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The effects of variable fluid property in a uniform tube with peristalsis

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IN

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The effects of variable fluid property in a uniform tube with variable viscosity

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A DISSERTATION SUBMITTED IN THE PARTIAL FULFILLMENT OF THE REQUIREMENTS FOR THE DEGREE OF THE MASTER OF PHILOSOPHY

We accept this dissertation as conforming to the required standard

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2008

Dedicated

To

My Loving Father and Mother

and

(Sweet Noor-Ul-Ain, Sana)

&

To

My

Most Respected Supervisor

Dr. Sohail Nadeem

Acknowledgement

All praise to Almighty Allah, the creator of all the creations in the universe, who created us in the structure of human beings as the best creature. Many thanks to Him who created us as Muslim and bless us with knowledge to differentiate between right and wrong.

I pay tribute to the Holy Prophet Hazrat Muhammad (Peace be upon him) whose life is forever a beacon of knowledge and guidance for whole humanity.

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Preface

Peristalsis is a well known process of a fluid transport that is used by many systems in the living body to propel or to mix the content of a tube. Such applications include urine transport from kidney to bladder, swallowing food through the esophagus, chyme motion in the gastrointestinal tract movement ovum in the female fallopian tube, vasomotion of small blood vessels and movement of spermatozoa in the human reproductive tract. There are many engineering process as well in which peristaltic pumps are used to handle a wide range of fluids particularly in chemical and pharmaceutical industries. This mechanism is also used in the transport of slurries corrosive fluids sanitary fluids and noxious fluids in the nuclear industry [1].

Mathematical and computer modeling of the peristaltic motion has attracted the attention of many researchers starting with the work of Shapiro [2]. Later on, Pozrikidis [3] used boundary integral method to study the peristaltic flow in a channel for Stokes flow and studied the relationship of molecular convective-transport to the mean pressure gradient. After the pioneering work of Shapiro and Pozrikidis, studies of Peristaltic flows in different flow geometry have been reported analytically, numerically and experimentally by number of researchers [6-14]. Recently, Mekheimer and Abd-elmaboud [11] studied the peristaltic flow with heat transfer analysis may be used to obtain information about the properties of tissues. Bio heat transfer phenomena is common in many biological processes as well as in some biomedical applications, such as in hypothermia treatment and RF ablation (radiofrequency ablation) [5]. Since most of the biochemical reactions in human body take place in a very narrow temperature range and the reaction rate is largely dependent on the local temperature, the heat transfer plays a major role in many processes in living systems.

The study of heat transfer analysis in connection with peristaltic motion has industrial and biological applications like sanitary fluid transport, blood pumps in heart lung machine and transport of corrosive fluids where the contact of the fluid with the machinery parts is prohibited. The interaction of peristalsis and heat transfer has been recognized and has received some attention [11, 12] as it is thought to be relevant in some important processes such as hemodialysis and oxygenation. Vajravelu et al [12] have investigated flow through vertical porous tube with peristalsis and heat transfer. They reported that the heat transfer at the wall is affected significantly by the amplitude of the peristaltic wave.

In most of the above mentioned studies, the fluid viscosity is assumed to be constant. This assumption is not valid every where. In general the coefficient of viscosity for real fluids are function of temperature and pressure. For many liquids, such as water oils and blood the variation in viscosity due to temperature change is more dominant than other effects. The pressure dependence viscosity is usually very small and thus can be neglected [8, 9]. All of the above mentioned studies adopt the assumption of constant viscosity in order to simplify the calculations. In fact, in many thermal transport processes the temperature distribution with in the flow field is never uniform, i.e., the fluid viscosity may change noticeably if a large temperature difference exists in the system. Therefore, it is highly desirable to include the effect of temperature dependent viscosity in momentum and thermal transport processes.

Considering the importance of heat transfer in peristalsis and keeping in mind the sensitivity of liquid viscosity to temperature. This dissertation has arranged in the following manner.

In chapter one, some basic definition of fluid and the governing equations of motion and energy have been derived.

Chapter two is devoted to the study of hydromagnetic flow of fluid with variable (space dependent) viscosity in a uniform tube with peristalsis. An analytical solution using regular perturbation has been discussed.

In chapter three, we discussed the analytical and numerical solutions for the peristaltic transport and heat transfer of a MHD Newtonian fluid with temperature dependent variable viscosity. Some interesting physical quantities have also been calculated and shown their graphical behavior.

Contents

Chapter 1

Basics definitions and equations

1.1 Introduction

This chapter deals with the basic definitions and equations which are relevant to the subsequent chapters. The momentum equations which governs the hydromagnetic flow of Newtonian fluid and energy equation for Newtonian fluid are derived in cylindrical coordinate system.

1.2 Basics definitions

1.2.1 Fluid mechanics

The branch of engineering science that is concerned with forces and energies generated by fluids at rest or in motion. The study of fluid mechanics involves applying the fundamental principles of mechanics and thermodynamics to develop physical understanding and analytic tools that engineers can use to design and evaluate equipment and processes involving fluids. The most common engineering fluids are air (gas), water (liquid) and steam (vapor). Generally in the fluid mechanics we study the behavior of liquids and gases.

1.2.2 Fluid

A fluid is defined as an isotropic substance that the individual particles of which deforms (flows) under the applications of a shear stress (stress along the tangent), no matter how small it is. It is a class of idealized materials includes liquids, gases, plasmas and to some extent, plastic solids.

1.2.3 Rheology

It is the study of non-Newtonian fluids under the influence of an applied stress.

1.2.4 Inviscid fluid

An inviscid fluid is defined as a fluid which has not only constant density but also zero viscosity under different temperature and stress conditions.

1.2.5 Viscous fluid

Viscous fluid is defined as a fluid which is assumed to have a constant density, but is allowed to have viscosity changes under different working conditions of temperature. These fluids are further categorized in Newtonian and non-Newtonian fluids.

1.2.6 Newtonian fluid

Newtonian fluids are non-viscous do not resist deformation and flow freely. It can also be defined as a fluid in which applied shear stress is directly proportional to deformation rate.

Mathematically it can be written as

$$\tau_{xy} = \mu \frac{du}{dy}, \tag{1.1}$$

where μ is the constant of proportionality and also known as the absolute viscosity. Water, air, gasoline, mineral spirits and light oils are examples of Newtonian fluids.

1.2.7 Non-Newtonian fluid

Non-Newtonian fluid is a fluid in which the viscosity changes with the applied strain rate. As a result, non-Newtonian fluids may not have a well defined viscosity. In such a fluid, the shear stress is directly proportional to the non-linear deformation rate.

Mathematically it can be written as

$$\tau_{xy} = \eta(\frac{du}{dy}), \tag{1.2}$$

where $\eta=(\frac{du}{dy})^n$ known as kinematic viscosity. Paints, duffing flour are examples of non-Newtonian fluids.

1.3 Types of flow model

1.3.1 Steady flow

If at a given point in space, the velocity of fluid particles passing through that point remains the same for all times the flow is termed as steady flow. For examples, flow through a conical pipe at a constant rate of discharge is a case of steady flow.

1.3.2 Unsteady flow

If the velocity of fluid particles passing through a point does not remains same for all times the flow is termed as unsteady flow. For example, flow through a long straight pipe at a changing rate of discharge is a case of unsteady flow.

1.3.3 Laminar flow

Laminar flow is characterized by smooth motion of one lamina of fluid past another. The velocity, pressure and other flow properties at each point in the fluid remain constant.

1.3.4 Turbulent flow

Turbulent flow characterized by irregular and nearly random motion super imposed on the main motion of the fluid. For example the trail of smoke leaving the burning cigarette.

1.3.5 Rotational flow

If in a given flow field, the velocity gradients exists and are continuous at each point and the curl of the velocity vector is not zero and has finite values at each point, then the flow in the field under consideration is known as rotational flow.

1.3.6 Irrotational flow

If in a given flow field, the velocity gradients exists and are continuous at each point and the curl of the velocity vector is zero then the flow in the field under consideration is known as irrotational flow.

1.3.7 Uniform flow

If all the particles in a fluid stream have the same velocity, both in magnitude and direction, the flow is known as uniform flow. The flow in a long straight pipe of constant diameter is an example of uniform flow.

1.3.8 Non-uniform flow

If at a given time the velocity profiles are not exactly same or if the average velocity changes from one cross-section to other, the flow is known non-uniform flow. The flow in a conical pipe is an example of non-uniform flow.

1.3.9 Compressible flow

Flow in which density of the fluid varies with space coordinates or time or both. It can also be defined as a flow in which variations in density are not negligible are termed as compressible flow i.e.

$$\rho = \rho(x, y, z, t) \neq \text{constant}.$$

1.3.10 Incompressible flow

A flow is considered incompressible if density of the fluid particles does not change during the flow i.e.

$$\rho = \rho(x, y, z, t) = \text{constant}.$$

1.4 Physical properties of fluid

1.4.1 Pressure

Pressure is a fluid property and is defined as the normal compressive force per unit area acting on a real or imaginary surface in the fluid.

Mathematically it can be written as

$$P = \frac{F}{A}, \tag{1.3}$$

where P is the pressure, F is the normal force and A is the area.

Pressure is a scaler quantity and has SI units of pascals, $1Pa = \frac{1N}{m^2}$.

1.4.2 Density

The density of a fluid for differential and finite-size system is defined as the mass per unit volume at a given temperature and pressure or stress conditions.

1.4.3 Stress

Stress is a measure of force per unit area with in a body.

1.4.4 Strain

Strain is the geometrical expression of the deformation caused by the action of stress on a physical body.

1.4.5 Viscosity

Viscosity is a physical property of fluid by virtue of which it offer resistance to flow or it can also be defined as a physical property of fluids associated with shearing deformation of fluid particles subjected to the action of applied forces.

1.4.6 Variable viscosity

Variable viscosity is the viscosity which does not remain constant. It varies with time as well as it may depend upon space coordinates, temperature and pressure etc.

1.4.7 Kinematic viscosity

Kinematic viscosity is defined as the ratio of the dynamic viscosity μ to the density ρ. Mathematically it can be expressed as

$$\nu = \frac{\mu}{\rho}. \tag{1.4}$$

1.4.8 Viscosity of liquids

Viscosity of liquids in general, decreases with increasing temperature. In liquids the additional forces between molecules become important. This leads to an additional contribution to the shear stress. Thus in liquids, viscosity is independent of pressure.

The viscosities (μ) of liquids generally vary approximately with absolute temperature T according to

$$\ln \mu = a - b \ln T. \tag{1.5}$$

1.4.9 Viscosity of gases

Viscosity in gases arises principally from the molecular diffusion that transports momentum between layers of flow. The kinetic theory of gases allows accurate prediction of the behavior of gaseous viscosity. The viscosity (μ) of many gases is approximated by the formula

$$\mu = \mu_0 \left(\frac{T}{T_0} \right)^n. \tag{1.6}$$

1.5 Thermodynamic properties

1.5.1 Heat

Heat is a form of energy which is transfered from one body to the other due to the difference of temperature between them.

1.5.2 Transmission of heat

If the objects are at different temperature. Heat can be transmitted from one object to the other by the following three processes.

Conduction

Conduction is the process in which heat is transmitted from one body to another by the interaction of atoms and electrons.

Convection

Transfer of heat by the actual movement of molecules from one place to other is called convection

Radiation

Radiation is the process of heat transfer in which heat energy reaches in the form of waves from one place to another without effecting the medium on its way.

1.5.3 Temperature

Temperature can be defined as degree of hotness and coldness of a body. It is usually denoted by T and its unit in SI system is Kelvin.

1.5.4 Flux

Consider a flow of a certain physical quantity (such as mass, energy, heat etc.). The flux is defined as a vector in the direction of the flow whose magnitude is given by the amount of quantity crossing a unit area normal to the flow in unit time.

1.5.5 Heat flux

Heat flux is defined as rate of heat transfer per unit cross-sectional area, and is denoted by Q.

1.5.6 Thermal conductivity

Thermal conductivity is the intensive property of a material that indicates its ability to conduct heat. It is denoted by k, its units in SI system are $(Jm^{-1}K^{-1}s^{-1})$ where $J = joule$, $m = meter$, $K = kelvin$, and $s = second$.

1.5.7 Coefficient of thermal conductivity

When a meter cube of a substance is maintained at a difference of temperature of $1K$, then the quantity of heat that reaches from one end to the other in one second is called the coefficient of thermal conductivity of that substance.

1.5.8 Fourier law of heat conduction

The law that rate of heat flow through a substances is proportional to the area normal to the direction of flow and to the negative of the rate of change of temperature with distance along the direction of flow. Mathematically, it can be written as

$$Q = -k\Delta T, \tag{1.7}$$

where k is thermal conductivity and T is the temperature. The minus sign indicates that heat flows in the direction of decreasing temperature.

1.5.9 Specific heat

The quantity of heat that causes $1K$ change in temperature in a substance of mass $1Kg$ is called specific heat. Its units in SI system are $(Jkg^{-1}K^{-1})$.

1.5.10 Thermal diffusivity

Thermal diffusivity is the ratio of thermal conductivity to volumetric heat capacity.

Mathematically, it can be expressed as

$$\alpha = \frac{k}{\rho c_p}, \tag{1.8}$$

where c_p is the specific heat, k is the thermal conductivity, ρ is the density. Its SI units are $\left(\frac{m^2}{s}\right)$.

1.5.11 Heat radiation

Heat radiation is electromagnetic radiation emitted from the surface of an object which is due to the object temperature.

1.5.12 Free convection

When heat is carried by the circulation of fluids due to buoyancy from density changes induced by heating itself this process is known as free convection.

1.5.13 Prandtl number

It is the ratio of kinematic viscosity and thermal diffusivity.

Mathematically, it can be written as

$$\mathrm{Pr} = \frac{\nu}{\alpha}. \tag{1.9}$$

In heat transfer problems its advantage is that it control the relative thickness of the momentum and thermal boundary layer.

1.5.14 Internal energy

Internal energy of a system is the energy content of the system due to its thermodynamic properties such as pressure and temperature.

1.5.15 Eckert number

The Eckert number is a number used in flow calculations. It expresses the relationship between a flows kinetic energy and enthalpy and is used to characterized dissipation.

1.5.16 Specific internal energy

Specific internal energy is defined as the internal energy of the system per unit mass of the system and has same dimension as enthalpy.

1.6 Peristalsis

Peristalsis is a mechanism to pump the fluid by means of moving contractions on the tube wall. There are various instances in physiology, where this phenomenon is used by the body to propel or mix the contents of a tube as in ureter, gastrointestinal tract, bile in the bile duct and other glandular ducts. Some worms use peristalsis as a means of locomotion. In engineering devices, like finger pump and roller pump work on this principle. Peristaltic transport of toxic liquid is used in the nuclear industries. The mechanism of peristaltic has been exploited for industrial applications like sanitary fluid transport, blood pumps in heart lung machine and transport of corrosive fluids where the contact of the fluid with the machinery parts is prohibited.

1.6.1 Volume flow rate

Volume flow rate is defined as the volume of flow which passes through a given surface per unit time.

1.6.2 Reynolds number

In the fluid mechanics, the Reynold number is the ratio of inertial forces to viscous forces. It is used to identify different flow regimes, such as turbulent flow or laminar flow.

1.7 Hydromagnetic fluids

The interaction of electrically conducting fluids with magnetic fields are known as hydromagnetic fluids. For example, the fluid can be ionized gases (commonly called plasmas), liquid metals, saltwater and sunspots that are caused by sun magnetic field.

1.7.1 Magnetic field

It is the region in which a magnetic force can be observed.

1.7.2 Electric field

A region in which a force would be exerted an electric charge. It is completely defined in magnitude and direction at any point by the force upon a unit positive charge situated at that point. It can be produced by electric charges or by changing magnetic fields.

1.7.3 Hartmann number

It is the measure of the ratio of magnetic body force to the viscous force.

1.7.4 Magnetic permeability

In electromagnetism, permeability is the degree of magnetization of a material that responds linearly to an applied magnetic field.

1.7.5 Electrical conductivity

Electrical conductivity is a measure of a material ability to conduct an electric current.

1.8 Continuity equation

In fluid dynamics, a continuity equation is an equation of conservation of mass. Its differential form is

$$\frac{\partial \rho}{\partial t} + \nabla.(\rho \mathrm{V}) = 0, \tag{1.10}$$

where V is the velocity and t is the time. For incompressible fluid, the density of any particle is constant, thus Eq. (1.10) takes the form

$$\nabla.\mathbf{V} = 0. \tag{1.11}$$

1.9 Navier-Stokes equations

The Navier-Stokes equations was introduced by Claude-Louis Navier and George Gabriel Stokes. These equations establish that changes in momentum in infinitesimal volumes of fluid are simply the sum of dissipative viscous forces (similar to friction) , changes in pressure, gravity and other forces acting inside the fluid.

