Analytical solution for MHD Peristaltic flow by an Adomian decomposition method

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A Dissertation Submitted in the Partial Fulfillment of the Requirements for the Degree of

MASTER OF PHILOSOPHY

IN

MATHEMATICS

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CERTIFICATE

A DISSERTATION SUBMITTED IN THE PARTIAL FULFILLMENT OF THE REQUIREMENTS FOR THE DEGREE OF THE MASTER OF PHILOSOPHY

We accept this dissertation as conforming to the required standard

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2008

Dedicated to
My Supervisor
Dr. Tasawar Hayat

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I filled with praise and glory to **Almighty Allah**, the sole creator and sustainer of the Universe to give me the strength, fortitude and abilities to learn. Countless daroods and salaam upon **Prophet Mohammad (P.B.U.H)** who is the source of knowledge, enlightenment guidance and shower of blessings for entire creation.

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Preface

Peristaltic motion is a form of fluid transport induced by a progressive wave of area contraction or expansion along the length of a distensible tube containing fluid. The investigations of peristaltic flows have not only promising applications in medical and engineering sciences but also are important in view of the interesting mathematical features offered by the equations governing the flow. In particular the peristaltic flows occur in the urine transport from kidney to the bladder through the ureter, chyme movement in the gastrointestinal tract, movement of ovum in the fallopian tube and in the vasomotion of small blood vessels such as arterioles, venules and capillaries. The roller and finger pumps also operate under the principle of peristaltic activity. Due to these facts various workers [1-10] are involved in the investigations of peristaltic flows.

It is common in the literature that there are situations where the fluid motion cannot be described by using the Navier-Stokes equations. The Navier-Stokes theory is inadequate when fluids of high molecular weights are considered. Such fluid belongs to the category of non-Newtonian fluid mechanics. Unlike the Newtonian fluids there is not a single constitutive equation by which one can predict the rheological properties of all non-Newtonian fluids. In view of this fact several constitutive equations of non-Newtonian fluids have been proposed. Amongst these there is one subclass of differential type fluids known as the third grade which we consider in this dissertation. The layout of this dissertation is as follows.

In chapter one we present relevant concepts of basic flows and fluids, continuity and linear momentum equations and the brief outlines of perturbation and Adomian decomposition methods.

Chapter two discusses the MHD peristaltic flow of a third grade fluid in a planar channel. Perturbation solution of the resulting non-linear problem is obtained. The obtained solution holds for small values of the Deborah number. The analysis of this chapter is review of a recent work by Hayat et al [11].

In chapter three we construct the solution of the problem considered in chapter two by Adomian decomposition method (ADM). The ADM solution is valid for all values of the Deborah number. Furthermore a comparison between perturbation and ADM solutions is given. The influence of pertinent flow parameters is seen by plotting graphs.

Contents

Chapter 1

Introduction

This chapter includes some basic definitions regarding fluids and flows. Laws of conservation of mass and linear momentum are also presented. Furthermore basic ideas of perturbation method and Adomian decomposition method (ADM) are highlighted.

1.1 Density

It is defined as mass per unit volume. For a homogeneous substance the density (ρ) in mathematical form can be expressed as

$$\rho = \frac{m}{V_1}, \tag{1.1}$$

where density ρ is measured in Kg/m^3, m is the mass of the substance measured in Kg and V_1 is the volume of the substance measured in meter cube. Density of a substance can be changed by changing either the pressure or the temperature. By increasing the pressure will always increase the density of the material but an increase in temperature causes a decrease in the density.

1.2 Pressure

The magnitude of force per unit area is known as pressure. It is written as

$$P = \frac{F}{A}, \tag{1.2}$$

in whish P is the pressure, F is magnitude of the normal force and A is the area..Pressure is a scalar quantity and has SI units of Pascals (Pa) i.e

$$1Pa = \frac{1N}{m^2}. \tag{1.3}$$

1.3 Viscosity

It is defined as" the ratio of shear stress to the rate of shear strain". Mathematical expression of viscosity is

$$Vis\cos ity = \frac{shear\ stress}{rate\ of\ shear\ strain}, \tag{1.4}$$

or for one-dimensional flow

$$\mu = \frac{T_{xy}}{du/dy}. \tag{1.5}$$

Viscosity plays an important role in experimental and mathematical analysis regarding flow, and has dimension

$$[\mu] = \left[\frac{M}{LT}\right]. \tag{1.6}$$

It is also known as dynamic viscosity.

1.3.1 Kinematic viscosity.

The ratio of dynamic viscosity (μ) to mass density (ρ) is called kinematic viscosity. It is denoted by ν and expressed in mathematical form as

$$\nu = \frac{\mu}{\rho}. \tag{1.7}$$

The viscosity of a fluid is highly dependent upon temperature. For liquids the kinematic viscosity decreases by increasing temperature and for gasses the kinematic viscosity increases with an increase in temperature.

1.4 Types of forces

Here we will define the surface and body forces.

1.4.1 Surface force

A force that acts across an internal or external surface element in a material body is called surface force. Surface force can be decomposed into two perpendicular components called pressure and stress force. Pressure force acts normally over an area while stress forces act tangentially over an area. Surface force is given by

$$f_s = P.A \tag{1.8}$$

where P is the pressure and A is the cross sectional area of the moving fluid. Frictional force is also an example of surface force.

1.4.2 Body force

A force that acts on the volume of a body is called body force. It can also be defined as an external force acting throughout the mass of a body. Simply we can say that it is a force which does not have direct contact with the body. The units of a body force are force per unit volume and can be expressed as

$$f = \frac{dF}{dV_1}. \tag{1.9}$$

In above equation F is the force and V_1 is the volume. Example of common body forces are gravity, and electromagnetic force .

1.5 Stress

A force that tends to deform a body is called stress. We can define stress as " Force per unit area".

$$Stress = \frac{Force}{Area}. \tag{1.10}$$

1.5.1 Shear stress

The component of stress parallel or tangential to a face of the material as opposed to normal stress when the stress is perpendicular to the face is called shear stress. Mathematically it is expressed by the following relations

$$\tau_n = Limt_{\delta A_n \to 0} \frac{\delta F_t}{\delta A_n}. \tag{1.11}$$

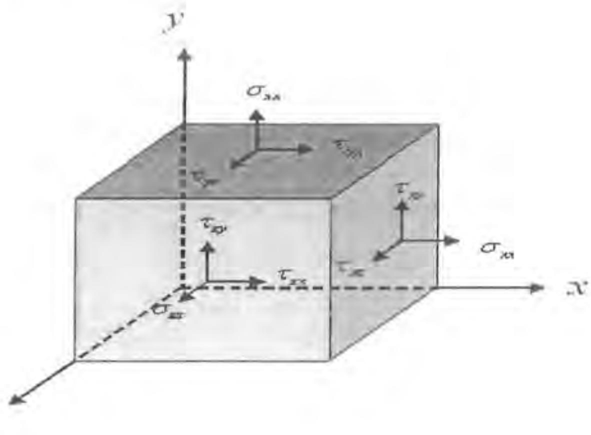
1.5.2 Normal stress

The component of stress perpendicular to the face of material are known as normal stress. Mathematically we write it as

$$\sigma_n = Limt_{\delta A_n \to 0} \frac{\delta F_n}{\delta A_n}. \tag{1.12}$$

The subscript "n" on the stress is included as a reminder that the stresses are associated with the surface δA, having an outward normal in the "n" direction. In rectangular coordinates we might consider the stresses acting on the planes whose outward drawn normal are in the x,y or z direction.

1.6 Stress tensor



The stress at a point can be specified by nine components (as shown in the above figure) and can be written in the form

$$\mathbf{T} = \begin{bmatrix} \sigma_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_{zz} \end{bmatrix} \tag{1.13}$$

1.7 Flow

Deformation in a body occurs when different forces act upon it. If this deformation increases continuously without a limit then this phenomenon is called a flow.

1.8 Types of flows

1.8.1 Steady flow

If the rate of the fluid that flows is constant then it is called steady flow. In such type of flow velocity, pressure and density of the particle at each point of the flow is independent of time. Mathematically we can write

$$\frac{\partial}{\partial t}(\mathbf{V}) = \frac{\partial}{\partial t}(p) = \frac{\partial}{\partial t}(\varrho) = 0. \tag{1.14}$$

1.8.2 Unsteady flow

A flow in which the fluid property at any time is time dependent, then it is called unsteady flow. For such flow one may write

$$\frac{\partial}{\partial t}(\eta) \neq 0, \tag{1.15}$$

where η is any fluid property.

1.8.3 Compressible flow

A flow in which the density of the fluid particle is not constant is called compressible flow. For a compressible flow

$$\frac{d}{dt}(\varrho) \neq 0, \tag{1.16}$$

where $\frac{d}{dt}$ is the material derivative.

1.8.4 Incompressible flow

If the density of the fluid particle is constant then the flow is called incompressible flow. For incompressible flow

$$\frac{d}{dt}(\varrho) = 0. \tag{1.17}$$

1.8.5 Creeping flow

When the motion of the fluid is very slow and is dominated by the viscous forces is called creeping flow. In such type of flow the Reynold number

$$R = \frac{\varrho\mu}{c} \tag{1.18}$$

is very small i.e $R << 1$. For such flow the inertia effects are ignored in comparison to the viscous resistance. Creeping flow at zero Reynolds number is knows as Stokes flow. The flow of ground water and oil through small channels or cracks are examples of creeping flow.

1.8.6 Turbulent flow

A flow (*gas or liquid*) in which the fluid undergoes irregular fluctuation or mixing is called turbulent flow. In such type of flow the speed of the fluid particle at a point changes continuously (both in magnitude and direction).The flow of blood in arteries, oil transport in pipe lines, flow of lava, ocean currents, the flow through turbines, the flow of winds, the flow in boat wakes and around aircraft wingtips are examples of turbulent flow.

1.8.7 Laminar flow

A flow (*gas or liquid*) in which the fluid travels smoothly or in regular paths, with no disruption between the layers is called laminar flow. In such type of flow each liquid particle has definite path and each individual particle do not cross each other. In laminar flow the velocity, pressure and other flow properties at each point in the fluid remains constant. Laminar flow occurs when the flow channel is relatively small. The fluid is moving slowly and its viscosity is relatively high. Oil flow through a thin tube or blood flow through capillaries are examples of laminar flow.

1.8.8 Rotational flow

When the fluid flows in such a way that the curl of the fluid velocity is not zero, then it is called rotational flow. Such flow may be expressed as

$$\omega = \nabla \times \mathbf{V} \neq 0, \tag{1.19}$$

where ω is the vorticity.

1.8.9 Irrotational flow

It is a type of flow of fluid in which the curl of the velocity vector is zero every where so that the circulation of the velocity about any closed curve vanishes. It is also called acyclic motion of fluid. For irrotational flow we can write

$$\omega = \nabla \times \mathbf{V} = 0. \tag{1.20}$$

1.9 Fluid

A substance deforming continuously when subjected to a shear stress regardless of how small the shear stress may be is known as fluid. This means that fluid will *flow* when subjected to a shear stress. Liquid, gases, plasma and to some extent plastic solids are examples of fluid.

1.10 Types of fluid

1.10.1 Ideal fluid

An ideal fluid is one that is incompressible and has no viscosity i.e $\mu = 0$.These fluids do not offer any resistance to the shear forces and hence does not exist in nature.

1.10.2 Real fluid

Those fluid in which viscosity is not equal to zero are known as real fluids. Such fluids give rise to fluid friction. These fluids appose the sliding of one particle past another. Real fluids are divided into two classes namely Newtonian fluid and non-Newtonian fluid.

1.10.3 Newtonian fluid

A fluid for which shear stress at each point is linearly proportional to its strain rate at that point is called Newtonian fluid. This concept was first deduced by Isaac Newton and is directly analogous to Hooke's law. For such types of fluid, the graph between shear stress and rate of deformation is a straight line. Mathematical expression of Newtonian fluid is given by

$$\tau_{yx} \propto \frac{du}{dy},$$

or

$$\tau_{yx} = \mu \frac{du}{dy}, \tag{1.21}$$

where τ_{xy} is the shear stress exerted by the fluid, μ is the constant of proportionality known as dynamic viscosity and $\frac{du}{dy}$ is the velocity gradient perpendicular to the direction of shear stress for a unidirectional and one dimensional flow. For two dimensional flow we can write

$$\tau_{yx} = \mu \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right). \tag{1.22}$$

For Newtonian fluid the viscosity depends only on temperature and pressure but not on the force acting upon it. Water, air, benzene, ethyl alcohol and solution of low molecular weight are some examples of Newtonian fluids.

1.10.4 Non-Newtonian fluid

These fluids do not obey the Newton's law of viscosity. For such fluids shear stress is directly and non-linearly proportional to the rate of strain. For such fluids the following expression holds

$$\tau_{yx} \propto \left(\frac{du}{dy} \right)^n, \quad n \neq 1, \tag{1.23}$$

or

$$\tau_{yx} = k \left(\frac{du}{dy} \right)^n, \tag{1.24}$$

or

$$\tau_{yx} = k \left(\frac{du}{dy} \right)^{n-1} \left(\frac{du}{dy} \right), \tag{1.25}$$

or

$$\tau_{yx} = \eta \left(\frac{du}{dy} \right), \tag{1.26}$$

whence

$$\eta = k \left(\frac{du}{dy} \right)^{n-1}. \tag{1.27}$$

Here η is called the apparent viscosity and n is the flow index . The above relation reduces to Newton's law of viscosity when $n = 1$ and $k = \mu$. Many industrial and biological fluids are non-Newtonian in their flow characteristics and are referred as rheological fluids. China clay, coal in water, oil-water emulsions, butter, paint, shampoo, blood, jams, jellies, soup and marmalades are examples of non-Newtonian fluids.

1.11 Constitutive equations

It is an expression which relates the stress components to the rate of shear strain or stress to the motion of the fluid. In general

$$\mathbf{T} = -P\mathbf{I} + \mathbf{S}. \tag{1.28}$$

Where $\mathbf{T}$ is the Cauchy stress tensor, P is the pressure, $\mathbf{I}$ is the identity tensor and $\mathbf{S}$ is the extra stress tensor. It is a common fact that all the non-Newtonian fluids cannot be described by a single constitutive equation. For different non-Newtonian fluids, various constitutive equations have been suggested in the literature. In this dissertation we will use the constitutive equation of a third grade fluid.

1.12 Law of conservation of mass (*continuity equation*)

Let us consider a specific mass of fluid in a control volume $\widetilde{V}$ which is arbitrarily chosen and whose surface $\widetilde{S}$ remain fixed in space. When the fluid flows, its size and shape changes but its mass remains unchanged. This is the principle of mass conservation which states that mass cannot be created or destroyed inside the control volume and is conserved at all time. Mathematically one can write

$$\frac{d}{dt}\int_{\widetilde{V}} \rho d\widetilde{V} = 0, \tag{1.29}$$

where ρ is the density of the fluid. Using Reynold's transport theorem one has

$$\frac{d}{dt}\int_{\widetilde{V}} \alpha d\widetilde{V} = \int_{\widetilde{V}} \left(\frac{\partial \alpha}{\partial t} + \operatorname{div}(\alpha \mathbf{V})\right) d\widetilde{V}, \tag{1.30}$$

where $\mathbf{V}$ is the velocity. By setting $\alpha = \rho$ and using Eq. (1.30) in Eq. (1.29) one gets

$$\frac{d}{dt}\int_{\widetilde{V}} \rho d\widetilde{V} = \int_{\widetilde{V}} \left(\frac{\partial \rho}{\partial t} + \operatorname{div}(\rho \mathbf{V})\right) d\widetilde{V} = 0. \tag{1.31}$$

Since the control volume $\widetilde{V}$ is arbitrary, so we get

$$\frac{\partial \rho}{\partial t} + \operatorname{div}(\rho \mathbf{V}) = 0. \tag{1.32}$$

Which is the continuity equation for unsteady compressible fluid.

1.12.1 Continuity equation for incompressible flow

For incompressible fluid we take ρ =constant no matter that the flow is steady or unsteady and therefore Eq. (1.32) reduces to

$$\nabla \cdot \mathbf{V} = 0. \tag{1.33}$$

1.12.2 Continuity equation in cylindrical coordinates

Here the continuity equation is of the form

$$\frac{\partial \rho}{\partial t} + \frac{1}{r}\frac{\partial}{\partial r}(r\rho V_r) + \frac{1}{r}\left(\frac{\partial}{\partial \theta}(\rho V_\theta)\right) + \frac{\partial}{\partial z}(\rho V_z) = 0, \tag{1.34}$$

which for an incompressible fluid becomes

$$\frac{1}{r}\frac{\partial}{\partial r}(rV_r) + \frac{1}{r}\left(\frac{\partial}{\partial \theta}V_\theta\right) + \frac{\partial}{\partial z}V_z = 0. \tag{1.35}$$

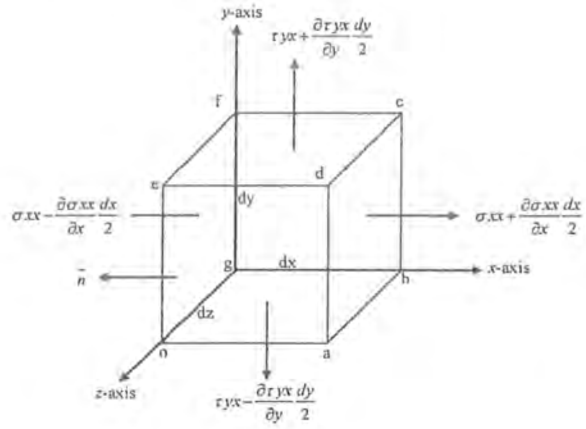
1.13 Law of conservation of momentum

According to Newton's second law of motion " The time rate of change of linear momentum is equal to the net force". To derive the differential form of the momentum equation, we shall apply Newton's second law to an infinitesimal fluid particle of mass dm i.e

$$d\mathbf{F} = dm\frac{d\mathbf{V}}{dt}, \tag{1.36}$$

$$d\mathbf{F} = dm\left[\frac{\partial \mathbf{V}}{\partial t} + (\mathbf{V}\cdot\nabla)\mathbf{V}\right], \tag{1.37}$$

where $\frac{d}{dt}$ is called the material derivative and $d\mathbf{F}$ is the net force acting on the infinitesimal system. Since the force acting on the system is classified as a body force and surface force including both normal forces and tangential (*shear*) forces. By considering the $x-component$ of the force, acting on a differential element of mass dm and volume dV_1 only those stresses that act in the $x - direction$ will give rise to the surface forces in the $x - direction$.



