Transient flows in porous medium

By

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DEPARTMENT OF MATHEMATICS
QUAID-I-AZAM UNIVERSITY
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CERTIFICATE

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We accept this dissertation as conforming to the required standard

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2008

Dedicated to

My loving parents

And my dedicated supervisor
Dr. Tasawar Hayat

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Preface

In recent years, considerable interest in the flows of non-Newtonian fluids has been stimulated due to their numerous industrial applications. Such applications include the processing of polymers, foams, pharmaceuticals, personal care products, clay suspensions and food products. Because of the difficulties posed by the complex rheology of such fluids, many constitutive equations expressing the behavior of non-Newtonian fluids have been proposed. These constitutive equations are usually classified under the categories of differential type, rate type and integral type models. The simplest subclass of differential type fluids is called second grade which can describe the normal stress effects and cannot predict the stress relaxation features. The Maxwell fluid which is a simplest subclass of rate type fluids can describe the stress relaxation phenomenon but it does not exhibit the retardation phenomenon which many concentrated polymers show. An Oldroyd-B fluid model can predict stress relaxation and retardation phenomena. The other subclasses of rate type fluids that contain more than one relaxation time and have been accorded proper attention by the researchers recently are the Burgers' and generalized Burgers' models. The constitutive equations of these fluids further add complexities in the momentum equation and the involved equations of non-Newtonian fluids are higher order than the Navier-Stokes equations.

On the other hand the flow through a porous medium is one of the most considerable and contemporary subjects, because it has great importance in geothermy, geophysics and technology. The study of flow in a porous medium has attracted the interest of many investigators in view of its applications in many engineering problems such as geothermal energy utilization, petroleum reservoirs, chemical catalytic convectors, storage of grain, fruits and vegetables, pollutant dispersions in aquifers, agricultural water distribution and combustion in situ in underground reservoirs for the enhancement of oil recovery. The flow of non-Newtonian fluids through a porous medium is also significant in biomechanics. To date majority of existing studies in a porous medium have been concerned with the Newtonian fluids. Despite the obvious relevance to industrial applications, little has been reported on the flow of non-Newtonian fluids in a porous medium. Only the key findings pertaining to the boundary value problems are mostly re-capitulated. Such attempts further narrowed down when modified Darcy's law is taken into account. These even become rare when transient flows with modified Darcy's law are considered.

The magnetohydrodynamic (MHD) flow problems of non-Newtonian fluids in porous media denote an idealization of many engineering applications. The Hall effect is important when the ratio between the electron-cyclotron frequency and the electron-atom collision frequency is high. In most MHD problems within small and moderate values of the magnetic field the Hall effect is neglected. However, the recent trend of MHD problems with strong magnetic field has widely appreciated the effects of Hall current in MHD generators, plasma studies and nuclear reactors. Such MHD flow problems in a porous medium pose challenges to cope with non-linearity of the governing equations and field coupling.

In view of the aforesaid observations, taking care of the non-Newtonian fluids with modified Darcy's law in a porous medium, Hall effect and transient phenomenon, an attempt is made in this thesis to develop the mathematical models that are competent to analyze analytical solutions. The analytical solutions still have their importance and even such solutions for Navier-Stokes equations are few because of the analytic difficulties associated with non-linear problems. Exact solutions are important not only in its own right as solutions of particular flows, but also serve as accuracy checks for the numerical solutions. Therefore, the outline of the thesis is as follows.

1. The aim of chapter one is to provide the review of the relevant existing literature. Basic equations and electromagnetic concepts are also presented.

2. An investigation of three unsteady flow problems of second grade fluid is presented in chapter two. The oscillatory flows with modified Darcy's law and Hall current are analyzed. Second grade fluids are used to describe the normal stress effects. Analytical results are derived by Fourier sine transform treatment. It is shown that the role of Hall and permeability parameters on the velocity is similar. It is also found that in a second grade fluid the magnetic parameter tends to decrease the velocity. These conclusions have been accepted for publication in **Int. J. Non-Linear Mech.**

3. Chapter three explores the Hall effects on the flow of an Oldroyd-B fluid in a porous medium for cylindrical geometry. Four unsteady problems are analyzed. Closed form

solutions have been derived for small and large times. Effects of Hall current, porosity and permeability of the porous medium for the Newtonian and Oldroyd-B fluids have been investigated in detail and shown graphically. It is noted that the steady state is achieved much earlier when there is no Hall effect. However in a non-porous medium, the steady state is achieved much later when compared with porous medium. These observations are published in **Computers and Mathematics with Applications 52 (2006) 269.**

4. Chapter four presents an analytic study of the accelerated flow of an Oldroyd-B fluid in a porous medium. Constant and variable accelerated flow problems are examined. Expressions of the velocity and adequate tangential stress are developed in each case. The obtained mathematical results are consistent with physical intuitions. As might be expected from physical consideration that velocity in variable accelerated flow is large in comparison to a constant accelerated flow. The contents of this chapter are in press in **Nonlinear Analysis: Real World Applications.**

5. It is emphasized quite recently that Burgers' fluid can characterize materials such as cheese, soil, asphalt and is important in the modeling of high temperature viscoelasticity of fine-grained polycrystalline olivine. In view of this, the object of chapter four is to extend the flow analysis of chapter three for a Burgers' fluid. Modified Darcy's law in a Burgers' fluid has been developed first time in the literature here. The influence of Hall current is noticed. Comparison between the results of an Oldroyd-B and Burgers' fluid is approached. It is observed that the velocity in Burgers' fluid is less than that of an Oldroyd-B fluid. The results of this chapter are published in **Transport Porous Media 68 (2007) 249.**

6. The magnetohydrodynamic flow of a generalized Burgers' fluid in a porous medium are investigated in chapter six. The corresponding modified Darcy's law has been first derived and then employed in the mathematical formulation. Three flow problems are investigated for the analytical solutions. Comparison of the velocity profiles in Newtonian, second grade, Maxwell, Oldroyd-B and generalized Burgers' fluid is established. The graphical results of the derived steady state solutions indicate that the velocity in an Oldroyd-B fluid is greatest and smallest in Newtonian fluid. It is also seen that there is a rapid increase

in the oscillations for the velocity in an Oldroyd-B and generalized Burgers' fluids when the magnetic parameter is increased. These points are in press in **Chaos, Solitons and Fractals**.

Chapter 1

Literature survey and involved equations

The material included here provides the review of the literature regarding second grade and the subclasses of rate type fluids. The involved electromagnetic concepts, equations and integral transforms in the subsequent chapters are also given.

1.1 Differential and rate type fluids

It is well known that the equations which can govern the flows of Newtonian fluids are called the Navier-Stokes equations. Such equations are inadequate in describing the flows of non-Newtonian fluids. The non-Newtonian fluids do undergo an increased or decreased viscosity change with increased flow. Undoubtedly the non-Newtonian fluids are well suited in industry and engineering when compared with that of the Newtonian fluids. Many industrial applications involve paints, glues, inks, soaps as well as suspensions such as coal-water slurries. These fluids display a behavior definitely different from that of Newtonian fluids. Unlike the Newtonian fluids, there is no constitutive equation that can predict the behavior of all the non-Newtonian fluids. Due to complex rheological characteristic of non-Newtonian fluids there are various models that have been proposed in the existing literature. Usually the classification of non-Newtonian fluids is given under the three categories namely the differential type, the rate type and the integral type. For incompressible fluids of differential type, apart from a constitutively indeterminate pressure, the stress only depends upon the velocity gradient and some of its higher time derivatives. During creeping phenomenon, the differential type fluids do not exhibit the stress relaxation. In general, for fluids of differential type of grade n, the equations of motion are of order $n+1$. For $n > 1$ the adherence boundary condition is sufficient to determine a unique solution. In view of mathematical simplicity, the differential type fluids have received much attention by the workers in the field. The simplest subclass of the differential type fluids is known as the second grade fluid. Much attention is concentrated in the literature on the flows dealing with second grade fluids. This is particularly so because of the fact that for this subclass one may hope to obtain an analytical solution. The constitutive equation of second grade fluid contains three material constants. Dunn and Fosdick [1] and Dunn and Rajagopal [2] have discussed in details the restrictions regarding these three material constants. In general the governing equations of non-Newtonian fluids are much complicated and of higher order than the Navier-Stokes equations. Therefore the no slip boundary condition is not sufficient to obtain the solution. In this situation one needs an extra condition(s) for a unique solution. This issue has been attented by Rajagopal [3] , Rajagopal and Kaloni [4] and Rajagopal and Gupta [5]. Ting [6] initiated the study of unidirectional transient flows of a hydrodynamic second grade fluid. Rajagopal [7] examined the four cases for unidirectional flows of a second grade fluid. At present, the literature dealing with the flows of second grade fluids in various situations is extensive. Some attempts in this direction have been made by Rajagopal [8], Benharbit and Siddiqui [9], Erdogan [10], Siddiqui and Kalani [11], Ariel [12–14] Fetecau and Fetecau [15], Bandellei et. al. [16], Bandelli and Rajagopal [17], Tan et al. [18], Junqi et al [19], Mingyu et al. [20], Fosdick and Bertstein [21], and Hayat et al. [22, 23]. In all these studies the flows have been considered in a non-porous medium. Moreover, Pop studied many flow problems in non-Newtonian fluid [24–26]. Quite recently, Tan and Masuoka [27] and Jordon and Puri [28] discussed the Stokes' first problem in a second grade fluid filling the porous half space.

On the other hand the simplest subclass of rate type fluids is called the Maxwell fluid. Although the Maxwell fluid model [29] can describe the stress relaxation, it cannot predict the shear thinning / shear thickening and normal stress effects. Due to simplicity, the Maxwell model is widely used by the researchers in the past. The Maxwell model is useful in the study of dilute polymeric fluids where the dimensionless relaxation time is much less than one [30].

Some recent investigations that explore the features of Maxwell fluid have been made by Hayat et al. [31–38], Fetecau and Fetecau [39–41], Fetecau and Zierep [42], Vieru et al. [43] and Tan et al. [44,45]. It should be further noted that Maxwell did not develop model for polymeric liquids. He recognized that such a fluid has a means of storing energy and a means for dissipating energy. The storing of energy is due to the fluid elasticity and dissipation of energy is because of the fluid viscosity. Recently, Rajagopal and Srinivasa [46] argued that Oldroyd-B fluid is one which stores energy like a linearized solid. The Oldroyd-B fluid model involves three material constants namely a viscosity, a relaxation time and a retardation time. This fluid model can describe the stress relaxation, creep and the normal stress differences but cannot predict the shear thinning / thickening effects. This fluid model is quite popular among the researchers. Some studies made regarding the flows of Oldroyd-B fluids include Fetecau [47,48], Fetecau and Fetecau [49,50], Rajagopal and Bhatnagar [51], Fetecau et al. [52], Hayat et al. [53,54], Chen et al. [55], Lozinski et al. [56], Phillips et al. [57], Huang et al. [58], Alves et al [59] and Khan et al. [60] Tan and Masuoka [61] also analyzed the Stokes first problem for an Oldroyd-B fluid filling the semi infinite porous space. Very recently Ravindran et al. [62] examined the steady flow of a Burgers' fluid in an orthgonal rheometer. In continuation Hayat et al [63] reported some simple flows of a Burgers' fluid.

1.2 Electromagnetic concepts

It is well known that the field of magnetohydrodynamics (MHD) is complex because it involves the solution of momentum equation characterizing fluid flow and Maxwell's equations for the magnetic field. In magnetofluid mechanics, Maxwell's equations are presented as follows:

$$\nabla.\mathbf{B} = 0, \tag{1.1}$$

$$\nabla.\mathbf{D} = 0, \tag{1.2}$$

$$\nabla \times \mathbf{H} = \mathbf{J}, \tag{1.3}$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}, \tag{1.4}$$

where **J** designates the current density and by Ohm's law it is given by

$$\mathbf{J} = \sigma[\mathbf{E} + \mathbf{V} \times \mathbf{B}] \tag{1.5}$$

in which σ is electric conductivity and magnetic flux density **B** is

$$\mathbf{B} = \mu_e \mathbf{H}, \tag{1.6}$$

$$\mathbf{D} = \epsilon \mathbf{E} \tag{1.7}$$

where μ_e the magnetic permeability and **E** is the electric field intensity.

Through combination of above equations we write

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{V} \times \mathbf{B}) + \nu_m \nabla^2 B. \tag{1.8}$$

Here **H** is replaced by $\mathbf{B}/\mu_e$ and $\nu_m = 1/\sigma\mu_e$. In the momentum equation, we have to include the electromagnetic force, $\mathbf{F}_m$, which is

$$\mathbf{F}_m = \mathbf{J} \times \mathbf{B} = \sigma[\mathbf{V} \times \mathbf{B}] \times \mathbf{B}. \tag{1.9}$$

1.3 Governing equations

Our interest in this thesis lies for an incompressible MHD fluid in a porous medium. The relevant fundamental equations are as follows.

1.3.1 Continuity equation

$$\nabla.\mathbf{V} = 0. \tag{1.10}$$

1.3.2 Momentum equation

$$\rho\frac{d\mathbf{V}}{dt} = -\nabla p + div\mathbf{S} + \mathbf{J} \times \mathbf{B} + \mathbf{R}. \tag{1.11}$$

In above equation $\mathbf{V}$ is the velocity field, ρ is the fluid density, p is pressure, $\mathbf{S}$ is an extra stress tensor, $\mathbf{R}$ is the Darcy resistance and the material derivative d/dt is

$$\frac{d}{dt} = \frac{\partial}{\partial t} + (\mathbf{V}.\nabla). \tag{1.12}$$

1.3.3 Generalized Ohms' law

If the Hall effects are present, Eq. (1.5) modifies to the following expression

$$\mathbf{J} + \frac{\omega_e \tau_e}{B_0} (\mathbf{J} \times \mathbf{B}) = \sigma[\mathbf{E} + \mathbf{V} \times \mathbf{B} + \frac{1}{en_e} \nabla p_e] \tag{1.13}$$

in which ω_e is the cyclotron frequency of electrons, τ_e is the electron collision time, e is the electron charge, p_e is the electron pressure and n_e is the number density of electrons. It should be pointed out that $\omega_e \tau_e \sim o(1)$, $\omega_i \tau_i << 1$ (ω_i and τ_i are respective cyclotron frequency and collision time for ions) and ion-slip and thermoelectric effects are not included.

1.4 Integral transforms

An integral transform T is defined by

$$(Tf)(u) = \int_{t_1}^{t_2} K(t, u) f(t) \, dt. \tag{1.14}$$

Note that the input of this transform is a function f, the output is another function Tf and K is called the kernel or nucleus of the transform. Through different choice of K, we have different transforms. Some kernels have an associated inverse kernel $K^{-1}(u, t)$ which yields an inverse transform

$$f(t) = \int_{u_1}^{u_2} K^{-1}(u, t) (Tf)(u) \, du. \tag{1.15}$$

1.4.1 Laplace transform

The Laplace transform of a function $f(x)$ is

$$f(p) = \int_0^\infty f(x) e^{-px} dx \tag{1.16}$$

which is obtained from $f(x)$ by multiplying by e^{-px} and integrating with respect to x from 0 to ∞. The function $f(p)$ defined in this way is obviously a function of the variable p. It is called the Laplace transform of the function $f(x)$. The function e^{-px} is called the kernel of transform. The inverse Laplace transform is given by

$$f(x) = \frac{1}{2\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} f(p)\, e^{px} dp. \tag{1.17}$$

1.4.2 Fourier sine transform

For any function $f(t)$ we define the Fourier sine transform $F(\omega)$ as

$$F(\omega) = \mathcal{F}(f)(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t)\, e^{-i\omega t} dt. \tag{1.18}$$

Utilizing Euler's formula one obtains

$$F(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) \cos\omega t dt - \frac{i}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) \sin\omega t dt. \tag{1.19}$$

For odd $f(t)$, the first integral must vanish to zero and the second may be simplified to give

$$F(\omega) = -i\sqrt{\frac{2}{\pi}} \int_{-\infty}^{\infty} f(t) \sin\omega t dt, \tag{1.20}$$

which is the Fourier sine transform. It is clear that transformed function $F(\omega)$ is also an odd function and a similar analysis of the general inverse transform yields a second sine transform, namely

$$f(t) = i\sqrt{\frac{2}{\pi}} \int_{-\infty}^{\infty} F(\omega) \sin\omega t d\omega \tag{1.21}$$

Note that the numerical factors in the transforms are defined uniquely only by their product, as for general continuous Fourier transforms. For this reason the imaginary units i and $-i$ can be omitted, with the more commonly seen forms of the Fourier sine transforms being

$$F(\omega) = \sqrt{\frac{2}{\pi}} \int_{0}^{\infty} f(t) \sin\omega t dt \tag{1.22}$$

and inverse transform is

$$f(t) = \sqrt{\frac{2}{\pi}} \int_0^\infty F(\omega) \sin \omega t d\omega. \tag{1.23}$$

Chapter 2

Oscillatory flows of second grade fluid in a porous space

This chapter is an analytical study with an aim to show the influence of Hall current on the three flow problems in a second grade fluid. These flows are induced by an oscillating pressure gradient and cosine and sine oscillations of a plate. The second grade fluid exhibits the normal stresses and fills the porous space. The flow modelling is based upon modified Darcy's law. The flow problems are solved analytically by the Fourier sine transform method. The results of velocity are calculated and discussed for the emerging flow parameters.

2.1 Governing equations

We consider an incompressible unidirectional flow of a second grade fluid in a porous medium.The flow considered is parallel to the x-axis. The fluid is electrically conducting in the presence of a uniform magnetic field applied transversely to the flow. The Hall effects are taken into account. The magnetic Reynolds number is taken small so that the induced magnetic field is neglected.

The unsteady flow in a porous medium is governed by the equation of motion (1.11), continuity equation (1.10) and the Maxwell equations (1.1)-(1.4). If the Hall term is retained, the current density $\mathbf{J}$ is given by equation (1.5).

For second grade fluid an extra stress tensor $\mathbf{S}$ is given in [7]

$$\mathbf{S} = \mu \mathbf{A}_1 + \alpha_1 \mathbf{A}_2 + \alpha_2 \mathbf{A}_{1,}^2 \tag{2.1}$$

where μ is the dynamic viscosity and $\alpha_i \; (i = 1, 2)$ are material constants satisfying $\alpha_1 \geq 0, \alpha_1 + \alpha_2 = 0$ [1, 2]. Note that $\alpha_1 \geq 0$ indicates that the free energy is minimum in equilibrium. Furthermore, $\alpha_1 + \alpha_2 = 0$ is required to satisfy the Clausius-Duhem inequality. The first two Rivlin-Ericksen tensors $\mathbf{A}_1$ and $\mathbf{A}_2$ are defined as

$$\begin{aligned} \mathbf{A}_1 &= (\text{grad } \mathbf{V}) + (\text{grad } \mathbf{V})^T, \\ \mathbf{A}_2 &= \frac{d\mathbf{A}_1}{dt} + \mathbf{A}_1 (\text{grad } \mathbf{V}) + (\text{grad } \mathbf{V})^T \mathbf{A}_1. \end{aligned}$$

The velocity field is of the form

$$\mathbf{V} = (u(y,t), 0, 0).$$

It is well known that in an unbounded porous medium Darcy's law holds for such viscous flows which have low speed. This law provides a relationship between the pressure drop induced by the frictional drag and velocity. The literature on the topic is quite extensive for viscous flows. Very little efforts have been devoted to mathematical macroscopic filtration models concerning viscoelastic flows in a porous medium. On the basis of Oldroyd constitutive equation, the following law describing both relaxation and retardation phenomenon in an unbounded porous medium has been suggested [27]

$$\left(1 + \lambda \frac{\partial}{\partial t}\right) \nabla p = -\frac{\mu}{k} \left(1 + \theta \frac{\partial}{\partial t}\right) \mathbf{V}_D. \tag{2.2}$$

In above equation λ and θ are the relaxation and retardation times respectively, k is the permeability, $\mathbf{V}_D \; (= \phi \mathbf{V})$ is the Darcian velocity and ϕ is the porosity. From equation (2.1), we have $S_{xx} = S_{yy} = \alpha_2 (\partial u / \partial y)^2$

$$S_{yx} = S_{xy} = \left(\mu + \alpha_1 \frac{\partial}{\partial t}\right) \frac{\partial u}{\partial y} \tag{2.3}$$

and $S_{zz} = S_{xz} = S_{yz} = 0$.

Since the constitutive equation for unidirectional flows of a second grade fluid is quite similar to that of an Oldroyd-B fluid. Note that equation (2.3) can be obtained from the constitutive equation of an Oldroyd-B fluid by taking $\lambda = 0$. Due to this analogy, equation (2.2) for a second grade fluid thus becomes

$$\nabla p = -\frac{\mu\phi}{k}\left(1+\theta\frac{\partial}{\partial t}\right)\mathbf{V}. \tag{2.4}$$

It should be pointed out here that Eqs. (2.2) and (2.4) respectively reduce into classical Darcy's law for $\lambda = 0$ and $\theta = 0$. Since the pressure gradient in equation (2.4) can also be interpreted as a measure of the resistance to flow in the bulk of the porous medium and $\mathbf{R}$ is a measure of the flow resistance offered by the solid matrix. Therefore $\mathbf{R}$ can be inferred from equation (2.4) to satisfy the following equation:

$$\mathbf{R} = -\frac{\mu\phi}{k}\left(1+\theta\frac{\partial}{\partial t}\right)\mathbf{V}. \tag{2.5}$$

Upon making use of the stated assumptions, the incompressibility condition (1.10) is automatically satisfied and equation (1.11) gives

$$\rho\frac{\partial u}{\partial t} = -\frac{\partial \widehat{p}}{\partial x} + \mu\frac{\partial^2 u}{\partial y^2} + \alpha_1\frac{\partial^3 u}{\partial y^2 \partial t} - \frac{\sigma B_0^2}{1-im}u - \frac{\mu\phi}{k}\left(1+\frac{\alpha_1}{\mu}\frac{\partial}{\partial t}\right)u, \tag{2.6}$$

where $m = \omega_e \tau_e$ is the Hall parameter , $\alpha_1 = \mu\theta$ and $\widehat{p}$ is the modified pressure given by $\widehat{p} = p - (2\alpha_1 + \alpha_2)\left(\frac{\partial u}{\partial y}\right)^2$ Here, we note that the viscoelasticity of fluid increases the order of the differential equation. In order to solve a well-posed problem, we usually require an additional boundary (initial) condition. This, however, may not be necessary in a specific problem. The comprehensive discussion regarding this issue of the boundary conditions has been given by Rajagopal [3] and Rajagopal and Gupta [5].

In the next three sections, we are now going to solve equation (2.6) for three problems.

2.2 Stokes' second problem

Here, the fluid is over an infinite non-conducting flat plate at $y = 0$. The fluid $(y > 0)$ is electrically conducting and magnetic field $\mathbf{B}_0$ is applied in the $y-$direction. No pressure gradient is applied and flow in the fluid is because of the plate oscillations for $t > 0$. The problem thus is of the following form

$$\rho\frac{\partial u}{\partial t}=\mu\frac{\partial^2 u}{\partial y^2}+\alpha_1\frac{\partial^3 u}{\partial y^2\partial t}-\frac{\sigma B_0^2}{1-im}u-\frac{\mu\phi}{k}\left(1+\frac{\alpha_1}{\mu}\frac{\partial}{\partial t}\right)u, \tag{2.7}$$

$$\begin{aligned} u(0,t) &= U_0\cos\omega t \text{ or } u(0,t)=U_0\sin\omega t;\ t>0, \\ u,\frac{\partial u}{\partial y} &\rightarrow 0 \text{ as } y\rightarrow\infty,\quad t\geq 0, \end{aligned} \tag{2.8}$$

$$u(y,0)=0,\quad y>0, \tag{2.9}$$

in which ω is the imposed frequency.

Note that the solution obtained by Laplace transform method holds for small times. In second grade fluid such solution in Stokes first problem does not satisfy the initial condition. This is not a trivial matter. Literature survey witnesses that such difficulty arises because of the incompatibility in the prescribed data.

We will find the solution using Fourier sine transform pair defined by equations (1.22) and(1.23).

2.2.1 When $u(0,t)=U_0\cos\omega t$

The transformed problem is

$$\frac{d\overline{u}}{dt}+P\overline{u}=\sqrt{\frac{2}{\pi}}\frac{\xi U_\circ}{\left(1+\frac{\alpha\phi}{k}+\alpha\xi^2\right)}\left(\nu\cos\omega t-\alpha\omega\sin\omega t\right), \tag{2.10}$$

$$\overline{u}(\xi,0)=0;\quad \xi>0, \tag{2.11}$$

where ν is the kinematic viscosity, $\alpha=\alpha_{1/\rho}$ and

$$P=\left(\nu\xi^2+\frac{\sigma B_0^2(1+im)}{\rho(1+m^2)}+\frac{\phi\nu}{k}\right)\left(1+\frac{\alpha\phi}{k}+\alpha\xi^2\right)^{-1}. \tag{2.12}$$

The solution of equations (2.10) and (2.11) is

$$\begin{aligned}
\overline{u}\left(\xi,t\right) &= \frac{2}{\pi}U_0\cos\omega t\frac{\xi\left(\left(1+\frac{\alpha\phi}{k}+\alpha\xi^2\right)\alpha\omega^2+\left(\nu^2\xi^2+\frac{\sigma B_\circ^2(1+im)}{\rho(1+m^2)}\nu+\frac{\phi\nu^2}{k}\right)\right)}{\left(\nu\xi^2+\frac{\sigma B_\circ^2(1+im)}{\rho(1+m^2)}+\frac{\phi\nu}{k}\right)^2+\omega^2\left(1+\frac{\alpha\phi}{k}+\alpha\xi^2\right)^2} \\
&+\frac{2}{\pi}U_\circ\omega\sin\omega t\frac{\xi\left(\nu-\frac{\alpha\sigma B_\circ^2(1+im)}{\rho(1+m^2)}\right)}{\left(\nu\xi^2+\frac{\sigma B_\circ^2(1+im)}{\rho(1+m^2)}+\frac{\phi\nu}{k}\right)^2+\omega^2\left(1+\frac{\alpha\phi}{k}+\alpha\xi^2\right)^2} \\
&-\frac{2}{\pi}U_\circ\frac{\xi\left(\left(1+\frac{\alpha\phi}{k}+\alpha\xi^2\right)\alpha\omega^2+\left(\nu^2\xi^2+\frac{\sigma B_\circ^2(1+im)}{\rho(1+m^2)}\nu+\frac{\phi\nu^2}{k}\right)\right)}{\left(\nu\xi^2+\frac{\sigma B_\circ^2(1+im)}{\rho(1+m^2)}+\frac{\phi\nu}{k}\right)^2+\omega^2\left(1+\frac{\alpha\phi}{k}+\alpha\xi^2\right)^2} \\
&\times\exp\left(-\frac{\left(\nu\xi^2+\frac{\sigma B_\circ^2(1+im)}{\rho(1+m^2)}+\frac{\phi\nu}{k}\right)t}{\left(1+\frac{\alpha\phi}{k}+\alpha\xi^2\right)}\right). \qquad (2.13)
\end{aligned}$$

Inverting Eq.(2.13) by means of Fourier sine transform we have

$$\begin{aligned}
u\left(y,t\right) &= \frac{2}{\pi}U_0\cos\omega t\int_0^\infty\frac{\xi\left(\left(1+\frac{\alpha\phi}{k}+\alpha\xi^2\right)\alpha\omega^2+\left(\nu^2\xi^2+\frac{\sigma B_\circ^2(1+im)}{\rho(1+m^2)}\nu+\frac{\phi\nu^2}{k}\right)\right)}{\left(\nu\xi^2+\frac{\sigma B_\circ^2(1+im)}{\rho(1+m^2)}+\frac{\phi\nu}{k}\right)^2+\omega^2\left(1+\frac{\alpha\phi}{k}+\alpha\xi^2\right)^2}\sin\left(\xi y\right)d\xi \\
&+\frac{2}{\pi}U_\circ\omega\sin\omega t\int_0^\infty\frac{\xi\left(\nu-\frac{\alpha\sigma B_\circ^2(1+im)}{\rho(1+m^2)}\right)}{\left(\nu\xi^2+\frac{\sigma B_\circ^2(1+im)}{\rho(1+m^2)}+\frac{\phi\nu}{k}\right)^2+\omega^2\left(1+\frac{\alpha\phi}{k}+\alpha\xi^2\right)^2}\sin\left(\xi y\right)d\xi \\
&-\frac{2}{\pi}U_\circ\int_0^\infty\frac{\xi\left(\left(1+\frac{\alpha\phi}{k}+\alpha\xi^2\right)\alpha\omega^2+\left(\nu^2\xi^2+\frac{\sigma B_\circ^2(1+im)}{\rho(1+m^2)}\nu+\frac{\phi\nu^2}{k}\right)\right)}{\left(\nu\xi^2+\frac{\sigma B_\circ^2(1+im)}{\rho(1+m^2)}+\frac{\phi\nu}{k}\right)^2+\omega^2\left(1+\frac{\alpha\phi}{k}+\alpha\xi^2\right)^2} \\
&\times\exp\left(-\frac{\left(\nu\xi^2+\frac{\sigma B_\circ^2(1+im)}{\rho(1+m^2)}+\frac{\phi\nu}{k}\right)t}{\left(1+\frac{\alpha\phi}{k}+\alpha\xi^2\right)}\right)\sin\left(\xi y\right)d\xi. \qquad (2.14)
\end{aligned}$$

The above equation can also be written as [15, 64]

$$
\begin{aligned}
u(y,t) &= U_0\cos\omega t \\
&-\frac{2}{\pi}U_0\cos\omega t\int_0^\infty \frac{\left(\begin{array}{c}\left(\frac{\sigma B_0^2(1+im)}{\rho(1+m^2)}+\frac{\phi\nu}{k}\right)\left(\nu\xi^2+\frac{\sigma B_0^2(1+im)}{\rho(1+m^2)}+\frac{\phi\nu}{k}\right)\\ +\omega^2\left(1+\frac{\alpha\phi}{k}\right)\left(1+\frac{\alpha\phi}{k}+\alpha\xi^2\right)\end{array}\right)}{\xi\left(\left(\nu\xi^2+\frac{\sigma B_0^2(1+im)}{\rho(1+m^2)}+\frac{\phi\nu}{k}\right)^2+\omega^2\left(1+\frac{\alpha\phi}{k}+\alpha\xi^2\right)^2\right)}\sin(\xi y)\,d\xi \\
&+\frac{2}{\pi}U_0\omega\sin\omega t\int_0^\infty \frac{\xi\left(\nu-\frac{\alpha\sigma B_0^2(1+im)}{\rho(1+m^2)}\right)}{\left(\nu\xi^2+\frac{\sigma B_0^2(1+im)}{\rho(1+m^2)}+\frac{\phi\nu}{k}\right)^2+\omega^2\left(1+\frac{\alpha\phi}{k}+\alpha\xi^2\right)^2}\sin(\xi y)\,d\xi \\
&-\frac{2}{\pi}U_0\int_0^\infty \frac{\xi\left(\left(1+\frac{\alpha\phi}{k}+\alpha\xi^2\right)\alpha\omega^2+\left(\nu^2\xi^2+\frac{\sigma B_0^2(1+im)}{\rho(1+m^2)}\nu+\frac{\phi\nu^2}{k}\right)\right)}{\left(\nu\xi^2+\frac{\sigma B_0^2(1+im)}{\rho(1+m^2)}+\frac{\phi\nu}{k}\right)^2+\omega^2\left(1+\frac{\alpha\phi}{k}+\alpha\xi^2\right)^2} \\
&\times\exp\left(-\frac{\left(\nu\xi^2+\frac{\sigma B_0^2(1+im)}{\rho(1+m^2)}+\frac{\phi\nu}{k}\right)t}{\left(1+\frac{\alpha\phi}{k}+\alpha\xi^2\right)}\right)\sin(\xi y)\,d\xi.
\end{aligned}
\tag{2.15}
$$

Note that starting solution $u(y,t)$ in Eq.(2.15) is a sum of three terms. The first four terms represent the steady state solution u_s (valid for large times) and fourth term gives the transient solution u_t. Clearly solution (2.15) satisfies Eqs.(2.10)$-$(2.12). The steady state solution is

$$
\begin{aligned}
u_s(y,t) &= U_0\cos\omega t-\frac{2}{\pi}U_0\cos\omega t \\
&\times\int_0^\infty \frac{\left(\begin{array}{c}\left(\frac{\sigma B_0^2(1+im)}{\rho(1+m^2)}+\frac{\phi\nu}{k}\right)\left(\nu\xi^2+\frac{\sigma B_0^2(1+im)}{\rho(1+m^2)}+\frac{\phi\nu}{k}\right)\\ +\omega^2\left(1+\frac{\alpha\phi}{k}\right)\left(1+\frac{\alpha\phi}{k}+\alpha\xi^2\right)\end{array}\right)}{\xi\left(\left(\nu\xi^2+\frac{\sigma B_0^2(1+im)}{\rho(1+m^2)}+\frac{\phi\nu}{k}\right)^2+\omega^2\left(1+\frac{\alpha\phi}{k}+\alpha\xi^2\right)^2\right)}\sin(\xi y)\,d\xi \\
&+\frac{2}{\pi}U_0\omega\sin\omega t\int_0^\infty \frac{\xi\left(\nu-\frac{\alpha\sigma B_0^2(1+im)}{\rho(1+m^2)}\right)}{\left(\nu\xi^2+\frac{\sigma B_0^2(1+im)}{\rho(1+m^2)}+\frac{\phi\nu}{k}\right)^2+\omega^2\left(1+\frac{\alpha\phi}{k}+\alpha\xi^2\right)^2}\sin(\xi y)\,d\xi.
\end{aligned}
\tag{2.16}
$$

In order to simplify equation (2.16) we use [64]

$$
\int_0^\infty \frac{x\sin(ax)}{(x^2+\epsilon b^2)^2+c^2}dx=\frac{\pi}{2c}\exp(-aA)\sin(aB), \tag{2.17}
$$

$$\int_0^\infty \frac{x\left(x^2+\epsilon b^2\right)\sin(ax)}{\left(x^2+\epsilon b^2\right)^2+c^2}dx = \frac{\pi}{2c}\exp\left(-aA\right)\cos\left(aB\right), \tag{2.18}$$

where

$$2A^2 = \sqrt{b^4+c^2}+\epsilon b^2,\ 2B^2 = \sqrt{b^4+c^2}-\epsilon b^2 \tag{2.19}$$

and

$$\epsilon = \pm 1.$$

After simplifying equation (2.16) can be written as

$$\begin{aligned} u_s\left(y,t\right) &= \frac{2}{\pi}U_0\cos\omega t\int_0^\infty \frac{\xi\left(\xi^2+\frac{\sigma B_0^2(1+im)\nu}{\rho(1+m^2)(\nu^2+\alpha^2\omega^2)}+\frac{\phi}{k}+\frac{\alpha\omega^2}{(\nu^2+\alpha^2\omega^2)}\right)\sin\left(\xi y\right)d\xi}{\left(\xi^2+\frac{\sigma B_0^2(1+im)\nu}{\rho(1+m^2)(\nu^2+\alpha^2\omega^2)}+\frac{\phi}{k}+\frac{\alpha\omega^2}{(\nu^2+\alpha^2\omega^2)}\right)^2+\frac{\omega^2\left(\nu-\frac{\alpha\sigma B_0^2(1+im)}{\rho(1+m^2)}\right)^2}{(\nu^2+\alpha^2\omega^2)^2}} \\ &+\frac{2}{\pi}U_0\frac{\omega\left(\nu-\frac{\alpha\sigma B_0^2(1+im)}{\rho(1+m^2)}\right)}{\left(\nu^2+\alpha^2\omega^2\right)} \\ &\times\sin\omega t\int_0^\infty \frac{\xi\sin\left(\xi y\right)d\xi}{\left(\xi^2+\frac{\sigma B_0^2(1+im)\nu}{\rho(1+m^2)(\nu^2+\alpha^2\omega^2)}+\frac{\phi}{k}+\frac{\alpha\omega^2}{(\nu^2+\alpha^2\omega^2)}\right)^2+\frac{\omega^2\left(\nu-\frac{\alpha\sigma B_0^2(1+im)}{\rho(1+m^2)}\right)^2}{(\nu^2+\alpha^2\omega^2)^2}}, \end{aligned}$$

or

$$\begin{aligned} u_s\left(y,t\right) &= \frac{2}{\pi}U_0\cos\omega t\int_0^\infty \frac{\xi\left(\xi^2+b^2\right)\sin\left(\xi y\right)d\xi}{\left(\xi^2+b^2\right)^2+c^2} \\ &+\frac{2c}{\pi}U_0\sin\omega t\int_0^\infty \frac{\xi\sin\left(\xi y\right)d\xi}{\left(\xi^2+b^2\right)^2+c^2}, \end{aligned}$$

or

$$u_s\left(y,t\right) = U_0e^{-Ay}\cos\left(\omega t-By\right), \tag{2.20}$$

whence

$$2A^2 = \frac{1}{\left(\nu^2+\alpha^2\omega^2\right)}\left(\begin{array}{c}\sqrt{\begin{array}{c}\left(\frac{\sigma B_0^2(1+im)\nu}{\rho(1+m^2)}+\frac{\phi}{k}\left(\nu^2+\alpha^2\omega^2\right)+\alpha\omega^2\right)^2 \\ +\omega^2\left(\nu-\frac{\alpha\sigma B_0^2(1+im)}{\rho(1+m^2)}\right)^2\end{array}} \\ +\left(\frac{\sigma B_0^2(1+im)\nu}{\rho(1+m^2)}+\frac{\phi}{k}\left(\nu^2+\alpha^2\omega^2\right)+\alpha\omega^2\right)\end{array}\right), \tag{2.21}$$

$$2B^2 = \frac{1}{(\nu^2+\alpha^2\omega^2)}\left(\begin{array}{c}\sqrt{\begin{array}{c}\left(\frac{\sigma B_0^2(1+im)\nu}{\rho(1+m^2)}+\frac{\phi}{k}\left(\nu^2+\alpha^2\omega^2\right)+\alpha\omega^2\right)^2\\ +\omega^2\left(\nu-\frac{\alpha\sigma B_0^2(1+im)}{\rho(1+m^2)}\right)^2\end{array}}\\ -\frac{\sigma B_0^2(1+im)\nu}{\rho(1+m^2)}+\frac{\phi}{k}\left(\nu^2+\alpha^2\omega^2\right)+\alpha\omega^2\end{array}\right). \tag{2.22}$$