These are one of the most useful set of equations because they describe the physics of large number of phenomena of academic and economic interest. They may be used to model weather, ocean currents, water flow in pipe, flow around an airfoil (wings) and motion stars inside a galaxy. These equations in both full and simplified forms are used in design aircraft and cars, the study of blood flow, the design of power stations, the analysis of the effects of pollution etc.

The most general form of the Navier-Stokes equations in an arbitrary control volume is

$$\rho\left(\frac{\partial \mathrm{V}}{\partial t} + (\mathrm{V}.\nabla)\,\mathrm{V}\right) = -\nabla p + \nabla S + f. \tag{1.12}$$

This is a statement of the conservation of momentum in a fluid, it is an application of Newton's second law of motion. Using the definition of the material or substantial derivative

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + \mathbf{V}.\nabla,$$

Eq. (1.12) can be written as

$$\rho \frac{D\mathbf{V}}{Dt} = -\nabla p + \nabla S + f. \tag{1.13}$$

The right side of the Eq. (1.13) is in effect a summation of body forces, ∇p and ∇S are gradients of surface forces and represent stresses inside the fluid, analogous to stresses in a solid.

1.10 Maxwell's equations

Maxwell's equations are a set of equations first presented by James Clerk Maxwell in the nineteenth century. He express (i) how electric charges produced electric fields, (ii) the experimental absence of magnetic monopoles, (iii) how electric currents and changing electric field produced magnetic fields (Ampere's circuital law) and (iv) how changing magnetic fields produced electric fields (Farady's law of induction) . These equations are as follows

Gauss's law

$$\nabla.E = \frac{\rho}{\epsilon_0}. \tag{1.14}$$

Gauss's law for magnetism

$$\nabla.\mathrm{B} = 0. \tag{1.15}$$

Farady's law of induction

$$\nabla \times E = -\frac{\partial \mathrm{B}}{\partial t}. \tag{1.16}$$

Ampere's circuital law

$$\nabla \times \mathbf{B} = \mu_m \mathbf{J} + \mu_m \epsilon_0 \frac{\partial E}{\partial t}. \tag{1.17}$$

In the above equations $\mathbf{B}$ is the magnetic field, E is the electric field, $\mathbf{J}$ is the current density, ρ is the electric charge density, μ_m is the magnetic permeability of the free space and ϵ_0 is the permittivity of the free space.

1.11 Equation of motion for hydromagnetic fluid

The continuity equation and the balance of linear momentum for hydromagnetic fluid in cylindrical coordinates system are

$$\operatorname{div} \mathbf{V} = 0, \tag{1.18}$$

$$\rho \frac{d\mathbf{V}}{dt} = -\nabla p + \operatorname{div} \mathbf{S} + J \times \mathbf{B}, \tag{1.19}$$

where

$$\mathbf{V} = \left(\bar{u}\left(\bar{r}, \bar{z}\right), 0, \bar{w}\left(\bar{r}, \bar{z}\right)\right), \tag{1.20}$$

is the velocity field, ρ is the density, p is the pressure, $\mathbf{S}$ is the extra stress tensor, J is the current density, $\mathbf{B} = H_0 + H_1$ is the total magnetic field and H_1 is the induced magnetic field assumed to be negligible.

For the viscous fluid the extra stress tensor S is defined as

$$S = -pI + \mu A_1, \tag{1.21}$$

where I is the identity tensor, A_1=L + L^T is the first Rivlin-Ericksen tensor L is the velocity gradient, μ is the dynamic viscosity.

The Ohm's law and the Maxwell's equations. with no displacement current are

$$J = \sigma\left(E + V \times \mathrm{B}\right), \quad \operatorname{div}\mathrm{B} = 0, \tag{1.22}$$

$$Curl\mathrm{B} = \mu_e J, \quad CurlE = -\frac{\partial \mathrm{B}}{\partial t}, \tag{1.23}$$

where σ is the electrical conductivity, μ_e is the magnetic permeability and E is the electric field.

For the model derivation, we consider:

∘ The quantities ρ,μ_e and σ are constants.

∘ There is no electric field E.

Based on these consideration, the magnetohydrodynamic force becomes

$$J \times \mathrm{B} = -\sigma\mu_e^2 H_0^2. \tag{1.24}$$

The first Rivlin Ericksen tensor is given as

$$A_1 = \begin{bmatrix} 2\frac{\partial \bar{u}}{\partial \bar{r}} & 0 & \frac{\partial \bar{u}}{\partial \bar{z}} + \frac{\partial \bar{w}}{\partial \bar{r}} \\ 0 & 2\frac{\bar{u}}{\bar{r}} & 0 \\ \frac{\partial \bar{u}}{\partial \bar{z}} + \frac{\partial \bar{w}}{\partial \bar{r}} & 0 & 2\frac{\partial \bar{w}}{\partial \bar{z}} \end{bmatrix}. \tag{1.25}$$

Using Eqs. (1.20) to (1.25) in Eqs. (1.18) and (1.19), we get

$$\frac{1}{\bar{r}}\frac{\partial\left(\bar{r}\bar{u}\right)}{\partial \bar{r}} + \frac{\partial \bar{w}}{\partial \bar{z}} = 0, \tag{1.26}$$

$$\begin{aligned} \frac{\partial \bar{P}}{\partial \bar{r}} &= \frac{\partial}{\partial \bar{r}}\left[[2\bar{\mu}(\bar{T})\frac{\partial \bar{u}}{\partial \bar{r}}\right] + \frac{2\,\bar{\mu}\left(\bar{T}\right)}{\bar{r}}\left(\frac{\partial \bar{u}}{\partial \bar{r}} - \frac{\bar{u}}{\bar{r}}\right) + \frac{\partial}{\partial \bar{z}}\left[\bar{\mu}\left(\bar{T}\right)\left(\frac{\partial \bar{u}}{\partial \bar{z}} + \frac{\partial \bar{w}}{\partial \bar{r}}\right)\right] \\ &\quad -\rho\left[\bar{u}\frac{\partial \bar{u}}{\partial \bar{r}} + \bar{w}\frac{\partial \bar{u}}{\partial \bar{r}}\right], \end{aligned} \tag{1.27}$$

$$\frac{\partial \bar{P}}{\partial \bar{z}} = \frac{\partial}{\partial \bar{z}}\left[2\bar{\mu}\left(\bar{T}\right)\frac{\partial \bar{w}}{\partial \bar{z}}\right] + \frac{1}{\bar{r}}\frac{\partial}{\partial \bar{r}}\left[\bar{\mu}(\bar{T})\bar{r}\left(\frac{\partial \bar{u}}{\partial \bar{z}} + \frac{\partial \bar{w}}{\partial \bar{r}}\right)\right] - \sigma\mu_m^2 H_o^2 \bar{w}$$
$$-\rho\left[\bar{u}\frac{\partial \bar{w}}{\partial \bar{r}} + \bar{w}\frac{\partial \bar{w}}{\partial \bar{z}}\right] + \rho g\alpha\left(\bar{T} - \bar{T}_0\right), \quad (1.28)$$

which are the required continuity and momentum equation in component form for hydromagnetic fluid.

In the above equations the viscosity is taken to be a function of temperature, the last term on the right hand side of Eq. (1.28) is due to convection.

1.12 Energy equation

In general, energy equation is expressed as

$$\rho\frac{de}{dt} = \tau.L - \operatorname{div} Q + \rho r, \quad (1.29)$$

where Q is the heat flux vector, $e = \rho c_p$ is the specific internal energy and r is the radiant heating.

According to Fourier law

$$Q = -k \operatorname{grad} \bar{T}, \quad (1.30)$$

where k is the constant of thermal conductivity and $\bar{T}$ is the temperature.

Since we are dealing with the two dimensional flow therefore we seek

$$\bar{T} = \bar{T}\left(r, z\right). \quad (1.31)$$

Using Eq. (1.31) in Eq. (1.30), we obtain

$$Q = -k\left[\frac{1}{\bar{r}}\frac{d\bar{T}}{d\bar{r}} + \frac{d^2\bar{T}}{d\bar{r}^2} + \frac{d^2\bar{T}}{d\bar{z}^2}\right]. \quad (1.32)$$

With the help of these Eqs.(1.30) to (1.32), Eq. (1.29) can be written as

$$\rho c_p \left[\bar{u}\frac{\partial \bar{T}}{\partial \bar{r}} + \bar{w}\frac{\partial \bar{T}}{\partial \bar{z}} \right] = k \left[\frac{\partial^2 \bar{T}}{\partial \bar{r}^2} + \frac{1}{\bar{r}}\frac{\partial \bar{T}}{\partial \bar{r}} + \frac{\partial^2 \bar{T}}{\partial \bar{z}^2} \right] + Q_0, \tag{1.33}$$

where Q_0 is the constant heat addition/absorption.

Chapter 2

Hydromagnetic flow of fluid with variable viscosity in a uniform tube with peristalsis

2.1 Introduction

The chapter deals with the effects of hydromagnetic fluid and variable viscosity in a uniform tube with peristalsis. The governing two dimensional problem has been formulated under the longwave length approximation. The problem is simplified using non-dimensional variables and then used the regular perturbation method to find the analytical solution for small viscosity parameter. This chapter is due to Hakeem et al [13] and a necessary calculations missing in the paper by Hakeem et al [13] are incorporated. At the end, we have presented the graphical results to see the physical behavior of various parameters appears in this chapter.

2.2 Formulation and analysis of the problem

We consider the creeping flow of an incompressible Newtonian fluid with variable viscosity through an axisymmetric form in a uniform tube thickness with a sinusoidal wave travelling down its wall. We also assume that the fluid is subjected to a constant transverse magnetic field. The induced magnetic field is negligible, which is justified for flow at small magnetic Reynolds number. The external electric field is zero and electric field due to polarization of charges is also negligible. Heat due to viscous and joule dissipation is neglected. Also we can neglect the gravity effects since gravity transverse to the flow in the small intestine and it does not interact with fluid particles. Height of the wall is

$$\bar{\mathbf{h}} = \mathbf{a} + \mathbf{b}\sin\frac{2\pi}{\lambda}\left(\bar{Z} - c\bar{t}\right), \tag{2.1}$$

where $\mathbf{a}$ is the radius of the tube at inlet, $\mathbf{b}$ is the wave amplitude, λ is the wavelength, c is the propagation velocity and $\bar{t}$ is the time. We are considering the cylindrical coordinates system $(\bar{R}, \bar{Z})$, where $\bar{Z} - axis$ lies along the centreline of the tube and $\bar{R}$ is transverse to it.

Introducing a wave frame $(\bar{r}, \bar{z})$ moving with velocity c away from the fixed frame $(\bar{R}, \bar{Z})$ by the transformations

$$\bar{z} = \bar{Z} - c\bar{t}, \qquad \bar{r} = \bar{R}, \tag{2.2}$$

$$\bar{w} = \bar{W} - c, \qquad \bar{u} = \bar{U}, \tag{2.3}$$

where $\bar{U}$, $\bar{W}$ and $\bar{u}$, $\bar{w}$ are the velocity components in the radial and axial directions in the fixed and moving coordinates respectively.

The governing equations (which are already derived in chapter one) are

$$\frac{1}{\bar{r}}\frac{\partial\left(\bar{r}\bar{u}\right)}{\partial\bar{r}} + \frac{\partial\bar{w}}{\partial\bar{z}} = 0, \tag{2.4}$$

$$
\begin{aligned}
\frac{\partial \bar{P}}{\partial \bar{r}} &= \frac{\partial}{\partial \bar{r}}\left[[2\bar{\mu}(\bar{r})\frac{\partial \bar{u}}{\partial \bar{r}}\right] + \frac{2\bar{\mu}(\bar{r})}{\bar{r}}\left(\frac{\partial \bar{u}}{\partial \bar{r}} - \frac{\bar{u}}{\bar{r}}\right) + \frac{\partial}{\partial \bar{z}}\left[\bar{\mu}(\bar{r})\left(\frac{\partial \bar{u}}{\partial \bar{z}} + \frac{\partial \bar{w}}{\partial \bar{r}}\right)\right] \\
&\quad -\rho\left[\bar{u}\frac{\partial \bar{u}}{\partial \bar{r}} + \bar{w}\frac{\partial \bar{u}}{\partial \bar{r}}\right], \qquad (2.5)
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial \bar{P}}{\partial \bar{z}} &= \frac{\partial}{\partial \bar{z}}\left[2\bar{\mu}(\bar{r})\frac{\partial \bar{w}}{\partial \bar{z}}\right] + \frac{1}{\bar{r}}\frac{\partial}{\partial \bar{r}}\left[\bar{\mu}(\bar{r})\bar{r}\left(\frac{\partial \bar{u}}{\partial \bar{z}} + \frac{\partial \bar{w}}{\partial \bar{r}}\right)\right] - \sigma\mu_m^2 H_o^2 \bar{w} \\
&\quad -\rho\left[\bar{u}\frac{\partial \bar{w}}{\partial \bar{r}} + \bar{w}\frac{\partial \bar{w}}{\partial \bar{z}}\right]. \qquad (2.6)
\end{aligned}
$$

The corresponding boundary conditions are

$$
\frac{\partial \bar{w}}{\partial \bar{r}} = 0, \qquad \bar{u} = 0 \quad \text{at} \quad \bar{r} = 0, \qquad (2.7a)
$$

$$
\bar{w} = -c, \qquad \bar{u} = -c\frac{d\bar{h}}{d\bar{z}} \qquad at \quad \bar{r} = \bar{h} = a + b\sin\frac{2\pi}{\lambda}(\bar{z}), \qquad (2.7b)
$$

where ρ is the density, $\bar{P}$ is the pressure, $\bar{\mu}(\bar{r})$ is the variable viscosity

It is convenient to non-dimensionalize the variables appearing in Eqs (2.1) to (2.7), introducing the wavenumber δ and the Hartmann number (M) as follows

$$
\begin{aligned}
R &= \frac{\bar{R}}{a},\ r = \frac{\bar{r}}{a},\quad Z = \frac{\bar{Z}}{\lambda},\quad z = \frac{\bar{z}}{\lambda},\ W = \frac{\bar{W}}{c},\ w = \frac{\bar{w}}{c}, \\
U &= \frac{\lambda\bar{U}}{ac},\ u = \frac{\lambda\bar{u}}{ac},\ P = \frac{a^2\bar{P}}{c\lambda\mu_0},\quad h = \frac{\bar{h}}{a} = 1 + \phi\sin 2\pi z, \\
\mu(r) &= \frac{\bar{\mu}(\bar{r})}{\mu_0},\ t = \frac{c\bar{t}}{\lambda},\quad \delta = \frac{a}{\lambda},\quad M = \sigma\mu_m H_0\sqrt{\frac{\sigma}{\mu_0}},\quad \mathrm{Re} = \frac{\rho c a_2}{\mu}, \qquad (2.8)
\end{aligned}
$$

where ϕ is the amplitude ratio $\left(\phi = \frac{a}{b}\right)$.

Using non-dimensional variables (2.8), Eqs. (2.4) to (2.7) take the following form

$$
\frac{1}{r}\frac{\partial(ru)}{\partial r} + \frac{\partial w}{\partial z} = 0, \qquad (2.9)
$$

$$\frac{\partial P}{\partial r} = \delta^2 \frac{\partial}{\partial r}\left[2\mu(r)\frac{\partial u}{\partial r}\right] + \left[\frac{2\delta^2 \mu(r)}{r}(\frac{\partial u}{\partial r} - \frac{u}{r})\right]$$
$$+\delta^2 \frac{\partial}{\partial z}\left[\mu(r)(\delta^2\frac{\partial u}{\partial r} + \frac{\partial w}{\partial z})\right] - \mathrm{Re}\,\delta^3(u\frac{\partial u}{\partial r} + w\frac{\partial u}{\partial z}), \tag{2.10}$$

$$\frac{\partial P}{\partial z} = \frac{1}{r}\frac{\partial}{\partial r}\left[\mu(r)\, r\left(\delta^2\frac{\partial u}{\partial z} + \frac{\partial w}{\partial r}\right)\right] + \delta^2\frac{\partial}{\partial z}\left[2\mu(r)\frac{\partial w}{\partial z}\right] - M^2 w$$
$$-\mathrm{Re}\,\delta\left(u\frac{\partial w}{\partial r} + w\frac{\partial w}{\partial z}\right), \tag{2.11}$$

$$\frac{\partial w}{\partial r} = 0, \qquad u = 0, \qquad \text{at} \quad r = 0, \tag{2.12a}$$

$$w = -1, \qquad u = -\frac{dh}{dz} \ \text{ at } \ r = h = 1 + \phi \sin 2\pi z. \tag{2.12b}$$

Using the long wavelength approximation ($\delta = 0$), Eqs. (2.10) and (2.11) reduces to

$$\frac{\partial P}{\partial r} = 0, \tag{2.13}$$

$$\frac{\partial P}{\partial z} = \frac{1}{r}\frac{\partial}{\partial r}\left[\mu(r)\, r\left(\frac{\partial w}{\partial r}\right)\right] - M^2 w. \tag{2.14}$$

The effects of viscosity variation on peristaltic flow can be investigated for any given function $\mu(r)$. For the present analysis, Hakeem [13] assume the viscosity variation in the dimensionless form as

$$\mu(r) = e^{-\alpha r} \quad \text{or} \quad \mu(r) = 1 - \alpha r \quad \text{for} \quad \alpha << 1. \tag{2.15}$$

This assumption is reasonable for the following physiological reasons because a normal person or animal of similar size consumes one to two liters of the fluid every day. On top of that, another six to seven liters fluid is recurred by a small intestine daily as secretions from salivary glands, stomach, pancreas, liver and small intestine itself. Also the viscosity of the gastric mucus (near the wall) varies between $1 - 10^{-2}$cp but the viscosity of the chyme varies between 10^{-3} to 10^{-6}cp.

