary nonzero parameter then the zero order deformation problem satisfies

$$(1-p)\,\mathcal{L}_2\left[\widetilde{\theta}\,(\eta,\xi,p)-\theta_0\,(\eta)\right]=p\hbar_7\mathcal{N}_7\left[\widetilde{f}\,(\eta,\xi,p)\,,\,\widetilde{\theta}\,(\eta,\xi,p)\right], \tag{7.8}$$

$$(1-p)\,\mathcal{L}_2\left[\widetilde{g}\,(\eta,\xi,p)-g_0\,(\eta)\right]=p\hbar_7\mathcal{N}_8\left[\widetilde{f}\,(\eta,\xi,p)\,,\,\widetilde{g}\,(\eta,\xi,p)\right], \tag{7.9}$$

$$\begin{aligned}\widetilde{\theta}\,(0,\xi,p) &= 1=-\widetilde{g}'\,(0,\xi,p)\,,\\ \widetilde{\theta}\,(\infty,\xi,p) &= \widetilde{g}\,(\infty,\xi,p)=0,\end{aligned} \tag{7.10}$$

where $p\,(\in[0,1])$ is an embedding parameter. For $p=0$ and $p=1$, we have

$$\begin{aligned}\widetilde{\theta}\,(\eta,\xi,0) &= \theta_0\,(\eta)\,, \quad \widetilde{\theta}\,(\eta,\xi,1)=\theta\,(\eta,\xi)\,.\\ \widetilde{g}\,(\eta,\xi,0) &= g_0\,(\eta)\,, \quad \widetilde{g}\,(\eta,\xi,1)=g\,(\eta,\xi)\,.\end{aligned} \tag{7.11}$$

The initial guesses $\theta_0\,(\eta)$ and $g_0\,(\eta)$ approach $\theta\,(\eta,\xi)$ and $g\,(\eta,\xi)$ respectively, as p varies from 0 to 1. By Taylor's series expansion

$$\begin{aligned}\widetilde{\theta}\,(\eta,\xi,p) &= \theta_0\,(\eta)+\sum_{m=1}^{\infty}\theta_m\,(\eta,\xi)\;p^m,\\ \widetilde{g}\,(\eta,\xi,p) &= g_0\,(\eta)+\sum_{m=1}^{\infty}g_m\,(\eta,\xi)\;p^m,\end{aligned} \tag{7.12}$$

$$\begin{aligned}\theta_m\,(\eta,\xi) &= \left.\frac{1}{m!}\frac{\partial^m\widetilde{\theta}\,(\eta,\xi,p)}{\partial p^m}\right|_{p=0},\\ g_m\,(\eta,\xi) &= \left.\frac{1}{m!}\frac{\partial^m\widetilde{g}\,(\eta,\xi,p)}{\partial p^m}\right|_{p=0},\end{aligned} \tag{7.13}$$

and the convergence of the series (7.12) depends upon $\hbar_2$ and $\hbar_7$. The values of $\hbar_2$ and $\hbar_7$ are chosen in such a way that the series (7.12) are convergent at $p=1$. Then by using Eq. (7.11) one obtains

$$\begin{aligned}\theta(\eta,\xi) &= \theta_0(\eta) + \sum_{m=1}^{\infty} \theta_m(\eta,\xi),\\ g(\eta,\xi) &= g_0(\eta) + \sum_{m=1}^{\infty} g_m(\eta,\xi).\end{aligned} \tag{7.14}$$

7.2.1 mth-order deformation problems

Here we have

$$\mathcal{L}_2\left[\theta_m(\eta,\xi) - \chi_m \theta_{m-1}(\eta,\xi)\right] = \hbar_7 \mathcal{R}_{m7}(\eta,\xi), \tag{7.15}$$

$$\mathcal{L}_2\left[g_m(\eta,\xi) - \chi_m g_{m-1}(\eta,\xi)\right] = \hbar_7 \mathcal{R}_{m8}(\eta,\xi), \tag{7.16}$$

$$\theta_m(0,\xi) = \theta_m(\infty,\xi) = g'_m(0,\xi) = g_m(\infty,\xi) = 0, \tag{7.17}$$

$$\mathcal{R}_{m7}(\eta,\xi) = \left[\begin{array}{c} \xi\widetilde{\theta}'' + \Pr\xi(1-\xi)\{\frac{\eta}{2}\widetilde{\theta}' - \xi\frac{\partial\widetilde{\theta}}{\partial\xi}\} + \frac{4\xi^2}{\delta}\widetilde{\theta} + 2\Pr\xi^2\left(\overline{f}\widetilde{\theta}' - \overline{f}'\widetilde{\theta}\right) \\ +\Pr E\left(\xi\overline{f}''^2 + \frac{12\xi^2}{\delta}\overline{f}''^2\right) + \Pr E\alpha\{\xi\overline{f}'\overline{f}''^2 - 2\xi\overline{f}\overline{f}''\overline{f}''' - \frac{24\xi^2}{\delta}\overline{f}\overline{f}'\overline{f}'' \\ (1-\xi)\left[\eta\frac{6\xi}{\delta}\overline{f}'\overline{f}'' + \frac{1}{2}\overline{f}''^2 + \frac{\eta}{2}\overline{f}''\overline{f}'''\right]\} \end{array}\right], \tag{7.18}$$

$$\mathcal{R}_{m8}(\eta,\xi) = \left[\begin{array}{c} \xi\widetilde{g}'' + \Pr\xi(1-\xi)\{\frac{\eta}{2}\widetilde{g}' - \xi\frac{\partial\widetilde{g}}{\partial\xi}\} + \frac{4\xi^2}{\delta}\widetilde{g} + 2\Pr\xi^2\left(\overline{f}\widetilde{g}' - \overline{f}'\widetilde{g}\right) \\ +\Pr E\left(\xi\overline{f}''^2 + \frac{12\xi^2}{\delta}\overline{f}''^2\right) + \Pr E\alpha\{\xi\overline{f}'\overline{f}''^2 - 2\xi\overline{f}\overline{f}''\overline{f}''' - \frac{24\xi^2}{\delta}\overline{f}\overline{f}'\overline{f}'' \\ (1-\xi)\left[\eta\frac{6\xi}{\delta}\overline{f}'\overline{f}'' + \frac{1}{2}\overline{f}''^2 + \frac{\eta}{2}\overline{f}''\overline{f}'''\right]\} \end{array}\right]. \tag{7.19}$$

The general solutions of Eqs. (7.15) and (7.16) are

$$\begin{aligned}\theta_m(\eta,\xi) &= \theta^\star_m(\eta,\xi) + C_4\exp(-\eta) + C_5\exp(\eta),\\ g_m(\eta,\xi) &= g^\star_m(\eta,\xi) + C_4\exp(-\eta) + C_5\exp(\eta),\end{aligned} \tag{7.20}$$

where $\theta^\star_m(\eta,\xi)$ and $g^\star_m(\eta,\xi)$ are the particular solutions and the constants are determined by the boundary conditions (7.17) which are given by

$$C_4 = -\theta^\star_m(0,\xi) \quad \text{(PST-case)}, \; C_4 = \left.\frac{\partial g^\star_m(\eta,\xi)}{\partial\eta}\right|_{\eta=0} \quad \text{(PHF-case)}, \; C_5 = 0. \tag{7.21}$$