Introducing the following dimensionless quantities

$$\begin{aligned}\widetilde{y} &= \sqrt{\frac{\omega}{2\nu}}y,\quad \widetilde{u}=\frac{u}{U_0},\quad \widetilde{t}=\omega t\\ \widetilde{\alpha} &= \frac{\alpha\omega}{\nu},\quad M^2=\frac{\sigma B_0^2}{\rho\omega},\quad \frac{1}{K}=\frac{\phi\nu}{k\omega}\end{aligned}$$

equation (2.20) takes the following form

$$u_s(y,t) = e^{-\widetilde{A}\widetilde{y}}\cos\left(\widetilde{t}-\widetilde{B}\widetilde{y}\right),$$

where

$$\widetilde{A} = \frac{1}{(1+\widetilde{\alpha}^2)}\left(\begin{array}{c}\sqrt{\left(\frac{M^2(1+im)}{(1+m^2)}+\frac{(1+\widetilde{\alpha}^2)}{K}+\widetilde{\alpha}\right)^2+\left(1-\frac{\widetilde{\alpha}M^2(1+im)}{(1+m^2)}\right)^2}\\ +\left(\frac{M^2(1+im)}{(1+m^2)}+\frac{(1+\widetilde{\alpha}^2)}{K}+\widetilde{\alpha}\right),\end{array}\right),$$

$$\widetilde{B} = \frac{1}{(1+\widetilde{\alpha}^2)}\left(\begin{array}{c}\sqrt{\left(\frac{M^2(1+im)}{(1+m^2)}+\frac{(1+\widetilde{\alpha}^2)}{K}+\widetilde{\alpha}\right)^2+\left(1-\frac{\widetilde{\alpha}M^2(1+im)}{(1+m^2)}\right)^2}\\ -\left(\frac{M^2(1+im)}{(1+m^2)}+\frac{(1+\widetilde{\alpha}^2)}{K}+\widetilde{\alpha}\right)\end{array}\right).$$

2.2.2 For $u(0,t) = U_0 \sin \omega t$

Employing the similar procedure as for $U_\circ \cos \omega t$, the starting solution and steady state solution in this case are respectively given by

$$\begin{aligned} u(y,t) &= \frac{2}{\pi}U_0 \sin \omega t \int_0^\infty \frac{\xi\left(\left(1+\frac{\alpha\phi}{k}+\alpha\xi^2\right)\alpha\omega^2+\left(\nu^2\xi^2+\frac{\sigma B_0^2(1+im)}{\rho(1+m^2)}\nu+\frac{\phi\nu^2}{k}\right)\right)}{\left(\nu\xi^2+\frac{\sigma B_\circ^2(1+im)}{\rho(1+m^2)}+\frac{\phi\nu}{k}\right)^2+\omega^2\left(1+\frac{\alpha\phi}{k}+\alpha\xi^2\right)^2}\sin(\xi y)\,d\xi \\ &-\frac{2}{\pi}U_\circ\omega\cos\omega t\int_0^\infty \frac{\xi\left(\nu-\frac{\alpha\sigma B_\circ^2(1+im)}{\rho(1+m^2)}\right)}{\left(\nu\xi^2+\frac{\sigma B_\circ^2(1+im)}{\rho(1+m^2)}+\frac{\phi\nu}{k}\right)^2+\omega^2\left(1+\frac{\alpha\phi}{k}+\alpha\xi^2\right)^2}\sin(\xi y)\,d\xi \\ &+\frac{2}{\pi}U_\circ\omega\left(\nu-\frac{\alpha\sigma B_\circ^2(1+im)}{\rho(1+m^2)}\right) \\ &\times\int_0^\infty \frac{\xi\exp\left(\frac{-\left(\nu\xi^2+\frac{\sigma B_\circ^2(1+im)}{\rho(1+m^2)}+\frac{\phi\nu}{k}\right)t}{\left(1+\frac{\alpha\phi}{k}+\alpha\xi^2\right)}\right)\sin(\xi y)\,d\xi}{\left(\nu\xi^2+\frac{\sigma B_\circ^2(1+im)}{\rho(1+m^2)}+\frac{\phi\nu}{k}\right)^2+\omega^2\left(1+\frac{\alpha\phi}{k}+\alpha\xi^2\right)^2}, \end{aligned} \tag{2.23}$$

$$\begin{aligned} u_s(y,t) &= U_\circ \sin\omega t - \frac{2}{\pi}U_0\sin\omega t \\ &\times\int_0^\infty \frac{\left(\begin{array}{c}\left(\frac{\sigma B_0^2(1+im)}{\rho(1+m^2)}+\frac{\phi\nu}{k}\right)\left(\nu\xi^2+\frac{\sigma B_0^2(1+im)}{\rho(1+m^2)}+\frac{\phi\nu}{k}\right) \\ +\omega^2\left(1+\frac{\alpha\phi}{k}\right)\left(1+\frac{\alpha\phi}{k}+\alpha\xi^2\right)\end{array}\right)}{\xi\left(\left(\nu\xi^2+\frac{\sigma B_0^2(1+im)}{\rho(1+m^2)}+\frac{\phi\nu}{k}\right)^2+\omega^2\left(1+\frac{\alpha\phi}{k}+\alpha\xi^2\right)^2\right)}\sin(\xi y)\,d\xi \\ &-\frac{2}{\pi}U_0\omega\cos\omega t \\ &\times\int_0^\infty \frac{\xi\left(\nu-\frac{\alpha\sigma B_0^2(1+im)}{\rho(1+m^2)}\right)}{\left(\nu\xi^2+\frac{\sigma B_0^2(1+im)}{\rho(1+m^2)}+\frac{\phi\nu}{k}\right)^2+\omega^2\left(1+\frac{\alpha\phi}{k}+\alpha\xi^2\right)^2}\sin(\xi y)\,d\xi. \end{aligned} \tag{2.24}$$

Equation (2.24) can also be simplified as

$$u_s(y,t) = U_0 e^{-Ay}\sin(\omega t - By). \tag{2.25}$$

The dimensionless form of above solution is

$$u_s(y,t) = e^{-\widetilde{A}\widetilde{y}} \sin\left(\widetilde{t} - \widetilde{B}\widetilde{y}\right).$$

2.3 Modified Stokes' second problem

In this section we consider an electrically conducting fluid between two infinite insulating plates at $y = 0$ and $y = d$. The lower plate is oscillating for $t > 0$ and the upper plate is at rest. The governing problem consists of equations (2.7), (2.8) and

$$u(d,t) = 0; \quad t > 0, \tag{2.26}$$

$$u(y,0) = 0, \quad 0 < y < d. \tag{2.27}$$

We will find the solution of the problem here by using finite Fourier sine transform pair defined by

$$\overline{u}_n(\xi,t) = \int_0^d u(y,t)\sin\left(\frac{n\pi}{d}y\right)dy, \tag{2.28}$$

$$u(y,t) = \frac{2}{d}\sum_{n=1}^{\infty}\overline{u}_n(\xi,t)\sin\left(\frac{n\pi}{d}y\right). \tag{2.29}$$

2.3.1 For $u(0,t) = U_0 \cos\omega t$

The governing problem in transformed domain is

$$\frac{d\overline{u}_n}{dt} + Q\overline{u}_n = \frac{\lambda_n U_o}{\left(1 + \frac{\alpha\phi}{k} + \alpha\lambda_n^2\right)}\left(\nu\cos\omega t - \alpha\omega\sin\omega t\right), \tag{2.30}$$

$$\overline{u}_n(\xi,0) = 0; \quad n = 1, 2, \ldots \tag{2.31}$$

in which

$$Q = \left(\nu\lambda_n^2 + \frac{\sigma B_0^2(1+im)}{\rho(1+m^2)} + \frac{\phi\nu}{k}\right)\left(1 + \frac{\alpha\phi}{k} + \alpha\lambda_n^2\right)^{-1}. \tag{2.32}$$

The solution of equation (2.30) subject to the condition (2.31) after using equation (2.29) is

$$
\begin{aligned}
u(y,t) &= \frac{2}{d}U_0\cos\omega t\sum_{n=1}^{\infty}\frac{\lambda_n\left(\begin{array}{c}\left(1+\frac{\alpha\phi}{k}+\alpha\lambda_n^2\right)\alpha\omega^2\\ +\left(\nu^2\lambda_n^2+\frac{\sigma B_o^2(1+im)}{\rho(1+m^2)}\nu+\frac{\phi\nu^2}{k}\right)\end{array}\right)\sin\left(\lambda_n y\right)}{\left(\left(\nu\lambda_n^2+\frac{\sigma B_o^2(1+im)}{\rho(1+m^2)}+\frac{\phi\nu}{k}\right)^2+\omega^2\left(1+\frac{\alpha\phi}{k}+\alpha\lambda_n^2\right)^2\right)}\\
&+\frac{2}{d}U_0\omega\sin\omega t\sum_{n=1}^{\infty}\frac{\lambda_n\left(\nu-\frac{\alpha\sigma B_o^2(1+im)}{\rho(1+m^2)}\right)\sin\left(\lambda_n y\right)}{\left(\left(\nu\lambda_n^2+\frac{\sigma B_o^2(1+im)}{\rho(1+m^2)}+\frac{\phi\nu}{k}\right)^2+\omega^2\left(1+\frac{\alpha\phi}{k}+\alpha\lambda_n^2\right)^2\right)}\\
&-\frac{2}{d}U_0\sum_{n=1}^{\infty}\frac{\lambda_n\left(\left(1+\frac{\alpha\phi}{k}+\alpha\lambda_n^2\right)\alpha\omega^2+\left(\nu^2\lambda_n^2+\frac{\sigma B_o^2(1+im)}{\rho(1+m^2)}\nu+\frac{\phi\nu^2}{k}\right)\right)}{\left(\left(\nu\lambda_n^2+\frac{\sigma B_o^2(1+im)}{\rho(1+m^2)}+\frac{\phi\nu}{k}\right)^2+\omega^2\left(1+\frac{\alpha\phi}{k}+\alpha\lambda_n^2\right)^2\right)}\\
&\times\exp\left(\frac{-\left(\nu\lambda_n^2+\frac{\sigma B_o^2(1+im)}{\rho(1+m^2)}+\frac{\phi\nu}{k}\right)t}{\left(1+\frac{\alpha\phi}{k}+\alpha\lambda_n^2\right)}\right)\sin\left(\lambda_n y\right) \qquad (2.33)
\end{aligned}
$$

or

$$
\begin{aligned}
u(y,t) &= U_0\left(1-\frac{y}{d}\right)\cos\omega t\\
&-\frac{2}{d}U_0\cos\omega t\sum_{n=1}^{\infty}\frac{\left(\begin{array}{c}\left(\frac{\sigma B_o^2(1+im)}{\rho(1+m^2)}+\frac{\phi\nu}{k}\right)\left(\nu\lambda_n^2+\frac{\sigma B_o^2(1+im)}{\rho(1+m^2)}+\frac{\phi\nu}{k}\right)\\ +\omega^2\left(1+\frac{\alpha\phi}{k}\right)\left(1+\frac{\alpha\phi}{k}+\alpha\lambda_n^2\right)\end{array}\right)\sin\left(\lambda_n y\right)}{\lambda_n\left(\left(\nu\lambda_n^2+\frac{\sigma B_o^2(1+im)}{\rho(1+m^2)}+\frac{\phi\nu}{k}\right)^2+\omega^2\left(1+\frac{\alpha\phi}{k}+\alpha\lambda_n^2\right)^2\right)}\\
&+\frac{2}{d}U_0\omega\sin\omega t\sum_{n=1}^{\infty}\frac{\lambda_n\left(\nu-\frac{\alpha\sigma B_o^2(1+im)}{\rho(1+m^2)}\right)\sin\left(\lambda_n y\right)}{\left(\left(\nu\lambda_n^2+\frac{\sigma B_o^2(1+im)}{\rho(1+m^2)}+\frac{\phi\nu}{k}\right)^2+\omega^2\left(1+\frac{\alpha\phi}{k}+\alpha\lambda_n^2\right)^2\right)}\\
&-\frac{2}{d}U_0\sum_{n=1}^{\infty}\frac{\lambda_n\left(\left(1+\frac{\alpha\phi}{k}+\alpha\lambda_n^2\right)\alpha\omega^2+\left(\nu^2\lambda_n^2+\frac{\sigma B_o^2(1+im)}{\rho(1+m^2)}\nu+\frac{\phi\nu^2}{k}\right)\right)}{\left(\left(\nu\lambda_n^2+\frac{\sigma B_o^2(1+im)}{\rho(1+m^2)}+\frac{\phi\nu}{k}\right)^2+\omega^2\left(1+\frac{\alpha\phi}{k}+\alpha\lambda_n^2\right)^2\right)}\\
&\times\exp\left(\frac{-\left(\nu\lambda_n^2+\frac{\sigma B_o^2(1+im)}{\rho(1+m^2)}+\frac{\phi\nu}{k}\right)t}{\left(1+\frac{\alpha\phi}{k}+\alpha\lambda_n^2\right)}\right)\sin\left(\lambda_n y\right), \qquad (2.34)
\end{aligned}
$$

$$
\begin{aligned}
u_s(y,t) &= U_0\left(1-\frac{y}{d}\right)\cos\omega t \\
&-\frac{2}{d}U_0\cos\omega t\sum_{n=1}^{\infty}\frac{\left[\begin{array}{c}\left(\frac{\sigma B_o^2(1+im)}{\rho(1+m^2)}+\frac{\phi\nu}{k}\right)\left(\nu\lambda_n^2+\frac{\sigma B_o^2(1+im)}{\rho(1+m^2)}+\frac{\phi\nu}{k}\right)\\+\omega^2\left(1+\frac{\alpha\phi}{k}\right)\left(1+\frac{\alpha\phi}{k}+\alpha\lambda_n^2\right)\end{array}\right]\sin(\lambda_n y)}{\lambda_n\left[\left(\nu\lambda_n^2+\frac{\sigma B_o^2(1+im)}{\rho(1+m^2)}+\frac{\phi\nu}{k}\right)^2+\omega^2\left(1+\frac{\alpha\phi}{k}+\alpha\lambda_n^2\right)^2\right]} \\
&+\frac{2}{d}U_0\omega\sin\omega t\sum_{n=1}^{\infty}\frac{\lambda_n\left[\nu-\frac{\alpha\sigma B_o^2(1+im)}{\rho(1+m^2)}\right]\sin(\lambda_n y)}{\left[\left(\nu\lambda_n^2+\frac{\sigma B_o^2(1+im)}{\rho(1+m^2)}+\frac{\phi\nu}{k}\right)^2+\omega^2\left(1+\frac{\alpha\phi}{k}+\alpha\lambda_n^2\right)^2\right]},
\end{aligned} \tag{2.35}
$$

where $\lambda_n = \frac{n\pi}{d}$.

2.3.2 For $u(0,t) = U_0\sin\omega t$

$$
\begin{aligned}
u(y,t) &= U_0\left(1-\frac{y}{d}\right)\sin\omega t \\
&-\frac{2}{d}U_0\sin\omega t\sum_{n=1}^{\infty}\frac{\left(\begin{array}{c}\left(\frac{\sigma B_o^2(1+im)}{\rho(1+m^2)}+\frac{\phi\nu}{k}\right)\left(\nu\lambda_n^2+\frac{\sigma B_o^2(1+im)}{\rho(1+m^2)}+\frac{\phi\nu}{k}\right)\\+\omega^2\left(1+\frac{\alpha\phi}{k}\right)\left(1+\frac{\alpha\phi}{k}+\alpha\lambda_n^2\right)\end{array}\right)\sin(\lambda_n y)}{\lambda_n\left(\left(\nu\lambda_n^2+\frac{\sigma B_o^2(1+im)}{\rho(1+m^2)}+\frac{\phi\nu}{k}\right)^2+\omega^2\left(1+\frac{\alpha\phi}{k}+\alpha\lambda_n^2\right)^2\right)} \\
&-\frac{2}{d}U_0\omega\cos\omega t\sum_{n=1}^{\infty}\frac{\lambda_n\left(\nu-\frac{\alpha\sigma B_o^2(1+im)}{\rho(1+m^2)}\right)\sin(\lambda_n y)}{\left(\left(\nu\lambda_n^2+\frac{\sigma B_o^2(1+im)}{\rho(1+m^2)}+\frac{\phi\nu}{k}\right)^2+\omega^2\left(1+\frac{\alpha\phi}{k}+\alpha\lambda_n^2\right)^2\right)} \\
&+\frac{2}{d}\omega U_0\sum_{n=1}^{\infty}\frac{\lambda_n\left(\nu-\frac{\alpha\sigma B_o^2(1+im)}{\rho(1+m^2)}\right)}{\left(\left(\nu\lambda_n^2+\frac{\sigma B_o^2(1+im)}{\rho(1+m^2)}+\frac{\phi\nu}{k}\right)^2+\omega^2\left(1+\frac{\alpha\phi}{k}+\alpha\lambda_n^2\right)^2\right)} \\
&\times\exp\left(\frac{-\left(\nu\lambda_n^2+\frac{\sigma B_o^2(1+im)}{\rho(1+m^2)}+\frac{\phi\nu}{k}\right)t}{\left(1+\frac{\alpha\phi}{k}+\alpha\lambda_n^2\right)}\right)\sin(\lambda_n y),
\end{aligned} \tag{2.36}
$$

$$
\begin{aligned}
u_s(y,t) &= U_0\left(1-\frac{y}{d}\right)\sin\omega t \\
&-\frac{2}{d}U_0\sin\omega t\sum_{n=1}^{\infty}\frac{\left(\begin{array}{c}\left(\frac{\sigma B_o^2(1+im)}{\rho(1+m^2)}+\frac{\phi\nu}{k}\right)\left(\nu\lambda_n^2+\frac{\sigma B_o^2(1+im)}{\rho(1+m^2)}+\frac{\phi\nu}{k}\right)\\+\omega^2\left(1+\frac{\alpha\phi}{k}\right)\left(1+\frac{\alpha\phi}{k}+\alpha\lambda_n^2\right)\end{array}\right)\sin(\lambda_n y)}{\lambda_n\left(\left(\nu\lambda_n^2+\frac{\sigma B_o^2(1+im)}{\rho(1+m^2)}+\frac{\phi\nu}{k}\right)^2+\omega^2\left(1+\frac{\alpha\phi}{k}+\alpha\lambda_n^2\right)^2\right)} \\
&-\frac{2}{d}U_0\omega\cos\omega t\sum_{n=1}^{\infty}\frac{\lambda_n\left(\nu-\frac{\alpha\sigma B_o^2(1+im)}{\rho(1+m^2)}\right)\sin(\lambda_n y)}{\left(\left(\nu\lambda_n^2+\frac{\sigma B_o^2(1+im)}{\rho(1+m^2)}+\frac{\phi\nu}{k}\right)^2+\omega^2\left(1+\frac{\alpha\phi}{k}+\alpha\lambda_n^2\right)^2\right)}. \quad (2.37)
\end{aligned}
$$

2.4 Plane Poiseuille flow

This section deals with the MHD flow of a fluid between two stationary insulating plates. For $t > 0$ the flow between the plates is due to a pressure gradient of the form [15]

$$\frac{\partial\widehat{p}}{\partial x}=-\rho\left[P_\circ+Q_\circ\cos\omega t\right] \quad \text{or} \quad \frac{\partial\widehat{p}}{\partial x}=-\rho\left[P_\circ+Q_\circ\sin\omega t\right]. \quad (2.38)$$

The governing problem here consists of equations (2.6), (2.26),(2.27) and

$$u(0,t)=0. \quad (2.39)$$

By means of Fourier sine transform the corresponding exact solutions to two cases given in equation (2.38) are respectively given by

$$
\begin{aligned}
u(y,t) &= \frac{4P_o}{d}\sum_{n=1}^{\infty}\frac{\sin(\lambda_{2n-1}y)}{\lambda_{2n-1}\left(\nu\lambda_{2n-1}^2+\frac{\sigma B_o^2(1+im)}{\rho(1+m^2)}+\frac{\phi\nu}{k}\right)} \\
&+\frac{4Q_o}{d}\sum_{n=1}^{\infty}\frac{\begin{pmatrix}\omega\left(1+\frac{\alpha\phi}{k}+\alpha\lambda_{2n-1}^2\right)\sin\omega t+\\ \left(\nu\lambda_{2n-1}^2+\frac{\sigma B_o^2(1+im)}{\rho(1+m^2)}+\frac{\phi\nu}{k}\right)\cos\omega t\end{pmatrix}\sin(\lambda_{2n-1}y)}{\lambda_{2n-1}\left(\omega^2\left(1+\frac{\alpha\phi}{k}+\alpha\lambda_{2n-1}^2\right)^2+\left(\nu\lambda_{2n-1}^2+\frac{\sigma B_o^2(1+im)}{\rho(1+m^2)}+\frac{\phi\nu}{k}\right)^2\right)} \\
&-\frac{4}{d}\sum_{n=1}^{\infty}\left(\frac{\frac{P_o}{\lambda_{2n-1}\left(\nu\lambda_{2n-1}^2+\frac{\sigma B_o^2(1+im)}{\rho(1+m^2)}+\frac{\phi\nu}{k}\right)}+Q_o\left(\nu\lambda_{2n-1}^2+\frac{\sigma B_o^2(1+im)}{\rho(1+m^2)}+\frac{\phi\nu}{k}\right)}{\lambda_{2n-1}\left(\omega^2\left(1+\frac{\alpha\phi}{k}+\alpha\lambda_{2n-1}^2\right)^2+\left(\nu\lambda_{2n-1}^2+\frac{\sigma B_o^2(1+im)}{\rho(1+m^2)}+\frac{\phi\nu}{k}\right)^2\right)}\right) \\
&\times\exp\left(\frac{-\left(\nu\lambda_{2n-1}^2+\frac{\sigma B_o^2(1+im)}{\rho(1+m^2)}+\frac{\phi\nu}{k}\right)t}{\left(1+\frac{\alpha\phi}{k}+\alpha\lambda_{2n-1}^2\right)}\right)\sin(\lambda_{2n-1}y), \qquad (2.40)
\end{aligned}
$$

$$
\begin{aligned}
u(y,t) &= \frac{4P_o}{d}\sum_{n=1}^{\infty}\frac{\sin(\lambda_{2n-1}y)}{\lambda_{2n-1}\left(\nu\lambda_{2n-1}^2+\frac{\sigma B_o^2(1+im)}{\rho(1+m^2)}+\frac{\phi\nu}{k}\right)} \\
&+\frac{4Q_o}{d}\sum_{n=1}^{\infty}\frac{\begin{pmatrix}\left(\nu\lambda_{2n-1}^2+\frac{\sigma B_o^2(1+im)}{\rho(1+m^2)}+\frac{\phi\nu}{k}\right)\sin\omega t\\ -\omega\left(1+\frac{\alpha\phi}{k}+\alpha\lambda_{2n-1}^2\right)\cos\omega t\end{pmatrix}\sin(\lambda_{2n-1}y)}{\lambda_{2n-1}\left(\omega^2\left(1+\frac{\alpha\phi}{k}+\alpha\lambda_{2n-1}^2\right)^2+\left(\nu\lambda_{2n-1}^2+\frac{\sigma B_o^2(1+im)}{\rho(1+m^2)}+\frac{\phi\nu}{k}\right)^2\right)} \\
&-\frac{4}{d}\sum_{n=1}^{\infty}\left(\frac{\frac{P_o}{\lambda_{2n-1}\left(\nu\lambda_{2n-1}^2+\frac{\sigma B_o^2(1+im)}{\rho(1+m^2)}+\frac{\phi\nu}{k}\right)}-Q_o\left(1+\frac{\alpha\phi}{k}+\alpha\lambda_{2n-1}^2\right)}{\lambda_{2n-1}\left\{\omega^2\left(1+\frac{\alpha\phi}{k}+\alpha\lambda_{2n-1}^2\right)^2+\left(\nu\lambda_{2n-1}^2+\frac{\sigma B_o^2(1+im)}{\rho(1+m^2)}+\frac{\phi\nu}{k}\right)^2\right\}}\right) \\
&\times\exp\left(\frac{-\left(\nu\lambda_{2n-1}^2+\frac{\sigma B_o^2(1+im)}{\rho(1+m^2)}+\frac{\phi\nu}{k}\right)t}{\left(1+\frac{\alpha\phi}{k}+\alpha\lambda_{2n-1}^2\right)}\right)\sin(\lambda_{2n-1}y). \qquad (2.41)
\end{aligned}
$$

2.5 Results and discussion

In this section, we present the graphical illustration of velocity profile for flow due to the oscillations of plate at $t > 0$. Emphasis has been focused to examine the difference between the velocity profiles for two kinds of fluids: a Newtonian fluid and second grade fluid. The emerging parameters here are the Hall parameter m, magnetic field parameter M and permeability of the porous medium K.

Figure 2.1 is prepared to show the effects of M on the velocity profiles in the two fluid cases. This figure depicts that with an increase of M the velocity profile decreases for both Newtonian and second grade fluids. However the effect of M in Newtonian fluid is more prominent in comparison to second grade fluid.

In order to illustrate the influence of K on the velocity profile, we made figure 2.2. It is evident from this figure that velocity profiles in both fluids increase by increasing K. This figure further indicates that the increase in velocity for Newtonian fluid is much when compared with that of second grade fluid.

The variations of m and M are given in figures 2.3-2.5. Figure 2.3 shows that velocity profiles in both fluids increase by increasing m. Moreover, boundary layer thickness also increases for large values of m. However the velocity in Newtonian fluid is found to be greater than that of a second grade fluid. The influence of M on the velocity profiles when $m = 0$ are shown in figures 2.4 and 2.5, respectively. It is found that here velocity profiles behave similar to that of $m \neq 0$. But the velocity profiles for $m = 0$ is smaller than that of $m \neq 0$.

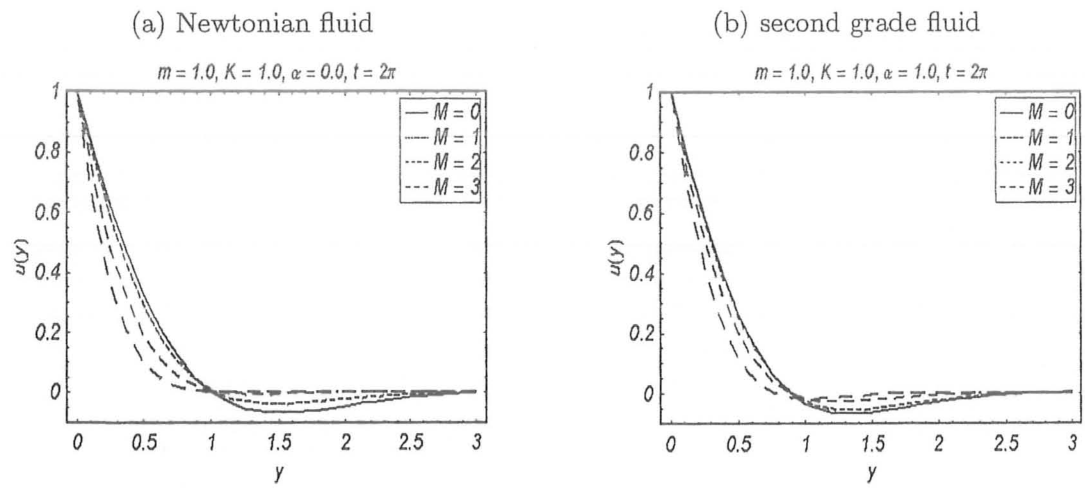

Figure 2.1: Profiles of the normalized steady state velocity $u(y)$ for various values of M.

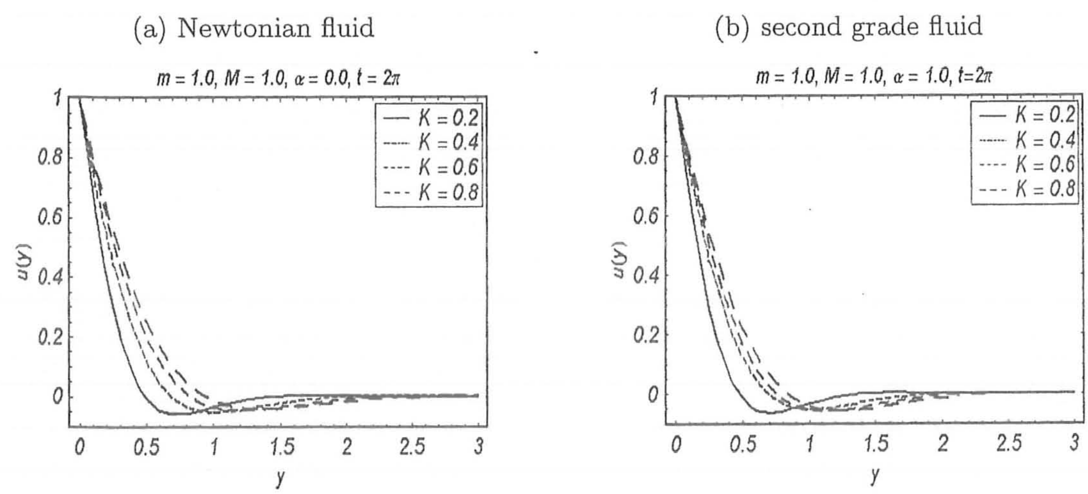

Figure 2.2: Profiles of the normalized steady state velocity $u(y)$ for various values of K.

(a) Newtonian fluid (b) second grade fluid

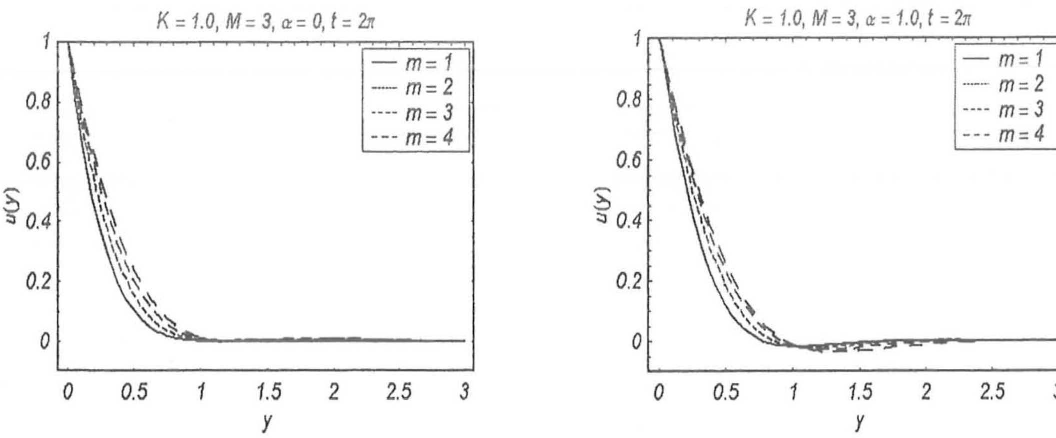

Figure 2.3: Profiles of the normalized steady state velocity $u(y)$ for various values of m.

(a) Newtonian fluid (b) second grade fluid

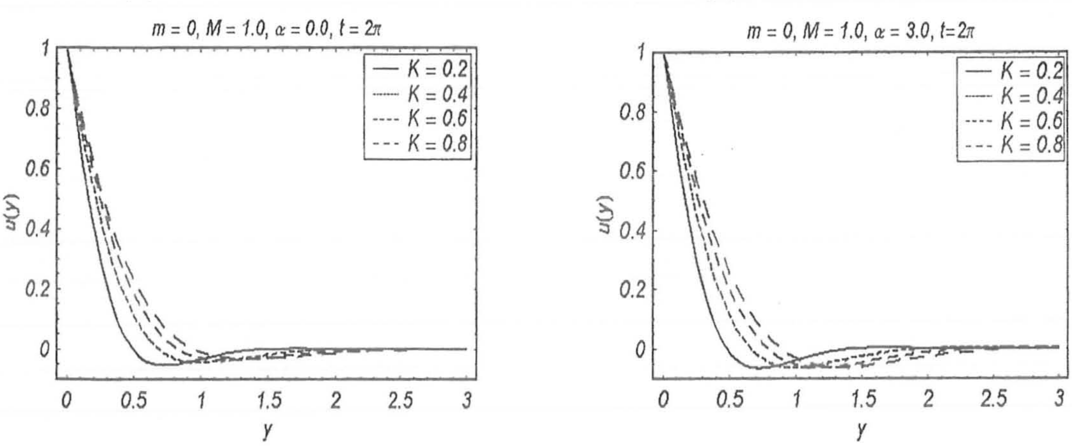

Figure 2.4: Profiles of the normalized steady state velocity $u(y)$ for various values of K in

absence of Hall parameter m.

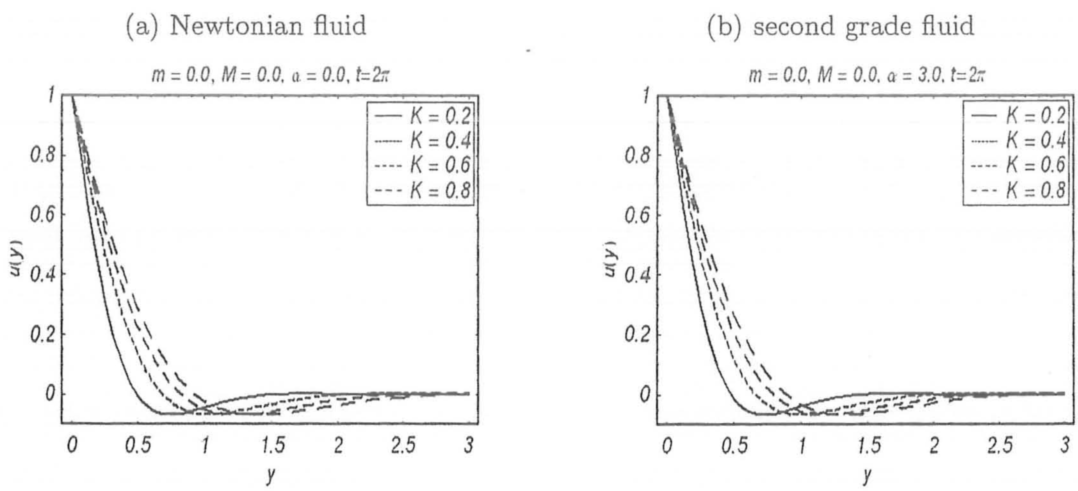

Figure 2.5: Profiles of the normalized steady state velocity $u(y)$ for various values of K in absence of M and m.

2.6 Concluding remarks

An analytical study is made of the three unsteady oscillatory flows of an electrically conducting second grade fluid through a porous medium. The system is stressed by strong transverse magnetic field. The analysis comprises the flow cases of a plate or between two plates. Based on modified Darcy's law, the governing equation is modeled. The exact solutions for velocity profiles are obtained and discussed. It is observed that, when the Hall parameter increases the velocity profiles increase, whereas when the porosity parameter increases the velocity decreases.

The presented study is more general than the existing studies. For example, the results of [5] can be recovered from equations (2.20) and (2.35) by taking $m = M = \phi = 0$. Also, the results for starting and steady solutions in [15] can be easily obtained by choosing $m = M = \phi = 0$. The steady solutions for Navier- Stokes fluid [65] are deduced when $m = M = \phi = \alpha = 0$.

The transient solutions in Stokes' second problem are

$$u_t(y,t) = -\frac{2}{\pi}U_0\int_0^{\infty}\frac{\xi\left(\left(1+\frac{\alpha\phi}{k}+\alpha\xi^2\right)\alpha\omega^2+\left(\nu^2\xi^2+\frac{\sigma B_0^2(1+im)}{\rho(1+m^2)}\nu+\frac{\phi\nu^2}{k}\right)\right)}{\left(\nu\xi^2+\frac{\sigma B_0^2(1+im)}{\rho(1+m^2)}+\frac{\phi\nu}{k}\right)^2+\omega^2\left(1+\frac{\alpha\phi}{k}+\alpha\xi^2\right)^2}$$
$$\times\exp\left(-\frac{\left(\nu\xi^2+\frac{\sigma B_0^2(1+im)}{\rho(1+m^2)}+\frac{\phi\nu}{k}\right)t}{\left(1+\frac{\alpha\phi}{k}+\alpha\xi^2\right)}\right)\sin(\xi y)\,d\xi, \tag{2.42}$$

$$u_t(y,t) = \frac{2}{\pi}U_o\omega\left(\nu-\frac{\alpha\sigma B_o^2(1+im)}{\rho(1+m^2)}\right)$$
$$\times\int_0^{\infty}\frac{\xi\exp\left(\frac{-\left(\nu\xi^2+\frac{\sigma B_0^2(1+im)}{\rho(1+m^2)}+\frac{\phi\nu}{k}\right)t}{\left(1+\frac{\alpha\phi}{k}+\alpha\xi^2\right)}\right)\sin(\xi y)\,d\xi}{\left(\nu\xi^2+\frac{\sigma B_o^2(1+im)}{\rho(1+m^2)}+\frac{\phi\nu}{k}\right)^2+\omega^2\left(1+\frac{\alpha\phi}{k}+\alpha\xi^2\right)^2}. \tag{2.43}$$

The results of transient solutions for Navier-Stokes fluid can be written by taking $m = M = \phi = \alpha = 0$ in Eqs. (2.42) and (2.43) and are given by

$$u_t(y,t) = -\frac{2}{\pi}\nu^2U_0\int_0^{\infty}\frac{\xi^3}{\nu^2\xi^4+\omega^2}\exp\left(-\nu\xi^2t\right)\sin(\xi y)\,d\xi, \tag{2.44}$$

$$u_t(y,t) = \frac{2}{\pi}U_o\omega\nu\int_0^{\infty}\frac{\xi\exp\left(-\nu\xi^2t\right)\sin(\xi y)\,d\xi}{\nu^2\xi^4+\omega^2}. \tag{2.45}$$

Note that the transient solutions given in above equations are different but are in simpler from than those given in [8] by the Laplace transform treatment.