2.3 Rate of volume flow

The instantaneous volume flow rate in the fixed coordinates system is given by

$$\hat{Q} = 2\pi \int_0^{\bar{h}} \bar{R}\bar{W} d\bar{R}, \tag{2.16}$$

where $\bar{h}$ is a function of $\bar{Z}$ and $\bar{t}$. Substituting Eqs. (2.2) and (2.3) into Eq. (2.16) and integrating yields

$$\hat{Q} = \bar{q} + \pi c \bar{h}, \tag{2.17}$$

where

$$\bar{q} = 2\pi \int_0^{\bar{h}} \bar{r}\bar{w} d\bar{r}, \tag{2.18}$$

is the volume flow rate in the moving coordinates system and is independent of time. Here $\bar{h}$ is the function of $\bar{z}$ alone and defined through Eq. (2.16). Using the dimensionless variables, we find

$$F = \frac{\bar{q}}{2\pi a^2 c} = \int_0^{h} rw dr. \tag{2.19}$$

The time-mean flow over a period $T = \frac{\lambda}{c}$ at a fixed $\bar{Z}$−position is defined as

$$\bar{Q} = \frac{1}{T} \int_0^{T} \hat{Q} d\bar{t}. \tag{2.20}$$

Invoking Eq. (2.16) into Eq. (2.20) and integrating, we get

$$\bar{Q} = \bar{q} + \pi c\left(a^2 + \frac{b^2}{2}\right), \tag{2.21}$$

which may be written as

$$\frac{\bar{Q}}{2\pi a^2 c} = \frac{\bar{q}}{2\pi a^2 c} + \frac{1}{2}\left(1 + \frac{\phi^2}{2}\right). \tag{2.22}$$

The dimensionless time-mean flow can be defined as $\Theta = \frac{\bar{Q}}{2\pi a^2 c}$.

Rewrite Eq. (2.22) as

$$\Theta = F + \frac{1}{2}\left(1 + \frac{\phi^2}{2}\right), \tag{2.24}$$

where F in the wave frame defined through (2.19).

2.4 Perturbation solution

For the solution of Eqs. (2.13) and (2.14), we look for a regular perturbation in term of small parameter α as follows

$$w = w_0 + \alpha w_1 + O(\alpha)^2, \tag{2.25a}$$

$$u = u_0 + \alpha u_1 + O(\alpha)^2, \tag{2.25b}$$

$$\frac{dp}{dz} = \frac{dp_0}{dz} + \alpha\frac{dp_1}{dz} + O(\alpha)^2, \tag{2.25c}$$

$$F = F_0 + \alpha F_1 + O(\alpha)^2. \tag{2.25d}$$

Substituting Eqs. (2.25a) to (2.25c) in Eqs. (2.12a) to (2.14) and comparing the like power of α, we have the following system of equations.

2.4.1 Zeroth order system

$$\frac{1}{r}\frac{\partial(ru_0)}{\partial r} + \frac{\partial w_0}{\partial z} = 0, \tag{2.26}$$

$$\frac{\partial P_0}{\partial r} = 0, \tag{2.27}$$

$$\frac{\partial P_0}{\partial z} = \frac{1}{r}\frac{\partial}{\partial r}\left[r\left(\frac{\partial w_0}{\partial r}\right)\right] - M^2 w_0, \tag{2.28}$$

$$\frac{\partial w_0}{\partial r} = 0, \qquad u_0 = 0 \quad at \quad r = 0, \tag{2.29}$$

$$w_0 = -1, \qquad u_0 = -\frac{dh}{dz} \ \ at \ \ r = h = 1 + \phi \sin 2\pi z. \tag{2.30}$$

2.4.2 First order system

$$\frac{1}{r}\frac{\partial (ru_1)}{\partial r} + \frac{\partial w_1}{\partial z} = 0, \tag{2.30}$$

$$\frac{\partial P_1}{\partial r} = 0, \tag{2.31}$$

$$\frac{\partial P_1}{\partial z} = \frac{1}{r}\frac{\partial}{\partial r}\left[-r^2\left(\frac{\partial w_0}{\partial r}\right) + r\left(\frac{\partial w_1}{\partial r}\right)\right] - M^2 w_1, \tag{2.32}$$

$$\frac{\partial w_1}{\partial r} = 0, \qquad u_1 = 0 \ \ , \ \text{at} \quad r = 0, \tag{2.33a}$$

$$w_1 = 0, \qquad u_1 = 0, \ \text{at} \quad r = h = 1 + \phi \sin 2\pi z. \tag{2.33b}$$

2.4.3 Solution of zeroth order system

From Eq. (2.27) it is obvious that $P_0 \neq P_0(r)$ and Eq. (2.28) can be written as

$$\frac{\partial^2 w_0}{\partial r^2} + \frac{1}{r}\frac{\partial w_0}{\partial r} - M^2 w_0 - \frac{dP_0}{dz} = 0, \tag{2.34}$$

we assume

$$w_0 = \acute{w}_0 - \frac{\frac{dP_0}{dz}}{M^2}, \tag{2.34a}$$

then Eq. (2.34) takes the form

$$\frac{\partial^2 \acute{w}_0}{\partial r^2} + \frac{1}{r}\frac{\partial \acute{w}_0}{\partial r} - M^2 \acute{w}_0 = 0. \tag{2.35}$$

Multiplication of r^2 on the both sides of Eq.(2.35) gives

$$r^2\frac{\partial^2 \acute{w}_0}{\partial r^2}+r\frac{\partial \acute{w}_0}{\partial r}-r^2M^2\acute{w}_0=0. \tag{2.36}$$

Eq. (2.36) is a modified Bessel equation of first kind, whose solution can be written as

$$\acute{w}_0\left(r\right)=C_1I_0\left(Mr\right)+C_2K_0\left(Mr\right), \tag{2.37}$$

where C_1 and C_2 are arbitrary constants.

Using Eq.(2.37) , Eq.(2.34a) takes the following form

$$w_0\left(r\right)=C_1I_0\left(Mr\right)+C_2K_0\left(Mr\right)-\frac{\frac{dP_0}{dz}}{M^2}, \tag{2.38}$$

With the help of boundary conditions (2.29a), and (2.29b), we obtain

$$C_1=\frac{\frac{dP_0}{dz}\frac{1}{M^2}-1}{I_0\left(Mh\right)},\quad C_2=0.$$

Thus solution (2.38) finally can be written as

$$w_0=\frac{\left(\frac{dp_0}{dz}-M^2\right)}{M^2I_0(Mh)}\left(I_0\left(Mr\right)-I_0\left(Mh\right)\right)-1. \tag{2.39}$$

The volume flow rate F_0 in the moving coordinates system is given by

$$F_0=\int_0^h rw_0dr. \tag{2.40}$$

Substituting Eq. (2.39) into Eq. (2.40) and solving the result for $\frac{dp_0}{dz}$, we get

$$\frac{dp_0}{dz}=\frac{M^4I_0\left(Mh\right)\left(2F_0+h^2\right)}{2MhI_1\left(Mh\right)-M^2h^2I_0\left(Mh\right)}+M^2. \tag{2.41}$$

2.4.4 Solution of the first order system

Using Eq. (2.39) into Eq. (2.32) and after simplifying, we obtain

$$r^2\frac{\partial^2 w_1}{\partial r^2}+r\frac{\partial w_1}{\partial r}-M^2r^2w_1=\frac{dP_1}{dz}r^2+g(z)\left[M^2r^3I_0(Mr)+Mr^2I_1(Mr)\right], \tag{2.42}$$

where

$$g(z)=\frac{\left(\frac{dp_0}{dz}-M^2\right)}{M^2I_0(Mh)}. \tag{2.43}$$

Dividing Eq. (2.42) by r^2 and then differentiating the resulting equation with respect to r, we get

$$\begin{aligned}\frac{\partial^3 w_1}{\partial r^3}-\frac{1}{r}\frac{\partial w_1}{\partial r}+\frac{1}{r}\frac{\partial^2 w_1}{\partial r^2}-M^2\frac{\partial w_1}{\partial r} &= g(z)\Big[M^2I_0(Mr)+M^2r\acute{I}_0(Mr)\\ &\quad +M\acute{I}_1(Mr)\Big],\end{aligned} \tag{2.44}$$

or

$$\begin{aligned}\frac{\partial^3 w_1}{\partial r^3}-\frac{1}{r}\frac{\partial w_1}{\partial r}+\frac{1}{r}\frac{\partial^2 w_1}{\partial r^2}-M^2\frac{\partial w_1}{\partial r} &= g(z)\Big[2M^2I_0(Mr)+M^3rI_1(Mr)\\ &\quad -\frac{M}{r}I_1(Mr)\Big].\end{aligned} \tag{2.45}$$

Now substituting

$$S=\frac{\partial w_1}{\partial r},$$

Eq. (2.45) take the form

$$\frac{\partial^2 S}{\partial r^2}-\frac{1}{r}S+\frac{1}{r}\frac{\partial S}{\partial r}-M^2S=g(z)\left[2M^2I_0(Mr)+M^3rI_1(Mr)-\frac{M}{r}I_1(Mr)\right] \tag{2.46}$$

Multiplying above equation by r^2 we have

$$\begin{aligned}r^2\frac{\partial^2 S}{\partial r^2}+r\frac{\partial S}{\partial r}-\left(M^2r^2+1\right)S &= g(z)\left[2M^2r^2I_0(Mr)+M^3r^3I_1(Mr)\right.\\ &\quad \left.-MrI_1(Mr)\right].\end{aligned} \tag{2.47}$$

Invoking

$$I_0(Mr)=\sum_{k=0}^{\infty}\frac{(Mr)^{2k}}{2^{2k}(k!)^2},\quad I_1(Mr)=\sum_{k=0}^{\infty}\frac{(Mr)^{2k+1}}{2^{2k+1}(k!)(k+1)!},\tag{2.48}$$

in Eq. (2.42), we obtain

$$r^2\frac{\partial^2 w_1}{\partial r^2}+r\frac{\partial w_1}{\partial r}-M^2r^2w_1=\frac{dP_1}{dz}r^2+\frac{g(z)}{M}\sum_{k=0}^{\infty}\frac{b_k}{(k+1)}(Mr)^{2k+3},\tag{2.49}$$

where

$$b_k=\frac{(2k+1)(2k+3)}{2^{2k+1}(\Gamma(k+1))^2(k+1)}\quad \text{for } k=0,1,2,3...\tag{2.50}$$

Eq. (2.49) is a non homogenous modified Bessel equation, its general solution is defined as

$$w_{1C}=C_3I_0(Mr)+C_4K_0(Mr),\tag{2.51}$$

where C_3, and C_4, are constants.

To get the particular solution of Eq. (2.49), we assume a solution of the form

$$w_{1p}=-\frac{1}{M^2}\frac{dp_1}{dz}+\frac{g(z)}{M}\sum_{k=0}^{\infty}\frac{a_k}{(2k+3)}(Mr)^{2k+3}.\tag{2.52}$$

Thus the complete solution of Eq. (2.49) can be written as

$$w_1=C_3I_0(Mr)+C_4K_0(Mr)-\frac{1}{M^2}\frac{dp_1}{dz}+\frac{g(z)}{M}\sum_{k=0}^{\infty}\frac{a_k}{(2k+3)}(Mr)^{2k+3}.\tag{2.53}$$

Using boundary conditions (2.33*a*) and (2.33*b*) in Eq. (2.53), we get

$$C_3=\frac{1}{M^2I_0(Mh)}\frac{dp_1}{dz}-\frac{g(z)}{MI_0(Mh)}\sum_{k=0}^{\infty}\frac{a_k}{(2k+3)}(Mh)^{2k+3},\quad C_4=0,$$

with the help of these constants Eq. (2.53) takes the following form

$$w_1 = \frac{\frac{dp_1}{dz}\left(I_0\left(Mr\right)-I_0\left(Mh\right)\right)}{M^2 I_0(Mh)} + \frac{\left(\frac{dp_0}{dz}-M^2\right)}{M^3 I_0(Mh)}\sum_{k=0}^{\infty}\frac{a_k\left(Mr\right)^{2k+3}}{2k+3} - \frac{\left(\frac{dp_0}{dz}-M^2\right)I_0\left(Mr\right)}{M^3\left(I_0(Mh)\right)^2}\sum_{k=0}^{\infty}\frac{a_k\left(Mr\right)^{2k+3}}{2k+3}. \tag{2.54}$$

The volume flow rate F_1 in the moving coordinates system is defined as

$$F_1 = \int_0^h r w_1 dr. \tag{2.55}$$

Substituting Eq. (2.54) into Eq. (2.55) and solving the result for $\frac{dP_1}{dz}$, yields

$$\frac{dP_1}{dz} = \frac{2F_1 M^4 I_0\left(Mh\right)}{2MhI_1\left(Mh\right)-\left(Mh\right)^2 I_0\left(Mh\right)} + A_1\sum_{k=0}^{\infty}\frac{a_k\left(Mh\right)^{2k+3}}{2k+3} + A_2\sum_{k=0}^{\infty}\frac{a_k\left(Mh\right)^{2k+5}}{2k+5}, \tag{2.56}$$

where $I_0\left(Mr\right)$ and $I_1\left(Mr\right)$ are known as modified Bessel functions of the first kind and a_k, A_1, and A_2 are constants which are defined as

$$a_0 = \frac{1}{2}, \quad a_k = \frac{b_k + a_{k-1}}{\left(2k+3\right)\left(2k+1\right)} \quad \text{for } k = 0, 1, 2, 3, 4.... \tag{2.57}$$

$$A_1 = \frac{M^3\left(2F_0+h^2\right)}{2MhI_1\left(Mh\right)-\left(Mh\right)^2 I_0\left(Mh\right)}, \tag{2.58a}$$

$$A_2 = \frac{M^3 I_0\left(Mh\right)\left(2F_0+h^2\right)}{\left[2MhI_1\left(Mh\right)-\left(Mh\right)^2 I_0\left(Mh\right)\right]^2}. \tag{2.58b}$$

Invoking Eqs. (2.39) and (2.54) into Eq. (2.25a) and using the relation

$$\frac{dP_0}{dz}=\frac{dP}{dz}-\alpha\frac{dP_1}{dz}+O\left(\alpha\right)^2.$$

neglecting terms greater than $O(\alpha)$, we get

$$\begin{aligned} w &= \frac{\left(\frac{dp_0}{dz}-M^2\right)}{M^2 I_0(Mh)}\left(I_0\left(Mr\right)-I_0\left(Mh\right)\right)-1+\alpha\left(\frac{\left(\frac{dp_0}{dz}-M^2\right)}{M^3 I_0(Mh)}\sum_{k=0}^{\infty}\frac{a_k\left(Mr\right)^{2k+3}}{2k+3}\right. \\ &\left.-\frac{\left(\frac{dp_0}{dz}-M^2\right)I_0\left(Mr\right)}{M^3\left(I_0(Mh)\right)^2}\sum_{k=0}^{\infty}\frac{a_k\left(Mr\right)^{2k+3}}{2k+3}\right). \end{aligned} \tag{2.59}$$

Substituting Eqs. (2.41) and (2.56) into Eq. (2.25c) using the relation

$$F_0=F-\alpha F_1+O\left(\alpha\right)^2.$$

where F is defined in Eq. (2.24) and neglecting the terms greater than $O(\alpha)$ we get

$$\begin{aligned} \frac{dP}{dz} &= \frac{M^4 I_0\left(Mh\right)\left(2\theta-\frac{\phi^2}{2}-1+h^2\right)}{2MhI_1\left(Mh\right)-\left(Mh\right)^2 I_0\left(Mh\right)}+M^2 \\ &+\alpha\left(B_1\sum_{k=0}^{\infty}\frac{a_k\left(Mh\right)^{2k+3}}{2k+3}+M_2\sum_{k=0}^{\infty}\frac{a_k\left(Mh\right)^{2k+5}}{2k+5}\right), \end{aligned} \tag{2.60}$$

where

$$B_1 = \frac{M^3\left(2\theta-\frac{\phi^2}{2}-1+h^2\right)}{2MhI_1\left(Mh\right)-\left(Mh\right)^2 I_0\left(Mh\right)}, \tag{2.61a}$$

$$B_2 = \frac{M^3 I_0\left(Mh\right)\left(2\theta-\frac{\phi^2}{2}-1+h^2\right)}{\left[2MhI_1\left(Mh\right)-\left(Mh\right)^2 I_0\left(Mh\right)\right]^2}. \tag{2.61b}$$

The non-dimensional pressure rise per wavelength ΔP_λ and friction force F_λ (on the wall) in the tube length λ in their non-dimensional forms are given by

$$\Delta P_\lambda = \int_0^1 \frac{dP}{dz} dz, \tag{2.62}$$

$$F_\lambda = \int_0^1 h^2 \left(-\frac{dP}{dz}\right) dz, \tag{2.63}$$

where $\frac{dP}{dz}$ is defined through Eq. (2.60).