In the next section, the linear non-homogeneous Eqs. (7.15) – (7.16) are solved using MATHEMATICA in the order $m = 1, 2, 3, ...$

7.3 Convergence of the analytic solution

Here we discuss the convergence of the series that contains the auxiliary parameters $\hbar_2$ and $\hbar_7$. The values of auxiliary parameters $\hbar_2$ and $\hbar_7$ helps as in the convergence region and rate of approximation for the homotopy analysis method. The explicit, analytic expressions of axisymmetric flow and heat transfer analysis contains two auxiliary parameters $\hbar_2$ and $\hbar_7$ respectively. In Figs. 4.1, 7.1 and 7.2 it is clear that the range for admissible values for $\hbar_2$ and $\hbar_7$ are $0 \leq \hbar_2 \leq 0.6$. and $-1 < \hbar_7 < 0$. And the series converges in the whole region of η, when $\hbar_2 = 0.2$ and $\hbar_7 = -0.2$ for both the prescribed surface heat flux (PHF case) and the prescribed surface temperature (PST case). It is also observed that the series $f(\eta, \xi)$ converges faster than that of $\theta(\eta, \xi)$ and $g(\eta, \xi)$ due to the fact that the non-linearity in the later case is stronger than the former. Thus, by means of choosing auxiliary parameters $\hbar_2$ and $\hbar_7$, we obtain an accurate analytic solution valid for all time $0 \leq \tau < \infty$ in whole region $0 \leq \eta < \infty$.

Fig. 7.1 PST case

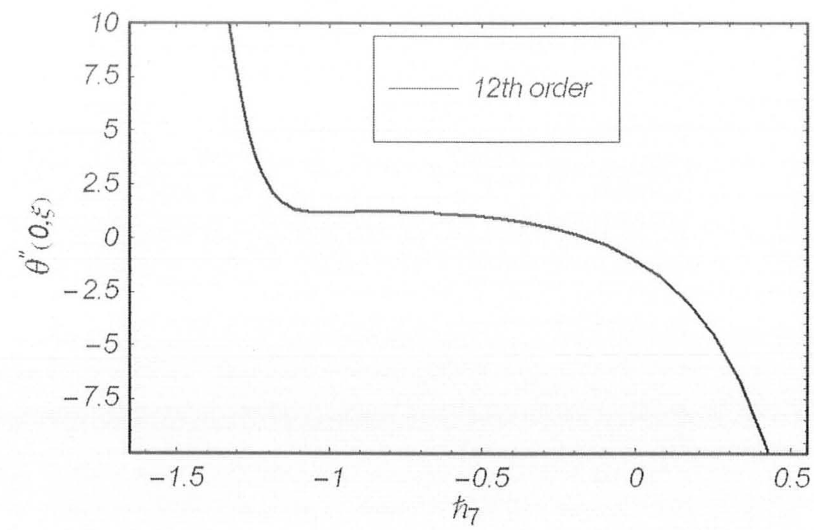


Fig. 7.1. $h-$curve for the non-dimensional temperature θ.

Fig. 7.2 PHF case

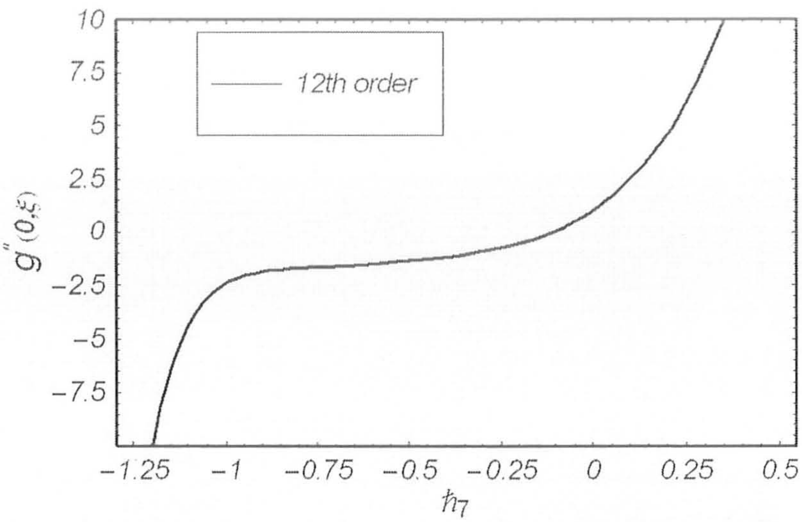


Fig. 7.2. $h-$curve for the non-dimensional temperature g.

7.4 Results and discussion

In this section, we discuss some results through graphs and attention has been focused on the emerging parameters $\Pr, E, \tau$ and α on the temperature distributions. For this purpose Figs. 7.3 – 7.10 have been displayed.

In these Figs. $\theta(\eta, \xi)$ is the temperature variations corresponding to the PST-case and $g(\eta, \xi)$ is temperature for the PHF-case.

The variations of various values of time on temperature for both PHF-case and PST-case are depicted in Figs. 7.3 and 7.4. It is obvious that by increasing time, the temperature and the thermal boundary layer increases in both cases. The effects of Prandtl number Pr on the temperature profiles are shown in Figs. 7.5 and 7.6. These Figs. show that the temperature decreases when the value of Prandtl number is increased. However, the boundary layer thickness increases in both cases. Influence of Eckert number E on the temperature field is observed from Figs. 7.7 – 7.8. It is noted that for large Eckert number E, the thermal boundary layer increases and also the temperature profile in both PHF and PST cases increases.

Figs. 7.9 and 7.10 are sketched to see the effect of second grade parameter α. These Figs. elucidate that both the temperature distribution and the thermal boundary layer increases when α increase.

Fig. 7.3 (PST-case)

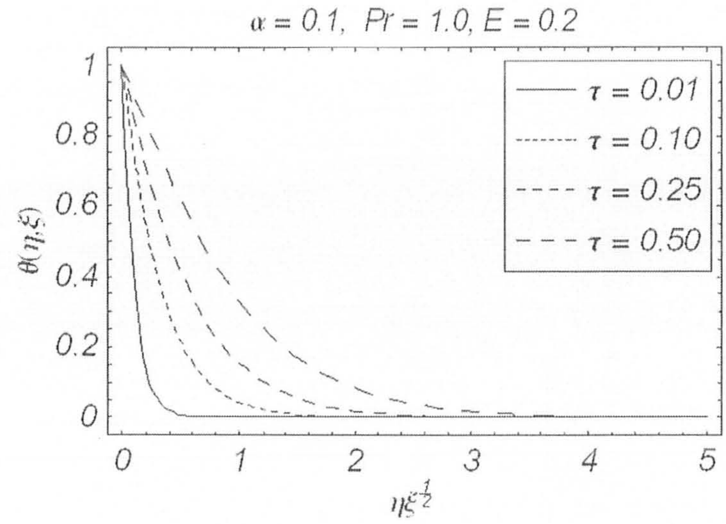


Fig. 7.3. Variation of temperature profile $\theta(\eta, \xi)$ with increasing time τ.