Chapter 3

Effect of Hall current on flows of an Oldroyd-B fluid through a porous medium for cylindrical geometries

In this chapter, the equations are developed for magnetohydrodynamic (MHD) flows of an Oldroyd-B fluid through a porous medium. These equations give rise to a mathematical description in which a modified Darcy's law for an Oldroyd-B fluid is taken into account with Hall effects. This particularly happens when magnetic field is high. Four characteristic examples for flows in pipe and cylinder are considered. These are

(i) starting flow in a circular cylinder moving parallel to its length,

(ii) starting flow in a circular pipe,

(iii) generalized flow in a circular pipe,

(iv) starting flow in a rotating cylinder.

The problems valid for a small magnetic Reynolds number are solved analytically by applying the Laplace transform method. Graphical results for the velocity are displayed and are discussed for the various parametric conditions.

3.1 Description of the basic equations

In our calculations we shall assume that an Oldroyd-B fluid is incompressible whose constitutive equation is

$$\mathbf{T} = -p\mathbf{I} + \mathbf{S}, \tag{3.1}$$

$$\mathbf{S}+\lambda\left(\frac{d\mathbf{S}}{dt} - \mathbf{LS} - \mathbf{SL}^{\mathsf{T}}\right) = \mu\left[\mathbf{A}_1 + \theta\left(\frac{d\mathbf{A}_1}{dt} - \mathbf{LA}_1 - \mathbf{A}_1\mathbf{L}^{\mathsf{T}}\right)\right], \tag{3.2}$$

where $\mathbf{T}$ is Cauchy stress, p the isotropic pressure, $\mathbf{I}$ the identity tensor, $\mathbf{S}$ the extra stress tensor, d/dt the material derivative, $\mathbf{L}$ the velocity gradient, $\mathbf{L}^{\mathsf{T}}$ the transpose of $\mathbf{L}$, μ the dynamic viscosity, t the time, λ the relaxation time and θ the retardation time. The expressions for first Rivlin-Ericksen tensor $\mathbf{A}_1$ and velocity gradient $\mathbf{L}$ are

$$\mathbf{A}_1 = \mathbf{L} + \mathbf{L}^{\mathsf{T}}, \quad \mathbf{L} = \nabla\mathbf{V}, \tag{3.3}$$

in which ∇ is the gradient operator, $\mathbf{V}$ the velocity of a fluid and $\lambda \geq \theta \geq 0$. The equations which govern the unsteady MHD flow with Hall effect are Maxwell (1.1)-(1.4), Generalized Ohm's law (1.13), conservation of mass for incompressible fluid (1.10), and equation of momentum (1.11). In the low magnetic Reynolds number consideration in which the induced magnetic field can be ignored and the imposed and induced electric fields are assumed negligible, the equation (1.13) simplifies to

$$\mathbf{J} \times \mathbf{B} = -\frac{\sigma B_0^2}{1 - im}\mathbf{V}, \tag{3.4}$$

where m $(= w_e \tau_e)$ denotes the Hall parameter.

In porous medium, the constitutive relationship between the pressure drop and velocity for an Oldroyd-B fluid [66, 67] is

$$\left(1 + \lambda\frac{\partial}{\partial t}\right)\nabla p = -\frac{\mu\phi}{k}\left(\mathbf{V} + \theta\frac{\partial\mathbf{V}}{\partial t}\right), \tag{3.5}$$

where ϕ is the porosity of the porous medium and k the permeability. For $\lambda = \theta = 0$, Eq. (3.5) describes the Darcy's law. Since the pressure gradient given in equation (3.5) is a measure of the flow resistance in the bulk of porous medium and $\mathbf{R}$ in Eq.(1.11) is interpreted as the flow resistance offered by the solid matrix. Therefore, $\mathbf{R}$ through equation (3.5) satisfies

$$\left(1+\lambda\frac{\partial}{\partial t}\right)\mathbf{R}=-\frac{\mu\phi}{k}\left(1+\theta\frac{\partial}{\partial t}\right)\mathbf{V}. \tag{3.6}$$

From equations (3.4),(1.11) and (3.6) one can write

$$\rho\left(1+\lambda\frac{\partial}{\partial t}\right)\frac{d\mathbf{V}}{dt}=\left(1+\lambda\frac{\partial}{\partial t}\right)div\mathbf{T}-\frac{\sigma B_0^2\left(1+im\right)}{1+m^2}\left(1+\lambda\frac{\partial}{\partial t}\right)\mathbf{V}-\frac{\mu\phi}{k}\left(1+\theta\frac{\partial}{\partial t}\right)\mathbf{V}. \tag{3.7}$$

Now we will discuss the four examples for flows mentioned above.

3.2 Starting flow in a circular cylinder

Here, an Oldroyd-B fluid is taken in a circular cylinder. The fluid is initially at rest and then starts suddenly because of the motion of the cylinder parallel to its length. Choosing $z-$axis as the axis of cylinder the velocity may be written as

$$\mathbf{V}=\left(0,0,w\left(r,t\right)\right), \tag{3.8}$$

in which w is the velocity component along the $z-$axis. Using above expression, the equation of mass is identically satisfied and equation (3.7) gives

$$\begin{aligned}\left(1+\lambda\frac{\partial}{\partial t}\right)\frac{\partial w}{\partial t} &= \nu\left(1+\theta\frac{\partial}{\partial t}\right)\left(\frac{\partial^2 w}{\partial r^2}+\frac{1}{r}\frac{\partial w}{\partial r}\right)-\frac{\sigma B_0^2\left(1+im\right)}{\rho\left(1+m^2\right)}\left(1+\lambda\frac{\partial}{\partial t}\right)w\\ &\quad -\frac{\nu\phi}{k}\left(1+\theta\frac{\partial}{\partial t}\right)w,\end{aligned} \tag{3.9}$$

where ν is the kinematic viscosity. To complete the formulation of the problem, we give the following boundary and initial conditions:

$$\begin{aligned} w\left(a,t\right) &= W, \quad t>0,\\ \frac{\partial w\left(0,t\right)}{\partial r} &= 0, \quad \text{for all } t,\\ \frac{\partial w\left(r,0\right)}{\partial t} &= w\left(r,0\right)=0, \quad 0\leq r<a,\end{aligned} \tag{3.10}$$

in which a is the radius of the cylinder and W the constant velocity at $r = a$.

We introduce the following dimensionless variables

$$w^* = \frac{w}{W}, \quad \xi = \frac{r}{a}, \quad \tau = \frac{\nu t}{a^2}, \quad \Lambda = \frac{\nu \lambda}{a^2}, \quad \Theta = \frac{\nu \theta}{a^2}, \quad M = \sqrt{\frac{\sigma}{\mu}} B_0 a, \quad \beta = \sqrt{\frac{\phi}{k}} a \tag{3.11}$$

The following problem is obtained by substituting the above dimensionless quantities into equations (3.9) and (3.10)

$$\begin{aligned}\left(1+\Lambda\frac{\partial}{\partial\tau}\right)\frac{\partial w}{\partial\tau} &= \left(1+\Theta\frac{\partial}{\partial\tau}\right)\left(\frac{\partial^2 w}{\partial\xi^2}+\frac{1}{\xi}\frac{\partial w}{\partial\xi}\right)-\frac{M^2(1+im)}{(1+m^2)}\left(1+\Lambda\frac{\partial}{\partial\tau}\right)w \\ &\quad -\beta^2\left(1+\Theta\frac{\partial}{\partial\tau}\right)w, \end{aligned} \tag{3.12}$$

$$\begin{aligned} w(1,\tau) &= 1, \quad \tau > 0, \\ \frac{\partial w(0,\tau)}{\partial\xi} &= 0, \quad \text{for all } \tau, \\ \frac{\partial w(\xi,0)}{\partial\tau} &= w(\xi,0) = 0, \quad 0 \le \xi < 1, \end{aligned} \tag{3.13}$$

where asterisks have been suppressed.

Recently, it has been shown by Erdogan [68, 69] that velocity expression in series form for large times can also be used for small times or vice versa. Therefore, following the same idea, we present the solutions for large and small times. The solution for small time has been obtained using Laplace transform method.

3.2.1 Large time solution

For steady state, the velocity distribution is given by

$$w(\xi) = \frac{I_0(P\xi)}{I_0(P)}, \tag{3.14}$$

where

$$P = \left[\frac{M^2(1+im)}{1+m^2} + \beta^2\right]^{1/2}$$

and I_0 is the modified Bessel function of first kind of order zero.

Let

$$w(\xi,\tau) = \frac{I_0(P\xi)}{I_0(P)} - f(\xi,\tau), \tag{3.15}$$

where $f(\xi,\tau)$ satisfies the following initial-boundary value problem

$$\begin{aligned}\left(1+\Lambda\frac{\partial}{\partial\tau}\right)\frac{\partial f}{\partial\tau} &= \left(1+\Theta\frac{\partial}{\partial\tau}\right)\left(\frac{\partial^2 f}{\partial\xi^2}+\frac{1}{\xi}\frac{\partial f}{\partial\xi}\right) - \frac{M^2(1+im)}{(1+m^2)}\left(1+\Lambda\frac{\partial}{\partial\tau}\right)f \\ &\quad -\beta^2\left(1+\Theta\frac{\partial}{\partial\tau}\right)f,\end{aligned} \tag{3.16}$$

$$\begin{aligned} f(1,\tau) &= 0, \quad \tau>0, \\ \frac{\partial f(0,\tau)}{\partial\xi} &= 0, \quad \text{for all } \tau, \\ f(\xi,0) &= \frac{I_0(P\xi)}{I_0(P)}, \\ \frac{\partial f(\xi,0)}{\partial\tau} &= 0. \end{aligned} \tag{3.17}$$

Now solving equations (3.16) and (3.17) we arrive at

$$f(\xi,\tau) = \sum_{n=1}^{\infty} B_n J_0(\lambda_n\xi)\,T_n(\tau), \tag{3.18}$$

where J_0 is the Bessel function of first kind of order zero and λ_n are the zeros of J_0 and

$$T_n(\tau) = -\frac{D_2}{D_1-D_2}e^{D_1\tau} + \frac{D_1}{D_1-D_2}e^{D_2\tau},$$

$$D_1 = \frac{(\delta_1-\eta_1)+(\delta_2-\eta_2)i}{2}, \quad D_2 = \frac{-(\delta_1+\eta_1)-(\delta_2+\eta_2)i}{2},$$

$$\delta_1 = \left[\frac{(\eta_1^2-\eta_2^2-\eta_3)+\sqrt{(\eta_1^2-\eta_2^2-\eta_3)^2+(2\eta_1\eta_2-\eta_4)^2}}{2}\right]^{\frac{1}{2}},$$

$$\delta_2 = \left[\frac{-(\eta_1^2-\eta_2^2-\eta_3)+\sqrt{(\eta_1^2-\eta_2^2-\eta_3)^2+(2\eta_1\eta_2-\eta_4)^2}}{2}\right]^{\frac{1}{2}},$$

$$\eta_1 = \left[1+(\beta^2+\lambda_n^2)\Theta+\frac{M^2}{1+m^2}\right]\frac{1}{\Lambda}, \quad \eta_2 = \frac{mM^2}{\Lambda(1+m^2)},$$

$$\eta_3 = \left[\left(\beta^2 + \lambda_n^2\right) + \frac{M^2}{1+m^2} \right] \frac{4}{\Lambda}, \quad \eta_4 = \frac{4mM^2}{\Lambda\left(1+m^2\right)}.$$

The values of B_n can be obtained by the initial condition for $f(\xi, \tau)$. Hence, the velocity distribution takes the form

$$w(\xi, \tau) = \frac{I_0(P\xi)}{I_0(P)} - 2\sum_{n=1}^{\infty} \frac{\lambda_n J_0(\lambda_n \xi) T_n(\tau)}{\left(P^2 + \lambda_n^2\right) J_1(\lambda_n)}, \tag{3.19}$$

where J_1 is the Bessel function of first kind of order one.

3.2.2 Small time solution

The Laplace transform $\overline{w}$ of w is defined as

$$\overline{w}(\xi, s) = \int_0^{\infty} w(\xi, \tau) e^{-s\tau} d\tau.$$

Therefore, the equations (3.12) and(3.13) reduces to

$$\overline{w}'' + \frac{1}{\xi}\overline{w}' - q^2\overline{w} = 0, \tag{3.20}$$

$$\begin{aligned} \overline{w}(1, s) &= \frac{1}{s}, \\ \frac{d\overline{w}(0, s)}{d\xi} &= 0, \end{aligned} \tag{3.21}$$

where

$$q = \left[\frac{1}{(1+\Theta s)} \left\{ (1+\Lambda s) \frac{M^2(1+im)}{1+m^2} + (1+\Theta s)\beta^2 + s(1+\Lambda s) \right\} \right]^{\frac{1}{2}}$$

and primes denote the differentiation with respect to ξ.

The solution of equations (3.20) and (3.21) is

$$\overline{w} = \frac{I_0(q\xi)}{sI_0(q)}. \tag{3.22}$$

Laplace inversion of equation (3.22) yields

$$w = \frac{1}{2\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} \frac{I_0(q\xi)}{sI_0(q)} e^{s\tau} ds. \tag{3.23}$$

In equation (3.23), $s = 0$ is a simple pole. Therefore, residue at $s = 0$ is

$$\text{Res}(0) = \frac{I_0(P\xi)}{I_0(P)}. \tag{3.24}$$

The other singular points of equation (3.23) are the zeros of

$$I_0(q) = 0.$$

Setting $q = i\lambda$, we find that

$$J_0(\lambda) = 0. \tag{3.25}$$

If λ_n, $n = 1, 2, \cdots, \infty$ are the zeros of equation (3.25), then s_{1n} and s_{2n} are the poles. These are simple poles and the residue at these poles are

$$\text{Res}(s_{1n}) = \frac{2\lambda_n (1 + \Theta s_{1n}) e^{s_{1n}\tau}}{s_{1n} l_1} \frac{J_0(\lambda_n \xi)}{J_1(\lambda_n)},$$

$$\text{Res}(s_{2n}) = \frac{2\lambda_n (1 + \Theta s_{2n}) e^{s_{2n}\tau}}{s_{2n} l_2} \frac{J_0(\lambda_n \xi)}{J_1(\lambda_n)},$$

where

$$s_{1n} = \frac{-\left[1 + (\lambda_n^2 + \beta^2)\Theta + \frac{M^2(1+im)}{(1+m^2)}\right] + \sqrt{\left[1 + (\lambda_n^2 + \beta^2)\Theta + \frac{M^2(1+im)}{(1+m^2)}\right]^2 - 4\Lambda(P^2 + \lambda_n^2)}}{2\Lambda},$$

$$s_{2n} = \frac{-\left[1 + (\lambda_n^2 + \beta^2)\Theta + \frac{M^2(1+im)}{(1+m^2)}\right] - \sqrt{\left[1 + (\lambda_n^2 + \beta^2)\Theta + \frac{M^2(1+im)}{(1+m^2)}\right]^2 - 4\Lambda(P^2 + \lambda_n^2)}}{2\Lambda},$$

$$l_1 = \left[1 + 2\Lambda s_{1n} + (\lambda_n^2 + \beta^2)\Theta + \frac{M^2(1+im)\Lambda}{(1+m^2)}\right],$$

$$l_2 = \left[1 + 2\Lambda s_{2n} + (\lambda_n^2 + \beta^2)\Theta + \frac{M^2(1+im)\Lambda}{(1+m^2)}\right].$$

Addition of Res(0) , Res(s_{1n}) and Res(s_{2n}) gives

$$w(\xi,\tau)=\frac{I_0(P\xi)}{I_0(P)}+2\sum_{n=1}^{\infty}\left\{\frac{(1+\Theta s_{1n})\,e^{s_{1n}\tau}}{s_{1n}l_1}+\frac{(1+\Theta s_{2n})\,e^{s_{2n}\tau}}{s_{2n}l_2}\right\}\frac{\lambda_n J_0(\lambda_n\xi)}{J_1(\lambda_n)}. \tag{3.26}$$

3.3 Starting flow in a circular pipe

Here, the unsteady flow is considered when the fluid is in a circular cylinder. Initially, the fluid is at rest. An application of constant pressure gradient suddenly starts the motion. With the help of equations (3.7) and (3.8), the equation for such a flow is of the following form:

$$\begin{aligned}\left(1+\lambda\frac{\partial}{\partial t}\right)\frac{\partial w}{\partial t} &= -\frac{1}{\rho}\frac{dp}{dz}+\nu\left(1+\theta\frac{\partial}{\partial t}\right)\left(\frac{\partial^2 w}{\partial r^2}+\frac{1}{r}\frac{\partial w}{\partial r}\right)\\ &\quad-\frac{\sigma B_0^2(1+im)}{\rho(1+m^2)}\left(1+\lambda\frac{\partial}{\partial t}\right)w-\frac{\nu\phi}{k}\left(1+\theta\frac{\partial}{\partial t}\right)w.\end{aligned} \tag{3.27}$$

The appropriate boundary and initial conditions are

$$\begin{aligned} w(a,t) &= \frac{\partial w(0,t)}{\partial r}=0, \quad \text{for all } t,\\ \frac{\partial w(r,0)}{\partial t} &= w(r,0)=0, \quad 0\le r<a.\end{aligned} \tag{3.28}$$

Using the dimensionless variables defined in equation (3.11) along with

$$z^*=\frac{z}{a}, \quad p^*=\frac{p}{(\mu W/a)}$$

in equations (3.27) and (3.28) and then solving the resulting problem by employing a similar procedure as in previous sections we have:

3.3.1 Large time solution

$$w(\xi,\tau)=-\frac{1}{P^2}\frac{dp}{dz}\left[1-\frac{I_0(P\xi)}{I_0(P)}\right]+2\frac{dp}{dz}\sum_{n=1}^{\infty}\frac{J_0(\lambda_n\xi)\,T_n(\tau)}{\lambda_n\left(P^2+\lambda_n^2\right)J_1(\lambda_n)}. \tag{3.29}$$

3.3.2 Small time solution

$$
\begin{aligned}
w(\xi,\tau) &= -\frac{dp}{dz}\left[\frac{1}{P^2}\left(1-\frac{I_0(P\xi)}{I_0(P)}\right)+\frac{e^{s_1\tau}}{s_1(s_1-s_2)}\left(1-\frac{I_0(q_1\xi)}{I_0(q_1)}\right)+\frac{e^{s_2\tau}}{s_2(s_2-s_1)}\left(1-\frac{I_0(q_2\xi)}{I_0(q_2)}\right)\right] \\
&+2\frac{dp}{dz}\sum_{n=1}^{\infty}\left\{\frac{(1+\Theta s_{1n})e^{s_{1n}\tau}}{l_1 l_3}+\frac{(1+\Theta s_{2n})e^{s_{2n}\tau}}{l_2 l_4}\right\}\frac{\lambda_n J_0(\lambda_n\xi)}{J_1(\lambda_n)},
\end{aligned}
\tag{3.30}
$$

where

$$
l_3=\left[\Lambda s_{1n}^3+\left(1+\Theta\beta^2+\frac{M^2(1+im)}{(1+m^2)}\right)s_{1n}^2+P^2 s_{1n}\right],
$$

$$
l_4=\left[\Lambda s_{2n}^3+\left(1+\Theta\beta^2+\frac{M^2(1+im)}{(1+m^2)}\right)s_{2n}^2+P^2 s_{2n}\right],
$$

$$
s_1=\frac{-\left[1+\Theta\beta^2+\frac{M^2(1+im)}{(1+m^2)}\right]+\sqrt{\left[1+\Theta\beta^2+\frac{M^2(1+im)}{(1+m^2)}\right]^2-4\Lambda P^2}}{2\Lambda},
$$

$$
s_2=\frac{-\left[1+\Theta\beta^2+\frac{M^2(1+im)}{(1+m^2)}\right]-\sqrt{\left[1+\Theta\beta^2+\frac{M^2(1+im)}{(1+m^2)}\right]^2-4\Lambda P^2}}{2\Lambda},
$$

$$
q_1=\left[\frac{\Lambda s_1^2+\left(1+\Theta\beta^2+\frac{M^2(1+im)\Lambda}{(1+m^2)}\right)s_1+P^2}{1+\Theta s_1}\right]^{1/2},
$$

$$
q_2=\left[\frac{\Lambda s_2^2+\left(1+\Theta\beta^2+\frac{M^2(1+im)\Lambda}{(1+m^2)}\right)s_2+P^2}{1+\Theta s_2}\right]^{1/2}.
$$

3.4 Generalized flow in a circular pipe

Here the flow geometry is same as in section 3.3 except that the fluid motion now is due to a constant pressure gradient and by the motion of the cylinder. Using the same dimensionless variables as defined in section 3.2, the governing problem takes the following form

$$
\begin{aligned}
\left(1+\Lambda\frac{\partial}{\partial\tau}\right)\frac{\partial w}{\partial\tau} &= -\frac{dp}{dz}+\left(1+\Theta\frac{\partial}{\partial\tau}\right)\left(\frac{\partial^2 w}{\partial\xi^2}+\frac{1}{\xi}\frac{\partial w}{\partial\xi}\right) \\
&-\frac{M^2(1+im)}{(1+m^2)}\left(1+\Lambda\frac{\partial}{\partial\tau}\right)w-\beta^2\left(1+\Theta\frac{\partial}{\partial\tau}\right)w,
\end{aligned}
\tag{3.31}
$$

$$
\begin{aligned}
w(1,\tau) &= 1, \quad \tau > 0, \\
\frac{\partial w(0,\tau)}{\partial \xi} &= 0, \text{ for all } \tau, \\
\frac{\partial w(\xi,0)}{\partial \tau} &= w(\xi,0) = 0, \quad 0 \leq \xi < 1.
\end{aligned} \tag{3.32}
$$

For the above problem the large and small times solutions are :

3.4.1 Large time solution

$$
\begin{aligned}
w(\xi,\tau) &= \frac{I_0(P\xi)}{I_0(P)} - \frac{1}{P^2}\frac{dp}{dz}\left[1 - \frac{I_0(P\xi)}{I_0(P)}\right] \\
&\quad -2\sum_{n=1}^{\infty} \frac{J_0(\lambda_n \xi) T_n(\tau)}{\lambda_n (P^2 + \lambda_n^2) J_1(\lambda_n)} \left[\lambda_n^2 - \frac{dp}{dz}\right].
\end{aligned} \tag{3.33}
$$

3.4.2 Small time solution

$$
\begin{aligned}
w(\xi,\tau) &= \frac{I_0(P\xi)}{I_0(P)} - \frac{dp}{dz}\left[\begin{array}{c} \frac{1}{P^2}\left(1 - \frac{I_0(P\xi)}{I_0(P)}\right) \\ +\frac{e^{s_1\tau}}{s_1(s_1-s_2)}\left(1 - \frac{I_0(q_1\xi)}{I_0(q_1)}\right) + \frac{e^{s_2\tau}}{s_2(s_2-s_1)}\left(1 - \frac{I_0(q_2\xi)}{I_0(q_2)}\right) \end{array}\right] \\
&\quad +2\sum_{n=1}^{\infty}\left\{\frac{(1+\Theta s_{1n}) e^{s_{1n}\tau}}{s_{1n} l_1} + \frac{(1+\Theta s_{2n}) e^{s_{2n}\tau}}{s_{2n} l_2}\right\} \frac{\lambda_n J_0(\lambda_n \xi)}{J_1(\lambda_n)} \\
&\quad +2\frac{dp}{dz}\sum_{n=1}^{\infty}\left\{\frac{(1+\Theta s_{1n}) e^{s_{1n}\tau}}{l_1 l_3} + \frac{(1+\Theta s_{2n}) e^{s_{2n}\tau}}{l_2 l_4}\right\} \frac{\lambda_n J_0(\lambda_n \xi)}{J_1(\lambda_n)}.
\end{aligned} \tag{3.34}
$$

3.5 Starting flow in a rotating cylinder

Here we consider the fluid in a circular cylinder. The fluid is initially at rest and suddenly sets in motion due to rotation of the cylinder. For such flow the velocity field is

$$
\mathbf{V} = (0, v(r,t), 0). \tag{3.35}
$$

Substitution of above equation into equation (3.7) yields

$$\begin{aligned}\left(1+\lambda\frac{\partial}{\partial t}\right)\frac{\partial v}{\partial t} &= \nu\left(1+\theta\frac{\partial}{\partial t}\right)\left(\frac{\partial^2 v}{\partial r^2}+\frac{1}{r}\frac{\partial v}{\partial r}-\frac{v}{r^2}\right)\\ &-\frac{\sigma B_0^2(1+im)}{\rho(1+m^2)}\left(1+\lambda\frac{\partial}{\partial t}\right)v-\frac{\nu\phi}{k}\left(1+\theta\frac{\partial}{\partial t}\right)v. \end{aligned} \tag{3.36}$$

The boundary and initial conditions for the problem can be written as

$$\begin{aligned} v(a,t) &= \Omega a, \quad t>0,\\ \frac{\partial v(0,t)}{\partial r} &= 0, \ \text{for all } t,\\ \frac{\partial v(r,0)}{\partial t} &= v(r,0)=0, \quad 0\le r<a, \end{aligned} \tag{3.37}$$

in which Ω is the angular velocity. Making use of dimensionless variables given in equation (3.11) together with

$$v^{*}=\frac{v}{\Omega a} \tag{3.38}$$

one produces the following problem after dropping asterisks

$$\begin{aligned}\left(1+\Lambda\frac{\partial}{\partial \tau}\right)\frac{\partial v}{\partial \tau} &= \left(1+\Theta\frac{\partial}{\partial \tau}\right)\left(\frac{\partial^2 v}{\partial \xi^2}+\frac{1}{\xi}\frac{\partial v}{\partial \xi}-\frac{v}{\xi^2}\right)\\ &-\frac{M^2(1+im)}{(1+m^2)}\left(1+\Lambda\frac{\partial}{\partial \tau}\right)v-\beta^2\left(1+\Theta\frac{\partial}{\partial \tau}\right)v, \end{aligned} \tag{3.39}$$

$$\begin{aligned} v(1,\tau) &= 1, \ \tau>0,\\ \frac{\partial v(0,\tau)}{\partial \xi} &= 0, \ \text{for all } \tau,\\ \frac{\partial v(\xi,0)}{\partial \tau} &= v(\xi,0)=0, \ 0\le\xi<1. \end{aligned} \tag{3.40}$$

The large and small times solutions of the above problem may be written as:

3.5.1 Large time solution

$$v(\xi,\tau)=\frac{I_1(P\xi)}{I_1(P)}+2\sum_{n=1}^{\infty}\frac{\lambda_n J_1(\lambda_n\xi)T_n(\tau)}{(P^2+\lambda_n^2)J_0(\lambda_n)}, \tag{3.41}$$

where I_1 is modified Bessel function of first kind of order one and λ_n are the zeros of J_1.

3.5.2 Small time solution

$$v(\xi,\tau)=\frac{I_1(P\xi)}{I_1(P)}-4\sum_{n=1}^{\infty}\left\{\frac{(1+\Theta s_{1n})e^{s_{1n}\tau}}{s_{1n}l_1}+\frac{(1+\Theta s_{2n})e^{s_{2n}\tau}}{s_{2n}l_2}\right\}\frac{\lambda_n J_1(\lambda_n\xi)}{[J_0(\lambda_n)-J_2(\lambda_n)]}. \quad (3.42)$$

3.6 Discussion of results

In this section, we present the graphical illustration of velocity profile for different flows namely, starting flow in a circular cylinder moving parallel to its length, starting flow in a circular pipe due to pressure gradient and starting flow in a rotating cylinder. Special attention has been given to examine the velocity profiles for two kinds of fluids: a Newtonian fluid and an Oldroyd-B fluid for different values of Hall parameter m and the constant of porous medium β when τ and M are fixed.

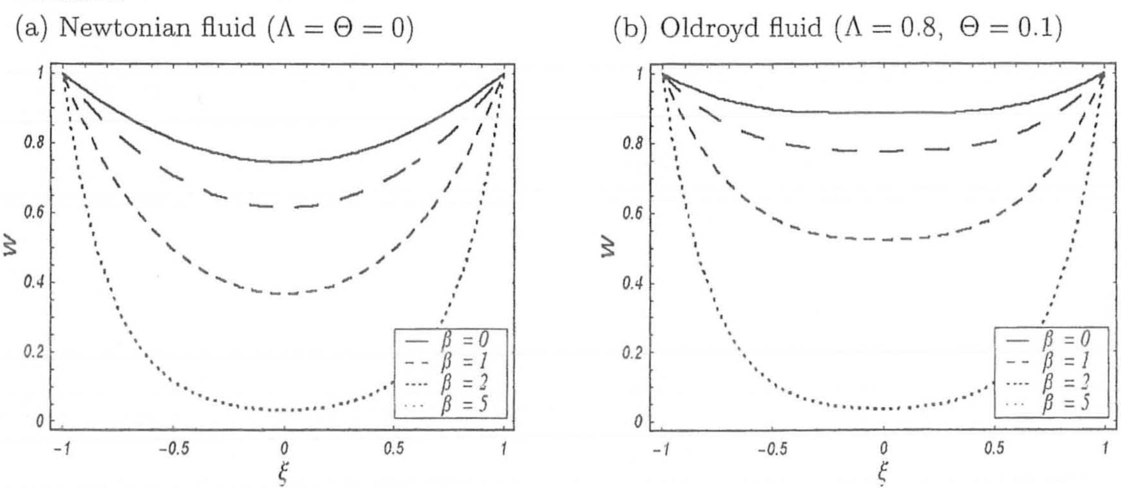

Figure 3.1 : Profiles of the normalized velocity $w(\xi,\tau)$ for various values of β when $\tau=M=1$ and $m=0$ are fixed.

(a) Newtonian fluid ($\Lambda = \Theta = 0$) (b) Oldroyd fluid ($\Lambda = 0.8,\ \Theta = 0.1$)

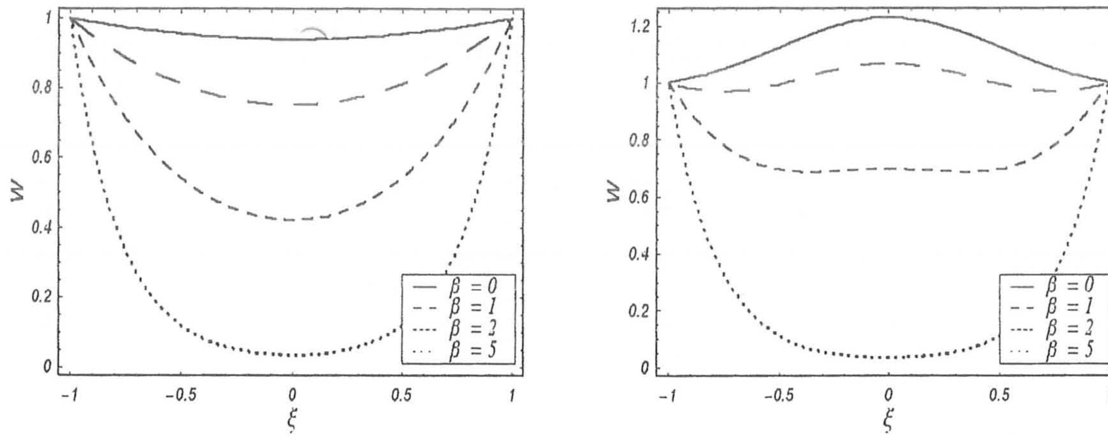

Figure 3.2 : Profiles of the normalized velocity $w\,(\xi, \tau)$ for various values of β when $\tau = M = 1$ and $m = 2$ are fixed.

(a) Newtonian fluid ($\Lambda = \Theta = 0$) (b) Oldroyd fluid ($\Lambda = 0.8,\ \Theta = 0.1$)

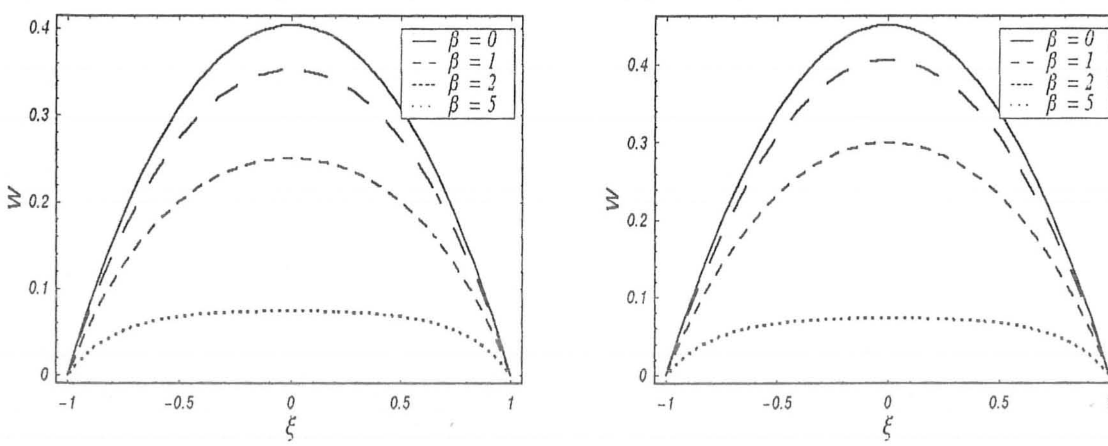

Figure 3.3 : Profiles of the normalized velocity $w\,(\xi, \tau)$ for various values of β when $\tau = M = 1$, $dp/dz = -2$ and $m = 0$ are fixed.

Figures 3.1 and 3.2 are prepared to show the effects of Hall parameter m and the constant of porous medium β on the velocity profiles of the flow due to the motion of the cylinder parallel to its length. From these figures it is noted that with an increase of β the velocity profile decreases for both a Newtonian fluid and an Oldroyd-B fluid. The effect of β on the velocity profiles of a Newtonian fluid is prominent as compared to an Oldroyd-B fluid. Moreover, from these figures, it is found that with an increase of Hall parameter, the velocity profiles increase for both the fluids. The permeability of the medium plays a similar role on the velocity as that of Hall parameter. Furthermore, it is observed that the steady state for an Oldroyd-B fluid in porous medium is achieved earlier when compared with the non-porous medium situation. The time to achieve the steady state for porous medium is about $\tau = 2.1$ (when $\Lambda = 0.8$, $\Theta = 0.1$, $m = M = 2$ and $\beta = 2$) while for non-porous medium it is about $\tau = 3$ (when $\Lambda = 0.8$, $\Theta = 0.1$, $m = M = 2$ and $\beta = 0$).

(a) Newtonian fluid ($\Lambda = \Theta = 0$) (b) Oldroyd fluid ($\Lambda = 0.8$, $\Theta = 0.1$)

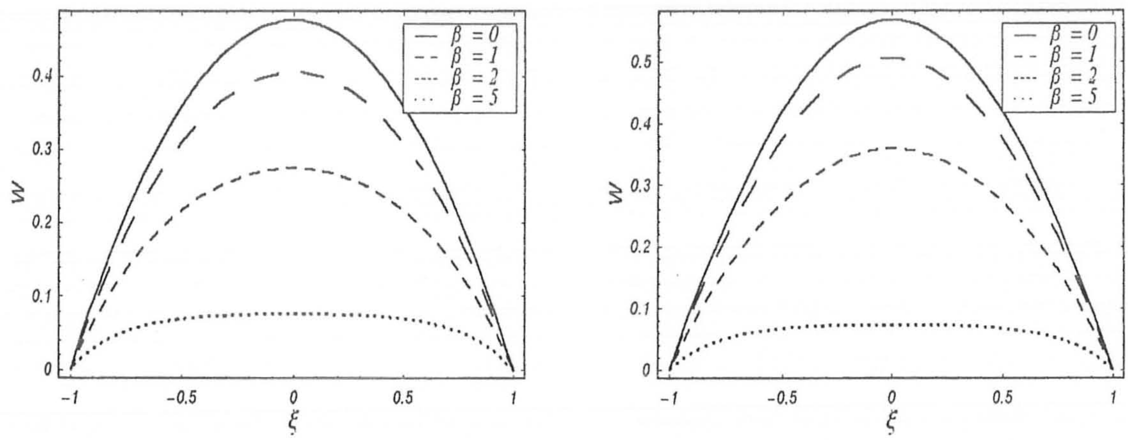

Figure 3.4 : Profiles of the normalized velocity $w(\xi, \tau)$ for various values of β when $\tau = M = 1$, $dp/dz = -2$ and $m = 2$ are fixed.

(a) Newtonian fluid ($\Lambda = \Theta = 0$) (b) Oldroyd fluid ($\Lambda = 0.8$, $\Theta = 0.1$)

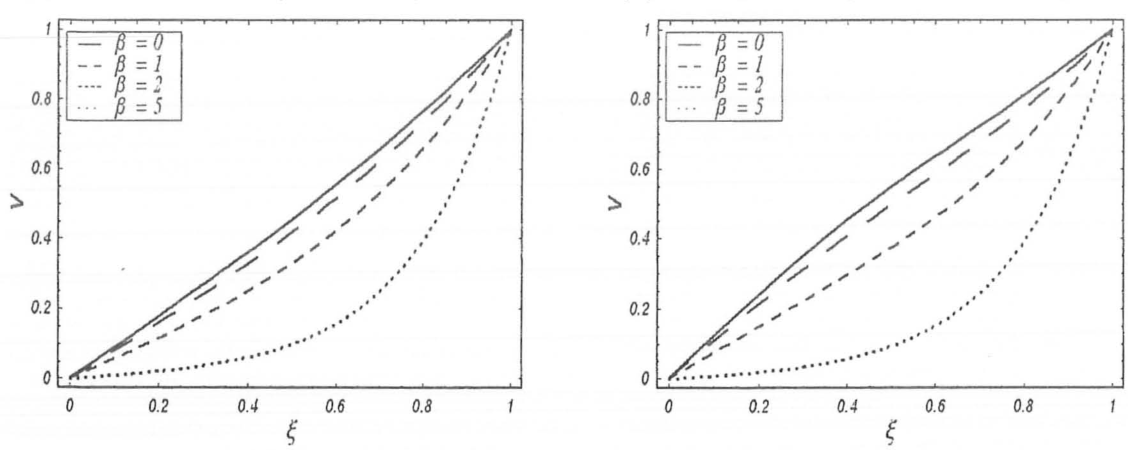

Figure 3.5 : Profiles of the normalized velocity $v(\xi, \tau)$ for various values of β when $\tau = M = 1$ and $m = 0$ are fixed.