2.5 Numerical Results and Discussion

We have used a regular perturbation series in term of the dimensionless viscosity parameter (α) to obtain an analytical solution to the field equations for peristaltic flow of a Newtonian fluid in an axisymmetric tube. To study the behavior of solutions, numerical calculations for several values of Hartmann number M, viscosity parameter α and amplitude ratio ϕ have been carried out using a digital computer. Also infinity in Eq. (2.60) is approximated to 9 since the variation in pressure gradient $\frac{dp}{dz}$ is negligible at $k > 9$ for all values of the parameters of interest and all values of z. The relation between pressure rise and flow rate given by Eq. (2.62) is plotted in Figures 2.1 (A) , 2.1 (B) and 2.2 (A) , The relation between friction force and flow rate given by Eq. (2.63) and plotted in Figures 2.2 (B) , 2.3 (A) and 2.3 (B) . It may be noted that the theory of long wavelength and zero Reynolds number of the present investigation remains applicable here, since the radius of the small compared with wavelength.

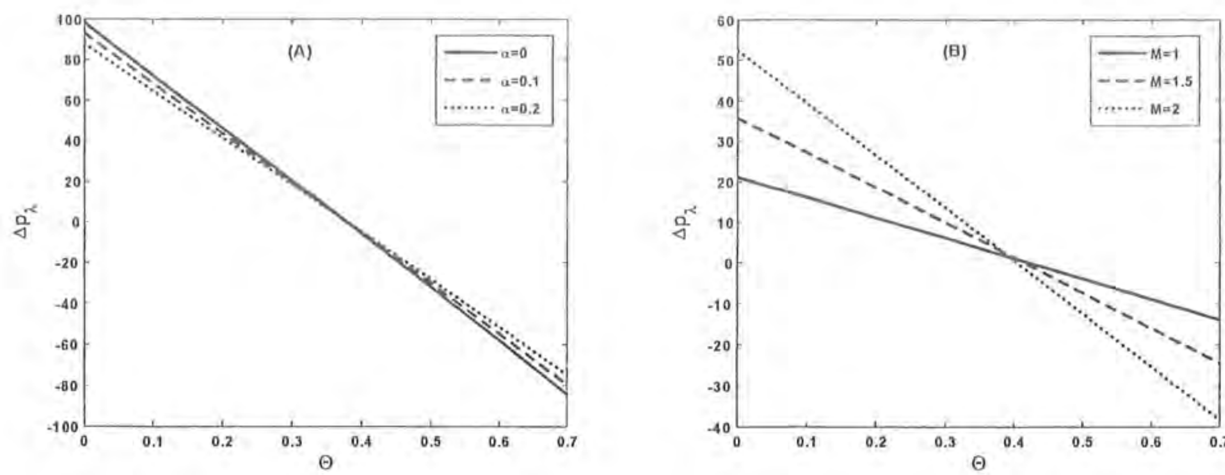


Figure 2.1. Pressure rise versus flow rate for (A) $\phi = 0.6$, $M = 3$. (B). $\alpha = 0.1$, $\phi = 0.6$.

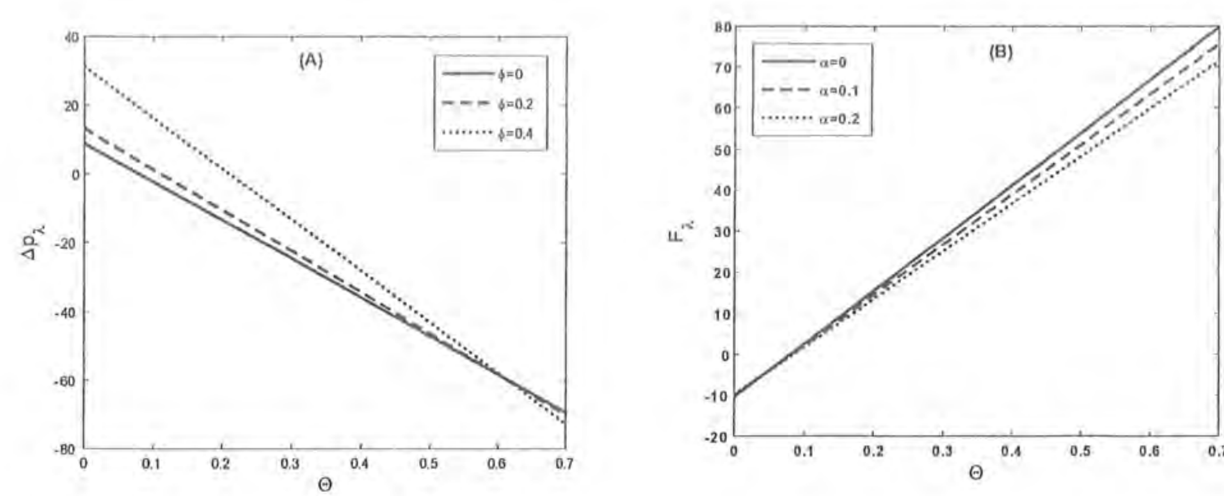


Figure 2.2. (A). Pressure rise versus flow rate for $\alpha = 0.1$, $M = 3$. (B).Friction force versus flow rate for $M = 3$, $\phi = 0.6$.

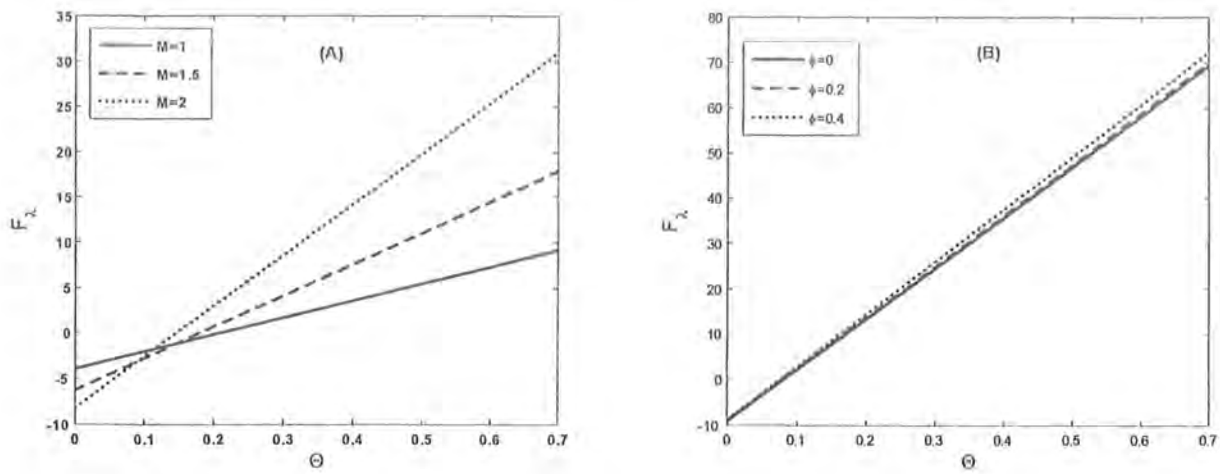


Figure 2.3. Friction force versus flow rate for (A) $\alpha = 0.1$, $\phi = 0.6$. (B) $\alpha = 0.1, M = 3$.

Figure 2.1 (A) represents the variation of dimensionless pressure rise ΔP_λ with time-mean flow rate Θ at $\alpha = 0, 0.1,\ 0.2$, $\phi = 0.6$, $M = 3$. It is clear that an increase in flow rate decreases the pressure rise at $0 \leq \Theta < 0.48$ for different values of viscosity parameter, otherwise it increases with increasing flow rate. Also pressure rise increases with decreasing viscosity parameter and it is independent of viscosity parameter at certain value of flow rate. Moreover the peristaltic pumping occurs at $0 \leq \Theta < 0.48$, otherwise augmented pumping occurs.

Figure 2.1 (B) represents the variation of dimensionless pressure rise ΔP_λ with time-mean flow rate Θ at $\alpha = 0.1, \phi = 0.6$, $M = 1, 1.5, 2$ which shows a linear relation between them and maximum pressure rise occur at zero flow rate for different values of Hartmann number. Also pressure rise increases as flow rate decreases at $0 \leq \Theta < 0.45$, $M = 1$, $0 \leq \Theta < 0.48$, $M = 1.5$ and $0 \leq \Theta < 0.5$, $M = 2$ other wise it increases with increasing flow rate. Furthermore, the pressure rise increases with increasing Hartmann number, and it is independent of Hartmann number variation at a certain value of flow rate. Moreover peristaltic pumping where $\Theta > 0$ (positive pumping) and $\Delta P_\lambda > 0$ (adverse pressure gradient) occurs at $0 \leq \Theta < 0.45$, $M = 1$,$0 \leq \Theta < 0.48$, $M =$

1.5 and $0 \leq \Theta < 0.5$, $M = 2$, otherwise augmented pumping occurs where $\Theta > 0$, (positive pumping) and $\Delta P_\lambda < 0$ (favorable pressure gradient).

Figure 2.2 (A) represents the variation of dimensionless pressure rise ΔP_λ with time-mean flow rate Θ at $\alpha = 0.1, \phi = 0, M = 3$ (no peristalsis), $\phi = 0.2$ (small occlusion), $\phi = 0.4$ (high occlusion). It is obvious that the pressure rise increases with increasing amplitude ratio. It is maximum at zero flow rate. Also, it is independent of flow amplitude ratio at certain values of flow rate. Furthermore, the peristaltic pumping occurs at $0 \leq \Theta < 0.23$, $\phi = 0$, $0 \leq \Theta < 0.26$, $\phi = 0.2$ and $0 \leq \Theta < 0.48$, $\phi = 0.4$, otherwise augmented pumping occurs.

In order to illustrate the effects of viscosity parameter, magnetic field and amplitude ratio on friction force Figures 2.2 (A), 2.3 (A, B) are plotted and it is observed that friction force has an opposite behaviour as compare to the pressure rise.

Chapter 3

Peristaltic transport and heat transfer of a MHD Newtonian fluid with variable viscosity

3.1 Introduction

In this chapter we have discussed the influence of heat transfer and magnetic field on the peristaltic flow of Newtonian fluid with variable viscosity, under the assumptions of long wavelength approximation. We have consider the temperature dependent viscosity and using well known Reynold model of viscosity. A perturbation series in dimensionless viscosity parameter ($\beta << 1$) is used to obtain explicit form for the velocity field, temperature field, relation between the flow rate and pressure gradient. The expression for the pressure rise, friction force, temperature and velocity are computed and plotted for different values of variable viscosity parameter β, Hartmann number M and amplitude ratio ϕ.

3.2 Mathematical formulation

We consider MHD flow of an electrically conducting Newtonian fluid through an axisymmetric tube of uniform cross-section with a sinusoidal wave travelling down its wall. We assume that the fluid is incompressible with uniform properties, i.e., density ρ and electrical conductivity σ are constant. The walls of the tube are set to a given temperature at $\bar{T} = \bar{T}_1$ and fluid viscosity is assumed to depend upon temperature given by Reynolds model. We assume that the fluid is subjected to a constant transverse magnetic field $\mathbf{B}$. A very small magnetic Reynolds number is assumed and hence the induced magnetic field can be neglected. When fluid moves into the magnetic field, two major physical effects arise. The first one is that an electric field $\mathbf{E}$ is induced in the flow. We will assume that there is no excess charge density therefore $\Delta.\mathbf{E} = \mathbf{0}$. Neglecting the induced magnetic field implies that $\Delta \times \mathbf{E} = \mathbf{0}$, and therefore the induced electric field is negligible. The second effect is dynamical in nature i.e. a Lorentz force $(\mathbf{J} \times \mathbf{B})$ where $\mathbf{J}$ is the current density, this force acts on the fluid and modifies its motion. These results transfers the energy from the electromagnetic field to the fluid. In present study, the relativistic effects are neglected and the current density $\mathbf{J}$ is given by Ohm's law as $\mathbf{J} = \sigma (\mathbf{V} \times \mathbf{B})$. The geometry of wall is presented as

$$\bar{\mathbf{h}} = \mathbf{a} + \mathbf{b} \sin \frac{2\pi}{\lambda} \left(\bar{Z} - c\bar{t} \right), \tag{3.1}$$

where $\mathbf{a}$ is the radius of the undisturbed tube, $\mathbf{b}$ is the amplitude of the peristaltic wave, λ is the wavelength, c is the wave propagation speed and $\bar{t}$ is the time. $\bar{R}$ and $\bar{Z}$ are the radial and axial coordinates of the tube with $\bar{Z}$ taken along the axis of the symmetry of the tube. We are considering the cylindrical coordinate system $(\bar{R}, \bar{Z})$, where $\bar{Z} - axis$ lies along the centreline of the tube and $\bar{R}$ is transverse to it. The wall of the tube is maintaining at temperature $\bar{T}_1$ and at the centerline, we have used symmetry condition on temperature and velocity.

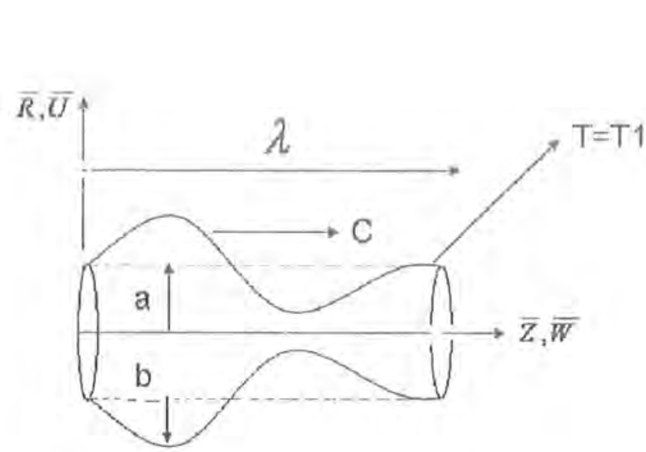


Figure 1: Geometry of peristaltic motion of heat conducting viscous fluid.

In the laboratory frame $(\bar{R}, \bar{Z})$, the flow is unsteady but if we introduce a wave frame $(\bar{r}, \bar{z})$ moving with velocity c away from the fixed frame, then the flow can be treated as steady.

The coordintates frames are related by the transformations

$$\bar{z} = \bar{Z} - c\bar{t}, \qquad \bar{r} = \bar{R}, \tag{3.2}$$

$$\bar{w} = \bar{W} - c, \qquad \bar{u} = \bar{U}, \tag{3.3}$$

where $\bar{U}$,$\bar{W}$ and $\bar{u}$, $\bar{w}$ are the velocity components in the radial and axial directions in the fixed and moving coordinates respectively.