Fig. 7.4 (PHF-case)

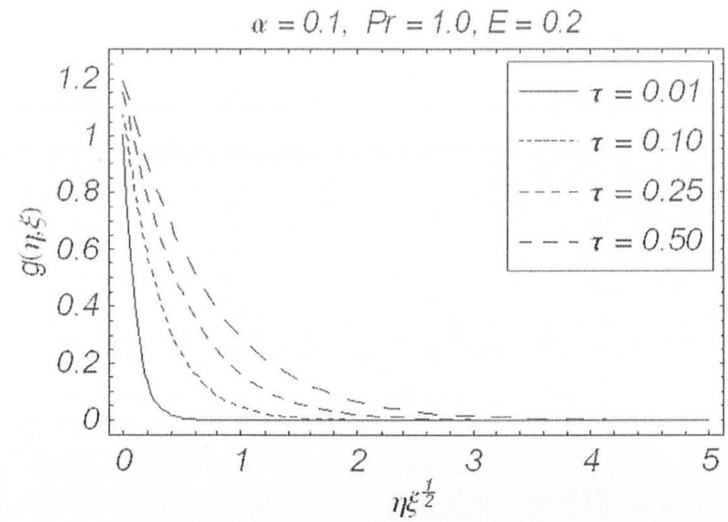


Fig. 7.4. Variation of temperature profile $g(\eta, \xi)$ with increasing time τ.

Fig. 7.5 (PST-case)

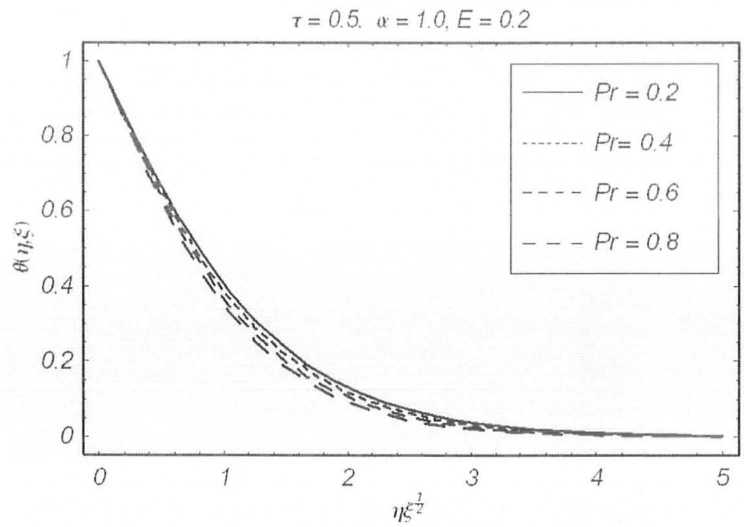


Fig. 7.5. Variation of temperature profile $\theta(\eta, \xi)$ with increasing Pr.

Fig. 7.6 (PHF-case)

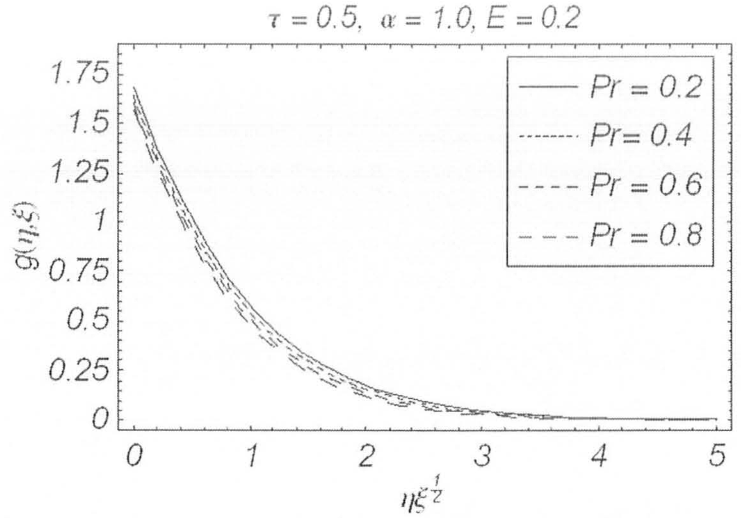


Fig. 7.6. Variation of temperature profile $g(\eta, \xi)$ with increasing Pr.

Fig. 7.7 (PST-case)

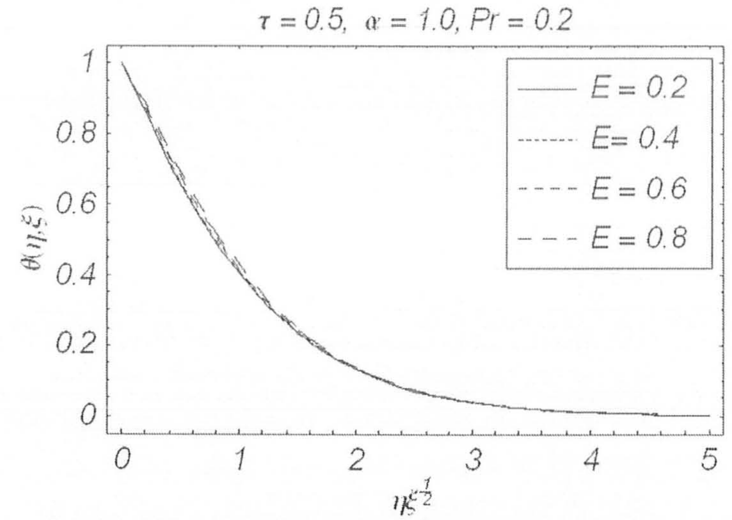


Fig. 7.7. Variation of temperature profile $\theta(\eta, \xi)$ with increasing E.

Fig. 7.8 (PHF-case)

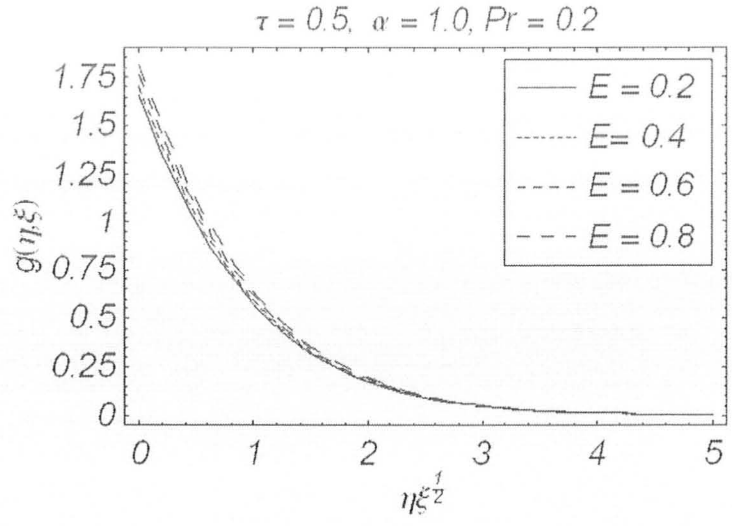


Fig. 7.8. Variation of temperature profile $g(\eta, \xi)$ with increasing E.