Figures 3.3 and 3.4 show the flow due to a constant pressure gradient in a porous medium and figures 3.5 and 3.6 are sketched for starting flow in a rotating cylinder. In all these figures, it is observed that the effects of β and Hall parameter m are similar to that in figures 3.1 and 3.2. For the case when the motion is generated due to constant pressure gradient, it is noted that the steady state time for an Oldroyd-B fluid in porous medium is about $\tau = 2.2$ (when $\Lambda = 0.8$, $\Theta = 0.1$, $m = M = 2$, $dp/dz = -2$ and $\beta = 2$) whereas for non-porous medium it is about $\tau = 3$ (when $\Lambda = 0.8$, $\Theta = 0.1$, $m = M = 2$, $dp/dz = -2$ and $\beta = 0$). The time to reach the steady state for the flow due to the rotation of cylinder is about $\tau = 1.3$ (when $\Lambda = 0.8$, $\Theta = 0.1$, $m = M = 2$ and $\beta = 2$) in porous medium and for non-porous medium it is about $\tau = 2.1$ (when $\Lambda = 0.8$, $\Theta = 0.1$, $m = M = 2$ and $\beta = 0$). It is further noted that steady state in case of Hall parameter is obtained much latter than that when no Hall parameter is present. For Hall parameter case, the steady state achieved is $\tau = 1.3$ when $\beta = 2$ whereas it is $\tau = 0.8$ when $m = 0$ and $\beta = 2$.

(a) Newtonian fluid ($\Lambda = \Theta = 0$) (b) Oldroyd fluid ($\Lambda = 0.8$, $\Theta = 0.1$)

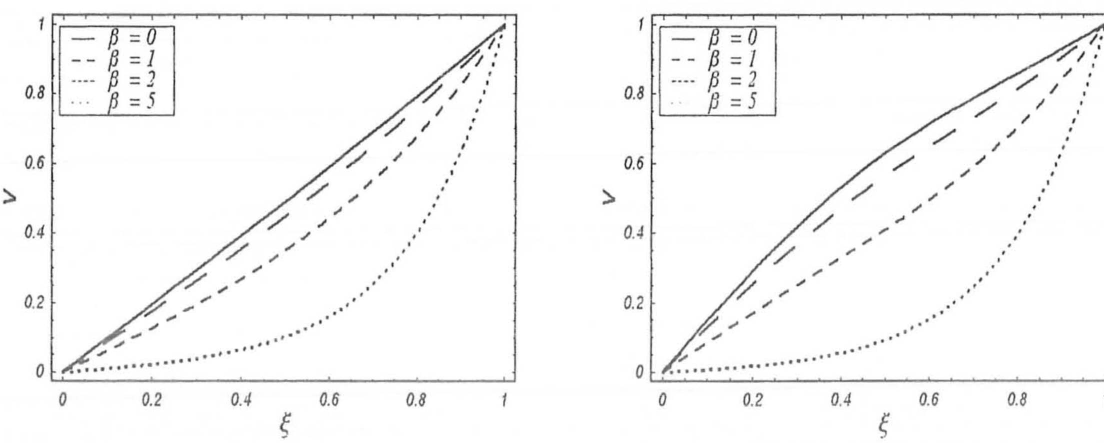

Figure 3.6 : Profiles of the normalized velocity $v(\xi, \tau)$ for various values of β when $\tau = M = 1$ and $m = 2$ are fixed.

3.7 Concluding remarks

In this study, a porous media model for an Oldroyd-B fluid is derived using modified Darcy's law. The modeled equation can be applied to problems with Hall effect. Four illustrative examples have been chosen to discuss the flow analysis from the modeled equation. In each case, the analytical solutions have been obtained in closed form. The presented analysis explores the influences of Hall current, porosity and permeability of the porous medium for the Newtonian and an Oldroyd-B fluids. The obtained results illustrate the novel facets of the model and emphasize their importance. The results categorically indicate the following findings:

- Increasing Hall parameter leads to an increase in the velocity for both a Newtonian and an Oldroyd-B fluids.

- On the velocity, the influence of permeability of the porous medium is similar to that of the Hall parameter. However, the effect of porosity of the medium is quite opposite.

- In presence of Hall parameter, the steady state is achieved much later when compared with the flows which hold in absence of Hall parameter.

- In porous medium, the velocity profiles attain steady state much quickly than those for a non-porous medium.

- For $m = \beta = 0$, the obtained results correspond to the results of reference [70]. This provides a useful check.

- The large time solutions (3.19), (3.29), (3.33) and (3.41) have been compared with the existing solutions in the literature. It is noted that for $M = \beta = 0$, the solutions (3.19) and (3.29) reduce to equations (3.21) and (4.8) in reference [47], respectively. Moreover, the solution (3.41) reduces to equation (31) in reference [48] when $M = \beta = 0$. It is further noted that the solution (3.33) is the sum of the solutions (3.19) and (3.29).

- The small time solutions (3.26), (3.30), (3.34) and (3.42) do not satisfy the initial condition. This is not surprising since the problems for which the boundary data are incompatible do not admit smooth solutions that satisfy both the initial and boundary conditions [16, 17, 71, 72].

Chapter 4

Accelerated flows of an Oldroyd-B fluid in a porous space

In this chapter, the two problems dealing with the unsteady unidirectional accelerated flows of an Oldroyd-B fluid in a porous medium are investigated. By using modified Darcy's law of an Oldroyd-B fluid, the problems governing the accelerated flows are modelled. Employing Fourier sine transform, the analytic solutions of the modelled equations are developed for the following two problems: (i) constant accelerated flow (ii) variable accelerated flow. Explicit expressions of the velocity field and adequate tangential stress are obtained in each case. The solutions for Newtonian, second grade and Maxwell fluids in a porous medium appear as the limiting cases of the present analysis.

4.1 Governing equations

The balance of linear momentum in a porous medium and continuity equation are given as in equations (1.11) and (1.10). The constitutive relationship for an incompressible Oldroyd-B fluid are given as in equations (3.1) and (3.2) We select the velocity of the following form

$$\mathbf{V} = (u(y,t), 0, 0). \tag{4.1}$$

With this choice of velocity the constraint of incompressibility is automatically satisfied. We also assume that extra stress tensor is function of y and t only i.e. S=S(y,t). Substituting equation (4.1) into equation (3.2) and keeping in mind that at $t = 0$ the fluid is at rest, we get $S_{xz} = S_{yy} = S_{yz} = S_{zz} = 0$ and

$$\left(1+\lambda\frac{\partial}{\partial t}\right) S_{xx} = 2\lambda S_{xy}\frac{\partial u}{\partial y} - 2\mu\theta\left(\frac{\partial u}{\partial y}\right)^2, \tag{4.2}$$

$$\left(1+\lambda\frac{\partial}{\partial t}\right) S_{xy} = \mu\left(1+\theta\frac{\partial}{\partial t}\right)\frac{\partial u}{\partial y}. \tag{4.3}$$

By analogy with an Oldroyd-B constitutive relationship, the phenomenological model (which relates pressure drop and velocity) given in equation (3.5) for an Oldroyd-B fluid is suggested [61, 73]. The Darcy's resistance **R** satisfies equation (3.6).

The balance of linear momentum gives

$$\rho\frac{\partial u}{\partial t} = -\frac{\partial p}{\partial x} + \frac{\partial S_{xy}}{\partial y} + R_x, \tag{4.4}$$

in which R_x is the $x-$component of **R**.

In absence of body force, the equations (4.3)-(4.4) and (3.5) and (3.6) give

$$\rho\left(1+\lambda\frac{\partial}{\partial t}\right)\frac{\partial u}{\partial t} = -\left(1+\lambda\frac{\partial}{\partial t}\right)\frac{\partial p}{\partial x} + \mu\left(1+\theta\frac{\partial}{\partial t}\right)\frac{\partial^2 u}{\partial y^2} - \frac{\mu\phi}{k}\left(1+\theta\frac{\partial}{\partial t}\right)u. \tag{4.5}$$

4.2 Flow due to constant accelerated plate

4.2.1 Calculation of velocity field

Consider an incompressible Oldroyd-B fluid over an infinite plate at $y = 0$. Initially the fluid as well as the plate is at rest. At $t = 0^+$ the plate starts to move with constant acceleration 'A' in the x-direction. In the absence of a pressure gradient in the flow direction, the governing equation and the appropriate boundary and initial conditions are

$$\left(1+\lambda\frac{\partial}{\partial t}\right)\frac{\partial u}{\partial t} = \nu\left(1+\theta\frac{\partial}{\partial t}\right)\frac{\partial^2 u}{\partial y^2} - \frac{\nu\phi}{k}\left(1+\theta\frac{\partial}{\partial t}\right)u, \tag{4.6}$$

$$u(0,t) = At, \quad t > 0, \tag{4.7}$$

$$u(y,t), \quad \frac{\partial u(y,t)}{\partial y} \to 0 \text{ as } y \to \infty, \; t > 0, \tag{4.8}$$

$$u(y,t) = 0, \quad \frac{\partial u(y,t)}{\partial t} = 0; \text{ when } t = 0, \quad y > 0, \tag{4.9}$$

where $(\nu = \mu/\rho)$ is the kinematic viscosity of the fluid.

In the following we shall use Fourier sine transform pair defined by equations (1.22)-(1.23). Taking the Fourier sine transform of equation (4.6) and using conditions (4.7)-(4.9), we find that $\bar{u}(\xi,t)$ satisfies the following problem

$$\lambda \frac{d^2\bar{u}}{dt^2} + \left[1 + \alpha\left(\xi^2 + \frac{\phi}{k}\right)\right]\frac{d\bar{u}}{dt} + \nu\left(\xi^2 + \frac{\phi}{k}\right)\bar{u} = \sqrt{\frac{2}{\pi}}\xi A(\nu t + \alpha), \quad \xi, t > 0, \tag{4.10}$$

$$\bar{u}(\xi,0) = \frac{d\bar{u}(\xi,0)}{dt} = 0, \quad \xi > 0, \tag{4.11}$$

where $\alpha = \nu\theta$.

The solution of equation (4.10) satisfying the initial conditions (4.11) has one of the following forms

$$\bar{u}(\xi,t) = \frac{\sqrt{2/\pi}\xi A}{(\xi^2 + \phi/k)}\left[\frac{r_2 r_3 e^{r_1 t} - r_1 r_4 e^{r_2 t}}{\nu(\xi^2 + \phi/k)(r_2 - r_1)}\lambda + t - \frac{1}{\nu(\xi^2 + \phi/k)}\right], \quad \text{if } \lambda < \theta, \tag{4.12}$$

$$\bar{u}(\xi,t) = \frac{\sqrt{2/\pi}\xi A}{(\xi^2 + \phi/k)}\left[\frac{e^{-\nu(\xi^2+\phi/k)t}}{\nu(\xi^2 + \phi/k)} + t - \frac{1}{\nu(\xi^2 + \phi/k)}\right], \quad \text{if } \lambda = \theta, \tag{4.13}$$

or

$$\bar{u}(\xi,t) = \frac{\sqrt{2/\pi}\xi A}{(\xi^2 + \phi/k)}\begin{cases} \frac{r_2 r_3 \exp(r_1 t) - r_1 r_4 \exp(r_2 t)}{\nu(\xi^2+\phi/k)(r_2 - r_1)}\lambda + t - \frac{1}{\nu(\xi^2+\phi/k)} & \text{for } \xi \notin \{a,b\} \\ \frac{[1+(\alpha-2\lambda\nu)(\xi^2+\phi/k)]t+2\lambda}{2\nu\lambda(\xi^2+\phi/k)} \exp\left[-\frac{1+\alpha(\xi^2+\phi/k)}{2\lambda}t\right] + t - \frac{1}{\nu(\xi^2+\phi/k)} & \\ \text{for } \xi \epsilon \{a,b\}, & \end{cases} \tag{4.14}$$

if $\lambda > \theta$. In the above relations

$$r_{1,2} = \frac{-\left[1+\alpha\left(\xi^2+\phi/k\right)\right] \pm \sqrt{\left[1+\alpha\left(\xi^2+\phi/k\right)\right]^2 - 4\lambda\nu\left(\xi^2+\phi/k\right)}}{2\lambda},$$

$$r_{3,4} = \frac{\left[1-\alpha\left(\xi^2+\phi/k\right)\right] \pm \sqrt{\left[1+\alpha\left(\xi^2+\phi/k\right)\right]^2 - 4\lambda\nu\left(\xi^2+\phi/k\right)}}{2\lambda},$$

$$a = \sqrt{\frac{1}{\left[\sqrt{\nu}\left(\sqrt{\lambda}+\sqrt{\lambda-\lambda_r}\right)\right]^2} - \frac{\phi}{k}} \text{ and } b = \sqrt{\frac{1}{\left[\sqrt{\nu}\left(\sqrt{\lambda}-\sqrt{\lambda-\lambda_r}\right)\right]^2} - \frac{\phi}{k}}.$$

Taking the inverse Fourier sine transform of equations (4.12)-(4.14), we find the following expressions of velocity field

$$\begin{aligned} u(y,t) &= At\exp\left(-y\sqrt{\frac{\phi}{k}}\right) - \frac{2A}{\nu\pi}\int_0^\infty \frac{\xi\sin(y\xi)}{\left(\xi^2+\phi/k\right)^2}d\xi \\ &+\frac{2A}{\nu\pi}\exp\left(-\frac{t}{2\lambda}\right)\int_0^\infty \exp\left[-\frac{\alpha\left(\xi^2+\phi/k\right)}{2\lambda}t\right] \\ &\left[ch\left(\frac{\beta t}{2\lambda}\right) + \frac{1+(\alpha-2\lambda\nu)\left(\xi^2+\phi/k\right)}{\beta}sh\left(\frac{\beta t}{2\lambda}\right)\right]\frac{\xi\sin(y\xi)}{\left(\xi^2+\phi/k\right)^2}d\xi, \end{aligned} \tag{4.15}$$

$$u(y,t) = At\exp\left(-y\sqrt{\frac{\phi}{k}}\right) - \frac{2A}{\nu\pi}\int_0^\infty \left\{1-e^{-\nu\left(\xi^2+\phi/k\right)t}\right\}\frac{\xi\sin(y\xi)}{\left(\xi^2+\phi/k\right)^2}d\xi, \tag{4.16}$$

and

$$\begin{aligned} u(y,t) &= At\exp\left(-y\sqrt{\frac{\phi}{k}}\right) - \frac{2A}{\nu\pi}\int_0^\infty \frac{\xi\sin(y\xi)}{\left(\xi^2+\phi/k\right)^2}d\xi + \\ &+\frac{2A}{\nu\pi}\exp\left(-\frac{t}{2\lambda}\right)\int_0^\infty B(\xi,t)\frac{\xi\sin(y\xi)}{\left(\xi^2+\phi/k\right)^2}d\xi \end{aligned} \tag{4.17}$$

if $\lambda < \lambda_r$, $\lambda = \lambda_r$ and $\lambda > \lambda_r$, respectively, and

$$B(\xi,t) = \exp\left(-\frac{\alpha(\xi^2+\phi/k)}{2\lambda}t\right)\begin{cases} ch\left(\frac{\beta t}{2\lambda}\right) + \frac{1+(\alpha-2\lambda\nu)(\xi^2+\phi/k)}{\beta} sh\left(\frac{\beta t}{2\lambda}\right) & \xi \in (0,a)\cup(b,\infty) \\ \frac{1+(\alpha-2\lambda\nu)(\xi^2+\phi/k)}{2\lambda}t+1 & \xi\in\{a,b\} \\ ch\left(\frac{\gamma t}{2\lambda}\right) + \frac{1+(\alpha-2\lambda\nu)(\xi^2+\phi/k)}{\gamma} sh\left(\frac{\gamma t}{2\lambda}\right) & \xi\in(a,b), \end{cases}$$

$\beta = \sqrt{\left[1+\alpha\left(\xi^2+\phi/k\right)\right]^2 - 4\lambda\nu\left(\xi^2+\phi/k\right)}$ and $\gamma = \sqrt{4\lambda\nu\left(\xi^2+\phi/k\right) - \left[1+\alpha\left(\xi^2+\phi/k\right)\right]^2}$.

4.2.2 Calculation of the tangential stress

The solution of the differential equation (4.3) with the initial condition

$$S_{xy}(y,0) = 0, \ y > 0 \tag{4.18}$$

is

$$\tau(y,t) = \frac{\mu}{\lambda}\exp\left(-\frac{t}{\lambda}\right)\int_0^t \exp\left(\frac{\tau}{\lambda}\right)\left(1+\theta\frac{\partial}{\partial\tau}\right)\frac{\partial v(y,\tau)}{\partial y}d\tau, \tag{4.19}$$

where $\tau(y,t) = S_{xy}(y,t)$.

Substituting equations (4.15)-(4.17) into equation (4.19), we get the shear stress under the form

$$\begin{aligned} \tau(y,t) &= -\mu A\sqrt{\frac{\phi}{k}}\exp\left(-y\sqrt{\frac{\phi}{k}}\right)\left[t+(\lambda-\theta)\left(e^{-t/\lambda}-1\right)\right] \\ &\quad -\frac{2\rho A}{\pi}\int_0^\infty \frac{\xi^2\cos(y\xi)}{\left(\xi^2+\phi/k\right)^2}d\xi + \frac{2\rho A}{\pi}\exp\left(-\frac{t}{2\lambda}\right)\int_0^\infty \exp\left[-\frac{\alpha\left(\xi^2+\phi/k\right)}{2\lambda}t\right] \\ &\quad \times\left[ch\left(\frac{\beta t}{2\lambda}\right)+\frac{1-\alpha\left(\xi^2+\phi/k\right)}{\beta}sh\left(\frac{\beta t}{2\lambda}\right)\right]\frac{\xi^2\cos(y\xi)}{\left(\xi^2+\phi/k\right)^2}d\xi, \end{aligned} \tag{4.20}$$

$$\tau(y,t) = -\mu At\sqrt{\frac{\phi}{k}}\exp\left(-y\sqrt{\frac{\phi}{k}}\right) - \frac{2\rho A}{\pi}\int_0^\infty\left\{1-e^{-\nu\left(\xi^2+\phi/k\right)t}\right\}\frac{\xi^2\cos(y\xi)}{\left(\xi^2+\phi/k\right)^2}d\xi \tag{4.21}$$

and

$$
\begin{aligned}
\tau(y,t) \;=\;& -\mu A\sqrt{\frac{\phi}{k}}\exp\left(-y\sqrt{\frac{\phi}{k}}\right)\left[t+(\lambda-\theta)\left(e^{-t/\lambda}-1\right)\right] \\
& -\frac{2\rho A}{\pi}\int_0^\infty \frac{\xi^2\cos(y\xi)}{\left(\xi^2+\phi/k\right)^2}d\xi \\
& +\frac{2\rho A}{\pi}\exp\left(-\frac{t}{2\lambda}\right)\int_0^a \exp\left[-\frac{\alpha\left(\xi^2+\phi/k\right)}{2\lambda}t\right] \\
& \left[\cos\left(\frac{\gamma t}{2\lambda}\right)+\frac{1-\alpha\left(\xi^2+\phi/k\right)}{\gamma}\sin\left(\frac{\gamma t}{2\lambda}\right)\right]\frac{\xi^2\cos(y\xi)}{\left(\xi^2+\phi/k\right)^2}d\xi + \\
& +\frac{2\rho A}{\pi}\exp\left(-\frac{t}{2\lambda}\right)\int_a^b \exp\left[-\frac{\alpha\left(\xi^2+\phi/k\right)}{2\lambda}t\right] \\
& \left[ch\left(\frac{\beta t}{2\lambda}\right)+\frac{1-\alpha\left(\xi^2+\phi/k\right)}{\beta}sh\left(\frac{\beta t}{2\lambda}\right)\right]\frac{\xi^2\cos(y\xi)}{\left(\xi^2+\phi/k\right)^2}d\xi + \\
& +\frac{2\rho A}{\pi}\exp\left(-\frac{t}{2\lambda}\right)\int_b^\infty \exp\left[-\frac{\alpha\left(\xi^2+\phi/k\right)}{2\lambda}t\right] \\
& \left[ch\left(\frac{\beta t}{2\lambda}\right)+\frac{1-\alpha\left(\xi^2+\phi/k\right)}{\beta}sh\left(\frac{\beta t}{2\lambda}\right)\right]\frac{\xi^2\cos(y\xi)}{\left(\xi^2+\phi/k\right)^2}d\xi,
\end{aligned}
\tag{4.22}
$$

corresponding to $\lambda < \theta$, $\lambda = \theta$ and $\lambda > \theta$.

4.3 Limiting cases

1. Making $\lambda \to 0$, into equations (4.15) and (4.20) we find that

$$
v(y,t) = At\exp\left(-y\sqrt{\frac{\phi}{k}}\right) - \frac{2A}{\nu\pi}\int_0^\infty\left\{1-\exp\left[\frac{-\nu\left(\xi^2+\phi/k\right)}{1+\alpha\left(\xi^2+\phi/k\right)}t\right]\right\}\frac{\xi\sin(y\xi)\,d\xi}{\left(\xi^2+\phi/k\right)^2}, \tag{4.23}
$$

and

$$
\begin{aligned}
\tau(y,t) &= -\mu A\sqrt{\frac{\phi}{k}}\exp\left(-y\sqrt{\frac{\phi}{k}}\right)(t+\theta) \qquad (4.24)\\
&-\frac{2\rho A}{\pi}\int_0^\infty\left\{1-\frac{1}{1+\alpha\left(\xi^2+\phi/k\right)}\exp\left[\frac{-\nu\left(\xi^2+\phi/k\right)}{1+\alpha\left(\xi^2+\phi/k\right)}t\right]\right\}\times\frac{\xi^2\cos(y\xi)}{\left(\xi^2+\phi/k\right)^2}d\xi,
\end{aligned}
$$

which are the similar solutions for a second grade fluid.

2. The velocity field

$$
\begin{aligned}
u(y,t) &= At\exp\left(-y\sqrt{\frac{\phi}{k}}\right)-\frac{2A}{\nu\pi}\int_0^\infty\frac{\xi\sin(y\xi)}{\left(\xi^2+\phi/k\right)^2}d\xi \qquad (4.25)\\
&+\frac{2A}{\nu\pi}\exp\left(-\frac{t}{2\lambda}\right)\int_0^c\left[ch\left(\frac{\delta t}{2\lambda}\right)+\frac{1-2\lambda\nu\left(\xi^2+\phi/k\right)}{\delta}sh\left(\frac{\delta t}{2\lambda}\right)\right]\frac{\xi\sin(y\xi)\,d\xi}{\left(\xi^2+\phi/k\right)^2}\\
&+\frac{2A}{\nu\pi}\exp\left(-\frac{t}{2\lambda}\right)\int_c^\infty\left[\cos\left(\frac{\chi t}{2\lambda}\right)+\frac{1-2\lambda\nu\left(\xi^2+\phi/k\right)}{\chi}\sin\left(\frac{\chi t}{2\lambda}\right)\right]\frac{\xi\sin(y\xi)\,d\xi}{\left(\xi^2+\phi/k\right)^2},
\end{aligned}
$$

as well as, the shear stress

$$
\begin{aligned}
\tau(y,t) &= -\mu A\sqrt{\frac{\phi}{k}}\exp\left(-y\sqrt{\phi/k}\right)\left[t+\lambda\left(e^{-t/\lambda}-1\right)\right]-\frac{2\rho A}{\pi}\int_0^\infty\frac{\xi^2\cos(y\xi)}{\left(\xi^2+\phi/k\right)^2}d\xi+\\
&\frac{2\rho A}{\nu\pi}\exp\left(-\frac{t}{2\lambda}\right)\int_0^c\left[ch\left(\frac{\delta t}{2\lambda}\right)+\frac{1}{\delta}sh\left(\frac{\delta t}{2\lambda}\right)\right]\frac{\xi^2\cos(y\xi)}{\left(\xi^2+\phi/k\right)^2}d\xi+\\
&\frac{2\rho A}{\nu\pi}\exp\left(-\frac{t}{2\lambda}\right)\int_c^\infty\left[\cos\left(\frac{\chi t}{2\lambda}\right)+\frac{1}{\chi}\sin\left(\frac{\chi t}{2\lambda}\right)\right]\frac{\xi^2\cos(y\xi)}{\left(\xi^2+\phi/k\right)^2}d\xi, \qquad (4.26)
\end{aligned}
$$

corresponding to a Maxwell fluid can be also obtained as the limiting cases of equations (4.17) and (4.22) for $\theta\to 0$. In the last relations

$$
c=\frac{1}{2}\sqrt{\frac{1-4\lambda\nu\phi/k}{\lambda\nu}},\ \delta=\sqrt{1-4\lambda\nu\left(\xi^2+\phi/k\right)}\text{ and }\chi=\sqrt{4\lambda\nu\left(\xi^2+\phi/k\right)-1}.
$$

3. By letting now $\theta\to 0$ into equations (4.23) and (4.24) or $\lambda\to 0$ into equations (4.25) and (4.26) we attain to the solutions (4.16) and (4.21) corresponding to a Newtonian fluid.

It is clear that for $\phi = 0$, all solutions reduce to those obtained in [52] corresponding to the flow induced by constantly accelerating plate in an Oldroyd-B fluid.

4.4 Flow induced by variable accelerated plate

Let us now consider a flow problem for which the plate at t=0^+ begins to move with a variable acceleration. The governing equation, the initial conditions and a part of the boundary conditions are same. Instead of the condition (4.7) we take the boundary condition

$$u(0,t) = At^2, \quad t > 0, \tag{4.27}$$

so that the corresponding ordinary differential equation becomes

$$\lambda \frac{d^2\bar{u}}{dt^2} + \left[1 + \alpha\left(\xi^2 + \phi/k\right)\right]\frac{d\bar{u}}{dt} + \nu\left(\xi^2 + \phi/k\right)\bar{u} = \sqrt{\frac{2}{\pi}}\xi A\left(\nu t^2 + 2\alpha t\right); \quad \xi, t > 0. \tag{4.28}$$

The general solution of above equation is

$$\bar{u}(\xi,t) = \frac{\sqrt{2/\pi}\xi A}{\left(\xi^2+\phi/k\right)}\begin{cases} 2\frac{1+(\alpha-\lambda\nu)\left(\xi^2+\phi/k\right)}{\nu^2\left(\xi^2+\phi/k\right)^2}\frac{r_1\exp(r_2 t)-r_2\exp(r_1 t)}{r_2-r_1} \\ +\frac{2}{\nu\left(\xi^2+\phi/k\right)}\frac{\exp(r_2 t)-\exp(r_1 t)}{r_2-r_1}+t^2- \\ \frac{2t}{\nu\left(\xi^2+\phi/k\right)}+2\frac{1+(\alpha-\lambda\nu)\left(\xi^2+\phi/k\right)}{\nu^2\left(\xi^2+\phi/k\right)^2} \quad \text{for} \quad \xi \notin \{a,b\} \\ \\ 2\left\{\begin{array}{c}\left[\begin{array}{c}\frac{1}{\nu\left(\xi^2+\phi/k\right)}- \\ \frac{1+(\alpha-\lambda\nu)\left(\xi^2+\phi/k\right)}{\nu^2\left(\xi^2+\phi/k\right)^2}\frac{1+\alpha\left(\xi^2+\phi/k\right)}{2\lambda}\end{array}\right]t- \\ \frac{1+(\alpha-\lambda\nu)\left(\xi^2+\phi/k\right)}{\nu^2\left(\xi^2+\phi/k\right)^2}\end{array}\right\}\exp\left[-\frac{1+\alpha\left(\xi^2+\phi/k\right)}{2\lambda}t\right] \\ +t^2-\frac{2t}{\nu\left(\xi^2+\phi/k\right)}+2\frac{1+(\alpha-\lambda\nu)\left(\xi^2+\phi/k\right)}{\nu^2\left(\xi^2+\phi/k\right)^2} \quad \text{for} \quad \xi \epsilon \{a,b\}, \end{cases} \tag{4.29}$$

we can write the velocity field under the next forms

$$
\begin{aligned}
u(y,t) &= At^2 \exp\left(-y\sqrt{\frac{\phi}{k}}\right) - \frac{4At}{\pi\nu}\int_0^\infty \frac{\xi \sin(y\xi)}{\left(\xi^2+\phi/k\right)^2} d\xi + \\
&\frac{4A}{\pi\nu^2}\int_0^\infty \frac{1+(\alpha-\lambda\nu)\left(\xi^2+\phi/k\right)}{\left(\xi^2+\phi/k\right)^3}\xi \sin(y\xi)\, d\xi \\
&-\frac{4A}{\pi\nu^2}\exp\left(-\frac{t}{2\lambda}\right)\int_0^\infty \frac{1+(\alpha-\lambda\nu)\left(\xi^2+\phi/k\right)}{\left(\xi^2+\phi/k\right)^3}\exp\left[-\frac{\alpha\left(\xi^2+\phi/k\right)}{2\lambda}t\right] \\
&\times\left[ch\left(\frac{\beta t}{2\lambda}\right)+\frac{1+\alpha\left(\xi^2+\phi/k\right)}{\beta}sh\left(\frac{\beta t}{2\lambda}\right)\right]\xi\sin(y\xi)\, d\xi
\end{aligned}
$$

$$
+\frac{8\lambda A}{\pi\nu}\exp\left(-\frac{t}{2\lambda}\right)\int_0^\infty \exp\left[-\frac{\alpha\left(\xi^2+\phi/k\right)}{2\lambda}t\right] sh\left(\frac{\beta t}{2\lambda}\right)\frac{\xi\sin(y\xi)\, d\xi}{\beta\left(\xi^2+\phi/k\right)^2} \quad \text{if } \lambda<\theta, \tag{4.30}
$$

$$
u(y,t) = At^2 \exp\left(-y\sqrt{\frac{\phi}{k}}\right) - \frac{4At}{\pi\nu}\int_0^\infty \frac{\xi \sin(y\xi)}{\left(\xi^2+\phi/k\right)^2} d\xi
$$

$$
+\frac{4A}{\pi\nu^2}\int_0^\infty \left\{1-\exp\left[-\nu\left(\xi^2+\frac{\phi}{k}\right)t\right]\right\}\frac{\xi\sin(y\xi)}{\left(\xi^2+\phi/k\right)^3}d\xi \quad \text{if } \lambda=\theta, \tag{4.31}
$$

$$
\begin{aligned}
u(y,t) &= At^2 \exp\left(-y\sqrt{\frac{\phi}{k}}\right) - \frac{4At}{\pi\nu}\int_0^\infty \frac{\xi \sin(y\xi)}{\left(\xi^2+\phi/k\right)^2} d\xi \qquad (4.32)\\
&+\frac{4A}{\pi\nu^2}\int_0^\infty \frac{1+(\alpha-\lambda\nu)\left(\xi^2+\phi/k\right)}{\left(\xi^2+\phi/k\right)^3}\xi \sin(y\xi)\, d\xi \\
&-\frac{4A}{\pi\nu^2}\exp\left(-\frac{t}{2\lambda}\right)\left\{
\begin{array}{l}
\int_0^\infty \frac{1+(\alpha-\lambda\nu)(\xi^2+\phi/k)}{(\xi^2+\phi/k)^3}\xi D(\xi,t)\sin(y\xi)\, d\xi \\
\qquad -\int_0^\infty E(\xi,t)\frac{\xi\sin(y\xi)}{(\xi^2+\phi/k)^2}d\xi
\end{array}
\right\}
\end{aligned}
$$

$$\text{if } \lambda>\theta$$

where

$$D(\xi,t)=\exp\left[-\frac{\alpha\left(\xi^2+\phi/k\right)}{2\lambda}t\right]\begin{cases} ch\left(\frac{\beta t}{2\lambda}\right)+\frac{1+\alpha\left(\xi^2+\phi/k\right)}{\beta}sh\left(\frac{\beta t}{2\lambda}\right) & \text{for } \xi\in(0,a)\cup(b,\infty)\\ 0 & \text{for } \xi\epsilon\{a,b\}\\ \cos\left(\frac{\gamma t}{2\lambda}\right)+\frac{1+\alpha\left(\xi^2+\phi/k\right)}{\gamma}\sin\left(\frac{\gamma t}{2\lambda}\right) & \text{for } \xi\epsilon(a,b)\end{cases}$$

and

$$E(\xi,t)=\exp\left[-\frac{\alpha\left(\xi^2+\phi/k\right)}{2\lambda}t\right]\begin{cases} \frac{2\lambda}{\beta}sh\left(\frac{\beta t}{2\lambda}\right) & \text{for } \xi\in(0,a)\cup(b,\infty)\\ \left[1-\frac{1+(\alpha-\lambda\nu)\left(\xi^2+\phi/k\right)}{\nu\left(\xi^2+\phi/k\right)}\frac{1+\alpha\left(\xi^2+\phi/k\right)}{2\lambda}\right]t & \\ -\frac{1+(\alpha-\lambda\nu)\left(\xi^2+\phi/k\right)}{\nu\left(\xi^2+\phi/k\right)} & \text{for } \xi\epsilon\{a,b\}\\ \frac{2\lambda}{\gamma}\sin\left(\frac{\gamma t}{2\lambda}\right) & \text{for } \xi\epsilon(a,b)\end{cases}$$

In the following, in order to determine the adequate stresses, we again introduce equations (4.30),(4.31) and (4.32) into equation (4.19). In order to determine the shear stress corresponding to the flow induced by a variable accelerating plate we write the velocity field (4.30) under the form

$$v(y,t)=At^2\exp\left(-y\sqrt{\frac{\phi}{k}}\right)+v_1(y,t)+v_2(y,t)+v_3(y,t)+v_4(y,t)$$

where

$$v_1(y,t) = -\frac{4At}{\pi\nu}\int_0^\infty\frac{\xi\sin(y\xi)}{\left(\xi^2+\phi/k\right)^2}d\xi, \tag{4.33}$$

$$v_2(y,t) = \frac{4A}{\pi\nu^2}\int_0^\infty\frac{1+(\alpha-\lambda\nu)\left(\xi^2+\phi/k\right)}{\left(\xi^2+\phi/k\right)^3}\xi\sin(y\xi)\,d\xi,$$

$$v_3(y,t)=\frac{4A}{\pi\nu^2}\int_0^\infty\frac{1+(\alpha-\lambda\nu)\left(\xi^2+\phi/k\right)}{\left(\xi^2+\phi/k\right)^3}\frac{r_1\exp(r_2t)-r_2\exp(r_1t)}{r_2-r_1}\xi\sin(y\xi)\,d\xi \tag{4.34}$$

$$v_4(y,t)=\frac{4A}{\pi\nu}\int_0^\infty\frac{\xi\sin(y\xi)}{\left(\xi^2+\phi/k\right)^2}\frac{\exp(r_2t)-\exp(r_1t)}{r_2-r_1}d\xi \tag{4.35}$$

and

$$
\begin{aligned}
r_1 r_2 &= \frac{\nu\left(\xi^2+\phi/k\right)}{\lambda}, \quad r_3 r_4 = \frac{\nu\left(\xi^2+\phi/k\right)\left(\lambda-\lambda_r\right)}{\lambda^2} \\
r_1 r_3 &= -\frac{\nu\left(\xi^2+\phi/k\right)\left(1+\lambda_r r_1\right)}{\lambda}, \quad r_2 r_4 = -\frac{\nu\left(\xi^2+\phi/k\right)\left(1+\lambda_r r_2\right)}{\lambda} \\
r_1 r_4 &= \frac{r_1+\nu\left(\xi^2+\phi/k\right)}{\lambda} = \frac{\nu\left(\xi^2+\phi/k\right)\left(\lambda-\lambda_r\right)-\lambda r_4}{\lambda^2} \\
r_2 r_3 &= \frac{r_2+\nu\left(\xi^2+\phi/k\right)}{\lambda} = \frac{\nu\left(\xi^2+\phi/k\right)\left(\lambda-\lambda_r\right)-\lambda r_3}{\lambda^2} \\
r_1+r_4 &= r_2+r_3 = -\alpha\left(\xi^2+\phi/k\right)/\lambda.
\end{aligned} \tag{4.36}
$$