Taking into account the magnetic Lorentz force and the energy transfer, the equations governing the flow of a viscous, Newtonian, MHD fluid are given by

$$\frac{1}{\bar{r}} \frac{\partial (\bar{r}\bar{u})}{\partial \bar{r}} + \frac{\partial \bar{w}}{\partial \bar{z}} = 0, \tag{3.4}$$

$$\frac{\partial \bar{P}}{\partial \bar{r}} = \frac{\partial}{\partial \bar{r}}\left[[2\bar{\mu}(\bar{T})\frac{\partial \bar{u}}{\partial \bar{r}}\right] + \frac{2\bar{\mu}\left(\bar{T}\right)}{\bar{r}}\left(\frac{\partial \bar{u}}{\partial \bar{r}} - \frac{\bar{u}}{\bar{r}}\right) + \frac{\partial}{\partial \bar{z}}\left[\bar{\mu}\left(\bar{T}\right)\left(\frac{\partial \bar{u}}{\partial \bar{z}} + \frac{\partial \bar{w}}{\partial \bar{r}}\right)\right]$$
$$-\rho\left[\bar{u}\frac{\partial \bar{u}}{\partial \bar{r}} + \bar{w}\frac{\partial \bar{u}}{\partial \bar{r}}\right], \quad (3.5)$$

$$\frac{\partial \bar{P}}{\partial \bar{z}} = \frac{\partial}{\partial \bar{z}}\left[2\bar{\mu}\left(\bar{T}\right)\frac{\partial \bar{w}}{\partial \bar{z}}\right] + \frac{1}{\bar{r}}\frac{\partial}{\partial \bar{r}}\left[\bar{\mu}(\bar{T})\bar{r}\left(\frac{\partial \bar{u}}{\partial \bar{z}} + \frac{\partial \bar{w}}{\partial \bar{r}}\right)\right] - \sigma\mu_m^2 H_o^2 \bar{w}$$
$$-\rho\left[\bar{u}\frac{\partial \bar{w}}{\partial \bar{r}} + \bar{w}\frac{\partial \bar{w}}{\partial \bar{z}}\right] + \rho g\alpha\left(\bar{T} - \bar{T}_0\right), \quad (3.6)$$

$$\rho c_p\left[\bar{u}\frac{\partial \bar{T}}{\partial \bar{r}} + \bar{w}\frac{\partial \bar{T}}{\partial \bar{z}}\right] = k\left[\frac{\partial^2 \bar{T}}{\partial \bar{r}^2} + \frac{1}{\bar{r}}\frac{\partial \bar{T}}{\partial \bar{r}} + \frac{\partial^2 \bar{T}}{\partial \bar{z}^2}\right] + Q_0. \quad (3.7)$$

The corresponding boundary conditions are the symmetry at the centerline and no-slip at the walls

$$\frac{\partial \bar{w}}{\partial \bar{r}} = 0, \qquad \bar{u} = 0 \quad \text{at} \quad \bar{r} = 0, \quad (3.6a)$$

$$\bar{w} = -c, \qquad \bar{u} = -c\frac{d\bar{h}}{d\bar{z}} \quad at \quad \bar{r} = \bar{h} = a + b\sin\frac{2\pi}{\lambda}\left(\bar{z}\right), \quad (3.6b)$$

$$\frac{\partial \bar{T}}{\partial \bar{r}} = 0, \qquad \text{at } \bar{r} = 0, \quad (3.7a)$$

$$\bar{T} = \bar{T}_1, \qquad \text{at } \bar{r} = \bar{h}, \quad (3.7b)$$

where ρ is the density, $\bar{P}$ is the pressure, $\bar{\mu}(\bar{T})$ is the temperature dependent viscosity, Q_0 is the constant heat addition/absorption, $\bar{T}$ is the temperature, k denotes the the thermal conductivity, c_p is the specific heat at constant pressure. The viscous dissipation is assumed to be negligible in the energy equation.

We non-dimensionalise the governing equations and boundary conditions by introducing the following non-dimensional parameters

$$\begin{aligned}
R &= \frac{\bar{R}}{a},\ r=\frac{\bar{r}}{a},\ \ Z=\frac{\bar{Z}}{\lambda},\ \ z=\frac{\bar{z}}{\lambda},\ W=\frac{\bar{W}}{c},\ \ w=\frac{\bar{w}}{c},\\
U &= \frac{\lambda\bar{U}}{ac},\ u=\frac{\lambda\bar{u}}{ac},\ \ P=\frac{a^2\bar{P}}{c\lambda\mu_0},\ \theta=\frac{(\bar{T}-\bar{T}_0)}{(\bar{T}_1-\bar{T}_0)},\ \ \mathrm{Pr}=\frac{\mu c_p}{k},\\
\mu(\theta) &= \frac{\bar{\mu}(\bar{T})}{\mu_0},\ t=\frac{c\bar{t}}{\lambda},\ \ \delta=\frac{a}{\lambda},\ \ M=\sigma\mu_m H_0\sqrt{\frac{\sigma}{\mu_0}},\ \ \mathrm{Re}=\frac{\rho c a_2}{\mu},\\
h &= \frac{\bar{h}}{a}=1+\phi\sin 2\pi z\ ,\ \beta_1=\frac{Q_0a^2}{k\,(T_1-T_0)},G_r=\frac{\alpha g a^3\,(T_1-T_0)}{\nu^2},
\end{aligned} \tag{3.8}$$

where ϕ is the amplitude ratio $\left(\phi=\frac{a}{b}\right)$, δ and M are the wave number and Hartmann number respectively. Using the above non-dimensional parameters in Eqs. (3.4) to (3.7), the non-dimensional system becomes

$$\frac{1}{r}\frac{\partial\,(ru)}{\partial r}+\frac{\partial w}{\partial z}=0, \tag{3.9}$$

$$\begin{aligned}
\frac{\partial P}{\partial r} &= \delta^2\frac{\partial}{\partial r}\left[2\mu\,(\theta)\,\frac{\partial u}{\partial r}\right]+\left[\frac{2\delta^2\mu\,(\theta)}{r}(\frac{\partial u}{\partial r}-\frac{u}{r})\right]\\
&+\delta^2\frac{\partial}{\partial z}\left[\mu\,(\theta)\,(\delta^2\frac{\partial u}{\partial z}+\frac{\partial w}{\partial r})\right]-\mathrm{Re}\,\delta^3(u\frac{\partial u}{\partial r}+w\frac{\partial u}{\partial z}),
\end{aligned} \tag{3.10}$$

$$\begin{aligned}
\frac{\partial P}{\partial z} &= \frac{1}{r}\frac{\partial}{\partial r}\left[\mu\,(\theta)\,r\left(\delta^2\frac{\partial u}{\partial z}+\frac{\partial w}{\partial r}\right)\right]+\delta^2\frac{\partial}{\partial z}\left[2\mu\,(\theta)\,\frac{\partial w}{\partial z}\right]-M^2w\\
&-\mathrm{Re}\,\delta\left(u\frac{\partial w}{\partial r}+w\frac{\partial w}{\partial z}\right)+G_r\theta,
\end{aligned} \tag{3.11}$$

$$\mathrm{Re}\,\mathrm{Pr}\,\delta\left[u\frac{\partial\theta}{\partial r}+w\frac{\partial\theta}{\partial z}\right]=\frac{\partial^2\theta}{\partial r^2}+\frac{1}{r}\frac{\partial\theta}{\partial r}+\delta^2\frac{\partial^2 z}{\partial r^2}+\beta_1, \tag{3.12}$$

$$\frac{\partial w}{\partial r}=0,\qquad u=0,\qquad \text{at}\quad r=0, \tag{3.13a}$$

$$w=-1,\qquad\qquad u=-\frac{dh}{dz}\ \text{ at }\ \ r=h=1+\phi\sin 2\pi z, \tag{3.13b}$$

$$\frac{\partial \theta}{\partial r} = 0, \qquad \text{at} \quad r = 0, \tag{3.13c}$$

$$\theta = 1, \text{ at } \quad r = h. \tag{3.13d}$$

Using the long wavelength approximation ($\delta = 0$) in Eqs. (3.10), (3.11) and (3.13), the appropriate equations describing the peristaltic flow with heat transfer reduce to

$$\frac{\partial P}{\partial r} = 0, \tag{3.14}$$

$$\frac{\partial P}{\partial z} = \frac{1}{r}\frac{\partial}{\partial r}\left[\mu(\theta)\, r\left(\frac{\partial w}{\partial r}\right)\right] - M^2 w + G_r \theta, \tag{3.15}$$

$$\frac{\partial^2 \theta}{\partial r^2} + \frac{1}{r}\frac{\partial \theta}{\partial r} + \beta_1 = 0. \tag{3.16}$$

Before we proceed towards finding the solutions of the above equations note that in the long wavelength limit ($\delta = 0$) the energy equation (3.16) is decoupled from the rest of the equations and thus can be solved independently. Also from the radial component of the momentum equation, we find that the pressure is independent of the radial direction. We will utilize these facts in finding the analytical and numerical solutions.

3.3 Rate of volume flow

The instantaneous volume flow rate in the fixed frame is given by

$$\hat{Q} = 2\pi \int_0^{\bar{h}} \bar{R}\bar{W} d\bar{R}, \tag{3.18}$$

where $\bar{h}$ is a function of $\bar{Z}$ and $\bar{t}$. Substituting from Eqs. (3.2) and (3.3) into Eq. (3.18) and then integrating yields

$$\hat{Q} = \bar{q} + \pi c \bar{h}, \tag{3.19}$$

where

$$\bar{q} = 2\pi \int_0^{\bar{h}} \bar{r}\bar{w} d\bar{r}. \tag{3.20}$$

The quantity $\bar{q}$ represents the volume flow rate in the moving coordinates system and is independent of time. Here $\bar{h}$ is the function of $\bar{z}$ alone and defined through Eq. (3.18). Using the dimensionless variables, we find

$$F = \frac{\bar{q}}{2\pi a^2 c} = \int_0^{h} rw dr. \tag{3.21}$$

The time-mean flow over a period $T = \frac{\lambda}{c}$ at a fixed $\bar{Z}-$position is defined as

$$\bar{Q} = \frac{1}{T} \int_0^{T} \hat{Q} d\bar{t}. \tag{3.22}$$

Invoking Eq. (3.18) into Eq. (3.22) and integrating, we obtain

$$\bar{Q} = \bar{q} + \pi c \left(a^2 + \frac{b^2}{2}\right), \tag{3.23}$$

which may be written as

$$\frac{\bar{Q}}{2\pi a^2 c} = \frac{\bar{q}}{2\pi a^2 c} + \frac{1}{2}\left(1 + \frac{\phi^2}{2}\right). \tag{3.24}$$

The dimensionless time-mean flow can be defined as

$$\sigma = \frac{\bar{Q}}{2\pi a^2 c}. \tag{3.25}$$

Eq. (3.24) can be written as

$$\sigma = F + \frac{1}{2}\left(1 + \frac{\phi^2}{2}\right), \tag{3.26}$$

where F in the wave frame defined through (3.21).

In the forthcoming analysis, we shall use the following exponential viscosity-temperature relation, also known as Reynolds model of variable viscosity.

$$\mu(\theta) = e^{-\beta\theta}, \ \mu(\theta) = 1 - \beta\theta \quad \text{for } \beta \ll 1, \tag{3.27}$$

where β represents the Reynold model viscosity parameter. The choice of μ is justified due to physiological applications. When dealing with bio-fluids, the viscosity of the fluid is not constants in all phenomenon, in some typical situations viscosity depends upon temperature. Here we have considered the well known temperature dependent viscosity model known as Reynold model of viscosity.

3.4 Analytical solution

In this section, we will present analytical solution of the system given in Eqs. (3.15) and (3.16) with boundary conditions $(3.13a)$ and $(3.13b)$. We will use regular perturbation technique to find the solutions. The temperature Eq. (3.16) yields

$$\theta(r) = -\frac{\beta_1 r^2}{4} + C_1 \ln(r) + C_2. \tag{3.28}$$

Using the boundary conditions $(3.13c)$ and $(3.13d)$, we get

$$C_1 = 0, \quad C_2 = 1 + \frac{\beta_1 h^2}{4}. \tag{3.29}$$

Thus Eq. (3.28) can be written as

$$\theta(r) = 1 + \frac{\beta_1}{4}\left(h^2 - r^2\right). \tag{3.30}$$

For the solution of Eqs. (3.14) and (3.15), we look for a regular perturbation in term of small parameter β as follows

$$w = w_0 + \beta w_1 + O(\beta)^2, \tag{3.31a}$$

$$u = u_0 + \beta u_1 + O(\beta)^2, \tag{3.31b}$$

$$\frac{dp}{dz} = \frac{dp_0}{dz} + \beta \frac{dp_1}{dz} + O(\beta)^2, \tag{3.31c}$$

$$F = F_0 + \beta F_1 + O(\beta)^2. \tag{3.31d}$$

Substituting from Eqs. (3.31a) to (3.31c) in Eqs. (3.12a) (3.12b) (3.15) and (3.16) and comparing the like power of β, we have the following system of equations

3.4.1 Zeroth order system

$$\frac{1}{r}\frac{\partial (r u_0)}{\partial r} + \frac{\partial w_0}{\partial z} = 0, \tag{3.32}$$

$$\frac{\partial P_0}{\partial r} = 0, \tag{3.33}$$

$$\frac{\partial P_0}{\partial z} = \frac{1}{r}\frac{\partial}{\partial r}\left[r\left(\frac{\partial w_0}{\partial r}\right)\right] - M^2 w_0 + G_r \theta, \tag{3.34}$$

$$\frac{\partial w_0}{\partial r} = 0, \qquad u_0 = 0 \quad \text{at} \quad r = 0, \tag{3.35a}$$

$$w_0 = -1, \qquad u_0 = -\frac{dh}{dz} \ \text{at} \ r = h = 1 + \phi \sin 2\pi z. \tag{3.35b}$$

3.4.2 First order system

$$\frac{1}{r}\frac{\partial (r u_1)}{\partial r} + \frac{\partial w_1}{\partial z} = 0, \tag{3.36}$$

$$\frac{\partial P_1}{\partial r} = 0, \tag{3.37}$$

$$\frac{\partial P_1}{\partial z} = \frac{1}{r}\frac{\partial}{\partial r}\left[-r\theta\left(\frac{\partial w_0}{\partial r}\right) + r\left(\frac{\partial w_1}{\partial r}\right)\right] - M^2 w_1, \tag{3.38}$$

$$\frac{\partial w_1}{\partial r} = 0, \qquad u_1 = 0 \ , \ \text{at} \quad r = 0, \tag{3.39a}$$

$$w_1 = 0, \qquad u_1 = 0, \text{ at } r = h = 1 + \phi \sin 2\pi z. \tag{3.39b}$$

3.4.3 Solution of zeroth order system

With the help of temperature solution (3.30), Eq.(3.34) can be written as

$$r^2 \frac{\partial^2 w_0}{\partial r^2} + r \frac{\partial w_0}{\partial r} - M^2 r^2 w_0 = \frac{dP_0}{dz} r^2 + \frac{G_r \beta_1 r^4}{4} - \frac{G_r \beta_1 r^2 h^2}{4} - G_r r^2. \tag{3.40}$$

Eq. (3.40) is a non homogenous modified Bessel equation, its complementary solution is defined as

$$w_{0C} = C_{11} I_0 (Mr) + C_{22} K_0 (Mr), \tag{3.41}$$

where $I_0 (Mr)$ and $I_1 (Mr)$ are known as modified Bessel functions of the first kind and C_{11}, and C_{22} are constants. To get the particular solution of Eq. (3.40) we assume a solution of the form

$$w_{0p} = -\frac{1}{M^2} \frac{dp_0}{dz} + \frac{G_r \beta_1 h^2}{4M^2} - \frac{G_r \beta_1}{M^4} - \frac{G_r \beta_1 r^2}{4M^2} + \frac{G_r}{M^2}. \tag{3.42}$$

Thus from Eqs.(3.41) and (3.42) , we get

$$w_0 = C_{11} I_0 (Mr) + C_{22} K_0 (Mr) - \frac{1}{M^2} \frac{dp_0}{dz} + \frac{G_r \beta_1 h^2}{4M^2} - \frac{G_r \beta_1}{M^4} - \frac{G_r \beta_1 r^2}{4M^2} + \frac{G_r}{M^2}. \tag{3.43}$$

Using Eqs. (3.35a) (3.35b) in Eq. (3.43) we have

$$C_{11} = \frac{1}{I_0(Mh)} \left(-1 + \frac{1}{M^2} \frac{dp_0}{dz} + \frac{G_r \beta_1}{M^4} - \frac{G_r}{M^2} \right), \quad C_{22} = 0. \tag{3.44}$$

Finally the solution can be written as

$$\begin{aligned} w_0 &= \frac{1}{I_0(Mh) M^2} \left(\frac{dp_0}{dz} - M^2 + K_8 \right) \left(I_0 (Mr) - I_0 (Mh) \right) - 1 \\ &\quad + \frac{G_r \beta_1}{4M^2} \left(h^2 - r^2 \right). \end{aligned} \tag{3.45}$$