If the stresses at the center of the differential element are taken to be σ_{xx}, τ_{yx} and τ_{zx} where σ_{xx} is the normal stress, τ_{yx} and τ_{zx} are the tangential (*shear*) stresses, then the net stress force acting in the $x - direction$ is

$$\begin{aligned} dF_{sx} &= \left[\left(\sigma_{xx} + \frac{\partial \sigma_{xx}}{\partial x}\frac{dx}{2}\right) - \left(\sigma_{xx} - \frac{\partial \sigma_{xx}}{\partial x}\frac{dx}{2}\right)\right] dydz \\ &+ \left[\left(\tau_{yx} + \frac{\partial \tau_{yx}}{\partial y}\frac{dy}{2}\right) - \left(\tau_{yx} - \frac{\partial \tau_{yx}}{\partial y}\frac{dy}{2}\right)\right] dxdz \\ &+ \left[\left(\tau_{zx} + \frac{\partial \tau_{zx}}{\partial z}\frac{dz}{2}\right) - \left(\tau_{zx} - \frac{\partial \tau_{zx}}{\partial z}\frac{dz}{2}\right)\right] dxdy, \end{aligned} \tag{1.38}$$

which upon simplification gives

$$dF_{sx} = \left(\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z}\right) dxdydz. \tag{1.39}$$

Let ρB_x be the body force per unit volume in the $x - direction$ then the net force (dF_x) in the $x - direction$ is

$$dF_x = dF_{sx} + dF_{Bx}, \tag{1.40}$$

or

$$dF_x = \left(\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z}\right) dxdydz + (\rho B_x)\, dxdydz,$$

or

$$dF_x = \left(\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} + \rho B_x \right) dxdydz. \tag{1.41}$$

Similarly the net forces in the y and z-directions are

$$dF_y = dF_{sy} + dF_{By}, \tag{1.42}$$

or

$$dF_y = \left(\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \tau_{zy}}{\partial z} + \rho B_y \right) dxdydz, \tag{1.43}$$

$$dF_z = \left(\frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \sigma_{zz}}{\partial z} + \rho B_z \right) dxdydz. \tag{1.44}$$

From Eqs. (1.36), (1.41), (1.43) and (1.44) we can write

$$\rho \frac{du}{dt} = \frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} + \rho B_x, \tag{1.45}$$

$$\rho \frac{dv}{dt} = \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \tau_{zy}}{\partial z} + \rho B_y, \tag{1.46}$$

$$\rho \frac{d\omega}{dt} = \frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \sigma_{zz}}{\partial z} + \rho B_z. \tag{1.47}$$

These are the equation of motion for any fluid satisfying the continuum assumption. In vector form the above equations can be combined into the following expression

$$\rho \frac{d\mathbf{V}}{dt} = \operatorname{div} \mathbf{T} + \rho \mathbf{B}, \tag{1.48}$$

where $\rho\mathbf{B}$ is the body force and the Cauchy stress tensor $\mathbf{T}$ is given by

$$\mathbf{T} = \begin{bmatrix} \sigma_{xx} & \tau_{yx} & \tau_{zx} \\ \tau_{xy} & \sigma_{yy} & \tau_{zy} \\ \tau_{xz} & \tau_{yz} & \sigma_{zz} \end{bmatrix}.$$

1.14 Magnetohydrodynamics (MHD)

It is the branch of engineering that deals with the study of an electrically conducting fluid in the presence of a magnetic field.

1.14.1 Magnetic field

The field around a magnet in which the magnet can exert force on a moving electric charge is called magnetic field.

1.14.2 Magnetic Reynold number (R_m)

The magnetic Reynold number is a dimensionless quantity that occurs in magnetohydrodynamics. It gives an estimate of the effect of magnetic advection to magnetic diffusion, or in other words we can say that it is the ratio of inertial forces to the magnetic forces. In mathematical form we can write

$$R_m = \frac{Inertial\ forces}{Magnetic\ forces}. \tag{1.49}$$

1.14.3 Small magnetic Reynolds number in MHD

Small magnetic Reynolds number means that the magnetic field associated with the induced current is negligible when compared with the imposed magnetic field. This assumption is taken to get the least effect of velocity **V** on **B**, for which

$$R_m = \frac{\mathbf{V}\ l}{\lambda} << 1 \tag{1.50}$$

where l is the charistaristic length and $\lambda = 1/\mu_0 \sigma$ is the magnetic diffusivity. Here σ is the electrical conductivity of fluid and μ_0 is the magnetic permeability.

1.15 Maxwell's equations

In absence of displacement current the Maxwell's equations are expressed in the form

$$\nabla \cdot \mathbf{E} = \frac{\rho_e}{\epsilon_0}, \qquad \text{(Guass law)} \tag{1.51}$$

$$\nabla \cdot \mathbf{B} = 0, \qquad \text{(Solenoid nature of } \mathbf{B}\text{)} \tag{1.52}$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}, \qquad \text{(Faraday's law)} \tag{1.53}$$

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t}, \qquad \text{(Ampere-Maxwell equation)} \tag{1.54}$$

where μ_0 and ϵ_0 are the magnetic permeability and the permittivity of the free surface. The charge density ρ_e plays no significant role in MHD and the current density $\mathbf{J}$ satisfies

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J}, \qquad \nabla \cdot \mathbf{J} = 0. \tag{1.55}$$

In above equations $\mathbf{E}$ is the electric field and $\mathbf{B}$ is the total magnetic field. The Lorentz force and Ohm's law are defined by the following expressions

$$\mathbf{F} = \mathbf{J} \times \mathbf{B}, \tag{1.56}$$

$$\mathbf{J} = \sigma \left(\mathbf{E} + \mathbf{V} \times \mathbf{B} \right). \tag{1.57}$$

1.16 Methods of solution

In general the resulting equations in the fluid mechanics are non linear and closed form solutions are not possible. Therefore in obtaining analytical solutions the following methods have been utilized.

1. Perturbation method,
2. Adomian decomposition method,

1.16.1 Perturbation method

Perturbation method is based on an assumption that there should be one or more small parameters to transform the non-linear problem into an infinite numbers of linear sub-problems. This method can be elaborated by considering following example

$$y^2 + \epsilon y - 1 = 0, \tag{1.58}$$

where ϵ is a pertubation quantity. Writing

$$y(\epsilon) = y_0 + \epsilon y_1 + \epsilon^2 y_2 + ... \tag{1.59}$$

where y_0, y_1, y_2 are independent of ϵ and y_0 is the solution of the problem for $\epsilon = 0$. We substitute Eq. (1.59) in Eq. (1.58) and expand for small ϵ, and collect coefficients of each power of ϵ. Since these equations must hold for all values of ϵ, each coefficient of ϵ must vanish independently. Thus we get a sequence of equations which are simpler than the original one and can be solved easily. The perturbation solution to Eq. (1.58) can be written as

$$y(\epsilon) = \begin{cases} 1 - \frac{\epsilon}{2} + \frac{\epsilon^2}{8} - \frac{\epsilon^4}{128} + O(\epsilon^6), \\ -1 - \frac{\epsilon}{2} - \frac{\epsilon^2}{8} + \frac{\epsilon^4}{128} + O(\epsilon^6). \end{cases} \tag{1.60}$$

1.16.2 Adomian decomposition method (ADM)

This method provides us a direct scheme for solving non-linear differential equations without any linearization, perturbation, and transformation. This method was developed by George Adomian and has been successfully applied to analyze the non-linear problems. A brief sketch of ADM is presented by considering a differential equation in the form

$$L(U) + R(U) + N(U) = g. \tag{1.61}$$

In above equation L is the highest-order derivative which is assumed to be invertible, R is a linear differential operator of less order than L, $N(U)$ indicates the non-linear terms, and g is the source term. Applying L^{-1} on both side of Eq. (1.61) we write

$$U = f - L^{-1}R(U) - L^{-1}N(U) \tag{1.62}$$

where the function f represents the terms arising from integrating the source term g by using the given conditions. According to ADM the solution $U(x)$ can be expressed in terms of the following series

$$U(x) = \sum_{n=0}^{\infty} U_n(x) \tag{1.63}$$

and the non-linear term $N(U)$ can be written as

$$N(U) = \sum_{n=0}^{\infty} A_n \tag{1.64}$$

in which A_n are specially generated Adomian polynomials for the specific non-linearity defined by

$$A_n = \frac{1}{n!}\frac{d^n}{d\lambda^n}\left[N\left(\sum_{i=0}^{\infty}\lambda^i U_i\right)\right]_{\lambda=0}. \tag{1.65}$$

Now the solution (1.62) becomes

$$U = \sum_{n=0}^{\infty} U_n = f - L^{-1}R\left(\sum_{n=0}^{\infty} U_n\right) - L^{-1}\sum_{n=0}^{\infty} A_n, \tag{1.66}$$

so that

$$U_{n+1} = -L^{-1}RU_n - L^{-1}A_n \qquad n \succeq 0 \tag{1.67}$$

$$\begin{aligned} U_0 &= f, \\ U_1 &= -L^{-1}RU_0 - L^{-1}A_0, \\ U_2 &= -L^{-1}RU_1 - L^{-1}A_1, \quad etc. \end{aligned} \tag{1.68}$$

Chapter 2

Peristaltic transport of a third grade fluid under the effect of a magnetic field

2.1 Introduction

This chapter deals with the peristaltic transport of an electrically conducting magnetohydrodynamic (MHD) third grade incompressible fluid in a planar channel in the presence of a uniform applied magnetic field. The channel walls are transversely displaced by an infinite, harmonic wave travelling in the longitudinal direction. The governing equations are simplified using long wavelength and small Deborah number approximations. Analytical solution for stream function and longitudinal pressure gradient are given upto second order. The obtained solution is graphically presented and the influence of various parameters of interest is analyzed. The work presented in this chapter is a review of research material in ref.[11].

2.2 Formulation of the problem

Let us consider a two-dimensional flow of an incompressible third grade fluid in a uniform channel of width $2d$. The fluid is electrically conducting under the application of a transverse uniform magnetic field $\mathbf{B}_0$. The induced magnetic field is neglected and electric field is assumed to be zero under the assumption of small magnetic Reynolds number. The Cartesian coordinates system is chosen in such a way that $\overline{X}$-axis lies along the central line of the channel and $\overline{Y}$-axis is normal to it. Assuming an infinite wave train travelling with velocity c along the walls of the symmetric channel the wall geometry is defined as

$$\overline{h}\left(\overline{X},\overline{t}\right) = d + b\sin\frac{2\pi}{\lambda}\left(\overline{X} - c\overline{t}\right) \tag{2.1}$$

in which b represents the wave amplitude, λ is the wave length and $\overline{t}$ is the time.

The equations which can govern the MHD flow are

$$\operatorname{div}\overline{\mathbf{V}} = 0, \tag{2.2}$$

$$\rho\frac{\mathbf{d}\overline{\mathbf{V}}}{\mathbf{d}\overline{t}} = \operatorname{div}\overline{\mathbf{T}} + \mathbf{J}\times\mathbf{B}, \tag{2.3}$$

where $d/d\overline{t}$ designates the material derivative, ρ the density, $\overline{\mathbf{V}}$ the velocity, $\mathbf{J}$ the current density, $\mathbf{B}=\mathbf{B}_0+\mathbf{B}_1$ the total magnetic field, $\mathbf{B}_0$ the applied magnetic field and $\mathbf{B}_1$ the induced magnetic field assumed negligible.The expression of Cauchy stress tensor is

$$\overline{\mathbf{T}} = -\overline{p}\overline{\mathbf{I}} + \overline{\mathbf{S}}. \tag{2.4}$$

Here $\overline{p}$ is a pressure, $\overline{\mathbf{I}}$ the identity tensor and the extra stress tensor $\overline{\mathbf{S}}$ for a third grade fluid is

$$\overline{\mathbf{S}} = \mu\overline{\mathbf{A}}_1 + \alpha_1\overline{\mathbf{A}}_2 + \alpha_2\overline{\mathbf{A}}_1^2 + \beta_3\left(tr\overline{\mathbf{A}}_1^2\right)\overline{\mathbf{A}}_1, \tag{2.5}$$

where μ, α_1, α_2 and β_3 are the material constants. The Rivlin–Ericksen tensors $\left(\overline{\mathbf{A}}_n\right)$ are

$$\overline{\mathbf{A}}_1 = \left(\operatorname{grad}\overline{\mathbf{V}}\right) + \left(\operatorname{grad}\overline{\mathbf{V}}\right)^T, \tag{2.6}$$

$$\overline{\mathbf{A}}_n = \frac{d}{dt}\overline{\mathbf{A}}_{n-1} + \overline{\mathbf{A}}_{n-1}\left(\operatorname{grad}\overline{\mathbf{V}}\right) + \left(\operatorname{grad}\overline{\mathbf{V}}\right)^T\overline{\mathbf{A}}_{n-1}, \qquad n > 1. \tag{2.7}$$

For the two dimensional flow the velocity is given by

$$\overline{\mathbf{V}} = \left[\overline{U}\left(\overline{X},\overline{Y},\overline{t}\right), \overline{V}\left(\overline{X},\overline{Y},\overline{t}\right), 0\right]. \tag{2.8}$$

In above definition $\overline{U}$ and $\overline{V}$ are the velocity components parallel to $\overline{X}$ and $\overline{Y}$ axes respectively. Furthermore there is no motion of the wall in the longitudinal direction. This assumption constraints the deformation of the wall; it does not necessary implies that the channel is rigid against longitudinal motions but is a convenient simplifications that can be justified by a more complete analysis. This assumption implies that for the no slip condition $\overline{U}=0$ at the wall.

Using Eq. (2.8) one has

$$\operatorname{grad}\overline{\mathbf{V}}=\begin{bmatrix}\frac{\partial\overline{U}}{\partial\overline{X}} & \frac{\partial\overline{U}}{\partial\overline{Y}} & 0\\ \frac{\partial\overline{V}}{\partial\overline{X}} & \frac{\partial\overline{V}}{\partial\overline{Y}} & 0\\ 0 & 0 & 0\end{bmatrix}, \tag{2.9}$$

$$\left(\operatorname{grad}\overline{\mathbf{V}}\right)^T=\begin{bmatrix}\frac{\partial\overline{U}}{\partial\overline{X}} & \frac{\partial\overline{V}}{\partial\overline{X}} & 0\\ \frac{\partial\overline{U}}{\partial\overline{Y}} & \frac{\partial\overline{V}}{\partial\overline{Y}} & 0\\ 0 & 0 & 0\end{bmatrix}, \tag{2.10}$$

$$\overline{\mathbf{A}}_1=\begin{bmatrix}2\frac{\partial\overline{U}}{\partial\overline{X}} & \frac{\partial\overline{U}}{\partial\overline{Y}}+\frac{\partial\overline{V}}{\partial\overline{X}} & 0\\ \frac{\partial\overline{U}}{\partial\overline{Y}}+\frac{\partial\overline{V}}{\partial\overline{X}} & 2\frac{\partial\overline{V}}{\partial\overline{Y}} & 0\\ 0 & 0 & 0\end{bmatrix}, \tag{2.11}$$

$$\overline{\mathbf{A}}_2=\frac{d\overline{\mathbf{A}}_1}{dt}+\overline{\mathbf{A}}_1\left(\operatorname{grad}\overline{\mathbf{V}}\right)+\left(\operatorname{grad}\overline{\mathbf{V}}\right)^T\overline{\mathbf{A}}_1, \tag{2.12}$$

where

$$\frac{d\overline{\mathbf{A}}_1}{dt}=\left(\frac{\partial}{\partial\overline{t}}+\overline{U}\frac{\partial}{\partial\overline{X}}+\overline{V}\frac{\partial}{\partial\overline{Y}}\right)\begin{bmatrix}2\frac{\partial\overline{U}}{\partial\overline{X}} & \frac{\partial\overline{U}}{\partial\overline{Y}}+\frac{\partial\overline{V}}{\partial\overline{X}} & 0\\ \frac{\partial\overline{U}}{\partial\overline{Y}}+\frac{\partial\overline{V}}{\partial\overline{X}} & 2\frac{\partial\overline{V}}{\partial\overline{Y}} & 0\\ 0 & 0 & 0\end{bmatrix},$$

$$\overline{\mathbf{A}}_1\left(\operatorname{grad}\overline{\mathbf{V}}\right)=\begin{bmatrix}2\left(\frac{\partial\overline{U}}{\partial\overline{X}}\right)^2+\frac{\partial\overline{V}}{\partial\overline{X}}\left(\frac{\partial\overline{U}}{\partial\overline{Y}}+\frac{\partial\overline{V}}{\partial\overline{X}}\right) & 2\frac{\partial\overline{U}}{\partial\overline{X}}\frac{\partial\overline{U}}{\partial\overline{Y}}+\frac{\partial\overline{V}}{\partial\overline{Y}}\left(\frac{\partial\overline{U}}{\partial\overline{Y}}+\frac{\partial\overline{V}}{\partial\overline{X}}\right) & 0\\ 2\frac{\partial\overline{V}}{\partial\overline{X}}\frac{\partial\overline{V}}{\partial\overline{Y}}+\frac{\partial\overline{U}}{\partial\overline{X}}\left(\frac{\partial\overline{U}}{\partial\overline{Y}}+\frac{\partial\overline{V}}{\partial\overline{X}}\right) & 2\left(\frac{\partial\overline{V}}{\partial\overline{Y}}\right)^2+\frac{\partial\overline{U}}{\partial\overline{Y}}\left(\frac{\partial\overline{U}}{\partial\overline{Y}}+\frac{\partial\overline{V}}{\partial\overline{X}}\right) & 0\\ 0 & 0 & 0\end{bmatrix}, \tag{2.13}$$

$$\left(\operatorname{grad}\overline{\mathbf{V}}\right)^T\overline{\mathbf{A}}_1=\begin{bmatrix}2\left(\frac{\partial\overline{U}}{\partial\overline{X}}\right)^2+\frac{\partial\overline{V}}{\partial\overline{X}}\left(\frac{\partial\overline{U}}{\partial\overline{Y}}+\frac{\partial\overline{V}}{\partial\overline{X}}\right) & 2\frac{\partial\overline{V}}{\partial\overline{X}}\frac{\partial\overline{V}}{\partial\overline{Y}}+\frac{\partial\overline{U}}{\partial\overline{X}}\left(\frac{\partial\overline{U}}{\partial\overline{Y}}+\frac{\partial\overline{V}}{\partial\overline{X}}\right) & 0\\ 2\frac{\partial\overline{U}}{\partial\overline{X}}\frac{\partial\overline{U}}{\partial\overline{Y}}+\frac{\partial\overline{V}}{\partial\overline{Y}}\left(\frac{\partial\overline{U}}{\partial\overline{Y}}+\frac{\partial\overline{V}}{\partial\overline{X}}\right) & 2\left(\frac{\partial\overline{V}}{\partial\overline{Y}}\right)^2+\frac{\partial\overline{U}}{\partial\overline{Y}}\left(\frac{\partial\overline{U}}{\partial\overline{Y}}+\frac{\partial\overline{V}}{\partial\overline{X}}\right) & 0\\ 0 & 0 & 0\end{bmatrix}, \tag{2.14}$$

$$\overline{\mathbf{A}}_1^2 = \begin{bmatrix} 4\left(\frac{\partial \overline{U}}{\partial \overline{X}}\right)^2 + \left(\frac{\partial \overline{U}}{\partial \overline{Y}} + \frac{\partial \overline{V}}{\partial \overline{X}}\right)^2 & 0 & 0 \\ 0 & 4\left(\frac{\partial \overline{V}}{\partial \overline{Y}}\right)^2 + \left(\frac{\partial \overline{U}}{\partial \overline{Y}} + \frac{\partial \overline{V}}{\partial \overline{X}}\right)^2 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \tag{2.15}$$

$$tr\left(\overline{\mathbf{A}}_1^2\right)\overline{\mathbf{A}}_1 = \left[4\left(\left(\frac{\partial \overline{U}}{\partial \overline{X}}\right)^2 + \left(\frac{\partial \overline{V}}{\partial \overline{Y}}\right)^2\right) + 2\left(\frac{\partial \overline{U}}{\partial \overline{Y}} + \frac{\partial \overline{V}}{\partial \overline{X}}\right)^2\right] \begin{bmatrix} 2\frac{\partial \overline{U}}{\partial \overline{X}} & \frac{\partial \overline{U}}{\partial \overline{Y}} + \frac{\partial \overline{V}}{\partial \overline{X}} & 0 \\ \frac{\partial \overline{U}}{\partial \overline{Y}} + \frac{\partial \overline{V}}{\partial \overline{X}} & 2\frac{\partial \overline{V}}{\partial \overline{Y}} & 0 \\ 0 & 0 & 0 \end{bmatrix}. \tag{2.16}$$