Introducing equations (4.33)-(4.35) into equation (4.19) we get for the last four terms

$$
\begin{aligned}
\tau_1(y,t) &= \frac{\mu}{\lambda}\exp\left(-\frac{t}{\lambda}\right)\int_0^t \exp\left(\frac{\tau}{\lambda}\right)\left(1+\lambda_r\partial_\tau\right)\partial_y v_1(y,\tau)\,d\tau \\
&= -\frac{4\rho A}{\pi}\left[t+\left(\lambda_r-\lambda\right)\left(1-e^{-t/\lambda}\right)\right]\int_0^\infty \frac{\xi^2\cos(y\xi)}{\left(\xi^2+\phi/k\right)^2}d\xi,
\end{aligned} \tag{4.37}
$$

$$
\tau_2(y,t) = \frac{4\rho A}{\pi\nu}\left(1-e^{-t/\lambda}\right)\int_0^\infty \frac{1+(\alpha-\lambda\nu)\left(\xi^2+\phi/k\right)}{\left(\xi^2+\phi/k\right)^3}\xi^2\cos(y\xi)\,d\xi, \tag{4.38}
$$

$$\begin{aligned}
\tau_3(y,t) &= -\frac{4\rho A}{\pi\nu}\frac{\lambda_r}{\lambda-\lambda_r}\int_0^\infty \frac{1+(\alpha-\lambda\nu)\left(\xi^2+\phi/k\right)}{\left(\xi^2+\phi/k\right)^3}\frac{r_1\exp(r_2t)-r_2\exp(r_1t)}{r_2-r_1}\xi^2\cos(y\xi)\,d\xi \\
&\quad -\frac{4\rho A}{\pi\nu}\frac{1}{\lambda-\lambda_r}\int_0^\infty \frac{1+(\alpha-\lambda\nu)\left(\xi^2+\phi/k\right)}{\left(\xi^2+\phi/k\right)^3}\frac{\exp(r_2t)-\exp(r_1t)}{r_2-r_1}\xi^2\cos(y\xi)\,d\xi \\
&\quad +\frac{4\rho A}{\pi\nu}\frac{\lambda_r}{\lambda-\lambda_r}\int_0^\infty \frac{1+(\alpha-\lambda\nu)\left(\xi^2+\phi/k\right)}{\left(\xi^2+\phi/k\right)^3}\frac{r_3\exp(r_2t)-r_4\exp(r_1t)}{r_2-r_1}\xi^2\cos(y\xi)\,d\xi \\
&= \frac{4\rho A}{\pi\nu}\frac{\lambda_r}{\lambda(\lambda-\lambda_r)}\int_0^\infty \frac{1+(\alpha-\lambda\nu)\left(\xi^2+\phi/k\right)}{\left(\xi^2+\phi/k\right)^3}\frac{\exp(r_2t)-\exp(r_1t)}{r_2-r_1}\xi^2\cos(y\xi)\,d\xi \\
&\quad -\frac{4\rho A}{\pi\nu}\frac{1}{\lambda-\lambda_r}\int_0^\infty \frac{1+(\alpha-\lambda\nu)\left(\xi^2+\phi/k\right)}{\left(\xi^2+\phi/k\right)^3}\frac{\exp(r_2t)-\exp(r_1t)}{r_2-r_1}\xi^2\cos(y\xi)\,d\xi \\
&= -\frac{4\rho A}{\pi\nu\lambda}\int_0^\infty \frac{1+(\alpha-\lambda\nu)\left(\xi^2+\phi/k\right)}{\left(\xi^2+\phi/k\right)^3}\frac{\exp(r_2t)-\exp(r_1t)}{r_2-r_1}\xi^2\cos(y\xi)\,d\xi \qquad (4.39)
\end{aligned}$$

and

$$\begin{aligned}
\tau_4(y,t) &= \frac{4\rho A}{\pi\nu}\frac{\lambda}{\lambda-\lambda_r}\int_0^\infty \frac{\xi^2\cos(y\xi)}{\left(\xi^2+\phi/k\right)^3}\frac{r_3\exp(r_2t)-r_4\exp(r_1t)}{r_2-r_1}d\xi \\
&\quad +\frac{4\rho A}{\pi\nu}\frac{\lambda_r}{\lambda-\lambda_r}\int_0^\infty \frac{\xi^2\cos(y\xi)}{\left(\xi^2+\phi/k\right)^3}\frac{r_2\exp(r_2t)-r_1\exp(r_1t)}{r_2-r_1}d\xi \\
&\quad +\frac{4\rho A}{\pi}\frac{\lambda_r}{\lambda-\lambda_r}\int_0^\infty \frac{\xi^2\cos(y\xi)}{\left(\xi^2+\phi/k\right)^2}\frac{\exp(r_2t)-\exp(r_1t)}{r_2-r_1}d\xi \\
&\quad +\frac{4\rho A}{\pi\nu}e^{-t/\lambda}\int_0^\infty \frac{\xi^2\cos(y\xi)}{\left(\xi^2+\phi/k\right)^3}d\xi \\
&= -\frac{4\rho A}{\pi\nu}\int_0^\infty \frac{\xi^2\cos(y\xi)}{\left(\xi^2+\phi/k\right)^3}\frac{r_2\exp(r_2t)-r_1\exp(r_1t)}{r_2-r_1}d\xi \\
&\quad +\frac{4\rho A}{\pi\nu}e^{-t/\lambda}\int_0^\infty \frac{\xi^2\cos(y\xi)}{\left(\xi^2+\phi/k\right)^3}d\xi. \qquad (4.40)
\end{aligned}$$

Now

$$\tau(y,t) = -\mu A\sqrt{\frac{\phi}{k}}\exp\left(-y\sqrt{\frac{\phi}{k}}\right)\left\{t^2-2(\lambda-\theta)\left[t-\lambda+\lambda\exp\left(-\frac{t}{\lambda}\right)\right]\right\}-$$

$$-\frac{4\rho A}{\pi}(t+\theta)\int_0^\infty\frac{\xi^2\cos(y\xi)}{(\xi^2+\phi/k)^2}d\xi+\frac{4\rho A}{\pi\nu}\int_0^\infty\frac{1+\alpha(\xi^2+\phi/k)}{(\xi^2+\phi/k)^3}\xi^2\cos(y\xi)\,d\xi-$$

$$-\frac{4\rho A}{\pi\nu}\exp\left(-\frac{t}{2\lambda}\right)\int_0^\infty\exp\left[-\frac{\alpha(\xi^2+\phi/k)}{2\lambda}t\right]$$

$$\left[ch\left(\frac{\beta t}{2\lambda}\right)+\frac{1+\alpha(\xi^2+\phi/k)}{\beta}sh\left(\frac{\beta t}{2\lambda}\right)\right]\frac{\xi^2\cos(y\xi)}{(\xi^2+\phi/k)^3}d\xi$$

$$-\frac{8\rho A}{\pi\nu}e^{-\frac{t}{2\lambda}}\int_0^\infty\frac{1+(\alpha-\nu\lambda)(\xi^2+\phi/k)}{(\xi^2+\phi/k)^3}\exp\left[-\frac{\alpha(\xi^2+\phi/k)}{2\lambda}t\right]\frac{1}{\beta}sh\left(\frac{\beta t}{2\lambda}\right)\xi^2\cos(y\xi)\,d\xi, \quad (4.41)$$

$$\tau(y,t) = -\mu At^2\sqrt{\frac{\phi}{k}}\exp\left(-y\sqrt{\frac{\phi}{k}}\right)-\frac{4\rho A}{\pi}t\int_0^\infty\frac{\xi^2\cos(y\xi)}{(\xi^2+\phi/k)^2}d\xi$$

$$+\frac{4\rho A}{\pi\nu}\exp\int_0^\infty\left\{1-\exp\left[-\nu\left(\xi^2+\frac{\phi}{k}\right)t\right]\right\}\frac{\xi^2\cos(y\xi)}{(\xi^2+\phi/k)^3}d\xi, \quad (4.42)$$

and

$$
\begin{aligned}
\tau(y,t) \;=\;& -\mu A\sqrt{\frac{\phi}{k}}\exp\left(-y\sqrt{\frac{\phi}{k}}\right)\left\{t^2-2(\lambda-\theta)\left[t-\lambda+\lambda\exp\left(-\frac{t}{\lambda}\right)\right]\right\} \\
& -\frac{4\rho A}{\pi}(t+\theta)\int_0^\infty\frac{\xi^2\cos(y\xi)}{\left(\xi^2+\phi/k\right)^2}d\xi+\frac{4\rho A}{\pi\nu}\int_0^\infty\frac{1+\alpha\left(\xi^2+\phi/k\right)}{\left(\xi^2+\phi/k\right)^3}\xi^2\cos(y\xi)\,d\xi \\
& -\frac{4\rho A}{\pi\nu}\exp\left(-\frac{t}{2\lambda}\right)\int_0^\infty D(\xi,t)\frac{\xi^2\cos(y\xi)}{\left(\xi^2+\phi/k\right)^3}d\xi \\
& -\frac{8\rho A}{\pi\nu}\exp\left(-\frac{t}{2\lambda}\right)\int_0^a\frac{\xi^2}{\beta}\frac{1+(\alpha-\lambda\nu)\left(\xi^2+\phi/k\right)}{\left(\xi^2+\phi/k\right)^3} \\
& \exp\left[-\frac{\alpha\left(\xi^2+\phi/k\right)}{2\lambda}t\right]sh\left(\frac{\beta t}{2\lambda}\right)\cos(y\xi)\,d\xi \\
& -\frac{8\rho A}{\pi\nu}\exp\left(-\frac{t}{2\lambda}\right)\int_a^b\frac{\xi^2}{\gamma}\frac{1+(\alpha-\lambda\nu)\left(\xi^2+\phi/k\right)}{\left(\xi^2+\phi/k\right)^3} \\
& \exp\left[-\frac{\alpha\left(\xi^2+\phi/k\right)}{2\lambda}t\right]\sin\left(\frac{\gamma t}{2\lambda}\right)\cos(y\xi)\,d\xi \\
& -\frac{8\rho A}{\pi\nu}\exp\left(-\frac{t}{2\lambda}\right)\int_b^\infty\frac{\xi^2}{\beta}\frac{1+(\alpha-\lambda\nu)\left(\xi^2+\phi/k\right)}{\left(\xi^2+\phi/k\right)^3} \\
& \exp\left[-\frac{\alpha\left(\xi^2+\phi/k\right)}{2\lambda}t\right]sh\left(\frac{\beta t}{2\lambda}\right)\cos(y\xi)\,d\xi \qquad (4.43)
\end{aligned}
$$

if $\lambda<\theta$, $\lambda=\theta$, $\lambda>\theta$ respectively.

Finally, it is clearly seen that making $\lambda=\theta$ into any one of equations (4.15), (4.17), (4.20), (4.22), (4.30), (4.32),(4.41) and (4.43), the solutions corresponding to a Newtonian fluid (4.16), (4.21),(4.31) and (4.42) are obtained.

4.5 Special cases

1. By letting $\lambda \to 0$ into equations (4.30) and (4.41), the velocity field

$$\begin{aligned} v(y,t) &= At^2 \exp\left(-y\sqrt{\frac{\phi}{k}}\right) - \frac{4At}{\pi\nu}\int_0^\infty \frac{\xi \sin(y\xi)}{\left(\xi^2+\phi/k\right)^2}d\xi + \\ &+\frac{4A}{\nu^2\pi}\int_0^\infty \left[1-\exp\left(\frac{-\nu\left(\xi^2+\phi/k\right)t}{1+\alpha\left(\xi^2+\phi/k\right)}\right)\right]\frac{\left[1+\alpha\left(\xi^2+\phi/k\right)\right]}{\left(\xi^2+\phi/k\right)^3}\xi\sin(y\xi)\,d\xi \end{aligned} \tag{4.44}$$

as well as the shear stress

$$\begin{aligned} \tau(y,t) &= -\mu A\sqrt{\frac{\phi}{k}}\exp\left(-y\sqrt{\frac{\phi}{k}}\right)\left(t^2+2\lambda_r t\right)+\frac{4\rho A}{\pi}\left(t+\lambda_r\right)\int_0^\infty \frac{\xi^2\cos(y\xi)}{\left(\xi^2+\phi/k\right)^2}d\xi + \\ &+\frac{4\rho A}{\nu\pi}\int_0^\infty\left[1-\exp\left(\frac{-\nu\left(\xi^2+\phi/k\right)t}{1+\alpha\left(\xi^2+\phi/k\right)}\right)\right]\frac{\xi^2\cos(y\xi)}{\left(\xi^2+\phi/k\right)^3}d\xi, \end{aligned} \tag{4.45}$$

corresponding to a second grade fluid are obtained.

2. Making now $\theta = 0$ into equations (4.32) and (4.43) we attain the similar solutions

$$\begin{aligned} v(y,t) &= At^2\exp\left(-y\sqrt{\frac{\phi}{k}}\right)-\frac{4At}{\pi\nu}\int_0^\infty\frac{\xi\sin(y\xi)}{\left(\xi^2+\phi/k\right)^2}d\xi+\frac{4A}{\pi\nu^2}\int_0^\infty\frac{1-\lambda\nu\left(\xi^2+\frac{\phi}{k}\right)}{\left(\xi^2+\phi/k\right)^3}\xi\sin(y\xi)\,d\xi \\ &-\frac{4A}{\pi\nu^2}\exp\left(-\frac{t}{2\lambda}\right)\int_0^c\left[ch\left(\frac{\delta t}{2\lambda}\right)+\frac{1}{\delta}sh\left(\frac{\delta t}{2\lambda}\right)\right]\frac{1-\lambda\nu\left(\xi^2+\phi/k\right)}{\left(\xi^2+\phi/k\right)^3}\xi\sin(y\xi)\,d\xi \\ &-\frac{4A}{\pi\nu^2}\exp\left(-\frac{t}{2\lambda}\right)\int_c^\infty\left[\cos\left(\frac{\chi t}{2\lambda}\right)+\frac{1}{\chi}\sin\left(\frac{\chi t}{2\lambda}\right)\right]\frac{1-\lambda\nu\left(\xi^2+\phi/k\right)}{\left(\xi^2+\phi/k\right)^3}\xi\sin(y\xi)\,d\xi \\ &-\frac{8\lambda A}{\pi\nu}\exp\left(-\frac{t}{2\lambda}\right)\int_0^c\frac{1}{\delta}sh\left(\frac{\delta t}{2\lambda}\right)\frac{\xi\sin(y\xi)}{\left(\xi^2+\phi/k\right)^2}d\xi- \\ &-\frac{8\lambda A}{\pi\nu}\exp\left(-\frac{t}{2\lambda}\right)\int_c^\infty\frac{1}{\chi}\sin\left(\frac{\chi t}{2\lambda}\right)\frac{\xi\sin(y\xi)}{\left(\xi^2+\phi/k\right)^2}d\xi \end{aligned} \tag{4.46}$$

and

$$\tau(y,t) = -\mu A\sqrt{\frac{\phi}{k}}\exp\left(-y\sqrt{\frac{\phi}{k}}\right)\left\{t^2-2\lambda\left[t-\lambda+\lambda\exp\left(-\frac{t}{2\lambda}\right)\right]\right\}$$
$$-\frac{4\rho A}{\pi}t\int_0^\infty \frac{\xi^2\cos(y\xi)}{(\xi^2+\phi/k)^2}d\xi+\frac{4\rho A}{\pi\nu}\int_0^\infty \frac{\xi^2\cos(y\xi)}{(\xi^2+\phi/k)^3}d\xi$$
$$-\frac{4\rho A}{\pi\nu}\exp\left(-\frac{t}{2\lambda}\right)\left\{\begin{array}{c}\int_0^c\left[ch\left(\frac{\delta t}{2\lambda}\right)+\frac{1}{\delta}sh\left(\frac{\delta t}{2\lambda}\right)\right]\frac{\xi^2\cos(y\xi)}{(\xi^2+\phi/k)^3}d\xi\\ \int_c^\infty\left[\cos\left(\frac{\chi t}{2\lambda}\right)+\frac{1}{\chi}\sin\left(\frac{\chi t}{2\lambda}\right)\right]\frac{\xi^2\cos(y\xi)}{(\xi^2+\phi/k)^3}d\xi\end{array}\right\}$$
$$+\frac{8\rho A}{\pi\nu}\lambda\exp\left(-\frac{t}{2\lambda}\right)\left\{\begin{array}{c}\int_0^c\frac{1}{\delta}sh\left(\frac{\delta t}{2\lambda}\right)\frac{\xi^2\cos(y\xi)}{(\xi^2+\phi/k)^2}d\xi\\ +\int_c^\infty\frac{1}{\chi}\sin\left(\frac{\chi t}{2\lambda}\right)\frac{\xi^2\cos(y\xi)}{(\xi^2+\phi/k)^2}d\xi\end{array}\right\}, \qquad (4.47)$$

for a Maxwell fluid.

3. Finally, the velocity field (4.31) and the shear stress (4.42), corresponding to Newtonian fluid, appear as limiting case of equations (4.44) and (4.45) (for $\alpha\to 0$) or equations (4.46) and (4.47) (for $\lambda\to 0$).

For $\phi=0$, the porous effects disappear and solutions corresponding to the flow induced by a plate of variable acceleration in an Oldroyd-B, Maxwell, second grade and Newtonian fluids are obtained.

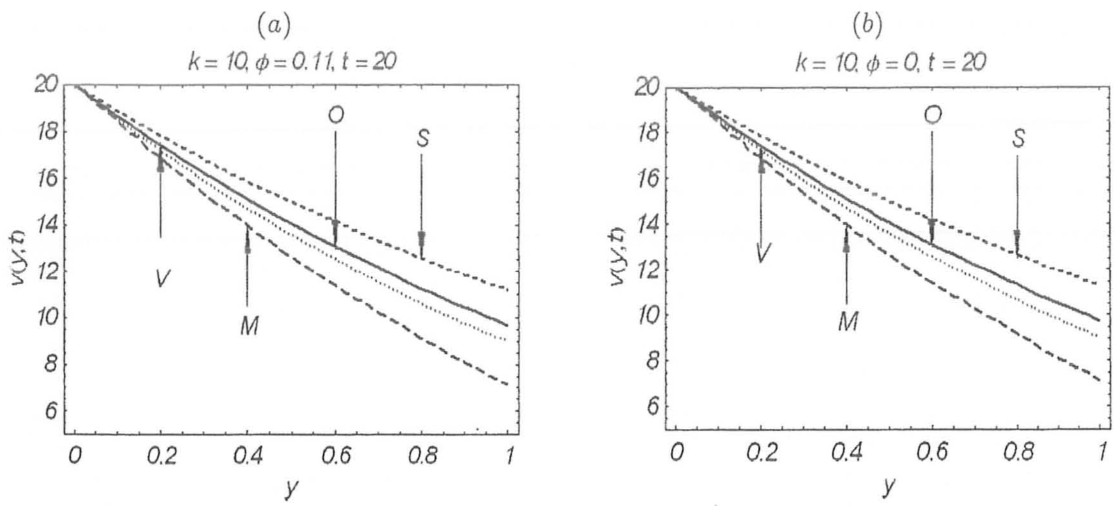

Figure 4.1: Velocity profiles u corresponding to a S=second grade, O=Oldroyd, V=viscous and M=Maxwell fluids corresponding to $A = 1.0, \nu = 0.11746,\ \lambda_r = 15, \lambda = 10$ (constant acceleration)

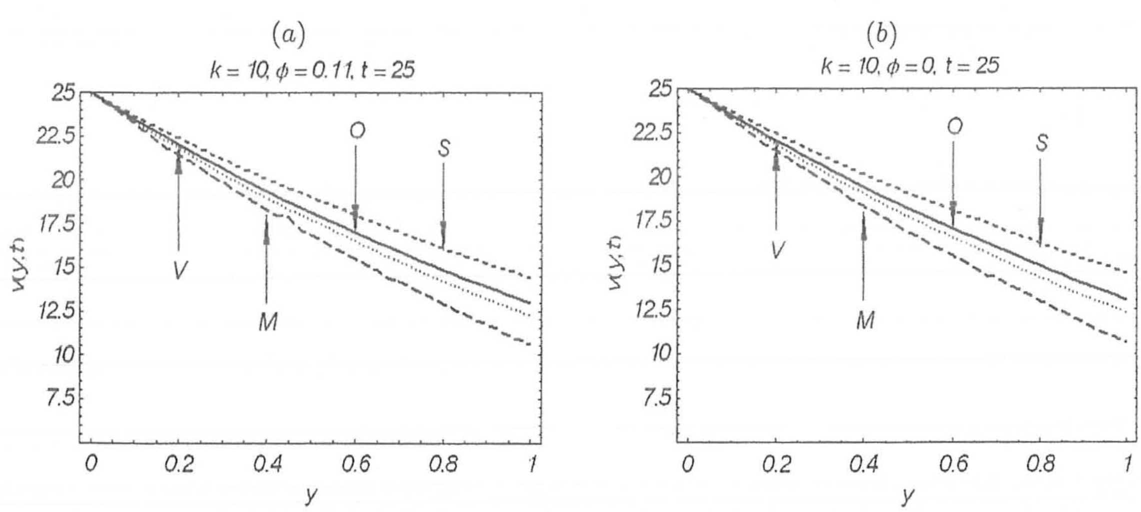

Figure 4.2: Velocity profiles u corresponding to a S=second grade, O=Oldroyd, V=viscous and M=Maxwell fluids corresponding to $A = 1.0, \nu = 0.11746,\ \lambda_r = 15, \lambda = 10$ (constant

acceleration)

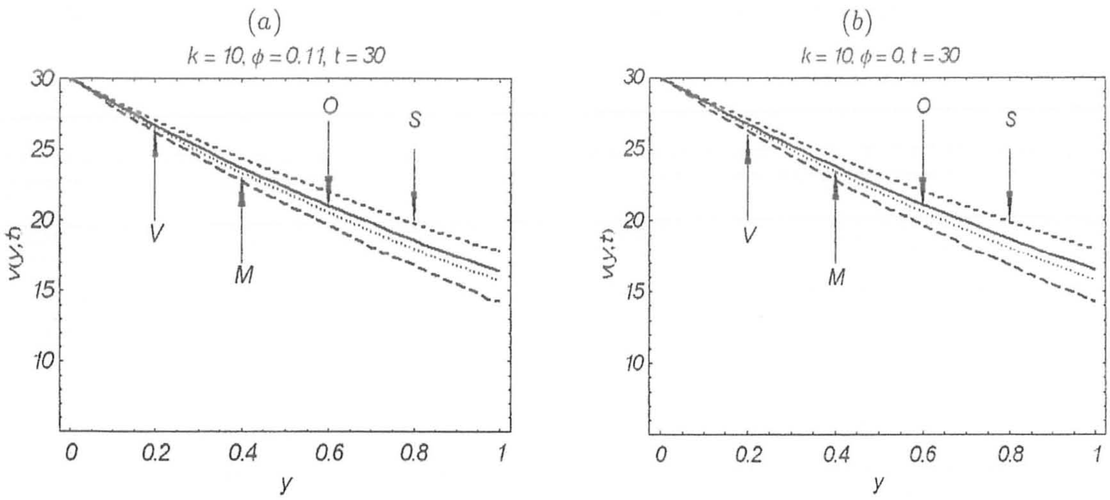

Figure 4.3: Velocity profiles u corresponding to a S=second grade, O=Oldroyd, V=viscous and M=Maxwell fluids corresponding to $A = 1.0, \nu = 0.11746, \lambda_r = 15, \lambda = 10$ (constant acceleration)

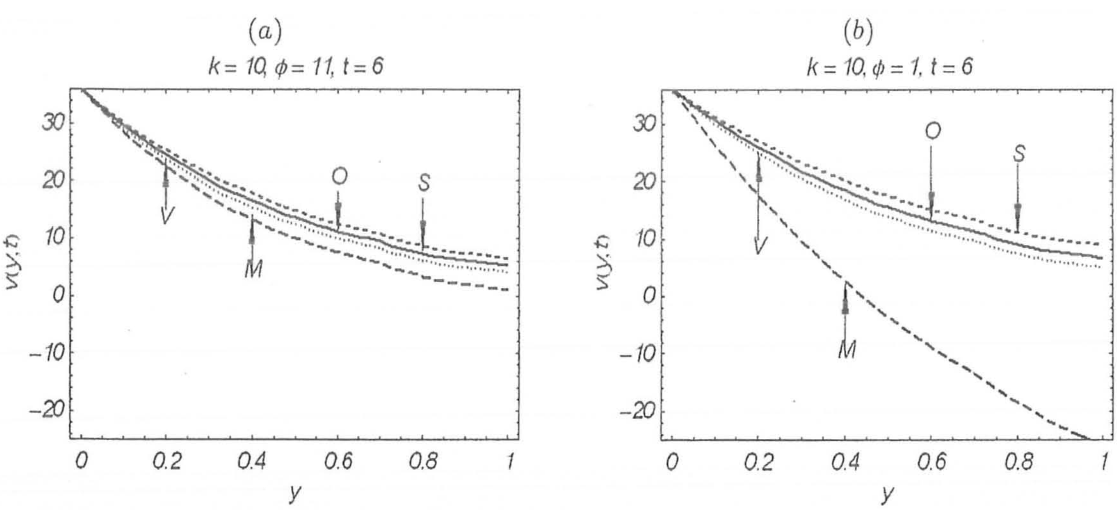

Figure 4.4: Velocity profiles u corresponding to a S=second grade, O=Oldroyd, V=viscous and

M=Maxwell fluids corresponding to $A = 1.0, \nu = 0.11746,\ \lambda_r = 2, \lambda = 1$ (variable acceleration)

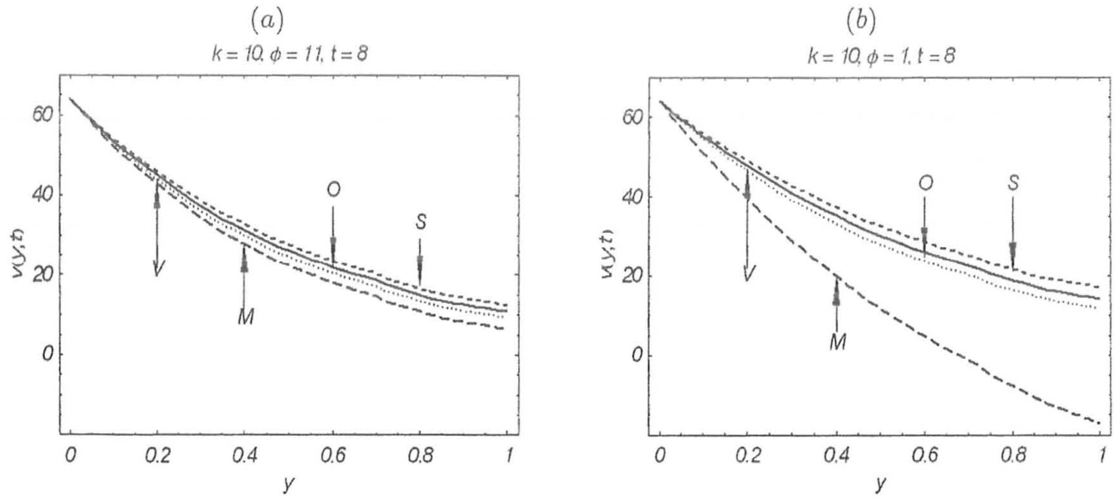

Figure 4.5: Velocity profiles u corresponding to a S=second grade, O=Oldroyd, V=viscous and M=Maxwell fluids corresponding to $A = 1.0, \nu = 0.11746,\ \lambda_r = 2, \lambda = 1$ (variable acceleration)

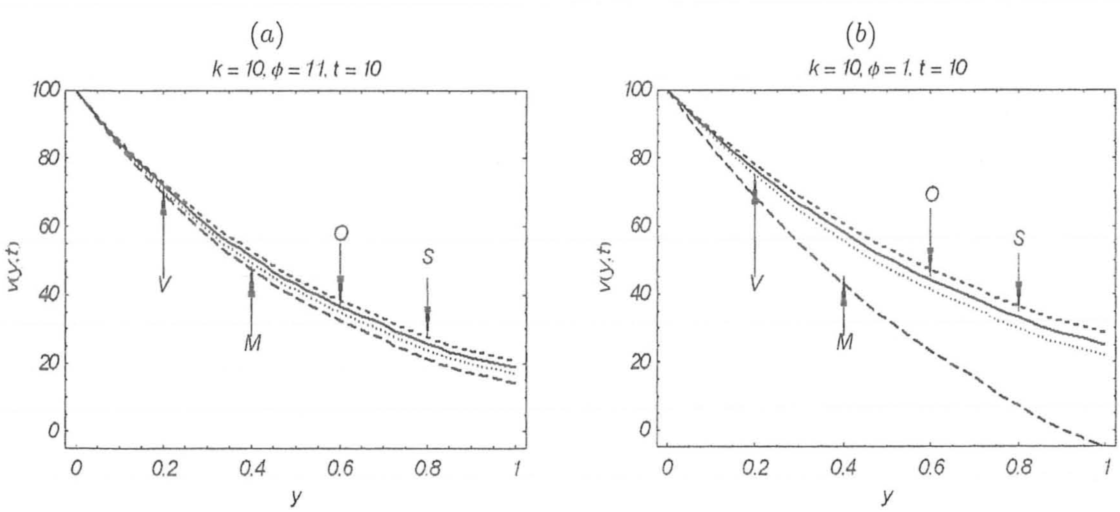

Figure 4.6: Velocity profiles u corresponding to a S=second grade, O=Oldroyd, V=viscous and M=Maxwell fluids corresponding to $A = 1.0, \nu = 0.11746,\ \lambda_r = 2, \lambda = 1$ (variable acceleration)

4.6 Results and discussion

This section includes the discussion relevant to the velocity profiles for (i) constant accelerated flow (ii) variable accelerated flow. Attention has been focussed to the variation of velocity fields in various fluids. Figures 4.1-4.3 have been sketched just to present the comparison among the velocity profiles of various fluid models with respect to "t". These Figures indicate that the velocity profile in second grade fluid is largest and smallest for Maxwell fluid. The velocity profile in an Oldroyd-B fluid is higher than that of Maxwell fluid. Also in absence of ϕ velocity profiles for all these fluid increase.

Figures 4.4-4.6 provide the comparison of the velocity profiles for the second grade, Maxwell, Oldroyd-B fluid and Newtonian fluids in variable accelerated flow. Qualitatively, the observations for variable accelerated flow are similar to that of constant accelerated flow. However, the velocity profiles in constant and variable accelerated flows are not similar quantitatively. Comparison shows that velocity profiles in variable accelerated flow are larger when compared to that of constant accelerated flow. It is also evident from these figures that for small values of ϕ the velocity profiles for all these fluids increase. However the velocity profiles in Maxwell fluid decreases drastically.

Chapter 5

Effects of Hall current on flows of a Burgers' fluid through a porous medium

The main goal of this chapter is to extend the flow analysis of chapter three to a Burgers's fluid model. We have obtained the analytical expressions of velocity distribution for large and small time. Earlier solutions of the considered problems are recovered. Comparison has been provided between Oldroyd-B and Burgers' fluids by plotting graphs. The physical interpretation is included. The observations seem to be consistent with physical intuitions.

5.1 The involved equations

For an incompressible flow of electrically conducting fluid, the relevant equations are (1.1)-(1.4) and (1.10). The Hall effect is taken into consideration and the current density **J** satisfies equation (1.13).

The constitutive equations for a Burgers' fluid may be put as [74]

$$\mathbf{T} = -p\mathbf{I} + \mathbf{S}, \qquad \mathbf{S} + \lambda\frac{\delta \mathbf{S}}{\delta t} + \beta\frac{\delta^2 \mathbf{S}}{\delta t^2} = \mu\left(1 + \theta\frac{\delta}{\delta t}\right)\mathbf{A}_1, \tag{5.1}$$

where p is the reaction stress due to constraint of incompressibility, $\mathbf{S}$ is the constitutively determined extra stress, $\mathbf{A}_1$ is the first Rivlin-Ericksen tensor, λ and β are the relaxation times, μ is the dynamic viscosity, $\theta\,(<\lambda)$ is the retardation time and

$$\frac{\delta \mathbf{S}}{\delta t}=\frac{d\mathbf{S}}{dt}-\mathbf{LS}-\mathbf{SL}^{\mathrm{T}} \tag{5.2}$$

is the upper convected time derivative and $\mathbf{L}$ is the velocity gradient. Also note that the model (5.1) includes as special cases of an Oldroyd-B model (for $\beta=0$), a Maxwell model (for $\beta=\theta=0$), a Newtonian fluid model (for $\beta=\theta=\lambda=0$) and a second grade fluid (if $\beta=\lambda=0$).

For Maxwell fluid, the following phenomenological model is available in the literature [66,75]

$$\left(1+\lambda\frac{\partial}{\partial t}\right)\nabla p=-\frac{\mu\phi}{k}\mathbf{V}, \tag{5.3}$$

which can be recovered from equation (3.5) for $\theta=0$.

Employing equations (3.5) and (5.3) we propose the following constitutive relationship between pressure drop and velocity for a Burgers' fluid as [74]

$$\left(1+\lambda\frac{\partial}{\partial t}+\beta\frac{\partial^2}{\partial t^2}\right)\nabla p=-\frac{\mu\phi}{k}\left(1+\theta\frac{\partial}{\partial t}\right)\mathbf{V}. \tag{5.4}$$

By considering the balance of forces acting on a volume element of fluid, the local volume average balance of linear momentum is given through equation (1.11) and the expression in [61,73,76,77].

$$\rho\left(\frac{\partial \mathbf{V}}{\partial t}+(\mathbf{V}.\nabla)\,\mathbf{V}\right)=-\nabla p+div\mathbf{S}+\mathbf{J}\times\mathbf{B}+\mathbf{R}. \tag{5.5}$$

Due to the volume averaging process, some information is lost, thus requiring supplementary empirical relation for the Darcy resistance. Recalling that the pressure gradient given in equation (5.4) is a measure of the resistance to the flow in the bulk of porous medium and $\mathbf{R}$ in equation (1.11) is measure of the flow resistance offered by the solid matrix, therefore, $\mathbf{R}$ through equation (1.11) satisfies [73]

$$\left(1+\lambda\frac{\partial}{\partial t}+\beta\frac{\partial^2}{\partial t^2}\right)\mathbf{R}=-\frac{\mu\phi}{k}\left(1+\theta\frac{\partial}{\partial t}\right)\mathbf{V}. \tag{5.6}$$

The extra stress and velocity are defined as

$$\mathbf{S}(r,t) = \begin{pmatrix} S_{rr} & S_{r\theta} & S_{rz} \\ S_{\theta r} & S_{\theta\theta} & S_{\theta z} \\ S_{zr} & S_{z\theta} & S_{zz} \end{pmatrix}, \qquad \mathbf{V}(r,t) = \begin{pmatrix} 0 \\ 0 \\ w \end{pmatrix}. \tag{5.7}$$

Using equation (5.7), the continuity equation (1.10) is satisfied identically and equations (1.13), (5.1)-(5.2) and (5.4) after using $\mathbf{S}(r,0) = \mathbf{0}$ (i.e. the fluid being at rest up to the moment $t = 0$) give

$$\rho\frac{\partial w}{\partial t} = -\frac{\partial p}{\partial z} + \frac{1}{r}\frac{\partial}{\partial r}(rS_{rz}) - \frac{\sigma B_0^2}{1-im}w - \frac{\mu\phi}{k}w, \tag{5.8}$$

$$\left(1+\lambda\frac{\partial}{\partial t}+\beta\frac{\partial^2}{\partial t^2}\right)S_{rz} = \mu\left(1+\theta\frac{\partial}{\partial t}\right)\frac{\partial w}{\partial r}, \tag{5.9}$$

$$S_{zz} + \lambda\left(\frac{\partial S_{zz}}{\partial t} - 2S_{rz}\frac{\partial w}{\partial r}\right) + \beta\left[\frac{\partial}{\partial t}\left(\frac{\partial S_{zz}}{\partial t} - 2S_{rz}\frac{\partial w}{\partial r}\right) - 2\frac{\partial S_{rz}}{\partial t}\frac{\partial w}{\partial r}\right] = -2\mu\theta\left(\frac{\partial w}{\partial r}\right)^2, \tag{5.10}$$

where $m = \omega_e\tau_e$ is the Hall parameter, $S_{rr} = S_{r\theta} = S_{\theta\theta} = S_{\theta z} = 0$ and r- and θ-components of momentum equation indicate that p is independent of r and θ and is at most a function of z and t.

Eliminating S_{rz} between equations (5.8) and (5.9), we get

$$\begin{aligned}\rho\left(1+\lambda\frac{\partial}{\partial t}+\beta\frac{\partial^2}{\partial t^2}\right)\frac{\partial w}{\partial t} &= -\left(1+\lambda\frac{\partial}{\partial t}+\beta\frac{\partial^2}{\partial t^2}\right)\frac{\partial p}{\partial z} + \mu\left(1+\theta\frac{\partial}{\partial t}\right)\left[\frac{\partial^2 w}{\partial r^2}+\frac{1}{r}\frac{\partial w}{\partial r}\right] \\ &\quad -\frac{\sigma B_0^2}{1-im}\left(1+\lambda\frac{\partial}{\partial t}+\beta\frac{\partial^2}{\partial t^2}\right)w - \frac{\mu\phi}{k}\left(1+\theta\frac{\partial}{\partial t}\right)w.\end{aligned} \tag{5.11}$$

5.2 Starting flow in a moving cylinder

Consider the MHD flow of an incompressible Burgers' fluid in a circular cylinder. Initially, the fluid is at rest and then cylinder motion is suddenly started. The z-axis is considered as the axis of the cylinder. The flow problem in absence of pressure gradient is

$$\rho\left(1+\lambda\frac{\partial}{\partial t}+\beta\frac{\partial^2}{\partial t^2}\right)\frac{\partial w}{\partial t} + \frac{\sigma B_0^2}{1-im}\left(1+\lambda\frac{\partial}{\partial t}+\beta\frac{\partial^2}{\partial t^2}\right)w + \frac{\mu\phi}{k}\left(1+\theta\frac{\partial}{\partial t}\right)w$$

$$= \mu \left(1 + \theta \frac{\partial}{\partial t}\right) \left[\frac{\partial^2 w}{\partial r^2} + \frac{1}{r}\frac{\partial w}{\partial r}\right], \tag{5.12}$$

$$\begin{aligned} w(a,t) &= W, \qquad t > 0, \\ \frac{\partial w(0,t)}{\partial r} &= 0, \qquad \text{for all } t, \\ \frac{\partial^2 w(r,0)}{\partial t^2} &= \frac{\partial w(r,0)}{\partial t} = w(r,0) = 0, \quad 0 \le r < a, \end{aligned} \tag{5.13}$$

where a is the cylinder radius and W the constant velocity at $r = a$.

On setting

$$\begin{aligned} w^* &= \frac{w}{W}, \quad r^* = \frac{r}{a}, \quad t^* = \frac{t}{(a^2/\nu)}, \quad \lambda^* = \frac{\lambda}{(a^2/\nu)}, \quad \lambda_r^* = \frac{\lambda_r}{(a^2/\nu)}, \\ \beta^* &= \frac{\beta}{(a^4/\nu^2)}, \quad M^2 = \frac{\sigma B_0^2}{(\mu/a^2)}, \quad \frac{1}{K} = \frac{\phi}{(k/a^2)} \end{aligned} \tag{5.14}$$

the non-dimensional governing problem after dropping the asterisks can be written as follows

$$\left(1 + \lambda\frac{\partial}{\partial t} + \beta\frac{\partial^2}{\partial t^2}\right)\frac{\partial w}{\partial t} + \frac{M^2}{1 - im}\left(1 + \lambda\frac{\partial}{\partial t} + \beta\frac{\partial^2}{\partial t^2}\right) w + \frac{1}{K}\left(1 + \lambda_r\frac{\partial}{\partial t}\right) w$$

$$= \left(1 + \lambda_r \frac{\partial}{\partial t}\right) \left[\frac{\partial^2 w}{\partial r^2} + \frac{1}{r}\frac{\partial w}{\partial r}\right], \tag{5.15}$$

$$\begin{aligned} w(1,t) &= 1, \qquad t > 0, \\ \frac{\partial w(0,t)}{\partial r} &= 0, \qquad \text{for all } t, \\ \frac{\partial^2 w(r,0)}{\partial t^2} &= \frac{\partial w(r,0)}{\partial t} = w(r,0) = 0, \quad 0 \le r < 1. \end{aligned} \tag{5.16}$$

5.2.1 Large time solution

For steady state, the velocity distribution is

$$w(r) = \frac{I_0(qr)}{I_0(q)}, \tag{5.17}$$

where

$$q=\left[\frac{M^2}{1-im}+\frac{1}{K}\right]^{1/2}$$

and I_0 is the modified Bessel function of first kind of order zero.