The volume flow rate F_0 in the moving coordinates system is given by

$$F_0 = \int_0^h r w_0 dr. \tag{3.46}$$

Substituting Eq. (3.45) into Eq. (3.46) and solving the result for $\frac{dp_0}{dz}$, yields

$$\frac{dp_0}{dz} = \frac{M^4 I_0 (Mh) (2F_0 + h^2 + K_5)}{2MhI_1 (Mh) - M^2 h^2 I_0 (Mh)} + K_6, \tag{3.47}$$

where

$$\begin{aligned}
K_1 &= \frac{1}{I_0 (Mh)} \left(-\beta_2 - \beta_3 h^2\right), \\
K_2 &= \frac{1}{I_0(Mh)} \left(K_3 \sum_{k=0}^{\infty} \frac{a_k (Mh)^{2k+4}}{2k+4} - K_4 \sum_{k=0}^{\infty} \frac{a_k (Mh)^{2k+2}}{2k+2} \right), \ K_3 = -\frac{\beta_1}{2M^2}, \\
K_4 &= \frac{\beta_1 h^2 + 4}{2}, \quad K_5 = -\frac{G_r \beta_1 h^4}{16M^2}, \quad K_6 = \frac{G_r \beta_1}{M^2} + G_r + M^2.
\end{aligned}$$

3.4.4 Solution of the first order system

With the help of Eqs. (3.30) and (3.45), Eq. (3.38) can be written as

$$\begin{aligned}
r^2 \frac{\partial^2 w_1}{\partial r^2} + r \frac{\partial w_1}{\partial r} - M^2 r^2 w_1 &= \frac{dP_1}{dz} r^2 - \frac{G_r \beta_1 r^2}{M^2} - \frac{G_r \beta_1^2 h^2 r^2}{4M^2} + \frac{G_r \beta_1^2 r^4}{2M^2} \\
&\quad - \frac{\beta_1 g(z)}{2M^2} \sum_{k=0}^{\infty} \frac{b_k}{(k+1)} (Mr)^{2k+4} + \\
&\quad \frac{(\beta_1 h^2 + 4) g(z)}{2} \sum_{k=0}^{\infty} \frac{b_k (Mr)^{2k+2}}{(k+2)}.
\end{aligned} \tag{3.48}$$

Eq. (3.48) is a non homogenous modified Bessel equation, its complementary solution is defined as

$$w_{1C} = C_{33} I_0 (Mr) + C_{44} K_0 (Mr), \tag{3.49}$$

where C_{33},and C_{44}, are constants.To get the particular solution of Eq. (3.48) we assume a solution of the form

$$\begin{aligned} w_{1p} &= -\frac{1}{M^2}\frac{dp_1}{dz}+\frac{G_r\beta_1}{M^4}+\frac{G_r\beta_1^2h^2}{4M^4}-\frac{G_r\beta_1^2r^2}{2M^4}-\frac{4G_r\beta_1^2}{2M^2} \\ &-\frac{\beta_1 g(z)}{2M^2}\sum_{k=0}^{\infty}\frac{a_k}{(2k+4)}(Mr)^{2k+4}+\frac{(\beta_1 h^2+4)g(z)}{2}\sum_{k=0}^{\infty}\frac{a_k(Mr)^{2k+2}}{(2k+2)}. \end{aligned} \quad (3.50)$$

The general solution can be written as

$$\begin{aligned} w_1 &= C_{33}I_0(Mr)+C_{44}K_0(Mr)-\frac{1}{M^2}\frac{dp_1}{dz}+\frac{G_r\beta_1}{M^4}+\frac{G_r\beta_1^2h^2}{4M^4}-\frac{G_r\beta_1^2r^2}{2M^4} \\ &-\frac{4G_r\beta_1^2}{2M^2}-\frac{\beta_1 g(z)}{2M^2}\sum_{k=0}^{\infty}\frac{a_k}{(2k+4)}(Mr)^{2k+4}+ \\ &\frac{(\beta_1 h^2+4)g(z)}{2}\sum_{k=0}^{\infty}\frac{a_k(Mr)^{2k+2}}{(2k+2)}, \end{aligned} \quad (3.51)$$

where

$$a_0=\frac{1}{4},\quad a_k=\frac{b_k}{(2k+2)(k+2)},b_k=\frac{(k+1)}{2^{2k+1}(k!)^2}.$$

With the help of Eqs. (3.39a) and (3.39b) , we obtain

$$C_{33}=\frac{1}{M^2}\frac{dp_1}{dz}+K_1+g(z)K_2,\quad C_{44}=0. \quad (3.52)$$

Therefore Eq. (3.51) takes the form

$$\begin{aligned} w_1 &= \frac{dP_1}{dz}\frac{1}{I_0(Mh)M^2}\left[\left(I_0(Mr)-I_0(Mh)+K_1I_0(Mr)\right]+g(z)K_2I_0(Mr)\right. \\ &+\beta_2+\beta_3r^2+g(z)K_3\sum_{k=0}^{\infty}\frac{a_k(Mr)^{2k+4}}{2k+4}+g(z)K_4\sum_{k=0}^{\infty}\frac{a_k(Mr)^{2k+2}}{2k+2}. \end{aligned} \quad (3.53)$$

The volume flow rate F_1 in the moving coordinates system is defined as

$$F_1=\int_0^h rw_1dr. \quad (3.54)$$

Substituting Eq. (3.53) into Eq. (3.54) and solving the result for $\frac{dP_1}{dz}$, yields

$$\frac{dP_1}{dz}=\frac{M^4 I_0(Mh)(2F_1-K_7)}{2MhI_1(Mh)-M^2h^2I_0(Mh)}+\frac{M^6 I_0(Mh)K_{77}(2F_0+h^2)}{(2MhI_1(Mh)-M^2h^2I_0(Mh))^2}+K_9, \quad (3.55)$$

where

$$\begin{aligned}
K_7 &= \frac{2K_1h}{M}I_1(Mh)+\beta_2h^2+\frac{\beta_3h^4}{2},\\
K_8 &= \frac{G_r\beta_1}{M^2}-G_r, \qquad K_9=\frac{2K_8K_{77}M^2}{(2MhI_1(Mh)-M^2h^2I_0(Mh))}, \ \beta_3=-\frac{G_r\beta_1^2}{2M^4},\\
K_{77} &= -\frac{1}{M^2}\left(K_2(Mh)I_1(Mh)+K_3\sum_{k=0}^{\infty}\frac{a_k(Mh)^{2k+6}}{(2k+4)(2k+6)}+K_4\sum_{k=0}^{\infty}\frac{a_k(Mh)^{2k+4}}{(2k+2)(2k+4)}\right)\\
\beta_2 &= \frac{G_r\beta_1}{M^4}+\frac{G_r\beta_1^2h^2}{4M^4}-\frac{2G_r\beta_1^2}{M^6}, \ g(z)=\frac{1}{I_0(Mh)M^2}\left(\frac{dp_0}{dz}-M^2+K_8\right).
\end{aligned}$$

Finally, substituting Eqs. (3.45) and (3.53) into Eq. (3.31a) and using the relation

$$\frac{dP_0}{dz}=\frac{dP}{dz}-\beta\frac{dP_1}{dz}+O(\beta)^2.$$

neglecting terms greater than $O(\beta)$, we get

$$\begin{aligned}
w &= \frac{1}{I_0(Mh)M^2}((\frac{dp}{dz}-M^2)+K_8))(I_0(Mr)-I_0(Mh))-1.\\
&+\frac{G_r\beta_1}{4M^2}(h^2-r^2)+\beta(K_1I_0(Mr)+g(z)K_2I_0(Mr)\\
&+\beta_2+\beta_3r^2+g(z)K_3\sum_{k=0}^{\infty}\frac{a_k(Mr)^{2k+4}}{2k+4}+g(z)K_4\sum_{k=0}^{\infty}\frac{a_k(Mr)^{2k+2}}{2k+2}). \quad (3.56)
\end{aligned}$$

Also using Eqs. (3.47) and (3.55) into Eq. (3.31c) and the relation

$$F_0=F-\beta F_1+O(\beta)^2.$$

where F is defined in Eq. (3.21) and neglecting the terms greater than $O(\beta)$, we get

$$\begin{aligned}\frac{dP}{dz} &= \frac{M^4 I_0(Mh)\left(2\theta - \frac{\phi^2}{2} - 1 + h^2 + K_{10}\right)}{2MhI_1(Mh) - M^2h^2I_0(Mh)} + K_6 \\ &+\beta\left(\frac{2M^6 I_0(Mh)\left(2\theta - \frac{\phi^2}{2} - 1 + h^2\right)K_{77}}{(2MhI_1(Mh) - M^2h^2I_0(Mh))^2} + K_9\right), \end{aligned} \tag{3.57}$$

where

$$K_{10} = K_5 - \beta K_7.$$

The non-dimensional pressure rise per wavelength ΔP_λ and friction force F_λ(on the wall) in the tube length λ in their non-dimensional forms are given by

$$\Delta P_\lambda = \int_0^1 \frac{dP}{dz}dz, \tag{3.58}$$

$$F_\lambda = \int_0^1 h^2\left(-\frac{dP}{dz}\right)dz, \tag{3.59}$$

where $\frac{dP}{dz}$ is defined through equation (3.57).

The corresponding stream function $\left(u = -\frac{1}{r}\frac{\partial\psi}{\partial r} \text{ and } w = \frac{1}{r}\frac{\partial\psi}{\partial r}\right)$ is

$$\begin{aligned}\psi(r,z) &= \frac{g(z)}{2M^2}\left(2MrI_0(Mr) - (Mr)^2 I_0(Mh)\right) - \frac{r^2}{2} + \frac{G_r\beta_1 r^2}{16M^2}\left(2h^2 - r^2\right) \\ &+\beta\left(\frac{1}{M}\left(K_1 r I_1(Mr) + g(z)K_2 r I_1(Mr)\right) + \frac{r^2}{4}\left(2\beta_2 + \beta_3 r^2\right) + \right. \\ &\left.\frac{g(z)}{M^2}\left(K_3\sum_{k=0}^{\infty}\frac{a_k (Mr)^{2k+6}}{(2k+4)(2k+6)} + K_4\sum_{k=0}^{\infty}\frac{a_k(Mr)^{2k+4}}{(2k+4)(2k+2)}\right)\right).\end{aligned}$$

3.5 Numerical computations

A finite difference technique is employed to check the results of the perturbation analysis and to indicate their validity. Recall that the system of equations and boundary conditions in the long wavelength limit are given by

$$\frac{\partial^2\theta}{\partial r^2}+\frac{1}{r}\frac{\partial\theta}{\partial r}+\beta_1=0, \tag{3.60}$$

$$\frac{\partial P}{\partial z}=\frac{1}{r}\frac{\partial}{\partial r}\left[\mu(\theta)\,r\left(\frac{\partial w}{\partial r}\right)\right]-M^2w+G_r\theta, \tag{3.61}$$

$$\frac{\partial\theta}{\partial r}=0,\quad \text{at}\quad r=0,\quad \theta=1,\ \text{at}\quad r=h, \tag{3.62}$$

$$\frac{\partial w}{\partial r}=0,\quad \text{at}\quad r=0,\ \ w=-1,\quad \text{at}\quad r=h. \tag{3.63}$$

We notice that in the long wavelength limit ($\delta=0$), the energy equation (3.60) is decouples from the axial velocity equations and thus can be solved independently which are already computed in previous section and represented by Eq. (3.30).

With the help of Eq. (3.30) and variable viscosity model $\mu(\theta)=1-\beta\theta$ and using the fact that $P\neq P(r)$, the axial momentum equation becomes after some algebra

$$\frac{\partial^2 w}{\partial r^2}\left[1-\beta_{22}-\frac{\beta_1\beta_{22}}{4}\left(h^2-r^2\right)\right]+\frac{1}{r}\frac{\partial w}{\partial r}\left[1-\beta_{22}-\frac{\beta_1\beta_{22}}{4}\left(h^2-r^2\right)+\right.$$
$$\left.\frac{r^2\beta_1\beta_{22}}{2}\right]-rM^2w=\frac{\partial P}{\partial z}-\frac{G_r\beta_1}{4}\left(h^2-r^2\right)-G_r. \tag{3.64}$$

We use finite difference method to solve the above equation treating it as an ordinary differential equation with the boundary conditions (3.63). The first step is to partition the domain $[0,h]$ into a number of sub-domain or intervals of length dx. We denote by x_i the interval end points or nodes, with $x_1=0$, and $x_{n+1}=h$. In general we have $x_i=(i-1)\,dx$ for $i=1,2,3....N$. We represent the axial velocity w at the ith node by W_i. The second step is to express the differential operators in discrete form. This can be accomplished using finite difference approximations to the differential operators. In this problem we will use the central difference approximation and replaced the derivatives by their discrete approximations

$$w'' = \frac{W_{i+1} - 2W_i + W_{i-1}}{(dx)^2}, \qquad w' = \frac{W_{i+1} + W_{i-1}}{2\,(dx)}. \tag{3.66}$$

Using Eq. (3.66) in Eq. (3.65) and after rearranging, we get a system of algebraic equations

$$A_i W_{i+1} + BW_i + \text{Ci}\, W_{i-1} = D_i \qquad i = 1, 2, 3.....N. \tag{3.67}$$

Finally, the resulting tridiagonal system is solved the using the famous Thomas-alogorithm.

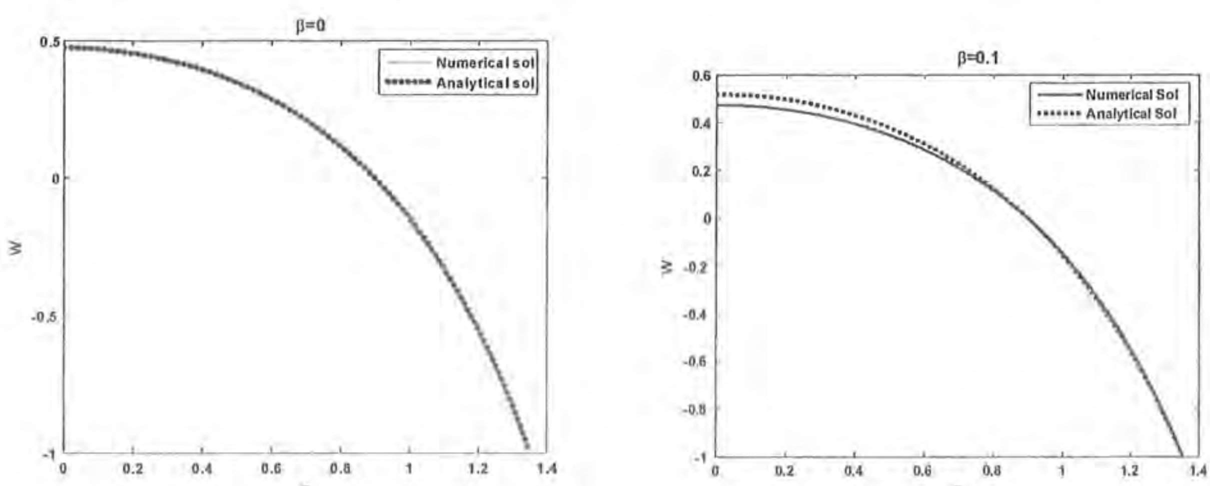


Figure 3.1. Comparison of analytical and Numerical velocity for $\frac{dP}{dz} = 0.4$, $M = 3$, $G_r = 2$, and $\beta_1 = 5$ for $\beta = 0, 0.1$

Figur.3.1 represents the comparison of analytical and numerical solutions of the axial velocity w. We find a very good agreement between the two results for the case ($\beta = 0$) which corresponds to a constant viscosity fluid and a fairly good agreement with variable viscosity coefficient for ($\beta = 0.1$)

3.6 Results and discussion

In this section results are presented and discussed for different physical quantities of interest. Pressure rise in the tube due to peristalsis is an important physical quantity. In order to better understand the effects of variable viscosity and magnetic field, the average pressure rise ΔP_λ is plotted against σ, the time averaged mean flow rate. Figures 3.2, 3.3 represents the average pressure rise ΔP_λ for different values of viscosity parameter β, Hartmann number M as well as for the different amplitude ratio ϕ, of the wave train traveling at the walls of the tube.