In absence of displacement current, the Maxwell equation and Ohm's law are expressed by the following relations

$$\operatorname{div} \mathbf{B} = \mathbf{0}, \quad curl\mathbf{B} = \mu_m \mathbf{J}, \quad \operatorname{curl} \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}, \quad \mathbf{J} = \sigma\left(\mathbf{E} + \mathbf{V} \times \mathbf{B}\right), \tag{2.17}$$

in where σ is the electrical conductivity, μ_m the magnetic permeability and $\mathbf{E}$ the electric field. In the present analysis the electric field is neglected and and the magnetic Reynold's number is taken small. Under these approximations we get

$$\mathbf{J} \times \mathbf{B} = -\sigma \mathbf{B}_0^2 \overline{U}. \tag{2.18}$$

Substitution of Eqs. (2.11) − (2.18) into Eqs. (2.2) − (2.5) gives

$$\frac{\partial \overline{U}}{\partial \overline{X}} + \frac{\partial \overline{V}}{\partial \overline{Y}} = 0, \tag{2.19}$$

$$\rho\left(\frac{\partial}{\partial \overline{t}} + \overline{U}\frac{\partial}{\partial \overline{X}} + \overline{V}\frac{\partial}{\partial \overline{Y}}\right)\overline{U} = -\frac{\partial \overline{p}\left(\overline{X}, \overline{Y}, \overline{t}\right)}{\partial \overline{X}} + \frac{\partial \overline{S}_{\overline{XX}}}{\partial \overline{X}} + \frac{\partial \overline{S}_{\overline{XY}}}{\partial \overline{Y}} - \sigma B_0^2 \overline{U}, \tag{2.20}$$

$$\rho\left(\frac{\partial}{\partial \overline{t}} + \overline{U}\frac{\partial}{\partial \overline{X}} + \overline{V}\frac{\partial}{\partial \overline{Y}}\right)\overline{V} = -\frac{\partial \overline{p}\left(\overline{X}, \overline{Y}, \overline{t}\right)}{\partial \overline{Y}} + \frac{\partial \overline{S}_{\overline{XY}}}{\partial \overline{X}} + \frac{\partial \overline{S}_{\overline{YY}}}{\partial \overline{Y}}, \tag{2.21}$$

$$\begin{aligned} \overline{S}_{\overline{XX}} &= 2\mu\overline{U}_{\overline{X}} + \alpha_1(2\overline{U}_{\overline{X}t} + 2\overline{U}\overline{U}_{\overline{XX}} + 2\overline{V}\overline{U}_{\overline{XY}} + 4\overline{U}_{\overline{X}}^2 + 2\overline{V}_{\overline{X}}^2 + 2\overline{V}_{\overline{X}}\overline{U}_{\overline{Y}}) \\ &\quad + \alpha_2\left(4\overline{U}_{\overline{X}}^2 + \overline{U}_{\overline{Y}}^2 + \overline{V}_{\overline{X}}^2 + 2\overline{V}_{\overline{X}}\overline{U}_{\overline{Y}}\right) \\ &\quad + \beta_3(8\overline{U}_{\overline{X}}^3 + 4\overline{U}_{\overline{X}}\overline{U}_{\overline{Y}}^2 + 8\overline{U}_{\overline{X}}\overline{V}_{\overline{Y}}^2 + 4\overline{U}_{\overline{X}}\overline{V}_{\overline{X}}^2 + 8\overline{U}_{\overline{Y}}\overline{V}_{\overline{X}}\overline{U}_{\overline{X}}), \end{aligned} \tag{2.22}$$

$$\begin{aligned}\overline{S}_{\overline{XY}} &= \mu(\overline{U}_{\overline{Y}}+\overline{V}_{\overline{X}})+\alpha_1(\overline{V}_{\overline{X}\overline{t}}+\overline{U}_{\overline{Y}\overline{t}}+\overline{U}\overline{U}_{\overline{XY}}+\overline{V}\overline{V}_{\overline{XY}}+\overline{V}\overline{U}_{\overline{YY}}\\&+\overline{U}\overline{V}_{\overline{XX}}+2\overline{U}_{\overline{X}}\overline{U}_{\overline{Y}}+2\overline{V}_{\overline{X}}\overline{V}_{\overline{Y}})+\alpha_2(2\overline{U}_{\overline{X}}\overline{V}_{\overline{X}}+2\overline{V}_{\overline{X}}\overline{V}_{\overline{Y}})\\&+\beta_3(4\overline{U}^2_{\overline{X}}\overline{U}_{\overline{Y}}+2\overline{U}^3_{\overline{Y}}+6\overline{V}^2_{\overline{X}}\overline{U}_{\overline{Y}}+4\overline{V}^2_{\overline{Y}}\overline{U}_{\overline{Y}}+6\overline{V}_{\overline{X}}\overline{U}^2_{\overline{Y}}+4\overline{V}_{\overline{X}}\overline{U}^2_{\overline{X}}\\&+2\overline{V}^3_{\overline{X}}+4\overline{V}^2_{\overline{Y}}\overline{V}_{\overline{X}}),\end{aligned} \tag{2.23}$$

$$\begin{aligned}\overline{S}_{\overline{YY}} &= 2\mu\overline{V}_{\overline{Y}}+\alpha_1(2\overline{V}_{\overline{Y}\overline{t}}+2\overline{U}\overline{V}_{\overline{XY}}+2\overline{V}\overline{V}_{\overline{YY}}+2\overline{U}^2_{\overline{Y}}+4\overline{V}^2_{\overline{Y}}+2\overline{V}_{\overline{X}}\overline{U}_{\overline{Y}})\\&+\alpha_2(\overline{U}^2_{\overline{Y}}+4\overline{V}^2_{\overline{Y}}+\overline{V}^2_{\overline{X}}+2\overline{V}_{\overline{X}}\overline{U}_{\overline{Y}})+\beta_3(8\overline{V}^3_{\overline{Y}}+8\overline{U}^2_{\overline{X}}\overline{V}_{\overline{Y}}\\&+4\overline{V}^2_{\overline{X}}\overline{V}_{\overline{Y}}+4\overline{U}^2_{\overline{Y}}\overline{V}_{\overline{Y}}+8\overline{U}_{\overline{Y}}\overline{V}_{\overline{X}}\overline{U}_{\overline{Y}}),\end{aligned} \tag{2.24}$$

where the subscripts denote the partial derivatives.

2.3 Transformation in the wave frame

Note that in the fixed coordinate system $(\overline{X},\overline{Y})$, the motion is time dependent. However in a coordinate system $(\overline{x},\overline{y})$ moving with the wave speed (c) in positive $\overline{x}$ direction, the boundary shape is stationary.

The transformations between laboratory and wave frames are

$$\overline{x}=\overline{X}-c\overline{t}, \qquad \overline{y}=\overline{Y}, \qquad \overline{u}=\overline{U}-c, \qquad \overline{v}=\overline{V}. \tag{2.25}$$

In above equation $(\overline{u},\overline{v})$ are the longitudinal and transverse velocity components in the moving coordinate system. Using the trasformations (2.25) one obtains

$$\frac{\partial}{\partial\overline{X}}=\frac{\partial}{\partial\overline{x}}, \qquad \frac{\partial}{\partial\overline{Y}}=\frac{\partial}{\partial\overline{y}}, \qquad \frac{\partial}{\partial\overline{t}}=-c\frac{\partial}{\partial\overline{x}} \tag{2.26}$$

Eqs. (2.19) – (2.21) in the wave frame take the form

$$\frac{\partial\overline{u}}{\partial\overline{x}}+\frac{\partial\overline{v}}{\partial\overline{y}}=0, \tag{2.27}$$

$$\rho(\overline{u}\frac{\partial}{\partial\overline{x}}+\overline{v}\frac{\partial}{\partial\overline{y}})\overline{u}=-\frac{\partial\overline{p}\,(\overline{x},\overline{y})}{\partial\overline{x}}+\frac{\partial\overline{S}_{\overline{xx}}}{\partial\overline{x}}+\frac{\partial\overline{S}_{\overline{xy}}}{\partial\overline{y}}-\sigma B_0^2\overline{u}, \tag{2.28}$$

$$\rho(\overline{u}\frac{\partial}{\partial\overline{x}}+\overline{v}\frac{\partial}{\partial\overline{y}})\overline{v}=-\frac{\partial\overline{p}\left(\overline{x},\overline{y}\right)}{\partial\overline{y}}+\frac{\partial\overline{S}_{\overline{xy}}}{\partial\overline{x}}+\frac{\partial\overline{S}_{\overline{yy}}}{\partial\overline{y}}, \tag{2.29}$$

$$\begin{aligned}\overline{S}_{\overline{xx}} &= 2\mu\overline{u}_{\overline{x}}+\alpha_1(2\overline{u}\overline{u}_{\overline{xx}}+2\overline{v}\overline{u}_{\overline{xy}}+4\overline{u}_{\overline{x}}^2+2\overline{v}_{\overline{x}}^2+2\overline{v}_{\overline{x}}\overline{u}_{\overline{y}})\\ &+\alpha_2\left(4\overline{u}_{\overline{x}}^2+\overline{u}_{\overline{y}}^2+\overline{v}_{\overline{x}}^2+2\overline{v}_{\overline{x}}\overline{u}_{\overline{y}}\right)\\ &+\beta_3(8\overline{u}_{\overline{x}}^3+4\overline{u}_{\overline{x}}\overline{u}_{\overline{y}}^2+8\overline{u}_{\overline{x}}\overline{v}_{\overline{y}}^2+4\overline{u}_{\overline{x}}\overline{v}_{\overline{x}}^2+8\overline{u}_{\overline{y}}\overline{v}_{\overline{x}}\overline{u}_{\overline{x}}),\end{aligned} \tag{2.30}$$

$$\begin{aligned}\overline{S}_{\overline{xy}} &= \mu(\overline{u}_{\overline{y}}+\overline{v}_{\overline{x}})+\alpha_1(\overline{u}\overline{u}_{\overline{xy}}+\overline{v}\overline{v}_{\overline{xy}}+\overline{v}\overline{u}_{\overline{yy}}\\ &+\overline{u}\overline{v}_{\overline{xx}}+2\overline{u}_{\overline{x}}\overline{u}_{\overline{y}}+2\overline{v}_{\overline{x}}\overline{v}_{\overline{y}})+\alpha_2(2\overline{u}_{\overline{x}}\overline{v}_{\overline{x}}+2\overline{v}_{\overline{x}}\overline{v}_{\overline{y}})\\ &+\beta_3(4\overline{u}_{\overline{x}}^2\overline{u}_{\overline{y}}+2\overline{u}_{\overline{y}}^3+6\overline{v}_{\overline{x}}^2\overline{u}_{\overline{y}}+4\overline{v}_{\overline{y}}^2\overline{u}_{\overline{y}}+6\overline{v}_{\overline{x}}\overline{u}_{\overline{y}}^2+4\overline{v}_{\overline{x}}\overline{u}_{\overline{x}}^2\\ &+2\overline{v}_{\overline{x}}^3+4\overline{v}_{\overline{y}}^2\overline{v}_{\overline{x}}),\end{aligned} \tag{2.31}$$

$$\begin{aligned}\overline{S}_{\overline{yy}} &= 2\mu\overline{v}_{\overline{y}}+\alpha_1(2\overline{u}\overline{v}_{\overline{xy}}+2\overline{v}\overline{v}_{\overline{yy}}+2\overline{u}_{\overline{y}}^2+4\overline{v}_{\overline{y}}^2+2\overline{v}_{\overline{x}}\overline{u}_{\overline{y}})\\ &+\alpha_2(\overline{u}_{\overline{y}}^2+4\overline{v}_{\overline{y}}^2+\overline{v}_{\overline{x}}^2+2\overline{v}_{\overline{x}}\overline{u}_{\overline{y}})+\beta_3(8\overline{v}_{\overline{y}}^3+8\overline{u}_{\overline{x}}^2\overline{v}_{\overline{y}}\\ &+4\overline{v}_{\overline{x}}^2\overline{v}_{\overline{y}}+4\overline{u}_{\overline{y}}^2\overline{v}_{\overline{y}}+8\overline{u}_{\overline{y}}\overline{v}_{\overline{x}}\overline{u}_{\overline{y}}),\end{aligned} \tag{2.32}$$

$$\overline{h}\left(\overline{x}\right)=d+b\sin\frac{2\pi}{\lambda}\left(\overline{x}\right). \tag{2.33}$$

Introducing the non-dimensional quantities

$$\begin{aligned}x &= \frac{2\pi\overline{x}}{\lambda},\quad y=\frac{\overline{y}}{d},\quad u=\frac{\overline{u}}{c},\quad v=\frac{\overline{v}}{c},\quad h=\frac{\overline{h}(\overline{x})}{d},\\ p &= \frac{2\pi d^2}{\lambda\mu c}\overline{p}(\overline{x}),\quad S=\frac{d}{\mu c}\overline{S}(\overline{x}),\quad \delta=\frac{2\pi d}{\lambda},\quad \mathrm{Re}=\frac{\rho c\alpha}{\mu},\\ \lambda_1 &= \frac{\alpha_1 c}{\mu d},\quad \lambda_2=\frac{\alpha_2 c}{\mu d},\quad \Gamma=\frac{\beta_3 c^2}{\mu d^2},\quad M=\sqrt{\frac{\sigma}{\mu}}B_0 d.\end{aligned} \tag{2.34}$$

And using Eq. (2.34) in Eqs. (2.27) – (2.32) one gets

$$\delta\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0, \tag{2.35}$$

$$\mathrm{Re}\left[\left(\delta u\frac{\partial}{\partial x}+v\frac{\partial}{\partial y}\right)u\right]=-\frac{\partial p}{\partial x}+\delta\frac{\partial S_{xx}}{\partial x}+\frac{\partial S_{xy}}{\partial y}-M^2(u+1), \tag{2.36}$$

$$\delta\,\mathrm{Re}\left[\left(\delta u\frac{\partial}{\partial x}+v\frac{\partial}{\partial y}\right)v\right]=-\frac{\partial p}{\partial y}+\delta^2\frac{\partial S_{xy}}{\partial x}+\delta\frac{\partial S_{yy}}{\partial y}, \tag{2.37}$$

$$\begin{aligned}S_{xx} &= 2\delta\frac{\partial u}{\partial x}+\lambda_1\left[2\left(\delta u\frac{\partial}{\partial x}+v\frac{\partial}{\partial y}\right)\left(\delta\frac{\partial u}{\partial x}\right)+4\delta^2\left(\frac{\partial u}{\partial x}\right)^2+2\delta\frac{\partial v}{\partial x}\left(\frac{\partial u}{\partial y}+\delta\frac{\partial v}{\partial x}\right)\right]\\ &+\lambda_2\left[4\delta^2\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial u}{\partial y}+\delta\frac{\partial v}{\partial x}\right)^2\right]\\ &+4\Gamma\delta\frac{\partial u}{\partial x}\left[2\delta^2\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial u}{\partial y}+\delta\frac{\partial v}{\partial x}\right)^2+2\left(\frac{\partial v}{\partial y}\right)^2\right],\end{aligned} \tag{2.38}$$

$$\begin{aligned}S_{xy} &= \left(\frac{\partial u}{\partial y}+\delta\frac{\partial v}{\partial x}\right)+\lambda_1\left[\left(\delta u\frac{\partial}{\partial x}+v\frac{\partial}{\partial y}\right)\left(\frac{\partial u}{\partial y}+\delta\frac{\partial v}{\partial x}\right)+2\delta\left(\frac{\partial u}{\partial x}\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\frac{\partial v}{\partial y}\right)\right]\\ &+2\Gamma\left[2\delta^2\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial u}{\partial y}+\delta\frac{\partial v}{\partial x}\right)^2+2\left(\frac{\partial v}{\partial y}\right)^2\right]\left(\frac{\partial u}{\partial y}+\delta\frac{\partial v}{\partial x}\right),\end{aligned} \tag{2.39}$$

$$\begin{aligned}S_{yy} &= 2\frac{\partial v}{\partial y}+\lambda_1\left[2\left(\delta u\frac{\partial}{\partial x}+v\frac{\partial}{\partial y}\right)\frac{\partial v}{\partial y}+4\left(\frac{\partial v}{\partial y}\right)^2+2\frac{\partial u}{\partial y}\left(\frac{\partial u}{\partial y}+\delta\frac{\partial v}{\partial x}\right)\right]\\ &+\lambda_2\left[4\left(\frac{\partial v}{\partial y}\right)^2+\left(\frac{\partial u}{\partial y}+\delta\frac{\partial v}{\partial x}\right)^2\right]\\ &+2\Gamma\left(\frac{\partial v}{\partial y}\right)\left[2\delta^2\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial u}{\partial y}+\delta\frac{\partial v}{\partial x}\right)^2+2\left(\frac{\partial v}{\partial y}\right)^2\right],\end{aligned} \tag{2.40}$$

where δ is the wave number, Re is the Reynolds number, $\lambda_1, \lambda_2, \Gamma$ are the material parameters and M is the Hartman number.Definig the stream function $\Psi(x,y)$ by

$$u=\frac{\partial \Psi}{\partial y}, \qquad v=-\delta\frac{\partial \Psi}{\partial x}, \tag{2.41}$$

the incompressibility condition (2.33) is identically satisfied and the Eqs. (2.36) – (2.40) takes the form

$$\delta\,\mathrm{Re}\left[\left(\frac{\partial\Psi}{\partial y}\frac{\partial}{\partial x}-\frac{\partial\Psi}{\partial x}\frac{\partial}{\partial y}\right)\frac{\partial\Psi}{\partial y}\right] = -\frac{\partial p}{\partial x}+\delta\frac{\partial S_{xx}}{\partial x}+\frac{\partial S_{xy}}{\partial y} -M^2\left(\frac{\partial\Psi}{\partial y}+1\right), \tag{2.42}$$

$$-\delta^3\,\mathrm{Re}\left[\left(\frac{\partial\Psi}{\partial y}\frac{\partial}{\partial x}-\frac{\partial\Psi}{\partial x}\frac{\partial}{\partial y}\right)\frac{\partial\Psi}{\partial x}\right] = -\frac{\partial p}{\partial y}+\delta^2\frac{\partial S_{xy}}{\partial x}+\delta\frac{\partial S_{yy}}{\partial y}, \tag{2.43}$$