Let

$$w(r,t)=\frac{I_0(qr)}{I_0(q)}-f(r,t). \tag{5.18}$$

In equation (5.18), $f(r,t)$ satisfies the following initial boundary value problem

$$\left(1+\lambda\frac{\partial}{\partial t}+\beta\frac{\partial^2}{\partial t^2}\right)\frac{\partial f}{\partial t}+\frac{M^2}{1-im}\left(1+\lambda\frac{\partial}{\partial t}+\beta\frac{\partial^2}{\partial t^2}\right)f+\frac{1}{K}\left(1+\theta\frac{\partial}{\partial t}\right)f$$

$$=\left(1+\theta\frac{\partial}{\partial t}\right)\left[\frac{\partial^2 f}{\partial r^2}+\frac{1}{r}\frac{\partial f}{\partial r}\right], \tag{5.19}$$

$$\begin{aligned} f(1,t) &= \frac{\partial f(0,t)}{\partial r}=0, \\ f(r,0) &= \frac{I_0(qr)}{I_0(q)}, \\ \frac{\partial f(r,0)}{\partial t} &= \frac{\partial^2 f(r,0)}{\partial t^2}=0. \end{aligned} \tag{5.20}$$

Now solving equation (5.19) subject to the boundary conditions (5.20) we arrive at

$$f(r,t)=\sum_{n=1}^{\infty}B_nJ_0(\alpha_n r)\,T_n(t), \tag{5.21}$$

where J_0 is the Bessel function of first kind of order zero and α_n are the zeros of J_0 and for $\beta\neq 0$

$$T_n(t)=\frac{D_{2n}D_{3n}e^{D_{1n}t}}{(D_{1n}-D_{2n})(D_{1n}-D_{3n})}+\frac{D_{1n}D_{3n}e^{D_{2n}t}}{(D_{2n}-D_{1n})(D_{2n}-D_{3n})}+\frac{D_{1n}D_{2n}e^{D_{3n}t}}{(D_{3n}-D_{1n})(D_{3n}-D_{2n})},$$

$$
\begin{aligned}
D_{1n} &= -\frac{a}{3\beta} - \frac{2^{1/3}\left(-a^2+3b_n\beta\right)}{3\beta D_n} + \frac{D_n}{3\left(2^{1/3}\right)\beta}, \\
D_{2n} &= -\frac{a}{3\beta} + \frac{\left(1+\sqrt{3}i\right)\left(-a^2+3b_n\beta\right)}{3\left(2^{2/3}\right)\beta D_n} - \frac{\left(1-\sqrt{3}i\right)D_n}{6\left(2^{1/3}\right)\beta}, \\
D_{3n} &= -\frac{a}{3\beta} + \frac{\left(1-\sqrt{3}i\right)\left(-a^2+3b_n\beta\right)}{3\left(2^{2/3}\right)\beta D_n} - \frac{\left(1+\sqrt{3}i\right)D_n}{6\left(2^{1/3}\right)\beta},
\end{aligned}
$$

$$
a = \lambda + \frac{\beta M^2}{1-im}, \quad b_n = 1 + \left(\frac{1}{K} + \alpha_n^2\right)\theta + \frac{\lambda M^2}{1-im}, \quad c_n = q^2 + \alpha_n^2,
$$

$$
D_n = \left[-2a^3 + 9ab_n\beta - 27c_n\beta^2 + \sqrt{4\left(-a^2+3b_n\beta\right)^3 + \left(-2a^3+9ab_n\beta-27c_n\beta^2\right)^2}\right]^{1/3}.
$$

The values of B_n can be obtained by the initial condition for $f(r,t)$. Hence, the velocity takes the following form

$$
w(r,t) = \frac{I_0(qr)}{I_0(q)} - 2\sum_{n=1}^{\infty} \frac{\alpha_n J_0(\alpha_n r) T_n(t)}{\left(q^2+\alpha_n^2\right) J_1(\alpha_n)}. \tag{5.22}
$$

In above equation (5.22) J_1 is the Bessel function of first kind of order one.

5.2.2 Small time solution

In this section we find the solution by Laplace transform method. If the Laplace transform of w is $\overline{w}$ then

$$
\overline{w}(r,s) = \int_0^\infty w(r,t)\, e^{-st} dt.
$$

Equation (5.15) and the boundary conditions (5.16) are transformed to the following problem

$$
\overline{w}'' + \frac{1}{r}\overline{w}' - \Gamma^2 \overline{w} = 0, \tag{5.23}
$$

$$
\begin{aligned}
\overline{w}(1,s) &= \frac{1}{s}, \\
\frac{d\overline{w}(0,s)}{dr} &= 0
\end{aligned} \tag{5.24}
$$

where

$$
\Gamma = \left[\frac{1}{(1+\theta s)}\left\{\beta s^3 + \left(\lambda + \frac{\beta M^2}{1-im}\right)s^2 + \left(1 + \frac{\lambda M^2}{1-im} + \frac{\theta}{K}\right)s + q^2\right\}\right]^{\frac{1}{2}}
$$

and prime denotes the differentiation with respect to r.

The solution of equation (5.23) satisfying the boundary conditions (5.24) is

$$\overline{w}=\frac{I_0\left(\Gamma r\right)}{sI_0\left(\Gamma\right)}. \tag{5.25}$$

Laplace inversion of equation (5.25) yields

$$w=\frac{1}{2\pi i}\int_{r-i\infty}^{r+i\infty}\frac{I_0\left(\Gamma r\right)}{sI_0\left(\Gamma\right)}e^{st}ds. \tag{5.26}$$

In equation (5.25), $s=0$ is a simple pole. Therefore residue at $s=0$ is

$$\operatorname{Re}s\left(0\right)=\frac{I_0\left(qr\right)}{I_0\left(q\right)}.$$

The other singular points of equation (5.26) are the zeros of

$$I_0\left(\Gamma\right)=0.$$

Setting $\Gamma=i\alpha$, we find that

$$J_0\left(\alpha\right)=0. \tag{5.27}$$

If α_n, $n=1,2,3,.....\infty$ are the zeros of the equation (5.27), then D_{1n}, D_{2n} and D_{3n}, defined in section (2.1) are the poles. These are the simple poles and the residue at these poles can be obtained as

$$\begin{aligned}
\operatorname{Re}s\left(D_{1n}\right) &= \frac{2\alpha_n\left(1+\theta D_{1n}\right)J_0\left(\alpha_n r\right)e^{D_{1n}t}}{D_{1n}\left[3\beta D_{1n}^2+2\left(\lambda+\frac{\beta M^2}{1-im}\right)D_{1n}+\left(1+\frac{\lambda M^2}{1-im}+\frac{\theta}{K}\right)+\theta\alpha_n^2\right]J_1\left(\alpha_n\right)},\\
\operatorname{Re}s\left(D_{2n}\right) &= \frac{2\alpha_n\left(1+\theta D_{2n}\right)J_0\left(\alpha_n r\right)e^{D_{2n}t}}{D_{2n}\left[3\beta D_{2n}^2+2\left(\lambda+\frac{\beta M^2}{1-im}\right)D_{2n}+\left(1+\frac{\lambda M^2}{1-im}+\frac{\theta}{K}\right)+\theta\alpha_n^2\right]J_1\left(\alpha_n\right)},\\
\operatorname{Re}s\left(D_{3n}\right) &= \frac{2\alpha_n\left(1+\theta D_{3n}\right)J_0\left(\alpha_n r\right)e^{D_{3n}t}}{D_{3n}\left[3\beta D_{3n}^2+2\left(\lambda+\frac{\beta M^2}{1-im}\right)D_{3n}+\left(1+\frac{\lambda M^2}{1-im}+\frac{\theta}{K}\right)+\theta\alpha_n^2\right]J_1\left(\alpha_n\right)}.
\end{aligned}$$

Adding Res(0), $\operatorname{Re}s\left(D_{1n}\right)$, $\operatorname{Re}s\left(D_{2n}\right)$ and $\operatorname{Re}s\left(D_{3n}\right)$, a complete solution is of the form

$$w(r,t)=\frac{I_0(qr)}{I_0(q)}+2\sum_{n=1}^{\infty}\left\{\frac{(1+\theta D_{1n})e^{D_{1n}t}}{D_{1n}l_{1n}}+\frac{(1+\theta D_{2n})e^{D_{2n}t}}{D_{2n}l_{2n}}+\frac{(1+\theta D_{3n})e^{D_{3n}t}}{D_{3n}l_{3n}}\right\}\frac{\alpha_n J_0(\alpha_n r)}{J_1(\alpha_n)}, \tag{5.28}$$

where

$$\begin{aligned}
l_{1n} &= \left[3\beta D_{1n}^2+2\left(\lambda+\frac{\beta M^2}{1-im}\right)D_{1n}+\left(1+\frac{\lambda M^2}{1-im}+\frac{\theta}{K}\right)+\theta\alpha_n^2\right],\\
l_{2n} &= \left[3\beta D_{2n}^2+2\left(\lambda+\frac{\beta M^2}{1-im}\right)D_{2n}+\left(1+\frac{\lambda M^2}{1-im}+\frac{\theta}{K}\right)+\theta\alpha_n^2\right],\\
l_{3n} &= \left[3\beta D_{3n}^2+2\left(\lambda+\frac{\beta M^2}{1-im}\right)D_{3n}+\left(1+\frac{\lambda M^2}{1-im}+\frac{\theta}{K}\right)+\theta\alpha_n^2\right].
\end{aligned}$$

5.3 Starting flow in a circular pipe

Let us consider Burgers' fluid in a circular cylinder initially at rest. The fluid motion is caused by a constant pressure gradient. The statement of the flow problem is

$$\rho\left(1+\lambda\frac{\partial}{\partial t}+\beta\frac{\partial^2}{\partial t^2}\right)\frac{\partial w}{\partial t}+\frac{\sigma B_0^2}{1-im}\left(1+\lambda\frac{\partial}{\partial t}+\beta\frac{\partial^2}{\partial t^2}\right)w+\frac{\mu\phi}{k}\left(1+\theta\frac{\partial}{\partial t}\right)w$$

$$+\frac{dp}{dz}=\mu\left(1+\theta\frac{\partial}{\partial t}\right)\left[\frac{\partial^2 w}{\partial r^2}+\frac{1}{r}\frac{\partial w}{\partial r}\right], \tag{5.29}$$

$$\begin{aligned}
\frac{\partial w(0,t)}{\partial r} &= w(a,t)=0, \qquad \text{for all } t,\\
\frac{\partial^2 w(r,0)}{\partial t^2} &= \frac{\partial w(r,0)}{\partial t}=w(r,0)=0, \quad 0\le r<a.
\end{aligned} \tag{5.30}$$

With the help of the dimensionless variables defined in equation (5.14) and

$$z^*=\frac{z}{a},\quad p^*=\frac{p}{(\mu/a)W}$$

the solutions of equations (5.29) and (5.30) are:

5.3.1 Large time solution

$$\frac{w(r,t)}{dp/dz} = -\frac{1}{q^2}\left[1-\frac{I_0(qr)}{I_0(q)}\right] + 2\sum_{n=1}^{\infty}\frac{J_0(\alpha_n r)\,T_n(t)}{\lambda_n\left(q^2+\alpha_n^2\right)J_1(\alpha_n)}. \tag{5.31}$$

5.3.2 Small time solution

$$\begin{aligned}\frac{w(r,t)}{dp/dz} &= -\frac{1}{q^2}\left(1-\frac{I_0(qr)}{I_0(q)}\right) - \frac{e^{s_1 t}}{s_1(s_1-s_2)(s_1-s_3)}\left(1-\frac{I_0(\xi_1 r)}{I_0(\xi_1)}\right)\\ &\quad -\frac{e^{s_2 t}}{s_2(s_2-s_1)(s_2-s_3)}\left(1-\frac{I_0(\xi_2 r)}{I_0(\xi_2)}\right) - \frac{e^{s_3 t}}{s_3(s_3-s_1)(s_3-s_2)}\left(1-\frac{I_0(\xi_3 r)}{I_0(\xi_3)}\right)\\ &\quad +2\sum_{n=1}^{\infty}\left\{\frac{(1+\theta D_{1n})e^{D_{1n}t}}{D_{1n}l_{1n}l_{4n}} + \frac{(1+\theta D_{2n})e^{D_{2n}t}}{D_{2n}l_{2n}l_{5n}} + \frac{(1+\theta D_{3n})e^{D_{3n}t}}{D_{3n}l_{3n}l_{6n}}\right\}\times\\ &\quad \frac{\alpha_n J_0(\alpha_n r)}{J_1(\alpha_n)},\end{aligned} \tag{5.32}$$

where

$$\begin{aligned} l_{4n} &= \left[\beta D_{1n}^3 + \left(\lambda+\frac{\beta M^2}{1-im}\right)D_{1n}^2 + \left(1+\frac{\lambda M^2}{1-im}+\frac{\theta}{K}\right)D_{1n} + q^2\right],\\ l_{5n} &= \left[\beta D_{2n}^3 + \left(\lambda+\frac{\beta M^2}{1-im}\right)D_{2n}^2 + \left(1+\frac{\lambda M^2}{1-im}+\frac{\theta}{K}\right)D_{2n} + q^2\right],\\ l_{6n} &= \left[\beta D_{3n}^3 + \left(\lambda+\frac{\beta M^2}{1-im}\right)D_{3n}^2 + \left(1+\frac{\lambda M^2}{1-im}+\frac{\theta}{K}\right)D_{3n} + q^2\right],\end{aligned}$$

$$\begin{aligned} s_1 &= -\frac{a}{3\beta} - \frac{2^{1/3}\left(-a^2+3b\beta\right)}{3\beta D} + \frac{D}{3\left(2^{1/3}\right)\beta},\\ s_2 &= -\frac{a}{3\beta} + \frac{\left(1+\sqrt{3}i\right)\left(-a^2+3b\beta\right)}{3\left(2^{2/3}\right)\beta D} - \frac{\left(1-\sqrt{3}i\right)D}{6\left(2^{1/3}\right)\beta},\\ s_3 &= -\frac{a}{3\beta} + \frac{\left(1-\sqrt{3}i\right)\left(-a^2+3b\beta\right)}{3\left(2^{2/3}\right)\beta D} - \frac{\left(1+\sqrt{3}i\right)D}{6\left(2^{1/3}\right)\beta},\end{aligned}$$

$$D = \left[-2a^3+9ab\beta-27q^2\beta^2+\sqrt{4\left(-a^2+3b\beta\right)^3+\left(-2a^3+9ab\beta-27q^2\beta^2\right)^2}\right]^{1/3},$$

$$b = 1+\frac{\theta}{K}+\frac{\lambda M^2}{1-im},$$

$$\xi_1 = \xi|_{s=s_1},\quad \xi_2 = \xi|_{s=s_2},\quad \xi_3 = \xi|_{s=s_3},$$

$$\xi = \left\{ \frac{\beta s^3 + \left(\lambda + \frac{\beta M^2}{1-im}\right) s^2 + \left(1 + \frac{\lambda M^2}{1-im} + \frac{\lambda_r}{K}\right) s + q^2}{1+\theta s} \right\}^{1/2}.$$

5.4 Generalized flow in a circular pipe

Here, the flow geometry is same as in section 5.3 but the fluid starts suddenly due to a constant pressure gradient and by the motion of the cylinder parallel to its length. The mathematical analysis corresponding to the flow situation is

$$\left(1+\lambda\frac{\partial}{\partial t}+\beta\frac{\partial^2}{\partial t^2}\right)\frac{\partial w}{\partial t}+\frac{M^2}{1-im}\left(1+\lambda\frac{\partial}{\partial t}+\beta\frac{\partial^2}{\partial t^2}\right)w+\frac{1}{K}\left(1+\theta\frac{\partial}{\partial t}\right)w$$

$$+\frac{dp}{dz}=\left(1+\theta\frac{\partial}{\partial t}\right)\left[\frac{\partial^2 w}{\partial r^2}+\frac{1}{r}\frac{\partial w}{\partial r}\right], \tag{5.33}$$

$$\begin{aligned} w(1,t) &= 1, \qquad t>0, \\ \frac{\partial w(0,t)}{\partial r} &= 0, \qquad \text{for all } t, \\ \frac{\partial^2 w(r,0)}{\partial t^2} &= \frac{\partial w(r,0)}{\partial t} = w(r,0)=0, \quad 0\le r<1. \end{aligned} \tag{5.34}$$

5.4.1 Large time solution

$$\begin{aligned} w(r,t) &= \frac{I_0(qr)}{I_0(q)} - \frac{1}{q^2}\frac{dp}{dz}\left[1-\frac{I_0(qr)}{I_0(q)}\right] \\ &\quad -2\sum_{n=1}^{\infty}\frac{\left(\alpha_n^2+dp/dz\right)J_0(\alpha_n r)\,T_n(t)}{\alpha_n\left(q^2+\alpha_n^2\right)J_1(\alpha_n)}. \end{aligned} \tag{5.35}$$

5.4.2 Small time solution

$$
\begin{aligned}
w(r,t) &= \frac{I_0(qr)}{I_0(q)} - \frac{dp}{dz}\left[\frac{1}{q^2}\left(1-\frac{I_0(qr)}{I_0(q)}\right) + \frac{e^{s_1 t}}{s_1(s_1-s_2)(s_1-s_3)}\left(1-\frac{I_0(\xi_1 r)}{I_0(\xi_1)}\right)\right. \\
&\left. + \frac{e^{s_2 t}}{s_2(s_2-s_1)(s_2-s_3)}\left(1-\frac{I_0(\xi_2 r)}{I_0(\xi_2)}\right) + \frac{e^{s_3 t}}{s_3(s_3-s_1)(s_3-s_2)}\left(1-\frac{I_0(\xi_3 r)}{I_0(\xi_3)}\right)\right] \\
&+2\sum_{n=1}^{\infty}\left\{\frac{(1+\theta D_{1n})e^{D_{1n}t}}{D_{1n}l_{1n}} + \frac{(1+\theta D_{2n})e^{D_{2n}t}}{D_{2n}l_{2n}} + \frac{(1+\theta D_{3n})e^{D_{3n}t}}{D_{3n}l_{3n}}\right\}\frac{\alpha_n J_0(\alpha_n r)}{J_1(\alpha_n)} \\
&+2\frac{dp}{dz}\sum_{n=1}^{\infty}\left\{\frac{(1+\theta D_{1n})e^{D_{1n}t}}{D_{1n}l_{1n}l_{4n}} + \frac{(1+\theta D_{2n})e^{D_{2n}t}}{D_{2n}l_{2n}l_{5n}} + \frac{(1+\theta D_{3n})e^{D_{3n}t}}{D_{3n}l_{3n}l_{6n}}\right\}\frac{\alpha_n J_0(\alpha_n r)}{J_1(\alpha_n)}.
\end{aligned}
\tag{5.36}
$$

5.5 Starting flow in a rotating cylinder

Here we consider the fluid in a circular cylinder. Initially, the fluid is at rest and suddenly sets in motion due to rotation of the cylinder. The velocity field is

$$
\mathbf{V} = (0, v(r,t), 0) \tag{5.37}
$$

and thus the governing problem is

$$
\begin{aligned}
&\rho\left(1+\lambda\frac{\partial}{\partial t}+\beta\frac{\partial^2}{\partial t^2}\right)\frac{\partial v}{\partial t} + \frac{\sigma B_0^2}{1-im}\left(1+\lambda\frac{\partial}{\partial t}+\beta\frac{\partial^2}{\partial t^2}\right)v + \frac{\mu\phi}{k}\left(1+\theta\frac{\partial}{\partial t}\right)v \\
&= \mu\left(1+\theta\frac{\partial}{\partial t}\right)\left[\frac{\partial^2 v}{\partial r^2}+\frac{1}{r}\frac{\partial v}{\partial r}-\frac{v}{r^2}\right],
\end{aligned}
\tag{5.38}
$$

$$
\begin{aligned}
v(a,t) &= \Omega a, \qquad t>0, \\
\frac{\partial v(0,t)}{\partial r} &= 0, \qquad \text{for all } t, \\
\frac{\partial^2 v(r,0)}{\partial t^2} &= \frac{\partial v(r,0)}{\partial t} = v(r,0) = 0, \quad 0 \le r < a
\end{aligned}
\tag{5.39}
$$

in which Ω is the angular velocity. Using equation (5.14) and

$$
v^{\star} = \frac{v}{\Omega a} \tag{5.40}
$$

we obtain

$$\left(1+\lambda\frac{\partial}{\partial t}+\beta\frac{\partial^2}{\partial t^2}\right)\frac{\partial v}{\partial t}+\frac{M^2}{1-im}\left(1+\lambda\frac{\partial}{\partial t}+\beta\frac{\partial^2}{\partial t^2}\right)v+\frac{1}{K}\left(1+\theta\frac{\partial}{\partial t}\right)v$$
$$=\left(1+\theta\frac{\partial}{\partial t}\right)\left[\frac{\partial^2 v}{\partial r^2}+\frac{1}{r}\frac{\partial v}{\partial r}-\frac{v}{r^2}\right], \tag{5.41}$$

$$\begin{aligned} v(1,t) &= 1, \qquad t>0,\\ \frac{\partial v(0,t)}{\partial r} &= 0, \qquad \text{for all } t,\\ \frac{\partial^2 v(r,0)}{\partial t^2} &= \frac{\partial v(r,0)}{\partial t}=v(r,0)=0, \quad 0\le r<1. \end{aligned} \tag{5.42}$$

The large and small times solutions of the above problem are:

5.5.1 Large time solution

$$v(r,t)=\frac{I_1(qr)}{I_1(q)}+2\sum_{n=1}^{\infty}\frac{\alpha_n J_1(\alpha_n r)\,T_n(t)}{(q^2+\alpha_n^2)\,J_0(\alpha_n)}, \tag{5.43}$$

where I_1 is modified Bessel function of first kind of order one and α_n are the zeros of J_1.

5.5.2 Small time solution

$$\begin{aligned} v(r,t) &= \frac{I_1(qr)}{I_1(q)}-4\sum_{n=1}^{\infty}\left\{\frac{(1+\theta D_{1n})e^{D_{1n}t}}{D_{1n}l_{1n}}+\frac{(1+\theta D_{2n})e^{D_{2n}t}}{D_{2n}l_{2n}}+\frac{(1+\theta D_{3n})e^{D_{3n}t}}{D_{3n}l_{3n}}\right\}\\ &\quad\times\frac{\alpha_n J_1(\alpha_n r)}{[J_0(\alpha_n)-J_2(\alpha_n)]}. \end{aligned} \tag{5.44}$$

5.6 Graphical results

This section includes several results obtained from the flows analyzed in this chapter. We interpret these results and verify that they are consistent physically. Special emphasis has been given to examine the velocity profiles for two kind of fluids: an Oldroyd-B fluid for which $\lambda\neq 0$, $\theta\neq 0$, $\beta\approx 0.0000001$ and a Burgers' fluid. The effects of various parameters on the velocity profiles especially, magnetic parameter M, Hall parameter m and rheological parameter β of the Burgers' fluid have been studied through several graphs.

Figures 5.1 and 5.2 are prepared for starting flow in a circular cylinder moving parallel to its length, 5.3 and 5.4 for starting flow in circular pipe and 5.5 and 5.6 for starting flow in a rotating cylinder. The effect of increasing magnetic parameter M is shown in figures 5.1 – 5.6 for both Oldroyd-B and Burgers' fluids with $m = 0$ and $m = 2$ and keeping the other parameters fixed. From all these figures, it is noted that an increase in magnetic parameter M reduces the velocity profiles monotonically due to the effect of the magnetic force against the direction of the flow for both Oldroyd-B and Burgers' fluids. This is according to the fact that magnetic field is responsible to reduce the velocity. Moreover, a comparison of these reveals that the effect of magnetic parameter M becomes more prominent on Burgers' fluid than Oldroyd-B fluid. From these figures, it is clearly seen that the velocity profiles for an Oldroyd-B fluid ($\lambda = 2,\ \theta = 1, \beta = 0$) are obviously larger than those for a Burgers' fluid ($\lambda = 2,\ \theta = 1, \beta = 0.8$). However, this result cannot be generalized with other chosen values of parameters because the behaviour of the rheological parameter β of the Burgers' fluid is non-monotonous.

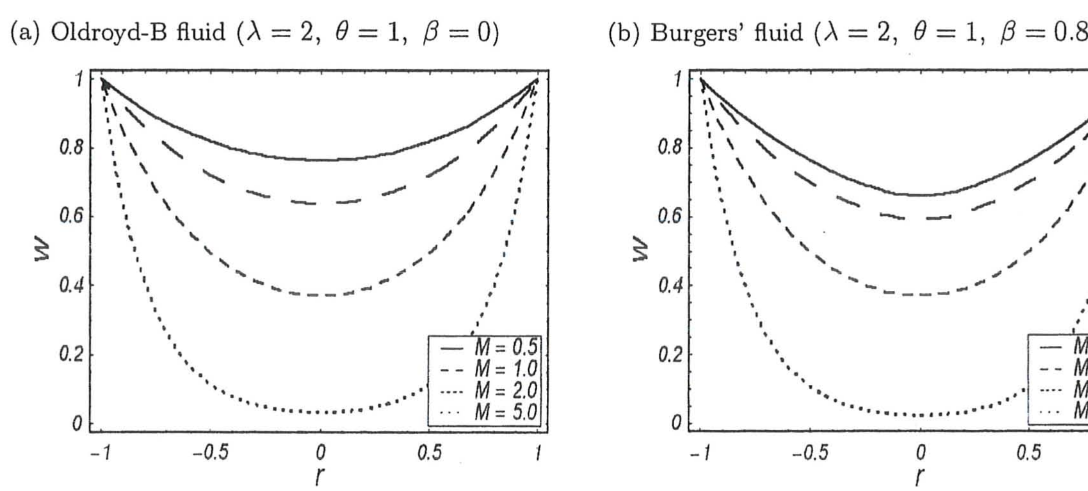

Figure 5.1 : Profiles of the normalized velocity $w(r, t)$ for various values of magnetic parameter M when $t = K = 1$ and $m = 0$ are fixed

(a) Oldroyd-B fluid ($\lambda = 2,\ \theta = 1,\ \beta = 0$) (b) Burgers' fluid ($\lambda = 2,\ \theta = 1,\ \beta = 0.8$)

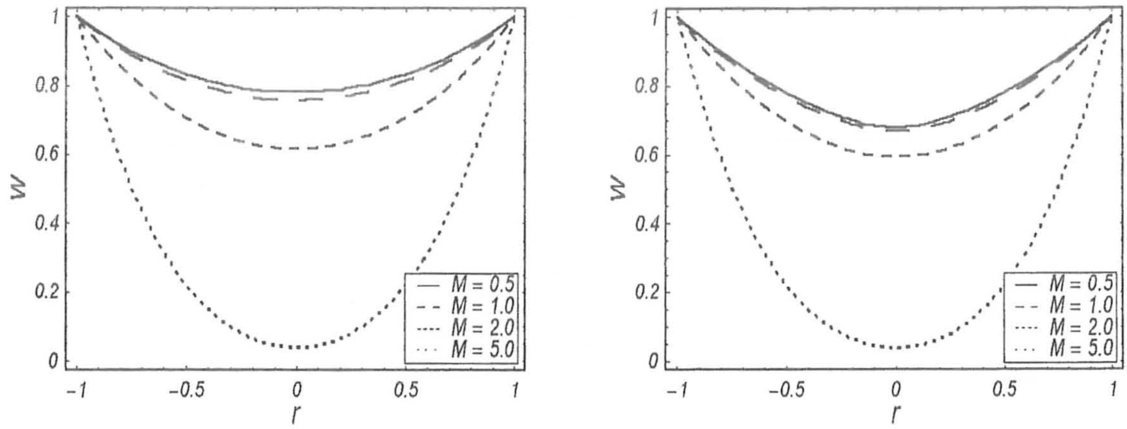

Figure 5.2 : Profiles of the normalized velocity $w(r,t)$ for various values of magnetic parameter M when $t = K = 1$ and $m = 2$ are fixed

The effect of the Hall parameter m on the velocity profiles with fixed values of other parameters for both fluids is also illustrated from these figures. As expected, the velocity increases by increasing m for both fluids as the effective conductivity decreases with increasing m, which reduces the magnetic damping force on velocity. Again from these figures, it is obvious that in the presence of Hall current the velocity profiles for an Oldroyd-B fluid are greater than those for a Burgers' fluid. Moreover, it can be easily seen from the governing equation (5.11) that increasing the permeability of the porous medium yields a similar effect as decreasing the magnetic field.

(a) Oldroyd-B fluid ($\lambda = 2,\ \theta = 1,\ \beta = 0$) (b) Burgers' fluid ($\lambda = 2,\ \theta = 1,\ \beta = 0.8$)

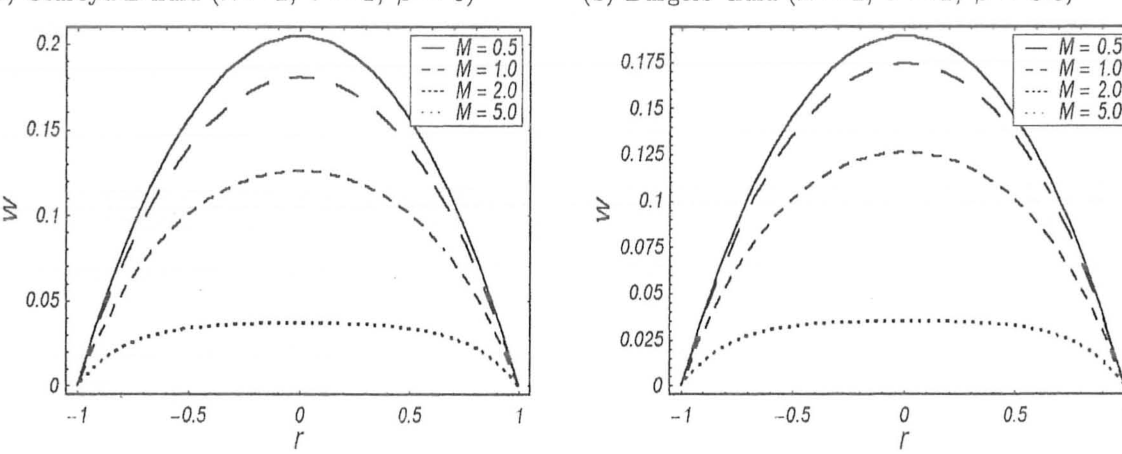

Figure 5.3 : Profiles of the normalized velocity $w(r,t)$ for various values of magnetic parameter M when $t = K = 1$ and $m = 0$ are fixed

(a) Oldroyd-B fluid ($\lambda = 2,\ \theta = 1,\ \beta = 0$) (b) Burgers' fluid ($\lambda = 2,\ \theta = 1,\ \beta = 0.8$)

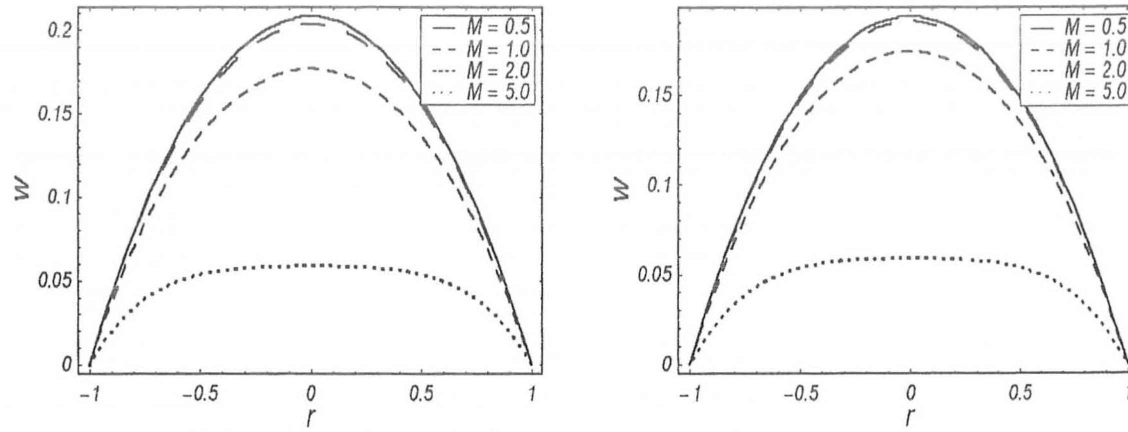

Figure 5.4 : Profiles of the normalized velocity $w(r,t)$ for various values of magnetic parameter M when $t = K = 1$ and $m = 2$ are fixed

(a) Oldroyd-B fluid ($\lambda = 2,\ \theta = 1,\ \beta = 0$) (b) Burgers' fluid ($\lambda = 2,\ \theta = 1,\ \beta = 0.8$)

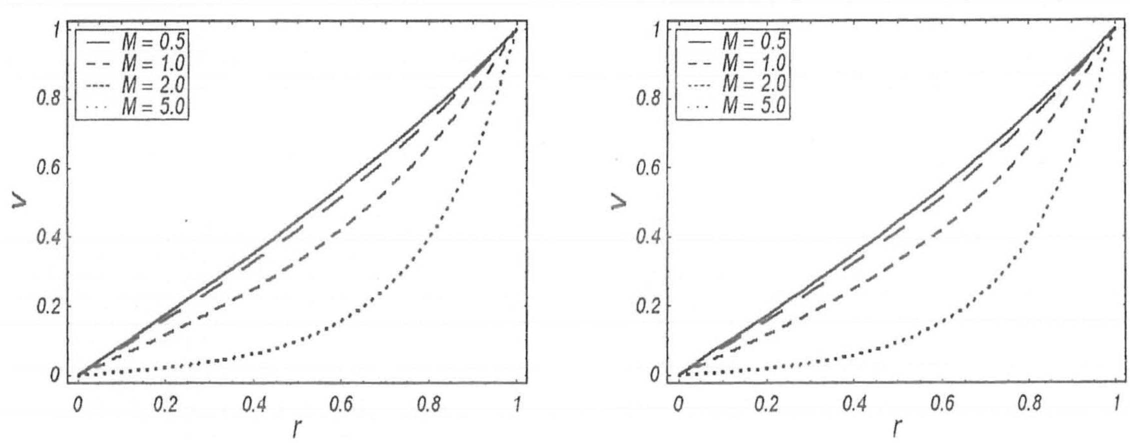

Figure 5.5 : Profiles of the normalized velocity $v(r,t)$ for various values of magnetic parameter M when $t = K = 1$ and $m = 0$ are fixed

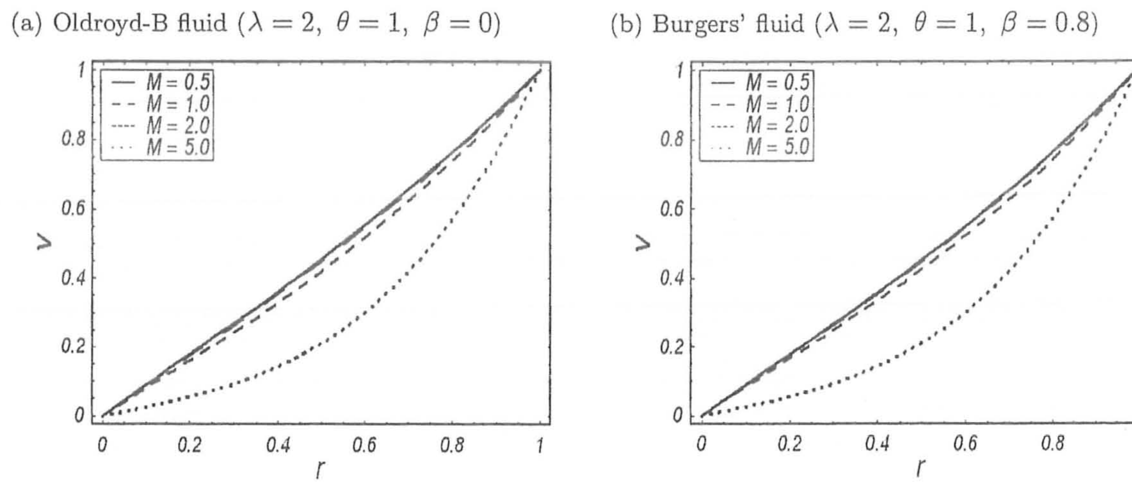

Figure 5.6 : Profiles of the normalized velocity $v\,(r,t)$ for various values of magnetic parameter M when $t = K = 1$ and $m = 2$ are fixed

5.7 Final remarks

In the present study, the four flow problems of an incompressible Burgers' fluid through a porous medium have been considered in the presence of a Hall current. It is found that th presence of a strong magnetic field considerably decreases the flow velocity. The study has been done using modified Darcy's law. It is important to appreciate that velocity profiles in case of Burgers' fluid are less than that of an Oldroyd-B fluid. Moreover, the presented analysis is more general and the results for several other fluid models (Oldroyd-B, Maxwell and Second grade) which are yet not available in the literature can be taken as the limiting cases.

Chapter 6

MHD oscillatory flows of a generalized Burgers' fluid in a porous medium

This chapter looks at the exact solutions for three unsteady MHD flow problems in a generalized Burgers' fluid occupying a porous medium. Modified Darcy's law for a generalized Burgers' fluid is introduced here first time in the literature. The fluid is electrically conducting under the influence of a uniform transverse magnetic field. The equations governing the magnetohydrodynamic (MHD) flows of a generalized Burgers' fluid in a porous medium are modeled. The MHD flows are induced by small amplitude plate oscillations and the imposed periodic pressure gradients. Closed form solutions are obtained for the velocity by using Fourier sine transform. Attention is focused upon the physical nature of the obtained solutions through graphs. Several existing results have been deduced in the limiting cases.