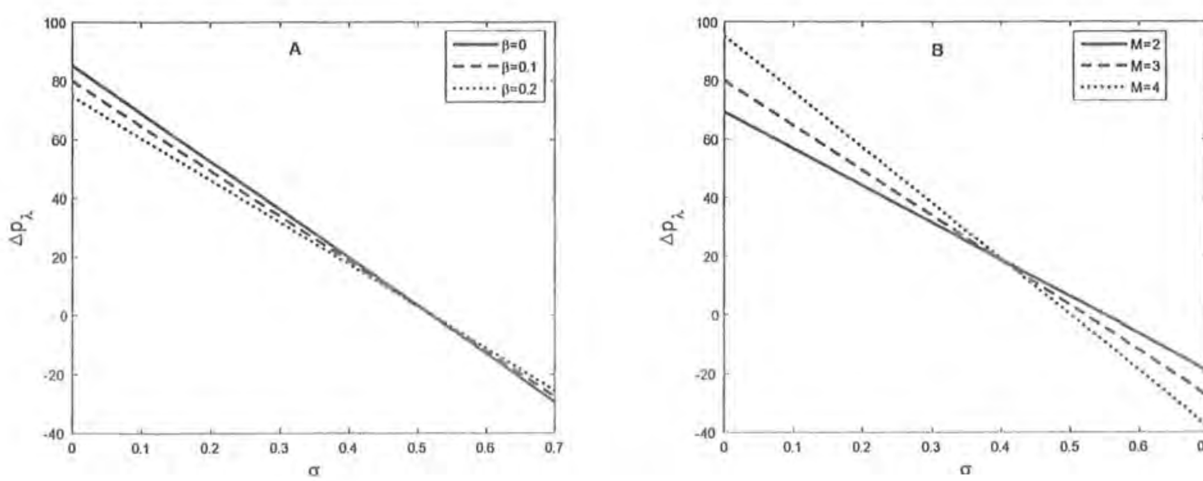


Figure 3.2. Pressure rise against flow rate (A) $\phi = 0.6$, $M = 3$, $\beta_1 = 5$, $G_r = 3$ (B) $\phi = 0.6$, $\beta = 0.1$, $\beta_1 = 5$, $G_r = 3$

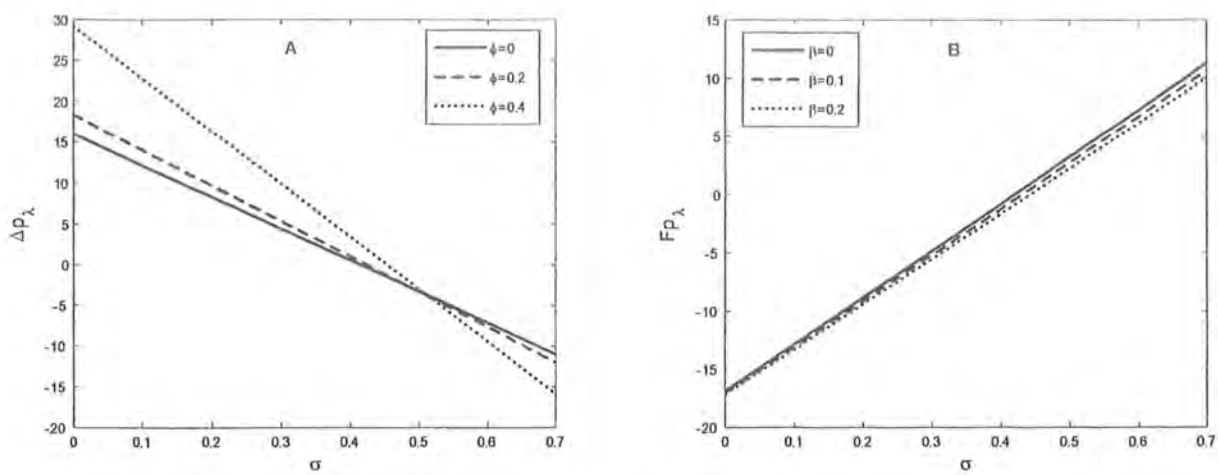


Figure 3.3. Pressure rise against flow rate (A) $\beta = 0.1$, $M = 3$, $\beta_1 = 5$, $G_r = 3$ (B) Friction force against flow rate for $\phi = 0.2$, $M = 3$, $\beta_1 = 5$, $G_r = 3$

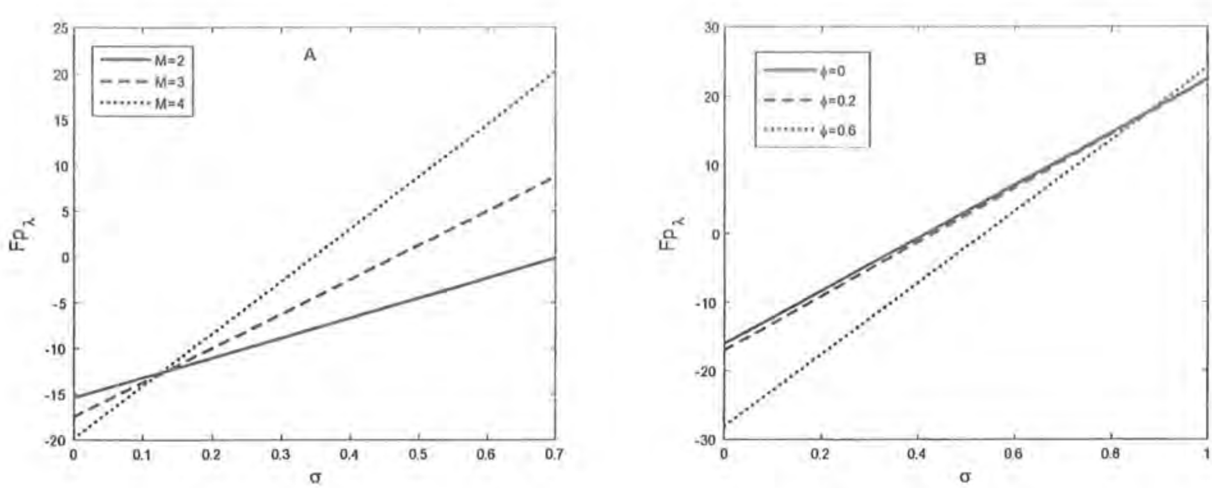


Figure 3.4. Friction force against flow rate (A) $\phi = 0.6$, $\beta = 0.3$, $\beta_1 = 5$, $G_r = 3$ (B)

$\beta = 0.1$, $M = 3$, $\beta_1 = 5$, $G_r = 3$

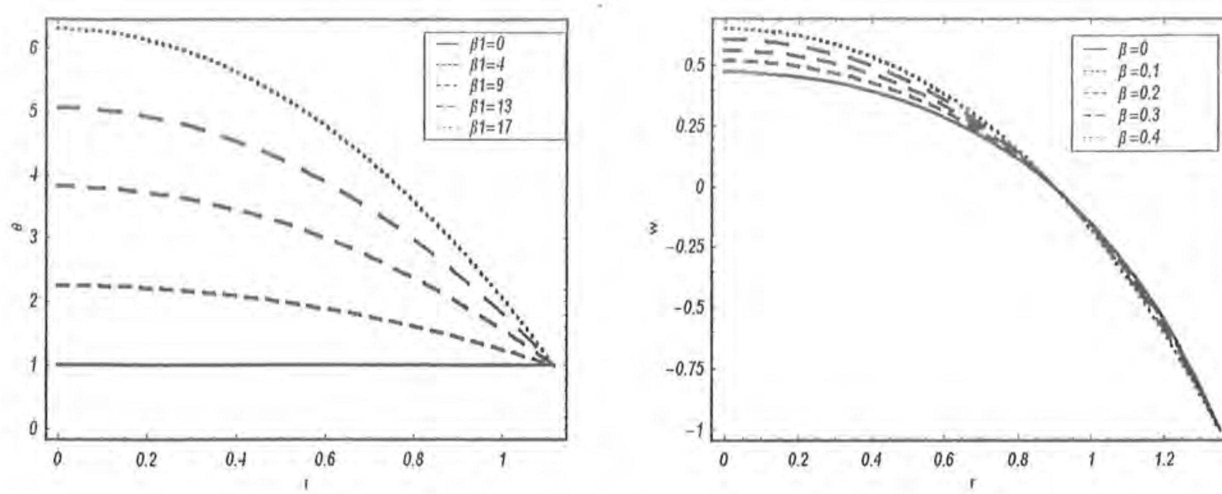


Figure 3.5. Temperature profile for $\phi = 0.2, z = 0.1$ and axial velocity for $\phi = 0.6, \beta_1 = 5, M = 3, z = 0.1, G_r = 2, \frac{dp}{dz} = 0.4$

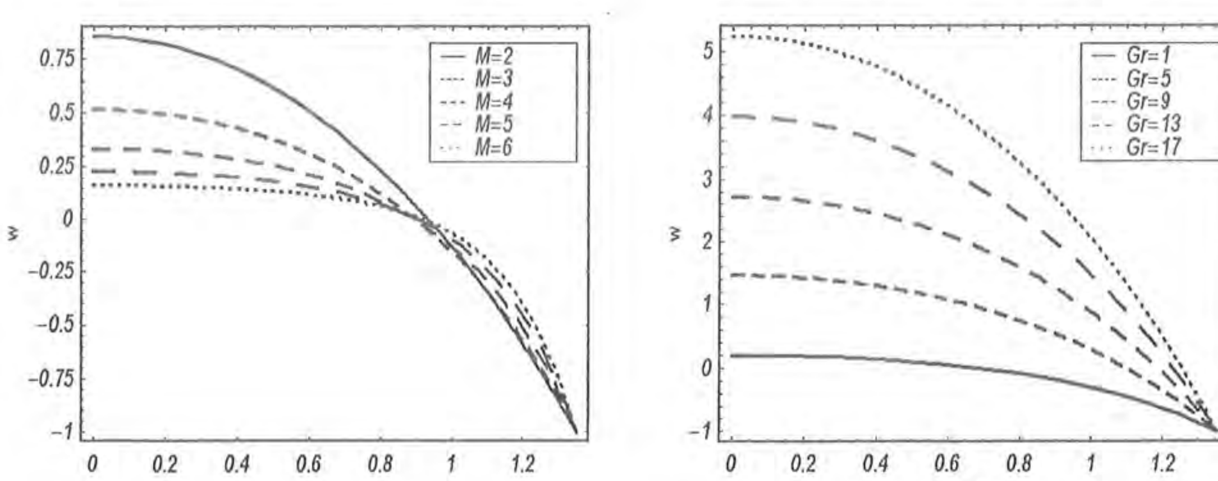


Figure 3.6. (i) Axial velocity for $\phi = 0.6, \beta_1 = 5, \beta = 0.1, z = 0.1, G_r = 2, \frac{dp}{dz} = 0.4$ (ii) $\phi = 0.6, \beta_1 = 5, \beta = 0.1, z = 0.1, M = 3, \frac{dp}{dz} = 0.4$

From Figure 3.2-A, it can be observed that the maximum pressure rise occurs at zero flow rate for different values of viscosity parameter β. The theoretical pressure rise increases as flow rate decreases and pressure rise decreases linearly with increasing time-mean flow. Moreover, it is observed that the pressure rise is maximum for constant viscosity ($\beta = 0$) case and it is decreases for increasing β. The average pressure rise ΔP_λ decreases as σ increased in peristaltic pumping region, $\sigma > 0$, and $\Delta P_\lambda > 0$, but it increases with σ in augmented pumping region, where $\sigma > 0$, and $\Delta P_\lambda < 0$.

Figure 3.2-B represents the pressure rise ΔP_λ for different magnetic parameter M, the Hartmann number. The pressure rise increases with M for $0 \leq \sigma < 0.4$ and for the range $\sigma \geq 0.4$ ΔP_λ decreases with the increase in Hartmann number. Furthermore, the peristaltic pumping occurs upto almost at $\sigma = 0.41$ for each Hartmann number, otherwise augmented pumping occurs.

Figure 3.3-A shows the effects of amplitude ratio on average pressure rise ΔP_λ plotted against the time-mean flow rate σ. It is observed that the behavior of pressure rise reverses at $\sigma = 0.5$. As seen from the graph, the pressure rise increases with increase in amplitude ratio ϕ upto $\sigma = 0.5$ and for $\sigma > 0.5$ pressure rise decreases with the increases in amplitude ratio. Furthermore the peristaltic pumping occurs in the region $0 \leq \sigma < 0.5$ and augmented pumping occurs otherwise.

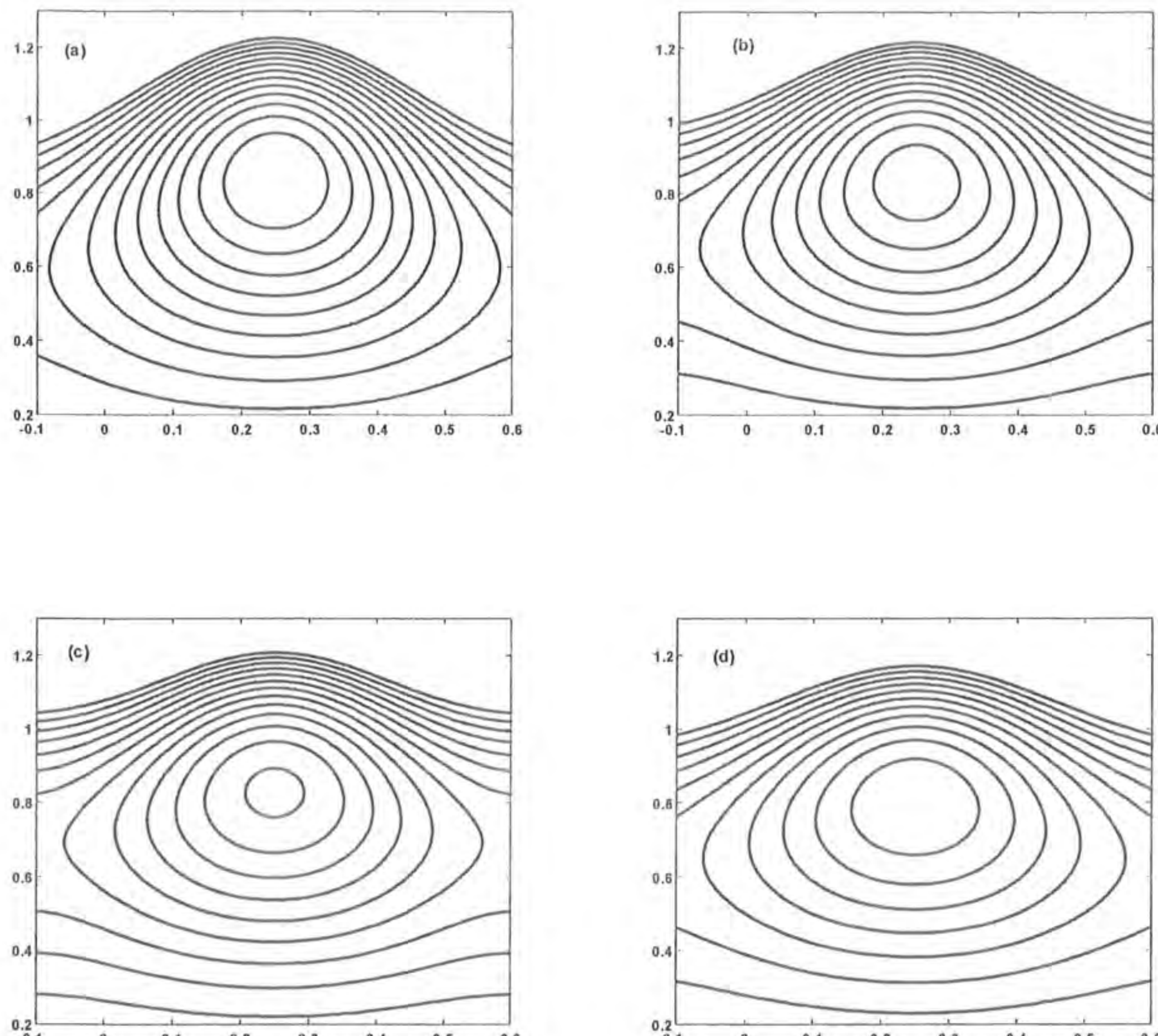
The friction force $F_{P\lambda}$ is plotted in Figure 3.3 $(B)$and 3.4 for different values of viscosity parameter β, Hartmann number M as well as for the different amplitude ratio ϕ. Increase in β results in the decrease of friction force and increase in magnetic parameter M decreases the friction in the range $0 \leq \sigma \leq 0.13$ and increases for $\sigma > 0.13$. The effects of the increase in amplitude ratio ϕ of the peristaltic wave results in the decrease of the friction force $F_{P\lambda}$ for $0 \leq \sigma \leq 0.85$. Overall, we observe that friction force $F_{P\lambda}$ has the opposite behavior as compared to pressure rise ΔP_λ.Figures 3.5, 3.6 represents the temperature and velocity profiles for different range of influential system parameters. The temperature field θ is plotted for different values of heat absorption parameter β_1. It is found that with increasing β_1, temperature field increases. Further the temperature field is maximum at the inlet region of the tube $(r = 0)$.