Fig. 7.9 (PST-case)

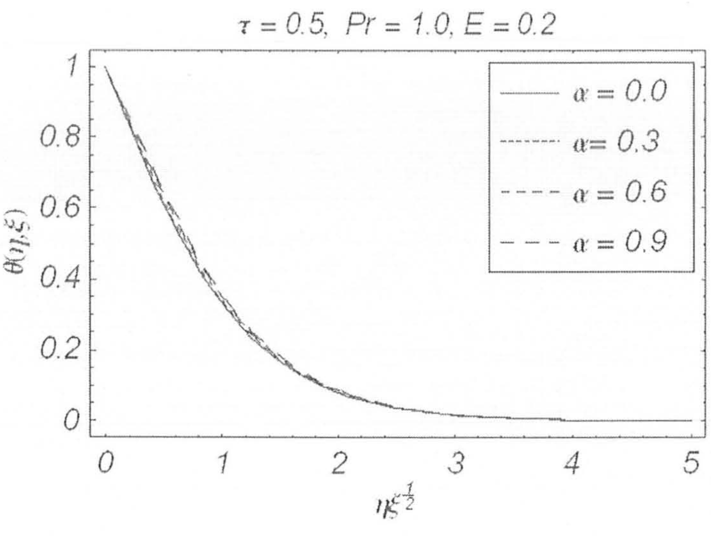


Fig. 7.9. Variation of temperature profile $\theta(\eta, \xi)$ with increasing α.

Fig. 7.10 (PHF-case)

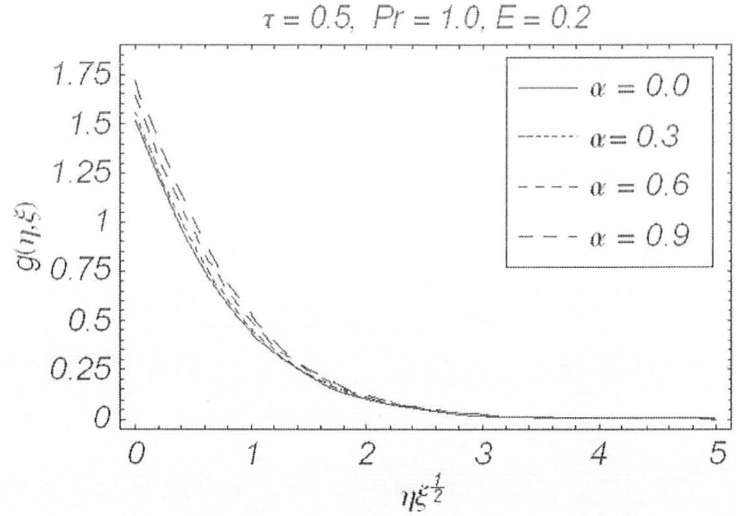


Fig. 7.10. Variation of temperature profile $g(\eta, \xi)$ with increasing α.

Chapter 8

Heat transfer analysis of unsteady boundary layer flow of second grade fluid over a planar stretching sheet

This chapter investigates the heat transfer analysis of the flow problem considered in chapter 5. The modeled non-linear problem is solved analytically using homotopy analysis method (HAM) subject to two heating processes (i) with prescribed surface temperature (PST-case) and (ii) with prescribed surface heat flux (PHF-case). The series solutions are obtained and the convergence of these solutions is explicitly discussed. Finally, results obtained are discussed through graphs.

8.1 Heat transfer analysis

Here, the physical model of the problem is same as in chapter in 5. Additionally heat transfer analysis is included. The energy equation corresponding to the unsteady boundary layer flow of a second grade fluid here is

$$\rho c_p\left(\frac{\partial T}{\partial t}+u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial y}\right)=k\frac{\partial^2 T}{\partial y^2}+\mu\left(\frac{\partial u}{\partial y}\right)^2+\alpha_1\left[\frac{\partial u}{\partial y}\frac{\partial^2 u}{\partial y\partial t}+u\frac{\partial u}{\partial y}\frac{\partial^2 u}{\partial x\partial y}+v\frac{\partial u}{\partial y}\frac{\partial^2 u}{\partial y^2}\right], \quad (8.1)$$

with the prescribed boundary conditions in Eqs.(6.4) and (6.5).

8.1.1 The prescribed surface temperature (PST case)

Through Eqs. (8.1) and (6.4) – (6.7) we obtain

$$\left[\begin{array}{c} \xi\theta'' + \Pr\xi\left(1-\xi\right)\left\{\frac{\eta}{2}\theta' - \xi\frac{\partial\theta}{\partial\xi}\right\} + +2\Pr\xi^2\left(f\theta' - f'\theta\right) + \Pr E\left(\xi + \frac{\alpha(1-\xi)}{2}\right)f''^2 \\ +\Pr E\alpha\left(\xi f' f''^2 - \xi f f'' f''' + (1-\xi)\left\{\xi f''\frac{\partial f''}{\partial\xi} - \frac{\eta}{2}f'' f'''\right\}\right) \end{array}\right] = 0. \quad (8.2)$$

8.1.2 The prescribed surface heat flux (PHF case)

From Eqs. (8.1) and (6.4) – (6.7) , we have

$$\left[\begin{array}{c} \xi g'' + \Pr\xi\left(1-\xi\right)\left\{\frac{\eta}{2}g' - \xi\frac{\partial g}{\partial\xi}\right\} + 2\Pr\xi^2\left(fg' - f'g\right) + \Pr E\left(\xi + \frac{\alpha(1-\xi)}{2}\right)f''^2 \\ +\Pr E\alpha\left(\xi f' f''^2 - \xi f f'' f''' + (1-\xi)\left\{\xi f''\frac{\partial f''}{\partial\xi} - \frac{\eta}{2}f'' f'''\right\}\right) \end{array}\right] = 0. \quad (8.3)$$

8.2 HAM solution

The temperature distributions $\theta(\eta,\xi)$ and $g(\eta,\xi)$ in terms of set of base functions of the form

$$\left\{\eta^k\xi^j\exp(-n\eta)\,\middle|\, k\geq 0, j\geq 0, n\geq 0\right\} \quad (8.4)$$

can be expressed through the following series

$$\theta(\eta,\xi) = \sum_{m=0}^{\infty}\theta_m(\eta,\xi)$$

$$g(\eta,\xi) = \sum_{m=0}^{\infty}g_m(\eta,\xi)$$

where

$$\begin{aligned} \theta_m(\eta,\xi) &= \sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}a_{3m,n}^{k,j}\eta^k\xi^j\exp(-n\eta), \\ g_m(\eta,\xi) &= \sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}a_{4m,n}^{k,j}\eta^k\xi^j\exp(-n\eta), \end{aligned} \quad (8.5)$$

in which $a_{3m,n}^{k,j}$ and $a_{4m,n}^{k,j}$ are the coefficients. Invoking the so-called *Rule of solution expressions* for $\theta(\eta,\xi)$ and $g(\eta,\xi)$ and Eqs. (5.7) , (8.2) and (8.3) the initial guesses $\theta_0(\eta)$ and $g_0(\eta)$ and linear operator are same as in the previous chapter and