$$\begin{aligned} S_{xx} = {} & 2\delta\frac{\partial^2\Psi}{\partial x\partial y} \\ & +\lambda_1\delta^2\left[2\left(\frac{\partial\Psi}{\partial y}\frac{\partial}{\partial x}-\frac{\partial\Psi}{\partial x}\frac{\partial}{\partial y}\right)\frac{\partial^2\Psi}{\partial x\partial y}+4\left(\frac{\partial^2\Psi}{\partial x\partial y}\right)^2-2\frac{\partial^2\Psi}{\partial x^2}\left(\frac{\partial^2\Psi}{\partial y^2}-\delta^2\frac{\partial^2\Psi}{\partial x^2}\right)\right] \\ & +\lambda_2\left[4\delta^2\left(\frac{\partial^2\Psi}{\partial x\partial y}\right)^2+\left(\frac{\partial^2\Psi}{\partial y^2}-\delta^2\frac{\partial^2\Psi}{\partial x^2}\right)^2\right] \\ & +4\Gamma\delta\frac{\partial^2\Psi}{\partial x\partial y}\left[2\delta^2\left(\frac{\partial^2\Psi}{\partial x\partial y}\right)^2+\left(\frac{\partial^2\Psi}{\partial y^2}-\delta^2\frac{\partial^2\Psi}{\partial x^2}\right)^2+2\delta^2\left(\frac{\partial^2\Psi}{\partial x\partial y}\right)^2\right], \end{aligned} \tag{2.44}$$

$$\begin{aligned} S_{xy} = {} & \left(\frac{\partial^2\Psi}{\partial y^2}-\delta^2\frac{\partial^2\Psi}{\partial x^2}\right) \\ & +\lambda_1\delta\left[\left(\frac{\partial\Psi}{\partial y}\frac{\partial}{\partial x}-\frac{\partial\Psi}{\partial x}\frac{\partial}{\partial y}\right)\left(\frac{\partial^2\Psi}{\partial y^2}-\delta^2\frac{\partial^2\Psi}{\partial x^2}\right)+2\left(\frac{\partial^2\Psi}{\partial y^2}-\delta^2\frac{\partial^2\Psi}{\partial x^2}\right)\frac{\partial^2\Psi}{\partial x\partial y}\right] \\ & 2\Gamma\left[2\delta^2\left(\frac{\partial^2\Psi}{\partial x\partial y}\right)^2+\left(\frac{\partial^2\Psi}{\partial y^2}-\delta^2\frac{\partial^2\Psi}{\partial x^2}\right)^2+2\delta^2\left(\frac{\partial^2\Psi}{\partial x\partial y}\right)^2\right]\times \\ & \left(\frac{\partial^2\Psi}{\partial y^2}-\delta^2\frac{\partial^2\Psi}{\partial x^2}\right) \end{aligned} \tag{2.45}$$

$$
\begin{aligned}
S_{yy} &= -2\delta\frac{\partial^2\Psi}{\partial x\partial y} \\
&+\lambda_1\left[-2\delta^2\left(\frac{\partial\Psi}{\partial y}\frac{\partial}{\partial x}-\frac{\partial\Psi}{\partial x}\frac{\partial}{\partial y}\right)\frac{\partial^2\Psi}{\partial x\partial y}+4\delta^2\left(\frac{\partial^2\Psi}{\partial x\partial y}\right)^2+2\left(\frac{\partial^2\Psi}{\partial y^2}-\delta^2\frac{\partial^2\Psi}{\partial x^2}\right)\frac{\partial^2\Psi}{\partial y^2}\right] \\
&+\lambda_2\left[4\delta^2\left(\frac{\partial^2\Psi}{\partial x\partial y}\right)^2+\left(\frac{\partial^2\Psi}{\partial y^2}-\delta^2\frac{\partial^2\Psi}{\partial x^2}\right)^2\right] \\
&-2\Gamma\delta\frac{\partial^2\Psi}{\partial x\partial y}\left[2\delta^2\left(\frac{\partial^2\Psi}{\partial x\partial y}\right)^2+\left(\frac{\partial^2\Psi}{\partial y^2}-\delta^2\frac{\partial^2\Psi}{\partial x^2}\right)^2+2\delta^2\left(\frac{\partial^2\Psi}{\partial x\partial y}\right)^2\right], \qquad (2.46)
\end{aligned}
$$

Eliminating p from Eq. (2.42) and (2.43) , yields the following equation

$$
\begin{aligned}
\delta\,\mathrm{Re}\left[\left(\frac{\partial\Psi}{\partial y}\frac{\partial}{\partial x}-\frac{\partial\Psi}{\partial x}\frac{\partial}{\partial y}\right)\nabla^2\Psi\right] &= \left[\left(\frac{\partial^2}{\partial y^2}+\delta^2\frac{\partial^2}{\partial x^2}\right)S_{xy}\right]+\delta\left[\frac{\partial^2}{\partial x\partial y}\left(S_{xx}-S_{xy}\right)\right] \\
&-M^2\frac{\partial^2\Psi}{\partial y^2}, \qquad (2.47)
\end{aligned}
$$

where

$$
\nabla^2=\left(\delta^2\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right).
$$

2.4 Volume flow rate

The instantaneous volume flow rate in the fixed frame is

$$
Q=\int_0^{\overline{H}}\overline{U}\left(\overline{X},\overline{Y},t\right)d\overline{Y}, \qquad (2.48)
$$

where $\overline{H}$ is a function of $\overline{X}$ and t and the above expression in the wave frame becomes

$$
q=\int_0^{\overline{h}}\overline{u}\left(\overline{x},\overline{y}\right)d\overline{y}, \qquad (2.49)
$$

where $\overline{h}$ is a function of $\overline{x}$ only. If we substitute Eq. (2.25) in Eq. (2.48) and use Eq. (2.49) the above two volume flow rates satisfy

$$
Q=q+c\overline{h}. \qquad (2.50)
$$

The time-mean flow over a period T at a fixed position $\overline{X}$ is

$$\overline{Q} = \frac{1}{T}\int_0^T Q dt. \tag{2.51}$$

Invoking Eq. (2.50) into Eq. (2.51), and then integrating we arrive at

$$\overline{Q} = q + dc. \tag{2.52}$$

Taking the dimensionless time-mean flows θ and F in the fixed and wave frame by

$$\theta = \frac{\overline{Q}}{dc}, \qquad \text{and} \qquad F = \frac{q}{dc}, \tag{2.53}$$

we get from Eq. (2.52) as

$$\theta = F + 1, \tag{2.54}$$

where

$$F = \int_0^h \left(\frac{\partial \Psi}{\partial y}\right) dy = \Psi(h) - \Psi(0). \tag{2.55}$$

Where F is the non–dimensional mean flow. The non–dimensional surface of the peristaltic wall $h(x)$ becomes

$$h(x) = 1 + \phi \ \sin x. \tag{2.56}$$

Here $\phi = (b/d)$ is the amplitude ratio and $0 < \phi < 1$. Choosing the zero value of the streamline along the center line $(y = 0)$ one has

$$\Psi(0) = 0, \tag{2.57}$$

and the shape of the wave is given by the streamline of value

$$\Psi(h) = F. \tag{2.58}$$

In wave frame the boundary conditions are prescribed as

$$\begin{aligned} \Psi &= 0 \quad \text{(by convention)} \quad at\ y = 0, \\ \frac{\partial^2 \Psi}{\partial y^2} &= 0 \quad \text{(by symmetry)} \quad at\ y = 0, \end{aligned} \tag{2.59}$$

$$\begin{aligned} \frac{\partial \Psi}{\partial y} &= -1 \quad \text{(no slip condition)} \quad at\ y = h, \\ \Psi &= F \qquad\qquad\qquad\qquad\qquad at\ y = h. \end{aligned} \tag{2.60}$$

2.5 Perturbation solution.

Equation (2.47) is highly complicated, non-linear and thus the closed form solution for arbitrary values of all parameters is difficult. Keeping this fact in view the analysis here is carried out for long wavelength approximation. This is a valid assumption especially for the flow of chyme in the small intestine. Mathematically for long wavelength approximation one takes

$$\delta = \frac{2\pi d}{\lambda} = 0. \tag{2.61}$$

Thus Eqs. (2.42)–Eq. (2.47) become

$$\frac{\partial p}{\partial x} = \frac{\partial^3 \Psi}{\partial y^3} + 2\Gamma \frac{\partial}{\partial y}\left(\frac{\partial^2 \Psi}{\partial y^2}\right)^3 - M^2\left(\frac{\partial \Psi}{\partial y} + 1\right), \tag{2.62}$$

$$\frac{\partial p}{\partial y} = 0, \tag{2.63}$$

$$S_{xy} = \frac{\partial^2 \Psi}{\partial y^2} + 2\Gamma\left(\frac{\partial^2 \Psi}{\partial y^2}\right)^3, \tag{2.64}$$

$$S_{xx} = S_{yy} = 0, \tag{2.65}$$

$$\frac{\partial^4 \Psi}{\partial y^4} = -2\Gamma \frac{\partial^2}{\partial y^2}\left(\frac{\partial^2 \Psi}{\partial y^2}\right)^3 + M^2 \frac{\partial^2 \Psi}{\partial y^2}, \tag{2.66}$$

Eq. (2.63) indicates that $p \neq p(y)$ so $\partial p/\partial x = dp/dx$. For perturbation solution we expand Ψ, p and F as

$$\begin{aligned} \Psi &= \Psi_0 + \Gamma\Psi_1 + \Gamma^2\Psi_2 + ..., \\ p &= p_0 + \Gamma p_1 + \Gamma^2 p_2 + ..., \\ F &= F_0 + \Gamma F_1 + \Gamma^2 F_2 + ... \end{aligned} \tag{2.67}$$

Putting the above equations into Eqs. (2.59), (2.60), (2.62), (2.64), (2.66) one can write the following systems.

2.5.1 System for Ψ_0

$$\frac{\partial^4\Psi_0}{\partial y^4} - M^2\frac{\partial^2\Psi_0}{\partial y^2} = 0, \tag{2.68}$$

$$\frac{dp_0}{dx} = \frac{\partial^3\Psi_0}{\partial y^3} - M^2\left(\frac{\partial\Psi_0}{\partial y} + 1\right), \tag{2.69}$$

$$\Psi_0 = 0, \qquad \frac{\partial^2\Psi_0}{\partial y^2} = 0 \qquad at\ y = 0, \tag{2.70(a)}$$

$$\frac{\partial\Psi_0}{\partial y} = -1, \qquad \Psi_0 = F_0 \qquad at\ y = h. \tag{2.70(b)}$$

2.5.2 System for Ψ_1

$$\frac{\partial^4\Psi_1}{\partial y^4} - M^2\frac{\partial^2\Psi_1}{\partial y^2} + 2\frac{\partial^2}{\partial y^2}\left[\left(\frac{\partial^2\Psi_0}{\partial y^2}\right)^3\right] = 0, \tag{2.71}$$

$$\frac{dp_1}{dx} = \frac{\partial^3\Psi_1}{\partial y^3} - M^2\left(\frac{\partial\Psi_1}{\partial y}\right) + 2\frac{\partial}{\partial y}\left[\left(\frac{\partial^2\Psi_0}{\partial y^2}\right)^3\right], \tag{2.72}$$

$$\Psi_1 = 0, \qquad \frac{\partial^2\Psi_1}{\partial y^2} = 0 \qquad at\ y = 0, \tag{2.73(a)}$$

$$\frac{\partial\Psi_1}{\partial y} = 0, \qquad \Psi_1 = F_1 \qquad at\ y = h. \tag{2.73(b)}$$

2.5.3 System for Ψ_2

$$\frac{\partial^4\Psi_2}{\partial y^4} - M^2\frac{\partial^2\Psi_2}{\partial y^2} + 6\frac{\partial^2}{\partial y^2}\left[\left(\frac{\partial^2\Psi_0}{\partial y^2}\right)^2\left(\frac{\partial^2\Psi_1}{\partial y^2}\right)\right] = 0, \tag{2.74}$$

$$\frac{dp_2}{dx} = \frac{\partial^3\Psi_2}{\partial y^3} - M^2\left(\frac{\partial\Psi_2}{\partial y}\right) + 6\frac{\partial}{\partial y}\left[\left(\frac{\partial^2\Psi_0}{\partial y^2}\right)^2\left(\frac{\partial^2\Psi_1}{\partial y^2}\right)\right], \tag{2.75}$$

$$\Psi_2 = 0, \qquad \frac{\partial^2\Psi_2}{\partial y^2} = 0 \qquad at\ y = 0, \tag{2.76(a)}$$

$$\frac{\partial\Psi_2}{\partial y} = 0, \qquad \Psi_2 = F_2 \qquad at\ y = h. \tag{2.76(b)}$$

2.5.4 Solution for Ψ_0

From Eqs. (2.68) and (2.70 a, b) one may get

$$\Psi_0 = \left(\frac{F_0 M + \tanh Mh}{hM - \tanh Mh}\right)\left(y - \frac{\sinh My}{M\cosh Mh}\right) - \frac{\sinh My}{M\cosh Mh}. \tag{2.77}$$

From Eqs. (2.69) and (2.77), we obtain

$$\frac{dp_0}{dx} = -M^2\left(\frac{F_0 M + \tanh Mh}{hM - \tanh Mh} + 1\right). \tag{2.78}$$

2.5.5 Solution for Ψ_1

If we use the zero order solution in Eq. (2.71) and then solve the resulting equation along with the corresponding boundary conditions we have

$$\begin{aligned}\Psi_1 &= y\left[H_0 M\cosh Mh + \left(\tfrac{dp_0}{dx}\right)^3\left(\tfrac{3\cosh 3Mh}{16M^4\cosh^3 Mh}\right) - \tfrac{3}{4M^4\cosh^2 Mh} - \tfrac{3h\sinh Mh}{4M^3\cosh^3 Mh}\right] \\ &\quad -H_0\sinh My - \left(\frac{dp_0}{dx}\right)^3\left[\tfrac{\sinh 3My}{16M^5\cosh^3 Mh} - \tfrac{3y\cosh My}{4M^4\cosh^3 Mh}\right].\end{aligned} \tag{2.79}$$

Making use of Eqs. (2.77) and (2.79) in Eq. (2.72) first and then solving we get

$$\frac{dp_1}{dx} = M^2\left(\frac{dp_0}{dx}\right)^3\left(\tfrac{3}{4M^4\cosh^2 Mh} + \tfrac{3h\sinh Mh}{4M^3\cosh^3 Mh} - \tfrac{3\cosh^3 Mh}{16M^4\cosh^3 Mh}\right) - M^3 H_0\cosh Mh, \tag{2.80}$$

$$H_0 = \frac{1}{(hM\cosh Mh - \sinh Mh)}\left[\begin{array}{c} F_1 - \left(\frac{dp_0}{dx}\right)^3 \\ \left(\frac{3h\cosh 3Mh}{16M^4\cosh^3 Mh} - \frac{\sinh 3Mh}{16M^5\cosh^3 Mh} - \frac{3h^2\sinh Mh}{4M^3\cosh^3 Mh}\right)\end{array}\right]. \tag{2.81}$$

2.5.6 Solution for Ψ_2

Equation (2.74) after using the zeroth and first order solutions takes the form

$$\begin{aligned}\Psi_2 &= y\left[\begin{array}{c} -\left(\frac{dp_0}{dx}\right)^2\left(\frac{9H_0\cosh 3Mh}{16M\cosh^2 Mh}-\frac{9H_0\cosh Mh}{4M\cosh^2 Mh}-\frac{9H_0 h\sinh Mh}{4\cosh^2 Mh}\right)-H_1 M\cosh Mh \\ -\left(\frac{dp_0}{dx}\right)^5\left\{\begin{array}{c}\frac{45\cosh 5Mh}{256M^6\cosh^5 Mh}-\frac{54\cosh 3Mh}{128M^6\cosh^5 Mh} \\ +\frac{225\cosh Mh}{64M^6\cosh^5 Mh}+\frac{261h\sinh Mh}{64M^5\cosh^5 Mh}-\frac{27h\sinh 3Mh}{64M^5\cosh^5 Mh}+\frac{9h^2\cosh Mh}{32M^4\cosh^5 Mh}\end{array}\right\}\end{array}\right] \\ &+H_1\sinh My+\left(\frac{dp_0}{dx}\right)^2\left(\frac{3H_0\sinh 3My}{16M^2\cosh^2 Mh}-\frac{9H_0 y\cosh My}{4M\cosh^2 Mh}\right)+\left(\frac{dp_0}{dx}\right)^5\times \\ &\left[\frac{9\sinh 5My}{256M^7\cosh^5 Mh}-\frac{12\sinh 3My}{128M^7\cosh^5 Mh}+\frac{225y\cosh My}{64M^6\cosh^5 Mh}-\frac{9y\cosh 3My}{64M^6\cosh^5 Mh}+\frac{9y^2\sinh My}{32M^5\cosh^5 Mh}\right].\end{aligned} \tag{2.82}$$

Substituting the values of Ψ_0, Ψ_1 and Ψ_2 from Eqs. (2.77), (2.79), and (2.82) in Eq. (2.75) we gets

$$\frac{dp_2}{dx}=\left(\begin{array}{c} -\left(\frac{dp_0}{dx}\right)^2\left(\frac{9H_0\cosh 3Mh}{16M\cosh^2 Mh}-\frac{9H_0\cosh Mh}{4M\cosh^2 Mh}-\frac{9H_0 h\sinh Mh}{4\cosh^2 Mh}\right)-H_1 M\cosh Mh \\ -\left(\frac{dp_0}{dx}\right)^5\left\{\begin{array}{c}\frac{45\cosh 5Mh}{256M^6\cosh^5 Mh}-\frac{54\cosh 3Mh}{128M^6\cosh^5 Mh} \\ +\frac{225\cosh Mh}{64M^6\cosh^5 Mh}+\frac{261h\sinh Mh}{64M^5\cosh^5 Mh}-\frac{27h\sinh 3Mh}{64M^5\cosh^5 Mh}+\frac{9h^2\cosh Mh}{32M^4\cosh^5 Mh}\end{array}\right\}\end{array}\right) \tag{2.83}$$

$$H_1=\frac{1}{(hM\cosh Mh-\sinh Mh)}\left[\begin{array}{c}\left(\frac{dp_0}{dx}\right)^2\left\{\frac{3H_0\sinh 3Mh}{16M^2\cosh^2 Mh}-\frac{9hH_0\cosh 3Mh}{16M\cosh^2 Mh}+\frac{9h^2H_0\sinh Mh}{4\cosh^2 Mh}\right\} \\ +\left(\frac{dp_0}{dx}\right)^5\times \\ \left\{\begin{array}{c}\frac{9\sinh 5Mh}{256M^7\cosh^5 Mh}-\frac{12\sinh 3Mh}{128M^7\cosh^5 Mh}-\frac{243h^2\sinh Mh}{64M^5\cosh^5 Mh} \\ +\frac{27h^2\sinh 3Mh}{64M^5\cosh^5 Mh}-\frac{9h^3\cosh Mh}{32M^4\cosh^5 Mh}-\frac{45h\cosh 5Mh}{256M^6\cosh^5 Mh} \\ +\frac{36h\cosh 3Mh}{128M^6\cosh^5 Mh}-F_2\end{array}\right\}\end{array}\right]. \tag{2.84}$$