The variation of the axial velocity is plotted against the radial coordinate r. The effect of variable viscosity parameter β is found to have a significant effect on the velocity profile. It is observed from the results that the velocity is maximum at $(r = 0)$ and it increases with the increase in β upto $r = 0.9$ and later on it decreases.

The influence of Hartmann number M and free convection parameter G_r is also studied. Figure 3.6 shows that with the increase in M the velocity decreases upto $r = 0.9$ and velocity is positive. After $r = 0.9$ the velocity becomes negative and we see the opposite effect of increasing Hartmann number. Further, the velocity is maximum for low Hartmann number. The effects of free convection parameter G_r is also given in Figure 3.6. It is observed from the figure that with the increase in G_r, results in the increase, of the axial velocity.

3.7 Trapping

Trapping is an interesting phenomenon in peristaltic motion. It is basically the formation of an internally circulating bolus of fluid by closed streamlines. This trapped bolus pushed a head along with the peristaltic wave. Figure 3.7(i) illustrates the streamline graphs for different values of time mean flow rate σ. It is observed that the size of trapped bolus decreases by increasing mean flow rate. It is also observed that the number of trapped bolus decreases by increasing flow rate. The streamlines for different values of amplitude ratio ϕ are shown in Figure 3.7(ii). It is evident from the figure that the trapped bolus increases by increasing the amplitude ratio. The effects of heat absorption parameter β_1 is illustrated in Figure 3.8(i). It is depicted that increasing β_1 the size of the bolus increases. Figure 3.8 (ii) shows the effects of the free convection parameter, G_r with a given fixed set of the other parameters. By increasing G_r the size of bolus increases. Figure 3.9(i) shows the effects of Hartmann number M. When we increase M the size of trapped bolus increases. To see the effects of viscosity paremeter β, Figure 3.9(ii) is plotted. It is observed that when we increase β, the size of trapped bolus increases


(a)
(b)
(c)
(d)

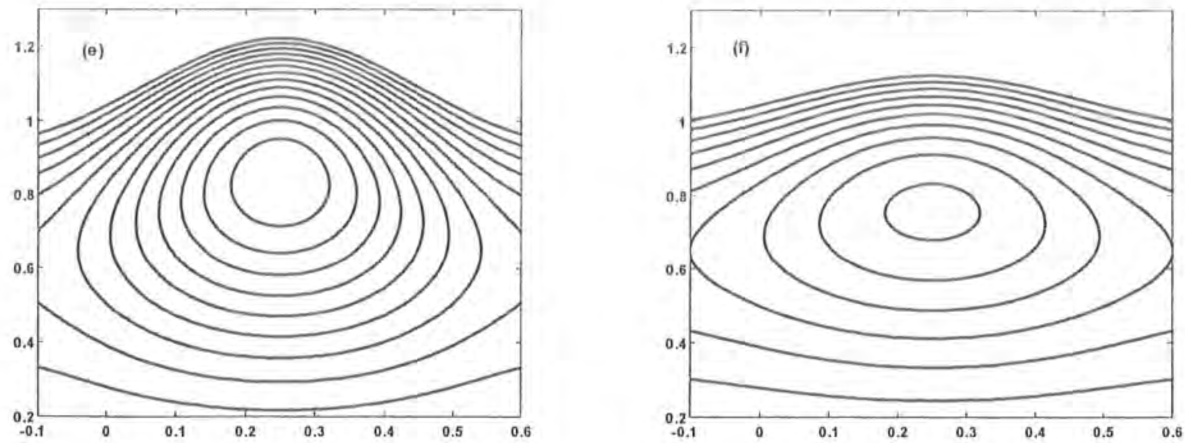
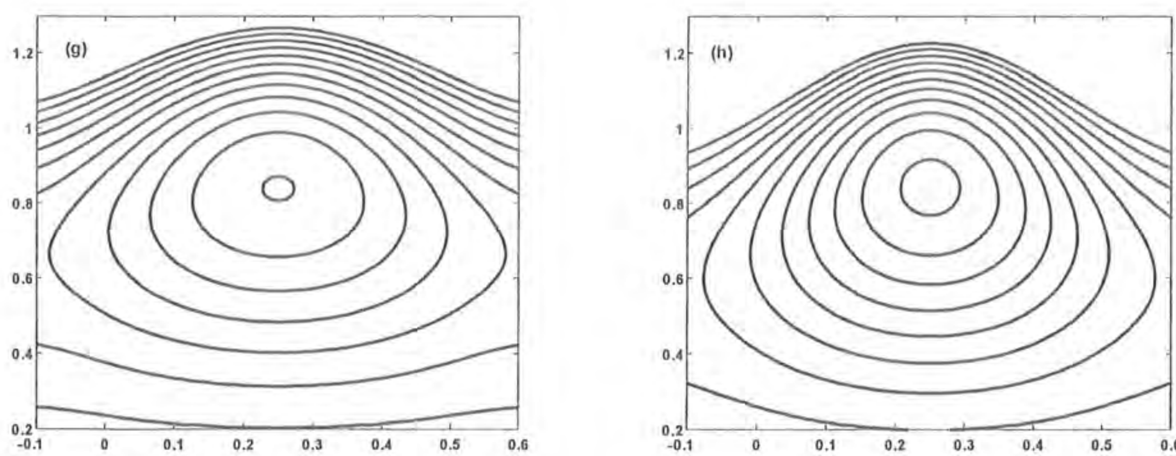


Figure 3.7(i). Streamlines for different values of $\sigma = .8, 1, 1.2$,(panels (a) to (c)) The other parameters are $\phi = 0.4$, $\beta_1 = 3$, $G_r = 0.6$, $M = 1.5$, $\beta = 0.4$. Figure 3.7(ii) . Streamlines for different values of $\phi = 0.2,\ 0.3,\ 0.4$,(panels (d) to (f)) The other parameters are $\sigma = 0.9$, $\beta_1 = 3$, $G_r = 0.6$, $M = 1.5$, $\beta = 0.4$.



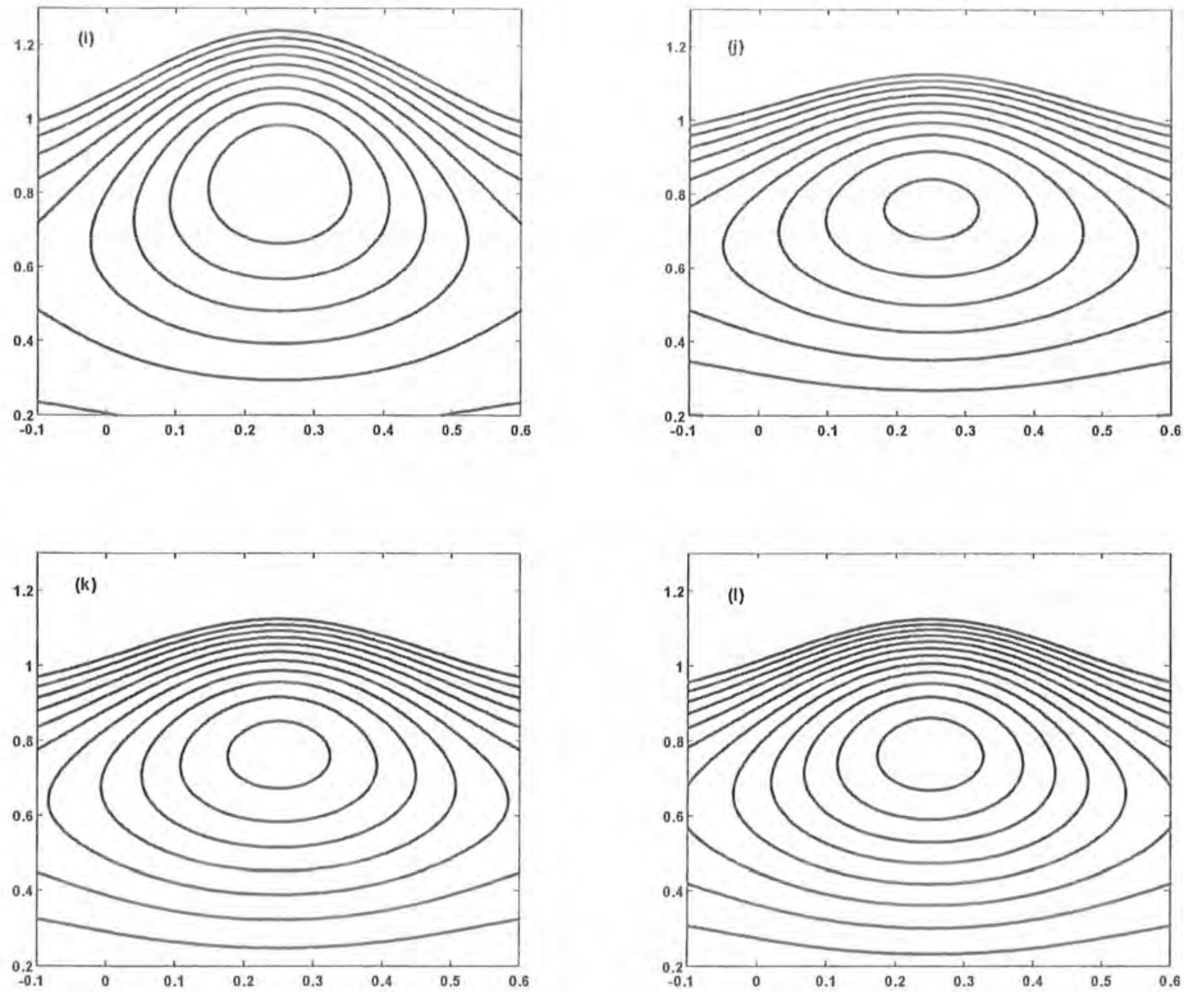
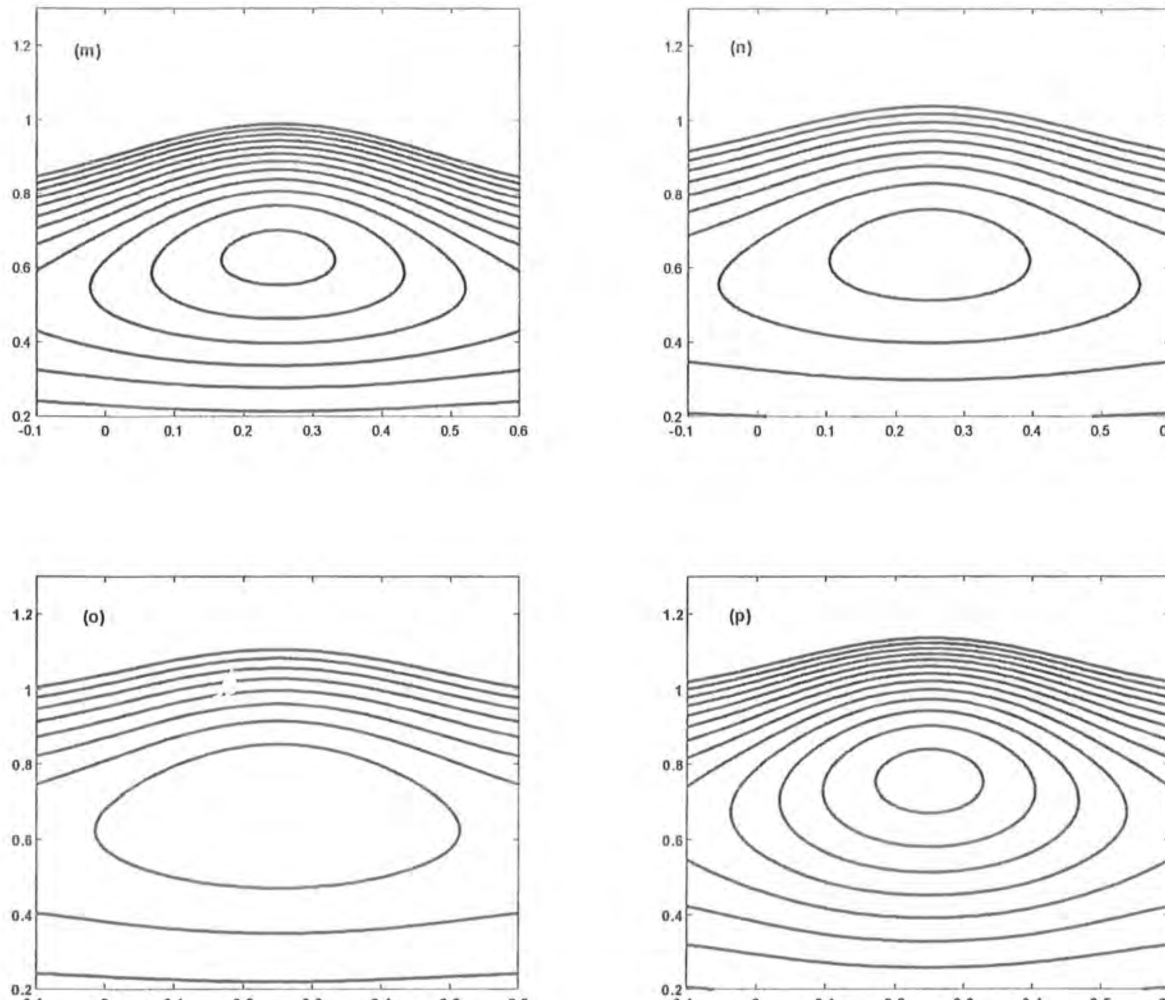


Figure 3.8(i). Streamlines for different values of $\beta_1 = 2, 2.5, 3$,(panels (g) to (i)) The other parameters are $\phi = 0.4$, $\sigma = 0.8$, $G_r = 0.6$, $M = 1.5$, $\beta = 0.4$. Figure 3.8(ii) . Streamlines for different values of $G_r = 0.7,\ 0.8,\ 0.9$,(panels (j) to (l)) The other parameters are $\sigma = 0.9$, $\beta_1 = 3$, $\phi = 0.2$, $M = 1.5$, $\beta = 0.4$.


(m)
(n)
(o)
(p)
1.2
1
0.8
0.6
0.4
0.2
-0.1
0
0.1
0.2
0.3
0.4
0.5
0.6

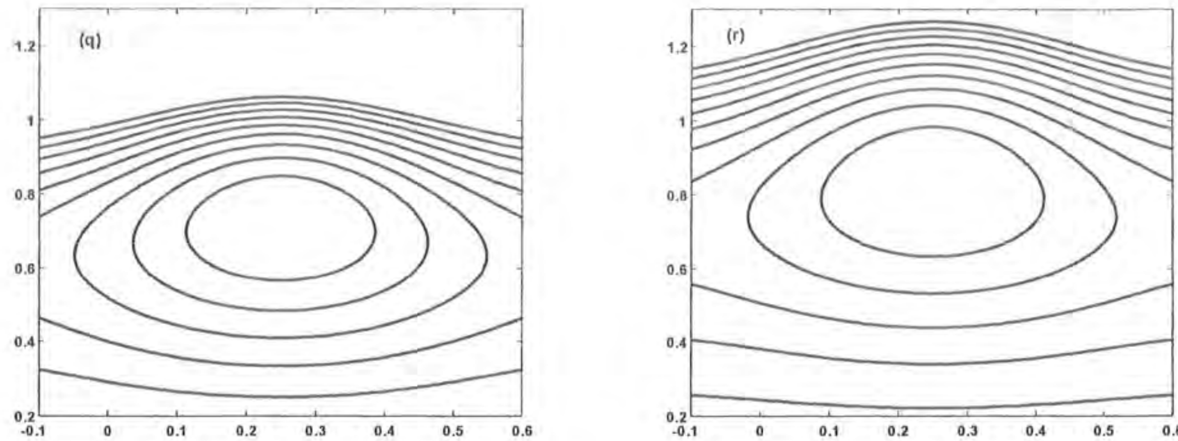


Figure 3.9(i). Streamlines for different values of $M = 0.8, 0.9, 1$,(panels (m) to (o)) The other parameters are $\phi = 0.1$, $\sigma = 1$, $G_r = 0.7$, $\beta_1 = 3$, $\beta = 0.5$. Figure 3.9(ii). Streamlines for different values of $\beta = 0.4, 0.5, 0.6$,(panels (p) to (r)) The other parameters are $\sigma = 0.9$, $\beta_1 = 3$, $\phi = 0.2$, $M = 1.5$, $G_r = 0.6$.

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