$$\mathcal{N}_9\left[\widetilde{f}(\eta,\xi;p),\widetilde{\theta}(\eta,\xi;p)\right]=\left[\begin{array}{l}\xi\widetilde{\theta}''+\Pr\xi(1-\xi)\{\frac{\eta}{2}\widetilde{\theta}'-\xi\frac{\partial\widetilde{\theta}}{\partial\xi}\}\\+\frac{4\xi^2}{\delta}\widetilde{\theta}+2\Pr\xi^2\left(\widetilde{f}\widetilde{\theta}'-\widetilde{f}'\widetilde{\theta}\right)\\+\Pr E\left(\xi\widetilde{f}''^2+\frac{\alpha(1-\xi)}{2}\widetilde{f}''^2\right)\\+\Pr E\alpha\{\xi\widetilde{f}'\widetilde{f}''^2-\xi\widetilde{f}\widetilde{f}''\widetilde{f}'''\\+(1-\xi)\left[\xi\widetilde{f}''\frac{\partial\widetilde{f}''}{\partial\xi}-\frac{\eta}{2}\widetilde{f}''\widetilde{f}'''\right]\}\end{array}\right],\tag{8.6}$$

$$\mathcal{N}_{10}\left[\widetilde{f}(\eta,\xi;p),\widetilde{g}(\eta,\xi;p)\right]=\left[\begin{array}{l}\xi\widetilde{g}''+\Pr\xi(1-\xi)\{\frac{\eta}{2}\widetilde{g}'-\xi\frac{\partial\widetilde{g}}{\partial\xi}\}\\+\frac{4\xi^2}{\delta}\widetilde{g}+2\Pr\xi^2\left(\widetilde{f}\widetilde{g}'-\widetilde{f}'\widetilde{g}\right)\\+\Pr E\left(\xi\widetilde{f}''^2+\frac{\alpha(1-\xi)}{2}\widetilde{f}''^2\right)+\\\Pr E\alpha\{\xi\widetilde{f}'\widetilde{f}''^2-\xi\widetilde{f}\widetilde{f}''\widetilde{f}'''\\+(1-\xi)\left[\xi\widetilde{f}''\frac{\partial\widetilde{f}''}{\partial\xi}-\frac{\eta}{2}\widetilde{f}''\widetilde{f}'''\right]\}\end{array}\right].\tag{8.7}$$

If $\hbar$ is the auxiliary nonzero parameter then the zero order deformation problems are

$$(1-p)\mathcal{L}_2\left[\widetilde{\theta}(\eta,\xi,p)-\theta_0(\eta)\right]=p\hbar_8\mathcal{N}_9\left[\widetilde{f}(\eta,\xi,p),\widetilde{\theta}(\eta,\xi,p)\right],\tag{8.8}$$

$$(1-p)\mathcal{L}_2\left[\widetilde{g}(\eta,\xi,p)-g_0(\eta)\right]=p\hbar_8\mathcal{N}_{10}\left[\widetilde{f}(\eta,\xi,p),\widetilde{g}(\eta,\xi,p)\right],\tag{8.9}$$

$$\begin{aligned}\widetilde{\theta}(0,\xi,p)&=1=-\widetilde{g}'(0,\xi,p),\\\widetilde{\theta}(\infty,\xi,p)&=\widetilde{g}(\infty,\xi,p)=0,\end{aligned}\tag{8.10}$$

where $p\,(\in[0,1])$ is an embedding parameter and $p=0$ and $p=1$ we have

$$\begin{aligned}\widetilde{\theta}(\eta,\xi,0)&=\theta_0(\eta),\quad\widetilde{\theta}(\eta,\xi,1)=\theta(\eta,\xi).\\\widetilde{g}(\eta,\xi,0)&=g_0(\eta),\quad\widetilde{g}(\eta,\xi,1)=g(\eta,\xi).\end{aligned}\tag{8.11}$$

The initial guesses $\theta_0(\eta)$ and $g_0(\eta)$ approach to $\theta(\eta,\xi)$ and $g(\eta,\xi)$, respectively, as p varies from 0 to 1. By Taylor's series expansion one can write

$$\begin{aligned}\widetilde{\theta}(\eta,\xi,p) &= \theta_0(\eta)+\sum_{m=1}^{\infty}\theta_m(\eta,\xi)\ p^m,\\ \widetilde{g}(\eta,\xi,p) &= g_0(\eta)+\sum_{m=1}^{\infty}g_m(\eta,\xi)\ p^m,\end{aligned}\tag{8.12}$$

$$\begin{aligned}\theta_m(\eta,\xi) &= \left.\frac{1}{m!}\frac{\partial^m\widetilde{\theta}(\eta,\xi,p)}{\partial p^m}\right|_{p=0},\\ g_m(\eta,\xi) &= \left.\frac{1}{m!}\frac{\partial^m\widetilde{g}(\eta,\xi,p)}{\partial p^m}\right|_{p=0},\end{aligned}\tag{8.13}$$

and the convergence of the series (8.13) depends upon $\hbar_3$ and $\hbar_8$. The values of $\hbar_3$ and $\hbar_8$ are chosen in such a way that the series (8.13) are convergent at $p=1$. Then by using Eq. (8.12) one obtains

$$\begin{aligned}\theta(\eta,\xi) &= \theta_0(\eta)+\sum_{m=1}^{\infty}\theta_m(\eta,\xi),\\ g(\eta,\xi) &= g_0(\eta)+\sum_{m=1}^{\infty}g_m(\eta,\xi).\end{aligned}\tag{8.14}$$

8.2.1 mth-order deformation problems

Here we first differentiate Eqs. (8.8) – (8.9) m times with respect to p then divide by $m!$ and setting $p=0$ we get

$$\mathcal{L}_2\left[\theta_m(\eta,\xi)-\chi_m\theta_{m-1}(\eta,\xi)\right]=\hbar_8\mathcal{R}_{m9}(\eta,\xi),\tag{8.15}$$

$$\mathcal{L}_2\left[g_m(\eta,\xi)-\chi_m g_{m-1}(\eta,\xi)\right]=\hbar_8\mathcal{R}_{m10}(\eta,\xi),\tag{8.16}$$

$$\theta_m(0,\xi)=\theta_m(\infty,\xi)=g'_m(0,\xi)=g_m(\infty,\xi)=0,\tag{8.17}$$