6.1 Governing equations

The Cauchy stress $\mathbf{T}$ in a generalized Burgers' fluid is [63]

$$\mathbf{T} = -p\mathbf{I} + \mathbf{S}, \tag{6.1}$$

$$\left(1+\lambda_1\frac{\delta}{\delta t}+\lambda_2\frac{\delta^2}{\delta t^2}\right)\mathbf{S}=\mu\left(1+\lambda_3\frac{\delta}{\delta t}+\lambda_4\frac{\delta^2}{\delta t^2}\right)\mathbf{A}, \tag{6.2}$$

where $-p\mathbf{I}$ indicates the indeterminate spherical stress, $\mathbf{A}=\mathbf{L}+\mathbf{L}^T$ is first Rivlin-Ericksen tensor, μ is the dynamic viscosity, $\mathbf{L}$ is the velocity gradient, λ_1 and λ_3 $(<\lambda_1)$ are the relaxation and retardation times, $\mathbf{S}$ is the extra stress tensor, λ_2, λ_4 are material constants and $\delta/\delta t$ is the upper convected time derivative defined by

$$\frac{\delta\mathbf{S}}{\delta t}=\frac{d\mathbf{S}}{dt}-\mathbf{LS}-\mathbf{SL}^T, \tag{6.3}$$

where d/dt is the material derivative and

$$\frac{\delta^2\mathbf{S}}{\delta t^2}=\frac{\delta}{\delta t}\left(\frac{\delta\mathbf{S}}{\delta t}\right). \tag{6.4}$$

The flows under consideration have the following velocity field

$$\mathbf{V}=u\left(y,t\right)\mathbf{i}, \tag{6.5}$$

where $\mathbf{i}$ and u are the unit vector and velocity parallel to the x-axis, respectively. The velocity field (6.5) automatically satisfies the incompressibility condition. Since u is a function of y and t, the stress field will also depend upon y and t. Now equation (6.2) together with the initial condition (the fluid being at rest up to the moment $t=0$)

$$\mathbf{S}\left(y,0\right)=\mathbf{0} \tag{6.6}$$

yields $S_{xz}=S_{yz}=S_{yy}=S_{zz}=0$ and

$$\begin{aligned} & S_{xx}+\lambda_1\left(\frac{\partial S_{xx}}{\partial t}-2S_{xy}\frac{\partial u}{\partial y}\right)+\lambda_2\left(\frac{\partial^2 S_{xx}}{\partial t^2}-4\frac{\partial S_{xy}}{\partial t}\frac{\partial u}{\partial y}-2S_{xy}\frac{\partial^2 u}{\partial t\partial y}\right)\\ = & -2\mu\lambda_3\left(\frac{\partial u}{\partial y}\right)^2-6\mu\lambda_4\left(\frac{\partial u}{\partial y}\right)\frac{\partial^2 u}{\partial t\partial y}, \end{aligned} \tag{6.7}$$

$$\left(1+\lambda_1\frac{\partial}{\partial t}+\lambda_2\frac{\partial^2}{\partial t^2}\right)S_{xy}=\mu\left(1+\lambda_3\frac{\partial}{\partial t}+\lambda_4\frac{\partial^2}{\partial t^2}\right)\frac{\partial u}{\partial y}. \tag{6.8}$$

The balance of linear momentum for MHD fluid in a porous medium is given by Eq. (1.11). Neglecting the displacement current, the Maxwell equations and Ohms' law are given in Eqs. (1.1)-(1.4) and (1.13).

On the basis of Oldroyd constitutive equation, the law for describing both relaxation and retardation phenomena in an unbounded porous medium is given in equation (3.6). By the analogy with constitutive equation (6.2), the following law for unidirectional flow of a generalized Burgers' fluid has been suggested:

$$\left(1+\lambda_1\frac{\partial}{\partial t}+\lambda_2\frac{\partial^2}{\partial t^2}\right)\nabla p=-\frac{\mu\phi}{k}\left(1+\lambda_3\frac{\partial}{\partial t}+\lambda_4\frac{\partial^2}{\partial t^2}\right)\mathbf{V} \tag{6.9}$$

and the flow resistance offered by the solid matrix $\mathbf{R}$ is

$$\left(1+\lambda_1\frac{\partial}{\partial t}+\lambda_2\frac{\partial^2}{\partial t^2}\right)\mathbf{R}=-\frac{\mu\phi}{k}\left(1+\lambda_3\frac{\partial}{\partial t}+\lambda_4\frac{\partial^2}{\partial t^2}\right)\mathbf{V}. \tag{6.10}$$

Upon making use of the stated assumptions, equation (1.11) yields

$$\begin{aligned}\left(1+\lambda_1\frac{\partial}{\partial t}+\lambda_2\frac{\partial^2}{\partial t^2}\right)\frac{\partial u}{\partial t} &= \nu\left(1+\lambda_3\frac{\partial}{\partial t}+\lambda_4\frac{\partial^2}{\partial t^2}\right)\frac{\partial^2 u}{\partial y^2}\\ &-\frac{\sigma B_0^2}{\rho}\left(1+\lambda_1\frac{\partial}{\partial t}+\lambda_2\frac{\partial^2}{\partial t^2}\right)u-\frac{\nu\phi}{k}\left(1+\lambda_3\frac{\partial}{\partial t}+\lambda_4\frac{\partial^2}{\partial t^2}\right)u,\end{aligned} \tag{6.11}$$

where the pressure gradient in the $x-$ direction has been ignored and ν is the kinematic viscosity.

6.2 Stokes' second problem

This section deals with the MHD flow of a generalized Burgers' fluid in a porous space $y > 0$. The fluid is bounded by a rigid boundary at $y = 0$. Initially, both fluid and boundary are at rest. For $t > 0$, the boundary starts to oscillate in its own plane. In absence of pressure gradient, the equation which governs the flow is (6.11). The appropriate boundary and initial conditions are

$$u(0,t)=U_0\cos\omega t \text{ or } u(0,t)=U_0\sin\omega t.\ t>0, \tag{6.12}$$

$$u(y,t)\to 0, \frac{\partial u}{\partial y}\to 0 \text{ as } y\to\infty,\ t\geq 0, \tag{6.13}$$

$$u(y,0)=\frac{\partial u(y,0)}{\partial t}=\frac{\partial^2 u(y,0)}{\partial t^2}=0,\ y\geq 0 \tag{6.14}$$

in which ω is the imposed frequency.

In order to find the solution we use the Fourier sine transform pair defined in equations (1.22) and (1.23).

6.2.1 When $u(0,t)=U_0\cos\omega t$ and $\lambda_2\neq 0$

Taking Fourier sine transform of equations (6.11) and (6.13)and then solving the resulting problem in the $\xi-$ plane, we have the following expression for the starting solution

$$\overline{u}(\xi,t)=\overline{u}_t(\xi,t)+\overline{u}_s(\xi,t), \tag{6.15}$$

where $\overline{u}_t(\xi,t)$ and $\overline{u}_s(\xi,t)$ indicate the transformed transient and steady state solutions, respectively and are given by

$$\begin{aligned}\overline{u}_t(\xi,t) &= \frac{-\sqrt{\frac{2}{\pi}}U_0\xi\nu\left(\left(m_2m_3-\omega^2\right)\left(F_0-\omega^2F_1\right)-\omega^2\left(m_2+m_3\right)F_2\right)e^{m_1t}}{\left(m_1-m_2\right)\left(m_1-m_3\right)F_3}\\ &+\frac{\sqrt{\frac{2}{\pi}}U_0\xi\nu\left(\left(\omega^2-m_1m_3\right)\left(F_0-\omega^2F_1\right)+\omega^2\left(m_1+m_3\right)F_2\right)e^{m_2t}}{\left(m_2-m_1\right)\left(m_2-m_3\right)F_3}\\ &-\frac{\sqrt{\frac{2}{\pi}}U_0\xi\nu\left(\left(m_1m_2-\omega^2\right)\left(F_0-\omega^2F_1\right)-\omega^2\left(m_1+m_2\right)F_2\right)e^{m_3t}}{\left(m_3-m_1\right)\left(m_3-m_2\right)F_3},\end{aligned} \tag{6.16}$$

$$\overline{u}_s(\xi,t)=\sqrt{\frac{2}{\pi}}U_0\nu\omega\sin\omega t\frac{\xi F_2}{F_3}+\sqrt{\frac{2}{\pi}}U_0\nu\cos\omega t\frac{\xi\left(F_0-\omega^2F_1\right)}{F_3} \tag{6.17}$$

in which

$$\begin{aligned}F_0 &= \nu\xi^2E_1+\frac{\sigma B_0^2}{\rho}+\frac{\nu\phi}{k},\ F_1=-E_2+\frac{\sigma B_0^2}{\rho}E_3+\frac{\nu\phi}{k}E_4,\\ F_2 &= E_5-\frac{\sigma B_0^2}{\rho}E_2,\ F_3=\left(\nu\xi^2+\frac{\sigma B_0^2}{\rho}+\frac{\nu\phi}{k}-\omega^2E_6\right)^2+\omega^2E_7^2\end{aligned}$$

$$
\begin{aligned}
E_1 &= \left(\left(1-\lambda_4\omega^2\right)^2+\left(\lambda_3\omega\right)^2\right),\ E_2=\left(\lambda_3-\lambda_1-\lambda_2\lambda_3\omega^2+\lambda_1\lambda_4\omega^2\right),\\
E_3 &= \left(\lambda_2+\lambda_4-\lambda_1\lambda_3-\lambda_2\lambda_4\omega^2\right),\ E_4=\left(2\lambda_4-\lambda_4^2\omega^2-\lambda_3^2\right),\\
E_5 &= \left(1-\omega^2\left(\lambda_2+\lambda_4\right)+\omega^2\left(\lambda_1\lambda_3+\lambda_2\lambda_4\omega^2\right)\right),\ E_6=\left(\lambda_1+\beta\xi^2+\frac{\sigma B_0^2}{\rho}\lambda_2+\frac{\phi}{k}\beta\right),\\
E_7 &= \left(1+\alpha\xi^2+\frac{\sigma B_0^2}{\rho}\lambda_1+\frac{\phi}{k}\alpha-\lambda_2\omega^2\right),
\end{aligned}
$$

$$
\begin{aligned}
m_1 &= -\frac{d}{3\lambda_2}-\frac{2^{1/3}(-d^2+3e\lambda_2)}{3\lambda_2\left(\begin{array}{c}-2d^3+9de\lambda_2-27f\lambda_2^2\\+\sqrt{4\left(-d^2+3e\lambda_2\right)^3+\left(-2d^3+9de\lambda_2-27f\lambda_2^2\right)}\end{array}\right)^{1/3}}\\
&\quad+\frac{\left(\begin{array}{c}-2d^3+9de\lambda_2-27f\lambda_2^2\\+\sqrt{4\left(-d^2+3e\lambda_2\right)^3+\left(-2d^3+9de\lambda_2-27f\lambda_2^2\right)}\end{array}\right)^{1/3}}{3\left(2\right)^{1/3}\lambda_2},\\
m_2 &= -\frac{d}{3\lambda_2}+\frac{\left(1+i\sqrt{3}\right)\left(-d^2+3e\lambda_2\right)}{3\left(2\right)^{2/3}\lambda_2\left(\begin{array}{c}-2d^3+9de\lambda_2-27f\lambda_2^2\\+\sqrt{4\left(-d^2+3e\lambda_2\right)^3+\left(-2d^3+9de\lambda_2-27f\lambda_2^2\right)}\end{array}\right)^{1/3}}\\
&\quad-\frac{\left(1-i\sqrt{3}\right)\left(\begin{array}{c}-2d^3+9de\lambda_2-27f\lambda_2^2\\+\sqrt{4\left(-d^2+3e\lambda_2\right)^3+\left(-2d^3+9de\lambda_2-27f\lambda_2^2\right)}\end{array}\right)^{1/3}}{6\left(2\right)^{1/3}\lambda_2},\\
m_3 &= -\frac{d}{3\lambda_2}+\frac{\left(1-i\sqrt{3}\right)\left(-d^2+3e\lambda_2\right)}{3\left(2\right)^{2/3}\lambda_2\left(\begin{array}{c}-2d^3+9de\lambda_2-27f\lambda_2^2\\+\sqrt{4\left(-d^2+3e\lambda_2\right)^3+\left(-2d^3+9de\lambda_2-27f\lambda_2^2\right)}\end{array}\right)^{1/3}}\\
&\quad-\frac{\left(1+i\sqrt{3}\right)\left(\begin{array}{c}-2d^3+9de\lambda_2-27f\lambda_2^2\\+\sqrt{4\left(-d^2+3e\lambda_2\right)^3+\left(-2d^3+9de\lambda_2-27f\lambda_2^2\right)}\end{array}\right)^{1/3}}{6\left(2\right)^{1/3}\lambda_2},
\end{aligned}
$$

$$
\begin{aligned}
d &= \lambda_1+\beta\xi^2+\frac{\sigma B_0^2}{\rho}\lambda_2+\frac{\phi}{k}\beta,\ e=1+\alpha\xi^2+\frac{\sigma B_0^2}{\rho}\lambda_1+\frac{\phi}{k}\alpha,\\
f &= \nu\xi^2+\frac{\sigma B_0^2}{\rho}+\frac{\phi}{k}\nu,\ \alpha=\nu\lambda_3,\ \beta=\nu\lambda_4.
\end{aligned}
$$

Fourier inversion of equations (6.16) and (6.17) yield

$$
\begin{aligned}
u_t(y,t) &= -\frac{2}{\pi}U_0\nu\int_0^\infty \frac{\xi\left(\left(m_2m_3-\omega^2\right)\left(F_0-\omega^2F_1\right)-\omega^2\left(m_2+m_3\right)F_2\right)e^{m_1t}\sin(\xi y)\,d\xi}{\left(m_1-m_2\right)\left(m_1-m_3\right)F_3} \\
&\quad +\frac{2}{\pi}U_0\nu\int_0^\infty \frac{\xi\left(\left(\omega^2-m_1m_3\right)\left(F_0-\omega^2F_1\right)+\omega^2\left(m_1+m_3\right)F_2\right)e^{m_2t}\sin(\xi y)\,d\xi}{\left(m_2-m_1\right)\left(m_2-m_3\right)F_3} \\
&\quad -\frac{2}{\pi}U_0\nu\int_0^\infty \frac{\xi\left(\left(m_1m_2-\omega^2\right)\left(F_0-\omega^2F_1\right)-\omega^2\left(m_1+m_2\right)F_2\right)e^{m_3t}\sin(\xi y)\,d\xi}{\left(m_3-m_1\right)\left(m_3-m_2\right)F_3}
\end{aligned}
\tag{6.18}
$$

$$
\begin{aligned}
u_s(y,t) &= \frac{2}{\pi}U_0\nu\omega\sin\omega t\int_0^\infty \frac{\xi F_2\sin(\xi y)\,d\xi}{F_3} \\
&\quad +\frac{2}{\pi}U_0\nu\cos\omega t\int_0^\infty \frac{\xi\left(F_0-\omega^2F_1\right)\sin(\xi y)\,d\xi}{F_3}.
\end{aligned}
\tag{6.19}
$$

Note that for large times $u_t(y,t)\to 0$ and $u_s(y,t)$ can be written as

$$
\begin{aligned}
u_s(y,t) &= \frac{2}{\pi}U_0\frac{\nu\omega\left(\begin{array}{c} 1-\left(\lambda_2+\lambda_4\right)\omega^2+\left(\lambda_1\lambda_3+\lambda_2\lambda_4\omega^2\right)\omega^2+ \\ \frac{\sigma B_0^2}{\rho}\left(\left(\lambda_1-\lambda_3\right)+\left(\lambda_2\lambda_3+\lambda_1\lambda_4\right)\omega^2\right)\end{array}\right)}{\nu^2\left(1-\lambda_4\omega^2\right)^2+\nu^2\omega^2\lambda_3^2}\sin\omega t\int_0^\infty \frac{\xi\sin(a\xi)\,d\xi}{\left(\xi^2+b^2\right)^2+c^2}+ \\
&\quad \frac{2}{\pi}U_0\cos\omega t\int_0^\infty \frac{\xi\left(\xi^2+b^2\right)\sin(a\xi)\,d\xi}{\left(\xi^2+b^2\right)^2+c^2},
\end{aligned}
$$

or

$$
u_s(y,t)=\frac{2}{\pi}U_0c\sin\omega t\int_0^\infty \frac{\xi\sin(a\xi)\,d\xi}{\left(\xi^2+b^2\right)^2+c^2}+\frac{2}{\pi}U_0\cos\omega t\int_0^\infty \frac{\xi\left(\xi^2+b^2\right)\sin(a\xi)\,d\xi}{\left(\xi^2+b^2\right)^2+c^2},
$$

or

$$
u_s(y,t)=\frac{2}{\pi}U_0c\sin\omega t\frac{\pi}{2c}\exp(-Ay)\sin(yB)+\frac{2}{\pi}U_0\cos\omega t\frac{\pi}{2c}\exp(-Ay)\cos(yB),
$$

or

$$
u_s(y,t)=U_0\exp(-Ay)\cos(\omega t-By), \tag{6.20}
$$

where

$$b^2 = \frac{1}{\nu\left(1-\lambda_4\omega^2\right)^2 + \nu\omega^2\lambda_3^2}\left(\begin{array}{c} \omega^2\left(\lambda_3-\lambda_1\right)+\omega^4\left(\lambda_1\lambda_4-\lambda_2\lambda_3\right) \\ +\frac{\phi}{k}\nu\left(\left(1-\lambda_4\omega^2\right)^2+\omega^2\lambda_3^2\right) \\ +\frac{\sigma B_0^2}{\rho}\left(1-\left(\lambda_2+\lambda_4\right)\omega^2+\left(\lambda_1\lambda_3+\lambda_2\lambda_4\omega^2\right)\omega^2\right) \end{array}\right),$$

$$c^2 = \frac{\nu^2\omega^2}{\left(\nu^2\left(1-\lambda_4\omega^2\right)^2+\nu^2\omega^2\lambda_3^2\right)^2}\left(\begin{array}{c} 1-\left(\lambda_2+\lambda_4\right)\omega^2+\left(\lambda_1\lambda_3+\lambda_2\lambda_4\omega^2\right)\omega^2 \\ +\frac{\sigma B_0^2}{\rho}\left(\left(\lambda_1-\lambda_3\right)+\left(\lambda_2\lambda_3+\lambda_1\lambda_4\right)\omega^2\right) \end{array}\right)^2,$$

$$\begin{aligned} 2A^2 = & \sqrt{\begin{array}{c} \left(\frac{\omega^2(\lambda_3-\lambda_1)+\omega^4(\lambda_1\lambda_4-\lambda_2\lambda_3)+\frac{\phi}{k}\nu\left((1-\lambda_4\omega^2)^2+\omega^2\lambda_3^2\right)+\frac{\sigma B_0^2}{\rho}\left(1-(\lambda_2+\lambda_4)\omega^2+\left(\lambda_1\lambda_3+\lambda_2\lambda_4\omega^2\right)\omega^2\right)}{\nu(1-\lambda_4\omega^2)^2+\nu\omega^2\lambda_3^2}\right)^2 \\ +\nu^2\omega^2\left(\frac{\left(1-(\lambda_2+\lambda_4)\omega^2+\left(\lambda_1\lambda_3+\lambda_2\lambda_4\omega^2\right)\omega^2+\frac{\sigma B_0^2}{\rho}\left((\lambda_1-\lambda_3)+(\lambda_2\lambda_3+\lambda_1\lambda_4)\omega^2\right)\right)}{\nu^2(1-\lambda_4\omega^2)^2+\nu^2\omega^2\lambda_3^2}\right)^2 \end{array}} \\ & +\frac{\begin{array}{c}\omega^2\left(\lambda_3-\lambda_1\right)+\omega^4\left(\lambda_1\lambda_4-\lambda_2\lambda_3\right)+\frac{\phi}{k}\nu\left(\left(1-\lambda_4\omega^2\right)^2+\omega^2\lambda_3^2\right) \\ +\frac{\sigma B_0^2}{\rho}\left(1-\left(\lambda_2+\lambda_4\right)\omega^2+\left(\lambda_1\lambda_3+\lambda_2\lambda_4\omega^2\right)\omega^2\right)\end{array}}{\nu\left(1-\lambda_4\omega^2\right)^2+\nu\omega^2\lambda_3^2}, \end{aligned}$$

$$\begin{aligned} 2B^2 = & \sqrt{\begin{array}{c} \left(\frac{\omega^2(\lambda_3-\lambda_1)+\omega^4(\lambda_1\lambda_4-\lambda_2\lambda_3)+\frac{\phi}{k}\nu\left((1-\lambda_4\omega^2)^2+\omega^2\lambda_3^2\right)+\frac{\sigma B_0^2}{\rho}\left(1-(\lambda_2+\lambda_4)\omega^2+\left(\lambda_1\lambda_3+\lambda_2\lambda_4\omega^2\right)\omega^2\right)}{\nu(1-\lambda_4\omega^2)^2+\nu\omega^2\lambda_3^2}\right)^2 \\ +\nu^2\omega^2\left(\frac{\left(1-(\lambda_2+\lambda_4)\omega^2+\left(\lambda_1\lambda_3+\lambda_2\lambda_4\omega^2\right)\omega^2+\frac{\sigma B_0^2}{\rho}\left((\lambda_1-\lambda_3)+(\lambda_2\lambda_3+\lambda_1\lambda_4)\omega^2\right)\right)}{\nu^2(1-\lambda_4\omega^2)^2+\nu^2\omega^2\lambda_3^2}\right)^2 \end{array}} \\ & -\frac{\begin{array}{c}\omega^2\left(\lambda_3-\lambda_1\right)+\omega^4\left(\lambda_1\lambda_4-\lambda_2\lambda_3\right)+\frac{\phi}{k}\nu\left(\left(1-\lambda_4\omega^2\right)^2+\omega^2\lambda_3^2\right) \\ +\frac{\sigma B_0^2}{\rho}\left(1-\left(\lambda_2+\lambda_4\right)\omega^2+\left(\lambda_1\lambda_3+\lambda_2\lambda_4\omega^2\right)\omega^2\right)\end{array}}{\nu\left(1-\lambda_4\omega^2\right)^2+\nu\omega^2\lambda_3^2}, \end{aligned}$$

In terms of E_i $(i = 1 - 5)$ the above expressions become

$$2A^2 = \sqrt{\left(\frac{\omega^2 E_2 + \frac{\phi}{k}\nu E_1 + \frac{\sigma B_0^2}{\rho}E_5}{\nu E_1}\right)^2 + \nu^2\omega^2\left(\frac{\left(E_5 - \frac{\sigma B_0^2}{\rho}E_2\right)}{\nu^2 E_1}\right)^2} + \frac{\omega^2 E_2 + \frac{\phi}{k}\nu E_1 + \frac{\sigma B_0^2}{\rho}E_5}{\nu E_1}, \tag{6.21}$$

$$2B^2 = \sqrt{\left(\frac{\omega^2 E_2 + \frac{\phi}{k}\nu E_1 + \frac{\sigma B_0^2}{\rho}E_5}{\nu E_1}\right)^2 + \nu^2\omega^2\left(\frac{\left(E_5 - \frac{\sigma B_0^2}{\rho}E_2\right)}{\nu^2 E_1}\right)^2} - \frac{\omega^2 E_2 + \frac{\phi}{k}\nu E_1 + \frac{\sigma B_0^2}{\rho}E_5}{\nu E_1}. \tag{6.22}$$

In order to obtain the equality (6.20) from (6.19) we have used the following two integrals [72]:

$$\int_0^\infty \frac{x\sin(ax)}{(x^2+\epsilon b^2)^2 + c^2}dx = \frac{\pi}{2c}\exp(-aA)\sin(aB),$$

$$\int_0^\infty \frac{x(x^2+\epsilon b^2)\sin(ax)}{(x^2+\epsilon b^2)^2 + c^2}dx = \frac{\pi}{2}\exp(-aA)\cos(aB)$$

where

$$2A^2 = \sqrt{b^4+c^2} + \epsilon b^2, 2B^2 = \sqrt{b^4+c^2} - \epsilon b^2$$

and

$$\epsilon = \pm 1.$$

.

Introducing

$$\tilde{y} = \sqrt{\frac{\omega}{2\nu}}y,\ \ \tilde{u} = \frac{u}{U_0},\ \ \tilde{t} = \omega t,\ \ \tilde{\lambda}_1 = \lambda_1\omega,\ \ \tilde{\lambda}_2 = \lambda_2\omega^2$$
$$\tilde{\lambda}_3 = \lambda_3\omega,\ \ \tilde{\lambda}_4 = \lambda_4\omega^2,\ \ M^2 = \frac{\sigma B_0^2}{\rho\omega},\ \ \frac{1}{K} = \frac{\nu\phi}{k\omega}$$

equation (6.20) takes the following form

$$\widetilde{u}_s(y,t) = \exp\left(-\widetilde{A}\widetilde{y}\right)\cos\left(\widetilde{t} - \widetilde{B}\widetilde{y}\right).$$

where

$$\widetilde{A} = \left(\sqrt{\left(\frac{\widetilde{E}_2 + \frac{1}{K}\widetilde{E}_1 + M^2\widetilde{E}_5}{\widetilde{E}_1}\right)^2 + \left(\frac{\widetilde{E}_5 + M^2\widetilde{E}_2}{\widetilde{E}_1}\right)^2} + \left(\frac{\widetilde{E}_2 + \frac{1}{K}\widetilde{E}_1 + M^2\widetilde{E}_5}{\widetilde{E}_1}\right)\right)^{\frac{1}{2}},$$

$$\widetilde{B} = \left(\sqrt{\left(\frac{\widetilde{E}_2 + \frac{1}{K}\widetilde{E}_1 + M^2\widetilde{E}_5}{\widetilde{E}_1}\right)^2 + \left(\frac{\widetilde{E}_5 + M^2\widetilde{E}_2}{\widetilde{E}_1}\right)^2} - \left(\frac{\widetilde{E}_2 + \frac{1}{K}\widetilde{E}_1 + M^2\widetilde{E}_5}{\widetilde{E}_1}\right)\right)^{\frac{1}{2}}.$$

$$\begin{aligned}
\widetilde{E}_1 &= \left(\left(1 - \widetilde{\lambda}_4\right)^2 + \widetilde{\lambda}_3^2\right), \quad \widetilde{E}_2 = \left(\widetilde{\lambda}_3 - \widetilde{\lambda}_1\right) + \left(\widetilde{\lambda}_1\widetilde{\lambda}_4 - \widetilde{\lambda}_2\widetilde{\lambda}_3\right), \\
\widetilde{E}_5 &= \left(1 - \left(\widetilde{\lambda}_2 + \widetilde{\lambda}_4\right) + \left(\widetilde{\lambda}_1\widetilde{\lambda}_3 + \widetilde{\lambda}_2\widetilde{\lambda}_4\right)\right).
\end{aligned}$$

6.2.2 For $u(0,t) = U_0 \cos\omega t$ and $\lambda_2 = 0$

Following the procedure of previous subsection one obtains

$$\begin{aligned}
u_t(y,t) &= \frac{2}{\pi}U_0\nu\int_0^\infty \frac{\xi\left(\begin{array}{c} r_2\left(F_0 - \omega^2\left(-F_4 + \frac{\sigma B_0^2}{\rho}F_5 + \frac{\nu\phi}{k}F_6\right)\right) \\ -\omega^2\left(F_7 - \frac{\sigma B_0^2}{\rho}F_4\right)\end{array}\right) e^{r_1 t}\sin(\xi y)\,d\xi}{(r_1 - r_2)\left((F_8 - \omega^2 F_9)^2 + \omega^2 F_{10}^2\right)} \\
&+ \frac{2}{\pi}U_0\nu\int_0^\infty \frac{\xi\left(\begin{array}{c} r_1\left(F_0 - \omega^2\left(-F_4 + \frac{\sigma B_0^2}{\rho}F_5 + \frac{\nu\phi}{k}F_6\right)\right) \\ -\omega^2\left(F_7 - \frac{\sigma B_0^2}{\rho}F_4\right)\end{array}\right) e^{r_2 t}\sin(\xi y)\,d\xi}{(r_2 - r_1)\left((F_8 - \omega^2 F_9)^2 + \omega^2 F_{10}^2\right)},
\end{aligned} \quad (6.23)$$

$$u_s(y,t) = \frac{2}{\pi}U_0\nu\omega\sin\omega t\int_0^\infty \frac{\xi\left(F_7-\frac{\sigma B_0^2}{\rho}F_4\right)\sin(\xi y)\,d\xi}{\left((F_8-\omega^2F_9)^2+\omega^2F_{10}^2\right)}$$
$$+\frac{2}{\pi}U_0\nu\cos\omega t\int_0^\infty \xi\frac{F_0-\omega^2\left(-F_4+\frac{\sigma B_0^2}{\rho}F_5+\frac{\nu\phi}{k}F_6\right)}{\left((F_8-\omega^2F_9)^2+\omega^2F_{10}^2\right)}\sin(\xi y)\,d\xi, \tag{6.24}$$

where

$$r_1 = \frac{-F_{10}+\sqrt{F_{10}^2-4F_8F_9}}{2F_9}, \tag{6.25}$$

$$r_2 = \frac{-F_{10}-\sqrt{F_{10}^2-4F_8F_9}}{2F_9}, \tag{6.26}$$

$$F_4 = \left(\lambda_3-\lambda_1+\lambda_1\lambda_4\omega^2\right),\ F_5=\lambda_4-\lambda_1\lambda_3,\ F_6=\left(2\lambda_4-\lambda_4^2\omega^2-\lambda_3^2\right),$$
$$F_7 = \left(1-\omega^2\lambda_4+\omega^2\lambda_1\lambda_3\right),\ F_8=\nu\xi^2+\frac{\sigma B_0^2}{\rho}+\frac{\nu\phi}{k},\ F_9=\lambda_1+\beta\xi^2+\frac{\phi}{k}\beta,$$
$$F_{10} = 1+\alpha\xi^2+\frac{\sigma B_0^2}{\rho}\lambda_1+\frac{\phi}{k}\alpha.$$

Adopting the same methodology of solution as for equation (6.19), equation (6.24) gives

$$\widetilde{u}_{s1}(y,t)=\exp\left(-\widetilde{A}_1\widetilde{y}\right)\cos\left(\widetilde{t}-\widetilde{B}_1\widetilde{y}\right), \tag{6.27}$$

where

$$\widetilde{A}_1=\left(\sqrt{\left(\frac{\widetilde{F}_4+\frac{1}{K}\widetilde{E}_1+M^2\widetilde{F}_7}{\widetilde{E}_1}\right)^2+\left(\frac{\widetilde{F}_7-M^2\widetilde{F}_4}{\widetilde{E}_1}\right)^2}+\left(\frac{\widetilde{F}_4+\frac{1}{K}\widetilde{E}_1+M^2\widetilde{F}_7}{\widetilde{E}_1}\right)\right)^{\frac{1}{2}},$$

$$\widetilde{B}_1=\left(\sqrt{\left(\frac{\widetilde{F}_4+\frac{1}{K}\widetilde{E}_1+M^2\widetilde{F}_7}{\widetilde{E}_1}\right)^2+\left(\frac{\widetilde{F}_7-M^2\widetilde{F}_4}{\widetilde{E}_1}\right)^2}-\left(\frac{\widetilde{F}_4+\frac{1}{K}\widetilde{E}_1+M^2\widetilde{F}_7}{\widetilde{E}_1}\right)\right)^{\frac{1}{2}},$$

$$\widetilde{F}_4=\left(\widetilde{\lambda}_3-\widetilde{\lambda}_1\right)+\left(\widetilde{\lambda}_1\widetilde{\lambda}_4\right),\ \widetilde{F}_7=\left(1-\widetilde{\lambda}_4+\widetilde{\lambda}_1\widetilde{\lambda}_3\right).$$

It is worthmentioning to note that for $\lambda_4=M=\phi=0$ the equation (6.27) reduces to the solutions of an Oldroyd-B fluid. Moreover, equation (6.27) recovers the results of second grade fluid [15] when $\lambda_1=\lambda_4=M=\phi=0$ and $\nu\lambda_3=\alpha_1/\rho\,(\alpha_1$ is the material parameter of second grade fluid$)$.

6.2.3 For $u(0,t) = U_0 \sin \omega t$ and $\lambda_2 \neq 0$

Employing the similar procedure as for the case of $U_0 \cos \omega t$, the transient and steady state solutions are

$$
\begin{aligned}
u_t(y,t) &= -\frac{2}{\pi} U_0 \nu \omega \int_0^\infty \frac{\xi \left(\left(\omega^2 - m_2 m_3 \right) F_2 - \left(m_2 + m_3 \right) \left(F_0 - \omega^2 F_1 \right) \right) e^{m_1 t} \sin(\xi y) \, d\xi}{\left(m_1 - m_2 \right) \left(m_1 - m_3 \right) F_3} \\
&\quad + \frac{2}{\pi} U_0 \nu \omega \int_0^\infty \frac{\xi \left(\left(m_1 m_3 - \omega^2 \right) F_2 + \left(m_1 + m_3 \right) \left(F_0 - \omega^2 F_1 \right) \right) e^{m_2 t} \sin(\xi y) \, d\xi}{\left(m_2 - m_1 \right) \left(m_2 - m_3 \right) F_3} \\
&\quad - \frac{2}{\pi} U_0 \nu \omega \int_0^\infty \frac{\xi \left(\left(\omega^2 - m_1 m_2 \right) F_2 - \left(m_1 + m_2 \right) \left(F_0 - \omega^2 F_1 \right) \right) e^{m_3 t} \sin(\xi y) \, d\xi}{\left(m_3 - m_1 \right) \left(m_3 - m_2 \right) F_3} \qquad (6.28)
\end{aligned}
$$

$$
\begin{aligned}
u_s(y,t) &= -\frac{2}{\pi} U_0 \nu \omega \cos \omega t \int_0^\infty \frac{\xi F_2 \sin(\xi y) \, d\xi}{F_3} \\
&\quad + \frac{2}{\pi} U_0 \nu \sin \omega t \int_0^\infty \frac{\xi \left(F_0 - \omega^2 F_1 \right) \sin(\xi y) \, d\xi}{F_3}. \qquad (6.29)
\end{aligned}
$$

The expression (6.29) in dimensionless variables now gives

$$
\widetilde{u}_s(y,t) = \exp\left(-\widetilde{A}\widetilde{y} \right) \sin\left(\widetilde{t} - \widetilde{B}\widetilde{y} \right). \qquad (6.30)
$$

6.2.4 For $u(0,t) = U_0 \sin \omega t$ and $\lambda_2 = 0$

Here we have

$$
\begin{aligned}
u_t(y,t) &= -\frac{2}{\pi} U_0 \nu \omega \int_0^\infty \frac{\xi \begin{pmatrix} r_2 \left(F_7 - \frac{\sigma B_0^2}{\rho} F_4 \right) + \\ \left(F_0 - \omega^2 \left(-F_4 + \frac{\sigma B_0^2}{\rho} F_5 + \frac{\nu \phi}{k} F_6 \right) \right) \end{pmatrix} e^{r_1 t} \sin(\xi y) \, d\xi}{\left(r_1 - r_2 \right) \left(\left(F_8 - \omega^2 F_9 \right)^2 + \omega^2 F_{10}^2 \right)} \\
&\quad + \frac{2}{\pi} U_0 \nu \omega \int_0^\infty \frac{\xi \begin{pmatrix} r_1 \left(F_7 - \frac{\sigma B_0^2}{\rho} F_4 \right) + \\ \left(F_0 - \omega^2 \left(-F_4 + \frac{\sigma B_0^2}{\rho} F_5 + \frac{\nu \phi}{k} F_6 \right) \right) \end{pmatrix} e^{r_2 t} \sin(\xi y) \, d\xi}{\left(r_1 - r_2 \right) \left(\left(F_8 - \omega^2 F_9 \right)^2 + \omega^2 F_{10}^2 \right)}, \qquad (6.31)
\end{aligned}
$$

$$u_s(y,t) = -\frac{2}{\pi}U_0\nu\omega\cos\omega t\int_0^\infty \frac{\xi\left(F_7 - \frac{\sigma B_0^2}{\rho}F_4\right)\sin(\xi y)\,d\xi}{\left((F_8-\omega^2F_9)^2+\omega^2F_{10}^2\right)}$$
$$+\frac{2}{\pi}U_0\nu\sin\omega t\int_0^\infty \frac{\xi\left(F_0-\omega^2\left(-F_4+\frac{\sigma B_0^2}{\rho}F_5+\frac{\nu\phi}{k}F_6\right)\right)\sin(\xi y)\,d\xi}{\left((F_8-\omega^2F_9)^2+\omega^2F_{10}^2\right)}. \quad (6.32)$$

After evaluating the above integrals, the solution in dimensionless variables is obtained as follows

$$\widetilde{u}_{s1}(y,t) = \exp\left(-\widetilde{A}_1\widetilde{y}\right)\sin\left(\widetilde{t}-\widetilde{B}_1\widetilde{y}\right). \quad (6.33)$$

The above equation also reduces to the result of second grade fluid [12] for $\lambda_1=\lambda_4=M=\phi=0$ and $\nu\lambda_3=\alpha_1/\rho$.