Now the expression for the stream function $\Psi(x, y)$ and the pressure gradient up to second order are

$$
\begin{aligned}
\Psi = {} & \left(\frac{F_0 M + \tanh Mh}{hM - \tanh Mh}\right)\left(y - \frac{\sinh My}{M\cosh Mh}\right) - \frac{\sinh My}{M\cosh Mh} \\
& + \Gamma \left[\begin{array}{l} y \left\{ \begin{array}{l} M\cosh Mh \left(\frac{F_1}{hM\cosh Mh - \sinh Mh} - \left(\frac{dp_0}{dx}\right)^3 H_0 \right) \\ + \left(\frac{dp_0}{dx}\right)^3 \left(\frac{3\cosh 3Mh}{16M^4\cosh^3 Mh} - \frac{3}{4M^4\cosh^2 Mh} - \frac{3h\sinh Mh}{4M^3\cosh^3 Mh} \right) \end{array} \right\} \\ - \sinh My \left\{ \frac{F_1}{hM\cosh Mh - \sinh Mh} - \left(\frac{dp_0}{dx}\right)^3 H_0 \right\} - \left(\frac{dp_0}{dx}\right)^3 \\ \left\{ \frac{\sinh 3My}{16M^5\cosh^3 Mh} - \frac{3y\cosh My}{4M^4\cosh^3 Mh} \right\} \end{array} \right] \\
& + \Gamma^2 \left[\begin{array}{l} y \left\{ \begin{array}{l} M\cosh Mh \left(\begin{array}{l} \frac{F_2}{hM\cosh Mh - \sinh Mh} - \left(\frac{dp_0}{dx}\right)^5 H_2 - \left(\frac{dp_0}{dx}\right)^2 \\ \left(\frac{F_1}{hM\cosh Mh - \sinh Mh} - \left(\frac{dp_0}{dx}\right)^3 H_0 \right) H_1 \end{array} \right) \\ - \left(\frac{dp_0}{dx}\right)^2 \left(\frac{9\cosh 3Mh}{16M\cosh^2 Mh} - \frac{9\cosh Mh}{4M\cosh^2 Mh} - \frac{9h\sinh Mh}{4\cosh^2 Mh} \right) \\ \left(\frac{F_1}{hM\cosh Mh - \sinh Mh} - \left(\frac{dp_0}{dx}\right)^3 H_0 \right) - \left(\frac{dp_0}{dx}\right)^5 \\ \left(\begin{array}{l} \frac{45\cosh 5Mh}{256M^6\cosh^5 Mh} - \frac{54\cosh 3Mh}{128M^6\cosh^5 Mh} + \frac{225\cosh Mh}{64M^6\cosh^5 Mh} \\ + \frac{261h\sinh Mh}{64M^5\cosh^5 Mh} - \frac{27h\sinh 3Mh}{64M^5\cosh^5 Mh} + \frac{9h^2\cosh Mh}{32M^4\cosh^5 Mh} \end{array} \right) \end{array} \right\} \\ - \sinh My \left\{ \begin{array}{l} \frac{F_2}{hM\cosh Mh - \sinh Mh} - \left(\frac{dp_0}{dx}\right)^5 H_2 - \left(\frac{dp_0}{dx}\right)^2 \\ \left(\frac{F_1}{hM\cosh Mh - \sinh Mh} - \left(\frac{dp_0}{dx}\right)^3 H_0 \right) H_1 \end{array} \right\} + \left(\frac{dp_0}{dx}\right)^2 \\ \left(\frac{3\sinh 3My}{16M^2\cosh^2 Mh} - \frac{9y\cosh My}{4M\cosh^2 Mh} \right) \left(\frac{F_1}{hM\cosh Mh - \sinh Mh} - \left(\frac{dp_0}{dx}\right)^3 H_0 \right) \\ + \left(\frac{dp_0}{dx}\right)^5 \left\{ \begin{array}{l} \frac{9\sinh 5My}{256M^7\cosh^5 Mh} - \frac{12\sinh 3My}{128M^7\cosh^5 Mh} + \frac{225y\cosh My}{64M^6\cosh^5 Mh} \\ - \frac{9y\cosh 3My}{64M^6\cosh^5 Mh} + \frac{9y^2\sinh My}{32M^5\cosh^5 Mh} \end{array} \right\} \end{array} \right],
\end{aligned}
\tag{2.85}
$$

and

$$\frac{dp}{dx} = -M^2\left(\frac{F_0 M+\tanh Mh}{hM-\tanh Mh}+1\right)$$
$$+\Gamma\left[\begin{array}{c} -M^3\cosh Mh\left\{\frac{F_1}{hM\cosh Mh-\sinh Mh}-\left(\frac{dp_0}{dx}\right)^3 H_0\right\} \\ +\left(\frac{dp_0}{dx}\right)^3\left(-\frac{3\cosh 3Mh}{16M^2\cosh^3 Mh}+\frac{3}{4M^2\cosh^2 Mh}+\frac{3h\sinh Mh}{4M\cosh^3 Mh}\right)\end{array}\right]$$
$$+\Gamma^2\left[\begin{array}{c} -M^3\cosh Mh\left(\begin{array}{c}\frac{F_2}{hM\cosh Mh-\sinh Mh}-\left(\frac{dp_0}{dx}\right)^5 H_2-\left(\frac{dp_0}{dx}\right)^2 \\ \left(\frac{F_1}{hM\cosh Mh-\sinh Mh}-\left(\frac{dp_0}{dx}\right)^3 H_0\right)H_1\end{array}\right) \\ +M^2\left(\frac{dp_0}{dx}\right)^2\left(\frac{9\cosh 3Mh}{16M\cosh^2 Mh}-\frac{9\cosh Mh}{4M\cosh^2 Mh}-\frac{9h\sinh Mh}{4\cosh^2 Mh}\right) \\ \left(\frac{F_1}{hM\cosh Mh-\sinh Mh}-\left(\frac{dp_0}{dx}\right)^3 H_0\right)+\left(\frac{dp_0}{dx}\right)^5 \\ \left(\begin{array}{c}\frac{45\cosh 5Mh}{256M^4\cosh^5 Mh}-\frac{54\cosh 3Mh}{128M^4\cosh^5 Mh}+\frac{225\cosh Mh}{64M^4\cosh^5 Mh} \\ +\frac{261h\sinh Mh}{64M^3\cosh^5 Mh}-\frac{27h\sinh 3Mh}{64M^3\cosh^5 Mh}+\frac{9h^2\cosh Mh}{32M^2\cosh^5 Mh}\end{array}\right)\end{array}\right] \quad (2.86)$$

$$H_2=\frac{1}{(hM\cosh Mh-\sinh Mh)}\left[\begin{array}{c}\frac{9\sinh 5Mh}{256M^7\cosh^5 Mh}-\frac{12\sinh 3Mh}{128M^7\cosh^5 Mh}-\frac{243h^2\sinh Mh}{64M^5\cosh^5 Mh} \\ +\frac{27h^2\sinh 3Mh}{64M^5\cosh^5 Mh}-\frac{9h^3\cosh Mh}{32M^4\cosh^5 Mh}-\frac{45h\cosh 5Mh}{256M^6\cosh^5 Mh} \\ +\frac{36h\cosh 3Mh}{128M^6\cosh^5 Mh}\end{array}\right]. \quad (2.87)$$

We have from Eq.(2.67)

$$F=F_0+\Gamma F_1+\Gamma^2 F_2.$$

Using the above equation in Eq.(2.86) and neglecting the terms greater than $O(\Gamma^2)$ we get

$$\frac{dp}{dx} = \frac{-M^3(F+h)\cosh Mh}{(hM\cosh Mh-\sinh Mh)}$$
$$+\Gamma\left[\begin{array}{c}\frac{M^8(F+h)^3}{(hM\cosh Mh-\sinh Mh)^3}\left\{-\frac{3\cosh Mh}{4M}-\frac{3h\sinh Mh}{4}+\frac{3\cosh 3Mh}{16M}\right\} \\ -\frac{M^{12}(F+h)^3\cosh^4 Mh}{(hM\cosh Mh-\sinh Mh)^4}H_0\end{array}\right]$$
$$+\Gamma^2\left[\begin{array}{c}\frac{M^{18}(F+h)^5\cosh^6 Mh}{(hM\cosh Mh-\sinh Mh)^6}H_0H_1-\frac{M^{18}(F+h)^5\cosh^6 Mh}{(hM\cosh Mh-\sinh Mh)^6}H_2 \\ +\frac{M^{17}(F+h)^5\cosh^3 Mh}{(hM\cosh Mh-\sinh Mh)^5}H_0\left\{\frac{9\cosh 3Mh}{16M}-\frac{9h\sinh Mh}{4}-\frac{9\cosh Mh}{4M}\right\} \\ -\frac{M^{13}(F+h)^5}{(hM\cosh Mh-\sinh Mh)^5}\left\{\begin{array}{c}\frac{45\cosh 5Mh}{256M^2}-\frac{54\cosh 3Mh}{128M^2}+\frac{225\cosh Mh}{64M^2} \\ +\frac{261h\sinh Mh}{64M}-\frac{27h\sinh 3Mh}{64M}+\frac{9h^2\cosh Mh}{32}\end{array}\right\}\end{array}\right].$$

(2.88)

The expressions of non-dimensional pressure rise per wave length $(\triangle P_\lambda)$ and frictional force (F_λ) are

$$\triangle P_\lambda = \int_0^{2\pi} \frac{dp}{dx} dx, \tag{2.89}$$

$$F_\lambda = \int_0^{2\pi} \left(-h\frac{dp}{dx}\right) dx. \tag{2.90}$$

2.6 Result and discussion

In this section we have discussed the graphical results to examine the effects of Deborah number (Γ), amplitude ratio (ϕ) and Hartman number (M) on the pressure gradient (dp/dx), pressure rise per wave length $(\triangle P_\lambda)$ and frictional force (F_λ).

In Figs. 2.1 and 2.2 we plot the pressure gradient against x with a wave length $x \in [0, 2\pi]$. These plot show the effect of Hartman number (M) and amplitude ratio (ϕ). It is noted that in the wider part of the channel, when $x \in [0, \pi]$ the pressure gradient is small and the flow can easily pass and there is no need of large pressure gradient. However in the narrow part of the channel when $x \in [\pi, 2\pi]$ a much large pressure gradient is needed to maintain the same flux to pass it,when compared with the wider part of the channel $x \in [0, \pi]$. It is also further observed that by increasing the amplitude ratio(ϕ) more pressure gradient is required to maintain the same flux.

Figs. $2.3-2.5$ display the behavior of pressure rise $\triangle P_\lambda$. For these Eq. (2.89) is numerically integrated. Graphs are displayed for different values of Hartman number (M), Deborah number (Γ) and amplitude ratio (ϕ). Pressure rise $\triangle P_\lambda$ is plotted against flow rate F. It is found that by increasing the Hartman number (M) the pressure riser per wave length decreases in magnitude. For Newtonian fluid (when Γ=0) there is a linear relationship between the pressure rise $\triangle P_\lambda$ and the flow rate F. For $\Gamma \neq 0$ the pumping curves are non-linear and falls below the Newtonian fluid. Moreover the pumping rate decreases with an increase in the amplitude ratio (ϕ).

Figs. $2.6-2.8$ illustrate the variations of frictional forces. In order to report such variations, Eq. (2.90) is integrated numerically. In these figures the variations of Hartman number (M), Deborah number (Γ) and amplitude ratio (ϕ) are seen. Here the frictional forces (F_λ) are plotted against the flow rate. It is noted that the frictional forces have opposite behaviour with respect to these three parameters when compared with the graphs of pressure rise $\triangle P_\lambda$ verses flow rate. i.e by increasing M, Γ and ϕ the frictional forces increase and resist the flow.

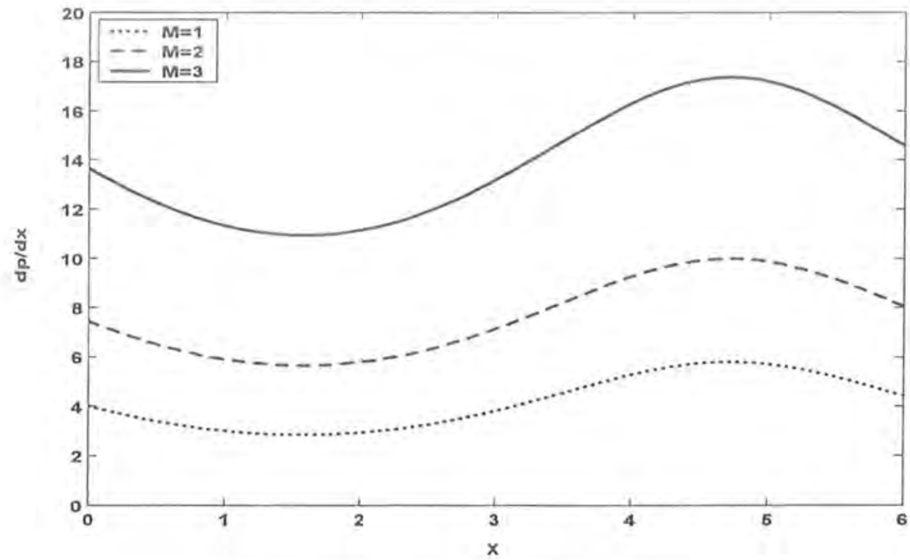


Fig.2.1 : $F = -2, \phi = 0.1, \Gamma = 0.001$

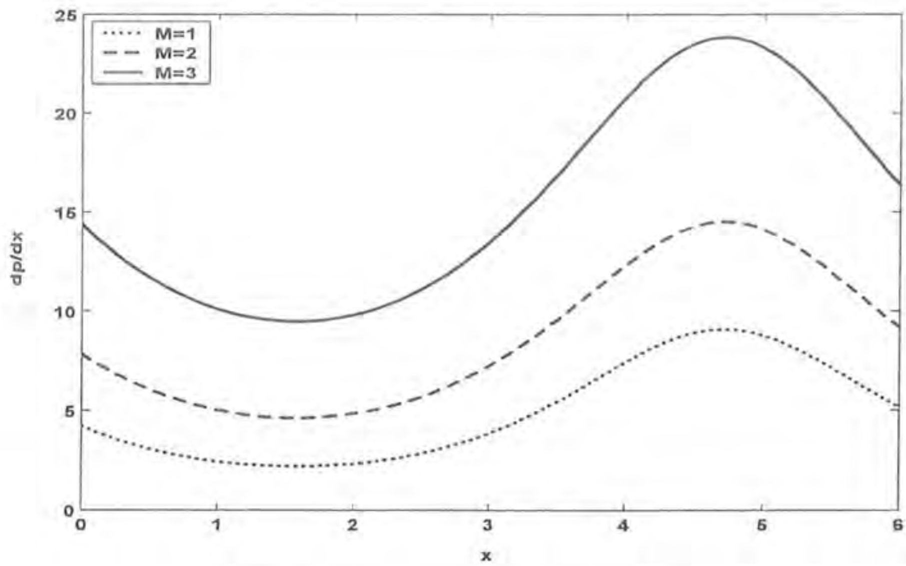


Fig. 2.2 : $F = -2, \phi = 0.2, \Gamma = 0.001$

Fig. 2.1 and 2.2 show the variation of pressure gradient dp/dx within wave length $x \in [0, 2\pi]$ for different values of Hartman number M and amplitude ratio ϕ by keeping other parameters fixed.

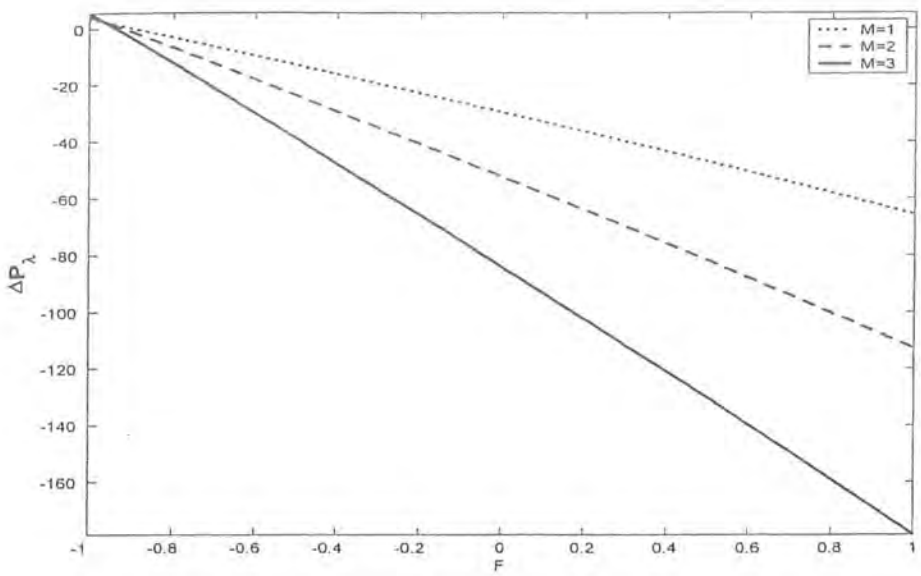


Fig. 2.3 : $\phi = 0.3$, $\Gamma = 0.001$

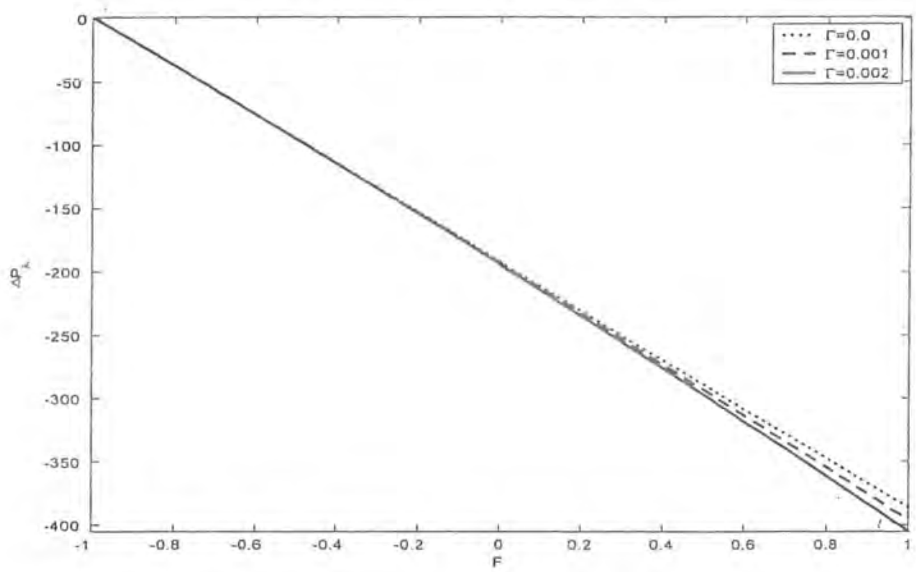


Fig. 2.4 : $\phi = 0.2$, $M = 5$

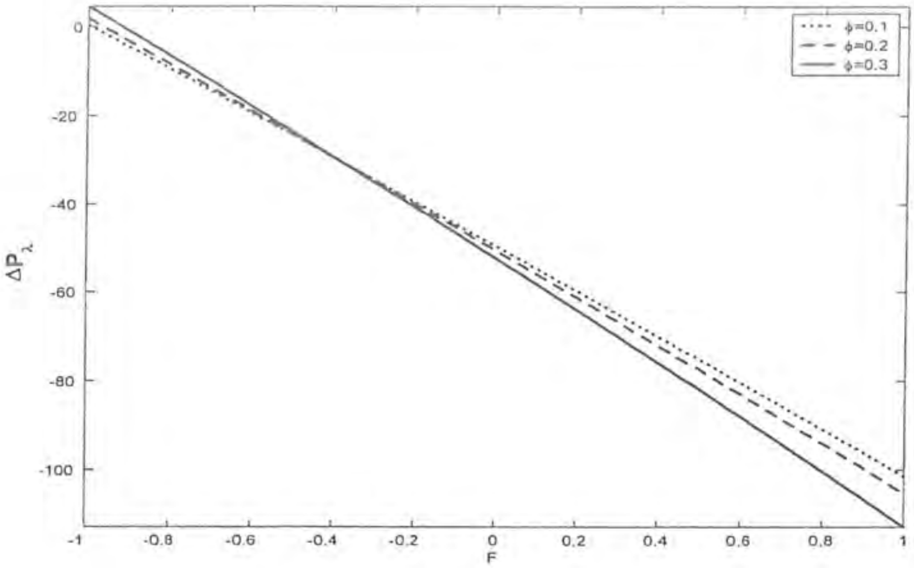


Fig. 2.5 : $M = 2$, $\Gamma = 0.001$

Figs. 2.3−2.5 show the variation of pressure rise per wavelength versus flow rate for different values of Hartman number M, Deborah number (Γ) and amplitude ratio ϕ by keeping other parameters fixed.