where

$$\mathcal{R}_{m9}(\eta,\xi)=\left[\begin{array}{c} \xi\frac{\partial^2\theta_{m-1}}{\partial\eta^2}+\Pr\xi(1-\xi)\left[\frac{\eta}{2}\frac{\partial\theta_{m-1}}{\partial\eta}-\xi\frac{\partial\theta_{m-1}}{\partial\xi}\right]+ \\ 2\Pr\xi^2\left[\sum_{k=0}^{m-1}f_{m-1-k}\frac{\partial\theta_k}{\partial\eta}-\sum_{k=0}^{m-1}\frac{\partial f_{m-1-k}}{\partial\eta}\theta_k\right] \\ +\Pr E\left[\xi\sum_{k=0}^{m-1}\frac{\partial^2 f_{m-1-k}}{\partial\eta^2}\frac{\partial^2 f_k}{\partial\eta^2}+\frac{\alpha(1-\xi)}{2}\sum_{k=0}^{m-1}\frac{\partial^2 f_{m-1-k}}{\partial\eta^2}\frac{\partial^2 f_k}{\partial\eta^2}\right] \\ +\Pr E\alpha\left[\begin{array}{c} \Pr E\alpha\{\xi\sum_{k=0}^{m-1}\sum_{l=0}^{k}f'_{m-1-k}f''_{k-l}f''_l \\ -\xi\sum_{k=0}^{m-1}\sum_{l=0}^{k}f_{m-1-k}f''_{k-l}f'''_l+ \\ (1-\xi)\left[\xi\sum_{k=0}^{m-1}f''_{m-1-k}\frac{\partial f''_k}{\partial\xi}-\frac{\eta}{2}\sum_{k=0}^{m-1}f''_{m-1-k}f'''\right] \end{array}\right] \end{array}\right], \tag{8.18}$$

$$\mathcal{R}_{m10}(\eta,\xi)=\left[\begin{array}{c} \xi\frac{\partial^2 g_{m-1}}{\partial\eta^2}+\Pr\xi(1-\xi)\left[\frac{\eta}{2}\frac{\partial g_{m-1}}{\partial\eta}-\xi\frac{\partial g_{m-1}}{\partial\xi}\right]+ \\ 2\Pr\xi^2\left[\sum_{k=0}^{m-1}f_{m-1-k}\frac{\partial g_k}{\partial\eta}-\sum_{k=0}^{m-1}\frac{\partial f_{m-1-k}}{\partial\eta}g_k\right] \\ +\Pr E\left[\xi\sum_{k=0}^{m-1}\frac{\partial^2 f_{m-1-k}}{\partial\eta^2}\frac{\partial^2 f_k}{\partial\eta^2}+\frac{\alpha(1-\xi)}{2}\sum_{k=0}^{m-1}\frac{\partial^2 f_{m-1-k}}{\partial\eta^2}\frac{\partial^2 f_k}{\partial\eta^2}\right] \\ +\Pr E\alpha\left[\begin{array}{c} \Pr E\alpha\{\xi\sum_{k=0}^{m-1}\sum_{l=0}^{k}f'_{m-1-k}f''_{k-l}f''_l \\ -\xi\sum_{k=0}^{m-1}\sum_{l=0}^{k}f_{m-1-k}f''_{k-l}f'''_l+ \\ (1-\xi)\left[\xi\sum_{k=0}^{m-1}f''_{m-1-k}\frac{\partial f''_k}{\partial\xi}-\frac{\eta}{2}\sum_{k=0}^{m-1}f''_{m-1-k}f'''\right] \end{array}\right] \end{array}\right]. \tag{8.19}$$

The general solutions of Eqs. (8.15) – (8.19) are

$$\begin{aligned} \theta_m(\eta,\xi) &= \theta_m^\star(\eta,\xi)+C_4\exp(-\eta)+C_5\exp(\eta), \\ g_m(\eta,\xi) &= g_m^\star(\eta,\xi)+C_4\exp(-\eta)+C_5\exp(\eta), \end{aligned} \tag{8.20}$$

where $\theta_m^\star(\eta,\xi)$ and $g_m^\star(\eta,\xi)$ are the particular solutions and the constants are determined by the boundary conditions (8.17) which are given by

$$C_4=-\theta_m^\star(0,\xi)\ \text{(PST-case)},\ C_4=\left.\frac{\partial g_m^\star(\eta,\xi)}{\partial\eta}\right|_{\eta=0}\quad\text{(PHF-case)}\ C_5=0. \tag{8.21}$$

In the next section, the linear non-homogeneous Eqs. (8.15) – (8.19) are solved using MATHEMATICA in the order $m=1,2,3,...$

8.3 Convergence

The convergence region and rate of approximation of the series solutions depends upon $\hbar_3$ and $\hbar_8$. In order to find the admissible values of these parameters, the $\hbar$-curves of velocity and temperature for both PST and PHF cases are displayed in Figs. 5.1, 8.1 and 8.2. It is clear from these Figs. that the range for $\hbar_3$ is $-2 < \hbar_3 < -0.5$ and for $\hbar_8$ the range is $-1.25 < \hbar_8 < -0.50$ for both PST and PHF cases.

Fig. 8.1 PST case

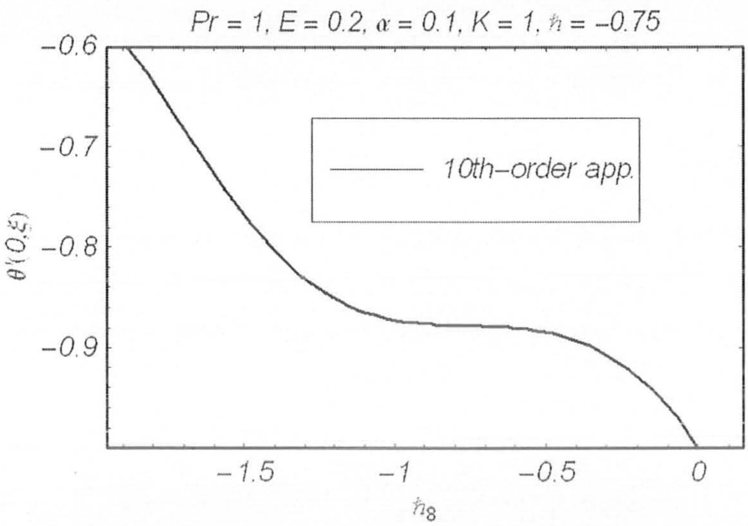


Fig. 8.1. $\hbar$ -curve for the non-dimensional temperature θ.

Fig. 8.2 PHF case

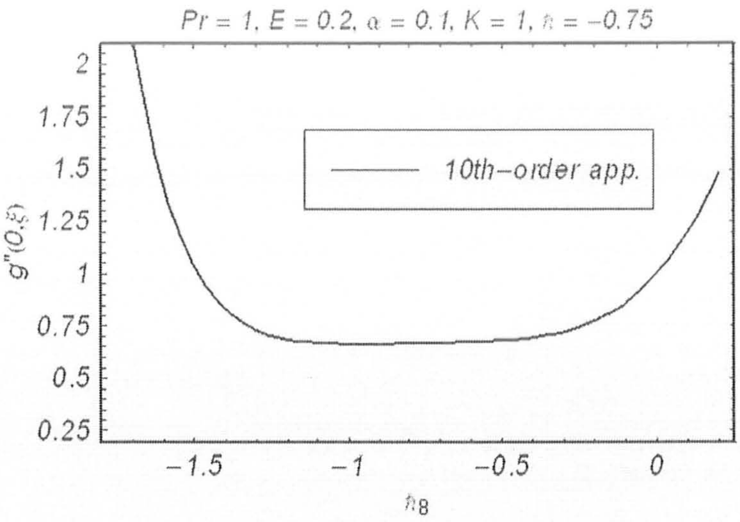


Fig. 8.2. $\hbar$ -curve for the non-dimensional temperature g.