6.3 Modified Stokes' second problem

Here, we consider the MHD fluid between two infinite parallel plates distant d apart. The lower plate at $y=0$ oscillates in its own plane for $t>0$ while the upper plate at $y=d$ is stationary. The problem which governs the flow consists of equations (6.11), (6.12) and

$$u(d,t)=0;\ t\in R, \quad (6.34)$$

$$u(y,0)=\frac{\partial u(y,0)}{\partial t}=\frac{\partial^2 u(y,0)}{\partial t^2}0,\ 0<y<d. \quad (6.35)$$

Following the same method of solution as in the previous section we have

6.3.1 For $u(0,t)=U_0\cos\omega t$ and $\lambda_2\neq 0$

$$u_t(y,t) = -\frac{2}{d}U_0\nu\sum_{n=1}^{\infty}\frac{\lambda_n\left((n_2n_3-\omega^2)\left(F_{11}-\omega^2F_1\right)-\omega^2(n_2+n_3)F_2\right)e^{n_1t}\sin(\lambda_n y)}{(n_1-n_2)(n_1-n_3)F_{12}}$$
$$+\frac{2}{d}U_0\nu\sum_{n=1}^{\infty}\frac{\lambda_n\left((\omega^2-n_1n_3)\left(F_{11}-\omega^2F_1\right)+\omega^2(n_1+n_3)F_2\right)e^{n_2t}\sin(\lambda_n y)}{(n_2-n_1)(n_2-n_3)F_{12}} \quad (6.36)$$
$$-\frac{2}{\pi}U_0\nu\sum_{n=1}^{\infty}\frac{\lambda_n\left((n_1n_2-\omega^2)\left(F_{11}-\omega^2F_1\right)-\omega^2(n_1+n_2)F_2\right)e^{n_3t}\sin(\lambda_n y)}{(n_3-n_1)(n_3-n_2)F_{12}},$$

$$u_s(y,t)=\frac{2}{d}U_0\nu\omega\sin\omega t\sum_{n=1}^{\infty}\frac{\lambda_n F_2\sin(\lambda_n y)}{F_{12}}+\frac{2}{d}U_0\nu\cos\omega t\sum_{n=1}^{\infty}\frac{\lambda_n\left(F_{11}-\omega^2F_1\right)\sin(\lambda_n y)}{F_{12}}, \quad (6.37)$$

where

$$F_{11}=\nu\lambda_n^2E_1+\frac{\sigma B_0^2}{\rho}+\frac{\nu\phi}{k},\ F_{12}=\left(\left(\nu\lambda_n^2+\frac{\sigma B_0^2}{\rho}+\frac{\nu\phi}{k}-\omega^2E_9\right)^2+\omega^2E_{10}^2\right),$$

$$E_9=\left(\lambda_1+\beta\lambda_n^2+\frac{\sigma B_0^2}{\rho}\lambda_2+\frac{\phi}{k}\beta\right),\quad E_{10}=\left(1+\alpha\lambda_n^2+\frac{\sigma B_0^2}{\rho}\lambda_1+\frac{\phi}{k}\alpha-\lambda_2\omega^2\right)$$

$$\begin{aligned}
n_1 &= -\frac{d_1}{3\lambda_2}-\frac{2^{1/3}(-d_1^2+3e_1\lambda_2)}{3\lambda_2\left(\begin{array}{c}-2d_1^3+9d_1e_1\lambda_2-27f_1\lambda_2^2\\+\sqrt{4\left(-d_1^2+3e_1\lambda_2\right)^3+\left(-2d_1^3+9d_1e_1\lambda_2-27f_1\lambda_2^2\right)}\end{array}\right)^{1/3}}\\
&+\frac{\left(\begin{array}{c}-2d_1^3+9d_1e_1\lambda_2-27f_1\lambda_2^2\\+\sqrt{4\left(-d_1^2+3e_1\lambda_2\right)^3+\left(-2d_1^3+9d_1e_1\lambda_2-27f_1\lambda_2^2\right)}\end{array}\right)^{1/3}}{3\,(2)^{1/3}\lambda_2},\\
n_2 &= -\frac{d_1}{3\lambda_2}+\frac{\left(1+i\sqrt{3}\right)\left(-d_1^2+3e_1\lambda_2\right)}{3\,(2)^{2/3}\lambda_2\left(\begin{array}{c}-2d_1^3+9d_1e_1\lambda_2-27f_1\lambda_2^2\\+\sqrt{4\left(-d_1^2+3e_1\lambda_2\right)^3+\left(-2d_1^3+9d_1e_1\lambda_2-27f_1\lambda_2^2\right)}\end{array}\right)^{1/3}}\\
&-\frac{\left(1-i\sqrt{3}\right)\left(\begin{array}{c}-2d_1^3+9d_1e_1\lambda_2-27f_1\lambda_2^2\\+\sqrt{4\left(-d_1^2+3e_1\lambda_2\right)^3+\left(-2d_1^3+9d_1e_1\lambda_2-27f_1\lambda_2^2\right)}\end{array}\right)^{1/3}}{6\,(2)^{1/3}\lambda_2},\\
n_3 &= -\frac{d_1}{3\lambda_2}+\frac{\left(1-i\sqrt{3}\right)\left(-d_1^2+3e_1\lambda_2\right)}{3\,(2)^{2/3}\lambda_2\left(\begin{array}{c}-2d_1^3+9d_1e_1\lambda_2-27f_1\lambda_2^2\\+\sqrt{4\left(-d_1^2+3e_1\lambda_2\right)^3+\left(-2d_1^3+9d_1e_1\lambda_2-27f_1\lambda_2^2\right)}\end{array}\right)^{1/3}}\\
&-\frac{\left(1+i\sqrt{3}\right)\left(\begin{array}{c}-2d_1^3+9d_1e_1\lambda_2-27f_1\lambda_2^2\\+\sqrt{4\left(-d_1^2+3e_1\lambda_2\right)^3+\left(-2d_1^3+9d_1e_1\lambda_2-27f_1\lambda_2^2\right)}\end{array}\right)^{1/3}}{6\,(2)^{1/3}\lambda_2},
\end{aligned}$$

$$d_1=\lambda_1+\beta\lambda_n^2+\frac{\sigma B_0^2}{\rho}\lambda_2+\frac{\phi}{k}\beta,\ e_1=1+\alpha\lambda_n^2+\frac{\sigma B_0^2}{\rho}\lambda_1+\frac{\phi}{k}\alpha,\ f_1=\nu\lambda_n^2+\frac{\sigma B_0^2}{\rho}+\frac{\phi}{k}\nu,$$

$$E_9=\lambda_1+\beta\lambda_n^2+\frac{\sigma B_0^2}{\rho}\lambda_2+\frac{\phi}{k}\beta,\quad E_{10}=1+\alpha\lambda_n^2+\frac{\sigma B_0^2}{\rho}\lambda_1+\frac{\phi}{k}\alpha-\lambda_2\omega^2.$$

6.3.2 For $u(0,t)=U_0\cos\omega t$ and $\lambda_2=0$

we have

$$\begin{aligned}u_t(y,t) &= \frac{2}{d}U_0\nu\sum_{n=1}^{\infty}\frac{\lambda_n\begin{pmatrix} r_4\left(F_{11}-\omega^2\left(-F_4+\frac{\sigma B_0^2}{\rho}F_5+\frac{\nu\phi}{k}F_6\right)\right)\\ -\omega^2\left(F_7-\frac{\sigma B_0^2}{\rho}F_4\right)\end{pmatrix}e^{r_3t}\sin(\lambda_ny)}{(r_3-r_4)\left((F_{13}-\omega^2F_{14})^2+\omega^2F_{15}^2\right)}\\ &+\frac{2}{d}U_0\nu\sum_{n=1}^{\infty}\frac{\lambda_n\begin{pmatrix} r_3\left(F_{11}-\omega^2\left(-F_4+\frac{\sigma B_0^2}{\rho}F_5+\frac{\nu\phi}{k}F_6\right)\right)\\ -\omega^2\left(F_7-\frac{\sigma B_0^2}{\rho}F_4\right)\end{pmatrix}e^{r_4t}\sin(\lambda_ny)}{(r_4-r_3)\left((F_{13}-\omega^2F_{14})^2+\omega^2F_{15}^2\right)}\end{aligned} \quad (6.38)$$

$$\begin{aligned}u_s(y,t) &= \frac{2}{d}U_0\nu\omega\sin\omega t\sum_{n=1}^{\infty}\frac{\lambda_n\left(F_7-\frac{\sigma B_0^2}{\rho}F_4\right)\sin(\lambda_ny)}{\left((F_{13}-\omega^2F_{14})^2+\omega^2F_{15}^2\right)}\\ &+\frac{2}{d}U_0\nu\cos\omega t\sum_{n=1}^{\infty}\frac{\lambda_n\left(F_{11}-\omega^2\left(-F_4+\frac{\sigma B_0^2}{\rho}F_5+\frac{\nu\phi}{k}F_6\right)\right)\sin(\lambda_ny)}{\left((F_{13}-\omega^2F_{14})^2+\omega^2F_{15}^2\right)}.\end{aligned} \quad (6.39)$$

where

$$r_3=\frac{-F_{15}+\sqrt{F_{15}^2-4F_{13}F_{14}}}{2F_{14}},\quad r_4=\frac{-F_{15}-\sqrt{F_{15}^2-4F_{13}F_{14}}}{2F_{14}},$$

$$F_{13}=\nu\lambda_n^2+\frac{\sigma B_0^2}{\rho}+\frac{\nu\phi}{k},\quad F_{14}=\lambda_1+\beta\lambda_n^2+\frac{\phi}{k}\beta,\quad F_{15}=1+\alpha\lambda_n^2+\frac{\sigma B_0^2}{\rho}\lambda_1+\frac{\phi}{k}\alpha.$$

Note that the results of second grade fluid can be obtained by choosing $\lambda_1=\lambda_4=M=\phi=0$ and $\nu\lambda_3=\alpha_1/\rho$ in equation (6.39).

6.3.3 For $u(0,t)=U_0\sin\omega t$ and $\lambda_2\neq0$

we obtain

$$
\begin{aligned}
u_t(y,t) &= -\frac{2}{d}U_0\nu\omega\sum_{n=1}^{\infty}\frac{\lambda_n\left(\left(\omega^2-n_2n_3\right)F_2-\left(n_2+n_3\right)\left(F_{11}-\omega^2F_1\right)\right)e^{n_1t}\sin\left(\lambda_ny\right)}{\left(n_1-n_2\right)\left(n_1-n_3\right)F_{12}}\\
&+\frac{2}{d}U_0\nu\omega\sum_{n=1}^{\infty}\frac{\lambda_n\left(\left(n_1n_3-\omega^2\right)F_2+\left(n_1+n_3\right)\left(F_{11}-\omega^2F_1\right)\right)e^{n_2t}\sin\left(\lambda_ny\right)}{\left(n_2-n_1\right)\left(n_2-n_3\right)F_{12}}\\
&-\frac{2}{d}U_0\nu\omega\sum_{n=1}^{\infty}\frac{\lambda_n\left(\left(\omega^2-n_1n_2\right)F_2-\left(n_1+n_2\right)\left(F_{11}-\omega^2F_1\right)\right)e^{n_3t}\sin\left(\lambda_ny\right)}{\left(n_3-n_1\right)\left(n_3-n_2\right)F_{12}}
\end{aligned}
\tag{6.40}
$$

$$
\begin{aligned}
u_s(y,t) &= -\frac{2}{d}U_0\nu\omega\cos\omega t\sum_{n=1}^{\infty}\frac{\lambda_nF_2\sin\left(\lambda_ny\right)}{F_{12}}+\\
&\frac{2}{d}U_0\nu\sin\omega t\sum_{n=1}^{\infty}\frac{\lambda_n\left(F_{11}-\omega^2F_1\right)\sin\left(\lambda_ny\right)}{F_{12}}.
\end{aligned}
\tag{6.41}
$$

6.3.4 For $u(0,t)=U_0\sin\omega t$ and $\lambda_2=0$

we get

$$
\begin{aligned}
u_t(y,t) &= -\frac{2}{d}U_0\nu\omega\sum_{n=1}^{\infty}\frac{\begin{array}{c}\lambda_n\left(r_4\left(F_7-\frac{\sigma B_0^2}{\rho}F_4\right)+F_{11}-\omega^2\left(-F_4+\frac{\sigma B_0^2}{\rho}F_5+\frac{\nu\phi}{k}F_6\right)\right)\\ \times e^{r_3t}\sin\left(\lambda_ny\right)\end{array}}{\left(r_3-r_4\right)\left(\left(F_{13}-\omega^2F_{14}\right)^2+\omega^2F_{15}^2\right)}\\
&+\frac{2}{d}U_0\nu\omega\sum_{n=1}^{\infty}\frac{\begin{array}{c}\lambda_n\left(r_3\left(F_7-\frac{\sigma B_0^2}{\rho}F_4\right)+F_{11}-\omega^2\left(-F_4+\frac{\sigma B_0^2}{\rho}F_5+\frac{\nu\phi}{k}F_6\right)\right)\\ \times e^{r_4t}\sin\left(\lambda_ny\right)\end{array}}{\left(r_3-r_4\right)\left(\left(F_{13}-\omega^2F_{14}\right)^2+\omega^2F_{15}^2\right)}
\end{aligned}
\tag{6.42}
$$

$$
\begin{aligned}
u_s(y,t) &= -\frac{2}{d}U_0\nu\omega\cos\omega t\sum_{n=1}^{\infty}\frac{\lambda_n\left(F_7-\frac{\sigma B_0^2}{\rho}F_4\right)\sin\left(\lambda_ny\right)}{\left(\left(F_{13}-\omega^2F_{14}\right)^2+\omega^2F_{15}^2\right)}\\
&+\frac{2}{d}U_0\nu\sin\omega t\sum_{n=1}^{\infty}\frac{\lambda_n\left(F_{11}-\omega^2\left(-F_4+\frac{\sigma B_0^2}{\rho}F_5+\frac{\nu\phi}{k}F_6\right)\right)}{\left(\left(F_{13}-\omega^2F_{14}\right)^2+\omega^2F_{15}^2\right)}\sin\left(\lambda_ny\right).
\end{aligned}
\tag{6.43}
$$

The above equation gives the solution of second grade fluid [15] for $\lambda_1 = \lambda_4 = M = \phi = 0$ and $\nu\lambda_3 = \alpha_1/\rho$.

6.4 Time-periodic plane Poiseuille flow

In this section the flow between the two stationary plates is induced by an oscillating pressure gradient in the x-direction. Initially the fluid and plates are at rest. The pressure gradient is

$$\frac{\partial p}{\partial x} = -\rho Q \cos \omega t \quad \text{or} \quad \frac{\partial p}{\partial x} = -\rho Q \sin \omega t. \tag{6.44}$$

The flow is governed by equation (6.35) and

$$\begin{aligned} &\left(1+\lambda_1\frac{\partial}{\partial t}+\lambda_2\frac{\partial^2}{\partial t^2}\right)\frac{\partial u}{\partial t}+\left(1+\lambda_1\frac{\partial}{\partial t}+\lambda_2\frac{\partial^2}{\partial t^2}\right)\frac{\partial p}{\partial x} \\ =\;& \nu\left(1+\lambda_3\frac{\partial}{\partial t}+\lambda_3\frac{\partial^2}{\partial t^2}\right)\frac{\partial^2 u}{\partial y^2} \\ &-\frac{\sigma B_0^2}{\rho}\left(1+\lambda_1\frac{\partial}{\partial t}+\lambda_2\frac{\partial^2}{\partial t^2}\right)u-\frac{\nu\phi}{k}\left(1+\lambda_3\frac{\partial}{\partial t}+\lambda_3\frac{\partial^2}{\partial t^2}\right)u, \end{aligned} \tag{6.45}$$

$$u(0,t) = u(d,t) = 0; \quad t \in R. \tag{6.46}$$

The solutions here are given by

6.4.1 When $\frac{\partial p}{\partial x} = -\rho Q \cos \omega t$ and $\lambda_2 \neq 0$

then

$$\begin{aligned} u_t(y,t) =\;& -\frac{4}{d}Q\sum_{n=1}^{\infty}\frac{\left(\left(q_2q_3-\omega^2\right)F_{16}-\omega^2\left(q_2+q_3\right)F_{17}\right)e^{q_1t}\sin\left(\lambda_{2n-1}y\right)}{\left(q_1-q_2\right)\left(q_1-q_3\right)F_{18}\lambda_{2n-1}} \\ &+\frac{4}{d}Q\sum_{n=1}^{\infty}\frac{\left(\left(\omega^2-q_1q_3\right)F_{16}+\omega^2\left(q_1+q_3\right)F_{17}\right)e^{q_2t}\sin\left(\lambda_{2n-1}y\right)}{\left(q_2-q_1\right)\left(q_2-q_3\right)F_{18}\lambda_{2n-1}} \\ &-\frac{4}{d}Q\sum_{n=1}^{\infty}\frac{\left(\left(q_1q_2-\omega^2\right)F_{16}-\omega^2\left(q_1+q_2\right)F_{17}\right)e^{q_3t}\sin\left(\lambda_{2n-1}y\right)}{\left(q_3-q_1\right)q_3-q_2F_{18}\lambda_{2n-1}}, \end{aligned} \tag{6.47}$$

$$u_s(y,t) = \frac{4Q}{d}\cos\omega t \sum_{n=1}^{\infty} \frac{F_{16}\sin(\lambda_{2n-1}y)}{\lambda_{2n-1}F_{18}} + \frac{4\omega Q}{d}\sin\omega t \sum_{n=1}^{\infty} \frac{F_{17}\sin(\lambda_{2n-1}y)}{\lambda_{2n-1}F_{18}}, \tag{6.48}$$

where

$$\begin{aligned}
q_1 &= -\frac{d_2}{3\lambda_2} - \frac{2^{1/3}(-d_2^2+3e_2\lambda_2)}{3\lambda_2\left(\begin{array}{c}-2d_2^3+9d_2e_2\lambda_2-27f_2\lambda_2^2\\ +\sqrt{4\left(-d_2^2+3e_2\lambda_2\right)^3+\left(-2d_2^3+9d_2e_2\lambda_2-27f_2\lambda_2^2\right)}\end{array}\right)^{1/3}} \\
&+ \frac{\left(\begin{array}{c}-2d_2^3+9d_2e_2\lambda_2-27f_2\lambda_2^2\\ +\sqrt{4\left(-d_2^2+3e_2\lambda_2\right)^3+\left(-2d_2^3+9d_2e_2\lambda_2-27f_2\lambda_2^2\right)}\end{array}\right)^{1/3}}{3(2)^{1/3}\lambda_2}, \\
q_2 &= -\frac{d_2}{3\lambda_2} + \frac{(1+i\sqrt{3})\left(-d_2^2+3e_2\lambda_2\right)}{3(2)^{2/3}\lambda_2\left(\begin{array}{c}-2d_2^3+9d_2e_2\lambda_2-27f_2\lambda_2^2\\ +\sqrt{4\left(-d_2^2+3e_2\lambda_2\right)^3+\left(-2d_2^3+9d_2e_2\lambda_2-27f_2\lambda_2^2\right)}\end{array}\right)^{1/3}} \\
&- \frac{(1-i\sqrt{3})\left(\begin{array}{c}-2d_2^3+9d_2e_2\lambda_2-27f_2\lambda_2^2\\ +\sqrt{4\left(-d_2^2+3e_2\lambda_2\right)^3+\left(-2d_2^3+9d_2e_2\lambda_2-27f_2\lambda_2^2\right)}\end{array}\right)^{1/3}}{6(2)^{1/3}\lambda_2}, \\
q_3 &= -\frac{d_1}{3\lambda_2} + \frac{(1-i\sqrt{3})\left(-d_2^2+3e_2\lambda_2\right)}{3(2)^{2/3}\lambda_2\left(\begin{array}{c}-2d_2^3+9d_2e_2\lambda_2-27f_2\lambda_2^2\\ +\sqrt{4\left(-d_2^2+3e_2\lambda_2\right)^3+\left(-2d_2^3+9d_2e_1\lambda_2-27f_2\lambda_2^2\right)}\end{array}\right)^{1/3}} \\
&- \frac{(1+i\sqrt{3})\left(\begin{array}{c}-2d_2^3+9d_2e_2\lambda_2-27f_2\lambda_2^2\\ +\sqrt{4\left(-d_2^2+3e_2\lambda_2\right)^3+\left(-2d_2^3+9d_2e_2\lambda_2-27f_2\lambda_2^2\right)}\end{array}\right)^{1/3}}{6(2)^{1/3}\lambda_2},
\end{aligned} \tag{6.49}$$

$$\begin{aligned}
F_{16} &= \left(\nu\lambda_{2n-1}^2E_5 + \frac{\sigma B_0^2}{\rho}E_{11} + \frac{\nu\phi}{k}E_5\right),\ F_{17} = \left(\nu\lambda_{2n-1}^2E_2 + E_{11} + \frac{\nu\phi}{k}E_2\right), \\
F_{18} &= \left(\nu\lambda_{2n-1}^2 + \frac{\sigma B_0^2}{\rho} + \frac{\nu\phi}{k} - \omega^2E_{12}\right)^2 + \omega^2E_{13}^2
\end{aligned}$$

$$
\begin{aligned}
E_{11} &= \left(1-\omega^2\lambda_2\right)^2 - \lambda_1^2\omega^2,\ E_{12} = \left(\lambda_1 + \beta\lambda_{2n-1}^2 + \frac{\sigma B_0^2}{\rho}\lambda_2 + \frac{\phi}{k}\beta\right) \\
E_{13} &= \left(1 + \alpha\lambda_{2n-1}^2 + \frac{\sigma B_0^2}{\rho}\lambda_1 + \frac{\phi}{k}\alpha - \lambda_2\omega^2\right) \qquad (6.50)
\end{aligned}
$$

$$
\begin{aligned}
d_2 &= \lambda_1 + \beta\lambda_{2n-1}^2 + \frac{\sigma B_0^2}{\rho}\lambda_2 + \frac{\phi}{k}\beta,\ e_2 = 1 + \alpha\lambda_{2n-1}^2 + \frac{\sigma B_0^2}{\rho}\lambda_1 + \frac{\phi}{k}\alpha, \\
f_2 &= \nu\lambda_{2n-1}^2 + \frac{\sigma B_0^2}{\rho} + \frac{\phi}{k}\nu. \qquad (6.51)
\end{aligned}
$$

6.4.2 When $\frac{\partial p}{\partial x} = -\rho Q\cos\omega t$ and $\lambda_2 = 0$

then

$$
\begin{aligned}
u_t(y,t) &= \frac{4}{d}Q\sum_{n=1}^{\infty}\frac{\begin{pmatrix} r_6\left(\nu\lambda_{2n-1}^2F_7 + \frac{\sigma B_0^2}{\rho}F_{19} + \frac{\nu\phi}{k}F_7\right) \\ -\omega^2\left(\nu\lambda_{2n-1}^2F_4 + F_{19} + \frac{\nu\phi}{k}F_4\right)\end{pmatrix} e^{r_5 t}\sin(\lambda_{2n-1}y)}{(r_5-r_6)\left((F_{20}-\omega^2F_{21})^2 + \omega^2F_{22}^2\right)\lambda_{2n-1}} \\
&+ \frac{4}{d}Q\sum_{n=1}^{\infty}\frac{\begin{pmatrix} r_5\left(\nu\lambda_{2n-1}^2F_7 + \frac{\sigma B_0^2}{\rho}F_{19} + \frac{\nu\phi}{k}F_7\right) \\ -\omega^2\left(\nu\lambda_{2n-1}^2F_4 + F_{19} + \frac{\nu\phi}{k}F_4\right)\end{pmatrix} e^{r_6 t}\sin(\lambda_{2n-1}y)}{(r_6-r_5)\left((F_{20}-\omega^2F_{21})^2 + \omega^2F_{22}^2\right)\lambda_{2n-1}} \qquad (6.52)
\end{aligned}
$$

$$
\begin{aligned}
u_s(y,t) &= \frac{4Q}{d}\cos\omega t\sum_{n=1}^{\infty}\frac{\left(\nu\lambda_{2n-1}^2F_7 + \frac{\sigma B_0^2}{\rho}F_{19} + \frac{\nu\phi}{k}F_7\right)}{\lambda_{2n-1}\left((F_{20}-\omega^2F_{21})^2 + \omega^2F_{22}^2\right)}\sin(\lambda_{2n-1}y) \\
&+ \frac{4\omega Q}{d}\sin\omega t\sum_{n=1}^{\infty}\frac{\left(\nu\lambda_{2n-1}F_4 + F_{19} + \frac{\nu\phi}{k}F_4\right)\sin(\lambda_{2n-1}y)}{\lambda_{2n-1}\left((F_{20}-\omega^2F_{21})^2 + \omega^2F_{22}^2\right)}, \qquad (6.53)
\end{aligned}
$$

where

$$
r_5 = \frac{-F_{22} + \sqrt{F_{22}^2 - 4F_{21}F_{20}}}{2F_{21}},\ r_6 = \frac{-F_{22} - \sqrt{F_{22}^2 - 4F_{21}F_{20}}}{2F_{21}},
$$

$$F_{19} = 1+\lambda_1^2\omega^2,\ F_{20} = \nu\lambda_{2n-1}^2 + \frac{\sigma B_0^2}{\rho} + \frac{\nu\phi}{k},$$
$$F_{21} = \lambda_1 + \beta\lambda_{2n-1}^2 + \frac{\phi}{k}\beta,\ F_{22} = 1 + \alpha\lambda_{2n-1}^2 + \frac{\sigma B_0^2}{\rho}\lambda_1 + \frac{\phi}{k}\alpha.$$

The solution of second grade fluid [15] can be deduced from equation (6.53) by taking $\lambda_1 = \lambda_4 = M = \phi = 0$ and $\nu\lambda_3 = \alpha_1/\rho$.

6.4.3 For $\frac{\partial p}{\partial x} = -\rho Q \sin\omega t$ and $\lambda_2 \neq 0$

we have

$$\begin{aligned} u_t(y,t) &= -\frac{4}{d}\omega Q\sum_{n=1}^{\infty}\frac{\left(\left(\omega^2 - q_2q_3\right)F_{17} - \left(q_2+q_3\right)F_{16}\right)e^{q_1t}\sin\left(\lambda_{2n-1}y\right)}{\left(q_1-q_2\right)\left(q_1-q_3\right)F_{18}\lambda_{2n-1}} \\ &+\frac{4}{d}\omega Q\sum_{n=1}^{\infty}\frac{\left(\left(q_2q_3 - \omega^2\right)F_{17} - \left(q_2+q_3\right)F_{16}\right)e^{q_2t}\sin\left(\lambda_{2n-1}y\right)}{\left(q_2-q_1\right)\left(q_2-q_3\right)F_{18}\lambda_{2n-1}} \\ &-\frac{4}{d}\omega Q\sum_{n=1}^{\infty}\frac{\left(\left(\omega^2 - q_1q_2\right)F_{17} - \left(q_1+q_2\right)F_{16}\right)e^{q_1t}\sin\left(\lambda_{2n-1}y\right)}{\left(q_3-q_1\right)\left(q_3-q_1\right)F_{18}\lambda_{2n-1}}, \end{aligned} \quad (6.54)$$

$$u_s(y,t) = \frac{4Q}{d}\sin\omega t\sum_{n=1}^{\infty}\frac{F_{16}\sin\left(\lambda_{2n-1}y\right)}{\lambda_{2n-1}F_{18}} - \frac{4\omega Q}{d}\cos\omega t\sum_{n=1}^{\infty}\frac{F_{17}\sin\left(\lambda_{2n-1}y\right)}{\lambda_{2n-1}F_{18}}. \quad (6.55)$$

6.4.4 For $\frac{\partial p}{\partial x} = -\rho Q \sin\omega t$ and $\lambda_2 = 0$

we get

$$
\begin{aligned}
u_t(y,t) &= -\frac{4}{d}\omega Q\sum_{n=1}^{\infty}\frac{\left(\begin{array}{c} r_6\left(\nu\lambda_{2n-1}^2F_4+F_{19}+\frac{\nu\phi}{k}F_4\right)\\ +\left(\nu\lambda_{2n-1}^2F_7+\frac{\sigma B_0^2}{\rho}F_{19}+\frac{\nu\phi}{k}F_7\right)\end{array}\right)e^{r_5t}\sin\left(\lambda_{2n-1}y\right)}{\left(r_5-r_6\right)\left(\left(F_{20}-\omega^2F_{21}\right)^2+\omega^2F_{22}^2\right)\lambda_{2n-1}}\\
&+\frac{4}{d}\omega Q\sum_{n=1}^{\infty}\frac{\left(\begin{array}{c} r_5\left(\nu\lambda_{2n-1}^2F_4+F_{19}+\frac{\nu\phi}{k}F_4\right)\\ +\left(\nu\lambda_{2n-1}^2F_7+\frac{\sigma B_0^2}{\rho}F_{19}+\frac{\nu\phi}{k}F_7\right)\end{array}\right)e^{r_6t}\sin\left(\lambda_{2n-1}y\right)}{\left(r_5-r_6\right)\left(\left(F_{20}-\omega^2F_{21}\right)^2+\omega^2F_{22}^2\right)\lambda_{2n-1}}, \qquad (6.56)
\end{aligned}
$$

$$
\begin{aligned}
u_s(y,t) &= \frac{4Q}{d}\sin\omega t\sum_{n=1}^{\infty}\frac{\left(\nu\lambda_{2n-1}^2F_7+\frac{\sigma B_0^2}{\rho}F_{19}+\frac{\nu\phi}{k}F_7\right)\sin\left(\lambda_{2n-1}y\right)}{\lambda_{2n-1}\left(\left(F_{20}-\omega^2F_{21}\right)^2+\omega^2F_{22}^2\right)}\\
&-\frac{4\omega Q}{d}\cos\omega t\sum_{n=1}^{\infty}\frac{\left(\nu\lambda_{2n-1}^2F_4+F_{19}+\frac{\nu\phi}{k}F_4\right)\sin\left(\lambda_{2n-1}y\right)}{\lambda_{2n-1}\left(\left(F_{20}-\omega^2F_{21}\right)^2+\omega^2F_{22}^2\right)}. \qquad (6.57)
\end{aligned}
$$

The above solutions yield the results of second grade fluid [15] when $\lambda_1=\lambda_4=M=\phi=0$ and $\nu\lambda_3=\alpha_1/\rho$.

6.5 Results and discussion

In this section we discuss the graphical results of velocity profiles due to the oscillations of the plate at $t>0$. The difference between velocity profiles of Oldroyd-B fluid and generalized Burgers' fluid is shown for different values of M and K.

Figure 6.1 is constructed to describe the effects of M on the velocity profiles in two fluid cases. It is evident from this figure that the oscillations rapidly increase for the velocity profiles in Oldroyd-B fluid and generalized Burgers' fluids by increasing M. The influence of magnetic field are more prominent on the velocity profiles in generalized Burgers' fluid when compared with an Oldroyd -B fluid.

Figure 6.2 elucidates to show the influence of K on the velocity profile in the presence of magnetic field parameter M . By increasing K the velocity profiles for both fluids increase. However the effects of K on the velocity are more prominent in an Oldroyd-B fluid when compared with generalized Burgers' fluid.

Figure 6.3 is displayed for the variation of K in the absence of M . It can be seen from this figure that the velocity also increases for both cases. This effect is more prominent in generalized Burgers' fluid than that of Oldroyd-B fluid.

The comparison of steady state solution u_s in various fluid models is shown in table 1. Clearly the value of u_s in Oldroyd-B fluid is greatest and smallest in viscous fluid when $M \neq 0$ and $K \neq 0$. Also the value of u_s in Maxwell fluid is greater than that of second grade and generalized Burgers' fluids. However, u_s is maximum for hydrodynamic second grade fluid and minimum in hydrodynamic generalized Burgers fluid when permeability of the porous medium is very very large. In this case, u_s for Maxwell fluid is large when compared with Newtonian and Oldroyd-B fluids. For $M = 0$ and $K \neq 0$ the behavior of u_s is similar to the case of $M \neq 0$ and $K \neq 0$. However, it is found that u_s for $M = 0$, $K \neq 0$ is large for all fluids except an Oldroyd-B fluid when compared with $M \neq 0$, $K \neq 0$. For $M = 1$ and $K \to \infty$, the behavior of u_s is similar to that of $M = 0$ and $K \to \infty$. But u_s for Oldroyd-B fluid is greatest than that of Newtonian and Maxwell fluids.

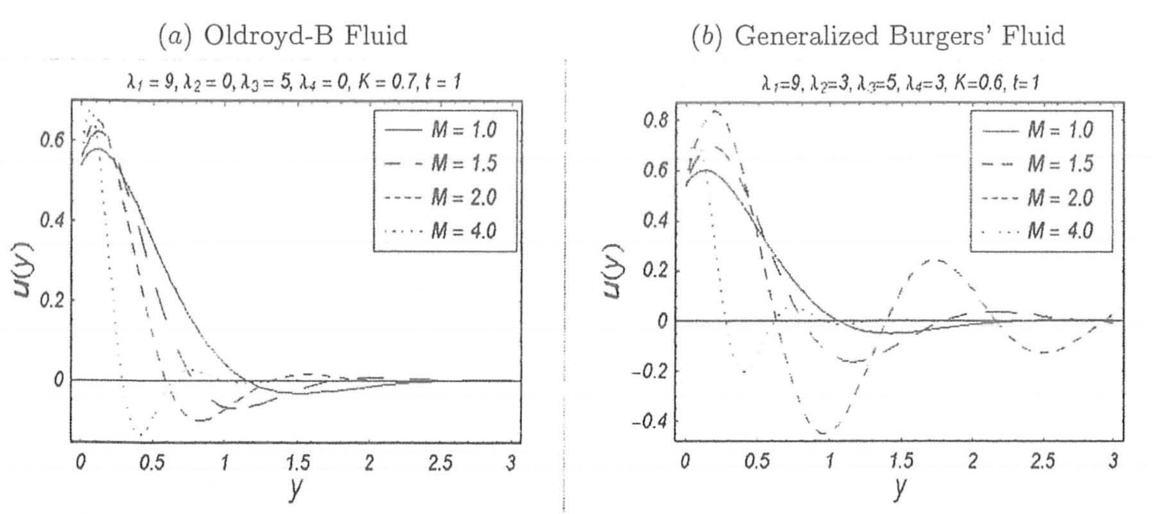

Figure 6.1: Profiles of normalized steady state velocity $u\,(y)$ for various values of M.

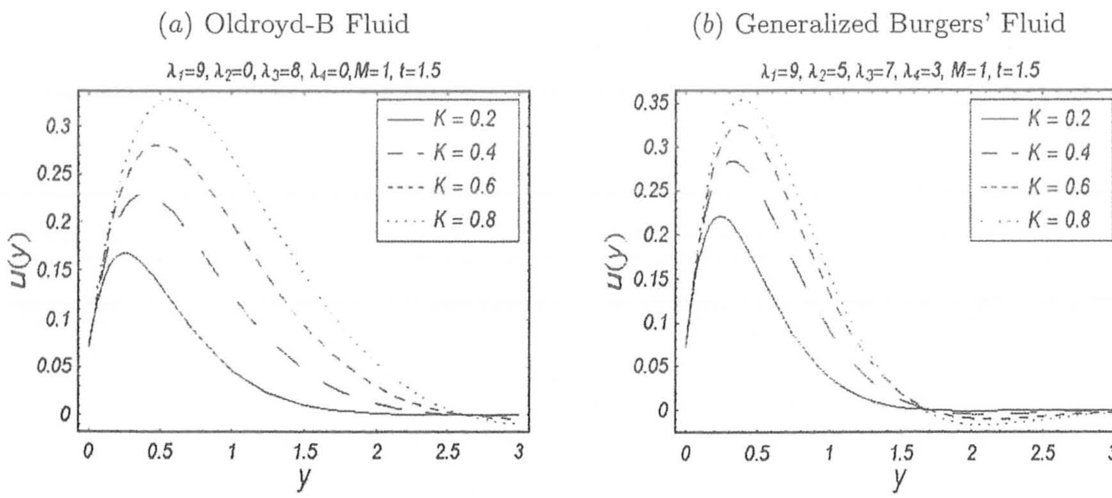

Figure 6.2: Profiles of normalized steady state velocity $u(y)$ for various values of K.

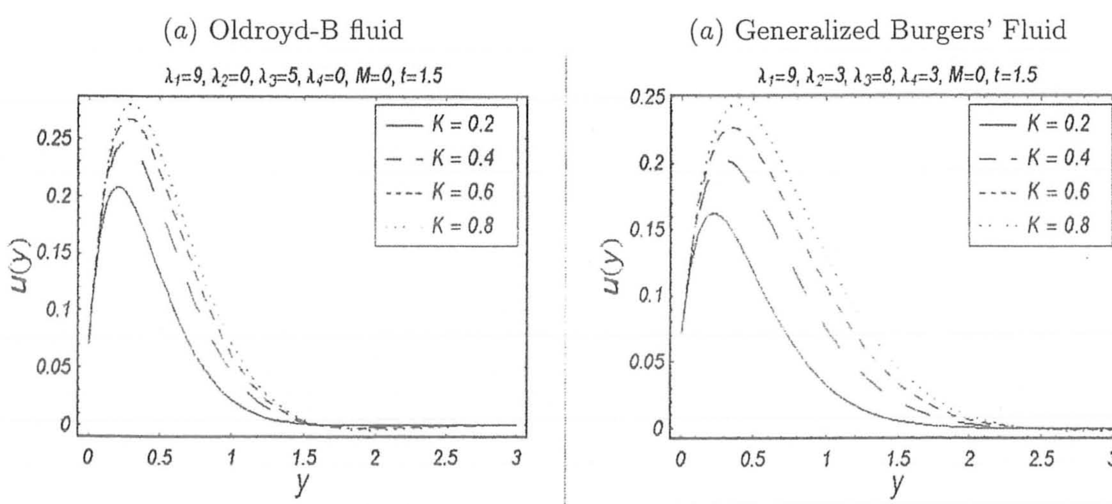

Figure 6.3: Profiles of normalized steady state velocity $u(y)$ for values of K and $M = 0$.

Type of Fluid	material constants	u_s for M=1, K=0.4	u_s for M=0, K→ ∞	u_s for M=0, K=0.4	u_s for M=1, K→ ∞
Newtonian	λ_i=0,i=1,2,3,4	0.131357	0.32771	0.158758	0.227262
Second Grade	$\lambda_1=\lambda_2=\lambda_4=0,\lambda_3=1.5$	0.156116	0.542339	0.184418	0.726136
Maxwell	$\lambda_2=\lambda_3=\lambda_4=0,\lambda_1=1$	0.195815	0.488669	0.198766	0.322845
Oldroyd-B	$\lambda_2=\lambda_4=0,\lambda_1=1,\lambda_3=1.5$	0.333288	0.338183	0.209639	0.444708
G.Burgers/	$\lambda_1=1,\lambda_2=2,\lambda_3=1.5,\lambda_4=3$	0.136538	0.301306	0.186846	0.203335

Table 1: Comparison of velocity in different fluids when $t = 1.5$ and $y = 0.5$

6.6 Final remarks

Mathematical modelling for MHD flow of generalized Burgers' fluid is given in a porous medium. The exact solutions for three flow problems are developed. The results for various fluids in a porous space can be obtained as the special cases of the present analysis by choosing appropriate values to the involved parameters. The comparison of the steady state velocity has been shown for five fluids. The existing results of second grade fluid [15] can be deduced by selecting $\lambda_1 = \lambda_2 = \lambda_4 = M = \phi = 0$ and $\nu\lambda_3 = \alpha_1/\rho$.

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