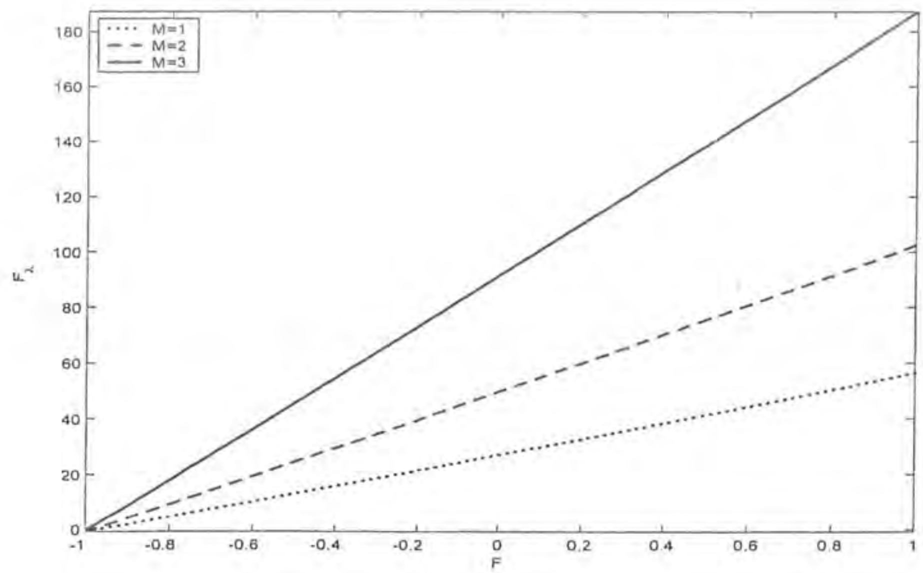


Fig. 2.6 : $\phi = 0.3$, $\Gamma = 0.001$

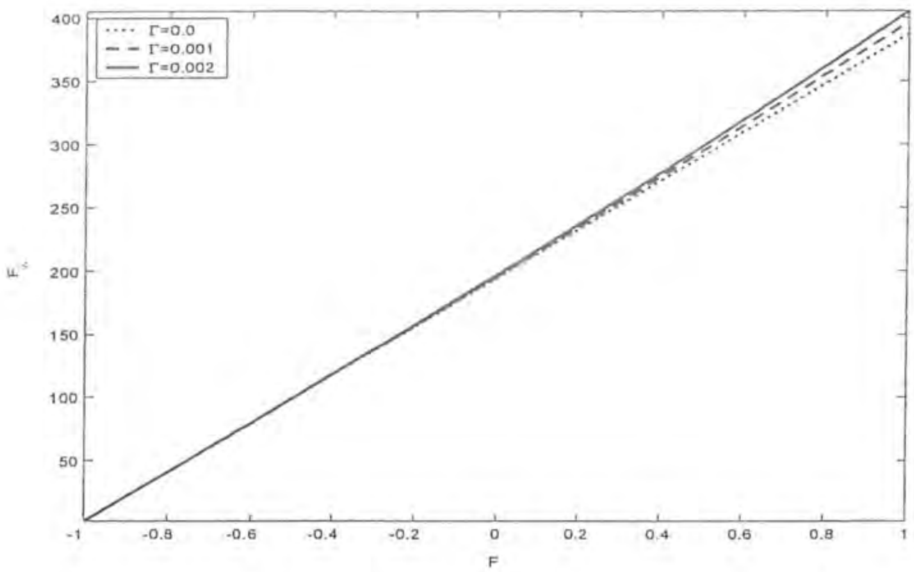


Fig. 2.7 : $\phi = 0.3, M = 5$

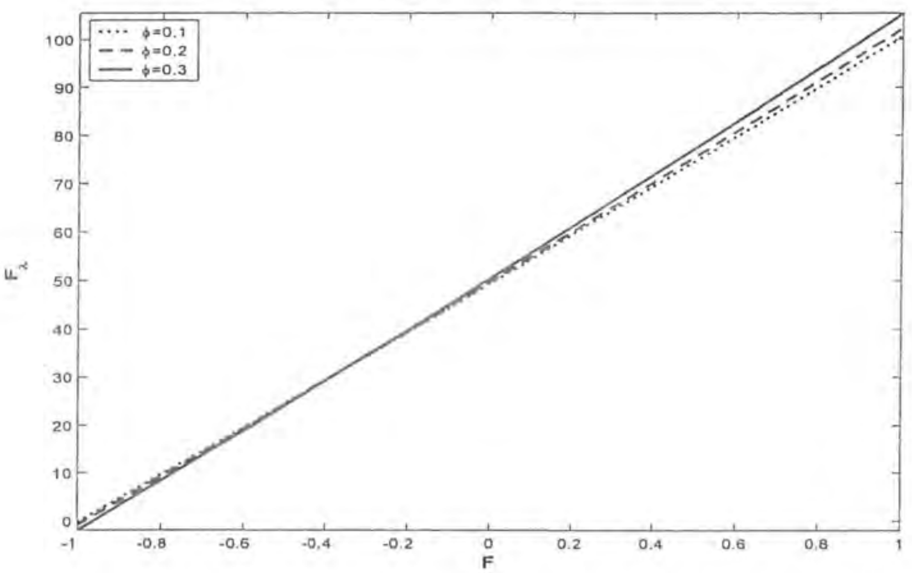


Fig. 2.8 : $M = 2, \Gamma = 0.001$

Figs. $2.6 - 2.8$ show the variation of frictional force per wavelength versus flow rate for different values of Hartman number M, Deborah number (Γ) and amplitude ratio ϕ by keeping other parameters fixed.

Chapter 3

Analytical solution for MHD Peristaltic flow by an Adomian decomposition method

The aim of this chapter is to provide an Adomian decomposition solution for MHD peristaltic flow of a third grade fluid. The non-linear problem is first solved and then comparison is provided between the solutions obtained by perturbation and Adomian decomposition methods.

3.1 Solution by Adomian decomposition method

In this section we consider the same problem as formulated in chapter two. Therefore the problem statement consist of Eqs. (2.60), (2.59).and (2.66). For Adomian decomposition solution, Eq. (2.66) in operator form can be written as

$$L\Psi = -2\Gamma \frac{\partial^2}{\partial y^2}\left(\frac{\partial^2 \Psi}{\partial y^2}\right)^3 + M^2 \frac{\partial^2 \Psi}{\partial y^2}, \tag{3.1}$$

where L is a fourth order differential operator i.e $\left(L = \frac{\partial^4 \Psi}{\partial y^4}\right)$ and is invertible so that

$$L^{-1}(\cdot) = \int_0^x \int_0^x \int_0^x \int_0^x (\cdot)\, dxdxdxdx. \tag{3.2}$$

By using Eq. (3.2) into Eq. (3.1) one can write

$$\Psi = C_1 + yC_2 + y^2C_3 + y^3C_4 - 2\Gamma L^{-1}\left[\frac{\partial^2}{\partial y^2}\left(\frac{\partial^2\Psi}{\partial y^2}\right)^3\right] + M^2L^{-1}\left[\frac{\partial^2\Psi}{\partial y^2}\right], \tag{3.3}$$

in which C_i $(i = 1 - 4)$ are function of integration.

Employing Adomian decomposition method we may express that

$$\Psi = \sum_{n=0}^{\infty}\Psi_n(x). \tag{3.4}$$

For $n > 0$ the values of $\Psi_n(x)$ can be calculated by recursive relations. From Eqs. (3.3) and (3.4) we get

$$\begin{aligned}\sum_{n=0}^{\infty}\Psi_n(x) &= C_1 + yC_2 + y^2C_3 + y^3C_4 \\ &\quad -2\Gamma L^{-1}\left[\frac{\partial^2}{\partial y^2}\left(\frac{\partial^2}{\partial y^2}\left\{\sum_0^{\infty}\Psi_n(x)\right\}\right)^3\right] \\ &\quad +M^2L^{-1}\left[\frac{\partial^2}{\partial y^2}\left\{\sum_0^{\infty}\Psi_n(x)\right\}\right].\end{aligned} \tag{3.5}$$

The expansion of Ψ up to second order is

$$\Psi = \sum_{n=0}^{\infty}\Psi_n(x) = \Psi_0 + \Psi_1 + \Psi_2. \tag{3.6}$$

Invoking above expression we get the following systems.

3.1.1 System for Ψ_0

$$L\Psi_0 = 0 = C_1 + yC_2 + y^2C_3 + y^3C_4, \tag{3.7}$$

$$\begin{aligned}\Psi_0 &= 0, \qquad \frac{\partial^2\Psi_0}{\partial y^2} = 0 \qquad at\ y = 0, \\ \frac{\partial\Psi_0}{\partial y} &= -1, \qquad \Psi_0 = F \qquad at\ y = h.\end{aligned} \tag{3.8}$$

3.1.2 System for Ψ_1

$$\Psi_1 = -2\Gamma L^{-1}\left[\frac{\partial^2}{\partial y^2}\left(\frac{\partial^2\Psi_0}{\partial y^2}\right)^3\right] + M^2L^{-1}\left[\frac{\partial^2\Psi_0}{\partial y^2}\right], \tag{3.9}$$

$$\begin{aligned}\Psi_1 &= 0, \qquad \frac{\partial^2\Psi_1}{\partial y^2}=0 \qquad at\ y=0,\\ \frac{\partial\Psi_1}{\partial y} &= 0, \qquad \Psi_1=0 \qquad at\ y=h.\end{aligned} \tag{3.10}$$

3.1.3 System for Ψ_2

$$\Psi_2 = -6\Gamma L^{-1}\left[\frac{\partial^2}{\partial y^2}\left\{\left(\frac{\partial^2\Psi_0}{\partial y^2}\right)^2\left(\frac{\partial^2\Psi_1}{\partial y^2}\right)\right\}\right] + M^2L^{-1}\left[\frac{\partial^2\Psi_1}{\partial y^2}\right], \tag{3.11}$$

$$\begin{aligned}\Psi_2 &= 0, \qquad \frac{\partial^2\Psi_2}{\partial y^2}=0 \qquad at\ y=0,\\ \frac{\partial\Psi_2}{\partial y} &= 0, \qquad \Psi_2=0 \qquad at\ y=h.\end{aligned} \tag{3.12}$$

3.1.4 Solution for Ψ_0

Substitution of boundary conditions (3.8) into Eq. (3.7) yields the following expression

$$C_1 = 0, \tag{3.13(a)}$$

$$C_3 = 0, \tag{3.13(b)}$$

$$C_2 + 2hC_3 + 3h^2C_4 = -1, \tag{3.13(c)}$$

$$C_1 + hC_2 + h^2C_3 + h^3C_4 = F. \tag{3.13(d)}$$

The solutions of Eqs. (3.13 (c)) and (3.13 (d)) are

$$C_2 = \frac{3F+h}{2h}, \tag{3.13(e)}$$

$$C4 = -\left(\frac{F+h}{2h^3}\right). \tag{3.13(f)}$$

Upon making use of values of C_i $(i = 1 - 4)$, Eq. (3.7) reduces to

$$\Psi_0 = \frac{y}{2h}\left[(F+h)\left(1-\frac{y^2}{h^2}\right)+2F\right]. \tag{3.14}$$

3.1.5 Solution for Ψ_1

By putting the value of Ψ_0 from Eq. (3.14) into Eq. (3.9) and then integrating we have

$$\begin{aligned}\Psi_1 &= \frac{(F+h)\,y^5\left[-h^6M^2+\left(108F^2+216Fh+108h^2\right)\Gamma\right]}{40h^9} \\ &\quad +C_5+yC_6+y^2C_7+y^3C_8. \end{aligned} \tag{3.15}$$

The above solution after using the boundary conditions (3.10) gives

$$C_5 = 0, \tag{3.16(a)}$$

$$C_7 = 0, \tag{3.16(b)}$$

$$\begin{aligned}&\frac{(F+h)\left[-h^6M^2+\left(108F^2+216Fh+108h^2\right)\Gamma\right]}{8h^5} \\ &+C_6+2hC_7+3h^2C_8=0,\end{aligned} \tag{3.16(c)}$$

$$\begin{aligned}&\frac{(F+h)\left[-h^6M^2+\left(108F^2+216Fh+108h^2\right)\Gamma\right]}{40h^4} \\ &+C_5+hC_6+h^2C_7+h^3C_8=0.\end{aligned} \tag{3.16(d)}$$

Solving Eqs. (3.16 (c)) and (3.16 (d)) one may write

$$C_6 = \frac{\left[-Fh^6M^2-h^7M^2+\left(108F^3+324F^2h+324Fh^2+108h^3\right)\Gamma\right]}{40h^5}, \tag{3.16(e)}$$

$$C_8 = \frac{\left[Fh^6M^2+h^7M^2-\left(108F^3+324F^2h+324Fh^2-108h^3\right)\Gamma\right]}{20h^7}. \tag{3.16(f)}$$

Now Eq. (3.15) after using the values of C_i $(i = 5 - 8)$ finally takes the form

$$\Psi_1 = (F+h)\left[108\,(F+h)^2\,\Gamma - h^6 M^2\right]\left[\frac{y}{40h^5} - \frac{y^3}{20h^7} + \frac{y^5}{40h^9}\right]. \tag{3.17}$$

3.1.6 Solution for Ψ_2

From Eqs. (3.11), (3.14) and (3.17) we arrive at

$$\Psi_2 = -6\Gamma L^{-1}\left[\frac{\partial^2}{\partial y^2}\left(L_1 y^3 + L_2 y^5\right)\right] + M^2 L^{-1}\left[L_3 y + L_4 y^3\right], \tag{3.18}$$

$$L1 = \frac{27\,(F+h)^3}{10h^{13}}\left[h^6 M^2 - 108\,(F+h)^2\,\Gamma\right],$$

$$L2 = \frac{-9\,(F+h)^3}{2h^{15}}\left[h^6 M^2 + 108\,(F+h)^2\,\Gamma\right],$$

$$L3 = \frac{3\,(F+h)}{10h^7}\left[h^6 M^2 - 108\,(F+h)^2\,\Gamma\right],$$

$$L4 = \frac{-\,(F+h)}{2h^9}\left[h^6 M^2 - 108\,(F+h)^2\,\Gamma\right].$$

Integrating above equations and then using the boundary conditions (3.12) we have

$$\Psi_2 = \frac{-\,(F+h)}{8400h^{15}}\left[A_1 y + A_2 y^3 + A_3 y^5 + A_4 y^7\right], \tag{3.19}$$

where

$$\begin{aligned}
A_1 &= -11h^{18}M^4 - 2808h^{12}\,(F+h)^2\,M^2\Gamma + 431568h^6\,(F+h)^4\,\Gamma^2,\\
A_2 &= 27h^{16}M^4 - 324h^{10}\,(F+h)^2\,M^2\Gamma - 279936h^4\,(F+h)^4\,\Gamma^2,\\
A_3 &= -21h^{14}M^4 + 9072h^8\,(F+h)^2\,M^2\Gamma - 734832h^2\,(F+h)^4\,\Gamma^2,\\
A_4 &= 5h^{12}M^4 - 5940h^6\,(F+h)^2\,M^2\Gamma + 583200\,(F+h)^4\,\Gamma^2.
\end{aligned}$$

The expression of stream function (3.6) now becomes

$$\Psi = A_5 y + A_6 y^3 + A_7 y^5 + A_8 y^7. \tag{3.20}$$

$$A_5 = \frac{F}{h} + \frac{(F+h)}{8400h^9}\left[\begin{array}{c} h^8\left\{4200 - 210h^2M^2 + 11h^4M^4\right\} \\ +216h^4(F+h)^2\left(105 + 13h^2M^2\right)\Gamma - 431568(F+h)^4\Gamma^2 \end{array}\right],$$

$$A_6 = \frac{(F+h)}{2800h^{11}}\left[\begin{array}{c} -h^8\left\{1400 - 140h^2M^2 + 9h^4M^4\right\} \\ +108h^4(F+h)^2\left(-140 + h^2M^2\right)\Gamma - 93312(F+h)^4\Gamma^2 \end{array}\right],$$

$$A_7 = \frac{(F+h)}{400h^{13}}\left[\begin{array}{c} h^8\left\{-10h^2M^2 + h^4M^4\right\} \\ -216h^4(F+h)^2\left(-5 + 2h^2M^2\right)\Gamma + 34992(F+h)^4\Gamma^2 \end{array}\right],$$

$$A_8 = \frac{-(F+h)}{1680h^{15}}\left[h^{12}M^4 - 1188h^6(F+h)^2M^2\Gamma + 166640(F+h)^4\Gamma^2\right].$$

The value of pressure gradient (2.62) is

$$\frac{dp}{dx} = \frac{\partial^3\Psi}{\partial y^3} + 2\Gamma\frac{\partial}{\partial y}\left(\frac{\partial^2\Psi}{\partial y^2}\right)^3 - M^2\left(\frac{\partial\Psi}{\partial y} + 1\right). \tag{3.21}$$

Putting the value of Ψ from Eq. (3.20) in Eq. (3.21) and simplifying the resulting expression we can write

$$\frac{dp}{dx} = \frac{-(F+h)}{8400h^3}A_9 - \frac{9(F+h)^3}{350h^7}A_{10}\Gamma + \frac{243(F+h)^5}{175h^{11}}A_{11}\Gamma^2, \tag{3.22}$$

where

$$\begin{aligned} A_9 &= 25200 + 10080h^2M^2 - 48h^4M^4 + 11h^6M^6, \\ A_{10} &= 1260 + 96h^2M^2 + 13h^4M^4, \\ A_{11} &= 144 + 37h^2M^2. \end{aligned}$$

3.2 Discussion of graphs

The purpose of this section is to discuss the graphs of velocity, stream function, pressure gradient and pressure rise. Particular attention has been focused to the variations of M and Γ.