8.4 Results and discussion

Here, the main purpose is just to see the variation of $\alpha, \Pr, E$ and τ on the temperature in both cases. For that we plotted Figs. 8.3 – 8.10. In these Figs. $g(\eta, \xi)$ is the temperature variation that corresponds to the PHF case and $\theta(\eta, \xi)$ is the temperature for PST case. The temperature and thermal boundary layer thickness increase by increasing time τ (see Figs. 8.3 and 8.4) for both PST and PHF cases. The effects of second grade parameter α and Eckert number E are similar to those of time τ but the effects of Prandtl number Pr are opposite to that of time τ and are shown in Figs. 8.5 – 8.10. Further, these Figs. indicate that for different values of second grade parameter α and Eckert number E the temperature profiles increase in both cases whereas the temperature decreases as we increase the values of Prandtle number Pr.

Fig. 8.3 PST-case

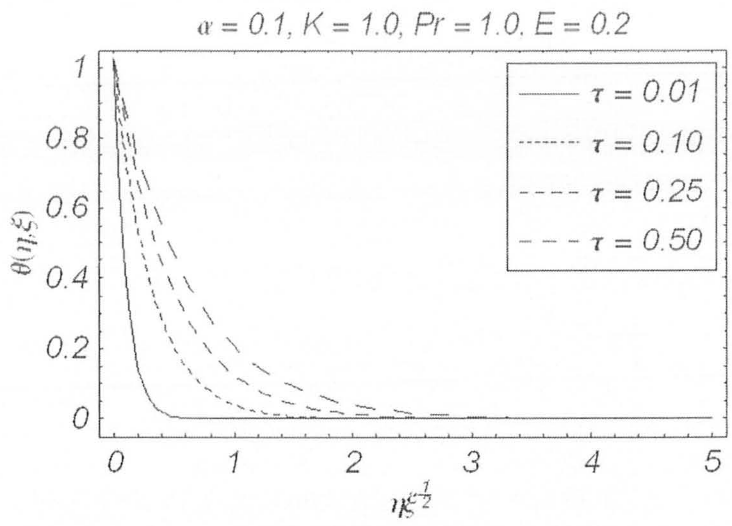


Fig. 8.3 The temperature profile $\theta(\eta, \xi)$ at different time τ.

Fig. 8.4 PHF-case

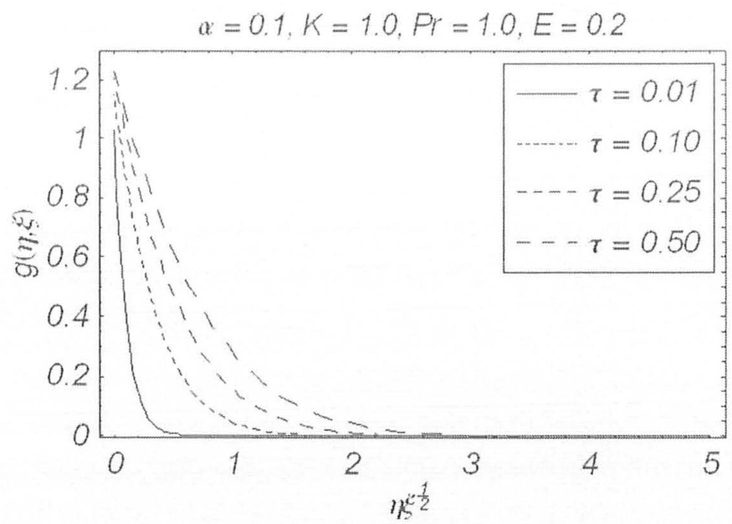


Fig. 8.4 The temperature profile $g(\eta, \xi)$ at different time τ.

Fig. 8.5 PST-case

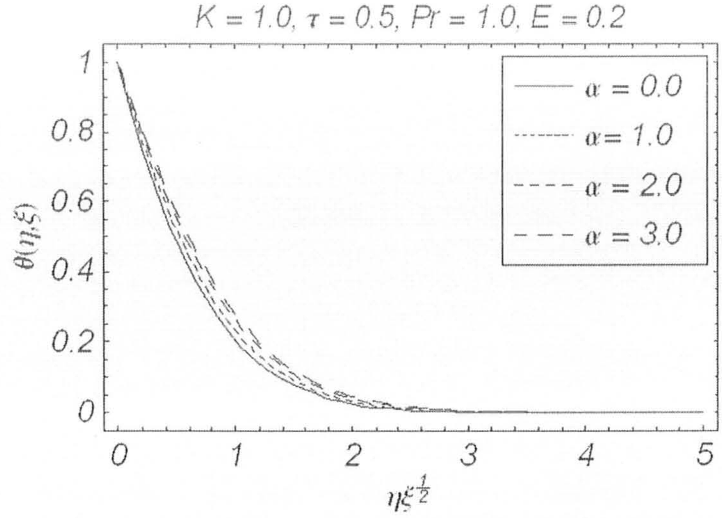


Fig. 8.5 The temperature profile $\theta(\eta, \xi)$ at different α.

Fig. 8.6 PHF-case

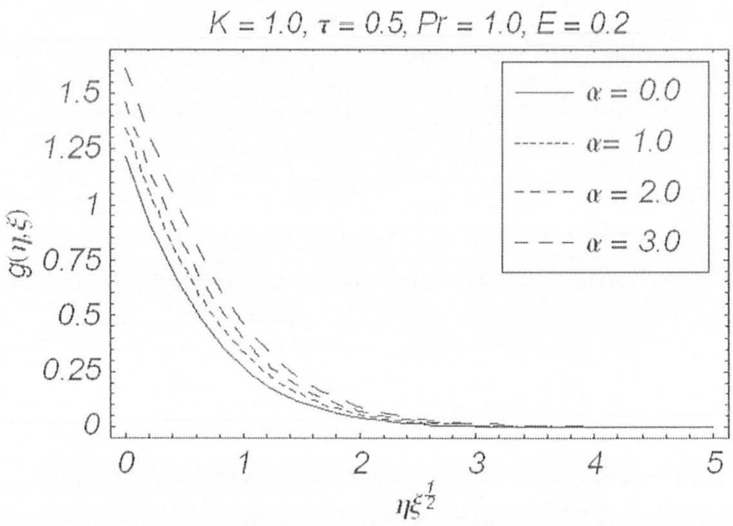


Fig. 8.6 The temperature profile $g(\eta, \xi)$ at different α.

Fig. 8.7 PST-case

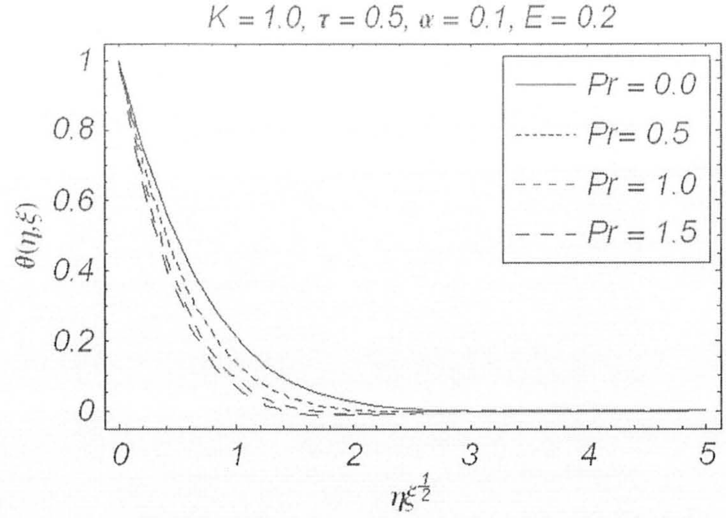


Fig. 8.7 The temperature profile $\theta(\eta, \xi)$ at different values of Pr.