In order to illustrate the variations of M on the velocity we plotted Figs.3.1 – 3.3. These figures depict that graphs for perturbation and ADM solutions coincide when $0 \leq M < 2$. However the difference in the graphs is more apparent for large values of M. It is noted that the ADM solution of velocity is large when compared with perturbation solution. Furthermore the velocity in both cases is similar near the boundary.

The variations of Γ on the velocity are seen in Figs.3.4 – 3.7. These figures show that perturbation and ADM solutions are identical when $\Gamma < 0.01$. However the difference between the solutions increases for $\Gamma > 0.01$ i.e the velocity profile for ADM solution is greater then the perturbation solution. It is also noted that a significant difference in both solutions occur at the center line of the channel.

In order to report the effects of M on the stream function we plotted Figs. 3.8 and 3.9 These figures elucidate that the perturbation and ADM solutions are similar when $M \leq 3$. The difference in these solutions is noted for $M > 3$.

Figs.3.10 – 3.12 show the influence of Γ on stream function Ψ. It is noted that the perturbation and ADM expressions of Ψ show similar result for small Γ. These solutions for Ψ are observed when Γ is large.

In Figs. 3.13 – 3.16 pressure gradient is plotted against x with a wave length $x \in [0, 2\pi]$, which plot the effects of Γ and M for comparison of preturbation and ADM solutions.

Fig. 3.13 and 3.14 show that for small Γ the solution obtained by preturbation method and ADM are identical but the difference occurs when $\Gamma > 0.001$. In Fig. 3.15 and 3.16 it is noted that the preturbation and ADM solutions coincide when $M \leq 2$ and the variation is seen for $M > 2$ Here one can say that the two methods are identical for small M.

In Figs. 3.17 – 3.20 pressure rise is plotted against flow field F. Perturbation and ADM solution are compared for different values of Γ and M. Figs.3.17 and 3.18 depict that for $\Gamma = 0.001$ both solutions are identical but variation is seen for large Γ. In Figs. 3.19 and 3.20 it is observed that the solution by two methods coincide for $M \leq 2$ and the variation occurs when $M > 2$.

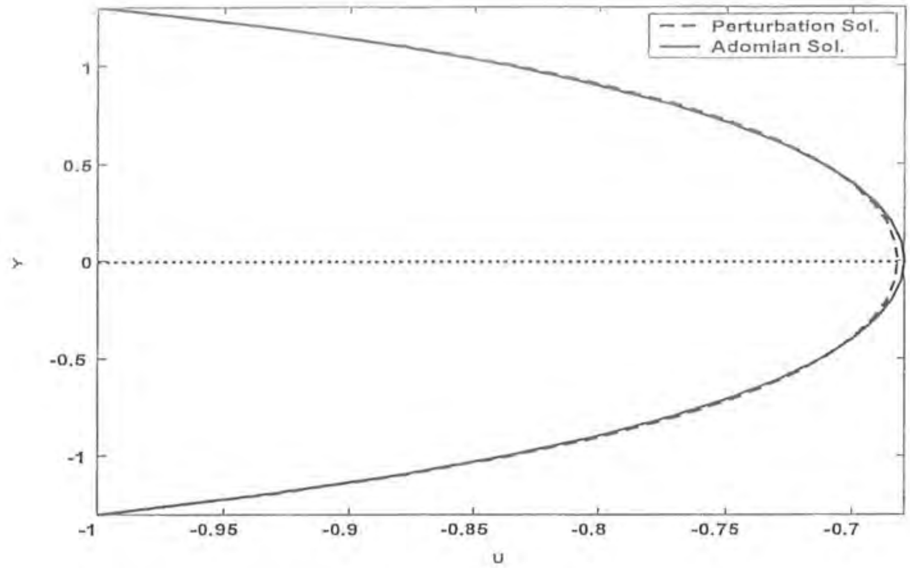


Fig. 3.1 : $M = 2, \phi = 0.2, \Gamma = 0.001$

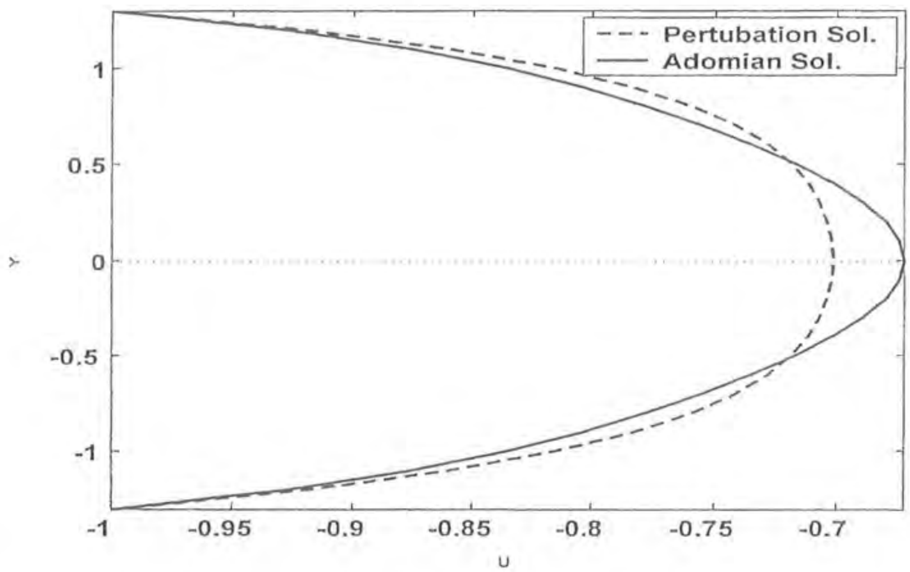


Fig. 3.2 : $M = 3, \phi = 0.2, \Gamma = 0.001$

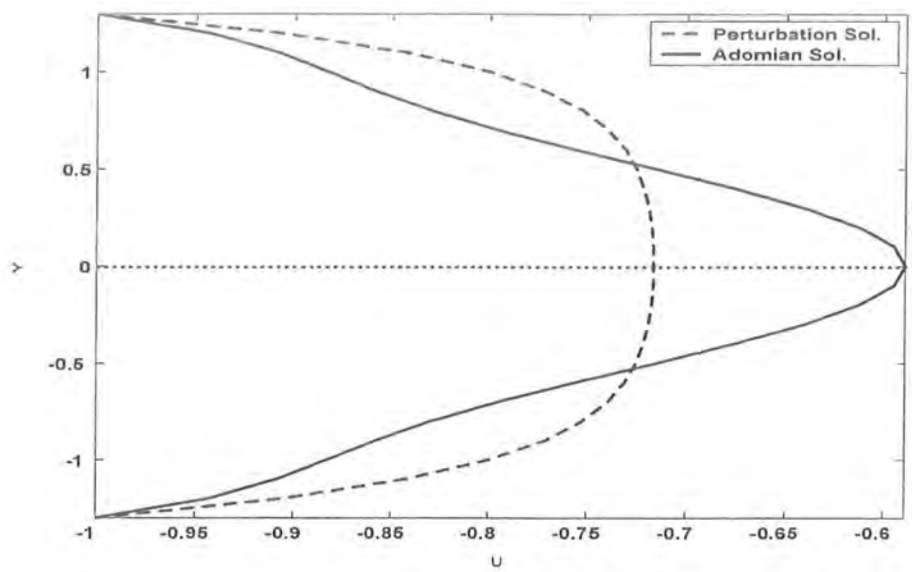


Fig. 3.3 : $M = 4, \phi = 0.2, \Gamma = 0.001$

Figs. $3.1 - 3.3$ show variation in velocity for perturbation and Adomian decomposition method for different values of Hartman number M and keeping other parameters fixed.

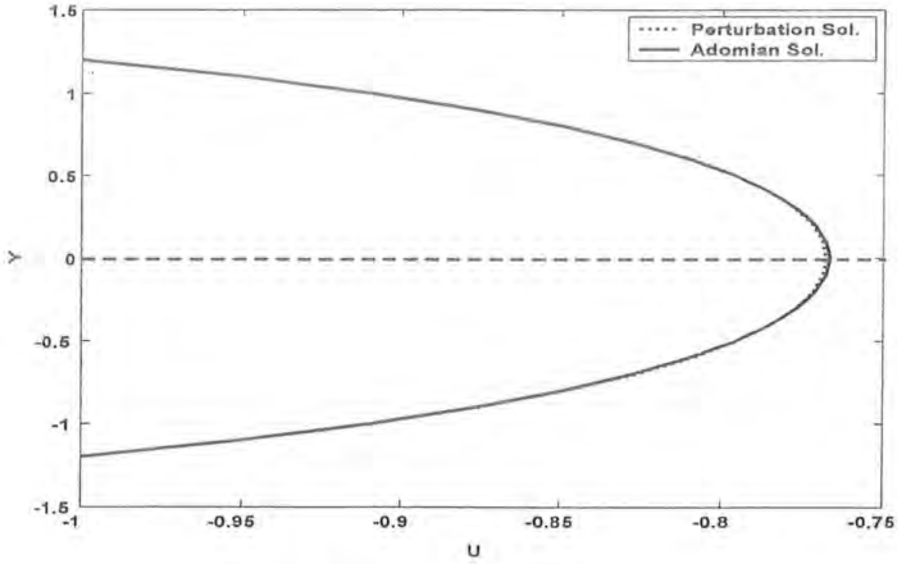


Fig. 3.4 : $M = 2, \phi = 0.2, \Gamma = 0.001$

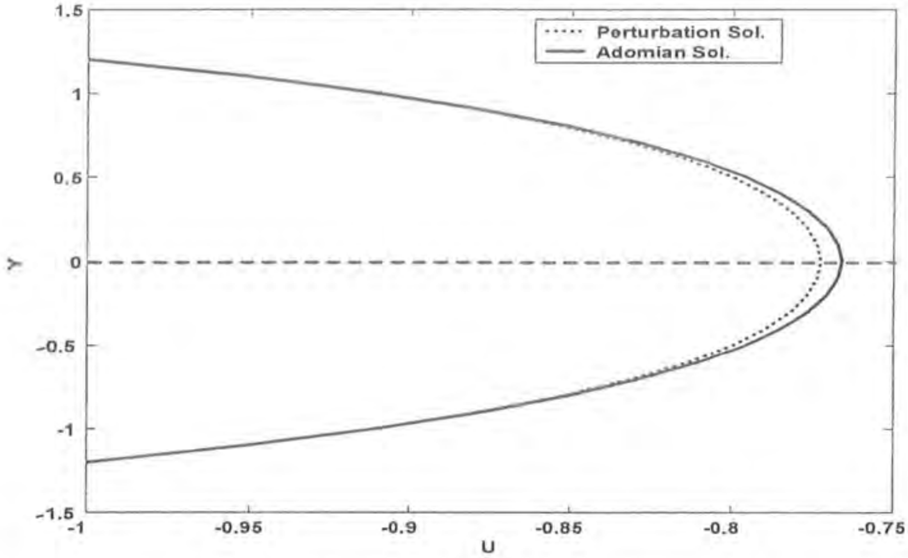


Fig. 3.5 : $M = 2, \phi = 0.2, \Gamma = 0.05$

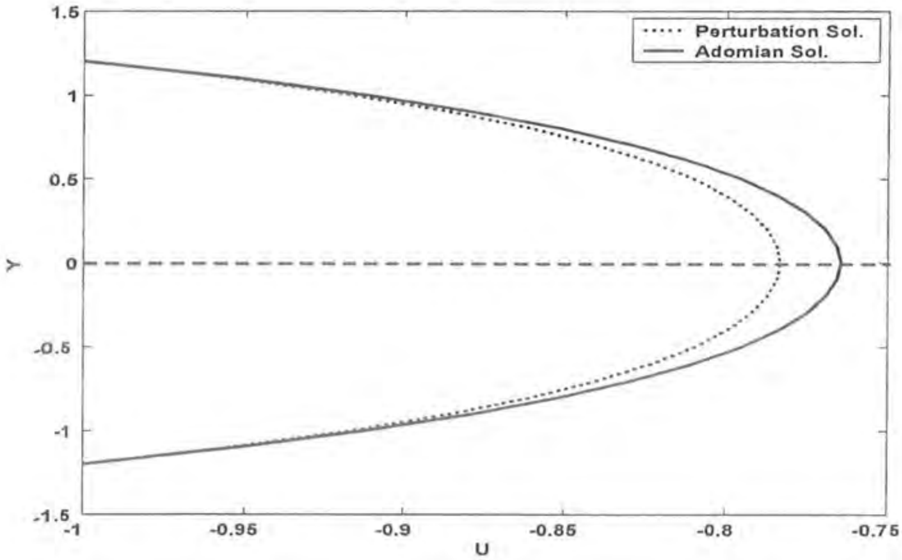


Fig. 3.6 : $M = 2, \phi = 0.2, \Gamma = 0.15$

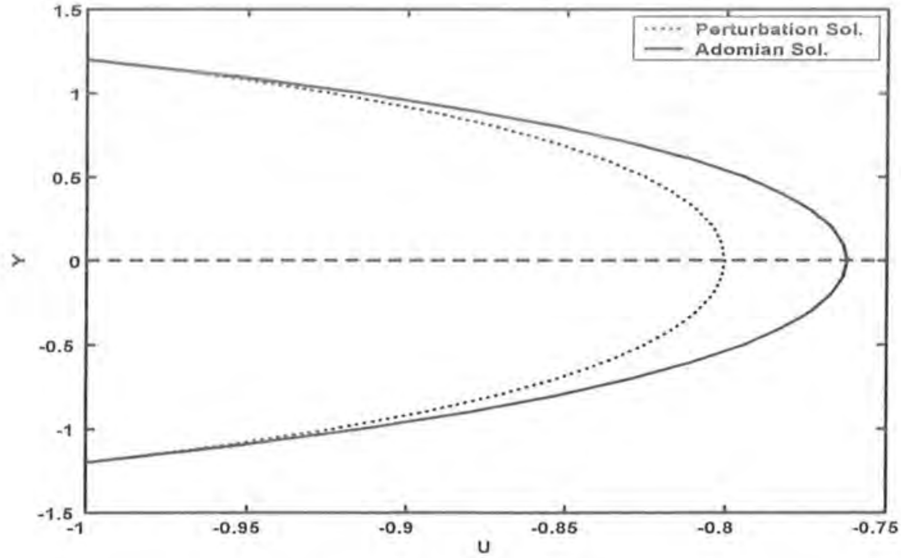


Fig. 3.7 : $M = 2, \phi = 0.2, \Gamma = 0.3$

Figs. 3.4−3.7 indicate variation in velocity by perturbation and Adomian decomposition method for different values of Γ (Deborah number) and keeping other parameters fixed.

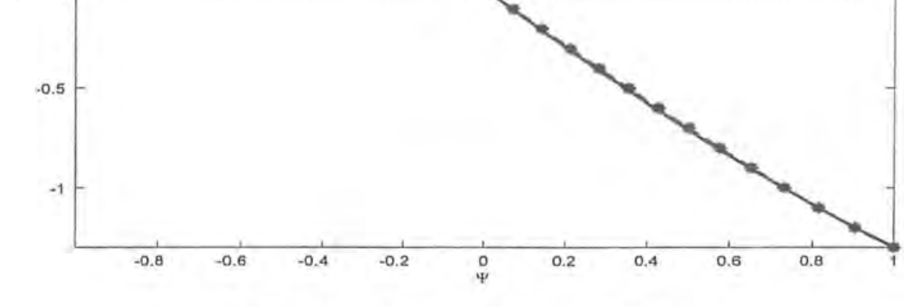


Fig. 3.8 : $M = 3, \phi = 0.2, \Gamma = 0.001$

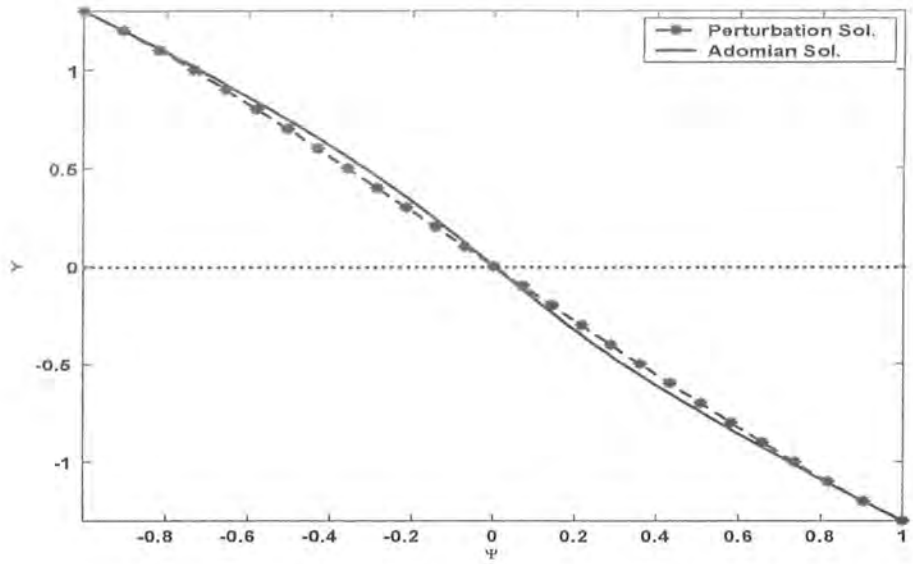


Fig. 3.9 : $M = 4, \phi = 0.2, \Gamma = 0.001$

Figs. 3.8 – 3.9 depict variation in stream function by perturbation and Adomian decomposition method for different values of Hartman number M and keeping other parameters fixed.

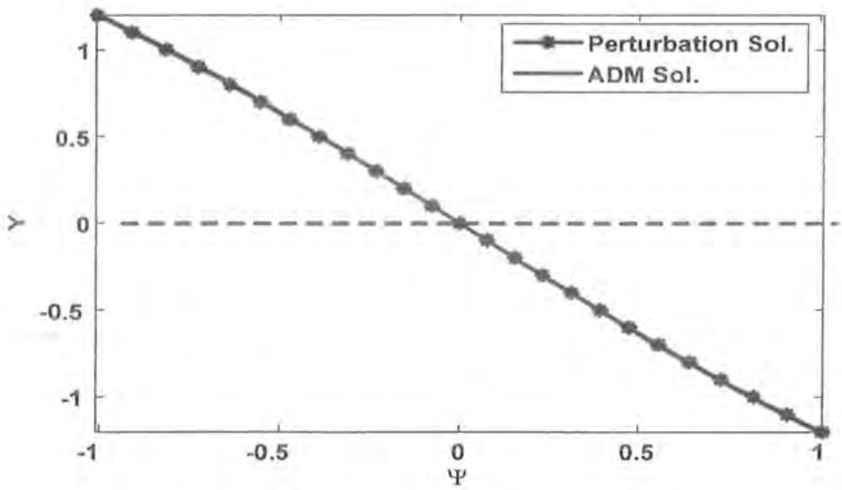


Fig. 3.10 : $M = 2, \phi = 0.2, \Gamma = 0.1$

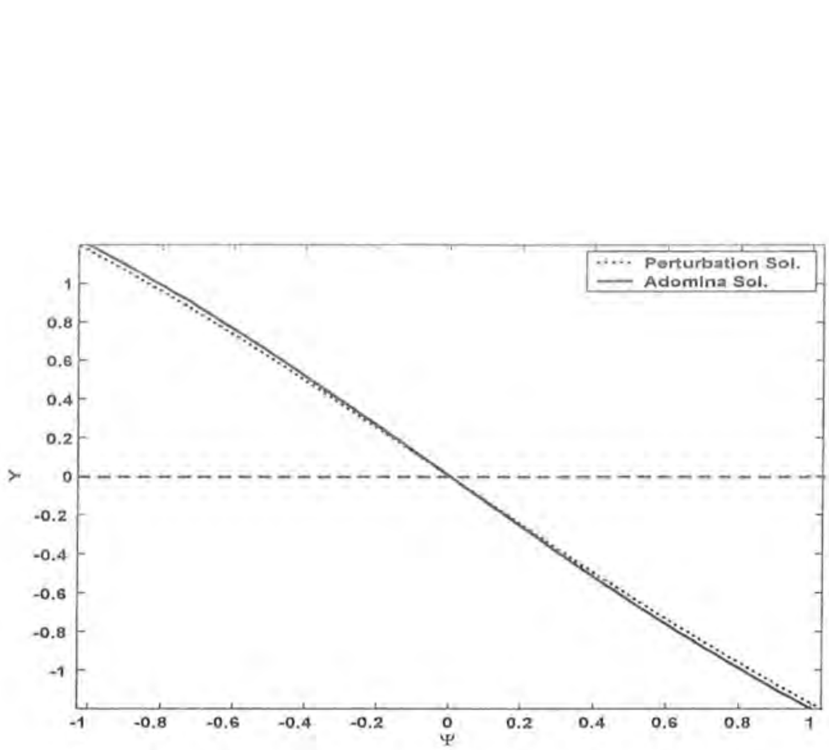


Fig. 3.11 : $M = 2, \phi = 0.2, \Gamma = 0.3$

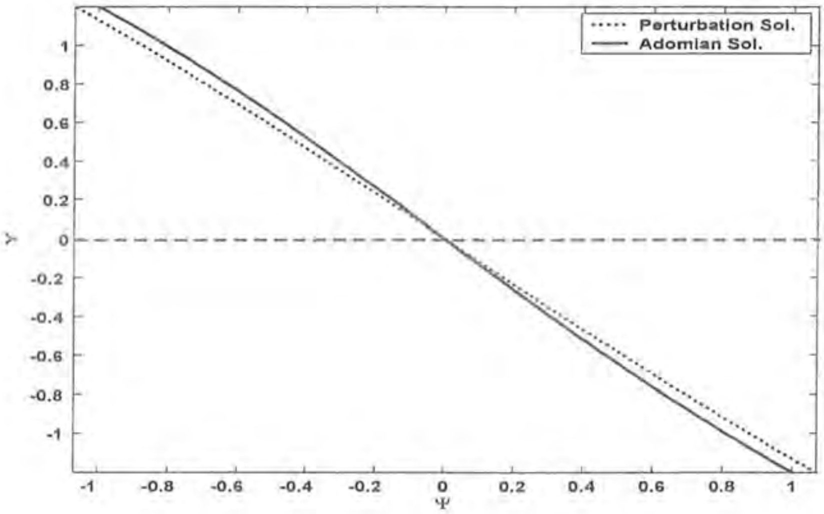


Fig. 3.12 : $M = 2, \phi = 0.2, \Gamma = 0.5$

Figs. 3.10 − 3.12 show variation in stream function by perturbation and Adomian decomposition method for different values of Γ (Deborah number) and keeping other parameters fixed.

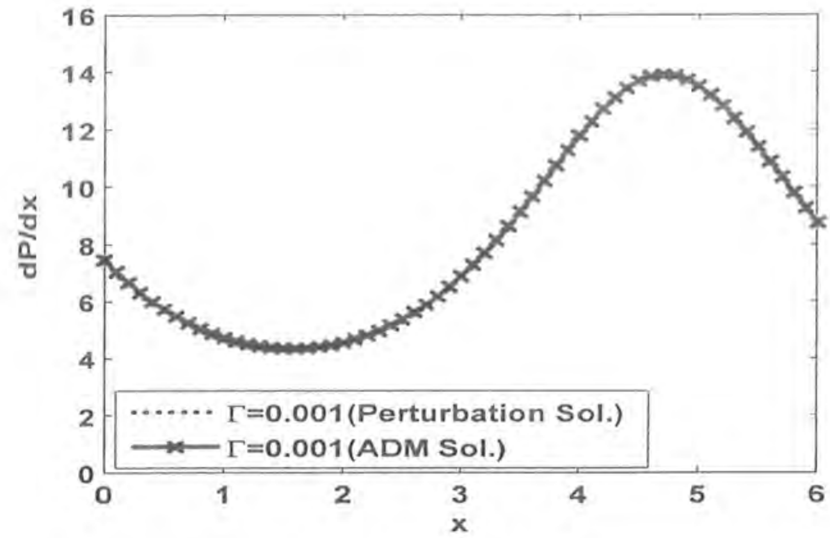


Fig. 3.13 : $M = 2, \phi = 0.2$

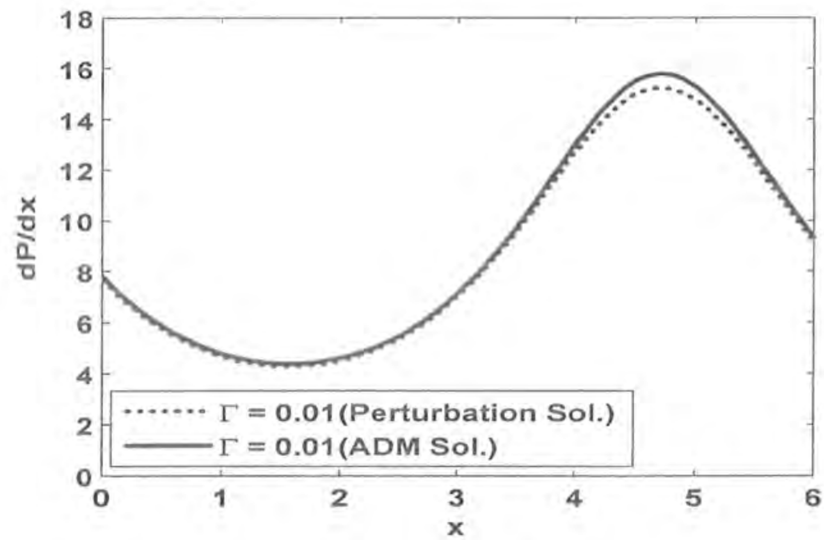


Fig. 3.14 : $M = 2, \phi = 0.2$

Figs. 3.13 and 3.14 show variation in pressure gradient by perturbation and Adomian decomposition method for different values of Γ (Deborah number) and keeping other parameters fixed.

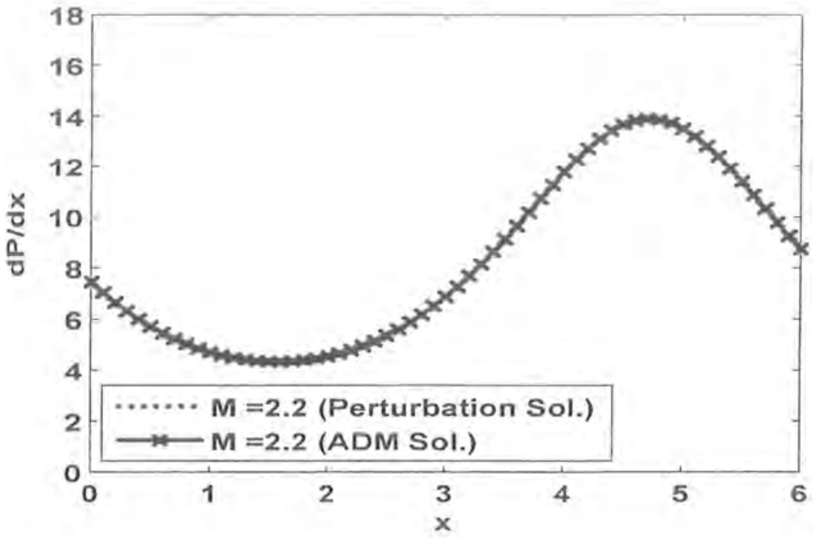


Fig. 3.15 : $\phi = 0.2$, $\Gamma = 0.001$

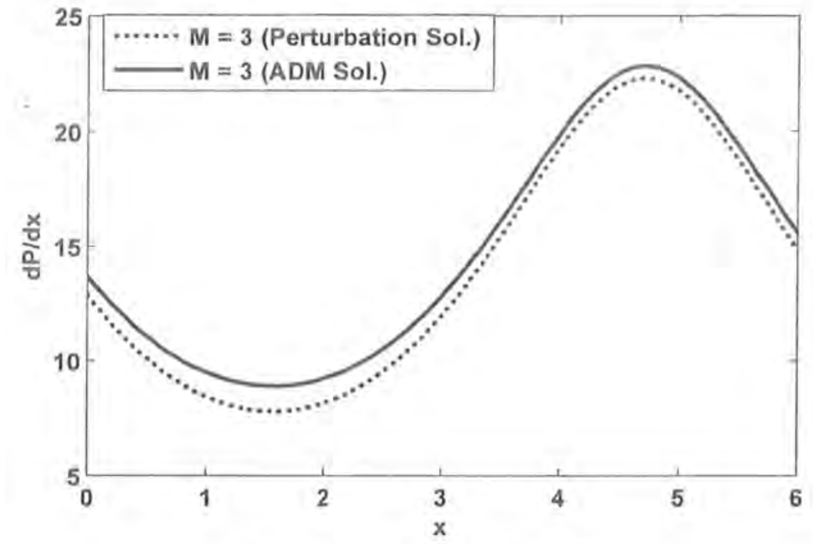


Fig. 3.16 : $\phi = 0.2$, $\Gamma = 0.001$

Figs. (3.15) and (3.16) show variation in pressure gradient by perturbation and Adomian decomposition method for different values of Hartman number M and keeping other parameters fixed.

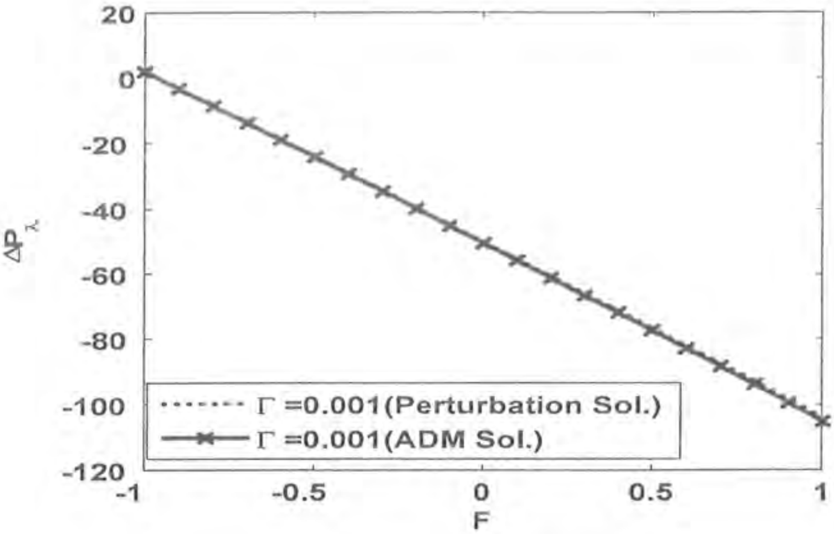


Fig. 3.17 : $M = 2.2$, $\phi = 0.2$

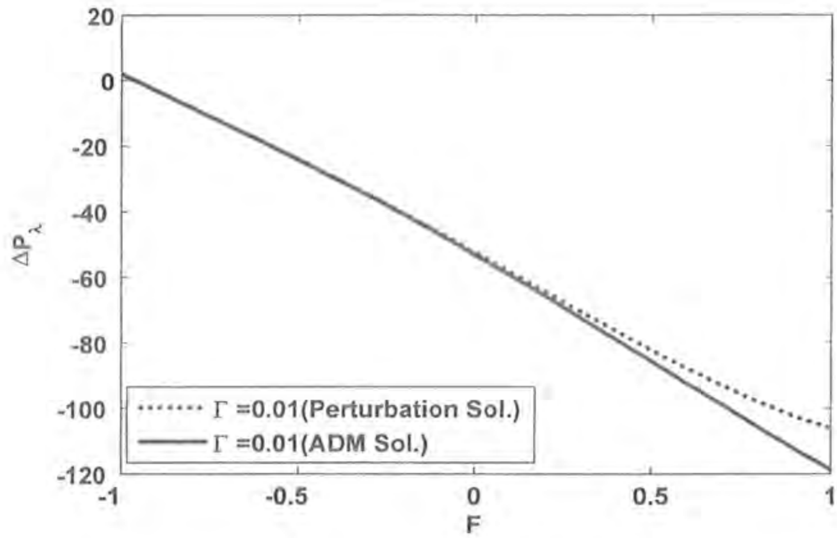


Fig. 3.18 : $M = 2.2$, $\phi = 0.2$

Figs. 3.17 and 3.18 depict variation in pressure rise by perturbation and Adomian decomposition method for different values of Γ (Deborah number) and keeping other parameters fixed.

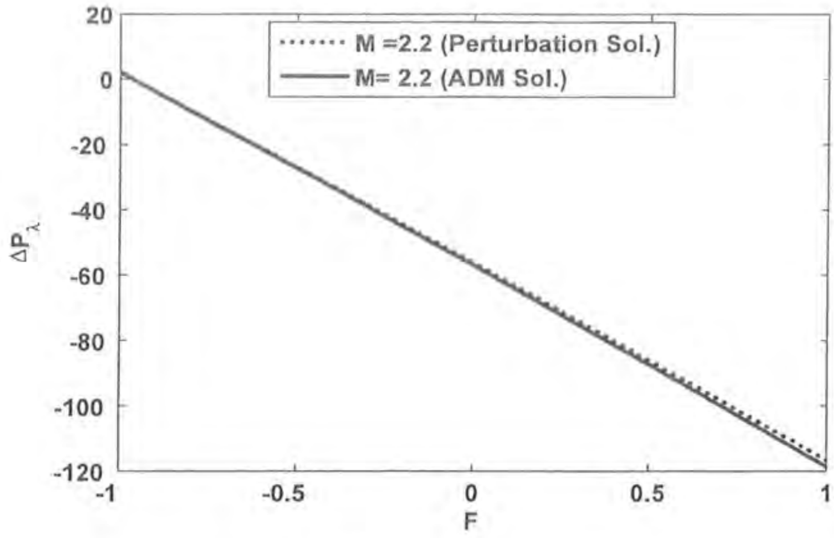


Fig. 3.19 : $\phi = 0.2$, $\Gamma = 0.001$

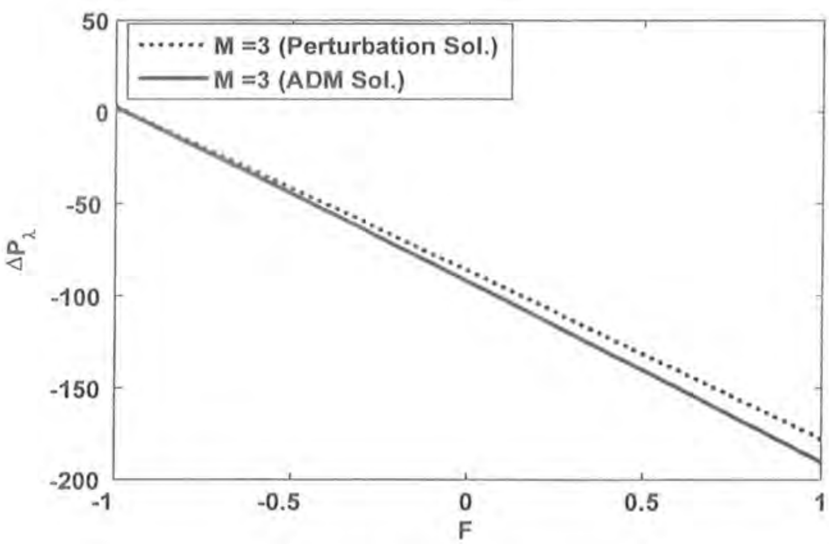


Fig. 3.20 : $\phi = 0.2$, $\Gamma = 0.001$

Figs. 3.19 and 3.20 show variation in pressure rise by perturbation and Adomian decomposition method for different values of Hartman number M and keeping other parameters fixed.

Bibliography

[1] G.Radhakrishnamacharya, Long wavelength approximation to peristaltic motion of a power law fluid, Rheol. Acta 21 (1982) 30 – 35.

[2] L.P.Srivastava, V.P.Srivastava, Peristaltic transport of blood : Casson model, J.Biomech. 17 (1984) 821 – 830.

[3] A.M.Siddiqui, W.H.Schwarz, Peristaltic flow of a second order fluid in tube, J.Non-Newtonian Fluid Mech. 53 (1994) 247 – 284.

[4] A.M.Siddiqui, W.H.Schwarz, Peristaltic motion of a third order fluid in a planar channel, Rheol. Acta 32 (1993) 47 – 56.

[5] T.Hayat, Y.Wang, A.M.Siddiqui, K.Hutter, Peristaltic motion of a Johnson-Segalman fluid in a planar channel, Math. Probl. Eng. 1 (2003) 1 – 23.

[6] T.Hayat, Y.Wang, A.M.Siddiqui, K.Hutter, S.Asghar, Peristaltic transport of a third order fluid in a circular cylindrical tube, Math. Methods Appl. Sci. 12 (2002) 1691 – 1706.

[7] D.Srinivasacharya, M.Mishra, A.R.Rao, Peristaltic pumping of a micropolar fluid in a tube, Acta Mech. 161 (2003) 165 – 178.

[8] M.Elshahed, M.H.Haroun, Peristaltic transport of a Johnson-Segalman fluid under effect of a magnetic field, Math.Probl. Eng. 6 (2005) 663 – 677.

[9] Kh.S.Mekheimer, Peristaltic transport of a couple stress fluid in a uniform and non-uniform channels, Biorheology 39 (2002) 755 – 765.

[10] Kh.S.Mekheimer, Peristaltic flow of blood under effect of a magnetic field in a non-uniform channels, Appl. Math. Comput. 153 (2004) 763 – 777.

[11] T.Hayat, A.Afsar, M. Khan, S.Asghar, Peristaltic transport of a third order fluid under the effect of a magnetic field, Computers and Mathematics with Application. 53 (2007) 1074 – 1087.