Fig. 8.8 PHF-case

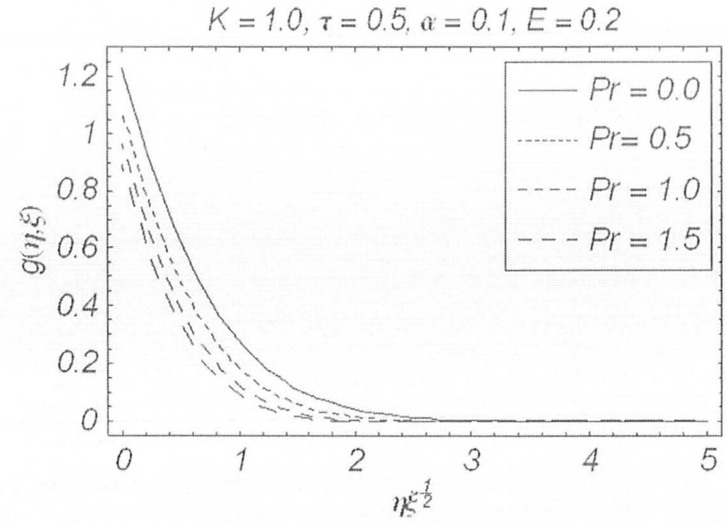


Fig. 8.8 The temperature profile $g(\eta, \xi)$ at different values of Pr.

Fig. 8.9 PST-case

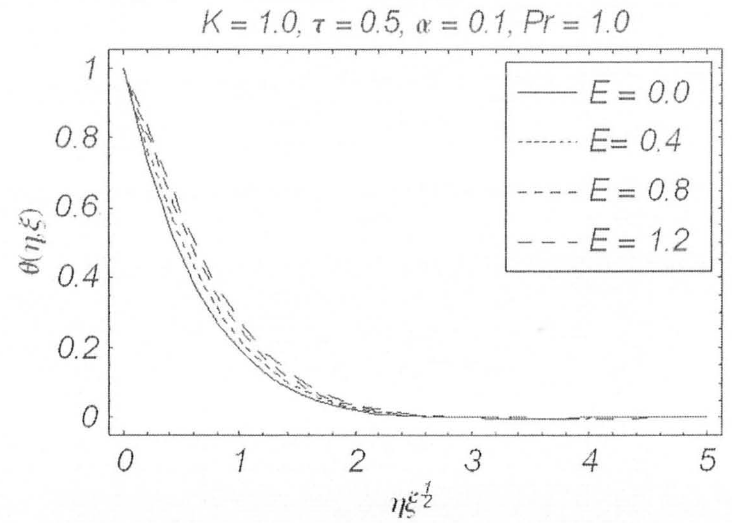


Fig. 8.9 The temperature profile $\theta(\eta, \xi)$ at different values of E.

Fig. 8.10 PHF-case

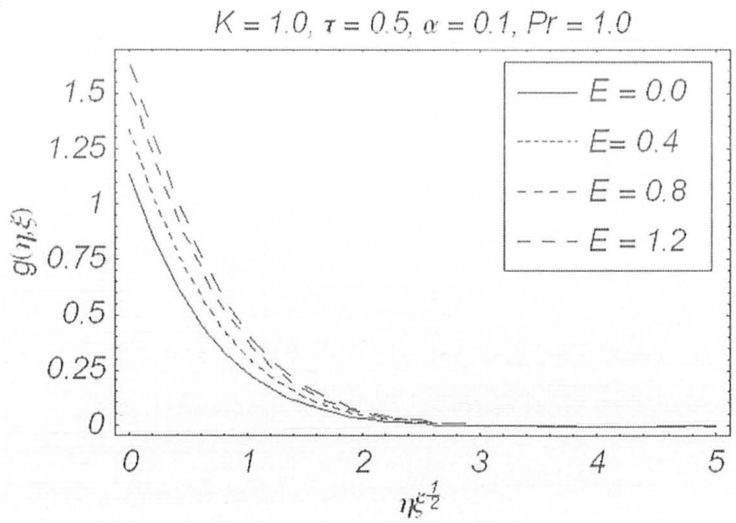


Fig. 8.10 The temperature profile $g(\eta, \xi)$ at different values of E.

Chapter 9

Conclusions

The work presented in this thesis deals with some non-linear problems for unsteady flow and heat transfer analysis in Newtonian and non-Newtonian fluids. Six chapters namely $3-8$ comprises such problems. The problem of axisymmetric flow of a viscous fluid over a stretching sheet is discussed in chapter 3. Chapter 4 presents the axisymmetric flow of a second grade fluid over a radially stretching sheet. In chapter 5 the series solution for the boundary layer flow of a second grade fluid caused by an impulsively stretching sheet is given. The heat transfer analysis of axisymmetric MHD flow of a Newtonian fluid in a porous medium is formulated and solved in chapter 6. The heat transfer analysis of axisymmetric and planar stretching in second grade are studied in chapters 7 and 8 respectively.

The main findings can be summarized as:

1 Unlike the perturbation technique, the used HAM does not require small or large parameter.

2 The convergence region of the obtained series can be well controlled by means of the auxiliary parameter.

3 HAM provide solutions in the form of infinite series and also a mechanism for finding the recurrence formulae for the coefficients of the series. The determined solution is regarded as an exact analytic solution if the convergence of the obtained series is explicitly discussed.

4 In the case of unsteady axisymmetric viscous fluid flow the r-component of velocity increases and the boundary layer thickness increases with an increase in time τ. However the z-component of velocity decreases.

5 The obtained solutions are valid for all time.

6 The magnitude of skin friction increases by increasing time.

7 The r-component of velocity increases and the z-component of velocity decreases when the second grade parameter is increased.

8 For large time, the r-component of velocity is much in second grade fluid when compared with that of Newtonian fluid..

9 The magnitude of skin friction coefficient decreases under the influence of second grade parameter and dimensionless time.

10 The heat transfer analysis for unsteady axisymmetric flow of viscous and second grade fluid is conducted first time in the literature.

11 In the case of viscous fluid the variations of time on the temperature profiles depicts that the temperature and thermal boundary layer thickness increases with an increase in time for both PST and PHF cases.

12 The effects of Prandtl number Pr are quite opposite to that of time.

13 The effect of Eckert number E, porosity parameter k and Hartmann number M are similar to time and opposite to Prandtl number.

14 By increasing Prandtl number Pr, the temperature profile decreases and the thermal boundary layer thickness increases for both PST and PHF cases.

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